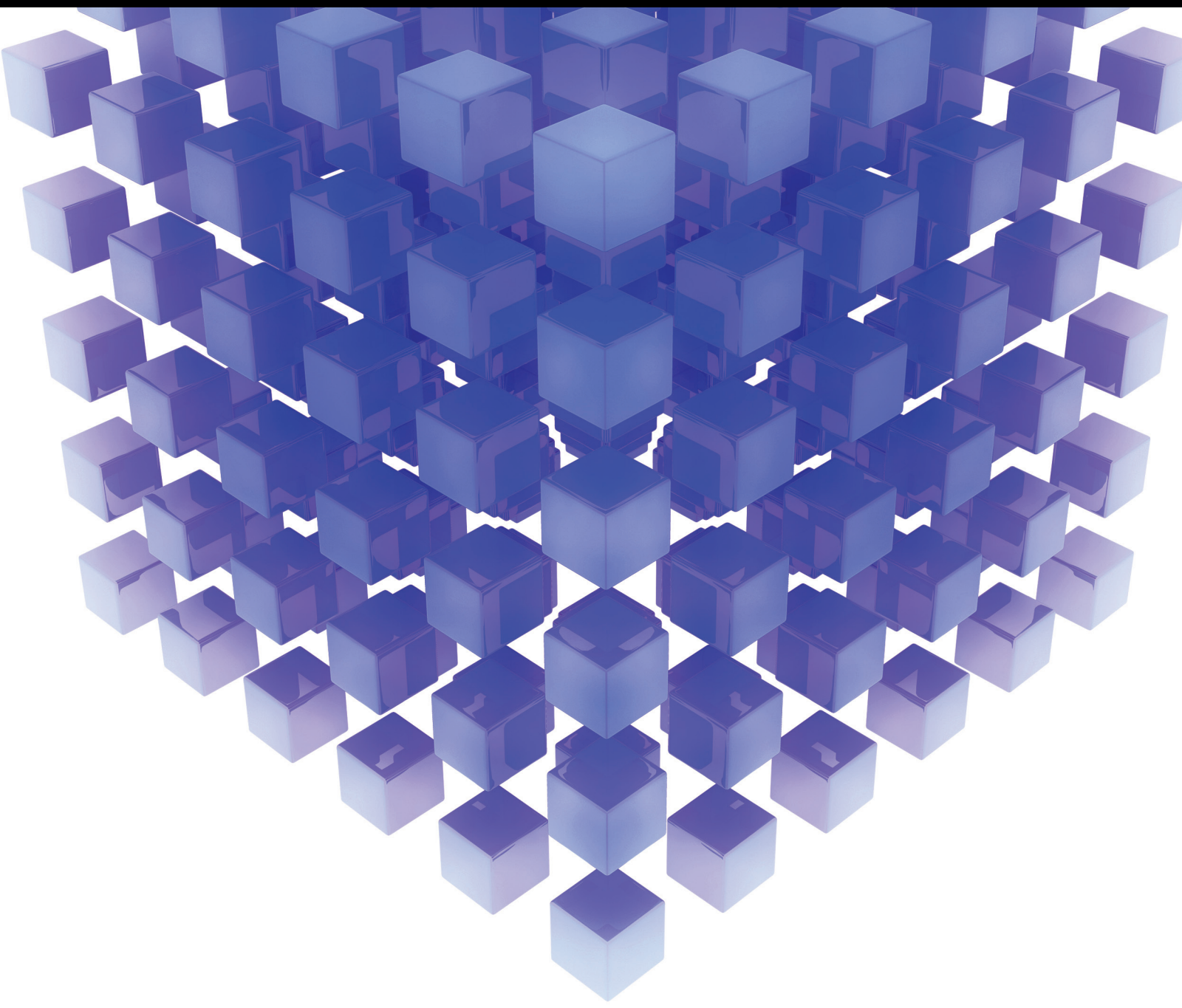
Mathematical Problems in Engineering

Partial Fractional Equations and Their Applications

Guest Editors: Abdon Atangana, Hossein Jafari, Samir B. Belhaouari, and Mustafa Bayram



Partial Fractional Equations and Their Applications

Mathematical Problems in Engineering

Partial Fractional Equations and Their Applications

Guest Editors: Abdon Atangana, Hossein Jafari, Samir B. Belhaouari, and Mustafa Bayram

Editorial Board

Contents

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 387205, 1 page
http://dx.doi.org/10.1155/2015/387205

Editorial

Partial Fractional Equations and Their Applications

Abdon Atangana,[1] Hossein Jafari,[2] Samir B. Belhaouari,[3] and Mustafa Bayram[4]

[1]*Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa*
[2]*Department of Mathematics, Faculty of Sciences, University of Mazandaran, P.O. Box 47416-95447, Babolsar, Iran*
[3]*Department of Mathematics and Applied Mathematics, Faculty of Sciences and Technology, Alfaisal University, P.O. Box 50927, Riyadh 11533, Saudi Arabia*
[4]*Department of Mathematical Engineering, Yildiz Technical University, Davutpasa, 34210 Istanbul, Turkey*

Correspondence should be addressed to Abdon Atangana; abdonatangana@yahoo.fr

Received 11 March 2015; Accepted 11 March 2015

In the past years, many physical problems in almost all branches of sciences and engineering were with success modelled using the concept of fractional order derivative. This subject has therefore fascinated consideration of many researchers around humankind. Particularly, the advantages of the concept of fractional calculus and their associated models via differential partial differential equations linking fractional order integrodifferential operators and their appliances have been previously intensively investigated with great success during the last years.

Despite the fact that the exploration of the phenomena is portrayed by interface of many organisms, the microreplications play an important role, and as result the computers turn out to be progressively scientific apparatus. The main reason for this special issue was to report and review the latest progress in the areas of partial differential equations. This special issue included in particular the following:

(i) fractional partial differential equations and their applications in sciences and engineering,

(ii) modelling and simulation of real world phenomena with partial differential equations,

(iii) analytical and numerical methods for partial differential equations,

(iv) new applications of iterations method,

(v) anomalous diffusion.

We have received many submissions, however, to keep the standard high only few were accepted for publications. This special issue includes papers on dissimilar features, for instance, analysis of Sturm-Liouville Eigen-problems, numerical methods and analysis, iterative methods, fractional boundary value problems, analytical method like Lie symmetry analysis, and integral transform operators.

Evidently, it is not imaginable to satisfactorily distinguish in this special issue all the strategy of up-to-date exploration in fractional partial differential equations with their applications; however, we assumed that this special isssue contents both hypothetical investigate and significant progress including contemporary challenging problems, new ideas, and open problems.

Acknowledgment

We acknowledge all the contributors to this special issue for their participations.

Abdon Atangana
Hossein Jafari
Samir B. Belhaouari
Mustafa Bayram

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 805763, 7 pages
http://dx.doi.org/10.1155/2015/805763

Research Article

Symmetry Analysis and Conservation Laws of a Generalized Two-Dimensional Nonlinear KP-MEW Equation

Khadijo Rashid Adem and Chaudry Masood Khalique

International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa

Correspondence should be addressed to Chaudry Masood Khalique; masood.khalique@nwu.ac.za

Received 2 August 2014; Accepted 20 September 2014

Academic Editor: Hossein Jafari

Lie symmetry analysis is performed on a generalized two-dimensional nonlinear Kadomtsev-Petviashvili-modified equal width equation. The symmetries and adjoint representations for this equation are given and an optimal system of one-dimensional subalgebras is derived. The similarity reductions and exact solutions with the aid of (G'/G)-expansion method are obtained based on the optimal systems of one-dimensional subalgebras. Finally conservation laws are constructed by using the multiplier method.

1. Introduction

Nonlinear evolution equations (NLEEs) have been widely used to describe natural phenomena of science and engineering. Therefore it is very important to find exact solutions of NLEEs. However, this is not an easy task. During the past few decades various integration techniques have been developed by the researchers to solve these NLEEs. Some of the well-known techniques used in the literature are the inverse scattering transform method [1], the homogeneous balance method [2], the Bäcklund transformation [3], the Weierstrass elliptic function expansion method [4], the Darboux transformation [5], the ansatz method [6, 7], Hirota's bilinear method [8], the (G'/G)-expansion method [9], the Jacobi elliptic function expansion method [10, 11], the variable separation approach [12], the sine-cosine method [13], the trifunction method [14, 15], the F-expansion method [16], the exp-function method [17], the multiple exp-function method [18], and the Lie symmetry method [19–25].

The purpose of this paper is to study one such NLEE, namely, the generalized two-dimensional nonlinear Kadomtsev-Petviashvili-modified equal width (KP-MEW) equation [26] that is given by

$$\left(u_t + \alpha\left(u^n\right)_x + \beta u_{xxt}\right)_x + \gamma u_{yy} = 0. \tag{1}$$

Here, in (1) α, β, γ and $n > 1$ are real valued constants. The solutions of (1) have been studied in various aspects. See, for example, the recent papers [26–28]. Wazwaz [26] used the tanh method and the sine-cosine method, for finding solitary waves and periodic solutions. Saha [27] used the theory of bifurcations of planar dynamical systems to prove the existence of smooth and nonsmooth travelling wave solutions. Wei et al. [28] used the qualitative theory of differential equations and obtained peakon, compacton, cuspons, loop soliton solutions, and smooth soliton solutions.

In this paper we obtain symmetry reductions of (1) using Lie group analysis [19–24] and based on the optimal systems of one-dimensional subalgebras. Furthermore, the (G'/G)-expansion method is employed to obtain some exact solutions of (1). In addition to this conservation laws will be derived for (1) using the multiplier method [29].

2. Symmetry Reductions and Exact Solutions of (1)

The vector field of the form

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + \xi^3 \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u}, \tag{2}$$

where ξ^i, $i = 1, 2, 3$, and η depend on x, y, t, and u, is a Lie point symmetry of (1) if

$$\mathrm{pr}^{(4)}X\left[\left(u_t + \alpha\left(u^n\right)_x + \beta u_{xxt}\right)_x + \gamma u_{yy}\right] = 0 \tag{3}$$

whenever $(u_t + \alpha(u^n)_x + \beta u_{xxt})_x + \gamma u_{yy} = 0$. Here $\mathrm{pr}^{(4)}X$ [20] denotes the fourth prolongation of X. Expanding (3) and splitting on the derivatives of u, we obtain an overdetermined system of linear partial differential equations. Solving this system one obtains the following four Lie point symmetries:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= \frac{\partial}{\partial t}, \\ X_4 &= y(1-n)\frac{\partial}{\partial y} + 2t(1-n)\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}. \end{aligned} \tag{4}$$

2.1. One-Dimensional Optimal System of Subalgebras. We now calculate the optimal system of one-dimensional subalgebras for (1) and use it to find the optimal system of group-invariant solutions for (1). We follow the method given in [20]. Recall that the adjoint transformations are given by

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\epsilon X_i\right)\right)X_j = {} & X_j - \epsilon\left[X_i, X_j\right] \\ & + \frac{1}{2}\epsilon^2\left[X_i\left[X_i, X_j\right]\right] - \cdots, \end{aligned} \tag{5}$$

where $[X_i, X_j]$ is the commutator defined by

$$\left[X_i, X_j\right] = X_iX_j - X_jX_i. \tag{6}$$

We present the commutator table of the Lie symmetries and the adjoint representations of the symmetry group of (1) on its Lie algebra in Tables 1 and 2, respectively. These two tables are then used to construct the optimal system of one-dimensional subalgebras for (1). As a result, after some calculations, one can obtain an optimal system of one-dimensional subalgebras given by $\{aX_1+bX_2+cX_3, dX_1+X_4\}$, where $a, d \in \mathbb{R}$, $b, c = 0, \pm 1$.

2.2. Symmetry Reductions and Exact Solutions of (1). In this subsection we use the optimal system of one-dimensional subalgebras calculated above to obtain symmetry reductions and exact solutions of the KP-MEW equation.

Case 1. Consider the following: $aX_1+bX_2+cX_3$; $a \in \mathbb{R}$, $b, c = \pm 1$.

The symmetry $aX_1 + bX_2 + cX_3$ gives rise to the following three invariants:

$$f = \frac{bt - cy}{b}, \qquad g = \frac{bx + ay}{b}, \qquad \phi = u. \tag{7}$$

Table 1: Commutator table of the Lie algebra of equation (1).

	X_1	X_2	X_3	X_4
X_1	0	0	0	0
X_2	0	0	0	$(1-n)X_2$
X_3	0	0	0	$(2-2n)X_3$
X_4	0	$-(1-n)X_2$	$-(2-2n)X_3$	0

Table 2: Adjoint table of the Lie algebra of equation (1).

Ad	X_1	X_2	X_3	X_4
X_1	X_1	X_2	X_3	X_4
X_2	X_1	X_2	X_3	$X_4 - \varepsilon(1-n)X_2$
X_3	X_1	X_2	X_3	$X_4 - \varepsilon(2-2n)X_3$
X_4	X_1	$e^{\varepsilon(1-n)}X_2$	$e^{\varepsilon(2-2n)}X_3$	X_4

Now treating ϕ as the new dependent variable and f and g as new independent variables, the KP-MEW equation (1) transforms to

$$\begin{aligned} & \left(2ac\gamma + b^2\right)\phi_{fg} + a^2\gamma\phi_{gg} + \alpha n\left(n - b^2\right)\phi^{n-2}\phi_g^2 \\ & \quad + b^2\alpha n\phi^{n-1}\phi_{gg} + b^2\beta\phi_{fggg} + c^2\gamma\phi_{ff} = 0, \end{aligned} \tag{8}$$

which is a nonlinear PDE in two independent variables. We now use the Lie point symmetries of (8) and transform it to an ordinary differential equation (ODE). Equation (8) has the two translational symmetries; namely,

$$\Gamma_1 = \frac{\partial}{\partial f}, \qquad \Gamma_2 = \frac{\partial}{\partial g}. \tag{9}$$

The combination $\Gamma_1+\Gamma_2$ of the two symmetries Γ_1 and Γ_2 yields the two invariants

$$z = f - g, \qquad F = \phi, \tag{10}$$

which gives rise to a group-invariant solution $F = F(z)$. Consequently using these invariants, (8) is transformed into the fourth-order nonlinear ODE:

$$\begin{aligned} & \gamma(a+c)^2F'' + \alpha n b^2F^{n-1}F'' - b^2F'' \\ & \quad + \alpha n b^2(n-1)F^{n-2}F'^2 - \beta b^2F'''' = 0. \end{aligned} \tag{11}$$

Integrating the above equation twice and taking the constants of integration to be zero we obtain a second-order ODE:

$$\left(\gamma(a+c)^2 - c^2\right)F + \alpha b^2F^n - \beta b^2F'' = 0. \tag{12}$$

Multiplying (12) by F', integrating once and taking the constant of integration to be zero, we obtain the first-order ODE:

$$\frac{1}{2}\left(\gamma(a+c)^2 - b^2\right)F^2 + \frac{\alpha b^2}{n+1}F^{n+1} - \frac{1}{2}\beta b^2F'^2 = 0. \tag{13}$$

One can integrate the above equation by separating the variables. After integrating and reverting back to the original variables, we obtain the following group-invariant solutions

of the KP-MEW equation (1) for arbitrary values of n in the following form:

$$u(x,y,t) = \left(\frac{(n+1)(b^2-\gamma(a+c)^2)}{2\beta b^2}\right)^{1/(n-1)} \operatorname{sech}^{2/(n-1)}[Q], \tag{14}$$

where

$$Q = \frac{\sqrt{\gamma(a+c)^2+b^2}}{2\sqrt{2}}\left(\frac{\sqrt{2}(1-n)}{c\sqrt{\beta}}z + (n-1)C\right), \qquad z = t - x - \frac{(a+c)}{b}y, \tag{15}$$

and C is a constant of integration. By taking $n = 2$, $\alpha = 1/2$, $\beta = 1$, $\gamma = 1$, $a = 1$, $b = 1$, $c = 1$, $t = 0$, and $C = 1$ in (14), the profile of the solution is given in Figure 1.

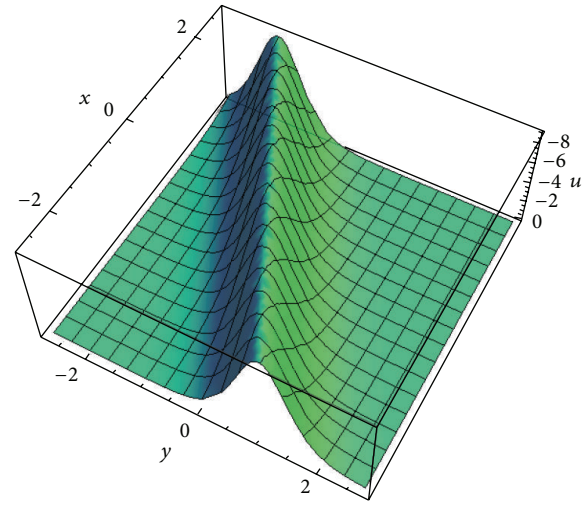


Figure 1: Profile of solution (14).

Case 2. Consider the following: $dX_1 + X_4$.

The symmetry $dX_1 + X_4$ gives rise to the three invariants:

$$f = \frac{t}{y^2}, \qquad g = \frac{d\ln y + nx - x}{n-1}, \qquad \phi = uy^{2/(n-1)}. \tag{16}$$

By treating ϕ as the new dependent variable and f and g as new independent variables, the KP-MEW equation (1) transforms to

$$\begin{aligned}
&\beta n^2\phi_{fggg} - 2\beta n\phi_{fggg} + 2d\gamma\phi_{gg} + 2n\gamma\phi + 4f^2\gamma\phi_{ff}\\
&\quad - 2f\gamma\phi_f - 3a_1\gamma\phi_g + \phi_{fg} - \alpha n\phi^{n-2}\phi_g^2\\
&\quad + \alpha n^4\phi^{n-2}\phi_g^2 + 3\alpha n^2\phi^{n-2}\phi_g^2 + 6fn^2\gamma\phi_f\\
&\quad - 4fn\gamma\phi_f - 3\alpha n^3\phi^{n-2}\phi_g^2 + \alpha n\phi^{n-1}\phi_{gg}\\
&\quad + \alpha n^3\phi^{n-1}\phi_{gg} - 8f^2n\gamma\phi_{ff} + 4f^2n^2\gamma\phi_{ff}\\
&\quad + 4df\gamma\phi_{fg} - 2\alpha n^2\phi^{n-1}\phi_{gg} + 2\gamma\phi + \beta\phi_{fggg}\\
&\quad + fn^2\phi_g - 2fn\phi_g - 4dfn\gamma\phi_{fg} - dfn\gamma\phi_g = 0.
\end{aligned} \tag{17}$$

Equation (17) has a single Lie point symmetry; namely,

$$\Gamma_1 = \frac{\partial}{\partial g}, \tag{18}$$

and this symmetry yields the two invariants

$$r = f, \qquad F = \phi, \tag{19}$$

which gives rise to a group-invariant solution $F = F(r)$ and consequently, using these invariants, (17) is then transformed to a second-order Cauchy-Euler ODE:

$$\begin{aligned}
&\left(2n^2r^2 - 4nr^2 + 2r^2\right)F'' + \left(3n^2r - 2nr - r\right)F'\\
&\quad + (1+n)F = 0.
\end{aligned} \tag{20}$$

Now solving this equation and reverting back to the original variables, we obtain the following solution of the KP-MEW equation (1):

$$u(x,y,t) = \frac{1}{y^{2/(n-1)}}\left(C_1 r^{-(n+1)/2(n-1)} + C_2 r^{-1/(n-1)}\right), \tag{21}$$

where $r = t/y^2$ and C_1 and C_2 are constants of integration.

3. (G'/G)-Expansion Method

In this section we use the (G'/G)-expansion method [9, 30] to obtain a few exact solutions of the KP-MEW equation (1) for $n = 2$ and $n = 3$.

Let us consider the solutions of (11) in the form

$$F(z) = \sum_{i=0}^{M} \mathscr{A}_i \left(\frac{G'(z)}{G(z)}\right)^i, \tag{22}$$

where $G(z)$ satisfies

$$G'' + \lambda G' + \mu G = 0 \tag{23}$$

and λ and μ are constants. The homogeneous balance method between the highest order derivative and highest order nonlinear term appearing in (11) determines the value of M and $\mathscr{A}_0, \ldots, \mathscr{A}_M$ are constants to be determined.

Consider $n = 2$. Application of the balancing procedure to fourth-order ODE (11) yields $M = 2$, so the solution of (11) is of the form

$$F(z) = \mathscr{A}_0 + \mathscr{A}_1\left(\frac{G'(z)}{G(z)}\right) + \mathscr{A}_2\left(\frac{G'(z)}{G(z)}\right)^2. \tag{24}$$

Substituting (23) and (24) into (11) leads to an overdetermined system of algebraic equations. Solving this system of algebraic equations with the aid of Maple, we obtain

$$\mathscr{A}_0 = \frac{b^2\alpha\lambda^2 A_2 + 8b^2\mu\alpha A_2 - 6\gamma a^2 - 12ac\gamma - 6\gamma c^2 + 6b^2}{12\alpha b^2},$$
$$\mathscr{A}_1 = \frac{6\beta\lambda}{\alpha},$$
$$\mathscr{A}_2 = \frac{6\beta}{\alpha}. \tag{25}$$

Now using the general solution of (23) in (24), we have the following three types of travelling wave solutions of the KP-MEW equation (1).

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution:

$$u(x,y,t) = \mathscr{A}_0 + \mathscr{A}_1\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right) + \mathscr{A}_2\left(-\frac{\lambda}{2} + \delta_1\frac{C_1\sinh(\delta_1 z) + C_2\cosh(\delta_1 z)}{C_1\cosh(\delta_1 z) + C_2\sinh(\delta_1 z)}\right)^2, \tag{26}$$

where $z = t - x - ((a+c)/b)y$, $\delta_1 = (1/2)\sqrt{\lambda^2 - 4\mu}$, and C_1 and C_2 are arbitrary constants.

The profile of the solution (26) is given in Figure 2.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution:

$$u(x,y,t) = \mathscr{A}_0 + \mathscr{A}_1\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right) + \mathscr{A}_2\left(-\frac{\lambda}{2} + \delta_2\frac{-C_1\sin(\delta_2 z) + C_2\cos(\delta_2 z)}{C_1\cos(\delta_2 z) + C_2\sin(\delta_2 z)}\right)^2, \tag{27}$$

where $z = t - x - ((a+c)/b)y$, $\delta_2 = (1/2)\sqrt{4\mu - \lambda^2}$, and C_1 and C_2 are arbitrary constants.

The profile of the solution (27) is given in Figure 3.

When $\lambda^2 - 4\mu = 0$, we obtain the rational function solution:

$$u(x,y,t) = \mathscr{A}_0 + \mathscr{A}_1\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right) + \mathscr{A}_2\left(-\frac{\lambda}{2} + \frac{C_2}{C_1 + C_2 z}\right)^2, \tag{28}$$

where $z = t - x - ((a+c)/b)y$ and C_1 and C_2 are arbitrary constants.

The profile of the solution (28) is given in Figure 4.

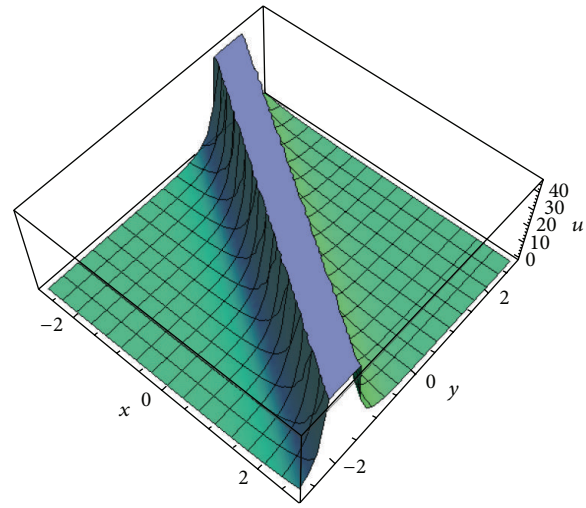


Figure 2: Profile of solution (26).

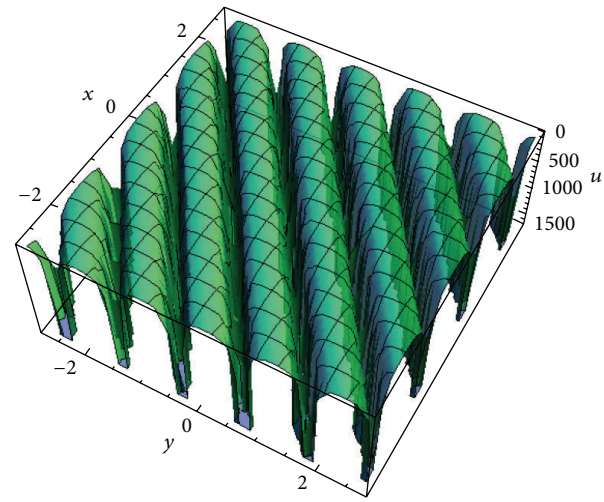


Figure 3: Profile of solution (27).

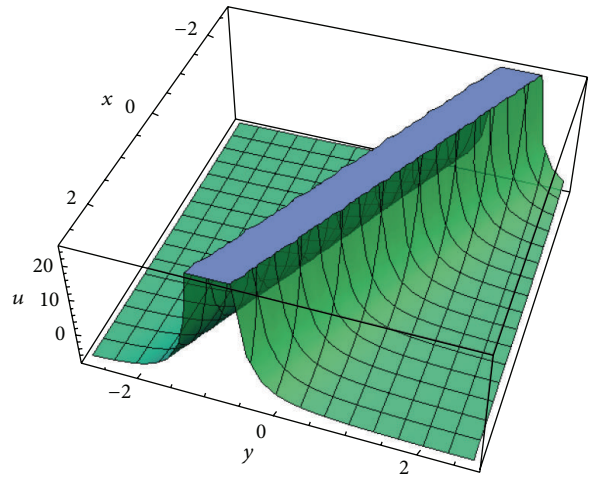


Figure 4: Profile of solution (28).

