and Abstract Applied Analysis

Special Issue

Topological and Variational Methods of Nonlinear Analysis and Their Applications

Guest Editors

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Contents

Editorial

TOPOLOGICAL AND VARIATIONAL METHODS OF NONLINEAR ANALYSIS AND THEIR APPLICATIONS

V. G. ZVYAGIN, YU. E. GLIKLIKH, AND V. V. OBUKHOVSKII

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This issue contains the papers selected from the talks of the international conference "Topological and Variational Methods of Nonlinear Analysis and their Applications" dedicated to the 85th Jubilee of Professor A. D. Myshkis and the 75th Jubilee of Professor Yu. G. Borisovich. The conference took place in Voronezh, Russia, 27 June–2 July, 2005.

Professor Anatoly Dmitrievich Myshkis is the founder of a number of new scientific directions in the theory of functional differential equations (1949–1951), partial differential equations, differential inclusions and multivalued dynamical systems, and many others. In particular, he was one of the first researchers who studied retarded-type equations, he introduced the notion of a generalized solution for a differential equation with set-valued discontinuous right-hand side, his studies of set-valued maps with aspheric values (1954) found in the recent decades very important and interesting applications in the theory of differential equations, inclusions and control systems, and so forth. For the series of papers in the theory of set-valued differential equations, he was awarded a prize by the Moscow Mathematical Society.

In his activities, A. D. Myshkis pays special attention to the problems of applications. It is also worth pointing out his contribution to approximate and numerical methods, difference equations and inequalities, turbulent systems, impulse impact systems, spectral problems with variable boundary, and his analysis of the influence of velocity forces on oscillatory stability. A. D. Myshkis pays much attention to the methodology of applied mathematics and in his works he expressed his original views about how engineers and other specialists should be taught mathematics.

The perennial educational work of A. D. Myshkis prompted him to write several textbooks in mathematics and mathematical physics. These textbooks became very popular and were translated into many languages.

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A. D. Myshkis is a member of editorial boards of well-known international journals such as "Nonlinear Analysis: Theory, Methods and Applications," "Functional Differential Equations," and "Journal of Difference Equations and Applications."

Professor Yuri Grigorievich Borisovich is worldwide known by his works on topological methods in nonlinear analysis and their applications to mathematical physics, control theory, geometry, and many other branches of mathematics. First we should mention a series of his works in the 50th, and 60th, of twentieth century on relative topological degree and relative rotation of vector fields that yielded the development of the degree theory for weakly continuous mappings in Banach spaces, for condensing operators (k-set contractions), and applications to functional-differential equations, partial differential equations, and so forth.

In the series of works with his collaborators in the 70th, a new version of degree theory for nonlinear Fredholm mappings was suggested that allowed one to cover Fredholm mappings with compact and condensing perturbations. This theory got plenty of applications, first of all to partial differential equations.

Another topic, where Professor Borisovich's influence is very well known, is the theory of set-valued mappings and differential inclusions. A lot of research and survey papers (in particular, in Russian Mathematical Surveys) and two monographs (joint with B. Gel'man, A. D. Myshkis, and V. Obukhovskii; the last one in 2005) were published by him and his collaborators on this subject.

Borisovich's text book "Introduction to Topology" (joint with N. Bliznyakov, T. Fomenko, and Y. Izrailevich) has been translated into many foreign languages and is one of the best introductory books on this subject.

In this issue we include papers on the themes where the ideas of variational and topological methods are applied. Here A. D. Myshkis and Yu. G. Borisovich made significant input. It is some parts of topology, topological index theory, functional analysis, global analysis and analysis on manifolds, stochastic analysis, hydrodynamics, and so forth.

V. G. Zvyagin
Yu. E. Gliklikh
V. V. Obukhovskii

THE KOLMOGOROV EQUATION IN THE STOCHASTIC FRAGMENTATION THEORY AND BRANCHING PROCESSES WITH INFINITE COLLECTION OF PARTICLE TYPES

R. YE. BRODSKII AND YU. P. VIRCHENKO

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The stochastic model for the description of the so-called fragmentation process in frameworks of Kolmogorov approach is proposed. This model is represented as the branching process with continuum set $(0,\infty)$ of particle types. Each type $r \in (0,\infty)$ corresponds to the set of fragments having the size r. It is proved that the branching condition of this process represents the basic equation of the Kolmogorov theory.

1. Introduction

There are various natural processes that represent the evolution during time of solid state media in the form of some successive subdivisions of all its connected parts to smaller parts having random forms and volumes, and, consequently, masses and/or chemical compositions. In statistical physics they are called the fragmentation processes. It is clear that such processes may have an adequate mathematical description only on the basis of some concepts of probability theory. Notice also that even the description of each separate random state of such a physical system, that is, the construction of the space Ω of elementary events, meets with some fatal troubles. Moreover, it is not clear what principles are necessary to use in order to construct adequate stochastic dynamics in the form of a random process in the space Ω. On the other hand, it seems unreasonable to think that the models of the great variety of physical fragmentation processes may be done on the basis of some relatively simple probabilistic scheme.

In the initial work of Kolmogorov on statistical fragmentation theory (Kolmogorov [2]), an approach to probabilistic description of fragmentation processes is proposed. It is based on the use of states characterizing the dynamical subdivision system at each specified time instant t by a random function $\tilde{N}(r,t)$ that takes values only in $\mathbb{N}_+$ and depends only on the unique nonnegative parameter r, which we will further call the

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fragment size. Each value of this function represents the number of fragments at time instant t with sizes being not greater than r. Therefore, in the framework of this approach, the mathematical model of fragmentation is represented by a random process $\{\widetilde{N}(r,t);\ t \in \mathbb{R}_+ = (0,\infty)\}$, $r \in \mathbb{R}_+$ with values in $\mathbb{R}_+^{\mathbb{N}_+}$. In Kolmogorov [2] a simple evolution equation for mathematical expectations $\mathbf{E}\widetilde{N}(r,t)$ (we consider the discrete time case) is derived. It has Markovian type and is constructed in terms of mathematical expectation $\mathbf{E}\widetilde{\nu}(r \mid r';t)$ of the other random function $\widetilde{\nu}(r \mid r';t) : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{N}_+ \mapsto \mathbb{N}_+$ that is the random number of fragments with sizes not greater than r and generated at time instant t from a specified randomly chosen fragment having the size r'. This equation has the following form:

$$\mathbf{E}\widetilde{N}(r,t+1) = \int_0^1 \mathbf{E}\widetilde{N}\left(\frac{r}{k},t\right) dS(k,t) \tag{1.1}$$

under the assumption that the function $\mathbf{E}\widetilde{\nu}(r \mid r';t) \equiv S(k,t)$ depends only on the fraction $k = r/r'$. Thus, the model formulated in Kolmogorov [2] is obtained on the basis of some phenomenological reasons as it is said in physical literature. These reasons are based on the concept of "the average field" that is often in use in statistical physics. Further, in Kolmogorov [2], it is proved that the integral limit theorem for the distribution function $\mathbf{E}\widetilde{N}(r,t)/\mathbf{E}\widetilde{N}(\infty,t)$ takes place under the assumption that the function $\mathbf{E}\widetilde{\nu}(k,t)$ does not depend on time and that its second "logarithmi" moment in the variable k is finite. It may be considered as the marginal one-dimensional probabilistic distribution of the random process $\{\widetilde{r}(t);\ t \in [0,\infty)\}$, with nonnegative trajectories $\widetilde{r}(t)$ and, physically, as the size of randomly chosen fragment from the whole system at time t.

Here we will not discuss the physical question of applicability of the abovementioned approach of the mathematical modelling to some real physical fragmentation processes. Our problem consists of the ground of (1.1) on the basis of an explicit construction of the random process $\{\widetilde{N}(r,t);\ t \in [0,\infty)\}$. The idea of such a ground has been stated in the cited work. But it seems that the consequent authors (see, e.g., the fundamental work (Filippov [1])) have not taken into account the great importance of this idea to realize it. From our point of view such an explicit construction of the mathematical model of a higher level, in the frameworks of which the main master (1.1) of Kolmogorov theory can be proved as a mathematical statement, may represent the important base for constructing more complicated (and, therefore, more adequate) models in the fragmentation theory.

2. Mathematical model description

Specify a number $\Delta > 0$. Further, divide the positive part $[0,\infty)$ of the real line into the sequence of disjoint half-open intervals $\langle [i\Delta,\ (i+1)\Delta);\ i \in \mathbb{N}_+\rangle$ being open from the right. Their union coincides with $[0,\infty)$. Introduce the random process $\mathfrak{N}_\Delta$ with discrete time and with values in the set $\mathbb{N}_+^{\mathbb{N}_+}$. The sampling space of this process consists of some random collections of functions $\{\langle \widetilde{\nu}_i(t);\ i \in \mathbb{N}_+\rangle;\ t \in \mathbb{N}_+\}$. Each function takes its values in $\mathbb{N}_+$. By its sense, each function $\widetilde{\nu}_i(t)$, $i = 0,1,2,\ldots$, represents the number of fragments, having random sizes, that belong to half-interval $[i\Delta,(i+1)\Delta)$. Define the process $\mathfrak{N}_\Delta$ as the Markov branching one with discrete time (see Sevast'ianov [3]) (the Markov chain). Generally speaking, it is inhomogeneous in time. Besides, it has an infinite collection $\mathbb{N}_+$ of *particle types*. The last words are taken from the terminology of branching random process theory. In our problem fragments with specified size r are the particles of some definite type from the point of view of this terminology.

Since the countable set $\mathbb{N}_+^{\mathbb{N}_+}$ is the process state space, then for each time instant $t \in \mathbb{N}_+$ the conditional probabilities

$$Q(m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+;t) = \Pr\{\widetilde{\nu}_j(t+1) = n_j,\ j \in \mathbb{N}_+ \mid \widetilde{\nu}_i(t) = m_i,\ i \in \mathbb{N}_+\} \tag{2.1}$$

of transitions form an infinite matrix when arguments $n_j, m_i \in \mathbb{N}_+, i,\ j \in \mathbb{N}_+$ are changed. The matrix (2.1) of transition conditional probabilities defines completely the Markov chain with countable set of states. In particular, it defines the evolution of one-dimensional marginal probability distribution of this chain

$$P(n_i, i \in \mathbb{N}_+;t) = \Pr\{\widetilde{\nu}_i(t) = n_i, i \in \mathbb{N}_+\}, \tag{2.2}$$

namely, it is defined uniquely by the Markov chain equation

$$P(n_j,\ j \in \mathbb{N}_+;t+1) = \sum_{\{m_i\}} P(m_i,\ i \in \mathbb{N}_+;t)Q(m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+;t), \tag{2.3}$$

where, here and below, the symbol of summation means that it is done with respect to all possible distributions of "filling numbers," that is, with respect to all collections $\langle m_i, i \in \mathbb{N}_+\rangle \in \mathbb{N}_+^{\mathbb{N}_+}$. For the matrix $Q(m_i, i \in \mathbb{N}_+ \mid n_j, j \in \mathbb{N}_+;t), n_j, m_i \in \mathbb{N}_+ i, j \in \mathbb{N}_+$, we will use also the shorter notation $Q(m_i \mid n_j;t)$. It is constructed for the Markov branching process by the following way. Define the function $q_l(k_j, j \in \mathbb{N}_+;t) \equiv q_l(k_j;t)$. It represents the probability of the event that describes the fact that a specified fragment with size l (i.e., its size r belongs to the half-interval $[l\Delta,(l+1)\Delta)$) gets at the time instant t to the set of fragments and this set is characterized by the collection of filling numbers $\langle k_j; j \in \mathbb{N}_+\rangle$. In this case, of course, this probability is not zero only if $k_j = 0$ at $j > l$. Thus, $q_l(k_j, j \in \mathbb{N}_+;t)$ is the probability of the fact that the random function $\widetilde{\nu}_{l,j}(t) : \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N}_+ \mapsto \mathbb{N}_+$ takes value k_j. The function is the number of fragments with sizes j that are formed from the specified fragment with size l at the time instant t; here, the second argument j is not greater than l. Further, we introduce the random function $\widetilde{\eta} : \mathbb{N}_+ \times \mathbb{N}_+ \times \mathbb{N} \times \mathbb{N}_+ \mapsto \mathbb{N}_+, \widetilde{\eta}_{l,j}(m;t) = \widetilde{\nu}^{(1)}_{l,j}(t) + \widetilde{\nu}^{(2)}_{l,j}(t) + \cdots + \widetilde{\nu}^{(m)}_{l,j}(t)$ for each pair $l, j \in \mathbb{N}_+$. It is the sum of $m \in \mathbb{N}$ statistically independent random functions $\widetilde{\nu}^{(1)}_{l,j}(t), \widetilde{\nu}^{(2)}_{l,j}(t), \ldots, \widetilde{\nu}^{(m)}_{l,j}(t)$ and it represents the set of filling numbers on sizes j of fragments formed by subdivision from m identical fragments having the size l at the time instant t. In such a definition of *the branching condition* that describes the disintegration of fragments having the size l, the individuality of each fragment is lost, that is, for each fixed fragment in the final state we do not take into account the fact, from which fragment of the size l appeared as a result of the disintegration. Due to the given definition of the random function $\tilde{\eta}_l(m, k_j,\ j \in \mathbb{N}_+; t)$, its probability distribution $q_l(m \mid k_j,\ j \in \mathbb{N}_+; t)$ is defined by the m-multiple convolution of the probability distribution collection $q_l(k_j^{(i)},\ j \in \mathbb{N}_+; t)$, $i = 1, \dots, m$,

$$q_l(m \mid k_j,\ j \in \mathbb{N}_+; t) = \sum_{\substack{k_j^{(i)} \geq 0, i=1,\dots,m, \\ k_j^{(1)} + \cdots + k_j^{(m)} = k_j, j \in \mathbb{N}_+}} \prod_{i=1}^{m} q_l\left(k_j^{(i)},\ j \in \mathbb{N}_+; t\right). \tag{2.4}$$

Indeed, the probability $q_l(m \mid k_j, j \in \mathbb{N}_+; t)$ is equal to zero if there exists $j \in \mathbb{N}_+,\ j > l$ such that the inequality $k_j \neq 0$ is valid.

At last, the matrix $Q(m_i \mid n_j; t)$ is determined by the formula

$$\begin{aligned} &Q(m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+; t) \\ &\quad = \sum_{k_{ij} \geq 0; i, j \in \mathbb{N}_+} \left[\prod_{i=0}^{\infty} q_i(m_i \mid k_{il},\ l \in \mathbb{N}_+; t)\right]\left[\prod_{j=0}^{\infty} \delta\left(n_j - \sum_{l: l \geq j} k_{lj}\right)\right], \end{aligned} \tag{2.5}$$

where $\delta(n - n') \equiv \delta_{n,n'}$ is the Kronecker symbol and the summation is done on all two-placed functions $k_{ij} : \mathbb{N}_+ \times \mathbb{N}_+ \mapsto \mathbb{N}_+$. The sense of the integer matrix is that it determines the fragment numbers with the size j that are formed from all fragments with size i.

The matrix $Q(m_i \mid n_j; t)$ and the probability distribution $P(n_j, j \in \mathbb{N}_+; 0)$ determine the random process $\mathfrak{N}_\Delta$ completely as well as (in particular) its characteristic functional $\Psi_\Delta[u] : \mathbb{S}_\infty^{\mathbb{N}_+}(\mathbb{R}_+) \mapsto \mathbb{C}$, the value of which is determined as

$$\Psi_\Delta[u] = \mathbf{E}\exp\left(i \sum_{t=0}^{\infty} \sum_{j=0}^{\infty} \tilde{\nu}_j(t) \int_{j\Delta}^{(j+1)\Delta} u_t(x)\,dx\right) \tag{2.6}$$

for each function sequence $u_t(x)$, $t = 0, 1, 2, \dots$ from the space $\mathbb{S}_\infty(\mathbb{R}_+)$ of compactly supported functions being infinitely differentiable on $\mathbb{R}_+$. Values of the functional exist due to the support compactness in x of the functions $u_t(x)$.

Now we give the definition of the random process $\mathfrak{N}$ with values in $\mathbb{R}_+^{\mathbb{N}_+}$ that describes the fragmentation. We will define it as the generalized random process generated by the process sequence $\mathfrak{N}_\Delta$ at $\Delta \to 0$.

Definition 2.1. Generalized random process $\mathfrak{N}$ with the characteristic functional $\Psi[u]$, determined by the limit

$$\Psi[u] = \lim_{\Delta\to 0} \Psi_\Delta[u] \tag{2.7}$$

for each function $u_t(x) \in \mathbb{S}_\infty^{\mathbb{N}_+}(\mathbb{R}_+)$, is called the random Kolmogorov fragmentation process.

3. Equation for the generating function

Introduce the space $\mathbb{S}_\infty(\mathbb{N}_+)$ of infinite bounded sequences where each of them has zero components beginning from a number. Further, we will imply that such sequences X have only nonnegative components. The set of all those sequences forms the cone in $\mathbb{S}_\infty(\mathbb{N}_+)$.

We also introduce the sequence $\mathbf{G}[X,\ t] = \langle g_l[X,t];\ l \in \mathbb{N}_+\rangle$ whose components are generating functions of probability distributions $q_l(k_j;t)$, $l \in \mathbb{N}_+$,

$$g_l[X,t] = \sum_{\{k_j\}} \left(\prod_{j=0}^{\infty} x_j^{k_j}\right) q_l(k_j, j \in \mathbb{N}_+;t). \tag{3.1}$$

Formally, they are functions of countable set of variables. However, due to the variable $\langle k_j; j \in \mathbb{N}_+\rangle$ in the probability distribution is a finite sequence, really, the function $g_l[X,t]$ depends only on finite components in X. Each lth function depends on l variables where l is determined by the maximal number j among nonzero components in $k_j \neq 0$.

Now compute the sums

$$\begin{aligned}
h_l^{(n)}[X,t] &= \sum_{\{k_j\}} \left(\prod_{j=0}^{\infty} x_j^{k_j}\right) q_l(n \mid k_j,\ j \in \mathbb{N}_+;t) \\
&= \sum_{\{k_j\}} \left(\prod_{j=0}^{\infty} x_j^{k_j}\right) \sum_{\substack{k_j^{(i)} \geq 0, i=1,\dots,n, \\ k_j^{(1)}+\cdots+k_j^{(n)}=k_j, j\in\mathbb{N}_+}} \prod_{i=1}^{n} q_l\left(k_j^{(i)},\ j \in \mathbb{N}_+;t\right) \\
&= \sum_{\{k_j\}} \sum_{\substack{k_j^{(i)} \geq 0, i=1,\dots,n, \\ k_j^{(1)}+\cdots+k_j^{(n)}=k_j, j\in\mathbb{N}_+}} \prod_{i=1}^{n} \left(\prod_{j=0}^{\infty} x_j^{k_j^{(i)}}\right) q_l\left(k_{j'}^{(i)},\ j' \in \mathbb{N}_+;t\right) \\
&= \sum_{k_j^{(i)} \geq 0; i=1,\dots,n, j\in\mathbb{N}_+} \prod_{i=1}^{n} \left(\prod_{j=0}^{\infty} x_j^{k_j^{(i)}}\right) q_l\left(k_{j'}^{(i)},\ j' \in \mathbb{N}_+;t\right) \\
&= \prod_{i=1}^{n} \left[\sum_{\{k_j^{(i)}\}} \left(\prod_{j=0}^{\infty} x_j^{k_j^{(i)}}\right) q_l\left(k_{j'}^{(i)}, j' \in \mathbb{N}_+;t\right)\right] = \prod_{i=1}^{n} g_l[X,t] = g_l^n[X,t].
\end{aligned} \tag{3.2}$$

Finally, compute the sum

$$
\begin{aligned}
& h[m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+; X, t] \\
&\quad = \sum_{\{n_j\}} \left(\prod_{j=0}^{\infty} x_j^{n_j} \right) Q(m_i,\ i \in \mathbb{N}_+ \mid n_{j'},\ j' \in \mathbb{N}_+; t) \\
&\quad = \sum_{\{n_j\}} \sum_{k_{ij} \geq 0;\ i,j \in \mathbb{N}_+} \left[\prod_{j=0}^{\infty} x_j^{n_j} \delta\left(n_j - \sum_{l: l \geq j} k_{lj} \right) \right] \left[\prod_{i=0}^{\infty} q_i(m_i \mid k_{il}, l \in \mathbb{N}_+; t) \right] \\
&\quad = \sum_{k_{ij} \geq 0;\ i,j \in \mathbb{N}_+} \left[\prod_{i=0}^{\infty} \left(\prod_{j=0}^{\infty} x_j^{k_{ij}} \right) q_i(m_i \mid k_{il}, l \in \mathbb{N}_+; t) \right] \\
&\quad = \prod_{i=0}^{\infty} \left[\sum_{k_{ij} \geq 0;\ j \in \mathbb{N}_+} \left(\prod_{j=0}^{\infty} x_j^{k_{ij}} \right) q_i(m_i \mid k_{il},\ l \in \mathbb{N}_+; t) \right] \\
&\quad = \prod_{i=0}^{\infty} h_i^{(m_i)}[X, t] = \prod_{i=0}^{\infty} g_i^{m_i}[X, t],
\end{aligned}
\tag{3.3}
$$

where we use the rule

$$
\prod_{j=0}^{\infty} x_j^{n_j} = \prod_{j=0}^{\infty} \prod_{i: i \geq j} x_j^{k_{ij}} = \prod_{i=0}^{\infty} \prod_{j: i \geq j} x_j^{k_{ij}},
\tag{3.4}
$$

and also we take into account that probabilities $q_i(m_i \mid k_{il},\ l \in \mathbb{N}_+; t)$ are not zero only if $k_{ij} = 0$ at $i < j$.

After these preparatory computations, introduce the generating function $\mathbf{H}_t[X]$ of the one-dimensional probability distribution $P(n_j,\ j \in \mathbb{N}_+; t)$ of the Markov chain at the time t according to the formula

$$
\mathbf{H}_t[X] = \sum_{\{n_j\}} \left(\prod_{j=0}^{\infty} x_j^{n_j} \right) P(n_j, j \in \mathbb{N}_+; t).
\tag{3.5}
$$

Then, applying the operation $\sum_{\{n_j\}} (\prod_{j=0}^{\infty} x_j^{n_j})$ to equation of motion (2.3) and using (3.3), we find the motion equation of the generating function $\mathbf{H}_t[X]$,

$$
\begin{aligned}
\mathbf{H}_{t+1}[X] &= \sum_{\{n_j\}} \left(\prod_{j=0}^{\infty} x_j^{n_j} \right) \sum_{\{m_i\}} P(m_i,\ i \in \mathbb{N}_+; t) Q(m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+; t) \\
&= \sum_{\{m_i\}} P(m_i,\ i \in \mathbb{N}_+; t) \sum_{\{n_j\}} \left(\prod_{j=0}^{\infty} x_j^{n_j} \right) Q(m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+; t) \\
&= \sum_{\{m_i\}} P(m_i,\ i \in \mathbb{N}_+; t) h[m_i,\ i \in \mathbb{N}_+ \mid n_j,\ j \in \mathbb{N}_+; X, t] \\
&= \sum_{\{m_i\}} P(m_i, i \in \mathbb{N}_+; t) \left(\prod_{i=0}^{\infty} g_i^{m_i}[X, t] \right) = \mathbf{H}_t[\mathbf{G}[X, t]],
\end{aligned}
\tag{3.6}
$$

where $\mathbf{G}[X, t] = \langle g_l[X, t]; l \in \mathbb{N}_+ \rangle$.

Thus, we have proved the following theorem.

THEOREM 3.1. *Generating function* $\mathbf{H}_t[X]$ *of the probability distribution* $P(n_j, j \in \mathbb{N}_+; t)$ *is governed by the equation*

$$\mathbf{H}_t[X] = \mathbf{H}_t[\mathbf{G}[X,t]] \tag{3.7}$$

that together with the initial condition $\mathbf{H}_0[X]$ *completely determine this distribution.*

4. Kolmogorov's master equation

On the basis of (3.7), we now obtain the evolution equation of mathematical expectations for the random process $\mathfrak{N}_\Delta$. For this we introduce the matrix $s_{lj}(t) = \mathbf{E}\widetilde{\nu}_{lj}(t)$ of mathematical expectations whose matrix elements are distinguished from zero only at $j \le l$. It is defined by the formula

$$s_{lj}(t) = \sum_{k_j=0}^{\infty} k_j q_l(k_{j'}, j' \in \mathbb{N}_+; t) = \left(\frac{\partial g_l[X,t]}{\partial x_j}\right)_{X\equiv 1}. \tag{4.1}$$

Further, the mathematical expectation $n_l(t) = \mathbf{E}\widetilde{\nu}_l(t)$ of the number $\widetilde{\nu}_l(t)$ of fragments with the size l at the time instant t is defined by the generating function $\mathbf{H}_t[X]$ by means of its partial derivative in x_l at the point $X \equiv 1$,

$$n_l(t) = \mathbf{E}\widetilde{\nu}_l(t) = \left(\frac{\partial \mathbf{H}_t[X]}{\partial x_l}\right)_{X\equiv 1}. \tag{4.2}$$

Then, on the basis of (3.7) and (4.1), we find

$$n_l(t+1) = \left(\frac{\partial \mathbf{H}_{t+1}[X]}{\partial x_l}\right)_{X\equiv 1} = \sum_{m=l}^{\infty} \left(\frac{\partial \mathbf{H}_t[\mathbf{G}[X,t]]}{\partial g_m[X,t]}\right)_{X\equiv 1} \left(\frac{\partial g_m[X,t]}{\partial x_l}\right)_{X\equiv 1}, \tag{4.3}$$

that is,

$$n_l(t+1) = \sum_{m=l}^{\infty} n_m(t) s_{ml}(t). \tag{4.4}$$

Now introduce the functions

$$N_l(t) = \sum_{k=0}^{l} n_k(t), \qquad S_{ml}(t) = \sum_{k=0}^{l} s_{mk}(t). \tag{4.5}$$

Then, by summing up (4.4) for all l, we derive the motion equation in terms of this function

$$\begin{aligned}
N_l(t+1) &= \sum_{k=0}^{l}\sum_{m=k}^{\infty} n_m(t)s_{mk}(t)\\
&= \sum_{k=0}^{l-1}\sum_{m=k}^{l-1} n_m(t)s_{mk}(t)+\sum_{k=0}^{l}\sum_{m=l}^{\infty} n_m(t)s_{mk}(t)\\
&= \sum_{m=0}^{l-1} n_m(t)\sum_{k=0}^{m} s_{mk}(t)+\sum_{m=l}^{\infty} n_m(t)S_{ml}(t)\\
&= \sum_{m=0}^{l-1} S_{mm}(t)\big[N_m(t)-N_{m-1}(t)\big]+\sum_{m=l}^{\infty} S_{ml}(t)\big[N_m(t)-N_{m-1}(t)\big],
\end{aligned} \tag{4.6}$$

where $N_{-1}(t)=0$.

At last, introduce the function $N_\Delta(r;t):\mathbb{R}_+\times\mathbb{N}_+\mapsto\mathbb{R}_+$,

$$N_\Delta(r;t)=N_l(t), \quad \text{if } r\le l\Delta<r+\Delta. \tag{4.7}$$

It is continuous from the left and it is equal to the average fragment number having sizes not greater than r. Besides, introduce the function $S_\Delta(r,r';t):\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{N}_+\mapsto\mathbb{R}_+$,

$$S_\Delta(r,r';t)=S_{ml}(t), \quad \text{if } r\le l\Delta<r+\Delta,\ r'\le m\Delta<r'+\Delta \tag{4.8}$$

being continuous from the left in both arguments r and r'. Then for $(l-1)\Delta<r\le l\Delta$, it follows from (4.6) that

$$\begin{aligned}
N_\Delta(r;t+1) &= \sum_{m=0}^{l-1} S_\Delta(r_m,\ r_m;t)\big[N_\Delta(r_m+\Delta;t)-N_\Delta(r_m;t)\big]\\
&\quad+\sum_{m=l}^{\infty} S_\Delta(r_m,\ r;t)\big[N_\Delta(r_m+\Delta;t)-N_\Delta(r_m;t)\big],
\end{aligned} \tag{4.9}$$

where $r_m=m\Delta$. The sums in the right-hand side of this equality may be considered as integral sums of the Riman-Stiltyes integral for step functions $N_\Delta(r;t)$ and $S_\Delta(r',r;t)$, that is,

$$N_\Delta(r;t+1)=\int_0^{r-0} S_\Delta(r',\ r';t)dN_\Delta(r';t)+\int_{r-0}^{\infty} S_\Delta(r',r;t)dN_\Delta(r';t). \tag{4.10}$$

Assuming that the function $S_\Delta(r',r;t)$ tends to a continuous function $S(r',r;t)$ as $\Delta\to 0$ and the function $N_\Delta(r;t)$ tends to a monotone nondecreasing function $N(r,t)$ and since discontinuity points of functions $S_\Delta(r',r;t)$ and $N_\Delta(r',t)$ in the argument r' coincide for every t, we may apply the second Helly theorem. It permits to realize the limit transition under the integral sign. In this case, we obtain the equation for evolution of average fragment number distribution in the form

$$N(r;t+1) = \int_0^{r-0} S(r',r';t)dN(r';t) + \int_{r-0}^{\infty} S(r',r;t)dN(r';t). \qquad (4.11)$$

Thus, the following statement takes place.

THEOREM 4.1. *Statistical characteristic* $N(r,\ t) = \mathbf{E}\tilde{N}(r,t)$ *of the generalized random process* $\mathfrak{N}$ *is governed by (4.11).*

At last, we show that (1.1) of the Kolmogorov theory is a particular case of (4.11). We suppose that the function $S(r',r;t)$ depends only on the ratio r/r', that is, $S(r',r;t) = S(r/r';t)$. In this case (4.11) is represented in the form

$$N(r;t+1) = S(1;t)N(r-0;t) + \int_{r-0}^{\infty} S\left(\frac{r}{r'};t\right)dN(r';t). \qquad (4.12)$$

Applying the integration by parts with the use of conditions $N(\infty;t) < \infty, S(0;t) = 0$, we get

$$N(r;t+1) = \int_{r-0}^{\infty} N(r';t)dS\left(\frac{r}{r'};t\right). \qquad (4.13)$$

Introducing the integration variable $k = r/r'$, we obtain

$$N(r;t+1) = \int_0^{1+0} N\left(\frac{r}{k};t\right)dS(k;t), \qquad (4.14)$$

Unlike (1.1), the latter takes into account the fact that the function $S(k;t)$ may have a step at the point $k = 1$.

5. Conclusion

We have shown how the Kolmogorov equation in statistical fragmentation theory may be justified in the framework of a certain probabilistic scheme. At the same time, even in the framework of the construction presented in the work, some general mathematical questions have been still unsolved. For example, it is necessary to clear up under what conditions the limit distribution of probabilistic distributions $q_l(k_j, j \in \mathbb{N}_+;t)$ exists and how it should be understood. The simplest situation when we try to answer this question is when this limit should be understood in weak sense. However, it is desirable that this weak limit nevertheless guarantees the existence of random realizations with probability 1. They should be regarded as some finite point random sets on $\mathbb{R}_+$.

It is necessary to find some conditions for distributions $q_l(k_j, j \in \mathbb{N}_+;t)$ that guarantee the existence of the limit mathematical expectation $\lim_{\Delta\to 0}\sum_{j\in\mathbb{N}_+:j\Delta<r}\mathbf{E}\tilde{\nu}_{lj}(t)$ such that it is a continuous function $S(r',r;t)$.

Finally, it is very important to prove the existence of the limit characteristic functional $\Psi[u]$ and, moreover, the existence of random trajectories of the process connected with this functional.

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NECESSARY AND SUFFICIENT CONDITIONS FOR GLOBAL-IN-TIME EXISTENCE OF SOLUTIONS OF ORDINARY, STOCHASTIC, AND PARABOLIC DIFFERENTIAL EQUATIONS

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We derive necessary and sufficient conditions for global-in-time existence of solutions of ordinary differential, stochastic differential, and parabolic equations. The conditions are formulated in terms of complete Riemannian metrics on extended phase spaces (conditions with two-sided estimates) or in terms of derivatives of proper functions on extended phase spaces (conditions with one-sided estimates).

1. Introduction

This is a survey paper with complete proofs of results obtained in [6, 7, 9–11]. We derive necessary and sufficient conditions for global-in-time existence of solutions of ordinary, stochastic, and parabolic differential equations. They are obtained as modifications of some well-known sufficient conditions (both with one-sided and two-sided estimates). In particular those modifications involve transition to extended phase spaces. We consider the general case of equations on smooth manifolds (mainly finite-dimensional). For ordinary differential equations we get necessary and sufficient conditions of both two-sided and one-sided sorts (in the latter case we also get a generalization to a certain infinite-dimensional case). For stochastic differential and parabolic equations we obtain necessary and sufficient conditions of one-sided sort for some classes of equations on finite-dimensional manifolds.

Recall that if all solutions of Cauchy problems of an ordinary differential equation with a smooth vector field in the right-hand side on a finite-dimensional manifold M exist on the entire line $(-\infty,\infty)$, the vector field and its flow are called complete. Below the solutions to Cauchy problems will be called the orbits of the flow or the integral curves of the vector field.

If the manifold M is compact, all continuous (in particular, all smooth) vector fields are complete. Indeed, in this case any Riemannian metric on M is complete, any continuous vector field is bounded, hence any integral curve has bounded length on any finite

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interval, that is, it is relatively compact. Thus, the flow of a smooth vector field is a diffeomorphism of M onto M at any time instant belonging to $(-\infty,+\infty)$, that is, the flow is a flow of diffeomorphisms.

In the case of noncompact manifolds (in particular, in linear spaces) the integral curves can get to infinity within some finite time interval and the problem of the flow completeness becomes nontrivial. Analogous situation takes place also for stochastic differential and parabolic equations.

Plenty of sufficient conditions for completeness of the flows of ordinary differential equations in linear spaces are well known. There exist two sorts of such conditions: with two-sided estimates and with one-sided estimates. The former is formulated in terms of estimates on the norm of the right-hand side and guarantees the existence of all integral curves for $t \in (-\infty,+\infty)$. Let us present some examples.

Let $X(t,x)$ be a smooth vector field on R^n. Consider the differential equation

$$\dot{x}(t) = X(t,x(t)). \tag{1.1}$$

The simplest examples of conditions with two-sided estimates are

(i) $\|X(t,x)\| < \psi(t)$ at all $x \in R^n$ and $t \in (-\infty,\infty)$ for some function $\psi > 0$ that is integrable on any finite interval (boundedness);

(ii) $\|X(t,x)\| < \psi(t)(1+\|x\|)$ with analogous ψ (linear growth).

The Wintner's theorem proves the completeness under the following conditions: $\|X(t,x)\| < \psi(t)L(\|x\|)$ where $\psi > 0$ is as above and $L : [0,\infty) \to (0,\infty)$ is a continuous function such that

$$\int_0^\infty \frac{1}{L(u)}du = \infty. \tag{1.2}$$

On nonlinear smooth manifolds analogous conditions are formulated in terms of norms generated by complete Riemannian metrics. Notice that under the conditions of Wintner's theorem we can take a certain smooth approximation of L (denote it also by L), such that (1.2) is valid for it, and introduce the new Riemannian metric on R^n by the formula $\langle\cdot,\cdot\rangle_x = (1/L(\|x\|)^2)(\cdot,\cdot)$ where $\langle\cdot,\cdot\rangle_x$ is the Riemanninan scalar product in the tangent space T_xR^n and $(\cdot,\cdot)$ is the Euclidean scalar product in R^n. From condition (1.2) one can easily derive that the new Riemannian metric is complete. Thus, the condition of Wintner's theorem means boundedness with respect to the new complete Riemannian metric in R^n. Notice also that the condition of linear growth is a particular case the Wintner's one and so for it there also exists a complete Riemannian metric with respect to which the right-hand side is uniformly bounded.

An example of conditions with one-sided estimates in R^n is as follows. Let $\varphi : R^n \to R$ be a smooth positive function such that $\varphi(x) \to \infty$ as $x \to \infty$. Then all integral curves exist for $t \in (t_0,\infty)$ (where t_0 is the initial time value of the curve) if $(X(t,x), \operatorname{grad}\varphi) < C$ at all $t \in (t_0,\infty)$, $x \in R^n$ for some real constant C. Notice that such conditions guarantee existence from any specified finite time instant to $+\infty$ but for $t \to -\infty$ the solutions may get to infinity within a finite time interval. Below we consider a modification of these conditions like $|(X(t,x), \operatorname{grad}\varphi)| < C$ for a certain $C > 0$ that guarantees existence of all integral curves on $(-\infty,+\infty)$. Such conditions we will also call the ones with one-sided estimates.

To describe the conditions with one-sided estimates on manifolds, notice that the property $\varphi(x) \to \infty$ as $x \to \infty$ means that φ is a so-called proper function on R^n, that is, its preimage of any relatively compact set in R is relatively compact in R^n (see general definition for proper functions on manifolds below). On the other hand, the product $(X(t,x), \operatorname{grad}\varphi)$ is equal to the derivative $X\varphi$ of φ in the direction of vector field X at $x \in R^n$. Thus, a condition with one-sided estimate on a smooth manifold M can be formulated as follows: let there exist a smooth proper positive function φ on M such that $|X\varphi| < C$ for some positive constant C at all $t \in (-\infty,\infty)$, $m \in M$. Then all integral curves of X exist on $(-\infty,\infty)$. Obviously for the condition $X\varphi < C$ we will get completeness in going only forward.

Analogous conditions with one-sided and two-sided estimates are known for completeness of flows of stochastic differential equations. An example of conditions with two-sided estimates is the well-known Itô condition of linear growth (see, e.g., [5]). In conditions of one-sided type for stochastic differential equations the operator of derivative in the direction of vector field in the right-hand side is replaced by the generator of stochastic flow, a special second-order differential operator. Among sufficient conditions of this sort we mention Elworthy's condition from [3, Theorem IX. 6A] and its particular case from Theorem 5.3 below. We discuss the stochastic case in more detail in Section 5.

For parabolic equations analogous sufficient conditions are also known. In particular, they can be obtained in the framework of stochastic approach to parabolic equations.

As it is mentioned above, in this paper we find modifications of sufficient conditions of completeness that make them necessary and sufficient. The structure of the paper is as follows. In Section 2 we deal with necessary and sufficient conditions of two-sided sort for completeness of smooth vector fields on finite-dimensional manifolds. Section 3 is devoted to the same problem but for conditions of one-sided sort. In Section 4 we obtain a generalization of conditions from Section 3 to a certain infinite-dimensional case. In Section 5 we get a necessary and sufficient condition of one-sided sort for completeness of a stochastic flow, continuous at infinity, on a finite-dimensional manifold. In Section 6, from the results of Section 5 we derive necessary and sufficient conditions for existence of global Feller semigroup for parabolic equations of some special type on finite-dimensional manifolds.

Preliminary information can be seen, for example, in [8].

2. Necessary and sufficient conditions of two-sided type for completeness of ODE flows

As we have mentioned in Section 1, under the conditions of Wintner's theorem it is possible to construct a new complete Riemannian metric on R^n with respect to which the right-hand side of the ODE becomes uniformly bounded. The same situation takes place also for many other sufficient conditions of two-sided sort. Below in Theorem 2.2 we proof that if on a complete Riemannian manifold the right-hand side is uniformly bounded, the flow of ODE is complete.

It turns out that the condition of boundedness of the right-hand side of ODE with respect to a complete Riemannian metric can be modified so that it becomes necessary and sufficient for completeness. This modification involves in particular transition to extended phase space.

Recall that in contemporary topology a map $f : X \to Y$ from a topological space X to a topological space Y is called *proper*, if the preimage of any relatively compact set from Y is relatively compact in X. According to this terminology we give the following.

Definition 2.1. A function $f : X \to R$ on the topological space X is called proper if the preimage of any relatively compact set from R is relatively compact in X.

Recall that in a complete Riemannian manifold and so in an Euclidean space (in particular, in R) a set is relatively compact if and only if it is bounded.

We should mention that in R^n a positive function f is proper if and only if $f(x) \to +\infty$ as $\|x\| \to +\infty$. On a smooth manifold the Riemannian distance of any complete Riemannian metric is a proper function. Below we sometimes will not specify a Riemannian metric on M and in this case the exact mathematical meaning of $x \to \infty$ for $x \in M$ is that x leaves every compact set, that is, $f(x) \to \infty$ for any proper function f on M.

Let M be a finite-dimensional smooth manifold and $X(t,m)$ be a smooth (jointly in $t \in R$ and $m \in M$) vector field on M.

Denote by $m(s) : M \to M$, $s \in R$ the flow of X. For any point $x \in M$ and time instant t the orbit $m(s)(t,m) = m_{t,m}(s)$ of the flow is the solution of

$$\dot{m}_{t,m}(s) = (s, m_{t,m}(s)), \tag{2.1}$$

with the initial condition

$$m_{t,m}(t) = m. \tag{2.2}$$

The orbits are also called the integral curves of X.

Consider the extended phase space $M^+ = R \times M$ and the vector field $X^+_{(t,m)} = (1, X(t, m))$ on it.

Theorem 2.2 (see [6]). *The flow of X on M is complete if and only if there exists a complete Riemannian metric on M^+ with respect to which the vector field X^+ is uniformly bounded.*

Proof. It is evident that the completeness of flow for X is equivalent to the completeness of flow for X^+.

Sufficiency. Let on M^+ there exist a complete Riemannian metric with respect to which the field X^+ is uniformly bounded. Then any integral curve of X^+ on any finite time interval has finite length and by completeness of the metric it is relatively compact, that is, the domain of solution is both closed and open in R. Hence it coincides with R.

Necessity. Let the vector field X be complete. Then X^+ is also complete. Since $X(t,m)$ is smooth by the hypothesis, X^+ is also smooth. Following [12], let us construct a certain proper function ψ on in the following way. Take on M a countable locally finite covering

$\mathcal{V} = \{V_i\}_{i\in N}$ where all V_i are open and relatively compact. This can be done by virtue of the paracompactness and the local compactness of M. Determine $\psi_i : M \to R$ by the formula

$$\psi_i(m) = \begin{cases} i & \text{if } m \in V_i, \\ 0 & \text{if } m \notin V_i. \end{cases} \tag{2.3}$$

By $\{\phi_i\}_{i=1}^{\infty}$ denote the smooth partition of unity corresponding to this covering. Define the function ψ on the entire M as $\psi(m) = \sum_{i=1}^{\infty} \psi_i(m)\phi_i(m)$. It is clear that $\psi(m)$ is smooth and proper by the construction.

In every tangent space $T_{(t,m)}(\{t\} \times M)$ to the submanifold $\{t\} \times M$ of M^+ introduce a scalar product that smoothly depends on (t,m). For example, one can take an arbitrary Riemannian metric on M and extend it by natural way. Now construct the Riemannian metric on M^+ by regarding the vectors of the field X^+ as being of unit length and orthogonal to the subspaces $T_{(m,t)}(M \times \{t\})$.

Consider the function $\Phi(t,m) = \psi(m^+_{(t,m)}(0))$ on M^+, where $m^+_{(t,m)}(s)$ is the integral curve of X^+ with initial condition $m^+_{(t,m)}(t) = m$ (the orbit of flow $m^+(s)$ corresponding to the vector field X^+ on M^+). Since by the hypothesis the integral curves of X^+ exist on $(-\infty,\infty)$, the function $\varphi : M^+ \to R$, given by the formula $\varphi(t,m) = \Phi(t,m) + t$, is obviously well-posed smooth and proper. It is also obvious that $X^+\varphi = 1$ ($X^+\varphi$ is the derivative of φ in the direction of X^+).

Now pick an arbitrary smooth function $g : M^+ \to R$ such that $g(t,m) > \max_{\|Y\|_1=1} \exp (Y\varphi)^2$, $Y \in T_{(t,m)}(\{t\} \times M)$. Such a function can by constructed, for example, as follows. For a relatively compact neighborhood of any point $(t',m') \in M^+$ there exists a constant that is greater than $\sup\max_{\|Y\|_1=1} \exp(Y\varphi)^2$, $Y \in T_{(t,m)}(\{t\} \times M)$ for all points (t,m) from this neighborhood. Taking into account paracompactness of M^+ and so, the existence of a smooth partition of unity (as above) from those constants we can glue the function ϕ defined on the whole of M^+.

At every point $(m,t) \in M^+$, define the inner product on $T_{(m,t)}M^+$ by the formula

$$\langle Y,Z\rangle_2 = g^2(t,m)\langle p_m Y, p_m Z\rangle_1 + p_x Y p_x Z, \tag{2.4}$$

where $Y,Z \in T_{(m,t)}M^+$ and p_m, p_X are orthogonal (in the metric $\langle\cdot,\cdot\rangle_1$) projections of $T_{(m,t)}M^+$ onto $T_{(t,m)}(\{t\} \times M)$ and X^+, respectively.

It is obvious that in the metric $\langle\cdot,\cdot\rangle_2$ the vector X is still orthogonal to the subspace $T_{(t,m)}(\{t\} \times M)$ and $\|X\|_2 = 1$. □

Lemma 2.3. *$\langle\cdot,\cdot\rangle_2$ is a complete Riemannian metric on M^+.*

Proof of Lemma 2.3. By Hopf-Rinow theorem (see, e.g., [2]) it is sufficient to show that any geodesic of the metric $\langle\cdot,\cdot\rangle_2$ exists on $(-\infty,\infty)$. It is enough to deal with the geodesics whose norm of velocity vector is equal to 1 (all others can be obtained from them by linear change of argument). Let $c(s)$ be such a geodesic, that is, $\|\dot c(s)\|_2 = 1$ for all s. It is easy to see that $(d/ds)\varphi(c(s)) = \dot c(s)\varphi = (p_m\dot c(s))\varphi + (p_x\dot c(s))\varphi$ (recall that here $\dot c(s)\varphi$ denotes the derivative of φ in the direction of $\dot c(s)$; for $(p_m\dot c(s))\varphi$ and $(p_x\dot c(s))\varphi$ the meaning is analogous). Since $\|\dot c(s)\|_2 = 1$ and the vectors $p_m\dot c(s)$ and $p_x\dot c(s)$ are orthogonal to each other in the metric $\langle\cdot,\cdot\rangle_2$, we have $\|p_m\dot{c}(s)\|_2 \le 1$, $\|p_x\dot{c}(s)\|_2 \le 1$. Hence

$$\left|\frac{d}{ds}\varphi(c(s))\right| \le \left|\frac{p_m\dot{c}(s)}{\|p_m\dot{c}(s)\|_2}\varphi\right| + \left|\frac{p_x\dot{c}(s)}{\|p_x\dot{c}(s)\|_2}\varphi\right| = \left|\frac{1}{g(c(s))}\frac{p_m\dot{c}(s)}{\|p_m\dot{c}(s)\|_1}\varphi\right| + |X^+\varphi| < 2 \tag{2.5}$$

by the constructions of functions g and φ.

Thus, the values of $\varphi(c(s))$ are bounded on any finite interval $s \in (a,b)$ and the set of points $c(s)$ for $s \in (a,b)$ is relatively compact since φ is proper. This proves the existence of geodesics on $(-\infty,\infty)$. □

As it is mentioned above, $\|X^+\|_2 = 1$. The theorem follows.

Remark 2.4. Let us emphasize that for the case of an autonomous smooth vector field X a complete metric on the manifold M, with respect to which X is uniformly bounded, may not exist.

Indeed, consider in R^n two differential equations $\dot{x} = \|x\|^2 \cdot x$ and $\dot{x} = -\|x\|^2 \cdot x$, where $\|x\|$ is the Euclidean norm of $x \in R^n$. It is well known that the field $-\|x\|^2 \cdot x$ is complete while the field $\|x\|^2 \cdot x$ is not complete: all its integral curves go to infinity within finite time interval. Nevertheless, those fields differ from each other only by the sign, that is, with respect to any Riemannian metric on M their norms are equal to each other.

3. Necessary and sufficient conditions of one-sided type for completeness of ODE flows

As well as in Section 2 consider a smooth manifold M with dimension $n < \infty$ and a smooth jointly in $t \in R$, $m \in M$ vector field $X = X(t,m)$ on M. The coordinate representation in a chart with respect to local coordinates $(q^1,\dots,q^n)$ takes the form $X = X^1(\partial/\partial q^1) + \cdots + X^n(\partial/\partial q^n)$. The vector field X can be also considered as the first-order differential operator on C^1-functions on M. For a function f the value of the above operator is given as $Xf = X^1(\partial f/\partial q^1) + \cdots + X^n(\partial f/\partial q^n)$, the derivative of f in the direction of vector field X. Let $\gamma(t)$ be an integral curve of X such that $\gamma(0) = m$. It is well known that Xf is represented in terms of $\gamma(t)$ as follows: $Xf(m) = (d/dt)f(\gamma(t))|_{t=0}$. The latter presentation is valid also in infinite-dimensional case where the use of coordinates is not applicable.

Consider the extended phase space $M^+ = R\times M$ with the natural projection $\pi^+ : M^+ \to M$, $\pi^+(t,m) = m$. As well as in Section 2 introduce the vector field $X^+_{(t,m)} = (1,X(t,m))$ on M^+. It is clear that its coordinate representation is given in the form $X^+ = \partial/\partial t + X^1(\partial/\partial q^1) + \cdots + X^n(\partial/\partial q^n)$. Hence the corresponding differential operator on the space of C^1-smooth functions on M^+ takes the form $\partial/\partial t + X$.

THEOREM 3.1 (see [10]). *A smooth vector field X on a finite-dimensional manifold M is complete if and only if there exists a smooth proper function $\varphi : M^+ \to R$ such that the absolute value of derivative $|X^+\varphi|$ of φ along X^+ is uniformly bounded, that is, $|X^+\varphi| = |(\partial/\partial t + X)\varphi| \le C$ at any $(t,m) \in M^+$ for a certain constant $C > 0$ that does not depend on (t,m).*

Proof

Sufficiency. Consider the flow $m^+(s) : M^+ \to M^+$, $s \in R$ with orbits $m^+(s)(t,m) = m^+_{(t,m)}(s)$ being the solutions of

$$\dot{m}^+_{(t,m)}(s) = X^+\left(m^+_{(t,m)}(s)\right) \tag{3.1}$$

with initial conditions

$$m^+_{(t,m)}(t) = (t,m). \tag{3.2}$$

Consider the derivative $X^+\varphi$ of φ along X^+ where φ is from the hypothesis. At $(t,m) \in M^+$ we get the equality

$$X^+\varphi(t,m) = \frac{d}{ds}\varphi\left(m^+_{(t,m)}(s)\right)|_{s=t}, \tag{3.3}$$

(see above) and under the hypothesis of our theorem

$$\left|\frac{d}{ds}\varphi\left(m^+_{(t,m)}(s)\right)|_{s=t}\right| \leq C. \tag{3.4}$$

Represent the values of φ along the orbit $m^+_{(t,m)}(s)$ as follows:

$$\varphi\left(m^+_{(t,x)}(s)\right) - \varphi(t,m) = \int_0^s \frac{d}{d\tau}\varphi\left(m^+_{(t,m)}(\tau)\right)d\tau. \tag{3.5}$$

From the last equality and from inequality (3.4) we evidently obtain that if s belongs to a finite interval, the values $\varphi(m^+_{(t,x)}(s))$ are bounded in R. Then since φ is proper, this means that the set $m^+_{(t,m)}(s)$ is relatively compact in M^+.

Recall that by the classical solution existence theorem the domain of any solution of ODE is an open interval in R. In particular, for $s > 0$ the solution of above Cauchy problem is well-posed for $s \in [t,\varepsilon)$. If $\varepsilon > 0$ is finite, then the corresponding values of the solution belong to a relatively compact set in M and so the solution is well-posed for $s \in [t,\varepsilon]$. The same arguments are valid also for $s < 0$. Thus, the domain is both open and closed and so it coincides with the entire real line $(-\infty,\infty)$.

Taking into account the construction of vector field X^+, we can represent the integral curves $m^+_{(t,m)}(s)$ in the form $m^+_{(t,m)}(s) = (s, m_{t,m}(s))$. Hence from global existence of integral curves of X^+ we obviously obtain the global existence of integral curves of X. So, the vector field X is complete.

Necessity. Let the vector field X be complete. Thus, all orbits $m_{t,m}(s)$ of the flow $m(s)$ are well-posed on the entire real line.

Consider the function $\varphi : M^+ \to R$ as in the proof of Theorem 2.2. As well as in the proof of Theorem 2.2 from completeness of X^+ it follows that φ is well-posed smooth and proper. Consider $X^+\varphi$. By the construction of the vector field X^+ and of the function φ, we get

$$X^+\varphi = X^+(\Phi(t,m)) + X^+t = 0 + 1 = 1. \tag{3.6}$$

Thus, we have proven the necessity part of our theorem for $C = 1$. This completes the proof. □

4. A generalization to infinite-dimensional case

Both Theorems 2.2 and 3.1 cannot be generalized to the infinite-dimensional case directly. For Theorem 2.2 the fatal difficulty is the absence of good enough infinite-dimensional analogy of Hopf-Rinow theorem. For Theorem 3.1 the main difficulty is the absence of continuous proper real-valued functions on infinite-dimensional manifolds. However it is possible to replace the set of functions, proper with respect to strong topology, by the one, proper with respect to a weaker topology so that an analogue of Theorem 3.1 takes place.

Let M be a Banach manifold that admits partition of unity of class C^p for a certain $p \geq 2$ (see [13]).

For the sake of convenience we consider charts on M as triples (U, V, φ), where V is an open ball in the model space, U is an open set in M, and $\varphi : V \to U$ is a homeomorphism.

Definition 4.1. A set Θ on M is called relatively weakly compact if there exists a finite collection of charts (U_i, V_i, φ_i) such that $\Theta \subset \bigcup_i U_i$ and for every i the set $\varphi_i^{-1}(\Theta \bigcap U_i) \subset V_i$ is bounded with respect to the norm of model space that contains V_i.

Remark 4.2. If the model space of M is a reflexive Banach space, then under some natural condition the relatively weakly compact set as in Definition 4.1 is relatively weakly compact with respect to the topology of weal convergence on M (see [15]). If M itself is a reflexive Banach space, then any relatively weakly compact set as in Definition 4.8 is weakly compact by standard definition of weak topology. These circumstances allow us to use the term "relatively weakly compact set" in the general case of Banach manifolds where (generally speaking) the weak topology is ill-posed.

Definition 4.3. A function $f : N \to R$ on a Banach space N is called weakly proper if for any relatively compact set in R its preimage is relatively weakly compact in N as in Definition 4.1

Let $X = X(t,m)$ be a smooth jointly in $t \in R$, $m \in M$ vector field on M. Consider the extended phase space $M^+ = R \times M$ and the vector field $X^+_{(t,m)} = (1, X(t,m))$ on it (cf. Sections 2 and 3).

Now we are in the position to prove the following generalization of Theorem 3.1.

THEOREM 4.4 (see [11]). *Let M be a Banach manifold that admits partition of unity of class C^p for a certain $p \geq 2$. A smooth vector field X on M is complete if and only if there exists a C^2-smooth weakly proper function $f : M^+ \to R$ on M^+ such that the absolute value of the derivative of f in the direction of X^+ is uniformly bounded, that is, $|X^+ f| \leq C$ for a certain constant $C > 0$ that does not depend on (t,m).*

Proof

Sufficiency. Let there exist f as in the hypothesis. Consider the flow $m^+(s) : M^+ \to M^+$, $s \in R$ of X^+. Its orbits $m^+(s)(t,m) = m^+_{t,m}(s)$ satisfy

$$m'^+_{(t,m)}(s) = X^+\big(m^+_{(t,m)}(s)\big) \tag{4.1}$$

with initial conditions

$$m^+_{(t,m)}(t) = (t,m). \tag{4.2}$$

Show the existence of all orbits on $s \in (-\infty,\infty)$. Consider the derivative $X^+ f$. At the point (t,m) the equality

$$X^+ f(t,m) = \frac{d}{ds} f\big(m^+_{(t,m)}(s)\big)_{|s=t} \tag{4.3}$$

holds and by the hypothesis

$$\left| \frac{d}{ds} f\big(m^+_{(t,m)}(s)\big)_{|s=t} \right| \leq C. \tag{4.4}$$

Thus, on any finite interval $[t,\varepsilon)$ the values $f(m^+_{(t,m)}(s))$ are bounded by the constant $C(\varepsilon - t)$. Then from Definitions 4.1 and 4.3 it follows that for $s \in (0,\varepsilon)$ there exists a finite number of charts (U_i, V_i, φ_i) such that the set $m^+_{(t,m)}(s)$ belongs to the union of U_i and the part of corresponding set in any V_i is bounded. In particular, the part in the last V_i is bounded and so there exists $\lim_{s\to\varepsilon}(m^+_{(t,m)}(s))$, that is, $m^+_{(t,m)}(s)$ does exist on the closed interval $[t,\varepsilon]$. As well as in finite-dimensional situation this means that the domain of $m^+_{(t,m)}(s)$ is the entire R. Obviously, $m^+_{(t,m)}(s) = (s, m_{(t,m)}(s))$. Hence from the completeness of X^+ it follows that X is also complete.

Necessity. Let X be complete, that is, all orbits $m_{(t,m)}(s)$ exist on the entire line. Then all orbits of the flow $m^+(s)$ also exist on the entire line.

Construct an open covering of M in the following way. For any $m \in M$ pick a chart (U_m, V_m, φ_m) such that $m \in U_m$. Pick also an open neighborhood $W_m \subset U_m$ of m such that $\varphi^{-1}(W_m) \subset V_m$ is bounded with respect to the norm of model space where V_m is contained. Notice that by the construction W_m is relatively weakly compact. Since M is paracompact and satisfies the second axiom of countability, we can choose from $\{W_m\}$ a countable locally finite subcovering $\{W_i\}$ (see [13]).

Define the functions $\psi_i : M \to R$ by the formula

$$\psi_i(m) = \begin{cases} i & m \in W_i, \\ 0 & m \notin W_i. \end{cases} \tag{4.5}$$

By $\theta_i(m)$, $i = 1,\ldots,\infty$, denote the C^p-smooth partition of unity, corresponding to $\{W_i\}$, that exists by the hypothesis. Define the function $\psi(m)$ on M by the formula

$$\psi(m) = \sum_{i=1}^{\infty} \theta_i(m)\psi_i(m). \tag{4.6}$$

By the construction $\psi(m)$ is C^p-smooth and weakly proper.

Now construct the C^p-smooth function $\Phi : M^+ \to R$ by assigning to the point $(t,m) \in M^+$ the value $\Phi(t,m) = \psi(m_{t,m}(0))$. Since X^+ is complete, $\Phi(t,m)$ is well-posed.

By its construction the function Φ takes constant values along the orbits of X^+. Indeed, for $m^+_{(t,m)}(s) = (s, m_{(t,m)}(s))$ the equality $m_{(s,m_{(t,m)}(s))}(0) = m_{t,m}(0)$ holds. Consider the function $f : M^+ \to R$, $f(t,m) = \Phi(t,m) + t$ that is C^p-smooth and weakly proper by the construction. Taking into account the construction of X^+ and f, we get

$$X^+ f = X^+\Phi(t,m) + X^+ t = 0 + 1 = 1. \tag{4.7}$$

Thus, any $C \geq 1$ can be chosen as the constant from the assertion of theorem that we are looking for. □

5. Stochastic case

The results of this section were announced in [9, 7].

Let M be a finite-dimensional noncompact manifold. Consider a smooth stochastic dynamical system (SDS) on M (see [3]) with the infinitesimal generator $\mathcal{A}(x)$. In local coordinates it is described in terms of a stochastic differential equation with C^∞-smooth coefficients in Itô or in Stratonovich form. Since the coefficients are smooth, we can pass from Stratonovich to Itô equation and vice versa.

Consider the one-point compactification $M \bigcup \{\infty\}$ of M where the system of open neighbourhoods of $\{\infty\}$ consists of complements to all compact sets of M. Denote by $\xi(s) : M \to M \bigcup \{\infty\}$ the stochastic flow of SDS. For any point $x \in M$ and time t the orbit $\xi_{t,x}(s)$ of this flow is the unique solution of the above-mentioned equation with initial conditions $\xi_{t,x}(t) = x$. As the coefficients of equation are smooth, this is a strong solution and a Markov diffusion process given on a certain random time interval. The point $\{\infty\}$ is the "cemetery" where the solution (defined on a random time interval) gets after explosion.

We refer the reader to [14] for more information on behavior of a diffusion process at infinity.

Recall that the generator $\mathcal{A}$ is a second-order differential operator without constant term (i.e., $\mathcal{A}1 = 0$ where 1 denotes the constant function identically equal to 1). In local coordinates one can find the matrix of its pure second-order term that is symmetric and so semipositive definite.

