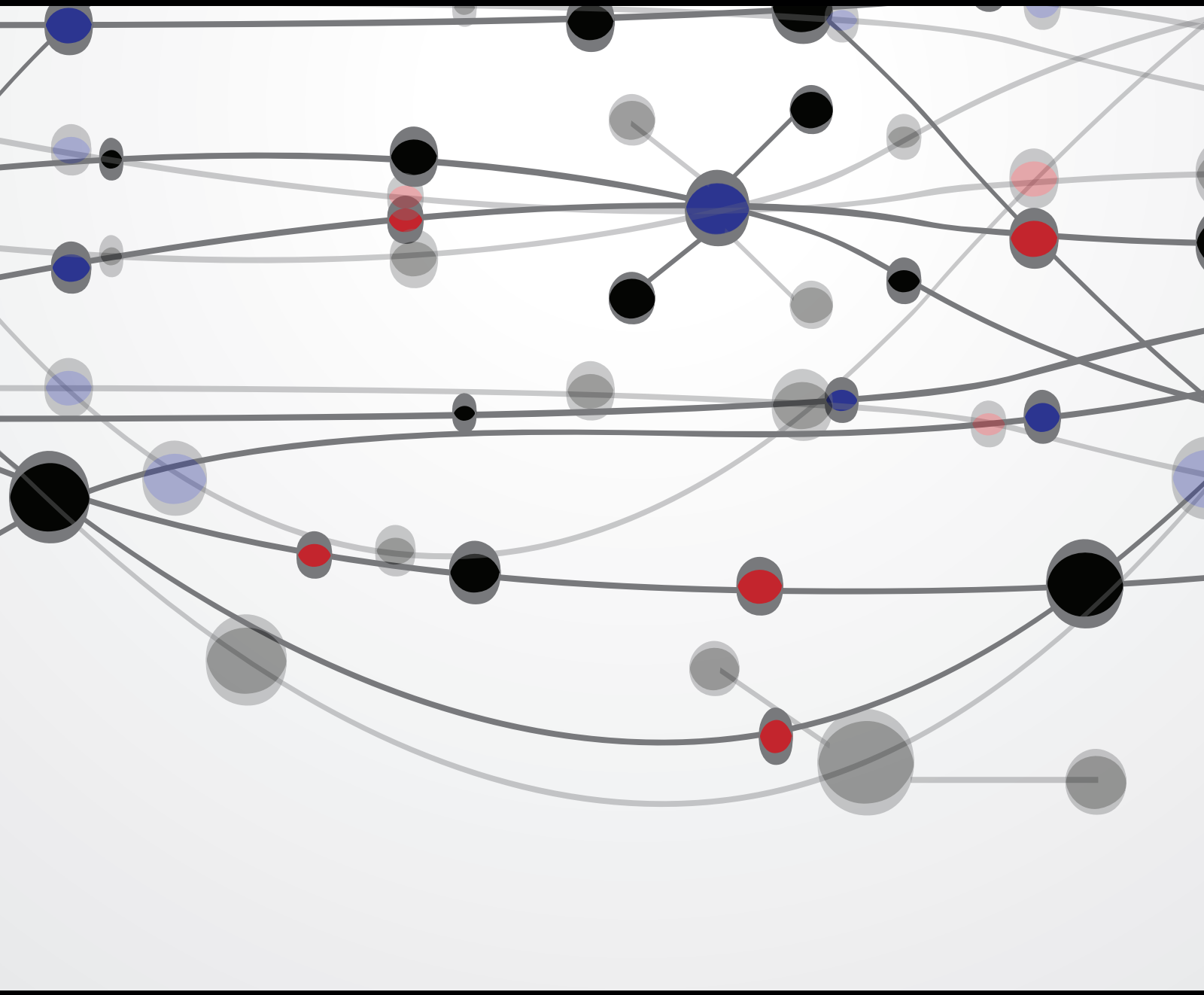
The Scientific World Journal

Theory and Application on Rough Set, Fuzzy Logic, and Granular Computing

Guest Editors: Xibei Yang, Weihua Xu, and Yanhong She



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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2015, Article ID 967350, 1 page
http://dx.doi.org/10.1155/2015/967350

Editorial

Theory and Application on Rough Set, Fuzzy Logic, and Granular Computing

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Received 8 June 2015; Accepted 8 June 2015

Recently, the rough set and fuzzy set theory have generated a great deal of interest among more and more researchers. Granular computing (GrC) is an emerging computing paradigm of information processing and an approach for knowledge representation and data mining. The purpose of granular computing is to seek for an approximation scheme which can effectively solve a complex problem at a certain level of granulation. This issue on the theory and application about rough set, fuzzy logic and granular computing, most of which are very meticulously performed reviews of the available current literature.

Four models of fuzzy or rough sets that are leading to a greater understanding of rough sets and fuzzy sets are discussed. These include multigranulation T-fuzzy rough sets, the so called approximation set of the interval set, the generalized interval-valued fuzzy rough set, and the δ-*cut* decision-theoretic rough set. Based on a kernelized information entropy model, an application on the fault detection and diagnosis for gas turbines is presented. The methods for reductions and their relevant algorithms are addressed in two manuscripts. Y. Zhang studies the distribution reduction in the inconsistent ordered information systems and further provides its algorithm. H. Ju et al. firstly give the model of δ-*cut* decision-theoretic rough set and then investigate the attribute reductions in this new decision-theoretic rough set model.

From the view of GrC, the optimistic multigranulation T-fuzzy rough set model was established based on multiple granulations under T-fuzzy approximation space by W. Xu. The manuscript of W. Li et al. improves the optimistic multigranulation T-fuzzy rough set deeply by investigating some further properties. And the relationships between multigranulation and classical T-fuzzy rough sets have been studied carefully. The interval set is a special fuzzy set, which describes uncertainty of an uncertain concept with its two crisp boundaries. Q. Zhang et al. review the similarity degrees between an interval-valued set and its two approximations and propose disadvantages of using upper approximation set or lower approximation as approximation sets of the uncertain set and present a new method for looking for a better approximation set of the interval set. T. Xue et al. also construct a novel model of the generalized fuzzy rough set under interval-valued fuzzy relation.

The aim of this special issue is to encourage researchers in related areas to discuss and communicate the latest advancements of rough set, fuzzy logic, and GrC, which covers both theoretical and practical results.

Xibei Yang
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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 617162, 13 pages
http://dx.doi.org/10.1155/2014/617162

Research Article

Fault Detection and Diagnosis for Gas Turbines Based on a Kernelized Information Entropy Model

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Received 28 April 2014; Accepted 19 June 2014; Published 28 August 2014

Academic Editor: Xibei Yang

Gas turbines are considered as one kind of the most important devices in power engineering and have been widely used in power generation, airplanes, and naval ships and also in oil drilling platforms. However, they are monitored without man on duty in the most cases. It is highly desirable to develop techniques and systems to remotely monitor their conditions and analyze their faults. In this work, we introduce a remote system for online condition monitoring and fault diagnosis of gas turbine on offshore oil well drilling platforms based on a kernelized information entropy model. Shannon information entropy is generalized for measuring the uniformity of exhaust temperatures, which reflect the overall states of the gas paths of gas turbine. In addition, we also extend the entropy to compute the information quantity of features in kernel spaces, which help to select the informative features for a certain recognition task. Finally, we introduce the information entropy based decision tree algorithm to extract rules from fault samples. The experiments on some real-world data show the effectiveness of the proposed algorithms.

1. Introduction

Gas turbines, mechanical systems operating on a thermodynamic cycle, usually with air as the working fluid, are considered as one kind of the most important devices in power engineering, where the air is compressed, mixed with fuel, and burnt in a combustor, with the generated hot gas expanded through a turbine to generate power, which is used for driving the compressor and for providing the means to overcome external loads. Gas turbines play an increasingly important role in the domains of mechanical drives in the oil and gas sectors, electricity generation in the power sector, and propulsion systems in the aerospace and marine sectors.

Safety and economy are always two fundamentally important factors in designing, producing, and operating gas turbine systems. Once a malfunction occurs to a gas turbine, a serious accident, even disaster, may take place. It was reported that about 25 accidents take place every year due to jet malfunctioning. In 1989, 111 were killed in a plane crash due to an engine fault. Although great progress has been made these years in the area of condition monitoring and fault diagnosis, how to predict and detect malfunctions is still an open problem for the complex systems. In some cases, such as offshore oil well drilling platforms, the main power system is self-monitoring without man on duty. So the reliability and stabilization are of critical importance to these systems. There are hundreds of offshore platforms with gas turbines providing electricity and powers in China. There is an urgent requirement to design and develop online remote monitoring and health management techniques for these systems.

More than two hundred sensors are installed in each gas turbine for monitoring the state of a gas turbine. The data gathered by these sensors reflects the state and trend of the system. If we build a center to monitor two hundred gas turbine systems, we should watch the data coming from more than forty thousand sensors. Obviously, it is infeasible to manually analyze them. Techniques on intelligent data analysis have been employed in gas turbine monitoring and diagnosis. In 2007, Wang et al. designed a conceptual system for remote monitoring and fault diagnosis of gas turbine-based power generation systems [1]. In 2008, Donat et al. discussed the issue of data visualization, data reduction,

and ensemble learning for intelligent fault diagnosis in gas turbine engines [2]. In 2009, Li and Nilkitsaranont described a prognostic approach to estimating the remaining useful life of gas turbine engines before their next major overhaul based on a combined regression technique with both linear and quadratic models [3]. In the same year, Bassily et al. proposed a technique, which assessed whether or not the multivariate autocovariance functions of two independently sampled signals coincide, to detect faults in a gas turbine [4]. In 2010, Young et al. presented an offline fault diagnosis method for industrial gas turbines in a steady-state using Bayesian data analysis. The authors employed multiple Bayesian models via model averaging for improving the performance of the resulted system [5]. In 2011, Yu et al. designed a sensor fault diagnosis technique for Micro-Gas Turbine Engine based on wavelet entropy, where wavelet decomposition was utilized to decompose the signal in different scales, and then the instantaneous wavelet energy entropy and instantaneous wavelet singular entropy are computed based on the previous wavelet entropy theory [6].

In recent years, signal processing and data mining techniques are combined to extract knowledge and build models for fault diagnosis. In 2012, Wu et al. studied the issue of bearing fault diagnosis based on multiscale permutation entropy and support vector machine [7]. In 2013, they designed a technique for defecting diagnostics based on multiscale analysis and support vector machines [8]. Nozari et al. presented a model-based robust fault detection and isolation method with a hybrid structure, where time-delay multilayer perceptron models, local linear neurofuzzy models, and linear model tree were used in the system [9]. Sarkar et al. [10] designed symbolic dynamic filtering by optimally partitioning sensor observation, and the objective is to reduce the effects of sensor noise level variation and magnify the system fault signatures. Feature extraction and pattern classification are used for fault detection in aircraft gas turbine engines.

Entropy is a fundamental concept in the domains of information theory and thermodynamics. It was first defined to be a measure of progressing towards thermodynamic equilibrium; then it was introduced in information theory by Shannon [11] as a measure of the amount of information that is missing before reception. This concept gets popular in both domains [12–16]. Now it is widely used in machine learning and data driven modeling [17, 18]. In 2011, a new measurement, called maximal information coefficient, was reported. This function can be used to discover the association between two random variables [19]. However, it cannot be used to compute the relevance between feature sets.

In this work, we will develop techniques to detect abnormality and analyze faults based on a generalized information entropy model. Moreover, we also describe a system for state monitoring of gas turbines on offshore oil well drilling platforms. First we will describe a system developed for remote and online condition monitoring and fault diagnosis of gas turbines installed on oil drilling platforms. As vast amount of historical records is gathered in this system, it is an urgent task to design algorithms for automatically online detecting abnormality of the data and analyze the data to obtain the causes and sources of faults. Due to the complexity of gas turbine systems, we focus on the gas-path subsystem in this work. The function of entropy is employed to measure the uniformity of exhaust temperatures, which is a key factor reflecting the health of the gas path of a gas turbine and also reflecting the performance of the gas turbine. Then we extract features from the healthy and abnormal records. An extended information entropy model is introduced to evaluate the quality of these features for selecting informative attributes. Finally, the selected features are used to build models for automatic fault recognition, where support vector machines [20] and C4.5 are considered. Real-world data are collected to show the effectiveness of the proposed techniques.

The remainder of the work is organized as follows. Section 2 describes the architecture of the remote monitoring and fault diagnosis center for gas turbines installed on the oil drilling platforms. Section 3 designs an algorithm for detecting abnormality of the exhaust temperatures. Then we extract features from the exhaust temperature data and select informative ones based on evaluating the information bottlenecks with extend information entropy in Section 4. Support vector machines and C4.5 are introduced for building fault diagnosis models in Section 5. In addition, numerical experiments are also described in this section. Finally, conclusions and future work are given in Section 6.

2. Framework of Remote Monitoring and Fault Diagnosis Center for Gas Turbine

Gas turbines are widely used as power and electric power sources. The structure of a general gas turbine is presented in Figure 1. This system transforms chemical energy into thermal power, then mechanical energy, and finally electric energy. Gas turbines are usually considered as the hearts of a lot of mechanical systems.

As the offshore oil well drilling platforms are usually unattended, an online and remote state monitoring system is much useful in this area, which can help find abnormality before serious faults occur. However, the sensor data cannot be sent into a center with ground based internet. The data can only be transmitted via telecommunication satellite, which was too expensive in the past. Now this is available.

The system consists of four subsystems: data acquisition and local monitoring subsystem (DALM), data communication subsystem (DAC), data management subsystem (DMS), and intelligent diagnosis system (IDS). The first subsystem gathers the outputs from different sensors and checks whether there is any abnormality in the system. The second one packs the acquired data and transforms them into the monitoring center. Users in the center can also send a message to this subsystem to ask for some special data if abnormality or fault occurs. The data management subsystem stores the historic information and also fault data and fault cases. A data compression algorithm is embedded in the system. As most of the historic data are useless for the final analysis, they will be compressed and removed for saving storage space. Finally, IDS watches the alarm information from different unit assemblies and starts the corresponding module to analyze the related information. This system gives

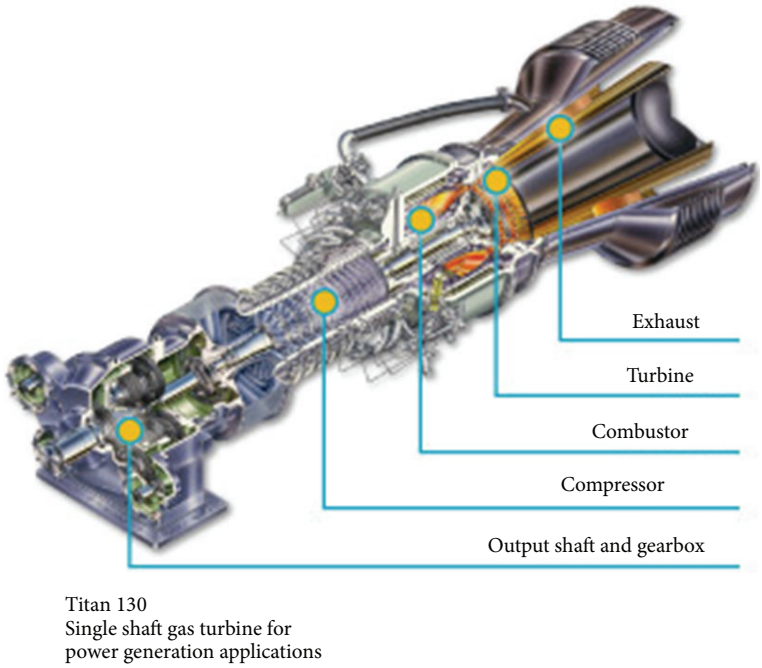


Figure 1: Prototype structure of a gas turbine.

some decision and explains how the decision has been made. The structure of the system is shown in Figure 2.

One of the webpages of the system is given in Figure 3, where we can see the rose figure of exhaust temperatures, and some statistical parameters varying with time are also presented.

3. Abnormality Detection in Exhaust Temperatures Based on Information Entropy

Exhaust temperature is one of the most critical parameters in a gas turbine as excessive turbine temperatures may lead to life reduction or catastrophic failures. In the current generation of machines, temperatures at the combustor discharge are too high for the type of instrumentation available. Exhaust temperature is also used as an indicator of turbine inlet temperature.

As the temperature profile out of a gas turbine is not uniform, a number of probes will help pinpoint disturbances or malfunctions in the gas turbine by highlighting the shifts in the temperature profile. Thus there are usually a set of thermometers fixed on the exhaust. If the system is normally operating, all the thermometers give similar outputs. However, if a fault occurs to some components of the turbine, different temperatures will be observed. The uniformity of exhaust temperatures reflects the state of the system. So we should develop an index to measure the uniformity of the exhaust temperatures. In this work, we consider the entropy function for it is widely used in measuring uniformity of random variables. However, to the best of our knowledge, this function has not been used in this domain.

Assume that there are n thermometers and their outputs are T_i, $i = 1, \ldots, n$, respectively. Then we define the uniformity of these outputs as

$$E(T) = -\sum_{i=1}^{n} \frac{T_i}{T} \log_2 \frac{T_i}{T}, \tag{1}$$

where $T = \sum_j T_j$. As $T_i \geq 0$, we define $0 \log 0 = 0$.

Obviously, we have $\log_2 n \geq E(T) \geq 0$. $E(T) = \log_2 n$ if and only if $T_1 = T_2 = \cdots = T_n$. In this case, all the thermometers produce the same output. So the uniformity of the sensors is maximal. In another extreme case, if $T_1 = T_2 = T_{i-1} = T_{i+1} \cdots = T_n = 0$ and $T_i = T$, then $E(T) = 0$.

It is notable that the value of entropy is independent of the values of thermometers, while it depends on the distribution of the temperatures. The entropy is maximal if all the thermometers output the same values.

Now we show two sets of real exhaust temperatures measured on an oil well drilling platform, where 13 thermometers are fixed. In the first set, the gas turbine starts from a time point and then runs for several minutes; finally the system stops.

Observing the curves in Figure 4, we can see that the 13 thermometers give the almost the same outputs at the beginning. In fact, the outputs are the room temperature in this case, as shown in Figure 6(a). Thus, the entropy reaches the peak value.

Some typical samples are presented in Figure 6, where the temperature distributions around the exhaust at time points $t = 5$,130,250,400, and 500 are given. Obviously, the distributions at $t = 130$,250, and 400 are not desirable. It can be derived that some abnormality occurs to the system. The entropy of temperature distribution is given in Figure 5.

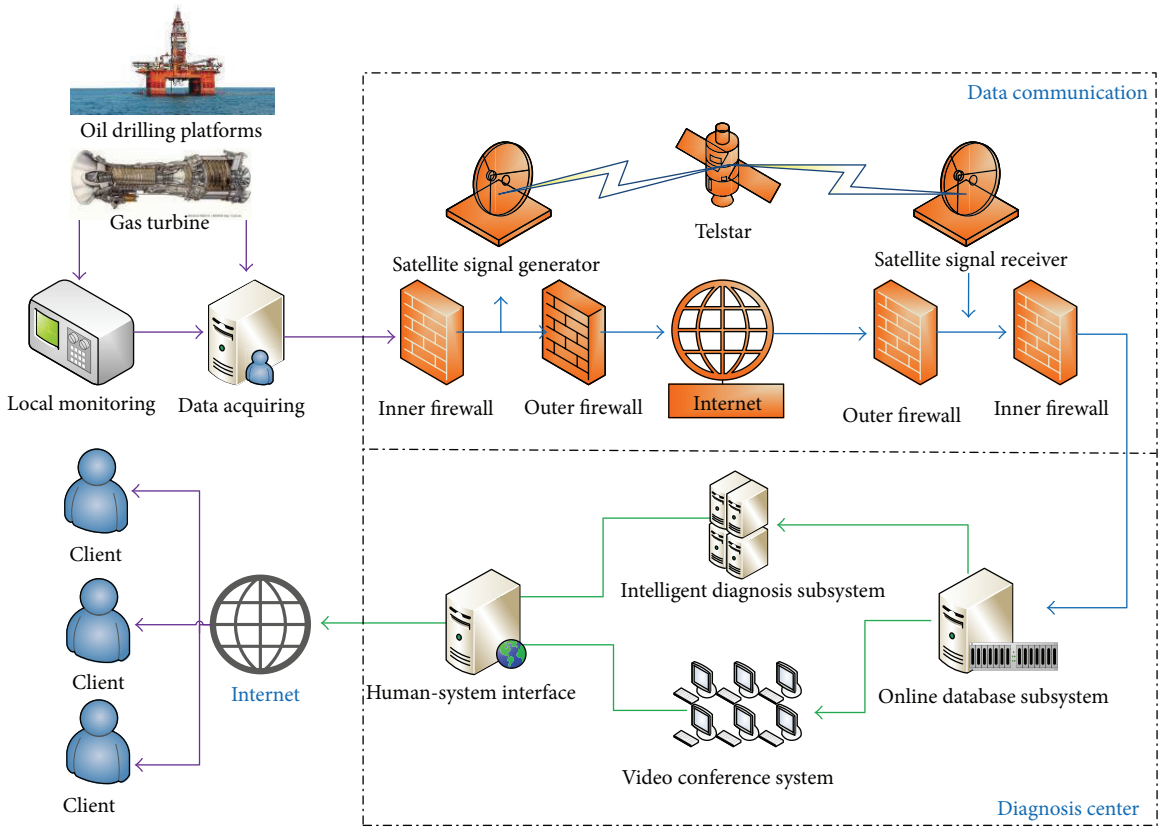


Figure 2: Structure of the remote system of condition monitoring and fault analysis.

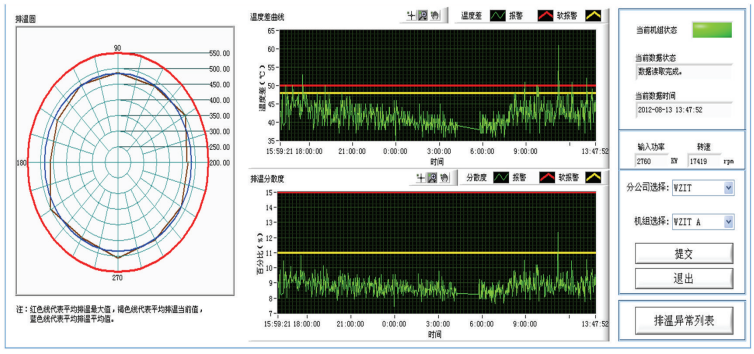


Figure 3: A typical webpage for monitoring of the subsystem.

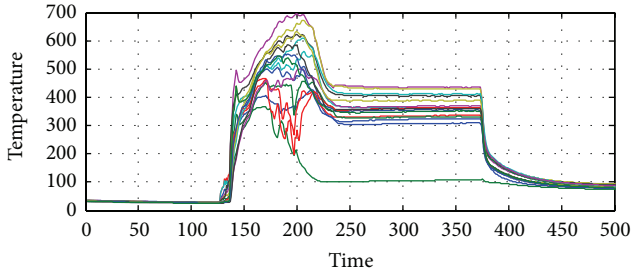


Figure 4: Exhaust temperatures from a set of thermometers.

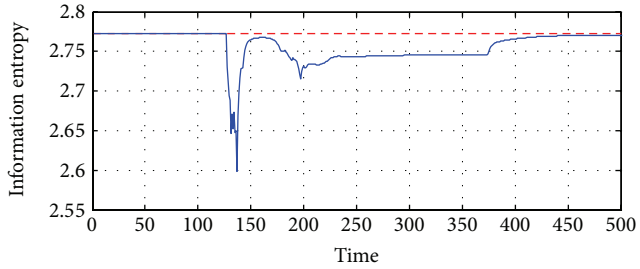
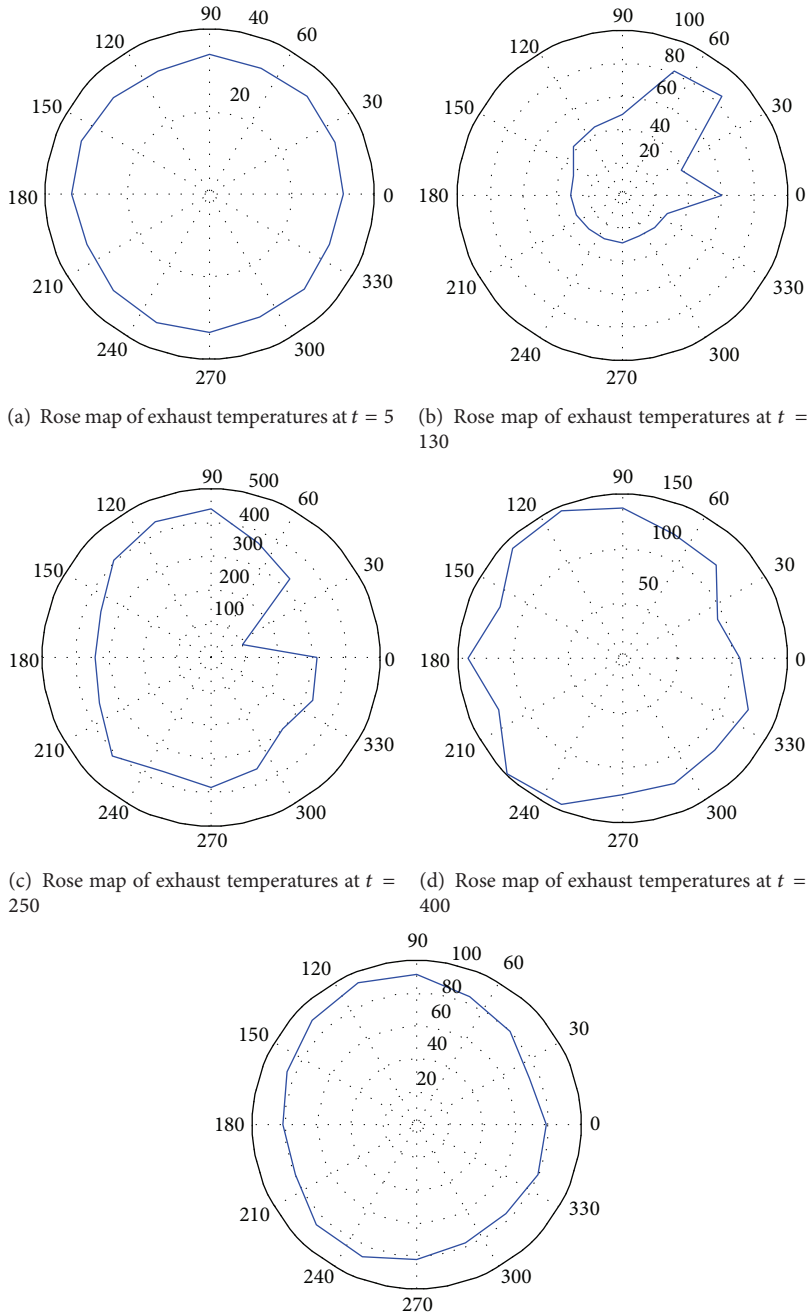


Figure 5: Uniformity of the temperatures (red dash line is the ideal case; blue line is the real case).



(a) Rose map of exhaust temperatures at $t = 5$

(b) Rose map of exhaust temperatures at $t = 130$

(c) Rose map of exhaust temperatures at $t = 250$

(d) Rose map of exhaust temperatures at $t = 400$

(e) Rose map of exhaust temperatures at $t = 500$

Figure 6: Samples of temperature distribution in different times.

Another example is also given in Figures 7 to 9. In this example, there is significant difference between the outputs of 13 thermometers even when the gas turbine is not running, just as shown in Figure 9(a). Thus the entropy of temperature distribution is a little lower than the ideal case, as shown in Figure 8. Besides, some representative samples are also given in Figure 9.

Considering the above examples, we can see that the function of entropy is an effective measurement of uniformity. It can be used to reflect the uniformity of exhaust temperatures. If the uniformity is less than a threshold, some faults possibly occur to the gas path of the gas turbine. Thus the entropy function is used as an index of the health of the gas path.

4. Fault Feature Quality Evaluation with Generalized Entropy

The above section gives an approach to detecting the abnormality in the exhaust temperature distribution. However, the function of entropy cannot distinguish what kind of faults

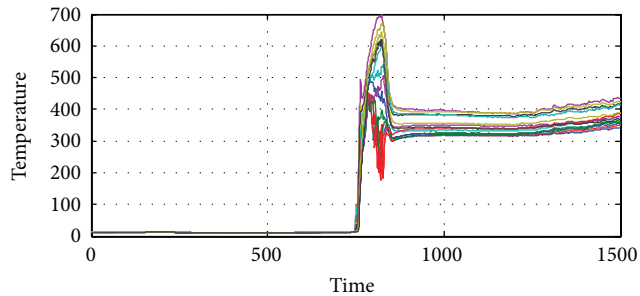


Figure 7: Exhaust temperatures from another set of thermometers.

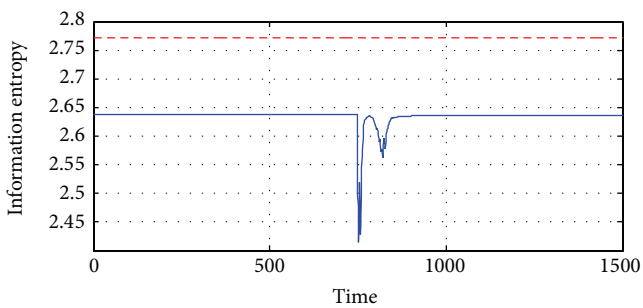


Figure 8: Entropy of the temperature distribution, where the red dash line is the ideal case and the blue one is the real case.

occurs to the system although it detects abnormality. In order to analyze why the temperature distribution is not uniform, we should develop some algorithms to recognize the fault.

Before training an intelligent model, we should construct some features and select the most informative subsets to represent different faults. In this section, we will discuss this issue.

Intuitively, we know that the temperatures of all thermometers reflect the state of the system. Besides, the temperature difference between neighboring thermometers also indicates the source of faults, which are considered as space neighboring information. Moreover, we know the temperature change of a thermometer necessarily gives hints to study the faults, which can be viewed as time neighboring information. In fact, the inlet temperature T_0 is also an important factor. In summary, we can use exhaust temperatures and their neighboring information along time and space to recognize different faults. If there are n ($n = 13$ in our system) thermometers, we can form a feature vector to describe the state of the exhaust system as

$$F = \{T_0, T_1, T_2, \ldots, T_n, T_1 - T_2, T_2 - T_3, \ldots, T_n - T_1, T_1', T_2', \ldots, T_n'\}, \tag{2}$$

where $T_i' = T_i(j) - T_i(j-1)$. $T_i(j)$ is the temperature at time j of the ith thermometer.

Apart from the above features, we can also construct other attributes to reflect the conditions of the gas turbine. In this work, we consider a gas turbine with 13 thermometers around the exhaust. So we can form a 40-attribute vector finally.

There are some questions whether all the extracted features are useful for final modeling and how we can evaluate the features and find the most informative features. In fact, there are a number of measures to estimate feature quality, such as dependency in the rough set theory [21], consistency [22], mutual information in the information theory [23], and classification margin in the statistical learning theory [24]. However, all these measures are computed in the original input space, while the effective classification techniques usually implement a nonlinear mapping of the original space to a feature space by a kernel function. In this case, we require a new measure to reflect the classification information of the feature space. Now we extend the traditional information entropy to measure it.

Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$. As to classification learning, each training sample x_i is associated with a decision y_i. As to an arbitrary subset $F' \subseteq F$ and a kernel function K, we can calculate a kernel matrix

$$K = \begin{bmatrix} k_{11} & \ldots & k_{1m} \\ \vdots & \ddots & \vdots \\ k_{m1} & \ldots & k_{mm} \end{bmatrix}, \tag{3}$$

where $k_{ij} = k(x_i, x_j)$. The Gaussian function is a representative kernel function:

$$k_{ij} = \exp\left(-\frac{\|x_i - x_j\|^2}{\sigma}\right). \tag{4}$$

A number of kernel functions have the properties (1) $k_{ij} \in [0, 1]$; (2) $k_{ij} = k_{ji}$.

Kernel matrix plays a bottleneck role in kernel based learning [25]. All the information that a classification algorithm can use is hidden in this matrix. In the same time, we can also calculate a decision kernel matrix as

$$D = \begin{bmatrix} d_{11} & \ldots & d_{1m} \\ \vdots & \ddots & \vdots \\ d_{m1} & \ldots & d_{mm} \end{bmatrix}, \tag{5}$$

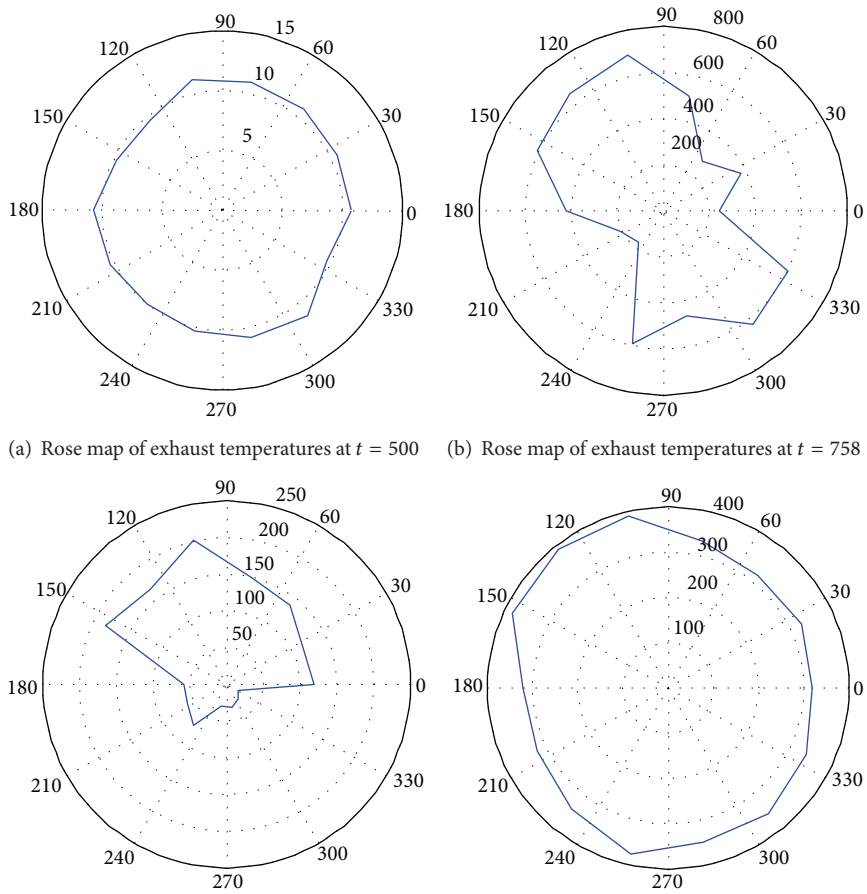
where $d_{ij} = 1$ if $y_i = y_j$; otherwise, $d_{ij} = 0$. In fact, the matrix D is a matching kernel.

Definition 1. Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$. $F' \subseteq F$, K is a kernel matrix over U in terms of F'. Then the entropy of F' is defined as

$$E(K) = -\frac{1}{m}\sum_{i=1}^{m}\log_2\frac{K_i}{m}, \tag{6}$$

where $K_i = \sum_{j=1}^{m} k_{ij}$.

As to the above entropy function, if we use Gaussian function as the kernel, we have $\log_2 m \geq E(K) \geq 0$. $E(K) = 0$ if and only if $k_{ij} = 1$ $\forall i, j$. $E(K) = \log_2 m$ if and only if $k_{ij} = 0$, $i \neq j$. $E(K) = 0$ means that any pair of samples cannot be



(a) Rose map of exhaust temperatures at $t = 500$

(b) Rose map of exhaust temperatures at $t = 758$

(c) Rose map of exhaust temperatures at $t = 820$

(d) Rose map of exhaust temperatures at $t = 1220$

Figure 9: Samples of temperature distribution in different moments.

distinguished with the current features, while $E(K) = \log_2 m$ means any pair of samples is different from each other. So they can be distinguished. These are two extreme cases. In real-world applications, part of samples can be discerned with the available features, while others are not. In this case, the entropy function takes value in the interval $[0, \log_2 m]$.

Moreover, it is easy to show that if $K_1 \subseteq K_2$, $E(K_1) \geq E(K_2)$, where $K_1 \subseteq K_2$ means $K_1(x_i, x_j) \leq K_2(x_i, x_j)$, $\forall i, j$.

Definition 2. Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$. $F_1, F_2 \subseteq F$. K_1 and K_2 are two kernel matrices induced by F_1 and F_2. K is a new function computed with $F_1 \cup F_2$. Then the joint entropy of F_1 and F_2 is defined as

$$E(K_1, K_2) = E(K) = -\frac{1}{m}\sum_{i=1}^{m}\log_2 \frac{K_i}{m}, \tag{7}$$

where $K_i = \sum_{j=1}^{m} k_{ij}$.

As to the Gaussian function, $K(x_i, x_j) = K_1(x_i, x_j) \times K_2(x_i, x_j)$. Thus $K \subseteq K_1$ and $K \subseteq K_2$. In this case, $E(K) \geq E(K_1)$ and $E(K) \geq E(K_2)$.

Definition 3. Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$.One has $F_1, F_2 \subseteq F$. K_1 and K_2 are two kernel matrices induced by F_1 and F_2. K is a new kernel function computed with $F_1 \cup F_2$. Knowning F_1, the condition entropy of F_2 is defined as

$$E(K_1 \mid K_2) = E(K) - E(K_1). \tag{8}$$

As to the Gaussian kernel, $E(K) \geq E(K_1)$ and $E(K) \geq E(K_2)$, so $E(K_1 \mid K_2) \geq 0$ and $E(K_2 \mid K_1) \geq 0$.

Definition 4. Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$. One has $F_1, F_2 \subseteq F$. K_1 and K_2 are two kernel matrices induced by F_1 and F_2. K is a new kernel function computed with $F_1 \cup F_2$. Then the mutual information of K_1 and K_2 is defined as

$$\mathrm{MI}(K_1, K_2) = E(K_1) + E(K_2) - E(K). \tag{9}$$

As to Gaussian kernel, $\mathrm{MI}(K_1, K_2) = \mathrm{MI}(K_2, K_1)$. If $K_1 \subseteq K_2$, we have $\mathrm{MI}(K_1, K_2) = E(K_2)$ and if $K_2 \subseteq K_1$, we have $\mathrm{MI}(K_1, K_2) = E(K_1)$.

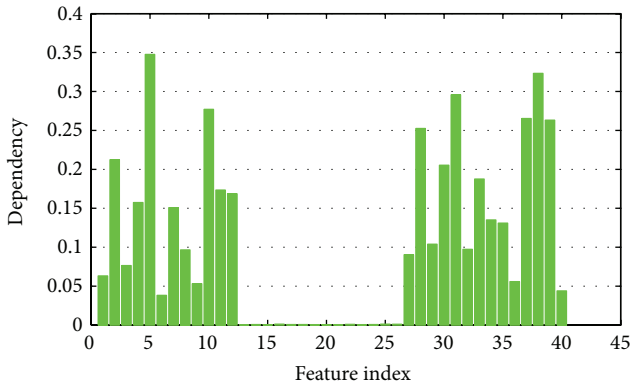


Figure 10: Fuzzy dependency between a single feature and decision.

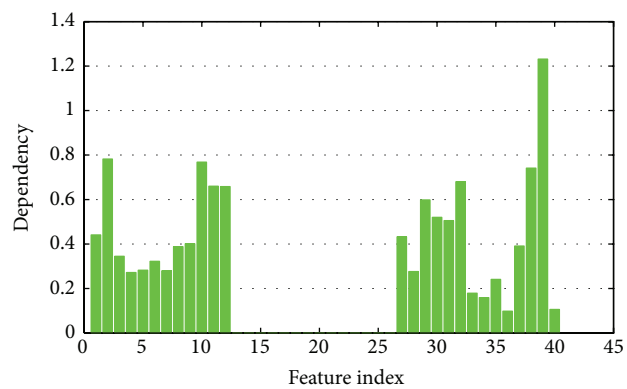


Figure 11: Kernelized mutual information between a single feature and decision.

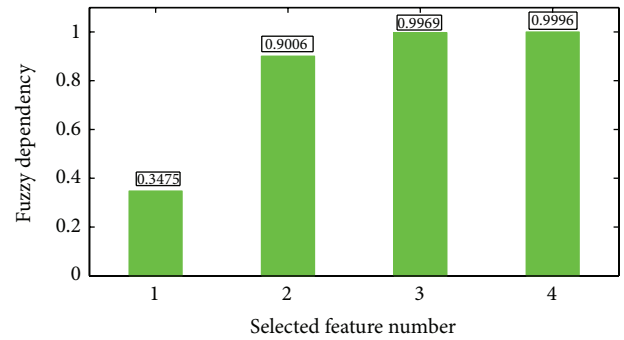


Figure 12: Fuzzy dependency between the selected features and decisions (Features 5, 37, 2, and 3 are selected sequentially).

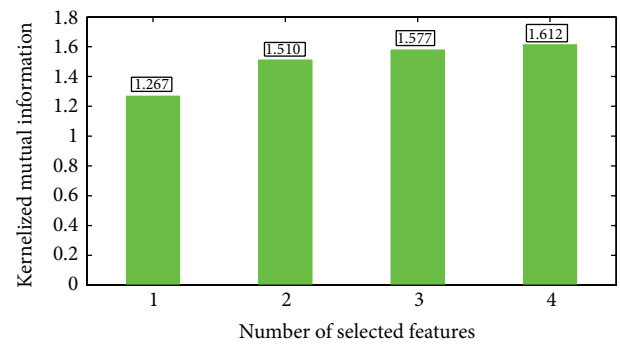


Figure 13: Kernelized mutual information between the selected features and decisions (Features 39, 31, 38, and 40 are selected sequentially).

Please note that if $F_1 \subseteq F_2$, we have $K_2 \subseteq K_1$. However, $K_2 \subseteq K_1$ does not mean $F_1 \subseteq F_2$.

Definition 5. Given a set of samples $U = \{x_1, x_2, \ldots, x_m\}$, each sample is described with n features $F = \{f_1, f_2, \ldots, f_n\}$. $F' \subseteq F$, K is a kernel matrix over U in terms of F', and D is the kernel matrix computed with the decision. Then the feature significance F' related to the decision is defined as

$$\mathrm{MI}(K, D) = E(K) + E(D) - E(K, D). \tag{10}$$

$\mathrm{MI}(K, D)$ measures the importance of feature subset F' in the kernel space to distinguish different classes. It can be understood as a kernelized version of Shannon information entropy, which is widely used feature evaluation selection. In fact, it is easy to derive the equivalence between this entropy function and Shannon entropy in the condition that the attributes are discrete and the matching kernel is used.

Now we show an example in gas turbine fault diagnosis. We collect 3581 samples from two sets of gas turbine systems. 1440 samples are healthy and the others belong to four kinds of faults: load rejection, sensor fault, fuel switching, and salt spray corrosion. The numbers of samples are 45, 588, 71, and 1437, respectively. Thirteen thermometers are installed in the exhaust. According to the approach described above, we form a 40-dimensional vector to represent the state of the exhaust. Obviously, the classification task is not understandable in such high dimensional space. Moreover, some features may be redundant for classification learning, which may confuse the learning algorithm and reduce modeling performance. So it is a key preprocessing step to select the necessary and sufficient subsets.

Here we compare the fuzzy rough set based feature evaluation algorithm with the proposed kernelized mutual information. Fuzzy dependency has been widely discussed and applied in feature selection and attribute reduction these years [26–28]. Fuzzy dependency can be understood as the average distance from the samples and their nearest neighbor belonging to different classes, while the kernelized mutual information reflects the relevance between features and decision in the kernel space.

Comparing Figures 10 and 11, significant difference is obtained. As to fuzzy rough sets, Feature 5 produces the largest dependency and then Feature 38. However, Feature 39 gets the largest mutual information, and Feature 2 is the second one. Thus different feature evaluation functions will lead to completely different results.

Figures 10 and 11 present the significance of single features. In applications, we should combine a set of features. Now we consider a greedy search strategy. Starting from an empty set and the best features are added one by one. In

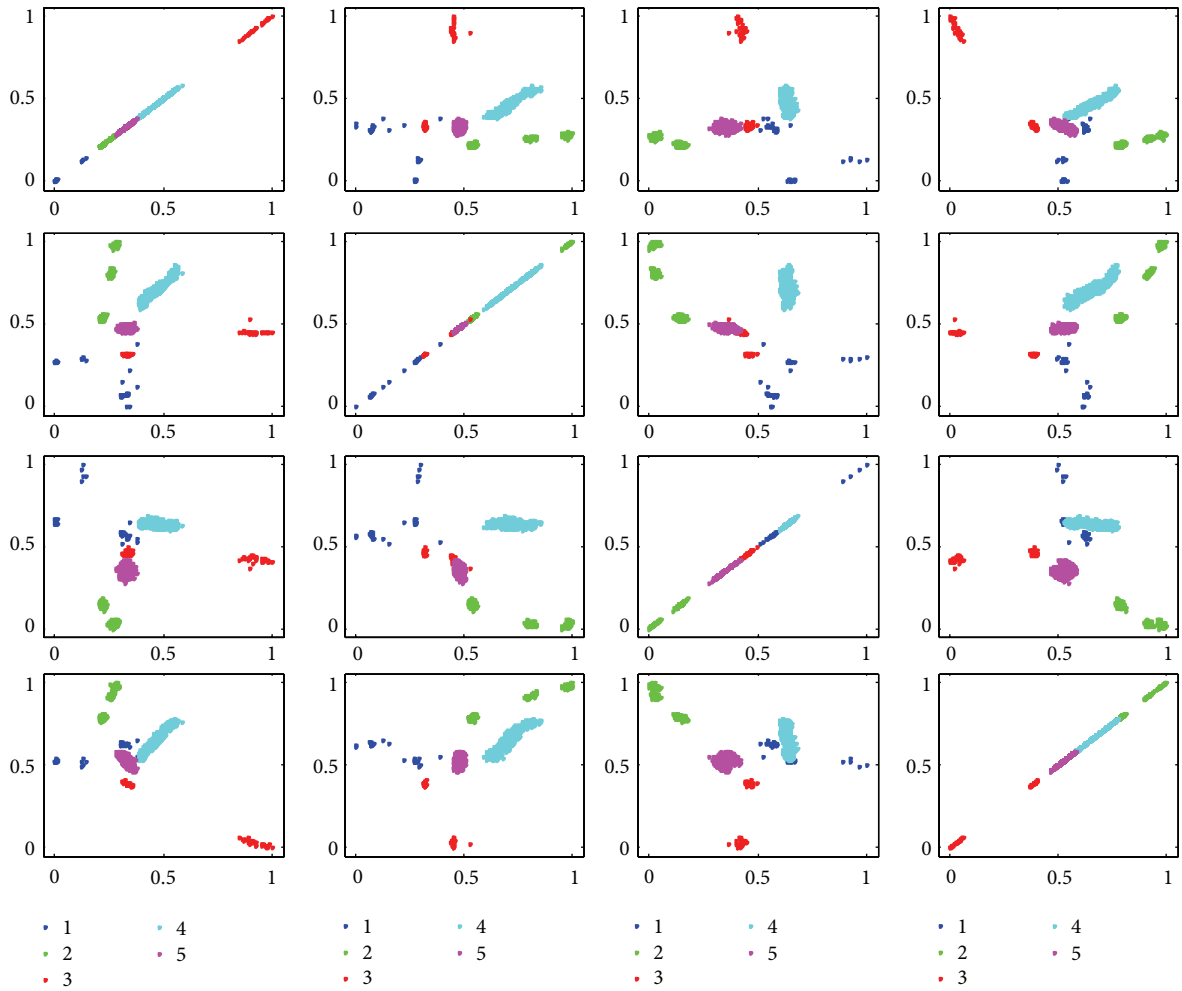


Figure 14: Scatter plots in 2D space expended with feature pairs selected by fuzzy dependency.

each round, we select a feature which produces the largest significance increment with the selected subset. Both fuzzy dependency and kernelized mutual information increase monotonically if new attributes are added. If the selected features are sufficient for classification, these two functions will keep invariant by adding any new attributes. So we can stop the algorithm if the increment of significance is less than a given threshold. The significances of the selected feature subset are shown in Figures 12 and 13, respectively.

In order to show the effectiveness of the algorithm, we give the scatter plots in 2D spaces, as shown in Figures 14 to 16, which are expended by the feature pairs selected by fuzzy dependency, kernelized mutual information, and Shannon mutual information. As to fuzzy dependency, we select Features 5, 37, 2, and 3. Then there are $4 \times 4 = 16$ combinations of feature pairs. The subplot in the ith row and jth column in Figure 14 gives the scatters of samples in 2D space expanded by the ith selected feature and the jth selected feature.

Observing the 2nd subplots in the first row of Figure 14, we can find that the classification task is nonlinear. The first class is dispersed and the third class is also located at different regions, which leads to the difficulty in learning classification models.

However, in the corresponding subplot of Figure 15, we can see that each class is relatively compact, which leads to a small intraclass distance. Moreover, the samples in five classes can be classified with some linear models, which also bring benefit for learning a simple classification model.

Comparing Figures 15 and 16, we can find that different classes are overlapped in feature spaces selected by Shannon mutual information or get entangled, which leads to the bad classification performance.