Consider $n = 3$. Again the application of the balancing procedure to fourth-order ODE yields $M = 1$, so the solution of (11) is of the form

$$F(z) = \mathscr{A}_0 + \mathscr{A}_1 \left(\frac{G'(z)}{G(z)} \right). \tag{29}$$

Solving this system of algebraic equations with the aid of Maple, we obtain

$$\begin{aligned} \beta &= \frac{\alpha}{2}\mathscr{A}_1^2, \\ \mathscr{A}_0 &= \frac{\lambda\sqrt{-a^2\gamma - 2ac\gamma + b^2 - c^2\gamma}}{\sqrt{\alpha b^2\lambda^2 - 4\alpha b^2\mu}}, \\ \mathscr{A}_1 &= \frac{2\sqrt{-a^2\gamma - 2ac\gamma + b^2 - c^2\gamma}}{\sqrt{\alpha b^2\lambda^2 - 4\alpha b^2\mu}}. \end{aligned} \tag{30}$$

Now using the general solution of (23) in (29), we have the following two types of travelling wave solutions of the KP-MEW equation (1).

When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function solution:

$$\begin{aligned} &u(x, y, t) \\ &= \mathscr{A}_0 + \mathscr{A}_1 \left(-\frac{\lambda}{2} + \delta_1 \frac{C_1 \sinh(\delta_1 z) + C_2 \cosh(\delta_1 z)}{C_1 \cosh(\delta_1 z) + C_2 \sinh(\delta_1 z)} \right), \end{aligned} \tag{31}$$

where $z = t - x - ((a + c)/b)y$, $\delta_1 = (1/2)\sqrt{\lambda^2 - 4\mu}$, and C_1 and C_2 are arbitrary constants.

When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function solution:

$$\begin{aligned} &u(x, y, t) \\ &= \mathscr{A}_0 + \mathscr{A}_1 \left(-\frac{\lambda}{2} + \delta_2 \frac{-C_1 \sin(\delta_2 z) + C_2 \cos(\delta_2 z)}{C_1 \cos(\delta_2 z) + C_2 \sin(\delta_2 z)} \right), \end{aligned} \tag{32}$$

where $z = t - x - ((a + c)/b)y$, $\delta_2 = (1/2)\sqrt{4\mu - \lambda^2}$, and C_1 and C_2 are arbitrary constants.

4. Conservation Laws of (1)

In this section we construct conservation laws for (1). The multiplier method [29, 30] will be used.

The zeroth-order multiplier $\Lambda(t, x, y, u)$ for (1) is given by

$$\begin{aligned} \Lambda = &-\frac{y^3}{6\gamma} f_1'(t) + xyf_1(t) - \frac{y^2}{2\gamma} f_2'(t) \\ &+ xf_2(t) + yf_3(t) + f_4(t), \end{aligned} \tag{33}$$

where $f_1(t)$, $f_2(t)$, $f_3(t)$, and $f_4(t)$ are arbitrary functions of t. Corresponding to the above multiplier we have the following conserved vectors of (1):

$$\begin{aligned} T_1^t = &\frac{1}{24\gamma}\{-12\gamma y f_1(t)u - 6\beta\gamma y f_1(t)u_{xx} \\ &+ 6\beta\gamma xy f_1(t)u_{xxx} + 12\gamma xy f_1(t)u_x \\ &- \beta y^3 f_1'(t)u_{xxx} - 2y^3 f_1'(t)u_x\}, \\ T_1^x = &-\frac{y}{24\gamma u}\{4\alpha n y^2 f_1'(t)u_x u^n - 24\alpha\gamma n x f_1(t)u_x u^n \\ &- \beta y^2 f_1''(t)u_{xx}u + 3\beta y^2 f_1'(t)u_{txx}u \\ &+ 2y^2 f_1'(t)u_t u - 12\beta\gamma f_1'(t)u_x u \\ &+ 6\beta\gamma x f_1'(t)u_{xx}u + 12\beta\gamma f_1(t)u_{tx}u \\ &- 18\beta\gamma x f_1(t)u_{txx}u - 12\gamma x f_1(t)u_t u \\ &+ 24\alpha\gamma f_1(t)u^{n+1} - 2y^2 f_1''(t)u^2 \\ &+ 12\gamma x f_1'(t)u^2\}, \\ T_1^y = &\frac{1}{6}\{3y^2 f_1'(t)u - 6\gamma x f_1(t)u \\ &+ 6\gamma xy f_1(t)u_y - y^3 f_1'(t)u_y\}; \\ T_2^t = &\frac{1}{8\gamma}\{-4\gamma f_2(t)u - 2\beta\gamma f_2(t)u_{xx} + 2\beta\gamma x f_2(t)u_{xxx} \\ &+ 4\gamma x f_2(t)u_x - \beta y^2 f_2'(t)u_{xxx} - 2y^2 f_2'(t)u_x\}, \\ T_2^x = &-\frac{1}{8\gamma u}\{4\alpha n y^2 f_2'(t)u_x u^n - 8\alpha\gamma n x f_2(t)u_x u^n \\ &- \beta y^2 f_2'' u_{xx}u + 3\beta y^2 f_2' u_{txx}u \\ &+ 2y^2 f_2'(t)u_t u - 4\beta\gamma f_2'(t)u_x u \\ &+ 2\beta\gamma x f_2'(t)u_{xx}u + 4\beta\gamma f_2(t)u_{tx}u \\ &- 6\beta\gamma x f_2(t)u_{txx}u - 4\gamma x f_2(t)u_t u \\ &+ 8\alpha\gamma f_2(t)u^{n+1} - 2y^2 f_2''(t)u^2 \\ &+ 4\gamma x f_2'(t)u^2\}, \\ T_2^y = &\frac{1}{2}\{2y f_2'(t)u + 2\gamma x f_2(t)u_y - y^2 f_2'(t)u_y\}; \\ T_3^t = &\frac{1}{4}\{\beta y f_3(t)u_{xxx} + 2y f_3(t)u_x\}, \\ T_3^x = &-\frac{y}{4u}\{-4\alpha n f_3(t)u_x u^n + \beta f_3'(t)u_{xx}u - 3\beta f_3(t)u_{txx}u \\ &- 2f_3(t)u_t u + 2f_3'(t)u^2\}, \\ T_3^y = &\gamma y f_3(t)u_y - \gamma f_3(t)u; \end{aligned}$$

$$T_4^t = \frac{1}{4}\left\{\beta f_4(t)\, u_{xxx} + 2 f_4(t)\, u_x\right\},$$

$$T_4^x = -\frac{1}{4u}\left\{-4\alpha n f_4(t)\, u_x u^n + \beta f_4'(t)\, u_{xx} u - 3\beta f_4(t)\, u_{txx} u - 2 f_4(t)\, u_t u + 2 f_4'(t)\, u^2\right\},$$

$$T_4^y = \gamma f_4(t)\, u_y. \tag{34}$$

Remark. The presence of the arbitrary functions in the multiplier leads to a family of infinitely many conservation laws for (1).

5. Concluding Remarks

In this paper we obtained the solutions of a generalized two-dimensional nonlinear Kadomtsev-Petviashvili-modified equal width equation by employing the Lie group analysis, the optimal systems of one-dimensional subalgebras, and the (G'/G)-expansion method. The solutions obtained are solitary waves and nontopological solitons. The conservation laws for the underlying equation were also derived by using the multiplier method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Soliton, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge University Press, Cambridge, UK, 1991.

[2] M. Wang, Y. Zhou, and Z. Li, "Application of a homogeneous balance method to exact solutions of nonlinear equations in mathematical physics," *Physics Letters A*, vol. 216, no. 1–5, pp. 67–75, 1996.

[3] C. H. Gu, *Soliton Theory and Its Application*, Zhejiang Science and Technology Press, Zhejiang, China, 1990.

[4] Y. Chen and Z. Yan, "The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations," *Chaos, Solitons and Fractals*, vol. 29, no. 4, pp. 948–964, 2006.

[5] V. B. Matveev and M. A. Salle, *Darboux Transformation and Soliton*, Springer, Berlin , Germany, 1991.

[6] J. Hu, "Explicit solutions to three nonlinear physical models," *Physics Letters A*, vol. 287, no. 1-2, pp. 81–89, 2001.

[7] J. Hu and H. Zhang, "A new method for finding exact traveling wave solutions to nonlinear partial differential equations," *Physics Letters A*, vol. 286, no. 2-3, pp. 175–179, 2001.

[8] R. Hirota, *The Direct Method in Soliton Theory*, vol. 155 of *Cambridge Tracts in Mathematics*, Cambridge University Press, Cambridge, UK, 2004.

[9] M. Wang, X. Li, and J. Zhang, "The (G'/G)-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics," *Physics Letters A*, vol. 372, no. 4, pp. 417–423, 2008.

[10] D. Lü, "Jacobi elliptic function solutions for two variant Boussinesq equations," *Chaos, Solitons & Fractals*, vol. 24, no. 5, pp. 1373–1385, 2005.

[11] Z. Yan, "Abundant families of Jacobi elliptic function solutions of the (2 + 1)-dimensional integrable Davey-Stewartson-type equation via a new method," *Chaos, Solitons and Fractals*, vol. 18, no. 2, pp. 299–309, 2003.

[12] S.-Y. Lou and J. Lu, "Special solutions from the variable separation approach: the Davey-Stewartson equation," *Journal of Physics, A: Mathematical and General*, vol. 29, no. 14, pp. 4209–4215, 1996.

[13] A.-M. Wazwaz, "The tanh and the sine-cosine methods for compact and noncompact solutions of the nonlinear Klein-Gordon equation," *Applied Mathematics and Computation*, vol. 167, no. 2, pp. 1179–1195, 2005.

[14] Z. Yan, "The new tri-function method to multiple exact solutions of nonlinear wave equations," *Physica Scripta*, vol. 78, no. 3, Article ID 035001, 5 pages, 2008.

[15] Z. Yan, "Periodic, solitary and rational wave solutions of the 3D extended quantum Zakharov-Kuznetsov equation in dense quantum plasmas," *Physics Letters, Section A: General, Atomic and Solid State Physics*, vol. 373, no. 29, pp. 2432–2437, 2009.

[16] M. Wang and X. Li, "Extended F-expansion method and periodic wave solutions for the generalized Zakharov equations," *Physics Letters A*, vol. 343, no. 1–3, pp. 48–54, 2005.

[17] S. Zhang, "Application of Exp-function method to high-dimensional nonlinear evolution equation," *Chaos, Solitons and Fractals*, vol. 38, no. 1, pp. 270–276, 2008.

[18] W.-X. Ma, T. Huang, and Y. Zhang, "A multiple exp-function method for nonlinear differential equations and its application," *Physica Scripta*, vol. 82, no. 6, Article ID 065003, 2010.

[19] G. W. Bluman and S. Kumei, *Symmetries and Differential Equations*, Applied Mathematical Sciences, Springer, New York, NY, USA, 1989.

[20] P. J. Olver, *Applications of Lie Groups to Differential Equations*, vol. 107 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 2nd edition, 1993.

[21] L. V. Ovsiannikov, *Group Analysis of Differential Equations*, Academic Press, New York, NY, USA, 1982.

[22] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 1, CRC Press, Boca Raton, Fla, USA, 1994.

[23] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 2, CRC Press, Boca Raton, Fla, USA, 1995.

[24] N. H. Ibragimov, *CRC Handbook of Lie Group Analysis of Differential Equations*, vol. 3, CRC Press, Boca Raton, Fla, USA, 1996.

[25] K. R. Adem and C. M. Khalique, "Exact solutions and conservation laws of a (2 + 1)-dimensional nonlinear KP-BBM equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 791863, 5 pages, 2013.

[26] A.-M. Wazwaz, "The tanh method and the sine-cosine method for solving the KP-MEW equation," *International Journal of Computer Mathematics*, vol. 82, no. 2, pp. 235–246, 2005.

[27] A. Saha, "Bifurcation of travelling wave solutions for the generalized KP-NEW equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 17, no. 9, pp. 3539–3551, 2012.

[28] M. Wei, S. Tang, H. Fu, and G. Chen, "Single peak solitary wave solutions for the generalized KP-MEW (2, 2) equation under

boundary condition," *Applied Mathematics and Computation*, vol. 219, no. 17, pp. 8979–8990, 2013.

[29] S. C. Anco and G. Bluman, "Direct construction method for conservation laws of partial differential equations. Part I: examples of conservation law classifications," *European Journal of Applied Mathematics*, vol. 13, no. 5, pp. 545–566, 2002.

[30] K. R. Adem and C. M. Khalique, "Conservation laws and traveling wave solutions of a generalized nonlinear ZK-BBM equation," *Abstract and Applied Analysis*, vol. 2014, Article ID 139513, 5 pages, 2014.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 457013, 13 pages
http://dx.doi.org/10.1155/2015/457013

Research Article

A New Approach and Solution Technique to Solve Time Fractional Nonlinear Reaction-Diffusion Equations

Inci Cilingir Sungu[1] and Huseyin Demir[2]

[1]*Department of Elementary School Mathematics Education, Education Faculty, Ondokuz Mayıs University, 55139 Samsun, Turkey*
[2]*Department of Mathematics, Arts and Science Faculty, Ondokuz Mayıs University, 55139 Samsun, Turkey*

Correspondence should be addressed to Huseyin Demir; hdemir@omu.edu.tr

Received 19 August 2014; Accepted 20 November 2014

Academic Editor: Samir B. Belhaouari

A new application of the hybrid generalized differential transform and finite difference method is proposed by solving time fractional nonlinear reaction-diffusion equations. This method is a combination of the multi-time-stepping temporal generalized differential transform and the spatial finite difference methods. The procedure first converts the time-evolutionary equations into Poisson equations which are then solved using the central difference method. The temporal differential transform method as used in the paper takes care of stability and the finite difference method on the resulting equation results in a system of diagonally dominant linear algebraic equations. The Gauss-Seidel iterative procedure then used to solve the linear system thus has assured convergence. To have optimized convergence rate, numerical experiments were done by using a combination of factors involving multi-time-stepping, spatial step size, and degree of the polynomial fit in time. It is shown that the hybrid technique is reliable, accurate, and easy to apply.

1. Introduction

The nonlinear reaction-diffusion equations have found numerous applications in pattern formation, in many branches of biology, chemistry, and physics [1–4]. Reaction-diffusion (RD) equations have also been applied to other areas of science and can be successfully modelled by the use of fractional order derivatives. [5–18]; for example, the RD equations are employed to describe the CO oxidation on Pt (110) [5], the study of Ca_2^+ waves on *Xenopus* oocytes [11], and the study of reentry in heart tissue [7, 13]. A great deal of effort has been expended over the last 10 years in attempting to find robust and stable numerical and analytical methods for solving fractional partial differential equations of physical interest. There has also been a wide variety of numerical methods, for example, finite difference techniques, finite element methods, spectral techniques, adaptive and nonadaptive algorithms, and so forth, which have been developed for RD's numerical solution [19, 20].

The differential transform method was used first by Zhou [21] who solved linear and nonlinear initial value problems in electric circuit analysis. This method constructs an analytical solution in the form of a polynomial. It is different from the traditional higher order Taylor series method, which requires symbolic computation of the necessary derivatives of the data functions. The Taylor series method computationally takes long time for large orders. The differential transform is an iterative procedure for obtaining analytic Taylor series solution of ordinary or partial differential equations. The method is well addressed in [22–26]. Recently, the application of differential transform method is successfully extended to obtain analytical approximate solutions to ordinary differential equations of fractional order. Fractional differential transform method (FDTM) is a method that Arikoglu and Ozkol [23] developed for solving linear and nonlinear integrodifferential equations of fractional order. This method solves problems with high accuracy while constructing semi-analytic solutions in the polynomial forms. FDTM is based on classical differential transform method, fractional power series, and Caputo fractional derivative [27]. Arikoglu and Ozkol [23] tested their approach on several examples and the results obtained are in good agreement with the existing

ones in the open literature. Momani et al. [28] presented a new generalization of the differential transform method that extended the application of the method to differential equations of fractional order. The new technique is named generalized differential transform method (GDTM) and is based on one-dimensional differential transform, generalized Taylor's formula [29], and Caputo fractional derivative [27].

Fractional partial differential equations (FPDEs) are also an interesting and an important topic. The fractional derivatives and integrals have been occurring in many physical and engineering problems with noninteger orders. Fractional calculus is based on the definition of the fractional derivatives and integrals. They play a major role in engineering, physics, and applied mathematics. FPDEs are used to model complex phenomena since the fractional order differential equations are naturally related to the systems with memory and nonlocal relations in space and time which exist in most physical phenomena. Fractional order differential equations are as stable as their integer order counterpart. One of the fundamental equations of physics is the Schrödinger equation which describes how the quantum state of physical system changes with time. The fractional Schrödinger equation provides us with a general point of view on the relationship between statistical properties of quantum mechanical path and structure of fundamental equations of quantum mechanics [30]. In other words, the fractional quantum mechanics includes the standard quantum mechanics as a particular case. So FPDEs are obtained by generalizing differential equations to an arbitrary order. There are three popular methods for seeking approximate solutions for FPDEs which are the finite difference, finite element and spectral methods. In the literature there are many papers on these three methods. In these papers, the authors proposed the use of least squares finite element solution and fully discrete Galerkin method to solve nonlinear space fractional partial differential equations [20, 31].

The differential transform is well suited to combine with other numerical techniques, as shown by Yu and Chen [32] who applied the hybrid method to solve the transient thermal stress distribution in a perfectly elastic isotropic annular fin. Kuo and Chen [33] employed the hybrid method to solve Burger's equation for flow systems with high Reynolds numbers. This method was also employed to analyze the dynamic response of an electrostatically actuated micro fixed-fixed beam [34].

In the current study, the hybrid generalized differential transform/finite difference method is used for solving time fractional nonlinear RD equations. The validity of the proposed approach has been confirmed by comparing the results derived in the literature using the GDTM method [19], homotopy perturbation method (HPM) [35], and fractional variational method (FVIM) [36].

There are several approaches of definitions for the fractional derivative. Among them, one is called Riemann-Liouville fractional derivatives and defined by

$$D^{\alpha} y(x) = \frac{d^m}{dx^m} J^{m-\alpha} y(x), \quad m-1 < \alpha \le m, \quad m \in \mathbb{N}, \quad x > 0. \tag{1}$$

Here J^{β} is the β-order Riemann-Liouville integral operator which is expressed as follows:

$$J^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \left[\int_0^x (x-t)^{\beta-1} f(t)\, d(t) \right], \quad \beta > 0. \tag{2}$$

If we use this definition, we must know the initial value of some fractional order derivative of the unknown function or we must have homogenous initial conditions. Unlike the Riemann-Liouville approach in the case of the Caputo derivative there are no restrictions on the initial conditions. Thus the following definition is used in this study:

$$D^{\alpha} y(x) = J^{m-\alpha} y^{(m)}(x), \quad m-1 < \alpha \le m, \quad m \in \mathbb{N}, \quad x > 0. \tag{3}$$

This operator is generally called "α-order Caputo differential operator" [37, 38].

For Caputo derivative we have

$$D^{\alpha} t^k = \begin{cases} 0, & k \le \alpha - 1 \\ \dfrac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, & k > \alpha - 1. \end{cases} \tag{4}$$

2. Generalized Differential Transform/Finite Difference Method

We firstly introduce the main features of GDTM [19, 23, 28, 39–41] according to the generalized differential transform of the kth derivative of a function of one variable defined as follows:

$$Y_{\alpha}(k) = \frac{1}{\Gamma(\alpha k + 1)} \left[\left(D^{\alpha}\right)^k y(x) \right] \Big|_{x=x_0}, \tag{5}$$

where $0 < \alpha \le 1$, $(D^{\alpha})^k = D^{\alpha} D^{\alpha} \cdots D^{\alpha}$ k times. In (5), $y(x)$ is the original function, $Y_{\alpha}(k)$ is the transformed function, and the differential inverse transform of $Y_{\alpha}(k)$ is defined as follows:

$$y(x) = \sum_{k=0}^{\infty} Y_{\alpha}(k) \left(x - x_0\right)^{\alpha k}. \tag{6}$$

In case of $\alpha = 1$, GDTM reduces to the classical DTM. From definitions (2) and (3) all fundamental properties of GDTM can be obtained easily [19, 39]. Since $\lim_{\alpha \to 1} D^{\alpha} u = Du$ has been proved from the definitions of fractional calculus, the fractional solutions $u_{\alpha}(x,t)$ reduce to the standard solution $u(x,t)$.

In this study, we use a hybrid method that is a combination of generalized, temporal differential transform and spatial finite difference methods to solve nonlinear fractional reaction diffusion equations.

We present a solution of a more general model of RD equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = D_1 \frac{\partial^2 u}{\partial x^2} + f(u), \quad 0 < x < a, \quad t > 0 \quad (0 < \alpha \le 1), \tag{7}$$

where D_1 is the diffusion coefficient and $f(u)$ is a nonlinear function. We consider two different forms of $f(u)$ which are called time fractional Fisher equation and time fractional FitzHugh-Nagumo equation.

We apply GDTM to discretize fractional order time derivative and central difference method to discretize derivatives in x direction, respectively. After transforming (7) using the GDTM, we get the following equation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\alpha i}(k+1) = D_1\frac{U_{\alpha(i+1)}(k)-2U_{\alpha i}(k)+U_{\alpha(i-1)}(k)}{h^2}+F\left(U_{\alpha i}\right), \tag{8}$$

where $U_{\alpha i}(k)$ is the transformed function of $u(x_i,t)$ and $F(U_{\alpha i})$ is the transformed function of $f(u)$. The region $0 < x < a$ is divided into several equal intervals and each interval has a width h. The time interval of interest is discretized using a time step Δt. After discretization of the equation we get solution at time Δt and these results are adopted as the initial values for the next time interval. This time-stepping procedure assists in obtaining a converged solution to a desired accuracy [39]. An appendix has been added to the paper to show that the error, if any, in the method is bounded. This implies stability of the scheme and by Lax equivalence theorem it thereby implies convergence.

The new algorithm has been developed to solve the nonlinear reaction diffusion equation and our aim of this approach is to combine the flexibility of differential transform and the efficiency of finite differences. This algorithm also provides an iterative procedure to calculate the numerical solutions; therefore, it is not necessary to carry out complicated symbolic computation. On applying the differential transform method with respect to time on the equation we are basically transforming the time-evolutionary equation to an elliptic type. In essence this means that the central finite difference approximation that is subsequently used on the transformed equation is a Poisson solver. The resulting system of linear algebraic equations is then diagonally dominant and hence the Gauss-Seidel iterative method used for solving the same has assured convergence as the coefficient matrix remains nonsingular throughout the computation. The algorithm used thus succeeds in segregating the time discretization from explicitly influencing the computation in the spatial domain and this presents a situation wherein the two can be handled independent of each other in the course of computation without having to bother about the stability of the solution if the differential transform part is properly handled. The latter is achieved deftly in the differential transform part of the algorithm by using the multistepping procedure as first enunciated by Yu and Chen [32] in their phenomenal work and used subsequently by Odibat et al. [29, 39]. In summary, this means that convergence is never in doubt in the algorithm but slow convergence can be if the time and spatial discretizations are badly handled. The convergence is optimized in the paper computationally by proper selection of time step in the differential transform part of the algorithm and then the spatial step size in the finite difference part of the algorithm. Next important step in the algorithm is the decision on the number of terms to be adopted in the inverse differential transform that gives us the solution of the problem as a power series in time. The aforementioned three vital components of the algorithm have been meticulously handled and a brief summary of the numerical experiment undertaken concerning the same is presented in a table. The numerical study recommends that the combination of 5, 10, and 50 spatial step size with a time step of 0.0005 or 0.001 assures the best rate of convergence if we take minimum ten terms in the time series.

3. Illustration of Generalized Differential Transform and Finite Difference Method

To show effectiveness of the proposed numerical solution using the temporal generalized differential transform and the spatial finite difference method and to give an understandable overview of the methodology, two examples of the reaction diffusion equations will be discussed in the following section. Then our results will be compared with published work of Rida et al. [19] in which GDTM was used to solve the same equations.

Example 1. The time fractional Fisher equation is

$$D_t^{\alpha}u = D_{xx}u + 6u(1-u), \quad x \in \Re, \quad t > 0, \quad (0 < \alpha \le 1). \tag{9}$$

In this example, we have the nonlinear function $f(u) = 6u(1-u)$. The initial condition used is

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{10}$$

Operating the generalized differential transform on (9) gives us the following equation:

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\alpha}(x,k+1) = \frac{\partial^2}{\partial x^2}U_{\alpha}(x,k) + 6U_{\alpha}(x,k) - 6\sum_{l=0}^{k}U_{\alpha}(x,k-l)\,U_{\alpha}(x,l), \tag{11}$$

where $U_{\alpha}(x,k)$ is the generalized differential transform of $u(x,t)$.