For a stochastic flow the generator plays the same role as the derivative in the direction of vector field in the right-hand side of an ordinary differential equation. The main result on completeness for stochastic flows here is analogous to Theorem 3.1 where the derivative in the direction of vector field X^+ is replaced with the corresponding generator. However, in the stochastic case there is an additional difficulty that for a flow with inverse time direction the generator does not coincide with the one for the flow itself. That is why we obtain a necessary and sufficient condition for completeness only for flows with additional assumption: the flow must be continuous at infinity (see the exact definition below).

Everywhere in this section we suppose $\mathcal{A}(x)$ to be autonomous and strictly elliptic (i.e., in a local coordinate system its pure second-order term is described by a nondegenerate, i.e., positive definite matrix). This assumption allows us to apply the machinery from [1]. Notice that using this machinery we can reduce the condition that the SDS is C^∞-smooth to the assumption that it is Hölder continuous.

Below we denote the probability space, where the flow is defined, by $(\Omega,\mathcal{F},\mathrm{P})$ and suppose that it is complete. We also deal with separable realizations of all processes.

Let $T > 0$ be a real number.

Definition 5.1. The flow $\xi(s)$ is complete on $[0,T]$ if $\xi_{t,x}(s)$ is a.s. in M for any couple (t,x) (with $0 \le t \le T$) and for all $s \in [t,T]$.

Definition 5.2. The flow $\xi(s)$ is complete if it is complete on any interval $[0,T] \subset R$.

We start with a certain sufficient condition for completeness of a stochastic flow analogous to conditions for completeness of ODE flows with one-sided estimates. It is a simple version of rather general sufficient condition from [3, Theorem IX. 6A].

Theorem 5.3. *Let there exist a smooth proper function φ on M such that $\mathcal{A}(t,m)\varphi < C$ for some $C > 0$ at all $t \in [0,+\infty)$ and $m \in M$. Then the flow $\xi(t,s)$ is complete.*

Proof. Consider the collection of sets $W_n = \varphi^{-1}([0,n))$ with the positive integers $1,2,\ldots,$ $n,\ldots.$ Since φ is proper, those sets are relatively compact and $\bigcup_n W_n = M$. Besides, by the construction $W_i \subset W_{i+1}$ $i = 1,2,\ldots,n,\ldots.$

Specify arbitrary $t \in [0,+\infty)$ and $m \in M$ and consider the orbit $\xi_{t,m}(s)$. Denote by τ_n the first entrance time of $\xi_{t,m}(s)$ in the boundary of W_n. Express $\varphi(\xi_{t,m}(s \wedge \tau_n))$ by Itô formula. Since W_n is relatively compact, Itô integral on the interval $[t,s \wedge \tau_n)$ is a martingale and so its expectation is equal to 0. Then

$$E\varphi(\xi_{t,m}(s \wedge \tau_n)) = \varphi(m) + \int_t^{s\wedge\tau_n} (\mathcal{A}\varphi)(\theta,\xi_{t,m}(\theta))\,d\theta < \varphi(m) + Cs, \tag{5.1}$$

since $\mathcal{A}(t,m)\varphi < C$ and $s \ge s \wedge \tau_n$.

Consider the set $\Omega_s^n = \{\omega \in \Omega \mid s < \tau_n\}$. Obviously,

$$n(1 - \mathrm{P}(\Omega_s^n)) < E\varphi(\xi_{t,m}(s \wedge \tau_n)), \tag{5.2}$$

since for $\omega \notin \Omega_s^n$ we get $\xi_{t,m}(s \wedge \tau_n,\omega) = \xi_{t,m}(\tau_n,\omega)$, that is, $\varphi(\xi_{t,m}(s \wedge \tau_n,\omega)) = n$. Thus,

$$1 - \mathrm{P}(\Omega_s^n) < \frac{\varphi(m) + Cs}{n}. \tag{5.3}$$

Hence $\lim_{n\to\infty}(1 - \mathrm{P}(\Omega_s^n)) = 0$. However by the construction $\lim_{n\to\infty}\Omega_s^n = \bigcup_{i=1}^n \Omega_s^i = \Omega$, that is, for any specified $s \ge t$ the value $\xi_{t,m}(s)$ exists with probability 1. $\square$

Let $\gamma(s)$ be a (not necessarily complete) stochastic flow.

Definition 5.4. We say that the flow $\gamma(s)$ is continuous at infinity if for any $0 \le t \le T$ and any compact $K \subset M$ the equality

$$\lim_{x \to +\infty} \mathrm{P}(\gamma_{t,x}(T) \in K) = 0 \tag{5.4}$$

holds.

One can easily see that continuity at infinity according to Definition 5.4 means that for any specified $t \in [0,+\infty)$ and for all $s \in [t,+\infty)$ the correspondence $(x,s) \mapsto \gamma_{t,x}(s)$ is continuous in probability at the point $(s,\{\infty\}) \in [t,\infty] \times (M \bigcup\{\infty\})$, see [16, 17] for details.

Our next task is to construct a special proper function associated to a complete stochastic flow $\xi(s)$.

Consider an expanding sequence of compact sets M_i such that $M_i \subset M_{i+1}$ for all i and $\bigcup_i M_i = M$. By T_i we denote an increasing sequence of real numbers tending to $+\infty$.

For $(t,x) \in [0,T_i] \times M_i$, the distribution function $\mu_{t,x,s}$ of random elements $\xi_{t,x}(s)$, $s \in [t,T_i]$, on M forms a weakly compact set of measures. Indeed, take an arbitrary sequence random element $\xi_{t_k,x_k}(s_k)$ and the corresponding measures μ_{t_k,x_k,s_k}. Since $[0,T_i] \times M_i \times [0,T_i]$ is compact, it is possible to select a subsequence t_{k_q}, x_{k_q}, s_{k_q} of the sequence t_k, x_k, s_k, converging to a certain t_0, x_0, s_0. It is a well-known fact that the function $Ef(\xi_{t,x}(s))$ is continuous jointly in t, x, s for any bounded continuous function $f : M \to R$. Then we obtain that

$$E(f(\xi_{t_{k_q},x_{k_q}}(s_{k_q}))) \longrightarrow E(f(\xi_{t_0,x_0}(s_0))), \tag{5.5}$$

that is, from any sequence of measures mentioned above it is possible to select a weakly converging subsequence.

Take a monotonically decreasing sequence of positive numbers $\varepsilon_i \to 0$ such that the series $\sum_{i=1}^{\infty}\sqrt{\varepsilon_i}$ converges. From Prokhorov's theorem it follows that for the measures corresponding to $\xi_{t,x}(s)$, $s \in [t,T_i]$, $(t,x) \in [0,T_i] \times M_i$ mentioned above, there exists a compact $\Xi_i \subset M$ such that for all $\mu_{t,x,s}$ the inequality $\mu_{t,x,s}(M\backslash\Xi_i) < \varepsilon_i$ holds. Construct an expanding system of compacts $\Theta_i \supset \bigcup_{k=0}^{i}\Xi_k$ for any i, being closures of open domains in M with smooth boundary and such that $\Theta_i \subset \Theta_{i+1}$ for any i and $\bigcup_i \Theta_i = M$. By the construction for $s \in [0,T_i]$, $(t,x) \in [0,T_i] \times M_i$ the relation $\mu_{t,x,s}(M\backslash\Theta_i) < \varepsilon_i$ holds. In particular, $\mu_{t,x,s}(\Theta_{i+1}\backslash\Theta_i) < \varepsilon_i$.

Choose neighborhoods $U_i \subset \widetilde{U}_i$ of the set Θ_i, that completely belong to Θ_{i+1}, and consider a smooth function ψ_i that equals 0 on U_i, equals 1 on $\Theta_{i+1}\backslash\widetilde{U}_i$, and takes values between 0 and 1 on $\widetilde{U}_i\backslash U_i$. Construct the function θ on M setting its value on $\Theta_{i+1}\backslash\Theta_i$ equal to $\psi_i(1/\sqrt{\varepsilon_i}) + (1-\psi_i)(1/\sqrt{\varepsilon_{i-1}})$. Notice that on $\Theta_{i+1}\backslash\Theta_i$ the values of θ are not greater than $1/\sqrt{\varepsilon_i}$.

Immediately from the construction we obtain the following.

Lemma 5.5. *For a complete flow $\xi(s)$ the function θ, constructed above, is smooth positive and proper.*

THEOREM 5.6. *If the flow $\xi(t)$ is complete, for any (t,x) and any $T > t$, the inequality $E\theta(\xi_{t,x}(s)) < \infty$ holds for each $s \in [t,T]$.*

Proof. Take i such that $[0,T] \subset [0,T_i]$, $t \in [0,T_i]$, and $x \in M_i$. Then $\mu_{t,x,s}(M \backslash \Theta_i) < \varepsilon_i$ or $\mu_{t,x,s}(\Theta_i) > (1-\varepsilon_i)$. By the construction the values of continuous function θ on compact Θ_i are bounded by constant $1/\sqrt{\varepsilon_{i-1}}$. Then also by the construction

$$E\theta(\xi_{t,x}(s)) \leq \frac{1}{\sqrt{\varepsilon_{i-1}}} + \sum_{k=i}^{\infty} \varepsilon_k \frac{1}{\sqrt{\varepsilon_k}} = \frac{1}{\sqrt{\varepsilon_{i-1}}} + \sum_{k=i}^{\infty} \sqrt{\varepsilon_k} < C < +\infty \tag{5.6}$$

for some positive constant C since by definition the series $\sum_{k=i+1}^{\infty} \sqrt{\varepsilon_k}$ converges. $\square$

COROLLARY 5.7. *The function $E\theta(\xi_{t,x}(s))$ is integrable in $s \in [t,T]$.*

Proof. From the construction in Theorem 5.6 it follows that for given t, x estimate (5.6) is valid with the same C for all $s \in [t,T]$. $\square$

Specify any $T > 0$ and consider the direct product $M^T = [0,T] \times M$. Denote by $\pi^T : M^T \to M$ the natural projection: $\pi^T(t,x) = x$.

THEOREM 5.8. *The function $u(t,x) = E\theta(\xi_{t,x}(T))$ on M^T is C^1-smooth in $t \in [0,T]$, C^2-smooth in $x \in M$ and satisfies*

$$\left(\frac{\partial}{\partial t} + \mathscr{A}\right) u = 0, \tag{5.7}$$

where $\mathscr{A}$ is the infinitesimal generator of the flow.

Proof. Since M is locally compact and paracompact, we can choose a countable locally finite open covering $\{V_i\}_{i=1}^{\infty}$ of M such that all V_i have compact closures. Consider a partition of unity $\{\varphi_i\}_{i=1}^{\infty}$ adapted to this covering. Then at any point $x \in M$ the equality $\theta(x) = \sum_{i=1}^{\infty} \varphi_i(x)\theta(x)$ holds.

Introduce the function $v_i(x) = \varphi_i(x)\theta(x)$ as well as the functions $u_i(t,x) = Ev_i(\xi_{t,x}(T))$ and $\theta_k(t,x) = \sum_{i=0}^{k} u_i(t,x)$. Notice that all $v_i(x)$ are smooth and bounded. Then any $v_i(t,x)$ satisfies the conditions of [5, Theorem 4, Chapter VIII] and so any $u_i(t,x)$ is C^1-smooth in t, C^2-smooth in x and satisfies the relation

$$\frac{\partial}{\partial t} u_i + \mathscr{A} u_i = 0. \tag{5.8}$$

Hence all functions $\theta_k(t,x)$, being finite sums of functions $u_i(t,x)$, are also C^1-smooth in t, C^2-smooth in x and satisfy

$$\frac{\partial}{\partial t} \theta_k + \mathscr{A} \theta_k = 0. \tag{5.9}$$

In addition it is evident that $\theta(t,x)$ is the limit of $\theta_k(t,x)$ at $k \to \infty$ and the functions $\theta_k(t,x)$ form an increasing locally bounded sequence. Then, since $\mathscr{A}$ is autonomous and strictly elliptic, the assertion of theorem follows from [1, Lemma 1.8]. $\square$

Theorem 5.9. *If a complete flow $\xi(s)$ is continuous at infinity, the function $u(t,x) = E\theta(\xi_{t,x}(T))$ on M^T is proper.*

Proof. Let $\xi(s)$ be continuous at infinity. To prove the properness of $u(t,x)$ it is sufficient to show that $u(t,x) \to \infty$ as $\theta(x) \to \infty$, that is, that for any $C > 0$ there exists $\Xi > 0$ such that $\theta(x) > \Xi$ yields $u(t,x) > C$ for any $t \in [0,T]$. Since θ is proper, $K = \theta^{-1}([0,2C])$ is compact. From formula (5.4) of the definition of continuity at infinity it follows that for any $t \in [0,T]$ there exists Ξ such that $\mathrm{P}(\xi_{t,x}(T) \notin K) > 1/2$ for $\theta(x) > \Xi$. Then $u(t,x) = E\theta(\xi_{t,x}(T)) > 2C \cdot 1/2 = C$. Since t is from compact set $[0,T]$ and $u(t,x)$ is continuous in t, this completes the proof. $\square$

On the manifold M^T consider the flow $\eta(s) = (s,\xi(s))$. Obviously, for $(t,x) \in M^T$ the trajectory of $\eta_{(t,x)}(s)$ satisfies the relation $\pi^T(\eta_{(t,x)}(s)) = \xi_{t,x}(s)$. It is clear that $\eta(s)$ is the flow generated by SDS with infinitesimal generator $\mathcal{A}^T$ determined by the formula

$$\mathcal{A}^T_{(t,x)} = \mathcal{A}(t,x) + \frac{\partial}{\partial t}. \tag{5.10}$$

Notice that $\mathcal{A}^T$ is a direct analogue of differentiation in the direction of X^+ in Theorem 3.1.

Theorem 5.10. *A flow $\xi(s)$ on M, continuous at infinity, is complete on $[0,T]$ if and only if there exists a positive proper function $u^T : M^T \to R$ on M^T that is C^1-smooth in $t \in [0,T]$, C^2-smooth in $x \in M$ and such that $\mathcal{A}^T u^T < C$ for a certain constant $C > 0$ at all points $(t,x) \in M^T$.*

Proof. Let there exist a smooth proper positive function $u^T(t,x)$ on M^T such that $\mathcal{A}^T u^T < C$ at all points of M^T. Then from Theorem 5.3 it follows that $\eta(s)$ is complete. Thus, $\xi(s)$ is also complete.

Let $\xi(s)$ be complete. Consider the function $\theta(x)$ on M introduced above and the function $u^T(t,x) = E\theta(\xi_{t,x}(T))$ on M^T. Since $\xi(s)$ is continuous at infinity, $u^T(t,x)$ is proper by Theorem 5.9. By Theorem 5.8 it is also C^1 in t, C^2 in x and satisfies the relation $((\partial/\partial t) + \mathcal{A})u^T = \mathcal{A}^T u^T = 0$. $\square$

Corollary 5.11. *A flow $\xi(s)$ on M, continuous at infinity, is complete if and only if for any $T > 0$ there exists a positive proper function $u^T : M^T \to R$ on M^T that is C^1-smooth in $t \in [0,T]$, C^2-smooth in $x \in M$ and such that $|\mathcal{A}^T u(t,x)| < C$ for a certain constant $C > 0$ at all points $(t,x) \in M^T$.*

6. Parabolic equations

Here by using stochastic approach to parabolic equations and the results of Section 5 we get a necessary and sufficient condition for existence of global Feller semigroup for some class of such equation (in particular, this class includes equations with the so-called C_0 property). We suppose that the second-order operator in the right-hand side of parabolic equation is autonomous and strictly elliptic. Under this assumption, on the one hand, the stochastic approach is applicable and on the other hand, the conditions of Section 5 are fulfilled for the corresponding stochastic flow.

Let M be a finite-dimensional (generally speaking) noncompact manifold. Consider on M a parabolic equation

$$\frac{\partial}{\partial t} u = \mathcal{A} u \tag{6.1}$$

with initial conditions

$$u(0,x) = u_0(x), \tag{6.2}$$

where $\mathcal{A}$ is an autonomous strictly elliptic operator with C^∞ coefficients without constant term (i.e., satisfying the property $\mathcal{A}1 = 0$), u_0 and u are smooth enough real-valued bounded functions.

In local coordinates on M, the operator $\mathcal{A}$ is represented in the form

$$\sum_{i=1}^{n} a^i \frac{\partial}{\partial q^i} + \sum_{i=1}^{n} b^i(\sigma^{kl}) \frac{\partial}{\partial q^i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma^{ij} \frac{\partial^2}{\partial q^i \partial q^j}. \tag{6.3}$$

Here $b_x(\sigma) = \sum_{i=1}^{n} b^i(\sigma^{kl}) \partial/\partial q^i$ is the so-called compensating term, depending on (σ^{kl}), that guarantees covariant transformation of the formula under changes of coordinates.

It is a well-known fact that under the above conditions on $\mathcal{A}$ the stochastic approach to investigation of parabolic equations is applicable in the following way. One can easily see that the matrix $(\sigma^{ij}(x))$ is a coordinate expression of a smooth symmetric $(2,0)$-tensor field on M. Since $\mathcal{A}$ is strictly elliptic, this matrix is not degenerate and taking at any $x \in M$ the inverse matrix $(\sigma_{ij}(x))$ one gets a smooth $(0,2)$-tensor field. Denote the latter field by σ_x. Thus, σ_x for any $x \in M$ is a symmetric bilinear form on the tangent space T_xM. Since $\mathcal{A}$ is strictly elliptic, this form at any $x \in M$ is positive definite and so the field σ_x can be considered as a Riemannian metric tensor on M. By Nash's theorem we can embed M with this metric isometrically into a certain Euclidean space R^k where k is large enough. Then the field of orthogonal projections $A_x : R^k \to T_xM$ is smooth and gives us the presentation of σ_x in the form

$$\sigma_x = A_x^* A_x, \tag{6.4}$$

where A_x^* is the conjugate operator.

The above construction yields the existence of a smooth stochastic dynamical system (SDS) on M (see [3]) whose infinitesimal generator is $\mathcal{A}$ and it is of the same type as in Section 5. In local coordinates it is described in terms of a stochastic differential equation with C^∞-smooth coefficients in Itô or in Stratonovich form with diffusion term A_x^*. In Itô form its drift is $a+b$. Since the coefficients are smooth, we can pass from Itô form to Stratonovich one and vice versa.

Denote by $\xi(s)$ the flow of above-mentioned SDS and by $\xi_{t,x}(s)$ its orbits (see Section 5). If $\xi(s)$ is complete, on the space of bounded measurable functions on M, there exists an operator semigroup $S(t,s)$ given for a function $f(x)$ by the formula

$$[S(t,s)f](x) = Ef(\xi_{t,x}(s)), \tag{6.5}$$

where E is the mathematical expectation. This is a Feller semigroup, that is, for any $t \geq 0$, $s \geq t$ the operators $S(t,s)$ are contracting and transform any continuous positive bounded function into one with the same properties. It is also well known that for continuous and bounded function $u_0(x)$ the continuous and bounded function

$$u(s,x) = [S(0,s)u_0](x) = Eu_0(\xi_{0,x}(s)) \tag{6.6}$$

is a generalized solution of (6.1)–(6.2). If $u(s,x)$ is smooth enough, it is a classical solution. By analytical methods it is shown that this solution is unique in the class of bounded measurable functions. See details, for example, in [4].

Thus, completeness of the stochastic flow $\xi(s)$ (i.e., global-in-time existence of solutions of the above-mentioned stochastic differential equation) is equivalent to global-in-time existence of solutions of (6.1)–(6.2).

As a corollary to Theorem 5.10 and Corollary 5.11 we obtain the following theorem.

Theorem 6.1. *If the flow $\xi(s)$ is continuous at infinity, the solutions of (6.1)–(6.2) exist globally in time if and only if for any $T > 0$ there exists a positive proper function $v^T : M^T \to R$ that is C^1-smooth in $t \in [0,T]$, C^2-smooth in $x \in M$ and such that $\mathcal{A}^T v^T(t,x) < C$ for a certain constant $C > 0$ at all points $(t,x) \in M^T$.*

Of course it is important to have conditions for global-in-time existence of solutions of (6.1)–(6.2) without referring to the properties of corresponding flow $\xi(s)$. For this purpose we select a smaller class of equations according to the following.

Definition 6.2 (see [1]). The flow $\xi(s)$ and the corresponding semigroup S are called to have C_0 property if for any compact $K \subset M$ the relation

$$\lim_{x \longrightarrow +\infty} \mathrm{P}(T_K(\xi_{t,x}) < T) = 0 \tag{6.7}$$

holds where $T_K(\xi_{t,x})$ is the first entrance time of $\xi_{t,x}$ in K.

It is well known that C_0 property is equivalent to the fact that the operators from semigroup S leave invariant the space $C_0(M)$ of continuous functions, tending to zero at infinity (see, e.g., [14, 16, 17] for details). Some conditions, under which C_0 property is satisfied, are found in [1].

Proposition 6.3. *Any flow with C_0 property is continuous at infinity.*

Proposition 6.3 follows from the obvious fact that $\mathrm{P}(T_K(\gamma_{t,x}) < T) \geq \mathrm{P}(\gamma_{t,x}(T) \in K)$. We refer the reader to [14, 16, 17] for some details on behavior of a stochastic flow at infinity and on relations between C_0 property and continuity at infinity.

From Proposition 6.3, Theorem 5.10, and Corollary 5.11 we get the following.

Theorem 6.4. *If operators (6.5) are C_0, the solutions of (6.1)–(6.2) exist globally in time if and only if for any $T > 0$ there exists a positive proper function $v^T : M^T \to R$ that is C^1-smooth in $t \in [0,T]$, C^2-smooth in $x \in M$ and such that $\mathcal{A}^T v^T(t,x) < C$ for a certain constant $C > 0$ at all points $(t,x) \in M^T$.*

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ON CALCULATION OF THE RELATIVE INDEX OF A FIXED POINT IN THE NONDEGENERATE CASE

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The paper is devoted to the calculation of the index of a zero and the asymptotic index of a linear completely continuous nonnegative operator. Also the case of a nonlinear completely continuous operator A whose domain and image are situated in a closed convex set Q of a Banach space is considered. For this case, we formulate the rules for calculating the index of an arbitrary fixed point and the asymptotic index under the assumption that the corresponding linearizations exist and the operators of derivative do not have eigenvectors with eigenvalue 1 in some wedges.

1. Introduction

Let Q be a closed convex set in a Banach space and let $A : Q \to Q$ be a completely continuous operator. In [4] the calculation problem of a fixed point index of a vector field $I - A$ was formulated. In the simplest case, when Q is a cone, this problem was investigated in the articles by Isaenko [3], Mukhamadiev and Sabirov [5], and Pokornyi [6] (see also references in [8]). Later, in [1, 2] Dancer presented the general formula for the fixed point index of a completely continuous operator A with its domain and image in an arbitrary closed convex set. However, the case of an asymptotic index was not considered. Note that for the case of a cone this problem was earlier considered in the articles by Pokornyi [6] and Pokornyi and Astaf'eva [7]. The present paper concerns the cases of a wedge and an arbitrary closed convex set. In the latter case, the calculation of a fixed point index is reduced to the index calculation with respect to a specially constructed wedge. We also show that for the infinity singular point one needs to take the wedge

$$W_\infty = \{h \in X : x + th \in Q \ (x \in Q,\ 0 \le t < \infty)\}. \tag{1.1}$$

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2. Index of a linear operator

Let X be a Banach space, W a wedge in X (this means that W is a closed subset of X such that $W + W \subseteq W$ and $\lambda W \subset W$ for $\lambda \geq 0$), and A a linear completely continuous operator such that

$$AW \subseteq W. \tag{2.1}$$

Let $L = W \cap (-W)$. Then L is the maximal subspace which is contained in W. From (2.1) and the linearity of A it follows that the inclusion

$$A(L) \subseteq L \tag{2.2}$$

holds.

Consider a quotient space X/L and a quotient mapping $[\cdot] : X \to X/L$. It is easy to check that the image $\widehat{W}$ of W under the quotient mapping $[\cdot]$ is a cone in X/L.

From (2.2) it follows that the operator A induces a linear mapping $\widehat{A}$ of the quotient space X/L into itself such that $\widehat{A}[x] = [Ax]$. And by (2.1) we have that the cone $\widehat{W}$ is invariant under the operator $\widehat{A}$, that is, $\widehat{A}$ is nonnegative in the quotient space X/L.

Theorem 2.1. *Let A be a linear completely continuous operator, acting in a Banach space X, and let W be a wedge that is invariant under the operator A. Suppose that $Ax \neq x$ for $x \in W$, $x \neq 0$. Then $\rho(\widehat{A}) \neq 1$ and*

$$\operatorname{ind}(0, I - A; W) = \operatorname{ind}(\infty, I - A; W) = \begin{cases} (-1)^{\beta(A|_L)} & \text{if } \rho(\widehat{A}) < 1, \\ 0 & \text{if } \rho(\widehat{A}) > 1, \end{cases} \tag{2.3}$$

where $A|_L$ is the restriction of the operator A to the space L, $\beta(A|_L)$ is the sum of multiplicities of eigenvalues of $A|_L$, greater than 1, and $\rho(\widehat{A})$ is the positive spectral radius of the operator $\widehat{A}$.

Proof. If $Ax \neq x$ for $x \in W$, $x \neq 0$, then zero and infinity singular points of the vector field $\Phi = I - A$ are isolated in W. Hence the relative indices $\operatorname{ind}(0, I - A; W)$ and $\operatorname{ind}(\infty, I - A; W)$ are well posed. In this case, by the definition of index at infinity (see, e.g., [4]), since the operator A has no more fixed points in W, it follows that $\operatorname{ind}(\infty, I - A; W) = \operatorname{ind}(0, I - A; W)$.

To calculate the index $\operatorname{ind}(0, \Phi; W)$ we will consider two possible cases: when the spectral radius $\rho(\widehat{A})$ of the operator $\widehat{A}$ is less than 1 and when it is greater than 1. The case $\rho(\widehat{A}) = 1$ is impossible. Indeed, if this is not true, then there exists an element $[x]^* \neq 0$ of the quotient space X/L such that $\widehat{A}[x]^* = [x]^*$. In other words, there exist $u \in W$, $u \notin L$, and $y \in L$ such that the equality $u - Au = y$ holds. Let us explore the solvability in L of $(I - A)w = y$. If it had a solution $w \in L$, then the vector $u - w$ would be an eigenvector of the operator A, corresponding to the eigenvalue 1 and would be in W, which would contradict our assumptions. If we supposed that the equation had no solutions in L, then the operator $I - A$ would be invertible in L, which is impossible.

In the case $\rho(\widehat{A}) < 1$, let us show that the vector field $\Phi x = x - Ax$ is linearly relatively homotopic on $S_W = \{x \in W : \|x\| = 1\}$ to the field $\Phi_1 x = x - AQ(x)$, where $Q : X \to L$ is a projection (in general, nonlinear) of X on L (see [4, Theorem 18.1]). To prove this fact assume the converse, that is, that

$$x = (1-\lambda)Ax + \lambda AQx \tag{2.4}$$

has a solution for some $x \in S_W$ and $\lambda \in [0,1]$. Now if we rewrite this equation for the operator $\widehat{A}$ and recall (2.2), we get

$$[x] = (1-\lambda)\widehat{A}[x] \quad ([x] \in \widehat{S}_W, \lambda \in [0,1]), \tag{2.5}$$

where $\widehat{S}_W = \{[x] \in \widehat{W} : \|[x]\| = 1\}$.

Under our assumption on the spectral radius of the operator $\widehat{A}$ this yields $[x] = 0$. In this case, we obtain $x = Qx$ and then (2.4) implies $x = Ax$ for $x \in S_W$, which contradicts the assumption of our theorem.

Thus the relative index $\operatorname{ind}(0, I - A; W)$ is equal to the relative index $\operatorname{ind}(0, I - AQ; W)$, which actually is the Leray-Schauder zero fixed point index of the restriction $A|_L$ of the operator A to the subspace L and we can calculate it by the well-known formula (see, e.g., [4, Theorem 21.1])

$$\operatorname{ind}(0, I - AQ; W) = (-1)^{\beta(A|_L)}, \tag{2.6}$$

where $\beta(A|_L)$ is the sum of multiplicities of eigenvalues of $A|_L$, greater than 1.

Now assume that $\rho(\widehat{A}) > 1$. In this case, there exists an element $[x]^*$ in the cone $\widehat{W}$ such that $\widehat{A}[x]^* = \rho[x]^*(\rho > 1)$. In other words, there exist an element $x^* \notin L$ of the wedge W (we assume that $\|x^*\| = 1$) and an element z of the subspace L such that $Ax^* = \rho x^* + z$.

Show that the vector field $\Phi x = x - Ax$ is linearly relatively homotopic on S_W to the field $\Phi_2 x = x - cx^*$, where the constant c will be defined later. Let us show that

$$x = (1-\lambda)Ax + \lambda cx^* \tag{2.7}$$

has no solutions for $x \in S_W$, $\lambda \in [0,1]$.

For $\lambda = 0$, (2.7) coincides with $x = Ax$. The latter equation has no solutions for $x \in S_W$. For $0 < \lambda \le 1$ from (2.7), it follows that there exists a real $t > 0$ such that

$$x \ge tx^*. \tag{2.8}$$

We claim that there exists the maximal of such reals: $\xi = \max_{x \in S_W, x \ge tx^*} t$. We argue by contradiction. The inequality $x^* \le x/t$ implies that if t tends to infinity, then $x^* \le 0$. On the other hand, x^* is an element of the wedge W, thus $x^* \ge 0$. Hence we get that $x^* \in L$, which contradicts its choice.

Further, (2.7) and (2.8) imply

$$x \ge (1-\lambda)\xi\rho x^* + \lambda cx^* = ((1-\lambda)\xi\rho + \lambda c)x^*. \tag{2.9}$$

Choose $c > \xi$ and observe that for this c the inequality $(1-\lambda)\xi\rho + \lambda c > \xi$ holds. Indeed, for $\lambda = 0$ this follows from $\rho > 1$. For $\lambda = 1$ this follows from our assumption for c. For other $\lambda \in (0,1)$ the inequality holds as the result of two previous cases.

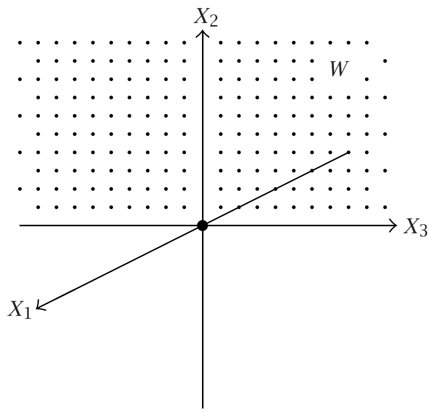

Figure 2.1

Therefore for the chosen c, the real ξ is not maximal among t such that $x \geq tx^*$. This contradiction proves the nondegeneracy of the linear homotopy connecting vector fields Φ and Φ_2. Consequently, $\operatorname{ind}(0,\Phi;W) = \operatorname{ind}(0,\Phi_2;W)$.

To calculate $\operatorname{ind}(0,\Phi_2;W)$ we will use the corollary of the Hahn-Banach theorem. According to it there exists a functional $l \in X^*$ such that $l(W) \geq 0$ and $l(x^*) = 1$ (then $\|l\| \geq 1$). Since (2.8), we obtain $\xi \leq l(x) \leq \|l\|$ for $x \in S_W$. And our assumption of c yields $c > \|l\| \geq 1$. Now we can show that the vector field $\Phi_2 x = x - cx^*$ is nondegenerate on $\mathrm{cl}B_W = \{x \in W : \|x\| \leq 1\}$. Indeed,

$$\|x - cx^*\| \geq \|cx^*\| - 1 = c - 1 > 0. \tag{2.10}$$

Hence, by the relative rotation property, $\operatorname{ind}(0,\Phi_2;W) = 0$. This completes the proof. □

In applications there usually exist a complement X_1 of the linear hull $L(W) = W - W$ of the wedge W to X and a complement X_2 of a maximal subspace $X_3 = W \cap (-W)$ in W to $L(W)$. Then X can be presented as the direct sum of subspaces (see Figure 2.1)

$$X = X_1 \dot{+} X_2 \dot{+} X_3. \tag{2.11}$$

From (2.1) and the linearity of the operator A it follows that the inclusions

$$A(X_2 \dot{+} X_3) \subseteq X_2 \dot{+} X_3, \qquad A(X_3) \subseteq X_3 \tag{2.12}$$

hold.

Assume that an intersection $W \cap X_2$ is not empty. Then it is easy to prove that this set is a cone K in X_2. It generates the order relation in X_2 by the following rule: $x \leq y$ if $y - x \in K$. The cone K can be set as $K = \{x \in X_2 : x \geq 0\}$.

It can be proved that under such decomposition of the space X, the wedge W is invariant under a linear operator A if and only if A is determined by the matrix

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{2.13}$$

where $a_{22} \geq 0$. Furthermore, one can show that zero and infinity singular points of such operator A are isolated in W if and only if 1 is not an eigenvalue of the operators a_{22} and a_{33}. In this case, Theorem 2.1 can be formulated in the following way.

Theorem 2.2. *Let A be a linear completely continuous operator, acting in a Banach space X, and let W be a wedge that is invariant under the operator A. Then A can be defined by matrix (2.13). If 1 is not an eigenvalue of the operators a_{22} and a_{33}, then zero and infinity singular points of the operator A are isolated in W and*

$$\operatorname{ind}(0, I - A; W) = \operatorname{ind}(\infty, I - A; W) = \begin{cases} (-1)^{\beta(a_{33})} & \text{if } \rho(a_{22}) < 1, \\ 0 & \text{if } \rho(a_{22}) > 1, \end{cases} \tag{2.14}$$

where $\rho(a_{22})$ is the spectral radius of the operator a_{22} and $\beta(a_{33})$ is the sum of multiplicities of eigenvalues of the operator a_{33}, greater than 1.

3. Index of a nonlinear operator

Let A be a nonlinear operator and let Q be a closed convex set of a Banach space X that is invariant under the operator A. In this part, we discuss the relative fixed point index calculations of A under the assumption that A has a Fréchet derivative at its fixed point. Recall that an operator A is called *differentiable at the point x_0 with respect to Q*, if

$$\lim_{\substack{x \in Q, \\ x \to x_0}} \frac{||Ax - Ax_0 - A'(x_0)x||}{||x||} = 0, \tag{3.1}$$

and *differentiable at infinity with respect to Q*, if

$$\lim_{\substack{x \in Q, \\ x \to \infty}} \frac{||Ax - A'(\infty)x||}{||x||} = 0, \tag{3.2}$$

where $A'(x_0)$ and $A'(\infty)$ are linear operators.

Let Q be an arbitrary closed convex set of a Banach space X. Assume that Q is invariant under a completely continuous operator A. Let x_0 be a fixed point of the operator A and let A have a Fréchet derivative at the point x_0 with respect to Q.

The following lemmas show that in the case when the fixed point x_0 is not the infinity singular point some wedge W_{x_0} comprising the set Q is invariant under the mapping of the derivative $A'(x_0)$, whereas in the case of the infinity singular point some wedge W_∞ lying in the set Q is invariant under the mapping of derivative $A'(\infty)$.

LEMMA 3.1. *Let A be a completely continuous operator, acting in a Banach space X, and let $Q \subseteq X$ be a closed convex set that is invariant under A. Let $x_0 \in Q$ be a fixed point of the operator A and let A have Fréchet derivative at x_0 with respect to Q. Then the wedge*

$$W_{x_0} = \mathrm{cl}\{h = t(x - x_0) : t \geq 0,\ x \in Q\} \tag{3.3}$$

is invariant under the mapping of the derivative $A'(x_0)$.

Proof. To prove that W_{x_0} is actually a wedge it is sufficient to show that W_{x_0} is closed under addition of its elements and their multiplication by nonnegative reals. If $h_1, h_2 \in W_{x_0}$, then $h_1 = t_1(x_1 - x_0)$, $h_2 = t_2(x_2 - x_0)$, where $x_1, x_2 \in Q$ and $0 \leq t_1, t_2 < \infty$. Thus for any α and β for which $0 \leq \alpha, \beta < \infty$, and $\alpha t_1 + \beta t_2 > 0$, we have

$$\begin{aligned} \alpha h_1 + \beta h_2 &= \alpha t_1(x_1 - x_0) + \beta t_2(x_2 - x_0) \\ &= (\alpha t_1 + \beta t_2)\left(\frac{\alpha t_1}{\alpha t_1 + \beta t_2}(x_1 - x_0) + \frac{\beta t_2}{\alpha t_1 + \beta t_2}(x_2 - x_0)\right) \\ &= (\alpha t_1 + \beta t_2)\left(\frac{\alpha t_1}{\alpha t_1 + \beta t_2}x_1 + \frac{\beta t_2}{\alpha t_1 + \beta t_2}x_2 - x_0\right). \end{aligned} \tag{3.4}$$

An element

$$\frac{\alpha t_1}{\alpha t_1 + \beta t_2}x_1 + \frac{\beta t_2}{\alpha t_1 + \beta t_2}x_2 \tag{3.5}$$

is a convex combination of elements $x_1, x_2 \in Q$ and therefore it is in Q. Thus (3.4) implies $\alpha h_1 + \beta h_2 \in W_{x_0}$. If $\alpha t_1 + \beta t_2 = 0$, then $\alpha h_1 + \beta h_2 = 0$ and thus $\alpha h_1 + \beta h_2 \in W_{x_0}$.

Let h be an arbitrary nonzero element of W_{x_0}. Then there exist $x \in Q$ and $t > 0$ such that $h = t(x - x_0)$. From the differentiability of the operator A at x_0 and the linearity of the operator $A'(x_0)$ it follows that

$$t(Ax - x_0) = A'(x_0)h + t\omega\left(\frac{h}{t}\right). \tag{3.6}$$

Since $A(Q) \subseteq Q$, the element in the left-hand side of this equality is in W_{x_0}. Taking the limit as $t \to \infty$, by the closedness of W_{x_0}, we get $A'(x_0)h \in W_{x_0}$. This completes the proof. □

LEMMA 3.2. *Let A be a completely continuous operator, acting in a Banach space X, and let $Q \subseteq X$ be a closed convex unbounded set that is invariant under A. Let A have Fréchet derivative at infinity with respect to Q. Then the wedge*

$$W_\infty = \{h \in X : x + th \in Q(x \in Q, 0 \leq t < \infty)\} \tag{3.7}$$

is invariant under the mapping of the derivative $A'(\infty)$.

Proof. To prove that W_∞ is invariant under the operator $A'(\infty)$, it suffices to show that there exists an element $x_* \in Q$ such that $x_* + tA'(\infty)h \in Q$ for any $h \in W_\infty$ and $0 \leq t < \infty$. Let $x_* \in Q$, then for any $\lambda > 0$, $t \geq 0$, $h \in W_\infty$ we have $x_* + \lambda th \in Q$. Since Q is invariant under the operator A, we get $A'(\infty)x_* + \lambda t A'(\infty)h + w(x_* + \lambda t h) \in Q$ for any $\lambda \geq 0$, $t \geq 0$, $h \in W_\infty$. Then, by the convexity of Q, for any $\lambda \geq 0$ we have

$$\left(1 - \frac{1}{\lambda}\right)x_* + \frac{1}{\lambda}A'(\infty)x_* + tA'(\infty)h + \frac{1}{\lambda}w(x_* + \lambda t h) \in Q. \tag{3.8}$$

Taking the limit as $\lambda \to \infty$, by the closedness of Q, we get $x_* + tA'(\infty)h \in Q$ for any $t \geq 0$, $h \in W_\infty$. This completes the proof. □

The following theorem specifies the main result of Dancer [2, Theorem 1].

Theorem 3.3. *Let A be a completely continuous operator, acting in a Banach space X, and let $Q \subseteq X$ be a closed convex set that is invariant under A. Let $x_0 \in \partial Q$ be a fixed point of the operator A and let A have Fréchet derivative at x_0 with respect to Q. Then the wedge*

$$W_{x_0} = \mathrm{cl}\,\{h = t(x - x_0) : t \geq 0,\ x \in Q\} \tag{3.9}$$

is invariant under the mapping of the derivative $A'(x_0)$.

If $A'(x_0)x \neq x$ for $x \in W_{x_0}$, $x \neq 0$, then the fixed point x_0 of the vector field $\Phi = I - A$ is isolated in Q and

$$\mathrm{ind}\,(x_0, I - A; Q) = \mathrm{ind}\,(0, I - A'(x_0); W_{x_0}). \tag{3.10}$$

Proof. Without loss of generality it can be assumed that $x_0 = 0$ (in the opposite case, the whole argument needs to be made for the operator $A(x_0 + x) - x_0$).

From the differentiability of the operator A at the point 0 it follows that there exists a linear operator $B = A'(0)$ such that

$$Ax = Bx + w(x), \tag{3.11}$$

where the operator w meets the condition

$$\lim_{\substack{x \in Q,\\ x \to 0}} \frac{\|w(x)\|}{\|x\|} = 0. \tag{3.12}$$

If $Bx \neq x$ for $x \in W_0$, $x \neq 0$, there exists a positive real $c > 0$ such that for any $x \in W_0$ the inequality

$$\|x - Bx\| \geq c\|x\| \tag{3.13}$$

holds.

Choose a real $r > 0$ such that inequalities

$$\|Ax - Bx\| \leq \frac{c}{2}\|x\|, \qquad \frac{cr}{(c + 2\|B\|)\rho_r} - 1 > 0 \tag{3.14}$$

hold for $x \in Q$, $\|x\| \leq r$, where $\rho_r = \sup_{x \in W_0,\ \|x\| = r} \rho(x, Q)$ and $\rho(x, Q)$ denotes the distance from the point x to the set Q.

Show that on the intersection of the sphere $S_r = \{x \in X : \|x\| = r\}$ of radius r with the set W_0 the vector field $\Phi' = I - B$ is linearly homotopic to the field $\Phi_\alpha x = x - AP_\alpha x$ where α meets the condition

$$0 < \alpha < \frac{cr}{(c + 2\|B\|)\rho_r} - 1, \tag{3.15}$$

and P_α is a projection (in general, nonlinear) on Q that has the following property:

$$\|x - P_\alpha x\| \le (1+\alpha)\rho(x,Q) \quad (x \in X) \tag{3.16}$$

(the existence of such projection follows from [4, Theorem 18.1]).

To prove this, consider the linear deformation

$$\Phi(\lambda, x) = x - (1-\lambda)Bx - \lambda AP_\alpha x \quad (x \in S_r \cap W_0,\ \lambda \in [0,1]) \tag{3.17}$$

that connects vector fields Φ' and Φ_α. From the convexity of W_0 and the invariance of the sets W_0 and $Q \subseteq W_0$ under operators B and A, respectively, it follows that the element $(1-\lambda)Bx + \lambda AP_\alpha x$ is in W_0 for any $x \in S_r \cap W_0$ and $\lambda \in [0,1]$.

The nondegeneracy of the deformation $\Phi(\lambda, x)$ for $x \in S_r \cap W_0$, $\lambda \in [0,1]$ follows from the inequalities

$$\begin{aligned}
&\|x - (1-\lambda)Bx - \lambda AP_\alpha x\| \\
&\quad \ge \|x - Bx\| - \lambda\|AP_\alpha x - Bx\| \\
&\quad \ge \|x - Bx\| - \|AP_\alpha x - BP_\alpha x\| - \|Bx - BP_\alpha x\| \\
&\quad \ge cr - \frac{c}{2}\big(r + (1+\alpha)\rho_r\big) - \|B\|(1+\alpha)\rho_r = \frac{cr}{2} - \left(\frac{c}{2} + \|B\|\right)(1+\alpha)\rho_r > 0.
\end{aligned} \tag{3.18}$$

Hence the vector fields Φ' and Φ_α are homotopic on $S_r \cap W_0$. Thus, by the first property of the relative rotation,

$$\operatorname{ind}(0, I - B; W_0) = \operatorname{ind}(0, I - AP_\alpha; W_0). \tag{3.19}$$

By definition, the relative index $\operatorname{ind}(0, I - AP_\alpha; W_0)$ is equal to the relative rotation $\gamma(I - AP_\alpha, B_r \cap W_0; W_0)$ of the vector field $I - AP_\alpha$ on the boundary of an open set $B_r \cap W_0$, where $B_r = \{x \in X : \|x\| < r\}$. By the additivity property of rotation,

$$\gamma(I - AP_\alpha, B_r \cap W_0; W_0) = \gamma(I - AP_\alpha, B_r \cap Q; W_0) + \gamma(I - AP_\alpha, B_r \cap (W_0 \setminus Q); W_0). \tag{3.20}$$

From the fact that AP_α has no fixed points beyond Q, it follows that $\gamma(I - AP_\alpha, B_r \cap (W_0 \setminus Q); W_0) = 0$. On the other hand, the relative rotation $\gamma(I - AP_\alpha, B_r \cap Q; W_0)$ can be considered as the rotation $\gamma(I - A, B_r \cap Q; Q)$ of the vector field $I - A$ on the boundary of the open set $B_r \cap Q$ with respect to Q. By the definition of relative index, this rotation coincides with $\operatorname{ind}(0, I - A; Q)$. This completes the proof. □

As it appears, the analogous statement is true for the case of asymptotic index.

Theorem 3.4. *Let A be a completely continuous operator, acting in a Banach space X, and let $Q \subseteq X$ be a closed convex unbounded set that is invariant under A. Let A have Fréchet derivative at infinity with respect to Q. Then the wedge*

$$W_\infty = \{h \in X : x + th \in Q \ (x \in Q, \ 0 \le t < \infty)\} \tag{3.21}$$

is invariant under the mapping of the derivative $A'(\infty)$.

If $W_\infty \neq \{0\}$ and $A'(\infty)x \neq x$ for $x \in W_\infty$, $x \neq 0$, then the infinity singular point of the vector field $\Phi = I - A$ is isolated in Q and

$$\operatorname{ind}(\infty, I - A; Q) = \operatorname{ind}\left(0, I - A'(\infty); W_\infty\right). \tag{3.22}$$

Proof. From the differentiability of the operator A at infinity it follows that there exists linear operator $B = A'(\infty)$ such that

$$Ax = Bx + w(x), \tag{3.23}$$

where the operator w meets the condition

$$\lim_{\substack{x \in Q, \\ x \to \infty}} \frac{\|w(x)\|}{\|x\|} = 0. \tag{3.24}$$

If $Bx \neq x$ for $x \in W_\infty$, $x \neq 0$, then there exists a positive real $c > 0$ such that for all $x \in W_\infty$ the inequality

$$\|x - Bx\| \ge c\|x\| \tag{3.25}$$

holds.

Choose a real $R > 0$ such that inequalities

$$\|Ax - Bx\| \le \frac{c}{2}\|x\|, \qquad \frac{cR}{2(1 + c + \|B\|)\rho_R} - 1 > 0 \tag{3.26}$$

hold for all $x \in Q$, $\|x\| \ge R$, where $\rho_R = \sup_{x \in Q, \ \|x\| = R} \rho(x, W_\infty)$ and $\rho(x, W_\infty)$ denotes the distance from the point x to the set W_∞.

Show that on the intersection of the sphere $S_R = \{x \in X : \|x\| = R\}$ of radius R with the set Q the vector field $\Phi = I - A$ is linearly homotopic to the field $\Phi_\alpha x = x - BP_\alpha$, where a real α meets the condition

$$0 < \alpha < \frac{cR}{2(1 + c + \|B\|)\rho_R} - 1, \tag{3.27}$$

and P_α is a projection (in general, nonlinear) on W_∞ that has the following property:

$$\|x - P_\alpha x\| \le (1 + \alpha)\rho(x, W_\infty) \quad (x \in X). \tag{3.28}$$

Consider the linear deformation

$$\Phi(\lambda, x) = x - (1 - \lambda)BP_\alpha x - \lambda Ax \quad (x \in S_R \cap Q, \ \lambda \in [0, 1]) \tag{3.29}$$

that connects vector fields Φ and Φ_α. Since the convexity of Q and the invariance of the sets $W_\infty \subseteq Q$ and Q under the operators B and A, respectively, it follows that the element $(1-\lambda)BP_\alpha x + \lambda Ax$ is in Q for any $x \in S_R \cap Q$ and $\lambda \in [0,1]$.

The nondegeneracy of $\Phi(\lambda,x)$ for $x \in S_R \cap Q$, $\lambda \in [0,1]$ follows from the inequalities

$$\begin{aligned}
&||x-(1-\lambda)BP_\alpha x-\lambda Ax|| \\
&\quad \geq ||x-BP_\alpha x||-\lambda||Ax-BP_\alpha x|| \\
&\quad \geq ||P_\alpha x-BP_\alpha x||-||x-P_\alpha x||-\|Ax-Bx\|-||Bx-BP_\alpha x|| \\
&\quad \geq c(R-(1+\alpha)\rho_R)-(1+\alpha)\rho_R-\frac{cR}{2}-\|B\|(1+\alpha)\rho_R \\
&\quad = \frac{cR}{2}-(1+c+\|B\|)(1+\alpha)\rho_R>0.
\end{aligned} \tag{3.30}$$

Hence the vector fields Φ and Φ_α are homotopic on $S_R \cap Q$. Thus, by the first property of the relative rotation,

$$\operatorname{ind}(\infty, I-A;Q) = \operatorname{ind}(\infty, I-BP_\alpha;Q). \tag{3.31}$$

By definition, the relative index $\operatorname{ind}(\infty, I-BP_\alpha;Q)$ is equal to the relative rotation $\gamma(I-BP_\alpha, B_R \cap Q;Q)$ of the vector field $I-BP_\alpha$ on the boundary of an open set $B_R \cap Q$, where $B_R = \{x \in X : \|x\| < R\}$. By the additivity property of rotation,

$$\gamma(I-BP_\alpha, B_R\cap Q;Q) = \gamma(I-BP_\alpha, B_R\cap W_\infty;Q)+\gamma(I-BP_\alpha, B_R\cap(Q\setminus W_\infty);Q). \tag{3.32}$$

From the fact that the operator BP_α has no fixed points beyond W_∞, it follows that the relative rotation $\gamma(I-BP_\alpha, B_R\cap(Q\setminus W_\infty);Q)$ is equal to zero. Finally, the relative rotation $\gamma(I-BP_\alpha, B_R\cap W_\infty;Q)$ can be considered as the rotation $\gamma(I-B, B_R\cap W_\infty;W_\infty)$ of the vector field $I-B$ on the boundary of open set $B_R\cap W_\infty$ with respect to W_∞. By the definition of relative index and since the operator B has no nonzero fixed points in W_∞, the latter rotation coincides with $\operatorname{ind}(0, I-B;W_\infty)$. This completes the proof. □

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THEOREM ON THE UNION OF TWO TOPOLOGICALLY FLAT CELLS OF CODIMENSION 1 IN $\mathbb{R}^n$

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In this paper we give a complete and improved proof of the "Theorem on the union of two $(n-1)$-cells." First time it was proved by the author in the form of reduction to the earlier author's technique. Then the same reduction by the same method was carried out by Kirby. The proof presented here gives a more clear reduction. We also present here the exposition of this technique in application to the given task. Besides, we use a modification of the method, connected with cyclic ramified coverings, that allows us to bypass referring to the engulfing lemma as well as to other multidimensional results, and so the theorem is proved also for spaces of any dimension. Thus, our exposition is complete and does not require references to other works for the needed technique.

1. Notations and statement of the result

Specify the standard coordinate system $Ox^1 \cdots x^n$ with the origin O and coordinate axes Ox^i in the space $\mathbb{R}^n$. The coordinate planes $Ox^1 \cdots x^i$ will be identified with $\mathbb{R}^i$. The unit disc in $\mathbb{R}^i$ is denoted by B^i. The semispaces $x^{n-1} \geq 0$ and $x^{n-1} \leq 0$ are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_-$, respectively, while semiplanes $\mathbb{R}^n_+ \cap \mathbb{R}^{n-1}$ and $\mathbb{R}^n_- \cap \mathbb{R}^{n-1}$ as $\mathbb{R}^{n-1}_+$ and $\mathbb{R}^{n-1}_-$, respectively. Semidiscs $B^{n-1} \cap \mathbb{R}^{n-1}_+$ and $B^{n-1} \cap \mathbb{R}^{n-1}_-$ are denoted by B^{n-1}_+ and B^{n-1}_-, respectively.

We will say that an embedding $q : B^i \to M^n$ of an i-disc in a topological n-manifold without boundary is *topologically flat* if one can extend it to an embedding in M^n of its neighborhood in $\mathbb{R}^n$. It is known that a topologically flat embedding of a disc into $\mathbb{R}^n$ is extendable to a homeomorphism of $\mathbb{R}^n$ onto itself. An embedding of a submanifold is *locally flat* if every point has a neighborhood in it that is homeomorphic to a disc and the embedding on this disc is topologically flat. Any locally flat embedding of a disc is topologically flat (see, e.g., [8]).

THEOREM 1.1. *Let an embedding $q : B^{n-1} \to \mathbb{R}^n$ be given, whose restrictions to both semidiscs B^{n-1}_+ and B^{n-1}_- are topologically flat. Then q is topologically flat.*

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We will denote the restriction of q onto $B_{\pm}^{n-1}$ as $q_{\pm}$, respectively.

Notice an important corollary to this theorem (first time mentioned by Cantrell [1]) that in the case $n \geq 4$ for embedding of an $(n-1)$-manifold into an n-manifold there are no isolated points where the condition of locally flatness is destroyed. If $n = 3$, it is not the case. The reason for this difference is the fact that for $n \geq 4$ an isolated singularity cannot exist on the boundary of an $(n-1)$-submanifold, and this is derived from the fact that in the former dimensions the arcs with only one singularity do not exist, (see [5, 6]) while it is well known that in the dimension 3 they do exist.

The proof of this theorem is based on a series of lemmas using the constructions of some elementary homeomorphisms described in Section 2. Here we introduce some notations.

Denote by Π_α the semiplane, bounded by subspace R^{n-2} and having the angle of α radians with $\mathbb{R}_+^{n-1} = \Pi_0$. ($\Pi_{-\pi} = \Pi_\pi = \mathbb{R}_-^{n-1}$.) $Q[\alpha,\beta]$, $\alpha < \beta$ will denote a closed domain between Π_α and Π_β ($Q[\alpha,\beta] = \cup_{\alpha \leq \gamma \leq \beta} \Pi_\gamma$), $Q(\alpha,\beta)$ denotes the interior of $Q[\alpha,\beta]$.

For a point $z \in \mathbb{R}^n$ we denote by x_z its projection onto $\mathbb{R}^{n-2}$ and by y_z its projection onto $\mathbb{R}^{n-1}$.

Consider the system of 2-planes P_x, $x \in \mathbb{R}^{n-2}$ orthogonal to $\mathbb{R}^{n-2}$ at the corresponding points x. Consider also in every plane P_x an orthonormal coordinate system with the origin x and axes xs and xt, the former is parallel and codirected with the axe Ox^{n-1}, and the latter is parallel and codirected with the axe Ox^n, s and t have the meaning of coordinate parameters. For a point $z \in \mathbb{R}^n$ we denote by s_z and t_z its coordinates in the plane P_{x_z}. At last, $C_x(r)$ will denote a circle with the radius r in the plane P_x centred at x. For a point $z \in \mathbb{R}^n$ we denote by r_z its distance from $\mathbb{R}^{n-2}$, that is, the radius r of a circle $C_{x_z}(r)$ passing through z.

2. Preliminary statements

The following two statements will help us to construct some elementary homeomorphisms of $\mathbb{R}^n$ that send every circle $C_x(r)$ onto itself piecewise linearly.

Statement 2.1. Let for some α a closed subset $M \subset \mathbb{R}^n$ be given such that in some neighborhood of B^{n-2} it does not intersect $\Pi_\alpha \setminus \mathbb{R}^{n-2}$ and lies on one side of Π_α (i.e., in $Q(\alpha,\alpha+\pi)$ or in $Q(\alpha-\pi,\alpha)$).

Then there exists a function $\varepsilon(z) > 0$, $z \in \Pi_\alpha \setminus \mathbb{R}^{n-2}$ (possibly in a smaller neighborhood of B^{n-2}), that is continuous, tending to zero as z is tending to a point in $\mathbb{R}^{n-2}$, and such that for any circle $C_x(r)$ in this neighborhood its arc with the length $\varepsilon(z)$, having an end in $z \in \Pi_\alpha$ and lying on one side of Π_α as M, does not intersect M.

The construction of $\varepsilon(z)$ is standard and evident, so that it may be omitted.

For $z \in \Pi_\alpha$ consider arcs of the circles $C_{x_z}(r_z)$ having one end at z and the length $\varepsilon(z)$, where this function is chosen according to Statement 2.1 for some set M, the arcs are taken on one side of Π_α, as M. The surface, described by the second ends of these arcs (i.e., not on Π_α) will be called the *fence* separating M from Π_α.

Note that every circle C_x (sufficiently close to B^{n-2}) intersects every Π_α and every fence exactly one time.

Statement 2.2. Let four sets A, B, C, D in a neighborhood of B^{n-2} be given so that each of them is either a Π_α or a fence, and the points of intersections of B and C with any circle $C_x(r)$ in this neighborhood are located between points of intersection of this circle with the sets A and D. Then there exists a homeomorphism of $\mathbb{R}^n$, identical outside a neighborhood of B^{n-2} and outside the domain between A and D (containing B and C), that sends B into C in a smaller neighborhood.

For the proof it is sufficient to construct a homeomorphism on every circle $C_x(r)$ in a small neighborhood of B^{n-2} that maps linearly the arc between A and B into the arc between A and C and simultaneously the arc between B and D into the arc between C and D, such that it is identical on the second arc between A and D. In some larger neighborhood one can continuously reduce this homeomorphism to the identity.

The homeomorphisms constructed as in this proof will be called *arcwise*. Note that arcwise homeomorphisms are naturally isotopic to the identity.

Before turning to our lemmas, let us introduce the following definitions.

Defintions 2.3. An embedding $\gamma : \Pi_\alpha \to \mathbb{R}^n$, being identity on $\mathbb{R}^{n-2}$, touches Π_β at points of B^{n-2} if for every $\varepsilon > 0$ one can find $\delta > 0$ so that $\gamma\Pi_\alpha \cap O_\delta(B^{n-2}) \subset Q[\beta - \varepsilon, \beta + \varepsilon]$.

Analogously, a sequence of points $z_n \in \mathbb{R}^n$ touches Π_α at a point $x \in B^{n-2}$ if for every $\varepsilon > 0$ there exists n_ε such that $z_n \in Q[\beta - \varepsilon, \beta + \varepsilon] \cap O_\varepsilon(x)$ for all $n > n_\varepsilon$.

3. Lemmas

Lemma 3.1. *Let an embedding $p_1 : B^n \setminus (B_-^{n-1} \setminus B^{n-2}) \to \mathbb{R}^n$ be identical on B^{n-2}, and for every $\alpha \in [-\pi + \pi/4, \pi - \pi/4]$ the set $p_1(\Pi_\alpha)$ touches Π_α. Let also $p_1 B_+^{n-1} \subset \mathbb{R}_+^n$.*

Then the cell $B_-^{n-1} \cup p_1 B_+^{n-1}$ is embedded topologically flat, that is, there is a homeomorphism $\bar{p}_1$ of $\mathbb{R}^n$ that maps B^{n-1} onto $B_-^{n-1} \cup p_1 B_+^{n-1}$. (The tangency of Π_α for $\alpha \in Q(-\pi/2, +\pi/2)$ is not essential and has only a technical role.)

Proof. First we will construct a mapping $w : \mathbb{R}^n \to \mathbb{R}^n$, that orthogonally projects B_-^{n-1} onto B^{n-2}, is homeomorphic outside B_-^{n-1}, and is identical on $\mathbb{R}_+^n$. Under the given conditions it is clear that the composition $w^{-1}\pi w$ coincides with p_1 on B_+^{n-1}. At the same time it occurs that this composition can be extended identically on B_-^{n-1}. The obtained extension is the homeomorphism $\bar{p}_1$ we are looking for.

The beginning of this construction of w is determined by the requirements that $w = 1$ on $\mathbb{R}_+^n$ and $w(y) = x_y$ for $y \in B_-^{n-1}$. Extend w identically to the points $y \in \mathbb{R}_-^{n-1}$ whose projections x_y onto $\mathbb{R}^{n-2}$ are situated outside B^{n-2}. If $x_y \in B^{n-2}$ and $y \in \mathbb{R}_-^{n-1} \setminus B_-^{n-1}$, we take as $w(y)$ the point that is obtained from y by the shift along the direction of the axe Ox^{n-1} in the distance equal to the intersection segment of B_-^{n-1} with the axe $x_y s$ in P_{x_y}. Thus we have constructed w on the space $\mathbb{R}^{n-1}$.