5. Diagnosis Modeling with Information Entropy Based Decision Tree Algorithm

After selecting the informative features, we now go to classification modeling. There are a great number of learning algorithms for building a classification model. Generalization capability and interpretability are the two most important criteria in evaluating an algorithm. As to fault diagnosis, a domain expert usually accepts a model which is consistent

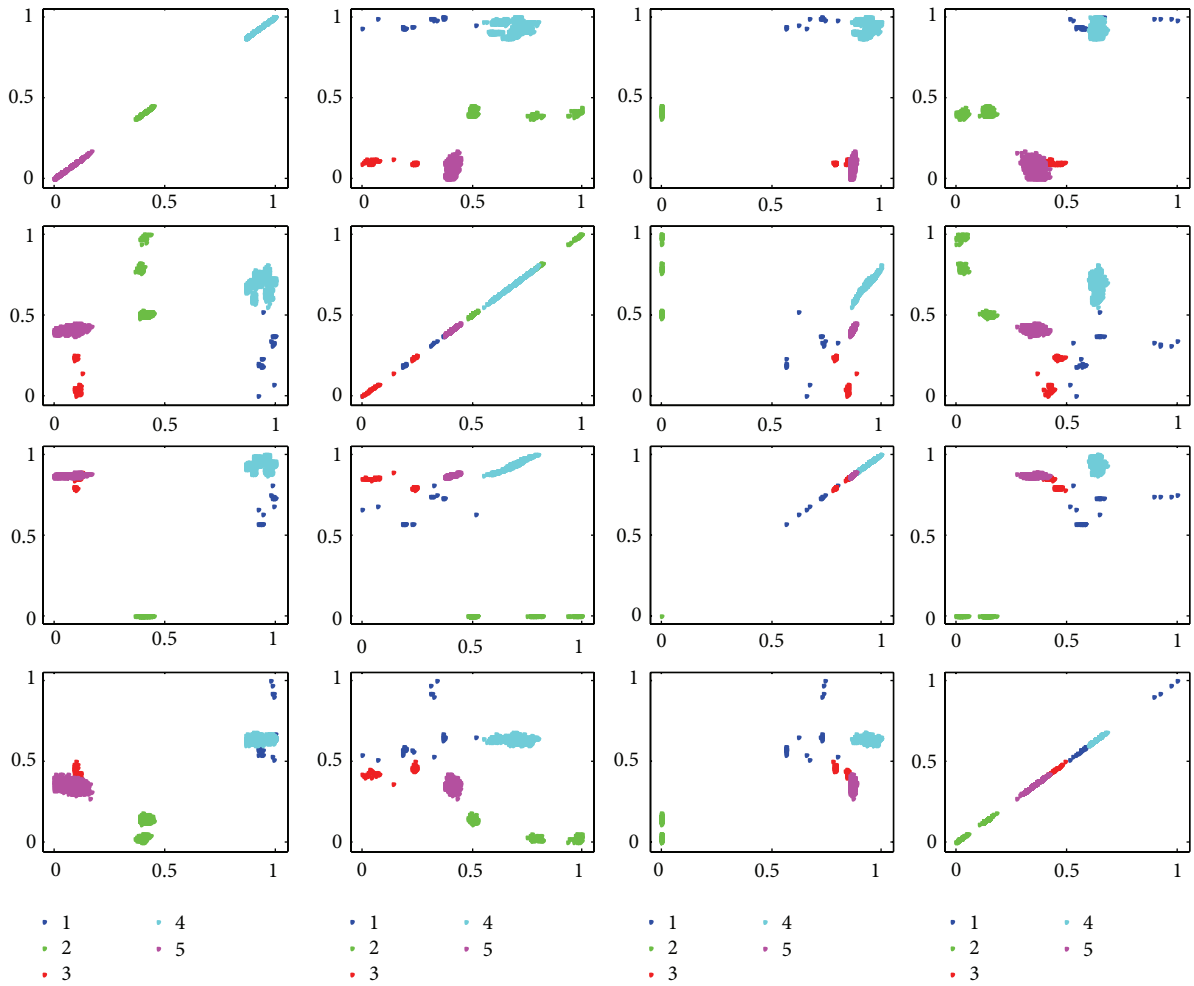


Figure 15: Scatter in 2D space expended with feature pairs selected by kernelized mutual information.

with his common knowledge. Thus, he expects the model is understandable; otherwise, he will not believe the outputs of the model. In addition, if the model is understandable, a domain expert can adapt it according to his prior knowledge, which makes the model suitable for different diagnosis objects.

Decision tree algorithms, including CART [29], ID3 [17], and C4.5 [18], are such techniques for training an understandable classification model. The learned model can be transformed into a set of rules. All these algorithms build a decision tree from training samples. They start from a root node and select one of the features to divide the samples with cuts into different branches according to their feature values. This procedure is interactively conducted until the branch is pure or a stopping criterion is satisfied. The key difference lies in the evaluation function in selecting attributes or cuts. In CART, splitting rules GINI and Twoing are adopted, while ID3 uses information gain and C4.5 takes information gain ratio. Moreover, C4.5 can deal with numerical attributes compared with ID3. Competent performance is usually observed with C4.5 in real-world applications compared with some popular algorithms, including SVM and Baysian net. In this work, we introduce C4.5 to train classification models. The pseudocode of C4.5 is formulated as follows.

Decision tree algorithm C4.5

Input: a set of training samples $U = \{x_1, x_2, \ldots, x_m\}$ with features $F = \{f_1, f_2, \ldots, f_n\}$

Stopping criterion τ

Output: decision tree T

(1) Check for sample set

(2) For each attribute f compute the normalized information gain ratio from splitting on a

(3) Let *f_best* be the attribute with the highest normalized information gain

(4) Create a decision node that splits on *f_best*

(5) Recurse on the sublists obtained by splitting on *f_best*, and add those nodes as children of node until stopping criterion τ is satisfied

(6) Output T.

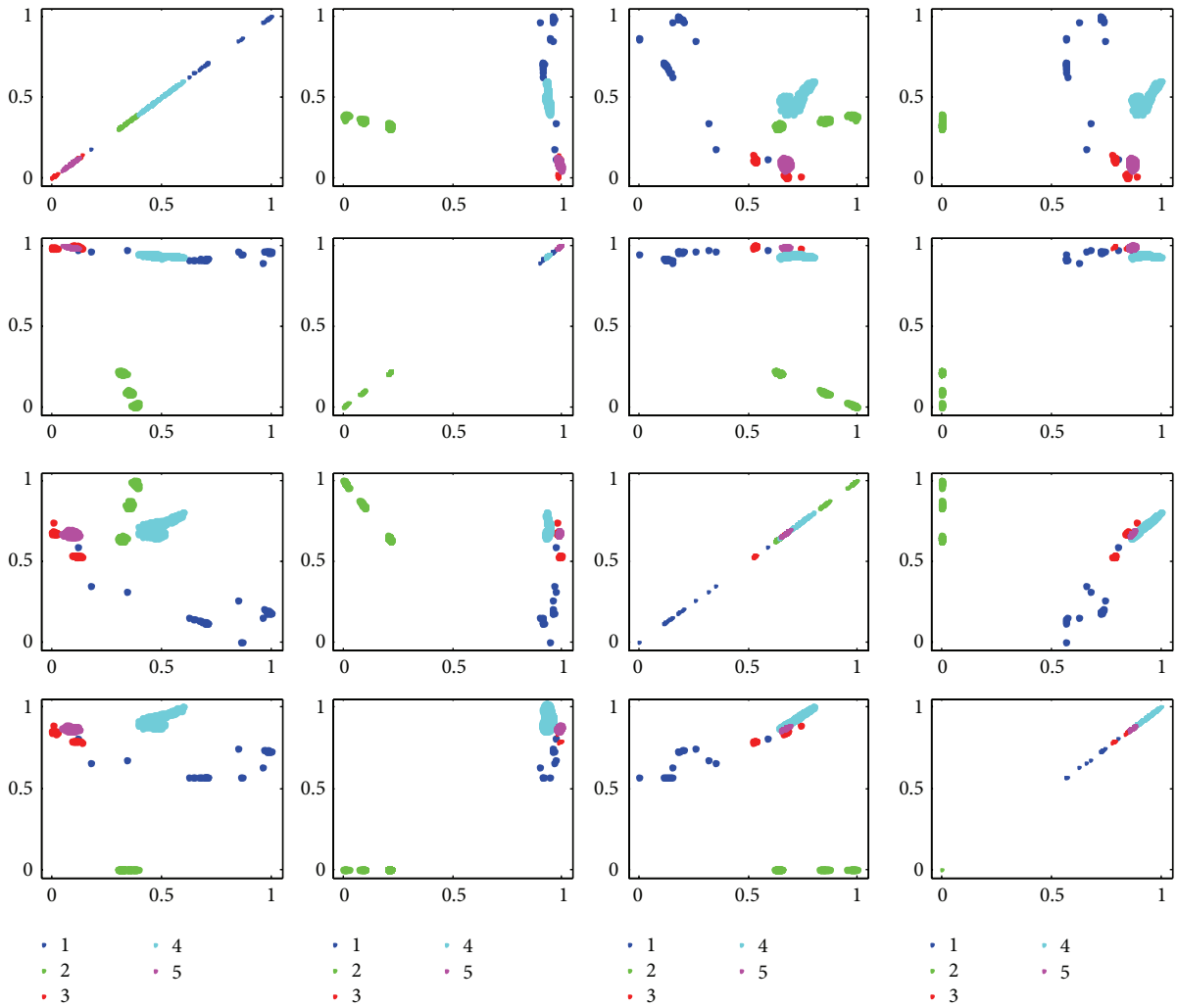


Figure 16: Scatter in 2D space expended with feature pairs selected by Shannon mutual information.

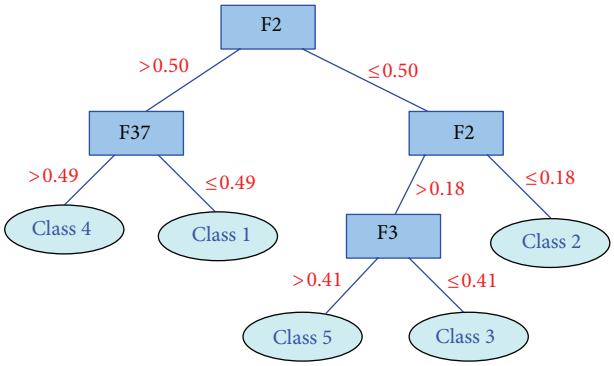


Figure 17: Decision tree trained on the features selected with fuzzy rough sets.

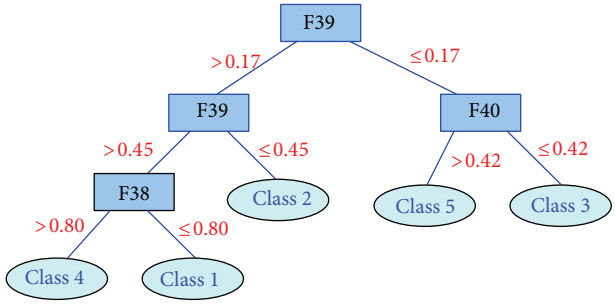


Figure 18: Decision tree trained on the features selected with kernelized mutual information.

We input the data sets into C4.5 and build the following two decision trees. Features 5, 37, 2, and 3 are included in the first dataset, and Features 39, 31, 38, and 40 are selected in the second dataset. The two trees are given in Figures 17 and 18, respectively.

We start from the root node to a leaf node along the branch, and then a piece of rule is extracted from the tree. As to the first tree, we can get five decision rules:

(1) if F2 > 0.50 and F37 > 0.49, then the decision is Class 4;

(2) if F2 > 0.50 and F37 ≤ 0.49, then the decision is Class 1;

(3) if 0.18 < F2 ≤ 0.50 and F3 > 0.41, then the decision is Class 5;

(4) if 0.18 < F2 ≤ 0.50 and F3 ≤ 0.41, then the decision is Class 3;

(5) if F2 ≤ 0.18, then the decision is Class 2.

As to the second decision tree, we can also obtain some rules as

(1) if F39 > 0.45 and F38 > 0.80, then the decision is Class 4;

(2) if F39 > 0.45 and F38 ≤ 0.80, then the decision is Class 1;

(3) if 0.17 < F39 ≤ 0.45, then the decision is Class 2;

(4) if F39 ≤ 0.17 and F40 > 0.42, then the decision is Class 5;

(5) if F39 ≤ 0.17, and F40 ≤ 0.42, then the decision is Class 3.

We can see the derived decision trees are rather simple and each can extract five pieces of rules. It is very easy for domain experts to understand the rules and even revise the rules. As the classification task is a little simple, the accuracy of each model is high to 97%. As new samples and faults are recorded by the system, more and more complex tasks may be stored. In that case, the model may become more and more complex.

6. Conclusions and Future Works

Automatic fault detection and diagnosis are highly desirable in some industries, such as offshore oil well drilling platforms, for such systems are self-monitoring without man on duty. In this work, we design an intelligent abnormality detection and fault recognition technique for the exhaust system of gas turbines based on information entropy, which is used in measuring the uniformity of exhaust temperatures, evaluating the significance of features in kernel spaces, and selecting splitting nodes for constructing decision trees. The main contributions of the work are two parts. First, we introduce the entropy function to measure the uniformity of exhaust temperatures. The measurement is easy to compute and understand. Numerical experiments also show its effectiveness. Second, we extend Shannon entropy for evaluating the significance of attributes in kernelized feature spaces. We compute the relevance between a kernel matrix induced with a set of attributes and the matrix computed with the decision variable. Some numerical experiments are also presented. Good results are derived.

Although this work gives an effective framework for automatic fault detection and recognition, the proposed technique is not tested on large-scale real tasks. We have developed a remote state monitoring and fault diagnosis system. Large scale data are flooding into the center. In the future, we will improve these techniques and develop a reliable diagnosis system.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgment

This work is partially supported by National Natural Foundation under Grants 61222210 and 61105054.

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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 307468, 9 pages
http://dx.doi.org/10.1155/2014/307468

Research Article

On Distribution Reduction and Algorithm Implementation in Inconsistent Ordered Information Systems

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Received 19 May 2014; Accepted 11 August 2014; Published 28 August 2014

Academic Editor: Weihua Xu

As one part of our work in ordered information systems, distribution reduction is studied in inconsistent ordered information systems (OISs). Some important properties on distribution reduction are studied and discussed. The dominance matrix is restated for reduction acquisition in dominance relations based information systems. Matrix algorithm for distribution reduction acquisition is stepped. And program is implemented by the algorithm. The approach provides an effective tool for the theoretical research and the applications for ordered information systems in practices. For more detailed and valid illustrations, cases are employed to explain and verify the algorithm and the program which shows the effectiveness of the algorithm in complicated information systems.

1. Introduction

In Pawlak's original rough set theory [1], partition or equivalence (indiscernibility) is an important and primitive concept. However, partition or equivalence relation, as the indiscernibility relation in Pawlak's original rough set theory, is still restrictive for many applications. To address this issue, several interesting and meaningful extensions to equivalence relation have been proposed in the past, such as neighborhood operators [2], tolerance relations [3], and others [4–10]. Moreover, the original rough set theory does not consider attributes with preference ordered domain, that is, criteria. In many real life practices, we often face problems in which the ordering of properties of the considered attributes plays a crucial role. One such type of problem is the ordering of objects. For this reason, Greco et al. proposed an extension rough set theory, called the dominance based rough set approach (DRSA), to take into account the ordering properties of criteria [11–16]. This innovation is mainly based on substitution of the indiscernibility relation by a dominance relation. Moreover, Greco et al. characterizes the DRSA and decision rules induced from rough approximations, while the usefulness of the DRSA and its advantages over the CRSA (classical rough set approach) are presented [11–16]. In DRSA, condition attributes are criteria and classes are preference ordered. Several studies have been made about properties and algorithmic implementations of DRSA [10, 17–19].

Nevertheless, only a limited number of methods using DRSA to acquire knowledge in inconsistent ordered information systems have been proposed and studied. Pioneering work on inconsistent ordered information systems with the DRSA has been proposed by Greco et al. [11–16], but they did not clearly point out the semantic explanation of unknown values. Shao and Zhang [20] further proposed an extension of the dominance relation in incomplete ordered information systems. Their work was established on the basis of the assumption that all unknown values are lost. Despite this, they did not mention the underlying concept of attribute reduction in inconsistent ordered decision system but they mentioned an approach to attribute reduction in consistent ordered information systems. Therefore, the purpose of this paper is to develop approaches to attribute reductions in inconsistent ordered information systems (IOIS). In this paper, theories and approaches of distribution reduction are investigated in inconsistent ordered information systems. Furthermore, algorithm of matrix computation of distribution reduction is introduced, from which we can provide a new approach to attributes reductions in inconsistent ordered information systems.

The rest of this paper is organized as follows. Some preliminary concepts are briefly recalled in Section 2. In Section 3, theories and approaches of distribution reduction are investigated in IOIS. In Section 4, we restate the definition of dominance matrix in ordered information systems and step the matrix algorithm for distribution reduction acquisition. Preparations are implemented to place the algorithm and the program is designed. The algorithm and the corresponding program we designed can provide a tool to theoretical research and applications of criterion based information system. Cases are employed to illustrate the algorithm and the program in Section 5. It is shown that the algorithm and program are effective in complicated information system. Furthermore conclusions on what we study in this paper are drawn to understand this paper briefly.

2. Ordered Information Systems

The following recalls necessary concepts and preliminaries required in the sequel of our work. Detailed description of the theory can be found in [11–16].

An information system with decisions is an ordered quadruple $\mathcal{I} = (U, A \cup D, F, G)$, where $U = \{x_1, x_2, \ldots, x_n\}$ is a nonempty finite set of objects; $A \cup D$ is a nonempty finite attributes set; $A = \{a_1, a_2, \ldots, a_p\}$ denotes the set of condition attributes; $D = \{d_1, d_2, \ldots, d_q\}$ denotes the set of decision attributes, $A \cap D = \phi$; $F = \{f_k \mid U \to V_k, k \le p\}$, $f_k(x)$ is the value of a_k on $x \in U$; V_k is the domain of a_k, $a_k \in A$; $G = \{g_{k'} \mid U \to V_{k'}, k' \le q\}$, $g_{k'}(x)$ is the value of $d_{k'}$ on $x \in U$, $V_{k'}$ is the domain of $d_{k'}$, $d_{k'} \in D$. In an information system, if the domain of an attribute is ordered according to a decreasing or increasing preference, then the attribute is a criterion. An information system is called an ordered information system (OIS) if all condition attributes are criterions.

Assume that the domain of a criterion $a \in A$is completely preordered by an outranking relation $\succeq_a$; then $x \succeq_a y$ means that x is at least as good as y with respect to criterion a. And we can say that x dominates y. In the following, without any loss of generality, we consider condition and decision criterions having a numerical domain; that is, $V_a \subseteq \mathcal{R}$ ($\mathcal{R}$ denotes the set of real numbers).

We define $x \succeq y$ by $f(x, a) \ge f(y, a)$ according to increasing preference, where $a \in A$ and $x, y \in U$. For a subset of attributes $B \subseteq A$, $x \succeq_B y$ means that $x \succeq_a y$ for any $a \in B$. That is to say x dominates y with respect to all attributes in B. Furthermore, we denote $x \succeq_B y$ by $x R_B^{\succeq} y$. In general, we indicate an ordered information system with decision by $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$. Thus the following definition can be obtained.

Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system with decisions, for $B \subseteq A$; denote

$$R_B^{\succeq} = \left\{\left(x_i, x_j\right) \in U \times U \mid f_l\left(x_i\right) \ge f_l\left(x_j\right), \forall a_l \in B\right\};$$
$$R_D^{\succeq} = \left\{\left(x_i, x_j\right) \in U \times U \mid g_m\left(x_i\right) \ge g_m\left(x_j\right), \forall d_m \in D\right\}. \tag{1}$$

$R_B^{\succeq}$ and $R_D^{\succeq}$ are called dominance relations of information system $\mathcal{I}^{\succeq}$.

Table 1: An ordered information system.

U	a_1	a_2	a_3	d
x_1	1	2	1	3
x_2	3	2	2	2
x_3	1	1	2	1
x_4	2	1	3	2
x_5	3	3	2	3
x_6	3	2	3	1

If we denote

$$\begin{aligned}[x_i]_B^{\succeq} &= \left\{x_j \in U \mid \left(x_j, x_i\right) \in R_B^{\succeq}\right\} \\ &= \left\{x_j \in U \mid f_l\left(x_j\right) \ge f_l\left(x_i\right), \forall a_l \in B\right\}, \\ [x_i]_D^{\succeq} &= \left\{x_j \in U \mid \left(x_j, x_i\right) \in R_D^{\succeq}\right\} \\ &= \left\{x_j \in U \mid g_m\left(x_j\right) \ge g_m\left(x_i\right), \forall d_m \in D\right\},\end{aligned} \tag{2}$$

then the following properties of a dominance relation are trivial.

Let $R_A^{\succeq}$ be a dominance relation. The following properties hold.

(1) $R_A^{\succeq}$ is reflexive and transitive, but not symmetric, so it is not an equivalence relation.

(2) If $B \subseteq A$, then $R_A^{\succeq} \subseteq R_B^{\succeq}$.

(3) If $B \subseteq A$, then $[x_i]_A^{\succeq} \subseteq [x_i]_B^{\succeq}$.

(4) If $x_j \in [x_i]_A^{\succeq}$, then $[x_j]_A^{\succeq} \subseteq [x_i]_A^{\succeq}$ and $[x_i]_A^{\succeq} = \cup\{[x_j]_A^{\succeq} \mid x_j \in [x_i]_A^{\succeq}\}$.

(5) $[x_j]_A^{\succeq} = [x_i]_A^{\succeq}$ if and only if $f(x_i, a) = f(x_j, a)$ ($\forall a \in A$).

(6) $\mathcal{I} = \cup\{[x]_A^{\succeq} \mid x \in U\}$ constitute a covering of U.

For any subset X of U, and A of $\mathcal{I}^{\succeq}$, define

$$\begin{aligned}\underline{R_A^{\succeq}}(X) &= \left\{x \in U[x]_A^{\succeq} \subseteq X\right\}, \\ \overline{R_A^{\succeq}}(X) &= \left\{x \in U \mid [x]_A^{\succeq} \cap X \ne \phi\right\}.\end{aligned} \tag{3}$$

$\underline{R_A^{\succeq}}(X)$ and $\overline{R_A^{\succeq}}(x)$ are said to be the lower and upper approximations of X with respect to a dominance relation $R_A^{\succeq}$. And the approximations have also some properties which are similar to those of Pawlak approximation spaces.

For an ordered information system with decisions $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$, if $R_A^{\succeq} \subseteq R_D^{\succeq}$, then this information system is consistent, otherwise, this information system is inconsistent (IOIS).

Example 1. An ordered information system is given in Table 1.

From the table, we have

$$[x_1]_A^{\succeq} = \{x_1, x_2, x_5, x_6\}; \qquad [x_2]_A^{\succeq} = \{x_2, x_5, x_6\};$$
$$[x_3]_A^{\succeq} = \{x_2, x_3, x_4, x_5, x_6\}; \qquad [x_4]_A^{\succeq} = \{x_4, x_6\};$$

$$[x_5]_A^{\succeq} = \{x_5\}; \qquad [x_6]_A^{\succeq} = \{x_6\};$$
$$[x_1]_d^{\succeq} = [x_5]_d^{\succeq} = \{x_1, x_5\};$$
$$[x_2]_d^{\succeq} = [x_4]_d^{\succeq} = \{x_1, x_2, x_4, x_5\};$$
$$[x_3]_d^{\succeq} = [x_6]_d^{\succeq} = \{x_1, x_2, x_3, x_4, x_5, x_6\}. \tag{4}$$

Obviously, by the above, we have $R_A^{\succeq} \not\subseteq R_d^{\succeq}$, so the system in Table 1 is inconsistent.

For a simple description, the following information system with decisions is based on dominance relations, that is, ordered information system.

3. Theories of Distribution Reduction in Inconsistent Ordered Information Systems

Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an information system with decisions, and $R_B^{\succeq}$, $R_D^{\succeq}$ dominance relations derived from condition attributes set $B \subseteq A$ and decision attributes set D, respectively. For $B \subseteq A$, denote

$$\frac{U}{R_B^{\succeq}} = \left\{[x_i]_B^{\succeq} \mid x_i \in U\right\},$$
$$\frac{U}{R_d^{\succeq}} = \{D_1, D_2, \ldots, D_r\},$$
$$\mu_B^{\succeq}(x) = \left(\frac{|D_1 \cap [x]_B^{\succeq}|}{|U|}, \frac{|D_2 \cap [x]_B^{\succeq}|}{|U|}, \ldots, \frac{|D_r \cap [x]_B^{\succeq}|}{|U|}\right),$$
$$\gamma_B^{\succeq}(x) = \max\left\{\frac{|D_1 \cap [x]_B^{\succeq}|}{|U|}, \frac{|D_2 \cap [x]_B^{\succeq}|}{|U|}, \ldots, \frac{|D_r \cap [x]_B^{\succeq}|}{|U|}\right\}, \tag{5}$$

where $[x]_B^{\succeq} = \{y \in U \mid (x, y) \in R_B^{\succeq}\}$. Furthermore, we let $\mu_B^{\succeq}(x)$ be a distribution function about attributions set B and $\gamma_B^{\succeq}(x)$ maximum distribution function about attributions set B.

Definition 2. Let $\alpha = (a_1, a_2, \ldots, a_n)$ and $\beta = (b_1, b_2, \ldots, b_n)$ be two vectors with n dimensions. If $a_i = b_i$, $(i = 1, 2, \ldots, n)$, we say that α is equal to β and is denoted by $\alpha = \beta$. If $a_i \leq b_i$, $(i = 1, 2, \ldots, n)$, we say that α is less than β and is denoted by $\alpha \leq \beta$. Otherwise, if it exists i_0 $(i_0 \in \{1, 2, \ldots, n\})$ such that $a_{i_0} > b_{i_0}$, we say α is not less than $\beta$and it is denoted by $\alpha \not\leq \beta$, such as $(1, 2, 3) \not\leq (1, 1, 4)$ and $(1, 1, 4) \not\leq (1, 2, 3)$.

From the above, we can have the following propositions immediately.

Proposition 3. *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an inconsistent information system.*

(1) *If $B \subseteq A$, then $\mu_A^{\succeq}(x) \leq \mu_B^{\succeq}(x)$, $\forall x \in U$.*

(2) *If $B \subseteq A$, then $\gamma_A^{\succeq}(x) \leq \gamma_B^{\succeq}(x)$, $\forall x \in U$.*

(3) *If $[y]_B^{\succeq} \supseteq [x]_B^{\succeq}$, then $\mu_B^{\succeq}(y) \leq \mu_B^{\succeq}(x)$, $\forall x, y \in U$.*

(4) *If $[y]_B^{\succeq} \supseteq [x]_B^{\succeq}$, then $\gamma_B^{\succeq}(y) \leq \gamma_B^{\succeq}(x)$, $\forall x, y \in U$.*

Definition 4. Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an inconsistent information system. If $\mu_B^{\succeq}(x) = \mu_A^{\succeq}(x)$, for all $x \in U$, we say that B is a distribution consistent set of $\mathcal{I}^{\succeq}$. If B is a distribution consistent set, and no proper subset of B is a distribution consistent set, then B is called a distribution consistent reduction of $\mathcal{I}^{\succeq}$.

Definition 5. Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an inconsistent information system. If $\gamma_B^{\succeq}(x) = \gamma_A^{\succeq}(x)$, for all $x \in U$, we say that B is a maximum distribution consistent set of $\mathcal{I}^{\succeq}$. If B is a maximum distribution set, and no proper subset of B is a maximum distribution consistent set, then B is called a maximum distribution consistent reduction of $\mathcal{I}^{\succeq}$.

Example 6. For the system in Table 1, if we denote

$$D_1 = [x_1]_d^{\succeq} = [x_5]_d^{\succeq},$$
$$D_2 = [x_2]_d^{\succeq} = [x_4]_d^{\succeq}, \tag{6}$$
$$D_3 = [x_3]_d^{\succeq} = [x_6]_d^{\succeq},$$

then we can have

$$\mu_A^{\succeq}(x_1) = \left(\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right);$$
$$\mu_A^{\succeq}(x_2) = \left(\frac{1}{6}, \frac{1}{3}, \frac{1}{2}\right);$$
$$\mu_A^{\succeq}(x_3) = \left(\frac{1}{6}, \frac{1}{2}, \frac{5}{6}\right);$$
$$\mu_A^{\succeq}(x_4) = \left(0, \frac{1}{6}, \frac{1}{3}\right);$$
$$\mu_A^{\succeq}(x_5) = \left(\frac{1}{6}, \frac{1}{6}, \frac{1}{6}\right); \tag{7}$$
$$\mu_A^{\succeq}(x_6) = \left(0, 0, \frac{1}{6}\right),$$
$$\gamma_A^{\succeq}(x_1) = \frac{2}{3}; \qquad \gamma_A^{\succeq}(x_2) = \frac{1}{2}; \qquad \gamma_A^{\succeq}(x_3) = \frac{5}{6};$$
$$\gamma_A^{\succeq}(x_4) = \frac{1}{3}; \qquad \gamma_A^{\succeq}(x_5) = \frac{1}{6}; \qquad \gamma_A^{\succeq}(x_6) = \frac{1}{6}.$$

When $B = \{a_2, a_3\}$, it can be easily checked that $[x]_A^{\succeq} = [x]_B^{\succeq}$, for all $x \in U$, so that$\mu_B^{\succeq}(x) = \mu_A^{\succeq}(x)$ and $\gamma_B^{\succeq}(x) = \gamma_A^{\succeq}(x)$ are true and $B = \{a_2, a_3\}$ is a distribution consistent set of $\mathcal{I}^{\succeq}$. Furthermore, we can examine that $\{a_2\}$ and $\{a_3\}$are not consistent sets of $\mathcal{I}^{\succeq}$. That is to say $B = \{a_2, a_3\}$ is a distribution reduction and is a maximum distribution reduction of $\mathcal{I}^{\succeq}$.

Moreover, it can easily be calculated that $B' = \{a_1, a_3\}$ and $B'' = \{a_1, a_2\}$ are not distribution consistent sets of $\mathcal{I}^{\succeq}$. Thus there exist only one distribution reduction and maximum distribution reduction of $\mathcal{I}^{\succeq}$ in the system of Table 1, which are $\{a_2, a_3\}$.

The distribution consistent set and the maximum distribution consistent set are related in the following theorem.

Theorem 7. *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system and $B \subseteq A$ is a distribution consistent set of $\mathcal{I}^{\succeq}$ if and only if B is a maximum distribution consistent set of $\mathcal{I}^{\succeq}$.*

Proof. It can be proved immediately from corresponding definitions and properties. From the definitions of distribution and maximum distribution consistent set, the key results of the implication is that $[x]_B^{\succeq} = [x]_A^{\succeq}$ always holds for any $x \in U$ while B is a distribution consistent set or maximum distribution consistent set. Thus, the theorem can be acquired immediately. □

Theorem 8. *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system.*

P: $B \subseteq A$ is a distribution consistent set of $\mathcal{I}^{\succeq}$.

Q: While $\mu_A^{\succeq}(y) \nleq \mu_A^{\succeq}(x)$, $[y]_B^{\succeq} \nsubseteq [x]_B^{\succeq}$ holds for any $x, y \in U$.

Then we have $P \Rightarrow Q$.

Proof. We will prove $\neg Q \Rightarrow \neg P$. Assume that when $\mu_A^{\succeq}(y) \nleq \mu_A^{\succeq}(x)$, $[y]_B^{\succeq} \nsubseteq [x]_B^{\succeq}$ does not hold and that implies $[y]_B^{\succeq} \subseteq [x]_B^{\succeq}$. So we can obtain $\mu_B^{\succeq}(y) \leq \mu_B^{\succeq}(x)$ by Proposition 3(3). On the other hand, since B is a distribution consistent set of $\mathcal{I}^{\succeq}$, we have $\mu_A^{\succeq}(x) = \mu_B^{\succeq}(x)$ and $\mu_A^{\succeq}(y) = \mu_B^{\succeq}(y)$. Hence we can get $\mu_A^{\succeq}(y) \leq \mu_A^{\succeq}(x)$, which is a contradiction. The theorem is proved. □

The distribution consistent set requires that the classification ability of the consistent remains the same with the original data table. That is, $B \subset A$, which is a distribution consistent set of A, must satisfy the fact that $[x]_B^{\succeq} = [x]_A^{\succeq}$ holds for any $x \in U$. This is very strict and other reductions studied in [21] may not reach this special condition.

4. Matrix Algorithm for Distribution Reduction Acquisition in Inconsistent Ordered Information Systems

In this section, the dominance matrices will be put as a restatement and matrices will be employed to realize the calculation of distribution reductions.

Definition 9. Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system, and $B \subset A$. Denote

$$M_B = \left(m_{ij}\right)_{n\times n}, \quad \text{where } m_{ij} = \begin{cases} 1, & x_j \in [x_i]_B^{\succeq}, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

The matrix M_B is called dominance matrix of attributes set $B \subseteq A$. If $|B| = l$, we say that the order of M_B is l.

Definition 10. Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system and M_B and M_C are dominance matrices of attributes sets $B, C \subseteq A$. The intersection of M_B and M_C is defined by

$$M_B \cap M_C = \left(m_{ij}\right)_{n\times n} \cap \left(m'_{ij}\right)_{n\times n} = \left(\min\left\{m_{ij}, m'_{ij}\right\}\right)_{n\times n}. \tag{9}$$

The intersection defined above can be implemented by the operator ".∗" in Matlab platform, $M_B \cap M_C = M_B. * M_C$, that is, the product of elements in corresponding positions. Then the following properties are obvious.

Proposition 11. *Let M_B, M_C be dominance matrices of attributes sets $B, C \subseteq A$; the following results always hold.*

(1) $m_{ii} = 1$.

(2) $M_{B\cup C} = M_B \cap M_C$.

From the above, we can see that a dominance relation of objects has one-one correspondence to a dominance matrix. The combination of dominance relations can be realized by the corresponding matrices and the dominance relations can be compared by the corresponding matrices from the following definitions.

Definition 12. Let $M_A = (\alpha_1, \alpha_2, \ldots, \alpha_n)^T$ and $M_B = (\beta_1, \beta_2, \ldots, \beta_n)^T$ be matrices with $n \times n$ dimensions and α_i and β_i row vectors, respectively. If $\alpha_i \leq \beta_i$ holds, for any $i \leq n$, we say that M_A is less than M_B and it is denoted by $M_A \leq M_B$.

By the definitions, dominance matrices have the following properties straightly.

Proposition 13. *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system and $B \subseteq A$. The dominance matrices with respect to A and B are, respectively, M_A and M_B. Then $M_A \leq M_B$.*

In the following, we give the preparation of matrix computation for distribution reductions in ordered information systems.

Proposition 14. *Let $\mathcal{I}^{\succeq} = (U, A\cup D, F, G)$ be an ordered information system and $U = \{x_1, x_2, \ldots, x_n\}$ and $A = \{a_1, a_2, \ldots, a_p\}$. Then*

$$M_A = \bigcap_{i=1}^{p} M_{\{a_i\}} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \tag{10}$$

and any vector $\alpha_i = (a_{i1}, a_{i2}, \ldots, a_{in})$ represents the dominance class of object x_i by the values 0 and 1, where 0 means the object not included in the class and 1 means the object included in the class.

Input: An inconsistent ordered information system $\mathcal{I}^{\succeq} = (U, A \cup D, V, f)$, where $U = \{x_1, x_2, \ldots, x_n\}$ and $A = \{a_1, a_2, \ldots, a_p\}$.
Output: All distribution reductions of $\mathcal{I}^{\succeq}$.
Step 1. Load the ordered information system and simplify the system by combining the objects with same values of every attribute,
Step 2. Classify by every single criterion and store then in separate matrices

$$M_{a_i} = \begin{pmatrix} a_{11}^i & a_{12}^i & \cdots & a_{1n}^i \\ a_{21}^i & a_{22}^i & \cdots & a_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^i & a_{n2}^i & \cdots & a_{nn}^i \end{pmatrix},$$

$$M_d = \begin{pmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{pmatrix}.$$

Step 3. Check the consistence of the information system

$$M_A = \bigcap_{i=1}^{p} M_{a_i} = M_{a_1}. * M_{a_2}. * \cdots . * M_{a_p} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}.$$

where ".$*$" is the operator in Matlab platform. If $M_A \leq M_d$, the system is consistent, terminate the algorithm.
Else the system is inconsistent, go to the next step.
Step 4. Acquire the consistent set. Let $B = \{b_1, b_2, \cdots, b_m\} \subset A$

$$M_B = \bigcap_{i=1}^{m} B_i = B_1. * B_2. * \cdots . * B_m. = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix}.$$

If $M_B = M_A$, B is a consistent set, store the set into the temporary storage cell. Else fetch another subset of A and repeat this step. Calculate till all subsets of A are verified, then go to the next step.
Step 5. Sort the consistent sets in the storage cell and find out the minimum consistent sets which are just the reductions. Output all reductions and terminate the algorithm.

Algorithm 1

Theorem 15. *Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system and $B \subseteq A$. B is a consistent set if and only if $M_B = M_A$.*

Proof. As is known, $[x]_A^{\succeq} \subseteq [x]_B^{\succeq}$ holds since $B \subseteq A$.

($\Rightarrow$) For B is a distribution consistent set, one can have $\mu_B = \mu_A$. Then, for any x and D_j, we have $|D_j \cap [x]_A^{\succeq}| = |D_j \cap [x]_B^{\succeq}|$. Since $[x]_A^{\succeq} \subseteq [x]_B^{\succeq}$, it is obvious that $[x]_A^{\succeq} = [x]_B^{\succeq}$. That is, the row vectors in M_B and M_A are correspondingly the same. Then $M_B = M_A$.

($\Leftarrow$) Since $M_B = M_A$, we can easily obtain that $[x]_A^{\succeq} = [x]_B^{\succeq}$ holds for any x and D_j. Then $|D_j \cap [x]_A^{\succeq}| = |D_j \cap [x]_B^{\succeq}|$ holds for any x and D_j. We can obtain that $\mu_B^{\succeq}(x) = \mu_A^{\succeq}(x)$ holds for any x. That is, B is a distribution consistent set.

To acquire reductions in inconsistent ordered information system, the matrices can be the only forms of storage in computing. And we illustrate the progress to calculate the reductions as shown in Algorithm 1. □

The algorithm and the distribution reduction allow us to calculate reductions which keep the classification ability the same with the original system in a brief way. And we do not need to acquire every approximation of the decisions.It shortens the computing time and provides an effective tool for knowledge acquisition in criterion based rough set theory. The flow chart of the Algorithm 1 can be designed and it is placed in Figure 1.

Analysis to Time Complexity of Algorithm 1. Let $\mathcal{I}^{\succeq} = (U, A \cup D, F, G)$ be an ordered information system. $U = \{x_1, x_2, \ldots, x_n\}$ is the simplified universe. The number of objects in original information system not being simplified is denoted by n_1. There are m condition attributes in A; that is, $|A| = m$. The number of compressed decision classes is r. We take a variable t_i to stand for the time complexity in an implementation. In the next, we can analyze the time complexity of Algorithm 1 step by step.

The time complexity to simplify the original information system is n_1^2 for any two objects being compared and is denoted by $t_1 = n_1^2$. Since $|U| = n$, $|A| = m$, and $|D| = 1$, the time complexities to be classified by condition attributes and decision D are, respectively, $t_2 = |U|^2 \times |A|$ and $t_3 = |U|^2$.

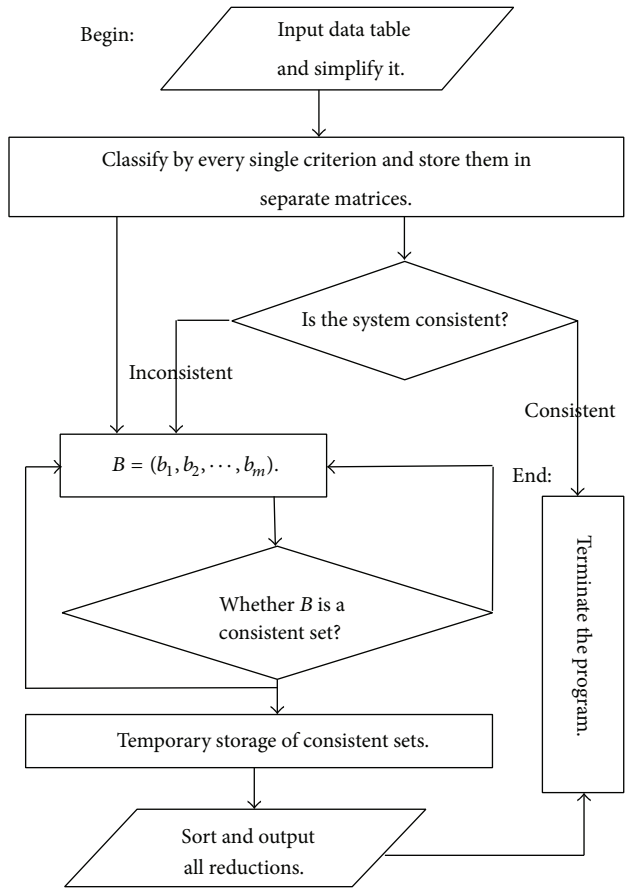

Figure 1: The flow chart of Algorithm 1.

For decision classes being merged by comparing classes of any two objects, the time complexity is $t_4 = |U|^2$. Now the consistency of the information system needs to be checked by comparing the condition class and decision class of any object. If the information system is consistent, the time complexity to check consistency is $|U|$. If the information system is inconsistent, the time complexity to check consistency is less than $|U|$. Thus, the time complexity to check consistency is no more than $|U|$; that is, it is presented as $t_5 \leq |U|$. Then, the possible and compatible distribution functions can be calculated and the time complexity is $t_6 = 2r \times |U|$. The time complexity to calculate each of these two functions is $r \times |U|$ and is denoted by $t_6^\sigma = t_6^\delta = r \times |U|$. The analysis to Step 1 is finished.

For Step 2, the time complexity to calculate possible and compatible distribution decision matrices, respectively, is denoted by $t_7^\sigma = t_7^\delta = |U|^2$. Thus, the time complexity to calculate distribution decision matrices is $t_7 = 2|U^2|$. The time complexity of Step 2 is completed.

The first two steps are preparations to calculate reductions. The next Step 3 to Step 5 are the steps which run the operations. There are $C_m^1 = m$ subsets $\{a_l\}$ and the dominance matrices are with dimensions $n \times n$. In addition, the representation C_m^i is the combinatorial number which means the number of selections to chose i elements from m ones. We consider that the judgement of a vector if it is zero runs one operation and the comparison of two vectors runs according to the dimension of the vectors. Therefore, the time complexities to compare M_d^σ and M_d^δ with $M_{\{a_l\}}$, respectively, are $|U|^2$. And the time complexity to compare every line vector of $M_{\{a_l\}}$ with zero is $|U|$. The possible and compatible distribution matrices are obtained by reassignment values n times. And the time complexities to process possible and compatible distribution matrices, respectively, are both n. Then, we have that the total time complexity of Step 3 is $t_8 = C_m^1 \times (3|U|^2 + 3|U|)$. The judgement in Step 4 just need to run according to the number of $\{a_l\}$ and the time complexity is $t_9 = 2C_m^1$.

Table 2

Names	Models	Parameters
CPU	Intel Core i3-380U	1.33 GHz
Memory	DDR3 SDRAM	2 × 2 GB 1333 MHz
Hard disk	TOSHIBA	320 GB
System	Windows 7	32 bit
Platform	Matlab	Leasehold

Since we just need to compute the intersection of nonzero 1st order possible (or compatible) distribution matrices, the

Table 3: $\mathcal{I}^{\succeq}$: An inconsistent ordered information system on animals sleep.

	$(U, C \cup \{d\})$	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	d
x_1:	African giant pouched rat	1	6.6	6.3	2	8.3	4.5	42	3	1	3
x_2:	Asian elephant	2547	4603	2.1	1.8	3.9	69	624	3	5	4
x_3:	Baboon	10.55	179.5	9.1	0.7	9.8	27	180	4	4	4
x_4:	Big brown bat	0.023	0.3	15.8	3.9	19.7	19	35	1	1	1
x_5:	Brazilian tapir	160	169	5.2	1	6.2	30.4	392	4	5	4
x_6:	Cat	3.3	25.6	10.9	3.6	14.5	28	63	1	2	1
x_7:	Chimpanzee	52.16	440	8.3	1.4	9.7	50	230	1	1	1
x_8:	Chinchilla	0.425	6.4	11	1.5	12.5	7	112	5	4	4
x_9:	Cow	465	423	3.2	0.7	3.9	30	281	5	5	5
x_{10}:	Eastern American mole	0.075	1.2	6.3	2.1	8.4	3.5	42	1	1	1
x_{11}:	Echidna	3	25	8.6	0	8.6	50	28	2	2	2
x_{12}:	European hedgehog	0.785	3.5	6.6	4.1	10.7	6	42	2	2	2
x_{13}:	Galago	0.2	5	9.5	1.2	10.7	10.4	120	2	2	2
x_{14}:	Goat	27.66	115	3.3	0.5	3.8	20	148	5	5	5
x_{15}:	Golden hamster	0.12	1	11	3.4	14.4	3.9	16	3	1	2
x_{16}:	Gray seal	85	325	4.7	1.5	6.2	41	310	1	3	1
x_{17}:	Ground squirrel	0.101	4	10.4	3.4	13.8	9	28	5	1	3
x_{18}:	Guinea pig	1.04	5.5	7.4	0.8	8.2	7.6	68	5	3	4
x_{19}:	Horse	521	655	2.1	0.8	2.9	46	336	5	5	5
x_{20}:	Lesser short-tailed shrew	0.005	0.14	7.7	1.4	9.1	2.6	21.5	5	2	4
x_{21}:	Little brown bat	0.01	0.25	17.9	2	19.9	24	50	1	1	1
x_{22}:	Man	62	1320	6.1	1.9	8	100	267	1	1	1
x_{23}:	Mouse	0.023	0.4	11.9	1.3	13.2	3.2	19	4	1	3
x_{24}:	Musk shrew	0.048	0.33	10.8	2	12.8	2	30	4	1	3
x_{25}:	N. American opossum	1.7	6.3	13.8	5.6	19.4	5	12	2	1	1
x_{26}:	Nine-banded armadillo	3.5	10.8	14.3	3.1	17.4	6.5	120	2	1	1
x_{27}:	Owl monkey	0.48	15.5	15.2	1.8	17	12	140	2	2	2
x_{28}:	Patas monkey	10	115	10	0.9	10.9	20.2	170	4	4	4
x_{29}:	Phanlanger	1.62	11.4	11.9	1.8	13.7	13	17	2	1	2
x_{30}:	Pig	192	180	6.5	1.9	8.4	27	115	4	4	4
x_{31}:	Rabbit	2.5	12.1	7.5	0.9	8.4	18	31	5	5	5
x_{32}:	Rat	0.28	1.9	10.6	2.6	13.2	4.7	21	3	1	3
x_{33}:	Red fox	4.235	50.4	7.4	2.4	9.8	9.8	52	1	1	1
x_{34}:	Rhesus monkey	6.8	179	8.4	1.2	9.6	29	164	2	3	2
x_{35}:	Rock hyrax (Hetero.b)	0.75	12.3	5.7	0.9	6.6	7	225	2	2	2
x_{36}:	Rock hyrax (Procavia hab)	3.6	21	4.9	0.5	5.4	6	225	3	2	3
x_{37}:	Sheep	55.5	175	3.2	0.6	3.8	20	151	5	5	5
x_{38}:	Tenrec	0.9	2.6	11	2.3	13.3	4.5	60	2	1	2
x_{39}:	Tree hyrax	2	12.3	4.9	0.5	5.4	7.5	200	3	1	3
x_{40}:	Tree shrew	0.104	2.5	13.2	2.6	15.8	2.3	46	3	2	2
x_{41}:	Vervet	4.19	58	9.7	0.6	10.3	24	210	4	3	4
x_{42}:	Water opossum	3.5	3.9	12.8	6.6	19.4	3	14	2	1	1

Table 4

a_1—Body weight in kg;
a_2—Brain weight in g;
a_3—Show wave ("nondreaming") sleep (hrs/day);
a_4—Paradoxical ("dreaming") sleep (hrs/day);
a_5—Total sleep (hrs/day);
a_6—Maximum life span (years);
a_7—Gestation time (days);
a_8—Predation index (1–5);
a_9—Sleep exposure index (1–5);
d—Overall danger index (1–5).

maximum time complexities can be analyzed in the next steps but not the true ones in computing. Therefore, the maximum time complexity relies on the number of attribute subsets $2^{|C|}$. The worst case is that no minimum reduction exists in the information system and all $2^{|C|}$ subsets are calculated in the algorithm. Thus, the maximum time complexity of Step 5 is $t_{10} = 2C_m^2 \times |U|^2$.

From the above analysis, we can know that the maximum time complexity of the main part in the algorithm (Step 3 to Step 5) is $t_{\text{main}} = t_8 + t_9 + t_{10} = |A| \times (|U|^2 + |U|)$.

Hence, the maximum time complexity of the main algorithm is approximately $O((|U|^2 + 2|U|) \times |A|)$.

5. Experimental Computing and Case Study

We design programs and employ two cases to demonstrate the effective of the method in the last section. This experimental computing program is running on a personal computer with the following hardware and software configuration. The configuration of the computer is a bit low but the program runs well and fast. It also shows the advantage of Algorithm 1 and the corresponding computing program (see Table 2).

An inconsistent ordered information system on animals sleep is presented in Table 3.

The information system is denoted by $\mathcal{I}^{\succeq} = (U, A \cup \{d\}, V, f)$, where A is the condition attribute set and d is the single dominance decision. There are 42 objects which represent the species of animals and 10 attributes with numerical values in the ordered information system. The animals' names are showed in Table 3 and the interpretations of the attributes will be listed as follows. The interpretations and the units of attributes are represented as shown in Table 4.

By the experimental computing program, the distribution reductions of the system can be calculated and they are represented in the following. The operating time to compute this case is 0.158581 seconds.

The distribution reductions are

$$\{a_1, a_3, a_4, a_6, a_7, a_8, a_9\},$$
$$\{a_2, a_3, a_4, a_6, a_7, a_8, a_9\},$$
$$\{a_1, a_2, a_3, a_4, a_6, a_7, a_8, a_9\},$$
$$\{a_1, a_3, a_4, a_5, a_6, a_7, a_8, a_9\},$$
$$\{a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}. \tag{11}$$

And it can be verified by taking the computer as an assistant that the above sets are reductions of the data table. Detailed progress of the verifying are not arranged here. From the results, we can easily see that the reductions studied in this paper are different from the ones approached in [25], since these reductions are $\{a_3, a_4, a_6, a_7\}$, $\{a_4, a_5, a_6, a_7\}$, $\{a_6, a_8, a_9\}$, and $\{a_1, a_2, a_8, a_9\}$. They are different kinds of reductions in ordered information systems and can adapt to different needs in practices. From the definition of different reductions, we can also easily obtain that possible and compatible reductions are usually subsets of distribution reduction. This is not strict and should be studied and verified separately and theatrically . And the work may be taken into account as one part of the future studies in our work.

Finally, we take other inconsistent ordered information system to acquire the distribution reduction, respectively. And the descriptions on the data tables are listed in Table 5.

From the results in Table 5, we can obtain that the algorithm and the program we studied in this paper could be effective and useful to acquire distribution reductions in practice. The numbers of objects and attributes can increase the computing time. But the matrices storage has the ability to shorten the memory and computing time. And it can be helpful in research theoretically and it is applicable.

6. Conclusions

As is known, many information systems are data tables considering criteria for various factors in practise. Therefore, it is meaningful to study the attribute reductions in inconsistent information system on the basis of dominance relations. In this paper, distribution reduction is restated in inconsistent ordered information systems. Some properties and theorems are studied and discussed. A fact is certified that the distribution reduction is equivalent to the maximum distribution reduction in ordered information systems. Theorems on distribution reduction are implemented to create preparations for reduction acquisition and the dominance matrix is also restated to acquire distribution reductions in criterion based information systems. The matrix algorithm for distribution reduction acquisition is stepped and programmed. The algorithm can provide an approach and the program can be effective for theoretical research on knowledge reductions in criterion based inconsistent information systems. Dominance matrices are the only relied parameters which need to be considered without others such as approximations and subinformation systems being brought in. Furthermore, cases are employed to illustrate the validity of the matrix method and the program, which shows that the effectiveness of the algorithm in complicated information systems.