Now we apply the central finite difference method to the derivatives with respect to x and this gives us

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}U_{\alpha i}(k+1) = \frac{U_{\alpha(i+1)}(k)-2U_{\alpha i}(k)+U_{\alpha(i-1)}(k)}{h^2} + 6U_{\alpha i}(k) - 6\sum_{l=0}^{k}U_{\alpha i}(k-l)\,U_{\alpha i}(l). \tag{12}$$

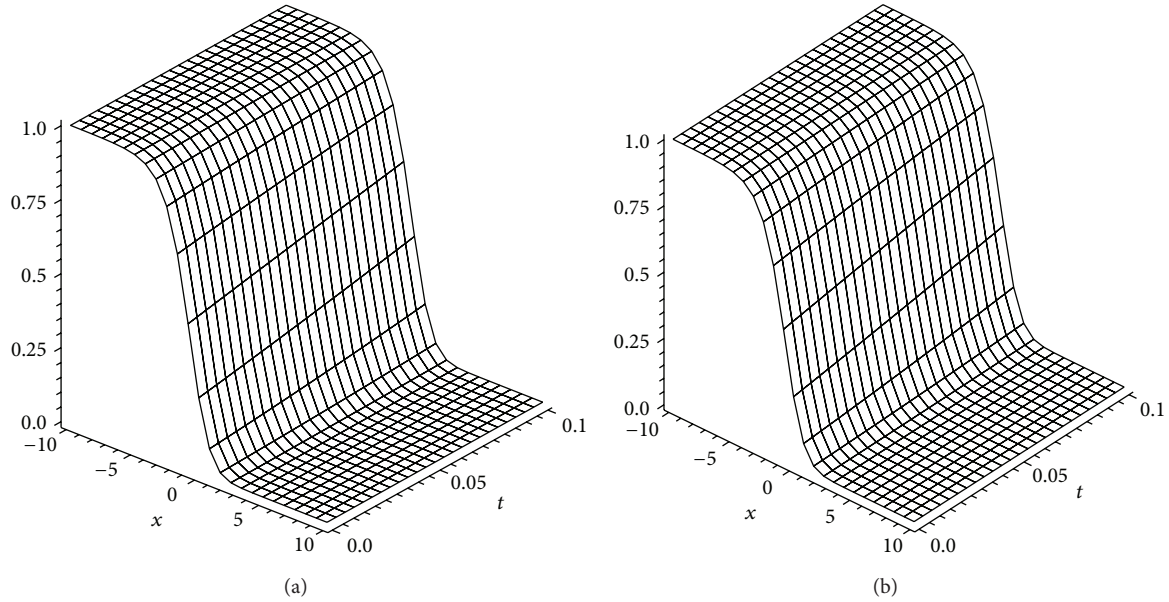


Figure 1: Numerical solution for the time fractional Fisher equation with $\alpha \to 1$ (a); comparison with the analytical solution (b).

Table 1: Some values of $U_{\alpha i}(k)$ of Example 1.

i	k		
	0	1	2
0	0.250000	$\frac{1.249791}{\Gamma(\alpha+1)}$	$\frac{3.126294}{\Gamma(2\alpha+1)}$
1	0.225644	$\frac{1.184363}{\Gamma(\alpha+1)}$	$\frac{3.406094}{\Gamma(2\alpha+1)}$
2	0.202649	$\frac{1.114004}{\Gamma(\alpha+1)}$	$\frac{3.618706}{\Gamma(2\alpha+1)}$

The initial condition on discretization yields

$$U_{\alpha i}(0) = \frac{1}{\left(1+e^{ih}\right)^2}. \tag{13}$$

Equation (12) is a recurrence relation. The time series solution of the given equation is then obtained by using (12) and (13) with $h = 0.1$ to obtain $U_{\alpha i}(k)$. Some of $U_{\alpha i}(k)$ are recorded in Table 1.

The time series solutions of (12) with the initial condition (13) are obtained as follows:

$$\begin{aligned} u(0,t) &= 0.250000 + \frac{1.249791}{\Gamma(\alpha+1)}t^{\alpha} + \frac{3.126294}{\Gamma(2\alpha+1)}t^{2\alpha} + \cdots \\ u(0.1,t) &= 0.225644 + \frac{1.184363}{\Gamma(\alpha+1)}t^{\alpha} + \frac{3.406094}{\Gamma(2\alpha+1)}t^{2\alpha} + \cdots \\ u(0.2,t) &= 0.202649 + \frac{1.114004}{\Gamma(\alpha+1)}t^{\alpha} + \frac{3.618706}{\Gamma(2\alpha+1)}t^{2\alpha} + \cdots \\ &\vdots \end{aligned} \tag{14}$$

The numerical calculation results are shown in Figures 1 and 2, respectively. Our results are in agreement with the published work of Rida et al. [19], who considered the same equation. An exact solution of the standard form of Fisher equation for $\alpha \to 1$ is

$$u(x,t) = \frac{1}{\left(1+e^{x-5t}\right)^2}. \tag{15}$$

The comparison of our results with the exact solution is shown in Figure 1, for $h = 0.1$, and quite clearly good agreement is found.

Approximate solutions are shown in Figure 2 for $\alpha = 0.99$ and $\alpha = 0.95$.

The influence of α on the function $u(x,t)$ is shown in Figure 3. This figure indicates a decrease in the fractional order α by choosing the fixed $x = 5$ that corresponds to an increase in the function and also indicates a slow diffusion for the values of $\alpha = 1$ and $\alpha = 0.9$ and a fast diffusion for the values of $\alpha = 0.8, 0.7, 0.6$, respectively. It is clearly seen that $u(x,t)$ increase for $\alpha = 1, 0.9, 0.8, 0.7, 0.6$ with the increases in t.

Numerical comparison between GDTM [19], HPM [35], FVIM [36], and hybrid method are found in Table 2 which shows hybrid method is more promising.

It is also found that the result is in complete agreement with the result of HPM [42, 43] and ADM [44] for $\alpha = 1$.

We investigate convergence criteria of our solutions for different values of h and n. To illustrate this, we compared our results with the analytical solution in case of $\alpha = 1$. Here n is order of differential transformation method and denotes the number of terms to be calculated.

In Figures 4, 5, and 6 the difference between the results obtained in this study and the results of the analytical solution is of the order of 10^{-5}. This is a pointer to the fact that there is convergence and is a restatement in numerical terms of what was shown in the Appendix.

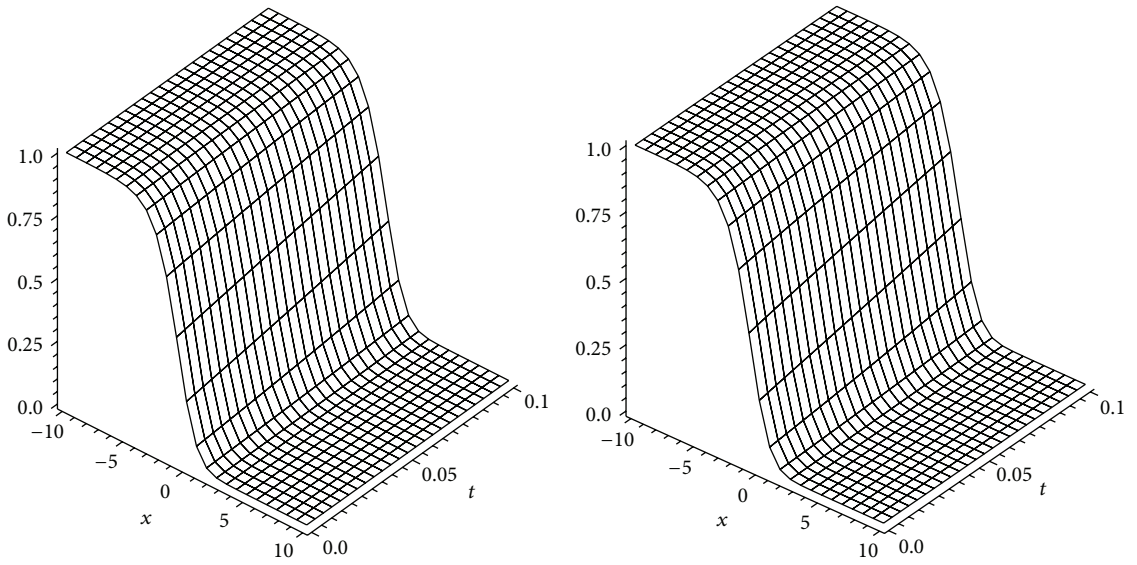


Figure 2: Numerical solution for the time fractional Fisher equation with (a) $\alpha \to 0.99$ and (b) $\alpha \to 0.95$.

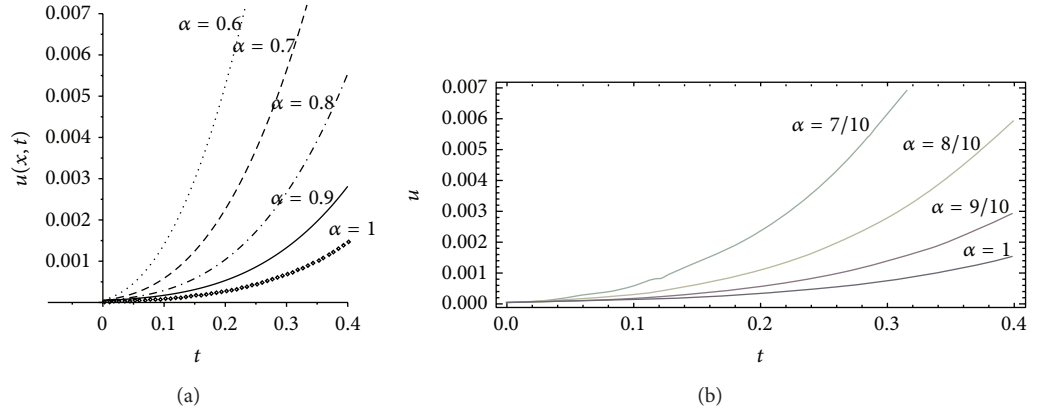


Figure 3: Approximate solution for the time fractional Fisher equation with different α values at $x = 5$ (a) present and (b) [35].

Table 2: Comparison of numerical results between different methods for the time fractional Fisher equation. GDTM: generalized differential transform method, Rida et al. [19], HPM: homotpy perturbation method, Khan et al. [35], and FVIM: fractional variational iteration method, Merdan [36].

			$\alpha = 0.75$			
t	x	U_{GDTM}	U_{HPM}	U_{FVIM}	Present	
0.02	2	0.0236384265	0.0241395370	0.0236296987	0.0241417516	
0.04	2	0.0325745492	0.0351552680	0.0325330330	0.0351447962	
0.06	2	0.0420810751	0.0489473130	0.0419776983	0.0488819720	
0.08	2	0.0521879221	0.0660928586	0.0519904364	0.0658898759	
			$\alpha = 1$			
t	x	U_{GDTM}	U_{HPM}	U_{FVIM}	Present	Exact
0.02	2	0.0169179992	0.0169279385	0.0169177486	0.0169293607	0.0169282151
0.04	2	0.0200377701	0.0201117284	0.0200357651	0.0201240442	0.0201217246
0.06	2	0.0235686494	0.0238370103	0.0235618826	0.0238628770	0.0238595181
0.08	2	0.0275106370	0.0281487518	0.0274945972	0.0282219465	0.0282178229

Table 3: Some values of $U_{\alpha i}(k)$ of Example 2.

i	k		
	0	1	2
0	0.5	$\frac{(0.125-0.25\mu)}{\Gamma(\alpha+1)}$	$\frac{(0.000011-0.000052\mu)}{\Gamma(2\alpha+1)}$
1	0.517670	$\frac{(0.124847-0.249687\mu)}{\Gamma(\alpha+1)}$	$\frac{(-0.002179+0.008772\mu-0.008824\mu^2)}{\Gamma(2\alpha+1)}$
2	0.535296	$\frac{(0.124384-0.248754\mu)}{\Gamma(\alpha+1)}$	$\frac{(-0.004379+0.017511\mu-0.017560\mu^2)}{\Gamma(2\alpha+1)}$

One important observation made from the computation is that when the number of mesh points was increased, less number of terms was required in the time series solution to have convergence for a predetermined accuracy. The hybrid method of the present study gives faster convergence than other traditional methods; for example, if we take $h = 0.02$ (mesh point is 50) then the solution converges for $n = 3$. We now consider another example.

Example 2. The time fractional FitzHugh-Nagumo equation is

$$D_t^\alpha u = D_x^2 u + u(1-u)(u-\mu), \quad \mu > 0, \quad 0 < \alpha \le 1, \quad x \in \Re, \quad t > 0. \tag{16}$$

In this type of equation, the nonlinear function depends on μ and it is $f(u) = u(1-u)(u-\mu)$. The initial condition is

$$u(x,0) = \frac{1}{\left(1+e^{-x/\sqrt{2}}\right)}. \tag{17}$$

Using the hybrid method on the above initial boundary value problem (IBVP), as done in the previous example, we get

$$\begin{aligned}
&\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{\alpha i}(k+1) \\
&= \frac{U_{\alpha(i+1)}(k) - 2U_{\alpha i}(k) + U_{\alpha(i-1)}(k)}{h^2} \\
&\quad - \mu U_{\alpha i}(k) + (1+\mu)\sum_{l=0}^{k} U_{\alpha i}(k-l)\,U_{\alpha i}(l) \\
&\quad - \sum_{s=0}^{k}\sum_{l=0}^{s} U_{\alpha i}(k-s)\,U_{\alpha i}(s-l)\,U_{\alpha i}(l), \\
&\quad U_{\alpha i}(0) = \frac{1}{1+e^{-ih/\sqrt{2}}}.
\end{aligned} \tag{18}$$

Using second order finite difference method, the boundary values were obtained as follows:

$$\begin{aligned}
U_{\alpha 0}(k) &= 3U_{\alpha 1}(k) - 3U_{\alpha 2}(k) + U_{\alpha 3}(k), \\
U_{\alpha N}(k) &= 3U_{\alpha(N-1)}(k) - 3U_{\alpha(N-2)}(k) + U_{\alpha(N-3)}(k).
\end{aligned} \tag{19}$$

Table 3 presents some of the $U_{\alpha i}(k)$'s.

The time series solution for the above IBVP at different times is

$$\begin{aligned}
&u(0,t) \\
&= 0.5 + \frac{(0.125-0.25\mu)}{\Gamma(\alpha+1)} t^\alpha \\
&\quad + \frac{(0.000011-0.000052\mu)}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots \\
&u(0.1,t) \\
&= 0.517670 + \frac{(0.124847-0.249687\mu)}{\Gamma(\alpha+1)} t^\alpha \\
&\quad + \frac{\left(-0.002179+0.008772\mu-0.008824\mu^2\right)}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots \\
&u(0.2,t) \\
&= 0.535296 + \frac{(0.124384-0.248754\mu)}{\Gamma(\alpha+1)} t^\alpha \\
&\quad + \frac{\left(-0.004379+0.01751\mu-0.017560\mu^2\right)}{\Gamma(2\alpha+1)} t^{2\alpha} + \cdots \\
&\vdots
\end{aligned} \tag{20}$$

Numerical solutions for the time fractional FitzHugh-Nagumo equation with various α values are shown in Figures 7 and 8. A comparison of the results in a limiting case wherein an analytical solution exists is shown in Figure 7. The results are in close agreement with those of Rida et al. [19] for the same equation.

For $\alpha \to 1$, it can easily be seen that the exact solution of FitzHugh-Nagumo equation is

$$u(x,t) = \frac{1}{1+e^{-(1/\sqrt{2})(x+((1-2\mu)/\sqrt{2})t)}}. \tag{21}$$

Figure 9 is prepared to show the influence of α on the function $u(x,t)$. It is clearly seen that $u(x,t)$ decrease for $\alpha = 1, 0.95, 0.85, 0.75, 0.65$ with the decreases in t.

As shown in the Table 4, our results show close agreement with the exact solution and agree with those of Rida et al. [19].

Table 4: Coefficients of $1, t, t^2$ for some i values and comparison with exact and Rida's solution $\mu = 0.7$ [19].

	Rida et al. [19]	Exact	Present
	$i = 0$		
Coef. of t^0	0.5	0.5	0.5
Coef. of t^1	−0.05	−0.05	−0.049999
Coef. of t^2	0.02	0	−0.000011
	$i = 1$		
Coef. of t^0	0.517670	0.517670	0.517670
Coef. of t^1	−0.049937	−0.0499937	−0.049933
Coef. of t^2	0.020327	−0.000176	−0.000181
	$i = 2$		
Coef. of t^0	0.535296	0.535296	0.535296
Coef. of t^1	−0.049937	−0.049937	−0.049743
Coef. of t^2	0.020602	−0.000351	−0.000352
	$i = 3$		
Coef. of t^0	0.552835	0.552835	0.552835
Coef. of t^1	−0.049441	−0.049441	−0.049430
Coef. of t^2	0.020821	−0.000522	−0.000542

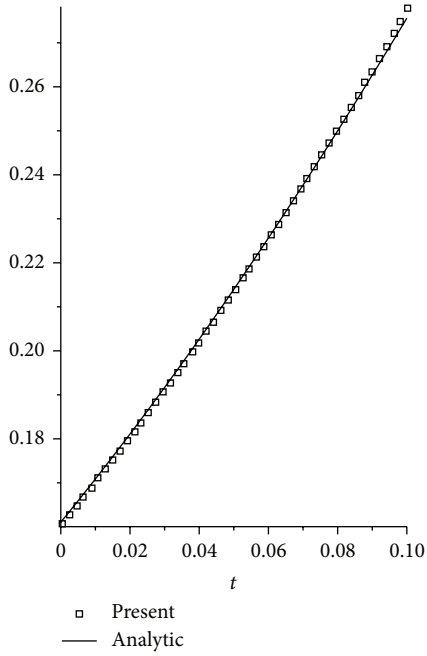


Figure 4: Comparison of present results for $h = 0.2$ and $n = 10$ with the analytical solution in case of $\alpha \to 1$ at $x = 0.4$.

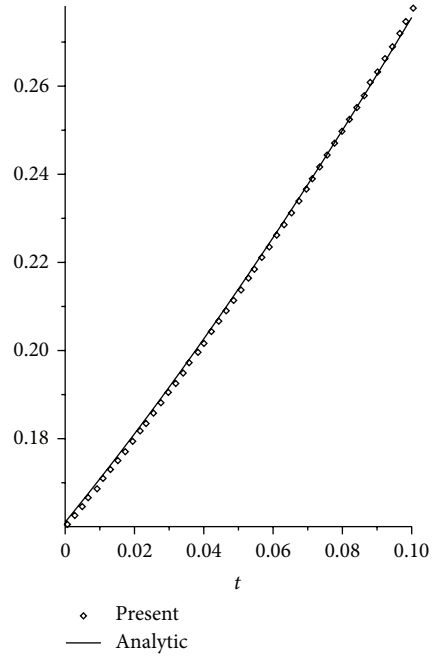


Figure 5: Comparison of present results for $h = 0.1$ and $n = 5$ with the analytical solution in case of $\alpha \to 1$ at $x = 0.4$.

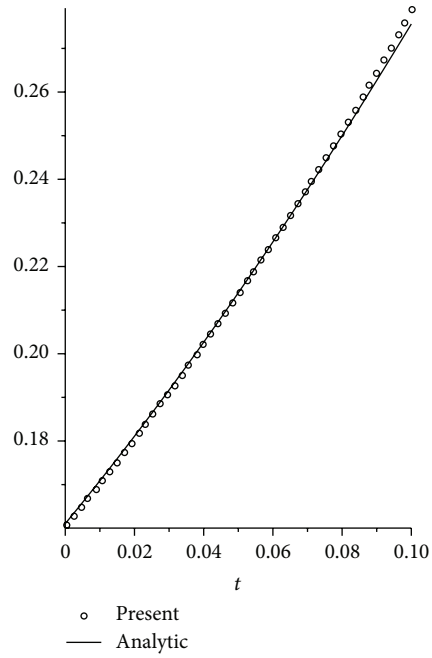


Figure 6: Comparison of present results for $h = 0.02$ and $n = 3$ with the analytical solution in case of $\alpha \to 1$ at $x = 0.4$.

From this table it is clear that the present work gives better approximation than GDTM as we increase n.

Numerical comparison between GDTM, FVIM, and hybrid method is shown in Table 5, which indicates hybrid method is more promising.

4. Conclusion

Many real physical problems can be best modelled with fractional differential equations but the fact is when the equation is nonlinear there are very few reliable methods. The numerical methods that can be used to solve fractional differential equations are known to have problems of convergence and stability. These aspects are well addressed in the paper by suggesting a new procedure that uses a combination of the generalized differential transform and central difference methods. The Appendix clearly spells out

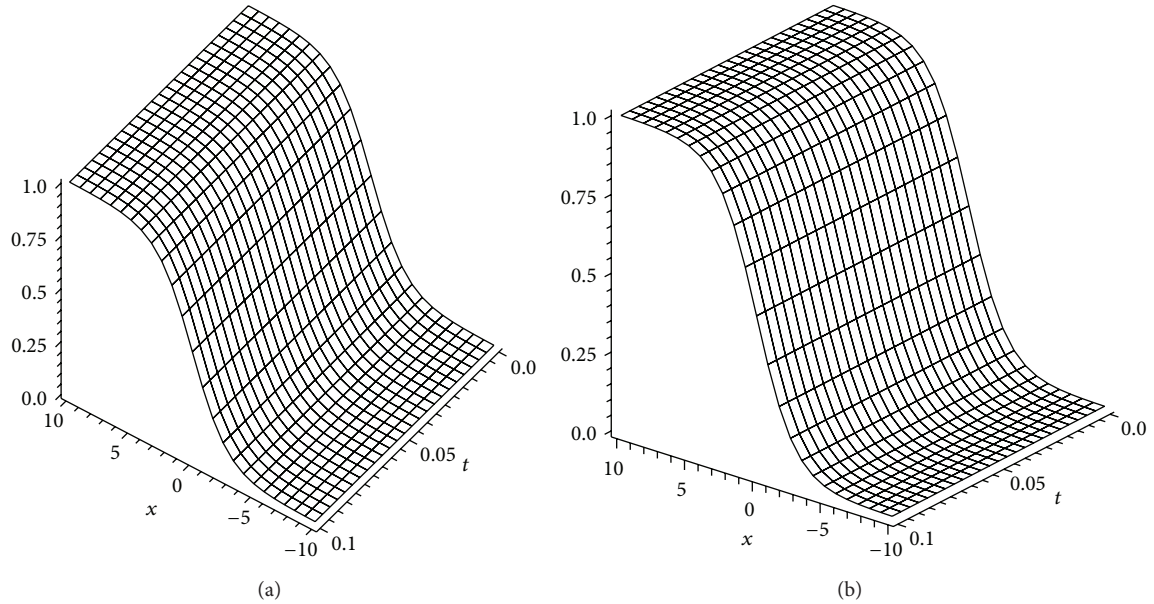


Figure 7: Numerical solution for the time fractional FitzHugh-Nagumo equation with $\alpha \to 1$ (a); comparison with the analytical solution (b) with $\mu = 0.5$.

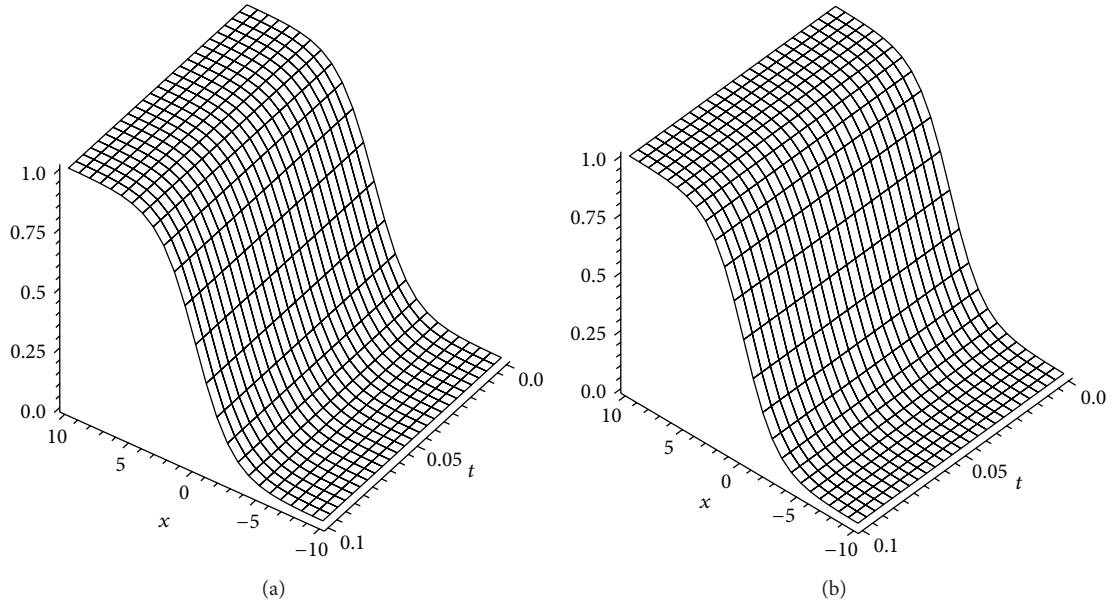


Figure 8: Numerical solution for the time fractional FitzHugh-Nagumo equation with $\alpha = 0.95$ (a) and $\alpha = 0.99$ (b).

the fact that the error as a result of discretization and computation is bounded and hence implies stability of the method. Lax equivalence theorem further implies convergence of the scheme. Two time fractional nonlinear reaction-diffusion equations considered for illustration of the hybrid method highlight the usefulness of the method in obtaining the solution of IBVPs involving time fractional derivatives. The control of convergence through a judicious choice of time and spatial step sizes and also the number of terms in the time series solution spells assured convergence. The segregation of the time domain from the spatial domain in the solution method ensures the fact that problem of stability does not arise. Diagonal dominance of the coefficient matrix in the system of linear algebraic equations resulting from the use of the central difference approximation in the Poisson equation ensures the fact that the matrix remains nonsingular during iterations and hence has assured convergence. An appropriate computational decision on the number of terms to be taken in the time series solution results in a convergent solution with fast convergence. Excellent comparison of the present results with the previous works on generalized differential transform method [19] and homotopy perturbation method

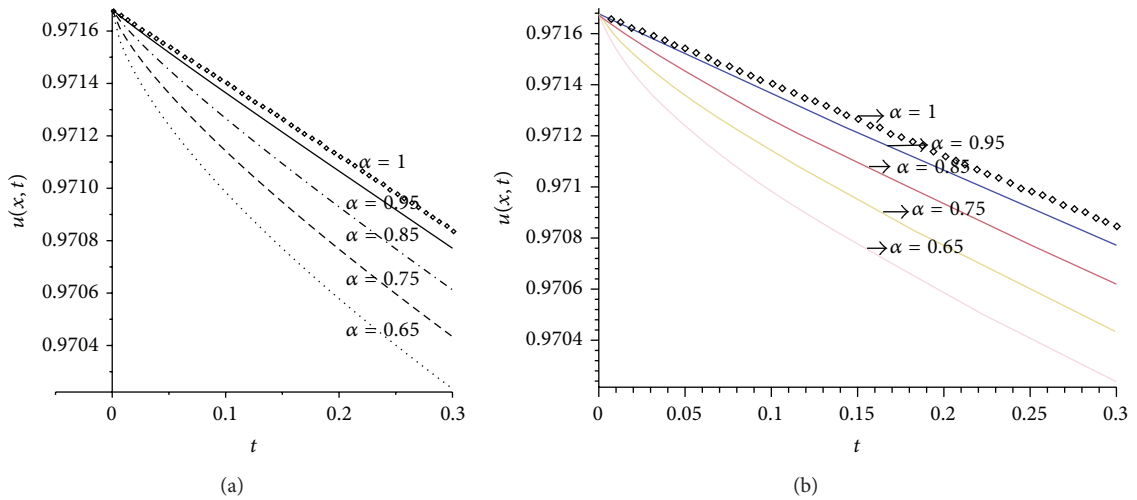


FIGURE 9: Approximate solution for the time fractional FitzHugh-Nagumo equation with different α values at $x = 5$ (a) present and (b) [36].