For every point $y \in \mathbb{R}_-^{n-1}$, we denote by L_y the straight line going through y and being parallel to the axe Ox^n. If x_y lies outside B^{n-2}, set $w = 1$ on L_y.

Let $y \in B^{n-1}$. Define that w sends L_y isometrically into the union of two rays in P_{x_y} starting at the point $x_y \in B^{n-2}$ with the angle $\alpha = \pm(\pi/2 - \pi/4 \cdot s_y)$ with respect to the axe $x_y s$ ($s_y < 0$ is the coordinate of y in P_{x_y}).

Notice that $\alpha \to \pi/2$, when $s_y \to 0$, that is, $y \to x_y$.

If $x_y \in B^{n-2}$ and $y \in \mathbb{R}^{n-1}_- \setminus B^{n-1}$, then w sends L_y isomorphically to the pair of rays in P_{x_y} starting at the point $w(y)$ with the angles $\alpha = \pm(\pi/2 - \pi/4 \cdot s_{y'})$, where y' is an intersection of the half-axe $x_y s \cap \mathbb{R}^n_-$ with the boundary of B^{n-1}.

Now w is well posed on the entire $\mathbb{R}^n$; it is continuous and identical on $\mathbb{R}^n_+$ and outside $\cup_{x\in B^{n-2}} P_x$. Also w retracts B^{n-1}_- onto B^{n-2} by the orthogonal projection and it is homeomorphic outside B^{n-1}_-.

It remains to note that a sequence of points z_n tends to a point $y \in B^{n-1}_-$ if and only if $w(y_n)$ tends to x_y, touching $\Pi_\alpha \cup \Pi_{-\alpha}$, where α is chosen according to the point y as above, that is, $\alpha_y = \pm(\pi/2 - \pi/4 \cdot s_y)$.

Indeed, take a spherical neighborhood V_ε with radius $\varepsilon > 0$ of a point x_y in the plane $x^{n-1} = 0$ and consider the set W_ε of points $z \in \mathbb{R}^n_-$ that are projected to V_ε. Let $U_\varepsilon(y)$ be the intersection of W_ε with the domain between two planes, being parallel to $x^{n-1} = 0$ and located on different sides of y in the distance ε. Let $U'_\varepsilon(x_y)$ be the intersection of W_ε with $Q(\alpha_y - \pi/2 \cdot \varepsilon, \alpha_y + \pi/2 \cdot \varepsilon) \cup Q(-\alpha_y - \pi/2 \cdot \varepsilon, -\alpha_y + \pi/2 \cdot \varepsilon)$. Then for every $\varepsilon' > 0$ one can find a $\varepsilon > 0$ such that $w(U_\varepsilon(y)) \subset U'_{\varepsilon'}$, and, conversely, for every $\varepsilon > 0$ one can find a ε' such that $w(U_\varepsilon(y)) \supset U'_{\varepsilon'}(x_y)$. Hence the sequence of points $z_n \in \mathbb{R}^n$ tends to $y \in Int\, B^{n-1}_-$ if and only if $w(z_n)$ tends to x_y and touches $\Pi_{\alpha_y} \cup \Pi_{-\alpha_y}$.

A sequence z_n tends to $y \in \partial B^{n-1}_- \setminus B^{n-2}$ if and only if $wz_n \to x_y$ and for every $\varepsilon > 0$ there exists n_0 such that for $n > n_0$ all z_n are located outside $Q(-\pi/2 - \pi/4 + \varepsilon, +\pi/2 + \pi/4 - \varepsilon)$. It is clear that the same property is fulfilled for the sequence $hw(z_n)$.

This proves that the homeomorphism $\widetilde{p}_1 = w^{-1} p_1 w$ is extended identically to B^{n-1}_-, as what was in demand. The constructed homeomorphism $\widetilde{p}_1$ coincides with the given p_1 on B^{n-1}_+ and is identical on B^{n-1}_+. Thus, the union of cells $B^{n-1}_- \cup p_1 B^{n-1}_+ = \bar{p}_1 B^n$ is embedded locally flat at least at the points of $B^{n-1} \setminus \partial B^{n-2}$. But then one can easily construct a homeomorphism of the whole space that sends $B^{n-1}_- \cup p_1 B^{n-1}_+$ into B^{n-1}_-. It is a standard construction (see [3]), which we leave as an exercise. So, the embedding of $B^{n-1}_- \cup p_1 B^{n-1}_+$ is topologically flat. □

LEMMA 3.2. *The assertion of Lemma 3.1 is true for the embedding $p_2 : Q[-\pi/2, \pi/2] \to Q[-\pi/2, \pi/2]$, for which $p_2 \Pi_\alpha$ touches Π_α with $\alpha \in (-\pi/2, \pi/2)$, $p_2 B^{n-1}_+ \subset Q(-\pi/4, \pi/4)$.*

Proof. Construct the arc homeomorphism $\rho : Q[-\pi, \pi] \to Q[-\pi/2, \pi/2]$, identical on $Q[-\pi/4, \pi/4]$, that sends linearly the arc of each circle $C_x(r)$ between the points of its intersections with Π_π and $\Pi_{\pi/4}$ to the arc between its intersections with $\Pi_{\pi/2}$, $\Pi_{\pi/4}$, and, analogously, sends the arc between $\Pi_{-\pi}$ and $\Pi_{-\pi/4}$ to the arc between $\Pi_{\pi/2}$ and $\Pi_{-\pi/4}$. It is clear that touching Π_α is transformed into touching $\rho \Pi_\alpha$. Then, the hypothesis of Lemma 3.1 is satisfied for $\rho^{-1} p_2 \rho$ that coincides with p_2 on B^{n-2}_+. Thus, the embedding of the cell $B^{n-1}_- \cup p_2 B^{n-2}_+$ is topologically flat. □

LEMMA 3.3. *Let an embedding $p_3 : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be given in some neighborhood of B^{n-2}, where the images of $\Pi_{-\pi/2}$, $\Pi_{\pi/2}$, and two more semiplanes Π_α, $-\pi/2 < \alpha < \pi/2$ (let them be for definiteness $\Pi_{-\pi/4}$ and Π_0) touch their preimages: $\Pi_{-\pi/2}, \Pi_{\pi/2}, \Pi_{-\pi/4}, \Pi_0$ at the points of B^{n-2}.*

Then there exists an isotopy $\phi_t : \mathbb{R}^n \to \mathbb{R}^n$, identical outside $p_3 \mathbb{R}^n_+$ and outside a given neighborhood of B^{n-2}, such that $\phi_0 = 1$ and $\bar{p}_3 \Pi_\alpha = \phi_1 p_3 \Pi_\alpha$ touches Π_α at the points of B^{n-2} for all $\alpha \in [-\pi/2, \pi/2]$. In particular, the condition of Lemma 3.2 for p_2 is fulfilled for $\bar{p}_3$.

Proof. At first note that if the hypothesis of touching is fulfilled for some dense set of $\alpha \in [-\pi/2, \pi/2]$, it will be fulfilled for all α from this interval. So, we will try to obtain its fulfillment for a countable dense set of values, namely for the set $\{\alpha = \pi d\}$, where d is a dyadic rational and $|d| \leq 1/2$.

Enumerate these numbers into a sequence α_k (starting with the four given values of α), and begin constructing a countable sequence of ε_k-isotopies $\phi^{(k)}$, each of which is identical on the ε_k-neighborhood of B^{n-2} and achieves the touching condition for the homeomorphism $\phi^{(k)} \cdots \phi^{(1)} p_3$ at the points of B^{n-2} for every next α_k without loosing this property for the preceding α_i.

In this construction, independently from the preceding steps, one may take the numbers ε_k arbitrarily small. Hence the sequence of isotopies $\phi_t^{(k)} \cdots \phi_t^{(1)}$ will tend to an isotopy, and in the limit the touching condition will be fulfilled for all $\alpha \in [-\pi/2, \pi/2]$.

It is sufficient to describe one step of the construction, say, for values of α given in the lemma. Next steps are absolutely analogous.

Let us show how to obtain this touching condition for $\alpha = \pi/4$. The construction of the isotopy in demand takes several steps. □

Step 1. Note that by applying the arcwise homeomorphisms one can get that the images of Π_α touch Π_α on one side; for example, that $p_3\Pi_0$ touches Π_0 and lies in $Q(0, \pi/2)$, and $p_3\Pi_{-\pi/4}$ touches $\Pi_{-\pi/4}$ and lies in $Q(-\pi/4, 0)$. (Certainly, it is sufficient for each touching condition to be fullfilled in a small neighborhood of B^{n-2}.) Let us show this for $p_3\Pi_0$.

Note that $\Pi_{3\pi/8}$ and $\Pi_{\pi/4}$ lie in a small neighborhood of B^{n-2} between $p_3\Pi_{\pi/2}$ and $p_3\Pi_0$ and do not intersect them except for $\mathbb{R}^{n-2}$.

Construct an arcwise homeomorphism τ', identical outside $Q(0, 3\pi/8)$, that sends $\Pi_{\pi/4}$ to the fence S_0 touching Π_0. Replace p_3 by $\widetilde{p}_3 = \tau' p_3$. All the hypotheses of the lemma remain true, but now $\widetilde{p}_3\Pi_0$ lies between the fence S_0 and $\Pi_{-\pi/16}$ (in a small neighborhood of B^{n-2}), since $\widetilde{p}_3\Pi_0$ touches Π_0.

Construct now an arcwise homeomorphism τ'', identical outside the domain in $\mathbb{R}^n_+$ bounded by S_0 and $\Pi_{-\pi/8}$, that sends $\Pi_{\pi/16}$ to Π_0. The lemma's hypotheses remain true after replacing $\widetilde{p}_3$ by $p_3^\circ = \tau'' \widetilde{p}_3$, but then $p_3^\circ\Pi_0$ lies between Π_0 and S_0, that is, on one side of Π_0.

Thus from the very beginning we may suppose that $p_3\Pi_0$ lies in $Q(0, \pi/4)$ and touches Π_0 as well as, analogously, that $p_3\Pi_{-\pi/4} \subset Q(-\pi/4, 0)$ and touches $\Pi_{-\pi/2}$. Moreover, $p_3\Pi_{\pi/2} \subset Q(\pi/4, \pi/2)$ and touches $\Pi_{\pi/2}$ as well as $p_3\Pi_{-\pi/2} \subset Q(-\pi/2, -\pi/4)$ and touches $\Pi_{-\pi/2}$.

Step 2. Construct an arcwise homeomorphism τ_1, identical outside $Q(0, \pi/2)$, that sends $\Pi_{\pi/4}$ to the fence S_1 touching Π_0, closely to Π_0 so that $p_3 S_1 \subset Q(0, \pi/4)$. Let $t_1 = p_3 \tau_1 p_3^{-1}$. Then $t_1 = 1$ outside $p_3 Q(0, \pi/2)$ and $t_1 p_3 \Pi_{\pi/4} \subset Q(0, \pi/4)$. Let $p_3' = t_1 p_3$.

Step 3. Construct now a fence S_2 touching $\Pi_{-\pi/4}$ closely so that $p_3' S_2$ lies in $Q(-\pi/4, 0)$.

Let τ_2 be an arcwise homeomorphism identical outside $Q(-\pi/4, \pi/4)$ that sends Π_0 to S_2. Let $t_2 = p_3' \tau_2 p_3'^{-1}$. Then $t_2 = 1$ outside $p_3' Q(-\pi/4, \pi/4)$, $t_2 p_3' \Pi_0 \subset Q(-\pi/4, 0)$, and $t_2 p_3' \Pi_{\pi/4} \subset Q(0, \pi/4)$. Let $p_3'' = t_2 p_3'$.

Step 4. Construct a fence S_2 that touches Π_0 and separates Π_0 from $p_3''\Pi_{\pi/4}$. Construct also another fence S_2' touching $\Pi_{\pi/4}$ and lying in $Q(0,\pi/4)$. Let τ_3 be an arcwise homeomorphism, identical outside of $Q(0,\pi/4)$, that sends S_2 to S_2'. Then $p_3''' = \tau_3 p_3''\Pi_{\pi/4}$ lies in the domain between $\Pi_{\pi/4}$ and S_2', that is, it touches $\Pi_{\pi/4}$ on one side.

Step 5. Consider now the homeomorphism $\bar{p}_3 = t_2^{-1}\pi_3''' = t_2^{-1}\tau_3 t_2 t_1 p_3$. It is clear that $\bar{p}_3 = p_3$ on $\Pi_{-\pi/2} \cup \Pi_{\pi/2} \cup \Pi_{-\pi/4} \cup \Pi_0$ and that $\bar{p}_3\Pi_{\pi/4}$ touches $\Pi_{\pi/4}$ on one side. Since $\bar{p}_3 \cdot p_3^{-1}$ is identical on $\Pi_{-\pi/2} \cup \Pi_{\pi/2}$, there is an isotopy ϕ_t, identical outside $Q(-\pi/2,\pi/2)$ and such that $\phi_0 = 1$, $\phi_1 = \bar{p}_3 p_3^{-1}$, that is, $\bar{p}_3 = \phi_1 p_3$.

It is clear that one can make this isotopy ε-small and identical outside the ε-neighborhood of B^{n-2} for any given ε. All conditions of the lemma are satisfied.

Lemma 3.4. *Let a homeomorphism $p_4 : \mathbb{R}^n \to \mathbb{R}^n$ be identical on $\mathbb{R}^{n-2}$ such that $p_4 Q_k \subset Q_k$, $0 \le k \le 3$ where $Q_0 = Q[-\pi/8,\pi/8]$ and Q_i, $1 \le i \le 3$ are obtained from Q_0 by consecutive turns by 90° counter-clockwise (from x^{n-1} to x^n).*

Then for every $\varepsilon > 0$ there exists a ε-isotopy $\psi_t : \mathbb{R}^n \to \mathbb{R}^n$, identical on $\mathbb{R}^{n-2}$ and outside a given neighborhood of B^{n-2}, and such that in a smaller neighborhood of B^{n-2} the homeomorphism $\bar{p}_4 = \psi_1 p_4$, restricted to $\mathbb{R}^n_+$, fulfills the conditions of Lemma 3.3 for p_3.

The proof of this lemma follows form an evident construction with arcwise homeomorphisms.

First of all, one may assume, as in the proof of the preceding lemma, that in a neighborhood of B^{n-2} the images of Π_{α_k} for $\alpha_k = k\pi/2$, $0 \le k \le 3$ touch Π_{α_k} at points of B^{n-2} and on the wishful side of Π_{α_k}.

Indeed, let $\Pi_{\pm\pi/4}$ and $\Pi_{\pm 3\pi/4}$ remain immovable. Move the semiplanes $\Pi_{+\pi/8}$ and $\Pi_{-\pi/8}$ by an arcwise homeomorphism σ_0, that is identical outside $Q[-\pi/4,\pi/4]$, to Π_0 and to a fence S_0 that touches Π_0, respectively.

By the same way, let an arcwise homeomorphism σ_1, identical outside $Q[\pi/4,3\pi/4]$, move the semiplane $\Pi_{\pi/2+\pi/8}$ to $\Pi_{\pi/2}$ and $\Pi_{\pi/2-\pi/8}$ to a fence S_1 touching $\Pi_{\pi/2}$. Let σ_2 be an arcwise homeomorphism, identical outside $Q[3\pi/4,5\pi/4]$, that sends $\Pi_{-\pi-\pi/8}$ to $\Pi_{-\pi}$ and $\Pi_{-\pi+\pi/8}$ to a fence S_2 touching $\Pi_{-\pi}$.

At last, let σ_3 be an arcwise homeomorphism that is identical outside $Q[-3\pi/4,-\pi/4]$ and sends $\Pi_{-\pi/2-\pi/8}$ and $\Pi_{-\pi/2+\pi/8}$ to $\Pi_{-\pi/2}$ and to a fence S_3 touching $\Pi_{-\pi/2}$, respectively.

Let σ be a homeomorphism that in each fourth-space, limited by semiplanes $\Pi_{\pm\pi/4}$ and $\Pi_{\pm 3\pi/4}$, coincides with the corresponding σ_k, $0 \le k \le 3$. Construct an arcwise homeomorphism $\tau : \mathbb{R}^n \to \mathbb{R}^n$ that sends the domain between Π_π and S_2 to the domain between $\Pi_{-\pi/2}$ and S_3 as well as the domain between $\Pi_{-\pi/2}$ and S_3 to the domain between $\Pi_{-\pi/4}$ and a fence touching $\Pi_{-\pi/4}$. Also, let $\tau = 1$ in $Q[-\pi/8,\pi-\pi/8]$ and on $\Pi_{\pm\pi/4} \cup \Pi_{\pm 3\pi/4}$.

It is clear that $\bar{p}_4 = \tau\sigma p_4$ satisfies the hypothesis of Lemma 3.3 for p_3. Namely, $\bar{p}_4\mathbb{R}^n_+ \subset \mathbb{R}^n_+$, $\bar{p}_4\Pi_{\pm\pi/2}$ touches $\Pi_{\pm\pi/2}$, $\bar{p}_4\Pi_0$ touches Π_0, and $\bar{p}_4\Pi_{-\pi/4}$ touches $\Pi_{-\pi/4}$; also, $\bar{p}_4$ is isotopic to p_4 by a small isotopy, since an arcwise homeomorphism is isotopic to the identity and its mesh does not supersede diameters of circles $C_x(r)$ and of their images on which it is not identical.

It should be pointed out that the homeomorphism τ, constructed above, is identical on $\Pi_{\pi/2+\pi/4}$.

Lemma 3.5. *Assume that there exists a homeomorphism* $p_5 : \mathbb{R}^n \to \mathbb{R}^n$ *such that* $p_5 Q[-\pi/8, \pi/8] \subset Q(-\pi/8, \pi/8)$.

There is an isotopy $\chi_t : \mathbb{R}^n \to \mathbb{R}^n$*, identical on* $\mathbb{R}^{n-2}$ *and outside a neighborhood of* B^{n-2}*, such that* $\chi_0 = 1$ *and* $\chi_1 p_5 = \bar{p}_5$ *satisfies the hypothesis of Lemma 3.4 for* p_4 *and coincides with* p_5 *on* $B^{n-1}_- \cup p_5 B^{n-1}_+$.

Proof. Consider the 4-sheeted covering $\nu : \mathbb{R}^n_2 \to \mathbb{R}^n_1$, branched over the subspace $\mathbb{R}^{n-2}_1 \subset \mathbb{R}^n_1$. (It is convenient to indicate distinction between the same objects in the domain and in the image spaces of the covering ν by means of lower indices.) Denote by $j : \mathbb{R}^n_2 \to \mathbb{R}^n_1$ the natural identification of both spaces. Let us concretize the covering by identifying every plane $P_{kx} \subset \mathbb{R}^n_i$, where $x \in \mathbb{R}^{n-2}_k$, $k = 1, 2$, with the complex line $\mathbb{C}^1$ (xs is the real and xt is the imaginary axes), and by representing ν as the function $z \mapsto e^{\mathbf{i}\cdot 3\varphi_z} z$, where $z = \rho_z e^{\mathbf{i}\varphi_z}$. Here $j = \nu$ on $\mathbb{R}^{n-1}_{2+}$.

According to the hypothesis of Lemma 3.4 the homeomorphism $p_5 : \mathbb{R}^n_1 \to \mathbb{R}^n_1$ is given. Consider the homeomorphism $\tilde{p}_5 : \mathbb{R}^n_2 \to \mathbb{R}^n_2$, covering p_5 ($\nu\tilde{p}_5 = p_5\nu$). We have $\nu\tilde{p}_5 = p_5 j$ on $\mathbb{R}^{n-1}_{2+}$.

Construct now a homeomorphism $\beta : \mathbb{R}^n_2 \to \mathbb{R}^n_2$, patching up the covering p so that $\nu\beta = j$ on $Q_2[-\pi/8, \pi/8]$. Namely, β is an arcwise homeomorphism, identical on $Q_2[-\pi/4, \pi/4]$, that sends $Q_2[-\pi/8, \pi/8]$ into $Q_2[-\pi/32, \pi/32]$. (One may analogously redefine β on other three quadrants, separated by planes $x^n = \pm x^{n-1}$, so that the mapping would remain a covering, but it is not important for us.)

As a result, $\nu\beta^{-1} = j$ on $Q_2[-\pi/8, \pi/8]$. The homeomorphism $\bar{p}_5 : \mathbb{R}^n_1 \to \mathbb{R}^n_1$, defined by equality $\bar{p}_5 = j\beta^{-1}\tilde{p}_5\beta j^{-1}$, coincides on $Q_1[-\pi/8, \pi/8]$ with $p_5 = \nu\beta\beta^{-1}\nu^{-1}p_5\nu\beta\beta^{-1}\nu^{-1}$. Moreover,

$$\bar{p}_5 Q_1[-k\pi/8, k\pi/8] \subset Q_1[-k\pi/8, k\pi/8], \quad 0 \le k \le 3. \tag{3.1}$$

So, $\bar{p}_5$ satisfies the hypothesis of Lemma 3.4 for the homeomorphism p_4, as what is required.

Besides, $\bar{p}_5$ is isotopic to the homeomorphism p_5 under isotopy that is identical on $\mathbb{R}^{n-2}_1$, since $\bar{p}_5 p_5^{-1}$ is identical on $Q_1[-\pi/8, \pi/8]$. □

4. Proof of the theorem

Since the embedding q_- is topologically flat, it can be extended to a homeomorphism of $\mathbb{R}^n$ and so we can assume that q is identical on B^{n-1}_-. Construct two fences $S_{-\pi}$ and S_π on two different sides of B^{n-1}_-, that are touching $\mathbb{R}^{n-1}$ from above and from below and separating B^{n-1}_- from qB^{n-1}_+. Then move them by an arcwise homeomorphism τ, identical on B^{n-1}_-, onto $\Pi_{-\pi/8}$ and onto $\Pi_{\pi/8}$, respectively, and replace q with $\tilde{q} = \tau q$. We obtain $\tilde{q}B^{n-1}_+ \subset Q(-\pi/8, \pi/8)$. Suppose that this is valid for q from the very beginning.

Since q_+ is topologically flat, it can be extended to a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ ($h|_{B^{n-1}_+} = q|_{B^{n-1}_+} \subset Q(-\pi/8, +\pi/8) \subset \mathbb{R}^n_+$). As $hB^{n-1}_+ \cap B^{n-1}_- = B^{n-2}$, applying as well as above the arcwise homeomorphisms, identical on B^{n-1}_-, one can wangle $hQ[-\pi/2, \pi/2] \subset Q(-\pi/2, \pi/2)$.

Thus the assertion of our theorem now takes the following form.

Theorem 4.1. *Suppose that there is a homeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$, for which $hB_+^{n-1} \subset Q(-\pi/8, \pi/8)$. Then the cell $B_-^{n-1} \cup hB_+^{n-1}$ is topologically flat. More precisely: there exists a homeomorphism $\bar{h} : \mathbb{R}^n \to \mathbb{R}^n$, identical on B_-^{n-1}, that coincides with h on B_+^{n-1}.*

The proof of this statement follows from the above sequence of lemmas as follows:

First, having constructed an arcwise homeomorphism γ, identical on $\mathbb{R}_-^n$, that sends $\Pi_{-\pi/8}$ and $\Pi_{\pi/8}$, respectively, to fences S_- and S_+, touching Π_0 from below and from above and separating Π_0 from $h^{-1}\Pi_{-\pi/8}$ and $h^{-1}\Pi_{\pi/8}$, we may replace h by a homeomorphism $h\gamma$ that coincides with h on B_+^{n-1} and moves $Q[-\pi/8, \pi/8]$ into $Q(-\pi/8, \pi/8)$. Then we obtain a homeomorphism satisfying the hypotheses of Lemma 3.5 for p_5.

By Lemma 3.5 we obtain a homeomorphism that coincides with the given h on B_+^{n-1} and satisfies the hypotheses of Lemma 3.4 for the homeomorphism p_4. By Lemma 3.4 we can construct a homeomorphism $\bar{p}_4$ that satisfies the condition for p_3 from Lemma 3.3, is isotopic to p_4, and is identical on $\Pi_{3\pi/4}$ by its construction.

Denote by D the semiball in $\Pi_{3\pi/4}$, bounded by B^{n-2}, and by γ the arcwise homeomorphism, constructed in Lemma 3.4, that is identical on D. Evidently, the cell $B_-^{n-1} \cup hB_+^{n-1}$ is topologically flat if and only if the same is true for the cell $D \cup hB_+^{n-1}$, if and only if it is so for $D \cup \gamma hB_+^{n-1}$, and if and only if it is so for $B_-^{n-1} \cup \gamma hB_+^{n-1}$, because these cells are obtained one from another by the application of some (arcwise) homeomorphisms of the space.

So, it is sufficient to prove that the cell $B_-^{n-1} \cup p_3B_+^{n-1}$ is topologically flat, where p_3 is the embedding given in Lemma 3.3.

According to Lemma 3.3 we can replace p_3 by an embedding, isotopic to p_3 under the isotopy, identical on B_-^{n-1}, that satisfies the conditions of Lemma 3.2. This isotopy sends the cell $B_-^{n-1} \cup p_3B_+^{n-1}$ to the cell $B_-^{n-1} \cup \bar{p}_3B_+^{n-1} = B_-^{n-1} \cup p_2B_+^{n-1}$ and we have to prove that the latter is locally flat. But this is the assertion of Lemma 3.2.

The theorem follows.

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BOURGIN-YANG-TYPE THEOREM FOR a-COMPACT PERTURBATIONS OF CLOSED OPERATORS. PART I. THE CASE OF INDEX THEORIES WITH DIMENSION PROPERTY

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A variant of the Bourgin-Yang theorem for a-compact perturbations of a closed linear operator (in general, unbounded and having an infinite-dimensional kernel) is proved. An application to integrodifferential equations is discussed.

1. Introduction

1.1. Goal. Among several different, but equivalent, formulations of the famous Borsuk-Ulam theorem, the following one is of our interest: if $f : S^n \to \mathbb{R}^n$ is a continuous odd map, then there exists an $x \in S^n$ such that $f(x) = f(-x) = 0$ (see [17] for other formulations, generalizations, and applications, and [11, 13] for a connection with the corresponding Brouwer degree results).

Under the "stronger" assumption that $f : S^n \to \mathbb{R}^m$, where $m < n$, one can expect that there are bigger coincidence sets. The results which measure the size of the set $A := \{x \in S^n \mid f(-x) = f(x)\}$ in topological terms, like dimension, (co)homology, genus (or other index theory), are usually called "Bourgin-Yang theorems." The simplest result in this direction (cf. [5, 19]) can be formulated as follows: (i) $\dim A(f) \geq n - m$ (covering or cohomological dimension) and (ii) $g(A(f)) \geq n - m + 1$, where $g(\cdot)$ stands for the genus with respect to the antipodal action (see Example 2.4). We refer to [17] for extensions of this result to more complicated (finite-dimensional) G-spaces, where G is a compact Lie group, as well as to index theories different from genus.

Holm and Spanier were the first to extend the Bourgin-Yang theorem to infinite dimensions (see [10], where the solution set to the equation $a(x) = f(x)$ was studied in the case a is a *proper* C^∞*-smooth Fredholm* operator and f is a compact map; both equivariant with respect to a free involution). It should be pointed out that the assumptions on a required in [10] allow a clear finite-dimensional reduction (the kernels and images in question are complementable). At the same time, the methods developed in [10] cannot be

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applied to treat the case when F is not Fredholm. The first step in this direction was done in recent papers [8, 9], where the author studied the situation when a is a continuous (resp., linear closed) linear operator without any restrictions with respect to $\dim\ker(a)$ (in fact, in these papers only, the "dimension part" of the Bourgin-Yang theorem was proved in the presence of the antipodal symmetry). The main new ingredient in [8, 9] allowing the author to go around the "complementability problem" is the application of the Michael selection theorem *respecting the antipodal symmetry* to the *multivalued* map a^{-1}. Observe, however, that the corresponding "equivariant selection theorem" was proved in [7] for free actions of a finite group—by no means to be extended to *nonfree actions of compact Lie groups*.

The *main goal* of our paper is to extend the results from [8–10] in several directions:

(i) a is an *arbitrary* closed linear map (in general, unbounded, and having an infinite-dimensional kernel) equivariant with respect to *arbitrary* compact Lie group representations;

(ii) f is a so-called a-compact G-equivariant map (see Definition 4.1);

(iii) the coincidence set is estimated in terms of an arbitrary index theory with the so-called "dimension property" (cf. [4, 17], [14, Chapter 5]).

To this end, based on the results from [1], we establish a general equivariant version of the Michael selection theorem (without any restrictions with respect to G-actions) which, in our opinion, is interesting in its own. This result allows us to construct for a an equivariant section taking bounded sets to the bounded ones (see Lemma 3.6). Using this lemma, we reduce the coincidence problem to the fixed point problem.

1.2. Overview. After Section 1, the paper is organized as follows. In Section 2, we briefly discuss "index theories." Section 3 is devoted to the proof of the equivariant Michael selection theorem and Lemma 3.6. After the reduction to the fixed point problem (see Section 4), we prove the main result (Theorem 4.3) in Section 5. In the last section, we give an application of the main result to integrodifferential equations. For the equivariant jargon, frequently used in this paper, we refer to [6].

2. Index theories

Convention and notations. Hereafter, G stands for a compact Lie group.

Without loss of generality, we will assume all Banach G-representations to be isometric.

Given a Banach G-representation E,

(i) S_R stands for the sphere in E of radius R centered at the origin;

(ii) $E^G = \{x \in E \mid gx = x, \text{ for all } g \in G\}$—the fixed point set.

Let us recall the standard construction of the join.

Definition 2.1. Let $X_1,\dots,X_n$ be topological spaces and $\Delta^{n-1} = \{(t_1,\dots,t_n) \in \mathbb{R}^n \mid 0 \le t_i \le 1, \sum_{i=1}^n t_i = 1\}$—the $(n-1)$-dimensional standard simplex. The *join* $X_1 * \cdots * X_n$ is the quotient space of the product $X_1 \times \cdots \times X_n \times \Delta^{n-1}$ under the following equivalence relation: $(x_1,\dots,x_n,t_1,\dots,t_n) \sim (x_1',\dots,x_n',t_1',\dots,t_n')$ if and only if $t_i = t_i'$ $(i = 1,\dots,n)$ and $x_i = x_i'$ whenever $t_i = t_i' > 0$.

It is convenient to denote a point of the join $X_1 * \cdots * X_n$ in the form of a formal convex combination: $\sum_{i=1}^n t_i x_i$.

If $X_1 = \cdots = X_n = X$, then write $J_n X$ for $X_1 * \cdots * X_n$.

If $X_1,\dots,X_n$ are G-spaces, then so is $X_1 * \cdots * X_n$ via $g \cdot \sum_{i=1}^n t_i x_i := \sum_{i=1}^n t_i g x_i$, $g \in G$.

Example 2.2. Obviously, $J_n S^0 = S^{n-1}$, $J_n S^1 = S^{2n-1}$, and $J_n S^3 = S^{4n-1}$. Also, if we consider S^0 (resp., S^1 and S^3) as free $\mathbb{Z}_2$—(resp., S^1- and $SU(2)$-spaces), then the action of $\mathbb{Z}_2$ on $J_n S^0$ (resp., S^1 on $J_n S^1$ and $SU(2)$ on $J_n S^3$) corresponds to the antipodal action (resp., scalar multiplication in $S^{2n-1} \subset \mathbb{C}^n$ and scalar multiplication in $S^{4n-1} \subset \mathbb{H}^n$, where $\mathbb{H}$ stands for the quaternions).

Following [4], [14, Chapter 5], [17], we give the following definition.

Definition 2.3. A function "ind" that assigns to every G-space A a number $\operatorname{ind}(A) \in \mathbb{N} \cup \{0\}$ or $\{\infty\}$ is called an *index theory* if it satisfies the following properties.

(i) $\operatorname{ind}(A) = 0$ if and only if $A = \varnothing$.

(ii) *Subadditivity*. If a G-space A is the union of two of its closed invariant subsets A_1 and A_2, then $\operatorname{ind}(A) \le \operatorname{ind}(A_1) + \operatorname{ind}(A_2)$.

(iii) *Continuity*. If A is a closed invariant subset of a G-space X, then there exists a closed invariant neighborhood $\mathcal{U}$ of A in X such that $\operatorname{ind}(A) = \operatorname{ind}(\mathcal{U})$.

(iv) *Monotonicity*. If A_1 and A_2 are two G-spaces and there exists an equivariant map $\varphi : A_1 \to A_2$, then $\operatorname{ind}(A_1) \le \operatorname{ind}(A_2)$.

In particular, (a) if $A_1 \subset A_2$, then $\operatorname{ind}(A_1) \le \operatorname{ind}(A_2)$, and (b) if $\varphi : A_1 \to A_2$ is an equivariant homeomorphism, then $\operatorname{ind}(A_1) = \operatorname{ind}(A_2)$.

Example 2.4 (genus). For a G-space A set $g(A) = k$ if there exist closed subgroups $H_1,\dots, H_k$ of G, $H_i \ne G$, $i = 1,\dots,k$, and a G-equivariant map $A \to G/H_1 * \cdots * G/H_k$, where k is minimal with this property (G acts on G/H_i by left translations). If such k does not exist, put $g(A) := \infty$. Also, $g(\varnothing) = 0$.

It is easy to check (see [3]) that the function g satisfies all the properties required for an index theory.

In fact, there is a "myriad" of nonequivalent index theories (mostly, cohomological (see [3] and references therein)).

In this paper, we are dealing with index theories satisfying an additional property (cf. [4], [14, Chapter 5], [17]). Namely, we have the following definition.

Definition 2.5 (dimension property). An index theory ind is said to satisfy the *dimension property* if there exists $d \in \mathbb{N}$ such that for any Banach G-representation E, one has $\operatorname{ind}(E^{kd} \cap S_1) = k$ for all invariant kd-dimensional subspaces E^{kd} of E satisfying $E^{kd} \cap E^G = \{0\}$.

As an immediate consequence of the dimension property, one has (cf. [4]) that $\operatorname{ind}(A) < \infty$ for any compact invariant subset $A \subset E$ of a Banach G-representation E with $A \cap E^G = \varnothing$. Although, in general, the genus does not satisfy the dimension property, there are some important (from the application point of view) classes of groups for which it does (see the examples following below).

Example 2.6. (i) If $G = \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p$ (p is prime), then the genus satisfies the dimension property with $d = 1$ (cf. [3]).

(ii) If $G = S^1 \times \cdots \times S^1$, then the genus satisfies the dimension property with $d = 2$ (cf. [3]).

Remark 2.7. Restricting the genus to free G-spaces, one can define a "restricted index theory" satisfying the dimension property with $d = 1 + \dim G$ (cf. [3]). Recall that if G acts freely on a finite-dimensional sphere, then G is either finite, or S^1, or S^3, or the normalizer of S^1 in S^3 (cf. [6, Chapter 4, Theorem 6.2]). All finite groups admitting a free action on a finite-dimensional sphere are described in [18].

3. Equivariant selection theorem

We begin this section with recalling the Michael selection theorem. To this end, we need several definitions.

Definition 3.1. (i) Let X and Y be topological spaces. It will be said that F is a multivalued map from X to Y if F associates with each point $x \in X$ a nonempty subset $F(x)$ of Y. If, in addition, X and Y are G-spaces, then F is called a multivalued G-map or a multivalued equivariant map, if $F(gx) = gF(x)$ for all $g \in G$ and $x \in X$, where $gF(x) = \{gy \mid y \in F(x)\}$.

(ii) A multivalued map F from X to Y is called *lower semicontinuous* (l.s.c.) if for any open subset $U \subset Y$, the set

$$F^{-1}(U) = \{x \in X \mid F(x) \cap U \neq \varnothing\} \tag{3.1}$$

is open in X.

Definition 3.2. (i) A continuous (single-valued) map $f : X \to Y$ is called a *selection* for a multivalued map F from X to Y if $f(x) \in F(x)$ for all $x \in X$.

(ii) Assume X and Y are G-spaces and F is a multivalued G-map. A selection f of F is called a G-selection if, in addition, f is a G-map.

The following fact is well known as the Michael selection theorem.

Theorem 3.3 (see [15]). *Let X be a paracompact space, Y a Banach space, and F an l.s.c. multivalued map from X to Y such that $F(x)$ is a nonempty, closed, convex set for all $x \in X$. Then F admits a selection.*

Below, we formulate and prove an equivariant version of the Michael selection theorem.

Theorem 3.4. *Let X be a paracompact G-space, Y a Banach G-representation, and F a multivalued l.s.c. G-map from X to Y such that for all $x \in X$, $F(x)$ is a closed, convex set. Then F admits a G-selection.*

Proof. According to the Michael selection theorem (Theorem 3.3), there exists a continuous selection $f : X \to Y$ of F. Let dg be the normalized Haar measure on G. Define a new single-valued map $\varphi : X \to Y$ by

$$\varphi(x) = \int_G g^{-1} f(gx)dg, \quad x \in X, \tag{3.2}$$

(the symbol on the right-hand side denotes the vector-valued integral with respect to the Haar measure).

We claim that φ is the desired G-selection of F. Indeed, since $f(gx) \in F(gx) = gF(x)$, we see that $g^{-1}f(gx) \in g^{-1}(gF(x)) = F(x)$ for all $g \in G$. Since $F(x)$ is a closed convex set, we infer that the closed convex hull $\overline{\operatorname{conv}(A_f)}$ of the set $A_f := \{g^{-1}f(gx) \mid g \in G\}$ is contained in $F(x)$. But the above integral belongs to $\overline{\operatorname{conv}(A_f)}$ (see [16, Part 1, Theorem 3.27]). This yields that $\varphi(x) \in F(x)$.

Continuity and equivariance of the map $\varphi : X \to Y$ can be easily derived from the corresponding properties of the integral presented in the following lemma □

Lemma 3.5 (see [1]). *Assume that V is a complete (in the sense of the natural uniformity induced from Z) convex invariant subset of a locally convex topological vector space Z on which a compact group G acts linearly. Let $C(G,V)$ denote the set of all continuous maps $f : G \to V$ endowed with the compact-open topology. Then the vector-valued Haar integral $\int : C(G,V) \to V$ is a well-defined continuous map satisfying the following properties:*

(a) *$\int {}_h f = \int f = \int f_h$ for any $f \in C(G,V)$ and any $h \in G$, where ${}_h f(g) = f(hg)$ and $f_h(g) = f(gh)$ for all $g \in G$;*

(b) *$\int g * f = g \int f$ for any $f \in C(G,V)$ and any $g \in G$, where the action $g * f$ of G on $C(G,V)$ is defined by $(g * f)(x) = gf(x)$, $x \in G$;*

(c) *$\int f = v_0$, if $f(G) = \{v_0\}$ for a point $v_0 \in V$.*

Also, assuming in addition that G is finite or Z is finite-dimensional, one can remove the completeness requirement on V.

Next, we will apply Theorem 3.4 to prove the existence of a special G-selection of a linear G-equivariant closed map of Banach G-representations.

Let E_1 and E_2 be Banach spaces, $a : D(a) \subset E_1 \to E_2$ a linear closed surjective map. Take the natural projection $p : E_1 \to E_1/\operatorname{Ker}(a) := \overline{E_1}$ and consider the (invertible) map $a_1 : D(a_1) \subset \overline{E_1} \to E_2$, where $D(a_1) := p(D(a))$ and $a_1([x]) := a(x)$. Put (see, e.g., [8, 9])

$$\beta(a) := \sup_{y \in E_2 \setminus \{0\}} \frac{\|a_1^{-1}(y)\|}{\|y\|} = \sup_{y \in E_2 \setminus \{0\}} \frac{\inf\{\|x\| \mid x \in E_1,\ a(x) = y\}}{\|y\|}. \tag{3.3}$$

Lemma 3.6. *Let E_1 and E_2 be Banach isometric G-representations, $a : D(a) \subset E_1 \to E_2$ a G-equivariant linear closed surjective map, and $k > \beta(a)$ (cf. (3.3)). Then there exists a G-equivariant continuous map $q : E_2 \to E_1$ satisfying the following conditions:*

(i) *$a(q(y)) = y$ for all $y \in E_2$;*

(ii) *$q(y) \le k\|y\|$ for all $y \in E_2$.*

Proof. Denote by a^{-1} a multivalued map from E_2 to E_1 "inverse" to a, that is, a^{-1} assigns to each $y \in E_2$ its full inverse image under a. Obviously, a^{-1} is a multivalued G-map with nonempty closed convex values. Moreover (cf. [2, Chapter 3], [8, 9]), a^{-1} is l.s.c. (even Lipschitzian with the Lipschitz constant $\beta(a)$).

Consider together with a^{-1} another multivalued map Φ from E_2 to E_1 defined by $\Phi(y) := B_{r(y)}[0]$, where $B_{r(y)}[0]$ is the closed ball of radius $r(y) = \beta(a)\|y\| + 1$ centered at the origin of E_1. Obviously, Φ is also G-equivariant. Put $F(y) := a^{-1}(y) \cap \Phi(y)$. Still F is a G-equivariant l.s.c map with nonempty closed convex values.

By Theorem 3.4, there exists a G-equivariant selection $q : E_2 \to E_1$ of F. By construction, q is as required. □

Remark 3.7. Lemma 3.6 is quite obvious in the case $\dim\ker(a) < \infty$. Indeed, one has a direct sum decomposition $E_1 = V \oplus \ker(a)$ and V is isomorphic to E_2 as a G-representation. However, in general, $\ker(a)$ is not complementable and, therefore, one can think of q as a nonlinear equivariant replacement for the corresponding G-isomorphism (the use of G-selections in this case seems to be unavoidable).

4. Main result: formulation and reduction to a fixed point problem

To formulate the main result of this paper (see Theorem 4.3), we need some preliminaries.

Definition 4.1. Let E_1, E_2 be Banach spaces, $a : D(a) \subset E_1 \to E_2$ a closed surjective linear map. A continuous map $g : X \subset E_1 \to E_2$ is said to be *a-compact* if the set $g(B \cap a^{-1}(A))$ is compact for any bounded sets $A \subset E_2$ and $B \subset X$ (the empty set is compact by definition).

To give a simple criterion for the a-compactness of g, recall that the graph norm makes $D(a)$ a Banach space, denoted by $\widetilde{E}$. Clearly, the embedding $j : \widetilde{E} \to E_1$ is continuous. Put $\widetilde{X} := j^{-1}(X)$ and consider the map $\widetilde{g} : \widetilde{X} \to E_2$ defined by $\widetilde{g}(x) = g(j(x))$.

Proposition 4.2. *Under the above notations, g is a-compact if and only if $\widetilde{g}$ is compact.*

As the proof of this proposition is straightforward, we omit it.

Here is our main result.

Theorem 4.3. *Take an index theory* ind *satisfying the dimension property with some natural number d (cf. Definitions 2.3 and 2.5). Let E_1, E_2 be Banach G-representations and $E_2^G = \{0\}$. Let, further, $a : D(a) \subset E_1 \to E_2$ be a closed surjective G-equivariant linear map such that E_1^G is a proper finite-dimensional subspace of* $\ker(a)$, *and denote by p the codimension of E_1^G in* $\ker(a)$ *(the case $p = \infty$ is not excluded). Let $f : D(f) \subset S_R \to E_2$ satisfy the following conditions:*

(i) $D(f) = D(a) \cap S_R$*;*

(ii) f *is G-equivariant;*

(iii) f *is a-compact.*

Denote by $N(a, f)$ the solution set to the equation

$$a(x) = f(x). \tag{4.1}$$

Then,

$$\operatorname{ind}\left(N(a,f)\right) \geq \frac{p}{d}. \tag{4.2}$$

The proof of Theorem 4.3 will be given in the next section. Here, by means of Lemma 3.6, we will reduce the study of (4.1) to a G-equivariant fixed point problem with a compact operator.

By assumption, E_1^G is finite-dimensional, hence we have a direct sum G-decomposition $E_1 = E_1^G \oplus \widetilde{E}_1$. Put $\widetilde{a} := a|_{\widetilde{E}_1 \cap D(a)}$—the restriction. Since, by assumption, $E_1^G \subset \ker(a)$, we still have that $\widetilde{a}$ is a closed G-equivariant surjective map. Let $q : E_2 \to \widetilde{E}_1$ be the map provided by Lemma 3.6 (applied to $\widetilde{a}$).

Next, define the map $g : D(a) \cap \widetilde{E}_1 \to E_2$ by

$$g(x) = \begin{cases} \dfrac{\|x\|}{R} f\left(\dfrac{Rx}{\|x\|}\right), & x \neq 0, \\ 0, & x = 0. \end{cases} \tag{4.3}$$

Further, take a direct sum G-decomposition $\ker(a) = E_1^G \oplus U$ $(\dim U = p)$, consider the Banach G-representation $E := E_2 \oplus U$ equipped with diagonal G-action and the norm $\|(y,u)\| = \|y\| + \|u\|$, and define the map $\alpha : E \to E_2$ by $\alpha(y,u) := g(q(y) + u)$. Since q and g are equivariant, so is α. Let us show that α is a compact map.

Take a bounded set $A \subset E$. Without loss of generality, one can assume that $A = A_1 \times U_1$ with $A_1 \subset E_2$ and $U_1 \subset U$. By Lemma 3.6(ii), the set $A_2 := \{q(y) + u \mid (y,u) \in A\}$ is also bounded. Obviously, $A_2 \subset a^{-1}(A_1)$. By the a-compactness of g, one concludes that the set $g(A_2) = \alpha(A)$ is compact.

Finally, take the unit sphere $S \subset E$ and consider the equation

$$\alpha(y,u) = y \quad (y,u) \in S. \tag{4.4}$$

Lemma 4.4. *Let $N(\alpha)$ be the solution set to (4.4), and define the map $\gamma : N(\alpha) \subset S \subset E \to S_R \subset E_1$ by $\gamma(y,u) := R((q(y)+u)/\|q(y)+u\|)$. Then*

(i) *γ is an equivariant homeomorphism onto its image;*
(ii) *$\gamma(N(\alpha)) \subset N(a,f)$.*

Statement (i) follows immediately from Lemma 3.6(i). To show statement (ii), take $(y_0,u_0) \in S$ being a solution to (4.4). Obviously, $z_0 := q(y_0) + u_0 \neq 0$. By direct computation,

$$f\left(\frac{Rz_0}{||z_0||}\right) = \frac{R}{||z_0||} y_0. \tag{4.5}$$

On the other hand, using the linearity of a, one obtains

$$a(z_0) = y_0. \tag{4.6}$$

Combining (4.5) and (4.6) yields $x_0 := Rz_0/\|z_0\| \in N(a,f)$.

5. Proof of the main result (Theorem 4.3)

Throughout this section, we keep the same notations as in the previous section (in particular, $\ker(a) = E_1^G \oplus U$ and $E := E_2 \oplus U$). The proof of Theorem 4.3 splits into three steps.

Step 1 (finite-dimensional case). Under the assumptions of Theorem 4.3, suppose that $\dim E < \infty$ and consider the equivariant map $\Phi : S \subset E \to E_2 \subset E$ defined by $\Phi(y,u) := \alpha(y,u) - y$. Then $N(\alpha) = \Phi^{-1}(0)$. By the continuity property of ind, there exists a closed neighborhood $\mathcal{U}$ of $N(\alpha)$ such that

$$\operatorname{ind}(N(\alpha)) = \operatorname{ind}(\mathcal{U}). \tag{5.1}$$

By the subadditivity property, one has

$$\operatorname{ind}(S) \leq \operatorname{ind}(\mathcal{U}) + \operatorname{ind}(\overline{S \setminus \mathcal{U}}). \tag{5.2}$$

Combining (5.1) and (5.2) yields

$$\operatorname{ind}(N(\alpha)) \geq \operatorname{ind}(S) - \operatorname{ind}(\overline{S \setminus \mathcal{U}}). \tag{5.3}$$

Observe that the equivariant map Φ takes $S \setminus \mathcal{U}$ to $E_2 \setminus \{0\}$. Therefore, by the monotonicity property,

$$\operatorname{ind}(\overline{S \setminus \mathcal{U}}) \leq \operatorname{ind}(E_2 \setminus \{0\}). \tag{5.4}$$

Further, $S \cap E_2$ is a G-retract of $E_2 \setminus \{0\}$, therefore, it follows from (5.4) and monotonicity property that

$$\operatorname{ind}(\overline{S \setminus \mathcal{U}}) \leq \operatorname{ind}(S \cap E_2). \tag{5.5}$$

Combining (5.3) and (5.5) yields

$$\operatorname{ind}(N(\alpha)) \geq \operatorname{ind}(S) - \operatorname{ind}(S \cap E_2). \tag{5.6}$$

Finally, using (5.6) and the dimension property of ind, one obtains

$$\operatorname{ind}(N(\alpha)) \geq \frac{\dim E}{d} - \frac{\dim E_2}{d}, \tag{5.7}$$

and the result follows in the considered case.

Step 2 (finite-dimensional kernel). Under the assumptions of Theorem 4.3, suppose that $\dim U < \infty$ and reduce this situation to the previous step.

Put $X := \overline{\mathrm{conv}(\alpha(S))} \subset E_2$. For any $\varepsilon > 0$, take the finite-dimensional G-equivariant Schauder projection $p_\varepsilon : X \to X$ (see, e.g., [12, pages 69–70]) satisfying the property

$$\|y - p_\varepsilon(y)\| < \varepsilon \quad (y \in X), \tag{5.8}$$

and put $\alpha_\varepsilon := p_\varepsilon \alpha$. Denote by $N(\alpha_\varepsilon)$ the solution set to the equation $\alpha_\varepsilon(y,u) = y$, $(y,u) \in S$.

Lemma 5.1. *Under the above notations,* $\mathrm{ind}(N(\alpha_\varepsilon)) \leq \mathrm{ind}(N(\alpha))$ *for all* ε *small enough.*

Proof. By continuity property of ind, there exists a closed invariant neighborhood $\mathcal{U} \supset N(\alpha)$ such that $\mathrm{ind}(\mathcal{U}) = \mathrm{ind}(N(\alpha))$. Since $N(\alpha)$ is compact, without loss of generality, one can assume that $\mathcal{U}$ is a uniform δ-neighborhood: $\mathcal{U} = \mathcal{U}_\delta(N(\alpha)) := \{z \in E \mid \|z - N(\alpha)\| < \delta\}$ for $\delta > 0$ small enough.

Let us show, first, that there exists $\varepsilon_0 > 0$ such that $N(\alpha_\varepsilon) \subset \mathcal{U}_\delta(N(\alpha))$ for all $0 < \varepsilon < \varepsilon_0$. Arguing indirectly, assume that for any $n \in \mathbb{N}$, there exists $(y_n, u_n) \in N(\alpha_{1/n})$ such that

$$\|(y_n, u_n) - N(\alpha)\| \geq \delta. \tag{5.9}$$

However, according to the definition of X and inequality (5.8), one has $\alpha(y_n, u_n) \in X$ and $\|y_n - \alpha(y_n, u_n)\| < 1/n$. Since X and the unit sphere of U are compact, without loss of generality, one can assume that $y_n \to y_*$ and $u_n \to u_*$. Moreover, $(y_*, u_*) \in S$. By passing to the limit, one obtains $\alpha(y_*, u_*) = y_*$ that contradicts (5.9).

Therefore, the statement of Lemma 5.1 follows from monotonicity property of ind. □

Return to the proof of Theorem 4.3 in the considered case. Take ε small enough and the Schauder projection p_ε satisfying (5.8). Let $\mathbb{R}^k \subset E_2$ be the invariant finite-dimensional subspace containing $p_\varepsilon(X)$. Put $\alpha'_\varepsilon := \alpha_\varepsilon|_{\mathbb{R}^k \oplus U}$ and let $N(\alpha'_\varepsilon)$ stand for the solution set to the equation $\alpha'_\varepsilon(y,u) = y$. Combining the result obtained at the previous step with the monotonicity property of ind, one obtains

$$\frac{p}{d} \leq \mathrm{ind}\left(N(\alpha'_\varepsilon)\right) \leq \mathrm{ind}\left(N(\alpha_\varepsilon)\right) \leq \mathrm{ind}\left(N(\alpha)\right). \tag{5.10}$$

Step 3 (infinite-dimensional kernel). Under the assumptions of Theorem 4.3, suppose that $p = \infty$ and take a finite-dimensional invariant subspace $V \subset U$ (cf. [20, Section 4 and Appendix C] or [21, page 57]). Put $E' := E_2 \oplus V$ and $\alpha_V := \alpha|_{E'}$. Denote by $N(\alpha_V)$ the solution set to the equation

$$\alpha_V(y,u) = y \quad (y \in E_2,\ u \in V). \tag{5.11}$$

By monotonicity property, $N(\alpha_V) \subset N(\alpha)$ implies that $\mathrm{ind}(N(\alpha)) \geq \mathrm{ind}(N(\alpha_V))$. However (see Step 2), $\mathrm{ind}(\alpha_V) \geq \dim V/d$. Bearing in mind that $\dim V$ can be chosen arbitrarily large (see again [20, Section 4 and Appendix C]), one obtains $\mathrm{ind}(N(\alpha)) = \infty$.

To complete the proof of Theorem 4.3, it remains to combine Steps 2 and 3 with Lemma 4.4 and the monotonicity property of ind.

Corollary 5.2. *Under the assumptions of Theorem 4.3, suppose that $p=\infty$. Then* $\dim N(a, f) = \infty$.

Proof. Arguing indirectly, assume that $\dim N(a,f)$ is finite. Then $N(a,f)$ is compact and, therefore, $\mathrm{ind}(N(a,f))$ is finite as well. The obtained contradiction completes the proof. □

6. Application

Let Λ be a finite-dimensional linear space (thought of as a parameter space) and $b : \mathbb{R}^n \times \Lambda \to \mathbb{R}^n$ a continuous map. Let, further, $C^{2\pi}_{[0,2\pi]}$ be the space of continuous functions $x : [0,2\pi] \to \mathbb{R}^n$ with $x(0) = x(2\pi)$ (equipped with the standard sup-norm). Put $E_1 := C^{2\pi}_{[0,2\pi]} \oplus \Lambda$ and $\|(x,\lambda)\|_{E_1} := \|x\| + \|\lambda\|$.

Consider the following problem.

Problem 6.1. *Given a real number $R > 0$, do there exist a differentiable 2π-periodic vector-function $x : \mathbb{R} \to \mathbb{R}^n$ and $\lambda \in \Lambda$ such that*

$$x'(t) = b(x(t),\lambda) - \frac{1}{2\pi}\int_0^{2\pi} b(x(s),\lambda)\,ds \quad \forall t \in \mathbb{R}, \tag{6.1}$$

and $\|(x,\lambda)\|_{E_1} = R$? In addition, what can be said about the topological structure of the solution set $N(b)$ to the above problem?

Assume, in addition, Λ is an (isometric) S^1-representation satisfying the condition

$(*)$ $\Lambda^{S^1} = \{0\}$.

In particular, $\dim\Lambda$ is even and we will assume that

$(**)$ $\dim\Lambda > 0$.

Identify $C^{2\pi}_{[0,2\pi]}$ with the space of continuous functions $x : S^1 \to \mathbb{R}^n$ and define on it the natural (isometric) S^1-representation: $(hx)(t) = x(t+\varphi)$, where $h = \exp(i\varphi) \in S^1$. Equip E_1 with the diagonal S^1-action.

Assume, further, the map b from Problem 6.1 to be S^1-invariant in the second variable, that is,

$(***)$ $b(x,h\lambda) = b(x,\lambda)$ for all $x \in \mathbb{R}^n$, $\lambda \in \Lambda$, $h \in S^1$.

Proposition 6.2. *Under the assumptions $(*)$, $(**)$, and $(***)$, one has the following genus estimate for $N(b)$:*

$$g(N(b)) \geq \frac{\dim\Lambda}{2} \tag{6.2}$$

(in particular, $N(b) \neq \emptyset$).

Proof. Observe, first, that by condition $(*)$,

$$E_1^{S^1} = \{(x(\cdot),0) \mid x(\cdot) \text{ is a constant function}\}. \tag{6.3}$$

Next, denote by $C_{[0.2\pi]}$ the space of continuous functions from $[0,2\pi]$ to $\mathbb{R}^n$ with the standard sup-norm, and let $d : D(d) \subset C^{2\pi}_{[0,2\pi]} \to C_{[0,2\pi]}$ be the differentiation operator, where

$$D(d) = \left\{x(\cdot) \in C^{[2\pi]}_{[0,2\pi]} \,\middle|\, x(\cdot) \text{ is smooth and } x'(0) = x'(2\pi)\right\}. \tag{6.4}$$

Obviously, d is closed and $\ker(d)$ coincides with the set of constant functions.

Consider now the operator $a : D(a) \subset E_1 \to C_{[0,2\pi]}$ defined by

$$a(x(\cdot),\lambda) := x'(\cdot). \tag{6.5}$$

Obviously, $D(a) = D(d) \oplus \Lambda$ and $\ker(a) = \ker(d) \oplus \Lambda$. Moreover, a is still a closed operator.

Put

$$E_2 := \operatorname{Im}(a) = \left\{y(\cdot) \in C_{[0,2\pi]} \,\middle|\, \int_0^{2\pi} y(s)ds = 0,\ y(0) = y(2\pi)\right\}. \tag{6.6}$$

By direct computation, E_2 is a closed S^1-invariant subset of $C^{2\pi}_{[0,2\pi]}$, and a is equivariant. Also, $E_2^{S^1} = \{0\}$.

Consider now a nonlinear continuous map f determined by the right-hand side of (6.1):

$$y(t) := f(x,\lambda)(t) = b(x(t),\lambda) - \frac{1}{2\pi}\int_0^{2\pi} b(x(s),\lambda)ds. \tag{6.7}$$

Obviously, $y(0) = y(2\pi)$ and $\int_0^{2\pi} y(s)ds = 0$. Hence, f takes E_1 to E_2. Moreover, since Λ is assumed to be finite-dimensional, the map f is a-compact. To check that f is S^1-equivariant, take $h = \exp(i\varphi) \in S^1$. Using condition $(***)$, we have

$$\begin{aligned} f(h(x(t),\lambda)) = f(h(x(t),h\lambda)) &= f(h(x(t),\lambda)) - \frac{1}{2\pi}\int g(h(x(s)),\lambda)ds \\ &= g(x(t+\varphi),\lambda) - \frac{1}{2\pi}\int_0^{2\pi} g(x(s+\varphi),\lambda)ds = hf(x(t),\lambda). \end{aligned} \tag{6.8}$$

To complete the proof of Proposition 6.2, take the sphere $S_R \subset E_1$ and apply Theorem 4.3 (cf. condition $(**)$ and Example 2.6(ii)). □

Remark 6.3. (i) In Proposition 6.2, one can take any index theory (for S^1) satisfying the dimension property. Also, the segment $[0,2\pi]$ is taken to simplify the presentation.

(ii) In this paper, we restrict ourselves with the simplest illustrative example. In forthcoming papers, more involved applications (in particular, admitting closed operators with *infinite-dimensional* kernels) will be considered.

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GANTMACHER-KREĬN THEOREM FOR 2 NONNEGATIVE OPERATORS IN SPACES OF FUNCTIONS

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The existence of the second (according to the module) eigenvalue λ_2 of a completely continuous nonnegative operator A is proved under the conditions that A acts in the space $L_p(\Omega)$ or $C(\Omega)$ and its exterior square $A \wedge A$ is also nonnegative. For the case when the operators A and $A \wedge A$ are indecomposable, the simplicity of the first and second eigenvalues is proved, and the interrelation between the indices of imprimitivity of A and $A \wedge A$ is examined. For the case when A and $A \wedge A$ are primitive, the difference (according to the module) of λ_1 and λ_2 from each other and from other eigenvalues is proved.

1. Introduction

In the monograph [3] the following statement was proved: *if the matrix* $\mathbf{A}$ *of a linear operator* A *in the space* $\mathbb{R}^n$ *is primitive along with its associated* $\mathbf{A}^{(j)}$ $(1 < j \leq k)$ *up to the order* k, *then the operator* A *has* k *positive simple eigenvalues* $0 < \lambda_k < \cdots < \lambda_2 < \lambda_1$, *with a positive eigenvector* e_1 *corresponding to the maximal eigenvalue* λ_1, *and an eigenvector* e_j, *which has exactly* $j-1$ *changes of sign, corresponding to* j*th eigenvalue* λ_j (see [3, page 310, Theorem 9]). Matrices with mentioned features are called henceforth *k-completely nonnegative*; in the most important case $k = n$ they are called *oscillatory*.