Table 5: Descriptions on the calculations.

Data name	Values	Objects	Conditions	Decisions	Reductions	Time	Operations
Body fat	Real	252	14	1	11	36.43723 s	10
Glass	Real	213	9	1	7	2.04624 s	10
Animal sleep	Real	42	9	1	5	0.13153 s	10

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (no. 61100116), the Natural Science Foundation Project of Jiangshu Province (no. BK2011492), and the Youth Foundation of Xuzhou Institute of Technology (no. xky2011201).

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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 927014, 18 pages
http://dx.doi.org/10.1155/2014/927014

Research Article

Further Study of Multigranulation T-Fuzzy Rough Sets

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Received 15 May 2014; Revised 22 June 2014; Accepted 7 July 2014; Published 17 August 2014

Academic Editor: Xibei Yang

The optimistic multigranulation T-fuzzy rough set model was established based on multiple granulations under T-fuzzy approximation space by Xu et al., 2012. From the reference, a natural idea is to consider pessimistic multigranulation model in T-fuzzy approximation space. So, in this paper, the main objective is to make further studies according to Xu et al., 2012. The optimistic multigranulation T-fuzzy rough set model is improved deeply by investigating some further properties. And a complete multigranulation T-fuzzy rough set model is constituted by addressing the pessimistic multigranulation T-fuzzy rough set. The full important properties of multigranulation T-fuzzy lower and upper approximation operators are also presented. Moreover, relationships between multigranulation and classical T-fuzzy rough sets have been studied carefully. From the relationships, we can find that the T-fuzzy rough set model is a special instance of the two new types of models. In order to interpret and illustrate optimistic and pessimistic multigranulation T-fuzzy rough set models, a case is considered, which is helpful for applying these theories to practical issues.

1. Introduction

Rough set theory, proposed by Pawlak [1], is an extension of the classical set theory and could be regarded as a mathematical and soft computing tool to handle imprecision, vagueness, and uncertainty in data analysis. This relatively new soft computing methodology has received great attention in recent years, and its effectiveness has been confirmed successful in applications in many science and engineering fields, such as pattern recognition, data mining, image processing, and medical diagnosis. Rough set theory is built on the basis of the classification mechanism; it is classified as the equivalence relation in a specific universe, and the equivalence relation constitutes a partition of the universe. A concept, or more precisely the extension of a concept, is represented by a subset of a universe of objects and is approximated by a pair of definable concepts of a logic language. The main idea of rough set theory is the use of a known knowledge in knowledge base to approximate the inaccurate and uncertain knowledge.

In recent years, the generalization of the rough set model is one of the most important research directions. On one hand, rough set theory is generalized by combining with other theories that deal with uncertain knowledge such as fuzzy set theory [2]. It has been acknowledged by different studies that fuzzy set theory and rough set theory are complementary in terms of handling different kinds of uncertainty. Fuzzy set theory deals with probabilistic uncertainty, connected with imprecision of stated perceptions and preferences. Rough set theory, in turn, deals with uncertainty following from ambiguity of information. The two types of uncertainty can be encountered together in real-life problems. For this reason, many approaches have been proposed to combine fuzzy set theory with rough set theory. Dübois and Prade [3] proposed concepts of rough fuzzy sets and fuzzy rough sets. A rough fuzzy set is a pair of fuzzy sets resulting from the approximation of a crisp set in a fuzzy approximation space, and a fuzzy rough set is a pair of fuzzy sets resulting from the approximation of a fuzzy set in a crisp approximation space. Some other researches about fuzzy rough set and rough fuzzy set from other directions have been discussed [4–10]. What is more, generalizations of fuzzy rough sets were defined by using a residual implication and a triangular norm on $[0, 1]$ to define the lower and upper approximation operators. Several authors also have proposed a kind of implication 4, weak fuzzy partitions on the universe. Wu et al. [11] characterized the (I, T)-fuzzy rough approximation operators. Morsi and Yakout researched axiomatics for fuzzy rough sets by

a triangular norm [12]. Mi et al. generalized fuzzy rough sets determined by a triangular norm [13].

On the other hand, rough set theory was discussed from the view of granular computing. In 1985, Hobbs proposed the concept of granularity [14], and Zadeh first explored the concept of granular computing between 1996 and 1997 [15]. They all think that information granules refer to pieces, classes, and groups into which complex information is divided in accordance with the characteristics and processes of the understanding and decision-making. Among the existing possibilities offered by granular computing, we may refer to fuzzy sets [16], rough sets [1], and vague sets [17], just to name some of the well-established alternatives. From the point of view of granular computing, Pawlak's rough set is based on a single granulation induced from an indiscernibility relation.

Actually, an attribute subset induces an equivalence relation; the partition formed by an equivalence relation can be regarded as a granulation. By using a finer granulation formed through combining two known granulations induced from two-attribute subsets to describe the target concept, the combination destroys the original granulation structure. In general, the above assumption cannot always be satisfied or required in practice. In order to apply the rough set theory, Qian and Liang extended Pawlak's single-granulation rough set model to a multiple granulation rough set model [18]. Since the multigranulation rough set was initially proposed by Qian et al. [19], later, many researchers have extended the multigranulation rough sets to the generalized multigranulation rough sets. Xu et al. developed a multigranulation fuzzy rough set model [20], a generalized multigranulation rough set approach [21], multigranulation rough sets based on tolerance relations [22], and a multigranulation rough set model in ordered information systems [23]; Yang et al. proposed the hierarchical structure properties of the multigranulation rough sets [24] and multigranulation rough set in incomplete information system [25] and presented a test cost sensitive multigranulation rough set model [26]; Lin et al. presented a neighborhood-based multigranulation rough set [27]; She and He explored the topological structures and the properties of multigranulation rough sets [28].

From the thought of multigranulation, optimistic multigranulation and pessimistic multigranulation are two of the most basic ways of research. In [29], authors only presented concepts of optimistic multigranulation fuzzy rough sets based on triangular norms. By analyzing the proposed definition in [29], there exists another perspective which is called pessimistic multigranulation. Authors in [29] did not investigate the pessimistic multigranulation fuzzy rough sets based on triangular norms, and relationships between optimistic multigranulation and single granulation fuzzy rough sets based on triangular norms were not presented either. Accordingly, from both optimistic multigranulation and pessimistic multigranulation perspectives, we generalize the multigranulation T-fuzzy rough set theory by using the concepts of a residual implication and a triangular norm on $[0,1]$. In this paper, we mainly improve the model proposed in [29] by discussing the further properties of optimistic multigranulation T-fuzzy rough sets, propose the multigranulation T-fuzzy rough set model from the perspective of pessimistic multigranulation and study its properties, and research relationships between multigranulation and classical T-fuzzy rough sets. These contents are not yet completed in [29], so this paper is an extended vision of [29]. The rest of this paper is organized as follows. In Section 2, we recall some concepts and properties to be used in this paper. In Sections 3 and 4, we presented the definition of the optimistic multigranulation T-fuzzy lower and upper approximation operators and proposed the pessimistic multigranulation T-fuzzy lower and upper approximation operators; basic properties about these two models are also studied. In Section 5, we get the relationship among these T-fuzzy approximation operators. We give the examples about the evaluation of fund projects in Section 6. Finally, Section 7 gets the conclusions.

2. Preliminaries

In this section, we review some basic concepts and properties about T-fuzzy rough sets. The notion of optimistic multigranulation T-fuzzy rough set is also introduced. The Cartesian product of U with U is denoted by $U \times U$. The classes of all fuzzy subsets of U are denoted by $F(U)$. Following, a binary operator T on the unit interval $I = [0,1]$ is said to be a triangular norm [30] if for all $a, b, c, d \in I$, we have

(i) $T(a,b) = T(b,a)$;

(ii) $T(a,1) = a$;

(iii) $a \le c, b \le d \Rightarrow T(a,b) \le T(c,d)$;

(iv) $T(T(a,b),c) = T(a,T(b,c))$.

A fuzzy relation R from U to U is a fuzzy subset of $U \times U$; that is, $R \in F(U \times U)$, and $R(x,y)$ is called the degree of relation between x and y. Consider the following:

(1) R is said to be reflexive on $U \times U \Leftrightarrow$ for all $x \in U$, $R(x,x) = 1$;

(2) R is said to be symmetric on $U \times U \Leftrightarrow$ for all $x \in U$, $R(x,y) = R(y,x)$;

(3) R is said to be T transitive on $U \times U \Leftrightarrow$ for all $x, y, z \in U$, $R(x,z) \ge T(R(x,y), R(y,z))$.

If R is reflexive, symmetric, and T transitive on $U \times U$, we then say that R is a T-fuzzy equivalence relation on U; if R is reflexive and symmetric on $U \times U$, we say that R is a T-fuzzy similarity relation on U.

A binary operator on I is given in the following

$$\theta(a,b) = \sup\{c \in I \mid T(a,c) \le b\}, \tag{1}$$

where θ is called the residual implication based on a triangular norm T.

For the sake of convenience, for any $X, Y \in F(U)$, $x \in U$, we will define several fuzzy sets as follows:

(1) $T(X,Y)(x) = T(X(x), Y(x))$;

(2) $\theta(X,Y)(x) = \theta(X(x), Y(x))$;

(3) $\theta(X,Y) = \bigwedge_{u \in U} \theta(X(u), Y(u))$;

(4) $\theta(1_x, \underline{\underline{a}}) = 1_{U \setminus \{x\}} \vee \underline{\underline{a}}$;

(5)

$$1_x(y) = \begin{cases} 1, & \text{if } y = x; \\ 0, & y \neq x; \end{cases} \tag{2}$$

(6)

$$1_Z(x) = \begin{cases} 1, & \text{if } x \in Z; \\ 0, & x \notin Z, \end{cases} \tag{3}$$

where $\underline{\underline{a}}$ is a constant fuzzy set and $Z \subseteq U$.

Consider a lower semicontinuous triangular norm T, for all $a, b, c \in I$; the residual implication based on the triangular norm T satisfies the following important properties:

(θ1) $\theta(a, 1) = 1, \theta(1, a) = a$;
(θ2) $a \leq b \Rightarrow \theta(c, a) \leq \theta(c, b)$;
(θ3) $a \leq b \Rightarrow \theta(a, c) \geq \theta(b, c)$;
(θ4) $T(\theta(a, c), \theta(c, b)) \leq \theta(a, b)$;
(θ5) $\theta(a \vee b, c) = \theta(a, c) \wedge \theta(b, c)$;
(θ6) $\theta(a, b \wedge c) = \theta(a, b) \wedge \theta(a, c)$;
(θ7) $a \leq b \Leftrightarrow \theta(a, b) = 1$;
(θ8) $\theta(a, \theta(b, c)) = \theta(b, \theta(a, c))$;
(θ9) $\theta(T(a, b), c) = \theta(a, \theta(b, c))$;
(θ10) $T(\theta(T(a, b), c), a) \leq \theta(b, c)$;
(θ11) $\bigwedge_{a \in I} \theta(T(b, \theta(c, a)), a) = \theta(b, c)$;
(θ12) $\theta(\theta(a, b), b) \geq a$;
(θ13) $\bigwedge_{b \in I} \theta(\theta(a, b), b) = a$;
(θ14) $T(\theta(a, b), c) \leq \theta(a, T(b, c))$;
(θ15) $\bigwedge_{b \in I} \theta(\theta(a, b), \theta(c, b)) = \theta(c, a)$;
(θ16) $\theta(a, b) \leq \theta(T(a, c), T(b, c))$;
(θ17) $\theta(a, b \vee c) = \theta(a, b) \vee \theta(a, c)$;
(θ18) $a \leq \theta(b, T(a, b))$;
(θ19) $\theta(a \wedge b, c) = \theta(a, c) \vee \theta(b, c)$;
(θ20) $\theta(a \wedge b, c) \geq \theta(a, c) \wedge \theta(b, c)$.

Definition 1 (see [13]). Let U be a finite and nonempty set called the universe, and let R be a T-fuzzy similarity relation from U to U. The pair (U, R) is called a T-fuzzy approximation space. For any $A \in F(U)$, we define two fuzzy set-theoretic operators from $F(U)$ to $F(U)$ as follows:

$$\begin{aligned} \underline{R}(A)(x) &= \bigwedge_{y \in U} \theta(R(x, y), A(y)), \\ \overline{R}(A)(x) &= \bigvee_{y \in U} T(R(x, y), A(y)), \\ & \qquad x \in U, \end{aligned} \tag{4}$$

where "$\bigvee$" means "max," "$\bigwedge$" means "min," and $\underline{R}$ and $\overline{R}$ are referred to as the T-fuzzy lower and upper approximation operators. The pair $(\underline{R}(A), \overline{R}(A))$ is called the T-fuzzy rough set of A.

Proposition 2 (see [13]). *Let (U, R) be a fuzzy approximation space, for all $A, B \in F(U)$, $(x, y) \in U \times U$; then, one has the following:*

(1) $\underline{R}(A) \subseteq A \subseteq \overline{R}(A)$;

(2) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$, $\overline{R}((A) \cup B) = \overline{R}(A) \cup \overline{R}(B)$;

(3) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$, $\underline{R}(A \cap B) \subseteq \underline{R}(A) \cap \overline{R}(B)$;

(4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$, $\overline{R}(A) \subseteq \overline{R}(B)$;

(5) $\underline{R}(\underline{R}(A)) = \underline{R}(A)$, $\overline{R}(\overline{R}(A)) = \overline{R}(A)$;

(6) $\overline{R}(\underline{R}(A)) = \underline{R}(A)$, $\underline{R}(\overline{R}(A)) = \overline{R}(A)$;

(7) $\overline{R}(A) = A \Leftrightarrow \underline{R}(A) = A$.

The above proposition reflects the relationships between $\underline{R}$ and $\overline{R}$. It is easy to prove that the T-fuzzy approximation operators in this section are really extensions of the approximation operators. In particular, if R is a crisp equivalence relation on U and $A \in P(U)$, then $\overline{R}(A)$ and $\underline{R}(A)$ coincide with the classical Pawlak rough set.

Authors in [29] proposed the model on optimistic multigranulation T-fuzzy rough set, which is presented in the following.

Definition 3 (see [29]). Let $K = (U, \mathbf{R})$ be a T-fuzzy approximation space, and let $\mathbf{R}$ be a family of T-fuzzy similarity relations from U to U, $R_{A_1}, R_{A_2}, \ldots, R_{A_n} \in \mathbf{R}$. For any $X \in F(U)$, the optimistic multigranulation T-fuzzy lower and upper approximations of X are defined as follows:

$$\begin{aligned} \underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X)(x) &= \bigvee_{i=1}^{n} \left(\bigwedge_{u \in U} \theta\left(R_{A_i}(u, x), X(u)\right) \right), \\ \overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X)(x) &= \bigwedge_{i=1}^{n} \left(\bigvee_{u \in U} T\left(R_{A_i}(u, x), X(u)\right) \right), \end{aligned} \tag{5}$$

where θ and T are defined in Section 2. $\underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}$ and $\overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}$ are referred to as the optimistic multigranulation T-fuzzy lower and upper approximation operators. The pair $(\underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X), \overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X))$ is called the optimistic multigranulation T-fuzzy rough set of X. If $\underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) = \overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X)$, then X is referred to as optimistic definable under the T-fuzzy approximation space; otherwise, X is referred to as optimistic undefinable or rough. The boundary of the optimistic multigranulation T-fuzzy rough set X is defined as

$$\mathrm{Bnd}^{\mathrm{O}}_{\sum_{i=1}^n A_i}(X) = \overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) \cap \left(\sim \underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) \right). \tag{6}$$

Example 4 (see [29]). Let (U, R_A, R_B) be a T-fuzzy approximation space, where $U = \{x_1, x_2, x_3, x_4, x_5\}$; then,

$$R_A = \begin{pmatrix} 1 & 0.4 & 0.8 & 0.5 & 0.5 \\ 0.4 & 1 & 0.4 & 0.4 & 0.4 \\ 0.8 & 0.4 & 1 & 0.5 & 0.5 \\ 0.5 & 0.4 & 0.5 & 1 & 0.6 \\ 0.5 & 0.4 & 0.5 & 0.6 & 1 \end{pmatrix},$$
$$R_B = \begin{pmatrix} 1 & 0.8 & 0.8 & 0.2 & 0.8 \\ 0.8 & 1 & 0.85 & 0.2 & 0.85 \\ 0.8 & 0.85 & 1 & 0.2 & 0.9 \\ 0.2 & 0.2 & 0.2 & 1 & 0.2 \\ 0.8 & 0.85 & 0.9 & 0.2 & 1 \end{pmatrix}. \tag{7}$$

Given $T(x, y) = \min(x, y)$, $X = (0.5, 0.3, 0.3, 0.6, 0.5)$.

It is not difficult to verify that the fuzzy relations R_A and R_B are both T-fuzzy similarity relations. So we can obtain the optimistic multigranulation T-fuzzy lower and upper approximations of X as follows:

$$\underline{\mathrm{OM}_{A+B}}(X) = (0.3, 0.3, 0.3, 0.6, 0.3),$$
$$\overline{\mathrm{OM}_{A+B}}(X) = (0.5, 0.4, 0.5, 0.6, 0.5). \tag{8}$$

Based on the model in Definition 3, we can conclude the relevant properties of optimistic multigranulation T-fuzzy rough sets accordingly.

3. Properties of Optimistic Multigranulation T-Fuzzy Rough Sets

In this section, we will study the properties of optimistic multigranulation T-fuzzy rough set which is on the rough approximation problem in a T-fuzzy approximation space.

Proposition 5. *Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be a T-fuzzy approximation space; let R_{A_i}, $i \in \{1, 2, 3, \ldots, n\}$, be the different T-fuzzy similarity relations, for all $x, y \in U$, $a, b \in I$, and $X, Y \in F(U)$. Then, the optimistic multigranulation T-fuzzy lower approximation has the following properties:*

(1) $\underline{OM_{\sum_{i=1}^n A_i}}(X) \subseteq X$;

(2) $\underline{OM_{\sum_{i=1}^n A_i}}(\underline{OM_{\sum_{i=1}^n A_i}}(X)) = \underline{OM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\underline{OM_{\sum_{i=1}^n A_i}}(X \cap Y) \subseteq \underline{OM_{\sum_{i=1}^n A_i}}(X) \cap \underline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(4) $X \subseteq Y \Rightarrow \underline{OM_{\sum_{i=1}^n A_i}}(X) \subseteq \underline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(5) $\underline{OM_{\sum_{i=1}^n A_i}}(X \cup Y) \supseteq \underline{OM_{\sum_{i=1}^n A_i}}(X) \cup \underline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(6) $\underline{OM_{\sum_{i=1}^n A_i}}(\Theta(1_x, \underline{\underline{a}}))(y) = \underline{OM_{\sum_{i=1}^n A_i}}(\Theta(1_y, \underline{\underline{a}}))(x) = \bigvee_{i=1}^n \theta(R_{A_i}(x, y), a)$;

(7) $\underline{OM_{\sum_{i=1}^n A_i}}(\theta(1_x, \underline{\underline{a}}))(x) = a$;

(8) $\underline{OM_{\sum_{i=1}^n A_i}}(\Theta(\underline{\underline{a}}, X)) = \Theta(\underline{\underline{a}}, \underline{OM_{\sum_{i=1}^n A_i}}(X))$;

(9) $\underline{OM_{\sum_{i=1}^n A_i}}(X, \underline{\underline{a}}) \subseteq \Theta(X, \underline{\underline{a}}) \subseteq \Theta(\underline{OM_{\sum_{i=1}^n A_i}}(X), \underline{\underline{a}})$;

(10) $\underline{OM_{\sum_{i=1}^n A_i}}(\underline{\underline{a}}) = \underline{\underline{a}}$;

(11) $\underline{OM_{\sum_{i=1}^n A_i}}(\underline{\underline{a}} \vee 1_Z)(x) = \bigvee_{i=1}^n \bigwedge_{u \notin Z} \theta(R_{A_i}(u, x), a)$;

(12) $\underline{\sum_{i=1}^n A_i}((\underline{\underline{\theta(a, b)}}) \vee 1_Z)(x) = \Theta((\underline{\underline{a}}), \underline{OM_{\sum_{i=1}^n A_i}}(b \vee 1_Z))$;

(13) $\bigwedge_{a \in I} \theta(\underline{OM_{\sum_{i=1}^n A_i}}(\Theta(1_x, \underline{\underline{a}}))(y), a) = \bigwedge_{i=1}^n R_{A_i}(x, y)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items hold when $R_A = R_B$. When $R_A \neq R_B$, (1)–(5) can be found in [29].

(6) Firstly, we can obtain

$$\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(1_x, \underline{\underline{a}}\right)\right)(y) \\
&= \bigwedge_{u \in U} \theta\left(R_A(u, x), \Theta\left(1_x, \underline{\underline{a}}\right)(u)\right) \\
&\quad \vee \bigwedge_{u \in U} \theta\left(R_B(u, x), \Theta\left(1_x, \underline{\underline{a}}\right)(u)\right) \\
&= \bigwedge_{u \in U} \theta\left(R_A(u, x), \Theta\left(1_x(u), a\right)\right) \\
&\quad \vee \bigwedge_{u \in U} \theta\left(R_B(u, x), \Theta\left(1_x(u), a\right)\right) \\
&= \left[\theta\left(R_A(x, y), a\right) \wedge \left(\bigwedge_{u \neq x} \theta\left(R_A(u, y), \theta(0, a)\right)\right)\right] \\
&\quad \vee \left[\theta\left(R_B(x, y), a\right) \wedge \left(\bigwedge_{u \neq x} \theta\left(R_B(u, y), \theta(0, a)\right)\right)\right] \\
&= \left[\theta\left(R_A(x, y), a\right) \wedge \left(\bigwedge_{u \neq x} \theta\left(R_A(u, y), 1\right)\right)\right] \\
&\quad \vee \left[\theta\left(R_B(x, y), a\right) \wedge \left(\bigwedge_{u \neq x} \theta\left(R_B(u, y), 1\right)\right)\right] \\
&= \left(\theta\left(R_A(x, y), a\right) \wedge 1\right) \vee \left(\theta\left(R_B(x, y), a\right) \wedge 1\right) \\
&= \theta\left(R_A(x, y), a\right) \vee \theta\left(R_B(x, y), a\right).
\end{aligned} \tag{9}$$

The proposition can be obtained by the symmetric and the above equation.

(7) It is easy to prove according to item (6).

(8) For any $x \in X$,

$$\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(\underline{\underline{a}}, X\right)\right)(x) \\
&= \bigwedge_{u \in U} \theta\left(R_A(u, x), \Theta\left(\underline{\underline{a}}, X\right)(u)\right) \\
&\quad \vee \bigwedge_{u \in U} \theta\left(R_B(u, x), \Theta\left(\underline{\underline{a}}, X\right)(u)\right) \\
&= \bigwedge_{u \in U} \theta\left(R_A(u, x), \theta(a, X(u))\right) \\
&\quad \vee \bigwedge_{u \in U} \theta\left(R_B(u, x), \theta(a, X(u))\right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{u\in U}\theta\left(a,\theta\left(R_A(u,x),X(u)\right)\right)\\
&\quad\vee \bigwedge_{u\in U}\theta\left(a,\theta\left(R_B(u,x),X(u)\right)\right)\\
&= \theta\left(a,\bigwedge_{u\in U}\theta\left(R_A(u,x),X(u)\right)\right)\\
&\quad\vee\theta\left(a,\bigwedge_{u\in U}\theta\left(R_B(u,x),X(u)\right)\right)\\
&= \theta\left(a,\bigwedge_{u\in U}\theta\left(R_A(u,x),X(u)\right)\right.\\
&\qquad\left.\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),X(u)\right)\right)\\
&= \theta\left(a,\underline{\mathrm{OM}_{A+B}}(X)(x)\right) = \Theta\left(\underline{a},\underline{\mathrm{OM}_{A+B}}(X)\right)(x).
\end{aligned}
\tag{10}
$$

(9) This item follows immediately from item (1) and $\theta(3)$.

(10) For any $x\in X$, we have

$$
\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\underline{a}\right)(x)\\
&\quad= \bigwedge_{u\in U}\theta\left(R_A(u,x),a\right)\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),a\right)\\
&\quad= \theta\left(\bigvee_{u\in U}R_A(u,x),a\right)\vee\theta\left(\bigvee_{u\in U}R_B(u,x),a\right)\\
&\quad= \theta(1,a) = a.
\end{aligned}
\tag{11}
$$

(11) For any $x\in U$, we can know

$$
\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\left(\underline{a}\right)\vee 1_Z\right)(x)\\
&\quad= \bigwedge_{u\in U}\theta\left(R_A(u,x),a\vee 1_Z(u)\right)\\
&\qquad\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),a\vee 1_Z(u)\right)\\
&\quad= \left[\bigwedge_{u\in Z}\theta\left(R_A(u,x),1\right)\wedge\bigwedge_{u\notin Z}\theta\left(R_A(u,x),a\right)\right]\\
&\qquad\vee\left[\bigwedge_{u\in Z}\theta\left(R_B(u,x),1\right)\wedge\bigwedge_{u\notin Z}\theta\left(R_B(u,x),a\right)\right]\\
&\quad= \bigwedge_{u\notin Z}\theta\left(R_A(u,x),a\right)\vee\bigwedge_{u\notin Z}\theta\left(R_B(u,x),a\right).
\end{aligned}
\tag{12}
$$

(12) For any $x\in U$, we have

$$
\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\left(\underline{\underline{\theta(a,b)}}\right)\vee 1_Z\right)(x)\\
&\quad= \bigwedge_{u\in U}\theta\left(R_A(u,x),\left(\underline{\underline{\theta(a,b)}}\vee 1_Z\right)(u)\right)\\
&\qquad\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),\left(\underline{\underline{\theta(a,b)}}\vee 1_Z\right)(u)\right)\\
&\quad= \left[\bigwedge_{u\in Z}\theta\left(R_A(u,x),1\right)\wedge\bigwedge_{u\notin Z}\theta\left(R_A(u,x),\theta(a,b)\right)\right]\\
&\qquad\vee\left[\bigwedge_{u\in U}\theta\left(R_B(u,x),1\right)\wedge\bigwedge_{u\notin Z}\theta\left(R_B(u,x),\theta(a,b)\right)\right]\\
&\quad= \bigwedge_{u\notin Z}\theta\left(R_A(u,x),\theta(a,b)\right)\vee\bigwedge_{u\notin Z}\theta\left(R_B(u,\mathrm{x}),\theta(a,b)\right)\\
&\quad= \theta\left(a,\bigwedge_{u\notin Z}\theta\left(R_A(u,x),b\right)\right)\vee\theta\left(a,\bigwedge_{u\notin Z}\theta\left(R_B(u,x),b\right)\right)\\
&\quad= \theta\left(a,\bigwedge_{u\notin Z}\theta\left(R_A(u,x),b\right)\vee\bigwedge_{u\notin Z}\theta\left(R_B(u,x),b\right)\right)\\
&\quad= \theta\left(a,\underline{\mathrm{OM}_{A+B}}\left(\left(\underline{b}\right)\vee 1_Z\right)(x)\right).
\end{aligned}
\tag{13}
$$

(13) According to item (6), we can have

$$
\begin{aligned}
&\bigwedge_{a\in I}\theta\left(\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(1_x,\underline{a}\right)(y),a\right)\right)\\
&\quad= \bigwedge_{a\in I}\theta\left(\theta\left(R_A(x,y),a\right)\vee\theta\left(R_B(x,y),a\right),a\right)\\
&\quad= \bigwedge_{a\in I}\left[\theta\left(\theta\left(R_A(x,y),a\right),a\right)\wedge\theta\left(\theta\left(R_B(x,y),a\right),a\right)\right]\\
&\quad= \left[\bigwedge_{a\in I}\theta\left(\theta\left(R_A(x,y),a\right),a\right)\right]\\
&\qquad\wedge\left[\bigwedge_{a\in I}\theta\left(\theta\left(R_B(x,y),a\right),a\right)\right]\\
&\quad= R_A(x,y)\wedge R_B(x,y).
\end{aligned}
\tag{14}
$$

□

Proposition 6. *Let $(U,R_{A_1},R_{A_2},\ldots,R_{A_n})$ be a T-fuzzy approximation space; let R_{A_i}, $i\in\{1,2,3,\ldots,n\}$, be the different T-fuzzy similarity relations. For all $x,y\in U$, $a,b\in I$, and $X,Y\in F(U)$, the optimistic multigranulation T-fuzzy upper approximation has the following properties:*

(1) $X\subseteq\overline{OM_{\sum_{i=1}^n A_i}}(X)$;

(2) $\overline{OM_{\sum_{i=1}^n A_i}}(\overline{OM_{\sum_{i=1}^n A_i}}(X)) = \overline{OM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\overline{OM_{\sum_{i=1}^n A_i}}(X\cup Y)\supseteq\overline{OM_{\sum_{i=1}^n A_i}}(X)\cup\overline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(4) $X\subseteq Y\Rightarrow\overline{OM_{\sum_{i=1}^n A_i}}(X)\subseteq\overline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(5) $\overline{OM_{\sum_{i=1}^n A_i}}(X\cap Y)\subseteq\overline{OM_{\sum_{i=1}^n A_i}}(X)\cap\overline{OM_{\sum_{i=1}^n A_i}}(Y)$;

(6) $\overline{OM_{\sum_{i=1}^n A_i}}(1_x)(y) = \overline{OM_{\sum_{i=1}^n A_i}}(1_y)(x) = \bigvee_{i=1}^n R_{A_i}(x,y)$;

(7) $\overline{OM_{\sum_{i=1}^n A_i}}(T(\underline{a},X)) = T(\underline{a},\overline{OM_{\sum_{i=1}^n A_i}}(X))$;

(8) $\overline{OM_{\sum_{i=1}^n A_i}}(\Theta(X,\underline{a}))\supseteq\Theta(X,\underline{a})\supseteq\Theta(\overline{OM_{\sum_{i=1}^n A_i}}(X),\underline{a})$;

(9) $\overline{OM_{\sum_{i=1}^{n} A_i}}(\underline{\underline{a}}) = \underline{\underline{a}}$;

(10) $\overline{OM_{\sum_{i=1}^{n} A_i}}(1_Z)(x) = \bigwedge_{i=1}^{n} \bigvee_{u\in Z} R_{A_i}(u,x)$;

(11) $\|\overline{OM_{\sum_{i=1}^{n} A_i}}(X)\| = \|X\|$, *where* $\|X\| = \sup_{u\in U} X(u)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items hold when $R_A = R_B$. When $R_A \neq R_B$, (1)–(5) can be found in [29].

(6) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{OM}_{A+B}}(1_x)(y) \\
&\quad = \bigvee_{u\in U} T(R_A(u,y), 1_x(u)) \wedge \bigvee_{u\in U} T(R_B(u,y), 1_x(u)) \\
&\quad = R_A(x,y) \wedge R_B(x,y).
\end{aligned} \tag{15}$$

Therefore, (6) can hold by the symmetric.

(7) For any $x \in X$,

$$\begin{aligned}
&\overline{\mathrm{OM}_{A+B}}\left(T\left(\underline{\underline{a}}, X\right)\right)(x) \\
&\quad = \bigvee_{u\in U} T\left(R_A(u,y), T\left(\underline{\underline{a}}, X\right)(u)\right) \\
&\qquad \wedge \bigvee_{u\in U} T\left(R_B(u,y), T\left(\underline{\underline{a}}, X\right)(u)\right) \\
&\quad = \bigvee_{u\in U} T(R_A(u,y), T(a, X(u))) \\
&\qquad \wedge \bigvee_{u\in U} T(R_B(u,y), T(a, X(u))) \\
&\quad = \bigvee_{u\in U} T(a, T(R_A(u,y), X(u))) \\
&\qquad \wedge \bigvee_{u\in U} T(a, T(R_B(u,y), X(u))) \\
&\quad = T\left(a, \bigvee_{u\in U} T(R_A(u,y), X(u))\right) \\
&\qquad \wedge T\left(a, \bigvee_{u\in U} T(R_B(u,y), X(u))\right) \\
&\quad = T\left(a, \bigvee_{u\in U} T(R_A(u,y), X(u)) \right. \\
&\qquad \left. \wedge \bigvee_{u\in U} T(R_B(u,y), X(u))\right) \\
&\quad = T\left(a, \overline{\mathrm{OM}_{A+B}}(X)\right)(x).
\end{aligned} \tag{16}$$

(8) It can be easily proved by item (1) and $\theta(2)$.

(9) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{OM}_{A+B}}\left(\underline{\underline{a}}\right)(x) \\
&\quad = \bigvee_{u\in U} T(R_A(u,x), a) \wedge \bigvee_{u\in U} T(R_B(u,x), a) \\
&\quad = T\left(\bigvee_{u\in U} R_A(u,x), a\right) \wedge T\left(\bigvee_{u\in U} R_B(u,x), X(u)\right) \\
&\quad = T(1, a) \\
&\quad = a.
\end{aligned} \tag{17}$$

(10) For any $Z \subseteq U$,

$$\begin{aligned}
&\overline{\mathrm{OM}_{A+B}}(1_Z)(x) \\
&\quad = \bigvee_{u\in U} T(R_A(u,x), (1_Z)(u)) \wedge \bigvee_{u\in U} T(R_B(u,x), (1_Z)(u)) \\
&\quad = \bigvee_{u\in Z} T(R_A(u,x), 1) \wedge \bigvee_{u\in Z} T(R_B(u,x), 1) \\
&\quad = \bigvee_{u\in Z} R_A(u,x) \wedge \bigvee_{u\in Z} R_B(u,x).
\end{aligned} \tag{18}$$

(11) Let $a = \|X\|$, so $X \subseteq \underline{\underline{a}}$. According to items (1) and (9), we can have

$$X \subseteq \overline{\mathrm{OM}_{A+B}}(X) \subseteq \overline{\mathrm{OM}_{A+B}}\left(\underline{\underline{a}}\right). \tag{19}$$

Therefore, $a = \|X\| \leq \|\overline{\mathrm{OM}_{A+B}}(X)\| \leq a$. □

Proposition 7. *Let* $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ *be a* T*-fuzzy approximation space; let* R_{A_i}, $i \in \{1, 2, 3, \ldots, n\}$, *be the different* T*-fuzzy similarity relations. For all* $x, y \in U$, $a, b \in I$, *and* $X, Y \in F(U)$, *the optimistic multigranulation* T*-fuzzy lower and upper approximation operators have the following properties:*

(1) $\underline{OM_{\sum_{i=1}^{n} A_i}}(X) = \bigwedge_{a\in I} \Theta(\overline{OM_{\sum_{i=1}^{n} A_i}}(\Theta(X, \underline{\underline{a}})), \underline{\underline{a}})$;

(2) $\overline{OM_{\sum_{i=1}^{n} A_i}}(X) = \bigwedge_{a\in I} \Theta(\underline{OM_{\sum_{i=1}^{n} A_i}}(\Theta(X, \underline{\underline{a}})), \underline{\underline{a}})$;

(3) $\underline{OM_{\sum_{i=1}^{n} A_i}}(\Theta(X, \underline{\underline{a}})) = \Theta(\overline{OM_{\sum_{i=1}^{n} A_i}}(X), \underline{\underline{a}})$;

(4) $\underline{OM_{\sum_{i=1}^{n} A_i}} \Theta(\overline{OM_{\sum_{i=1}^{n} A_i}}(X), \underline{\underline{a}}) = \Theta(\overline{OM_{\sum_{i=1}^{n} A_i}}(X), \underline{\underline{a}})$;

(5) $\theta(X, \underline{OM_{\sum_{i=1}^{n} A_i}}(Y)) = \theta(\overline{OM_{\sum_{i=1}^{n} A_i}}(X), Y)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items

hold when $R_A = R_B$. When $R_A \neq R_B$, the proposition can be proved as follows.

(1) For any $x \in U$,

$$\begin{aligned}
&\bigwedge_{a\in I}\Theta\left(\overline{\mathrm{OM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right),\underline{\underline{a}}\right)(x)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigvee_{u\in U}T\left(R_A(u,x),\theta\left(X(u),a\right)\right)\\
&\qquad\qquad\wedge\bigvee_{u\in U}T\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\theta\left(\bigvee_{u\in U}T\left(R_A(u,x),\theta\left(X(u),a\right)\right),a\right)\\
&\qquad\vee\bigwedge_{a\in I}\theta\left(\bigvee_{u\in U}T\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\right)\\
&\quad=\bigwedge_{a\in I}\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(\theta\left(X(u),a\right),a\right)\right)\\
&\qquad\vee\bigwedge_{a\in I}\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(\theta\left(X(u),a\right),a\right)\right)\\
&\quad=\bigwedge_{u\in U}\theta\left(R_A(u,x),X(u)\right)\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),X(u)\right)\\
&\quad=\underline{\mathrm{OM}_{A+B}}(X)(x).
\end{aligned}\tag{20}$$

(2) For any $x \in U$,

$$\begin{aligned}
&\bigwedge_{a\in I}\Theta\left(\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right),\underline{\underline{a}}\right)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right)\\
&\qquad\qquad\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\theta\left(\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right),a\right)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\left(\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\right)\\
&\quad=\bigwedge_{a\in I}\theta\left(\bigwedge_{u\in U}\theta\left(T\left(R_A(u,x),X(u)\right),a\right),a\right)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\left(\bigwedge_{u\in U}\theta\left(T\left(R_B(u,x),X(u)\right),a\right),a\right)\\
&\quad=\bigwedge_{a\in I}\theta\left(\theta\left(\bigvee_{u\in U}T\left(R_A(u,x),X(u)\right),a\right),a\right)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\left(\theta\left(\bigvee_{u\in U}T\left(R_B(u,x),X(u)\right),a\right),a\right)\\
&\quad=\bigvee_{u\in U}T\left(R_A(u,x),X(u)\right)\wedge\bigvee_{u\in U}T\left(R_B(u,x),X(u)\right)\\
&\quad=\overline{\mathrm{OM}_{A+B}}(X)(x).
\end{aligned}\tag{21}$$

(3) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right)(x)\\
&\quad=\bigwedge_{a\in I}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right)\\
&\qquad\vee\bigwedge_{a\in I}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right)\\
&\quad=\bigwedge_{a\in I}\theta\left(T\left(R_A(u,x),X(u)\right),a\right)\\
&\qquad\vee\bigwedge_{a\in I}\theta\left(T\left(R_B(u,x),X(u)\right),a\right)\\
&\quad=\theta\left(\bigvee_{a\in I}T\left(R_A(u,x),X(u)\right),a\right)\\
&\qquad\vee\theta\left(\bigvee_{a\in I}T\left(R_B(u,x),X(u)\right),a\right)\\
&\quad=\theta\Bigg(\bigvee_{a\in I}T\left(R_A(u,x),X(u)\right)\\
&\qquad\qquad\wedge\bigvee_{a\in I}T\left(R_B(u,x),X(u)\right),a\Bigg)\\
&\quad=\theta\left(\overline{\mathrm{OM}_{A+B}}(X)(x),a\right).
\end{aligned}\tag{22}$$

(4) We can have, by item (3) in Proposition 7 and item (2) in Proposition 6,

$$\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\Theta\left(\overline{\mathrm{OM}_{A+B}}(X),\underline{\underline{a}}\right)\right)\\
&\quad=\Theta\left(\overline{\mathrm{OM}_{A+B}}\left(\overline{\mathrm{OM}_{A+B}}(X)\right),\underline{\underline{a}}\right)=\Theta\left(\overline{\mathrm{OM}_{A+B}}(X),\underline{\underline{a}}\right).
\end{aligned}\tag{23}$$

(5) According to $\theta 6$ and $\theta 9$, we can have

$$\begin{aligned}
&\theta\left(X,\underline{\mathrm{OM}_{A+B}}(Y)\right)\\
&\quad=\bigwedge_{u\in U}\theta\left(X(u),\underline{\mathrm{OM}_{A+B}}(Y)(u)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{u\in U}\theta\Bigg(X(u), \bigwedge_{v\in U}\theta\left(R_A(v,u), Y(v)\right) \\
&\qquad \vee \bigwedge_{v\in U}\theta\left(R_B(v,u), Y(v)\right)\Bigg) \\
&= \bigwedge_{u\in U}\theta\left(X(u), \bigwedge_{v\in U}\theta\left(R_A(v,u), Y(v)\right)\right) \\
&\quad \vee \bigwedge_{u\in U}\theta\left(X(u), \bigwedge_{v\in U}\theta\left(R_B(v,u), Y(v)\right)\right) \\
&= \bigwedge_{u\in U}\bigwedge_{v\in U}\theta\left(T\left(R_A(v,u), X(u)\right), Y(v)\right) \\
&\quad \vee \bigwedge_{u\in U}\bigwedge_{v\in U}\theta\left(T\left(R_B(v,u), X(u)\right), Y(v)\right) \\
&= \bigwedge_{v\in U}\theta\left(\bigvee_{u\in U}T\left(R_A(v,u), X(u)\right), Y(v)\right) \\
&\quad \vee \bigwedge_{v\in U}\theta\left(\bigvee_{u\in U}T\left(R_B(v,u), X(u)\right), Y(v)\right) \\
&= \bigwedge_{v\in U}\theta\Bigg(\bigvee_{u\in U}T\left(R_A(v,u), X(u)\right) \\
&\qquad \wedge \bigvee_{u\in U}T\left(R_B(v,u), X(u)\right), Y(v)\Bigg) \\
&= \bigwedge_{v\in U}\theta\left(\overline{\mathrm{OM}_{A+B}}(X)(v), Y(v)\right).
\end{aligned}
\tag{24}
$$

□

4. Model and Properties of Pessimistic Multigranulation T-Fuzzy Rough Sets

In Sections 2 and 3, we introduced the model and properties of optimistic multigranulation T-fuzzy rough sets. Now, we begin to study a new kind of multigranulation T-fuzzy rough sets called the pessimistic multigranulation rough set in the T-fuzzy approximation space.

Definition 8. Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be a T-fuzzy approximation space. For any $X \in F(U)$, we can define the pessimistic multigranulation T-fuzzy lower and upper approximations of X as follows:

$$
\begin{aligned}
\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X)(x) &= \bigwedge_{i=1}^{n}\left(\bigwedge_{u\in U}\theta\left(R_{A_i}(u,x), X(u)\right)\right), \\
\overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X)(x) &= \bigvee_{i=1}^{n}\left(\bigvee_{u\in U}T\left(R_{A_i}(u,x), X(u)\right)\right),
\end{aligned}
\tag{25}
$$

where "$\bigvee$" means "max," "$\bigwedge$" means "min," and θ and T are defined in Section 2. $\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}$ and $\overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}$ are referred to as the pessimistic multigranulation T-fuzzy lower and T upper approximation operators. The pair $(\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X), \overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X))$ is called the pessimistic multigranulation T-fuzzy rough set of X. If $\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X) = \overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X)$, then X is referred to as pessimistic definable under the T-fuzzy approximation space; otherwise, X is referred to as pessimistic undefinable. The boundary of the pessimistic multigranulation T-fuzzy rough set X is defined as

$$
\mathrm{Bnd}^P_{\sum_{i=1}^n A_i}(X) = \overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X) \cap \left(\sim \underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X)\right). \tag{26}
$$

Example 9 (continued from Example 4). From Definition 8, we can compute pessimistic multigranulation lower and upper approximations of X over the T-fuzzy similar relations R_A and R_B as

$$
\begin{aligned}
\underline{\mathrm{PM}_{A+B}} &= \{0.3, 0.3, 0.3, 0.3, 0.3\}, \\
\overline{\mathrm{PM}_{A+B}} &= \{0.5, 0.5, 0.5, 0.6, 0.6\}.
\end{aligned}
\tag{27}
$$

From the definition of the pessimistic multigranulation T-fuzzy lower and upper approximations, it is possible to deduce the following properties of the pessimistic multigranulation T-fuzzy lower and upper approximation operators.

Proposition 10. *Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be a T-fuzzy approximation space; let R_{A_i} $(i \in \{1, 2, 3, \ldots, n\})$ be the different T-fuzzy similarity relations. For all $x, y \in U$, $a, b \in I$, and $X, Y \in F(U)$, the pessimistic multigranulation T-fuzzy lower approximation has the following properties:*

(1) $\underline{PM_{\sum_{i=1}^n A_i}}(X) \subseteq X$;

(2) $\underline{PM_{\sum_{i=1}^n A_i}}(\underline{PM_{\sum_{i=1}^n A_i}}(X)) \subseteq \underline{PM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\underline{PM_{\sum_{i=1}^n A_i}}(X \cap Y) = \underline{PM_{\sum_{i=1}^n A_i}}(X) \cap \underline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(4) $X \subseteq Y \Rightarrow \underline{PM_{\sum_{i=1}^n A_i}}(X) \subseteq \underline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(5) $\underline{PM_{\sum_{i=1}^n A_i}}(X \cup Y) \supseteq \underline{PM_{\sum_{i=1}^n A_i}}(X) \cup \underline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(6) $\underline{PM_{\sum_{i=1}^n A_i}}(\Theta(1_x, \underline{\underline{a}}))(y) = \underline{PM_{\sum_{i=1}^n A_i}}(\Theta(1_y, \underline{\underline{a}}))(x) = \bigvee_{i=1}^n \theta(R_{A_i}(x, y), a)$;

(7) $\underline{PM_{\sum_{i=1}^n A_i}}(\theta(1_x, \underline{\underline{a}}))(x) = a$;

(8) $\underline{PM_{\sum_{i=1}^n A_i}}(\Theta(\underline{\underline{a}}, X)) = \Theta(\underline{\underline{a}}, \underline{PM_{\sum_{i=1}^n A_i}}(X))$;

(9) $\underline{PM_{\sum_{i=1}^n A_i}}(X, \underline{\underline{a}}) \subseteq \Theta(X, \underline{\underline{a}}) \subseteq \Theta(\underline{PM_{\sum_{i=1}^n A_i}}(X), \underline{\underline{a}})$;

(10) $\underline{PM_{\sum_{i=1}^n A_i}}(\underline{\underline{a}}) = \underline{\underline{a}}$;

(11) $\underline{PM_{\sum_{i=1}^n A_i}}((\underline{\underline{a}}) \vee 1_Z)(x) = \bigvee_{i=1}^n \bigwedge_{u\notin Z}\theta(R_{A_i}(u, x), a)$;

(12) $\sum_{i=1}^n A_i((\underline{\underline{\theta(a,b)}}) \vee 1_Z)(x) = \Theta((\underline{\underline{a}}), \underline{PM_{\sum_{i=1}^n A_i}}(b \vee 1_Z))$;

(13) $\bigwedge_{a\in I}\theta(\underline{PM_{\sum_{i=1}^n A_i}}(\Theta(1_x, \underline{\underline{a}}))(y), a) = \bigwedge_{i=1}^n R_{A_i}(x, y)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items

hold when $R_A = R_B$. When $R_A \neq R_B$, the proposition can be proved as follows.

(1) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}(X)(x) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), X(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), X(u)\right) \\
&\quad \leq \theta\left(R_A(x,x), X(x)\right) \wedge \theta\left(R_B(x,x), X(x)\right) = X(x).
\end{aligned} \tag{28}$$

(2) According to item (1), it obviously holds.

(3) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}(X \cap Y)(x) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), X(u) \wedge Y(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), X(u) \wedge Y(u)\right) \\
&\quad = \left[\bigwedge_{u\in U}\theta\left(R_A(u,x), X(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_A(u,x), Y(u)\right)\right] \\
&\qquad \wedge \left[\bigwedge_{u\in U}\theta\left(R_B(u,x), X(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), Y(u)\right)\right] \\
&\quad = \left[\bigwedge_{u\in U}\theta\left(R_A(u,x), X(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), X(u)\right)\right] \\
&\qquad \wedge \left[\bigwedge_{u\in U}\theta\left(R_A(u,x), Y(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), Y(u)\right)\right] \\
&\quad = \underline{\mathrm{PM}_{A+B}}(X)(x) \cap \underline{\mathrm{PM}_{A+B}}(Y)(x).
\end{aligned} \tag{29}$$

(4) For any $x \in U$, we have $X(x) \leq Y(x)$ by $X \subseteq Y$. So

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}(X)(x) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), X(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), X(u)\right) \\
&\quad \leq \bigwedge_{u\in U}\theta\left(R_A(u,x), Y(u)\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), Y(u)\right) \\
&\quad = \underline{\mathrm{PM}_{A+B}}(Y)(x).
\end{aligned} \tag{30}$$

(5) It is easy to prove according to item (4).

(6) First of all, we have

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\Theta\left(1_x, \underline{a}\right)\right)(y) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,y), \left(\Theta\left(1_x, \underline{a}\right)(u)\right)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,y), \left(\Theta\left(1_x, \underline{a}\right)(u)\right)\right) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,y), \left(\theta\left(1_x(u), a\right)\right)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,y), \left(\theta\left(1_x(u), a\right)\right)\right) \\
&\quad = \bigwedge_{u=x}\theta\left(R_A(x,y), \left(\theta(1,a)\right)\right) \\
&\qquad \wedge \bigwedge_{u=x}\theta\left(R_B(x,y), \left(\theta(1,a)\right)\right) \\
&\quad = \theta\left(R_A(x,y), a\right) \wedge \theta\left(R_B(x,y), a\right).
\end{aligned} \tag{31}$$

By the symmetric and the above equation, item (6) can be proved.

(7) It can be verified by item (6).

(8) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\Theta\left(\underline{a}, X\right)\right)(x) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), \Theta\left(\underline{a}, X\right)(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), \Theta\left(\underline{a}, X\right)(u)\right) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), \theta(a, X(u))\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), \theta(a, X(u))\right) \\
&\quad = \bigwedge_{u\in U}\theta\left(a, \theta\left(R_A(u,x), X(u)\right)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(a, \theta\left(R_B(u,x), X(u)\right)\right) \\
&\quad = \theta\left(a, \bigwedge_{u\in U}\theta\left(R_A(u,x), X(u)\right)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), X(u)\right) \\
&\quad = \Theta\left(a, \underline{\mathrm{PM}_{A+B}}(X)\right)(x).
\end{aligned} \tag{32}$$

(9) It is easy to prove by item (1) and $\theta 3$.