[35] and fractional variational iteration method [36] provides confidence in the methodology adopted for the solution of time fractional differential equations.

Appendix

Estimation of Bounds on Truncation Error

Consider the fractional differential equations (9) and (16) in a general form as

$$D_t^{\alpha} u = D_{xx} u + f(u), \quad t \geq 0, \quad x \in \Re \quad (0 < \alpha \leq 1). \tag{A.1}$$

The differential transform of (A.1) at the spatially discretized points x_i gives us

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{\alpha}(x_i, k+1) = D_{xx} U_{\alpha}(x_i, k) + F(U_{\alpha}(x_i, k)), \quad k = 0, 1, 2, \ldots, \tag{A.2}$$

where $U_{\alpha}(x_i, k)$ is the differential transformed function of $u(x_i, t)$ and $F(U_{\alpha})$ is that of $f(u)$. Let us further denote $u(x_i, t_k)$ as $u_i(t_k)$ and its transform by $U_{\alpha i}(k)$. In this notation, (A.2) reads as

$$\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)} U_{\alpha i}(k+1) = D_{xx} U_{\alpha i}(k) + F(U_{\alpha i}(k)), \quad k = 0, 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, N. \tag{A.3}$$

We now follow Jang et al. [45] and move on to arrive at an estimate on the bounds for the truncation error in a general way by considering the Taylor series expansion of $U_{\alpha i}(t_k + \Delta t)$:

$$U_{\alpha i}(t_k + \Delta t) = U_{\alpha i}(t_k) + \Delta t \frac{dU_{\alpha i}}{dt}(t_k) + \cdots + \frac{(\Delta t)^n}{n!} \frac{d^n U_{\alpha i}}{dt^n}(t_k) + \frac{(\Delta t)^{n+1}}{(n+1)!} \frac{d^{n+1} U_{\alpha i}}{dt^{n+1}}(\xi), \quad k = 0, 1, 2, \ldots, \quad i = 0, 1, 2, \ldots, N, \tag{A.4}$$

where $(1/(n+1)!)(d^{n+1}U_{\alpha i}/dt^{n+1})(\xi)(\Delta t)^{n+1}$, $t_k < \xi < t_{k+1}$ is the remainder. Let $w_i(t_k)$ denote an approximate solution to $U_{\alpha i}(t_k)$ that satisfies

$$\begin{aligned} w_i(t_k + \Delta t) &= w_i(t_k) + \Delta t \phi(t_k, w_i(t_k); \Delta t) \\ &= U_{\alpha i}(t_k) + \Delta t \frac{dU_{\alpha i}}{dt}(t_k) \\ &\quad + \cdots + \frac{(\Delta t)^n}{n!} \frac{d^n U_{\alpha i}}{dt^n}(t_k). \end{aligned} \tag{A.5}$$

Clearly the difference between $U_{\alpha i}(t_k + \Delta t)$ and $w_i(t_k + \Delta t)$ is of $O[(\Delta t)^{n+1}]$. Thus, the local error is

$$\left| U_{\alpha i}(t_k + \Delta t) - w_i(t_k + \Delta t) \right| = \frac{1}{(n+1)!} \frac{d^{n+1} U_{\alpha i}}{dt^{n+1}}(\xi) (\Delta t)^{n+1}. \tag{A.6}$$

Let us now suppose that

$$\operatorname{Max}_r \left| \frac{1}{r!} \frac{d^r U_{\alpha i}}{dt^r}(\xi) \right| = \beta, \quad r = n+1, n+2, \ldots, \infty. \tag{A.7}$$

Using (A.7) in (A.6), we get

$$\left| U_{\alpha i}(t_k + \Delta t) - w_i(t_k + \Delta t) \right| < \beta (\Delta t)^{n+1}. \tag{A.8}$$

Table 5: Comparison of numerical results between different methods for the time fractional FitzHugh-Nagumo equation. GDTM: generalized differential transform method, Rida et al. [19], and FVIM: fractional variational iteration method, Merdan, [36].

			$\alpha = 0.7, \mu = 0.6$		
t	x	U_{GDTM}	U_{FVIM}	Present	
0.2	0	0.49150484	0.48896821	0.48191202	
—	0.25	0.53566881	0.53312591	0.53520376	
—	0.5	0.57927974	0.57677042	0.57880163	
—	0.75	0.62168644	0.61924846	0.62118883	
—	1	0.66230919	0.65997604	0.74220689	
			$\alpha = 1, \mu = 0.6$		
t	x	U_{GDTM}	U_{FVIM}	Present	Exact
0.2	0	0.49510000	0.49450005	0.49473894	0.49500016
—	0.25	0.53922189	0.53862130	0.53911415	0.53911409
—	0.5	0.58273747	0.58214558	0.58262443	0.58262371
—	0.75	0.62500118	0.62442684	0.62488468	0.62488385
—	1	0.66544142	0.66489240	0.66761495	0.66532300
			$\alpha = 0.7, \mu = 0.6$		
t	x	U_{GDTM}	U_{FVIM}	Present	
0	0.2	0.53529653	0.53529653	0.53529653	
0.05	—	0.53199661	0.53163167	0.53193245	
0.1	—	0.52999926	0.52903570	0.52983082	
0.15	—	0.52833265	0.52663211	0.52803767	
0.2	—	0.52685856	0.52431371	0.52642106	
			$\alpha = 1, \mu = 0.6$		
t	x	U_{GDTM}	U_{FVIM}	Present	Exact
0	0.2	0.53529653	0.53529653	0.53529653	0.53529653
0.05	—	0.53405919	0.53402165	0.53405262	0.53405254
0.1	—	0.53283474	0.53268452	0.53280831	0.53280813
0.15	—	0.53162316	0.53128507	0.53156363	0.53156331
0.2	—	0.53042446	0.52982323	0.53031869	0.53031809

We now consider more numbers of terms in Taylor expansion (A.4); that is,

$$U_{\alpha i}(t_k + \Delta t) = U_{\alpha i}(t_k) + \Delta t \frac{dU_{\alpha i}}{dt}(t_k) + \cdots + \frac{(\Delta t)^{n+m}}{(n+m)!} \frac{d^{n+m}U_{\alpha i}}{dt^{n+m}}(t_k) + \frac{(\Delta t)^{n+m+1}}{(n+m+1)!} \frac{d^{n+m+1}U_{\alpha i}}{dt^{n+m+1}}(\xi). \tag{A.9}$$

As done earlier, let us denote by $\widetilde{w}_i(t_k + \Delta t)$ the following expression:

$$\widetilde{w}_i(t_k + \Delta t) = \widetilde{w}_i(t_k) + (\Delta t)\,\varphi(t_k, \widetilde{w}_i(t_k); \Delta t) = U_{\alpha i}(t_k) + \Delta t \frac{dU_{\alpha i}}{dt}(t_k) + \cdots + \frac{(\Delta t)^{n+m}}{(n+m)!} \frac{d^{n+m}U_{\alpha i}}{dt^{n+m}}(t_k). \tag{A.10}$$

Again, as earlier, let us suppose that

$$\operatorname{Max}_s \left| \frac{1}{s!} \frac{d^s U_{\alpha i}}{dt^s}(\xi) \right| = \widetilde{\beta}, \quad s = n+m+1, n+m+2, \ldots, \infty. \tag{A.11}$$

Similar to (A.8), we now get

$$\left| U_{\alpha i}(t_k + \Delta t) - \widetilde{w}_i(t_k + \Delta t) \right| < \widetilde{\beta} (\Delta t)^{n+m+1}. \tag{A.12}$$

We so far addressed the local error due to two different truncations in the time series. In what follows we estimate the bounds on the cumulative error that includes the error discussed above.

Let $y_i(t_k)$ denote the solution of (A.2). The local error in $w_i(t_k)$ relative to $y_i(t_k)$ is

$$\left| y_i(t_k + \Delta t) - w_i(t_k + \Delta t) \right| \le \left| y_i(t_k + \Delta t) - \widetilde{w}_i(t_k + \Delta t) \right| + \left| \widetilde{w}_i(t_k + \Delta t) - w_i(t_k + \Delta t) \right|. \tag{A.13}$$

Since $\widetilde{w}_i(t_k + \Delta t)$ is a better approximation than $w_i(t_k + \Delta t)$, we may assume that

$$\left| y_i(t_k + \Delta t) - \widetilde{w}_i(t_k + \Delta t) \right| \ll 1. \tag{A.14}$$

In view of (A.14), we now have

$$\left|y_i\left(t_k+\Delta t\right)-w_i\left(t_k+\Delta t\right)\right| \doteq \left|\widetilde{w}_i\left(t_k+\Delta t\right)-w_i\left(t_k+\Delta t\right)\right|. \tag{A.15}$$

Using (A.8) and noting that $(\Delta t)^{n+1}$ is quite small in (A.15), we may take β to be

$$\beta \doteq \frac{\left|\widetilde{w}_i\left(t_k+\Delta t\right)-w_i\left(t_k+\Delta t\right)\right|}{(\Delta t)^{n+1}}. \tag{A.16}$$

Thus $\beta(\Delta t)^{n+1}=\varepsilon$ is the bound on the tolerance in the to-be-obtained solution. When using different number of terms in the Taylor series expansion, earlier we denoted the solutions using a time step Δt by $w_i(t_k+\Delta t)$ and $\widetilde{w}_i(t_k+\Delta t)$, respectively.

The paper uses an adaptive step size in computing the results. This is because such a procedure succeeds in keeping the error bounded and ensures convergence as a consequence of Lax equivalence theorem. To see what the adaptive step size produces and to show how such a procedure keeps the error bounded we start with the premise that Δt is the most appropriate step size for the problem. This step size is determined using the definition of inverse differential transform:

$$u_i\left(\Delta t\right)=\sum_{k=0}^{\infty}U_{\alpha i}\left(k\right)\left(\Delta t\right)^k. \tag{A.17}$$

In our actual calculation we will not be able to consider infinite number of terms. We consider "n" terms in respect of w_i and "$n+m$" terms in respect of $\widetilde{w}_i$.

Thus

$$\begin{aligned}&\left|\widetilde{w}_i\left(t_k+\Delta t\right)-w_i\left(t_k+\Delta t\right)\right|\\&=\sum_{j=0}^{n+m}U_{\alpha i}\left(j\right)\left(\Delta t\right)^j-\sum_{j=0}^{n}U_{\alpha i}\left(j\right)\left(\Delta t\right)^j\\&=\sum_{j=n+1}^{n+m}U_{\alpha i}\left(j\right)\left(\Delta t\right)^j.\end{aligned} \tag{A.18}$$

To write down a simpler expression we change the summation index from j to $p=n+j$. So we have from (A.18) the following:

$$\left|\widetilde{w}_i\left(t_k+\Delta t\right)-w_i\left(t_k+\Delta t\right)\right|=\sum_{p=1}^{m}U_{\alpha i}\left(n+p\right)\left(\Delta t\right)^{n+p}. \tag{A.19}$$

Thus, (A.16) on using (A.19) may be written as

$$\beta \doteq \left|\sum_{p=1}^{m}U_{\alpha i}\left(n+p\right)\left(\Delta t\right)^{p-1}\right|. \tag{A.20}$$

Using yet another step size Δt, also relation (A.20) is satisfied. Let $\overline{w}_i(k)$ be the solution using Δt_1. So from (A.8) we now have

$$\left|U_{\alpha i}\left(k+1\right)-w_i\left(k+1\right)\right|<\beta\left(\Delta t_1\right)^{n+1}<\varepsilon. \tag{A.21}$$

Using (A.20) and (A.21), we may write

$$\Delta t_1<\left(\frac{\varepsilon}{\left|\sum_{p=1}^{m}U_{\alpha i}\left(n+p\right)\left(\Delta t\right)^{p-1}\right|}\right)^{1/(n+1)}, \tag{A.22}$$

for $m=1$,

$$\Delta t_1<\left(\frac{\varepsilon}{\left|U_{\alpha i}\left(n+1\right)\right|}\right)^{1/(n+1)}. \tag{A.23}$$

Thus, the above proceedings tell us that if criterion (A.23) is satisfied, then the error is bounded. In effect, this means that the scheme is convergent in lieu of Lax equivalence theorem. In our computations Δt has been always chosen to satisfy inequality (A.23).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are thankful to Ondokuz Mayıs University, Samsun, Turkey, for providing financial support to carry out this work under a major research project (Grant no. pyo.fen.1901.13.003).

References

[1] W. Hundsdorfer and J. G. Verwer, *Numerical Solution of Time Dependent Advection-Diffusion-Reaction Equations*, Springer, Berlin, Germany, 2003.

[2] Y. Kuramoto, *Chemical Oscillations Waves and Turbulence*, Dover, Mineola, NY, USA, 2003.

[3] J. D. Murray, *Mathematical Biology. II*, vol. 18 of *Interdisciplinary Applied Mathematics*, Springer, New York, NY, USA, 3rd edition, 2003.

[4] H. Wilhelmsson and E. Lazzaro, *Reaction-Diffusion Problems in the Physics of Hot Plasmas*, Institute of Physics Publishing, Philadelphia, Pa, USA, 2001.

[5] M. Bär, N. Gottschalk, M. Eiswirth, and G. Ertl, "Spiral waves in a surface reaction: model calculations," *The Journal of Chemical Physics*, vol. 100, no. 2, pp. 1202–1214, 1994.

[6] D. Barkley, "A model for fast computer simulation of waves in excitable media," *Physica D: Nonlinear Phenomena*, vol. 49, no. 1-2, pp. 61–70, 1991.

[7] F. H. Fenton, E. M. Cherry, H. M. Hastings, and S. J. Evans, "Multiple mechanisms of spiral wave breakup in a model of cardiac electrical activity," *Chaos*, vol. 12, no. 3, pp. 852–892, 2002.

[8] M. Gosak, M. Marhl, and M. Perc, "Spatial coherence resonance in excitable biochemical media induced by internal noise," *Biophysical Chemistry*, vol. 128, no. 2-3, pp. 210–214, 2007.

[9] A. Karma, "Meandering transition in two-dimensional excitable media," *Physical Review Letters*, vol. 65, no. 22, pp. 2824–2827, 1990.

[10] J. P. Keener, "A geometrical theory for spiral waves in excitable media," *SIAM Journal on Applied Mathematics*, vol. 46, no. 6, pp. 1039–1056, 1986.

[11] J. P. Keener, *Mathematical Physiology, Interdisciplinary Applied Mathematics*, Springer, New York, NY, USA, 1998.

[12] V. M. Kenkre and M. N. Kuperman, "Applicability of the Fisher equation to bacterial population dynamics," *Physical Review E*, vol. 67, no. 5, Article ID 051921, 5 pages, 2003.

[13] V. Krinsky and A. Pumir, "Models of defibrillation of cardiac tissue," *Chaos*, vol. 8, no. 1, pp. 188–203, 1998.

[14] N. F. Otani, "A primary mechanism for spiral wave meandering," *Chaos*, vol. 12, no. 3, pp. 829–842, 2002.

[15] M. Perc, "Spatial coherence resonance in excitable media," *Physical Review E*, vol. 72, no. 3, Article ID 016207, 2005.

[16] M. Perc, "Stochastic resonance on excitable small-world networks via a pacemaker," *Physical Review E*, vol. 76, no. 6, Article ID 066203, 2007.

[17] M. Perc, "Effects of small-world connectivity on noise-induced temporal and spatial order in neural media," *Chaos, Solitons & Fractals*, vol. 31, no. 2, pp. 280–291, 2007.

[18] J. J. Tyson, "What everyone should know about the Belousov-Zhabotinsky reaction," in *Frontiers in Mathematical Biology*, vol. 100 of *Lecture Notes in Biomathematics*, pp. 569–587, Springer, Berlin, Germany, 1994.

[19] S. Z. Rida, A. M. El-Sayed, and A. A. Arafa, "On the solutions of time-fractional reaction-diffusion equations," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 12, pp. 3847–3854, 2010.

[20] Y. Zheng and Z. Zhao, "A fully discrete Galerkin method for a nonlinear space-fractional diffusion equation," *Mathematical Problems in Engineering*, vol. 2011, Article ID 171620, 20 pages, 2011.

[21] J. K. Zhou, *Differential Transformation and Its Applications for Electrical Circuits*, Huazhang University Press, Wuhan, China, 1986, (Chinese).

[22] I. H. Abdel-Halim Hassan, "Comparison differential transformation technique with Adomian decomposition method for linear and nonlinear initial value problems," *Chaos, Solitons and Fractals*, vol. 36, no. 1, pp. 53–65, 2008.

[23] A. Arikoglu and I. Ozkol, "Solution of boundary value problems for integro-differential equations by using differential transform method," *Applied Mathematics and Computation*, vol. 168, no. 2, pp. 1145–1158, 2005.

[24] F. Ayaz, "Solutions of the system of differential equations by differential transform method," *Applied Mathematics and Computation*, vol. 147, no. 2, pp. 547–567, 2004.

[25] N. Bildik, A. Konuralp, F. Orak Bek, and S. Kucukarslan, "Solution of different type of the partial differential equation by differential transform method and Adomian's decomposition method," *Applied Mathematics and Computation*, vol. 172, no. 1, pp. 551–567, 2006.

[26] H. Liu and Y. Song, "Differential transform method applied to high index differential-algebraic equations," *Applied Mathematics and Computation*, vol. 184, no. 2, pp. 748–753, 2007.

[27] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent. Part II," *Geophysical Journal of the Royal Astronomical Society*, vol. 13, no. 5, pp. 529–539, 1967.

[28] S. Momani, Z. Odibat, and V. S. Erturk, "Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation," *Physics Letters A*, vol. 370, no. 5-6, pp. 379–387, 2007.

[29] Z. M. Odibat, C. Bertelle, M. A. Aziz-Alaoui, and G. H. Duchamp, "A multi-step differential transform method and application to non-chaotic or chaotic systems," *Computers & Mathematics with Applications*, vol. 59, no. 4, pp. 1462–1472, 2010.

[30] N. Laskin, "Fractional Schrödinger Equation," *Physical Review E*, vol. 66, no. 5, Article ID 056108, 2002.

[31] G. J. Fix and J. P. Roop, "Least squares finite-element solution of a fractional order two-point boundary value problem," *Computers & Mathematics with Applications*, vol. 48, no. 7-8, pp. 1017–1033, 2004.

[32] L.-T. Yu and C.-K. Chen, "Application of the hybrid method to the transient thermal stresses response in isotropic annular fins," *Journal of Applied Mechanics*, vol. 66, no. 2, pp. 340–347, 1999.

[33] B.-L. Kuo and C.-K. Chen, "Application of a hybrid method to the solution of the nonlinear burgers' equation," *Journal of Applied Mechanics, Transactions ASME*, vol. 70, no. 6, pp. 926–929, 2003.

[34] C. K. Chen, H. Y. Lai, and C. C. Liu, "Application of hybrid differential transformation/finite difference method to nonlinear analysis of micro fixed-fixed beam," *Microsystem Technologies*, vol. 15, no. 6, pp. 813–820, 2009.

[35] N. A. Khan, M. Ayaz, L. Jin, and A. Yildirim, "On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type," *International Journal of Physical Sciences*, vol. 6, no. 10, pp. 2483–2496, 2011.

[36] M. Merdan, "Solutions of time-fractional reaction-diffusion equation with modified Riemann-Liouville derivative," *International Journal of Physical Sciences*, vol. 7, no. 15, pp. 2317–2326, 2012.

[37] C. Li and G. Peng, "Chaos in Chen's system with a fractional order," *Chaos, Solitons & Fractals*, vol. 22, no. 2, pp. 443–450, 2004.

[38] Y. Luchko and R. Gorenflo, "An operational method for solving fractional differential equations with the Caputo derivatives," *Acta Mathematica Vietnamica*, vol. 24, no. 2, pp. 207–233, 1999.

[39] Z. Odibat and S. Momani, "A generalized differential transform method for linear partial differential equations of fractional order," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 194–199, 2008.

[40] Z. M. Odibat and N. T. Shawagfeh, "Generalized Taylor's formula," *Applied Mathematics and Computation*, vol. 186, no. 1, pp. 286–293, 2007.

[41] D. Nazari and S. Shahmorad, "Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 883–891, 2010.

[42] D. Agırseven and T. Ozis, "An analytical study for Fisher type equations by using homotopy perturbation method," *Computers & Mathematics with Applications*, vol. 60, no. 3, pp. 602–609, 2010.

[43] M. Dehghan, J. M. Heris, and A. Saadatmandi, "Application of semi-analytic methods for the Fitzhugh-Nagumo equation,

which models the transmission of nerve impulses," *Mathematical Methods in the Applied Sciences*, vol. 33, no. 11, pp. 1384–1398, 2010.

[44] A.-M. Wazwaz and A. Gorguis, "An analytic study of Fisher's equation by using Adomian decomposition method," *Applied Mathematics and Computation*, vol. 154, no. 3, pp. 609–620, 2004.

[45] M.-J. Jang, C.-L. Chen, and Y.-C. Liy, "On solving the initial-value problems using the differential transformation method," *Applied Mathematics and Computation*, vol. 115, no. 2-3, pp. 145–160, 2000.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 212760, 8 pages
http://dx.doi.org/10.1155/2015/212760

Research Article

On Analysis of Fractional Navier-Stokes Equations via Nonsingular Solutions and Approximation

Emile Franc Doungmo Goufo and Stella Mugisha

Department of Mathematical Sciences, University of South Africa, Florida Science Campus, Florida, Gauteng 0003, South Africa

Correspondence should be addressed to Emile Franc Doungmo Goufo; dgoufef@unisa.ac.za

Received 16 June 2014; Accepted 24 July 2014

Academic Editor: Abdon Atangana

Until now, all the investigations on fractional or generalized Navier-Stokes equations have been done under some restrictions on the different values that can take the fractional order derivative parameter β. In this paper, we analyze the existence and stability of nonsingular solutions to fractional Navier-Stokes equations of type ($\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \mathrm{Re}^{-1}(-\nabla)^{\beta}\mathbf{u} = \mathbf{f}$ in $\Omega \times (0, T]$) defined below. In the case where $\beta = 2$, we show that the stability of the (quadratic) convergence, when exploiting Newton's method, can only be ensured when the first guess $\mathscr{U}^0$ is sufficiently near the solution $\mathscr{U}$. We provide interesting well-posedness and existence results for the fractional model in two other cases, namely, when $1/2 < \beta < 1$ and $\beta \geq (1/2) + (3/4)$.

1. Introduction

Navier-Stokes equations (NSEs) are considered as the foundation of fluid mechanics and, strangely enough, are rarely recorded in their entirety. They describe motions of Newtonian fluid flows irrespective of whether they are laminar or turbulent. NSEs are usually obtained according to the following considerations [1].

(i) Constitutive relations governing the fluid are isotropic, which means independent of the orientation of the coordinate system axes.

(ii) At rest the fluid obeys the laws of statics.

(iii) The equation of the fluid is objective; that is, tensors are used. It is independent of the Galilean reference frame in which it is expressed, and independent of the observer.

Although existence and characterization of smooth or nonsingular solutions of NSEs have been comprehensively investigated in numerous works [2–7], they remain open problems. Fractional calculus applied to differential equations has captured a huge amount of attention recently [8–12] and it was shown that its applications lie in modeling many problems and phenomena in applied technology, engineering, and sciences including applied mathematics, computing, physics, biology, chemistry, economic, and other domains of applications.

2. A Brief Overview on Fractional Order Derivatives

There is a huge amount of works concerning various definitions of fractional derivatives, but two of them have emerged as the most popular and most applicable on dynamical systems: the Caputo derivative and Riemann-Liouville derivative. Caputo derivative is expressed as

$$\begin{aligned} {}_0^C D_x^{\beta}(u(x)) &= \frac{\partial^{\beta}}{\partial x^{\beta}} u(x) \\ &= \frac{1}{\Gamma(n-\beta)} \int_0^x (x-t)^{n-\beta-1} \frac{d^n u(t)}{dt^n} dt. \end{aligned} \tag{1}$$

For the fractional derivative in the sense of Riemann-Liouville, we have the following definition:

$$D_x^\beta (u(x)) = \frac{\partial^\beta}{\partial x^\beta} u(x) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^x (x-t)^{n-\beta-1} u(t)\, dt. \quad (2)$$

In applied sciences in general and dynamical system in particular, each of these fractional derivatives presents some advantages and disadvantages [8, 13]. For example, Caputo derivatives are defined only for differentiable functions while functions that have no first order derivative might have fractional derivatives of all orders less than one in the sense of Riemann-Liouville [11]. Another example is the fact that the Riemann-Liouville derivative of a constant is not zero unlike Caputo's derivative which gives zero when applying to a constant function. However Caputo's derivative demands higher conditions of regularity for differentiability. Then, to compute the fractional derivative of a function in the Caputo sense, we have to calculate first its derivative. Fortunately for Riemann-Liouville derivative, the author in [14] proposed the following alternative definition:

$$D_x^\beta (u(x)) = \frac{\partial^\beta}{\partial x^\beta} u(x) = \frac{1}{\Gamma(n-\beta)} \frac{d^n}{dt^n} \int_0^x (x-t)^{n-\beta-1} \{u(t) - u(0)\}\, dt; \quad (3)$$

the above modified Riemann-Liouville derivative seems to have advantages of both the standard Riemann-Liouville and Caputo fractional derivatives: it is defined for arbitrary continuous (nondifferentiable) functions and the fractional derivative of a constant is equal to zero. However from its definition we do not actually give a fractional derivative of a function say $u(x)$ but the fractional derivative of $u(x) - u(0)$ can lead to fractional derivative that is not defined at the origin for some function for which $u(0)$ does not exist. We note that Caputo and Riemann-Liouville may have their disadvantages but still remain the best definition of the fractional derivative according to their easy applicability. We need the following definition.