Naturally, there arises a problem whether it is possible to extend this statement to operators in infinite-dimensional spaces, for example, to linear integral operators. This problem practically has not been studied in full volume. However, in the monograph [3], Gantmacher and Kreĭn have thoroughly studied the linear integral operators

$$Kx(t) = \int_a^b k(t,s)x(s)\,ds \tag{1.1}$$

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acting in the space $L_2([a,b])$ with continuous kernels $k(t,s)$, for which the matrices $\|k(t_i,t_j)\|_1^n$ ($n = 1,2,\ldots$) for any points $t_1,\ldots,t_n \in [a,b]$, among which at least one is interior, are oscillatory. Such kernels, named in [3] *oscillatory*, form quite full analogue to oscillatory matrices. In [3], for the integral operators with continuous oscillatory kernels, it was proved that there exists a converging-to-zero sequence of positive simple eigenvalues $\lambda_1 > \lambda_2 > \cdots > \lambda_n > \cdots$ with eigenfunctions $e_n(t)$ that has exactly $n-1$ changes of sign, corresponding to the nth eigenvalue λ_n (see [3, page 211]).

In connection with the formulated Gantmacher-Kreĭn theorem, there arises a natural question on the possibility of spreading the statements about k-completely-nonnegative matrices from [3] onto the integral operators with k-completely-nonnegative kernels, that is, the kernels $k(t,s)$, for which the matrices $\|k(t_i,t_j)\|_1^n$ ($n = 1,2,\ldots,k$) for any points $t_1,\ldots,t_n \in [a,b]$, among which at least one is interior, are oscillatory. The answer to this question is positive. Moreover, this statement was actually proved exactly in [3].

However, here arises a question how substantial the condition of continuity of the kernel $k(t,s)$ is in these statements and how substantial the assumption that the problem is regarded in the space of functions, defined exactly on the interval $[a,b]$, is. And of course the natural question arises whether it is possible to obtain similar statements for abstract (not necessarily integral) operators in an arbitrary Banach spaces.

In the present paper we study 2-completely-nonnegative (or otherwise bi-nonnegative) operators in the spaces $L_p(\Omega)$ ($1 \le p \le \infty$) and $C(\Omega)$. As the authors believe, the natural machinery for the examination of such operators is a crossway from studying an operator A in one of the spaces $L_p(\Omega)$ and $C(\Omega)$ to the study of the operators $A \otimes A$ and $A \wedge A$, acting, respectively, in the spaces $L_p \otimes L_p = L_p(\Omega \times \Omega)$ and $L_p \wedge L_p = L_p^a(\Omega \times \Omega)$ (the latter is a subspace of the space $L_p \otimes L_p = L_p(\Omega \times \Omega)$, consisting of antisymmetric functions, i.e., functions $x(t,s)$, for which $x(t,s) = -x(s,t)$).

2. Tensor and exterior square of the spaces $L_p(\Omega)$ and $C(\Omega)$

Let $(\Omega, \mathfrak{A}, \mu)$ be a triple consisting of some set Ω, some σ-algebra $\mathfrak{A}$ of "measurable" subsets and some σ-finite and σ-additive measure on $\mathfrak{A}$. We will be interested in the space $L_p(\Omega)$ of functions, integrable on Ω with the power p for $1 \le p < \infty$ or measurable and substantially bounded for $p = \infty$, the analogous space $L_p(\Omega \times \Omega)$ of functions, integrable on $\Omega \times \Omega$ with the power p for $1 \le p < \infty$ or essentially bounded for $p = \infty$ and, finally, the subspace $L_p^a(\Omega \times \Omega)$ of the space $L_p(\Omega \times \Omega)$ of antisymmetric functions. Henceforth let p be a fixed number from $[1,\infty]$.

We start with observing the following facts:

(1) *the space $L_p^a(\Omega \times \Omega)$ is one of the tensor products of the space $L_p(\Omega)$ by itself*, and, respectively,

(2) *the space $L_p^a(\Omega \times \Omega)$ is one of the exterior products of the space $L_p(\Omega)$ by itself.*

The first of these statements means the following.

(a) For arbitrary functions $x_1, x_2 \in L_p(\Omega)$ their $\odot$-product $x_1 \odot x_2(t_1,t_2) = x_1(t_1)x_2(t_2)$ belongs to the space $L_p(\Omega \times \Omega)$, with

$$\|x_1(t_1)x_2(t_2)\| = \|x_1(t_1)\| \|x_2(t_2)\|. \tag{2.1}$$

(b) The linear hull of the set of all $\odot$-products of functions from $L_p(\Omega)$, that is, the set of all functions of the form

$$x(t_1,t_2) = \sum_i x_1^i(t_1)x_2^i(t_2) \tag{2.2}$$

is dense in the space $L_p(\Omega\times\Omega)$.

The second statement means the following.

(a) The $\wedge$-product of arbitrary functions $x_1,x_2 \in L_p(\Omega)$ with $x_1 \wedge x_2(t_1,t_2) = x_1(t_1)x_2(t_2) - x_1(t_2)x_2(t_1)$ also belongs to the space $L_p(\Omega\times\Omega)$, and it is obvious that

$$\begin{gathered} x_1 \wedge x_2(t_1,t_2) = -x_1 \wedge x_2(t_2,t_1), \\ \|x_1 \wedge x_2(t_1,t_2)\| \le 2\|x_1(t_1)\|\|x_2(t_2)\|. \end{gathered} \tag{2.3}$$

(b) The linear hull of the set of all $\wedge$-products of the functions from $L_p(\Omega)$ is dense in the space $L_p^a(\Omega\times\Omega)$.

The space $L_p^a(\Omega\times\Omega)$ is isomorphic in the category of Banach spaces to the space $L_p(W)$, where W is the subset $\Omega\times\Omega$, for which the sets $W\cap\widetilde{W}$ and $(\Omega\times\Omega)\setminus(W\cup\widetilde{W})$ have zero measure; here $\widetilde{W} = \{(t_2,t_1) : (t_1,t_2)\in W\}$ (such sets do always exist). Really, extending the functions from $L_p(W)$ as antisymmetric functions from W to $\Omega\times\Omega$, we obtain the set of all the functions from $L_p^a(\Omega\times\Omega)$. Further, setting the norm of a function in $L_p(W)$ to be equal to the norm of its extension, we get that the spaces $L_p^a(\Omega\times\Omega)$ and $L_p(W)$ are isomorphic in the category of normed spaces.

The general scheme of the interrelations between the spaces $L_p(\Omega)\oplus L_p(\Omega)$, $L_p(\Omega\times\Omega)$, $L_p^a(\Omega\times\Omega)$, and $L_p(W)$ can be represented by the diagram

$$L_p(\Omega)\otimes L_p(\Omega) \xrightarrow{\otimes} L_p(\Omega\times\Omega) \xrightarrow{\mathbf{a}} L_p^a(\Omega\times\Omega) = L_p(W), \tag{2.4}$$

where $\mathbf{a}$ is the antisymmetrization operator acting from $L_p(\Omega\times\Omega)$ to $L_p^a(\Omega\times\Omega)$ according to the rule

$$\mathbf{a}x(t_1,t_2) = \frac{x(t_1,t_2) - x(t_2,t_1)}{2}. \tag{2.5}$$

Let us examine some examples of constructing the set W for different sets Ω.

(1) Let $\Omega = [a,b]$; then $\Omega\times\Omega = [a,b]^2$, and as W we may regard the triangle, defined by inequalities $a \le t_1 \le t_2 \le b$. Really, in this case $\widetilde{W}$ is defined by the inequalities $a \le t_2 \le t_1 \le b$, $\Omega\times\Omega = [a,b]^2 = W\cup\widetilde{W}$ and $W\cap\widetilde{W} = w_0$, where w_0, defined by the inequalities $a \le t_1 = t_2 \le b$, is a set of zero measure.

(2) Consider another example. Let $\Omega = [a,b]^2$. Then $\Omega\times\Omega = [a,b]^4$. Define on the space $\mathbb{R}^2$ the following order relation: $(t_1,t_2) \le (s_1,s_2)$, if $t_1 \le s_1$. Introduce $W = \{(t_1,t_2,t_3,t_4) \in [a,b]^4 : (t_1,t_2) < (t_3,t_4)\}$ and $\widetilde{W} = \{(t_1,t_2,t_3,t_4) \in [a,b]^4 : (t_3,t_4) < (t_1,t_2)\}$. As we see, $\Omega\times\Omega = W\cup\widetilde{W}\cup w_0$, where $w_0 = \{(t_1,t_2,t_3,t_4) \in [a,b]^4 : (t_1,t_2) = (t_3,t_4)\}$ is a set of zero measure in the 4-dimensional space. After thorough examination of the inequalities defining the set W, one obtains

$W = \{a \le t_1 < t_3 \le b;\ a \le t_2,\ t_4 \le b\}$. The geometrical interpretation of this construction is as follows: the 4-dimensional cube is divided by the hyperflat $t_1 = t_3$ into two symmetric parts. Notice that the cube can be divided in such a way with the help of another surfaces as well, for example, with the help of the hyperflat $t_1 + t_2 = t_3 + t_4$.

(3) Suppose that on the set Ω a relation with the following properties is given:
(a) almost all the elements of Ω are comparable;
(b) $\mu(\{t_1, t_2 \in \Omega : t_1 \preceq t_2\} \cap \{t_1, t_2 \in \Omega : t_2 \preceq t_1\}) = 0$. Then we can define the sets $W = \{(t_1, t_2) \in \Omega^2 : t_1 \preceq t_2\}$ and $\widetilde{W} = \{(t_1, t_2) \in \Omega^2 : t_2 \preceq t_1\}$, possessing the necessary properties.

Now let (Ω, τ) be some compact Hausdorff topological space and let $C(\Omega)$ be the space of all continuous functions on Ω. Then the set $\Omega \times \Omega$, with the topology $\tau \times \tau$ given on it, is also a compact Hausdorff space. Denote by $C(\Omega \times \Omega)$ the space of all continuous functions on $\Omega \times \Omega$, and by $C^a(\Omega \times \Omega)$—the subspace $C(\Omega \times \Omega)$ of all antisymmetric functions on $\Omega \times \Omega$. Just as in the case of Lebesgue spaces, the space $C(\Omega \times \Omega)$ is one of the tensor products of the space $C(\Omega)$ by itself, and the space $C^a(\Omega \times \Omega)$ is one of the exterior products of the space $C(\Omega)$ by itself. In other words, for them the analogues of statements (1) and (2) for Lebesgue spaces are true. Further, the space $C^a(\Omega \times \Omega)$ is isomorphic to the space $C_0(W)$ (here W is a subset $\Omega \times \Omega$, for which $W \cap \widetilde{W} = \Delta$, $\Delta = \{(t,t) : t \in \Omega\}$, $W \cup \widetilde{W} = \Omega \times \Omega$, and $C_0(W)$ is a subspace of the space $C(W)$, consisting of all functions $x(t_1, t_2)$, for which $x(t_1, t_1) = 0$). In particular, the following diagram is true:

$$C(\Omega) \otimes C(\Omega) \xrightarrow{\otimes} C(\Omega \times \Omega) \xrightarrow{\mathbf{a}} C^a(\Omega \times \Omega) = C_0(W) \subset C(W). \tag{2.6}$$

Sometimes Lebesgue spaces and the space of all continuous functions have to be examined at the same time. In this case it is natural to require continuous functions to be measurable. This means that the topology τ, σ-algebra $\mathfrak{A}$, and the measure μ on Ω must be related to each other in the following way: $\mathfrak{A}$ contains all the closed sets from τ and the measure μ is regular, that is, for any $A \in \mathfrak{A}$ and any number $\epsilon > 0$ there exist a closed set F and an open set G such that $F \subset A \subset G$ and $\mu(G \setminus F) < \epsilon$. In this case the space $C(\Omega)$ is associated with a closed subspace of the space $L_\infty(\Omega)$.

3. Tensor and exterior squares of linear operators in the spaces $L_p(\Omega)$ and $C(\Omega)$

Let A and B be continuous linear operators acting in the space $L_p(\Omega)$. These operators generate the operator $A \otimes B$ in the space $L_p(\Omega \times \Omega)$ as follows: on degenerate functions it is defined by the equality

$$(A \otimes B)x(t_1, t_2) = \sum_j A x_1^j(t_1) \cdot B x_2^j(t_2) \qquad \left(x(t_1, t_2) = \sum_j x_1^j(t_1) \cdot x_2^j(t_2) \right), \tag{3.1}$$

and on arbitrary functions it is defined by extension via continuity from the subspace of degenerate functions onto the whole of $L_p(\Omega \times \Omega)$. The possibility of such an extension is due to the density of the set of all degenerate functions in the space $L_p(\Omega \times \Omega)$ and to the fact that the operator $A \otimes B$ is bounded on the subspace of degenerate functions of the space $L_p(\Omega \times \Omega)$; the latter comes out from the following observations.

Let

$$x(t_1,t_2) = \sum_j x_1^j(t_1)x_2^j(t_2) \tag{3.2}$$

be a degenerate function from $L_p(\Omega \times \Omega)$. It is obvious that for almost every $t_2 \in \Omega$ this function is measurable as a function of $t_1 \in \Omega$. Moreover, it can be regarded as a linear combination of functions $x_1^j(t_1)$ with coefficients $x_2^j(t_2)$. Therefore

$$A_{(1)}x(t_1,t_2) = \sum_j Ax_1^j(t_1) \cdot x_2^j(t_2); \tag{3.3}$$

the obtained function turns out to be measurable with respect to all the variables (the operator's A index (1) means that it is used for the variable t_1). Because of the Fubini theorem, the following estimate is true:

$$||A_{(1)}x(t_1,t_2)|| \leq \|A\| \left\| \sum_j x_1^j(t_1) \cdot x_2^j(t_2) \right\|. \tag{3.4}$$

Further, the function $\sum_j Ax_1^j(t_1)x_2^j(t_2)$ is measurable with respect to all the variables, it is measurable as a function of $t_2 \in \Omega$ for almost every $t_1 \in \Omega$, and it can also be regarded as a linear combination of functions $x_2^j(t_2)$ with coefficients $Ax_1^j(t_1)$. Therefore, after using the operator B for the variable t_2,

$$B_{(2)}A_{(1)}x(t_1,t_2) = \sum_j Ax_1^j(t_1) \cdot Bx_2^j(t_2); \tag{3.5}$$

the obtained function turns out to be measurable with respect to all the variables. Applying the Fubini theorem again, we get the estimate

$$||B_{(2)}A_{(1)}x(t_1,t_2)|| \leq \|A\| \, \|B\| \left\| \sum_j x_1^j(t_1) \cdot x_2^j(t_2) \right\|. \tag{3.6}$$

We may conventionally write down the value of the operator $A \otimes B$ in the form of

$$(A \otimes B)x(t_1,t_2) = A_{(1)}B_{(2)}x(t_1,t_2) = B_{(2)}A_{(1)}x(t_1,t_2), \tag{3.7}$$

for arbitrary functions from $L_p(\Omega \times \Omega)$ as well. However, with such a separate usage of the operators A and B, there arises a question of measurability of the function $B_{(2)}x(t_1,t_2)$ for the variable t_1. To avoid this trouble, we have to use the procedure of extension by continuity from the subspace of degenerate functions, where, as shown above, the mentioned trouble does not arise.

In the case of the space $C(\Omega \times \Omega)$, formula (3.7) and the estimate

$$||(A \otimes B)x(t_1,t_2)|| \leq \|A\| \, \|B\| \, ||x(t_1,t_2)|| \tag{3.8}$$

arising from it are obvious, since the function that is continuous with respect to all the variables is continuous also in each variable separately with the fixed another one.

Further in this work we will examine exclusively the tensor square $A \otimes A$ of the operator A.

Let us examine the operator $A \wedge A : L_p^a(\Omega \times \Omega) \to L_p^a(\Omega \times \Omega)$, defined as the restriction of the operator $A \otimes A$ onto the subspace $L_p^a(\Omega \times \Omega)$. It is obvious that for degenerate antisymmetric functions the operator $A \wedge A$ can be defined by the equality

$$(A \wedge A)x(t_1,t_2) = \sum_j Ax_1^j(t_1) \wedge Ax_2^j(t_2), \quad x(t_1,t_2) = \sum_j x_1^j(t_1) \wedge x_2^j(t_2). \tag{3.9}$$

Decompose $L_p(\Omega \times \Omega)$ into the direct sum of subspaces invariant with respect to $A \otimes A$:

$$L_p(\Omega \times \Omega) = L_p^a(\Omega \times \Omega) \oplus L_p^s(\Omega \times \Omega), \tag{3.10}$$

where $L_p^s(\Omega \times \Omega)$ is the subspace of all symmetric functions from $L_p(\Omega \times \Omega)$. The operator $A \otimes A$ can be represented in the block form

$$A \otimes A = \begin{pmatrix} A \wedge A & 0 \\ 0 & (A \otimes A)|_{L_p^s} \end{pmatrix}. \tag{3.11}$$

Further, it will be useful to compare the operator A with the antisymmetrization of its tensor square $\mathbf{a} \circ (A \otimes A) : L_p(\Omega \times \Omega) \to L_p^a(\Omega \times \Omega) \subset L_p(\Omega \times \Omega)$, where $\mathbf{a}$ is the antisymmetrization operator, defined by formula (2.5). Taking into account that the antisymmetrization operator leaves antisymmetric functions without changes, we conclude that the restriction of $\mathbf{a} \circ (A \otimes A)$ onto the subspace $L_p^a(\Omega \times \Omega)$ coincides with $A \wedge A$.

The space $C(\Omega \times \Omega)$ can also be decomposed into the direct sum of subspaces invariant with respect to $A \otimes A$:

$$C(\Omega \times \Omega) = C^a(\Omega \times \Omega) \oplus C^s(\Omega \times \Omega), \tag{3.12}$$

where $C^s(\Omega \times \Omega)$ is the subspace of all symmetric functions from $C(\Omega \times \Omega)$. It is easy to see that the operator $A \otimes A : C(\Omega \times \Omega) \to C(\Omega \times \Omega)$ can also be represented in the block form. The operators $A \wedge A : C^a(\Omega \times \Omega) \to C^a(\Omega \times \Omega)$ and $\mathbf{a} \circ (A \otimes A) : C(\Omega \times \Omega) \to C^a(\Omega \times \Omega)$ are defined in the same way.

4. Spectrum of the tensor square of linear operators in the spaces $L_p(\Omega)$ and $C(\Omega)$

As usual, we will denote by $\sigma(A)$ the spectrum of the operator A, and by $\sigma_p(A)$ the point spectrum, that is, the set of all eigenvalues of the operator A. We will denote by $\sigma_{eb}(A)$ the Browder essential spectrum of the operator A, that is, the set of all points $\lambda \in \sigma(A)$, such that at least one of the following conditions holds:

(1) $R(A - \lambda I)$ is not closed;
(2) λ is a limit point of $\sigma(A)$;
(3) $\bigcup_{n \geq 0} \ker(A - \lambda I)^n$ is of infinite dimension.

Thus $\sigma(A) \setminus \sigma_{eb}(A)$ will be the set of all isolated finite-dimensional eigenvalues of the operator A, (for more detailed information see [6, 7]).

In the papers by Ichinose [4–7] there have been obtained the results, representing the spectra and the parts of the spectra of the tensor product of linear bounded operators in terms of the spectra and parts of the spectra of the given operators under the natural suppositions that

(a) the tensor product of linear bounded operators $A \otimes B$ can be extended from the set of degenerate functions, and the extension is also a linear bounded operator in $L_p(\Omega \times \Omega)$ and $C(\Omega \times \Omega)$ respectively;

(b) the adjoint spaces $L_{p'}(\Omega \times \Omega)$ and $rca(\Omega \times \Omega)$ have the same property.

In fact these statements have been proved in the previous part. The explicit formulae, expressing the set of all isolated finite dimensional eigenvalues and the Browder essential spectrum of the operator $A \otimes A$ in terms of the parts of the spectrum of the given operator are obtained, for example, in [4, page 110, Theorem 4.2]. In particular, Ichinose proved, that for the tensor square of a linear bounded operator A the following equalities hold:

$$\sigma(A \otimes A) = \sigma(A)\sigma(A); \tag{4.1}$$

$$\sigma(A \otimes A) \setminus \sigma_{eb}(A \otimes A) = \big(\sigma(A) \setminus \sigma_{eb}(A)\big)\big(\sigma(A) \setminus \sigma_{eb}(A)\big) \setminus \big(\sigma_{eb}(A)\sigma(A)\big); \tag{4.2}$$

$$\sigma_{eb}(A \otimes A) = \sigma_{eb}(A)\sigma(A). \tag{4.3}$$

Besides, for an arbitrary $\lambda \in (\sigma(A \otimes A) \setminus \sigma_{eb}(A \otimes A))$ the following equality holds:

$$\ker(A \otimes A - \lambda I \otimes I) = \ker(A - \lambda_i I) \otimes \ker\big(A - \lambda_j I\big), \tag{4.4}$$

where $\lambda_i, \lambda_j \in (\sigma(A) \setminus \sigma_{eb}(A))$ such that $\lambda = \lambda_i \cdot \lambda_j$.

In the finite-dimensional case, when the matrix $A \otimes A$ appears to be a tensor square of the matrix A, Stephanos's result ([13], see also [10]) tells that the set of all eigenvalues of the operator $A \otimes A$ is the set of all the possible products of the form $\{\lambda_i \lambda_j\}$, where $\{\lambda_i\}$ is the set of all eigenvalues of the operator A, repeated according to multiplicity. Thus the property

$$\sigma_p(A \otimes A) = \sigma_p(A)\sigma_p(A) \tag{4.5}$$

is widely known. In the infinite-dimensional case the analogous formula, expressing $\sigma_p(A \otimes A)$ in terms of the parts of the spectrum of the operator A, seems to be unknown. That is why further we will be interested in the case of a completely continuous operator A. For a completely continuous operator the following equalities are true:

$$\big(\sigma(A) \setminus \sigma_{eb}(A)\big) \setminus \{0\} = \sigma_p(A) \setminus \{0\}; \qquad \sigma_{eb}(A) = \{0\} \text{ or } \varnothing. \tag{4.6}$$

So, from (4.2) we can get the complete information about the nonzero eigenvalues of the tensor square of a completely continuous operator:

$$\sigma_p(A \otimes A) \setminus \{0\} = \sigma_p(A)\sigma_p(A) \setminus \{0\}. \tag{4.7}$$

Here zero can be either a finite- or infinite-dimensional eigenvalue of $A \otimes A$, or a point of the essential spectrum. That is why, even for the case of a completely continuous operator, formula (4.4) in general is incorrect.

5. Spectrum of the exterior square of linear operators in the spaces $L_p(\Omega)$ and $C(\Omega)$

For the exterior square, which is the restriction of the tensor square, the following inclusions are true:

$$\sigma(A \wedge A) \subset \sigma(A \otimes A); \tag{5.1}$$

$$\sigma_p(A \wedge A) \subset \sigma_p(A \otimes A). \tag{5.2}$$

In the finite-dimensional case, it is known that the matrix $A \wedge A$ in a suitable basis appears to be the second-order associated matrix to the matrix A, and we conclude that all the possible products of the type $\{\lambda_i \lambda_j\}$, where $i < j$, form the set of all eigenvalues of the exterior square $A \wedge A$, repeated according to multiplicity (see [3, Theorem 23, page 80]).

In the infinite-dimensional case we can also obtain some information concerning eigenvalues of the exterior square of a linear bounded operator.

Theorem 5.1. *Let X be either $L_p(\Omega)$ or $C(\Omega)$ and let $\{\lambda_i\}$ be the set of all eigenvalues of the operator $A : X \to X$, repeated according to multiplicity. Then all the possible products of the form $\{\lambda_i \lambda_j\}$, where $i < j$, will be the eigenvalues of the exterior square $A \wedge A$.*

Proof. Let $\lambda_i, \lambda_j \in \sigma_p(A)$. Then there exist functions $x(t)$, $y(t)$ from X, such that $(A - \lambda_i I)x(t) = 0$ and $(A - \lambda_j I)y(t) = 0$. Let us examine the value of the operator $A \wedge A - \lambda_i \lambda_j I \wedge I$ on the degenerate antisymmetric function $(x \wedge y)(t_1, t_2)$:

$$\begin{aligned}(A \wedge A - \lambda_i \lambda_j I \wedge I)x \wedge y &= Ax \wedge Ay - \lambda_i \lambda_j x \wedge y \\ &= Ax \wedge Ay - \lambda_i x \wedge Ay + \lambda_i x \wedge Ay - \lambda_i \lambda_j x \wedge y \\ &\text{[because of the linearity of the exterior product]} \\ &= (Ax - \lambda_i x) \wedge Ay + \lambda_i x \wedge (Ay - \lambda_j y) = 0.\end{aligned} \tag{5.3}$$

From this we see that $\lambda_i \lambda_j \in \sigma_p(A \wedge A)$. □

However, just as in the case of the tensor square of an operator, in order to obtain a statement, analogous to the finite-dimensional case, we need the additional assumption about complete continuity. For nonzero eigenvalues of the exterior square of a complete continuous operator, the following statement holds.

Theorem 5.2. *Let X be either $L_p(\Omega)$ or $C(\Omega)$ and let $\{\lambda_i\}$ be the set of all eigenvalues of an absolutely continuous operator $A : X \to X$, repeated according to multiplicity. Then all the possible products of the type $\{\lambda_i \lambda_j\}$, where $i < j$, form the set of all the possible (except, probably, zero) eigenvalues of the exterior square $A \wedge A$, repeated according to multiplicity.*

Proof. The inclusion $\{\lambda_i \lambda_j\}_{i<j} \subset \sigma_p(A \wedge A)$, that is, each product of the form $\lambda_i \lambda_j$, where $i < j$, is an eigenvalue of $A \wedge A$, comes out from Theorem 5.1.

Now we will prove the reverse inclusion: $\sigma_p(A \wedge A) \subset \{\lambda_i \lambda_j\}_{i<j}$. As it was shown above in formulae (4.7) and (5.2),

$$\sigma_p(A \wedge A) \setminus \{0\} \subset \sigma_p(A \otimes A) \setminus \{0\} = \sigma_p(A)\sigma_p(A) \setminus \{0\}, \tag{5.4}$$

that is, the operator $A \wedge A$ has no other eigenvalues, except products of the form $\lambda_i\lambda_j$. Enumerate the set of pairs $\{(i,j)\}i,\ j = 1,2,\dots$. In this way we get a numeration of $\{\lambda_i\lambda_j\}$—the set of all eigenvalues of $A \otimes A$, repeated according to multiplicity. Decompose the obtained finite or countable set of indices Λ in the following way:

$$\Lambda = \Lambda_1 \cup \Lambda_2 \cup \Lambda_3, \tag{5.5}$$

where the set Λ_1 includes the numbers of those pairs (i,j) for which $i < j$, Λ_2 includes those pairs for which $i = j$, and Λ_3 includes those pairs for which $i > j$. The set of all eigenvalues of $A \otimes A$, repeated according to multiplicity, will be then decomposed into three parts:

$$\{\lambda_\alpha\}_{\alpha\in\Lambda} = \{\lambda_\alpha\}_{\alpha\in\Lambda_1} \cup \{\lambda_\alpha\}_{\alpha\in\Lambda_2} \cup \{\lambda_\alpha\}_{\alpha\in\Lambda_3}. \tag{5.6}$$

As it was shown in Section 3, the operator $A \otimes A$ has a block structure, and so $\sigma_p(A \otimes A)$ can be decomposed into two subsets:

$$\sigma_p(A \otimes A) = \sigma_p(A \wedge A) \cup \sigma_p(A \otimes A|_{X^s}), \tag{5.7}$$

where X^s is the subset of all symmetric functions from X. In order to prove that the eigenvalues of $A \otimes A$, belonging to the sets $\{\lambda_\alpha\}_{\alpha\in\Lambda_2}$ and $\{\lambda_\alpha\}_{\alpha\in\Lambda_3}$, will not be the eigenvalues of $A \wedge A$, it is enough to show that they will be the eigenvalues of $(A \otimes A)|_{X^s}$. Indeed, let $x_i(t) \in X$ be an eigenfunction of the operator A, corresponding to the eigenvalue λ_i. Let us examine the value of the operator $(A \otimes A - \lambda_i^2 I \wedge I)$ on the function $x_i(t_1)x_i(t_2) \in X^s(\Omega \times \Omega)$:

$$\begin{aligned}
&(A \otimes A - \lambda_i^2 I \otimes I)x_i(t_1)x_i(t_2)\\
&\quad = Ax_i(t_1)Ax_i(t_2) - \lambda_i^2 x_i(t_1)x_i(t_2)\\
&\quad = Ax_i(t_1)Ax_i(t_2) - \lambda_i x_i(t_1)Ax_i(t_2) + \lambda_i x_i(t_1)Ax_i(t_2) - \lambda_i^2 x_i(t_1)x_i(t_2)\\
&\quad = (Ax_i - \lambda_i x_i)(t_1)Ax_i(t_2) + \lambda_i x_i(t_1)(Ax_i - \lambda_i x_i)(t_2) = 0.
\end{aligned} \tag{5.8}$$

From this we see that $\lambda_i^2 \in \sigma_p((A \otimes A)|_{X^s})$. In an analogous way we can prove that a product of the form $\lambda_i\lambda_j$ will also be an eigenvalue of $(A \otimes A)|_{X^s}$ (with the corresponding symmetric function $x_i(t_1)x_j(t_2) + x_j(t_1)x_i(t_2)$). □

It is obvious that the spectrum of the operator $\mathbf{a} \circ (A \otimes A)$ coincides with the spectrum of $A \wedge A$.

6. Generalization of the Gantmacher-Kreĭn theorems in the case of 2-totally-nonnegative operators in the spaces $L_p(\Omega)$ and $C(\Omega)$

Let us prove some generalizations of the Gantmacher-Kreĭn theorems in the case of operators in the spaces $L_p(\Omega)$ and $C(\Omega)$, using the Kreĭn-Rutman theorem (see, e.g., [14]) about completely continuous operators leaving invariant an almost-reproducing cone K in a Banach space (for such operators we have the following property of the spectral radius: $\rho(A) \in \sigma_p(A)$).

Theorem 6.1. *Let X be either $L_p(\Omega)$ or $C(\Omega)$, and, respectively, let $\widetilde{X}$ be either $L_p(W)$ or $C_0(W)$. Let a completely continuous operator $A : X \to X$ with $\rho(A) > 0$ leave invariant an almost-reproducing cone K in X and there is only one eigenvalue on the spectral circle $\lambda = \rho(A)$. Let its exterior square $A \wedge A : \widetilde{X} \to \widetilde{X}$ leave invariant an almost-reproducing cone $\widetilde{K}$ in $\widetilde{X}$, besides $\rho(A \wedge A) > 0$, and there is also only one eigenvalue on the spectral circle $\lambda = \rho(\wedge^j A)$. Then the operator A has a positive eigenvalue $\lambda_1 = \rho(A)$, and its second (according to the module) eigenvalue λ_2 is also positive.*

Proof. Enumerate eigenvalues of a completely continuous operator A, repeated according to multiplicity, in order of decrease of their modules:

$$|\lambda_1| < |\lambda_2| < |\lambda_3| \geq \cdots . \tag{6.1}$$

Applying the Kreĭn-Rutman theorem to A, we get $\lambda_1 = \rho(A) > 0$. Now applying the Kreĭn-Rutman theorem to the operator $A \wedge A$ (that obviously is also completely continuous), we get $\rho(A \wedge A) \in \sigma_p(A \wedge A)$.

As it follows from the statement of Theorem 5.2, the exterior square of the operator A has no other nonzero eigenvalues, except all the possible products of the form $\lambda_i\lambda_j$, where $i < j$. So, we get a conclusion that $\rho(A \wedge A) > 0$ can be represented in the form of the product $\lambda_i\lambda_j$ with some values of the indices i, j, $i < j$, and from the fact that eigenvalues are numbered in a decreasing order, it follows that $\rho(A \wedge A) = \lambda_1\lambda_2$. Therefore $\lambda_2 = \rho(A \wedge A)/\lambda_1 > 0$. □

To this end a nonnegative linear operator A is called *indecomposable* (see [2]) if it does not have any invariant components. For a linear operator which is indecomposable and nonnegative with respect to the cone of nonnegative functions in $L_p(\Omega)$ ($C(\Omega)$) and such that $\rho(A) > 0$, the positiveness and the simplicity of the first eigenvalue $\rho(A)$ is proved, for example, in [1, 2, 9, 11, 12]. An indecomposable operator A is called *primitive* if its peripheral spectrum consists of the single point $\rho(A)$ and is called *imprimitive* if its peripheral spectrum contains more than one point. For imprimitive operators that are nonnegative with respect to the cone of nonnegative functions in $L_p(\Omega)$ ($C(\Omega)$), the invariance of the spectrum of the operator A with respect to some rotation is proved in [1, 8]. (In [1] an analogue of the classical Frobenius theorem on the general form of primitive and imprimitive matrices is proved for compact indecomposable integral operators. This statement holds also true for arbitrary, not necessarily integral, compact indecomposable operators.) Call a cone K in $L_p(\Omega)$ ($C(\Omega)$) *assumed* if an indecomposable, nonnegative with-respect-to-the-cone-K, and completely continuous operator A has the properties, proved in [1], that is, $\rho(A)$ is a positive simple eigenvalue of A; and if A has h eigenvalues, equal in modulus to $\rho(A)$, then each of them is simple and they coincide with the hth roots of $\rho(A)^h$. We will call the operator A *2-totally nonnegative* if A and $A \wedge A$ are nonnegative with respect to some assumed cones K and $\widetilde{K}$ in $L_p(\Omega)$ ($C(\Omega)$) and $L_p(W)$ ($C_0(W)$, resp.) and primitive.

Let us prove a generalization of one of the statements of Kreĭn and Gantmakher in the case of 2-totally-nonnegative operators in the spaces $L_p(\Omega)$ and $C(\Omega)$.

Theorem 6.2. *Let X be either $L_p(\Omega)$ or $C(\Omega)$, and, respectively, let $\widetilde{X}$ be either $L_p(W)$ or $C_0(W)$. Let a completely continuous operator $A : X \to X$ with $\rho(A) > 0$ be nonnegative with respect to an assumed cone K in X and indecomposable, and let the operator $A \wedge A : \widetilde{X} \to \widetilde{X}$ with $\rho(A \wedge A) > 0$ be nonnegative with respect to an assumed cone $\widetilde{K}$ in $\widetilde{X}$ and also indecomposable. Let $h(A)$ and $h(A \wedge A)$ be the indices of imprimitivity of A and $A \wedge A$ respectively. Then*

(a) *either $h(A) = 1$ and $h(A \wedge A)$ is arbitrary, or $h(A) = 3$ and $h(A \wedge A) = 3$;*

(b) *if $h(A) = 1$ then the operator A has two possible simple eigenvalues λ_1, λ_2, with*

$$\rho(A) = \lambda_1 > \lambda_2 \geq |\lambda_3| \geq \cdots ; \tag{6.2}$$

(c) *if $h(A) = h(A \wedge A) = 1$, then λ_2 is different according to the module from the other eigenvalues;*

(d) *if $h(A) = 1$ and $h(A \wedge A) > 1$, then the operator A has $h(A \wedge A)$ eigenvalues λ_2, $\lambda_3, \ldots, \lambda_{h(A\wedge A)+1}$, equal in modulus to λ_2, each of them is simple, and they coincide with the $h(A \wedge A)$th roots of $\lambda_2^{h(A\wedge A)}$.*

Proof. (a) First we will prove that if a completely continuous nonnegative operator A is imprimitive with $h(A) = 2$, then its exterior square can not be nonnegative. Really, according to the theorem of indecomposable operators, we have that there are two eigenvalues $\rho(A) > 0$ and $-\rho(A)$ on the spectral circle of the operator A. As it follows, there is only one negative eigenvalue $-\rho^2(A)$ on the spectral circle of $A \wedge A$ and that is impossible if $A \wedge A$ is nonnegative.

Let $h(A) > 3$ and let $A \wedge A$ be nonnegative. Prove that $A \wedge A$ is decomposable (i.e., it has invariant components). Suppose the opposite: let $A \wedge A$ be indecomposable. Then $\rho(A \wedge A) = \rho(A)^2$ and all the other eigenvalues on the spectral circle of $A \wedge A$ are simple. On the other hand, from Theorem 5.2 and imprimitivity of A, it follows that all the eigenvalues of $A \wedge A$, situated on the spectral circle, can be represented as couple products of different $h(A)$th roots of $\rho(A)^{h(A)}$. Let us examine $\lambda_j = \rho(A)e^{2\pi(j-1)i/h(A)}$ ($j = 1,\ldots,h(A)$)—all the eigenvalues of A, situated on the spectral circle. It is obvious that $\lambda_2\lambda_{h(A)} = \lambda_3\lambda_{h(A)-1} = \cdots = \lambda_k\lambda_{h(A)-(k-2)} = \cdots = \rho(A)^2$. As it follows, $\rho(A \wedge A) = \rho(A)^2$ is not a simple eigenvalue of $A \wedge A$.

Prove that if A is imprimitive with its index $h(A) = 3$ and its exterior square is indecomposable, then $A \wedge A$ is also imprimitive with $h(A \wedge A) = 3$. Indeed, in this case there are three eigenvalues $\lambda_1 = \rho(A)$, $\lambda_2 = \rho(A)e^{2\pi i/3}$, $\lambda_3 = \rho(A)e^{4\pi i/3}$ on the spectral circle of the operator A and there are also three eigenvalues $\lambda_1\lambda_2 = \rho(A)^2e^{2\pi i/3}$, $\lambda_1\lambda_3 = \rho(A)^2e^{4\pi i/3}$, and $\lambda_2\lambda_3 = \rho(A)e^{2\pi i/3}\rho(A)e^{4\pi i/3} = \rho(A)^2$ that coincide with the 3rd roots of $(\rho(A)^2)^3$, on the spectral circle of $A \wedge A$.

(b) The existence and the positiveness of the first and the second eigenvalues follow from Theorem 6.1. The simplicity of λ_2 follows from the equality $\lambda_2 = \rho(A \wedge A)/\rho(A)$ and the simplicity of eigenvalues $\rho(A)$ and $\rho(A \wedge A)$.

(c) In the case of $h(A) = h(A \wedge A) = 1$ the distinction according to the module of λ_2 from other eigenvalues is obvious.

(d) In case of $h(A \wedge A) > 1$ from Theorem 5.2 and the properties of the peripheral spectrum of the imprimitive operator $A \wedge A$ it follows that for an eigenvalue $\lambda_j, j = 2,\ldots,h(A \wedge A)+1$ the next equality holds: $\lambda_j = \rho(A \wedge A)e^{2\pi(j-1)i/h(A\wedge A)}/\rho(A)$. $\square$

Consider an example of operator for which the conditions of Theorem 6.2 are satisfied and $h(A) = 3$. Let the operator $A : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \tag{6.3}$$

It is obvious that this operator is nonnegative with respect to the cone of nonnegative vectors in $\mathbb{R}^3$ and imprimitive with $h(A) = 3$. In the basis, which consists of exterior products of the given basic vectors, the matrix of the exterior square of operator $A \wedge A$ coincides with the second associated matrix, that is, it can be represented in the following form:

$$\mathbf{A} \wedge \mathbf{A} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6.4}$$

It is obvious that $A \wedge A$ is imprimitive with $h(A \wedge A) = 3$. It is also obvious that it leaves invariant the cone of vectors $(1,0,0)$, $(0,-1,0)$, and $(0,0,1)$.

Given an operator A in the finite-dimensional space $\mathbb{R}^3$, it is not difficult, using the standard scheme, to define a linear integral operator with the same properties, acting in $L_p(\Omega)$ or $C(\Omega)$.

7. Linear integral operators in the spaces $L_p(\Omega)$ and $C(\Omega)$

Let us examine a linear integral operator A with kernel $k(t,s)$, acting in the space $L_p(\Omega)$. Observing that

$$\begin{aligned} (A \otimes A)x(t_1,t_2) &= A_{(1)}A_{(2)}x(t_1,t_2) \\ &= \int_\Omega k(t_1,s_1)\left(\int_\Omega k(t_2,s_2)x(s_1,s_2)\,d\mu\right)d\mu \\ &= \int_\Omega\int_\Omega k(t_1,s_1)k(t_2,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2), \end{aligned} \tag{7.1}$$

we conclude that the tensor square of the operator A is a linear integral operator with kernel $k(t_1,s_1)k(t_2,s_2)$, acting in the space $L_p(\Omega \times \Omega)$. Thus the exterior square of A is a restriction of the integral operator with kernel $k(t_1,s_1)k(t_2,s_2)$ onto the subspace $L_p^a(\Omega \times \Omega)$.

Let us examine the value of the operator $\mathbf{a} \circ (A \otimes A)$ on an arbitrary function $x(t_1,t_2)$ from $L_p(\Omega \times \Omega)$:

$$\begin{aligned}
\mathbf{a} \circ \Big(\int_{\Omega\times\Omega} & k(t_1,s_1)k(t_2,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2)\Big) \\
&= \frac{1}{2}\int_{\Omega\times\Omega} k(t_1,s_1)k(t_2,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&\quad - \frac{1}{2}\int_{\Omega\times\Omega} k(t_2,s_1)k(t_1,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&= \frac{1}{2}\int_{\Omega\times\Omega} \begin{vmatrix} k(t_1,s_1) & k(t_1,s_2) \\ k(t_2,s_1) & k(t_2,s_2) \end{vmatrix} x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2).
\end{aligned} \tag{7.2}$$

Since the values of the operators $\mathbf{a} \circ (A \otimes A)$ and $A \wedge A$ on antisymmetric functions do coincide, it is obvious that

$$\begin{aligned}
\int_{\Omega\times\Omega} & k(t_1,s_1)k(t_2,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&= \mathbf{a} \circ \int_{\Omega\times\Omega} k(t_1,s_1)k(t_2,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&= \frac{1}{2}\int_{\Omega\times\Omega} \begin{vmatrix} k(t_1,s_1) & k(t_1,s_2) \\ k(t_2,s_1) & k(t_2,s_2) \end{vmatrix} x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2),
\end{aligned} \tag{7.3}$$

where $x(t_1,t_2)$ is an arbitrary function from $L_p^a(\Omega \times \Omega)$.

Further, we will call the kernel $\begin{vmatrix} k(t_1,s_1) & k(t_1,s_2) \\ k(t_2,s_1) & k(t_2,s_2) \end{vmatrix}$ *the second associated kernel* $k(t,s)$ and will denote it by $(k \wedge k)(t_1,t_2,s_1,s_2)$.

Since $\Omega \times \Omega = W \cup \widetilde{W}$, we have that

$$\begin{aligned}
(A \wedge A)&x(t_1,t_2) \\
&= \frac{1}{2}\int_W (k\wedge k)(t_1,t_2,s_1,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&\quad + \frac{1}{2}\int_{\widetilde{W}} (k\wedge k)(t_1,t_2,s_1,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2) \\
&= \cdots .
\end{aligned} \tag{7.4}$$

Exchanging the places of s_1 and s_2 in the second integral, we get

$$\cdots = \int_W (k\wedge k)(t_1,t_2,s_1,s_2)x(s_1,s_2)\,d(\mu\otimes\mu)(s_1,s_2). \tag{7.5}$$

Now we can consider the exterior square of the operator A, acting in the space $L_p(\Omega)$, as a linear integral operator, acting in $L_p(W)$, with the kernel equal to the second associated to $k(t,s)$. The same reasoning is true for the space $C(\Omega)$.

A nonnegative kernel $k(t,s)$ is called *indecomposable* if for any measurable set $D \in \Omega$, $0 < \mu(D) < \mu(\Omega)$, there exist measurable sets $A \in D$, $B \in \Omega \setminus D$, such that $\mu(A) > 0$,

$\mu(B) > 0$ and $k(t,s) > 0$ almost everywhere on $A \times B$. A nonnegative integral operator A with a kernel $k(t,s)$ is indecomposable if and only if the kernel $k(t,s)$ is indecomposable (see [8]). In [1], for an indecomposable nonnegative linear integral operator, the positiveness and the simplicity of the eigenvalue $\rho(A)$ is proved (see [1, page 6, Theorem 2]).

An indecomposable kernel $k(t,s)$ is called *imprimitive*, if there exists a decomposition of the set Ω in $n > 1$ nonintersecting sets Ω_j $(j = 1,\dots,n)$ of a positive measure: $\Omega = \cup_{j=1}^{n}\Omega_j$, for which $k(t,s) = 0$ with $t \in \Omega_j$, $s \notin \Omega_{j+1}$ for any $j = 1,\dots,n$ (in the case of $j = n$ we presuppose $j+1$ to be equal to 1). Otherwise the kernel $k(t,s)$ is called *primitive*. A compact integral operator A is imprimitive (resp., primitive) if and only if its kernel is imprimitive (primitive). A kernel $k(t,s)$ is called 2-*totally nonnegative*, if both $k(t,s)$ and $(k \wedge k)(t_1,t_2,s_1,s_2)$ are primitive. Further, we will examine on the set W the second associated kernel and the conditions implied to it. The index of imprimitivity of $(k \wedge k)(t_1,t_2,s_1,s_2)$ will be denoted by $h(k \wedge k)$.

Let the kernel $k(t,s)$ of a completely continuous integral operator A in the space $L_p(\Omega)$ or $C(\Omega)$ be nonnegative and primitive. Let its second associated kernel be also nonnegative and primitive. Then Theorem 6.2 implies that *the second, according to the module, eigenvalue of the operator A λ_2 is positive, simple, and different in modulus from the other eigenvalues*.

Note that in this reasoning the kernel is not presupposed to be continuous, but only assume that the operator A acts in the space $L_p(\Omega)$ or $C(\Omega)$.

8. Concluding remarks

All the results given in the present paper can be easily spread on k-totally-nonnegative operators $(k > 2)$, acting in the space $L_p(\Omega)$ or $C(\Omega)$. Similar results will be also true for other spaces, for example, for some ideal spaces, the Lorenz-Martzinkevich space, and so forth.

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AN ORIENTED COINCIDENCE INDEX FOR NONLINEAR FREDHOLM INCLUSIONS WITH NONCONVEX-VALUED PERTURBATIONS

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We suggest the construction of an oriented coincidence index for nonlinear Fredholm operators of zero index and approximable multivalued maps of compact and condensing type. We describe the main properties of this characteristic, including applications to coincidence points. An example arising in the study of a mixed system, consisting of a first-order implicit differential equation and a differential inclusion, is given.

1. Introduction

The necessity of studying coincidence points of Fredholm operators and nonlinear (compact and condensing) maps of various classes arises in the investigation of many problems in the theory of partial differential equations and optimal control theory (see, e.g., [3, 4, 7, 13, 17, 18, 20–22]). The use of topological characteristics of coincidence degree type is a very effective tool for solving such type of problems. For inclusions with linear Fredholm operators, a number of such topological invariants was studied in the works [7, 8, 13, 18, 19]. In the present paper, we suggest the general construction of an oriented coincidence index for nonlinear Fredholm operators of zero index and approximable multivalued maps of compact and condensing type. A nonoriented analogue of such index was described earlier in the authors work [17].

The paper is organized in the following way. In Section 2, we give some preliminaries. In Section 3, we present the construction of the oriented coincidence index, first for a finite-dimensional case, and later, on that base, we develop the construction in the case of a compact triplet. In Section 4, using the technique of fundamental sets, we give the most general construction of the oriented index for a condensing triplet and describe its main properties, including its application to the existence of coincidence points. In Section 5, we consider an example of a condensing triplet arising in the study of a mixed system, consisting of a first-order implicit differential equation and a differential inclusion.

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2. Preliminaries

By the symbols E, E', we will denote real Banach spaces. Everywhere, by Y we will denote an open bounded set $U \subset E$ (case (i)) or $U_* \subset E \times [0,1]$ (case (ii)). We recall some notions (see, e.g., [3]).

Definition 2.1. A C^1-map $f : Y \to E'$ is Fredholm of index $k \geq 0$ ($f \in \Phi_k C^1(Y)$) if for every $y \in Y$ the Frechet derivative $f'(y)$ is a linear Fredholm map of index k, that is, $\dim \operatorname{Ker} f'(y) < \infty$, $\dim Co\ker f'(y) < \infty$, and

$$\dim \operatorname{Ker} f'(y) - \dim Co\ker f'(y) = k. \tag{2.1}$$

Definition 2.2. A map $f : \overline{Y} \to E'$ is proper if $f^{-1}(\mathcal{K})$ is compact for every compact set $\mathcal{K} \subset E'$.

We recall now the notion of oriented Fredholm structure on Y.

An atlas $\{(Y_i,\Psi_i)\}$ on Y is said to be *Fredholm* if, for each pair of intersecting charts (Y_i,Ψ_i) and (Y_j,Ψ_j) and every $y \in Y_i \cap Y_j$, it is

$$(\Psi_j \circ \Psi_i^{-1})'(\Psi_i(y)) \in CG(\widetilde{E}), \tag{2.2}$$

where $\widetilde{E}$ is the corresponding model space, and $CG(\widetilde{E})$ denotes the collection of all linear invertible operators in $\widetilde{E}$ of the form $i+k$, where i is the identity map and k is a compact linear operator.

The set $CG(\widetilde{E})$ is divided into two connected components. The component containing the identity map will be denoted by $CG^+(\widetilde{E})$.

Two Fredholm atlases are said to be equivalent if their union is still a Fredholm atlas. The class of equivalent atlases is called a *Fredholm structure.*

A Fredholm structure on U is associated to a Φ_0C^1-map $f : U \to E'$ if it admits an atlas $\{(Y_i,\Psi_i)\}$ with model space E' for which

$$(f \circ \Psi_i^{-1})'(\Psi_i(y)) \in LC(E') \tag{2.3}$$

at each point $y \in U$, where $LC(E')$ denotes the collection of all linear operators in E' of the form: identity plus a compact map. Let us note that each Φ_0C^1-map $f : U \to E'$ generates a Fredholm structure on U associated to f.

A Fredholm atlas $\{(Y_i,\Psi_i)\}$ on Y is said to be oriented if for each intersecting charts (Y_i,Ψ_i) and (Y_j,Ψ_j) and every $y \in Y_i \cap Y_j$, it is true that

$$(\Psi_j \circ \Psi_i^{-1})'(\Psi_i(y)) \in CG^+(E). \tag{2.4}$$

Two oriented Fredholm atlases are called orientally equivalent if their union is an oriented Fredholm atlas on Y. The equivalence class with respect to this relation is said to be the *oriented Fredholm structure on Y*.

We will need also the following result (see [3]).

Proposition 2.3. *Let $f \in \Phi_k C^1(Y)$; $K \subset Y$ a compact set. Then there exist an open neighborhood $\mathbb{O}$, $K \subset \mathbb{O} \subset Y$, and a finite-dimensional subspace $E'_n \subset E'$ such that*

$$f^{-1}(E'_n) \cap \mathbb{O} = M^{n+k}, \tag{2.5}$$

where M^{n+k} is an $n+k$ dimensional manifold. Moreover, the restriction $f_{|\mathbb{O}}$ is transversal to E'_n, that is, $f'(x)E + E'_n = E'$ for each $x \in \mathbb{O}$.

We describe now some notions of the theory of multivalued maps that will be used in the sequel (details can be found, e.g., in [1, 2, 9, 12]).

Let $(X, d_X), (Z, d_Z)$ be metric spaces.

Given a subset A and $\varepsilon > 0$, we denote by $O_\varepsilon(A)$ the ε-neighborhood of A. Let $K(Z)$ denote the collection of all nonempty compact subsets of Z. Given a multivalued map (multimap) $\Sigma : X \multimap K(Z)$, a continuous map, $\sigma_\varepsilon : X \to Z$, $\varepsilon > 0$, is said to be an *ε-approximation* of Σ if for every $x \in X$, there exists $x' \in O_\varepsilon(x)$ such that $\sigma_\varepsilon(x) \in O_\varepsilon(\Sigma(x'))$.

It is clear that the notion can be equivalently expressed saying that

$$\sigma_\varepsilon(x) \in O_\varepsilon(\Sigma(O_\varepsilon(x))) \tag{2.6}$$

for all $x \in X$, or that

$$\Gamma_{\sigma_\varepsilon} \subset (\Gamma_\Sigma), \tag{2.7}$$

where $\Gamma_{\sigma_\varepsilon}, \Gamma_\Sigma$ denote the graphs of σ_ε and Σ, respectively, while the metric in $X \times Z$ is defined in a natural way as

$$d((x,z),(x',z')) = \max\{d_X(x,x'), d_Z(z,z')\}. \tag{2.8}$$

The fact that σ_ε is an ε-approximation of the multimap Σ will be denoted by $\sigma_\varepsilon \in a(\Sigma,\varepsilon)$.

A multimap $\Sigma : X \multimap K(Z)$ is said to be *upper semicontinuous (u.s.c.)* if for every open set $V \subset Z$, the set $\Sigma_+^{-1}(V) = \{x \in X : \Sigma(x) \subset V\}$ is open in X.

An u.s.c. multimap $\Sigma : X \multimap K(Z)$ is *closed* if its graph Γ_Σ is a closed subset of $X \times Z$.

We can summarize some properties of ε-approximations in the following statement (see [9]).

Proposition 2.4. *Let $\Sigma : X \multimap K(Z)$ be an u.s.c. multimap.*

(i) *Let X_1 be a compact subset of X. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma,\delta)$ implies $\sigma_{|X_1} \in a(\Sigma_{|X_1},\varepsilon)$.*

(ii) *Let X be compact, Z_1 a metric space, and $\varphi : Z \to Z_1$ a continuous map. Then, for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma,\delta)$ implies $\varphi \circ \sigma \in a(\varphi \circ \Sigma,\varepsilon)$.*

(iii) *Let X be compact, $\Sigma_* : X \times [0,1] \to K(Z)$ an u.s.c. multimap. Then, for every $\lambda \in [0,1]$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma_* \in a(\Sigma_*,\delta)$ implies that $\sigma_*(\cdot,\lambda) \in a(\Sigma_*(\cdot,\lambda),\varepsilon)$.*

(iv) *Let Z_1 be a metric space, $\Sigma_1 : X \multimap K(Z_1)$ an u.s.c. multimap. Then for every $\varepsilon > 0$, there exists $\delta > 0$ such that $\sigma \in a(\Sigma,\delta)$ and $\sigma_1 \in a(\Sigma_1,\delta)$ imply that $\sigma \times \sigma_1 \in a(\Sigma \times \Sigma_1,\varepsilon)$, where $(\sigma \times \sigma_1)(x) = \sigma(x) \times \sigma_1(x)$, $(\Sigma \times \Sigma_1)(x) = \Sigma(x) \times \Sigma_1(x)$.*

In the sequel, we will use the following important property of ε-approximations.

Proposition 2.5. *Let X, X', Z be metric spaces; $f : X \to X'$ a continuous map; $\Sigma : X \multimap K(Z)$ an u.s.c. multimap; $\varphi : Z \to X'$ a continuous map. Suppose that $X_1 \subseteq X$ is a compact subset such that*

$$\operatorname{Coin}(f, \varphi \circ \Sigma) \cap X_1 = \varnothing, \tag{2.9}$$

where $\operatorname{Coin}(f, \varphi \circ \Sigma) = \{x \in X : f(x) \in \varphi \circ \Sigma(x)\}$ *is the coincidence points set. If $\varepsilon > 0$ is sufficiently small and $\sigma_\varepsilon \in a(\Sigma, \varepsilon)$, then*

$$\operatorname{Coin}(f, \varphi \circ \sigma_\varepsilon) \cap X_1 = \varnothing. \tag{2.10}$$

Proof. Suppose, to the contrary, that there are sequences $\{x_n\} \subset X_1$ and $\varepsilon_n \to 0$, $\varepsilon_n > 0$, such that

$$f(x_n) = \varphi\sigma_{\varepsilon_n}(x_n) \tag{2.11}$$

for a sequence $\sigma_{\varepsilon_n} \in a(\Sigma, \varepsilon_n)$.

From Proposition 2.4(i) and (ii) we can deduce that, without loss of generality, the maps $\varphi\sigma_{\varepsilon_n|X_1}$ form a sequence of δ_n-approximations of $\varphi\Sigma_{|X_1}$, with $\delta_n \to 0$ and hence

$$(x_n, \varphi\sigma_{\varepsilon_n}(x_n)) \in O_{\delta_n}(\Gamma_{\varphi\Sigma_{|X_1}}). \tag{2.12}$$

The graph of the u.s.c. multimap $\varphi\Sigma_{|X_1}$ is a compact set (see, e.g., [12, Theorem 1.1.7]), hence we can assume, without loss of generality, that

$$(x_n, \varphi\sigma_{\varepsilon_n}(x_n)) \xrightarrow[n\to\infty]{} (x_0, y_0) \in \Gamma_{\varphi\Sigma_{|X_1}}, \tag{2.13}$$

that is, $y_0 \in \varphi\Sigma(x_0)$. Passing to the limit in (2.11), we obtain that $f(x_0) = y_0 \in \varphi\Sigma(x_0)$, that is, $x_0 \in \operatorname{Coin}(f, \varphi\Sigma)$, giving the contradiction. □

To present the class of multimaps which will be considered, we recall some notions.

Definition 2.6 (see, e.g., [1, 9, 10, 15]). A nonempty compact subset A of a metric space Z is said to be *aspheric* (or UV^∞, or ∞-*proximally connected*) if for every $\varepsilon > 0$, there exists δ, $0 < \delta < \varepsilon$, such that for each $n = 0, 1, 2, \ldots$, every continuous map $g : S^n \to O_\delta(A)$ can be extended to a continuous map $\widetilde{g} : B^{n+1} \to O_\varepsilon(A)$, where $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ and $B^{n+1} = \{x \in \mathbb{R}^{n+1} : \|x\| \le 1\}$.

Definition 2.7 (see [11]). A nonempty compact space A is said to be an R_δ*-set* if it can be represented as the intersection of a decreasing sequence of compact, contractible spaces.

Definition 2.8 (see [9]). An u.s.c. multimap $\Sigma : X \to K(Z)$ is said to be a J*-multimap* ($\Sigma \in J(X, Z)$) if every value $\Sigma(x)$, $x \in X$, is an aspheric set.

We will use the notions of absolute retract (AR-space) and of absolute neighborhood retract (ANR-space) (see, e.g., [5, 9]).

Proposition 2.9 (see [9]). *Let Z be an ANR-space. In each of the following cases, an u.s.c. multimap $\Sigma : X \to K(Z)$ is a J-multimap: for each $x \in X$, the value $\Sigma(x)$ is*
(a) *a convex set;*
(b) *a contractible set;*
(c) *an R_δ-set;*
(d) *an AR-space.*
In particular, every continuous map $\sigma : X \to Z$ is a J-multimap.

The next statement describes the approximation properties of J-multimaps.

Proposition 2.10 (see [9, 10, 15]). *Let X be a compact ANR-space; Z a metric space; $\Sigma \in J(X,Z)$. Then*
(i) *the multimap Σ is approximable; that is, for every $\varepsilon > 0$, there exists $\sigma_\varepsilon \in a(\Sigma,\varepsilon)$;*
(ii) *for each $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for every δ $(0 < \delta < \delta_0)$ and for every two δ-approximations $\sigma_\delta, \sigma'_\delta \in a(\Sigma,\delta)$, there exists a continuous map (homotopy) $\sigma_* : X \times [0,1] \to Z$ such that*
 (a) *$\sigma_*(\cdot,0) = \sigma_\delta$, $\sigma_*(\cdot,1) = \sigma'_\delta$;*
 (b) *$\sigma_*(\cdot,\lambda) \in a(\Sigma,\varepsilon)$ for all $\lambda \in [0,1]$.*

Definition 2.11. Denote by $CJ(X,X')$ the collection of all multimaps $G : X \to K(X')$ of the form $G = \varphi \circ \Sigma$, where $\Sigma \in J(X,Z)$ for some metric space Z, $\varphi : Z \to X'$ is a continuous map. The composition $\varphi \circ \Sigma$ will be called the representation (or decomposition, cf. [9]) of G. Denote $G = (\varphi \circ \Sigma) \in CJ(X,X')$.

It has to be noted that a multimap can admit different representations (see [9]).

3. Oriented coincidence index for compact triplets

We will start from the following notion.

Definition 3.1. The map $f : \overline{Y} \to E'$, the multimap $G = (\varphi \circ \Sigma) \in CJ(X,X')$, and the space $\overline{Y}$ form a *compact triplet* $(f,G,\overline{Y})_C$ if the following conditions are satisfied:
(h1) f is a continuous proper map, $f_{|Y} \in \Phi_k C^1(Y)$ with $k = 0$ in case (i), $k = 1$ in case (ii), and the Fredholm structure on Y generated by f is oriented;
(h2) G is compact, that is, $G(\overline{Y})$ is a relatively compact subset of E';
(h3) $\mathrm{Coin}(f,G) \cap \partial Y = \emptyset$.