(10) For any $x \in U$,

$$\begin{aligned}
\underline{\mathrm{PM}_{A+B}}\left(\underline{a}\right)(x) &= \bigwedge_{u\in U}\theta\left(R_A(u,x), a\right) \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), a\right) \\
&= \theta\left(\bigvee_{u\in U} R_A(u,x), a\right) \wedge \theta\left(\bigvee_{u\in U} R_B(u,x), a\right) \\
&= \theta(1, a) = a.
\end{aligned} \tag{33}$$

(11) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\left(\underline{a}\right) \vee 1_Z\right)(x) \\
&\quad = \bigwedge_{u\in U} \theta\left(R_A(u,x), \left(\left(\underline{a}\right) \vee 1_Z\right)(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U} \theta\left(R_B(u,x), \left(\left(\underline{a}\right) \vee 1_Z\right)(u)\right) \\
&\quad = \bigwedge_{u\in U} \theta\left(R_A(u,x), a \vee 1_Z(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U} \theta\left(R_B(u,x), a \vee 1_Z(u)\right) \\
&\quad = \bigwedge_{u\in Z} \theta\left(R_A(u,x), 1\right) \vee \bigwedge_{u\notin Z} \theta\left(R_A(u,x), a\right) \\
&\qquad \vee \bigwedge_{u\in Z} \theta\left(R_B(u,x), 1\right) \vee \bigwedge_{u\notin Z} \theta\left(R_B(u,x), a\right) \\
&\quad = \bigwedge_{u\notin Z} \theta\left(R_A(u,x), a\right) \vee \bigwedge_{u\notin Z} \theta\left(R_B(u,x), a\right).
\end{aligned} \tag{34}$$

(12) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\left(\underline{\underline{\theta(a,b)}}\right) \vee 1_Z\right)(x) \\
&\quad = \bigwedge_{u\in U} \theta\left(R_A(u,x), \left(\left(\underline{\underline{\theta(a,b)}}\right) \vee 1_Z\right)(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U} \theta\left(R_B(u,x), \left(\left(\underline{\underline{\theta(a,b)}}\right) \vee 1_Z\right)(u)\right) \\
&\quad = \bigwedge_{u\in U} \theta\left(R_A(u,x), \left(\theta(a,b) \vee 1_Z(u)\right)\right) \\
&\qquad \wedge \bigwedge_{u\in U} \theta\left(R_B(u,x), \left(\theta(a,b) \vee 1_Z(u)\right)\right) \\
&\quad = \bigwedge_{u\notin Z} \theta\left(R_A(u,x), \theta(a,b)\right) \wedge \bigwedge_{u\notin Z} \theta\left(R_B(u,x), \theta(a,b)\right) \\
&\quad = \bigwedge_{u\notin Z} \theta\left(a, \theta\left(R_A(u,x), b\right)\right) \wedge \bigwedge_{u\notin Z} \theta\left(a, \theta\left(R_B(u,x), b\right)\right) \\
&\quad = \theta\left(a, \bigwedge_{u\notin Z} \theta\left(R_A(u,x), b\right) \wedge \bigwedge_{u\notin Z} \theta\left(R_B(u,x), b\right)\right) \\
&\quad = \theta\left(a, \underline{\mathrm{PM}_{A+B}}\left(\left(\underline{b}\right) \vee 1_Z\right)\right)(x).
\end{aligned} \tag{35}$$

(13) We can have, by item (6),

$$\begin{aligned}
&\bigwedge_{a\in I} \theta\left(\underline{\mathrm{PM}_{A+B}}\left(\Theta\left(1_x, \underline{a}\right)\right)(y), a\right) \\
&\quad = \bigwedge_{a\in I} \theta\left(\theta\left(R_A(x,y), a\right) \wedge \theta\left(R_B(x,y), a\right), a\right) \\
&\quad \geq \bigwedge_{a\in I} \theta\left(\theta\left(R_A(x,y), a\right), a\right) \wedge \bigwedge_{a\in I} \theta\left(\theta\left(R_B(x,y), a\right), a\right) \\
&\quad = R_A(x,y) \wedge R_B(x,y).
\end{aligned} \tag{36}$$

□

Proposition 11. *Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be a T-fuzzy approximation space; let R_{A_i} $(i \in \{1,2,3,\ldots,n\})$ be the different T-fuzzy similarity relations. For all $x, y \in U$, $a, b \in I$, and $X, Y \in F(U)$, the pessimistic multigranulation T-fuzzy upper approximation has the following properties:*

(1) $X \subseteq \overline{PM_{\sum_{i=1}^n A_i}}(X)$;

(2) $\overline{PM_{\sum_{i=1}^n A_i}}(\overline{PM_{\sum_{i=1}^n A_i}}(X)) \supseteq \overline{PM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\overline{PM_{\sum_{i=1}^n A_i}}(X \cup Y) = \overline{PM_{\sum_{i=1}^n A_i}}(X) \cup \overline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(4) $X \subseteq Y \Rightarrow \overline{PM_{\sum_{i=1}^n A_i}}(X) \subseteq \overline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(5) $\overline{PM_{\sum_{i=1}^n A_i}}(X \cap Y) \subseteq \overline{PM_{\sum_{i=1}^n A_i}}(X) \cap \overline{PM_{\sum_{i=1}^n A_i}}(Y)$;

(6) $\overline{PM_{\sum_{i=1}^n A_i}}(1_x)(y) = \overline{PM_{\sum_{i=1}^n A_i}}(1_y)(x) = \bigvee_{i=1}^n R_{A_i}(x,y)$;

(7) $\overline{PM_{\sum_{i=1}^n A_i}}(T(\underline{\underline{a}}, X)) = T(\underline{\underline{a}}, \overline{PM_{\sum_{i=1}^n A_i}}(X))$;

(8) $\overline{PM_{\sum_{i=1}^n A_i}}(\Theta(X, \underline{\underline{a}})) \supseteq \Theta(X, \underline{\underline{a}}) \supseteq \Theta(\overline{PM_{\sum_{i=1}^n A_i}}(X), \underline{\underline{a}})$;

(9) $\overline{PM_{\sum_{i=1}^n A_i}}(\underline{\underline{a}}) = \underline{\underline{a}}$;

(10) $\overline{PM_{\sum_{i=1}^n A_i}}(1_Z)(x) = \bigwedge_{i=1}^n \bigvee_{u\in Z} R_{A_i}(u,x)$;

(11) $\|\overline{PM_{\sum_{i=1}^n A_i}}(X)\| = \|X\|$, *where* $\|X\| = \sup_{u\in U} X(u)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items hold when $R_A = R_B$. When $R_A \neq R_B$, the proposition can be proved as follows.

(1) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}(X)(x) \\
&\quad = \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right) \\
&\quad \geq T\left(R_A(x,x), X(x)\right) \vee T\left(R_B(x,x), X(x)\right) = X(x).
\end{aligned} \tag{37}$$

(2) This item can be proved by item (1).

(3) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}(X \cup Y)(x) \\
&\quad = \bigvee_{u\in U} T\left(R_A(u,x), X(u) \vee Y(u)\right) \\
&\qquad \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u) \vee Y(u)\right)
\end{aligned}$$

$$= \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right)$$
$$\vee \bigvee_{u\in U} T\left(R_A(u,x), Y(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), Y(u)\right)$$
$$= \overline{\mathrm{PM}_{A+B}}(X)(x) \cup \overline{\mathrm{PM}_{A+B}}(Y)(x). \tag{38}$$

(4) Since $X \subseteq Y$, for any $x \in X$, we can have $X(x) \leq Y(x)$. Thus,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}(X)(x) \\
&= \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right) \\
&\leq \bigvee_{u\in U} T\left(R_A(u,x), Y(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), Y(u)\right) \\
&= \overline{\mathrm{PM}_{A+B}}(Y)(x).
\end{aligned} \tag{39}$$

(5) It is easy to prove by item (4).

(6) According to Definition 8, we have

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}(1_x)(y) \\
&= \bigvee_{u\in U} T\left(R_A(u,y), (1_x)(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,y), (1_x)(u)\right) \\
&= T\left(R_A(x,y), 1\right) \vee T\left(R_B(x,y), 1\right) \\
&= R_A(x,y) \vee R_B(x,y).
\end{aligned} \tag{40}$$

We can conclude that $\overline{\mathrm{PM}_{A+B}}(1_x)(y) = \overline{\mathrm{PM}_{A+B}}(1_y)(x) = R_A(x,y) \vee R_B(x,y)$ by the symmetric and the above equation.

(7) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}\left(T\left(\underline{a}, X\right)\right)(x) \\
&= \bigvee_{u\in U} T\left(R_A(u,x), \left(T\left(\underline{a}, X\right)\right)(u)\right) \\
&\quad \vee \bigvee_{u\in U} T\left(R_B(u,x), \left(T\left(\underline{a}, X\right)\right)(u)\right) \\
&= \bigvee_{u\in U} T\left(R_A(u,x), T(a, X(u))\right) \\
&\quad \vee \bigvee_{u\in U} T\left(R_B(u,x), T(a, X(u))\right) \\
&= T\left(a, \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right)\right) \\
&\quad \vee T\left(a, \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right)\right) \\
&= T\left(a, \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right)\right. \\
&\qquad \left. \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right)\right) \\
&= T\left(\underline{a}, \overline{\mathrm{PM}_{A+B}}(X)\right)(x).
\end{aligned} \tag{41}$$

(8) It directly follows from item (1) and $\theta 3$.

(9) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}\left(\underline{a}\right)(x) \\
&= \bigvee_{u\in U} T\left(R_A(u,x), a\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), a\right) \\
&= T\left(\bigvee_{u\in U} R_A(u,x), a\right) \vee T\left(\bigvee_{u\in U} R_B(u,x), a\right) = a.
\end{aligned} \tag{42}$$

(10) For any $x \in U$,

$$\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}(1_Z)(x) \\
&= \bigvee_{u\in U} T\left(R_A(u,x), X(u)\right) \vee \bigvee_{u\in U} T\left(R_B(u,x), X(u)\right) \\
&= \bigvee_{u\in Z} T\left(R_A(u,x), X(u)\right) \vee \bigvee_{u\in Z} T\left(R_B(u,x), X(u)\right) \\
&= \bigvee_{u\in Z} R_A(u,x) \vee \bigvee_{u\in Z} R_B(u,x).
\end{aligned} \tag{43}$$

(11) Let $a = \|X\|$, so $X \subseteq \underline{a}$. According to items (1) and (9), we can have

$$X \subseteq \overline{\mathrm{PM}_{A+B}}(X) \subseteq \overline{\mathrm{PM}_{A+B}}\left(\underline{a}\right). \tag{44}$$

Therefore, $a = \|X\| \leq \|\overline{\mathrm{PM}_{A+B}}(X)\| \leq a$. □

Proposition 12. *Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be a T-fuzzy approximation space, and let R_{A_i} $(i \in \{1,2,3,\ldots,n\})$ be the different T-fuzzy similarity relations. For all $x, y \in U$, $a, b \in I$, and $X, Y \in F(U)$, the pessimistic multigranulation T-fuzzy lower and upper approximation operators have the following properties:*

(1) $\bigwedge_{a\in I} \Theta(\overline{PM_{\sum_{i=1}^n A_i}}(\Theta(X,\underline{a})), \underline{a}) = \underline{PM_{\sum_{i=1}^n A_i}}(X)$;

(2) $\bigwedge_{a\in I} \Theta(\underline{PM_{\sum_{i=1}^n A_i}}(\Theta(X,\underline{a})), \underline{a}) = \overline{PM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\underline{PM_{\sum_{i=1}^n A_i}}(\Theta(X,\underline{a})) = \Theta(\overline{PM_{\sum_{i=1}^n A_i}}(X), \underline{a})$;

(4) $\theta(X, \underline{PM_{\sum_{i=1}^n A_i}}(Y)) = \theta(\overline{PM_{\sum_{i=1}^n A_i}}(X), Y)$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U, R_A, R_B) for convenience. All items

hold when $R_A = R_B$. When $R_A \neq R_B$, the proposition can be proved as follows.

(1) For any $x \in U$,

$$\begin{aligned}
&\bigwedge_{a\in I}\Theta\left(\overline{\mathrm{PM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right),\underline{\underline{a}}\right)(x)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigvee_{u\in U}T\left(R_A(u,x),\theta\left(X(u),a\right)\right)\\
&\qquad\qquad\vee\bigvee_{u\in U}T\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigvee_{u\in U}T\left(R_A(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\Bigg(\bigvee_{u\in U}T\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(\theta\left(X(u),a\right),a\right)\right)\\
&\qquad\wedge\bigwedge_{a\in I}\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(\theta\left(X(u),a\right),a\right)\right)\\
&\quad=\bigwedge_{u\in U}\theta\left(R_A(u,x),X(u)\right)\wedge\bigwedge_{u\in U}\theta\left(R_B(u,x),X(u)\right)\\
&\quad=\underline{\mathrm{PM}_{A+B}}(X)(x).
\end{aligned}\tag{45}$$

(2) For any $x \in U$,

$$\begin{aligned}
&\bigwedge_{a\in I}\Theta\left(\underline{\mathrm{PM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right),\underline{\underline{a}}\right)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right)\Bigg)\\
&\qquad\wedge\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\qquad\vee\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(T\left(R_A(u,x),X(u)\right),a\right),a\Bigg)\\
&\qquad\vee\bigwedge_{a\in I}\theta\Bigg(\bigwedge_{u\in U}\theta\left(T\left(R_B(u,x),X(u)\right),a\right),a\Bigg)\\
&\quad=\bigwedge_{a\in I}\theta\Bigg(\theta\Bigg(\bigvee_{u\in U}T\left(R_A(u,x),X(u)\right),a\Bigg),a\Bigg)\\
&\qquad\vee\bigwedge_{a\in I}\theta\Bigg(\theta\Bigg(\bigvee_{u\in U}T\left(R_B(u,x),X(u)\right),a\Bigg),a\Bigg)\\
&\quad=\bigvee_{u\in U}T\left(R_A(u,x),X(u)\right)\vee\bigvee_{u\in U}T\left(R_B(u,x),X(u)\right)\\
&\quad=\overline{\mathrm{PM}_{A+B}}(X)(x).
\end{aligned}\tag{46}$$

(3) For any $x \in U$,

$$\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\Theta\left(X,\underline{\underline{a}}\right)\right)(x)\\
&\quad=\bigwedge_{a\in I}\theta\left(R_A(u,x),\theta\left(X(u),a\right)\right)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\left(R_B(u,x),\theta\left(X(u),a\right)\right)\\
&\quad=\bigwedge_{a\in I}\theta\left(T\left(R_A(u,x),X(u)\right),a\right)\\
&\qquad\wedge\bigwedge_{a\in I}\theta\left(T\left(R_B(u,x),X(u)\right),a\right)\\
&\quad=\theta\Bigg(\bigvee_{a\in I}T\left(R_A(u,x),X(u)\right),a\Bigg)\\
&\qquad\wedge\theta\Bigg(\bigvee_{a\in I}T\left(R_B(u,x),X(u)\right),a\Bigg)\\
&\quad=\theta\Bigg(\bigvee_{a\in I}T\left(R_A(u,x),X(u)\right)\vee\bigvee_{a\in I}T\left(R_B(u,x),X(u)\right),a\Bigg)\\
&\quad=\theta\left(\overline{\mathrm{PM}_{A+B}}(X)(x),a\right).
\end{aligned}\tag{47}$$

(4) According to $\theta 6$ and $\theta 9$, we can obtain

$$\begin{aligned}
&\theta\left(X,\underline{\mathrm{PM}_{A+B}}(Y)\right)\\
&\quad=\bigwedge_{u\in U}\theta\left(X(u),\underline{\mathrm{PM}_{A+B}}(Y)(u)\right)\\
&\quad=\bigwedge_{u\in U}\theta\Bigg(X(u),\bigwedge_{v\in U}\theta\left(R_A(v,u),Y(v)\right)\\
&\qquad\qquad\wedge\bigwedge_{v\in U}\theta\left(R_B(v,u),Y(v)\right)\Bigg)\\
&\quad=\bigwedge_{u\in U}\theta\Bigg(X(u),\bigwedge_{v\in U}\theta\left(R_A(v,u),Y(v)\right)\Bigg)\\
&\qquad\wedge\bigwedge_{u\in U}\theta\Bigg(X(u),\bigwedge_{v\in U}\theta\left(R_B(v,u),Y(v)\right)\Bigg)\\
&\quad=\bigwedge_{u\in U}\bigwedge_{v\in U}\theta\left(T\left(R_A(v,u),X(u)\right),Y(v)\right)\\
&\qquad\wedge\bigwedge_{u\in U}\bigwedge_{v\in U}\theta\left(T\left(R_B(v,u),X(u)\right),Y(v)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \bigwedge_{v\in U}\theta\left(\bigvee_{u\in U}T\left(R_A(v,u),X(u)\right),Y(v)\right)\\
&\quad\wedge\bigwedge_{v\in U}\theta\left(\bigvee_{u\in U}T\left(R_B(v,u),X(u)\right),Y(v)\right)\\
&= \bigwedge_{v\in U}\theta\left(\bigvee_{u\in U}T\left(R_A(v,u),X(u)\right)\right)\\
&\quad\vee\bigvee_{u\in U}T\left(R_B(v,u),X(u)\right),Y(v)\\
&= \bigwedge_{v\in U}\theta\left(\overline{\mathrm{PM}_{A+B}}(X)(v)\right),Y(v).
\end{aligned}
\tag{48}
$$

Then, this proposition is proved. □

5. Relationships between Multigranulation and Classical T-Fuzzy Rough Sets

Based on the T-fuzzy similarity relation, after the discussion about the properties of the optimistic and pessimistic multigranulation T-fuzzy rough sets, we will investigate the relationships among the two types of multigranulation T-fuzzy rough sets and the classical T-fuzzy rough set in this section.

By the definitions of the optimistic and pessimistic multigranulation T-fuzzy rough set operators, for all $X \in F(U)$, the relationship can be easily obtained as

$$
\begin{aligned}
\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X) &\subseteq \underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) \subseteq X\\
&\subseteq \overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) \subseteq \overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X).
\end{aligned}
\tag{49}
$$

Note that if (U,R) is a T-fuzzy approximation space, then $\underline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X) = \underline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) = \underline{R}(X)$ and $\overline{\mathrm{OM}_{\sum_{i=1}^n A_i}}(X) = \overline{\mathrm{PM}_{\sum_{i=1}^n A_i}}(X) = \overline{R}(X)$. So in the special case of a T-fuzzy approximation space, both optimistic and pessimistic T-fuzzy lower and upper approximations can degenerate into the standard T-fuzzy lower and upper approximations.

Proposition 13. *Let $(U,R_{A_1},R_{A_2},\ldots,R_{A_n})$ be T-fuzzy approximation space, and let R_{A_i} ($i\in\{1,2,3,\ldots,n\}$) be the different T fuzzy similarity relations. For all $x,y\in U$, $a,b\in I$, and $X,Y\in F(U)$, one has the following:*

(1) $\overline{OM_{\sum_{i=1}^n A_i}}(\underline{PM_{\sum_{i=1}^n A_i}}(X)) = \underline{PM_{\sum_{i=1}^n A_i}}(X)$;

(2) $\underline{OM_{\sum_{i=1}^n A_i}}(\overline{PM_{\sum_{i=1}^n A_i}}(X)) = \overline{PM_{\sum_{i=1}^n A_i}}(X)$;

(3) $\overline{PM_{\sum_{i=1}^n A_i}}(\underline{PM_{\sum_{i=1}^n A_i}}(X)) \subseteq \underline{OM_{\sum_{i=1}^n A_i}}(X)$;

(4) $\underline{PM_{\sum_{i=1}^n A_i}}(\overline{PM_{\sum_{i=1}^n A_i}}(X)) \supseteq \overline{OM_{\sum_{i=1}^n A_i}}(X)$;

(5) $\theta(\overline{PM_{\sum_{i=1}^n A_i}}(X),\underline{OM_{\sum_{i=1}^n A_i}}(Y)) = \theta(\overline{PM_{\sum_{i=1}^n A_i}}(X),Y)$;

(6) $\theta(\overline{OM_{\sum_{i=1}^n A_i}}(X),\overline{PM_{\sum_{i=1}^n A_i}}(Y)) = \theta(X,\overline{PM_{\sum_{i=1}^n A_i}}(Y))$;

(7) $\underline{OM_{\sum_{i=1}^n A_i}}(\Theta(\underline{a},\overline{PM_{\sum_{i=1}^n A_i}}(X))) = \Theta(\underline{a},\overline{PM_{\sum_{i=1}^n A_i}}(X))$.

Proof. We only need to prove the proposition in a T-fuzzy approximation space (U,R_A,R_B) for convenience. All items hold when $R_A = R_B$. When $R_A \neq R_B$, the proposition can be proved as follows.

(1) For any $x\in U$,

$$
\begin{aligned}
&\overline{\mathrm{OM}_{A+B}}\left(\underline{\mathrm{PM}_{A+B}}(X)\right)(x)\\
&\quad= \bigvee_{u\in U}T\left(R_A(u,x),\underline{\mathrm{PM}_{A+B}}X(u)\right)\\
&\qquad\wedge\bigvee_{u\in U}T\left(R_B(u,x),\underline{\mathrm{PM}_{A+B}}X(u)\right)\\
&\quad= \left[\bigvee_{u\in U}T\left(R_A(u,x),\bigwedge_{v\in U}\theta\left(R_A(v,u),X(v)\right)\right.\right.\\
&\qquad\qquad\left.\left.\wedge\bigwedge_{v\in U}\theta\left(R_B(v,u),X(v)\right)\right)\right]\\
&\qquad\wedge\left[\bigvee_{u\in U}T\left(R_B(u,x),\bigwedge_{v\in U}\theta\left(R_A(v,u),X(v)\right)\right.\right.\\
&\qquad\qquad\left.\left.\wedge\bigwedge_{v\in U}\theta\left(R_B(v,u),X(v)\right)\right)\right]\\
&\quad\le \bigvee_{u\in U}T\left(R_A(u,x),\bigwedge_{v\in U}\theta\left(R_A(v,u),X(v)\right)\right)\\
&\qquad\wedge\bigvee_{u\in U}T\left(R_B(u,x),\bigwedge_{v\in U}\theta\left(R_B(v,u),X(v)\right)\right)\\
&\quad= \bigvee_{u\in U}\bigwedge_{v\in U}T\left(R_A(u,x),\theta\left(R_A(v,u),X(v)\right)\right)\\
&\qquad\wedge\bigvee_{u\in U}\bigwedge_{v\in U}T\left(R_B(u,x),\theta\left(R_B(v,u),X(v)\right)\right)\\
&\quad\le \bigvee_{u\in U}\bigwedge_{v\in U}T\left(R_A(u,x),\right.\\
&\qquad\qquad\left.\theta\left(T\left(R_A(v,x),R_A(x,u)\right),X(v)\right)\right)\\
&\qquad\wedge\bigvee_{u\in U}\bigwedge_{v\in U}T\left(R_B(u,x),\right.\\
&\qquad\qquad\left.\theta\left(T\left(R_B(v,x),R_B(x,u)\right),X(v)\right)\right)\\
&\quad\le \bigwedge_{v\in U}\theta\left(R_A(v,x),X(v)\right)\wedge\bigwedge_{v\in U}\theta\left(R_B(v,x),X(v)\right)\\
&\quad= \underline{\mathrm{PM}_{A+B}}(X)(x).
\end{aligned}
\tag{50}
$$

On the other hand, $\underline{\mathrm{PM}_{A+B}}(X) \subseteq \overline{\mathrm{OM}_{A+B}}(\underline{\mathrm{PM}_{A+B}}(X))$. Therefore, $\underline{\mathrm{PM}_{A+B}}(X) = \overline{\mathrm{OM}_{A+B}}(\underline{\mathrm{PM}_{A+B}}(X))$.

(2) For any $x\in U$,

$$
\begin{aligned}
&\underline{\mathrm{OM}_{A+B}}\left(\overline{\mathrm{PM}_{A+B}}(X)\right)(x)\\
&\quad= \bigwedge_{u\in U}\theta\left(R_A(u,x),\overline{\mathrm{PM}_{A+B}}X(u)\right)\\
&\qquad\vee\bigwedge_{u\in U}\theta\left(R_B(u,x),\overline{\mathrm{PM}_{A+B}}X(u)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \bigwedge_{u\in U}\theta\Big(R_A(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v)) \\
&\qquad \vee \bigvee_{v\in U}T(R_B(v,u),X(v))\Big) \\
&\quad \vee \bigwedge_{u\in U}\theta\Big(R_B(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v)) \\
&\qquad \vee \bigvee_{v\in U}T(R_B(v,u),X(v))\Big) \\
&\geq \bigwedge_{u\in U}\theta\Big(R_A(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v))\Big) \\
&\quad \vee \bigwedge_{u\in U}\theta\Big(R_B(u,x), \bigvee_{v\in U}T(R_B(v,u),X(v))\Big) \\
&= \bigwedge_{u\in U}\bigvee_{v\in U}\theta(R_A(u,x),T(R_A(v,u),X(v))) \\
&\quad \vee \bigwedge_{u\in U}\bigvee_{v\in U}\theta(R_B(u,x),T(R_B(v,u),X(v))) \\
&\geq \bigvee_{u\in U}\bigwedge_{v\in U}\theta(R_A(u,x), \\
&\qquad T(T(R_A(v,x),R_A(x,u)),X(v))) \\
&\quad \vee \bigvee_{u\in U}\bigwedge_{v\in U}\theta(R_B(u,x), \\
&\qquad T(T(R_B(v,x),R_B(x,u)),X(v))) \\
&\geq \bigvee_{v\in U}T(R_A(v,x),X(v)) \vee \bigvee_{v\in U}T(R_B(v,x),X(v)) \\
&= \overline{\mathrm{PM}_{A+B}}(X)(x).
\end{aligned}
\tag{51}
$$

On the other hand, $\underline{\mathrm{OM}_{A+B}}(\overline{\mathrm{PM}_{A+B}}(X)) \subseteq \overline{\mathrm{PM}_{A+B}}(X)$. Therefore, $\underline{\mathrm{OM}_{A+B}}(\overline{\mathrm{PM}_{A+B}}(X)) = \overline{\mathrm{PM}_{A+B}}(X)$.

(3) For any $x \in U$,

$$
\begin{aligned}
&\overline{\mathrm{PM}_{A+B}}\left(\underline{\mathrm{PM}_{A+B}}(X)\right)(x) \\
&\quad = \bigvee_{u\in U}T\left(R_A(u,x), \underline{\mathrm{PM}_{A+B}}X(u)\right) \\
&\qquad \vee \bigvee_{u\in U}T\left(R_B(u,x), \underline{\mathrm{PM}_{A+B}}X(u)\right) \\
&\quad = \bigvee_{u\in U}T\Big(R_A(u,x), \bigwedge_{v\in U}\theta(R_A(v,u),X(v)) \\
&\qquad \wedge \bigwedge_{v\in U}\theta(R_B(v,u),X(v))\Big) \\
&\qquad \vee \bigvee_{u\in U}T\Big(R_B(u,x), \bigwedge_{v\in U}\theta(R_A(v,u),X(v)) \\
&\qquad \wedge \bigwedge_{v\in U}\theta(R_B(v,u),X(v))\Big) \\
&\quad \leq \bigvee_{u\in U}T\Big(R_A(u,x), \bigwedge_{v\in U}\theta(R_A(v,u),X(v))\Big) \\
&\qquad \vee \bigvee_{u\in U}T\Big(R_B(u,x), \bigwedge_{v\in U}\theta(R_B(v,u),X(v))\Big) \\
&\quad = \bigvee_{u\in U}\bigwedge_{v\in U}T(R_A(u,x),\theta(R_A(v,u),X(v))) \\
&\qquad \vee \bigvee_{u\in U}\bigwedge_{v\in U}T(R_B(u,x),\theta(R_B(v,u),X(v))) \\
&\quad \leq \bigvee_{u\in U}\bigwedge_{v\in U}T(R_A(u,x), \\
&\qquad \theta(T(R_A(v,x),R_A(x,u)),X(v))) \\
&\qquad \vee \bigvee_{u\in U}\bigwedge_{v\in U}T(R_B(u,x), \\
&\qquad \theta(T(R_B(v,x),R_B(x,u)),X(v))) \\
&\quad \leq \bigwedge_{v\in U}\theta(R_A(v,x),X(v)) \vee \bigwedge_{v\in U}\theta(R_B(v,x),X(v)) \\
&\quad = \underline{\mathrm{OM}_{A+B}}(X)(x).
\end{aligned}
\tag{52}
$$

(4) For any $x \in U$,

$$
\begin{aligned}
&\underline{\mathrm{PM}_{A+B}}\left(\overline{\mathrm{PM}_{A+B}}(X)\right)(x) \\
&\quad = \bigwedge_{u\in U}\theta\left(R_A(u,x), \overline{\mathrm{PM}_{A+B}}(X)(u)\right) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), \overline{\mathrm{PM}_{A+B}}(X)(u)\right) \\
&\quad = \bigwedge_{u\in U}\theta\Big(R_A(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v)) \\
&\qquad \vee \bigvee_{v\in U}T(R_B(v,u),X(v))\Big) \\
&\qquad \wedge \bigwedge_{u\in U}\theta\Big(R_B(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v)) \\
&\qquad \vee \bigvee_{v\in U}T(R_B(v,u),X(v))\Big) \\
&\quad \geq \bigwedge_{u\in U}\theta\Big(R_A(u,x), \bigvee_{v\in U}T(R_A(v,u),X(v))\Big)
\end{aligned}
$$

$$\wedge \bigwedge_{u\in U}\theta\left(R_B(u,x), \bigvee_{v\in U} T(R_B(v,u), X(v))\right)$$
$$= \bigwedge_{u\in U}\bigvee_{v\in U}\theta(R_A(u,x), T(R_A(v,u), X(v)))$$
$$\wedge \bigwedge_{u\in U}\bigvee_{v\in U}\theta(R_B(u,x), T(R_B(v,u), X(v)))$$
$$\geq \bigvee_{u\in U}\bigwedge_{v\in U}\theta(R_A(u,x), T(T(R_A(v,x), R_A(x,u)), X(v)))$$
$$\wedge \bigvee_{u\in U}\bigwedge_{v\in U}\theta(R_B(u,x), T(T(R_B(v,x), R_B(x,u)), X(v)))$$
$$\geq \bigvee_{v\in U} T(R_A(v,x), X(v)) \wedge \bigvee_{v\in U} T(R_B(v,x), X(v))$$
$$= \overline{\mathrm{OM}_{A+B}}(X)(x). \tag{53}$$

(5) According to item (5) in Proposition 7 and item (2) in Proposition 13, we can obtain

$$\theta\left(\overline{\mathrm{PM}_{A+B}}(X), \underline{\mathrm{OM}_{A+B}}(Y)\right) = \theta\left(\underline{\mathrm{OM}_{A+B}}\overline{\mathrm{PM}_{A+B}}(X), Y\right) = \theta\left(\overline{\mathrm{PM}_{A+B}}(X), Y\right). \tag{54}$$

(6) This property can be proved by item (5).

(7) According to item (8) in Proposition 5 and item (2) in Proposition 13, we can have

$$\underline{\mathrm{OM}_{A+B}}\Theta\left(\underline{a}, \overline{\mathrm{PM}_{A+B}}(X)\right) = \Theta\left(\underline{a}, \underline{\mathrm{OM}_{A+B}}\overline{\mathrm{PM}_{A+B}}(X)\right) = \Theta\left(\underline{a}, \overline{\mathrm{PM}_{A+B}}(X)\right). \tag{55}$$

According to the properties, we can get the relation as follows:

$$\underline{\mathrm{PM}}(X) \subseteq \overline{\mathrm{PM}}(\underline{\mathrm{PM}}(X)) \subseteq \underline{\mathrm{OM}}(X) \subseteq X \subseteq \overline{\mathrm{OM}}(X) \subseteq \underline{\mathrm{PM}}\left(\overline{\mathrm{PM}}(X)\right) \subseteq \overline{\mathrm{PM}}(X). \tag{56}$$

□

Proposition 14. *Let $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$ be T-fuzzy approximation space; let R_{A_i} $(i \in \{1,2,3,\ldots,n\})$ be the different T fuzzy similarity relations, for all $x, y \in U$, $a, b \in I$, and $X, Y \in F(U)$. Then, consider the following:*

(1) $\underline{OM_{\sum_{i=1}^n A_i}}(X) = \bigcup_{i=1}^n \underline{R_{A_i}}(X)$ $\overline{OM_{\sum_{i=1}^n A_i}}(X) = \bigcap_{i=1}^n \overline{R_{A_i}}(X)$;

(2) $\underline{PM_{\sum_{i=1}^n A_i}}(X) = \bigcap_{i=1}^n \underline{R_{A_i}}(X)$ $\overline{PM_{\sum_{i=1}^n A_i}}(X) = \bigcup_{i=1}^n \overline{R_{A_i}}(X)$;

(3) $\underline{PM^{\succcurlyeq}_{\sum_{i=1}^n A_i}}(X) \subseteq \underline{R^{\succcurlyeq}_{A_i}}(X) \subseteq \underline{OM^{\succcurlyeq}_{\sum_{i=1}^n A_i}}(X)$;

(4) $\overline{PM^{\succcurlyeq}_{\sum_{i=1}^n A_i}}(X) \supseteq \overline{R^{\succcurlyeq}_{A_i}}(X) \supseteq \overline{OM^{\succcurlyeq}_{\sum_{i=1}^n A_i}}(X)$;

Proof. This proposition can be easily proved by Definitions 1, 3, and 8. □

6. Case Study

Let us consider a fund investment issue. There are ten fund projects x_i $(i = 1, 2, \ldots, 10)$ that can be considered. They can be evaluated from the view of profit factors. Profit factors can be classified into five factors, which are market, technology, management, environment, and production. Table 1 is an evaluation information table about fund investment given by an expert, where $U = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $AT = \{$Market, Technology, Management, Environment, Production$\}$. For convenience, a_1, a_2, a_3, a_4, and a_5 will stand for market, technology, management, environment, and production, respectively.

Now, we can use the following similarity functions to calculate the similarity relation between the objects x_i, x_j as

$$R_{AT}(x_i, x_j) = \begin{cases} \sum_{k=1}^{m}\left(1 - 4\cdot\dfrac{\left|x_i^{a_k} - x_j^{a_k}\right|}{\left|\max a_k - \min a_k\right|}\right)(m)^{-1}, & \dfrac{\left|x_i^{a_k} - x_j^{a_k}\right|}{\left|\max a_k - \min a_k\right|} \leq 0.25; \\ 0, & \text{otherwise.} \end{cases} \tag{57}$$

Let $A_1 = \{$Market, Technology$\} = \{a_1, a_2\}$, $A_2 = \{$Management, Environment$\} = \{a_3, a_4\}$, and $A_3 = \{$Production$\} = \{a_5\}$. So we can get three different T-fuzzy similarity relations as follows:

$$R_{A_1}(x_i, x_j) = \begin{cases} \left(\left(1 - 4\cdot\dfrac{\left|x_i^{a_1} - x_j^{a_1}\right|}{\left|\max a_1 - \min a_1\right|}\right) + \left(1 - 4\cdot\dfrac{\left|x_i^{a_2} - x_j^{a_2}\right|}{\left|\max a_2 - \min a_2\right|}\right)\right) \times (2)^{-1}, & \dfrac{\left|x_i^{a_k} - x_j^{a_k}\right|}{\left|\max a_k - \min a_k\right|} \leq 0.25 \quad (k = 1, 2); \\ 0, & \text{otherwise,} \end{cases}$$

$$R_{A_2}(x_i, x_j) = \begin{cases} \left(\left(1 - 4 \cdot \dfrac{\left|x_i^{a_3} - x_j^{a_3}\right|}{\left|\max a_3 - \min a_3\right|}\right) + \left(1 - 4 \cdot \dfrac{\left|x_i^{a_4} - x_j^{a_4}\right|}{\left|\max a_4 - \min a_4\right|}\right)\right) \times (2)^{-1}, & \dfrac{\left|x_i^{a_k} - x_j^{a_k}\right|}{\left|\max a_k - \min a_k\right|} \le 0.25 \quad (k = 3, 4); \\ 0, & \text{otherwise}, \end{cases}$$

$$R_{A_3}(x_i, x_j) = \begin{cases} \dfrac{\left(1 - 4 \cdot \left|x_i^{a_5} - x_j^{a_5}\right|\right)}{\left|\max a_5 - \min a_5\right|}, & \dfrac{\left|x_i^{a_5} - x_j^{a_5}\right|}{\left|\max a_5 - \min a_5\right|} \le 0.25; \\ 0, & \text{otherwise}. \end{cases} \tag{58}$$

From Table 1, we can get the T-fuzzy similarity matrices as follows:

$$R_{A_1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0.674 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0.727 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.727 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.674 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.778 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.778 & 0 & 1 \end{bmatrix},$$

$$R_{A_2} = \begin{bmatrix} 1 & 0 & 0 & 0.133 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0.471 & 0 & 0 & 0 & 0 & 0.479 & 0 & 0 \\ 0 & 0.471 & 1 & 0.547 & 0.471 & 0 & 0 & 0 & 0 & 0 \\ 0.133 & 0 & 0.547 & 1 & 0.479 & 0 & 0 & 0.094 & 0 & 0 \\ 0 & 0 & 0.471 & 0.479 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.479 & 0 & 0.094 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.667 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.667 & 0 & 1 \end{bmatrix},$$

$$R_{A_3} = \begin{bmatrix} 1 & 0 & 0 & 0.077 & 0.282 & 0 & 0 & 0.897 & 0.179 & 0 \\ 0 & 1 & 0.385 & 0 & 0 & 0.179 & 0 & 0 & 0 & 0.795 \\ 0 & 0.385 & 1 & 0.487 & 0 & 0.795 & 0 & 0 & 0 & 0.590 \\ 0.077 & 0 & 0.487 & 1 & 0 & 0.692 & 0 & 0.179 & 0 & 0.077 \\ 0.282 & 0 & 0 & 0 & 1 & 0 & 0 & 0.179 & 0.897 & 0 \\ 0 & 0179 & 0.795 & 0.692 & 0 & 1 & 0 & 0 & 0 & 0.385 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0.897 & 0 & 0 & 0.179 & 0.179 & 0 & 0 & 1 & 0.077 & 0 \\ 0.179 & 0 & 0 & 0 & 0.897 & 0 & 0 & 0.077 & 1 & 0 \\ 0 & 0.795 & 0.590 & 0.077 & 0 & 0.385 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{59}$$

Taking $T(x, y) = \min(x, y)$, the residual implication of T is

$$\theta(x, y) = \begin{cases} 1, & x \le y; \\ y, & x > y. \end{cases} \tag{60}$$

Assume that the comprehensive evaluation of a customer for these fund projects is a fuzzy set $X = (0.5, 0.6, 0.3, 0.8, 0.5, 0.2, 0.4, 0.7, 0.2, 0, 3)$. Then, the T-fuzzy lower and upper approximations of X are

$$\underline{R}(A_1)(X) = (0.5, 0.6, 0.3, 0.3, 0.5, 0.2, 0.4, 0.3, 0.2, 0.3),$$

$$\overline{R}(A_1)(X) = (0.5, 0.6, 0.727, 0.8, 0.5, 0.2, 0.4, 0.7, 0.2, 0.7),$$

Table 1: An information system about fund investment.

U	Market	Technology	Management	Environment	Production
x_1	73	88	75	85	74
x_2	86	84	79	60	54
x_3	84	71	81	68	60
x_4	87	69	79	74	65
x_5	68	87	83	76	81
x_6	71	62	80	91	62
x_7	92	52	75	39	43
x_8	55	72	75	62	73
x_9	60	55	65	72	82
x_{10}	55	68	72	62	56

$$\begin{aligned}
&\underline{R}(A_2)(X) \\
&\quad = (0.5, 0.6, 0.3, 0.3, 0.3, 0.2, 0.4, 0.3, 0.2, 0.3), \\
&\overline{R}(A_2)(X) \\
&\quad = (0.5, 0.6, 0.547, 0.8, 0.5, 0.2, 0.4, 0.7, 0.2, 0.667), \\
&\underline{R}(A_2)(X) \\
&\quad = (0.5, 0.3, 0.2, 0.2, 0.2, 0.2, 0.4, 0.5, 0.2, 0.2), \\
&\overline{R}(A_2)(X) \\
&\quad = (0.7, 0.6, 0.487, 0.8, 0.5, 0.692, 0.4, 0.7, 0.5, 0.6).
\end{aligned} \tag{61}$$

Furthermore, we can get the optimistic and pessimistic multigranulation T-fuzzy lower and upper approximations of X, respectively,

$$\begin{aligned}
&\underline{\mathrm{OM}_{A_1+A_2+A_3}}(X) \\
&\quad = (0.5, 0.6, 0.3, 0.3, 0.5, 0.2, 0.4, 0.5, 0.2, 0.3), \\
&\overline{\mathrm{OM}_{A_1+A_2+A_3}}(X) \\
&\quad = (0.5, 0.6, 0.487, 0.8, 0.5, 0.2, 0.4, 0.7, 0.2, 0.6), \\
&\underline{\mathrm{PM}_{A_1+A_2+A_3}}(X) \\
&\quad = (0.5, 0.3, 0.2, 0.2, 0.2, 0.2, 0.4, 0.3, 0.2, 0.2), \\
&\overline{\mathrm{PM}_{A_1+A_2+A_3}}(X) \\
&\quad = (0.7, 0.6, 0.727, 0.8, 0.5, 0.692, 0.4, 0.7, 0.5, 0.7).
\end{aligned} \tag{62}$$

From the above three granulations A_1, A_2, and A_3, the projects must support optimistically the customer's comprehensive evaluation based on the degrees (0.5, 0.6, 0.3, 0.3, 0.5, 0.2, 0.4, 0.5, 0.2, 0.3) and may support optimistically the customer's comprehensive evaluation based on the degrees (0.5, 0.6, 0.487,0.8, 0.5, 0.2, 0.4, 0.7, 0.2, 0.6); the projects must support pessimistically the customer's comprehensive evaluation based on the degrees (0.5, 0.3,0.2, 0.2, 0.2, 0.2, 0.4, 0.3, 0.2, 0.2) and may support optimistically the customer's comprehensive evaluation based on the degrees (0.7, 0.6, 0.727, 0.8, 0.5, 0.692, 0.4, 0.7, 0.5, 0.7).

7. Conclusions

In this paper, we mainly presented the pessimistic multigranulation rough set model from the pessimistic multigranulation perspective by using T-fuzzy similarity relations in terms of triangular norms and studied the properties of optimistic and pessimistic multigranulation T-fuzzy lower and upper approximation operators. In the T-fuzzy approximation space $(U, R_{A_1}, R_{A_2}, \ldots, R_{A_n})$, the definitions of optimistic and pessimistic multigranulation T-fuzzy lower and upper approximation operators were recalled and proposed, respectively. It was obvious that the T-fuzzy lower and upper approximation operators which are defined on (U, R) were special cases of those of the two types of models. Furthermore, many interesting properties of the optimistic and the pessimistic multigranulation T-fuzzy rough sets models with respect to triangular norm have been explored. What is more, we researched the relationships among these approximation operators. The constructions of two new types of multigranulation rough set models over T-fuzzy similarity relations were meaningful in terms of the generalization of rough set theory. Finally, the models were illustrated by a case study about the evaluation of fund projects.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by Natural Science Foundation of China (no. 61105041) and National Natural Science Foundation of CQ CSTC (no. cstc 2013jcyjA40051).

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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 317387, 12 pages
http://dx.doi.org/10.1155/2014/317387

Research Article

Approximation Set of the Interval Set in Pawlak's Space

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Received 21 May 2014; Revised 19 July 2014; Accepted 19 July 2014; Published 11 August 2014

Academic Editor: Xibei Yang

The interval set is a special set, which describes uncertainty of an uncertain concept or set $\mathbb{Z}$ with its two crisp boundaries named upper-bound set and lower-bound set. In this paper, the concept of similarity degree between two interval sets is defined at first, and then the similarity degrees between an interval set and its two approximations (i.e., upper approximation set $\overline{R}(\mathbb{Z})$ and lower approximation set $\underline{R}(\mathbb{Z})$) are presented, respectively. The disadvantages of using upper-approximation set $\overline{R}(\mathbb{Z})$ or lower-approximation set $\underline{R}(\mathbb{Z})$ as approximation sets of the uncertain set (uncertain concept) $\mathbb{Z}$ are analyzed, and a new method for looking for a better approximation set of the interval set $\mathbb{Z}$ is proposed. The conclusion that the approximation set $R_{0.5}(\mathbb{Z})$ is an optimal approximation set of interval set $\mathbb{Z}$ is drawn and proved successfully. The change rules of $R_{0.5}(\mathbb{Z})$ with different binary relations are analyzed in detail. Finally, a kind of crisp approximation set of the interval set $\mathbb{Z}$ is constructed. We hope this research work will promote the development of both the interval set model and granular computing theory.

1. Introduction

Since the twenty-first century, researchers have done more and more research on uncertain problems [1]. It is an important research topic on how to effectively deal with uncertain data and how to acquire more knowledge and rules from the big data. At the same time, many methods for acquiring uncertain knowledge from uncertain information systems appeared gradually. In 1965, fuzzy sets theory was proposed by Zadeh [2]. In 1982, rough sets theory was proposed by Pawlak [3]. In 1990, quotient space theory was presented by L. Zhang and B. Zhang [4]. In 1993, interval sets and interval sets algebra were presented by Yao [5, 6].

Rough set theory is a mathematical tool to handle the uncertain information, which is imprecise, inconsistent, or incomplete. The basic thought of rough set is to obtain concepts and rules through classification of relational database and discover knowledge by the classification induced by equivalence relations; then approximation sets of the target concept are obtained with many equivalence classes. Rough set is a useful tool to handle uncertain problems, as well as fuzzy set theory, probability theory, and evidence theory. Because rough set theory has novel ideas and its calculation is easy and simple, it has been an important technology in intelligent information processing [7–9]. The key issue of rough set is building a knowledge space which is a partition of the domain U and is induced by an equivalence relation. In the knowledge space, two certain sets named upper approximation set and lower approximation set are used to describe the target concept X as its two boundaries. If knowledge granularity in knowledge space is coarser, then the border region of described target concept is wider and approximate accuracy is relatively lower. On the contrary, if knowledge granularity in knowledge space is finer, then the border region is narrower and approximate accuracy is relatively higher.

The interval set theory is an effective method for describing ambiguous information [10–12] and can be used in uncertain reasoning as well as the rough set [13–15]. The interval set not only can be used to describe the partially known concept, but also can be used to study the approximation set of the uncertain target concept. So, the interval set is a more general model for processing the uncertain information [16]. The interval set is described by two sets named upper bound

and lower bound [17]. The elements in lower bound certainly belong to target concept, and the elements in upper bound probably belong to target concept. When the boundary region has no element, the interval set degenerates into a usual set [5], while, in a certain knowledge granularity space, target concept may be uncertain. To solve this problem, in this paper, the approximate representation of interval set is discussed in detail in Pawlak's approximation space. And then, the upper approximation set of interval set and lower approximation set of interval set are defined, respectively. The change rules of the approximation set of interval set with the different knowledge granularity in Pawlak's approximation space are analyzed.

In this paper, an approximation set of the target concept $\mathbb{Z}$ is built in a certain knowledge space induced by many conditional attributes, and we find that this approximation set may have better similarity degree with the target concept $\mathbb{Z}$ than that of $\underline{R}(\mathbb{Z})$ or $\overline{R}(\mathbb{Z})$. Therefore, an interval set is translated into a fuzzy set at first in this paper. And then, according to the different membership degrees of different elements in boundary region, an approximation set of interval set $\mathbb{Z}$ is obtained by cut-set with some threshold. And then, the decision-making rules can be obtained through the approximation set instead of $\mathbb{Z}$ in current knowledge granularity space. In addition, the change rules of similarity between a target concept $\mathbb{Z}$ and its approximation sets are analyzed in detail.

The method used is getting the approximation of interval sets with a special approximation degree. With this method, we can use certain sets to describe an interval set in Pawlak's space. Our motivation is to get a mathematical theory model, which can be helpful to promote interval sets development in knowledge acquisition.

The rest of this paper is organized as follows. In Section 2, the related basic concepts and preliminary knowledge are reviewed. In Section 3, the concept of similarity degree between two interval sets is defined. The approximation set of interval set and 0.5-approximation set are proposed in Section 4. The change rules of similarity degree between the approximation sets and the target concept $\mathbb{Z}$ with the different knowledge granularity spaces are discussed in Section 5. This paper is concluded in Section 6.

2. Preliminaries

In order to introduce the approximation set of interval set more easily, many basic concepts will be reviewed at first.

Definition 1 (interval set [17]). An interval set is a new collection, and it is described by two sets named upper bound and lower bound. The interval set can be defined as follows. Let U be a finite set which is called universal set, and then let 2^U be the power set of U and let interval set $\mathbb{Z}$ be a subset of 2^U. In mathematical form, interval set $\mathbb{Z}$ is defined as $\mathbb{Z} = [Z_l, Z_u] = \{Z \in 2^U \mid Z_l \subseteq Z \subseteq Z_u\}$. If $Z_l = Z_u$, $\mathbb{Z}$ is a usual classical set.

In order to better explain the interval set, there is an example [17, 18] as follows. Let U be all papers submitted to a conference. After being reviewed, there are 3 kinds of results. The first kind of results is the set of papers certainly accepted and represented by Z_l. The second kind of results is the set of papers that need to be further reviewed and represented by $Z_u - Z_l$. The last kind of results is the set of papers rejected and represented by $U - Z_u$. Although every paper just can be rejected or accepted, no one knows the final result before further evaluation. Through reviewing, the set of papers accepted by the conference is described as $[Z_l, Z_u]$.

Definition 2 (indiscernibility relation [4, 19]). For any attribute set $R \subseteq A$, let us define one unclear binary relationship $\mathrm{IND}(R) = \{(x, y) \mid (x, y) \in U^2, \forall b \in R \rightarrow b(x) = b(y)\}$.

Definition 3 (information table of knowledge expression system [4, 20]). A knowledge expression system can be described as $S = \langle U, A, V, f\rangle$. U is the domain, and $A = C \cup D$ is the set of all attributes. Subset C is a set of conditional attributes, and D is a set of decision-making attributes. $V = \cup_{r\in A} V_r$ is the set of attribute values. V_r describes the range of attribute values r where $r \in A$. $f : U \times A \rightarrow V$ is an information function which describes attribute values of object x in U.