Definition 1 (partial derivatives of fractional order). Assume now that $u(x)$ is a function of n variables x_i, $i = 1, \ldots, n$ also of class C on $D \in \mathbb{R}_n$. We define as partial derivative of order β for $u(x)$ with respect to x_i the function

$$a\partial_{\underline{x}}^\beta u = \frac{1}{\Gamma(m-\beta)} \int_a^{x_i} (x_i - t)^{m-\beta-1} \partial_{x_i}^m u(x_j)\Big|_{x_j} dt \quad (4)$$

if it exists, where $\partial_{x_i}^m$ is the usual partial derivative of integer order m.

In this paper, we consider the fractional derivative $(-\nabla)^\beta$ in the sense of Caputo to obtain and analyze the generalized (fractional) NSE in 3D of the form

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathrm{Re}^{-1} (-\nabla)^\beta \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times (0, T], \quad (5)$$

with $\beta > 0$ a real number. The system is subject to the incompressibility condition,

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times (0, T], \quad (6)$$

the homogeneous no-slip boundary condition,

$$\mathbf{u} = 0 \quad \text{on } \Gamma_{\text{rigid}} \times (0, T], \quad (7)$$

the inflow condition

$$\mathbf{u}^{\text{in}} = \mathbf{u} \quad \text{on } \Gamma_{\text{in}}, \quad (8)$$

and the initial condition,

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}) \quad \text{in } \Omega, \quad (9)$$

where $\Omega \subseteq \mathbb{R}^3$, $\mathbf{u} = (u, v, w) = (u(t, x, y, z), v(t, x, y, z), w(t, x, y, z))$ is the fluid velocity vector field with components u, v, and w at the point (x, y, z) and time t. Note that $x = x(t)$, $y = y(t)$, $z = z(t)$, $u = dx/dt$, $v = dy/dt$, and $w = dz/dt$. Here $\mathbf{f}$ is the body force per unit mass (note that we will assume that $\mathbf{f} = 0$). Γ_{rigid} and Γ_{in} are the rigid part and the inflow part of the boundary Γ, respectively.

3. Classical Model of Navier-Stokes Equations

3.1. Model Description and Definitions. The theory of mechanics of continuous media, also known as continuum mechanics, allows the description of the constitutive equations laws that describe the deformations of fluid medium. These laws, in combination with the general conservation principles (conservation of mass and of momentum), form the system of partial differential equations, which are equal in number to the number of unknowns of the system. Namely, for 3D motion there are four dependent variables: u, v, w, and p, and four independent variables: x, y, z, and t. This leads to the following classical Navier-Stokes equations, obtained from the generalized version (5) by putting $\beta = 2$, and of which we discuss a possible treatment using the method of finite elements technique:

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = \mathbf{f} \quad \text{in } \Omega \times (0, T], \quad (10)$$

with the incompressibility condition, boundary condition, inflow condition, and initial condition remaining the same as (6), (7), (8), and (9), respectively. In addition we assume that Ω does not change in time.

The finite element discretization of the Navier-Stokes problem is based on the variational formulation, and the use of Sobolev spaces is needed for the mathematical treatment of the variational formulation of the model. We use subspaces of the usual Hilbert space

$$L^2(\Omega) = \left\{ f : \int_\Omega |f|^2 \, d\mathbf{x} < \infty \right\} \quad (11)$$

of square-integrable functions on Ω, where integration is in the sense of Lebesgue:

$$L_0^2(\Omega) = \left\{ f : f \in L^2(\Omega), (f, 1) = 0 \right\}, \quad (12)$$

and the corresponding inner products and norms

$$(f,g)=\int_\Omega fg d\mathbf{x},\qquad \|f\|_0=(f,f)^{1/2}. \tag{13}$$

Next, for any nonnegative integer k, we define the Sobolev space

$$H^k(\Omega)=\left\{f: f\in L^2(\Omega), D^s f\in L^2(\Omega), \text{for } s=1,\dots,k\right\} \tag{14}$$

of square-integrable functions, all of whose derivatives of order up to k are also square integrable, where D^s denotes any and all derivatives of order s. $H^k(\Omega)$ comes with the norm

$$\|f\|_k=\left(\|f\|_0^2+\sum_{s\le k, s\neq 0}\|D^s f\|_0^2\right)^{1/2}. \tag{15}$$

The following definitions can now be stated:

$$H^0(\Omega)=L^2(\Omega)$$

$$H^1(\Omega)=\left\{f: f\in L^2(\Omega), \partial_i f\in L^2(\Omega), 1\le i\le 3\right\}$$

$$\|\nabla f\|_0=(\nabla f,\nabla f)^{1/2} \tag{16}$$

$$\|f\|_1=\left(\|f\|_0^2+\|\nabla f\|_0^2\right)^{1/2}=\left(\|f\|_0^2+\sum_{i=1}^3\left\|\frac{\partial f}{\partial x_i}\right\|_0^2\right)^{1/2}.$$

Of particular interest is the subspace of $H_0^1(\Omega)$ of $H^1(\Omega)$ defined by

$$H_0^1(\Omega)=\left\{f: f\in H^1(\Omega), f=0 \text{ on } \Gamma\right\}, \tag{17}$$

whose elements vanish on the boundary Γ.

For functions belonging to $H^1(\Omega)$, the seminorm

$$|f|_1=\left(\sum_{i=1}^3\left\|\frac{\partial f}{\partial x_i}\right\|_0^2\right)^{1/2} \tag{18}$$

defines a norm equivalent to (16). The proof of this statement is not our aim in this dissertation, but it can be found in Braess [15]. Thus for such functions, (18) may be used instead of (16).

We denote by $H^{-1}(\Omega)$ the dual space consisting of bounded linear functionals on $H_0^1(\Omega)$; that is, $f\in H^{-1}(\Omega)$ implies that $(f,w)\in\mathbb{R}$ for all $w\in H_0^1(\Omega)$. A norm for $H^{-1}(\Omega)$ is given by

$$\|f\|_{-1}=\sup_{0\neq w\in H_0^1(\Omega)}\frac{(f,w)}{|w|_1}. \tag{19}$$

Since the velocity field $\mathbf{u}=\mathbf{u}(u,v,w)=(\mathbf{u}_i)_{i=1,2,3}$ is a vector valued function, we use the spaces

$$\begin{aligned}
\mathbf{H}^k(\Omega)&=H^k(\Omega)^3=\left\{\mathbf{u}:\mathbf{u}_i\in H^k(\Omega) \text{ for } i=1,2,3\right\},\\
\mathbf{H}_0^1(\Omega)&=H_0^1(\Omega)^3=\left\{\mathbf{u}:\mathbf{u}_i\in H_0^1(\Omega) \text{ for } i=1,2,3\right\},\\
\mathbf{H}^{-1}(\Omega)&=H^{-1}(\Omega)^3=\left\{\mathbf{u}:\mathbf{u}_i\in H^{-1}(\Omega) \text{ for } i=1,2,3\right\}.
\end{aligned} \tag{20}$$

For $k\ge 0$, $\mathbf{H}^k(\Omega)$ is equipped with the norm

$$\|\mathbf{u}\|_k=\left(\sum_{i=1}^3\|\mathbf{u}_i\|_k^2\right)^{1/2}. \tag{21}$$

Alternatively, for functions belonging to $\mathbf{H}_0^1(\Omega)$, we may use

$$|\mathbf{u}|_1=\left(\sum_{i=1}^3|\mathbf{u}_i|_1^2\right)^{1/2}. \tag{22}$$

The inner product for functions belonging to $\mathbf{L}^2(\Omega)=\mathbf{H}^0(\Omega)=L^2(\Omega)^3$ is also given by

$$(\mathbf{u},\mathbf{w})=\int_\Omega \mathbf{u}\cdot\mathbf{w}d\mathbf{x}. \tag{23}$$

Before stating the weak variational formulation for our model, let us discuss some preliminaries concerning the existence and uniqueness of a solution of the Navier-Stokes equations.

3.2. Existence and Uniqueness for a Solution of Navier-Stokes Equations. From the mathematical point of view, two questions concerning the Navier-Stokes equations are of main interest. Given a set of data which are sufficiently smooth,

(1) does a solution of (10), (6)–(9) exist?

(2) if a solution exists, is it unique?

First, we have to clarify the notion of a *solution* of (10), (6)–(9). There exist several concepts of the notion of a solution of the above system, the most important of which are the *classical solution* and the *weak solution*.

Definition 2 (classical solution). A pair $(\mathbf{u},p)$ is called a classical solution of the Navier-Stokes problem (10), (6)–(9) if

(1) $(\mathbf{u},p)$ satisfies the Navier-Stokes problem (10), (6)–(9),

(2) $\mathbf{u}$ and p are infinitely many times differentiable with respect to space and time variables.

Then, according to John and Kaya [16], *the existence of a classical solution of* (10), (6)–(9) *cannot yet be proven, but if a classical solution exists, it is unique*.

To define a weak solution, we first need to transform (10) into a weak form by

(i) multiplying (10) with a suitable vector valued function φ (test function),

(ii) integrating over $\Omega\times(0,T]$,

(iii) applying integration by parts (Green's theorem).

The last step is possible only if there are some restrictions on the domain. For the test function φ, one requires

(i) $\varphi\in C_{0,\mathrm{div}}^\infty(\Omega)$ for *each time* t, *where* $C_{0,\mathrm{div}}^\infty(\Omega)=\{\mathbf{f}: \mathbf{f}\in C_0^\infty(\Omega), \nabla\cdot\mathbf{f}=0\}$,

(ii) φ is infinitely differentiable with respect to time,

(iii) $\varphi(\cdot, T) = \mathbf{0}$.

This gives the weak formulation of the Navier-Stokes equations

$$\int_0^T \left[-(\mathbf{u}, \varphi_t) + (\mathbf{u} \cdot \nabla \mathbf{u}, \varphi) + \mathrm{Re}^{-1} (\nabla \mathbf{u}, \nabla \varphi) \right] dt = \int_o^T (\mathbf{f}, \varphi)\, dt + (\mathbf{u}_0, \varphi(\cdot, 0)), \tag{24}$$

which has the following features:

(i) there is no time derivative of **u,**

(ii) there is no second order spatial derivative with respect to **u,**

(iii) the pressure vanishes, since Green's formula yields

$$(\nabla p, \varphi) = \int_{\partial\Omega} p\varphi \cdot \mathbf{n} ds - (p, \nabla \cdot \varphi) = 0 \tag{25}$$

because $\varphi \cdot \mathbf{n} = 0$ on $\partial\Omega$ and $\nabla \cdot \varphi = 0$.

Definition 3 (weak solution). A function **u** is called weak solution of the Navier-Stokes equations if

(1) **u** satisfies (24) for all test functions φ with the properties on φ given above,

(2) **u** has the following regularity:

$$\mathbf{u} \in L^2\left(0, T; H^1_{0,\mathrm{div}}(\Omega)\right) \cap L^\infty\left(0, T; L^2_{\mathrm{div}}(\Omega)\right), \tag{26}$$

where the subscript div means *space of divergence-free functions*; for instance,

$$C^\infty_{0,\mathrm{div}}(\Omega) = \left\{\mathbf{f} : \mathbf{f} \in C^\infty_0(\Omega), \nabla \cdot \mathbf{f} = 0\right\},$$
$$L^2\left(0, T; H^1_0(\Omega)\right) = \left\{ f(\mathbf{x}, t) : \int_0^T \|f\|_0^2\, dt < \infty \right\}. \tag{27}$$

More generally

$$L^q(t_0, t_1; X) = \left\{ f(\mathbf{x}, t) : \int_{t_0}^{t_1} \|f\|_X^q\, dt < \infty \right\} \quad \text{for any } q \in [1, \infty) \tag{28}$$

is the space of strongly measurable maps $f : [t_o, t_1] \to X$, such that

$$\|f\|_{L^q(t_0,t_1;X)} = \left(\int_{t_0}^{t_1} \|f\|_X^q\, dt \right)^{1/q} < \infty \quad \text{for } q \in [1, \infty) \tag{29}$$

and X is a Banach space. Furthermore

$$L^\infty(t_0, t_1; X) = \left\{ f(\mathbf{x}, t) : \operatorname{ess\,sup}_{t_0 \le t \le t_1} \|f\|_X < \infty \right\} \tag{30}$$

with

$$\|f\|_{L^\infty(t_0,t_1;X)} = \operatorname{ess\,sup}_{t_0 \le t \le t_1} \|f\|_X < \infty \quad \text{for } q = \infty. \tag{31}$$

It is obvious that all these spaces are needed for the weak formulation given in the next section.

The existence of a weak solution of (10), (6)–(9) was proved in 1934 by Leray [17]. The weak solution is unique if every other weak solution satisfies an additional regularity assumption, Serrin's condition; see Serrin [7] or John and Kaya [16]. But it is not known in 3D if every weak solution possesses such additional condition.

According to the same article [16], *the existence of a weak solution of the Navier-Stokes equations can be proven in arbitrary domains, but the uniqueness cannot yet be proven.*

The answer to the question of *uniqueness of the weak solution in 3D or existence of a classical solution in 3D* is one of the major mathematical challenges of this century (John and Kaya [16]). There is a prize of one million US Dollars for people who can answer these questions.

3.3. The Stationary Case. We now consider our domain Ω defined in the first chapter and assume that all the variables in the system (10), (6)–(9) are independent of time. We therefore obtain the following stationary Navier-Stokes problem:

$$\mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mathrm{Re}^{-1} \nabla^2 \mathbf{u} = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0, \quad \text{in } \Omega \tag{32}$$

with the following boundary conditions:

$$\mathbf{u}|_{\Gamma_{\mathrm{rigid}}} = 0, \qquad \mathbf{u}|_{\Gamma_{\mathrm{in}}} = \mathbf{u}^{\mathrm{in}} \tag{33}$$

having the following weak formulation.

Find functions $\mathbf{u} \in \mathbf{u}^{\mathrm{in}} + \mathbf{H}^1_0(\Omega)$ and $p \in L^2_0(\Omega)$ such that

$$a(\mathbf{u}, \mathbf{v}) + c(\mathbf{u}, \mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega)$$
$$b(q, \mathbf{u}) = 0 \quad \forall q \in L^2_0(\Omega), \tag{34}$$

discretized as follows.

Find a pair $(\mathbf{u}^h, p^h) \in \mathbf{u}^{\mathrm{in}} + \mathbf{V}^h_0 \times S^h_0$ such that

$$a(\mathbf{u}^h, \mathbf{v}^h) + c(\mathbf{u}^h, \mathbf{u}^h, \mathbf{v}^h) + b(p^h, \mathbf{v}^h) = (\mathbf{f}, \mathbf{v}^h) \quad \forall \mathbf{v}^h \in \mathbf{V}^h_0$$
$$b(q^h, \mathbf{u}^h) = 0 \quad \forall q^h \in S^h_0. \tag{35}$$

For the sake of a stable approximation as $h \to 0$, it is crucial that we relate the continuous and discrete spaces by the following hypotheses.

Hypothesis **H1** (approximation property of $\mathbf{V}^h_0$). There exists an operator $r^h \in \mathscr{L}([H^2(\Omega) \cap H^1_o(\Omega)]^2; \mathbf{V}^h_0)$ and an integer l such that

$$\|\varphi - r^h \varphi\|_1 \le C h^m \|\varphi\|_{m+1} \quad \forall \varphi \in \mathbf{H}^{m+1}(\Omega), \quad 1 \le m \le l. \tag{36}$$

Hypothesis **H2** (approximation property of Q^h). There exists an operator $s^h \in \mathscr{L}(L^2(\Omega); Q^h)$ such that

$$\|q - s^h q\|_0 \le C h^m \|q\|_m \quad \forall q \in H^m(\Omega), \quad 0 \le m \le l. \tag{37}$$

Hypothesis **H3** (uniform inf-sup condition). For each $q^h \in S_0^h$, there exists a $\mathbf{v}^h \in \mathbf{V}_0^h$ such that

$$b\left(q^h, \mathbf{v}^h\right) = \left\|q^h\right\|_0^2$$
$$\left|\mathbf{v}^h\right|_1 \leq C \left\|q^h\right\|_0, \tag{38}$$

where the constant $C > 0$ is independent of h, q^h and v^h; $\mathscr{L}(Y, W)$ is the space of linear operators from Y to W; $\|\cdot\|_0$ and $\|\cdot\|_m$ are the standard norms in $L^2(\Omega)$ and $H^m(\Omega)$, respectively; $|\cdot|_1$ is the standard seminorm in $H^1(\Omega)$.

We assume that the bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ and the approximating subspaces $\mathbf{V}_0^h$ and S_0^h satisfy all the hypotheses **H1**, **H2**, **H3** required for suitable approximations. For any $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, the system (35) has a solution $(\mathbf{u}^h, p^h)$. It is well known (see Girault and Raviart [4] or Gunzburger [18]) that the solution is unique for "sufficiently small" data $\mathbf{f}$ or "sufficiently small" Reynolds number, Re. In fact if we set.

$$\mathbf{Z} = \left\{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : b(q, \mathbf{v}) = 0 \quad \forall q \in L_0^2(\Omega)\right\} \tag{39}$$

the space of divergence-free functions, and

$$\mathcal{N} = \sup_{\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbf{Z}} \frac{c(\mathbf{w}, \mathbf{u}, \mathbf{v})}{|\mathbf{u}|_1 |\mathbf{v}|_1 |\mathbf{w}|_1} \tag{40}$$

then given $\mathbf{f} \in \mathbf{H}^{-1}(\Omega)$, and if $(\mathrm{Re})^2 \mathcal{N} \|\mathbf{f}\|_{-1} < 1$, then the problem (34) has a unique solution $(\mathbf{u}, p) \in \mathbf{Z} \times L_0^2(\Omega)$.

From the previous assertion, we can state one of the fundamental properties (see Joseph [5]) of the solutions of the Navier-Stokes equations (32) and (33) which reads as follows: *when the viscosity is large (or the Reynolds number is small), all solutions of the Navier-Stokes equations tend to a single basic flow*. So what is the final destiny of all these uniquely determined solutions of the Navier-Stokes equations? For large values of the Reynolds number, the final set of flows which evolve from a given set of initial fields is generally "turbulent."

As just proven, the Navier-Stokes equations have in general more than one solution, unless the data satisfies very stringent requirements. However, it can also be shown that in many practical examples these solutions are mostly isolated; that is, there exists a neighborhood in which each solution is unique. Furthermore, it can be established (Girault and Raviart [4]) that the solutions depend continuously on the Reynolds number, Re (which is inversely proportional to the kinematic viscosity). Thus as the Reynolds number varies along an interval, each solution of the Navier-Stokes equations describes an isolated branch. In particular, this means that the bifurcation phenomenon can be rare or occurs infrequently (Girault and Raviart [4]). This situation, frequently encountered in practice, is expressed mathematically by the notion of branches of nonsingular solutions. So the solutions of the problem (32) and (33) are "in general" nonsingular.

But when the Reynolds number Re is large (small viscosity), compared to the other parameters of the fluid, there arises a boundary layer in the neighborhood of $\partial\Omega$ where the viscosity predominates while it is negligible in the interior of Ω. At the same time, the transition to turbulence occurs. Thus the solutions of the Navier-Stokes equations are seriously discontinuous at certain values of the Reynolds number, Re.

In the coming sections, we show that, under hypotheses **H1**, **H2**, **H3**, the problem (35) possesses a branch of nonsingular solutions that, as $h \to 0$, converges to a given branch of nonsingular solutions of (34). This convergence depends on the Reynolds number Re. We also use a finite element method, the Newton method, to show that the stability of such a convergence is guaranteed for small variations of Re.

3.4. Nonsingular Solutions and Approximation. Let Z and $\mathscr{Z}$ be two Banach spaces and Λ a compact interval of the real line $\mathbb{R}$. We are given a $\mathscr{C}^p$-mapping ($p \geq 1$)

$$F : (\mathrm{Re}, \mathscr{U}) \in \Lambda \times Z \longrightarrow F(\mathrm{Re}, \mathscr{U}) \in \mathscr{Z} \tag{41}$$

and we want to solve the equation

$$F(\mathrm{Re}, \mathscr{U}) = 0; \tag{42}$$

that is, we want to find pairs $(\mathrm{Re}, \mathscr{U}) \in \Lambda \times Z$ which are solutions of (42).

Let $\{(\mathrm{Re}, \mathscr{U}(\mathrm{Re})); \mathrm{Re} \in \Lambda\}$ be a branch of solutions of (42). This means that

$$\mathrm{Re} \longrightarrow \mathscr{U}(\mathrm{Re}) \text{ is a continuous function from } \Lambda \text{ into } Z,$$
$$F(\mathrm{Re}, \mathscr{U}(\mathrm{Re})) = 0. \tag{43}$$

Moreover, we suppose that these solutions are *nonsingular* in the sense that

$$D_{\mathscr{U}} F(\mathrm{Re}, \mathscr{U}(\mathrm{Re})) \text{ is an isomorphism from } Z \text{ onto } \mathscr{Z}$$
$$\forall\, \mathrm{Re} \in \Lambda. \tag{44}$$

As an immediate consequence of (44), it follows from the implicit function theorem (see [4]) that $\mathrm{Re} \to \mathscr{U}(\mathrm{Re})$ is a $\mathscr{C}^p$-function from Λ into Z.

Let us show that our problem for the Navier-Stokes equations (34) fits into the above framework. We first set

$$Z = \mathscr{Z} = \mathbf{H}^1(\Omega) \times L_0^2(\Omega), \tag{45}$$

and we introduce the intermediate space

$$Y = \mathbf{H}^{-1}(\Omega) \times \left\{g \in \mathbf{H}^{1/2}(\Gamma); \int_\Gamma g \cdot \mathbf{n} dx = 0\right\}. \tag{46}$$

Next we define a linear operator T as follows: given $(\mathbf{f}_*, g_*) \in Y$, we denote by $(\mathbf{u}_*, p_*) = T(\mathbf{f}_*, g_*) \in Z$ the solution of the Dirichlet problem for the Stokes equations:

$$-\nabla^2 \mathbf{u}_* + \nabla p_* = \mathbf{f}_* \quad \text{in } \Omega$$
$$\nabla \cdot \mathbf{u}_* = 0 \quad \text{in } \Omega \tag{47}$$
$$\mathbf{u}_*|_\Gamma = g_*.$$

Finally, with the data $(\mathbf{f}, g) \in Y$, we associate a $\mathscr{C}^\infty$-mapping G from $\mathbb{R}_+ \times Z$ into Y, defined for a $\mathscr{U} = (\mathbf{w}, q) \in Z$, by

$$G : (\mathrm{Re}, \mathscr{U}) \longrightarrow G(\mathrm{Re}, \mathscr{U}) = \left(\mathrm{Re}\,(\mathbf{w} \cdot \nabla \mathbf{w} - \mathbf{f}), -g\right) \quad (48)$$

and we set

$$F(\mathrm{Re}, \mathscr{U}) = \mathscr{U} + TG(\mathrm{Re}, \mathscr{U}). \quad (49)$$

It is clear that $(\mathbf{f}, g) = (\mathbf{f}, 0) \in Y$ and (48) becomes

$$G(\mathrm{Re}, \mathscr{U}) = (\mathrm{Re}\,(\mathbf{w} \cdot \nabla \mathbf{w} - \mathbf{f}), 0) \quad (50)$$

or simply

$$G(\mathrm{Re}, \mathscr{U}) = \mathrm{Re}\,(\mathbf{w} \cdot \nabla \mathbf{w} - \mathbf{f}). \quad (51)$$

Now we may state the lemma.

Lemma 4. *The pair $(\mathbf{u}, p) \in \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$ is a solution of problem (32) and (33) if and only if $(\mathrm{Re}, \mathscr{U})$, with $\mathscr{U} = (\mathbf{u}, \mathrm{Re}\, p)$, is a solution of (42), where the spaces Z and $\mathscr{Z}$ are defined by (45) and the compound mapping F is defined by (49) and (50).*

Proof. If $(\mathbf{u}, p)$ is a solution of the problem (32) and (33) then

$$\begin{aligned} -\nabla^2 \mathbf{u} + \nabla(\mathrm{Re}\, p) &= \mathrm{Re}\,(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}) \quad \text{in } \Omega \\ \nabla \cdot \mathbf{u} &= 0 \quad \text{in } \Omega \\ \mathbf{u} &= 0 \quad \text{on } \Gamma_{\text{rigid}}. \end{aligned} \quad (52)$$

From both (47) and $(\mathbf{u}_*, p_*) = T(f_*, g_*)$, applied to $f_* = \mathrm{Re}(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u})$ and $p_* = \mathrm{Re}\, p$, we can write

$$(\mathbf{u}, \mathrm{Re}\, p) = T(\mathrm{Re}\,(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}), 0) \quad (53)$$

or

$$\begin{aligned} &(\mathbf{u}, \mathrm{Re}\, p) - T(\mathrm{Re}\,(\mathbf{f} - \mathbf{u} \cdot \nabla \mathbf{u}), 0) = 0 \\ &\left(\text{since the operator } T \text{ is linear}\right). \end{aligned} \quad (54)$$

Then, (51) yields

$$\mathscr{U} + TG(\mathrm{Re}, \mathscr{U}) = 0 \quad (55)$$

and (49) yields

$$F(\mathrm{Re}, \mathscr{U}) = 0, \quad (56)$$

which means

$$(\mathrm{Re}, \mathscr{U}) \text{ is a solution of } (39). \quad (57)$$

From Lemma 4, it is clear that if $(\mathbf{u}, p)$ is a solution of the problem (32) and (33), then $(\mathrm{Re}, \mathscr{U})$, where $\mathscr{U} = (\mathbf{u}, \mathrm{Re}\, p)$, is a nonsingular solution of (42). □

We may now state the following proposition.