Let us mention that from hypotheses (h1), (h2), it follows that the coincidence points set $Q = \mathrm{Coin}(f,G)$ is compact.

3.1. The case of a finite-dimensional triplet.

Given a triplet $(f,G,\overline{Y})_C$, from Proposition 2.3 we know that there exist an open neighborhood $\mathcal{O} \subset Y$ of the set $Q = \mathrm{Coin}(f,G)$ and an n-dimensional subspace $E'_n \subset E'$ such that $f^{-1}(E'_n) \cap \mathcal{O} = M$, a manifold which is n-dimensional in case (i) and $(n+1)$-dimensional in case (ii).

Now, suppose that the multimap $G = \varphi \circ \Sigma$ is finite-dimensional, that is, that there exists a finite-dimensional subspace $E'_m \subset E'$ such that $G(\overline{Y}) \subset E'_m$. We can assume, without loss of generality, that $E'_m \subset E'_n$. Then clearly $Q \subset M$. Let us mention also that the orientation on Y induces the orientation on M.

A compact triplet $(f,G,\overline{Y})_C$ such that G is finite-dimensional will be denoted by $(f,G,\overline{Y})_{C_m}$ and will be called *finite-dimensional*.

LEMMA 3.2. *For $(f,G=(\varphi\circ\Sigma),\overline{Y})_{C_m}$, let $O_\varkappa$ be a $\varkappa$-neighborhood of Q. Then, $\Sigma_{|\overline{O}_\varkappa}$ is approximable provided that $\varkappa>0$ is sufficiently small.*

Proof. Consider an open bounded set N satisfying the following conditions:

(i) $Q\subset N\subset\overline{N}\subset M$;

(ii) $\overline{N}$ is a compact ANR-space.

Let us note that as N we can take a union of a finite collection of balls with centers in Q.

Let us take $\varkappa>0$ so that $O_\varkappa\subset N$. Then the statement follows from Propositions 2.10(i) and 2.4(i). □

Now, let the neighborhood $O_\varkappa$ be chosen so that Σ is approximable on $\overline{O}_\varkappa$. From Proposition 2.5, we know that

$$\operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon)\cap\partial O_\varkappa=\varnothing \tag{3.1}$$

provided that $\sigma_\varepsilon\in a(\Sigma_{|\overline{O}_\varkappa},\varepsilon)$ and $\varepsilon>0$ is sufficiently small.

So, we can consider the following map of pairs of spaces:

$$f-\varphi\circ\sigma_\varepsilon:(\overline{O}_\varkappa,\partial O_\varkappa)\longrightarrow(E'_n,E'_n\backslash 0). \tag{3.2}$$

Now we are in position to give the following notion.

Definition 3.3. The oriented coincidence index of a finite-dimensional triplet $(f,G=(\varphi\circ\Sigma),\overline{U})_{C_m}$ is defined by the equality

$$(f,G=(\varphi\circ\Sigma),\overline{U})_{C_m}:=\deg(f-\varphi\circ\sigma_\varepsilon,\overline{O}_\varkappa), \tag{3.3}$$

where $\varkappa>0$ and $\varepsilon>0$ are taken small enough and the right-hand part of equality (3.3) denotes the Brouwer topological degree.

Now we will demonstrate that the given definition is consistent, that is, the coincidence index does not depend on the choice of an ε-approximation σ_ε and the neighborhood $O_\varkappa$.

LEMMA 3.4. *Let σ_ε and $\sigma'_\varepsilon\in a(\Sigma_{|\overline{O}_{\varkappa}.},\varepsilon)$ be two approximations. Then*

$$\deg(f-\varphi\circ\sigma_\varepsilon,\overline{O}_\varkappa)=\deg(f-\varphi\circ\sigma'_\varepsilon,\overline{O}_\varkappa) \tag{3.4}$$

provided that $\varepsilon>0$ is sufficiently small.

Proof. Let us take any neighborhood N' of Q such that $Q\subset N'\subset\overline{N'}\subset O_\varkappa$ and $\overline{N'}$ is an ANR-space. Then, by Propositions 2.4(i) and 2.5, we know that we can take $\varepsilon>0$ small enough to provide that $\sigma_{\varepsilon|\overline{N'}}$ and $\sigma'_{\varepsilon|\overline{N'}}$ are δ_0-approximations of $\Sigma_{|\overline{N'}}$ and

$$\begin{aligned}\operatorname{Coin}(f,\varphi\circ\sigma_\varepsilon)\cap(\overline{O}_\varkappa\backslash N')&=\varnothing,\\ \operatorname{Coin}(f,\varphi\circ\sigma'_\varepsilon)\cap(\overline{O}_\varkappa\backslash N')&=\varnothing.\end{aligned} \tag{3.5}$$

Since $\Sigma_{|\overline{N'}}$ is approximable, we can assume that $\varepsilon > 0$ is chosen so small that there exists a map $\gamma : \overline{N'} \times [0,1] \to Z$ with the following properties:

(i) $\gamma(\cdot,0) = \sigma_{\varepsilon|\overline{N'}}, \gamma(\cdot,1) = \sigma'_{\varepsilon|\overline{N'}}$;

(ii) $\gamma(\lambda,\cdot) \in a(\Sigma_{|\overline{N'}},\delta_1)$ for each $\lambda \in [0,1]$, where δ_1 is arbitrary small;

(iii) $\operatorname{Coin}(f,\varphi \circ \gamma(\cdot,\lambda)) \cap \partial N' = \varnothing$ for all $\lambda \in [0,1]$ (see Propositions 2.10(ii) and 2.5).

Each map $f - \varphi \circ \gamma(\cdot,\lambda)$, $\lambda \in [0,1]$, transforms the pair $(\overline{N'},\partial N')$ into the pair $(E_n, E_n\backslash 0)$ for each $\lambda \in [0,1]$, and by the homotopy property of the Brouwer degree we have $\deg(f - \varphi \circ \sigma_\varepsilon,\overline{N'}) = \deg(f - \varphi \circ \sigma'_\varepsilon,\overline{N'})$. Further from (3.5) and the additive property of the Brouwer degree, we have

$$\begin{aligned} \deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{O}_\varkappa\right) &= \deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{N'}\right), \\ \deg\left(f - \varphi \circ \sigma'_\varepsilon,\overline{O}_\varkappa\right) &= \deg\left(f - \varphi \circ \sigma'_\varepsilon,\overline{N'}\right) \end{aligned} \tag{3.6}$$

proving equality (3.4). □

Now, if $O_{\varkappa'} \subset O_\varkappa$, the equality

$$\deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{O}_{\varkappa'}\right) = \deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{O}_\varkappa\right), \tag{3.7}$$

where $\varepsilon > 0$ is sufficiently small, follows easily from Propositions 2.4(i), 2.5, and the additive property of the Brouwer degree.

At last, let us mention also the independence of the construction on the choice of the transversal subspace E'_n. In fact, if we take two subspaces E'_{n_0} and E'_{n_1}, we may assume, without loss of generality, that $E'_{n_0} \subset E'_{n_1}$. As earlier, we assume that $G(\overline{U}) \subset E'_m \subset E'_{n_0} \subset E'_{n_1}$. Then, from the construction, we obtain two manifolds M^{n_0}, M^{n_1}, $M^{n_0} \subset M^{n_1}$ and two neighborhoods $O^{n_0}_\varkappa \subset M^{n_0}$, $O^{n_1}_\varkappa \subset M^{n_1}$, $O^{n_0}_\varkappa \subset O^{n_1}_\varkappa$ for $\varkappa > 0$ sufficiently small. Now, take $\varepsilon > 0$ small enough to provide that the degrees $\deg(f - \varphi \circ \sigma_\varepsilon,\overline{O^{n_1}_\varkappa})$ and $\deg(f - \varphi \circ \sigma_\varepsilon,\overline{O^{n_0}_\varkappa})$ are well defined. Then the equality

$$\deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{O^{n_1}_\varkappa}\right) = \deg\left(f - \varphi \circ \sigma_\varepsilon,\overline{O^{n_0}_\varkappa}\right) \tag{3.8}$$

follows from the map restriction property of Brouwer degree.

Now, let us mention the main properties of the defined characteristic. Directly from Definition 3.3 and Proposition 2.5, we deduce the following statement.

Theorem 3.5 (the coincidence point property). *If* $\operatorname{Ind}(f,G,\overline{U})_{C_m} \neq 0$, *then* $\varnothing \neq \operatorname{Coin}(f, G) \subset U$.

To formulate the topological invariance property of the coincidence index, we will give the following definition.

Definition 3.6. Two finite-dimensional triplets $(f_0,G_0 = (\varphi_0 \circ \Sigma_0),\overline{U}_0)_{C_m}$ and $(f_1,G_1 = (\varphi_1 \circ \Sigma_1),\overline{U}_1)_{C_m}$ are said to be homotopic,

$$\left(f_0,G_0 = (\varphi_0 \circ \Sigma_0),\overline{U}_0\right)_{C_m} \sim \left(f_1,G_1 = (\varphi_1 \circ \Sigma_1),\overline{U}_1\right)_{C_m}, \tag{3.9}$$

if there exists a finite-dimensional triplet $(f_*, G_*, \overline{U_*})_{C_m}$, where $U_* \subset E \times [0,1]$ is an open set, such that

(a) $U_i = U_* \cap (E \times \{i\})$, $i = 0,1$;

(b) $f_{*|\overline{U_i}} = f_i$, $i = 0,1$;

(c) G_* has the form

$$G_*(x,\lambda) = \varphi_*(\Sigma_*(x,\lambda),\lambda), \tag{3.10}$$

where $\Sigma_* \in J(\overline{U_*}, Z)$, $\varphi_* : Z \times [0,1] \to E'$, is a continuous map, and

$$\Sigma_{*|\overline{U_i}} = \Sigma_i, \quad \varphi_{*|Z\times\{i\}} = \varphi_i, \quad i = 0,1. \tag{3.11}$$

Theorem 3.7 (the homotopy invariance property). *If*

$$(f_0, G_0, \overline{U}_0)_{C_m} \sim (f_1, G_1, \overline{U}_1)_{C_m}, \tag{3.12}$$

then

$$\left| \operatorname{Ind}(f_0, G_0, \overline{U}_0)_{C_m} \right| = \left| \operatorname{Ind}(f_1, G_1, \overline{U}_1)_{C_m} \right|. \tag{3.13}$$

Proof. Let $(f_*, G_*, \overline{U_*})_{C_m}$ be a finite-dimensional triplet connecting the triplets $(f_0, G_0, \overline{U}_0)_{C_m}$ and $(f_1, G_1, \overline{U}_1)_{C_m}$. Let $O_{*\varkappa} \subset U_*$ be a $\varkappa$-neighborhood of $Q_* = \operatorname{Coin}(f_*, G_*)$, where $\varkappa > 0$ is sufficiently small.

Take $\sigma_{*\varepsilon} \in a(\Sigma_{*|\overline{O_{*\varkappa}}}, \varepsilon)$ for $\varepsilon > 0$ sufficiently small. Applying Propositions 2.4 and 2.5, we can verify that the map $\varphi_* \circ \sigma_{*\varepsilon} : \overline{O_{*\varkappa}} \to E'$, $\varphi_* \circ \sigma_{*\varepsilon}(x,\lambda) = \varphi_*(\sigma_{*\varepsilon}(x,\lambda),\lambda)$ is a δ'-approximation of $G_{*|\overline{O_{*\varkappa}}}$ for $\delta' > 0$ arbitrary small and, moreover,

$$\operatorname{Coin}(f_*, \varphi_* \circ \sigma_{*\varepsilon}) \cap \partial O_{*\varkappa} = \varnothing \tag{3.14}$$

and $\varphi_* \circ \sigma_{*\varepsilon|\overline{O_{\varkappa i}}}$, for $\overline{O_{\varkappa i}} = \overline{O_{*\varkappa}} \cap U_i$, $i = 0,1$, are δ''-approximations of $G_{i|\overline{O_{\varkappa i}}}$, $i = 0,1$, where $\delta'' > 0$ is arbitrary small.

Denoting $\sigma_{*\varepsilon|\overline{O_{\varkappa i}}} = \sigma_i$, $i = 0,1$, we have

$$\left| \deg(f_0 - \varphi_0 \circ \sigma_0, \overline{O_{\varkappa 0}}) \right| = \left| \deg(f_1 - \varphi_1 \circ \sigma_1, \overline{O_{\varkappa 1}}) \right| \tag{3.15}$$

(see [22]), proving the theorem. □

Remark 3.8. If the Fredholm map f is constant under the homotopy, that is, U_* has the form $U_* = U \times [0,1]$, where $U \subset E$ is an open set and $f_*(x,\lambda) = f(x)$ for all $\lambda \in [0,1]$, where $f \in \Phi_0 C^1(U)$, then

$$\deg(f - \varphi_0 \circ \sigma_0, \overline{U}) = \deg(f - \varphi_1 \circ \sigma_1, \overline{U}) \tag{3.16}$$

(see [21, 22]). Hence

$$\operatorname{Ind}(f, G_0, \overline{U})_{C_m} = \operatorname{Ind}(f, G_1, \overline{U})_{C_m}. \tag{3.17}$$

From Definition 3.3 and the additive property of the Brouwer degree, we obtain the following property of the oriented coincidence index.

Theorem 3.9 (additive dependence on the domain property). *Let U_0 and U_1 be disjoint open subsets of an open bounded set $U \subset E$ and let $(f,G,\overline{U})_{C_m}$ be a finite-dimensional triplet such that*

$$\operatorname{Coin}(f,G) \cap \left(\overline{U} \setminus (U_0 \cup U_1)\right) = \emptyset. \tag{3.18}$$

Then

$$\operatorname{Ind}(f,G,\overline{U})_{C_m} = \left(f,G,\overline{U_0}\right)_{C_m} + \left(f,G,\overline{U_1}\right)_{C_m}. \tag{3.19}$$

3.2. The case of a compact triplet. Now, we want to define the oriented coincidence degree for the general case of a compact triplet $(f,G=(\varphi \circ \Sigma),\overline{U})_C$.

From the properness property of f and the compactness of G, one can easily deduce the following statement.

Proposition 3.10. *Let $(f,G,\overline{U})_C$ be a compact triplet; $\Lambda : \overline{Y} \to K(E')$ a multimap defined as*

$$\Lambda(y) = f(y) - G(y). \tag{3.20}$$

Then, for every closed subset $Y_1 \subset \overline{Y}$, the set $\Lambda(Y_1)$ is closed.

From the above assertion, it follows that, given a compact triplet $(f,G,\overline{U})_C$, there exists $\delta > 0$ such that

$$B_\delta(0) \cap \Lambda(\partial U) = \emptyset, \tag{3.21}$$

where $B_\delta(0) \subset E'$ is a δ-neighborhood of the origin.

Let us take a continuous map $i_\delta : \overline{G(\overline{U})} \to E_m$, where $E_m \subset E$ is a finite-dimensional subspace, with the property that

$$\|i_\delta(v) - v\| < \delta \tag{3.22}$$

for each $v \in \overline{G(\overline{U})}$. As i_δ, we can choose the Schauder projection (see, e.g., [14]).

Now, if G has the representation $G = \varphi \circ \Sigma$, consider the finite-dimensional multimap $G_m = i_\delta \circ \varphi \circ \Sigma$. From (3.21) and (3.22), it follows that f, G_m and $\overline{U}$ form a finite-dimensional triplet $(f,G_m,\overline{U})_{C_m}$.

We can now define the oriented coincidence index for a compact triplet in the following way.

Definition 3.11. The oriented coincidence index for a compact triplet $(f,G=(\varphi \circ \Sigma),\overline{U})_C$ is defined by the equality

$$\operatorname{Ind}(f,G,\overline{U})_C := \operatorname{Ind}\left(f,G_m,\overline{U}\right)_{C_m}, \tag{3.23}$$

where $G_m = i_\delta \circ \varphi \circ \Sigma$ and the map i_δ satisfies condition (3.22).

To prove the consistency of the given definition, it is sufficient to mention that, given two different maps $i_\delta^0, i_\delta^1 : \overline{G(\overline{U})} \to E'_m$ satisfying property (3.22), we have the homotopy of the corresponding finite-dimensional triplets

$$(f, G_m^0, \overline{U})_{C_m} \sim (f, G_m^1, \overline{U})_{C_m}, \tag{3.24}$$

where $G_m^k = i_\delta^k \circ \varphi \circ \Sigma$, $i = 0, 1$. (It is clear that the finite-dimensional space E'_m can be taken the same for both maps i_δ^0, i_δ^1.)

In fact, the homotopy is realized by the multimap $G_* : \overline{U} \times [0,1] \to K(E'_m)$, defined as

$$G_*(x,\lambda) = \varphi_*(\Sigma(x,\lambda)), \qquad \text{where } \varphi_*(z,\lambda) = (1-\lambda)i_\delta^0\varphi(z) + \lambda i_\delta^1\varphi(z). \tag{3.25}$$

So, from Remark 3.8, it follows that

$$\operatorname{Ind}(f, G_m^0, \overline{U})_{C_m} = \operatorname{Ind}(f, G_m^1, \overline{U})_{C_m}. \tag{3.26}$$

Applying Proposition 3.10 and Theorem 3.5, we can deduce the following coincidence point property.

Theorem 3.12. *If* $\operatorname{Ind}(f, G, \overline{U})_C \neq 0$, *then* $\varnothing \neq \operatorname{Coin}(f, G) \subset U$.

The definition of homotopy for compact triplets $(f, G_0, \overline{U})_C \sim (f, G_1, \overline{U})_C$ has the same form as in Definition 3.6 with the only difference that the connected triplet $(f_*, G_*, \overline{U_*})$ is assumed to be compact.

Taking a finite-dimensional approximation of $G_* = \varphi_* \circ \Sigma_*$ as $G_{*m} = i_\delta \circ \varphi_* \circ \Sigma_*$ and applying Theorem 3.7 and Definition 3.11, we obtain the following homotopy invariance property.

Theorem 3.13. *If* $(f, G_0, \overline{U})_C \sim (f, G_1, \overline{U})_C$, *then*

$$|\operatorname{Ind}(f_0, G_0, \overline{U}_0)_C| = |\operatorname{Ind}(f_1, G_1, \overline{U}_1)_C|. \tag{3.27}$$

Again, if f and U are constant, we have the equality

$$\operatorname{Ind}(f, G_0, \overline{U})_C = \operatorname{Ind}(f, G_1, \overline{U})_C. \tag{3.28}$$

An analog of the additive dependence on the domain property (see Theorem 3.9) for compact triplets also holds.

4. Oriented coincidence index for condensing triplets

In this section, we extend the notion of the oriented coincidence index to the case of condensing triplets. At first we recall some notions (see, e.g., [12]). Denote by $P(E')$ the collection of all nonempty subsets of a Banach space E'. Let $(\mathcal{A}, \geq)$ be a partially ordered set.

Definition 4.1. A map $\beta : P(E') \to \mathcal{A}$ is called a *measure of noncompactness* (MNC) in E' if

$$\beta(\overline{co}D) = \beta(D) \quad \text{for every } D \in P(E'). \tag{4.1}$$

An MNC β is called

(i) *monotone* if $D_0, D_1 \in P(E')$, $D_0 \subseteq D_1$, implies $\beta(D_0) \leq \beta(D_1)$;

(ii) *nonsingular* if $\beta(\{a\} \cup D) = \beta(D)$ for every $a \in E'$, $D \in P(E')$;

(iii) *real* if $A = \overline{\mathbb{R}_+} = [0, +\infty]$ with the natural ordering, and $\beta(D) < +\infty$ for every bounded set $D \in P(E')$.

Among the known examples of MNC satisfying all the above properties we can consider the *Hausdorff MNC*

$$\chi(D) = \inf\{\varepsilon > 0 : D \text{ has a finite } \varepsilon\text{-net}\}, \tag{4.2}$$

and the *Kuratowski MNC*

$$\alpha(D) = \inf\{d > 0 : D \text{ has a finite partition with sets of diameter less than } d\}. \tag{4.3}$$

Let again $Y = U \subset E$, or $U_* \subset E \times [0,1]$, open bounded sets, $f : \overline{Y} \to E'$ a map; $G : \overline{Y} \to K(E')$ a multimap, β an MNC in E'.

Definition 4.2. Maps f, G and the space $\overline{Y}$ form a *β-condensing triplet* $(f, G, \overline{Y})_\beta$ if they satisfy conditions (h1) and (h3) in Definition 3.1, and (h2$_\beta$) a multimap $G = \varphi \circ \Sigma \in CJ(\overline{Y}, E')$ is β-condensing with respect to f, that is,

$$\beta(G(\Omega)) \not\geq \beta(f(\Omega)) \tag{4.4}$$

for every $\Omega \subseteq \overline{Y}$ such that $G(\Omega)$ is not relatively compact.

Our target is to define the coincidence index for a β-condensing triplet $(f, G, \overline{U})_\beta$. To this aim, let us recall the following notion (see, e.g., [1, 7, 8, 12, 16]).

Definition 4.3. A convex, closed subset $T \subset E'$ is said to be fundamental for a triplet $(f, G, \overline{Y})_\beta$ if

(i) $G(f^{-1}(T)) \subseteq T$;

(ii) for any point $y \in \overline{Y}$, the inclusion $f(y) \in \overline{co}(G(y) \cup T)$ implies that $f(y) \in T$.

The entire space E' and the set $\overline{co}G(\overline{Y})$ are natural examples of fundamental sets for $(f, G, \overline{U})_\beta$.

It is easy to verify the following properties of a fundamental set.

Proposition 4.4. (a) *The set* Coin(f, G) *is included in* $f^{-1}(T)$ *for each fundamental set* T *of* $(f, G, \overline{U})_\beta$.

(b) *Let* T *be a fundamental set of* $(f, G, \overline{U})_\beta$, *and* $P \subset T$, *then the set* $\widetilde{T} = \overline{co}(G(f^{-1}(T)) \cup P)$ *is also fundamental.*

(c) *Let* $\{T_\alpha\}$ *be a system of fundamental sets of* $(f, G, \overline{U})_\beta$. *The set* $T = \cap_\alpha T_\alpha$ *is also fundamental.*

Proposition 4.5. *Each β-condensing triplet* $(f, G, \overline{U})_\beta$, *where* β *is a monotone, nonsingular MNC, admits a nonempty, compact fundamental set* T.

Proof. Consider the collection $\{T_\alpha\}$ of all fundamental sets of $(f,G,\overline{U})_\beta$ containing an arbitrary point $a \in E'$. This collection is nonempty since it contains E'. Then, taking $T = \cap_\alpha T_\alpha \neq \emptyset$, we obviously have

$$T = \overline{co}(G(f^{-1}(T)) \cup \{a\}), \tag{4.5}$$

and hence

$$\beta(f(f^{-1}(T))) \leq \beta(T) = \beta(G(f^{-1}(T))), \tag{4.6}$$

so $G(f^{-1}(T))$ is relatively compact and T is compact. □

Everywhere from now on, we assume that the MNC β is monotone and nonsingular.

Now, if T is a nonempty compact fundamental set of a β-condensing triplet $(f,G = (\varphi \circ \Sigma),\overline{Y})_\beta$, let $\rho : E' \to T$ be any retraction. Consider the multimap $\widetilde{G} = \rho \circ \varphi \circ \Sigma \in CJ(\overline{Y},E')$. From Proposition 4.4(a), it follows that

$$\operatorname{Coin}(f,\widetilde{G}) = \operatorname{Coin}(f,G). \tag{4.7}$$

Hence, f, $\widetilde{G}$, and $\overline{Y}$ form a compact triplet $(f,G,\overline{Y})_C$. We will say that $(f,G,\overline{Y})_C$ is a *compact approximation* of the triplet $(f,G,\overline{Y})_\beta$.

Definition 4.6. The oriented coincidence index of a β-condensing triplet $(f,G,\overline{U})_\beta$ is defined by the equality

$$\operatorname{Ind}(f,G,\overline{U})_\beta := \operatorname{Ind}\left(f,\widetilde{G},\overline{U}\right)_C, \tag{4.8}$$

where $(f,\widetilde{G},\overline{U})_C$ is a compact approximation of $(f,G,\overline{U})_\beta$.

To prove the consistency of the above definition, consider two nonempty, compact fundamental sets T_0 and T_1 of the triplet $(f,G = \varphi \circ \Sigma,\overline{U})_\beta$ with retractions $\rho_0 : E' \to T_0$ and $\rho_1 : E' \to T_1$, respectively.

If $T_0 \cap T_1 = \emptyset$, then by Proposition 4.4(a) and (c), $\operatorname{Coin}(f,\widetilde{G}_0) = \operatorname{Coin}(f,\widetilde{G}_1) = \operatorname{Coin}(f,\widetilde{G}) = \emptyset$, where $\widetilde{G}_i = \rho_i \circ \varphi \circ \Sigma$, $i = 0,1$. Hence, by Theorem 3.12, $\operatorname{Ind}(f,\widetilde{G}_0,\overline{U})_C = \operatorname{Ind}(f,\widetilde{G}_1,\overline{U})_C = 0$. Otherwise, we can assume, without loss of generality, that $T_0 \subseteq T_1$. In this case, consider the map $\overline{\varphi} : Z \times [0,1] \to E'$, given by $\overline{\varphi}(z,\lambda) = \rho_1 \circ (\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z))$ and the multimap $\overline{G} \in CJ(\overline{U} \times [0,1],E')$, $\overline{G}(x,\lambda) = \overline{\varphi}(\Sigma(x),\lambda)$.

The compact triplet $(f,\overline{G},\overline{U} \times [0,1])_C$ realizes the homotopy

$$\left(f,\widetilde{G}_0,\overline{U}\right)_C \sim \left(f,\widetilde{G}_1,\overline{U}\right)_C. \tag{4.9}$$

Indeed, the only fact that we need to verify is that

$$\operatorname{Coin}(\overline{f},\overline{G}) \cap (\partial U \times [0,1]) = \emptyset, \tag{4.10}$$

where $\overline{f}(x,\lambda) \equiv f(x)$ is the natural extension.

To the contrary, suppose that there exists $(x,\lambda) \in \partial U \times [0,1]$ such that

$$f(x) = \rho_1 \circ (\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z)) \tag{4.11}$$

for some $z \in \Sigma(x)$. But in this case, $x \in f^{-1}(T_1)$ and hence $\varphi(z) \in T_1$. Since also $\rho_0 \circ \varphi(z) \in T_1$, we have

$$\lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z) \in T_1 \tag{4.12}$$

and so

$$f(x) = \lambda\varphi(z) + (1-\lambda)\rho_0 \circ \varphi(z) \in \overline{co}(G(x) \cup T_0) \tag{4.13}$$

and we obtain $f(x) \in T_0$ and $x \in f^{-1}(T_0)$, implying $\varphi(z) \in T_0$ and $\rho_0 \circ \varphi(z) = \varphi(z)$. We conclude that $f(x) = \varphi(z) \in G(x)$ giving the contradiction.

Definition 4.7. Two β-condensing triplets $(f_0,G_0,\overline{U}_0)_\beta$ and $(f_1,G_1,\overline{U}_1)_\beta$ are said to be homotopic:

$$(f_0,G_0,\overline{U}_0)_\beta \sim (f_1,G_1,\overline{U}_1)_\beta, \tag{4.14}$$

if there exists a β-condensing triplet $(f_*,G_*,\overline{U}_*)_\beta$ satisfying conditions (a), (b), (c) of Definition 3.6.

THEOREM 4.8 (the homotopy invariance property). *If*

$$(f_0,G_0,\overline{U}_0)_\beta \sim (f_1,G_1,\overline{U}_1)_\beta, \tag{4.15}$$

then

$$|\operatorname{Ind}(f_0,G_0,\overline{U}_0)_\beta| = |\operatorname{Ind}(f_1,G_1,\overline{U}_1)_\beta|. \tag{4.16}$$

Proof. Let T_* be a nonempty compact fundamental set of the triplet $(f_*,G_* = (\varphi_* \circ \Sigma_*),\overline{U_*})$ connecting $(f_0,G_0,\overline{U}_0)_\beta$ with $(f_1,G_1,\overline{U}_1)_\beta$. It is easy to see that T_* is fundamental also for the triplets $(f_k,G_k,\overline{U}_k)_\beta$, $k=0,1$. Let $\rho_* : E' \to T_*$ be any retraction, and $(f_*,\widetilde{G}_* = \rho_* \circ \varphi_* \circ \sigma_*,\overline{U}_*)_C$ the corresponding compact approximation of $(f_*,G_*,\overline{U}_*)_\beta$. Then $(f_*,\widetilde{G}_*,\overline{U}_*)_C$ realizes a compact homotopy connecting the triplets $(f_k,\rho_* \circ \varphi_k \circ \Sigma_k,\overline{U}_k)_C$, $k=0,1$ which are compact approximations of $(f_k,G_k,\overline{U}_k)_\beta$, $k=0,1$, respectively.

By Theorem 3.13, we have

$$|\operatorname{Ind}(f_0,\rho_* \circ \varphi_0 \circ \Sigma_0,\overline{U}_0)_C| = |\operatorname{Ind}(f_1,\rho_* \circ \varphi_1 \circ \Sigma_1,\overline{U}_1)_C| \tag{4.17}$$

giving the desired equality (4.16). □

Remark 4.9. Let us mention that in case of invariable f and $\overline{U}$:

$$\begin{gathered} U_* = U \times [0,1] \\ f_*(x,\lambda) \equiv f(x), \quad \forall\lambda \in [0,1], \end{gathered} \tag{4.18}$$

the condition of β-condensivity for a triplet $(f, G_{*,}\overline{U} \times [0,1])_\beta$ may be weakened: for the existence of a nonempty, compact fundamental set T, it is sufficient to demand that

$$\beta(G_*(\Omega \times [0,1])) \not\geq \beta(f(\Omega)) \tag{4.19}$$

for every $\Omega \subseteq \overline{U}$ such that $G_*(\Omega \times [0,1])$ is not relatively compact.

In fact, it is enough to notice that in this case $f_*^{-1}(T) = f^{-1}(T) \times [0,1]$ and to follow the line of reasoning of Proposition 4.5.

Taking into consideration the corresponding property of compact triplets, we can precise the above property of homotopy invariance.

If $(f, G_{*,}\overline{U} \times [0,1])_\beta$ is a β-condensing triplet, where G_* has the form (c) of Definition 3.6, then

$$\operatorname{Ind}(f, G_0, \overline{U})_\beta = \operatorname{Ind}(f, G_1, \overline{U})_\beta, \tag{4.20}$$

where $G_k = G_*(\cdot, \{k\})$, $k = 0, 1$.

From relation (4.7) and Theorem 3.12, the following theorem follows immediately.

Theorem 4.10 (coincidence point property). *If* $\operatorname{Ind}(f, G, \overline{U})_\beta \neq 0$, *then* $\emptyset \neq \operatorname{Coin}(f, G) \subset U$.

As an example of application of Theorems 4.8 and 4.10, consider the following coincidence point result.

Theorem 4.11. *Let* $f \in \Phi_0 C^1(E, E')$ *be odd;* $G \in CJ(E, E')$ *β-condensing with respect to f on bounded subsets of E, that is,* $\beta(G(\Omega)) \not\geq \beta(f(\Omega))$ *for every bounded set* $\Omega \subset E$ *such that* $G(\Omega)$ *is not relatively compact.*

If the set of solutions of one-parameter family of operator inclusions

$$f(x) \in \lambda G(x) \tag{4.21}$$

is a priori bounded, then $\operatorname{Coin}(f, G) \neq \emptyset$.

Proof. From the condition it follows that there exists a ball $\mathcal{B} \subset E$ centered at the origin whose boundary $\partial\mathcal{B}$ does not contain solutions of (4.21). Let $\varphi \circ \Sigma$ be a representation of G. If $G_* : \overline{\mathcal{B}} \times [0,1] \to K(E')$ has the form

$$G_*(x, \lambda) = \varphi_*(\Sigma(x), \lambda), \quad \varphi_*(z, \lambda) = \lambda\varphi(z), \quad (z, \lambda) \in \overline{\mathcal{B}} \times [0,1], \tag{4.22}$$

then f, G_*, and $\overline{\mathcal{B}} \times [0,1]$ form a β-condensing triplet $(f, G_{*,}\overline{\mathcal{B}} \times [0,1])_\beta$.

In fact, suppose that $\beta(G_*(\Omega)) \geq \beta(f(\Omega))$ for some $\Omega \subset \overline{\mathcal{B}}$. Since $G_*(\Omega \times [0,1]) = \overline{co}(G(\Omega) \cup \{0\})$, we have $\beta(G(\Omega)) \geq \beta(f(\Omega))$ implying that $G(\Omega)$, and hence $G_*(\Omega \times [0,1])$, is relatively compact.

So the triplet $(f, G_{*,}\overline{\mathcal{B}} \times [0,1])_\beta$ induces a homotopy connecting the triplets $(f, G, \overline{\mathcal{B}})_\beta$ and $(f, 0, \overline{\mathcal{B}})_\beta$. Since the triplet $(f, 0, \overline{\mathcal{B}})_\beta$ is finite-dimensional, from the odd condition on f and the odd field property of the Brouwer degree, it follows that $(f, 0, \overline{\mathcal{B}})_\beta$ is an odd number.

Then, from the equality $\mathrm{Ind}(f,G,\overline{\mathcal{B}})_\beta = \mathrm{Ind}(f,0,\overline{\mathcal{B}})_\beta$, it follows that $\mathrm{Ind}(f,G,\overline{\mathcal{B}})_\beta \neq 0$ and we can apply the coincidence point property. □

In conclusion of this section, let us formulate the additive dependence on the domain property for β-condensing triplets.

THEOREM 4.12. *Let U_0 and U_1 be disjoint open subsets of an open bounded set $U \subset E$. If $(f,G,\overline{U})_\beta$ is a β-condensing triplet such that*

$$\mathrm{Coin}(f,G) \cap (\overline{U} \setminus (U_0 \cup U_1)) = \varnothing, \tag{4.23}$$

then,

$$\mathrm{Ind}(f,G,\overline{U})_\beta = \mathrm{Ind}\left(f,G,\overline{U}_0\right)_\beta + \mathrm{Ind}\left(f,G,\overline{U}_1\right)_\beta. \tag{4.24}$$

5. Example

Consider a mixed problem of the following form:

$$A(t,x(t),x'(t)) = B(t,x(t),x'(t),y(t)), \tag{5.1}$$

$$y'(t) \in C(t,x(t),y(t)), \tag{5.2}$$

$$x(0) = x_0, \qquad y(0) = y_0, \tag{5.3}$$

where $A : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, $B : [0,a] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are continuous maps; $C : [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \multimap \mathbb{R}^m$ is a multimap, and $x_0 \in \mathbb{R}^n$; $y_0 \in \mathbb{R}^m$.

By a solution of problem (5.1)–(5.3), we mean a pair of functions (x,y), where $x \in C^1([0,a];\mathbb{R}^n)$, $y \in AC([0,a];\mathbb{R}^m)$ satisfy initial conditions (5.1), (5.3) for all $t \in [0,a]$ and inclusion (5.2) for a.a. $t \in [0,a]$.

It should be noted that problem (5.1)–(5.3) may be treated as the law of evolution of a system $x(t)$, whose dynamics is described by the implicit differential equation (5.1) and the control $y(t)$ is the subject of the feedback relation (5.2). Our aim is to show that, under appropriate conditions, the problem of solving problem (5.1)–(5.3) can be reduced to the study of a condensing triplet of the above-mentioned form (see Section 4).

Consider the following condition:

(A) for each $(t,u,v) \in [0,a] \times \mathbb{R}^n \times \mathbb{R}^n$, there exist continuous partial derivatives $A'_u(t,u,v)$, $A'_v(t,u,v)$, and moreover, $\det A'_v(t,u,v) \neq 0$.

PROPOSITION 5.1. *Under condition (A), a map $f : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n) \times \mathbb{R}^n$ defined as*

$$f(x)(t) = \left(A(t,x(t),x'(t)),x(0)\right) \tag{5.4}$$

is a Fredholm map of index zero, whose restriction to each closed bounded set $D \subset C^1([0,a];\mathbb{R}^n)$ is proper.

Proof. (i) At first, let us prove that f is a Fredholm map of index zero. It is sufficient to show that the map $\widetilde{f} : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$, $\widetilde{f}(x)(t) = A(t,x(t),x'(t))$ is Fredholm of index n.

Let us note that $\widetilde{f}$ is a C^1 map and, moreover, its derivative can be written explicitly:

$$(\widetilde{f}'(x)h)(t) = A'_u(t,x(t),x'(t))h(t) + A'_v(t,x(t),x'(t))h'(t) \tag{5.5}$$

for $h \in C^1([0,a];\mathbb{R}^n)$. The linear operator $\widetilde{f}'(x) : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ is a Fredholm operator of index n. In fact, introducing the auxiliary operators

$$\begin{gathered}
\widetilde{f}'_u(x) : C^1([0,a];\mathbb{R}^n) \longrightarrow C([0,a];\mathbb{R}^n), \\
(\widetilde{f}'_u(x)h)(t) = A'_u(t,x(t),x'(t))h(t), \quad t \in [0,a], \\
\widetilde{f}'_v(x) : C^1([0,a];\mathbb{R}^n) \longrightarrow C([0,a];\mathbb{R}^n), \\
(\widetilde{f}'_v(x)h)(t) = A'_v(t,x(t),x'(t))h'(t), \quad t \in [0,a],
\end{gathered} \tag{5.6}$$

we can write

$$\widetilde{f}'(x)h = \widetilde{f}'_u(x)h + \widetilde{f}'_v(x)h. \tag{5.7}$$

The operator $\widetilde{f}'_u(x)$ is completely continuous since it can be represented as the composition of a completely continuous inclusion map $i : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ and a continuous linear operator $M : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ $(Mh)(t) = A'_u(t,x(t),x'(t))h(t)$. Now, it is sufficient to show that the operator $\widetilde{f}'_v(x)$ is a Fredholm operator of index n.

Let us represent this operator as the composition of the differentiation operator $d/dt : C^1([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ and the operator $L : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$, $(Lz)(t) = A'_v(t,x(t),x'(t))z(t)$. It is well known that the operator d/dt is a Fredholm operator of index n. Since the matrix $A'_v(t,x(t),x'(t))$ is invertible, the operator L is invertible too. Hence, the operators $\widetilde{f}'_v(x)$ and, therefore, $\widetilde{f}'(x)$ are Fredholm of index n and $f'(x)$ is a Fredholm map of index zero. So, f is a nonlinear Fredholm map of index zero.

(ii) Now, let $D \subset C^1([0,a];\mathbb{R}^n)$ be a closed bounded set. Denoting the restriction of $\widetilde{f}$ on D by the same symbol, let us demonstrate its properness. Let $\mathcal{K} \subset C([0,a];\mathbb{R}^n)$ be any compact set, and let $\{x_n\}_{n\in\mathbb{N}} \subset \widetilde{f}^{-1}(\mathcal{K})$ be an arbitrary sequence. Without loss of generality, we may assume that $\widetilde{f}(x_n) \to z \in \mathcal{K}$. Since the sequence $\{x_n\}$ is bounded in $C^1([0,a];\mathbb{R}^n)$ we may also assume, without loss of generality, that the sequence $\{x_n\}$ tends, in $C([0,a];\mathbb{R}^n)$, to some $\omega \in C([0,a];\mathbb{R}^n)$. Further, from the representation

$$A(t,\omega(t),x'_n(t)) = A(t,x_n(t),x'_n(t)) + [A(t,\omega(t),x'_n(t)) - A(t,x_n(t),x'_n(t))] \tag{5.8}$$

it follows that the sequence $z_n = A(\cdot,\omega(\cdot),x'_n(\cdot))$ tends to z in $C([0,a];\mathbb{R}^n)$. From the inverse mapping theorem it follows that $x'_n = \Psi(z_n)$, where $\Psi : C([0,a];\mathbb{R}^n) \to C([0,a];\mathbb{R}^n)$ is a continuous map, implying that x'_n tends to $\Psi(z)$ in $C([0,a];\mathbb{R}^n)$. So, the sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in the space $C^1([0,a];\mathbb{R}^n)$ and, hence, the set $\widetilde{f}^{-1}(\mathcal{K})$ is compact. The properness of f easily follows. $\square$

Now we will describe the assumptions on the map B and the multimap C.

Denoting by the symbol $Kv(\mathbb{R}^m)$ the collection of all nonempty compact convex subsets of $\mathbb{R}^m$, we suppose that the multimap $C : [0,a] \times \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ satisfies the following conditions:

(C1) the multifunction $C(\cdot,u,w) : [0,a] \to Kv(\mathbb{R}^m)$ has a measurable selection for all $(u,w) \in \mathbb{R}^n \times \mathbb{R}^m$;

(C2) the multimap $C(t,\cdot,\cdot) : \mathbb{R}^n \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ is upper semicontinuous for a.a. $t \in [0,a]$;

(C3) the multimap C is uniformly continuous in the second argument, in the following sense: for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$C(t,\overline{u},w) \subset W_\varepsilon\big(C(t,u,w)\big) \quad \forall (t,w) \in [0,a] \times \mathbb{R}^m \tag{5.9}$$

whenever $\|\overline{u} - u\| < \delta$ (where W_ε denotes the ε-neighborhood of a set);

(C4) there exists a function $\gamma \in L^1_+([0,a])$ such that

$$\big|\big|C(t,u,w)\big|\big| := \sup\big\{\|c\| : c \in C(t,u,w)\big\} \le \gamma(t)\big(1 + \|u\| + \|w\|\big). \tag{5.10}$$

For a given function $x \in C^1([0,a];\mathbb{R}^n)$ consider the multimap $C_x : [0,a] \times \mathbb{R}^m \to Kv(\mathbb{R}^m)$ defined as $C_x(t,w) = C(t,x(t),w)$. From [12, Theorem 1.3.5], it follows that for each $w \in \mathbb{R}^m$ the multifunction $C_x(\cdot,w)$ admits a measurable selection. Furthermore, from (C2) and (C3), it follows that for a.a. $t \in [0,a]$ the multimap $C_x(t,w)$ depends upper semicontinuously on (x,w). Applying known results on existence, topological structure, and continuous dependence of solutions for Carathéodory-type differential inclusions (see, e.g., [2, 6, 12]) we conclude the following.

Proposition 5.2. *For each given $x \in C^1([0,a];\mathbb{R}^n)$, the set Π_x of the Carathéodory solutions of the Cauchy problem*

$$\begin{aligned} y'(t) &\in C\big(t,x(t),y(t)\big), \\ y(0) &= y_0 \end{aligned} \tag{5.11}$$

is an R_δ-set in $C([0,a];\mathbb{R}^m)$. Moreover, the multimap $\Pi : C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^m))$, $\Pi(x) = \Pi_x$ is upper semicontinuous.

Now, we will assume that the maps A and B satisfy the following Lypschitz-type condition:

(AB) there exists a constant q, $0 \le q < 1$, such that

$$\big|B(t,u,v,w) - B(t,u,\overline{v},w)\big| \le q\big|A(t,u,v) - A(t,u,\overline{v})\big| \tag{5.12}$$

for all $t \in [0,a]$, $u,v,\overline{v} \in \mathbb{R}^n$, $w \in \mathbb{R}^m$.

Consider the continuous map $\widetilde{\sigma}: C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m) \to C([0,a];\mathbb{R}^n)$ defined as

$$\widetilde{\sigma}(x,y)(t) = B\big(t,x(t),x'(t),y(t)\big) \tag{5.13}$$

and the multimap $\widetilde{\Sigma}: C^1([0,a];\mathbb{R}^n) \to K(C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m))$, $\widetilde{\Sigma}(x) = \{x\} \times \Pi(x)$.

From Propositions 5.2 and 2.9, it follows that $\widetilde{\Sigma}$ is a J-multimap, and hence the composition $\widetilde{G} = \widetilde{\sigma} \circ \widetilde{\Sigma}: C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^n))$ is a CJ-multimap. It is clear that the set $\widetilde{G}(x)$ consists of all functions of the form $B(t,x(t),x'(t),y(t))$, where $y \in \Pi(x)$.

Define now the CJ-multimap $G: C^1([0,a];\mathbb{R}^n) \to K(C([0,a];\mathbb{R}^n) \times \mathbb{R}^n)$ by

$$G(x) = \widetilde{G}(x) \times \{x_0\}. \tag{5.14}$$

The solvability of problem (5.1)–(5.3) is equivalent to the existence of a coincidence point $x \in C^1([0,a];\mathbb{R}^n)$ for the pair (f,G).

If $U \subset C^1([0,a];\mathbb{R}^n)$ is an open bounded set, then to show that $(f,G,\overline{U})$ form a condensing triplet with respect to the Kuratowski MNC, it is sufficient to prove the following statement.

Proposition 5.3. *The triplet $(\widetilde{f},\widetilde{G},\overline{U})$ is α-condensing with respect to the Kuratowski MNC α in the space $C([0,a];\mathbb{R}^n)$.*

Proof. Take any subset $\Omega \subset \overline{U}$, and let $\alpha(\widetilde{f}(\Omega)) = d$. From the definition of Kuratowski MNC, it follows that taking an arbitrary $\varepsilon > 0$ we may find a partition of the set $\widetilde{f}(\Omega)$ into subsets $\widetilde{f}(\Omega_i)$, $i = 1,\dots,s$, such that $\operatorname{diam}(\widetilde{f}(\Omega_i)) \le d + \varepsilon$. Since the embedding $C^1([0,a];\mathbb{R}^n) \hookrightarrow C([0,a];\mathbb{R}^n)$ is completely continuous, the image Ω_C of Ω under this embedding is relatively compact. It is known (see, e.g., [2, 12]) that an u.s.c. compact-valued multimap sends compact sets to compact sets, then we can conclude that the set $\Pi(\Omega)$ is relatively compact. It means that taking a fixed $\delta > 0$ and any Ω_i, we may divide the sets Ω_{iC} and $\Pi(\Omega)$ into a finite number of subsets Ω_{ijC}, $j = 1,\dots,p_i$, and balls $D_{ik}(z_{ik})$, $k = 1,\dots,r_i$, centered at $z_{ik} \in C([0,a];\mathbb{R}^m)$, respectively, such that for each $t \in [0,a]$; $u_1(\cdot),u_2(\cdot) \in \Omega_{ijC}$, $v \in \mathbb{R}^n$; $w_1(\cdot),w_2(\cdot) \in D_{ik}(z_{ik})$, we have

$$\begin{gathered} \big|A\big(t,u_1(t),v\big) - A\big(t,u_2(t),v\big)\big| < \delta, \\ \big|B\big(t,u_1(t),v,w_1(t)\big) - B\big(t,u_2(t),v,w_2(t)\big)\big| < \delta. \end{gathered} \tag{5.15}$$

Now, the set $\widetilde{G}(\Omega)$ is covered by a finite number of sets Γ_{ijk}, $i = 1,\dots,s$; $j = 1,\dots,p_i$; $k = 1,\dots,r_i$ of the form

$$\Gamma_{ijk} = \big\{B\big(\cdot,x(\cdot),x'(\cdot),y(\cdot)\big) : x \in \Omega_{ijC},\ y \in D_{ik}(z_{ik})\big\}. \tag{5.16}$$

Let us estimate the diameters of these sets. Taking arbitrary $x_1, x_2 \in \Omega_{ijC}$ and $y_1, y_2 \in D_{ik}(z_{ik})$ and applying (5.15), and condition (AB), for any $t \in [0,a]$, we have

$$\begin{aligned}
&|B(t,x_1(t),x_1'(t),y_1(t)) - B(t,x_2(t),x_2'(t),y_2(t))| \\
&\quad < |B(t,x_1(t),x_1'(t),z_{ik}(t)) - B(t,x_2(t),x_2'(t),z_{ik}(t))| + 2\delta \\
&\quad \leq |B(t,x_1(t),x_1'(t),z_{ik}(t)) - B(t,x_1(t),x_2'(t),z_{ik}(t))| \\
&\qquad + |B(t,x_1(t),x_2'(t),z_{ik}(t)) - B(t,x_2(t),x_2'(t),z_{ik}(t))| + 2\delta \\
&\quad \leq q|A(t,x_1(t),x_1'(t)) - A(t,x_1(t),x_2'(t))| + 3\delta \\
&\quad \leq q|A(t,x_1(t),x_1'(t)) - A(t,x_2(t),x_2'(t))| \\
&\qquad + q|A(t,x_2(t),x_2'(t)) - A(t,x_1(t),x_2'(t))| + 3\delta \\
&\quad < q(d+\varepsilon) + q\delta + 3\delta.
\end{aligned} \tag{5.17}$$

Now, if $q = 0$, it means, by the arbitrariness of the choice of $\delta > 0$, that $\alpha(\widetilde{G}(\Omega)) = 0$ and then the triplet $(\widetilde{f}, \widetilde{G}, \overline{U})$, and therefore $(f, G, \overline{U})$, is compact. Otherwise, let us take $\varepsilon > 0$ and $\delta > 0$ so small that

$$q\varepsilon + (q+3)\delta < (1-q)d. \tag{5.18}$$

Then, $q(d+\varepsilon) + q\delta + 3\delta = \mu d$, where $0 < \mu < 1$ and, hence $\operatorname{diam}\Gamma_{ijk} \leq \mu d$, implying that

$$\alpha(\widetilde{G}(\Omega)) \leq \mu\alpha(\widetilde{f}(\Omega)). \tag{5.19}$$

□

The proved statement implies that the coincidence index theory, developed in the previous sections, can be applied to the study of the solvability of problem (5.1)–(5.3). Moreover, it is easy to see that the coincidence point set $\operatorname{Coin}(f,G)$ of a condensing triplet $(f,G,\overline{U})_\beta$ is a compact set. In case when problem (5.1)–(5.3) is a model for a control system, this approach can be used also to obtain the existence of optimal solutions. As an example, we can consider the following statement.

Proposition 5.4. *Under the above conditions, suppose that the map A is odd: $A(t,-u,-v) = A(t,u,v)$ for all $t \in [0,a]$; $u, v \in \mathbb{R}^n$ and the set of functions $x \in C^1([0,a];\mathbb{R}^n)$ satisfying the family of relations*

$$\begin{gathered}
A(t,x(t),x'(t)) = \lambda B(t,x(t),x'(t),y(t)), \quad \lambda \in [0,1], \\
y'(t) \in C(t,x(t),y(t)), \\
x(0) = x_0, \qquad y(0) = y_0
\end{gathered} \tag{5.20}$$

is a priori bounded. Then, there exists a solution (x_*, y_*) of problem (5.1)–(5.3) minimizing a given lower-semicontinuous functional

$$l : C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m) \longrightarrow \mathbb{R}_+. \tag{5.21}$$

Proof. The application of Theorem 4.11 yields that the set $Q = \mathrm{Coin}(f, G)$ is nonempty and compact. It remains only to notice that the set of solutions $\{(x, y)\}$ of (5.1)–(5.3) is closed and it is contained in the compact set $Q \times \Pi(Q) \subset C^1([0,a];\mathbb{R}^n) \times C([0,a];\mathbb{R}^m)$. □

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FLOW OF ELECTRORHEOLOGICAL FLUID UNDER CONDITIONS OF SLIP ON THE BOUNDARY

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We study a mathematical model describing flows of electrorheological fluids. A theorem of existence of a weak solution is proved. For this purpose the approximating-topological method is used.

1. Introduction

Electrorheological fluids are smart materials that are concentrated suspensions of polarizable particles in a nonconducting dielectric liquid. In moderately large electric fields the particles form chains along the field lines and these chains then aggregate into the form of columns. These chainlike and columnar structures yield dramatic changes in the rheological properties of the suspensions. The fluid becomes anisotropic, the apparent viscosity (the resistance to flow) in the direction, orthogonal to that of the electric field, abruptly increases, while the apparent viscosity in the direction of the electric field changes not so drastically.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, in which a fluid flows, $n \in \{2,3\}$. Let the boundary S of Ω be Lipschitz continuous. As it is well known, the stationary movement of any fluid is described by the equation in Cauchy form:

$$\sum_{j=1}^{n} u_j \frac{\partial u_i}{\partial x_j} - \sum_{j=1}^{n} \frac{\partial \sigma_{ij}}{\partial x_j} = F_i, \tag{1.1}$$

where $x \in \Omega$, $u = (u_1,\dots,u_n)$ is the velocity field of the fluid, $\{\sigma_{ij}\}_{i,j=1}^{n}$ is the stress tensor, $F = (F_1,\dots,F_n)$ is the volume force. We also add the condition of incompressibility to (1.1):

$$\operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0. \tag{1.2}$$

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We will consider the following constitutive equation (see [2]):

$$\sigma_{ij}(p,u) = -p\delta_{ij} + 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u), \tag{1.3}$$

where δ_{ij} are the components of the unit tensor, $\varepsilon_{ij}(u)$ are the components of the rate of strain tensor, $\varepsilon_{ij}(u) = (1/2)(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$, p is the spherical part of the stress tensor, φ is the viscosity function, $I(u) = \sum_{i,j=1}^{n}(\varepsilon_{ij}(u))^2$,

$$\mu(u,E)(x) = \left(\frac{\alpha\theta + u(x)}{\alpha\sqrt{n} + |u(x)|}, \frac{E(x)}{|E(x)|}\right)^2_{\mathbb{R}^n}, \tag{1.4}$$

α is a small positive constant, and $\mathbb{R}^n \ni \theta = (1,\ldots,1)$, $E = (E_1,\ldots,E_n)$ is the electric field strength.

We consider the condition of slip on S (see [4, 6]). Let $f = (f_1,\ldots,f_n)$ be an external surface force acting on the fluid,

$$f_i = \sum_{j=1}^{n}\left[-p\delta_{ij} + 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u)\right]\eta_j|_S, \tag{1.5}$$

where $\eta = (\eta_1,\ldots,\eta_n)$ is the unit outward normal to S. We represent f in the form

$$f(s) = f^\eta(s) + f^\tau(s) \quad \forall s \in S, \tag{1.6}$$

where f^η and f^τ are the normal and the tangent vectors:

$$\begin{gathered} f^\eta(s) = f_\eta(s)\eta(s), \qquad f_\eta(s) = \sum_{i=1}^{n} f_i(s)\eta_i(s), \\ f^\tau(s) = f(s) - f^\eta(s) = \sum_{i=1}^{n} f_{\tau i}(s)e_i, \qquad f_{\tau i}(s) = f_i(s) - f_\eta(s)\eta_i(s), \end{gathered} \tag{1.7}$$

$\{e_1,\ldots,e_n\}$ is an orthonormal basis in $\mathbb{R}^n$.

For the field u, a similar decomposition is valid.

The slip conditions on the boundary are the following [4]:

$$u_\eta(s) = 0 \quad \forall s \in S, \tag{1.8}$$

$$f^\tau(s) = -\chi\left(f_\eta(s), |u^\tau(s)|^2\right)u^\tau(s) \quad \forall s \in S \tag{1.9}$$

(by $|\cdot|$ we denote the norm in Euclidian space $\mathbb{R}^n$). Instead of (1.9) we will consider the regularized condition

$$f^\tau(s) = -\chi\Big(f_{r\eta}(s), |u^\tau(s)|^2\Big)u^\tau(s) \quad \forall s \in S, \tag{1.10}$$

$$\begin{aligned} f_{r\eta}(p,u) &= \left[-Pp + \sum_{i,j=1}^{n} 2\varphi(I(Pu),|E|,\mu(Pu,E))\varepsilon_{ij}(Pu)\eta_i\eta_j\right]\Bigg|_S, \\ Pv(x) &= \int_{\mathbb{R}^n} \omega(|x-\acute{x}|)v(\acute{x})d\acute{x}, \quad x \in \overline{\Omega}, \end{aligned} \tag{1.11}$$

where $\omega \in C^\infty(\mathbb{R}_+)$, $\operatorname{supp}\omega \in [0,a]$, $a \in \mathbb{R}_+$, $\omega(z) \geqslant 0$ at $z \in \mathbb{R}_+$, $\int_{\mathbb{R}^n}\omega(|x|)dx = 1$.

Here $\mathbb{R}_+$ is the set of nonnegative numbers.

Equation (1.10) means that the model of slip is not local, this is natural from the physical view point (see [1]).

We assume that

$$\int_\Omega p(x)dx = 0. \tag{1.12}$$

Let us describe the concept of a weak solution of (1.1)–(1.3), (1.8), (1.10), (1.12). We introduce some Hilbert spaces (see [4]):

$$\begin{aligned} Z &= \left\{v : v \in H^1(\Omega)^n,\ v_\eta|_S = 0, \int_\Omega [\operatorname{div} v](x)dx = 0\right\}, \\ W &= \{v : v \in Z,\ \operatorname{div} v = 0\}. \end{aligned} \tag{1.13}$$

The expression

$$(u,v)_Z = \sum_{i,j=1}^{n}\int_\omega \varepsilon_{ij}[u](x)\varepsilon_{ij}[v](x)dx + \sum_{i=1}^{n}\int_S u_{\tau i}(s)v_{\tau i}(s)ds \tag{1.14}$$

defines a scalar product on Z (and in W).

Multiplying (1.1) by a function h in $L_2(\Omega)^n$, and using Green's formula and (1.3), (1.8), (1.10), we see that

$$\begin{aligned} &\sum_{i,j=1}^{n}\int_\omega 2\varphi(I(u)(x), |E(x)|, \mu(u,E)[x])\varepsilon_{ij}[u](x)\varepsilon_{ij}[h](x)dx \\ &\quad + \sum_{i=1}^{n}\int_S \chi[f_{rn}(p(s),u(s)), |u_\tau(s)|^2]u_{\tau i}(s)h_{\tau i}(s)ds \\ &\quad + \sum_{i,j=1}^{n}\int_\omega u_j(x)\frac{\partial u_i}{\partial x_j}(x)h_i(x)dx - \int_\omega p(x)[\operatorname{div} h](x)dx = \sum_{i=1}^{n}\int_\omega F_i(x)h_i(x)dx \end{aligned} \tag{1.15}$$

(here we suppose that $F \in L_2(\Omega)^n$).

Definition 1.1. A couple of functions $(u,p) \in W \times L_2(\Omega)$ is a weak solution of problem (1.1)–(1.3), (1.8), (1.10), (1.12) if it satisfies equality (1.15) for all $h \in Z$.

The following conditions are imposed on the functions φ and χ.

(C1) There are positive constants a_1 and a_2 such that

$$a_1 \leqslant \varphi(y_1,y_2,y_3) \leqslant a_2 \quad \forall (y_1,y_2,y_3) \in \mathbb{R}_+^2 \times [0,1]. \tag{1.16}$$

(C2) The function $\varphi(y_1,\cdot,y_3) : \mathbb{R}_+ \to \mathbb{R}$ is measurable in y_2 for all specified $(y_1,y_3) \in \mathbb{R}_+ \times [0,1]$.

(C3) The function $\varphi(\cdot,y_2,\cdot) : \mathbb{R}_+ \times [0,1] \to \mathbb{R}$ is jointly continuous in (y_1,y_3) for all $y_2 \in \mathbb{R}_+$.

(C4) The function $y_1 \mapsto \varphi(y_1^2,y_2,y_3)y_1$ is not decreasing at nonnegative values of y_1.

(C5) There are positive constants b_1 and b_2 such that

$$b_1 \leqslant \chi(z_1,z_2) \leqslant b_2 \quad \forall (z_1,z_2) \in \mathbb{R} \times \mathbb{R}_+. \tag{1.17}$$

(C6) The function $\chi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ is continuous.

Note that conditions (C1)–(C6) have a physical meaning (see [2, 4]).

The main result of this paper is the following theorem.

Theorem 1.2. *Suppose that conditions (C1)–(C6) are satisfied. Then there exists a weak solution of problems (1.1)–(1.3), (1.8), (1.10), (1.12).*

For the proof of Theorem 1.2 we use the approximating-topological method [8]. For this purpose, in the beginning, we determine an equivalent operational treatment of the problem under consideration. After that for the obtained operational equation, we introduce an approximating family of equations depending on a parameter δ and by use of Skrypnik's version of the topological degree [7], on the basis of a priori estimates, we prove existence of solutions of the approximating equations. As a result, making limiting transition for $\delta \to 0$, we obtain the solvability of problem (1.1)–(1.3), (1.8), (1.10), (1.12).

2. Operational treatment

Let us introduce some notations. By X^* we denote the space, conjugate to some Banach space X, $\langle g,y\rangle$ denotes the action of the functional $g \in X^*$ on the element $y \in X$, X^m is the topological product of m copies of the space X.