Definition 4 (upper approximation set and lower approximation set of rough set [3]). A knowledge-expression system is described as $S = \langle U, A, V, f\rangle$. For any $X \subseteq U$ and $R \subseteq A$, upper approximation set $\overline{R}(X)$ and lower approximation set $\underline{R}(X)$ of rough set X on R are defined as follows:

$$\begin{aligned} \overline{R}(X) &= \cup\left\{Y_i \mid Y_i \in \frac{U}{\mathrm{IND}(R)} \wedge Y_i \cap X \neq \phi\right\}, \\ \underline{R}(X) &= \cup\left\{Y_i \mid Y_i \in \frac{U}{\mathrm{IND}(R)} \wedge Y_i \subseteq X\right\}, \end{aligned} \tag{1}$$

where $U/\mathrm{IND}(R) = \{X \mid (X \subseteq U \wedge \forall_{x\in X, y\in X, b\in R}(b(x) = b(y)))\}$ is the classification of equivalence relation R on U. Upper approximation set and lower approximation set of rough set X on R can be defined in another form as follows:

$$\begin{aligned} \underline{R}(X) &= \{x \mid x \in U \wedge [x]_R \subseteq X\}, \\ \overline{R}(X) &= \{x \mid x \in U \wedge [x]_R \cap X \neq \phi\}, \end{aligned} \tag{2}$$

where $[x]_R \in U/\mathrm{IND}(R)$ and $[x]_R$ is an equivalence class of x on relation R. $\underline{R}(X)$ is a set of objects which certainly belong to U according to knowledge R; $\overline{R}(X)$ is a set of objects which possibly belong to U according to knowledge R. Let $\mathrm{BN}_R(X) = \overline{R}(X) - \underline{R}(X)$ be called *boundary region* of target concept X on relation R. Let $\mathrm{POS}_R(X) = \underline{R}(X)$ be called *positive region* of target concept X on relation R. Let $\mathrm{NEG}_R(X) = U - \overline{R}(X)$ be called *negative region* of target concept X on relation R. $\mathrm{BN}_R(X)$ is a set of objects which just possibly belong to target concept X.

Definition 5 (similarity degree between two sets [20]). Let A and B be two subsets of domain U, which means $A \subseteq U$, $B \subseteq U$. Defining a mapping $S : U \times U \rightarrow [0, 1]$, that is,

$(A, B) \rightarrow S(A, B)$, $S(A, B)$ is the similarity degree between A and B, if $S(A, B)$ satisfies the following conditions.

(1) For any $A, B \subseteq U$, $0 \leqslant S(A, B) \leqslant 1$ (boundedness).

(2) For any $A, B \subseteq U$, $S(A, B) = S(B, A)$ (symmetry).

(3) For any $A, B \subseteq U$, $S(A, A) = 1$; $S(A, B) = 0$ if and only if $A \cap B = \phi$.

Any formula satisfying (1), (2), and (3) is a similarity degree formula between two sets. Zhang et al. [20] gave out a similarity degree formula

$$S(A, B) = \frac{|A \cap B|}{|A \cup B|}, \tag{3}$$

where $|\cdot|$ represents the number of elements in finite subset. Obviously, this formula satisfies (1), (2), and (3).

Definition 6 (similarity degree between two interval sets). Let $\mathbb{Z} = [Z_l, Z_u] = \{Z \in 2^U \mid Z_l \subseteq Z \subseteq Z_u\}$ be an interval set and let $\mathbb{N} = [N_l, N_u] = \{N \in 2^U \mid N_l \subseteq N \subseteq N_u\}$ be also an interval set. Similarity degree between two interval sets can be defined as follows:

$$\mathbb{S}(\mathbb{Z}, \mathbb{N}) = \frac{|Z_l \cap N_l|}{2|Z_l \cup N_l|} + \frac{|Z_u \cap N_u|}{2|Z_u \cup N_u|}. \tag{4}$$

$\mathbb{S}(\mathbb{Z}, \mathbb{N})$ accords with Definition 5.

Definition 7 (upper approximation set and lower approximation set of an interval set). Let $\mathbb{Z} = [Z_l, Z_u] = \{Z \in 2^U \mid Z_l \subseteq Z \subseteq Z_u\}$ be an interval set. Let R be an equivalence relation on domain U. Upper approximation set of this interval set $\mathbb{Z}$ is defined as $\overline{R}(\mathbb{Z}) = [\overline{R}(Z_l), \overline{R}(Z_u)]$. Lower approximation set of this interval set $\mathbb{Z}$ is defined as $\underline{R}(\mathbb{Z}) = [\underline{R}(Z_l), \underline{R}(Z_u)]$.

Figures 1 and 2 are probably helpful to understand Definition 7. In Figure 1, the outer circle standing for a set Z_u and inner circle standing for a set Z_l represent an interval set $\mathbb{Z}$, and each block represents an equivalence class. The black region represents $\overline{R}(Z_l)$, and the whole colored region (black and gray region) represents $\overline{R}(Z_u)$. In Figure 2, the outer circle standing for a set Z_u and inner circle standing for a set Z_l represent an interval set $\mathbb{Z}$, and each block represents an equivalence class. The black region represents $\underline{R}(Z_l)$, and the whole colored region (black and gray region) represents $\underline{R}(Z_u)$.

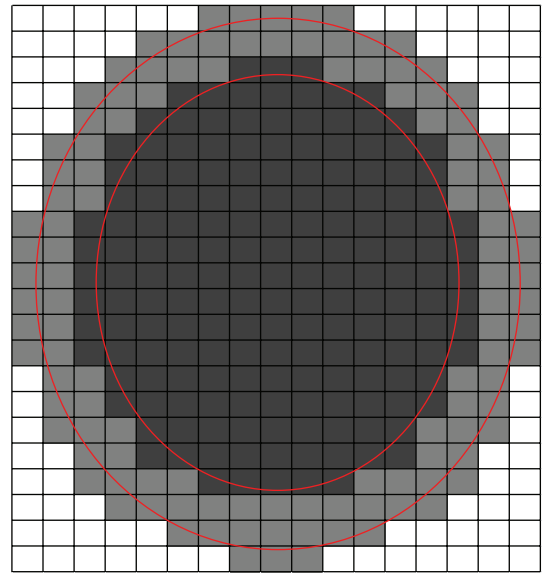

Figure 1: Upper approximation set of an interval set.

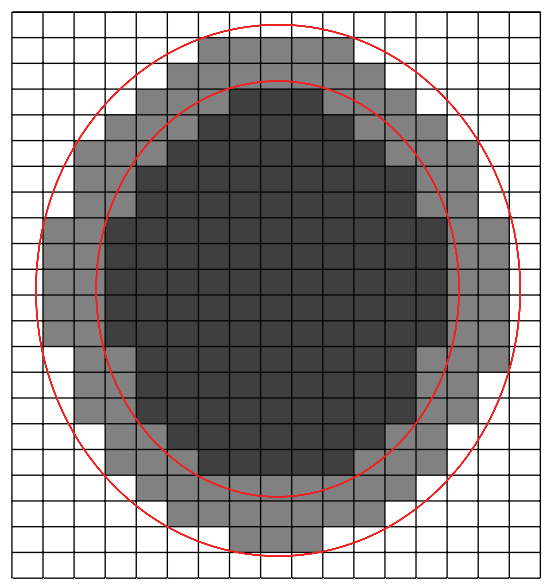

Figure 2: Lower approximation set of an interval set.

3. Approximation Set $R_\lambda(\mathbb{Z})$ of an Interval Set $\mathbb{Z}$

If $\overline{R}(\mathbb{Z})$ stands for the upper approximation set of the interval set $\mathbb{Z}$, then the similarity degree between $\mathbb{Z}$ and $\overline{R}(\mathbb{Z})$ can be defined as follows:

$$\begin{aligned}\mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z})) &= \frac{S(Z_l, \overline{R}(Z_l))}{2} + \frac{S(Z_u, \overline{R}(Z_u))}{2} \\ &= \frac{|Z_l \cap \overline{R}(Z_l)|}{2|Z_l \cup \overline{R}(Z_l)|} + \frac{|Z_u \cap \overline{R}(Z_u)|}{2|Z_u \cup \overline{R}(Z_u)|}.\end{aligned} \tag{5}$$

If $\underline{R}(\mathbb{Z})$ stands for the lower approximation set of the interval set $\mathbb{Z}$, then the similarity degree between $\mathbb{Z}$ and $\underline{R}(\mathbb{Z})$ is defined as follows:

$$\begin{aligned}\mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z})) &= \frac{S(Z_l, \underline{R}(Z_l))}{2} + \frac{S(Z_u, \underline{R}(Z_u))}{2} \\ &= \frac{|Z_l \cap \underline{R}(Z_l)|}{2|Z_l \cup \underline{R}(Z_l)|} + \frac{|Z_u \cap \underline{R}(Z_u)|}{2|Z_u \cup \underline{R}(Z_u)|}.\end{aligned} \tag{6}$$

If the knowledge space keeps unchanged, is there a better approximation set of the target concept Z? In this paper, the better approximation sets of target concept will be proposed.

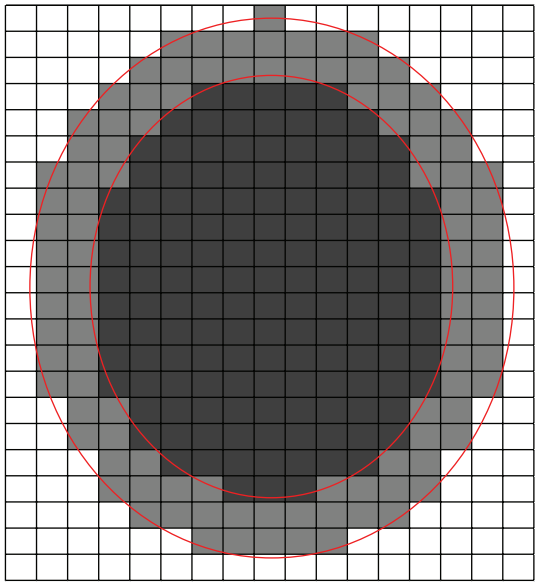

Figure 3: 0.5-approximation set of an interval set.

Let U be a nonempty set of objects. Let $Z \subseteq U$, $x \in Z$, and the membership degree of x belonging to set Z is defined as

$$\mu_Z^R(x) = \frac{|Z \cap [x]_R|}{|[x]_R|}. \tag{7}$$

Obviously, $0 \leqslant \mu_Z^R(x) \leqslant 1$.

Definition 8 (λ-approximation set of set Z [20]). Let U be a nonempty set of objects, and let knowledge space be $U/\mathrm{IND}(R)$. Let $Z \subseteq U$, $x \in Z$, and the membership degree belonging to set Z is

$$\mu_Z^R(x) = \frac{|Z \cap [x]_R|}{|[x]_R|}. \tag{8}$$

If $R_\lambda(Z) = \{x \in Z \mid \mu_Z^R(x) \geqslant \lambda,\ 1 \geqslant \lambda > 0\}$, then $R_\lambda(Z)$ is called λ-approximation set of set Z.

Definition 9 (λ-approximation set of set $\mathbb{Z}$). Let $\mathbb{Z} = [Z_l, Z_u] = \{Z \in 2^U \mid Z_l \subseteq Z \subseteq Z_u\}$ and $R_\lambda(\mathbb{Z}) = [R_\lambda(Z_l), R_\lambda(Z_u)]$; then $R_\lambda(\mathbb{Z})$ is called λ-approximation set of the interval set $\mathbb{Z}$.

Figure 3 is probably helpful to understand Definition 9. In Figure 3, the outer circle standing for a set Z_u and inner circle standing for a set Z_l represent an interval set $\mathbb{Z}$, and each block represents an equivalence class. The black region represents $R_{0.5}(Z_l)$, and the whole colored region (black and gray region) represents $R_{0.5}(Z_u)$.

4. Approximation Set $R_{0.5}(\mathbb{Z})$ of an Interval Set $\mathbb{Z}$

Lemma 10 (see [20]). *Let a, b, c, and d be all real numbers. If $0 < a < b$, $0 < c < d$, then $a/b < (a+d)/(b+c)$.*

Lemma 11 (see [20]). *Let a, b, c, and d be all real numbers. In the numbers, $0 < a < b$, $0 < c < d$. If $a/b \geqslant c/d$, then $a/b \leqslant (a-c)/(b-d)$. If $a/b \leqslant c/d$, then $a/b \geqslant (a-c)/(b-d)$.*

In order to better understand the similarity degree between $R_{0.5}(\mathbb{Z})$ and $\mathbb{Z}$, Theorems 12 and 13 are presented as follows.

Theorem 12. *Let U be a finite domain, let $\mathbb{Z}$ be an interval set on U, and let R be an equivalence relation on U. Then, $S(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant S(\mathbb{Z}, \underline{R}(\mathbb{Z}))$.*

For example, let $U/R = \{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$, $Z_u = \{x_1, x_2, x_3, x_4, x_5\}$, $Z_l = \{x_2, x_3, x_4\}$. Then, $\underline{R}(Z_u) = \{x_1, x_2, x_3, x_4\}$, $\overline{R}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $R_{0.5}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\underline{R}(Z_l) = \{x_3, x_4\}$, $\overline{R}(Z_l) = \{x_1, x_2, x_3, x_4\}$, $R_{0.5}(Z_l) = \{x_1, x_2, x_3, x_4\}$.

And then we can have $\mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z})) = (2/(2\times 3)) + (4/(2\times 5)) = 11/15$, $\mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z})) = (5/(2\times 6)) + (3/(2\times 4)) = 19/24$, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = (1/2) + (3/(2\times 4)) = 7/8$, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z}))$.

Proof. According to Definition 6,

$$\begin{aligned}
\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) &= \frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} + \frac{|Z_u \cap R_{0.5}(Z_u)|}{2|Z_u \cup R_{0.5}(Z_u)|}, \\
\mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z})) &= \frac{|Z_l \cap \underline{R}(Z_l)|}{2|Z_l \cup \underline{R}(Z_l)|} + \frac{|Z_u \cap \underline{R}(Z_u)|}{2|Z_u \cup \underline{R}(Z_u)|}.
\end{aligned} \tag{9}$$

(1) There we first prove

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap \underline{R}(Z_l)|}{2|Z_l \cup \underline{R}(Z_l)|}. \tag{10}$$

For all $x \in R_{0.5}(Z_l)$, we have $\mu_{Z_l}^R(x) \geqslant 0.5$. That is,

$$\mu_{Z_l}^R(x) = \frac{|[x]_R \cap Z_l|}{|[x]_R|} \geqslant 0.5. \tag{11}$$

Because R is an equivalence relation on U, the classifications induced by R can be denoted as $[x_1]_R, [x_2]_R, \ldots, [x_n]_R$. Then, $R_{0.5}(Z_l) = \{x \mid \mu_{Z_l}^R(x) \geqslant 0.5\} = \{x \mid \mu_{Z_l}^R(x) = 1\} \cup \{x \mid 0.5 \leqslant \mu_X^R(x) < 1\}$. Obviously, $\{x \mid \mu_{Z_l}^R(x) = 1\} = \underline{R}(Z_l)$, and then let $\{x \mid 0.5 \leqslant \mu_{Z_l}^R(x) < 1\} = [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_k}]_R$. So, $Z_l \cap R_{0.5}(Z_l) = Z_l \cap (\underline{R}(Z_l) \cup [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_k}]_R)$. Because the intersection sets between any two elements in $\underline{R}(Z_l), [x_{i_1}]_R, [x_{i_2}]_R, \ldots, [x_{i_k}]_R$ are empty sets, we can get that

$$\begin{aligned}
|Z_l \cap R_{0.5}(Z_l)| &= |Z_l \cap \underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| \\
&\quad + |Z_l \cap [x_{i_2}]_R| + \cdots + |Z_l \cap [x_{i_k}]_R| \\
&= |\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| \\
&\quad + \cdots + |Z_l \cap [x_{i_k}]_R|.
\end{aligned} \tag{12}$$

Because $Z_l \cup R_{0.5}(Z_l) = Z_l \cup ([x_{i_1}]_R - Z_l) \cup ([x_{i_2}]_R - Z_l) \cup \cdots \cup ([x_{i_k}]_R - Z_l)$ and the intersection set between any two elements in $Z_l, ([x_{i_1}]_R - Z_l), ([x_{i_2}]_R - Z_l), \ldots, ([x_{i_k}]_R - Z_l)$ is empty, we have that $|Z_l \cup R_{0.5}(Z_l)| = |Z_l| + |([x_{i_1}]_R - Z_l)| + |([x_{i_2}]_R - Z_l)| + \cdots + |([x_{i_k}]_R - Z_l)|$. So,

$$\begin{aligned}
&\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \\
&\quad = \left(|\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| \right. \\
&\qquad \left. + \cdots + |Z_l \cap [x_{i_k}]_R|\right) \\
&\qquad \times \left(|Z_l| + |([x_{i_1}]_R - Z_l)| + |([x_{i_2}]_R - Z_l)| \right. \\
&\qquad \left. + \cdots + |([x_{i_k}]_R - Z_l)|\right)^{-1}.
\end{aligned} \tag{13}$$

Because

$$\begin{aligned}
\mu_{Z_l}^R(x_{i_1}) &= \frac{|[x_{i_1}]_R \cap Z_l|}{|[x_{i_1}]_R|} \\
&= \frac{|[x_{i_1}]_R \cap Z_l|}{|[x_{i_1}]_R \cap Z_l| + |[x_{i_1}]_R - Z_l|} \geqslant 0.5,
\end{aligned} \tag{14}$$

we have $|[x_{i_1}]_R \cap Z_l| \geqslant |[x_{i_1}]_R - Z_l|$. In the same way, according to $|[x_{i_2}]_R \cap Z_l| \geqslant |[x_{i_2}]_R - Z_l|, \ldots, |[x_{i_k}]_R \cap Z_l| \geqslant |[x_{i_k}]_R - Z_l|$ and Lemma 10, we can easily get

$$\begin{aligned}
&\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \\
&\quad = \left(|\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| \right. \\
&\qquad \left. + \cdots + |Z_l \cap [x_{i_k}]_R|\right) \\
&\qquad \times \left(|Z_l| + |([x_{i_1}]_R - Z_l)| + |([x_{i_2}]_R - Z_l)| \right. \\
&\qquad \left. + \cdots + |([x_{i_k}]_R - Z_l)|\right)^{-1} \\
&\quad \geqslant \frac{|\underline{R}(Z_l)|}{|Z_l|}.
\end{aligned} \tag{15}$$

Therefore,

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap \underline{R}(Z_l)|}{2|Z_l \cup \underline{R}(Z_l)|}. \tag{16}$$

(2) In a similar way with (1), we can have the inequality

$$\frac{|Z_u \cap R_{0.5}(Z_u)|}{2|Z_u \cup R_{0.5}(Z_u)|} \geqslant \frac{|Z_u \cap \underline{R}(Z_u)|}{2|Z_u \cup \underline{R}(Z_u)|}. \tag{17}$$

From (1) and (2), we have $S(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant S(\mathbb{Z}, \underline{R}(\mathbb{Z}))$. So, Theorem 12 has been proved completely. □

Theorem 12 shows that the similarity degree between an interval set $\mathbb{Z}$ and its approximation set $R_{0.5}(\mathbb{Z})$ is better than the similarity degree between $\mathbb{Z}$ and its lower approximation set $\underline{R}(\mathbb{Z})$.

Theorem 13. *Let U be a finite domain, let $\mathbb{Z}$ be an interval set on U, and let R be an equivalence relation on U. If*

$$\begin{aligned}
\frac{|Z_l|}{|\overline{R}(Z_l)|} &> \frac{|Z_l - R_{0.5}(Z_l)|}{|\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l|}, \\
\frac{|Z_u|}{|\overline{R}(Z_u)|} &> \frac{|Z_u - R_{0.5}(Z_u)|}{|\overline{R}(Z_u) - R_{0.5}(Z_u) - Z_u|},
\end{aligned} \tag{18}$$

then $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z}))$.

For example, let $U/R = \{\{x_1\}, \{x_2, x_3, x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}\}\}$, $Z_u = \{x_1, x_2, x_7\}$, $Z_l = \{x_1, x_2\}$. Then, $\overline{R}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $\underline{R}(Z_u) = \{x_1\}$, $R_{0.5}(Z_u) = \{x_1\}$, $\underline{R}(Z_l) = \{x_1\}$, $\overline{R}(Z_l) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $R_{0.5}(Z_l) = \{x_1\}$,

$$\begin{aligned}
\frac{|Z_l|}{|\overline{R}(Z_l)|} &= \frac{2}{6} = \frac{1}{3} > \frac{|Z_l - R_{0.5}(Z_l)|}{|\overline{R}(Z_l) - Z_l - R_{0.5}(Z_l)|} = \frac{1}{4}, \\
\frac{|Z_u|}{|\overline{R}(Z_u)|} &= \frac{3}{10} > \frac{|Z_u - R_{0.5}(Z_u)|}{|\overline{R}(Z_u) - Z_u - R_{0.5}(Z_u)|} = \frac{2}{7}.
\end{aligned} \tag{19}$$

And then we can have $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = (1/(2\times 2)) + (1/(2\times 3)) = 5/12$, $\mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z})) = (1/(2\times 6)) + (3/(2\times 10)) = 7/30$, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z}))$.

Proof. According to Definition 6,

$$\begin{aligned}
\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) &= \frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} + \frac{|Z_u \cap R_{0.5}(Z_u)|}{2|Z_u \cup R_{0.5}(Z_u)|}, \\
\mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z})) &= \frac{|Z_l \cap \overline{R}(Z_l)|}{2|Z_l \cup \overline{R}(Z_l)|} + \frac{|Z_u \cap \overline{R}(Z_u)|}{2|Z_u \cup \overline{R}(Z_u)|}.
\end{aligned} \tag{20}$$

(1) Let $\overline{R}(Z_l) - R_{0.5}(Z_l) = [x_{j_1}]_R \cup [x_{j_2}]_R \cup \cdots \cup [x_{j_s}]_R$, and the intersection sets between any two elements in $[x_{j_1}]_R, [x_{j_2}]_R, \ldots, [x_{j_s}]_R$ are empty sets. Because

$$0 < \mu_{Z_l}^R(x_{j_1}) = \frac{|[x_{j_1}]_R \cap Z_l|}{|[x_{j_1}]_R|} < 0.5, \tag{21}$$

it is obvious that $[x_{j_1}]_R \cap Z_l \neq \phi$. In the same way, we have $[x_{j_2}]_R \cap Z_l \neq \phi, \ldots, [x_{j_s}]_R \cap Z_l \neq \phi$. Then we have $Z_l \cap R_{0.5}(Z_l) = Z_l - ([x_{j_1}]_R \cap Z_l) - ([x_{j_2}]_R \cap Z_l) - \cdots - ([x_{j_s}]_R \cap Z_l) = Z_l - (Z_l - R_{0.5}(Z_l))$. Because the intersection sets between any two elements in $[x_{j_1}]_R \cap Z_l, [x_{j_2}]_R \cap Z_l, \ldots, [x_{j_s}]_R \cap Z_l$ are empty sets, we have $Z_l \cap R_{0.5}(Z_l) = Z_l - (Z_l - R_{0.5}(Z_l))$, $|Z_l \cap R_{0.5}(Z_l)| = |Z_l| - |Z_l - R_{0.5}(Z_l)|$, and $Z_l \cup R_{0.5}(Z_l) = \overline{R}(Z_l) - (([x_{j_1}]_R - Z_l) \cup ([x_{j_2}]_R - Z_l) \cup \cdots \cup ([x_{j_s}]_R - Z_l))$. Because the intersection sets between any two elements in $([x_{j_1}]_R - Z_l), ([x_{j_2}]_R - Z_l), \ldots, ([x_{j_s}]_R - Z_l)$ are empty sets, $Z_l \cup R_{0.5}(Z_l) = \overline{R}(Z_l) - (\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l)$

and $|Z_l \cup R_{0.5}(Z_l)| = |\overline{R}(Z_l)| - |(\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l)|$ are held. So,

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} = \frac{|Z_l| - |Z_l - R_{0.5}(Z_l)|}{|\overline{R}(Z_l)| - |(\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l)|}. \tag{22}$$

For

$$\frac{|Z_l|}{|\overline{R}(Z_l)|} > \frac{|Z_l - R_{0.5}(Z_l)|}{|\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l|}, \tag{23}$$

according to Lemma 11, we have

$$\frac{|Z_l|}{|\overline{R}(Z_l)|} \leqslant \frac{|Z_l| - |Z_l - R_{0.5}(Z_l)|}{|\overline{R}(Z_l)| - |(\overline{R}(Z_l) - R_{0.5}(Z_l) - Z_l)|}; \tag{24}$$

that is to say,

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap \overline{R}(Z_l)|}{|Z_l \cup \overline{R}(Z_l)|}. \tag{25}$$

Therefore, we have

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap \overline{R}(Z_l)|}{2|Z_l \cup \overline{R}(Z_l)|}. \tag{26}$$

(2) In a similar way with (1), we can easily obtain the conclusion that

$$\frac{|Z_u \cap R_{0.5}(Z_u)|}{2|Z_u \cup R_{0.5}(Z_u)|} \geqslant \frac{|Z_u \cap \overline{R}(Z_u)|}{2|Z_u \cup \overline{R}(Z_u)|} \tag{27}$$

when

$$\frac{|Z_u|}{|\overline{R}(Z_u)|} > \frac{|Z_u - R_{0.5}(Z_u)|}{|\overline{R}(Z_u) - R_{0.5}(Z_u) - Z_u|}. \tag{28}$$

According to (1) and (2), the inequality $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, \overline{R}(\mathbb{Z}))$ is held. So, Theorem 13 has been proved successfully. □

Theorem 13 shows that, under some conditions, the similarity degree between an interval set $\mathbb{Z}$ and its approximation set $R_{0.5}(\mathbb{Z})$ is better than the similarity degree between $\mathbb{Z}$ and its lower approximation set $\overline{R}(\mathbb{Z})$.

Theorem 14. *Let U be a finite domain, $\mathbb{Z}$ an interval set on U, and R an equivalence relation on U. If $1 \geqslant \lambda > 0.5$, then $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, R_\lambda(\mathbb{Z})) \geqslant S(\mathbb{Z}, \underline{R}(\mathbb{Z}))$.*

For example, let $U/R = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}\}$, $Z_u = \{x_1, x_2, x_3, x_4, x_5\}$, $Z_l = \{x_1, x_2, x_3\}$. Then, $\overline{R}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $R_{0.5}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\underline{R}(Z_u) = \{x_1, x_2, x_3\}$, $R_{0.75}(Z_u) = \{x_1, x_2, x_3\}$, $\underline{R}(Z_l) = \{x_1, x_2, x_3\}$, $\overline{R}(Z_l) = \{x_1, x_2, x_3\}$, and $R_{0.5}(Z_l) = \{x_1, x_2, x_3\}$, $R_{0.75}(Z_l) = \{x_1, x_2, x_3\}$.

And then we can have $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = (3/(2 \times 3)) + (5/(2 \times 6)) = 11/12$, $\mathbb{S}(\mathbb{Z}, R_{0.75}(\mathbb{Z})) = (3/(2 \times 3)) + (3/(2 \times 5)) = 4/5$, $\mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z})) = (3/(2 \times 3)) + (3/(2 \times 6)) = 3/4$, $\mathbb{S}(\mathbb{Z}, \underline{R}(\mathbb{Z})) < \mathbb{S}(\mathbb{Z}, R_{0.75}(\mathbb{Z})) < \mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z}))$. This example is in accordance with the theorem.

Proof. (1) For all $x \in R_{0.5}(Z_l)$, then $\mu^R_{Z_l}(x) \geqslant 0.5$, which means

$$\mu^R_{Z_l}(x) = \frac{|[x]_R \cap Z_l|}{|[x]_R|} \geqslant 0.5. \tag{29}$$

Because $R_{0.5}(Z_l) = \{x \mid \mu^R_{Z_l}(x) \geqslant 0.5\} = \{x \mid \mu^R_{Z_l}(x) = 1\} \cup \{x \mid 0.5 \leqslant \mu^R_{Z_l}(x) < 1\}$, we can easily get $\{x \mid \mu^R_{Z_l}(x) = 1\} = \underline{R}(Z_l)$. Let $\{x \mid 0.5 \leqslant \mu^R_{Z_l}(x) < 1\} = [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_k}]_R$ and $R_\lambda(Z_l) = \{x \mid \mu^R_{Z_l}(x) \geqslant \lambda > 0.5\} = \{x \mid \mu^R_{Z_l}(x) = 1\} \cup \{x \mid 0.5 < \lambda \leqslant \mu^R_{Z_l}(x) < 1\}$, and then we can get $R_\lambda(Z_l) \subseteq R_{0.5}(Z_l)$. To simplify the proof, let $\{x \mid 0.5 < \lambda \leqslant \mu^R_{Z_l}(x) < 1\} = [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_q}]_R$ and $q \leqslant k$ in this paper. So, $Z_l \cap R_\lambda(Z_l) = Z_l \cap (\underline{R}(Z_l) \cup [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_q}]_R)$. Because the intersection sets between any two elements in $\underline{R}(Z_l), [x_{i_1}]_R, [x_{i_2}]_R, \ldots, [x_{i_k}]_R$ are empty sets, we can easily get that $|Z_l \cap R_{0.5}(Z_l)| = |Z_l \cap \underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| + \cdots + |Z_l \cap [x_{i_k}]_R| = |\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| + \cdots + |Z_l \cap [x_{i_k}]_R|$. And

$$\begin{aligned}
&\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \\
&\quad = \Big(|\underline{R}(Z_l)| + \left|Z_l \cap \left[x_{i_1}\right]_R\right| + \cdots + \left|Z_l \cap \left[x_{i_q}\right]_R\right| \\
&\qquad + \left|Z_l \cap \left[x_{i_{q+1}}\right]_R\right| + \cdots + \left|Z_l \cap \left[x_{i_k}\right]_R\right|\Big) \\
&\qquad \times \Big(|Z_l| + \left|\left(\left[x_{i_1}\right]_R - Z_l\right)\right| + \cdots + \left|\left(\left[x_{i_q}\right]_R - Z_l\right)\right| \\
&\qquad + \left|\left(\left[x_{i_{q+1}}\right]_R - Z_l\right)\right| + \cdots + \left|\left(\left[x_{i_k}\right]_R - Z_l\right)\right|\Big)^{-1},
\end{aligned} \tag{30}$$

and we have

$$\begin{aligned}
&\frac{|Z_l \cap R_\lambda(Z_l)|}{|Z_l \cup R_\lambda(Z_l)|} \\
&\quad = \Big(|\underline{R}(Z_l)| + \left|Z_l \cap \left[x_{i_1}\right]_R\right| + \left|Z_l \cap \left[x_{i_2}\right]_R\right| \\
&\qquad + \cdots + \left|Z_l \cap \left[x_{i_q}\right]_R\right|\Big) \\
&\qquad \times \Big(|Z_l| + \left|\left(\left[x_{i_1}\right]_R - Z_l\right)\right| + \left|\left(\left[x_{i_2}\right]_R - Z_l\right)\right| \\
&\qquad + \cdots + \left|\left(\left[x_{i_q}\right]_R - Z_l\right)\right|\Big)^{-1}.
\end{aligned} \tag{31}$$

And because $|Z_l \cap [x_{i_{q+1}}]_R| + \cdots + |Z_l \cap [x_{i_k}]_R| \geqslant |([x_{i_{q+1}}]_R - Z_l)| + \cdots + |([x_{i_k}]_R - Z_l)|$, according to Lemma 10, the inequality

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap R_\lambda(Z_l)|}{|Z_l \cup R_\lambda(Z_l)|} \geqslant \frac{|Z_l \cap \underline{R}(Z_l)|}{|Z_l \cup \underline{R}(Z_l)|} \tag{32}$$

is held. Therefore, we have

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{2|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|Z_l \cap R_\lambda(Z_l)|}{2|Z_l \cup R_\lambda(Z_l)|} \geqslant \frac{|Z_l \cap \underline{R}(Z_l)|}{2|Z_l \cup \underline{R}(Z_l)|}. \tag{33}$$

(2) The inequality

$$\frac{|Z_u \cap R_{0.5}(Z_u)|}{2|Z_u \cup R_{0.5}(Z_u)|} \geqslant \frac{|Z_u \cap R_\lambda(Z_u)|}{2|Z_u \cup R_\lambda(Z_u)|} \geqslant \frac{|Z_u \cap \underline{R}(Z_u)|}{2|Z_u \cup \underline{R}(Z_u)|} \tag{34}$$

can be easily proved in a similar way to (1).

According to (1) and (2), the inequality $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, R_\lambda(\mathbb{Z})) \geqslant S(\mathbb{Z}, \underline{R}(\mathbb{Z}))$ is held. □

Based on Theorems 13 and 14, Corollary 15 can be obtained easily as follows.

Corollary 15. *Let U be a finite domain, $\mathbb{Z}$ an interval set on U, and R an equivalence relation on U. $\mathbb{Z} \subseteq 2^U$. If*

$$\begin{aligned} \frac{|Z_l|}{|\overline{R}(Z_l)|} &> \frac{|Z_l - R_\lambda(Z_l)|}{|\overline{R}(Z_l) - R_\lambda(Z_l) - Z_l|}, \\ \frac{|Z_u|}{|\overline{R}(Z_u)|} &> \frac{|Z_u - R_\lambda(Z_u)|}{|\overline{R}(Z_u) - R_\lambda(Z_u) - Z_u|}, \end{aligned} \tag{35}$$

then $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \geqslant S(\mathbb{Z}, R_\lambda(\mathbb{Z})) \geqslant S(\mathbb{Z}, \overline{R}(\mathbb{Z}))$.

Theorem 16. *Let U be a finite domain, $\mathbb{Z}$ an interval set on U, and R an equivalence relation on U. if $0.5 \leqslant \lambda_1 < \lambda_2 \leqslant 1$, then $\mathbb{S}(\mathbb{Z}, R_{\lambda_1}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, R_{\lambda_2}(\mathbb{Z}))$.*

For example, $U/R = \{\{x_1\}, \{x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9, x_{10}\}\}$, $Z_u = \{x_1, x_2, x_3, x_5, x_6, x_7\}$, $Z_l = \{x_1, x_2, x_3, x_5, x_6\}$. Then, $\overline{R}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$, $\underline{R}(Z_u) = \{x_1, x_2, x_3\}$, $R_{0.6}(Z_u) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $R_{0.8}(Z_u) = \{x_1, x_2, x_3\}$, $\overline{R}(Z_l) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $\underline{R}(Z_l) = \{x_1, x_2, x_3\}$, $R_{0.6}(Z_l) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, $R_{0.8}(Z_l) = \{x_1, x_2, x_3\}$.

And then we can have $\mathbb{S}(\mathbb{Z}, R_{0.6}(\mathbb{Z})) = (5/(2\times6)) + (5/(2\times7)) = 65/84$, $\mathbb{S}(\mathbb{Z}, R_{0.8}(\mathbb{Z})) = (3/(2\times5)) + (3/(2\times7)) = 18/35$, $\mathbb{S}(\mathbb{Z}, R_{0.6}(\mathbb{Z})) > \mathbb{S}(\mathbb{Z}, R_{0.8}(\mathbb{Z}))$. This example is in accordance with the theorem.

Proof. (1) For any $x \in R_{\lambda_1}(Z_l)$, we have

$$\mu^R_{Z_l}(x) = \frac{|[x]_R \cap Z_l|}{|[x]_R|} \geqslant \lambda_1 \geqslant 0.5. \tag{36}$$

Because R is an equivalence relation on U, all the classifications induced by R can be denoted by $[x_1]_R, [x_2]_R, \ldots, [x_n]_R$. We have $R_{\lambda_1}(Z_l) = \{x \mid \mu^R_{Z_l}(x) \geqslant \lambda_1\} = \{x \mid \mu^R_{Z_l}(x) = 1\} \cup \{x \mid \lambda_1 \leqslant \mu^R_{Z_l}(x) < 1\}$ and $\{x \mid \mu^R_{Z_l}(x) = 1\} = \underline{R}(Z_l)$ as well as $\{x \mid \lambda_1 \leqslant \mu^R_{Z_l}(x) < 1\} = [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_u}]_R$. We can also get $R_{\lambda_2}(Z_l) = \{x \mid \mu^R_{Z_l}(x) \geqslant \lambda_2\} = \{x \mid \mu^R_{Z_l}(x) = 1\} \cup \{x \mid \lambda_2 \leqslant \mu^R_{Z_l}(x) < 1\}$. Let $\{x \mid \lambda_2 \leqslant \mu^R_{Z_l}(x) < 1\} = [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_v}]_R$, where $0.5 \leqslant \lambda_1 < \lambda_2 \leqslant 1$ and $v \leqslant u$. So, $Z_l \cap R_{\lambda_1}(Z_l) = Z_l \cap (\underline{R}(Z_l) \cup [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_v}]_R \cup \cdots \cup [x_{i_u}]_R)$, $Z_l \cap R_{\lambda_2}(Z_l) = Z_l \cap (\underline{R}(Z_l) \cup [x_{i_1}]_R \cup [x_{i_2}]_R \cup \cdots \cup [x_{i_v}]_R)$.

And because the intersection sets between any two elements in $\underline{R}(Z_l), [x_{i_1}]_R, [x_{i_2}]_R, \ldots, [x_{i_u}]_R$ are empty sets, we have

$$\begin{aligned} |Z_l \cap R_{\lambda_1}(Z_l)| &= |Z_l \cap \underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| \\ &\quad + |Z_l \cap [x_{i_2}]_R| + \cdots + |Z_l \cap [x_{i_v}]_R| \\ &\quad + \cdots + |Z_l \cap [x_{i_u}]_R| \\ &= |\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + |Z_l \cap [x_{i_2}]_R| \\ &\quad + \cdots + |Z_l \cap [x_{i_v}]_R| + \cdots + |Z_l \cap [x_{i_u}]_R|, \\ |Z_l \cap R_{\lambda_2}(Z_l)| &= |\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| \\ &\quad + |Z_l \cap [x_{i_2}]_R| + \cdots + |Z_l \cap [x_{i_v}]_R|. \end{aligned} \tag{37}$$

According to $Z_l \cup R_{\lambda_1}(Z_l) = Z_l \cup ([x_{i_1}]_R - Z_l) \cup ([x_{i_2}]_R - Z_l) \cup \cdots \cup ([x_{i_v}]_R - Z_l) \cup \cdots \cup ([x_{i_u}]_R - Z_l)$ and because the intersection sets between any two elements in $Z_l, ([x_{i_1}]_R - Z_l), ([x_{i_2}]_R - Z_l), \ldots, ([x_{i_v}]_R - Z_l), \ldots, ([x_{i_u}]_R - Z_l)$ are empty sets, we easily have

$$\begin{aligned} |Z_l \cup R_{\lambda_1}(Z_l)| &= |Z_l| + |([x_{i_1}]_R - Z_l)| \\ &\quad + |([x_{i_2}]_R - Z_l)| + \cdots + |([x_{i_v}]_R - Z_l)| \\ &\quad + \cdots + |([x_{i_u}]_R - Z_l)|, \\ |Z_l \cup R_{\lambda_2}(Z_l)| &= |Z_l| + |([x_{i_1}]_R - Z_l)| + |([x_{i_2}]_R - Z_l)| \\ &\quad + \cdots + |([x_{i_v}]_R - Z_l)|. \end{aligned} \tag{38}$$

Therefore,

$$\begin{aligned} &\frac{|Z_l \cap R_{\lambda_1}(Z_l)|}{|Z_l \cup R_{\lambda_1}(Z_l)|} \\ &\quad = \Big(|\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + \cdots + |Z_l \cap [x_{i_v}]_R| \\ &\qquad + |Z_l \cap [x_{i_{v+1}}]_R| + \cdots + |Z_l \cap [x_{i_u}]_R|\Big) \\ &\qquad \times \Big(|Z_l| + |([x_{i_1}]_R - Z_l)| + \cdots + |([x_{i_v}]_R - Z_l)| \\ &\qquad + |([x_{i_{v+1}}]_R - Z_l)| + \cdots + |([x_{i_u}]_R - Z_l)|\Big)^{-1}, \\ &\frac{|Z_l \cap R_{\lambda_2}(Z_l)|}{|Z_l \cup R_{\lambda_2}(Z_l)|} \\ &\quad = \frac{|\underline{R}(Z_l)| + |Z_l \cap [x_{i_1}]_R| + \cdots + |Z_l \cap [x_{i_v}]_R|}{|Z_l| + |([x_{i_1}]_R - Z_l)| + \cdots + |([x_{i_v}]_R - Z_l)|}. \end{aligned} \tag{39}$$

According to

$$\mu_{Z_l}^R\left(x_{i_{v+1}}\right) = \frac{\left|\left[x_{i_{v+1}}\right]_R \cap Z_l\right|}{\left|\left[x_{i_{v+1}}\right]_R\right|} = \frac{\left|\left[x_{i_{v+1}}\right]_R \cap Z_l\right|}{\left|\left[x_{i_{v+1}}\right]_R \cap Z_l\right| + \left|\left[x_{i_{v+1}}\right]_R - Z_l\right|} \geqslant \lambda_1 \geqslant 0.5, \tag{40}$$

we have $|[x_{i_{v+1}}]_R \cap Z_l| \geqslant |[x_{i_{v+1}}]_R - Z_l|$. According to $|[x_{i_{v+2}}]_R \cap Z_l| \geqslant |[x_{i_{v+2}}]_R - Z_l|, \ldots, |[x_{i_u}]_R \cap Z_l| \geqslant |[x_{i_u}]_R - Z_l|$, we have $|Z_l \cap [x_{i_{v+1}}]_R| + \cdots + |Z_l \cap [x_{i_u}]_R| \geqslant |([x_{i_{v+1}}]_R - Z_l)| + \cdots + |([x_{i_u}]_R - Z_l)|$. And based on Lemma 10, we can easily have

$$\begin{aligned}
&\frac{\left|Z_l \cap R_{\lambda_2}(Z_l)\right|}{\left|Z_l \cup R_{\lambda_2}(Z_l)\right|} \\
&= \frac{\left|\underline{R}(Z_l)\right| + \left|Z_l \cap \left[x_{i_1}\right]_R\right| + \cdots + \left|Z_l \cap \left[x_{i_v}\right]_R\right|}{\left|Z_l\right| + \left|\left(\left[x_{i_1}\right]_R - Z_l\right)\right| + \cdots + \left|\left(\left[x_{i_v}\right]_R - Z_l\right)\right|} \\
&\leqslant \left(\left|\underline{R}(Z_l)\right| + \left|Z_l \cap \left[x_{i_1}\right]_R\right| + \cdots + \left|Z_l \cap \left[x_{i_v}\right]_R\right| \right. \\
&\quad \left. + \left|Z_l \cap \left[x_{i_{v+1}}\right]_R\right| + \cdots + \left|Z_l \cap \left[x_{i_u}\right]_R\right|\right) \\
&\quad \times \left(\left|Z_l\right| + \left|\left(\left[x_{i_1}\right]_R - Z_l\right)\right| + \cdots + \left|\left(\left[x_{i_v}\right]_R - Z_l\right)\right| \right. \\
&\quad \left. + \left|\left(\left[x_{i_{v+1}}\right]_R - Z_l\right)\right| + \cdots + \left|\left(\left[x_{i_u}\right]_R - Z_l\right)\right|\right)^{-1} \\
&= \frac{\left|Z_l \cap R_{\lambda_1}(Z_l)\right|}{\left|Z_l \cup R_{\lambda_1}(Z_l)\right|}.
\end{aligned} \tag{41}$$

Therefore,

$$\frac{\left|Z_l \cap R_{\lambda_2}(Z_l)\right|}{2\left|Z_l \cup R_{\lambda_2}(Z_l)\right|} \leqslant \frac{\left|Z_l \cap R_{\lambda_1}(Z_l)\right|}{2\left|Z_l \cup R_{\lambda_1}(Z_l)\right|}. \tag{42}$$

(2) In the same way as (1), the inequality

$$\frac{\left|Z_u \cap R_{\lambda_2}(Z_u)\right|}{2\left|Z_u \cup R_{\lambda_2}(Z_u)\right|} \leqslant \frac{\left|Z_u \cap R_{\lambda_1}(Z_u)\right|}{2\left|Z_u \cup R_{\lambda_1}(Z_u)\right|} \tag{43}$$

is held.

According to (1) and (2), the inequality $\mathbb{S}(\mathbb{Z}, R_{\lambda_1}(\mathbb{Z})) \geqslant \mathbb{S}(\mathbb{Z}, R_{\lambda_2}(\mathbb{Z}))$ is held. So, the proof of Theorem 16 has been completed successfully. □

Theorems 14 and 16 show that the similarity degree between an interval set $\mathbb{Z}$ and its approximation set $R_\lambda(\mathbb{Z})$ is a monotonically decreasing function with the parameter λ, and the similarity degree reaches its maximum value when $\lambda = 0.5$.

5. The Change Rules of Similarity in Different Knowledge Granularity Spaces

In different Pawlak's approximation spaces with different knowledge granularities, the change rules of the uncertainty of rough set are a key issue [21, 22]. Many researchers try to discover the change rules of uncertainty in rough set model [23, 24]. And we also find many change rules of uncertain concept in different knowledge spaces in our other papers [20]. In this paper, we continue to discuss the change rules of the similarity degree $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z}))$ in Pawlak's approximation spaces with different knowledge granularities. In this paper, we focus on discussing how the similarity degree between $\mathbb{Z}$ and $R_{0.5}(\mathbb{Z})$ changes when the granules are divided into more subgranules in Pawlak's approximation space. In other words, it is an important issue concerning how $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z}))$ changes with different knowledge granularities in Pawlak's approximation space.

Let $[x_1]_R, [x_2]_R, \ldots, [x_n]_R$ be classifications of U under equivalence relation R. Let $[x_1]_{R'}, [x_2]_{R'}, \ldots, [x_n]_{R'}$ be classifications of U under equivalence relation R'. If $R' \subseteq R$, then $[x_i]_{R'} \subseteq [x_i]_R$ $(1 \leq i \leq n)$. And then, U/R' is called a refinement of U/R, which is written as $U/R' \preceq U/R$. If $\exists x_j \in U$, then $[x_j]_{R'} \subset [x_j]_R$. And then, U/R' is called a strict refinement of U/R, which is written as $U/R' \prec U/R$.

Next, we will analyze the relationship between $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z}))$ and $\mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$. Let $U/R' \prec U/R$; in other words, for all $x \in U$, $[x]_{R'} \subseteq [x]_R$ is always satisfied, and $\exists y \in U$, $[y]_{R'} \subset [y]_R$. And then, there must be two or more granules in U/R' whose union is $[y]_R$. To simplify the proof, we suppose that there is just only one granule which is divided into two subgranules, denoted by $[x_{i_t}^1]_{R'}$ and $[x_{i_t}^2]_{R'}$ in U/R', and other granules keep unchanged.

There are 9 cases, and only 6 cases are possible.

Theorem 17. *Let U be a finite domain, $\mathbb{Z}$ an interval set on U, and R and R' two equivalence relations on U. Let $[x_{i_t}]_R$ be one granule which is divided into two subgranules marked as $[x_{i_t}^1]_{R'}$ and $[x_{i_t}^2]_{R'}$. If*

$$\begin{aligned}
S\left(Z_l, R_{0.5}(Z_l)\right) &\geqslant \frac{\left|\left[x_{i_t}^2\right] \cap Z_l\right|}{\left|\left[x_{i_t}^2\right] - Z_l\right|}, \\
S\left(Z_u, R_{0.5}(Z_u)\right) &\geqslant \frac{\left|\left[x_{i_t}^2\right] \cap Z_u\right|}{\left|\left[x_{i_t}^2\right] - Z_u\right|}.
\end{aligned} \tag{44}$$

Then, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \leqslant \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$.

Proof. There are 6 possible cases which will be discussed one by one as follows.

(1) $[x_{i_t}]_R$ is contained in both positive region of Z_l and positive region of Z_u. In this case, obviously $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$ is held.

(2) $[x_{i_t}]_R$ is contained in both positive region of Z_l and negative region of Z_u. In this case, obviously, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$ is held.

(3) $[x_{i_t}]_R$ is contained in both negative region of Z_l and negative region of Z_u. In this case, obviously, $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) = \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$ is held.

(4) $[x_{i_t}]_R$ is contained in both negative region of Z_l and boundary region of Z_u. In this case,

$$\frac{|R_{0.5}(Z_l)\cap Z_l|}{2|R_{0.5}(Z_l)\cup Z_l|}=\frac{|R'_{0.5}(Z_l)\cap Z_l|}{2|R'_{0.5}(Z_l)\cup Z_l|} \tag{45}$$

is held obviously. Next, we discuss the relationship between

$$\frac{|R_{0.5}(Z_u)\cap Z_u|}{2|R_{0.5}(Z_u)\cup Z_u|},\quad \frac{|R'_{0.5}(Z_u)\cap Z_u|}{2|R'_{0.5}(Z_u)\cup Z_u|}. \tag{46}$$

Let $R_{0.5}(Z_u)=\underline{R}(Z_u)\cup[x_{i_1}]_R\cup[x_{i_2}]_R\cup\cdots\cup[x_{i_k}]_R$. Let $\mathrm{BN}_R(Z_u)=[x_{i_1}]_R\cup[x_{i_2}]_R\cup\cdots\cup[x_{i_m}]_R$ where $m\geqslant k$. When $[x_{i_t}]_R$ is in boundary region of Z_u, we should further discuss this situation. To simplify the proof, we suppose that there is just only one granule marked as $[x_{i_t}]_R$ in U/R which is divided into two subgranules marked as $[x^1_{i_t}]_{R'}$ and $[x^2_{i_t}]_{R'}$ in U/R'. And the other granules keep unchanged.

(a) If $k<t\leqslant m$, then $[x_{i_t}]_R\not\subset R_{0.5}(Z_u)$.