Proposition 5. *Assume that hypotheses* **H1**, **H2,** *and* **H3** *hold. Let $\{(\mathrm{Re}, (\mathbf{u}(\mathrm{Re}), \mathrm{Re}\, p(\mathrm{Re})); \mathrm{Re} \in \Lambda\}$ be a branch of nonsingular solutions of the Navier-Stokes problem (34). Then there exists a neighborhood $\mathscr{O}$ of the origin in $\mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ and for $h \leq h_0$ sufficiently small a unique $\mathscr{C}^\infty$ branch $\{(\mathrm{Re}, (\mathbf{u}^h(\mathrm{Re}), \mathrm{Re}\, p^h(\mathrm{Re})); \mathrm{Re} \in \Lambda\}$ of nonsingular solutions of problem (35) such that*

$$\begin{aligned} &\left(\mathbf{u}^h(\mathrm{Re}), \mathrm{Re}\, p^h(\mathrm{Re})\right) \in \left(\mathbf{u}(\mathrm{Re}), \mathrm{Re}\, p(\mathrm{Re})\right) + \mathscr{O} \\ &\qquad \forall\, \mathrm{Re} \in \Lambda. \end{aligned} \quad (58)$$

Moreover, we have the convergence property

$$\limsup_{h \to 0,\, \mathrm{Re} \in \Lambda} \left\{ \left| \mathbf{u}^h(\mathrm{Re}) - \mathbf{u}(\mathrm{Re}) \right|_1 + \left\| p^h(\mathrm{Re}) - p(\mathrm{Re}) \right\|_0 \right\} = 0. \quad (59)$$

In addition, if the mapping $\mathrm{Re} \to (\mathbf{u}(\mathrm{Re}), p(\mathrm{Re}))$ is continuous from Λ into $H^{m+1}(\Omega)^3 \times H^m(\Omega)$ for some integer m with $1 \leq m \leq l$, we have for all $\mathrm{Re} \in \Lambda$

$$\begin{aligned} &\left| \mathbf{u}^h(\mathrm{Re}) - \mathbf{u}(\mathrm{Re}) \right|_1 + \left\| p^h(\mathrm{Re}) - p(\mathrm{Re}) \right\|_0 \\ &\qquad \leq K h^m. \end{aligned} \quad (60)$$

Our goal is not to prove this proposition, but to use the results. A similar proof of this proposition can be found in Girault and Raviart [4].

It is also possible to derive an L^2-estimate for the velocity. But the following regularity must be satisfied:

$$\begin{aligned} &\text{The mapping } (\phi, \mu) \longmapsto \nabla \mu - \mathrm{Re}^{-1} \nabla^2 \phi \text{ is an isomorphism} \\ &\text{from } \left[\mathbf{H}^2(\Omega) \cap \mathscr{S}\right] \times \left[H^1(\Omega) \cap L_0^2(\Omega)\right] \text{ onto } \mathbf{L}^2(\Omega), \end{aligned} \quad (61)$$

where $\mathscr{S}$ is a closed subspace of Z.

Proposition 6. *We retain the hypotheses of Proposition 5 and we assume that (61) holds. If the mapping $\mathrm{Re} \to (\mathbf{u}(\mathrm{Re}), p(\mathrm{Re}))$ is continuous from Λ into $\mathbf{H}^{m+1}(\Omega) \times H^m(\Omega)$ for some integer $m \in [1, l]$, then we have the following L^2-estimate for all $\mathrm{Re} \in \Lambda$:*

$$\left\| \mathbf{u}^h(\mathrm{Re}) - \mathbf{u}(\mathrm{Re}) \right\|_0 \leq K h^{m+1}. \quad (62)$$

These propositions show that the convergence of finite element approximations is guaranteed, and the stability constants depend on Re.

3.5. Stability of Newton's Method. The method discussed here is one of the finite element methods that is intended to solve the Navier-stokes equations. We saw in Lemma 4 that it suffices to investigate equations of type (42):

$$F(\mathrm{Re}, \mathscr{U}) = 0, \quad (63)$$

where F is defined by (49) and is a $\mathscr{C}^P$-mapping ($P \geq 1$) defined on $\Lambda \times Z$ (with Λ a compact interval of the real line $\mathbb{R}$ and $Z = \mathbf{H}^1(\Omega) \times L_0^2(\Omega)$). For simplicity, let us set and fix

$$\lambda = \mathrm{Re} \in \Lambda \tag{64}$$

and assume that $\mathscr{U} = \mathscr{U}(\lambda)$ is a nonsingular solution of (63). Then

$$\begin{aligned} &F(\lambda, \mathscr{U}) = 0, \\ &D_{\mathscr{U}} F(\lambda, \mathscr{U}) \text{ is an isomorphism from } Z \text{ onto } \mathscr{Z}, \end{aligned} \tag{65}$$

where $\mathscr{Z}$ is a Banach space.

Since $\mathscr{U}$ is an isolated solution of (63) and since F is at least differentiable, an efficient way to approximate $\mathscr{U}$ is by the Newton method. The *Newton algorithm* reads as follows.

Starting from an initial guess $\mathscr{U}^0$, construct the sequence $\{\mathscr{U}^n\}_n$ in Z by

$$\mathscr{U}^{n+1} = \mathscr{U}^n - \left[D_{\mathscr{U}} F(\lambda, \mathscr{U}^n)\right]^{-1} \cdot F(\lambda, \mathscr{U}^n) \quad n \geq 0 \tag{66}$$

or equivalently

$$D_{\mathscr{U}} F(\lambda, \mathscr{U}^n) \cdot \left(\mathscr{U}^{n+1} - \mathscr{U}^n\right) = -F(\lambda, \mathscr{U}^n). \tag{67}$$

As $D_{\mathscr{U}} F(\lambda, \mathscr{U})$ is a linear operator, each step of Newton's method requires the solution of a different problem relative to $D_{\mathscr{U}} F(\lambda, \mathscr{U}^n)$. If this is too costly, the simplest alternative is to replace (66) by

$$\mathscr{U}^{n+1} = \mathscr{U}^n - \left[D_{\mathscr{U}} F\left(\lambda, \mathscr{U}^0\right)\right]^{-1} \cdot F(\lambda, \mathscr{U}^n) \quad n \geq 0, \tag{68}$$

or equivalently

$$D_{\mathscr{U}} F\left(\lambda, \mathscr{U}^0\right) \cdot \left(\mathscr{U}^{n+1} - \mathscr{U}^n\right) = -F(\lambda, \mathscr{U}^n). \tag{69}$$

The drawback of Newton's method is that the stability of its (quadratic) convergence (see [4]) can only be ensured when the first guess $\mathscr{U}^0$ is sufficiently near the solution $\mathscr{U}$. If this solution is part of a branch of nonsingular solutions, and if we know the solution at a neighboring point, say $\mathscr{U}(\lambda - \nabla\lambda)$ for an adequate increment $\nabla\lambda$, then we can derive from this value the first guess to start Newton's algorithm.

Since $\lambda \mapsto \mathscr{U}(\lambda)$ is a branch of nonsingular solution of (63), then F is a $\mathscr{C}^P$-mapping ($P \geq 2$), so is the mapping $\mathscr{U}(\lambda)$ and we can differentiate both sides of (63):

$$\begin{aligned} &D_{\mathscr{U}} F(\lambda, \mathscr{U}(\lambda)) \cdot \left(\frac{d\mathscr{U}(\lambda)}{d\lambda}\right) + D_\lambda F(\lambda, \mathscr{U}(\lambda)) = 0 \\ &\qquad\qquad \forall \lambda \in \Lambda; \end{aligned} \tag{70}$$

that is, we find a first order differential equation of the form

$$\frac{d\mathscr{U}(\lambda)}{d\lambda} = -\phi(\lambda), \tag{71}$$

where

$$\phi(\lambda) = \left[D_{\mathscr{U}} F(\lambda, \mathscr{U}(\lambda))\right]^{-1} D_\lambda F(\lambda, \mathscr{U}(\lambda)). \tag{72}$$

The simplest way to solve (71) is to use the one-step, explicit Euler's method; this brings us to choose

$$\mathscr{U}^0(\lambda) = \mathscr{U}(\lambda - \Delta\lambda) - \phi(\lambda - \Delta\lambda)\,\Delta\lambda. \tag{73}$$

In other words $\mathscr{U}^0(\lambda)$ is defined by

$$\begin{aligned} &D_{\mathscr{U}} F(\lambda - \Delta\lambda, \mathscr{U}(\lambda - \Delta\lambda)) \cdot \left(\mathscr{U}^0(\lambda) - \mathscr{U}(\lambda - \Delta\lambda)\right) \\ &\quad = -D_\lambda F(\lambda - \Delta\lambda, \mathscr{U}(\lambda - \Delta\lambda)) \cdot \Delta\lambda. \end{aligned} \tag{74}$$

Let us estimate the difference $\mathscr{U}(\lambda) - \mathscr{U}^0(\lambda)$. From (71), we infer that

$$\mathscr{U}(\lambda) = \mathscr{U}(\lambda - \Delta\lambda) - \int_{\lambda - \Delta\lambda}^{\lambda} \phi(\xi)\, d\xi. \tag{75}$$

Subtracting (73) from this equality yields

$$\begin{aligned} \mathscr{U}(\lambda) - \mathscr{U}^0(\lambda) &= -\left[\int_{\lambda - \Delta\lambda}^{\lambda} \phi(\xi)\, d\xi - \phi(\lambda - \Delta\lambda) \cdot \Delta\lambda\right] \\ &= -\int_{\lambda - \Delta\lambda}^{\lambda} \phi'\left(\theta_\xi\right) \cdot (\xi - \lambda + \Delta\lambda)\, d\xi. \end{aligned} \tag{76}$$

Hence

$$\left\|\mathscr{U}(\lambda) - \mathscr{U}^0(\lambda)\right\|_Z \leq \left[\frac{(\Delta\lambda)^2}{2}\right] \max_{\theta \in (\lambda - \Delta\lambda, \lambda)} \left\|\phi'(\theta)\right\|_Z. \tag{77}$$

Thus $\|\mathscr{U}(\lambda) - \mathscr{U}^0(\lambda)\|_Z$ is $\mathbf{O}((\Delta\lambda)^2)$ and if $\Delta\lambda$ is small enough, the solutions $\mathscr{U}$ and $\mathscr{U}^0$ stay close, characterizing the stability, and $\mathscr{U}^0$ defined by (73) is an adequate starting value for Newton's algorithm.

4. Existence Results: Discussion and Concluding Remark

We have analyzed existence of nonsingular solution to NSEs of type (5), with $\beta = 2$, and used Newton's method to study its stability. It happens that the stability of its (quadratic) convergence, when exploiting Newton's method, can only be ensured when the first guess $\mathscr{U}^0$ is sufficiently near the solution $\mathscr{U}$. Now what happens for the system (5) with a general $\beta > 0$? Does a nonsingular solution exist? If yes, in which spaces? We know [19] that there is global existence for the fractional NSEs (5), with only $1/2 < \beta < 1$ and the system is well-posed in some supercritical Besov spaces as well as in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$. It has also been shown [20] that any classical solution to NSEs (5) with $\beta \geq (1/2) + (3/4)$ is always global in time. Thus, it is clear that investigations on the system of NSEs of type (5) are far from being complete and the work developed here will contribute to future possible investigation in order to study, for a general parameter β, the existence and well-posedness of the fractional Navier-Stokes equations in some Banach spaces.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] B. Mohammadi, *Analysis of the K-EPSILON Turbulence*, Wiley-Masson Series, 1997.

[2] R. E. Boisvert, *Group analysis of the Navier-Stokes equations [Ph.D Dissertation]*, Georgia Institute of Technology, Atlanta, Ga, USA, 1982.

[3] R. E. Boisvert, W. F. Ames, and U. N. Srivastava, "Group properties and new solutions of Navier-Stokes equations," *Journal of Engineering Mathematics*, vol. 17, no. 3, pp. 203–221, 1983.

[4] V. Girault and P. Raviart, *Finite Element Methods for Navier-Stokes Equations, Theory and Algorithms*, vol. 5 of *Springer Series in Computational Mathematics*, Springer, Berlin, Germany, 1986.

[5] D. D. Joseph, *Stability of Fluid Motions I*, vol. 27 of *Springer tracts in Natural Philosophy*, Springer, Berlin, Germany, 1976.

[6] J. M. Manale, "Group classification of the two-dimensional Navier-Stokes-type equations," *International Journal of Non-Linear Mechanics*, vol. 35, no. 4, pp. 627–644, 2000.

[7] J. Serrin, "The initial value problem for the Navier-Stokes equations," in *Nonlinear Problems*, R. E. Langer, Ed., pp. 69–98, University of Wisconsin Press, Madison, Wis, USA, 1963.

[8] A. Atangana and A. Secer, "A note on fractional order derivatives and table of fractional derivatives of some special functions," *Abstract and Applied Analysis*, vol. 2013, Article ID 279681, 8 pages, 2013.

[9] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent," *Journal of the Royal Australian Historical Society*, vol. 13, part 2, pp. 529–539, 1967.

[10] E. F. Doungmo Goufo and S. Mugisha, "Mathematical solvability of a Caputo fractional polymer degradation model using further generalized functions," *Mathematical Problems in Engineering*, vol. 2014, Article ID 392792, 5 pages, 2014.

[11] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[12] J. Wu, "Generalized MHD equations," *Journal of Differential Equations*, vol. 195, no. 2, pp. 284–312, 2003.

[13] A. Kadem and A. Kilicman, "The approximate solution of fractional Fredholm integrodifferential equations by variational iteration and homotopy perturbation methods," *Abstract and Applied Analysis*, vol. 2012, Article ID 486193, 10 pages, 2012.

[14] G. Jumarie, "On the representation of fractional Brownian motion as an integral with respect to $(\mathrm{d}t)^a$," *Applied Mathematics Letters*, vol. 18, no. 7, pp. 739–748, 2005.

[15] D. Braess, *Finite Elements—Theory, Fast Solvers, and Applications in Solid Mechanics*, Cambridge University Press, Cambridge, UK, 1997.

[16] V. John and S. Kaya, "A finite element variational multiscale method for the Navier-Stokes equations," *SIAM Journal on Scientific Computing*, vol. 26, no. 5, pp. 1485–1503, 2005.

[17] J. Leray, "On the motion of viscous liquid filling space," *Acta Mathematica*, vol. 63, 1934.

[18] M. D. Gunzburger, *Finite Element Methods for Viscous Incompressible Flows: A Guide to Theory*, Academic Press, London, UK, 1989.

[19] X. Yu and Z. Zhai, "Well-posedness for fractional Navier-Stokes equations in the largest critical spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$," *Mathematical Methods in the Applied Sciences*, vol. 35, no. 6, pp. 676–683, 2012.

[20] J. Wu, "The generalized incompressible Navier-Stokes equations in Besov spaces," *Dynamics of Partial Differential Equations*, vol. 1, no. 4, pp. 381–400, 2004.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 753936, 9 pages
http://dx.doi.org/10.1155/2015/753936

Research Article

Fractional Heat Conduction Models and Thermal Diffusivity Determination

Monika Žecová and Ján Terpák

Institute of Control and Informatization of Production Processes, Faculty of BERG, Technical University of Košice, 042 00 Košice, Slovakia

Correspondence should be addressed to Monika Žecová; monika.zecova@tuke.sk

Received 16 October 2014; Revised 7 December 2014; Accepted 16 December 2014

Academic Editor: Abdon Atangana

The contribution deals with the fractional heat conduction models and their use for determining thermal diffusivity. A brief historical overview of the authors who have dealt with the heat conduction equation is described in the introduction of the paper. The one-dimensional heat conduction models with using integer- and fractional-order derivatives are listed. Analytical and numerical methods of solution of the heat conduction models with using integer- and fractional-order derivatives are described. Individual methods have been implemented in MATLAB and the examples of simulations are listed. The proposal and experimental verification of the methods for determining thermal diffusivity using half-order derivative of temperature by time are listed at the conclusion of the paper.

1. Introduction

Unsteady heat conduction process, described by partial differential equation, was first formulated by Jean Baptiste Joseph Fourier (1768–1830). In 1807, he wrote an article "Partial differential equation for heat conduction in solids." The issue of heat conduction was addressed by other scientists as well, such as Adolf Fick (1829–1901) [1, 2], James Clerk Maxwell (1831–1879) [3–5], Albert Einstein (1879–1955) [6], Lorenzo Richards (1904–1993) [7], and Geoffrey Taylor (1886–1975) [8, 9].

The various analytical and numerical methods are used to solve the Fourier heat conduction equation (FHCE) [10, 11]. In the case of heat conduction in materials with nonstandard structure, such as polymers, granular and porous materials, and composite materials, a standard description is insufficient and required the creation of more adequate models with using derivatives of fractional-order [12–15]. The causes are mainly memory systems and ongoing processes [16–20], roughness, or porosity of the material [21–23] and also fractality and chaotic behavior of systems [24–28].

The more adequate models of processes subsequently require new methods to determine the parameters of these models. In the case of FHCE, the basic parameter of this equation is thermal diffusivity, which characterizes the dynamics of temperature changes in the substance. Measurement of thermal diffusivity can be realized by many ways. The latest methods for determining thermal diffusivity are mainly *laser flash method* [29, 30], *Kennedy transient heat flow method* [31–33], *single rectangular pulse heating method* [34], and *thermal wave method* [35, 36].

The issue of research and development methods and tools for processes modeling with using fractional-order derivatives is very actual, since it means a qualitatively new level of modeling. Important authors of the first articles were Fourier (1768–1830), Abel, Leibniz (1646–1716), Grünwald (1838–1920), and Letnikov (1837–1888). Mathematicians like Liouville (1809–1882) [37, 38] and Riemann (1826–1866) [39] made major contributions to the theory of fractional calculus. Nowadays the fractional calculus interests many scientists and engineers from different fields, such as mechanics, physics, chemistry, and control theory [40].

At the present time, there are a number of analytical [41–48] and numerical solutions of fractional heat conduction equation. In the case of numerical methods different methods are developed based on the random walk models [49–52],

the finite difference method (FDM) [53–55], the finite element method [56–59], numerical quadrature [60–62], the method of Adomian decomposition [63, 64], Monte Carlo simulation [65, 66], matrix approach [12, 13, 67], or the matrix transform method [68, 69]. The finite difference method is an extended method where an explicit [53, 70, 71], an implicit [54, 72–74], and a Crank-Nicolson scheme [55, 75] are used. For the Crank-Nicolson scheme, the literature describes the use of Grünwald-Letnikov definition only for a spatial derivative [73, 76–78].

2. Models of Heat Conduction Processes

Heat conduction is a molecular transfer of thermal energy in solids, liquids, and gases due to the temperature difference. The process of heat conduction takes place between the particles of the substance to touch directly each other and has different temperature. Existing models of heat conduction processes are divided according to various criterions. We consider a division of models into two groups, namely models with using derivatives of integer- and fractional-order.

Models with using derivatives of integer order are the nonstationary and stationary models. Nonstationary models are described by Fourier heat conduction equation, where the temperature T (K) is a function of spatial coordinate x (m) and time τ (s). In the case of one-dimensional heat conduction, it has the following form [11, 79]:

$$\frac{\partial T(x,\tau)}{\partial \tau} = \left(\sqrt{a}\right)^2 \frac{\partial^2 T(x,\tau)}{\partial x^2} \quad \text{for } 0 < x < L,\ \tau > 0, \tag{1}$$

$$T(0,\tau) = T_1, \quad T(L,\tau) = T_2 \quad \text{for } \tau > 0,$$
$$T(x,0) = f(x) \quad \text{for } 0 \le x \le L, \tag{2}$$

where $a = \lambda/(\rho c_p)$ is thermal diffusivity ($m^2 \cdot s^{-1}$), ρ is density ($kg \cdot m^{-3}$), c_p is specific heat capacity ($J \cdot kg^{-1} \cdot K^{-1}$), and λ is thermal conductivity ($W \cdot m^{-1} \cdot K^{-1}$).

Heat conduction model with using derivatives of fractional-order for various one-dimensional geometric cases was expressed by the following Oldham-Spanier equation [22]:

$$\frac{\partial T(x,\tau)}{\partial x} + \frac{1}{\sqrt{a}} \frac{\partial^{1/2}\left[T(x,\tau) - T_0\right]}{\partial \tau^{1/2}} + \frac{g\left[T(x,\tau) - T_0\right]}{x + R} = 0, \tag{3}$$

where g is a geometric factor and R is a radius of curvature.

In the case of one-dimensional heat conduction planar wall ($g = 0$), (3) will take the following form:

$$\frac{\partial^{1/2}\left[T(x,\tau) - T_0\right]}{\partial \tau^{1/2}} = -\sqrt{a} \frac{\partial T(x,\tau)}{\partial x}. \tag{4}$$

A more general formulation of the task for modeling not only one-dimensional heat conduction is based on the model in which, on the left-hand side of (1) instead of the first derivative with respect to time, the derivative of order α occurs; that is, we can find it in the form [42, 79]

$$\frac{\partial^\alpha u}{\partial \tau^\alpha} = (b)^2 \frac{\partial^2 u}{\partial x^2} \quad \text{for } 0 < x < L,\ \tau > 0, \tag{5}$$

$$u(0,\tau) = U_1, \quad u(L,\tau) = U_2 \quad \text{for } \tau > 0,$$
$$u(x,0) = f(x) \quad \text{for } 0 \le x \le L, \tag{6}$$

where b represents a constant coefficient with the unit $m \cdot s^{-\alpha/2}$.

Fractional-order models can also be described by the following equation, where α and β are of arbitrary order [12, 13, 67]:

$$\frac{\partial^\alpha T(x,\tau)}{\partial \tau^\alpha} = \left(\sqrt{a}\right)^2 \frac{\partial^\beta T(x,\tau)}{\partial |x|^\beta}. \tag{7}$$

3. Solutions

One-dimensional heat conduction models using integer and fractional derivatives can be solved by analytical and numerical methods.

3.1. Analytical Methods of Solution. Analytical methods can be used for solving problems in a bounded, semibounded, or unbounded interval.

Analytical solution of heat conduction model (1) for a bounded interval $\langle 0, L\rangle$ has the following shape [11, 42]:

$$T(x,\tau) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) \exp\left(-\left(\frac{n\pi\sqrt{a}}{L}\right)^2 \tau\right) c_k. \tag{8}$$

Analytical solution for a fractional diffusion-wave equation (5) has the form

$$T(x,\tau) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right) E_\alpha\left(-\left(\frac{n\pi b}{L}\right)^2 \tau^\alpha\right) c_k. \tag{9}$$

For models (1) and (5),

$$c_k = \frac{2}{L} \int_0^L \left[f(\xi) - \frac{1}{L}\left(T_2 - T_1\right)\xi - T_1\right] \sin\frac{k\pi\xi}{L} d\xi. \tag{10}$$

We developed and derived the coefficient c_k for the form of the function $f(\xi) = a_0 + a_1\xi + a_2\xi^2$, in order to implement simulations for different initial conditions (constant, straight line, and parabola). Coefficient c_k has this final shape [79]:

$$c_k = \frac{2}{nL}\left[T_2 - a_0 - a_1 L + a_2\left(\frac{2}{n^2} - L^2\right)\right](-1)^k - T_1 + a_0 - a_2\frac{2}{n^2}, \tag{11}$$

where $n = k\pi/L$.

3.2. Numerical Methods of Solution. The best known numerical methods include finite element method, finite difference method, and boundary element methods.

Finite difference methods according to the type of differential expression can be divided into explicit, implicit, and Crank-Nicolson scheme.