Determine several mappings as follows:

$$\begin{gathered}
A : Z \longrightarrow Z^*, \quad \langle A(u),h\rangle = \sum_{i,j=1}^{n} \int_\omega 2\varphi(I(u),|E|,\mu(u,E))\varepsilon_{ij}(u)\varepsilon_{ij}(h)dx, \\
K : Z \times L_2(\Omega) \longrightarrow Z^*, \\
\langle K(u,p),h\rangle = \sum_{i=1}^{n} \int_S \chi[f_{rn}(p(s),u(s)),|u_\tau(s)|^2]u_{\tau i}(s)h_{\tau i}(s)ds, \\
M : Z \longrightarrow Z^*, \quad \langle M(u),h\rangle = \sum_{i,j=1}^{n} \int_\omega u_j(x)\frac{\partial u_i}{\partial x_j}(x)h_i(x)dx.
\end{gathered} \tag{2.1}$$

Take

$$D : Z \longrightarrow L_2(\Omega), \qquad D(u) = \operatorname{div} u. \tag{2.2}$$

Identifying $L_2(\Omega)^n$ and $(L_2(\Omega)^n)^*$ we obtain

$$D^* : L_2(\Omega)^n \equiv \left(L_2(\Omega)^n\right)^* \longrightarrow Z^*, \qquad \langle D^*(p), h\rangle = \int_\Omega p(x)[\operatorname{div} h](x)dx. \tag{2.3}$$

It is obvious that the set of weak solutions of problem (1.1)–(1.3), (1.8), (1.10), (1.12) coincides with the set of couples $(u,p) \in W \times L_2(\Omega)$ that satisfy the following operational equation:

$$A(u) + K(u,p) + M(u) - D^*(p) = F. \tag{2.4}$$

3. Properties of operators

Everywhere below the expressions $\mathbf{v}_k \xrightarrow[k\to\infty]{} \mathbf{v}_0$ and $\mathbf{v}_k \rightharpoonup_{k\to\infty} \mathbf{v}_0$ will denote strong and weak convergences, respectively, of the sequence $\{\mathbf{v}_k\}_{k=1}^\infty$ to an element $\mathbf{v}_0$. The case when the sequence $\{\mathbf{v}_k\}_{k=1}^\infty$ does not converge to $\mathbf{v}_0$ in strong sense is denoted as $\mathbf{v}_k \not\xrightarrow[k\to\infty]{} \mathbf{v}_0$.

Lemma 3.1. *The following statements hold.*

(1) *The operator A is bounded and demicontinuous (the latter means that if $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, then $\langle A(u_k), h\rangle \xrightarrow[k\to\infty]{} \langle A(u_0), h\rangle$ $\forall h \in Z$).*

(2) *For the operator*

$$\langle \mathcal{A}(u,v), h\rangle = \sum_{i,j=1}^{n} \int_\omega 2\varphi(I(u), |E|, \mu(v,E))\varepsilon_{ij}(u)\varepsilon_{ij}(h)dx, \tag{3.1}$$

the following inequality holds

$$\langle \mathcal{A}(u_1,v) - \mathcal{A}(u_2,v), u_1 - u_2\rangle \geqslant 0 \quad \forall u_1,u_2,v \in Z. \tag{3.2}$$

(3) *Specify an element $u \in Z$. Then from any bounded sequence $\{v_k\}_{k=1}^\infty$ in Z it is possible to choose a subsequence $\{v_{k_l}\}_{l=1}^\infty$ such that $\mathcal{A}(u, v_{k_l}) \xrightarrow[l\to\infty]{} \mathcal{A}(u, v_0)$ in Z^*.*

Proof. (a) Boundedness of the operator A is obvious. Let us show that it is demicontinuous. So, let $u_k \xrightarrow[k\to\infty]{} u_0$ in Z. Clearly, there is a subsequence $\{u_{k_l}\}_{l=1}^\infty$, for which

$$I(u_{k_l})[x] \xrightarrow[l\to\infty]{} I(u_0)[x], \quad u_{k_l}[x] \xrightarrow[l\to\infty]{} u_0[x] \quad \text{for a.e. } x \in \Omega. \tag{3.3}$$

Assume that $\langle A(u_k), \check{h}\rangle \not\xrightarrow[k\to\infty]{} \langle A(u_0), \check{h}\rangle$ for some $\check{h} \in Z$. Without loss of generality for a sequence $\{u_{k_l}\}_{l=1}^\infty$ the following inequality holds with some $\zeta > 0$:

$$\left| \langle A(u_{k_l}), \check{h}\rangle - \langle A(u_0), \check{h}\rangle \right| > \zeta. \tag{3.4}$$

We estimate

$$
\begin{aligned}
&|\langle A(u_{k_l})-A(u_0),\check{h}\rangle| \\
&\quad\leqslant \left|\sum_{i,j=1}^{n} 2\int_{\Omega}\varphi(I(u_{k_l}),|E|,\mu(u_{k_l},E))\varepsilon_{ij}(u_{k_l}-u_0)\varepsilon_{ij}(\check{h})dx\right| \\
&\qquad+\left|\sum_{i,j=1}^{n} 2\int_{\Omega}[\varphi(I(u_{k_l}),|E|,\mu(u_{k_l},E))-\varphi(I(u_0),|E|,\mu(u_0,E))]\varepsilon_{ij}(u_0)\varepsilon_{ij}(\check{h})dx\right| \\
&\quad\leqslant 2a_2\|u_{k_l}-u_0\|_Z\|\check{h}\|_Z \\
&\qquad+2\left\{\int_{\Omega}[\varphi(I(u_{k_l}),|E|,\mu(u_{k_l},E))-\varphi(I(u_0),|E|,\mu(u_0,E))]^2 I(u_0)dx\right\}^{1/2}\|\check{h}\|_Z.
\end{aligned}
\tag{3.5}
$$

The first component in the last part of inequality (3.5) tends to zero because of the convergence of $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, the second component tends to zero by the Lebesgue theorem. Hence, $\langle A(u_{k_l})-A(u_0),\check{h}\rangle \xrightarrow[l\to\infty]{} 0$, that contradicts inequality (3.4).

(b) Introduce the notation $\phi(u,v)=\varphi(I(u),|E|,\mu(v,E))$. With the help of the Cauchy inequality and condition (C4) we obtain that

$$
\begin{aligned}
&\langle \mathcal{A}(u_1,v)-\mathcal{A}(u_2,v),u_1-u_2\rangle \\
&\quad=\sum_{i,j=1}^{n} 2\int_{\Omega}[\phi(u_1,v)\varepsilon_{ij}(u_1)-\phi(u_2,v)\varepsilon_{ij}(u_2)][\varepsilon_{ij}(u_1-u_2)]dx \\
&\quad=2\int_{\Omega}[\phi(u_1,v)I(u_1)+\phi(u_2,v)I(u_2)]dx \\
&\qquad-\sum_{i,j=1}^{n} 2\int_{\Omega}[\phi(u_1,v)\varepsilon_{ij}(u_1)\varepsilon_{ij}(u_2)-\phi(u_2,v)\varepsilon_{ij}(u_1)\varepsilon_{ij}(u_2)]dx \\
&\quad\geqslant 2\int_{\Omega}[\phi(v,u_1)I^{1/2}(u_1)-\phi(u_2,v)I^{1/2}(u_2)](I^{1/2}(u_1)-I^{1/2}(u_2))dx\geqslant 0.
\end{aligned}
\tag{3.6}
$$

(c) Let $\{v_k\}_{k=1}^{\infty}$ be a bounded sequence in $Z, u\in Z$. There is a subsequence $\{v_{k_l}\}_{l=1}^{\infty}$ and an element $v_0\in Z$ such that

$$
v_{k_l}(x)\xrightarrow[l\to\infty]{} v_0(x)\quad \text{for a.e. } x\in\Omega. \tag{3.7}
$$

We have

$$
\begin{aligned}
&\|\mathcal{A}(u,v_{k_l})-\mathcal{A}(u,v_0)\|_{Z^*} \\
&\quad=\sup_{\|h\|_Z=1}\sum_{i,j=1}^{n} 2\int_{\omega}[\varphi(I(u),E,\mu(v_{k_l},E))\varepsilon_{ij}(u)-\varphi(I(u),E,\mu(v_0,E))\varepsilon_{ij}(u)]\varepsilon_{ij}(h)dx \\
&\quad\leqslant 2n\left\{\int_{\omega}[\varphi(I(u),E,\mu(v_{k_l},E))-\varphi(I(u),E,\mu(v_0,E))]^2 I(u)dx\right\}^{1/2}.
\end{aligned}
\tag{3.8}
$$

The last expression in inequality (3.8) tends to zero by conditions (C1)–(C3) and by the Lebesgue theorem. □

LEMMA 3.2. *The operator K possesses the following properties.*

(1) *For any sequence $\{u^k, h^k, p^k\}_{k=1}^{\infty}$ from $Z \times Z \times L_2(\Omega)^n$, for which $(u^k, h^k, p^k) \xrightarrow[k\to\infty]{} (u^0, h^0, p^0)$, there is a subsequence $\{u^{k_l}, h^{k_l}, p^{k_l}\}_{l=1}^{\infty}$ such that $\langle K(u^{k_l}, p^{k_l}), h^{k_l}\rangle \xrightarrow[l\to\infty]{} \langle K(u^0, p^0), h^0\rangle$ (from this property, in particular, it follows that the operator K is bounded and demicontinuous).*

(2) *The operator $K(\cdot, T(\cdot))$ is compact for any operator $T : Z \to L_2(\Omega)$.*

Proof. (a) So, let the limits from the condition take place. Using the fact that the embedding $Z \hookrightarrow L_2(S)^n$ is compact, we take a subsequence $\{u^{k_l}, h^{k_l}\}_{l=1}^{\infty}$ such that

$$(u^{k_l}, h^{k_l}) \xrightarrow[l\to\infty]{} (u^0, h^0) \quad \text{in } L_2(S)^n \times L_2(S)^n, \tag{3.9}$$

$$(u^{k_l}[s], h^{k_l}[s]) \xrightarrow[l\to\infty]{} (u^0[s], h^0[s]) \quad \text{in } \mathbb{R}^{2n} \quad \text{for a.e. } s \in S. \tag{3.10}$$

Extend the functions of the sequence $\{p^k\}_{k=1}^{\infty}$ (and the function p^0) onto the entire space $\mathbb{R}^n$. For this purpose we take $\widetilde{p}^k(x) = p^k(x)$ if $x \in \Omega$, otherwise $\widetilde{p}^k(x) = 0$. It is obvious that $\widetilde{p}^{k_l} \xrightarrow[l\to\infty]{} \widetilde{p}^0$ in $L_2(\mathbb{R}^n)$. Thus we have

$$\int_{\mathbb{R}^n} \omega(|\xi - \acute{x}|)\widetilde{p}^{k_l}(\acute{x})d\acute{x} = P(p^{k_l})[\xi] \xrightarrow[l\to\infty]{} P(p^0)[\xi] \quad \forall \xi \in S. \tag{3.11}$$

Similarly

$$\begin{aligned} &\left[-Pp^{k_l} + \sum_{i,j=1}^{n} 2\varphi(I(Pu^{k_l}), |E|, \mu(Pu^{k_l}, E))\varepsilon_{ij}(Pu^{k_l})\eta_i\eta_j\right](s) \\ &\quad = f_{r\eta}(p^{k_l}, u^{k_l})[s] \xrightarrow[l\to\infty]{} f_{r\eta}(p^0, u^0)[s] \end{aligned} \tag{3.12}$$

for all $s \in S$.

We estimate

$$\begin{aligned} &|\langle K(u^{k_l}, p^{k_l}), h^{k_l}\rangle - \langle K(u^0, p^0), h^0\rangle| \\ &\quad \leqslant \sum_{i=1}^{n} \left| \int_S \left\{\chi\left(f_{r\eta}(p^{k_l}, u^{k_l}), |u_\tau^{k_l}|^2\right) - \chi\left(f_{r\eta}(p^0, u^0), |u_\tau^0|^2\right)\right\} u_{\tau i}^0 h_{\tau i}^0 ds\right| \\ &\qquad + \sum_{i=1}^{n} \left| \int_S \left\{\chi\left(f_{r\eta}(p^{k_l}, u^{k_l}), |u_\tau^{k_l}|^2\right)\right\} (u_{\tau i}^0 - u_{\tau i}^{k_l}) h_{\tau i}^0 ds\right| \\ &\qquad + \sum_{i=1}^{n} \left| \int_S \left\{\chi\left(f_{r\eta}(p^{k_l}, u^{k_l}), |u_\tau^{k_l}|^2\right)\right\} u_{\tau i}^{k_l} (h_{\tau i}^0 - h_{\tau i}^{k_l}) ds\right|. \end{aligned} \tag{3.13}$$

Using condition (C5) and (3.11), (1.2) we conclude that the first summand in the right-hand side of inequality (3.13) tends to zero by the Lebesgue theorem, the other summands tend to zero by (3.9).

(b) We will use Gelfand's criterion, having reformulated it according to the following lemma.

Lemma 3.3. *The subset $\mathfrak{M}$ of a separable Banach space $\mathscr{X}$ is relatively compact, if from any sequence of functionals $\{\mathbf{f}_k\}_{k=1}^{\infty}$ belonging to $\mathscr{X}^*$ and such that*

$$\mathbf{f}_k(y) \xrightarrow[k\to\infty]{} 0 \quad \forall y \in \mathscr{X}, \tag{3.14}$$

it is possible to take a subsequence $\{\mathbf{f}_{k_l}\}_{l=1}^{\infty}$ such that (3.14) is valid for it uniformly for all y from $\mathfrak{M}$.

The proof of Lemma 3.3 is similar to that of [3, Theorem 3(1.IX), page 274], minor changes are connected with transition to a subsequence in the formulation of the statement.

Now, let $T : Z \to L_2(\Omega)$ be an operator, let $\mathfrak{M}$ be some bounded set from Z, and let $h_k \xrightharpoonup[k\to\infty]{} 0$ in the space $(Z^*)^* \equiv Z$. For all u from $\mathfrak{M}$ we have

$$\langle h_k, K(u,Tu)\rangle \leqslant b_2\|u\|_{L_2(S)^n}\big\|h_k\big\|_{L_2(S)^n}. \tag{3.15}$$

However, the embedding $Z \hookrightarrow L_2(S)^n$ is compact, therefore for some subsequence $\{h_{k_l}\}_{l=1}^{\infty}$ we obtain that $\langle h_{k_l}, K(u,Tu)\rangle \xrightarrow[l\to\infty]{} 0$ uniformly for all u from $\mathfrak{M}$. Hence, the set $K(\mathfrak{M}, T(\mathfrak{M}))$ is relatively compact as it was required to show. □

Lemma 3.4. *Let M_δ, $\delta > 0$, be an approximation of the operator M, that is*

$$M_\delta : Z \longrightarrow Z^*, \quad \langle M_\delta(u), h\rangle = \frac{1}{1+\delta^{1/4}\|u\|_{L_4(\Omega)^n}} \sum_{i,j=1}^{n} \int_\Omega u_j \frac{\partial u_i}{\partial x_j} h_i dx. \tag{3.16}$$

(1) *The operator M_δ is compact.*
(2) *Take any sequences $\{u^k\}_{k=1}^{\infty}$ and $\{h^k\}_{k=1}^{\infty}$ in the space Z such that $u^k \xrightharpoonup[k\to\infty]{} u^0$ in Z, $h^k \xrightharpoonup[k\to\infty]{} h^0$ in Z. Then there are subsequences $\{u^{k_l}\}_{l=1}^{\infty}$ and $\{h^{k_l}\}_{l=1}^{\infty}$ such that $\langle M_\delta(u^{k_l}), h^{k_l}\rangle \xrightarrow[l\to\infty]{} \langle M_\delta(u^0), h^0\rangle$.*

Proof. (1) The proof of boundedness and continuity of the operator M_δ is standard. The property of compactness is shown similarly to item (2) of Lemma 3.2. Here the following inequality is in use:

$$\langle M_\delta(u), h\rangle \leqslant \frac{\|u\|_{L_4(\Omega)^n}\|u\|_{H^1(\Omega)^n}}{1+\delta^{1/4}\|u\|_{L_4(\Omega)^n}} \|h\|_{L_4(\Omega)^n}. \tag{3.17}$$

(2) Let $u^k \xrightharpoonup[k\to\infty]{} u^0$ and $h^k \xrightharpoonup[k\to\infty]{} h^0$ in Z. It follows from here that there are the subsequences $\{u^{k_l}\}_{l=1}^{\infty}$ and $\{h^{k_l}\}_{l=1}^{\infty}$ such that

$$(u^{k_l}, h^{k_l}) \xrightarrow[l\to\infty]{} (u^0, h^0) \quad \text{in } (L_4(\Omega)^n)^2. \tag{3.18}$$

Thus

$$
\begin{aligned}
&|\langle M_\delta(u^0),h^0\rangle - \langle M_\delta(u^{k_l}),h^{k_l}\rangle| \\
&\quad\leqslant \left|\frac{1}{1+\delta^{1/4}||u^0||_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega u_j^0\left(\frac{\partial u_i^0}{\partial x_j}-\frac{\partial u_i^{k_l}}{\partial x_j}\right)h_i^0\,dx\right| \\
&\qquad+\left|\frac{1}{1+\delta^{1/4}||u^0||_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega (u_j^0-u_j^{k_l})\frac{\partial u_i^{k_l}}{\partial x_j}h_i^0\,dx\right| \\
&\qquad+\left|\frac{1}{1+\delta^{1/4}||u^0||_{L_4(\Omega)^n}}\sum_{i,j=1}^n\int_\omega u_j^{k_l}\frac{\partial u_i^{k_l}}{\partial x_j}(h_i^0-h_i^{k_l})\,dx\right| \\
&\qquad+\left|\left(\frac{1}{1+\delta^{1/4}||u^0||_{L_4(\Omega)^n}}-\frac{1}{1+\delta^{1/4}||u^{k_l}||_{L_4(\Omega)^n}}\right)\sum_{i,j=1}^n\int_\omega u_j^{k_l}\frac{\partial u_i^{k_l}}{\partial x_j}h_i^{k_l}\,dx\right|.
\end{aligned}
\tag{3.19}
$$

The first summand on the right-hand side of (3.19) tends to zero since $u^{k_l}\xrightarrow[l\to\infty]{} u^0$ in Z, the other summands tend to zero by (3.18). □

4. Approximation equation and an a priori estimate

For any $\delta>0$ we introduce an auxiliary equation in the unknown function u^δ:

$$A^+(u^\delta)+A(u^\delta)+K_\delta(u^\delta)+M_\delta(u^\delta)+\delta^{-1}D^*D(u^\delta)=F, \tag{4.1}$$

where

$$
\begin{aligned}
\langle K_\delta(u),h\rangle &= \sum_{i=1}^n\int_S\chi\left(f_{rn}(-\delta^{-1}Du,u),|u_\tau|^2\right)u_{\tau i}h_{\tau i}\,ds, \\
\langle A^+(u),h\rangle &= \delta\sum_{i,j=1}^n\int_\omega \varepsilon_{ij}(u)\varepsilon_{ij}(h)\,dx.
\end{aligned}
\tag{4.2}
$$

Let $\langle K^+(u),h\rangle=(b_1/2)\sum_{i=1}^n\int_S u_{\tau i}h_{\tau i}\,ds$.

Lemma 4.1. *For the following family of the operational equations, depending on the parameter $t\in[0,1]$:*

$$
\begin{aligned}
\Lambda_t^\delta(u^\delta) &= A^+(u^\delta)+K^+(u^\delta) \\
&\quad+t\left(A(u^\delta)+K_\delta(u^\delta)-K^+(u^\delta)+M_\delta(u^\delta)+\delta^{-1}D^*D(u^\delta)-F\right)=0,
\end{aligned}
\tag{4.3}
$$

the estimate

$$||u^\delta||_Z\leqslant C(\|F\|_{Z^*},a_1,b_1,n,\Omega) \tag{4.4}$$

holds for $0<\delta\leqslant c(a_1,b_1,n,\Omega)$. Here c and C are variables that depend only on the specified parameters.

Proof. For $t = 0$ there is only a zero solution, in this case estimation (4.4) is obvious. Let u^δ be a solution of (4.3) for some $t \in (0,1]$. Apply both sides of (4.3) to u^δ. We obtain

$$\langle A^+(u^\delta) + K^+(u^\delta), u^\delta \rangle \geqslant 0, \tag{4.5}$$

thus

$$\langle A(u^\delta), u^\delta\rangle + \langle K_\delta(u^\delta), u^\delta\rangle - \langle K^+(u^\delta), u^\delta\rangle + \langle M_\delta(u^\delta), u^\delta\rangle + \delta^{-1}\langle D^*D(u^\delta), u^\delta\rangle \leqslant \langle \mathbf{f}, u^\delta\rangle. \tag{4.6}$$

Notice that

$$\langle A(u^\delta), u^\delta\rangle + \langle K_\delta(u^\delta), u^\delta\rangle - \langle K^+(u^\delta), u^\delta\rangle \geqslant \min(2a_1, b_1/2)\|u^\delta\|_Z^2. \tag{4.7}$$

At the same time

$$\begin{aligned}
|\langle M_\delta(u^\delta), u^\delta\rangle| &= \frac{1}{1+\delta^{1/4}\|u^\delta\|_{L_4(\Omega)^n}}\left|\sum_{i,j=1}^n \int_\omega u_j^\delta \frac{\partial u_i^\delta}{\partial x_j} u_i^\delta\, dx\right| \\
&= \frac{1}{1+\delta^{1/4}\|u^\delta\|_{L_4(\Omega)^n}}\left|\left(-\frac{1}{2}\int_\Omega D(u^\delta)|u^\delta|^2 dx + \frac{1}{2}\sum_{i=1}^n \int_S u_{\tau i}^\delta \eta_i |u^\delta|^2 ds\right)\right| \\
&\leqslant \frac{n}{2}\|D(u^\delta)\|_{L_2(\Omega)} \frac{\|u^\delta\|_{L_4(\Omega)^n}^2}{1+\delta^{1/4}\|u^\delta\|_{L_4(\Omega)^n}} \leqslant \frac{n}{2}\|D(u^\delta)\|_{L_2(\Omega)} \frac{\|u^\delta\|_{L_4(\Omega)^n}^2}{\delta^{1/4}\|u^\delta\|_{L_4(\Omega)^n}} \\
&\leqslant \frac{\|D(u^\delta)\|_{L_2(\Omega)}}{\delta^{1/2}} \delta^{1/4}\frac{n}{2}\vartheta\|u^\delta\|_Z \leqslant \frac{\|D(u^\delta)\|_{L_2(\Omega)}^2}{2\delta} + \delta^{1/2}\frac{n^2}{8}\vartheta^2\|u^\delta\|_Z^2
\end{aligned} \tag{4.8}$$

(ϑ is the norm of the operator of embedding of the space Z into the space $L_4(\Omega)^n$).

It is easy to see that

$$\begin{gathered}
\delta^{-1}\langle D^*D(u^\delta), u^\delta\rangle = \delta^{-1}\langle D(u^\delta), D(u^\delta)\rangle = \delta^{-1}\|D(u^\delta)\|_{L_2(\Omega)}^2, \\
\|F\|_{Z^*}\|u^\delta\|_Z \leqslant \frac{l^2}{2}\|F\|_{Z^*}^2 + \frac{1}{2l^2}\|u^\delta\|_Z^2.
\end{gathered} \tag{4.9}$$

Now for sufficiently large l and small δ we obtain the required estimate (4.4). □

5. Existence of a solution of the approximation equation

For the proof of existence of a solution of the approximation equation we apply the method of topological degree for generalized monotonous maps (see [7]). We show that the family Λ_t^δ carries out homotopy of the maps Λ_0^δ and Λ_1^δ. For this purpose, we will notice first that from the a priori estimate (4.4) it follows that there is a sphere of nonzero radius R, with the center at zero, such that on its boundary there are no solutions of the equation $\Lambda_t^\delta(u^\delta) = 0 (t \in [0,1])$.

For our purpose it is necessary to prove first the following statements.

(a) For any sequence $\{u_k\}_{k=1}^{\infty}$ on the border of the sphere $\partial\Theta_R$, and for any sequence $\{t_k\}_{k=1}^{\infty}$ of points from the interval $[0,1]$ such that $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, $\Lambda_{t_k}^{\delta}(u_k) \xrightarrow[k\to\infty]{} 0$ in Z^*, and $\langle \Lambda_{t_k}^{\delta}(u_k), u_k - u_0\rangle \xrightarrow[k\to\infty]{} 0$, the convergence $u_k \xrightarrow[k\to\infty]{} u_0$ in Z also takes place.

(b) For any sequence $\{u_k\}_{k=1}^{\infty}$ from the closure of the sphere $\overline{\Theta}_R$ such that $u_k \xrightarrow[k\to\infty]{} u_0$ in Z, for any sequence of points $\{t_k\}_{k=1}^{\infty}$ ($t_k \in [0,1]$) such that $t_k \xrightarrow[k\to\infty]{} t_0$, there is a limit $\Lambda_{t_k}^{\delta}(u_k) \xrightarrow[k\to\infty]{} \Lambda_{t_0}^{\delta}(u_0)$ in Z^*.

Proof. (a) Assume the contrary, that is, let $u_k \not\xrightarrow[k\to\infty]{} u_0$ in Z. Then for some subsequence $\{u_{k_l}\}_{l=1}^{\infty}$ and some fixed number $\epsilon > 0$ the following inequality:

$$\|u_{k_l} - u_0\|_Z > \epsilon \tag{5.1}$$

holds. We will show that this inequality is not valid.

From the hypothesis of statement (a) we have

$$\lim_{l\to\infty} \langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}), u_{k_l} - u_0\rangle = \lim_{l\to\infty} \langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_{k_l}) + \Lambda_{t_0}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_0), u_{k_l} - u_0\rangle = 0. \tag{5.2}$$

Without loss of generality we may suppose that the subsequence $\{t_{k_l}\}_{l=1}^{\infty}$ is such that $t_{k_l} \xrightarrow[l\to\infty]{} t_0 \in [0,1]$. All operators, that we have determined, are bounded. Hence

$$\begin{aligned} &\lim_{l\to\infty} \langle \Lambda_{t_{k_l}}^{\delta}(u_{k_l}) - \Lambda_{t_0}^{\delta}(u_{k_l}), u_{k_l} - u_0\rangle \\ &\quad = \lim_{l\to\infty} \langle (t_{k_l} - t_0)[A(u_{k_l}) + K_\delta(u_{k_l}) - K^+(u_{k_l}) \\ &\qquad\qquad + M_\delta(u_{k_l}) + \delta^{-1}D^*D(u_{k_l}) - F], u_{k_l} - u_0\rangle = 0. \end{aligned} \tag{5.3}$$

At the same time, from the properties of the operator, that we have proved, it follows that from the sequence $\{u_{k_l}\}_{l=1}^{\infty}$ it is possible to take a subsequence $\{u_{k_{l_m}}\}_{m=1}^{\infty}$ such that

$$\begin{gathered} \lim_{m\to\infty} \langle A(u_{k_{l_m}}) - \mathcal{A}(u_{k_{l_m}}, u_0), u_{k_{l_m}} - u_0\rangle = 0, \\ \lim_{m\to\infty} \langle \mathcal{A}(u_{k_{l_m}}, u_0) - A(u_0), u_{k_{l_m}} - u_0\rangle \geqslant 0, \\ \lim_{m\to\infty} \langle K_\delta(u_{k_{l_m}}) - K_\delta(u_0) + K^+(u_{k_{l_m}}) - K^+(u_0) + M_\delta(u_{k_{l_m}}) - M_\delta(u_0), u_{k_{l_m}} - u_0\rangle = 0. \end{gathered} \tag{5.4}$$

Clearly,

$$\lim_{m\to\infty} \langle \delta^{-1}D^*D(u_{k_{l_m}} - u_0), u_{k_{l_m}} - u_0\rangle \geqslant 0. \tag{5.5}$$

From (5.3)–(5.5) it follows that

$$\lim_{m\to\infty} \langle [A^+ + K^+](u_{k_{l_m}} - u_0), u_{k_{l_m}} - u_0\rangle \leqslant 0. \tag{5.6}$$

The limiting relation (5.6) contradicts (5.1), since the operator $A^+ + K^+$ is strictly monotonous:

$$\langle [A^+ + K^+](u-w), u-w\rangle \geqslant \min\left(\delta, \frac{b_1}{2}\right)\|u-w\|_Z^2 \quad \forall u,w \in Z. \tag{5.7}$$

Proof of statement (b). Obviously for any $h \in Z$ the equality

$$\lim_{k\to\infty}\langle \Lambda_{t_k}^{\delta}(u_k) - \Lambda_{t_0}^{\delta}(u_0), h\rangle = \lim_{k\to\infty}\langle \Lambda_{t_k}^{\delta}(u_k) - \Lambda_{t_0}^{\delta}(u_k), h\rangle + \lim_{k\to\infty}\langle \Lambda_{t_0}^{\delta}(u_k) - \Lambda_{t_0}^{\delta}(u_0), h\rangle \tag{5.8}$$

holds. The first summand in equality (5.8) tends to zero since $t_k \xrightarrow[k\to\infty]{} t_0$, the second summand tends to zero since all operators in (5.8) are demicontinuous. □

Thus, the maps Λ_0^{δ} and Λ_1^{δ} are homotopic, at the same time the degree $deg(\Lambda_0^{\delta}, \Theta_R, 0)$ is an odd number, as Λ_0^{δ} is an odd map. Therefore, the solutions of the equation $\Lambda_1^{\delta}(u^{\delta}) = 0$ exist for sufficiently small δ, as it was required to show.

6. Limiting transition

Take a sequence $\delta_k \xrightarrow[k\to\infty]{} 0$ and assign to every δ_k the solution $u_k \in Z$ of the equation $\Lambda_1^{\delta_k}(u_k) = 0$. We have

$$\delta_k^{-1} D^* D(u_k) = F - A^+(u_k) - A(u_k) - K_{\delta_k}(u_k) - M_{\delta_k}(u_k). \tag{6.1}$$

From the results of [5] it follows that the operator D^* carries the out isomorphism of the spaces

$$\begin{gathered}\mathcal{P}_0 = \left\{\rho \in L_2(\Omega) : \int_{\Omega}\rho(x)dx = 0\right\},\\ W_0 = \{\mathbf{f} \in Z^* : \langle \mathbf{f}, u\rangle = 0 \ \forall u \in W\}.\end{gathered} \tag{6.2}$$

From the a priori estimate and (6.1) it follows that there are elements $u_0 \in Z, p_0 \in L_2(\Omega)$ such that

$$\begin{gathered} u_k \xrightharpoonup[k\to\infty]{} u_0 \quad \text{in } Z,\\ u_k \xrightarrow[k\to\infty]{} u_0 \quad \text{in norm of } L_2(\Omega)^n, \text{ a.e. in } \Omega,\\ u_k \xrightarrow[k\to\infty]{} u_0 \quad \text{in norm of } L_2(S)^n, \text{ a.e. on } S,\\ D(u_k) \xrightarrow[k\to\infty]{} 0 \quad \text{in } L_2(\Omega),\\ \delta_k^{-1} D(u_k) \xrightharpoonup[k\to\infty]{} -p_0 \quad \text{in } \mathcal{P}_0. \end{gathered} \tag{6.3}$$

Let

$$\begin{aligned}\Upsilon(u_k,v) = \langle A^+(u_k) + A(u_k) - \mathscr{A}(v,u_k) + K_{\delta_k}(u_k) - K(u_0,p_0) \\ + M_{\delta_k}(u_k) - M(u_0) + \delta_k^{-1}D^*D(u_k) + D^*(p_0), u_k - v\rangle.\end{aligned} \tag{6.4}$$

Taking into account Lemmas 3.1–3.4 and (6.3), we obtain

$$\begin{gathered}\lim_{k\to\infty} \langle A(u_k^v) - \mathscr{A}(v,u_k^v), u_k^v - v\rangle \geqslant 0, \\ \lim_{k\to\infty} \langle K_{\delta_k}(u_k^v) - K(u_0,p_0) + M_{\delta_k}(u_k^v) - M(u_0), u_k^v - v\rangle = 0, \\ \lim_{k\to\infty} \langle \delta_k^{-1}D^*D(u_k^v) + D^*(p_0), u_k^v - v\rangle = 0, \\ \lim_{k\to\infty} \langle A^+(u_k^v), u_k^v - v\rangle = 0.\end{gathered} \tag{6.5}$$

It follows from relations (6.5) that

$$\lim_{k\to\infty} \Upsilon(u_k^v, v) \geqslant 0 \quad \forall v \in Z. \tag{6.6}$$

On the other hand,

$$\begin{aligned}\lim_{k\to\infty} \Upsilon(u_k^v,v) &= \lim_{k\to\infty} \langle A^+(u_k^v), u_k^v - v\rangle \\ &\quad + \lim_{k\to\infty} \langle A(u_k^v) + K_{\delta_k}(u_k^v) + M_{\delta_k}(u_k^v big) + \delta_k^{-1}D^*D(u_k^v), u_k^v - v\rangle \\ &\quad + \lim_{k\to\infty} \langle -\mathscr{A}(v,u_k^v) - K(u_0,p_0) - M(u_0) + D^*(p_0), u_k^v - v\rangle \\ &= \langle F - \mathscr{A}(v,u_0) - K(u_0,p_0) - M(u_0) + D^*(p_0), u_0 - v\rangle \quad \forall v \in Z.\end{aligned} \tag{6.7}$$

Now, take $v = u_0 - \gamma h$, where $\gamma > 0$, $h \in Z$, and pass to the limit as $\gamma \to 0$. From relations (6.6) and (6.7) we get the inequality

$$\langle F - A(u_0) - K(u_0,p_0) - M(u_0) + D^*(p_0), h\rangle \geqslant 0 \quad \forall h \in Z. \tag{6.8}$$

Since h is arbitrary, after replacing h with $-h$, we obtain that $(u = u_0,\ p = p_0)$ is the solution of (2.4).

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THE INTEGRAL LIMIT THEOREM IN THE FIRST PASSAGE PROBLEM FOR SUMS OF INDEPENDENT NONNEGATIVE LATTICE VARIABLES

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The integral limit theorem as to the probability distribution of the random number ν_m of summands in the sum $\sum_{k=1}^{\nu_m} \xi_k$ is proved. Here, $\xi_1, \xi_2, \ldots$ are some nonnegative, mutually independent, lattice random variables being equally distributed and ν_m is defined by the condition that the sum value exceeds at the first time the given level $m \in \mathbb{N}$ when the number of terms is equal to ν_m.

1. Introduction

The following problem arises in some applications of the theory of random processes.

Let $\{\xi(t);\ t \in \mathbb{R}_+ = [0,\infty)\}$ be a stationary ergodic random process such that it has nonnegative trajectories with probability one. Consider the random process $\{J[t;\xi];\ t \in [0,\infty)\}$,

$$J[t;\xi] = \int_0^t \xi(s)ds \tag{1.1}$$

that is defined as a functional on the process $\{\xi(t);\ t \in [0,\infty)\}$. It is naturally implied that the process $\{\xi(t)\}$ is measurable with probability one. Further, let each trajectory of the process $\{\xi(t)\}$ with probability one have no temporal interval, where it equals to zero. It is required to calculate the probability distribution of the random time τ that is a solution of

$$J[\tau;\xi] = E \tag{1.2}$$

for the given value $E > 0$. It is a well-posed random variable. First, if there exists a solution of (1.1), then it is unique under restrictions pointed out. It is because the function $J[t;\xi]$ increases for each realization $\xi(t)$ if the integral (1.1) is defined at $t \in [0,\infty)$ for it. Then its graph may cross the level E only once and the constancy interval is absent in this situation. Second, due to the ergodicity of the stationary process $\{\xi(t);\ t\in[0,\infty)\}$, the following equality:

$$\lim_{t\to\infty} t^{-1}J[t;\xi] = \mathbf{E}\xi(s), \quad s\in[0,\infty), \tag{1.3}$$

is fulfilled with probability one. Then, choosing an $\varepsilon\in(0,\mathbf{E}\xi(s))$ (further, the symbol $\mathbf{E}$ denotes the mathematical expectation everywhere), there exists almost surely such a random number θ for this ε and for given realization $\xi(t)$ when it is fulfilled $t^{-1}J[t;\xi] > \mathbf{E}\xi(s)-\varepsilon$ at $t>\theta$. Then $J[t;\xi] > t(\mathbf{E}\xi(s)-\varepsilon)$ and, consequently, there exists a solution of (1.2) with the probability one.

The calculation problem of probability distribution for the random variable τ defined by (1.2) arises, for example, in the control theory of stochastic systems and in the reliability theory (Homenko [4]), in the statistical theory of material destruction (Virchenko [6, 7]), in the statistical radiophysics (Mazmanishvili [5]). Note, we may expect that the probability distribution pointed out has a universal behaviour in some sense when E tends to infinity. It is due to the ergodicity of the process $\{\xi(t);\ t\in[0,\infty)\}$ and if tending to the limit (1.3) is sufficiently fast. In this case, the integral in (1.2) may be considered as the sum of large number of weakly dependent, equally distributed random variables approximately equal to $T\mathbf{E}\xi(s)$ with overwhelming probability. This circumstance makes the study of the probability distribution of the random variable τ a very important problem from the viewpoint of mentioned applications.

The problem, described above, admits some natural generalizations. This is important for its study since, in frameworks of more general problem setting, this may find such its particular cases that are more simple from the analytical viewpoint. The condition of almost sure absence of the interval where $\xi(t)=0$ for each process trajectory is not optional. If we define the variable τ by

$$\tau = \inf\left\{t;\ \int_0^t \xi(s)ds \geq E\right\}, \tag{1.4}$$

then we may ignore this condition. Arguments guaranteeing the nonemptiness of the set where the inequality is fulfilled are the same as above in the proof of solution existence for (1.2).

Moreover, it may set the analogous problem for random processes with discrete time, that is, for random stationary ergodic sequences $\{\xi_k;\ k\in\mathbb{N}\}$ for which $\xi_k\geq 0$. Such a problem arises in the mathematical statistics, that is, in the so-called *sequential statistical analysis*. In the case of sequences, it is necessary to introduce (Wald [9], Basharinov [1]) the process $\{J_n[\xi];\ n\in\mathbb{N}\}$ with realizations

$$J_n[\xi] = \sum_{k=1}^{n} \xi_k \tag{1.5}$$

and the random variable ν_E defined as

$$\nu_E = \min\{n;\ J_n[\xi]\geq E\}. \tag{1.6}$$

In particular, such a problem makes sense in the case when the sequence $\{\xi_k,\ k \in \mathbb{N}\}$ presents the collection of independent, equally distributed, nonnegative random variables. Just this case is investigated in our work under the additional condition. Namely, we assume that random variables ξ_k are lattice.

2. The problem setting

As it was mentioned above, the calculation problem of the probability distribution $P_m(n) = \Pr\{\nu_m = n\}$ of a random variable ν_m was considered by A. Wald when he developed the sequential statistical analysis. The random variable ν_m is determined by the formula

$$\nu_m = \min\{l;\ \eta_l \geq m\}, \tag{2.1}$$

where $m \in \mathbb{N}$, $n \in \mathbb{N}$, $\eta_l = \sum_{k=1}^{l} \xi_k$, $l \in \mathbb{N}$ and $\xi_1, \xi_2, \ldots$ is a sequence of independent values, $\xi_k \in \{0,1\}$, $k = 1,2,\ldots$ with the success probability equal to $p = \Pr\{\xi = 1\} > 0$.

Probabilities $P_m(n)$ are approximated by the following way:

$$\Pr\{\nu_m = n;\ np/m \in [x, x+dx)\} \sim f(x)dx \tag{2.2}$$

in the limit $m \to \infty$, $n \to \infty$. Here, $x = np/m$, $dx = p/m$, and

$$f(x) = \left(\frac{m}{2\pi(1-p)x}\right)^{1/2} \exp\left[-\frac{m}{2(1-p)}\left(x^{1/2} - x^{-1/2}\right)^2\right] \tag{2.3}$$

is the density of the probability distribution of a suitable continuous random variable.

Further, this result was spread (Virchenko [8]) for the general case of arbitrary sequences $\xi_1, \xi_2, \ldots$ of statistically independent and equally distributed, nonnegative, lattice random variables under the following restrictions on the probability distribution $p_k = \Pr\{\xi = k\}$ of their typical representative ξ. First, it is nontriviality condition $1 > p_0 > 0$ for the distribution p_k. Second, the fast decrease $\lim_{k\to\infty}(p_k)^{1/k} = \rho^{-1} < 1$ of the probabilities must take place. If $m, n \to \infty$ and $m = n\mathbf{E}\xi + O(n^{1/2})$, then the probability distribution of the variable ν_m is approximated by the formula

$$\Pr\{\nu_m = n;\ n\mathbf{E}\xi/m \in [x, x+dx)\} \sim f(x)dx \tag{2.4}$$

similar to (1.2) with the density

$$f(x) = \left(\frac{m\mathbf{E}\xi}{2\pi x\mathbf{D}\xi}\right)^{1/2} \exp\left[-\frac{m\mathbf{E}\xi}{2\mathbf{D}\xi}\left(x^{1/2} - x^{-1/2}\right)^2\right]. \tag{2.5}$$

Equations (2.2) and (2.4) can be considered as some local limit theorems for probability distribution of the variable ν_m. They present some asymptotic formulas provided $m \to \infty$ at the mentioned restrictions concerning the n variation. But these formulas are defined only up to a factor ~ 1 in this limit. In connection with this fact a natural further step is to prove of the corresponding integral theorem at $m, n \to \infty$ relative to the distribution of $P_m(n)$. Such a theorem determines uniquely the limit probability distribution unlike the local one.

Here, we will prove the integral theorem of probability distribution for the random variable

$$\zeta_m = \left(\frac{\nu_m \mathbf{E}\xi}{m} - 1\right)\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2} \tag{2.6}$$

centering the variable ν_m.

3. The integral representation

In this section we obtain the integral representation for the probability $P_m(n) \equiv \Pr\{\nu_m = n\}$. It will be used further for proving the integral theorem.

At first, we make use the following decomposition of the event $\{\nu_m = n\}$ on events $\{\eta_{n-1} = k\}$, $k = 0,\dots,m-1$ that are pairwise disjoint,

$$\begin{aligned}\{\nu_m = n\} = \{\eta_{n-1} < m,\ \eta_n \geq m\} &= \bigcup_{k=0}^{m-1}\{\eta_{n-1} = k,\ \xi_n \geq m-k\} \\ &= \bigcup_{k=0}^{m-1}\{\eta_{n-1} = k\} \cap \{\xi_n \geq m-k\}.\end{aligned} \tag{3.1}$$

We obtain the following formula for probability $P_m(n)$:

$$P_m(n) = \sum_{k=0}^{m-1}\Pr\{\eta_{n-1} = k\}\Pr\{\xi_n \geq m-k\}, \tag{3.2}$$

based on the decomposition and using the total probability formula with the independence condition for variables $\xi_1,\xi_2,\dots,\xi_n$.

Now, we introduce the generating function $F(z)$ of probability distribution of the typical representative ξ of the sequence $\xi_1,\xi_2,\dots$:

$$F(z) = \sum_{k=0}^{\infty} z^k \Pr\{\xi = k\}. \tag{3.3}$$

Taking into account the independence condition for random variables $\xi_1,\xi_2,\dots,\xi_n$, we have

$$F_n(z) \equiv \sum_{k=0}^{\infty} z^k \Pr\{\eta_n = k\} = [F(z)]^n, \quad n \in \mathbb{N}. \tag{3.4}$$

Using this fact, we express the following generating function:

$$G_n(z) = \sum_{m=1}^{\infty} z^m P_m(n) \tag{3.5}$$

via the function $F(z)$. For this, we multiply (3.2) by z^m and sum all those equalities over $m \in \mathbb{N}$. As a result, we get

$$\begin{aligned} G_n(z) &= \sum_{m=1}^{\infty} z^m \sum_{k=0}^{m-1} \Pr\{\eta_{n-1} = k\} \Pr\{\xi_n \geq m-k\} \\ &= \sum_{k=0}^{\infty} z^k \Pr\{\eta_{n-1} = k\} \sum_{m=k+1}^{\infty} z^{m-k} \Pr\{\xi_n \geq m-k\} \\ &= z[F(z)]^{n-1} \sum_{l=1}^{\infty} \Pr\{\xi_n = l\} \sum_{m=0}^{l-1} z^m = \frac{z}{1-z}(1-F(z))[F(z)]^{n-1}. \end{aligned} \tag{3.6}$$

The function $G_n(z)$ is analytical into the unit disk $\{z : |z| < 1\}$. This follows from the fact that the function $F(z)$ is always analytical into the closed unit disk since the power series defining $F(z)$ converges uniformly in it,

$$|F(z)| \leq \sum_{k=0}^{\infty} |z|^k p_k \leq \sum_{k=0}^{\infty} p_k = 1. \tag{3.7}$$

The probability $P_m(n)$ is defined as the mth coefficient of the Taylor series of the function $G_n(z)$. Therefore, the probability $P_m(n)$ may be represented by the Cauchy formula

$$P_m(n) = \frac{1}{2\pi i} \oint_C z^{-(m+1)} G_n(z) dz = \frac{1}{2\pi i} \oint_C \frac{(1-F(z))}{1-z} [F(z)]^{n-1} \frac{dz}{z^m}, \tag{3.8}$$

where $C = \{z;\ |z| = r\}$, $r < 1$ is a closed countour with the positive going around.

We formulate the obtained result in the form of the separate statement.

Theorem 3.1. *Let $\xi_1, \xi_2, \ldots$ be a sequence of statistically independent and equally distributed, nonnegative, lattice random variables. Let p_k, $k = 1, 2, \ldots$, be an arbitrary probability distribution of its typical representative ξ. Then the probability $P_m(n)$ is defined by (3.8).*

4. The extremal property of the holomorphic function $H(z)$ with nonnegative coefficients

In this section we prove the theorem on the module maximum of the holomorphic function $H(z)$ that has nonnegative coefficients in its expansion in the power series. This property will be used hereinafter for the proof of the limit theorem.

Theorem 4.1 (Fedoryuk [2]). *Let*

$$H(z) = \sum_{k=0}^{\infty} a_k z^k, \quad z \in \mathbb{C}, \tag{4.1}$$

be a holomorphic function in the circle $C_\rho = \{z : |z| \leq \rho\} \subset \mathbb{C}$ of the radius ρ having nonnegative coefficients $\{a_k \geq 0;\ k = 0, 1, 2, \ldots\}$. If

(a) $a_0 > 0$;

(b) *there exists an integer* $j \geq 2$, $j \in \mathbb{N}$ *such that* $a_k = 0$ *at* $k \neq jn$, $n \in \mathbb{N}$, *then the module of function* $H(z)$ *reaches the maximum on the positive part of real axe.*

Consider the point $z_* \in \{z : |z| = r_*\}$ where $\max\{|H(z)|;\ |z| = r_*\}$ is reached on the circle with the zero center and with the radius $r_* \in (0,\rho]$.

Prove that the point z_* coincides with r_* provided $a_0 > 0$. Assume that $z_* \neq |z_*| = r_*$. We will show that there exists such an integer $j \geq 2$ for which the following formula:

$$H(z) = \sum_{k=0}^{\infty} a_{jk} z^{jk} \tag{4.2}$$

is valid in this case. Thus, we will come to the contradiction with the condition (b) of the theorem formulation.

Due to the non-negativity of a_k, we have the following inequality for any $z = r_* e^{i\varphi}$:

$$|H(z)| = \left| \sum_{k=0}^{\infty} a_k r_*^k e^{ik\varphi} \right| \leq \sum_{k=0}^{\infty} a_k r_*^k. \tag{4.3}$$

On the other hand, since the maximum is reached at the point z_* on the circle $\{z : |z| = r_*\}$, then the equality is realized in the obtained inequality. Because of the fact that $a_k \geq 0$, it is possible only if all the summands have the same argument, that is, $e^{in\varphi} = 1$ at all $k = 0,1,2,\ldots$ if $a_k \neq 0$. This follows from the condition $a_0 > 0$. Let $k_1, k_2, \ldots$ be integers corresponding to nonzero coefficients in the series of $H(z)$, that is, $1 < k_1 < k_2 < \cdots$ and $e^{ik_n\varphi} = 1$, $n \in \mathbb{N}$. Then there exist the integers $l_n \in \mathbb{N}$ such that $k_n\varphi = 2\pi l_n$, $l_n \leq k_n$, $n = 1,2,\ldots$. Therefore, we obtain $k_n = (k_1/l_1)l_n$ for all $n \in \mathbb{N}$. The rational number k_1/l_1 is represented as irreducible fraction $k_1/l_1 = j/l$ where $j \geq 2$ and $l \geq 1$ are relatively prime numbers. The latter is valid since at $j = 1$ we have $k_n = l_n$ and, consequently, $\varphi = 2\pi$ that is not true. Then $k_n = jl_n/l$, that is, l_n is divided by l, $l_n = lm_n$, $m_n \in \mathbb{N}$. Therefore, $k_n = jm_n$, that is, the function $H(z)$ has form (4.2).

5. The integral limit theorem

Main Theorem 5.1. *Let the probability distribution* $\{p_k;\ k \in \mathbb{N}_+\}$ *satisfy the collection of following conditions:*

(a) $p_0 > 0, p_0 \neq 1$;

(b) *the number* $j \geq 2$, $j \in \mathbb{N}$, *such that* $p_k = 0$ *at* $k \neq jl$, $l \in \mathbb{N}$, *is absent (this condition indicates that the unit is the proper (minimal) step of the lattice probability distribution of the random variable* ξ*);*

(c) *the fast decrease takes place*

$$\lim_{k\to\infty} (p_k)^{1/k} = \rho^{-1} < 1. \tag{5.1}$$

Then the limit formula

$$\lim_{m\to\infty} \Pr\{a \leq \zeta_m < b\} = (2\pi)^{-1/2} \int_a^b \exp\{-x^2/2\}dx \tag{5.2}$$

for the probability distribution of random variable

$$\zeta_m = \left(\frac{\nu_m \mathbf{E}\xi}{m} - 1\right)\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2} \tag{5.3}$$

is valid provided $m \to \infty$ uniformly in a and b $(-\infty \le a < b \le +\infty)$.

Our proof of the theorem will be affected in several steps.

(A) We find the expression of the characteristic function of the variable ζ_m. The characteristic function of the variable ν_m is expressed in the following form in terms of the integral representation (3.8):

$$\begin{aligned}\mathbf{E}e^{it\nu_m} &= \sum_{n=1}^{\infty} e^{itn} P_m(n) = \frac{1}{2\pi i}\oint_C \frac{1-F(z)}{z^m(1-z)} dz \sum_{n=1}^{\infty} e^{itn} \left[F(z)\right]^{n-1} \\ &= \frac{e^{it}}{2\pi i}\oint_C \frac{1-F(z)}{z^m(1-z)(1-e^{it}F(z))} dz.\end{aligned} \tag{5.4}$$

Then the formula

$$\mathbf{E}e^{it\zeta_m} = \exp\left[-it\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}\right]\mathbf{E}\exp\left[i\nu_m\left(\frac{t(\mathbf{E}\xi)^{3/2}}{\sqrt{m\mathbf{D}\xi}}\right)\right] = K_m \oint_C h(z)dz \tag{5.5}$$

is valid. Here,

$$\begin{aligned}K_m &= \frac{1}{(2\pi i)}\exp\left[-it\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}\left(1 - \frac{\mathbf{E}\xi}{m}\right)\right], \\ h(z) &= \frac{1-F(z)}{z^m(1-z)}\left(1 - F(z)\exp\left[it\frac{(\mathbf{E}\xi)^{3/2}}{\sqrt{m\mathbf{D}\xi}}\right]\right)^{-1}.\end{aligned} \tag{5.6}$$

(B) We calculate the limit value of the integral (5.5) provided $m \to \infty$. For this, we introduce the auxiliary circumferential circuit $C' = \{z : |z| = r'\}$ having the negative going around. The value r' meets the condition $1 < r' < \rho$. The latter is possible in view of the condition (b) of the theorem. For a given small real number $\varepsilon > 0$, we draw directed segments $L_+ = \langle (r^2 - \varepsilon^2)^{1/2} + i\varepsilon, (r'^2 - \varepsilon^2)^{1/2} + i\varepsilon\rangle$ and $L_- = \langle (r'^2 - \varepsilon^2)^{1/2} - i\varepsilon, (r^2 - \varepsilon^2)^{1/2} - i\varepsilon\rangle$. Here, they are characterized by ordered pairs representing their initial and finish points. Further, we cut out small arcs from circuits C and C' included between intersection points of these circuits with segments $L_\pm$. As a result, we get contours C_ε and C'_ε with the preserved direction of going around on them, corresponding to going around on contours C and C'.

We consider the closed circuit L that consists of the sequential passage of circuits C_ε, L_-, C'_ε, L_+. It is negatively oriented. Therefore,

$$\oint_L h(z)dz = -2\pi i \sum_{z_k(m)} \left(\operatorname{Res} h(z)\right)_{z=z_k(m)}, \tag{5.7}$$

where the summation is done on the set of poles $\{z_k(m)\}$ that are solutions of

$$F(z) = \exp\left[-it\frac{(\mathbf{E}\xi)^{3/2}}{\sqrt{m\mathbf{D}\xi}}\right], \tag{5.8}$$

depending on parameter m.

Since the function $F(z)$ is analytical in a small neighborhood of the point $z = 1$ and $(dF(z)/dz)_{z=1} = \mathbf{E}\xi \neq 0$ in view of $p_0 \neq 1$, then the inverse analytical function $y(z)$ exists when the variable z is being changed in this neighborhood. It is defined by $F(z) = y$ and the condition $y(1) = 1$ (it is clear that it is impossible to guarantee the uniqueness of the solution in general case without the condition pointed out. For instance, if $p_{2k+1} = 0$, $k = 0,1,2,\dots$, then there is a solution satisfying the condition $y(1) = -1$ together with the mentioned solution). Further, in view of the condition (b) of the theorem formulation and Theorem 4.1, we may state that, for any circle centered at zero, the function $F(z)$ reaches its module maximum on the positive part of real axe when it is varied on the circle. Therefore, the solution $z = 1$ of $F(z) = 1$ is unique. It is valid due to the inequality $|F(z)| \le F(|z|) \le F(1) = 1$. Here, the equality is reached only if $|z| = 1$. But if this condition holds, then $F(1) > F(z)$ provided $z \neq 1$.

Thus, in a sufficiently small neighborhood of the point $z = 1$, there exists the unique inverse function $y(z)$ of the function $F(z)$. Therefore, since the right-hand side of (5.8) tends to 1 provided $m \to \infty$, then, under sufficiently large value m, there exists the unique solution $z(m)$ of this equation. In this connection, formula (5.7) takes the form

$$\oint_L h(z)dz = -2\pi i\big(\operatorname{Res} h(z)\big)_{z=z(m)}. \tag{5.9}$$

The integral in the left-hand side of this equality is decomposed into the sum of four integrals

$$\oint_L h(z)dz = \oint_{C_\varepsilon} h(z)dz + \int_{L_+} h(z)dz + \oint_{C'_\varepsilon} h(z)dz + \int_{L_-} h(z)dz. \tag{5.10}$$

We go to the limit $\varepsilon \to +0$. Then integrals over segments L_+ and L_- compensate each other due to the integration in opposite directions. Furthermore, circuits C_ε and C'_ε turn into circuits C and C' under such a passage to the limit. Further, the integral over the circuit C' tends to zero when $m \to \infty$,

$$\lim_{m\to\infty} K_m \oint_{C'} h(z)dz = 0, \tag{5.11}$$

because $|z| = r' > 1$ on this circuit and, therefore, $|h(z)| < \operatorname{const} r'^{-m}$. Due to these facts, we find from (5.9)

$$\lim_{m\to\infty}\lim_{\varepsilon+0} K_m \oint_L h(z)dz = \lim_{m\to\infty} K_m \oint_C h(z)dz = -2\pi i \lim_{m\to\infty} K_m\big(\operatorname{Res} h(z)\big)_{z=z(m)}. \tag{5.12}$$

Now we proceed to the limit $m \to \infty$ in formula (5.5) taking into account the limiting relationship (5.12). As a result, we get

$$\lim_{m\to\infty} \mathbf{E}e^{it\zeta_m} = \lim_{m\to\infty} \left[F'(z(m))\right]^{-1} \exp\left[-it\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}\right] \cdot \frac{1-F(z(m))}{z(m)^m(1-z(m))}. \tag{5.13}$$

(Note that the obtained formula makes sense only if $p_0 < 1$, otherwise, $F(z) \equiv 1$ and $F' = 0$.)

(C) We calculate the limit in (5.13). For this, we represent $z(m)$ as the expansion in half-integer powers

$$z(m) = 1 + \frac{w_1}{\sqrt{m}} + \frac{w_2}{m} + o(m^{-1}). \tag{5.14}$$

Correspondingly, we represent the generating function $F(z)$ with the same accuracy in the following form taking into account that $F(1) = 1$, $F'(1) = \mathbf{E}\xi$, and $F''(1) = \mathbf{D}\xi - \mathbf{E}\xi + (\mathbf{E}\xi)^2$,

$$F(z(m)) = 1 + \mathbf{E}\xi\left(\frac{w_1}{\sqrt{m}} + \frac{w_2}{m}\right) + \frac{w_1^2}{2m}\left(\mathbf{D}\xi - \mathbf{E}\xi + (\mathbf{E}\xi)^2\right) + o(m^{-1}). \tag{5.15}$$

Using (5.8), we get

$$F(z(m)) = 1 - it\frac{(\mathbf{E}\xi)^{3/2}}{\sqrt{m\mathbf{D}\xi}} - t^2\frac{(\mathbf{E}\xi)^3}{2m\mathbf{D}\xi} + o(m^{-1}). \tag{5.16}$$

Equating coefficients at powers $m^{-1/2}$ and m^{-1} in (5.15) and (5.16), we find the expression for w_1,

$$w_1 = -it\left(\frac{\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2} \tag{5.17}$$

and the equation for w_2,

$$w_2\mathbf{E}\xi + \frac{w_1^2}{2}\left(\mathbf{D}\xi - \mathbf{E}\xi + (\mathbf{E}\xi)^2\right) = -\frac{t^2(\mathbf{E}\xi)^3}{2\mathbf{D}\xi} \tag{5.18}$$

from which it follows

$$w_2 = -\frac{t^2(\mathbf{E}\xi)^2}{2\mathbf{D}\xi} - \frac{w_1^2}{2\mathbf{E}\xi}\left(\mathbf{D}\xi - \mathbf{E}\xi + (\mathbf{E}\xi)^2\right) = \frac{t^2}{2}\left(1 - \frac{\mathbf{E}\xi}{\mathbf{D}\xi}\right). \tag{5.19}$$

The substitution of these expressions into (5.14) gives us the formula for $z(m)$,

$$z(m) = 1 - it\left(\frac{\mathbf{E}\xi}{m\mathbf{D}\xi}\right)^{1/2} + \frac{t^2}{2m\mathbf{D}\xi}(\mathbf{D}\xi - \mathbf{E}\xi) + o(m^{-1}). \tag{5.20}$$

Find an asymptotic formula for $z(m)^{-m}$. Since

$$\begin{aligned}\ln z(m)^{-m} &= m\left(it\left(\frac{\mathbf{E}\xi}{m\mathbf{D}\xi}\right)^{1/2} - \frac{t^2}{2m\mathbf{D}\xi}(\mathbf{D}\xi - \mathbf{E}\xi) - \frac{t^2\mathbf{E}\xi}{2m\mathbf{D}\xi}\right) + o(m^{-1}) \\ &= it\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2} - \frac{t^2}{2} + o(m^{-1}),\end{aligned} \tag{5.21}$$

then

$$z(m)^{-m} = \exp\left[it\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2} - \frac{t^2}{2} + o(m^{-1})\right]. \tag{5.22}$$

Passing to the limit in the expression $(1 - F(z(m)))/(1 - z(m))$, we can change it by $F'(z(m))$ in accordance with L'Hospital rule. Then the direct substitution of this expression and expression (5.22) into (5.13) gives the limit of the characteristic function

$$\lim_{m\to\infty} \mathbf{E}e^{it\zeta_m} = \exp\{-t^2/2\}. \tag{5.23}$$

Using the theorem of the connection between the characteristic functions sequence convergence and the convergence of corresponding sequence of probability distributions (Gnedenko [3]), we obtain the theorem statement.