(1) If $[x^1_{i_t}]_{R'}\subseteq R'_{0.5}(Z_u)$, $[x^2_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$. From the proof of Theorem 12, we know

$$\begin{aligned}&\frac{|Z_u\cap R_{0.5}(Z_u)|}{|Z_u\cup R_{0.5}(Z_u)|}\\&\quad=\Big(|\underline{R}(Z_u)|+|Z_u\cap[x_{i_1}]_R|+|Z_u\cap[x_{i_2}]_R|\\&\qquad+\cdots+|Z_u\cap[x_{i_k}]_R|\Big)\\&\qquad\times\Big(|Z_u|+|([x_{i_1}]_R-Z_u)|+|([x_{i_2}]_R-Z_u)|\\&\qquad\quad+\cdots+|([x_{i_k}]_R-Z_u)|\Big)^{-1}.\end{aligned} \tag{47}$$

Because $[x^1_{i_t}]_{R'}\cup[x^2_{i_t}]_{R'}=[x_{i_t}]_R$, $[x^1_{i_t}]_{R'}\subseteq R'_{0.5}(Z_u)$, and $[x^2_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$, we have

$$\begin{aligned}&\frac{|Z_u\cap R'_{0.5}(Z_u)|}{|Z_u\cup R'_{0.5}(Z_u)|}\\&\quad=\Big(|\underline{R}'(Z_u)|+|Z_u\cap[x_{i_1}]_{R'}|+|Z_u\cap[x_{i_2}]_{R'}|\\&\qquad+\cdots+|Z_u\cap[x_{i_k}]_{R'}|+|Z_u\cap[x^1_{i_t}]_{R'}|\Big)\\&\qquad\times\Big(|Z_u|+|([x_{i_1}]_{R'}-Z_u)|+|([x_{i_2}]_{R'}-Z_u)|\\&\qquad\quad+\cdots+|([x_{i_k}]_{R'}-Z_u)|+|([x^1_{i_t}]_{R'}-Z_u)|\Big)^{-1}\\&\quad=\Big(|\underline{R}(Z_u)|+|Z_u\cap[x_{i_1}]_R|+|Z_u\cap[x_{i_2}]_R|\\&\qquad+\cdots+|Z_u\cap[x_{i_k}]_R|+|Z_u\cap[x^1_{i_t}]_{R'}|\Big)\\&\qquad\times\Big(|Z_u|+|([x_{i_1}]_R-Z_u)|+|([x_{i_2}]_R-Z_u)|\\&\qquad\quad+\cdots+|([x_{i_k}]_R-Z_u)|+|([x^1_{i_t}]_{R'}-Z_u)|\Big)^{-1}.\end{aligned} \tag{48}$$

For $[x^1_{i_1}]_{R'}\subseteq R'_{0.5}(Z_u)$, $|[x^1_{i_t}]_{R'}\cap Z_u|/(|[x^1_{i_t}]_{R'}\cap Z_u|+|[x^1_{i_t}]_{R'}-Z_u|)\geqslant 0.5$, which means $|[x^1_{i_t}]_{R'}\cap Z_u|\geqslant|[x^1_{i_t}]_{R'}-Z_u|$. According to Lemma 10, we have

$$\frac{|R_{0.5}(Z_u)\cap Z_u|}{2|R_{0.5}(Z_u)\cup Z_u|}\leqslant\frac{|R'_{0.5}(Z_u)\cap Z_u|}{2|R'_{0.5}(Z_u)\cup Z_u|}. \tag{49}$$

(2) If $[x^1_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$, $[x^2_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$, then

$$\frac{|R_{0.5}(Z_u)\cap Z_u|}{2|R_{0.5}(Z_u)\cup Z_u|}=\frac{|R'_{0.5}(Z_u)\cap Z_u|}{2|R'_{0.5}(Z_u)\cup Z_u|}. \tag{50}$$

Because $[x_{i_t}]_R\not\subseteq R_{0.5}(Z_u)$, the case that $[x^1_{i_1}]_{R'}\subseteq R'_{0.5}(Z_u)$ and $[x^2_{i_1}]_{R'}\subseteq R'_{0.5}(Z_u)$ is impossible.

(b) If $1\leqslant t\leqslant k$, then $[x_{i_t}]_R\subseteq R_{0.5}(Z_u)$.

(1) If $[x^1_{i_t}]_{R'}\subseteq R'_{0.5}(Z_u)$ and $[x^2_{i_t}]_{R'}\subseteq R'_{0.5}(Z_u)$, then we can easily have

$$\frac{|R_{0.5}(Z_u)\cap Z_u|}{2|R_{0.5}(Z_u)\cup Z_u|}=\frac{|R'_{0.5}(Z_u)\cap Z_u|}{2|R'_{0.5}(Z_u)\cup Z_u|}. \tag{51}$$

(2) If $[x^1_{i_t}]_{R'}\subseteq R'_{0.5}(Z_u)$ and $[x^2_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$, then

$$\frac{|R_{0.5}(Z_u)\cap Z_u|}{|R_{0.5}(Z_u)\cup Z_u|}\geqslant\frac{|[x^2_{i_t}]\cap Z_u|}{|[x^2_{i_t}]-Z_u|}. \tag{52}$$

Because $[x^2_{i_t}]_{R'}\not\subseteq R'_{0.5}(Z_u)$, we have the following.

(i) If $[x^2_{i_t}]_{R'}\cap Z_u=\phi$, then we have $|Z_u\cap[x^1_{i_t}]_R|=|Z_u\cap[x_{i_t}]_R|$ and $|([x^1_{i_t}]_{R'}-Z_u)|<|([x_{i_t}]_R-Z_u)|$. Therefore,

$$\begin{aligned}&\frac{|Z_u\cap R'_{0.5}(Z_u)|}{|Z_u\cup R'_{0.5}(Z_u)|}\\&\quad=\Big(|\underline{R}'(Z_u)|+|Z_u\cap[x_{i_1}]_{R'}|+|Z_u\cap[x_{i_2}]_{R'}|\\&\qquad+\cdots+|Z_u\cap[x^1_{i_t}]_{R'}|+\cdots+|Z_u\cap[x_{i_k}]_{R'}|\Big)\\&\qquad\times\Big(|Z_u|+|([x_{i_1}]_{R'}-Z_u)|+|([x_{i_2}]_{R'}-Z_u)|\\&\qquad\quad+\cdots+|([x^1_{i_t}]_{R'}-Z_u)|+\cdots+|([x_{i_k}]_{R'}-Z_u)|\Big)^{-1}\\&\quad=\Big(|\underline{R}(Z_u)|+|Z_u\cap[x_{i_1}]_R|+|Z_u\cap[x_{i_2}]_R|\\&\qquad+\cdots+|Z_u\cap[x_{i_t}]_R|+\cdots+|Z_u\cap[x_{i_k}]_R|\Big)\end{aligned}$$

$$\begin{aligned}
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_R-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_R-Z_u\right)\right|\right.\\
&\quad\left.+\cdots+\left|\left(\left[x_{i_t}^1\right]_{R'}-Z_u\right)\right|+\cdots+\left|\left(\left[x_{i_k}\right]_R-Z_u\right)\right|\right)^{-1}\\
&>\left(\left|\underline{R}(Z_u)\right|+\left|Z_u\cap\left[x_{i_1}\right]_R\right|+\left|Z_u\cap\left[x_{i_2}\right]_R\right|\right.\\
&\quad\left.+\cdots+\left|Z_u\cap\left[x_{i_k}\right]_R\right|\right)\\
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_R-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_R-Z_u\right)\right|\right.\\
&\quad\left.+\cdots+\left|\left(\left[x_{i_k}\right]_R-Z_u\right)\right|\right)^{-1}\\
&=\frac{\left|Z_u\cap R_{0.5}(Z_u)\right|}{\left|Z_u\cup R_{0.5}(Z_u)\right|}.
\end{aligned}\tag{53}$$

(ii) If $[x_{i_t}^1]_{R'}\subseteq Z_u$, then $|[x_{i_t}^1]_{R'}\cap Z_u|=|[x_{i_t}^1]_{R'}|$. Therefore,

$$\begin{aligned}
&\frac{\left|Z_u\cap R'_{0.5}(Z_u)\right|}{\left|Z_u\cup R'_{0.5}(Z_u)\right|}\\
&=\left(\left|\underline{R}'(Z_u)\right|+\left|Z_u\cap\left[x_{i_1}\right]_{R'}\right|+\left|Z_u\cap\left[x_{i_2}\right]_{R'}\right|\right.\\
&\quad\left.+\cdots+\left|Z_u\cap\left[x_{i_t}^1\right]_{R'}\right|+\cdots+\left|Z_u\cap\left[x_{i_k}\right]_{R'}\right|\right)\\
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_{R'}-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_{R'}-Z_u\right)\right|\right.\\
&\quad\left.+\cdots+\left|\left(\left[x_{i_t}^1\right]_{R'}-Z_u\right)\right|+\cdots+\left|\left(\left[x_{i_k}\right]_{R'}-Z_u\right)\right|\right)^{-1}\\
&=\left(\left|\underline{R}(Z_u)\right|+\left|Z_u\cap\left[x_{i_1}\right]_R\right|+\left|Z_u\cap\left[x_{i_2}\right]_R\right|\right.\\
&\quad\left.+\cdots+\left|\left[x_{i_t}^1\right]_{R'}\right|+\cdots+\left|Z_u\cap\left[x_{i_k}\right]_R\right|\right)\\
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_R-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_R-Z_u\right)\right|\right.\\
&\quad+\cdots+\left|\left[x_{i_{t-1}}\right]_R-Z_u\right|\\
&\quad\left.+\left|\left[x_{i_{t+1}}\right]_R-Z_u\right|\cdots+\left|\left(\left[x_{i_k}\right]_R-Z_u\right)\right|\right)^{-1}\\
&=\frac{\left|Z_u\cap R_{0.5}(Z_u)\right|-\left|\left[x_{i_t}^2\right]_{R'}\cap Z_u\right|}{\left|Z_u\cup R_{0.5}(Z_u)\right|-\left|\left[x_{i_t}^2\right]_{R'}-Z_u\right|}.
\end{aligned}\tag{54}$$

Because

$$\frac{\left|Z_u\cap R_{0.5}(Z_u)\right|}{\left|Z_u\cup R_{0.5}(Z_u)\right|}\geqslant\frac{\left|\left[x_{i_t}^2\right]\cap Z_u\right|}{\left|\left[x_{i_t}^2\right]-Z_u\right|},\tag{55}$$

according to Lemma 11, we have

$$\frac{\left|R_{0.5}(Z_u)\cap Z_u\right|}{\left|R_{0.5}(Z_u)\cup Z_u\right|}\leqslant\frac{\left|R'_{0.5}(Z_u)\cap Z_u\right|}{\left|R'_{0.5}(Z_u)\cup Z_u\right|}.\tag{56}$$

(iii) If $[x_{i_t}^1]_{R'}\subseteq \mathrm{BN}_{R'}(Z_u)$ and $[x_{i_t}^2]_{R'}\subseteq \mathrm{BN}_{R'}(Z_u)$, because $[x_{i_t}^1]_{R'}\subseteq R'_{0.5}(Z_u)$ and $[x_{i_t}^2]_{R'}\not\subseteq R'_{0.5}(Z_u)$, we have

$$\begin{aligned}
&\frac{\left|Z_u\cap R'_{0.5}(Z_u)\right|}{\left|Z_u\cup R'_{0.5}(Z_u)\right|}\\
&=\left(\left|\underline{R}'(Z_u)\right|+\left|Z_u\cap\left[x_{i_1}\right]_{R'}\right|+\left|Z_u\cap\left[x_{i_2}\right]_{R'}\right|\right.\\
&\quad\left.+\cdots+\left|Z_u\cap\left[x_{i_t}^1\right]_{R'}\right|+\cdots+\left|Z_u\cap\left[x_{i_k}\right]_{R'}\right|\right)\\
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_{R'}-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_{R'}-Z_u\right)\right|\right.\\
&\quad\left.+\cdots+\left|\left(\left[x_{i_t}^1\right]_{R'}-Z_u\right)\right|+\cdots+\left|\left(\left[x_{i_k}\right]_{R'}-Z_u\right)\right|\right)^{-1}\\
&=\left(\left|\underline{R}(Z_u)\right|+\left|Z_u\cap\left[x_{i_1}\right]_R\right|+\left|Z_u\cap\left[x_{i_2}\right]_R\right|\right.\\
&\quad\left.+\cdots+\left|\left[x_{i_t}^1\right]_{R'}\right|+\cdots+\left|Z_u\cap\left[x_{i_k}\right]_R\right|\right)\\
&\times\left(|Z_u|+\left|\left(\left[x_{i_1}\right]_R-Z_u\right)\right|+\left|\left(\left[x_{i_2}\right]_R-Z_u\right)\right|\right.\\
&\quad\left.+\cdots+\left|\left[x_{i_t}^1\right]_{R'}-Z_u\right|\cdots+\left|\left(\left[x_{i_k}\right]_R-Z_u\right)\right|\right)^{-1}\\
&=\frac{\left|Z_u\cap R_{0.5}(Z_u)\right|-\left|\left[x_{i_t}^2\right]_{R'}\cap Z_u\right|}{\left|Z_u\cup R_{0.5}(Z_u)\right|-\left|\left[x_{i_t}^2\right]_{R'}-Z_u\right|}.
\end{aligned}\tag{57}$$

Because

$$\frac{\left|Z_u\cap R_{0.5}(Z_u)\right|}{\left|Z_u\cup R_{0.5}(Z_u)\right|}\geqslant\frac{\left|\left[x_{i_t}^2\right]\cap Z_u\right|}{\left|\left[x_{i_t}^2\right]-Z_u\right|},\tag{58}$$

according to Lemma 11, we have

$$\frac{\left|R_{0.5}(Z_u)\cap Z_u\right|}{\left|R_{0.5}(Z_u)\cup Z_u\right|}\leqslant\frac{\left|R'_{0.5}(Z_u)\cap Z_u\right|}{\left|R'_{0.5}(Z_u)\cup Z_u\right|}.\tag{59}$$

According to (a) and (b) above, we have $\mathbb{S}(\mathbb{Z},R_{0.5}(\mathbb{Z}))\leqslant\mathbb{S}(\mathbb{Z},R'_{0.5}(\mathbb{Z}))$ when

$$\frac{\left|R_{0.5}(Z_u)\cap Z_u\right|}{\left|R_{0.5}(Z_u)\cup Z_u\right|}\geqslant\frac{\left|\left[x_{i_t}^2\right]\cap Z_u\right|}{\left|\left[x_{i_t}^2\right]-Z_u\right|}.\tag{60}$$

(5) $[x_{i_t}]_R$ is contained in boundary region of Z_l and positive region of Z_u. In this case,

$$\frac{\left|R_{0.5}(Z_u)\cap Z_u\right|}{\left|R_{0.5}(Z_u)\cup Z_u\right|}=\frac{\left|R'_{0.5}(Z_u)\cap Z_u\right|}{\left|R'_{0.5}(Z_u)\cup Z_u\right|}.\tag{61}$$

Next, we discuss the relationship between

$$\begin{gathered}
\frac{\left|R_{0.5}(Z_l)\cap Z_l\right|}{\left|R_{0.5}(Z_l)\cup Z_l\right|},\\
\frac{\left|R'_{0.5}(Z_l)\cap Z_l\right|}{\left|R'_{0.5}(Z_l)\cup Z_l\right|}.
\end{gathered}\tag{62}$$

Similar to (a) and (b) in (4), when

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \geqslant \frac{|[x_{i_t}^2] \cap Z_l|}{|[x_{i_t}^2] - Z_l|}, \tag{63}$$

we can get

$$\frac{|Z_l \cap R_{0.5}(Z_l)|}{|Z_l \cup R_{0.5}(Z_l)|} \leqslant \frac{|Z_l \cap R'_{0.5}(Z_l)|}{|Z_l \cup R'_{0.5}(Z_l)|}. \tag{64}$$

So, in the condition, we can draw a conclusion that $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \leqslant \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$.

(6) $[x_{i_t}]_R$ is contained in boundary region of Z_l and boundary region of Z_u.

According to the proofs of (4) and (5), if $S(Z_l, R_{0.5}(Z_l)) \geqslant |[x_{i_t}^2] \cap Z_l|/|[x_{i_t}^2] - Z_l|$ and $S(Z_u, R_{0.5}(Z_u)) \geqslant |[x_{i_t}^2] \cap Z_u|/|[x_{i_t}^2] - Z_u|$, we easily have $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z})) \leqslant \mathbb{S}(\mathbb{Z}, R'_{0.5}(\mathbb{Z}))$.

From (1), (2), (3), (4), (5), and (6), Theorem 17 is proved successfully. □

Theorem 17 shows that, under some conditions, the similarity degree between an interval set $\mathbb{Z}$ and its approximation set $R_{0.5}(\mathbb{Z})$ is a monotonically increasing function when the knowledge granules in U/R are divided into many finer subgranules in U/R', where U/R' is a refinement of U/R.

6. Conclusion

With the development of uncertain artificial intelligence, the interval set theory attracts more and more researchers and gradually develops into a complete theory system. The interval set theory has been successfully applied to many fields, such as machine learning, knowledge acquisition, decision-making analysis, expert system, decision support system, inductive inference, conflict resolution, pattern recognition, fuzzy control, and medical diagnostics systems. It is an important tool of granular computing as well as the rough set which is one of the three main tools of granular computing [25, 26]. In the interval set theory, the target concept is approximately described by two certain sets, that is, the upper bound and lower bound. In other words, the essence of this theory is that we deal with the uncertain problems with crisp set theory method. Many researches have been completed on extended models of the interval set, but the theories nearly cannot present better approximation set of the interval set $\mathbb{Z}$. In this paper, the approximation set $R_{0.5}(\mathbb{Z})$ of target concept $\mathbb{Z}$ in current knowledge space is proposed from a new viewpoint and related properties are analyzed in detail.

In this paper, the interval set is transformed into a fuzzy set at first, and then the uncertain elements in boundary region are classified by cut-set with some threshold. Next, the approximation set $R_{0.5}(\mathbb{Z})$ of the interval set $\mathbb{Z}$ is defined and the change rules of $\mathbb{S}(\mathbb{Z}, R_{0.5}(\mathbb{Z}))$ in different knowledge granularity spaces are analyzed. These researches show that $R_{0.5}(\mathbb{Z})$ is a better approximation set of $\mathbb{Z}$ than both $\overline{R}(\mathbb{Z})$ and $\underline{R}(\mathbb{Z})$. Finally, a kind of crisp approximation set of interval set is proposed in this paper. These researches present a new method to describe uncertain concept from a special viewpoint, and we hope these results can promote the development of both uncertain artificial intelligence and granular computing and extend the interval set model into more application fields. It is an important research issue concerning discovering more knowledge and rules from the uncertain information [27]. The fuzzy set and the rough set have been used widely [28–32]. Recently, the interval set theory is applied to many important fields, such as software testing [33], the case generation based on interval combination [34], and incomplete information table [35–38]. In the future research, we will focus on acquiring the approximation rules from uncertain information systems based on the approximation sets of an interval set.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (nos. 61272060, 61309014, and 61379114), The Natural Science Foundation of Chongqing of China (nos. cstc2012jjA40047, cstc2012jjA40032, and cstc2013jcyjA40063), and the Science & Technology Research Program of Chongqing Education Committee of China (no. KJ130534).

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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 783940, 14 pages
http://dx.doi.org/10.1155/2014/783940

Research Article

A Novel Method of the Generalized Interval-Valued Fuzzy Rough Approximation Operators

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Received 1 May 2014; Revised 19 June 2014; Accepted 5 July 2014; Published 4 August 2014

Academic Editor: Yanhong She

Rough set theory is a suitable tool for dealing with the imprecision, uncertainty, incompleteness, and vagueness of knowledge. In this paper, new lower and upper approximation operators for generalized fuzzy rough sets are constructed, and their definitions are expanded to the interval-valued environment. Furthermore, the properties of this type of rough sets are analyzed. These operators are shown to be equivalent to the generalized interval fuzzy rough approximation operators introduced by Dubois, which are determined by any interval-valued fuzzy binary relation expressed in a generalized approximation space. Main properties of these operators are discussed under different interval-valued fuzzy binary relations, and the illustrative examples are given to demonstrate the main features of the proposed operators.

1. Introduction

Rough set theory proposed by Pawlak [1] is an extension of set theory for the study of intelligent systems characterized by inexact, uncertain, or insufficient information. The core of the rough set theory and its applications is to define a pair of lower and upper approximation operators, and an equivalence relation is a key and primitive notion in Pawlak's rough set model [1]. This equivalence relation is the key concept of Pawlak's rough set model, but also a very strict condition, which may limit the applicability of the rough set model [2, 3]. To solve this problem, several authors have generalized the notion of the approximation operators by using nonequivalent binary relations. The most important research is the amalgamation of fuzzy set theory and rough set theory [2, 4–7] as well as the rough set theory based on generalized binary relations [8–13]. Pawlak first discussed the relation between rough sets and fuzzy sets in [6]. Dubois and Prade [4] proposed the fuzzy rough set theory by amalgamating the fuzzy set theory with the rough set theory. In addition, based on the definition of neighborhood operators, Yao [8–11] studied the rough set theory based on the generalized binary relation, that is, the generalized rough set theory. Recently, Wu et al. [14–19] defined the generalized fuzzy rough set theory based on the study of the fuzzy rough set theory and the generalized rough set theory, and Zhu [12] studied generalized rough sets based on relations.

A rough set model is composed of two parts: the approximation space and the approximated object. Rough set theory comes with a lot of extensions and generalizations. Yao et al. researched the generalized rough sets by considering sets and relations of the approximation space and the approximated object [9, 16]. In Pawlak's rough set model [6], the relation of approximation spaces is a classical binary equivalence relation and the approximated object is a set. If the equivalence relation is weakened to a general binary relation, the equivalence relation is a special case of the general binary relation. The set theory is generalized to the form of the fuzzy set theory, so that the classical set theory is a special case of the fuzzy set theory. These relationships are outlined in Figure 1.

Most researches on the fuzzy rough set theory focus on point-valued fuzzy sets and point-valued fuzzy binary relations. But the fuzzy notion described by using point values may lose some available information in the real-life information systems sometimes. If the description is done

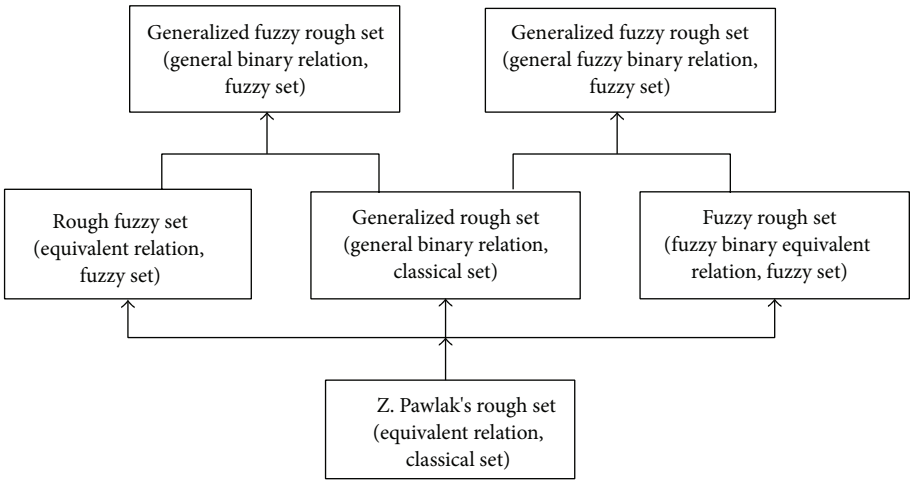

Figure 1: Select dependencies in rough set theory.

by interval values, it may acquire a better effectiveness than that by using point ones, for example, a self-evolving interval type-2 fuzzy neural network with online structure and parameter learning [20], encoding words into interval type-2 fuzzy sets using an interval approach [21], and corrections to aggregation using the linguistic weighted average and interval type-2 fuzzy sets [22]. Gong et al. [23] proposed a kind of interval-valued rough fuzzy set model based on an equivalent relation and applied the model to acquire rules from the interval-valued fuzzy information systems. It is very significant to apply the interval-valued fuzzy set in researching the rough set theory. Yeung et al. [24] generalized the fuzzy rough sets by means of arbitrary fuzzy relations and presented a general framework for the study of fuzzy rough sets by using both constructive and axiomatic approaches. Wu et al. [25] generalized the concept of fuzzy rough sets to interval type-2 fuzzy environments and proposed a method of attribute reduction within the interval type-2 fuzzy rough set framework. Xue et al. [26] generalized interval-valued fuzzy rough approximation operators. Zhang et al. [27] studied the characterization of generalized interval-valued fuzzy rough sets on two universes of discourse. The positive approximation and converse approximation in interval-valued fuzzy rough sets have been studied in [28]. Zhang and Jiang [29] proposed a note on interval-valued fuzzy rough sets and interval-valued intuitionistic fuzzy sets. Zhang et al. [30] proposed a general frame for intuitionistic fuzzy rough sets. Xu et al. [31] studied an axiomatic approach of interval-valued intuitionistic fuzzy rough sets based on interval-valued intuitionistic fuzzy approximation operators. Zhang and Tian [32] studied interval-valued intuitionistic fuzzy rough sets based on implicators. Wu and Zhou [33] studied intuitionistic fuzzy topologies based on intuitionistic fuzzy reflexive and transitive relations. Zhang et al. [34] proposed a variable-precision-dominance-based rough set approach to interval-valued information systems. Liang and Liu [35] studied three-way decisions with interval-valued decision-theoretic rough sets. Dai et al. [36] proposed an uncertainty measurement for interval-valued decision systems based on extended conditional entropy. Zhang et al. [37] studied multiconfidence rule acquisition and confidence-preserved attribute reduction in interval-valued decision systems. Ma and Hu [38] studied topological and lattice structures of L-fuzzy rough sets determined by lower and upper sets. Hao and Li [39] discussed the relationship between L-fuzzy rough set and L-topology. Zhang et al. [40] studied the union and intersection operations of rough sets based on various approximation spaces. She and He [41–43] studied rough approximation operators on R0-algebras (nilpotent minimum algebras) with an application in formal logic L, the rough consistency measures of logic theories, and approximate reasoning in rough logic and the structure of the multigranulation rough set model as well. Yang et al. [44] studied the combination of interval-valued fuzzy set and soft set. In terms of these researches above, a number of important conclusions are drawn, which exhibit great significance to research the rough fuzzy set theory. However the generalized interval-valued fuzzy rough set theory under the generalized relations needs to be further investigated.

In this paper, we further study the generalized fuzzy rough approximation operators defined in [16]. In particular, from the viewpoint of constructive approach, we reconstruct the lower approximation operator on the premise of the fact that the upper approximation operator is not changed and expand it to interval environments. It is proved that the lower approximation operator is equivalent to the generalized interval Dubois fuzzy rough approximation operator in the approximation space formed by arbitrary binary interval-valued fuzzy relations. Also, properties of the operators are discussed under the different binary interval-valued fuzzy relations.

The rest of the paper is organized as follows. In Section 2, we give some basic notions of interval-valued fuzzy sets and interval-valued fuzzy relations. In Section 3, we study the generalized fuzzy rough approximation operators defined in [16]. In Section 4, from the viewpoint of constructive and interval approach, we reconstruct new lower and upper approximation operators of the generalized interval-valued

fuzzy rough sets. In Section 5, we prove some properties of the generalized interval-valued fuzzy rough approximation operators and the presented scheme by the extensive analysis results. In Section 6, we bring forward some conclusions and highlight further work.

2. Basic Concepts of Interval-Valued Fuzzy Sets and Interval-Valued Fuzzy Relations

In this section, we introduce some basic notions and properties related to interval-valued fuzzy sets which will be used in this paper. We first review an interval-valued subset originated by [28]. We first review some basic concepts.

Let I be a closed unit interval; that is, $I = [0,1]$. $[I] = \{[a^-, a^+] : a^- \le a^+, a^-, a^+ \in I\}$ is the set of all interval-valued subsets of I. $a = [a^-, a^+] \in [I]$ is an interval value. When $a^- = a^+$, the interval-valued $a = [a^-, a^+]$ becomes a real number in $[I]$. In particular, real numbers return intervals of zero length, say $1 = [1,1]$ and $0 = [0,0]$.

Definition 1. Let $a, b \in [I]$. $a \le b$ if and only if $a^- \le b^-$, $a^+ \le b^+$; $a = b$ if and only if $a^- = b^-, a^+ = b^+$; $a < b$ if and only if $a \le b$ and $a \ne b$.

Definition 2. Let $a, b \in [I]$. $a \nleq b$ indicates that a is not less than or equal to b; $a \nless b$ indicates that a is not less than b; $a \ngeq b$ indicates that a is not greater than or equal to b; $a \ngtr b$ indicates that a is not greater than b.

According to the order relation defined in Definition 1, different elements in $[I]$ may not exhibit order relations, so Definition 2 becomes necessary.

Definition 3. Let $a_i \in [I], b_i \in I, i \in J, J = \{1, 2, \cdots, m\}$; one defines

$$\begin{gathered}
\bigvee_{i\in J} b_i = \sup\{b_i : i \in J\}, \qquad \bigwedge_{i\in J} b_i = \inf\{b_i : i \in J\}, \\
\bigvee_{i\in J} a_i = \vee\{a_i : i \in J\} = \left[\bigvee_{i\in J} a_i^-, \bigvee_{i\in J} a_i^+\right], \\
\bigwedge_{i\in J} a_i = \wedge\{a_i : i \in J\} = \left[\bigwedge_{i\in J} a_i^-, \bigwedge_{i\in J} a_i^+\right], \\
\sim a = 1 - a = \left[1 - a^+, 1 - a^-\right].
\end{gathered} \tag{1}$$

Obviously, $([I], \le)$ is a complete lattice, and the triple $([I], \vee, \wedge)$ is an algebraic system, which is derived by $([I], \le)$ with the maximal element $[1,1]$ and the minimum element $[0,0]$.

Definition 4. Let U be a finite and nonempty universe of discourse; then a mapping $A : U \to [I]$ is called an interval-valued fuzzy set on U. All interval-valued fuzzy sets on U are denoted by $F^I(U)$. In particular, when $A = U, A(x) = [1,1]$, for all $x \in U$, and when $A = \emptyset$, $A(x) = [0,0]$, for all $x \in U$.

Similar to fuzzy sets, the operators $\subseteq, \cap, \cup$, and complement of interval-valued fuzzy sets are defined as follows. For all $A, B \in F^I(U)$, $A \subseteq B$ means $A(x) \le B(x)$ and for all $x \in U$, $(A \cap B)(x) = A(x) \wedge B(x)$, $(A \cup B)(x) = A(x) \vee B(x)$, and $(\sim A)(x) = 1 - A(x)$.

Definition 5. Let $\alpha \in [I], A \in F^I(U)$. αA is called numerical product of α and A and is defined as $(\alpha A)(x) = \alpha \wedge A(x)$, for all $x \in U$.

Definition 6. Let $\alpha \in [I], A \in F^I(U)$. $A_\alpha = \{x \in U : A(x) \ge \alpha\}$ is called α-cut set of A and $A_{\overline{\alpha}} = \{x \in U : A(x) > \alpha\}$ is called strong α-cut set of A.

Theorem 7 (the decomposition theorem of the interval-valued fuzzy sets). *Let $A \in F^I(U)$; then*

$$A = \bigcup_{\alpha\in[I]} \alpha A_\alpha, \qquad A = \bigcup_{\alpha\in[I]} \alpha A_{\overline{\alpha}}. \tag{2}$$

Proof. For all $x \in U$,

$$\begin{aligned}
&\left(\bigcup_{\alpha\in[I]} \alpha A_\alpha\right)(x) \\
&\quad = \bigvee_{\alpha\in[I]} (\alpha \wedge A_\alpha(x)) \\
&\quad = \left(\bigvee_{\alpha\le A(x)} (\alpha \wedge A_\alpha(x))\right) \vee \left(\bigvee_{\alpha\nleq A(x)} (\alpha \wedge A_\alpha(x))\right) \\
&\quad = \bigvee_{\alpha\le A(x)} (\alpha \wedge A_\alpha(x)) \\
&\quad = \bigvee_{\alpha\le A(x)} \alpha \\
&\quad = A(x).
\end{aligned} \tag{3}$$

Then $A = \cup_{\alpha\in[I]} \alpha A_\alpha$.

Similarly, one can show that $A = \cup_{\alpha\in[I]} \alpha A_{\overline{\alpha}}$. □

Definition 8. Let U and W be two finite and nonempty universes of discourse. Then the mapping $IR : U \times W \to [I]$ is called an interval-valued fuzzy relation from U to W, where $U \times W = \{(x, y) : x \in U, y \in W\}$. When $U = W$, IR is called an interval-valued fuzzy relation on U.

Remark 9. Obviously, an interval-valued fuzzy relation IR from U to W is an interval-valued fuzzy set denoted by $IR \in F^I(U \times W)$. So Definitions 4, 5, and 6 and Theorem 7 are still true in the interval-valued fuzzy relation. For example, $IR \in F^I(U \times W)$ is an interval-valued fuzzy relation. If we see it as an interval-valued fuzzy set, then $IR_\alpha = \{(x, y) \in U \times W : R(x, y) \ge \alpha\}$.

Definition 10. Let IR be an interval-valued fuzzy relation from U to W; then IR is said to be serial if and only if for all $x \in U$, there exists $y \in W$ such that $IR(x, y) = [1, 1]$.

Definition 11. Let IR be an interval-valued fuzzy relation on U; then IR is reflexive if and only if $IR(x, x) = [1, 1]$, for all $x \in U$; IR is symmetric if and only if $IR(x, y) = IR(y, x)$, for all $x, y \in U$; IR is transitive if and only if $IR(x, z) \geq \vee_{y\in U}(IR(x, y) \wedge IR(y, z))$, for all $x, z \in U$; IR is Euclidean if and only if $IR(y, z) \geq \vee_{x\in U}(IR(x, y) \wedge IR(x, z))$, for all $y, z \in U$.

One can prove that the binary relation obtained by calculating α-cut set or strong α-cut set to an interval-valued fuzzy relation, for all $\alpha \in [I]$, still satisfies the corresponding definition of Definition 11 under the classical binary relation; that is, if IR is, respectively, reflexive, symmetric, and transitive, then $IR_\alpha(IR_{\overline{\alpha}})$ is, respectively, reflexive, symmetric, and transitive under the classical binary relation.

3. Generalized Fuzzy Rough Approximation Operators

Definition 12. Let U and W be two finite universes of discourse. If R is an arbitrary binary fuzzy relation from U to W, then the triple (U, W, R) is called a generalized fuzzy approximation space.

Definition 13. Let (U, W, R) be a generalized fuzzy approximation space, for all $x \in U$; one defines $R(x) = \{(y, R(x, y)) : y \in W\}$.

$R(x)$ is the row of the fuzzy relation which includes x, and obviously $R_\alpha(x) = (R(x))_\alpha$.

Definition 14. Let (U, W, R) be a generalized fuzzy approximation space, for all $\alpha, \beta \in [0, 1]$, $A \in F(W)$,

$$\begin{aligned} \underline{R}_\alpha\left(A_\beta\right) &= \left\{x \in U : R_\alpha(x) \subseteq A_\beta\right\}, \\ \overline{R}_\alpha\left(A_\beta\right) &= \left\{x \in U : R_\alpha(x) \cap A_\beta \neq \emptyset\right\}. \end{aligned} \tag{4}$$

$\underline{R}_\alpha(A_\beta)$ and $\overline{R}_\alpha(A_\beta)$ are called (α, β) lower and upper approximations of A with respect to (U, W, R).

Definition 15. Let (U, W, R) be a generalized fuzzy approximation space, for all $A \in F(W)$. One defines

$$\underline{R}(A) = \bigcup_{\alpha\in I} \alpha \underline{R}_{\overline{1-\alpha}}(A_\alpha), \qquad \overline{R}(A) = \bigcup_{\alpha\in I} \alpha \overline{R}_\alpha(A_\alpha). \tag{5}$$

The pair $(\underline{R}(A), \overline{R}(A))$ is called the generalized fuzzy rough set of A on (U, W, R), and the operators $\underline{R}(A)$ and $\overline{R}(A)$ are called the generalized fuzzy rough lower and upper approximation operators, respectively.

The dual properties are quite useful in proving the properties of the approximation operators. When one intends to prove two dual properties, it suffices to prove one of them, which simplifies the proof procedure. The properties of the lower and upper approximation operators are characterized as follows.

Theorem 16. *Let (U, W, R) be a generalized fuzzy approximation space. Then for all $A \in F(W)$, $\underline{R}(A) = \sim \overline{R}(\sim A)$ and $\overline{R}(A) = \sim \underline{R}(\sim A)$.*

Proof. (1) Note that $\cap_{\alpha\in I}(\underline{\underline{\alpha}} \cup \underline{R}_{1-\alpha}(A_{\overline{\alpha}})) = \cup_{\alpha\in I}(\underline{\underline{\alpha}} \cap \underline{R}_{\overline{1-\alpha}}(A_\alpha))$. Here $\underline{\underline{\alpha}} \in F(U)$ and $\underline{\underline{\alpha}}(x) = \alpha$ for all $x \in U$.

For all $x \in U$, $x \in \underline{R}_{1-\alpha}(A_{\overline{\alpha}})$, that is, $\underline{R}_{1-\alpha}(A_{\overline{\alpha}})(x) = 1$, if and only if $R_{1-\alpha}(x) \subseteq A_{\overline{\alpha}}$, such that for all $y \in W$ if $R(x, y) \geq 1 - \alpha$, then $A(y) > \alpha$; that is, for all $y \in W$, $R(x, y) < 1 - \alpha$ or $A(y) > \alpha$.

Hence we have $\wedge_{y\in W}((1 - R(x, y)) \vee A(y)) > \alpha$.

For the second case, $x \notin \underline{R}_{1-\alpha}(A_{\overline{\alpha}})$; that is, $\underline{R}_{1-\alpha}(A_{\overline{\alpha}})(x) = 0$ if and only if $\wedge_{y\in W}((1-R(x, y))\vee A(y)) \leq \alpha$. Since $\wedge_{y\in W}((1-R(x, y)) \vee A(y)) \in I$, there exists $\alpha \in I$, so that $\alpha = \wedge_{y\in W}((1 - R(x, y)) \vee A(y))$.

Hence,

$$\begin{aligned} &\bigcap_{\alpha\in I}\left(\underline{\underline{\alpha}} \cup \underline{R}_{1-\alpha}\left(A_{\overline{\alpha}}\right)\right)(x) \\ &\quad = \bigwedge_{\alpha\in I}\left(\alpha \vee \underline{R}_{1-\alpha}\left(A_{\overline{\alpha}}\right)(x)\right) \\ &\quad = \bigwedge_{y\in W}\left(\left(1 - R(x, y)\right) \vee A(y)\right). \end{aligned} \tag{6}$$

Similarly,

$$\begin{aligned} &\bigcup_{\alpha\in I}\left(\underline{\underline{\alpha}} \cap \underline{R}_{\overline{1-\alpha}}\left(A_\alpha\right)\right)(x) \\ &\quad = \bigvee_{\alpha\in I}\left(\alpha \wedge \underline{R}_{\overline{1-\alpha}}\left(A_\alpha\right)(x)\right) \\ &\quad = \bigwedge_{y\in W}\left(\left(1 - R(x, y)\right) \vee A(y)\right). \end{aligned} \tag{7}$$

Therefore, $\cap_{\alpha\in I}(\underline{\underline{\alpha}} \cup \underline{R}_{1-\alpha}(A_{\overline{\alpha}})) = \cup_{\alpha\in I}(\underline{\underline{\alpha}} \cap \underline{R}_{\overline{1-\alpha}}(A_\alpha))$.

(2) Now, we prove the validity of the relationship $\underline{R}(A) = \sim \overline{R}(\sim A)$. In view of Definition 14, from Theorem 3.2(1) of [18], it follows that

$$\begin{aligned} \sim \overline{R}(\sim A) &= \sim \bigcup_{\alpha\in I}\left(\alpha\overline{R}_\alpha\left((\sim A)_\alpha\right)\right) \\ &= \sim \bigcup_{\alpha\in I}\left(\alpha\overline{R}_\alpha\left(\sim A_{\overline{1-\alpha}}\right)\right) \\ &= \bigcap_{\alpha\in I}\left(\underline{\underline{1-\alpha}} \cup \left(\sim\overline{R}_\alpha\left(\sim A_{\overline{1-\alpha}}\right)\right)\right) \\ &= \bigcap_{\alpha\in I}\left(\underline{\underline{1-\alpha}} \cup \underline{R}_\alpha\left(A_{\overline{1-\alpha}}\right)\right) \end{aligned}$$

$$= \bigcap_{\alpha\in I}\left(\underline{\underline{\alpha}} \cup \underline{R}_{1-\alpha}\left(A_{\overline{\alpha}}\right)\right)$$
$$= \bigcup_{\alpha\in I}\left(\alpha \underline{R}_{\overline{1-\alpha}}\left(A_{\alpha}\right)\right)$$
$$= \underline{R}\left(A\right). \tag{8}$$

Hence, $\underline{R}(A) =\sim \overline{R}(\sim A)$.
Similarly, $\overline{R}(A) =\sim \underline{R}(\sim A)$. □

Suppose that $y_1 = \alpha \vee \underline{R}_{1-\alpha}(A_{\overline{\alpha}})(x)$, $y_2 = \alpha \wedge \underline{R}_{\overline{1-\alpha}}(A_{\alpha})(x)$ and $\beta = \wedge_{y\in W}((1 - R(x, y)) \vee A(y))$; when the variable x is a certain value, the variables y_1 and y_2 are functions of the variable α. Refer to Figure 2 for the pertinent detail.

In the proof of Theorem 16, we show that the equation $\underline{R}(A) =\sim \overline{R}(\sim A)$ holds when the minimum of function y_1 is equal to the maximum of function y_2, such that $\cap_{\alpha\in I}(\underline{\underline{\alpha}} \cup \underline{R}_{1-\alpha}(A_{\overline{\alpha}})) = \cup_{\alpha\in I}(\alpha \underline{R}_{\overline{1-\alpha}}(A_{\overline{\alpha}}))$; thus, $\underline{R}(A) =\sim \overline{R}(\sim A)$ holds. In [16], the lower approximation operator makes function y_2 equal to zero at the point β, which makes the maximum of function y_2 approach β, but it does not exist. In this paper, the lower and upper approximation operators in Definition 15 have a better duality.

4. Generalized Interval-Valued Fuzzy Rough Sets

Definition 17. Let U and W be two finite universes of discourse. If IR is an arbitrary binary interval-valued fuzzy relation from U to W, then the triple (U, W, IR) is called a generalized interval-valued fuzzy approximation space. In particular, when $U = W$, the space is denoted by (U, IR).

Definition 18. Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, for all $x \in U$,

$$IR\left(x\right) = \left\{\left(y, IR\left(x, y\right)\right) : y \in W\right\}. \tag{9}$$

Definition 19. Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$, for all $x \in U$; one defines

$$\begin{aligned}\underline{RIF}\left(A\right)\left(x\right) &= \bigwedge_{y\in W}\left(\sim IR\left(x, y\right) \vee A\left(y\right)\right)\\ &= \left[\bigwedge_{y\in W}\left(\left(1 - IR(x, y)^{+}\right) \vee A(y)^{-}\right),\right.\\ &\qquad \left.\bigwedge_{y\in W}\left(\left(1 - IR(x, y)^{-}\right) \vee A(y)^{+}\right)\right],\\ \overline{RIF}\left(A\right)\left(x\right) &= \bigvee_{y\in W}\left(IR\left(x, y\right) \wedge A\left(y\right)\right)\\ &= \left[\bigvee_{y\in W}\left(IR(x, y)^{-} \wedge A(y)^{-}\right),\right.\\ &\qquad \left.\bigvee_{y\in W}\left(IR\left(x, y\right)^{+} \wedge A(y)^{+}\right)\right].\end{aligned} \tag{10}$$

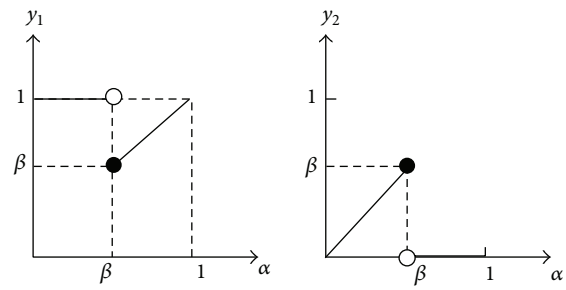

Figure 2: The coordinate frame of functions y_1 and y_2.

The pair $(\underline{RIF}(A), \overline{RIF}(A))$ is called the generalized interval-valued fuzzy rough set of A with respect to the approximation space (U, W, IR). The operators $\underline{RIF}$ and $\overline{RIF}$ are called the generalized interval-valued fuzzy rough lower and upper approximation operators, respectively.

Definition 20. Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, for all $\alpha, \beta \in [I]$, $A \in F^I(W)$; one defines

$$\begin{aligned}\underline{RIF}'_{\alpha}\left(A_{\beta}\right) &= \left\{x \in U : IR_{\alpha}\left(x\right) \subseteq A_{\beta}\right\},\\ \overline{RIF}'_{\alpha}\left(A_{\beta}\right) &= \left\{x \in U : IR_{\alpha}\left(x\right) \cap A_{\beta} \neq \emptyset\right\}.\end{aligned} \tag{11}$$

$\underline{RIF}'_{\alpha}(A_{\beta})$ and $\overline{RIF}'_{\alpha}(A_{\beta})$ are called the α, β lower and upper approximations of A with respect to (U, W, IR), respectively.

Definition 21. Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; one defines

$$\begin{aligned}\underline{RIF}'\left(A\right) &= \bigcup_{\alpha\in[I]}\alpha \underline{RIF}'_{\overline{1-\alpha}}\left(A_{\alpha}\right),\\ \overline{RIF}'\left(A\right) &= \bigcup_{\alpha\in[I]}\alpha \overline{RIF}'_{\alpha}\left(A_{\alpha}\right).\end{aligned} \tag{12}$$

The pair $(\underline{RIF}', \overline{RIF}')$ is called the generalized interval-valued fuzzy rough set of A with respect to the approximation space (U, W, IR). The operators $\underline{RIF}'$ and $\overline{RIF}'$ are called the generalized interval-valued fuzzy rough lower and upper approximation operators.

Remark 22. The approximation operators introduced in Definition 20 extend the generalized Dubois fuzzy rough approximation operators from numeric value to intervals.

The approximation operators defined in Definition 21 provide the same type of generalization. The approximation operators defined in Definition 21 show the inherent relationship between Pawlak's rough set and interval-valued fuzzy rough sets.

Lemma 23. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; then for all $\alpha \in [I]$, $\overline{RIF}'_\alpha(A_\alpha) \subseteq (\overline{RIF}(A))_\alpha$.*

Proof. We observe that, for all $x \in U$, if $x \in \overline{RIF}'_\alpha(A_\alpha)$, then $IR_\alpha(x) \cap A_\alpha \neq \emptyset$. This means that there exist $y \in W$, $IR(x, y) \geq \alpha$, and $A(y) \geq \alpha$.

By the interval-valued operations of Definition 3, we have $\vee_{y\in W}(IR(x, y) \wedge A(y)) \geq \alpha$, so $x \in (\overline{RIF}(A))_\alpha$.

Therefore, $\overline{RIF}'_\alpha(A_\alpha) \subseteq (\overline{RIF}(A))_\alpha$. □

Now, we prove that the reverse of Lemma 23 does not hold. Based on the interval-valued operations, which are defined in Definition 3, there exists $y \in W, IR(x, y) \wedge A(y) \geq \alpha$; that is, there exists $y \in W$, so that $IR(x, y) \geq \alpha$ and $A(y) \geq \alpha$ cannot be deduced by $\vee_{y\in W}(IR(x, y) \wedge A(y)) \geq \alpha$.

Next, we give an example illustrating that the relationship $(\overline{RIF}(A))_\alpha \subseteq \overline{RIF}'_\alpha(A_\alpha)$ does not hold.

Example 24. Suppose that (U, W, IR) is a generalized interval-valued fuzzy approximation space,

$$U = \{x_1, x_2\}, \qquad W = \{y_1, y_2, y_3\}, \qquad \alpha = [0.2, 0.6],$$
$$R(x_1) = \frac{[0.3, 0.5]}{y_1} + \frac{[0.1, 0.7]}{y_2} + \frac{[0.1, 0.4]}{y_3},$$
$$A = \frac{[0.2, 0.5]}{y_1} + \frac{[0.1, 0.9]}{y_2} + \frac{[0.2, 0.3]}{y_3}, \tag{13}$$

since

$$\overline{RIF}(A)(x_1) = \bigvee_{y\in W}(R(x_1, y) \wedge A(y)) = [0.2, 0.7] \geq [0.2, 0.6], \quad x_1 \in \left(\overline{RIF}(A)\right)_\alpha. \tag{14}$$

On the other hand, since $IR_\alpha(x_1) = A_\alpha = \emptyset$, we have $IR_\alpha(x_1) \cap A_\alpha = \emptyset$ and we get $x_1 \notin \overline{RIF}'_\alpha(A_\alpha)$. This shows that $x_1 \in (\overline{RIF}(A))_\alpha$, but $x_1 \notin \overline{RIF}'_\alpha(A_\alpha)$. So $(\overline{RIF}(A))_\alpha \nsubseteq \overline{RIF}'_\alpha(A_\alpha)$.

Lemma 25. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; then, for all $\alpha \in [I]$, $\underline{RIF}'_{\overline{1-\alpha}}(A_\alpha) \subseteq (\underline{RIF}(A))_\alpha$.*

Proof. Note that, for all $x \in U$, if $x \in \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)$ then $IR_{\overline{1-\alpha}}(x) \subseteq A_\alpha$. This means that, for all $y \in W$, if $IR(x, y) > 1 - \alpha$ then $A(y) \geq \alpha$; that is, for all $y \in W$, $IR(x, y) \leq 1 - \alpha$ or $A(y) \geq \alpha$. By the interval-valued operations as in Definition 3, we have $\wedge_{y\in W}(\sim IR(x, y) \vee A(y)) \geq \alpha$; that is, $\underline{RIF}(A)(x) \geq \alpha$, so $x \in (\underline{RIF}(A))_\alpha$. □

Now we prove that the reverse of Lemma 25 is not true. Here we use similar reasoning as already used in Lemma 23. $\sim IR(x, y) \vee A(y) \geq \alpha$ means that $(1 - IR(x, y))^- \vee A(y)^- \geq \alpha^-$ and $(1 - IR(x, y))^+ \vee A(y)^+ \geq \alpha^+$, which cannot deduce that $1 - IR(x, y) \geq \alpha$ or $A(y) \geq \alpha$ in Definition 3. Thus for all $y \in W$, $\sim IR(x, y) \vee A(y) \geq \alpha$, we cannot deduce that $IR(x, y) \leq 1 - \alpha$ or $A(y) \geq \alpha$, and note that, for all $y \in W$, $\sim IR(x, y) \vee A(y) \geq \alpha$ if and only if $\wedge_{y\in W}(\sim IR(x, y) \vee A(y)) \geq \alpha$. Therefore for all $y \in W$, $IR(x, y) \leq 1 - \alpha$ or $A(y) \geq \alpha$ cannot hold.

Next, we show that $(\underline{RIF}(A))_\alpha \subseteq \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)$ does not hold.

Example 26. Suppose that (U, W, IR) is a generalized interval-valued fuzzy approximation space,

$$U = \{x_1, x_2\}, \qquad W = \{y_1, y_2, y_3\}, \qquad \alpha = [0.3, 0.6],$$
$$IR(x_1) = \frac{[0.3, 0.8]}{y_1} + \frac{[0.4, 0.9]}{y_2} + \frac{[0.5, 0.6]}{y_3},$$
$$A = \frac{[0.4, 0.5]}{y_1} + \frac{[0.3, 0.4]}{y_2} + \frac{[0.4, 0.9]}{y_3}, \tag{15}$$

since

$$\underline{RIF}(A)(x_1) = \bigwedge_{y\in W}(\sim IR(x_1, y) \vee A(y)) = [0.3, 0.6] \geq [0.3, 0.6], \quad x_1 \in \left(\underline{RIF}(A)\right)_\alpha. \tag{16}$$

On the other hand, since $IR_{\overline{1-\alpha}}(x_1) = y_2, A_\alpha = y_3$, we have $IR_{\overline{1-\alpha}}(x_1) \nsubseteq A_\alpha$; hence $x_1 \notin \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)$. This shows that $x_1 \in (\underline{RIF}(A))_\alpha$, but $x_1 \notin \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)$. Therefore $(\underline{RIF}(A))_\alpha \nsubseteq \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)$.