Explicit Scheme. Explicit scheme for solving the heat conduction model defined by (1) in the case of homogeneous material has the form

$$T_{m,p} = MT_{m-1,p-1} + T_{m,p-1} - 2MT_{m,p-1} + MT_{m+1,p-1}, \quad (12)$$

where module M is determined by the relation

$$M = \left(\frac{\sqrt{a}}{\Delta x}\right)^2 \quad \Delta\tau \le 0,5, \quad (13)$$

and in the case of nonhomogeneous material, it has the following form:

$$\begin{aligned} T_{m,p} &= M_{m-1}T_{m-1,p-1} + T_{m,p-1} \\ &\quad - \left(M_{m-1} + M_m\right)T_{m,p-1} + M_m T_{m+1,p-1}, \end{aligned} \quad (14)$$

where module M_m is

$$M_m = \left(\frac{\sqrt{a_m}}{\Delta x}\right)^2 \quad \Delta\tau \le 0,5. \quad (15)$$

Implicit Scheme. In the case of the implicit scheme, the temperature at a given point is calculated for a homogeneous body according to the following formula:

$$-MT_{m-1,p} + (1+2M)T_{m,p} - MT_{m+1,p} = T_{m,p-1}, \quad (16)$$

and for a nonhomogeneous body, it has the following formula:

$$\begin{aligned} &- M_{m-1}T_{m-1,p} + \left(1 + M_{m-1} + M_m\right)T_{m,p} \\ &\quad - M_m T_{m+1,p} = T_{m,p-1}. \end{aligned} \quad (17)$$

Crank-Nicolson Scheme. For a homogeneous body, it has the form

$$\begin{aligned} T_{m,p} &= \frac{M}{2}\left(T_{m-1,p} - 2T_{m,p} + T_{m+1,p}\right) \\ &\quad + \frac{M}{2}\left(T_{m-1,p-1} - 2T_{m,p-1} + T_{m+1,p-1}\right) + T_{m,p-1} \end{aligned} \quad (18)$$

and for a nonhomogeneous body it has the form

$$\begin{aligned} T_{m,p} &= \frac{1}{2}\left(M_{m-1}T_{m-1,p} - \left(M_{m-1} + M_m\right)T_{m,p}\right. \\ &\qquad \left. + M_m T_{m+1,p}\right) \\ &\quad + \frac{1}{2}\left(M_{m-1}T_{m-1,p-1} - \left(M_{m-1} + M_m\right)T_{m,p-1}\right. \\ &\qquad \left. + M_m T_{m+1,p-1}\right) \\ &\quad + T_{m,p-1}. \end{aligned} \quad (19)$$

Numerical Methods of Fractional-Order. For solving numerical methods of fractional-order, we use Grünwald-Letnikov definition with using the principle of "short memory" [16, 80]:

$$\frac{\partial^\alpha T(x,\tau)}{\partial\tau^\alpha} = \frac{\sum_{j=0}^{N_f} bc_j T\left(x, \tau - j\Delta\tau\right)}{\Delta\tau^\alpha}, \quad (20)$$

where L is the "length memory," τ is the time step, and the value of $N(f)$ will be determined by the following relation:

$$\begin{aligned} N(f) &= \min\left\{\left[\frac{\tau}{\Delta\tau}\right], \left[\frac{L}{\Delta\tau}\right]\right\}, \\ bc_0 &= 1, \qquad bc_j = \left(1 - \frac{1+\alpha}{j}\right)\cdot bc_{j-1}, \\ &\qquad\qquad \text{where } j \ge 1. \end{aligned} \quad (21)$$

Explicit Scheme. Explicit scheme for the heat conduction model using derivative of *fractional-order* (5) for a homogeneous material has the form

$$\begin{aligned} T_{m,p} &= MT_{m-1,p-1} - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \\ &\quad - 2MT_{m,p-1} + MT_{m+1,p-1} \end{aligned} \quad (22)$$

and for a nonhomogeneous material it has the form

$$\begin{aligned} T_{m,p} &= M_{m-1}T_{m-1,p-1} - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \\ &\quad - \left(M_{m-1} + M_m\right)T_{m,p-1} + M_m T_{m+1,p-1}. \end{aligned} \quad (23)$$

Implicit Scheme. Fractional shape for a homogeneous body is given by the following relation:

$$\begin{aligned} &- MT_{m-1,p} + (1+2M)T_{m,p} - MT_{m+1,p} \\ &\quad = -\sum_{j=1}^{N_f} bc_j T_{m,p-j} \end{aligned} \quad (24)$$

and for a nonhomogeneous body, it has the following relation:

$$\begin{aligned} &- M_{m-1}T_{m-1,p} + \left(1 + M_{m-1} + M_m\right)T_{m,p} \\ &\quad - M_m T_{m+1,p} = -\sum_{j=1}^{N_f} bc_j T_{m,p-j}. \end{aligned} \quad (25)$$

Crank-Nicolson Scheme. The fractional shape for a homogeneous body has the form

$$\begin{aligned} T_{m,p} &= \frac{M}{2}\left(T_{m-1,p} - 2T_{m,p} + T_{m+1,p}\right) \\ &\quad + \frac{M}{2}\left(T_{m-1,p-1} - 2T_{m,p-1} + T_{m+1,p-1}\right) \\ &\quad - \sum_{j=1}^{N_f} bc_j T_{m,p-j} \end{aligned} \quad (26)$$

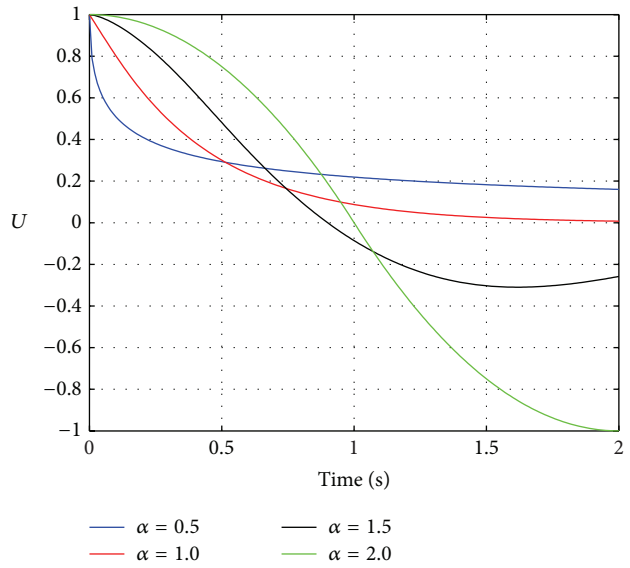


Figure 1: Comparison of analytical method for four different derivatives of temperature according to the time.

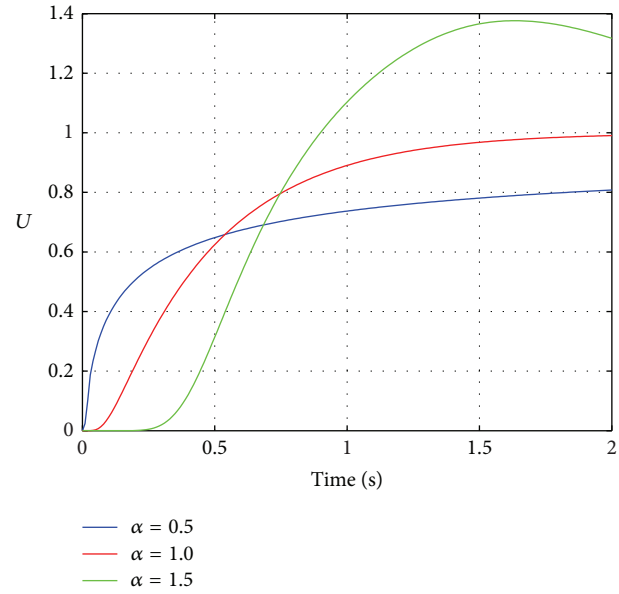


Figure 2: Crank-Nicolson scheme for derivatives of 0.5, 1, and 1.5.

and for a nonhomogeneous body it has the form

$$T_{m,p} = \frac{1}{2}\left(M_{m-1}T_{m-1,p} - \left(M_{m-1} + M_m\right)T_{m,p} + M_m T_{m+1,p}\right) + \frac{1}{2}\left(M_{m-1}T_{m-1,p-1} - \left(M_{m-1} + M_m\right)T_{m,p-1} + M_m T_{m+1,p-1}\right) - \sum_{j=1}^{N_f} bc_j T_{m,p-j}. \quad (27)$$

4. Simulations

Implementation of the one-dimensional heat conduction model was realized in the programming environment MATLAB. Two toolboxes for the one-dimensional heat conduction model with using integer- and fractional-order derivatives have been created. All implemented functions are published at Mathworks, Inc., MATLAB Central File Exchange as Heat Conduction Toolbox and Fractional Heat Conduction Toolbox [81, 82].

Simulations of heat conduction model for analytical solution have been implemented for four different derivatives temperatures according to time, namely, for the derivative order of 0.5, 1, 1.5, and 2 (Figure 1). The model input parameters were set as follows: initial temperature in the shape of parabolic function $f(x) = 2x - x^2$, boundary condition of the 1st kind for $U_1 = U_2 = 0$, total time simulation 2 s, time step 0.01 s, number of items' sum 100, distance 2 m, number of points 21, and coefficient for material properties 1 m·s$^{-\alpha/2}$.

Simulation with a heat conduction model for explicit, implicit, and Crank-Nicolson scheme was performed with the time step 0.01 s and order of the derivative of 1.5. Input parameters of the model were chosen as follows: initial value $U(x) = 0$, boundary condition of the 1st kind for $U_1 = U_2 = 1$, total time simulation 2 s, time step 0.01 s, number of items' sum 100, distance 2 m, number of points 21, and coefficient for material properties 1 m·s$^{-3/4}$.

From the numerical methods, we have chosen Crank-Nicolson scheme, in which we can see what effect a different order of the derivative has on the temperatures course (Figure 2).

In Figure 3, we see the comparison of courses of individual numerical methods and analytical solution.

5. Proposal Method for Thermal Diffusivity Determination

The method is based on the method of calculation of heat flows:

$$i_Q = \sqrt{c_p \rho \lambda}\cdot {}_0D_\tau^{1/2} g(\tau), \quad g(\tau) = T_w(\tau) - T_0. \quad (28)$$

Determination of the heat flow i_Q is possible in two ways: namely,

(i) from the gradient of the two measured temperatures (T_1, T_2),

$$i_Q = -\lambda \frac{d}{dx} T_1(\tau), \quad (29)$$

(ii) from the half-order derivative of one measured temperature (T_1),

$$i_Q = \frac{\lambda}{\sqrt{a}} \frac{d^{1/2}}{d\tau^{1/2}} \left[T_1(\tau) - T_0\right]. \quad (30)$$

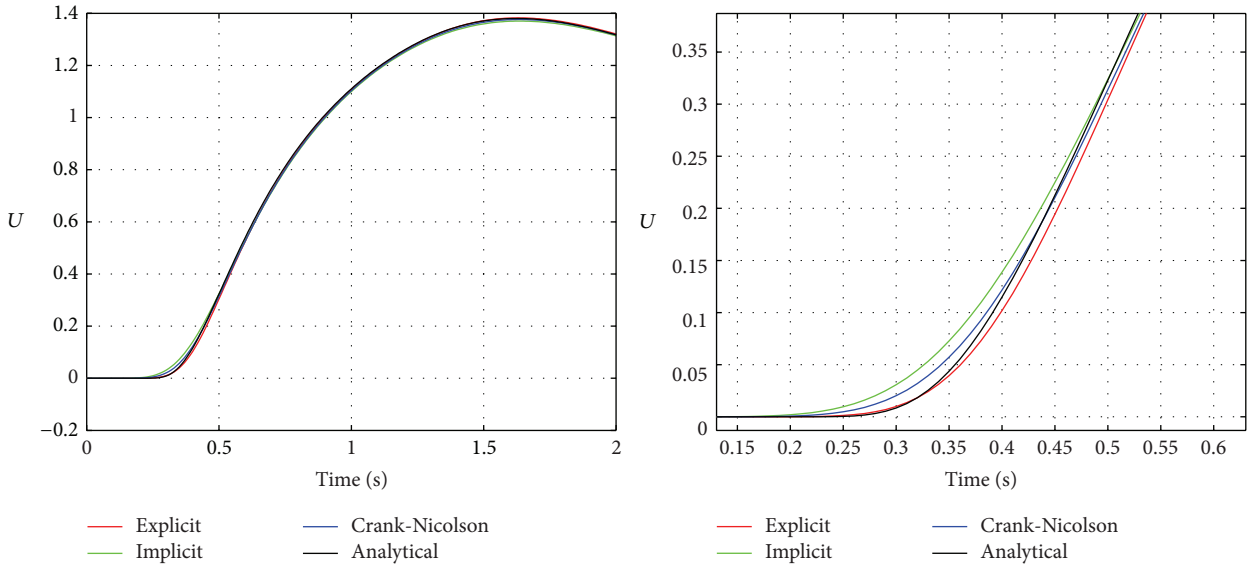


Figure 3: Comparison of analytical solution and numerical methods for the derivative of 1.5.

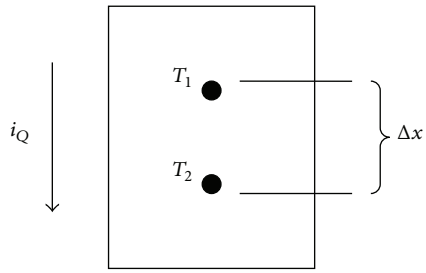


Figure 4: Measured temperatures.

Share of half-order derivative and gradient of temperature is proportional to the square root of the thermal diffusivity:

$$\sqrt{a} = \frac{\left(d^{1/2}/d\tau^{1/2}\right)\left[T_1(\tau) - T_0\right]}{-(d/dx)\,T_1(\tau)}. \tag{31}$$

The differential form of (31) is shown in the shape

$$\sqrt{a} = \frac{\Delta\tau^{-1/2}\sum_{j=0}^{N(f)} bc_j\left[T_{1,p-j} - T_0\right]}{\Delta x^{-1}\left[T_{2,p} - T_{1,p}\right]}. \tag{32}$$

For the numerical calculation of the first derivative of temperature according to the coordinate, respectively, temperature gradient (31) is sufficient to measure two temperatures (Figure 4).

The calculation of thermal diffusivity is based on the ratio half-order derivative of temperature according to the time to the temperature gradient (Figure 5) which is observed based on the values of two neighbouring temperatures in space obtained from simulations.

More previous values of temperature in time are used for the calculation of the half-order derivative, as in the case of the first derivative, which uses only one previous value [83].

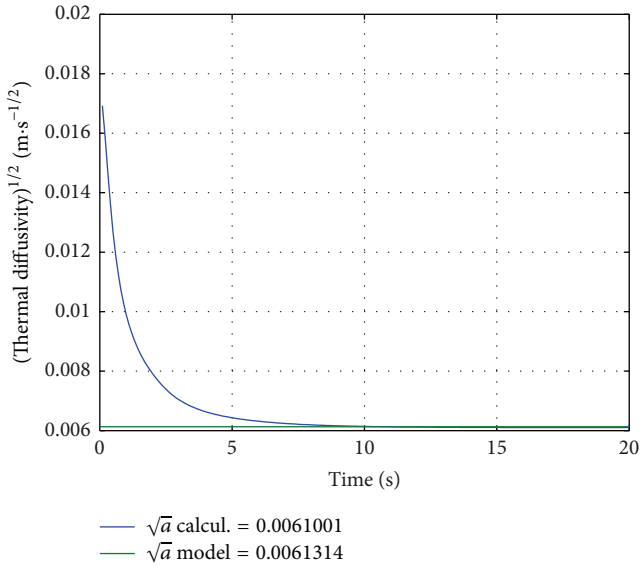


Figure 5: Rate of half-order temperature derivative to the temperature gradient.

The method was tested on the model using Crank-Nicolson scheme on a brass sample. The value of thermal diffusivity for a brass is 3.7594×10^{-5} m^2·s^{-1}. The initial temperature of simulation was determined on 20°C, boundary condition of the 1st kind for 20°C and 100°C, with a time step of the simulation 0.01 s. Input parameters of the brass: density 8,400 kg·m^{-3}, specific heat capacity 380 J·kg^{-1}·K^{-1}, and thermal conductivity 120 W·m^{-1}·K^{-1}.

In Figure 6, we can see the effect of time step to calculate the square root of thermal diffusivity.

The calculation accuracy of determining the value of the square root of thermal diffusivity depends on the number of

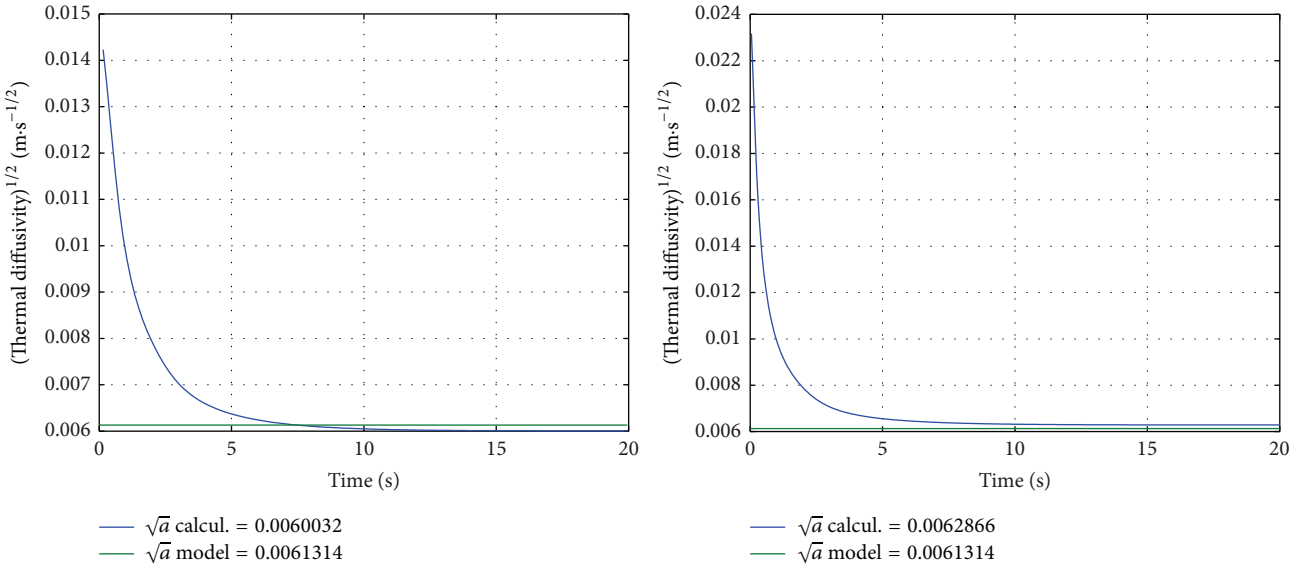


Figure 6: Crank-Nicolson scheme for a time step of 0.15 s and 0.05 s.

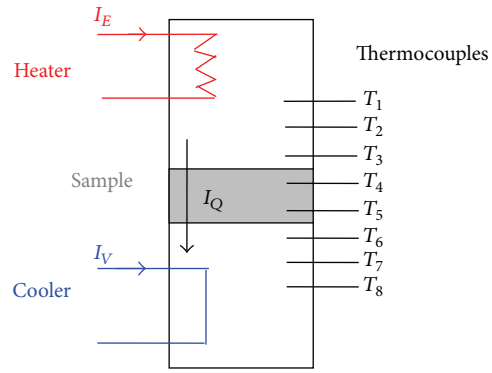


Figure 7: Scheme of HT11C.

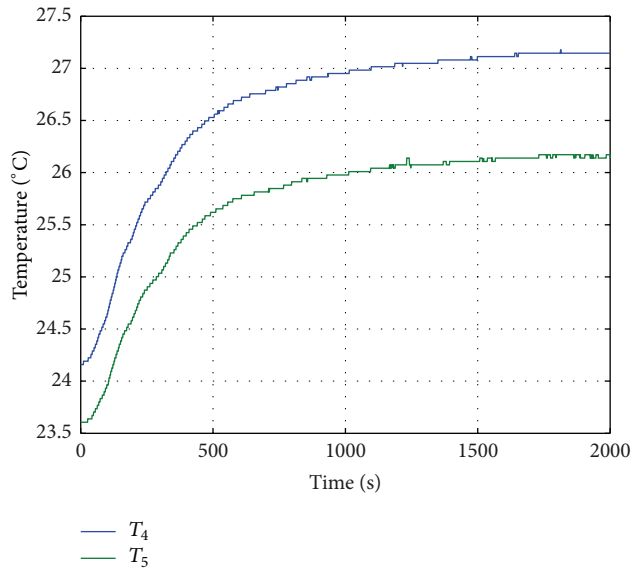


Figure 8: Experimental measurements of temperatures T_4 and T_5 for the brass.

previous values of temperatures in time and also from the selected time step.

Reducing the number of previous values of temperatures leads to higher inaccuracy of calculation.

6. Experimental Verification

The method has been verified on the experimental measurements. Measurements were carried out on the devices HT10XC and HT11C. Module HT11C is a physical model of one-dimensional heat conduction [84]. It consists of a heating and cooling section between which is inserted the sample of material (Figure 7).

Brass sample was used in the form of a cylinder with a diameter of 25 mm and a height of 30 mm. Contact areas of the sample were coated with a thin layer of thermal paste to minimize the transient thermal resistance. Module HT11C uses the thermocouples of type K in the temperature range from 0 to 133°C and the distance among them is 15 mm. The device HT10XC with HT11C module is connected via USB to a PC. The software that comes with the device allows setting conditions of the experiment and the measurement data saving to a file.

Experimental measurements which are referred to in this paper were carried out under the following conditions: namely, heater power 1.3 W, the water flow in the cooler 0.5 L/min, and the time step for recording of measured data 1.0 s. A unit jump in the heater power from 1.3 to 3.3 W was realized after stabilizing the temperatures. The transition from one steady state to another is shown in Figure 8.

On Figure 9 is determined the square root of thermal diffusivity from the measured values of the device HT11C

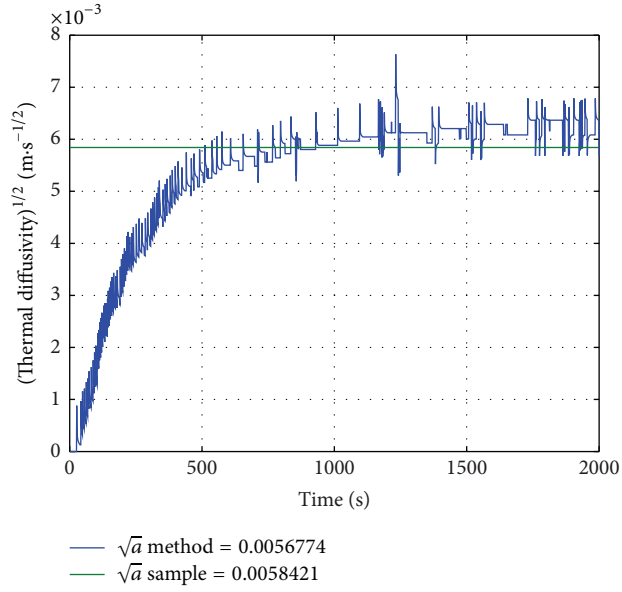


Figure 9: Thermal diffusivity.

(Figure 8), that is, from the ratio of half-order derivative the temperature according to the time to the temperature gradient.

The value of thermal diffusivity of the used brass sample for equipment HT11C was $3.2233 \times 10^{-5}\,\mathrm{m}^2\cdot\mathrm{s}^{-1}$ and it corresponds to the square root of thermal diffusivity $0.0056774\,\mathrm{m}\cdot\mathrm{s}^{-1/2}$. Brass sample was also measured on the device LFA [85] and the value of thermal diffusivity was $3.4130 \times 10^{-5}\,\mathrm{m}^2\cdot\mathrm{s}^{-1}$, which corresponds to the square root of thermal diffusivity $0.0058421\,\mathrm{m}\cdot\mathrm{s}^{-1/2}$. Calculated relative error between the measured values of the thermal diffusivity of the brass sample on HT11C and LFA is 5.5591% [79].

7. Conclusion

Benefits of this work are mainly the developed analytical and numerical methods for solving one-dimensional heat conduction using integer and fractional derivatives, which are implemented in the form of libraries functions in MATLAB. Another benefit is the designed, implemented, and verified method of determining thermal diffusivity using the half-order derivative of temperature according to the time on the experimental equipment HT10XC with module HT11C.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work was supported by the Slovak Research and Development Agency under the Contract no. APVV-0482-11 and by Projects VEGA 1/0729/12, 1/0552/14, and 1/0529/15.

References

[1] A. Fick, "Ueber diffusion," *Annalen der Physik*, vol. 170, no. 1, pp. 59–86, 1855.

[2] A. Fick, "On liquid diffusion," *Philosophical Magazine and Journal of Science*, vol. 10, pp. 31–39, 1855.

[3] J. C. Maxwell, "Scientific worthies," *Nature*, vol. 8, no. 203, pp. 397–399, 1873.

[4] J. C. Maxwell, *Treatise on Electricity and Magnetism*, vol. 1, Oxford at the Clarendon Press, 2nd edition, 1881.

[5] J. C. Maxwell, *An Elementary Treatise on Electricity*, Clarendon Press, Oxford, UK, 2nd edition, 1888.

[6] A. Einstein, "Uber die yon der molekularkinetischen Fliissigkeiten snspendierten Teilchen," *Annals of Physics*, vol. 17, pp. 549–560, 1905, (English translation in *Investigations on the Theory of Brownian Movement by Albert Einstein*, edited with notes by R. Fiirth, Methuen, London, UK, pp. 1–35, 1926).

[7] L. A. Richards, "Capillary conduction of liquids through porous mediums," *Journal of Applied Physics*, vol. 1, no. 5, pp. 318–333, 1931.

[8] G. Taylor, "Dispersion of soluble matter in solvent flowing slowly through a tube," *Proceedings of the Royal Society of London, Series A*, vol. 219, pp. 186–203, 1953.

[9] T. N. Narasimhan, "Fourier's heat conduction equation: history, influence, and connections," *Proceedings of the Indian Academy of Sciences, Earth and Planetary Sciences*, vol. 108, no. 3, pp. 117–148, 1999.

[10] D. W. Hahn and M. N. Ozisik, *Heat Conduction*, John Wiley & Sons, 2012.

[11] P. V. O'Nell, *Advanced Engineering Mathematics*, Cengage Learning, Toronto, Canada, 2012.

[12] I. Podlubny, A. Chechkin, T. Skovranek, Y. Chen, and B. M. Vinagre Jara, "Matrix approach to discrete fractional calculus II. Partial fractional differential equations," *Journal of Computational Physics*, vol. 228, no. 8, pp. 3137–3153, 2009.

[13] I. Podlubny, T. Skovranek, B. M. Vinagre Jara, I. Petras, V. Verbitsky, and Y. Chen, "Matrix approach to discrete fractional calculus III: non-equidistant grids, variable step length and distributed orders," *Philosophical Transactions of the Royal Society A*, vol. 371, no. 1990, pp. 1–15, 2013.

[14] J. S. Leszczynski, *An Introduction to Fractional Mechanics*, Czestochowa University of Technology, 2011.

[15] D. Sierociuk, A. Dzieliński, G. Sarwas, I. Petras, I. Podlubny, and T. Skovranek, "Modelling heat transfer in heterogeneous media using fractional calculus," *Philosophical Transactions of the Royal Society A*, vol. 371, no. 1990, Article ID 20120146, 2013.