6. The Wald representation

Consider the random variable

$$\zeta_m = \left(\frac{\nu_m\mathbf{E}\xi}{m} - 1\right)\left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}. \tag{6.1}$$

Denote

$$\rho_m = \frac{\nu_m\mathbf{E}\xi}{m} = 1 + \zeta_m\left(\frac{\mathbf{D}\xi}{m\mathbf{E}\xi}\right)^{1/2}. \tag{6.2}$$

Since the sequence $\{\zeta_m;\ m = 1,2,\dots\}$ is bounded with probability one when $m \to \infty$, then

$$\rho_m^{\pm 1/2} = 1 \pm \frac{1}{2}\zeta_m\left(\frac{\mathbf{D}\xi}{m\mathbf{E}\xi}\right)^{1/2} + O(m^{-1}). \tag{6.3}$$

Consequently,

$$\rho_m^{1/2} - \rho_m^{-1/2} = \zeta_m\left(\frac{\mathbf{D}\xi}{m\mathbf{E}\xi}\right)^{1/2} + O(m^{-1}) \tag{6.4}$$

and, therefore, it follows that

$$\zeta_m = \left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}(\rho_m^{1/2} - \rho_m^{-1/2}) + O(m^{-1/2}) \equiv g(\rho_m) + O(m^{-1/2}), \tag{6.5}$$

where the function

$$g(x) = \left(\frac{m\mathbf{E}\xi}{\mathbf{D}\xi}\right)^{1/2}(x^{1/2} - x^{-1/2}) \tag{6.6}$$

has the inverse one.

From the formula (5.2) it follows that the limit distribution density $f_\zeta(x)$ of the random variable ζ_m equals to

$$f_\zeta(x) = (2\pi)^{-1/2}\exp\left(-x^2/2\right). \tag{6.7}$$

Then we have

$$f_\rho(x) = g'(x) f_\zeta(g(x)). \tag{6.8}$$

It is valid by the transformation of the probability distribution density of the continuous random variable ζ to the probability distribution density of the random variable being the function $\rho = g(\zeta)$. The density $f_\rho(x)$ approximates asymptotically the probability distribution density of the variable ρ_m. We have the following formula for it:

$$f_\rho(x) = \left(\frac{m\mathbf{E}\xi}{8\pi x\mathbf{D}\xi}\right)^{1/2}(1 + x^{-1})\exp\left(-\frac{m\mathbf{E}\xi}{2\mathbf{D}\xi}(x^{1/2} - x^{-1/2})^2\right). \tag{6.9}$$

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ON A CERTAIN FUNCTIONAL EQUATION IN THE ALGEBRA OF POLYNOMIALS WITH COMPLEX COEFFICIENTS

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Many analytical problems can be reduced to determining the number of roots of a polynomial in a given disc. In turn, the latter problem admits further reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane. However, this procedure requires complicated coefficient transformations. In the present paper we suggest a *direct* method to evaluate the number of roots of a polynomial with complex coefficients in a disc, based on studying a certain equation in the algebra of polynomials. An application for computing the rotation of plane polynomial vector fields is also given.

1. Functional equations: basic properties of solutions

Let

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_n \neq 0, \tag{1.1}$$

$$F(z) = b_0 + b_1 z + \cdots + b_n z^n + b_{n+1} z^{n+1}, \quad b_0 \neq 0,\ b_{n+1} \neq 0, \tag{1.2}$$

be polynomials with complex coefficients $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_{n+1}$ of degree n and $n+1$, respectively. Assume the polynomials f and F to satisfy the functional equation

$$(a+bz)f(z) + (c+dz)f^*(z) = F(z), \tag{1.3}$$

where a, b, c, d are certain complex numbers and the polynomial f^* is defined by

$$f^*(z) = \bar{a}_0 z^n + \bar{a}_1 z^{n-1} + \cdots + \bar{a}_n = z^n \overline{f\left(\frac{1}{\bar{z}}\right)}. \tag{1.4}$$

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Consider along with (1.3) the following functional equation:

$$g(z)\cdot f(z)=(\bar{a}z+\bar{b})F(z)-(c+dz)F^*(z), \tag{1.5}$$

where

$$g(z)=\alpha z^2+\beta z+\bar{\alpha},\quad \alpha=\bar{a}b-\bar{c}d,\ \beta=|a|^2+|b|^2-|c|^2-|d|^2, \tag{1.6}$$

$$F^*(z)=\bar{b}_0z^{n+1}+\bar{b}_1z^n+\cdots+\bar{b}_{n+1}. \tag{1.7}$$

Lemma 1.1. *If the polynomials f and F satisfy the functional equation (1.3), then they also satisfy the functional equation (1.5).*

Conversely, if at least one of the numbers α, β is different from zero and f, F satisfy (1.5), then they satisfy (1.3) as well.

Proof. Let polynomials (1.1) and (1.2) satisfy (1.3). By the definition of f^* and F^*, one has

$$(\bar{a}z+\bar{b})f^*(z)+(\bar{c}z+\bar{d})f(z)=F^*(z). \tag{1.8}$$

Multiplying (1.3) (resp., (1.8)) by $(\bar{a}z+\bar{b})$ (resp., by $(c+dz)$) and taking the difference of the obtained expressions, one obtains (1.5), where the coefficients α and β are defined by (1.6). Thus the first implication is established.

Conversely, assume that f and F satisfy (1.5), and at least one of α, β is different from zero. Since the coefficient β of g is real (cf. [4]), one has

$$z^2\overline{g\left(\frac{1}{\bar{z}}\right)}=g(z). \tag{1.9}$$

Therefore, it follows from (1.5) that

$$g(z)f^*(z)=(a+bz)F^*(z)-(\bar{c}z+\bar{d})F(z). \tag{1.10}$$

Multiplying (1.5) (resp., (1.10)) by $(a+bz)$ (resp., by $(c+dz)$) and summing up the obtained expressions, one arrives at the following equality:

$$g(z)\big[(a+bz)f(z)+(c+dz)f^*(z)\big]=g(z)F(z). \tag{1.11}$$

Since $g(z)\not\equiv 0$ and the algebra of polynomials does not contain zero divisors, it follows that f and F satisfy (1.3).

The lemma is completely proved. □

Assume that the polynomials f and F satisfy (1.3). It follows from (1.1)–(1.3) that (1.3) is equivalent to the following system:

$$aa_k+c\bar{a}_{n-k}+ba_{k-1}+d\bar{a}_{n-k+1}=b_k,\quad k=0,1,\dots,n+1, \tag{1.12}$$

where we put $a_k=0$ for $k<0$ and $k>n$.

Similarly, (1.5) is equivalent to the following system:

$$\bar{\alpha}a_k + \beta a_{k-1} + \alpha a_{k-2} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 0,1,\dots,n+2, \tag{1.13}$$

where $a_k = b_k = 0$ for $k < 0$ and $a_k = b_{k+1} = 0$ for $k > n$.

Thus, under the assumption that a, b, c, and d satisfy the condition $|\alpha| + |\beta| > 0$, system (1.3) is equivalent to (1.12) as well as to (1.13).

Below we will list some properties of solutions to (1.3).

(1) (a) The coefficients a, b, c, d along with the polynomial f determine the polynomial F uniquely.
(b) If the polynomials f and F are defined by (1.1) and (1.2) and satisfy (1.3), then the coefficients a, b, c, d satisfy the conditions

$$|a| + |c| > 0, \qquad |b| + |d| > 0. \tag{1.14}$$

(c) If a collection (a,b,c,d,f,F) of numbers a, b, c, d, and polynomials f, F satisfy (1.3), then so is the collection $(\lambda a, \lambda b, \bar{\lambda}c, \bar{\lambda}d, f/\lambda, F)$ for any complex number $\lambda \neq 0$.

(2) Given a polynomial F and numbers a, b, c, and d, satisfying $|\alpha| + |\beta| > 0$, there exists a unique f satisfying (1.3). Indeed, if $\alpha \neq 0$, then the first $n+1$ equations of system (1.13) completely determine the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f. If, however, $\alpha = 0$ and $\beta \neq 0$, then all the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f are completely determined by $n+1$ equations of system (1.13) starting with the second one.

(3) It follows from (1.5) that the roots of $g(z)$ turn out to be the roots of $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$. Also, if $\alpha \neq 0$ and $\beta^2 \neq 4|\alpha|^2$, then $z_0, z_1 = 1/\bar{z}_0$, where

$$z_0 = \frac{-\beta + \sqrt{\beta^2 - 4|\alpha|^2}}{2\alpha}, \tag{1.15}$$

are the roots of $g(z)$. Therefore, F and the coefficients a, b, c, d are connected by the following relations:

$$(\bar{a}z_j + \bar{b})F(z_j) - (c + dz_j)F^*(z_j) = 0, \quad j = 0,1. \tag{1.16}$$

If $\alpha \neq 0$ and $\beta^2 = 4|\alpha|^2$, then $z_1 = z_0$ is a multiple root of $g(z)$, and, therefore,

$$\begin{gathered}(\bar{a}z_0 + \bar{b})F(z_0) - (c + dz_0)F^*(z_0) = 0, \\ (\bar{a}z_0 + \bar{b})F'(z_0) + \bar{a}F(z_0) - (c + dz_0)F^{*'}(z_0) - dF^*(z_0) = 0.\end{gathered} \tag{1.17}$$

Finally, if $\alpha = 0$ and $\beta \neq 0$, then the linear function $g(z) = \beta z$ has the only root $z_0 = 0$. Hence it follows from (1.5) (see also (1.10)) that

$$\begin{gathered}\bar{b}F(0) - cF^*(0) = 0, \\ \bar{d}F(0) - aF^*(0) = 0.\end{gathered} \tag{1.18}$$

2. Functional equations: solubility conditions

Given the polynomial (1.2), consider the solubility problem for the functional equation (1.3) with respect to unknown coefficients a, b, c, d and a polynomial f. To treat the above problem, we will use the necessary conditions for the solubility of (1.3) given by (3.1)–(3.8) (depending on whether the root z_0 of $g(z)$ satisfies $0 < |z_0| < 1$, $|z_0| = 1$, or $z_0 = 0$).

Assume z_0 and z_1 to be given and consider system (1.16) with respect to unknown a, b, c, d. We will try to find a solution to (1.16) in such a way that (1.3) will have a solution with respect to f. Also, given z_0, we will follow the same way regarding system (1.17).

To describe the solubility conditions for the functional equation (1.3), it is convenient to introduce the notion of a regular point.

A point z is called *regular* with respect to the polynomial F if the following conditions are satisfied:

$$\begin{gathered} F(z)\cdot F^*(z) \neq 0, \quad \left|F(z)\right| \neq \left|F^*(z)\right|, \quad \text{for } |z| \neq 1, \\ (n+1)\left|F(z)\right|^2 \neq 2\Re\left[\overline{F(z)}F'(z)z\right], \quad \text{for } |z| = 1. \end{gathered} \tag{2.1}$$

Observe that the notion of a regular point is introduced with respect to the unit circle. It follows immediately from the definition of a regular point that if z_0 is regular, then so is $z_1 = 1/\bar{z}_0$ and vice versa.

According to the definition of polynomial F^*, the rational function

$$A(z) = \frac{F^*(z)}{F(z)} \tag{2.2}$$

satisfies the identity

$$\overline{A(z)}\cdot A\left(\frac{1}{\bar{z}}\right) \equiv 1. \tag{2.3}$$

In addition, $|A(z)| \neq 1$ for all regular points $z, |z| \neq 1$.

Assume that z_0 is a regular point of F, $A_0 = A(z_0)$, $A_1 = 1/\bar{A}_0$, and σ_0, σ_1 are arbitrary complex numbers. Consider the linear system

$$\begin{gathered} \bar{a}z_0 + \bar{b} = \sigma_0 A_0, \qquad \bar{a} + \bar{b}\bar{z}_0 = \sigma_1 A_1, \\ c + dz_0 = \sigma_0, \qquad c\bar{z}_0 + d = \sigma_1, \end{gathered} \tag{2.4}$$

with unknown a, b, c, and d. It should be pointed out that any solution to system (2.4) is also a solution to (1.16) for $z_0 \neq 0$, as well as a solution to (1.18) for $z_0 = 0$.

For $|z_0| < 1$ system, (2.4) has the unique solution

$$\begin{gathered} \bar{a}\left(1 - |z_0|^2\right) = \sigma_1 A_1 - \bar{z}_0\sigma_0 A_0, \qquad \bar{b}\left(1 - |z_0|^2\right) = \sigma_0 A_0 - \bar{z}_0\sigma_1 A_1, \\ c\left(1 - |z_0|^2\right) = \sigma_0 - z_0\sigma_1, \qquad d\left(1 - |z_0|^2\right) = \sigma_1 - \bar{z}_0\sigma_0. \end{gathered} \tag{2.5}$$

With the above a, b, c, and d on hands, the coefficients α, β from formula (1.6) satisfy the equalities

$$\alpha\left(1-|z_0|^2\right)=\bar{z}_0\left(|A_1|^2-1\right)\left(|\sigma_0 A_0|^2-|\sigma_1|^2\right), \tag{2.6}$$

$$\beta\left(1-|z_0|^2\right)=-\left(|A_1|^2-1\right)\left(|\sigma_0 A_0|^2-|\sigma_1|^2\right). \tag{2.7}$$

Theorem 2.1. *Let z_0 with $|z_0|<1$ be a regular point of the polynomial F, and assume the numbers σ_0 and σ_1 to satisfy the condition*

$$|\sigma_0 A_0| \neq |\sigma_1|. \tag{2.8}$$

Let, further, a, b, c, and d be defined by (2.5). Then the functional equation (1.3) admits a solution $f(z)$.

Proof. To begin with, consider the case $z_0 \neq 0$. It follows from (2.8) and (2.6) that $\alpha \neq 0$. Formulae (2.4) and (2.3) provide that z_0 and $z_1 = 1/\bar{z}_0$ are the roots of the polynomial $g(z)$ (1.6):

$$\begin{gathered} g(z_0)=(a+bz_0)(\bar{a}z_0+\bar{b})-(c+dz_0)(\bar{c}z_0+\bar{d})=\bar{\sigma}_1\bar{A}_1\sigma_0A_0-\sigma_0\bar{\sigma}_1=\bar{\sigma}_1\sigma_0-\sigma_0\bar{\sigma}_1=0, \\ g(z_1)=z_1^2\overline{g(z_0)}=0. \end{gathered} \tag{2.9}$$

On the other hand, the numbers z_0 and z_1 are the roots of the polynomial

$$G(z)=(\bar{a}z+\bar{b})F(z)-(c+dz)F^*(z). \tag{2.10}$$

Therefore, by the Bezout theorem, the rational function

$$f(z)=\frac{G(z)}{g(z)} \tag{2.11}$$

is, in fact, a polynomial satisfying the functional equation (1.5). Now the statement of the theorem in the considered case follows from Lemma 1.1.

Assume now $z_0=0$. The condition (2.8) and equalities (2.6), (2.7) yield $\alpha=0$ and $\beta \neq 0$, that is, $g(z)=\beta z$ is a linear function. In addition, $z=0$ is a root of the polynomial $G(z)$. Therefore, $f(z)=G(z)/g(z)$ is a polynomial satisfying (1.5), and again the statement of the theorem in the considered case follows from Lemma 1.1.

Theorem 2.1 is completely proved. □

Assume now that a regular point z_0 of the polynomial F belongs to the unit circle $|z|=1$. Given numbers c and d, consider system (1.18) with unknown a, b. By solving system (1.18) one obtains

$$\bar{a}F^2=\Delta\cdot c+(FF^*+z\Delta)d, \qquad \bar{b}F^2=(FF^*-z\Delta)c-z^2\Delta d, \tag{2.12}$$

where

$$\begin{gathered}\Delta = z^n\Delta_0, \quad \Delta_0 = (n+1)|F|^2 - 2\Re[\bar{F}F'z], \\ F = F(z), \quad F' = F'(z), \quad F^* = F^*(z), \quad z = z_0.\end{gathered} \tag{2.13}$$

Take the coefficients a, b, c, and d satisfying (2.12) and define the polynomial $g(z)$ by means of formula (1.6). Formula (2.12) provides the following relations for α and β:

$$z\alpha|F|^4 = \Delta_0\big[|F|^2(|c|^2 - |d|^2) - \Delta_0|c+zd|^2\big], \tag{2.14}$$

$$\beta|F|^4 = 2\Delta_0\big[\Delta_0|c+zd|^2 - |F|^2(|c|^2 - |d|^2)\big]. \tag{2.15}$$

Theorem 2.2. *Let $z_0, |z_0| = 1$ be a regular point of the polynomial F and let the numbers c, d satisfy the condition:*

$$\Delta_0\big|c+z_0d\big|^2 \neq \big|F(z_0)\big|^2(|c|^2 - |d|^2). \tag{2.16}$$

If the coefficients a, b, c, and d satisfy relation (2.12), then the functional equation (1.3) has the unique solution (a,b,c,d,f).

Proof. It follows from (1.6) and (2.15)–(3.1) that $\beta = -2\alpha z_0 \neq 0$. Since the coefficient β is real and $z_0\bar{z}_0 = 1$, one obtains the equality $\bar{\alpha} = \alpha z_0^2$, that is, $g(z) \equiv \alpha(z - z_0)^2$. At the same time, z_0 is a multiple root to the polynomial $G(z) = (\bar{a}z + \bar{b})F(z) - (c + dz)F^*(z)$. Therefore, the rational function $f(z) = G(z)/g(z)$ is, in fact, a polynomial satisfying the functional equation (1.5). To complete the proof of Theorem 2.1 it remains to apply Lemma 1.1. The theorem follows. □

Combining Theorems 2.1 and 2.2 with property (2) of solutions to (1.3) one can effectively compute the numbers a, b, c, d and the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f. Indeed, assume, for instance, that $z = 0$ is a regular point of the polynomial F. Then the regularity condition for the point $z = 0$ along with condition (2.8) take the form

$$b_0 \cdot b_{n+1} \neq 0, \qquad \big|b_0\big| \neq \big|b_{n+1}\big|, \qquad \big|cb_{n+1}\big| \neq \big|db_0\big|; \tag{2.17}$$

also, the equalities (2.6) and (2.7) take the form

$$\bar{a}b_{n+1} = d\bar{b}_0, \qquad \bar{b}b_0 = c\bar{b}_{n+1}. \tag{2.18}$$

In this case the leading coefficient α of the polynomial $g(z)$, that is determined by a, b, c, d, is equal to zero. Hence, in order to determine unknown coefficients $a_0, a_1, \dots, a_n$ from system (1.13), one has

$$\begin{gathered}0 = \bar{b}b_0 - c\bar{b}_{n+1}, \\ \beta a_0 = \bar{a}b_0 - d\bar{b}_{n+1} + \bar{b}b_1 - c\bar{b}_n, \\ \beta a_{k-1} = \bar{a}b_{k-1} - d\bar{b}_{n-k+2} + \bar{b}b_k - c\bar{b}_{n-k+1}, \quad k = 2, \dots, n, \\ \beta a_n = \bar{a}b_n - d\bar{b}_1 + \bar{b}b_{n+1} - c\bar{b}_0, \\ 0 = \bar{a}b_{n+1} - d\bar{b}_0.\end{gathered} \tag{2.19}$$

Observe that the first and the last equations in (2.19) coincide with (2.18). According to (2.17) and (2.18), one obtains the following relation for the coefficient β of the polynomial $g(z)$:

$$\beta = \left(|b_{n+1}|^2 - |b_0|^2\right)\left(\frac{|c|^2}{|b_0|^2} - \frac{|d|^2}{|b_{n+1}|^2}\right) \neq 0. \tag{2.20}$$

Thus the coefficients $a_0, a_1, \dots, a_n$ of the polynomial f can be uniquely determined from system (2.19).

It follows from Theorems 2.1 and 2.2 that the existence of a regular point for the polynomial F is a sufficient condition for the existence of a solution to the functional equation (1.3).

It turns out that the existence of a regular point for the polynomial F is intimately connected to the linear (in)dependence of the polynomials F and F^* in the complex linear space of (complex) polynomials. To be more precise, *there exists a regular point for F if and only if F and F^* are linearly independent.* This statement is a direct consequence of the following lemma.

Lemma 2.3. *The following conditions are equivalent:*

(a) *F and F^* are linearly independent in the complex linear space of (complex) polynomials;*

(b) *the identity $|F(z)| \equiv |F^*(z)|$ is satisfied;*

(c) *the identity*

$$2\operatorname{Re}\left[\overline{F(w)}F'(w)w\right] \equiv (n+1)|F(w)|^2 \quad \forall |w| = 1 \tag{2.21}$$

is satisfied.

Proof. Assume (a) is satisfied: $F^* = C \cdot F$ for some nonzero complex number C. Then, according to the definition of the polynomial F^*, one has the following equality for the coefficients $b_0, b_{n+1} : \bar{b}_0 = Cb_{n+1}$, $\bar{b}_{n+1} = Cb_0$, from which it follows that $|C| = 1$, and therefore, $|F^*(z)| \equiv |F(z)|$.

Thus (a) implies (b).

Assume, further, $|F(z)| \equiv |F^*(z)|$. Applying to this identity the change of coordinates $z = rw$, $r \geq 0$, $|w| = 1$ and using the definition of F^*, one obtains

$$r^{2n+2}F(r^{-1}w)\overline{F(r^{-1}w)} \equiv F(rw)\overline{F(rw)}. \tag{2.22}$$

Differentiating the last identity with respect to the real argument r at the point $r = 1$ we obtain (c).

Thus (b) implies (c).

Finally, assume that (c) is satisfied and show that F and F^* are linearly dependent. Consider the function $\overline{F(w)}/F(w)$ on the unit circle $|w| = 1$, where $F(w) \neq 0$, and show that this function has a continuous extension over the unit circle. Assume that F vanishes at some point w_0 belonging to the unit circle. Then we have the following representation: $F(z) = (z - w_0)^k F_1(z)$, where $k \geq 1$ is an integer, $F_1(w_0) \neq 0$.

The conditions

$$\lim_{w\to w_0}\frac{\bar{w}-\bar{w}_0}{w-w_0}=-\frac{1}{w_0^2},\qquad |w|=1, \tag{2.23}$$

yield

$$\lim_{w\to w_0}\frac{\overline{F(w)}}{F(w)}=(-1)^k w_0^{-2k}\frac{\overline{F_1(w_0)}}{F_1(w_0)},\qquad |w|=1, \tag{2.24}$$

providing the continuity of the function $\overline{F(w)}/F(w)$.

Differentiating the function $w^{n+1}\overline{F(w)}/F(w)$, $w=\exp(it)$ with respect to t at the points where $F(w)\neq 0$ and using condition (c), one obtains

$$\frac{d}{dt}\frac{w^{n+1}\overline{F(w)}}{F(w)}=iw^{n+1}\left\{\frac{(n+1)\overline{F(w)}-\overline{F'(w)w}}{F(w)}-\frac{\overline{F(w)}F'(w)w}{F^2(w)}\right\}=0. \tag{2.25}$$

The above equality along with the continuity of the function $w^{n+1}\overline{F(w)}/F(w)$ on the unit circle yield that the latter function is, in fact, constant, that is $w^{n+1}\overline{F(w)}=C\cdot F(w)$ or, equivalently, $F^*(w)=C\cdot F(w)$, $|w|=1$. Therefore, the coefficients of the polynomials $C\cdot F$ and F^* coincide. The linear dependence of the polynomials F and F^* is established and the proof of Lemma 2.3 is complete. □

3. An algorithm for computing the number of roots in the unit circle

In what follows we will be interested in the case when the coefficients a, b, c, d satisfy the following additional condition:

$$|a+bw|\geq |c+dw|,\qquad |w|=1. \tag{3.1}$$

Lemma 3.1. *Linear functions $a+bz$, $c+dz$ satisfy condition (3.1) if and only if the numbers $\alpha=\bar{a}b-\bar{c}d$, $\beta=|a|^2+|b|^2-|c|^2-|d|^2$ satisfy the inequality*

$$2|\alpha|\leq\beta. \tag{3.2}$$

Proof. The equality

$$|a+bz|^2-|c+dz|^2=|a|^2+2\Re(\bar{a}bz)+|b|^2|z|^2-|c|^2-2\Re(\bar{c}dz)-|d|^2|z|^2 \tag{3.3}$$

yields, for $z=e^{it}$, $t\in[0,2\pi]$,

$$|a+bz|^2-|c+dz|^2=\beta+2\Re[\alpha e^{it}]. \tag{3.4}$$

Combining this with the equality

$$\min_t \Re[\alpha e^{it}]=-|\alpha|, \tag{3.5}$$

one obtains the equivalence of conditions (3.1) and (3.2). The lemma is proved. □

Observe that Lemma 3.1 allows one to verify effectively the validity of condition (3.1) for the coefficients a, b, c, d determined by regular points of the polynomial F.

LEMMA 3.2. *Let $z_0, |z_0| \leq 1$, be a regular point of the polynomial F. Assume that the numbers a, b, c, d are determined by equalities (2.5) with*

$$(|A_0| - 1)(|\sigma_0 A_0| - |\sigma_1|) > 0, \quad |z_0| < 1, \tag{3.6}$$

or that they satisfy (2.12) with

$$\Delta_0 \Big[\Delta_0 |c + z_0 d|^2 - |F(z_0)|^2 (|c|^2 - |d|^2) \Big] > 0, \quad |z_0| = 1. \tag{3.7}$$

Then a, b, c, d satisfy (3.1).

The statement following below provides an important property of solution (a,b,c,d,f) to the functional equation (1.3).

THEOREM 3.3. *Assume that the polynomial F does not contain roots on the unit circle $|z| = 1$. Suppose, further, that the coefficients a, b, c, d satisfy condition (3.1) and*

$$|ad - bc| + \beta > 0, \quad \beta = |a|^2 + |b|^2 - |c|^2 - |d|^2. \tag{3.8}$$

Then the polynomial f as well as any polynomial of the parameterized family

$$G_\lambda(z) = (a + bz) f(z) + \lambda (c + dz) f^*(z), \quad 0 \leq \lambda \leq 1, \tag{3.9}$$

does not contain roots on the unit circle $|z| = 1$.

Proof. Arguing indirectly, one obtains the existence of numbers $w, |w| = 1$, and $\lambda \in [0,1]$ such that

$$G_\lambda(w) = (a + bw) f(w) + \lambda (c + dw) f^*(w) = 0. \tag{3.10}$$

Since $\bar{w}w = 1$, one has

$$f^*(w) = w^n \overline{f\left(\frac{1}{\bar{w}}\right)} = w^n \overline{f(w)}, \tag{3.11}$$

from which it follows that

$$(a + bw) f(w) + \lambda (c + dw) w^n \overline{f(w)} = 0. \tag{3.12}$$

By condition,

$$G_1(w) = (a + bw) f(w) + (c + dw) w^n \overline{f(w)} = F(w) \neq 0, \tag{3.13}$$

therefore, $f(w) \neq 0$ and $0 \leq \lambda < 1$. Now, using the equality (3.12), we obtain

$$|a + bw| = \lambda |c + dw|, \tag{3.14}$$

from which it follows (see Lemma 3.1) that $|a+bw|=0$, $|c+dw|=0$. The latter equalities yield $|a|=|b|$, $|c|=|d|$, $ad-bc=0$ that contradicts condition (3.8) and the result follows. □

Denote by $\kappa(F)$ the number of roots of the polynomial F (counted according to their multiplicity) belonging to the open unit disc $|z|<1$. As noted, the computing $\kappa(F)$ maybe a reduction to the generalized Rauss-Hurwitz problem of determining the number of roots of a polynomial in a semiplane (see, for instance, [1–3]). The above results allow us to construct an iterative process for computing $\kappa(F)$, namely, the following theorem.

Theorem 3.4. *Assume that the polynomial F does not have roots on the unit circle $|z|=1$. If the polynomials F and F^* are linearly dependent, then*

$$\kappa(F)=\frac{n+1}{2}, \tag{3.15}$$

otherwise

$$\kappa(F)=\kappa(G_0), \tag{3.16}$$

where $G_0(z)=(a+bz)f(z)$ and the collection (a,b,c,d,f) satisfying (3.1), (3.8) is a solution to the functional equation (1.3).

Proof. Assume the polynomials F and F^* to be linearly dependent, that is $F^*(z)\equiv CF(z)$. Then the following presentation takes place:

$$F(z)=b_{n+1}(z-z_1)^{\alpha_1}\cdots(z-z_m)^{\alpha_m}\left(z-\frac{1}{\bar{z}_1}\right)^{\alpha_1}\cdots\left(z-\frac{1}{\bar{z}_m}\right)^{\alpha_m}, \tag{3.17}$$

where

$$|z_s|<1,\quad s=1,\dots,m,\qquad 2(\alpha_1+\cdots+\alpha_m)=n+1. \tag{3.18}$$

From this it follows that the number $n+1$ is even and

$$\kappa(F)=\alpha_1+\cdots+\alpha_m=\frac{(n+1)}{2}. \tag{3.19}$$

Assume now the polynomials F and F^* to be linearly independent. Then, by Lemma 2.3, F admits a regular point $z_0, |z_0|\le 1$. Therefore, by Lemmas 1.1, 2.3, and 3.2, there exists a collection (a,b,c,d,f) satisfying conditions (3.1), (3.8) and being a solution to the functional equation (1.3). Combining Theorem 3.3 and the Rouche theorem one obtains $\kappa(F)=\kappa(G_0)$, where $G_0(z)=(a+bz)f(z)$.

The proof of Theorem 3.4 is complete. □

It is easy to see that the number $\kappa(G_0)$ satisfies the equality

$$\kappa(G_0)=\kappa(f)+\varepsilon, \tag{3.20}$$

where $\varepsilon=1$ for $|a|<|b|$ and $\varepsilon=0$, otherwise. A simple argument shows that if the numbers a, b, c, d are determined by a regular point z_0, then ε can be evaluated according to the formulae

$$\varepsilon = \frac{1+\operatorname{sign}\left(\left|F^*(z_0)\right| - \left|F(z_0)\right|\right)}{2}, \quad \left|z_0\right| < 1, \qquad \varepsilon = \frac{1-\operatorname{sign}(\Delta_0)}{2}, \quad \left|z_0\right| = 1. \tag{3.21}$$

Thus, under the assumptions of Theorems 2.1 and 2.2, one has

$$\kappa(F) = \frac{1+\operatorname{sign}\left(\left|F^*(z_0)\right| - \left|F(z_0)\right|\right)}{2} + \kappa(f), \quad \left|z_0\right| < 1, \tag{3.22}$$

$$\kappa(F) = \frac{1-\operatorname{sign}(\Delta_0)}{2} + \kappa(f), \quad \left|z_0\right| = 1. \tag{3.23}$$

Formulae (3.22) and (3.23) give rise to a recurrent procedure for the computation of κ. Indeed, they allow one to compute $\kappa(F)$, where F is a polynomial of degree $n+1$, based on $\kappa(f)$, where f is a polynomial of degree n and its coefficients are completely determined by coefficients of F.

Observe that if $z_0 = 0$, then formula (3.22) takes the form

$$\kappa(F) = \frac{1+\operatorname{sign}\left(\left|b_{n+1}\right| - \left|b_0\right|\right)}{2} + \kappa(f). \tag{3.24}$$

Here the coefficients of f can be determined from system (2.19).

4. Criterion for the absence of roots on the unit circle

Given the coefficients of the polynomials F and F^*, one can construct the following $(2n+2)\times(2n+2)$ matrix:

$$M_F = \begin{pmatrix} b_0 & b_1 & \cdot & b_{n+1} & \cdot & 0 \\ \bar{b}_{n+1} & \bar{b}_n & \cdot & \bar{b}_0 & \cdot & 0 \\ 0 & b_0 & \cdot & b_n & \cdot & 0 \\ 0 & \bar{b}_{n+1} & \cdot & \bar{b}_0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdot & b_1 & \cdot & b_{n+1} \\ 0 & 0 & \cdot & \bar{b}_{n+1} & \cdot & \bar{b}_0 \end{pmatrix}. \tag{4.1}$$

The matrix M_F coincides with the Sylvester matrix of F and F^* (up to a permutation of its lines). Therefore, $\det(M_F)$ coincides (up to a sign) with the resultant $R(F,F^*)$ of the polynomials F and F^*. Let z_s, $s = 1,\dots,n+1$ be all the roots of F (counted according to their multiplicities). By condition, $F(0) = b_0 \neq 0$, therefore, all the roots are different from zero. By definition of the polynomial F^*, the numbers $\bar{z}_s^{-1}$, $s = 1,\dots,n+1$, are roots of F^*. Hence (cf. [5]), $R(F,F^*)$ can be represented as follows:

$$R(F,F^*) = \bar{b}_0^{n+1} b_{n+1}^{n+1} \prod_{s,t}\left(z_s - \frac{1}{\bar{z}_t}\right). \tag{4.2}$$

Formula (4.2) gives rise to the following criteria for the polynomial F to have no roots on the unit circle.

THEOREM 4.1. *Let* $\det(M_F)$ *be different from zero. Then* F *does not have roots on the unit circle* $|z| = 1$.

Assume that a collection $(a,b,1,1,f)$ of the numbers $a,b,c=d=1$ and a polynomial f satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b - 1 = 0$. Using a, b define the following square matrix of order $2n+2$:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ -\bar{b} & 1 & -\bar{a} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 1 & -a & 1 & -b & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & -\bar{b} & 1 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -a & \cdot & 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdot & -\bar{b} & 1 & -\bar{a} & 1 \\ 0 & 0 & 0 & 0 & \cdot & 1 & -a & 1 & -b \\ 0 & 0 & 0 & 0 & \cdot & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.3}$$

It is easy to see that $\det(P)$ satisfies the following condition:

$$\det P = (-1)^{n-1}\beta^{n-1}(1 - |a|^2), \tag{4.4}$$

where $\beta = |a|^2 + |b|^2 - 2$.

Let M_f be a square matrix of order $2n$ determined by the polynomial f. Using the matrices M_F, M_f, and P one can express a connection between the coefficients of polynomials F and f, given in system (2.19), by the following matrix equality:

$$PM_F = \begin{pmatrix} b_0 & b_1 & b_2 & & \cdot & & & 0 \\ 0 & & & & & & & 0 \\ 0 & & & & \beta M_f & & & 0 \\ \cdot & & & & & & & \cdot \\ 0 & & & & \cdot & \bar{b}_2 & \bar{b}_1 & \bar{b}_0 \end{pmatrix}. \tag{4.5}$$

Consider the matrix written in the right-hand side of (4.5) more intently: (i) its first and last lines coincide with the corresponding lines of the matrix M_F; (ii) the remaining entries (except for zeros related to the first and the last column) are filled out by the elements of the matrix $\beta \cdot M_F$. Therefore, equality (4.5) connects $\det(M_F)$ and $\det(M_f)$ as follows.

THEOREM 4.2. *Let* $c = d = 1$ *and assume that the numbers* a, b *along with coefficients* $a_0,\ldots,a_n$ *of the polynomial* f *satisfy (2.19),* $\beta \neq 0$. *Then one has*

$$\det M_F = (-1)^{n+1}\left(|b_{n+1}|^2 - |b_0|^2\right)\beta^n \det M_f. \tag{4.6}$$

Proof. Combining equality (4.5), the above formula for $\det(P)$ with the standard determinant properties yields

$$\det M_F = \frac{|b_0|^2 \det(\beta M_f)}{\det P} = (-1)^{n+1}\left(|b_{n+1}|^2 - |b_0|^2\right)\beta^n \det M_f. \tag{4.7}$$

Assume a collection $(a,b,1,1,f,F)$ of the numbers $a,b,c=d=1$ and polynomials f,F to satisfy the functional equation (1.3) and the condition $\alpha \equiv \bar{a}b-1=0$. Consider a sequence of collections $(a^k,b^k,1,1,F^k)$ of numbers $a^k,b^k,c^k=d^k=1$ and polynomials

$$F^k(z)=b_{0k}+b_{1k}z+\cdots+b_{kk}z^k, \quad b_{0k}\neq 0,\ b_{kk}\neq 0, \tag{4.8}$$

of degree k, satisfying the functional equation

$$(a^k+b^kz)F^k(z)+(1+z)(F^k)^*(z)=F^{k+1}(z), \quad k=n,n-1,\dots,1, \tag{4.9}$$

and the condition

$$\alpha^k \equiv \overline{a^k}b^k-1=0, \tag{4.10}$$

where $a^n=a$, $b^n=b$ $F^n=f$, $F^{n+1}=F$. By Theorem 2.1, if $z=0$ is a regular point of the polynomial

$$F^{k+1}(z)=b_{0k+1}+b_{1k+1}z+\cdots+b_{k+1,k+1}z^{k+1}, \tag{4.11}$$

that is,

$$b_{0k+1}\cdot b_{k+1,k+1}\neq 0, \quad |b_{0k+1}|\neq|b_{k+1,k+1}|, \tag{4.12}$$

then the system of (4.9), (4.10) has the unique solution (a^k,b^k,F^k). Moreover,

$$\overline{a^k}=\frac{\overline{b_{0k+1}}}{b_{k+1,k+1}}, \qquad \overline{b^k}=\frac{1}{a^k}. \tag{4.13}$$

This along with formula (3.24) justify the following relation:

$$\kappa(F)=\sum_{k=1}^{n+1}\frac{1+\operatorname{sign}(|b_{kk}|-|b_{0k}|)}{2}. \tag{4.14}$$

□

5. Application for computing the rotation of a plane vector field

(1) Consider a vector field $\Phi(x,y)=\{p(x,y),q(x,y)\}$, where

$$p(x,y)=\sum_{k,j}a_{kj}x^ky^j, \qquad q(x,y)=\sum_{k,j}b_{kj}x^ky^j \tag{5.1}$$

are polynomials in real variables x and y with real coefficients. Assume that $\Phi(x,y)\neq 0$, $(x,y)\in S=\{(x,y):x^2+y^2=1\}$. We are interested in computing the rotation $\gamma(\Phi,S)$.

Recall the definition of rotation. Consider the complex presentation of the field Φ:

$$p+iq=\exp(i\theta(t))\,|p+iq|, \quad p+iq=p(\cot t,\sin t)+iq(\cos t,\sin t), \quad t\in[0,2\pi], \tag{5.2}$$

where $\theta(t)$ is a continuous function. Then (cf. [4])

$$\gamma(\Phi,S) := \frac{1}{2\pi}\big(\theta(2\pi)-\theta(0)\big). \tag{5.3}$$

The following statement reduces the computation of rotation of a plane vector field to the computation of the number of roots in the unit disc of some polynomial.

Lemma 5.1. *Given a (polynomial) plane vector field Φ, there exists a unique pair (m,F), where m is an integer and F is a polynomial in complex variable with complex coefficients, $F(0)\neq 0$, satisfying the following condition:*

$$F(z) = z^m\big(p(x,y)+iq(x,y)\big), \quad z = x+iy,\ |z| = 1. \tag{5.4}$$

Proof. Take a polynomial $P(x,y) = p(x,y)+iq(x,y)$ in real variables x, y. The change of variables

$$(x,y) \longrightarrow \left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z}\right) \tag{5.5}$$

determines the rational function in complex variable z:

$$R(z) = P\left(\frac{1+z^2}{2z}, \frac{i(1-z^2)}{2z}\right). \tag{5.6}$$

The function $R(z)$ can be represented in the form

$$R(z) = \frac{F(z)}{z^m}, \tag{5.7}$$

where $F(z)$ is a polynomial satisfying the condition $F(0)\neq 0$ and m is an integer. Since

$$x = \frac{1+z^2}{2z}, \quad y = \frac{i(1-z^2)}{2z}, \quad z = x+iy,\ |z| = 1, \tag{5.8}$$

the pair (m,F) satisfies (5.4).

To complete the proof of Lemma 5.1, it remains to establish the uniqueness of the pair satisfying (5.4). Suppose that (m_1,F_1) is another pair satisfying

$$F_1(z) = z^{m_1}\big(p(x,y)+iq(x,y)\big), \quad z = x+iy,\ |z| = 1, \tag{5.9}$$

and $F_1(0)\neq 0$. Assuming, without loss of generality, that $m_1 \geq m$, one obtains the following equalities for F_1 and $z^{m_1-m}F$:

$$F_1(z) = z^{m_1}P(x,y) = z^{m_1-m}z^mP(x,y) = z^{m_1-m}F(z) \tag{5.10}$$

for $z = x+iy$, $|z| = 1$. From this it follows that F_1 and $z^{m_1-m}F$ coincide. Further, by assumption, $F(0)\neq 0$, $F_1(0)\neq 0$, hence $m_1 = m$. Thus $F_1 = F$ and Lemma 5.1 is completely proved. □

Let Φ and F be as in Lemma 5.1. Then $\gamma(\Phi,S)$ and $\kappa(F)$ satisfy the following relation:

$$\kappa(F) = m + \gamma(\Phi,S). \tag{5.11}$$

(2) Assume a field Φ to be given in a parametric form: $\Phi(t) = \{p(t),q(t)\}$, where $p(t)$, $q(t)$ are real continuous 2π-periodic functions. Suppose $\Phi(t) \neq 0$, $t \in [0,2\pi]$. The field Φ may be considered as the one defined on the unit circle S, by assigning to each point $x = \cos t$, $y = \sin t$ the vector $\{p(t),q(t)\}$. Therefore, the rotation $\gamma(\Phi,S)$ is correctly defined on S. Assuming the functions $p(t)$, $q(t)$ to be smooth enough, one can assign to the field Φ the Fourier series of the complex function $P(t) = p(t) + iq(t)$:

$$P(t) = \sum_{k=-\infty}^{\infty} c_k \exp(ikt). \tag{5.12}$$

Since the series (5.12) converges uniformly and Φ does not vanish, there exists an integer N such that for all $t \in [0,2\pi]$ the following estimate is true:

$$\left| P(t) - \sum_{k=-N}^{N} c_k \exp(ikt) \right| < |P(t)|. \tag{5.13}$$

Set $m := \max\{-k : |c_k| > 0,\ |k| \leq N\}$ and consider the polynomial

$$F(z) = \sum_{k=0}^{N+m} c_{k-m} z^k. \tag{5.14}$$

Using the same arguments as in Section 1 it is easy to check that $\gamma(\Phi,S)$ and $\kappa(F)$, where F is defined by (5.14), satisfy (5.11).

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THE MAPPINGS OF DEGREE 1

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The maps of the form $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$, called 1-degree maps, are introduced and investigated. For noncommutative algebras and modules over them 1-degree maps give an analogy of linear maps and differentials. Under some conditions on the algebra $\mathcal{A}$, contractibility of the group of 1-degree isomorphisms is proved for the module $l_2(\mathcal{A})$. It is shown that these conditions are fulfilled for the algebra of linear maps of a finite-dimensional linear space. The notion of 1-degree map gives a possibility to define a nonlinear Fredholm map of $l_2(\mathcal{A})$ and a Fredholm manifold modelled by $l_2(\mathcal{A})$. 1-degree maps are also applied to some problems of Markov chains.

1. Motivation

For the last several decades various algebras and algebra modules have been intensively investigated. Module maps are traditionally called linear if they preserve addition and multiplication by elements of the algebra (for noncommutative algebra—left or right multiplication). For nonlinear maps of modules it is easy to give a definition of the derivative similar to Freche derivative in linear spaces. In the case of noncommutative algebra, even simplest nonlinear maps, for example, power maps of the algebra (i.e., here a "one-dimensional" module) do not have such derivatives, however there exists analogue of the differential, containing only first power of the "argument increment." Namely,

$$
\begin{gathered}
(x+\Delta x)^2 - x^2 = x\cdot\Delta x + \Delta x\cdot x + (\Delta x)^2, \\
(x+\Delta x)^3 - x^3 = x^2\cdot\Delta x + x\cdot\Delta x\cdot x + \Delta x\cdot x^2 + (\Delta x)^2\cdot x + \Delta x\cdot x\cdot\Delta x + x\cdot(\Delta x)^2 + (\Delta x)^3,
\end{gathered}
\tag{1.1}
$$

and so forth.

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This was the reason for the author to introduce and investigate the maps as $f_1(\Delta x) = x\cdot\Delta x + \Delta x\cdot x$ and $f_2(\Delta x) = x^2\cdot\Delta x + x\cdot\Delta x\cdot x + \Delta x\cdot x^2$.

2. 1-degree maps of algebras

Let $\mathcal{A}$ be an algebra.

Definition 2.1 [2–10]. The map $f:\mathcal{A}\to\mathcal{A}$ is called a mapping of degree 1 (1-degree map for short) if $f(x) = \sum_{i=1}^{n} a_i\cdot x\cdot b_i$ for some $n\in N$ and $a_1,a_2,\dots,a_n,b_1,b_2,\dots,b_n\in\mathcal{A}$.

By $d_1(\mathcal{A})$ denote the set of 1-degree maps.

Obviously, for a commutative algebra $\mathcal{A}$ a 1-degree map is a trivial multiplication by a specified element. That is why $\mathcal{A}$ will denote further a noncommutative algebra.

Let $\mathrm{Map}(\mathcal{A})$ be the algebra of all maps from $\mathcal{A}$ to $\mathcal{A}$ with usual addition and multiplication by number and with map composition as element's multiplication.

Let A be $\mathcal{A}$ without the element's multiplication. A is a linear space over some number field F. Let $L(A)$ denote the algebra of its linear operators. It is obvious that $d_1(\mathcal{A})\subset L(A)\subset \mathrm{Map}(\mathcal{A})$.

Definition 2.2. We say that $\mathcal{A}$ is *1-algebra* if $d_1(\mathcal{A}) = L(A)$.

Theorem 2.3. *If $\mathcal{A} = L(F^n)$ for some field F, then $\mathcal{A}$ is a 1-algebra.*

Proof. It is sufficient to prove that $L(A)\subset d_1(\mathcal{A})$. Let $X\in\mathcal{A}$. Since $\mathcal{A} = L(F^n)$, then $\mathcal{A}$ is isomorphic to the algebra of square F-matrices, that is, A is n^2-dimensional linear space with matrices

$$P_{ij} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \tag{2.1}$$

as a basis. (Matrix P_{ij} consists of zeroes except 1 at the intersection of ith row and jth column.) If

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & & \vdots \\ x_{n1} & \cdots & x_{nn} \end{pmatrix}, \tag{2.2}$$

$X = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}P_{ij}$ and $f(X) = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij}f(P_{ij})$ for $f\in L(A)$. If $x_{ij}\neq 0$, then $P_{ij} = (1/x_{ij})P_{ii}XP_{jj} = ((1/x_{ij})P_{ii})XP_{jj} = P_{ii}X((1/x_{ij})P_{jj})$. ($P_{ij}$ can be obtained from X in another way as well.)

Let

$$f(P_{ij}) = \begin{pmatrix} a_{11}^{ij} & \cdots & a_{1n}^{ij} \\ \vdots & & \vdots \\ a_{n1}^{ij} & \cdots & a_{nn}^{ij} \end{pmatrix} = \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}^{ij} P_{kl}. \tag{2.3}$$

For any $i, j \in \{1, 2, \ldots, n\}$ we have $P_{kl} = P_{ki}P_{ij}P_{jl}$. Hence

$$\begin{aligned} f(X) &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} \sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}^{ij} P_{ki}P_{ij}P_{jl} \\ &= \sum_{x_{ij}\neq 0}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}^{ij} P_{ki} x_{ij} \frac{1}{x_{ij}} P_{ii} X P_{jj} P_{jl} \\ &= \sum_{x_{ij}\neq 0}\sum_{k=1}^{n}\sum_{l=1}^{n} a_{kl}^{ij} P_{ki} P_{ii} X P_{jj} P_{jl}. \end{aligned} \tag{2.4}$$

If $x_{ij} = 0$, then $P_{ii}XP_{jj}$ is a nil-matrix. So,

$$f(X) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} (a_{kl}^{ij} P_{ki} P_{ii}) X (P_{jj} P_{jl}) = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} (a_{kl}^{ij} P_{ki}) X (P_{jl}), \tag{2.5}$$

that is, $f \in d_1(\mathscr{A})$. Therefore $d_1(\mathscr{A}) = L(A)$. The theorem follows. □

Replace the multiplication xy in $\mathscr{A}$ by $x \odot y = yx$ and denote the new algebra by $\widetilde{\mathscr{A}}$. There is an algebra homomorphism (epimorphism) $h : \mathscr{A} \otimes \widetilde{\mathscr{A}} \to d_1(\mathscr{A}), h(\sum_{i=1}^{n} a_i \otimes b_i) = f$, where $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$. $\operatorname{Ker} h$ is not trivial. Some conditions, under which $\sum_{i=1}^{n} a_i \otimes b_i \in \operatorname{Ker} h$, are stated below.

Theorem 2.4. *If $\mathscr{A} = L(E)$ for some linear space E over any field F, then*

(1) *$a \otimes b \in \operatorname{Ker} h$ if and only if $a = 0$ or $b = 0$;*

(2) *for $a, b, c, d \neq 0$ one gets $a \otimes b - c \otimes d \in \operatorname{Ker} h$ if and only if $\exists \lambda \in F$ is such that $a = \lambda c$ and $b = (1/\lambda)d$.*

Proof. (1) If $a = 0$ or $b = 0$, then, obviously, for any $x \in \mathscr{A}$ one gets $axb = 0$, that is, $a \otimes b \in \operatorname{Ker} h$. If $a \neq 0$ and $b \neq 0$, then there exist $e_1, e_2 \in E$ such that $b(e_1) \neq 0$ and $a(e_2) \neq 0$. Since $\mathscr{A}$ is the algebra of *all* linear maps from E to E, there is $x \in \mathscr{A}$ such that $x(b(e_1)) = e_2$. So, $(axb)(e_1) = (ax)(b(e_1)) = a(x(b(e_1))) = a(e_2) \neq 0$. Therefore, $axb \neq 0 \in \mathscr{A}$, thus $a \otimes b \notin \operatorname{Ker} h$.

(2) Suppose that $a, b, c, d \neq 0$ but there is no $\lambda \in F$ with properties formulated in this theorem. (If such λ exists, then, evidently, $a \otimes b - c \otimes d \in \operatorname{Ker} h$.) Consider the following cases.

(2.1) There exists $\alpha \in F$ such that $\alpha \neq 0$, $a = \alpha c$, $b \neq (1/\alpha)d$. As $b \neq (1/\alpha)d$, b, $d \neq 0$, we can find $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) \neq (1/\alpha)d(e_1)$, and at least one element $b(e_1)$ or $d(e_1)$ is not equal to zero. As $a, c \neq 0$, $a = \alpha c$, we can find $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) = \alpha c(e_2) \neq 0$.

If $b(e_1) = 0$, then for any $x \in \mathcal{A}(axb)(e_1) = 0$. But again as far as $\mathcal{A} = L(E)$ is the algebra of *all* linear maps from E to E, there exists $x \in \mathcal{A}$ such that $x(d(e_1)) = e_2$ and $(cxd)(e_1) = c(x(d(e_1))) = c(e_2) \neq 0$. So, $(axb)(e_1) \neq (cxd)(e_1)$ and $axb \neq cxd$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $d(e_1) = 0$, then for any $x \in \mathcal{A}(cxd)(e_1) = 0$ and we take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$. We obtain $(axb)(e_1) = a(x(b(e_1))) = a(e_2) \neq 0$, that is, $(axb)(e_1) \neq (cxd)(e_1)$, $axb \neq cxd$, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $b(e_1) \neq 0$, $d(e_1) \neq 0$, we take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$, $x(d(e_1)) = \beta e_2$, where $\beta \neq 0$, $\alpha \neq \beta$. Therefore $(axb)(e_1) = a(x(b(e_1))) = a(e_2) = \alpha c(e_2) = \alpha c((1/\beta)x(d(e_1))) = (\alpha/\beta)c(x(d(e_1))) = (\alpha/\beta)(cxd)(e_1) \neq (cxd)(e_1)$ and $axb \neq cxd$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

(2.2) There exists $\alpha \in F$ such that $\alpha \neq 0$, $a \neq \alpha c$, $b = (1/\alpha)d$. As $b = (1/\alpha)d \neq 0$, there exists $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) = (1/\alpha)d(e_1) \neq 0$. As $a, c \neq 0$, $a \neq \alpha c$, there exists $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) \neq \alpha c(e_2)$, and at least one element $a(e_2)$ or $c(e_2)$ is not equal to zero. Let us take $x \in \mathcal{A}$ such that $x(b(e_1)) = e_2$.

If one element, either $a(e_2)$, or $c(e_2)$ is equal to zero, then, obviously, $(axb)(e_1) \neq (cxd)(e_1)$ because only one of these elements is equal to zero.

If $a(e_2) \neq 0$, $c(e_2) \neq 0$, then $(axb)(e_1) = a(x(b(e_1))) = a(e_2) \neq \alpha c(e_2) = \alpha c(x(b(e_1))) = \alpha c(x((1/\alpha)d(e_1))) = \alpha(1/\alpha)c(x(d(e_1))) = (cxd)(e_1)$, hence $axb \neq cxd$. Thus $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

(2.3) $a \neq \alpha c$, $b \neq (1/\alpha)d$ for each $\alpha \in F$, $\alpha \neq 0$. As $b \neq 0$, there exists $e_1 \in E$ such that $e_1 \neq 0$, $b(e_1) \neq 0$.

If $d(e_1) = 0$, then take $e_2 \in E$ such that $e_2 \neq 0$, $a(e_2) \neq 0$ that exists because $a \neq 0$ and $x \in \mathcal{A}$ is such that $x(b(e_1)) = e_2$. Then $(axb)(e_1) = a(e_2) \neq 0 = (bxd)(e_1)$, therefore $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $d(e_1) \neq 0$ and there exists $\gamma \in F$ such that $\gamma \neq 0$, $b(e_1) = \gamma d(e_1)$, then take $e_2 \in E$ and $x \in \mathcal{A}$ such that $e_2 \neq 0$, $a(e_2) \neq (1/\gamma)c(e_2)$, $x(b(e_1)) = e_2$. We obtain $(axb)(e_1) = a(e_2) \neq (1/\gamma)c(e_2) = (1/\gamma)c(x(b(e_1))) = (1/\gamma)c(x(\gamma d(e_1))) = (1/\gamma)\gamma c(x(d(e_1))) = (cxd)(e_1)$. Thus $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$.

If $b(e_1)$ and $d(e_1)$ are linearly independent (therefore $b(e_1) \neq 0$ and $d(e_1) \neq 0$), then take $e_2 \in E$, $x \in \mathcal{A}$ such that $e_2 \neq 0$, $x(b(e_1)) = e_2$. If $a(x(b(e_1))) \neq c(x(d(e_1)))$, then $axb \neq cxd$ and $a \otimes b - c \otimes d \notin \operatorname{Ker} h$. If it appears to be $a(x(b(e_1))) = c(x(d(e_1)))$, then replace x by $\tilde{x}$ such that $\tilde{x}(b(e_1)) = x(b(e_1))$, $\tilde{x}(d(e_1)) = \delta x(d(e_1))$ where $\delta \neq 0$, $\delta \neq 1$. Thus we obtain $(a\tilde{x}b)(e_1) = a(\tilde{x}(b(e_1))) = a(x(b(e_1))) = c(x(d(e_1))) = c((1/\delta)\tilde{x}(d(e_1))) = (1/\delta)c(\tilde{x}(d(e_1))) \neq c(\tilde{x}(d(e_1)))$. This yields $a\tilde{x}b \neq c\tilde{x}d$, that is, $a \otimes b - c \otimes d \notin \operatorname{Ker} h$. The theorem is proved. □

Let $\mathcal{A}$ be an involutive algebra.

Definition 2.5 [8, 9]. The elements $f, f^* \in d_1(\mathcal{A})$ are called *conjugate* if

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i, \qquad f^*(x) = \sum_{i=1}^{n} b_i^* \cdot x \cdot a_i^*. \tag{2.6}$$

Operation $f \leftrightarrow f^*$ is an involution for $d_1(\mathcal{A})$. It is clear that $f^*(x^*) = (f(x))^*$ for all $x \in \mathcal{A}$.

Definition 2.6 [8, 9]. An element $f \in d_1(\mathcal{A})$ is called *self-conjugate* if $f = f^*$.

Obviously, for self-conjugate $f \in d_1(\mathcal{A})$ and for all $x \in \mathcal{A}$ one gets $f(x^*) = (f(x))^*$. In addition $f \in d_1(\mathcal{A})$ is self-conjugate if and only if for all $i \in \{1,2,\dots,n\}$ either $b_i = a_i^*$ or there exists $j \in \{1,2,\dots,n\}$, $j \neq i$ such that $a_j = b_i^*$, $b_j = a_i^*$.

Definition 2.7 [8, 9]. Self-conjugate $f \in d_1(\mathcal{A})$ is called positive if

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot a_i^*, \quad \textit{for some } \ a_1, a_2, \dots, a_n \in \mathcal{A}. \tag{2.7}$$

This notation is chosen since for C^*-algebra $\mathcal{A}$ a positive map sends positive elements of algebra $\mathcal{A}$ to positive ones. Indeed, for C^*-algebra $\mathcal{A}$ one of definitions of positive elements is $x = yy^*$ for a certain $y \in \mathcal{A}$. If $f \in d_1(\mathcal{A})$ is positive and $x \in \mathcal{A}$ is positive, then

$$f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot a_i^* = \sum_{i=1}^{n} a_i \cdot (yy^*) \cdot a_i^* = \sum_{i=1}^{n} (a_i y) \cdot (y^* a_i^*) = \sum_{i=1}^{n} (a_i y) \cdot (a_i y)^*. \tag{2.8}$$

Since for C^*-algebra the cone P of positive elements is convex, then $f(x) \in P$.

Now consider some properties of eigenvalues and eigenelements for self-conjugate and positive 1-degree maps. It is useful to note that characteristic values of self-conjugate maps are real. The next three theorems were announced in [9].

Theorem 2.8. *If $\mathcal{A}$ is a C^*-algebra, $f \in d_1(\mathcal{A})$ is a positive map, $\lambda \in R$ is a characteristic value of f, and there exists a positive characteristic element $x \in \mathcal{A}$, corresponding to the characteristic value λ, then $\lambda \geq 0$.*

Proof. Suppose that $\mathcal{A}$ is a C^*-algebra, $f \in d_1(\mathcal{A})$ is a positive map, $x \in \mathcal{A}$ is a positive element ($x \neq 0$), $\lambda \in R, f(x) = \lambda x$. If $\lambda < 0$, then $-\lambda > 0$ and $-\lambda x$ is positive, that is, $\lambda x \in (-P)$, where P is the cone of positive elements. But $\lambda x = f(x) \in P$. Hence $\lambda x \in P \cap (-P) = \{0\}$. It is impossible since for $\lambda \neq 0$, $x \neq 0$, $\lambda x \neq 0$. Therefore the assumption $\lambda < 0$ is false and $\lambda \geq 0$. □

Theorem 2.9. *If $\mathcal{A}$ is an involutive algebra, $f \in d_1(\mathcal{A})$ is a self-conjugate map, $\lambda \in R$ is a characteristic value of f, then there exists a self-conjugate characteristic element $x \in \mathcal{A}(x \neq 0)$ corresponding to the characteristic value λ.*

Proof. For any $x \in \mathcal{A}$ there exists a unique representation as the sum of "real" and "imaginary" parts $x = x_1 + ix_2$, where $x_1, x_2 \in \mathcal{A}$ are self-conjugate elements ($x_1 = (1/2)(x + x^*), x_2 = (1/2i)(x - x^*)$). Therefore $f(x) = \lambda x \Leftrightarrow f(x_1 + ix_2) = \lambda(x_1 + ix_2) \Leftrightarrow f(x_1) + if(x_2) = \lambda x_1 + i\lambda x_2$.

The elements λx_1, λx_2 are self-conjugate. As f, x_1, x_2 are self-conjugate, $f(x_1)$, $f(x_2)$ are self-conjugate as well. Since the representation as the sum of "real" and "imaginary" parts is unique, then $f(x_1) = \lambda x_1$, $f(x_2) = \lambda x_2$. For $x \neq 0$ at least one element x_1 or x_2 is not equal to zero. This completes the proof. □

Theorem 2.10. *If $\mathcal{A}$ is a C^*-algebra, $f \in d_1(\mathcal{A})$ is a positive map, and there is a characteristic value $\lambda < 0$ of f, then there exist positive elements x_1, x_2, $a \in \mathcal{A}$ such that $f(x_1) - \lambda x_1 = f(x_2) - \lambda x_2 = a$.*

Proof. A positive map is self-conjugate. From Theorem 2.9 it follows that there is a self-conjugate element $x \in \mathcal{A}$ such that $x \neq 0$ and $f(x) = \lambda x$. If $\mathcal{A}$ is C^*-algebra and $x \in \mathcal{A}$ is a self-conjugate element, then there exists unique couple of positive or equal to zero elements $x_1, x_2 \in \mathcal{A}$ such that $x = x_1 - x_2$, $x_1x_2 = x_2x_1 = 0$.