Theorem 27. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; then $\overline{RIF}(A) = \overline{RIF}'(A)$.*

Proof. According to Theorem 7, we have $\overline{RIF}(A) = \cup_{\alpha\in[I]}\alpha(\overline{RIF}(A))_\alpha$, and from Lemma 23, we see that $\overline{RIF}'_\alpha(A_\alpha) \subseteq (\overline{RIF}(A))_\alpha$, for all $\alpha \in [I]$.

Therefore, $\overline{RIF}'(A) \subseteq \overline{RIF}(A)$.

Next we prove that $\overline{RIF}(A) \subseteq \overline{RIF}'(A)$.

In fact, for all $x \in U, y \in W$, there exists $\alpha = IR(x, y) \wedge A(y) \in [I]$, such that $y \in IR_\alpha(x) \cap A_\alpha$. We observe that $y \in IR_\alpha(x) \cap A_\alpha$ means that $IR_\alpha(x) \cap A_\alpha \neq \emptyset$, which can

deduce that $x \in \overline{RIF}'_\alpha(A_\alpha)$; that is, $\overline{RIF}'_\alpha(A_\alpha)(x) = 1$; hence $\alpha \wedge \overline{RIF}'_\alpha(A_\alpha)(x) = \alpha = IR(x, y) \wedge A(y)$.

So, for arbitrary value of y, $\overline{RIF}'(A)(x) = \vee_{\alpha\in[I]}(\alpha \wedge \overline{RIF}'_\alpha(A_\alpha)(x)) \geq \vee_{y\in W}(IR(x, y) \wedge A(y)) = \overline{RIF}(A)(x)$, which yields $\overline{RIF}(A) \subseteq \overline{RIF}'(A)$.

Therefore, $\overline{RIF}(A) = \overline{RIF}'(A)$. □

Theorem 28. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; then $\underline{RIF}(A) = \underline{RIF}'(A)$.*

Proof. In view of Theorem 7, we have $\underline{RIF}(A) = \cup_{\alpha\in[I]} \alpha(\underline{RIF}(A))_\alpha$, and from Lemma 25, $\underline{RIF}'_{\overline{1-\alpha}}(A_\alpha) \subseteq (\underline{RIF}(A))_\alpha$, for all $\alpha \in [I]$.

Then $\underline{RIF}'(A) \subseteq \underline{RIF}(A)$.

Now we prove that $\underline{RIF}(A) \subseteq \underline{RIF}'(A)$. For all $x \in U$, suppose that

$$\alpha_1 = \left[0, \bigwedge_{y\in W}(\sim IR(x, y) \vee A(y))^+\right],$$
$$\alpha_2 = \left[\bigwedge_{y\in W}(\sim IR(x, y) \vee A(y))^-, \bigwedge_{y\in W}(\sim IR(x, y) \vee A(y))^-\right]. \tag{17}$$

(1) We verify that $x \in \underline{RIF}'_{\overline{1-\alpha_1}}(A_{\alpha_1})$. Let

$$\alpha_1 = \left[0, \bigwedge_{y\in W}\left(\left(1 - IR(x, y)^-\right) \vee A(y)^+\right)\right],$$
$$1 - \alpha_1 = \left[\bigvee_{y\in W}\left(IR(x, y)^- \wedge \left(1 - A(y)^+\right)\right), 1\right]. \tag{18}$$

Note that, for all $y_0 \in W$, $y_0 \in IR_{\overline{1-\alpha_1}}(x)$, such that $IR(x, y_0) > 1 - \alpha_1$, and from $(1 - \alpha_1)^+ = 1$, we have $IR(x, y_0)^- > \vee_{y\in W}(IR(x, y)^- \wedge (1 - A(y)^+))$.

Further from $y_0 \in W$, we have $IR(x, y_0)^- > IR(x, y_0)^- \wedge (1 - A(y_0)^+)$.

Therefore we obtain that $IR(x, y_0)^- > 1 - A(y_0)^+$ and $1 - A(y_0)^+ = IR(x, y_0)^- \wedge (1 - A(y_0)^+)$.

Because $IR(x, y_0)^- \wedge (1 - A(y_0)^+) \leq \vee_{y\in W}(IR(x, y)^- \wedge (1 - A(y)^+))$, we have $1 - A(y_0)^+ \leq \vee_{y\in W}(IR(x, y)^- \wedge (1 - A(y)^+))$, such that $A(y_0)^+ \geq \wedge_{y\in W}((1 - IR(x, y)^-) \vee A(y)^+)$, and from $\alpha_1^- = 0$, we get $A(y_0) \geq \alpha_1$; that is, $y_0 \in A_{\alpha_1}$.

So, for arbitrary value of y_0, $IR_{\overline{1-\alpha_1}}(x) \subseteq A_{\alpha_1}$; that is, $x \in \underline{RIF}'_{\overline{1-\alpha_1}}(A_{\alpha_1})$.

(2) Similar to the proof shown in (1), we have $x \in \underline{RIF}'_{\overline{1-\alpha_2}}(A_{\alpha_2})$. Note that

$$\begin{aligned}
&\underline{RIF}'(A)(x) \\
&= \bigvee_{\alpha\in[I]}\left(\alpha \wedge \underline{RIF}'_{\overline{1-\alpha}}(A_\alpha)(x)\right) \\
&\geq \left(\alpha_1 \wedge \underline{RIF}'_{\overline{1-\alpha_1}}(A_{\alpha_1})(x)\right) \vee \left(\alpha_2 \wedge \underline{RIF}'_{\overline{1-\alpha_2}}(A_{\alpha_2})(x)\right) \\
&= \alpha_1 \vee \alpha_2 \\
&= \left[\bigwedge_{y\in W}(\sim IR(x, y) \vee A(y))^-, \bigwedge_{y\in W}(\sim IR(x, y) \vee A(y))^+\right] \\
&= \underline{RIF}(A)(x).
\end{aligned} \tag{19}$$

For any x, $\underline{RIF}(A) \subseteq \underline{RIF}'(A)$.

Therefore, $\underline{RIF}(A) = \underline{RIF}'(A)$. □

Remark 29. According to Theorems 27 and 28, $\underline{RIF}'$ and $\overline{RIF}'$ satisfy the property of duality.

Theorem 30. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space, $A \in F^I(W)$; then $\sim \underline{RIF}(A) = \overline{RIF}(\sim A)$, $\sim\overline{RIF}(A) = \underline{RIF}(\sim A)$.*

Proof. We observe that, for all $x \in U$,

$$\begin{aligned}
&\overline{RIF}(\sim A)(x) \\
&= \left[\bigvee_{y\in W}(IR(x, y) \wedge (\sim A)(y))^-, \bigvee_{y\in W}(IR(x, y) \wedge (\sim A)(y))^+\right] \\
&= \left[\bigvee_{y\in W}\left(IR(x, y)^- \wedge \left(1 - A(y)^+\right)\right), \bigvee_{y\in W}\left(IR(x, y)^+ \wedge \left(1 - A(y)^-\right)\right)\right] \\
&= 1 - \left[\bigwedge_{y\in W}\left(\left(1 - IR(x, y)^+\right) \vee A(y)^-\right), \bigwedge_{y\in W}\left(\left(1 - IR(x, y)^-\right) \vee A(y)^+\right)\right] \\
&= 1 - \left[\bigwedge_{y\in W}\left((1 - IR(x, y))^- \vee A(y)^-\right), \bigwedge_{y\in W}\left((1 - IR(x, y))^+ \vee A(y)^+\right)\right]
\end{aligned}$$

$$= 1 - \left[\bigwedge_{y\in W} ((1 - IR(x,y)) \vee A(y))^{-}, \bigwedge_{y\in W} ((1 - IR(x,y)) \vee A(y))^{+} \right]$$
$$= \sim \underline{RIF}(A)(x). \tag{20}$$

Hence, $\sim \underline{RIF}(A) = \overline{RIF}(\sim A)$.
Similarly, $\sim \overline{RIF}(A) = \underline{RIF}(\sim A)$. □

5. Properties of the Approximation Operators

Theorem 31. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space; then the lower approximation operator $\underline{RIF}'$ and the upper approximation operator $\overline{RIF}'$ satisfy the following properties.*

For all $A, B \in F^{I}(W), a \in [I]$,

(1) $\underline{RIF}'(A \cup \underline{\underline{a}}) = \underline{RIF}'(A) \cup \underline{\underline{a}}$,
$\overline{RIF}'(A \cap \underline{\underline{a}}) = \overline{RIF}'(A) \cap \underline{\underline{a}}$;

(2) $\overline{RIF}'(A \cup B) = \overline{RIF}'(A) \cup \overline{RIF}'(B)$,
$\underline{RIF}'(A \cap B) = \underline{RIF}'(A) \cap \underline{RIF}'(B)$;

(3) *if* $A \subseteq B$ *then* $\underline{RIF}'(A) \subseteq \underline{RIF}'(B)$,
and $\overline{RIF}'(A) \subseteq \overline{RIF}'(B)$;

(4) $\underline{RIF}'(A \cup B) \supseteq \underline{RIF}'(A) \cup \underline{RIF}'(B)$,
$\overline{RIF}'(A \cap B) \subseteq \overline{RIF}'(A) \cap \overline{RIF}'(B)$.

Here $\underline{\underline{a}}$ is a constant interval-valued fuzzy set; that is, $\underline{\underline{a}}(x) = a$, for all $x \in U$ and $x \in W$.

Proof. (1) We prove that $\underline{RIF}'(A \cup \underline{\underline{a}}) = \underline{RIF}'(A) \cup \underline{\underline{a}}$.

For all $x \in U$, let

$$D_1 = \{\alpha \in [I] : \forall y \in IR_{\overline{1-\alpha}}(x), A(y) \geq \alpha \text{ or } a \geq \alpha\},$$
$$D_2 = \{\alpha \in [I] : \forall y \in IR_{\overline{1-\alpha}}(x), A(y) \vee a \geq \alpha\}, \tag{21}$$

where $\forall x$ is "for all x" and $\exists x$ is "there exists x," which are the same as follows.

Obviously, $D_1 \subseteq D_2, \vee_{\alpha\in D_1}\alpha \leq \vee_{\alpha\in D_2}\alpha$. Set $D_3 = D_2 - D_1$, for all $\beta \in D_3$; two cases appear:

$$\beta^{+} \leq a^{+}, \qquad \beta^{-} \leq \bigwedge_{y\in R_{\overline{1-\alpha}}(x)} A(y)^{-} \tag{22}$$

or

$$\beta^{-} \leq a^{-}, \qquad \beta^{+} \leq \bigwedge_{y\in R_{\overline{1-\alpha}}(x)} A(y)^{+}. \tag{23}$$

For the first case, suppose $b = [\wedge_{y\in R_{\overline{1-\alpha}}(x)} A(y)^{-}, \wedge_{y\in R_{\overline{1-\alpha}}(x)} A(y)^{+}]$, because $a, b \in D_1$; we have $\beta \leq a \vee b \leq \vee_{\alpha\in D_1}\alpha$. The proof for the second case is similar.

For arbitrary β, it is easy to see that $\vee_{\alpha\in D_3}\alpha \leq \vee_{\alpha\in D_1}\alpha$, so $\vee_{\alpha\in D_2}\alpha = \vee_{\alpha\in D_1}\alpha$:

$$\begin{aligned}
&\underline{RIF}'(A \cup \underline{\underline{a}})(x) \\
&= \bigvee_{\alpha\in[I]} \left(\alpha \wedge \underline{RIF}'_{\overline{1-\alpha}}(A \cup \underline{\underline{a}})_{\alpha}(x)\right) \\
&= \vee \left\{\alpha \in [I] : IR_{\overline{1-\alpha}}(x) \subseteq (A \cup \underline{\underline{a}})_{\alpha}\right\} \\
&= \vee \{\alpha \in [I] : \forall y \in IR_{\overline{1-\alpha}}(x), A(y) \vee a \geq \alpha\} \\
&= \vee \{\alpha \in [I] : \forall y \in IR_{\overline{1-\alpha}}(x), A(y) \geq \alpha \text{ or } a \geq \alpha\} \\
&= a \vee (\vee \{\alpha \in [I] : \forall y \in IR_{\overline{1-\alpha}}(x), A(y) \geq \alpha\}) \\
&= a \vee \underline{RIF}'(A)(x) \\
&= (\underline{\underline{a}} \cup \underline{RIF}'(A))(x).
\end{aligned} \tag{24}$$

Hence, $\underline{RIF}'(A \cup \underline{\underline{a}}) = \underline{RIF}'(A) \cup \underline{\underline{a}}$.
Similarly, $\overline{RIF}'(A \cap \underline{\underline{a}}) = \overline{RIF}'(A) \cap \underline{\underline{a}}$.

(2) We verify $\overline{RIF}'(A \cup B) = \overline{RIF}'(A) \cup \overline{RIF}'(B)$.
For all $x \in U$, let

$$D_4 = \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), A(y) \geq \alpha \text{ or } B(y) \geq \alpha\};$$
$$D_5 = \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), A(y) \vee B(y) \geq \alpha\}, \tag{25}$$

$\vee_{\alpha\in D_5}\alpha = \vee_{\alpha\in D_4}\alpha$ holds by using similar arguments as in (1).

We observe that

$$\begin{aligned}
&\overline{RIF}'(A \cup B)(x) \\
&= \bigvee_{\alpha\in[I]} \left(\alpha \wedge \overline{RIF}'_{\alpha}(A \cup B)_{\alpha}(x)\right) \\
&= \vee \{\alpha \in [I] : IR_{\alpha}(x) \cap (A \cup B)_{\alpha} \neq \emptyset\} \\
&= \vee \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), (A \cup B)(y) \geq \alpha\} \\
&= \vee \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), A(y) \vee B(y) \geq \alpha\} \\
&= \vee \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), A(y) \geq \alpha \text{ or } B(y) \geq \alpha\} \\
&= (\vee \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), A(y) \geq \alpha\}) \\
&\quad \vee (\vee \{\alpha \in [I] : \exists y \in IR_{\alpha}(x), B(y) \geq \alpha\}) \\
&= \left(\bigvee_{\alpha\in[I]} \left(\alpha \wedge \overline{RIF}'_{\alpha}(A)_{\alpha}(x)\right)\right) \\
&\quad \vee \left(\bigvee_{\alpha\in[I]} \left(\alpha \wedge \overline{RIF}'_{\alpha}(B)_{\alpha}(x)\right)\right) \\
&= \overline{RIF}'(A)(x) \vee \overline{RIF}'(B)(x) \\
&= \left(\overline{RIF}'(A) \cup \overline{RIF}'(B)\right)(x),
\end{aligned} \tag{26}$$

which yields that $\underline{RIF}'(A \cup B) = \underline{RIF}'(A) \cup \underline{RIF}'(B)$.

Similarly, $\underline{RIF}'(A \cap B) = \underline{RIF}'(A) \cap \underline{RIF}'(B)$.

(3) We prove that if $A \subseteq B$ then $\underline{RIF}'(A) \subseteq \underline{RIF}'(B)$. If $A \subseteq B$, then $A_\alpha \subseteq B_\alpha$. According to Definition 20 and Theorem 3.2(4) of [18], we have $\underline{RIF}'_{1-\alpha}(A_\alpha) \subseteq \underline{RIF}'_{1-\alpha}(B_\alpha)$, so $\cup_{\alpha\in[I]}\alpha\underline{RIF}'_{1-\alpha}(A_\alpha) \subseteq \cup_{\alpha\in[I]}\alpha\underline{RIF}'_{1-\alpha}(B_\alpha)$; that is, $\underline{RIF}'(A) \subseteq \underline{RIF}'(B)$.

Similarly, if $A \subseteq B$, then $\overline{RIF}'(A) \subseteq \overline{RIF}'(B)$.

(4) From (3), one immediately obtains (4). □

Remark 32. From Theorem 31 (1), one can see that $\underline{RIF}'(W) = U$, $\overline{RIF}'(\emptyset) = \emptyset$.

Theorem 33. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space; then the following conditions are equivalent:*

(1) *IR is serial;*

(2) *$\underline{RIF}'(A) \subseteq \overline{RIF}'(A)$, for all $A \subseteq F^I(W)$;*

(3) *$\overline{RIF}'(W) = U$;*

(4) *$\underline{RIF}'(\emptyset) = \emptyset$.*

Theorem 34. *Let (U, IR) be a generalized interval-valued fuzzy approximation space; then the following conditions are equivalent:*

(1) *IR is reflexive;*

(2) *$\underline{RIF}'(A) \subseteq A$, for all $A \subseteq F^I(W)$;*

(3) *$A \subseteq \overline{RIF}'(A)$, for all $A \subseteq F^I(W)$.*

Lemma 35. *Let (U, W, IR) be a generalized interval-valued fuzzy approximation space; then the following properties hold:*

(1) *$\overline{RIF}'(1_{\{y\}})(x) = IR(x, y)$, $(x, y) \in U \times W$;*

(2) *$\underline{RIF}'(1_{W\setminus\{y\}})(x) = 1 - IR(x, y)$,$(x, y) \in U \times W$.*

Here 1_A is an interval-valued fuzzy set which gets interval value $[1, 1]$ in the set A and interval value $[0, 0]$ in the set $\sim A$, respectively.

Remark 36. The proofs of Theorems 33 and 34 as well as Lemma 35 are similar to Theorems 3.8, 3.9, and 3.7 in [16], respectively; it suffices to change point values to interval values in the proof.

Lemma 37. *Let (U, IR) be a generalized interval-valued fuzzy approximation space and A is an interval-valued fuzzy set on U; then $\forall\alpha \in [I]$, $\overline{RIF}'_\alpha(A_\alpha) \subseteq (\overline{RIF}'(A))_\alpha$, and $\underline{RIF}'_{1-\alpha}(A_\alpha) \subseteq (\underline{RIF}'(A))_\alpha$.*

Proof. Clearly,

$$\begin{aligned}\overline{RIF}'_\alpha(A_\alpha) &= \left(\alpha\overline{RIF}'_\alpha(A_\alpha)\right)_\alpha \\ &\subseteq \left(\alpha\overline{RIF}'_\alpha(A_\alpha) \cup \left(\bigcup_{\beta\in[I]-\alpha}\beta\overline{RIF}'_\beta(A_\beta)\right)\right)_\alpha \\ &= \left(\bigcup_{\beta\in[I]}\beta\overline{RIF}'_\beta(A_\beta)\right)_\alpha \\ &= \left(\overline{RIF}'(A)\right)_\alpha.\end{aligned} \tag{27}$$

Hence, $\overline{RIF}'_\alpha(A_\alpha) \subseteq (\overline{RIF}'(A))_\alpha$.

Similarly, $\underline{RIF}'_{1-\alpha}(A_\alpha) \subseteq (\underline{RIF}'(A))_\alpha$. □

Remark 38. $\overline{RIF}'_\alpha(A_\alpha) = (\overline{RIF}'(A))_\alpha$ and $\underline{RIF}'_{1-\alpha}(A_\alpha) = (\underline{RIF}'(A))_\alpha$ hold for the fuzzy rough set in Lemma 37, but these are not true for the interval-valued fuzzy rough set. The reason is that the two interval values cannot always be comparable. Next, we give two examples to visualize this effect.

Example 39. Suppose that (U, IR) is a generalized interval-valued fuzzy approximation space, where $U = \{x_1, x_2\}$, $IR(x_1) = [0.5, 0.5]/x_1 + [0.3, 0.7]/x_2$, $A = [0.9, 1]/x_1 + [0.7, 0.9]/x_2$, $\alpha = [0.4, 0.6]$, $\beta_1 = [0.5, 0.5]$, and $\beta_2 = [0.3, 0.7]$. Since $IR_\alpha(x_1) = \emptyset$, $A_\alpha = \{x_1, x_2\}$, we have $IR_\alpha(x_1) \cap A_\alpha = \emptyset$. Hence $x_1 \notin \overline{RIF}'_\alpha(A_\alpha)$.

On the other hand, since $IR_{\beta_1}(x_1) = \{x_1\}$, $A_{\beta_1} = \{x_1, x_2\}$, $IR_{\beta_2}(x_1) = \{x_2\}$, $A_{\beta_2} = \{x_1, x_2\}$, we see that $x_1 \in \overline{RIF}'_{\beta_1}(A_{\beta_1})$ and $x_1 \in \overline{RIF}'_{\beta_2}(A_{\beta_2})$ get $\beta_1 \wedge \overline{RIF}'_{\beta_1}(A_{\beta_1})(x_1) = \beta_1$ and $\beta_2 \wedge \overline{RIF}'_{\beta_2}(A_{\beta_2})(x_1) = \beta_2$.

Note that $\vee_{\beta\in[I]}(\beta \wedge \overline{RIF}'_\beta(A_\beta)(x_1)) \geq \beta_1 \vee \beta_2 > \alpha$; then $\overline{RIF}'(A)(x_1) > \alpha$. Hence, $x_1 \in (\overline{RIF}'(A))_\alpha$.

Example 40. Suppose that (U, IR) is a generalized interval-valued fuzzy approximation space, where $U = \{x_1, x_2, x_3\}$, $IR(x_1) = [0.3, 0.7]/x_1 + [0.1, 0.8]/x_2 + [0.5, 0.5]/x_3$, $A = [0.5, 0.7]/x_1 + [0.6, 1]/x_2 + [0.7, 1]/x_3$, $\alpha = [0.4, 0.8]$, $\beta_1 = [0.2, 1]$, $\beta_2 = [0.5, 0.6]$.

Since $IR_{\overline{1-\alpha}}(x_1) = \{x_1\}$, $A_\alpha = \{x_2, x_3\}$, we have $IR_{\overline{1-\alpha}}(x_1) \not\subseteq A_\alpha$; hence, $x_1 \notin \underline{RIF}'_{1-\alpha}(A_\alpha)$.

On the other hand, since $IR_{\overline{1-\beta_1}}(x_1) = \{x_2\}$, $A_{\beta_1} = \{x_2, x_3\}$, $IR_{\overline{1-\beta_2}}(x_1) = \{x_3\}$, $A_{\beta_2} = \{x_2, x_3\}$, we conclude that $x_1 \in \underline{RIF}'_{1-\beta_1}(A_{\beta_1})$ and $x_1 \in \underline{RIF}'_{1-\beta_2}(A_{\beta_2})$ get $\beta_1 \wedge \underline{RIF}'_{1-\beta_1}(A_{\beta_1})(x_1) = \beta_1$ and $\beta_2 \wedge \underline{RIF}'_{1-\beta_2}(A_{\beta_2})(x_1) = \beta_2$.

Note that $\vee_{\beta\in[I]}(\beta \wedge \underline{RIF}'_{1-\beta}(A_\beta)(x_1)) \geq \beta_1 \vee \beta_2 > \alpha$; then $\underline{RIF}'(A)(x_1) > \alpha$.

Therefore, $x_1 \in (\underline{RIF}'(A))_\alpha$.

Theorem 41. *Let (U, IR) be a generalized interval-valued fuzzy approximation space; then the following conditions are equivalent:*

(1) *IR is transitive;*

(2) *$\underline{RIF}'(A) \subseteq \underline{RIF}'(\underline{RIF}'(A))$, for all $A \in F^I(U)$;*

(3) *$\overline{RIF}'(A) \subseteq \overline{RIF}'(\overline{RIF}'(A))$, for all $A \in F^I(U)$.*

Proof. (1) $\Rightarrow$ (2) For all $A \in F^I(U)$, from Definition 20, Lemma 37, and Theorem 3.6 of [18], we have

$$\begin{aligned}\underline{RIF}'\left(\underline{RIF}'(A)\right) &= \bigcup_{\alpha\in[I]} \alpha \underline{RIF}'_{1-\alpha}\left(\left(\underline{RIF}'(A)\right)_\alpha\right)\\ &\supseteq \bigcup_{\alpha\in[I]} \alpha \underline{RIF}'_{1-\alpha}\left(\left(\underline{RIF}'_{1-\alpha}(A_\alpha)\right)\right)\\ &\supseteq \bigcup_{\alpha\in[I]} \alpha \underline{RIF}'_{1-\alpha}(A_\alpha)\\ &= \underline{RIF}'(A).\end{aligned} \tag{28}$$

Hence $\underline{RIF}'(A) \subseteq \underline{RIF}'(\underline{RIF}'(A))$.

(3) $\Rightarrow$ (1) For all $x, y, z \in U$, let

$$\begin{aligned}D_6 &= \left\{\alpha \in [I] : \exists u \in U, IR(x,u) \geq \alpha,\right.\\ &\qquad \left.\overline{RIF}'\left(1_{\{z\}}\right)(u) = IR(u,z) \geq \alpha\right\},\\ D_7(y) &= IR(x,y) \wedge IR(y,z), \qquad \forall y \in U.\end{aligned} \tag{29}$$

For all $y \in U$, suppose that $\alpha = D_7(y) = IR(x,y) \wedge IR(y,z)$; then $IR(x,y) \geq \alpha, IR(y,z) \geq \alpha$; hence $\alpha \in D_6$ and by the arbitrary y, $\vee_{y\subseteq U} D_7(y) \leq \vee_{\alpha\in D_6}\alpha$.

For all $\alpha \in D_6$, there exists $y \in U$, such that $\alpha \leq IR(x,y) \wedge IR(y,z)$. For arbitrary α, $\vee_{y\subseteq U} D_7(y) \geq \vee_{\alpha\in D_6}\alpha$, so $\vee_{y\subseteq U} D_7(y) = \vee_{\alpha\in D_6}\alpha$. We observe that

$$\begin{aligned}&\overline{RIF}'\left(\overline{RIF}'\left(1_{\{z\}}\right)\right)(x)\\ &\quad= \vee\left\{\alpha \in [I] : x \in \overline{RIF}'_\alpha\left(\overline{RIF}'\left(1_{\{z\}}\right)\right)_\alpha\right\}\\ &\quad= \vee\left\{\alpha \in [I] : IR_\alpha(x) \cap \left(\overline{RIF}'\left(1_{\{z\}}\right)\right)_\alpha \neq \emptyset\right\}\\ &\quad= \vee\left\{\alpha \in [I] : \exists u \in U, IR(x,u) \geq \alpha,\right.\\ &\qquad\qquad \left.\left(\overline{RIF}'\left(1_{\{z\}}\right)\right)(u) = IR(u,z) \geq \alpha\right\}\\ &\quad= \bigvee_{y\in U}\left(IR(x,y) \wedge IR(y,z)\right).\end{aligned} \tag{30}$$

Hence, by Lemma 35 (1), we have $\overline{RIF}'(\overline{RIF}'(1_{\{z\}}))(x) \leq \overline{RIF}'(1_{\{z\}})(x) = IR(x,z)$; then $IR(x,z) \geq \vee_{y\in U}(IR(x,y) \wedge IR(y,z))$. Therefore, IR is transitive.

(2) $\Leftrightarrow$ (3) This conclusion follows immediately from the duality. □

Remark 42. In [18], if IR is symmetric, then the approximation operators satisfy $A \subseteq \underline{R}\overline{R}(A)$ and $\overline{R}\underline{R}(A) \subseteq A$ for all $A \subseteq U$; if IR is Euclidean, then the approximation operators satisfy $\overline{R}(A) \subseteq \underline{R}\overline{R}(A)$ and $\overline{R}\underline{R}(A) \subseteq \underline{R}(A)$ for all $A \subseteq U$. These properties do not hold in the interval-valued fuzzy rough sets. Next, we give a counterexample to show it.

Example 43. Suppose that (U, IR) is a generalized interval-valued fuzzy approximation space, $U = \{x_1, x_2, x_3\}$, and

$$\begin{aligned}IR &= \begin{pmatrix}[0.9,1] & [0.6,0.6] & [0.1,0.6]\\ [0.6,0.6] & [0.8,0.9] & [0.1,0.9]\\ [0.1,0.6] & [0.1,0.9] & [0.1,1]\end{pmatrix},\\ A &= \frac{[0.3,0.4]}{x_1} + \frac{[0.4,0.6]}{x_2} + \frac{[0.2,0.8]}{x_3},\\ \overline{RIF}'(A) &= \frac{[0.4,0.6]}{x_1} + \frac{[0.4,0.8]}{x_2} + \frac{[0.1,0.8]}{x_3},\\ \underline{RIF}'\left(\overline{RIF}'(A)\right) &= \frac{[0.4,0.6]}{x_1} + \frac{[0.1,0.6]}{x_2} + \frac{[0.1,0.9]}{x_3}.\end{aligned} \tag{31}$$

From Definition 11, IR is symmetric and Euclidean, but $\overline{RIF}'(A) \subseteq \underline{RIF}'(\overline{RIF}'(A))$ and $A \subseteq \underline{RIF}'(\overline{RIF}'(A))$ do not hold. According to the duality, $\overline{RIF}'(\underline{RIF}'(A)) \subseteq \underline{RIF}'(A)$ and $\overline{RIF}'(\underline{RIF}'(A)) \subseteq A$ are not true.

Theorem 44. *Let (U, IR) be a generalized interval-valued fuzzy approximation space; then the following conditions are equivalent:*

(1) $IR(x,z) \leq \wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,z)), x, z \in U$;

(2) $\overline{RIF}'(\underline{RIF}'(A)) \subseteq \underline{RIF}'(A), A \in F^I(U)$;

(3) $\overline{RIF}'(A) \subseteq \underline{RIF}'(\overline{RIF}'(A)), A \in F^I(U)$.

Proof. (2) $\Rightarrow$ (1) For all $x, z \in U$, we have

$$\begin{aligned}&\overline{RIF}'\left(\underline{RIF}'\left(1_{U\setminus\{z\}}\right)\right)(x)\\ &= \vee\left\{\alpha \in [I] : x \in \overline{RIF}'_\alpha\left(\underline{RIF}'\left(1_{U\setminus\{z\}}\right)\right)_\alpha\right\}\\ &= \vee\left\{\alpha \in [I] : IR_\alpha(x) \cap \left(\underline{RIF}'\left(1_{U\setminus\{z\}}\right)\right)_\alpha \neq \emptyset\right\}\\ &= \vee\left\{\alpha \in [I] : \exists y \in U, IR(x,y) \geq \alpha, \underline{RIF}'\left(1_{U\setminus\{z\}}\right)(y) \geq \alpha\right\}\\ &= \vee\left\{\alpha \in [I] : \exists y \in U, IR(x,y) \geq \alpha, 1 - IR(y,z) \geq \alpha\right\}\\ &= \vee\left\{\alpha \in [I] : \exists y \in U, IR(x,y) \wedge (1 - IR(y,z)) \geq \alpha\right\}\\ &= \bigvee_{y\in U}\left(IR(x,y) \wedge (1 - IR(y,z))\right).\end{aligned} \tag{32}$$

By Lemma 35 (2), we get $\underline{RIF}'(1_{U\setminus\{z\}})(x) = 1 - IR(x,z)$. Hence, $\vee_{y\in U}(IR(x,y) \wedge (1 - IR(y,z))) \leq 1 - IR(x,z)$.

At the same time, we have

$$\begin{aligned}IR(x,z) &\leq 1 - \bigvee_{y\in U}\left(IR(x,y) \wedge (1 - IR(y,z))\right)\\ &= [1,1] - \left[\bigvee_{y\in U}\left(IR(x,y)^- \wedge \left(1 - IR(y,z)^+\right)\right)\right.,\end{aligned}$$

$$\bigvee_{y\in U}\left(IR(x,y)^{+}\wedge\left(1-IR(y,z)^{-}\right)\right)\Bigg]$$
$$=\Bigg[1-\bigvee_{y\in U}\left(IR(x,y)^{+}\wedge\left(1-IR(y,z)^{-}\right)\right),$$
$$1-\bigvee_{y\in U}\left(IR(x,y)^{-}\wedge\left(1-IR(y,z)^{+}\right)\right)\Bigg]$$
$$=\Bigg[\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{+}\right)\vee IR(y,z)^{-}\right),$$
$$\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{-}\right)\vee IR(y,z)^{+}\right)\Bigg]$$
$$=\bigwedge_{y\in U}\left(\left(1-IR(x,y)\right)\vee IR(y,z)\right). \tag{33}$$

Therefore (1) has been proven.

(1) $\Rightarrow$ (3) First we prove that

$$\overline{RIF}'_{\alpha}(A_{\alpha})\subseteq \underline{RIF}'_{\overline{1-\alpha}}\left(\overline{RIF}'_{\alpha}(A_{\alpha})\right). \tag{34}$$

Note that, for all $x\in U$, if $x\in \overline{RIF}'_{\alpha}(A_{\alpha})$, then $IR_{\alpha}(x)\cap A_{\alpha}\neq\emptyset$; that is, there exists $z\in IR_{\alpha}(x)\cap A_{\alpha}$. So we have $IR(x,z)\geq\alpha$ and $A(z)\geq\alpha$.

Since

$$\alpha\leq IR(x,z)\leq\bigwedge_{y\in U}\left(\left(1-IR(x,y)\right)\vee IR(y,z)\right)$$
$$=\Bigg[\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{+}\right)\vee IR(y,z)^{-}\right),$$
$$\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{-}\right)\vee IR(y,z)^{+}\right)\Bigg], \tag{35}$$

we have

$$\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{+}\right)\vee IR(y,z)^{-}\right)\geq\alpha^{-},$$
$$\bigwedge_{y\in U}\left(\left(1-IR(x,y)^{-}\right)\vee IR(y,z)^{+}\right)\geq\alpha^{+}. \tag{36}$$

Hence, for all $y\in U$, $IR(x,y)^{+}\leq 1-\alpha^{-}$ or $IR(y,z)^{-}\geq\alpha^{-}$ and $IR(x,y)^{-}\leq 1-\alpha^{+}$ or $IR(y,z)^{+}\geq\alpha^{+}$ imply that if $IR(x,y)^{+}>1-\alpha^{-}$, then $IR(y,z)^{-}\geq\alpha^{-}$, and if $IR(x,y)^{-}>1-\alpha^{+}$, then $IR(y,z)^{+}\geq\alpha^{+}$. It follows that $IR(x,y)^{+}>1-\alpha^{-}$ and $IR(x,y)^{-}>1-\alpha^{+}$ imply that $IR(y,z)^{-}\geq\alpha^{-}$ and $IR(y,z)^{+}\geq\alpha^{+}$; therefore, $IR(x,y)>1-\alpha$ implies that $IR(y,z)\geq\alpha$, because $IR(x,y)>1-\alpha$ and $IR(y,z)\geq\alpha$ are equivalent to $y\in IR_{\overline{1-\alpha}}(x)$ and $z\in IR_{\alpha}(y)$, respectively. If $y\in IR_{\overline{1-\alpha}}(x)$, then $z\in IR_{\alpha}(y)$, and since $A(z)\geq\alpha$, $y\in IR_{\overline{1-\alpha}}(x)$ implies that $z\in IR_{\alpha}(y)\cap A_{\alpha}$; that is, $IR_{\alpha}(y)\cap A_{\alpha}\neq\emptyset$. So $y\in IR_{\overline{1-\alpha}}(x)$ implies that $y\in\overline{RIF}'_{\alpha}(A_{\alpha})$.

For arbitrary y, it follows that $IR_{\overline{1-\alpha}}(x)\subseteq\overline{RIF}'_{\alpha}(A_{\alpha})$. Hence, $x\in\underline{RIF}'_{\overline{1-\alpha}}(\overline{RIF}'_{\alpha}(A_{\alpha}))$. Then, for arbitrary x, we obtain $\overline{RIF}'_{\alpha}(A_{\alpha})\subseteq\underline{RIF}'_{\overline{1-\alpha}}(\overline{RIF}'_{\alpha}(A_{\alpha}))$.

By Lemma 37, it follows that

$$\begin{aligned}\underline{RIF}'\left(\overline{RIF}'(A)\right)&=\bigcup_{\alpha\in[I]}\alpha\underline{RIF}'_{\overline{1-\alpha}}\left(\left(\overline{RIF}'(A)\right)_{\alpha}\right)\\&\supseteq\bigcup_{\alpha\in[I]}\alpha\underline{RIF}'_{\overline{1-\alpha}}\left(\overline{RIF}'_{\alpha}(A_{\alpha})\right)\\&\supseteq\bigcup_{\alpha\in[I]}\alpha\overline{RIF}'_{\alpha}(A_{\alpha})\\&=\overline{RIF}'(A).\end{aligned} \tag{37}$$

Therefore $\overline{RIF}'(A)\subseteq\underline{RIF}'(\overline{RIF}'(A))$.

(2) $\Leftrightarrow$ (3) This conclusion follows immediately from the duality. □

Theorem 45. *Let (U,IR) be a generalized interval-valued fuzzy approximation space; then the following conditions are equivalent:*

(1) $\wedge_{y\in U}((1-IR(x,y))\vee IR(y,x))=1, x\in U$;

(2) $IR(x,y)=0$ *or* $IR(y,x)=1, x,y\in U$;

(3) $\overline{RIF}'(\underline{RIF}'(A))\subseteq A, A\in F^{I}(U)$;

(4) $A\subseteq\underline{RIF}'(\overline{RIF}'(A)), A\in F^{I}(U)$.

Proof. (1) $\Leftrightarrow$ (2) We observe that, for all $x\in U$, $\wedge_{y\in U}((1-IR(x,y))\vee IR(y,x))=1$ if and only if $((1-IR(x,y)^{+})\vee IR(y,x)^{-})=1$ and $((1-IR(x,y)^{-})\vee IR(y,x)^{+})=1$; namely, for all $x\in U$, $IR(x,y)^{+}=0$ or $IR(y,x)^{-}=1$ and $IR(x,y)^{-}=0$ or $IR(y,x)^{+}=1$.

On the one hand, if $IR(x,y)^{+}=0$, then $IR(x,y)^{-}=0$; we have $IR(x,y)=0$. If $IR(x,y)^{+}\neq 0$, then $IR(y,x)^{-}=1$ and $IR(y,x)^{+}=1$; we have $IR(y,x)=1$.

Hence, $\wedge_{y\in U}((1-IR(x,y))\vee IR(y,x))=1$ if and only if for all $y\in U$, $IR(x,y)=0$ or $IR(y,x)=1$.

(2) $\Rightarrow$ (4) We first prove that $A_{\alpha}\subseteq\underline{RIF}'_{\overline{1-\alpha}}(\overline{RIF}'_{\alpha}(A_{\alpha}))$.

Suppose that $x\in A_{\alpha}$, for all $y\in U$, $IR(x,y)=0$ or $IR(y,x)=1$ if and only if $IR(x,y)\neq 0$ deduces $IR(y,x)=1$. It follows that if $IR(x,y)>1-\alpha$, then $IR(y,x)\geq\alpha$; that is, $x\in IR_{\alpha}(y)$. Further since $x\in A_{\alpha}$, $IR(x,y)>1-\alpha$ implies that $x\in IR_{\alpha}(y)\cap A_{\alpha}$. So $IR_{\alpha}(y)\cap A_{\alpha}\neq\emptyset$. Note that $IR(x,y)>1-\alpha$ and $IR_{\alpha}(y)\cap A_{\alpha}\neq\emptyset$ are equivalent to $y\in IR_{\overline{1-\alpha}}(x)$ and $y\in\overline{RIF}'_{\alpha}(A_{\alpha})$, respectively; we have if $IR(x,y)>1-\alpha$, then $x\in IR_{\alpha}(y)\cap A_{\alpha}$. It shows that if $y\in IR_{\overline{1-\alpha}}(x)$, then $y\in\overline{RIF}'_{\alpha}(A_{\alpha})$. By the arbitrary y, $IR_{\overline{1-\alpha}}(x)\subseteq\overline{RIF}'_{\alpha}(A_{\alpha})$ holds; namely, $x\in\underline{RIF}'_{\overline{1-\alpha}}(\overline{RIF}'_{\alpha}(A_{\alpha}))$. For any x, $A_{\alpha}\in\underline{RIF}'_{\overline{1-\alpha}}(\overline{RIF}'_{\alpha}(A_{\alpha}))$ holds.

On the other hand, in view of Lemma 37, we have

$$\begin{aligned}\underline{RIF}'\left(\overline{RIF}'(A)\right) &= \bigcup_{\alpha\in[I]} \alpha \underline{RIF}'_{1-\alpha}\left(\left(\overline{RIF}'(A)\right)_\alpha\right)\\ &\supseteq \bigcup_{\alpha\in[I]} \alpha \underline{RIF}'_{1-\alpha}\left(\overline{RIF}'_\alpha(A_\alpha)\right)\\ &\supseteq \bigcup_{\alpha\in[I]} \alpha A_\alpha\\ &= A.\end{aligned} \tag{38}$$

(3) $\Rightarrow$ (1) For all $x, z \in U$, from the proof of "(2) $\Rightarrow$ (1)" in Theorem 44, we know that $\vee_{y\in U}(IR(x,y)\wedge(1-IR(y,z))) = \overline{RIF}'(\underline{RIF}'(1_{U/\{x\}}))(z)$.

Furthermore, since $\overline{RIF}'(\underline{RIF}'(1_{U/\{x\}}))(z) \leq 1_{U/\{x\}}(z)$, it follows that $\vee_{y\in U}(IR(x,y) \wedge (1 - IR(y,z))) \leq 1_{U/\{x\}}(z)$; namely, $\wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,z)) \geq 1 - 1_{U/\{x\}}(z)$.

When $z = x$, we have $\wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,x)) \geq 1$. Because the value of $\wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,x))$ is restricted in $[I]$, we have $\wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,x)) = 1$.

(3) $\Leftrightarrow$ (4) This conclusion follows immediately from the duality. □

Theorem 46. *Let (U, IR) be a generalized interval-valued fuzzy approximation space.*

(1) *If IR is reflexive and transitive, then $\underline{RIF}'(A) = \underline{RIF}'(\underline{RIF}'(A))$ and $\overline{RIF}'(A) = \overline{RIF}'(\overline{RIF}'(A))$, for all $A \in F^I(U)$.*

(2) *If IR is reflexive and $IR(x,z) \leq \wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,z))$, for all $x, z \in U$, then $\underline{RIF}'(A) = \overline{RIF}'(\underline{RIF}'(A))$ and $\overline{RIF}'(A) = \underline{RIF}'(\overline{RIF}'(A))$, for all $A \in F^I(U)$.*

Proof. Theorem 46 is proved easily by Theorems 34, 41, and 44. □

According to duality and Theorem 46, one can obtain the next corollary.

Corollary 47. *Suppose that (U, IR) is a generalized interval-valued fuzzy approximation space.*

(1) *If IR is reflexive and transitive, then*

$$\begin{aligned}&\sim\underline{RIF}'(A) = \overline{RIF}'\left(\sim\underline{RIF}'(A)\right) = \overline{RIF}'\left(\overline{RIF}'(\sim A)\right), \quad \forall A \in F^I(U),\\ &\sim\overline{RIF}'(A) = \underline{RIF}'\left(\sim\overline{RIF}'(A)\right) = \underline{RIF}'\left(\underline{RIF}'(\sim A)\right), \quad \forall A \in F^I(U).\end{aligned} \tag{39}$$

(2) *If IR is reflexive and $IR(x,z) \leq \wedge_{y\in U}((1 - IR(x,y)) \vee IR(y,z))$, for all $x, z \in U$, then*

$$\begin{aligned}&\sim\underline{RIF}'(A) = \underline{RIF}'\left(\sim\underline{RIF}'(A)\right) = \underline{RIF}'\left(\overline{RIF}'(\sim A)\right), \quad \forall A \in F^I(U),\\ &\sim\overline{RIF}'(A) = \overline{RIF}'\left(\sim\overline{RIF}'(A)\right) = \overline{RIF}'\left(\underline{RIF}'(\sim A)\right), \quad \forall A \in F^I(U).\end{aligned} \tag{40}$$

6. Conclusion and Future Work

In this paper, we proposed two types of the generalized interval-valued fuzzy approximation operators by integrating the generalized rough set theory and interval-valued fuzzy sets as well as fuzzy relations. The equivalence of these two types of the generalized interval-valued fuzzy approximation operators has been examined. Furthermore, we also demonstrated the duality of the lower and upper generalized interval-valued fuzzy approximation operators and discussed the properties of the generalized interval-valued fuzzy approximation operators under different interval-valued fuzzy relations.

In this paper, one can prove that the binary relation obtained by calculating α-cut set or strong α-cut set to an interval-valued fuzzy relation, for all $\alpha \in [I]$, still satisfies the corresponding definition of Definition 11 under the classical binary relation; that is, if IR is reflexive, symmetric, and transitive, respectively, then $IR(IRa)$ is reflexive, symmetric, and transitive, respectively, under the classical binary relation. Thus, if IR can satisfy the above functions, this technology can be applied in reasoning, learning, and decision-making. In Sections 4 and 5, the definitions and theorems provide some theoretical bases for reasoning, learning, and decision-making.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is supported by the National Natural Science Foundation of China under Grant no. 61273018, the Foundation and Advanced Technology Research Program of Henan Province of China under Grant no. 132300410174, and the Key Scientific and Technological Project of Education Department of Henan Province of China under Grant no. 14A520082.

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Hindawi Publishing Corporation
The Scientific World Journal
Volume 2014, Article ID 382439, 12 pages
http://dx.doi.org/10.1155/2014/382439

Research Article

δ-Cut Decision-Theoretic Rough Set Approach: Model and Attribute Reductions

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Received 5 May 2014; Accepted 6 July 2014; Published 22 July 2014

Academic Editor: Weihua Xu

Decision-theoretic rough set is a quite useful rough set by introducing the decision cost into probabilistic approximations of the target. However, Yao's decision-theoretic rough set is based on the classical indiscernibility relation; such a relation may be too strict in many applications. To solve this problem, a δ-cut decision-theoretic rough set is proposed, which is based on the δ-cut quantitative indiscernibility relation. Furthermore, with respect to criterions of decision-monotonicity and cost decreasing, two different algorithms are designed to compute reducts, respectively. The comparisons between these two algorithms show us the following: (1) with respect to the original data set, the reducts based on decision-monotonicity criterion can generate more rules supported by the lower approximation region and less rules supported by the boundary region, and it follows that the uncertainty which comes from boundary region can be decreased; (2) with respect to the reducts based on decision-monotonicity criterion, the reducts based on cost minimum criterion can obtain the lowest decision costs and the largest approximation qualities. This study suggests potential application areas and new research trends concerning rough set theory.

1. Introduction

Decision-theoretic rough set (DTRS) was proposed by Yao et al. in the early 1990s [1, 2]. Decision-theoretic rough set introduces Bayesian decision procedure and loss function into rough set. In decision-theoretic rough set, the pair of thresholds α and β, which are used to describe the tolerance of approximations, can be directly calculated by minimizing the decision costs with Bayesian theory. Following Yao's pioneer works, many theoretical and applied results related to decision-theoretic rough set have been obtained; see [3–13] for more details.

In decision-theoretic rough set, Pawlak's indiscernibility relation is a basic concept [14–19], and it is an intersection of some equivalence relations in knowledge base. It should be noticed that, in [20], Zhao et al. have made a further investigation about indiscernibility relation and proposed another two indiscernibility relations, which are referred to as weak indiscernibility and δ-cut quantitative indiscernibility relations, respectively. Correspondingly, Pawlak's indiscernibility relation is called the strong indiscernibility relation. By comparing such three binary relations, it is proven that the δ-cut quantitative indiscernibility relation is a generalization of both strong and weak indiscernibility relations. Therefore, it is interesting to construct δ-cut decision-theoretic rough set based on δ-cut quantitative indiscernibility relation. This is what will be discussed in this paper.

Furthermore, attribute reduction is one of the most fundamental and important topics in rough set theory and has drawn attention from many researchers. As far as attribute reduction in decision-theoretic rough set, the properties of nonmonotonicity and decision cost should be concerned. (1) On the one hand, as we all know, in Pawlak's rough set model, the positive region is monotonic with respect to the set inclusion of attributes. However, the monotonicity property of the decision regions with respect to the set inclusion of

attributes does not hold in the decision-theoretic rough set model [21, 22]. To fill such a gap, Yao and Zhao proposed the definition of decision-monotonicity criterion based attribute reduction [23]; (2) on the other hand, decision cost is a very important notion in decision-theoretic rough set model; to deal with the minimal decision cost, Jia et al. proposed a fitness function and designed a heuristic algorithm [24].

As a generalization of decision-theoretic rough set, in our δ-cut decision-theoretic rough set, we conduct the attribute reductions from above two aspects. Firstly, we introduce the notion of decision-monotonicity criterion into attribute reduction and design a significance to measure attributes; secondly, to deal with the minimum decision cost problem, we regard it as an optimization problem and apply the generic algorithm to obtain a reduct with the lowest decision cost.

To facilitate our discussions, we present the basic knowledge, such as Pawlak's rough set, δ-cut quantitative rough set, and Yao's decision-theoretic rough set in Sections 2 and 3. In Section 4, we propose a new δ-cut decision-theoretic rough set and present several related properties. In Section 5, we discuss the attribute reductions by considering two criterions. The paper ends with conclusions in Section 6.

2. Indiscernibility Relations and Rough Sets

2.1. Strong Indiscernibility Relation. An information system is a pair $S = (U, AT)$, in which universe U is a finite set of the objects; AT is a nonempty set of the attributes, such that for all $a \in AT$, and V_a is the domain of a. For all $x \in U$, $a(x)$ denotes the value of x on a. Particularly, when $AT = C \cup D$ and $C \cap D = \emptyset$ (C is the set of conditional attributes and D is the set of decisional attributes), the information system is also called decision system.

Each nonempty subset $A \subseteq AT$ determines a strong indiscernibility relation $IND(A)$ as follows:

$$IND(A) = \left\{(x, y) \in U^2 : a(x) = a(y), \forall a \in A\right\}. \quad (1)$$

A strong indiscernibility relation with respect to A is denoted as $IND(A)$. Two objects in U satisfy $IND(A)$ if and only if they have the same values on all attributes in A; it is an equivalence relation. $IND(A)$ partitions U into a family of disjoint subsets $U/IND(A)$ called a quotient set of U:

$$\frac{U}{IND(A)} = \left\{[x]_A : x \in U\right\}, \quad (2)$$

where $[x]_A$ denotes the equivalence class determined by x with respect to A; that is,

$$[x]_A = \left\{y \in U : (x, y) \in IND(A)\right\}. \quad (3)$$

Definition 1. Let S be an information system, let A be any subset of AT, and let X be any subset of U. The lower approximation of X denoted as $\underline{A}_S(X)$ and the upper approximation of X denoted as $\overline{A}_S(X)$, respectively, are defined by

$$\begin{aligned} \underline{A}_S(X) &= \left\{x \in U : [x]_A \subseteq X\right\}; \\ \overline{A}_S(X) &= \left\{x \in U : [x]_A \cap X \neq \emptyset\right\}. \end{aligned} \quad (4)$$

The pair $[\underline{A}_S(X), \overline{A}_S(X)]$ is referred to as Pawlak's rough set of X with respect to the set of attributes A.