We have $f(x) = f(x_1 - x_2) = \lambda(x_1 - x_2) = \lambda x_1 - \lambda x_2$. Therefore $f(x_1) - \lambda x_1 = f(x_2) - \lambda x_2$. Denote this element by a. It follows from Theorem 2.8 that $x_1 \neq 0$, $x_2 \neq 0$. Hence both x_1 and x_2 are positive. As x_1, x_2, f, $(-\lambda)$ are positive, then $f(x_1)$, $f(x_2)$, $(-\lambda x_1)$, $(-\lambda x_2)$, and a are positive. This completes the proof. □

If $\mathcal{A}$ is a normed algebra, then the standard norm on $d_1(\mathcal{A})$ is also given, since $d_1(\mathcal{A}) \subset L(A)$, that is, for $f \in d_1(\mathcal{A})$,

$$\|f\| = \sup_{x \neq 0} \frac{\|f(x)\|}{\|x\|}. \tag{2.9}$$

Obviously, $\|f^*\| = \|f\|$ and if $f(x) = \sum_{i=1}^{n} a_i \cdot x \cdot b_i$, then $\|f\| \leq \sum_{i=1}^{n} \|a_i\| \cdot \|b_i\|$.

As far as $L(A)$ is closed, $\overline{d_1(\mathcal{A})} \subset L(A)$, where $\overline{d_1(\mathcal{A})}$ is the closure of $d_1(\mathcal{A})$.

Definition 2.11. A normed algebra $\mathcal{A}$ is $\overline{1}$-*algebra* if $\overline{d_1(\mathcal{A})} = L(A)$. Obviously, if $\mathcal{A}$ is a normed 1-algebra, then $\mathcal{A}$ is a $\overline{1}$-algebra.

Definition 2.12. If $\mathcal{A}$ is an involutive algebra, then the maps $f, f^* \in \overline{d_1(\mathcal{A})}$ are called *conjugate* if $f(x) = \lim_{n\to\infty} f_n(x)$, $f^*(x) = \lim_{n\to\infty} f_n^*(x)$ for some $f_1, f_2, \dots, f_n, \dots \in d_1(\mathcal{A})$.

3. 1-degree maps of modules

(A) Let $\mathcal{M}_1$, $\mathcal{M}_2$ be free finitely generated $\mathcal{A}$-modules,

$$\mathcal{M}_1 = \mathcal{A}^n, \qquad \mathcal{M}_2 = \mathcal{A}^m, \qquad F : \mathcal{M}_1 \longrightarrow \mathcal{M}_2, \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_m \end{pmatrix}, \qquad f_i = P_i \circ F : \mathcal{M}_1 \longrightarrow \mathcal{A}, \tag{3.1}$$

where $P_i : \mathcal{M}_2 \to \mathcal{A}$ is the projection of $\mathcal{M}_2 = \mathcal{A} \oplus \mathcal{A} \oplus \cdots \oplus \mathcal{A}$ onto ith term.

Definition 3.1. $F : \mathcal{M}_1 \to \mathcal{M}_2$ is called *1-degree* ($\overline{1}$-*degree*) *map* if for each $i \in \{1,2,\dots,m\}$ the relation $f_i(x_1,x_2,\dots,x_n) = f_{i1}(x_1) + f_{i2}(x_2) + \cdots + f_{in}(x_n)$ holds, where $f_{i1}, f_{i2}, \dots, f_{in} \in d_1(\mathcal{A})(\in \overline{d_1(\mathcal{A})})$.

Assign the following matrix to a 1-degree or a $\overline{1}$-degree map:

$$F \longleftrightarrow \begin{pmatrix} f_{11} & f_{12} & \cdots & f_{1n} \\ f_{21} & f_{22} & \cdots & f_{2n} \\ \vdots & \vdots & & \vdots \\ f_{m1} & f_{m2} & \cdots & f_{mn} \end{pmatrix}. \tag{3.2}$$

Definition 3.2. If $\mathcal{A}$ is an involutive algebra, then the 1-degree or $\overline{1}$-degree maps $F = (f_{ij})_{m\times n}$, $F^* = (f_{ij}^*)_{m\times n}$ are called *conjugate*.

As well as it is for algebra maps, $F^*(x^*) = (F(x))^*$.

If the modules $\mathcal{M}_1,\mathcal{M}_2$ are normed (e.g., for a normed algebra $\mathcal{A}$ the norm can be defined as $\|x\| = \sqrt{\sum_i \|x_i\|^2}$), then $\|F\|$ is defined in the standard way

$$\|F\| = \sup_{x\neq 0} \frac{||F(x)||}{\|x\|}. \tag{3.3}$$

The straightforward computation gives

$$\|F\| \leq \sqrt{\sum_{i=1}^{m}\left(\sum_{j=1}^{n}||f_{ij}||\right)^2} \leq \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n}||f_{ij}||^2} \tag{3.4}$$

and $\|F\| = \|F^*\|$.

(B) Let $\mathcal{A}$ be a C^*-algebra and let $l_2(\mathcal{A})$ be a Hilbert $\mathcal{A}$-module,

$$F : l_2(\mathcal{A}) \longrightarrow l_2(\mathcal{A}), \qquad F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}, \tag{3.5}$$

where $f_i = P_i \circ F$, as above. Since $F(x) \in l_2(\mathcal{A})$ it has to be $\|F(x)\|^2 = \sum_{i=1}^{\infty} \|f_i(x)\|^2 < \infty$ for any $x \in l_2(\mathcal{A})$. This is fulfilled, in particular, if $\sum_{i=1}^{\infty} \|f_i\|^2 < \infty$. In this case

$$||F(x)||^2 \leq \sum_{i=1}^{\infty} ||f_i||^2 \cdot \|x\|^2 , \tag{3.6}$$

therefore

$$\|F\| \leq \sqrt{\sum_{i=1}^{\infty} ||f_i||^2} < \infty \quad \text{for } \|F\| = \sup_{x\neq 0} \frac{||F(x)||}{\|x\|}. \tag{3.7}$$

Definition 3.3. A bounded map $F : l_2(\mathcal{A}) \to l_2(\mathcal{A})$ is called *1-degree map* if for each $i \in N$ $f_i(x) = f_i(x_1,x_2,\ldots) = \sum_{j=1}^{\infty} f_{ij}(x_j)$, where $f_{ij} \in d_1(\mathcal{A})$ for any $j \in N$,

$$F \longleftrightarrow \begin{pmatrix} f_{11} & f_{12} & \cdots \\ f_{21} & f_{22} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.8}$$

Since $\lim_{n\to\infty}\sum_{j=1}^{n} f_{ij}(x_j)$ must exist for all $x=(x_1,x_2,\dots)\in l_2(\mathcal{A})$, we assume that for each $i\in N$ there exists a constant $M_{F,i}>0$ such that $\|f_{ij}\|\le M_{F,i}$ for each $j\in N$. Then

$$\begin{aligned}\left\|f_i(x)\right\|^2 &= \left\|\sum_{j=1}^{\infty} f_{ij}(x_j)\right\|^2 \le \sum_{j=1}^{\infty}\left\|f_{ij}\right\|^2\cdot\left\|x_j\right\|^2\\ &\le \sum_{j=1}^{\infty} M_{F,i}^2\cdot\left\|x_j\right\|^2 = M_{F,i}^2\sum_{j=1}^{\infty}\left\|x_j\right\|^2\end{aligned}\tag{3.9}$$

and $\|f_i\|\le M_{F,i}$. Assume that $\sum_{i=1}^{\infty} M_{F,i}^2<\infty$ for each 1-degree map F. Then for each 1-degree map $F\|F\|<\infty$.

By $d_1(l_2(\mathcal{A}))$ denote the set of all 1-degree maps from $l_2(\mathcal{A})$ to $l_2(\mathcal{A})$ and by $\overline{d_1(l_2(\mathcal{A}))}$ denote its closure. We say that $F: l_2(\mathcal{A})\to l_2(\mathcal{A})$ is a $\overline{1}$-degree map if $F\in\overline{d_1(l_2(\mathcal{A}))}$. Obviously, $F\in\overline{d_1(l_2(\mathcal{A}))}\leftrightarrow f_{ij}\in\overline{d_1(\mathcal{A})}$ for all $i,j\in N$.

Let $e_1=(1,0,0,\dots)$, $e_2=(0,1,0,\dots)$, $e_3=(0,0,1,\dots)$, and so forth be basic elements in $l_2(\mathcal{A})$ and let $L_{n,\mathcal{A}}^{\perp}$ be a submodule of the module $l_2(\mathcal{A})$ generated by elements $e_{n+1},e_{n+2},\dots$.

Definition 3.4. Following [14], we say that a map $F\in\overline{d_1(l_2(\mathcal{A}))}$ is *compact* if $\lim_{n\to\infty}\|F|_{L_{n,\mathcal{A}}^{\perp}}\|=0$.

Definition 3.5. A map $F\in\overline{d_1(l_2(\mathcal{A}))}$ is called *Fredholm* $\overline{1}$*-map* if $F=H+C$, where $H,C\in\overline{d_1(l_2(\mathcal{A}))}$, H is an isomorphism, and C is a compact map.

The set $C_{\overline{1}}(l_2(\mathcal{A}))$ of all compact $\overline{1}$-degree maps from $l_2(\mathcal{A})$ to $l_2(\mathcal{A})$ is an ideal in the algebra $\overline{d_1(l_2(\mathcal{A}))}$.

Definition 3.6. A manifold M modelled by $l_2(\mathcal{A})$ is called *Fredholm* $\overline{1}$*-manifold* if M has an atlas with transformation functions having the "derivatives" in the form $I+C$, where I is the identity map, $C\in C_{\overline{1}}(l_2(\mathcal{A}))$.

By $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ denote the group of $\overline{1}$-degree isomorphisms of $l_2(\mathcal{A})$.

Theorem 3.7. *If $\mathcal{A}$ is C^*-$\overline{1}$-algebra, then $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ is contractible.*

Proof. Let E be the Banach space obtained from $l_2(\mathcal{A})$ by ignoring the element multiplication in $\mathcal{A}$, where $\|\cdot\|_E=\|\cdot\|_{l_2(\mathcal{A})}$.

For C^*-$\overline{1}$-algebra $\mathcal{A}$ we get $\overline{d_1(l_2(\mathcal{A}))}=B(E)$ and $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))=\mathrm{GL}(E)$. Hence $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ is an open subset of Banach space $B(E)$. Therefore, by Milnor's theorem [13] $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ has the homotopy type of a CW-complex and by Whitehead's theorem [16] the strong and weak homotopy trivialities of $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ are equivalent. So, it is sufficient to prove the weak homotopy triviality.

The proof technique repeats that for $\mathrm{GL}(H)$ in [11] or for $\mathrm{GL}(l_2(\mathcal{A}))$ in [12]. First we use Atiyah's theorem on small balls: if f is a continuous map from n-dimensional sphere S^n to $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$, then f is homotopic to f' so that $f'(S^n)$ is a finite polyhedron lying in $\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ together with the homotopy.

Then we use from [11, Lemma 3]: there exist a decomposition $l_2(\mathcal{A})$ in $H'\oplus H_1$, where $H'\cong H_1\cong l_2(\mathcal{A})$, and a continuous map $f'': S^n\to\mathrm{GL}_{\overline{1}}(l_2(\mathcal{A}))$ homotopic to f' such that $f''(s)(x)=x$ for all $s\in S^n$, $x\in H'$.

Next we prove the statement similar to [11, Lemma 7], and [12, Lemma 7.1.4]: $V = \{g \in \mathrm{GL}_{\bar{1}}(l_2(\mathcal{A}))\ |g|_{H'} = Id_{H'},\ g(H_1) = H_1\}$ is contractible to 1 in $\mathrm{GL}_{\bar{1}}(l_2(\mathcal{A}))$. Here we use the condition from the hypothesis of our theorem that $\mathcal{A}$ is C^*-$\bar{1}$-algebra. If so, then for any $u \in \mathrm{GL}_{\bar{1}}(l_2(\mathcal{A}))$, u^{-1} and all linear combinations of u and u^{-1} belong to $\mathrm{GL}_{\bar{1}}(l_2(\mathcal{A}))$ as well. The construction of homotopy is the same as in [11, 12].

Since $H' \cong l_2(\mathcal{A})$, we can decompose H' into the sum $H' = H_2 \oplus H_3 \oplus \cdots H_i \cong l_2(\mathcal{A})$, then we get $l_2(\mathcal{A}) = H_1 \oplus H_2 \oplus H_3 \oplus \cdots$.

Let $g \in V$ on, then the matrix of g related to the above decomposition is $g = \mathrm{diag}(u, 1, 1, 1, \ldots) = \mathrm{diag}(u, u^{-1}u, 1, u^{-1}u, 1, \ldots)$, where $u = g|_{H_1}$.

For $t \in [0, \pi/2]$ we get $g_t|_{H_1} = g|_{H_1} = u$ and for each $i \in N$,

$$\begin{aligned} g_t|_{H_{2i}\oplus H_{2i+1}} &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot \cos^2 t + u^{-1}\sin^2 t & (u-1)\sin t \cos t \\ (1-u^{-1})\sin t \cos t & u\sin^2 t + 1 \cdot \cos^2 t \end{pmatrix}. \end{aligned} \tag{3.10}$$

For $t \in [\pi/2, \pi]$ and for each $i \in N$

$$\begin{aligned} g_t|_{H_{2i-1}\oplus H_{2i}} &= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \cdot \cos^2 t + u\sin^2 t & (u^{-1}-1)\sin t \cos t \\ (1-u)\sin t \cos t & u^{-1}\sin^2 t + 1 \cdot \cos^2 t \end{pmatrix}, \end{aligned} \tag{3.11}$$

$g_0 = g, g_{\pi/2} = \mathrm{diag}(u, u^{-1}, u, u^{-1}, u, \ldots)$ by both formulas of homotopies, and $g_\pi = \mathrm{diag}(1, 1, 1, 1, \ldots)$.

Since all these maps are linear isomorphisms in $L(E)$, then by the hypothesis of the theorem they are $\bar{1}$-degree isomorphisms.

At last we repeat [12, Lemma 7.1.5]: the set

$$W = \{g \in \mathrm{GL}_{\bar{1}}(l_2(\mathcal{A}))\ |g|_{H'} = Id_{H'}\} \tag{3.12}$$

is contractible to V. (In [12] this is proved for $\mathrm{GL}(l_2(\mathcal{A}))$.) This concludes the proof. □

4. Polynomials over a noncommutative algebra and nonlinear Fredholm maps

Let $\mathcal{A}$ be a noncommutative algebra, $x \notin \mathcal{A}$, $x^1 = x$, $x^k \cdot x^l = x^{k+l}$ for all $k, l \in N$. Let us give the recurrent definition of a monomial in x.

Definition 4.1. (1) Every element of $\mathcal{A}$ is a *0-degree monomial.*

(2) x is a *1-degree monomial.*

(3) If m_1 is a k-degree monomial for any $k \in N$ and m_2 is an l-degree monomial for any $l \in N$, then $m_1 \cdot m_2$ is a *$(k+l)$-degree monomial.*

(4) The other monomials do not exist.

Definition 4.2. A finite sum of n-degree monomials is called a *homogeneous n-degree polynomial.*

Definition 4.3. A finite sum of homogeneous polynomials is called an all linear*n-degree polynomial*, where n is the maximal degree of the terms.

The set P of all polynomials forms a graded algebra $P = \oplus_{n=1}^{\infty} P_n$, where P_n is the set of homogeneous n-degree polynomials. Obviously, $P_0 \cong \mathcal{A}, P_1 \cong \mathcal{A} \otimes \widetilde{\mathcal{A}}$.

Definition 4.4. The map corresponding to the homogeneous n-degree polynomial is called an *n-degree map* and the map corresponding to any polynomial is called a *polynomial map*.

The set $d(\mathcal{A})$ of all polynomial maps forms a graded algebra as well: $d(\mathcal{A}) = \oplus_{n=1}^{\infty} d_n(\mathcal{A})$, where $d_n(\mathcal{A})$ is the set of n-degree maps. There is a natural map of graded algebras $h : P \to d(\mathcal{A})$ assigning to the polynomial its corresponding map. For $n = 1$ it was shown in Section 2 that $h\,|_{P_1} : P_1 \cong \mathcal{A} \otimes \widetilde{\mathcal{A}} \to d_1(\mathcal{A})$ is not injective. Polynomial maps defined above are maps that have 1-degree "derivatives."

We will obtain the definition of a monomial in several variables $x_1, x_2, \dots, x_n, \dots$ if in Definition 4.1 we replace (2) by "x_i is a 1-degree monomial for every $i \in N$." Then Definitions 4.2 and 4.3 give the notions of polynomials in $x_1, x_2, \dots, x_n, \dots$ and of corresponding polynomial maps from $l_2(\mathcal{A})$ to $\mathcal{A}$.

Definition 4.5. The map $F : U \to l_2(\mathcal{A})$ is called *polynomial* if f_i are polynomial maps for all $i \in N$, where $U \subset l_2(\mathcal{A})$ is an open domain and

$$F = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix}. \tag{4.1}$$

The "derivative"

$$F'(x) = \begin{pmatrix} df_1 \\ df_2 \\ \vdots \end{pmatrix} \tag{4.2}$$

of a polynomial map F is a 1-degree map for every $x \in U$.

Definition 4.6. The polynomial map $F : U \to l_2(\mathcal{A})$ is called a *Fredholm map* if $F'(x)$ is a Fredholm 1-degree map for each $x \in U$.

5. A certain application of 1-degree maps

In probability theory a discrete Markov chain is described by the transition matrix consisting of elements from $[0,1]$. Markov's theorem proclaims that if some power of this matrix does not contain zeroes, then limit transition probabilities exist. We assume that kth power of the transition matrix does not contain zeroes and so the question is arising: how the elements of transition matrix that can be changed for kth power of matrix remain without zeroes.

Assume that the transition matrix is an $n \times n$-matrix and consider the set of the real $n \times n$-matrices as a normed involutive algebra with transposition as the involution and the norm $\|A\| = n \cdot \max_{1\le i,j\le n} |a_{ij}|$, (equivalent to the norm $\|A\| = \sup_{x\neq 0} \|Ax\|/\|x\|$ but more suitable for the calculation). The inequality $\|AB\| \le \|A\| \cdot \|B\|$ for the normed algebra is fulfilled.

Let $P = (p_{ij})_{n\times n}$ be a transition matrix, let $P^k = \left(p_{ij}^{(k)}\right)_{n\times n}$ be the kth power of P and $f(P) = P^k$. Using the equality $f(P+\Delta P) - f(P) = (P+\Delta P)^k - P^k = df + o(\Delta P)$ we approximate the difference at the left-hand side by the "differential" df and instead of the inequality $\|\Delta f\| = \|f(P+\Delta P) - f(P)\| < \varepsilon$ we solve the inequality $\|df\| < \varepsilon$. Here

$$df = P^{k-1} \cdot \Delta P + P^{k-2} \cdot \Delta P \cdot P + \cdots + P \cdot \Delta P \cdot P^{k-2} + \Delta P \cdot P^{k-1} \tag{5.1}$$

is a d_1-map of ΔP,

$$\begin{aligned} o(\Delta P) = \big[&P^{k-2} \cdot (\Delta P)^2 + P^{k-3} \cdot (\Delta P)^2 \cdot P + \cdots + P \cdot (\Delta P)^2 \cdot P^{k-3} \\ &+ P^{k-3} \cdot \Delta P \cdot P \cdot \Delta P + \cdots + \Delta P \cdot P \cdot \Delta P \cdot P^{k-3}\big] \\ &+ \big[P^{k-3} \cdot (\Delta P)^3 + \cdots\big] + \cdots + (\Delta P)^k. \end{aligned} \tag{5.2}$$

Taking into account the estimate

$$\begin{aligned} \|df\| &\le \left\|P^{k-1}\right\| \cdot \|\Delta P\| + \left\|P^{k-2}\right\| \cdot \|\Delta P\| \cdot \|P\| \\ &\quad + \cdots + \|P\| \cdot \|\Delta P\| \cdot \left\|P^{k-2}\right\| + \|\Delta P\| \cdot \left\|P^{k-1}\right\| \\ &\le \|P\|^{k-1} \cdot \|\Delta P\| + \|P\|^{k-2} \cdot \|\Delta P\| \cdot \|P\| \\ &\quad + \cdots + \|P\| \cdot \|\Delta P\| \cdot \left\|P^{k-2}\right\| + \|\Delta P\| \cdot \left\|P^{k-1}\right\| = k \cdot \|P\|^{k-1} \cdot \|\Delta P\|, \end{aligned} \tag{5.3}$$

it is sufficient to solve the inequality $k \cdot \|P\|^{k-1} \cdot \|\Delta P\| < \varepsilon$. Thus we get

$$\|\Delta P\| < \delta_1(\varepsilon) = \frac{\varepsilon}{k \cdot \|P\|^{k-1}}, \tag{5.4}$$

therefore for all $i, j \in \{1,2,\dots,n\}$,

$$|\Delta p_{ij}| < \frac{\varepsilon}{n \cdot k \cdot \|P\|^{k-1}}. \tag{5.5}$$

Further we will not reject $o(\Delta P)$ and obtain more exact estimate:

$$\begin{aligned} \|o(\Delta P)\| \le \big[&\|P\|^{k-2} \cdot \|\Delta P\|^2 + \|P\|^{k-3} \cdot \|\Delta P\|^2 \cdot \|P\| + \cdots \\ &+ \|P\| \cdot \|\Delta P\|^2 \cdot \|P\|^{k-3} + \|\Delta P\|^2 \cdot \|P\|^{k-2} \\ &+ \|P\|^{k-3} \cdot \|\Delta P\| \cdot \|P\| \cdot \|\Delta P\| + \cdots + \|\Delta P\| \cdot \|P\| \cdot \|\Delta P\| \cdot \|P\|^{k-3}\big] \\ &+ \big[\|P\|^{k-3} \cdot \|\Delta P\|^3 + \cdots\big] + \cdots + \|\Delta P\|^k \\ = {}& C_k^2 \cdot \|P^{k-2} \cdot \|\Delta P\|^2 + C_k^3 \cdot \|P\|^{k-3} \cdot \|\Delta P\|^3 + \cdots + C_k^s \cdot \|P\|^{k-s} \cdot \|\Delta P\|^s \\ &+ \cdots + \|\Delta P\|^k = \big(\|P\| + \|\Delta P\|\big)^k - \|P\|^k - k \cdot \|P\|^{k-1} \cdot \|\Delta P\|. \end{aligned} \tag{5.6}$$

Let $\varphi(x) = x^k$ be a real function. Then for $x = \|P\|$, $\Delta x = \|\Delta P\|$ we get

$$\begin{gathered}\Delta\varphi = \varphi(x+\Delta x) - \varphi(x) = \varphi\big(\|P\| + \|\Delta P\|\big) - \varphi\big(\|P\|\big) = \big(\|P\| + \|\Delta P\|\big)^k - \|P\|^k,\\ d\varphi = \varphi'(x)\Delta x = kx^{k-1}\Delta x = k\cdot\|P\|^{k-1}\cdot\|\Delta P\|,\\ \big(\|P\| + \|\Delta P\|\big)^k - \|P\|^k - k\cdot\|P\|^{k-1}\cdot\|\Delta P\| = \Delta\varphi - d\varphi = o(\Delta x) = o\big(\|\Delta P\|\big).\end{gathered}\tag{5.7}$$

Consider $o(\Delta x)$ in the form $(\varphi''(x+t\cdot\Delta x)/2)(\Delta x)^2$, where $0 < t < 1$. For $\varphi''(x) = k(k-1)x^{k-2}$, we obtain $\varphi''(x+t\cdot\Delta x) = k(k-1)(x+t\cdot\Delta x)^{k-2}$.

Since P and $P+\Delta P$ are transition matrices, that is, their elements are probabilities, then $|\Delta p_{ij}| \le 1$ for all $i,j \in \{1,2,\dots,n\}$ and $\|\Delta P\| \le n$. Thus for $x = \|P\|$, $\Delta x = \|\Delta P\|$ we get $x \ge 0$, $0 < \Delta x \le n$, $x + t\cdot\Delta x \le x + \Delta x \le x + n$, and $(x+t\cdot\Delta x)^{k-2} \le (x+n)^{k-2}$. Therefore $\|o(\Delta P)\| \le (k(k-1)/2)(\|P\|+n)^{k-2}\|\Delta P\|^2$ and

$$\|(P+\Delta P)^k - P^k\| \le \|df\| + \big|\big|o(\Delta P)\big|\big| \le k\cdot\|P\|^{k-1}\cdot\|\Delta P\| + \frac{k(k-1)}{2}\big(\|P\|+n\big)^{k-2}\|\Delta P\|^2.\tag{5.8}$$

Now we have to solve the square inequality

$$k\cdot\|P\|^{k-1}\cdot\|\Delta P\| + \frac{k(k-1)}{2}\big(\|P\|+n\big)^{k-2}\|\Delta P\|^2 < \varepsilon\tag{5.9}$$

for $\|\Delta P\|$. Taking into account $\|\Delta P\| \ge 0$ we obtain

$$\|\Delta P\| < \delta_2(\varepsilon) = \frac{\sqrt{\big(k\|P\|^{k-1}\big)^2 + 2\varepsilon k(k-1)\big(\|P\|+n\big)^{k-2}} - k\|P\|^{k-1}}{k(k-1)\big(\|P\|+n\big)^{k-2}}.\tag{5.10}$$

Since $\|\Delta P\| = n\cdot\max_{1\le i,j\le n}|\Delta p_{ij}|$, we get for all $i,j \in \{1,2,\dots,n\}$,

$$|\Delta p_{ij}| < \frac{\sqrt{\big(k\|P\|^{k-1}\big)^2 + 2\varepsilon k(k-1)\big(\|P\|+n\big)^{k-2}} - k\|P\|^{k-1}}{nk(k-1)\big(\|P\|+n\big)^{k-2}}.\tag{5.11}$$

This is a sufficient condition for $\|\Delta f\| < \varepsilon$.

The number ε must be chosen so that all elements of the matrix $(P+\Delta P)^k$ are strictly positive, that is, they differ from the corresponding elements of P^k less than by $m = \min_{1\le i,j\le n} p_{ij}^{(k)} > 0$. Hence it must be $\|df\| = \|(P+\Delta P)^k - P^k\| < nm$, therefore ε has to be chosen so that $\varepsilon \le nm$, for example, $\varepsilon = nm$.

In this case the first formula yields $|\Delta p_{ij}| < m/k\|P\|^{k-1}$ and the second formula yields

$$|\Delta p_{ij}| < \frac{\sqrt{\big(k\|P\|^{k-1}\big)^2 + 2nmk(k-1)\big(\|P\|+n\big)^{k-2}} - k\|P\|^{k-1}}{nk(k-1)\big(\|P\|+n\big)^{k-2}}\tag{5.12}$$

for all $i,j \in \{1,2,\dots,n\}$. The formulae (5.5) and (5.12) were announced in [10].

6. Conclusion

Let us ascertain the interdependence between the notions defined in this paper and some other ones.

(1) Let f be a polynomial in x with coefficients from $\mathcal{A}$, then df is a 1-degree polynomial in Δx, where the coefficients are also polynomials in x. So, $d : P \to P \otimes \widetilde{P}$ (see Section 2 for the description of what $\widetilde{\mathcal{A}}$ with respect to $\mathcal{A}$ is). Let $g : P \otimes \widetilde{P} \to P$ be an algebra homomorphism defined as $g(f_1 \otimes f_2) = f_1 \cdot f_2$. Then $\delta = g \circ d : P \to P$ is a derivation of the algebra P, that is, $\delta(f_1 \cdot f_2) = \delta(f_1) \cdot f_2 + f_1 \cdot \delta(f_2)$ for all $f_1, f_2 \in P$. If f is an n-degree polynomial, then $\delta(f)$ is also an $(n-1)$-degree polynomial.

(2) For a noncommutative operator algebra $\mathcal{A}$ consider the algebra homomorphisms $\mu_A : \mathcal{F}_1 \to \mathcal{A}$ and $\mu_{\overset{1}{A}_1,\dots,\overset{n}{A}_n} : \mathcal{F}_n \to \mathcal{A}$ that are defined, for example, in [15]. Here $A, \overset{1}{A}_1, \dots, \overset{n}{A}_n \in \mathcal{A}$ are called generators, $\mathcal{F}_1$ is a space of one-place symbols, $\mathcal{F}_n$ is a space of n-place symbols, numbers over letters are called Feynman numbers, and the set of letters with numbers over them is called Feynman set. The one-place or n-place symbols are real or complex functions of one or n variables, respectively. These homomorphisms μ_A and $\mu_{\overset{1}{A}_1,\dots,\overset{n}{A}_n}$ may be called "evaluation homomorphisms" because they give a possibility to find in $\mathcal{A}$ the values $f(A)$ or $f(\overset{1}{A}_1,\dots,\overset{n}{A}_n)$ for different f and specified A or $\overset{1}{A}_1,\dots,\overset{n}{A}_n$. For example, if f is a polynomial in one or several variables, it is necessary to substitute A or $\overset{1}{A}_1,\dots,\overset{n}{A}_n$ for variables of f.

If f is a polynomial in $2n+1$ variables, $f = f(y_1,\dots,y_n,z,x_1,\dots,x_n) = \sum_{i=1}^{n} y_i \cdot z \cdot x_i$, then for $a_1,\dots,a_n,c,b_1,\dots,b_n \in \mathcal{A}$ we can evaluate $f(\overset{1}{b}_1,\dots,\overset{n}{b}_n,\overset{n+1}{c},\overset{n+2}{a}_1,\dots,\overset{2n+1}{a}_n)$. Specifying $a_1,\dots,a_n,b_1,\dots,b_n$ and replacing c by $\mathcal{A}$, we obtain the 1-degree map corresponding to the 1-degree polynomial in x of the form $\sum_{i=1}^{n} a_i \cdot x \cdot b_i$. Obviously, for another polynomial it is necessary to use another homomorphism and the quantity of generators depends on the quantity of polynomial terms.

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SURGERY AND THE RELATIVE INDEX IN ELLIPTIC THEORY

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This is a survey article featuring the general index locality principle introduced by the authors, which can be used to obtain index formulas for elliptic operators and Fourier integral operators in various situations, including operators on stratified manifolds and manifolds with singularities.

1. Introduction

This is a survey article intended as an elementary introduction to the general index locality principle introduced in [17]. We discuss it and give some examples of its consequences and applications showing that this principle often proves to be a powerful tool for obtaining index formulas in various situations. Having in mind the introductory character of the article, we try to keep the exposition at a level as elementary as possible and often give only the simplest versions of results. Proofs can be found elsewhere; we provide only references. A lack of reference usually means that more detailed explanations can be found in [17].

A detailed account of the history of the problem is also contained in [17]. Here we only cite papers where this is specifically needed in the text.

1.1. Elliptic operators. We start by recalling elementary notions of elliptic theory. Let M be a smooth compact manifold, and let D be a differential operator on M. In local coordinates, one has

$$D = \sum_{|\alpha|\le m} a_\alpha(x)\left(-i\frac{\partial}{\partial x}\right)^{\alpha}. \tag{1.1}$$

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The *principal symbol* (characteristic polynomial) of D, defined in local coordinates by the formula

$$\sigma(D) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha, \tag{1.2}$$

is an invariantly defined function on the cotangent bundle T^*M.

Definition 1.1. The operator D is said to be *elliptic* if $\sigma(D)$ is everywhere invertible on T_0^*M (i.e., for $\xi \neq 0$).

The following theorem is the main assertion concerning the analytic properties of elliptic operators.

THEOREM 1.2 (finiteness theorem). *If the operator D is elliptic, then it is Fredholm in the Sobolev spaces:*

$$D : H^s(M) \longrightarrow H^{s-m}(M). \tag{1.3}$$

Recall that D is said to be Fredholm if the following properties hold:
(i) $\operatorname{Im} D$ is closed;
(ii) $\dim\ker D < \infty$;
(iii) $\dim\ker D^* < \infty$.

1.2. The index. Suppose that D is an elliptic operator on M. Then $\dim\ker D$ and $\dim\ker D^*$ are not invariant under homotopies of D in the class of elliptic operators, but their difference $\operatorname{ind} D$ is already a homotopy invariant and *hence is of interest*.

The famous Atiyah-Singer theorem expresses the "infinite-dimensional" homotopy invariant $\operatorname{ind} D$ in terms of topological invariants of the principal symbol $\sigma(D)$.

However, things become far more complicated if we abandon the smooth compact setting and consider operators on singular or noncompact manifolds.

In this connection, a natural task is to give methods that can help one to solve the index problem (possibly reducing it to the Atiyah-Singer case) in situations not covered by the Atiyah-Singer theorem. The list of such situations includes
(i) boundary value problems (for which the index formula was obtained by Atiyah and Bott [5]);
(ii) elliptic operators on noncompact manifolds;
(iii) elliptic operators on manifolds with singularities;
(iv) quantized canonical transformations (Fourier integral operators);
and possibly many other situations as well.

2. Surgery and the superposition principle

There are numerous methods that can be applied in index problems. However, we focus our attention only on one method, namely, *surgery*, that is, cutting and pasting of manifolds and elliptic operators, together with the associated *superposition principle* valid for the relative index (or index increment) resulting from surgeries.

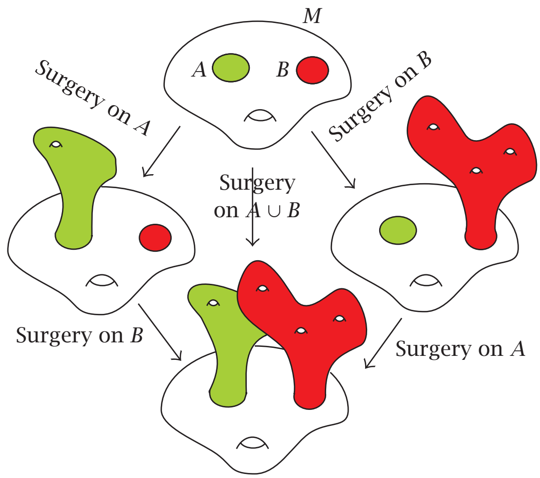

Figure 2.1. Commuting surgeries.

We will

(i) explain the superposition principle using a simple example;
(ii) give the general statement of the superposition principle;
(iii) provide some further examples and applications.

2.1. Example: operators on compact closed manifolds. Let D be an elliptic differential operator on a compact closed manifold M, and let $A, B \subset M$ be disjoint closed subsets of M (see Figure 2.1). Let us modify the operator D on A via some surgery. Namely, we cut away the piece A from M and replace it by some other (smooth) piece and continue the operator from the rest of M into the new piece in such a way that the resulting operator is elliptic. (Of course, we need to assume that this is possible.) Let us denote this new operator by D_A. The index of the new operator does not necessarily coincide with the index of D, and hence we see that our surgery at A results in the index increment (relative index)

$$\triangle_A \overset{\text{def}}{=} \operatorname{ind} D_A - \operatorname{ind} D. \tag{2.1}$$

In a similar way, we can modify the operator over B with the help of some surgery, thus obtaining a new operator D_B and the index increment

$$\triangle_B \overset{\text{def}}{=} \operatorname{ind} D_B - \operatorname{ind} D. \tag{2.2}$$

These processes are completely independent: when we modify the operator over A, we do not touch anything away from A (in particular, on B) and vice versa. Hence we can apply both modifications (surgeries) simultaneously, and the result for the operator will be the same as if we applied first one surgery and then the other, their order being irrelevant (see

Figure 2.1). The resulting operator will be denoted by $D_{A\cup B}$ and the index increment by

$$\triangle_{A\cup B} \stackrel{\text{def}}{=} \operatorname{ind} D_{A\cup B} - \operatorname{ind} D. \tag{2.3}$$

It is natural to ask how this "total" increment is related to the "partial" increments $\triangle_A$ and $\triangle_B$. The answer is exactly as it should be.

Lemma 2.1 (superposition principle).

$$\triangle_{A\cup B} = \triangle_A + \triangle_B. \tag{2.4}$$

Proof. This follows from the *local index formula*

$$\operatorname{ind} D = \int_M \alpha(x), \tag{2.5}$$

where the *local index density* $\alpha(x)$ at a point x depends only on $\sigma(D)$ and its derivatives in the fiber T_x^*M. Indeed, it suffices to note that, say,

$$\triangle_A = \int_A \big(\alpha(x) - \alpha'(x)\big), \tag{2.6}$$

where α' is the local index density corresponding to D_A, since $\alpha = \alpha'$ outside A. The desired formula follows since the integral is an additive set function. □

Remark 2.2. The superposition principle means that index increments stemming from independent surgeries behave additively.

2.2. General elliptic operators. Beautiful as it is, the superposition principle on smooth closed manifolds is generally not of much help when computing the index for two reasons.

(1) The index formula for the case of smooth closed manifold is already known (the Atiyah-Singer formula).
(2) The proof given above is hardly satisfactory, since it relies on the fact that we already know the (local) index formula.

Hence we wish to generalize this principle to cases beyond the Atiyah-Singer theorem and, moreover, invent a proof that does not rely on the presence of any *a priori* known index formula.

Thus the problem is as follows: *describe a sufficiently general framework in which the superposition principle for index increments is valid.*

One possible solution to this problem is to consider the class of *general elliptic operators* introduced by Atiyah [4]. Let us recall the relevant definitions.

Definition of general elliptic operators. Let X be a Hausdorff compactum, and let $C(X)$ be the algebra of continuous functions on X. Further, let H_1 and H_2 be Hilbert $*$-modules over $C(X)$, that is, Hilbert spaces equipped with a $*$-action of the C^*-algebra $C(X)$.

Definition 2.3. An operator

$$A : H_1 \longrightarrow H_2 \tag{2.7}$$

is called a general elliptic operator if the following two conditions are satisfied:

(i) A is Fredholm;

(ii) A almost commutes with the action of $C(X)$:

$$\varphi A - A\varphi \in \mathcal{K}(H_1, H_2) \quad \forall \varphi \in C(X), \tag{2.8}$$

where $\mathcal{K}(H_1, H_2)$ is the set of compact linear operators from H_1 to H_2.

Surgery and the superposition principle. To state the superposition principle, we should first define the notion of surgery for general elliptic operators. This is however intuitively clear. Let D_1 and D_2 be two general elliptic operators (with the same underlying compactum X). Next, let $A \subset X$ be a closed set.

Definition 2.4. We say that D_1 and D_2 are obtained from each other by a *modification* (or *surgery*) on A if for each function $\varphi \in C(X)$ whose support does not meet A (i.e., $\operatorname{supp} \varphi \cap A = \varnothing$) one has

$$\varphi D_1 \varphi \equiv \varphi D_2 \varphi \quad \text{modulo compact operators.} \tag{2.9}$$

In this case, we write $D_1 \xrightarrow{A} D_2$ or $D_2 \xrightarrow{A} D_1$.

This definition, however, needs further clarification: we did not assume that D_1 and D_2 act in the same spaces, so how can we compare $\varphi D_1 \varphi$ and $\varphi D_2 \varphi$? Let us give necessary explanations (which prove to be a bit technical).

If H is a Hilbert $*$-module over $C(X)$, then the notion of support $\operatorname{supp} u \subset X$ is well defined for all $u \in H$ in a natural way: a point x does *not* belong to the support of u if $\varphi u = 0$ for all $\varphi \in C(X)$ supported in a sufficiently small neighborhood of x.

We introduce the following notation: by $H_K \subset H$ we denote the closure of the set of elements $u \in H$ supported in K. Now if

$$D_1 : H_1 \longrightarrow G_1, \qquad D_2 : H_2 \longrightarrow G_2 \tag{2.10}$$

are general elliptic operators, then it makes sense to say that they are obtained from each other by a surgery on A if some isomorphisms

$$H_{1U} \longrightarrow H_{2U}, \qquad G_{1U} \longrightarrow G_{2U} \tag{2.11}$$

of $*$-modules over $C(X)$, where $U = X \setminus A$ is the complement of A, are given and fixed.

Definition 2.5 (definition of surgery on A revisited). We write

$$D_1 \xrightarrow{A} D_2 \tag{2.12}$$

if the diagram

$$
\begin{array}{ccc}
H_{1U} & \xrightarrow{\varphi D_1 \varphi} & G_{1U} \\
\downarrow & & \downarrow \\
H_{2U} & \xrightarrow{\varphi D_1 \varphi} & H_{2U}
\end{array}
\tag{2.13}
$$

commutes modulo compact operators for each $\varphi \in C(X)$ such that

$$\operatorname{supp} \varphi \cap A = \varnothing. \tag{2.14}$$

With this definition of surgery for general elliptic operators, the following superposition theorem holds for the index increments.

Theorem 2.6. *Let*

$$
\begin{array}{ccc}
D & \xrightarrow{A} & D_A \\
{\scriptstyle B}\downarrow & & \downarrow{\scriptstyle B} \\
D_B & \xrightarrow[A]{} & D_{AB}
\end{array}
\tag{2.15}
$$

be a commutative diagram (A diagram of surgeries is said to be commutative if the underlying isomorphisms of Hilbert spaces over $X \setminus (A \cap B)$ *form a commutative diagram.) of independent surgeries* ($A \cap B = \varnothing$) *of general elliptic operators over* $C(X)$. *Then*

$$\triangle_{AB} = \triangle_A + \triangle_B. \tag{2.16}$$

2.3. Operators in collar spaces. The theorem given in the preceding subsections does not cover applications related to Fourier integral operators (which do not almost commute with multiplication by functions). Furthermore, strictly speaking, it applies only to zero-order operators, since operators of positive order (in particular, any differential operators) do not compactly (or even boundedly) commute with continuous functions. So it is a good idea to devise a slightly different framework for the superposition principle, including the preceding as a special case.

This was done in [17], and we describe the corresponding results very briefly. The main ideas of the approach are as follows.

(1) We actually do not need arbitrary spaces X in applications of the superposition principle. If X is a compactum and $A, B \subset X$ are closed disjoint subsets, then there always exists a continuous mapping

$$f : X \longrightarrow [-1,1] \tag{2.17}$$

such that $A \subset f^{-1}(-1)$ and $B \subset f^{-1}(1)$. The mapping f induces the structure of a $C([-1,1])$-module on every $C(X)$-module, and so we can always assume that $X = [-1,1]$, $A = \{-1\}$, and $B = \{1\}$.

(2) Instead of $C([-1,1])$-modules one considers $C^\infty([-1,1])$-modules (for brevity referred to as *collar spaces*), which permit one to cover the case of positive-order operators (in particular, differential operators).

(3) Finally, instead of single operators one considers families of operators depending on a small parameter such that the "support of the kernel" for these operators tends to the diagonal as the parameter tends to zero. Thus for each given parameter value the operators need not be local; they are only "local in the limit." This permits one to consider a wider class of operators, including Fourier integral operators on manifolds with singularities, while the superposition principle remains true.

Let us give some more details; the reader uninterested in these details can skip the remaining part of the section and proceed to examples and applications.

Collar spaces. Collar spaces are a natural framework in which one can deal with surgeries and prove a rather general relative index theorem. They were introduced in [14, 16–18].

Consider the algebra $C^\infty([-1,1])$ of smooth functions $\varphi(t)$, $t \in [-1,1]$, on the interval $[-1,1]$ with topology given by the standard system of seminorms

$$\|\varphi\|_k = \sup_{t\in[-1,1]} \left|\varphi^{(k)}(t)\right|. \tag{2.18}$$

The multiplication in $C^\infty([-1,1])$ is defined pointwise. Obviously, this is a unital topological algebra with unit $\mathbf{1}$ being the function identically equal to 1 for all $t \in [-1,1]$.

Definition 2.7. A *collar space* is a separable Hilbert space H equipped with the structure of a module over the commutative algebra $C^\infty([-1,1])$ (the action is continuous, and the unit function $\mathbf{1} \in C^\infty([-1,1])$ acts as the identity operator in H).

Elliptic operators in collar spaces. For operators in function spaces for which the Schwartz kernel theorem holds, there is an important notion of the *support of the kernel*, which is a closed subset of the direct product of the set where the functions are defined by itself. Although operators in collar spaces cannot be described as integral operators in general, the notion of the *support of an operator* defined as a subset of the square $[-1,1]\times[-1,1]$ is meaningful and proves useful in studying various questions pertaining to the relative index.

Definition 2.8. Let $A : H_1 \to H_2$ be a continuous linear operator in collar spaces and $K \subset [-1,1]\times[-1,1]$ a closed subset. One says that *the support of A is contained in K* if

$$\operatorname{supp} Ah \subset K(\operatorname{supp} h) \tag{2.19}$$

for every $h \in H$. In formula (2.19), K is treated as a self-multimapping of the interval $[-1,1]$:

$$Kx \overset{\text{def}}{=} \{y \in [-1,1] \mid (x,y) \in K\}. \tag{2.20}$$

The intersection of all closed sets K with property (2.19) is called the *support* of A and denoted by $\operatorname{supp} A$.

Let $\Delta \subset [-1,1] \times [-1,1]$ be the diagonal

$$\Delta = \{(x,x) \mid x \in [-1,1]\}, \tag{2.21}$$

and let

$$\Delta_\varepsilon = \{(x,y) \in [-1,1] \times [-1,1] \mid |x-y| < \varepsilon\} \tag{2.22}$$

be the *ε-neighborhood of* Δ.

Definition 2.9. A *proper operator* in collar spaces H_1 and H_2 is a family of continuous linear operators

$$A_\delta : H_1 \longrightarrow H_2 \tag{2.23}$$

with parameter $\delta > 0$ such that
  (i) A_δ continuously depends on δ in the uniform operator topology;
  (ii) for each $\varepsilon > 0$ there is a $\delta_0 > 0$ such that

$$\operatorname{supp} A_\delta \subset \Delta_\varepsilon \quad \text{for } \delta < \delta_0. \tag{2.24}$$

Remark 2.10. Condition (2.24) can be restated as follows: for $\delta < \delta_0$, one has

$$\operatorname{supp} A_\delta h \subset U_\varepsilon(\operatorname{supp} h) \tag{2.25}$$

for every $h \in H_1$, where $U_\epsilon(F)$ is the ε-neighborhood of a set F.

Now we can give the definition of elliptic operators in collar spaces.

Definition 2.11. An *elliptic operator* in collar spaces H and G is a proper operator

$$D_\delta : H \longrightarrow G \tag{2.26}$$

such that D_δ is Fredholm for each δ and has an almost inverse $D_\delta^{[-1]}$ such that the family $D_\delta^{[-1]}$ is also a proper operator.

Here, as usual, the almost inverse of a bounded operator A is defined as an operator $A^{[-1]}$ such that the products $AA^{[-1]}$ and $A^{[-1]}A$ differ from the identity operators by compact operators in the corresponding spaces.

Definition 2.12. Let $F \subset [-1,1]$ be an open subset. One says that proper operators A_1 and A_2 *coincide on* F if for each compact subset $K \subset F$ the following condition is satisfied: there is a number $\delta_0 = \delta_0(K) > 0$ such that

$$A_{1\delta} h = A_{2\delta} h, \tag{2.27}$$

whenever $\delta < \delta_0$ and $\operatorname{supp} h \subset K$.

We note that for (2.27) to be well defined one has to assume that some isomorphisms of parts of the underlying Hilbert spaces where the operators act are given.

In the conditions of Definition 2.12, we say that A_1 is obtained from A_2 by a *modification* on $[-1,1]\backslash F$ (or A_1 *coincides with* A_2 on F) and write $A_1 \overset{F}{=} A_2$ or $A_1 \xleftrightarrow{[-1,1]\backslash F} A_2$.

Superposition principle. Now we are in a position to state the main theorem of this subsection.

THEOREM 2.13 [15, 17]. *Suppose that the following commutative diagram of modifications of elliptic operators in collar spaces holds:*

$$\begin{tikzcd} D \arrow[r, leftrightarrow, "-1"] \arrow[d, leftrightarrow, "1"'] & D_- \arrow[d, leftrightarrow, "1"] \\ D_+ \arrow[r, leftrightarrow, "-1"] & D_\pm \end{tikzcd} \tag{2.28}$$

Then

$$\operatorname{ind}(D) - \operatorname{ind}(D_-) = \operatorname{ind}(D_+) - \operatorname{ind}(D_\pm). \tag{2.29}$$

A detailed proof of this theorem (which however occupies less than two pages) can be found in [15].

3. Examples and applications

We consider examples from the following areas:
(i) elliptic operators on manifolds with singularities;
(ii) elliptic operators on noncompact manifolds;
(iii) boundary value problems;
(iiii) Fourier integral operators.

3.1. Elliptic operators on manifolds with conical singularities.

Cone-degenerate operators. Let M be a manifold with conical point a and base Λ of the cone (Figure 3.1; for definitions, e.g., see [6] or [13]).

Cone-degenerate differential operators on M near the conical point have the form of finite sums:

$$D = \sum a_{\alpha j}(\omega, r)\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha}\left(ir\frac{\partial}{\partial r}\right)^{j}, \tag{3.1}$$

where r is the distance from the conical point and ω is a coordinate on the base of the cone. The operator family

$$\sigma_c(D) = \sum a_{\alpha j}(\omega, 0)\left(-i\frac{\partial}{\partial\omega}\right)^{\alpha} p^{j} \tag{3.2}$$

on Ω is called the *conormal symbol* of D. Cone-degenerate operators are considered in weighted Sobolev spaces $H^{s,\gamma}(M)$ (for the definition, e.g., see [6]), and the ellipticity condition for cone-degenerate operators is that the interior principal symbol is invertible outside the zero section of the ("compressed" [11]) cotangent bundle of M minus the zero section and the conormal symbol is invertible on the weight line $\operatorname{Im} p = \gamma$. (In what follows, we for simplicity always assume that $\gamma = 0$.)

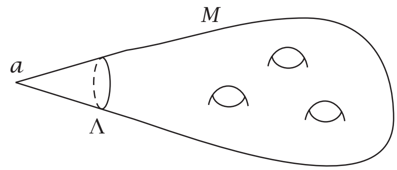

Figure 3.1. A manifold with conical singularities.

The index formula. Let M be a manifold with conical point a, and let D be an elliptic operator on M. Consider the problem of finding $\operatorname{ind} D$. This problem can be solved with the help of the index increment superposition principle under certain symmetry conditions on the interior principal symbol of D. Namely, the following theorem holds.

THEOREM 3.1. *Suppose that the interior principal symbol of D satisfies the symmetry condition*

$$\sigma(D)(\omega, r, q, -p) = f_1 \sigma(D)(\omega, r, q, p) f_2, \tag{3.3}$$

where f_1 and f_2 are bundle isomorphisms on M. Then

$$\operatorname{ind} D = \frac{1}{2}\left(\operatorname{ind} 2D + \operatorname{ind} D_c\right), \tag{3.4}$$

where $2D$ is the elliptic operator on the double of M whose principal symbol is obtained by clutching with the use of symmetry conditions and D_c is an operator on the spindle $S\Lambda$ explicitly constructed from the conormal symbol of D.

Thus the index of D is represented as the sum of two terms, one of which depends only on the interior principal symbol and can be expressed by the Atiyah-Singer index theorem (applied to the operator $2D$ on the compact closed manifold $2M$) and the other depends only on the conormal symbol.

This theorem was first obtained in [19] under the slightly stronger symmetry condition

$$\sigma_c(D)(p) = f_1 \sigma_c(D)(p_0 - p) f_2, \tag{3.5}$$

(where f_1 and f_2 are bundle automorphisms) imposed on the conormal rather than interior symbol. In this case, the second term in the index formula can be represented as a sum of multiplicities of poles of the operator family $\sigma_c(D)(p)^{-1}$ in a certain strip in the complex plane. For the general case, for example, see [17]; here the second term is expressed as the spectral flow of some homotopy of $\sigma_c(D)(p)$ to $f_1\sigma_c(D)(p_0 - p) f_2$.

The proof of this index formula is given by surgery in conjunction with the superposition principle; the nontrivial part of the surgery diagram is shown in Figure 3.2.

3.2. Elliptic operators on noncompact manifolds. Let X_0 and X_1 be noncompact manifolds, and let D_0 and D_1 be Fredholm elliptic operators in certain L^2 spaces on these manifolds. Suppose that these manifolds coincide at infinity. Namely, there are compact

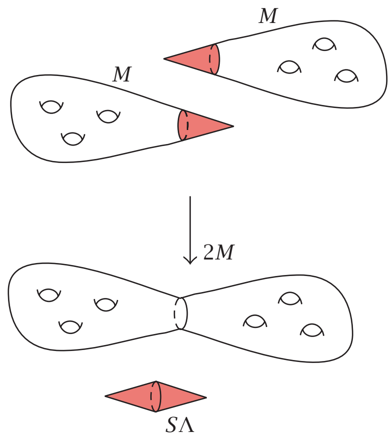

Figure 3.2. A surgery proving the index theorem for manifolds with conical singularities.

sets $K_j \subset X_j$ and a measure-preserving diffeomorphism

$$X_0 \setminus K_0 \overset{a}{\simeq} X_1 \setminus K_1 \tag{3.6}$$

such that

$$D_1 = \Phi D_0 \Psi^{-1}, \tag{3.7}$$

where Φ and Ψ are vector bundle isomorphisms over a.

We can compactify both manifolds by cutting the noncompact ends away along some compact hypersurface H and then glueing the same compact "cap" to both manifolds and continuing the operators in the same way to the cap.

The new elliptic operators on the new compact manifolds $\widetilde{X}_1$ and $\widetilde{X}_2$ thus obtained will be denoted by D_0 and D_1.

The superposition theorem implies the following assertion.

Theorem 3.2. *One has*

$$\operatorname{ind} D_1 - \operatorname{ind} D_0 = \operatorname{ind} \widetilde{D}_1 - \operatorname{ind} \widetilde{D}_0. \tag{3.8}$$

This theorem was proved for the special case of Dirac operators on complete Riemannian manifolds by Gromov and Lawson [8] and later extended to a more general class of operators by Anghel [3].

3.3. Boundary value problems. The index increment superposition principle has also well-known manifestations in boundary value problems.

Let M be a compact manifold with boundary ∂M, and let D be an elliptic operator on M; in local coordinates near the boundary,

$$D = \sum_{|\alpha|+\beta \leq m} a_{\alpha\beta}(x,t)\left(-i\frac{\partial}{\partial x}\right)^{\alpha}\left(-i\frac{\partial}{\partial t}\right)^{\beta}, \tag{3.9}$$

where we assume that the boundary is given by the equation $\partial M = \{t = 0\}$ and the interior of the manifold corresponds to positive t.

We consider classical boundary value problems of the form

$$\begin{aligned} Du &= f, \\ Bu|_{\partial M} &= g. \end{aligned} \tag{3.10}$$

For short, we denote such a problem by (D,B). Recall the ellipticity conditions for the problem (D,B). To obtain these conditions, one freezes the coefficients of the equation at some point $(x,0) \in \partial M$, drops away lower-order terms, and makes the Fourier transform with respect to the variables tangent to the boundary. Thus we obtain the ordinary differential operator

$$\widetilde{D}(x,\xi) = \sum_{|\alpha|+\beta = m} a_{\alpha\beta}(x,0)\xi^{\alpha}\left(-i\frac{\partial}{\partial t}\right)^{\beta} \tag{3.11}$$

on the half-line $\mathbb{R}_+$. This operator depends on the parameters $(x,\xi) \in T_0^*\partial M$.

Let $L_+ \equiv L_+(x,\xi)$ be the subspace of initial data at $t = 0$ for solutions of $\widetilde{D}v = 0$ decaying as $t \to \infty$.

Condition 3.3 (Shapiro-Lopatinskii). We require that $\sigma(B)\,|_{L_+}$ be an isomorphism for $\xi \neq 0$.

The main analytic theorem of the theory of boundary value problems is as follows.

THEOREM 3.4. *If the boundary value problem (D,B) satisfies the Shapiro-Lopatinskii condition, then it is Fredholm.*

Now we are in a position to state two relative index theorems for boundary value problems.

Let D_1 and D_2 be two elliptic operators coinciding near ∂M, and let B be a boundary operator satisfying the Shapiro-Lopatinskii conditions with respect to one (and hence both) of the operators.

The superposition principle implies the following assertion.

THEOREM 3.5. *One has*

$$\operatorname{ind}(D_1,B) - \operatorname{ind}(D_2,B) = \operatorname{ind} D, \tag{3.12}$$

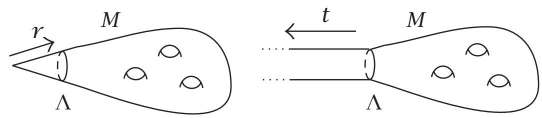

Figure 3.3. Cylindrical coordinates.

where D is an elliptic operator on M such that

$$\sigma(D) = \sigma(D_1)\sigma(D_2)^{-1} \tag{3.13}$$

and D acts as a system bundle isomorphism near ∂M.

Another relative index theorem deals, on the opposite, with the case of one operator and two boundary conditions.

Let D be an elliptic operator on M, and let B_1 and B_2 be two boundary operators satisfying the Shapiro-Lopatinskii condition.

Then it follows from the superposition principle that the theorem below holds.

Theorem 3.6. *One has*

$$\operatorname{ind}(D,B_1) - \operatorname{ind}(D,B_2) = \operatorname{ind} C, \tag{3.14}$$

where C is an elliptic operator on ∂M with

$$\sigma(C) = \sigma(B_1)|_{L_+}(\sigma(B_2)|_{L_+})^{-1}. \tag{3.15}$$

These two theorems are known as Agranovich and Agranovich-Dynin theorems (see [1, 2]). Surgery, in conjunction with the superposition principle, provides new, elementary proofs.

3.4. Quantized canonical transformations. The index problem for quantized canonical transformations (Fourier integral operators) was posed by Weinstein [20, 21], and its solution was obtained for smooth manifolds by Epstein and Melrose [7] in a particular case and by Leichtnam et al. [9] in the general case.

The superposition principle permits one to derive an index formula for quantized contact transformations on singular manifolds.

Let us briefly describe the construction of Fourier integral operators on a manifold with conical singularities.

Let M be a manifold with a conical singular point α and base Λ of the cone. (We assume for simplicity that there is only one conical point.) We will use the cylindrical model, that is, pass to from the coordinate r to the cylindrical coordinate t by the formula $r = e^{-t}$ (see Figure 3.3).

Quantized canonical transformations are obtained by the quantization of classical transformations, so let us say a few words about the latter.

Classical transformations. We will quantize homogeneous canonical (contact) transformations:

$$g : T_0^* M \longrightarrow T_0^* M. \tag{3.16}$$

Transformations associated with the conical structure should be continuous up to $r = 0$. In the t-coordinate, this corresponds to "exponential stabilization of the coefficients" as $t \to \infty$. For simplicity, we impose an even stronger condition that the coefficients "are independent of t for sufficiently large t." Stated precisely, this means the following.

Condition 3.7 (stabilization). The transformation g commutes with translations for along the t-axis for $t \gg 0$.

In other words,

$$g|_{t\gg 0} = g_\infty : T_0^* C \longrightarrow T_0^* C, \tag{3.17}$$

where $C = \mathbb{R} \times \Lambda$ is the infinite cylinder with base Λ and g_∞ commutes with translations.

Quantization. The quantized transformation is given by the Fourier integral operator associated with the graph $L_g \subset T_0^* M \times T_0^* M$ of the classical transformation g. This graph is a Lagrangian manifold, and we make the following assumption.

Assumption 3.8. The quantization condition (e.g., see [10, 12]) is satisfied for L_g (i.e., the *Maslov index* is zero on L_g).

Then the quantized canonical transformation

$$T_g : L^2(M) \longrightarrow L^2(M) \tag{3.18}$$

is defined in the usual manner as the Fourier integral operator with amplitude 1 associated with the Lagrangian manifold L_g. To ensure appropriate behavior near the conical point, we require that T_g commutes with translations along the t-axis for large t. This can be done in view of the similar condition imposed on g. We will assume that T_g is elliptic.

The index theorem. We impose the simplest symmetry condition on the classical transformation.

Condition 3.9. The transformation g_∞ commutes with the inversion $(t, p) \mapsto (-t, -p)$.

Then (in fact, after some homotopies) two copies of g can be glued into a canonical transformation $2g : T_0^* 2M \to T_0^* 2M$ of the double of the cotangent bundle $T_0^* M$.

Surgery and the superposition principle give the following formula for the index of the quantized canonical transformation.

Theorem 3.10. *The index of an elliptic quantized canonical transformation is given by the formula*

$$\operatorname{ind} T_g = \frac{1}{2}\left(\operatorname{ind} T_{2g} + \operatorname{ind} T_{g_\infty}\right), \tag{3.19}$$

where

(i) T_{2g} *is a quantized canonical transformation on the smooth closed manifold* $2M$*;*

(ii) T_{g_∞} *is a quantized canonical transformation on the cylinder* C*.*

Remark 3.11. (1) The index $\operatorname{ind} T_{2g}$ is computable by Epstein-Melrose theorem.

(2) In special cases $\operatorname{ind} T_{g_\infty}$ can be computed as the sum of multiplicities of poles of an operator family (called the conormal symbol) associated with T_g.

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