2.2. Weak Indiscernibility Relation. In the definition of strong indiscernibility relation, we can observe that two objects in U satisfy $IND(A)$ if and only if they have the same values on all attributes in A; such case may be too strict to be used in many applications. To address this issue, Zhao and Yao proposed a notion which is called weak indiscernibility relation. The semantic interpretation of weak indiscernibility relation is that two objects are considered as indistinguishable if and only if they have the same values on at least one attribute in A.

In an information system S, for any subset of AT, a weak indiscernibility relation can be defined as follows [20]:

$$WIND(A) = \left\{(x, y) \in U^2 : a(x) = a(y), \exists a \in A\right\}. \quad (5)$$

From the description of the weak indiscernibility relation we can find that a weak indiscernibility relation $WIND(A)$ with respect to A only requires that two objects have the same values on at least one attribute in A. A weak indiscernibility relation is reflexive and symmetric, but not necessarily transitive. Such a relation is known as a compatibility or a tolerance relation.

Definition 2. Let S be an information system; for all $A \subseteq AT$, for all $X \subseteq U$, the lower and upper approximations of X based on weak indiscernibility relation, denoted as $\underline{A}_W(X)$ and $\overline{A}_W(X)$, respectively, are defined by

$$\begin{aligned} \underline{A}_W(X) &= \left\{x \in U : [x]_A^W \subseteq X\right\}; \\ \overline{A}_W(X) &= \left\{x \in U : [x]_A^W \cap X \neq \emptyset\right\}, \end{aligned} \quad (6)$$

where $[x]_A^W = \{y \in U : (x, y) \in WIND(A)\}$ is the set of objects, which are weak indiscernibility with x in terms of set of attributes A.

2.3. δ-Cut Quantitative Indiscernibility Relation. The strong and weak indiscernibility relations represent the two extreme cases, which include many levels of indiscernibility. With respect to a nonempty set of attributes $A \subseteq AT$, a δ-cut quantitative indiscernibility relation is defined as a mapping from $U \times U$ to the unit interval $[0, 1]$.

Definition 3 (see [20]). Let S be an information system; for all $A \subseteq AT$, the δ-cut quantitative indiscernibility relation $ind_\delta(A)$ is defined by

$$ind_\delta(A) = \left\{(x, y) \in U^2 : \frac{\left|\{a \in A : a(x) = a(y)\}\right|}{|A|} \geq \delta\right\}, \quad (7)$$

where $|\cdot|$ denotes the cardinality of a set.

By the definition of δ-cut quantitative indiscernibility relation, we can obtain the lower and upper approximations as in the following definition.

Definition 4. Let S be an information system; for all $A \subseteq AT$, for all $X \subseteq U$, the δ-cut quantitative indiscernibility based lower and upper approximations are denoted by $\underline{A}_\delta(X)$ and $\overline{A}_\delta(X)$, respectively:

$$\begin{aligned} \underline{A}_\delta(X) &= \left\{x \in U : [x]_A^\delta \subseteq X\right\}; \\ \overline{A}_\delta(X) &= \left\{x \in U : [x]_A^\delta \cap X \neq \emptyset\right\}, \end{aligned} \tag{8}$$

where $[x]_A^\delta = \{y \in U : (x, y) \in ind_\delta(A)\}$ is the set of objects, which are δ-cut indiscernibility with x in terms of set of attributes A.

3. Decision-Theoretic Rough Set

The Bayesian decision procedure deals with making a decision with minimum risk based on observed evidence. Yao and Zhou introduced a more general rough set model called a decision-theoretic rough set (DTRS) model [25–27]. In this section, we briefly introduce the original DTRS model. According to the Bayesian decision procedure, the DTRS model is composed of two states and three actions. The set of states is given by $\Omega = \{X, \sim X\}$ indicating that an object is in X or not, respectively. The probabilities for these two complement states can be denoted as $P(X \mid [x]_A) = |X \cap [x]_A|/|[x]_A|$ and $P(\sim X \mid [x]_A) = 1 - P(X \mid [x]_A)$. The set of actions is given by $\mathscr{A} = \{a_P, a_B, a_N\}$, where a_P, a_B, and a_N represent the three actions in classifying an object x, namely, deciding that x belongs to the positive region, deciding that x belongs to the boundary region, and deciding that$x$ belongs to the negative region, respectively. The loss functions are regarding the risk or cost of actions in different states. Let λ_{PP}, λ_{BP}, and λ_{NP} denote the cost incurred for taking actions a_P, a_B, and a_N, respectively, when an object belongs to X, and let λ_{PN}, λ_{BN}, and λ_{NN} denote the cost incurred for taking the same actions when an object belongs to $\sim X$.

According to the loss functions, the expected costs associated with taking different actions for objects in $[x]_A$ can be expressed as follows:

$$\begin{aligned} R_P &= R(a_P \mid [x]_A) \\ &= \lambda_{PP} \cdot P(X \mid [x]_A) + \lambda_{PN} \cdot P(\sim X \mid [x]_A); \\ R_B &= R(a_B \mid [x]_A) \\ &= \lambda_{BP} \cdot P(X \mid [x]_A) + \lambda_{BN} \cdot P(\sim X \mid [x]_A); \\ R_N &= R(a_N \mid [x]_A) \\ &= \lambda_{NP} \cdot P(X \mid [x]_A) + \lambda_{NN} \cdot P(\sim X \mid [x]_A). \end{aligned} \tag{9}$$

The Bayesian decision procedure leads to the following minimum-risk decision rules:

(P) if $R_P \leq R_B$ and $R_P \leq R_N$, then this decides that x belongs to the positive region;

(B) if $R_B \leq R_P$ and $R_B \leq R_N$, then this decides that x belongs to the boundary region;

(N) if $R_N \leq R_P$ and $R_N \leq R_B$, then this decides that x belongs to the negative region.

Consider a special kind of loss functions with $\lambda_{PP} \leq \lambda_{BP} \leq \lambda_{NP}$ and $\lambda_{NN} \leq \lambda_{BN} \leq \lambda_{PN}$; that is to say, the loss of classifying an object x belonging to X into the positive region is no more than the loss of classifying x into the boundary region, and both of these losses are strictly less than the loss of classifying x into the negative region. The reverse order of losses is used for classifying an object not in X. We further assume that a loss function satisfies the following condition:

$$(\lambda_{PN} - \lambda_{BN}) \cdot (\lambda_{NP} - \lambda_{BP}) > (\lambda_{BP} - \lambda_{PP}) \cdot (\lambda_{BN} - \lambda_{NN}). \tag{10}$$

Based on the above two assumptions, we have the following simplified rules:

($P1$) if $P(X \mid [x]_A) \geq \alpha$, then this decides that x belongs to the positive region;

($B1$) if $\beta < P(X \mid [x]_A) < \alpha$, then this decides that x belongs to the boundary region;

($N1$) if $P(X \mid [x]_A) \leq \beta$, then this decides that x belongs to the negative region,

where

$$\begin{aligned} \alpha &= \frac{\lambda_{PN} - \lambda_{BN}}{(\lambda_{PN} - \lambda_{BN}) + (\lambda_{BP} - \lambda_{PP})}; \\ \beta &= \frac{\lambda_{BN} - \lambda_{NN}}{(\lambda_{BN} - \lambda_{NN}) + (\lambda_{NP} - \lambda_{BP})}, \end{aligned} \tag{11}$$

with $1 \geq \alpha \geq \beta \geq 0$.

Using these three decision rules, for all $A \subseteq AT$ and for all $X \subseteq U$, we get the following probabilistic approximations:

$$\begin{aligned} \underline{A}_{(\alpha,\beta)}(X) &= \{x \in U : P(X \mid [x]_A) \geq \alpha\}; \\ \overline{A}_{(\alpha,\beta)}(X) &= \{x \in U : P(X \mid [x]_A) > \beta\}. \end{aligned} \tag{12}$$

The pair $[\underline{A}_{(\alpha,\beta)}(X), \overline{A}_{(\alpha,\beta)}(X)]$ is referred to as decision-theoretic rough set of X with respect to the set of attributes A. Therefore, the positive region of X can be expressed as $\mathrm{POS}_{(\alpha,\beta)}(X) = \underline{A}_{(\alpha,\beta)}(X)$, the boundary region of X is $\mathrm{BND}_{(\alpha,\beta)}(X) = \overline{A}_{(\alpha,\beta)}(X) - \underline{A}_{(\alpha,\beta)}(X)$, and the negative region of X is $\mathrm{NEG}_{(\alpha,\beta)}(X) = U - \overline{A}_{(\alpha,\beta)}(X)$.

4. δ-Cut Decision-Theoretic Rough Set

As the discussion in Section 3, we can observe that the classical decision-theoretic rough set is based on the strong indiscernibility relation which is too strict since it requires that the two objects have the same values on all attributes. In this section, we introduce the concept of δ-cut indiscernibility relation into the decision-theoretic rough set model.

4.1. Definition of δ-Cut Decision-Theoretic Rough Set

Definition 5. Let S be an information system; for all $A \subseteq AT$, for all $X \subseteq U$, the decision-theoretic lower and upper approximations based on the δ-cut quantitative

indiscernibility relation, denoted as $\underline{A}^{\delta}_{(\alpha,\beta)}(X)$ and $\overline{A}^{\delta}_{(\alpha,\beta)}(X)$, respectively, are defined by

$$\begin{aligned} \underline{A}^{\delta}_{(\alpha,\beta)}(X) &= \left\{x \in U : P\left(X \mid [x]^{\delta}_{A}\right) \geq \alpha\right\}; \\ \overline{A}^{\delta}_{(\alpha,\beta)}(X) &= \left\{x \in U : P\left(X \mid [x]^{\delta}_{A}\right) > \beta\right\}. \end{aligned} \tag{13}$$

The pair $[\underline{A}^{\delta}_{(\alpha,\beta)}(X), \overline{A}^{\delta}_{(\alpha,\beta)}(X)]$ is referred to as a δ-cut decision-theoretic rough set of X with respect to the set of attributes A.

After obtaining the lower and upper approximations, the probabilistic positive, boundary, and negative regions are defined by

$$\begin{aligned} \mathrm{POS}^{\delta}_{(\alpha,\beta)}(X) &= \underline{A}^{\delta}_{(\alpha,\beta)}(X); \\ \mathrm{BND}^{\delta}_{(\alpha,\beta)}(X) &= \overline{A}^{\delta}_{(\alpha,\beta)}(X) - \underline{A}^{\delta}_{(\alpha,\beta)}(X); \\ \mathrm{NEG}^{\delta}_{(\alpha,\beta)}(X) &= U - \mathrm{POS}^{\delta}_{(\alpha,\beta)}(X) \cup \mathrm{BND}^{\delta}_{(\alpha,\beta)}(X) \\ &= U - \overline{A}^{\delta}_{(\alpha,\beta)}(X). \end{aligned} \tag{14}$$

Let DS be a decision system and let $\pi_D = \{D_1, D_2, \ldots, D_t\}$ be a partition of the universe U, which is defined by the decision attribute D, representing t classes. By the definition of quantitative decision-theoretic rough set, the lower and upper approximations of the partition can be expressed as follows:

$$\begin{aligned} \underline{A}^{\delta}_{(\alpha,\beta)}(\pi_D) &= \left(\underline{A}^{\delta}_{(\alpha,\beta)}(D_1), \underline{A}^{\delta}_{(\alpha,\beta)}(D_2), \ldots, \underline{A}^{\delta}_{(\alpha,\beta)}(D_t)\right); \\ \overline{A}^{\delta}_{(\alpha,\beta)}(\pi_D) &= \left(\overline{A}^{\delta}_{(\alpha,\beta)}(D_1), \overline{A}^{\delta}_{(\alpha,\beta)}(D_2), \ldots, \overline{A}^{\delta}_{(\alpha,\beta)}(D_t)\right). \end{aligned} \tag{15}$$

For this t-classes problem, it can be regarded as t two-class problems; following this approach, the positive region, boundary region, and negative region of all the decision classes can be expressed as follows:

$$\begin{aligned} \mathrm{POS}^{\delta}_{(\alpha,\beta)}(\pi_D) &= \bigcup_{i=1}^{t} \mathrm{POS}^{\delta}_{(\alpha,\beta)}(D_i); \\ \mathrm{BND}^{\delta}_{(\alpha,\beta)}(\pi_D) &= \bigcup_{i=1}^{t} \mathrm{BND}^{\delta}_{(\alpha,\beta)}(D_i); \\ \mathrm{NEG}^{\delta}_{(\alpha,\beta)}(\pi_D) &= U - \mathrm{POS}^{\delta}_{(\alpha,\beta)}(\pi_D) \cup \mathrm{BND}^{\delta}_{(\alpha,\beta)}(\pi_D). \end{aligned} \tag{16}$$

Based on the notions of the three regions in δ-cut decision-theoretic rough set model, three important rules should be concerned, that is, positive rule, boundary rule, and negative rule. Similar to Yao's decision-theoretic rough set, when $\alpha > \beta$, for all $D_i \in \pi_D$, we can obtain the following decision rules, that is, tie-break:

(δ-P) if $P(D_i \mid [x]^{\delta}_{A}) \geq \alpha$, then this decides that $x \in \mathrm{POS}^{\delta}_{(\alpha,\beta)}(D_i)$;

(δ-B) if $\beta < P(D_i \mid [x]^{\delta}_{A}) < \alpha$, then this decides that $x \in \mathrm{BND}^{\delta}_{(\alpha,\beta)}(D_i)$;

(δ-N) if $P(D_i \mid [x]^{\delta}_{A}) \leq \beta$, then this decides that $x \in \mathrm{NEG}^{\delta}_{(\alpha,\beta)}(D_i)$.

Let DS be a decision system, $\delta \in (0, 1]$; for all $D_i \in \pi_D$, the Bayesian expected costs of decision rules can be expressed as follows:

(i) (δ-P) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{POS}(D_i)} (\lambda_{PP} \cdot P(D_i \mid [x]^{\delta}_{A}) + \lambda_{PN} \cdot P(\sim D_i \mid [x]^{\delta}_{A}))$;

(ii) (δ-N) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{NEG}(D_i)} (\lambda_{NP} \cdot P(D_i \mid [x]^{\delta}_{A}) + \lambda_{NN} \cdot P(\sim D_i \mid [x]^{\delta}_{A}))$;

(iii) (δ-B) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{BND}(D_i)} (\lambda_{BP} \cdot P(D_i \mid [x]^{\delta}_{A}) + \lambda_{BN} \cdot P(\sim D_i \mid [x]^{\delta}_{A}))$.

Considering the special case where we assume zero cost for a correct classification, that is, $\lambda_{PP} = \lambda_{NN} = 0$, the decision costs of rules can be simply expressed as follows:

(i) (δ-$P1$) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{POS}(D_i)} \lambda_{PN} \cdot P(\sim D_i \mid [x]^{\delta}_{A})$;

(ii) (δ-$N1$) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{NEG}(D_i)} \lambda_{NP} \cdot P(D_i \mid [x]^{\delta}_{A})$;

(iii) (δ-$B1$) cost: $\sum_{D_i \in \pi_D} \sum_{x \in \mathrm{BND}(D_i)} (\lambda_{BP} \cdot P(D_i \mid [x]^{\delta}_{A}) + \lambda_{BN} \cdot P(\sim D_i \mid [x]^{\delta}_{A}))$.

For any subset of conditional attributes, the overall cost of all decision rules can be denoted as $\mathrm{COST}(A)$, such that

$$\begin{aligned} \mathrm{COST}(A) &= \mathrm{COST}^{\mathrm{POS}}_{A} + \mathrm{COST}^{\mathrm{NEG}}_{A} + \mathrm{COST}^{\mathrm{BND}}_{A} \\ &= \sum_{D_i \in \pi_D} \sum_{x \in \mathrm{POS}(D_i)} \lambda_{PN} \cdot P\left(\sim D_i \mid [x]^{\delta}_{A}\right) \\ &\quad + \sum_{D_i \in \pi_D} \sum_{x \in \mathrm{NEG}(D_i)} \lambda_{NP} \cdot P\left(D_i \mid [x]^{\delta}_{A}\right) \\ &\quad + \sum_{D_i \in \pi_D} \sum_{x \in \mathrm{BND}(D_i)} \left(\lambda_{BP} \cdot P\left(D_i \mid [x]^{\delta}_{A}\right) \right. \\ &\quad \left. + \lambda_{BN} \cdot P\left(\sim D_i \mid [x]^{\delta}_{A}\right)\right). \end{aligned} \tag{17}$$

4.2. Related Properties

Proposition 6. *Let S be an information system; if $\lambda_{PN} = \lambda_{NP} = 1$ and $\lambda_{PP} = \lambda_{NN} = \lambda_{BP} = \lambda_{BN} = 0$, $\forall X \subseteq U$, one has*

$$\begin{aligned} \underline{A}^{\delta}_{(\alpha,\beta)}(X) &= \underline{A}_{\delta}(X); \\ \overline{A}^{\delta}_{(\alpha,\beta)}(X) &= \overline{A}_{\delta}(X). \end{aligned} \tag{18}$$

Proof. In this proposition, we suppose that there is a unit misclassification cost if an object in X is classified into the negative region or if an object in $\sim X$ is classified into the positive region; otherwise there is no cost; that is, $\lambda_{PN} = \lambda_{NP} = 1$ and $\lambda_{PP} = \lambda_{NN} = \lambda_{BP} = \lambda_{BN} = 0$. By the computational processes of α and β, we have $\alpha = 1$ and

$\beta = 0$ and by the definition of δ-cut decision-theoretic rough set, we can observe that

$$\begin{aligned}\underline{A}^{\delta}_{(\alpha,\beta)}(X) &= \left\{x \in U : P\left(X \mid [x]^{\delta}_{A}\right) \geq 1\right\} \\ &= \left\{x \in U : \frac{\left|X \cap [x]^{\delta}_{A}\right|}{\left|[x]^{\delta}_{A}\right|} \geq 1\right\} \\ &= \left\{x \in U : [x]^{\delta}_{A} \subseteq X\right\} \\ &= \underline{A}_{\delta}(X).\end{aligned} \tag{19}$$

Similarly, it is not difficult to prove $\overline{A}^{\delta}_{(\alpha,\beta)}(X) = \overline{A}_{\delta}(X)$. □

Proposition 7. *Let S be an information system; for all $A \subseteq AT$, for all $X \subseteq U$, one has*

$$\begin{aligned}&\underline{A}^{\delta}_{(\alpha,\beta)}(X) \supseteq \underline{A}_{\delta}(X); \\ &\overline{A}^{\delta}_{(\alpha,\beta)}(X) \subseteq \overline{A}_{\delta}(X).\end{aligned} \tag{20}$$

Proof. For all $x \in \underline{A}_{\delta}(X)$ and by Definition 4, we have $[x]^{\delta}_{A} \subseteq X$; that is to say, $P(X \mid [x]^{\delta}_{A}) = |[x]^{\delta}_{A} \cap X|/|[x]^{\delta}_{A}| = 1$; since $\alpha \in (0, 1]$, then $P(X \mid [x]^{\delta}_{A}) \geq \alpha$; by the probability, we have that $x \in \underline{A}^{\delta}_{(\alpha,\beta)}(X)$ holds obviously, and it follows that $\underline{A}^{\delta}_{(\alpha,\beta)}(X) \supseteq \underline{A}_{\delta}(X)$.

Similarly, it is not difficult to prove $\overline{A}^{\delta}_{(\alpha,\beta)}(X) \subseteq \overline{A}_{\delta}(X)$. □

Propositions 6 and 7 show the relationships between δ-cut decision-theoretic rough set and classical δ-cut quantitative rough set. The details are given as follows: the classical δ-cut quantitative indiscernibility lower approximation is included into the δ-cut decision-theoretic lower approximation and the δ-cut decision-theoretic upper approximation is included into the classical δ-cut quantitative indiscernibility upper approximation. Particularly, with some limitations, the δ-cut decision-theoretic rough set can degenerate to the classical δ-cut quantitative rough set. As the discussion above, we can observe that the δ-cut decision-theoretic rough set is a generalization of classical δ-cut quantitative rough set, and it can increase lower approximation and decrease upper approximation.

Proposition 8. *Let S be an information system; if $\delta = 1$, then, for all $A \subseteq AT$, for all $X \subseteq U$, one has*

$$\begin{aligned}&\underline{A}^{\delta}_{(\alpha,\beta)}(X) = \underline{A}_{\alpha,\beta}(X); \\ &\overline{A}^{\delta}_{(\alpha,\beta)}(X) = \overline{A}_{\alpha,\beta}(X).\end{aligned} \tag{21}$$

Proof. It is not difficult to prove this proposition by Definitions 3 and 5 and the definition of decision-theoretic rough set. □

Proposition 8 shows the relationships between δ-cut decision-theoretic rough set and Yao's decision-theoretic rough set. The details are the following: if we set the value of δ with 1, the lower and upper approximations based on our decision-theoretic rough set are equal to those based on Yao's decision-theoretic rough set. By Proposition 8 we can observe that our decision-theoretic rough set is also a generalization of Yao's decision-theoretic rough set.

5. Attribute Reductions in Quantitative Decision-Theoretic Rough Set

5.1. Decision-Monotonicity Criterion Based Reducts. In Pawlak's rough set theory, attribute reduction is an important concept which has been addressed by many researchers all around the world. In classical rough set, the reduct is a minimal subset of attributes which is independent and has the same power as all of the attributes. The positive region, the boundary region, and the negative region are monotonic with respect to the set inclusion of attributes in classical rough set theory. However, in decision-theoretic rough set model, the monotonicity property of the decision regions with respect to the set inclusion of attributes does not hold. To solve such a problem, Yao and Zhao have proposed a decision-monotonicity criterion [23]. The decision-monotonicity criterion requires two things. Firstly, the criterion requires that by reducing attributes a positive rule is still a positive rule of the same decision. Secondly, the criterion requires that by reducing attributes a boundary rule is still a boundary rule or is upgraded to a positive rule with the same decision. Following their work, it is not difficult to introduce the decision-monotonicity criterion into our δ-cut decision-theoretic rough set. The detailed definition is shown in Definition 9 as follows.

Definition 9. Let $DS = (U, C \cup D)$ be a decision system, $\delta \in (0, 1]$, and let A be any subset of conditional attributes; A is referred to as a decision-monotonicity reduct in DS if and only if A is the minimal set of conditional attributes, which preserves $\underline{C}^{\delta}_{(\alpha,\beta)}(D_i) \subseteq \underline{A}^{\delta}_{(\alpha,\beta)}(D_i)$, for each $D_i \in \pi_D$.

Let DS be a decision system, $\delta \in (0, 1]$, and let A be any subset of conditional attributes and $a_i \in A$; we define the following coefficients:

$$\begin{aligned}&DM^{\text{sig}}_{\text{in}}(a_i, A, \delta) \\ &\quad = \frac{\sum_{D_i \in \pi_D}\left(\underline{A}^{\delta}_{(\alpha,\beta)}(D_i) \odot \underline{A - \{a_i\}}^{\delta}_{(\alpha,\beta)}(D_i)\right)}{m \cdot t}; \\ &DM^{\text{sig}}_{\text{out}}(a_i, A, \delta) \\ &\quad = \frac{\sum_{D_i \in \pi_D}\left(\underline{A}^{\delta}_{(\alpha,\beta)}(D_i) \odot \underline{A \cup \{a_i\}}^{\delta}_{(\alpha,\beta)}(D_i)\right)}{m \cdot t},\end{aligned} \tag{22}$$

where m and t are the numbers of objects and decision classes, respectively, and

$$A \odot B = \begin{cases} |B - A| & A \subseteq B, \\ -|A - B| & \text{otherwise}. \end{cases} \tag{23}$$

Input: Decision system $DS = (U, C \cup D)$, threshold δ;
Output: A decision-monotonicity reduct *red*.
Step 1. $B \leftarrow \emptyset$, $M \leftarrow C$, compute $\underline{C}^{\delta}_{(\alpha,\beta)}(D_i)$, $D_i \in \pi_D$;
Step 2. Compute the decision-monotonicity significance for each $a_i \in C$ with $DM^{\text{sig}}_{\text{in}}(a_i, C, \delta)$;
Step 3. $B \leftarrow a_j$ where
$DM^{\text{sig}}_{\text{in}}(a_j, C, \delta) = \max\{DM^{\text{sig}}_{\text{in}}(a_i, C, \delta) : a_i \in C\}$;
Step 4. **While** $M \neq \emptyset$ **do**
$\forall a_i \in C - B$, compute $DM^{\text{sig}}_{\text{out}}(a_i, B, \delta)$;
Select the maximal $DM^{\text{sig}}_{\text{out}}(a_j, B, \delta)$ and corresponding attribute a_j;
If $DM^{\text{sig}}_{\text{out}}(a_j, B, \delta) > 0$
$B = B \cup \{a_j\}$;
End
$M = M - \{a_j\}$;
End
Step 5. $\forall a_i \in B$
If $DM^{\text{sig}}_{\text{in}}(a_i, C, \delta) \geq 0$
$B = B - \{a_i\}$;
End
Step 6. $red = B$.

Algorithm 1: Heuristic algorithm for attribute reduction based on decision-monotonicity criterion.

Input: Decision system $DS = (U, C \cup D)$, threshold δ;
Output: A optimal cost reduct *red*.
Step 1. Create an initial random population (number = 40);
Step 2. Evaluation the population;
Step 3. **While** Number of generations < 100 **do**
Select the fittest chromosomes in the population;
Perform crossover on the selected chromosomes to create offspring;
Perform mutation on the selected chromosomes;
Evaluate the new population;
End
Step 4. Selected the fittest chromosome form current population and output it as *red*.

Algorithm 2: Genetic algorithm for attribute reduction based on cost minimum criterion.

Based on these measures, we can design a heuristic algorithm to compute the decision-monotonicity reduct; the details are shown as in Algorithm 1.

5.2. Cost Minimum Criterion Based Reducts. Cost is one of the important features of the δ-cut decision-theoretic rough set. In Section 4.1 we have discussed the cost issue of our δ-cut decision-theoretic rough set. However, in the reduction process, from the viewpoint of cost criterion, we want to obtain a reduct with smaller or smallest cost. Similar to the decision-monotonicity criterion, it is not difficult to introduce the cost criterion into our rough set model.

Definition 10. Let $DS = (U, C \cup D)$ be a decision system, $\delta \in (0, 1]$, and let A be any subset of conditional attributes; A is referred to as a cost reduct in DS if and only if A is the minimal set of conditional attributes, which satisfies $\text{COST}(A) \leq \text{COST}(C)$, and, for each set $B \subset A$, $\text{COST}(B) > \text{COST}(A)$.

In this definition, we want to find a subset of conditional attributes so that the overall decision cost will be decreased or unchanged based on the reduct. In most situations, it is better for the decider to obtain a smaller or smallest cost in the decision procedure. We propose an optimization problem with the objective of minimizing the cost values; the minimum cost can be denoted as follows [3]:

$$\min \text{COST}(A). \tag{24}$$

Then the optimization problem is described as finding a proper attributes set to make the whole decision cost minimum. Therefore, in the following, we will present a genetic algorithm to compute cost minimum based reducts. The details of genetic algorithm are described in Algorithm 2.

Table 1: Data sets description.

ID	Data sets	Samples	Features	Decision classes
1	Annealing	798	38	5
2	Dermatology	366	34	6
3	Soybean	307	35	4
4	Zoo	101	17	7

Table 2: The decision rules between raw data and decision-monotonicity criterion based reducts (Annealing).

δ	(P) rules		(B) rules		(N) rules	
	Raw	Reduct	Raw	Reduct	Raw	Reduct
0.1	319.2 ± 412.1	**650.4 ± 311.2**	558.6 ± 385.5	**221.4 ± 356.5**	3112 ± 252.3	**3118 ± 233.4**
0.2	399 ± 420.6	**650.4 ± 311.2**	1117 ± 1077	**590.4 ± 582.1**	2473 ± 878.2	**2749 ± 516.0**
0.3	399 ± 420.6	**576.6 ± 356.5**	638.4 ± 733.3	**442.8 ± 622.3**	2952 ± 757.0	**2971 ± 498.1**
0.4	638.4 ± 336.5	**724.2 ± 336.5**	159.6 ± 336.5	**73.80 ± 233.4**	3192 ± 0.000	3192 ± 0.000
0.5	239.4 ± 385.5	**429.0 ± 388.9**	638.4 ± 504.7	**442.8 ± 516.0**	3112 ± 252.3	**3118 ± 233.4**
0.6	319.2 ± 412.1	**583.4 ± 346.6**	798.0 ± 995.3	**503.6 ± 947.5**	2873 ± 770.9	**2903 ± 686.7**
0.7	363.5 ± 384.7	**523.7 ± 310.4**	611.3 ± 519.9	**438.2 ± 480.2**	3015 ± 335.2	**3028 ± 307.4**
0.8	379.3 ± 312.8	**648.9 ± 221.6**	729.8 ± 592.7	**437.9 ± 638.3**	2881 ± 417.8	**2903 ± 519.3**
0.9	713.2 ± 34.81	**727.5 ± 52.72**	161.1 ± 54.24	**119.4 ± 96.12**	3116 ± 39.46	**3143 ± 50.72**
1.0	798 ± 0	798 ± 0	0 ± 0	0 ± 0	**3192 ± 0**	3118 ± 0
Mean values	456.8 ± 311.9	**631.2 ± 253.2**	541.2 ± 519.9	**334.4 ± 470.6**	2992 ± 370.4	**3024 ± 327.8**

Table 3: The decision rules between raw data and decision-monotonicity criterion based reducts (Dermatology).

δ	(P) rules		(B) rules		(N) rules	
	Raw	Reduct	Raw	Reduct	Raw	Reduct
0.1	0 ± 0	**1.100 ± 3.478**	**1134 ± 952.1**	1154 ± 848.9	**1061 ± 952.1**	1041 ± 847.3
0.2	0 ± 0	**0.500 ± 1.269**	**512.4 ± 602.6**	534.7 ± 583.3	**1683 ± 602.6**	1661 ± 582.9
0.3	0 ± 0	**0.200 ± 0.426**	512.3 ± 715.5	**448.8 ± 650.2**	1683 ± 715.5	**1747 ± 650.1**
0.4	0 ± 0	0.000 ± 0.000	878.2 ± 915.9	**860.4 ± 822.4**	1317 ± 915.9	**1335 ± 822.4**
0.5	0 ± 0	**0.4000 ± 0.699**	534.0 ± 703.5	**505.9 ± 643.8**	1662 ± 703.5	**1689 ± 643.5**
0.6	0 ± 0	**2.000 ± 0.943**	688.5 ± 754.3	**671.3 ± 779.7**	1507 ± 754.3	**1522 ± 779.3**
0.7	29.20 ± 5.473	**33.20 ± 12.35**	539.0 ± 522.5	**536.0 ± 525.2**	1627 ± 522.5	1627 ± 521.6
0.8	139.9 ± 5.953	139.9 ± 5.953	458.6 ± 314.6	458.6 ± 314.6	1597 ± 313.8	1597 ± 313.8
0.9	212.9 ± 9.036	219.9 ± 9.036	292.4 ± 124.3	292.4 ± 124.3	1691 ± 117.7	1691 ± 117.7
1.0	328.2 ± 1.932	328.2 ± 1.932	66.8 ± 24.14	66.8 ± 24.14	1801 ± 23.73	1801 ± 23.73
Mean values	71.02 ± 2.239	**71.84 ± 3.608**	561.7 ± 562.9	**552.8 ± 531.6**	1563 ± 562.2	**1571 ± 530.3**

5.3. Experimental Analyses. In this subsection, by experimental analyses, we will illustrate the differences between Algorithms 1 and 2. All the experiments have been carried out on a personal computer with Windows 7, Intel Core 2 DuoT5800 CPU (4.00 GHz), and 4.00 GB memory. The programming language is Matlab 2012b.

We download four public data sets from UCI Repository of Machine Learning Databases, which are described in Table 1. In the experiment, 10 different groups of loss functions are randomly generated.

Tables 2, 3, 4, and 5 show the experimental results of (P) rules, (B) rules, and (N) rules. The number of these rules is equivalent to the number of objects in positive region, boundary region, and negative region, respectively. This is mainly because each object in positive/boundary/negative region can induce a (P)/(B)/(N) decision rule.

Based on these four tables, it is not difficult to draw the following conclusions.

(1) With respect to the original data set, decision-monotonicity reducts can generate more (P) rules; this is mainly because the condition of decision-monotonicity reducts requires that, by reducing attributes, a positive rule is still a positive rule, or a boundary rule is upgraded to a positive rule. This mechanism not only keeps the original (P) rules unchanged, but also increases the (P) rules.

(2) With respect to the original data set, decision-monotonicity reducts can generate less (B) rules; this is mainly because the second condition of decision-monotonicity reducts requires that, by reducing attributes, a boundary rule is still a boundary rule or is upgraded to a positive rule; that is to say, the number of (B) rules may be equal to or less than those of original data set.

In order to compare the differences between decision-monotonicity criterion based reducts and cost minimum criterion based reducts, we conduct the experiments from three aspects, that is, decision costs, approximation qualities, and running times. On the one hand, Figure 1 shows the costs comparisons between these two attribute reduction algorithms; on the other hand, Tables 6, 7, 8, and 9 show the differences between decision-monotonicity criterion based reducts and cost minimum criterion based reducts in approximation qualities and running times, respectively.

In Figure 1, each subfigure is corresponding to a data set. In each subfigure, the x-coordinate pertains to different

Table 4: The decision rules between raw data and decision-monotonicity criterion based reducts (Soybean).

δ	(P) rules		(B) rules		(N) rules	
	Raw	Reduct	Raw	Reduct	Raw	Reduct
0.1	244.3 ± 128.7	**295.7 ± 20.52**	94.4 ± 149.0	**52.1 ± 98.76**	**889.3 ± 96.77**	880.2 ± 95.80
0.2	280.4 ± 43.59	**285.1 ± 30.39**	**85.2 ± 101.5**	93.2 ± 96.18	**862.4 ± 97.91**	849.7 ± 98.14
0.3	277.9 ± 8.212	**282.3 ± 14.11**	76.3 ± 73.26	**69.3 ± 78.64**	873.8 ± 67.97	**876.4 ± 69.86**
0.4	269.2 ± 2.821	**272.6 ± 12.33**	63.8 ± 19.17	**59.0 ± 27.68**	**895.0 ± 17.98**	825.9 ± 19.50
0.5	275.2 ± 5.827	**289.2 ± 15.64**	**81.1 ± 67.07**	112.9 ± 153.8	**871.7 ± 67.63**	825.9 ± 155.0
0.6	304.7 ± 2.312	**305.5 ± 1.581**	9.90 ± 14.65	**9.30 ± 14.96**	**913.4 ± 15.04**	913.2 ± 14.98
0.7	302.9 ± 3.755	**305.7 ± 1.059**	5.70 ± 4.423	**2.80 ± 2.573**	919.4 ± 2.413	**919.5 ± 1.958**
0.8	298.2 ± 1.932	**304.8 ± 4.638**	17.5 ± 4.836	**5.70 ± 9.956**	912.3 ± 3.713	917.5 ± 5.797
0.9	307 ± 0	307 ± 0	0 ± 0	0 ± 0	921 ± 0	921 ± 0
1.0	307 ± 0	307 ± 0	0 ± 0	0 ± 0	921 ± 0	921 ± 0
Mean values	286.6 ± 19.72	**295.5 ± 10.03**	43.39 ± 43.40	**40.43 ± 48.26**	**897.9 ± 36.92**	892.1 ± 46.11

Table 5: The decision rules between raw data and decision-monotonicity criterion based reducts (Zoo).

δ	(P) rules		(B) rules		(N) rules	
	Raw	Reduct	Raw	Reduct	Raw	Reduct
0.1	0 ± 0	**43 ± 0**	141.4 ± 70.62	**69.60 ± 59.90**	565.6 ± 70.72	**594.4 ± 59.90**
0.2	0 ± 0	**43 ± 0**	153 ± 55.23	**116 ± 66.97**	554 ± 55.23	548 ± 66.97
0.3	0 ± 0	**43 ± 0**	210.7 ± 197.1	**131 ± 127.1**	496.3 ± 197.1	**532.1 ± 127.1**
0.4	0 ± 0	**43 ± 0**	246.3 ± 216.9	**163.7 ± 158.5**	460.7 ± 216.9	**500.3 ± 158.5**
0.5	1.9 ± 3.143	**2.6 ± 3.406**	178.6 ± 173.9	**167.8 ± 126.4**	526.5 ± 174.1	**536.6 ± 124.3**
0.6	36.5 ± 3.689	36.5 ± 3.689	102.4 ± 68.08	102.4 ± 68.08	568.1 ± 69.54	568.1 ± 69.54
0.7	66.2 ± 8.377	**67.5 ± 8.657**	66.6 ± 39.14	**62.60 ± 43.91**	574.2 ± 38.29	**576.9 ± 41.53**
0.8	78.7 ± 8.795	78.7 ± 8.795	46.3 ± 20.95	46.30 ± 20.95	582 ± 15.24	582 ± 15.24
0.9	95.0 ± 0	95.0 ± 0	11.40 ± 0.9661	11.40 ± 0.9661	600.6 ± 0.9661	600.6 ± 0.9661
1.0	101 ± 0	101 ± 0	0 ± 0	0 ± 0	606 ± 0	606 ± 0
Mean values	37.93 ± 2.40	**55.33 ± 2.45**	115.7 ± 84.28	**87.17 ± 67.28**	553.4 ± 83.80	**564.5 ± 66.39**

Table 6: The comparison between decision-monotonicity criterion based reducts and cost based reducts (Annealing).

δ	Approximation qualities		Run times (s)	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
0.1	**0.8150 ± 0.3899**	0.4000 ± 0.5164	**12.43 ± 0.3997**	311.3 ± 25.18
0.2	**0.8150 ± 0.3899**	0.6401 ± 0.3762	**12.11 ± 0.0070**	261.8 ± 63.63
0.3	**0.7226 ± 0.4467**	0.6855 ± 0.2790	**12.12 ± 0.0143**	186.5 ± 26.66
0.4	**0.9075 ± 0.2925**	0.8397 ± 0.1122	**12.12 ± 0.0190**	205.6 ± 22.68
0.5	0.5376 ± 0.4874	**0.7153 ± 0.1350**	**12.12 ± 0.0076**	223.3 ± 29.21
0.6	0.7311 ± 0.4343	**0.8429 ± 0.0509**	**36.03 ± 39.38**	200.2 ± 8.255
0.7	0.6563 ± 0.3890	**0.9244 ± 0.0368**	**43.91 ± 40.84**	209.6 ± 40.87
0.8	0.8132 ± 0.2777	**0.9754 ± 0.0070**	**50.46 ± 42.55**	270.1 ± 67.62
0.9	0.9117 ± 0.0661	**0.9984 ± 0.0021**	**53.12 ± 36.45**	359.5 ± 106.0
1.0	1.0000 ± 0.0000	1.0000 ± 0.0000	**25.94 ± 32.02**	389.1 ± 129.9
Mean values	0.7910 ± 0.3174	**0.8022 ± 0.1516**	**27.04 ± 19.17**	261.7 ± 52.00

values of δ, whereas the y-coordinate concerns the values of costs. Through an investigation of Figure 1, it is not difficult to observe that, in all the ten used values of δ, the decision costs of cost minimum criterion based reducts are the same or lower than those obtained by decision-monotonicity criterion based reducts.

Tables 6 to 9 show the differences between decision-monotonicity criterion based reducts and cost minimum criterion based reducts in approximation qualities and running times, respectively. It is not difficult to note that, from the viewpoint of approximation qualities, the approximation qualities of decision-monotonicity criterion based reducts are

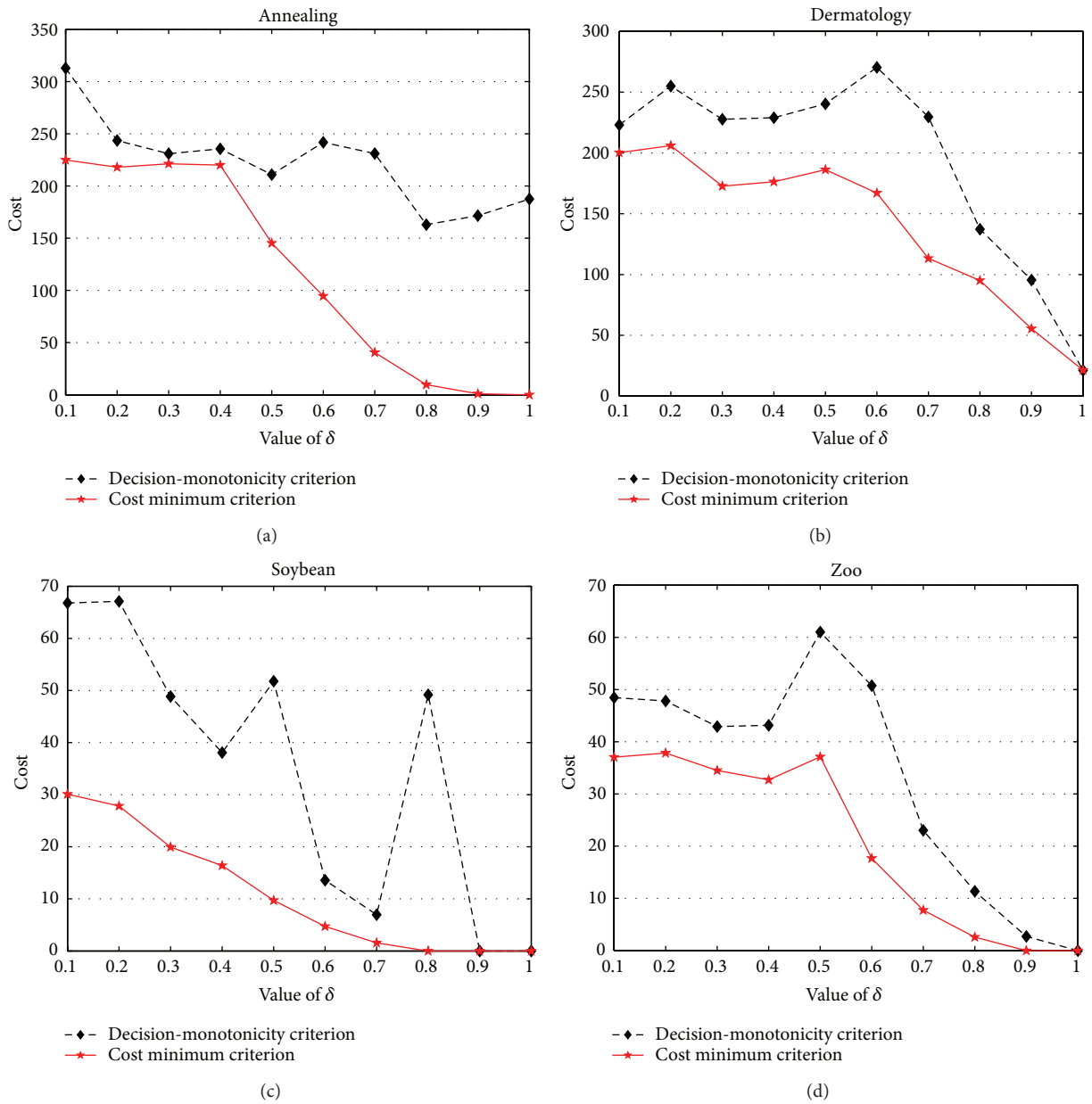

Figure 1: Reducts' costs comparison between decision-monotonicity and cost minimum criterions.

larger than those of cost minimum criterion based reducts at times. However, in most cases, the approximation qualities of cost minimum criterion based reducts are larger than those of decision-monotonicity criterion based reducts. From the point of running times, it is easy to observe that the run times of genetic algorithm are greater than those of heuristic algorithm.

To sum up, we can draw the following conclusions.

(1) From the viewpoint of decision monotonicity, our heuristic algorithm based on decision-monotonicity criterion can generate more (P) rules and less (B) rules with respect to the original data set. Such approach not only increases the certainties which are expressed by (P) rules and (N) rules, but also decreases the uncertainty coming from (B) rules.

(2) From the viewpoint of decision costs, the generic algorithm based on cost minimum criterion can obtain the lowest decision costs and the largest approximation qualities by comparing with heuristic algorithm based on decision-monotonicity criterion. However, such approach loses the property of decision monotonicity and it wastes larger running times than heuristic algorithm.

6. Conclusion

In this paper, we have developed a generalized framework of decision-theoretic rough set, which is referred

Table 7: The comparison between decision-monotonicity criterion based reducts and cost based reducts (Dermatology).

δ	Approximation qualities		Run times (s)	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
0.1	0.0030 ± 0.0095	**0.0913 ± 0.1191**	**2.524 ± 0.9840**	60.14 ± 12.11
0.2	0.0014 ± 0.0035	**0.1795 ± 0.0996**	**3.531 ± 2.9403**	47.88 ± 14.01
0.3	0.0005 ± 0.0012	**0.2197 ± 0.0345**	**4.444 ± 4.6011**	42.23 ± 3.884
0.4	0.0000 ± 0.0000	**0.2128 ± 0.0283**	**2.183 ± 0.1003**	43.43 ± 3.094
0.5	0.0011 ± 0.0019	**0.2014 ± 0.0166**	**6.589 ± 7.1960**	42.18 ± 3.991
0.6	0.0055 ± 0.0026	**0.3954 ± 0.1061**	**18.14 ± 0.2819**	46.62 ± 6.518
0.7	0.0907 ± 0.0338	**0.5462 ± 0.0488**	**17.71 ± 0.1935**	44.46 ± 5.502
0.8	0.3822 ± 0.0163	**0.5402 ± 0.0680**	**16.13 ± 0.7679**	49.32 ± 6.926
0.9	0.5817 ± 0.0247	**0.7533 ± 0.0358**	**13.91 ± 0.1070**	51.12 ± 10.22
1.0	0.8967 ± 0.0053	**0.8975 ± 0.0053**	**13.36 ± 0.0263**	58.57 ± 2.361
Mean values	0.1963 ± 0.0099	**0.4037 ± 0.0562**	**9.853 ± 1.719**	48.80 ± 6.863

Table 8: The comparison between decision-monotonicity criterion based reducts and cost based reducts (Soybean).

δ	Approximation qualities		Run times (s)	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
0.1	**0.9632 ± 0.0669**	0.9013 ± 0.0584	**6.976 ± 3.1845**	22.13 ± 4.443
0.2	0.9287 ± 0.0990	**0.9459 ± 0.0691**	**7.672 ± 2.3621**	23.36 ± 3.463
0.3	0.9195 ± 0.0460	**0.9492 ± 0.0578**	**7.638 ± 1.7622**	27.05 ± 6.543
0.4	0.8879 ± 0.0402	**0.9866 ± 0.0129**	**7.247 ± 2.2257**	28.93 ± 5.668
0.5	0.9420 ± 0.0510	**0.9948 ± 0.0054**	**2.217 ± 0.9900**	31.09 ± 8.096
0.6	**0.9951 ± 0.0052**	0.9896 ± 0.0103	**7.869 ± 0.1669**	33.91 ± 9.279
0.7	0.9958 ± 0.0035	**0.9964 ± 0.0047**	**6.398 ± 2.7240**	41.46 ± 6.722
0.8	0.9928 ± 0.0151	**1.0000 ± 0.0000**	**2.738 ± 2.6772**	37.79 ± 8.939
0.9	1.0000 ± 0.0000	1.0000 ± 0.0000	**7.529 ± 0.2508**	32.97 ± 9.095
1.0	1.0000 ± 0.0000	1.0000 ± 0.0000	**7.299 ± 0.2905**	35.63 ± 12.27
Mean values	0.9625 ± 0.0327	**0.9764 ± 0.0218**	**6.3586 ± 1.1663**	31.435 ± 7.453

Table 9: The comparison between decision-monotonicity criterion based reducts and cost based reducts (Zoo).

δ	Approximation qualities		Run times (s)	
	Algorithm 1	Algorithm 2	Algorithm 1	Algorithm 2
0.1	**0.4257 ± 0.0000**	0.2644 ± 0.2690	**0.0938 ± 0.0163**	4.2872 ± 0.4856
0.2	**0.4257 ± 0.0000**	0.2911 ± 0.3063	**0.0915 ± 0.0053**	4.4342 ± 0.1583
0.3	**0.4257 ± 0.0000**	0.3762 ± 0.2777	**0.0961 ± 0.0032**	4.7973 ± 0.4250
0.4	**0.4257 ± 0.0000**	0.3257 ± 0.2638	**0.0877 ± 0.0065**	4.4749 ± 0.4790
0.5	0.0257 ± 0.0337	**0.3277 ± 0.3271**	**0.2077 ± 0.1580**	4.1559 ± 0.3482
0.6	0.3614 ± 0.0365	**0.7129 ± 0.0417**	**0.3875 ± 0.0067**	4.4908 ± 0.5544
0.7	0.6683 ± 0.0857	**0.8554 ± 0.0896**	**0.3477 ± 0.0757**	5.2780 ± 1.1886
0.8	0.7792 ± 0.0871	**0.9564 ± 0.0344**	**0.3747 ± 0.0244**	6.6658 ± 1.6919
0.9	0.9406 ± 0.0000	**1.0000 ± 0.0000**	**0.3857 ± 0.0048**	6.4638 ± 1.4479
1.0	1.0000 ± 0.0000	1.0000 ± 0.0000	**0.3799 ± 0.0143**	6.5578 ± 1.8033
Mean values	0.5478 ± 0.0243	**0.6110 ± 0.1610**	**0.2452 ± 0.0315**	5.1806 ± 0.8582

to as a δ-cut decision-theoretic rough set. Different from Yao's decision-theoretic rough set model, our model is constructed based on δ-cut quantitative indiscernibility relation, and it can degenerate to Yao's decision-theoretic rough set with some limitation. Based on the proposed model, we discussed the attribute reductions from two criterions; the experiments show that, on the one hand, the decision-monotonicity criterion based reducts can generate more positive rules and less boundary rules; on the other hand, the cost minimum criterion based reducts can obtain the lowest decision costs with high approximation qualities.

The present study is the first step towards δ-cut decision-theoretic rough set. The following are challenges for further research.

(1) δ-cut decision-theoretic rough set approach to complicated data type, such as interval-valued data, is one of the challenges; incomplete data may be an interesting topic.

(2) The threshold learning of δ in this paper is also a serious challenge.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Natural Science Foundation of China (nos. 61100116, 61272419, 61373062, and 61305058), Natural Science Foundation of Jiangsu Province of China (nos. BK2011492, BK2012700, and BK20130471), Qing Lan Project of Jiangsu Province of China, Postdoctoral Science Foundation of China (no. 2014M550293), Key Laboratory of Intelligent Perception and Systems for High-Dimensional Information (Nanjing University of Science and Technology), Ministry of Education (no. 30920130122005), and Natural Science Foundation of Jiangsu Higher Education Institutions of China (nos. 13KJB520003 and 13KJD520008).

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