[16] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[17] I. Podlubny, "Fractional-order models: a new stage in modelling and control," in *Proceedings of the IFAC Conference: System Structure and Control*, pp. 231–235, Nantes, France, July 1998.

[18] S. Sakakibara, "Properties of vibration with fractional derivatives damping of order 1/2," *JSME International Journal, Series C*, vol. 40, no. 3, pp. 393–399, 1997.

[19] P. J. Torvik and R. L. Bagley, "On the appearance of the fractional derivative in the behavior of real materials," *Journal of Applied Mechanics*, vol. 51, no. 2, pp. 294–298, 1984.

[20] S. G. Samko, A. A. Kilbas, and O. I. Maričev, *Fractional Integrals and Derivatives and Some of Their Applications*, Nauka i technika, Minsk, Belarus, 1987.

[21] K. B. Oldham, "Semiintegral electroanalysis: analog implementation," *Analytical Chemistry*, vol. 45, no. 1, pp. 39–47, 1973.

[22] K. B. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Academic Press, New York, NY, USA, 1974.

[23] K. B. Oldham and C. G. Zoski, "Analogue instrumentation for processing polarographic data," *Journal of Electroanalytical Chemistry*, vol. 157, no. 1, pp. 27–51, 1983.

[24] B. Mandelbrot, "Some noises with $1/f$ spectrum, a bridge between direct current and white noise," *IEEE Transactions on Information Theory*, vol. 13, no. 2, pp. 289–298, 1967.

[25] A. Oustaloup, "From fractality to non integer derivation through recursivity, a property common to these, two concepts: a fundamental idea for a new process control," in *Proceedings of the 12th IMACS World Congress*, pp. 203–208, Paris, France, 1988.

[26] A. Oustaloup, *La Derivation non Entiere*, Hermes, Paris, France, 1995, (French).

[27] L. O. Chua, L. Pivka, and C. W. Wu, "A universal circuit for studying chaotic phenomena," *Philosophical Transactions of the Royal Society of London, Series A: Mathematical, Physical Sciences and Engineering*, vol. 353, no. 1701, pp. 65–84, 1995.

[28] I. Petras, *Fractional-Order Nonlinear Systems*, Higher Education Press, Beijing, China, 2011.

[29] W. J. Parker, R. J. Jenkins, C. P. Butler, and G. L. Abbott, "Method of determining thermal diffusivity, heat capacity and thermal conductivity," *Journal of Applied Physics*, vol. 32, no. 9, p. 1961, 1961.

[30] A. M. Hofmeister, "Thermal diffusivity of garnets at high temperature," *Physics and Chemistry of Minerals*, vol. 33, no. 1, pp. 45–62, 2006.

[31] J. Madsen and J. Trefny, "Transient heat pulse technique for low thermal diffusivity solids," *Journal of Physics E: Scientific Instruments*, vol. 21, no. 4, pp. 363–366, 1988.

[32] C. S. McMenamin, D. F. Brewer, T. E. Hargreaves et al., "An improved analysis technique for heat pulse measurements," *Physica B: Physics of Condensed Matter*, vol. 194–196, no. 1, pp. 21–22, 1994.

[33] H. Bougrine, J. F. Geys, S. Dorbolo et al., "Simultaneous measurements of thermal diffusivity, thermal conductivity and thermopower with application to copper and ceramic superconductors," *The European Physical Journal B*, vol. 13, no. 3, pp. 437–443, 2000.

[34] K. Kobayasi, "Simultaneous measurement of thermal diffusivity and specific heat at high temperatures by a single rectangular pulse heating method," *International Journal of Thermophysics*, vol. 7, no. 1, pp. 181–195, 1986.

[35] B. Abeles, G. D. Cody, and D. S. Beers, "Apparatus for the measurement of the thermal diffusivity of solids at high temperatures," *Journal of Applied Physics*, vol. 31, no. 9, pp. 1585–1592, 1960.

[36] A. Kaźmierczak-Bałata, J. Bodzenta, and D. Trefon-Radziejewska, "Determination of thermal-diffusivity dependence on temperature of transparent samples by thermal wave method," *International Journal of Thermophysics*, vol. 31, no. 1, pp. 180–186, 2010.

[37] J. Liouville, "Memoire sur le calcul des differentielles a indices quelconques," *Journal de l'École Polytechnique*, vol. 13, p. 163, 1832.

[38] J. Liouville, "Mémoire sur l'intégration de l'équation $(mx_2+nx+p)\mathrm{d}^2\mathrm{y}/\mathrm{dx}^2 + (qx+r)\mathrm{dy}/\mathrm{dx} + sy = 0$ à l'aide des différentielles à indices quelconques," *Journal d l'Ecole Polytechnique*, vol. 13, pp. 163–186, 1832.

[39] B. Riemann, *Versuch einer allgemeinen Auffassung der Integration und Differentiation: The Collected Works of Bernhard Riemann*, edited by H. Weber, Dover, New York, NY, USA, 2nd edition, 1953.

[40] S. Westerlund and L. Ekstam, "Capacitor theory," *IEEE Transactions on Dielectrics and Electrical Insulation*, vol. 1, no. 5, pp. 826–839, 1994.

[41] R. Gorenflo, Y. Luchko, and F. Mainardi, "Wright functions as scale-invariant solutions of the diffusion-wave equation," *Journal of Computational and Applied Mathematics*, vol. 118, no. 1-2, pp. 175–191, 2000.

[42] O. P. Agrawal, "Solution for a fractional diffusion-wave equation defined in a bounded domain," *Nonlinear Dynamics*, vol. 29, pp. 145–155, 2002.

[43] F. Liu, V. V. Anh, I. Turner, and P. Zhuang, "Time fractional advection-dispersion equation," *Journal of Applied Mathematics and Computing*, vol. 13, no. 1-2, pp. 233–245, 2003.

[44] F. Mainardi, Y. Luchko, and G. Pagnini, "The fundamental solution of the space-time fractional diffusion equation," *Fractional Calculus & Applied Analysis*, vol. 4, no. 2, pp. 153–192, 2001.

[45] F. Huang and F. Liu, "The time fractional diffusion equation and the advection-dispersion equation," *The ANZIAM Journal*, vol. 46, no. 3, pp. 317–330, 2005.

[46] W. Wyss, "The fractional diffusion equation," *Journal of Mathematical Physics*, vol. 27, no. 11, pp. 2782–2785, 1986.

[47] W. R. Schneider and W. Wyss, "Fractional diffusion and wave equations," *Journal of Mathematical Physics*, vol. 30, no. 1, pp. 134–144, 1989.

[48] J.-S. Duan, "Time- and space-fractional partial differential equations," *Journal of Mathematical Physics*, vol. 46, no. 1, Article ID 013504, 013504, 8 pages, 2005.

[49] R. Gorenflo, F. Mainardi, D. Moretti, and P. Paradisi, "Time fractional diffusion: a discrete random walk approach," *Nonlinear Dynamics*, vol. 29, pp. 129–143, 2002.

[50] R. Gorenflo and A. Vivoli, "Fully discrete random walks for space-time fractional diffusion equations," *Signal Processing*, vol. 83, no. 11, pp. 2411–2420, 2003.

[51] F. Liu, S. Shen, V. Anh, and I. Turner, "Analysis of a discrete non-Markovian random walk approximation for the time fractional diffusion equation," *The ANZIAM Journal*, vol. 46, pp. 488–504, 2004/05.

[52] Q. Liu, F. Liu, I. Turner, and V. Anh, "Approximation of the Lévy-Feller advection-dispersion process by random walk and finite difference method," *Journal of Computational Physics*, vol. 222, no. 1, pp. 57–70, 2007.

[53] S. Shen and F. Liu, "Error analysis of an explicit finite difference approximation for the space fractional diffusion equation with insulated ends," *The ANZIAM Journal*, vol. 46, pp. 871–887, 2005.

[54] P. Zhuang and F. Liu, "Implicit difference approximation for the time fractional diffusion equation," *Journal of Applied Mathematics & Computing*, vol. 22, no. 3, pp. 87–99, 2006.

[55] C. Tadjeran and M. M. Meerschaert, "A second-order accurate numerical method for the two-dimensional fractional diffusion equation," *Journal of Computational Physics*, vol. 220, no. 2, pp. 813–823, 2007.

[56] J. P. Roop, "Computational aspects of FEM approximation of fractional advection dispersion equations on bounded domains in R^2," *Journal of Computational and Applied Mathematics*, vol. 193, no. 1, pp. 243–268, 2006.

[57] G. J. Fix and J. P. Roof, "Least squares finite-element solution of a fractional order two-point boundary value problem," *Computers & Mathematics with Applications*, vol. 48, no. 7-8, pp. 1017–1033, 2004.

[58] V. J. Ervin and J. P. Roop, "Variational formulation for the stationary fractional advection dispersion equation," *Numerical Methods for Partial Differential Equations*, vol. 22, no. 3, pp. 558–576, 2006.

[59] V. J. Ervin and J. P. Roop, "Variational solution of fractional advection dispersion equations on bounded domains in $\mathbb{R}^d$," *Numerical Methods for Partial Differential Equations*, vol. 23, no. 2, pp. 256–281, 2007.

[60] P. Kumar and O. P. Agrawal, "An approximate method for numerical solution of fractional differential equations," *Signal Processing*, vol. 86, no. 10, pp. 2602–2610, 2006.

[61] O. P. Agrawal, "A numerical scheme for initial compliance and creep response of a system," *Mechanics Research Communications*, vol. 36, no. 4, pp. 444–451, 2009.

[62] L. Yuan and O. P. Agrawal, "A numerical scheme for dynamic systems containing fractional derivatives," *Journal of Vibration and Acoustics*, vol. 124, no. 2, pp. 321–324, 2002.

[63] S. Momani and Z. Odibat, "Numerical solutions of the space-time fractional advection-dispersion equation," *Numerical Methods for Partial Differential Equations*, vol. 24, no. 6, pp. 1416–1429, 2008.

[64] H. Jafari and V. Daftardar-Gejji, "Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition," *Applied Mathematics and Computation*, vol. 180, no. 2, pp. 488–497, 2006.

[65] D. Fulger, E. Scalas, and G. Germano, "Monte Carlo simulation of uncoupled continuous-time random walks yielding a stochastic solution of the space-time fractional diffusion equation," *Physical Review E—Statistical, Nonlinear, and Soft Matter Physics*, vol. 77, no. 2, Article ID 021122, 2008.

[66] M. M. Marseguerra and A. Zoia, "Monte Carlo evaluation of FADE approach to anomalous kinetics," *Mathematics and Computers in Simulation*, vol. 77, no. 4, pp. 345–357, 2008.

[67] I. Podlubny, "Matrix approach to discrete fractional calculus," *Fractional Calculus & Applied Analysis*, vol. 3, no. 4, pp. 359–386, 2000.

[68] M. Ilić,, I. W. Turner, and D. P. Simpson, "A restarted Lanczos approximation to functions of a symmetric matrix," *IMA Journal of Numerical Analysis*, vol. 30, no. 4, pp. 1044–1061, 2010.

[69] Q. Yang, I. Turner, F. Liu, and M. Ilić, "Novel numerical methods for solving the time-space fractional diffusion equation in two dimensions," *SIAM Journal on Scientific Computing*, vol. 33, no. 3, pp. 1159–1180, 2011.

[70] S. B. Yuste and L. Acedo, "An explicit finite difference method and a new von Neumann-type stability analysis for fractional diffusion equations," *SIAM Journal on Numerical Analysis*, vol. 42, no. 5, pp. 1862–1874, 2005.

[71] R. Lin, F. Liu, V. Anh, and I. Turner, "Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation," *Applied Mathematics and Computation*, vol. 212, no. 2, pp. 435–445, 2009.

[72] F. Liu, P. Zhuang, V. Anh, and I. Turner, "A fractional-order im plicit difference approximation for the space-time fractional diffusion equation," *ANZIAM Journal*, vol. 47, pp. 48–68, 2006.

[73] S. Chen, F. Liu, P. Zhuang, and V. Anh, "Finite difference approximations for the fractional Fokker-Planck equation," *Applied Mathematical Modelling*, vol. 33, no. 1, pp. 256–273, 2009.

[74] T. A. Langlands and B. I. Henry, "The accuracy and stability of an implicit solution method for the fractional diffusion equation," *Journal of Computational Physics*, vol. 205, no. 2, pp. 719–736, 2005.

[75] C. Tadjeran, M. M. Meerschaert, and H.-P. Scheffler, "A second-order accurate numerical approximation for the fractional diffusion equation," *Journal of Computational Physics*, vol. 213, no. 1, pp. 205–213, 2006.

[76] E. Sousa, "Numerical approximations for fractional diffusion equations via splines," *Computers and Mathematics with Applications*, vol. 62, no. 3, pp. 938–944, 2011.

[77] C. Celik and M. Duman, "Crank-Nicolson difference scheme for the fractional diffusion equation with the Rieszfractional derivative," *Journal of Computational Physics*, vol. 231, no. 1, pp. 1743–1750, 2012.

[78] D. Wang, A. Xiao, and W. Yang, "Crank-Nicolson difference scheme for the coupled nonlinear Schrodinger equations with the Riesz space fractional derivative," *Journal of Computational Physics*, vol. 242, pp. 670–681, 2013.

[79] M. Zecova, *Modelovanie procesov prenosu tepla vedenfm (Modeling of the heat conduction processes) [Ph.D. thesis]*, TUKE, Kosice, Slovakia, 2014, (Slovak).

[80] I. Petras, "Fractional derivatives, fractional integrals, and fractional differential equations in Matlab," in *Engineering Education and Research Using MATLAB*, pp. 239–265, 2011.

[81] M. Zecova and J. Terpak, Heat Conduction Toolbox—File Exchange—MATLAB Central, 2014, http://www.mathworks.com/matlabcentral/fileexchange/43146-heat-conduction-toolbox.

[82] M. Zecova and J. Terpak, *Fractional Heat Conduction Toolbox*, File exchange, MATLAB Central, 2014, http://www.mathworks.com/matlabcentral/fileexchange/45491-fractional-heat-conduction-toolbox.

[83] M. Zecova, J. Terpak, and L. Dorcak, "The modeling of heat conduction using integer-and fractional-order derivatives," in *Proceedings of the 15th International Carpathian Control Conference (ICCC '14)*, pp. 710–715, Velke Karlovice, Czech Republic, May 2014.

[84] "HT10XC Computer Controlled Heat Transfer Teaching Equipment," 2014, http://discoverarmfield.com/en/products/view/ht10xc/computer-controlled-heat-transfer-teaching-equipment.

[85] "LFA 427-Laser Flash Apparatus (2000°C)-NETZSCH," 2013, http://www.netzsch.com/n87885.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 131690, 6 pages
http://dx.doi.org/10.1155/2015/131690

Research Article

Sumudu Transform Method for Analytical Solutions of Fractional Type Ordinary Differential Equations

Seyma Tuluce Demiray,[1] Hasan Bulut,[1] and Fethi Bin Muhammad Belgacem[2]

[1] *Department of Mathematics, Firat University, 23119 Elazig, Turkey*
[2] *Department of Mathematics, Faculty of Basic Education, PAAET, 92400 Al-Ardiya, Kuwait*

Correspondence should be addressed to Seyma Tuluce Demiray; seymatuluce@gmail.com

Received 26 June 2014; Revised 5 August 2014; Accepted 5 August 2014

Academic Editor: Abdon Atangana

We make use of the so-called Sumudu transform method (STM), a type of ordinary differential equations with both integer and noninteger order derivative. Firstly, we give the properties of STM, and then we directly apply it to fractional type ordinary differential equations, both homogeneous and inhomogeneous ones. We obtain exact solutions of fractional type ordinary differential equations, both homogeneous and inhomogeneous, by using STM. We present some numerical simulations of the obtained solutions and exhibit two-dimensional graphics by means of Mathematica tools. The method used here is highly efficient, powerful, and confidential tool in terms of finding exact solutions.

1. Introduction

Computation and analysis of solutions for nonlinear fractional differential equations now span a half-century or more and play a crucial role in several theoretical and applied sciences such as, but certainly not limited to, theoretical biology and ecology, solid state physics, viscoelasticity, fiber optics, signal processing and electric control theory, stochastic based finance, and hemo-, hydro-, and thermodynamics [1–7]. In particular, modified Kudryashov method [8], generalized Kudryashov method (GKM) [9], adomian decomposition method (ADM) [10], and homotopy decomposition method (HDM) [11] have all been used to seek solutions to fractional differential equations.

The Sumudu transform method (STM) was initiated in 1993, by Watugala who used it to solve engineering control problems [12, 13]. Subsequently, he expanded the Sumudu transform to two variables [14]. The first application of the STM to partial differential equations and the establishment of the inverse formula were done by Weerakoon [15, 16]. Sumudu transform based solutions to convolution type integral equations and discrete dynamic systems were later obtained by Asiru [17–19]. Following in this trend, Belgacem et al. established applications of the STM [20, 21]. Most of the STM applications in the first decade of the 21st century concentrated on integer type differential equations, and although Belgacem's practical interests for applying to fractional differential equations started as early as 2008 in various local presentations and conference talks, concrete results in this direction started appearing only in the second decade, with various teams of research [22], including the Bulut et al. team [23]. In fact, the Bulut et al. teams of research have been most active in the integer type differential equations area using the STM [24, 25]. Also, Atangana and Kilicman have tackled Sumudu transform method for nonlinear fractional heat-like equations [26].

In this work, our aim is to exhibit exact solutions of some homogeneous and nonhomogeneous fractional ordinary differential equations by using the STM. For paper layout, we first recall basic features of fractional calculus, in Section 2. In Section 3, we remind the reader of the properties of the STM. In Section 4, we apply the STM to obtain new exact solutions of both homogeneous and inhomogeneous fractional ordinary differential equations.

2. Basic Features of Fractional Calculus

In this section, we primarily introduce main features of fractional calculus following notations [1–7], making consensus among them. Fractional derivatives and integrals are most commonly introduced and used in the sense of Abel-Riemann (A-R), described by

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma[m-\alpha]} \dfrac{d}{dt^m} \displaystyle\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \\ \qquad\qquad m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), \qquad \alpha = m, \end{cases} \tag{1}$$

where $m \in \mathbb{Z}^+$, $\alpha \in R^+$. D^{α} is derivative operator and

$$D^{-\alpha} f(t) = \frac{1}{\Gamma[\alpha]} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad 0 < \alpha \le 1. \tag{2}$$

Also, according to A-R, an integral of fractional order is described by performing the integration operator J^{α} in the following format:

$$J^{\alpha} f(t) = \frac{1}{\Gamma[\alpha]} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad t > 0,\ \alpha > 0. \tag{3}$$

Following Podlubny [3] set-up, we have

$$\begin{aligned} J^{\alpha} t^n &= \frac{\Gamma[1+n]}{\Gamma[1+n+\alpha]} t^{n+\alpha}, \\ D^{\alpha} t^n &= \frac{\Gamma[1+n]}{\Gamma[1+n-\alpha]} t^{n-\alpha}. \end{aligned} \tag{4}$$

Another important and basic definition was first established by Caputo [1], given by

$${}^{C}D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma[m-\alpha]} \displaystyle\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \\ \qquad\qquad m-1 < \alpha < m, \\ \dfrac{d^m}{dt^m} f(t), \quad \alpha = m. \end{cases} \tag{5}$$

A basic property of the Caputo fractional derivative is that (see for instance [22])

$$J^{\alpha}\left[{}^{C}D^{\alpha} f(t)\right] = f(t) - \sum_{k=0}^{\infty} f^{(k)}(0^+) \frac{t^k}{k!}. \tag{6}$$

3. Basic Features of the Sumudu Transform Method (STM)

The Sumudu transform is obtained over the set of functions [20] as

$$\begin{aligned} A = \Big\{ f(t) \mid \exists M, \tau_1,\ \tau_2 > 0,\ \left|f(t)\right| < M e^{|t|/\tau_i}, \\ \text{if}\ \ t \in (-1)^j \times [0,\infty) \Big\} \end{aligned} \tag{7}$$

by

$$F(u) = S\left[f(t)\right] = \int_0^{\infty} f(ut)\, e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \tag{8}$$

Theorem 1. *If $F(u)$ is Sumudu transform of $f(t)$, then the Sumudu transform of the derivatives with integer order is as follows [20–23]:*

$$S\left[\frac{d^n f(t)}{dt^n}\right] = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k \left.\frac{d^n f(t)}{dt^n}\right|_{t=0}\right]. \tag{9}$$

Proof. The Sumudu transform of the first derivative of $f(t)$, $f'(t) = df(t)/dt$, is given by

$$\begin{aligned} S\left[\frac{df(t)}{dt}\right] &= \int_0^{\infty} \frac{df(ut)}{dt} e^{-t} dt = \lim_{p\to\infty} \int_0^p \frac{df(ut)}{dt} e^{-t} dt \\ &= \lim_{p\to\infty}\left[\frac{1}{u} e^{-t/u} f(t)\Big|_0^p + \frac{1}{u^2}\int_0^p e^{-t/u} f(t)\, dt\right] \\ &= \lim_{p\to\infty}\left[\frac{1}{u} e^{-t/u} f(t)\Big|_0^p \right. \\ &\qquad \left. + \frac{1}{u}\left(\frac{1}{u}\int_0^p e^{-t/u} f(t)\, dt\right)\right] \\ &= \lim_{p\to\infty}\left[-\frac{1}{u} f(0) + \frac{1}{u}\left(\frac{1}{u}\int_0^p e^{-t/u} f(t)\, dt\right)\right] \\ &= -\frac{1}{u} f(0) + \frac{1}{u} F(u). \end{aligned} \tag{10}$$

Proceeding in the same manner, we get the Sumudu transform of the second order derivative as

$$S\left[\frac{d^2 f(t)}{dt^2}\right] = \frac{1}{u^2}\left[F(u) - f(0) - u\left.\frac{df(t)}{dt}\right|_{t=0}\right]. \tag{11}$$

To finish the proof of Theorem 1, we proceed by induction in the same way only to reach the general formula for the Sumudu transform of any integer n-order derivative [23]; namely,

$$S\left[\frac{d^n f(t)}{dt^n}\right] = u^{-n}\left[F(u) - \sum_{k=0}^{n-1} u^k \left.\frac{d^n f(t)}{dt^n}\right|_{t=0}\right]. \tag{12}$$

□

Theorem 2. *If $F(u)$ is the Sumudu transform of $f(t)$, then the Sumudu transform of the Riemann-Liouville fractional derivative is given by (see [22])*

$$\begin{aligned} S\left[D^{\alpha} f(t)\right] = u^{-\alpha}\left[F(u) - \sum_{k=1}^{n} u^{\alpha-k}\left[D^{\alpha-k}\left(f(t)\right)\right]_{t=0}\right], \\ -1 < n-1 \le \alpha < n. \end{aligned} \tag{13}$$

At the moment, we harness the properties developments of the STM and utilize them for finding exact solutions of fractional ordinary differential equations.

We consider the general linear fractional ordinary differential equation (FODE) as follows:

$$\frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}} = \frac{\partial^2 y(t)}{\partial t^2} + \frac{\partial y(t)}{\partial t} + y(t) + c, \tag{14}$$

subject to the initial condition

$$y(0) = f(0). \tag{15}$$

When we get Sumudu transform of (14) taking into consideration (12) and (13), we obtain Sumudu transform of (14) as follows:

$$S\left[\frac{\partial^{\alpha} y(t)}{\partial t^{\alpha}}\right] = S\left[\frac{\partial^{2} y(t)}{\partial t^{2}}\right] + S\left[\frac{\partial y(t)}{\partial t}\right] + S[y(t)] + S[c],$$

$$u^{-\alpha}\left[F(u) - \sum_{k=1}^{n} u^{\alpha-k}\left[D^{\alpha-k}(f(t))\right]_{t=0}\right] = \frac{1}{u^{2}}\left[F(u) - f(0) - u\left.\frac{\partial f(t)}{\partial t}\right|_{t=0}\right] + \frac{1}{u}[F(u) - f(0)] + F(u) + c,$$

$$F(u) - \sum_{k=1}^{n} u^{\alpha-k}\left[D^{\alpha-k}(f(t))\right]_{t=0} = u^{\alpha-2}\left[F(u) - f(0) - u\left.\frac{\partial f(t)}{\partial t}\right|_{t=0}\right] + u^{\alpha-1}[F(u) - f(0)] + u^{\alpha}F(u) + cu^{\alpha},$$

$$F(u) = u^{\alpha-2}F(u) - u^{\alpha-2}f(0) + \sum_{k=1}^{n} u^{\alpha-k}\left[D^{\alpha-k}(f(t))\right]_{t=0} - u^{\alpha-1}\left.\frac{\partial f(t)}{\partial t}\right|_{t=0} + u^{\alpha-1}F(u) - u^{\alpha-1}f(0) + u^{\alpha}F(u) + cu^{\alpha},$$

$$F(u) - u^{\alpha-2}F(u) - u^{\alpha-1}F(u) - u^{\alpha}F(u) = -u^{\alpha-2}f(0) + \sum_{k=1}^{n} u^{\alpha-k}\left[D^{\alpha-k}(f(t))\right]_{t=0} - u^{\alpha-1}\left.\frac{\partial f(t)}{\partial t}\right|_{t=0} - u^{\alpha-1}f(0) + cu^{\alpha},$$

$$F(u) = \frac{1}{1 - u^{\alpha-2} - u^{\alpha-1} - u^{\alpha}} \times \left[g(u) - u^{\alpha-1}f(0) - u^{\alpha-2}f(0) + cu^{\alpha}\right], \tag{16}$$

where $g(u)$ is described by $\sum_{k=1}^{n} u^{\alpha-k}[D^{\alpha-k}(f(t))]_{t=0} - u^{\alpha-1}(\partial f(t)/\partial t)|_{t=0}$.

When we get inverse Sumudu transform of (16) by using inverse transform table in [21, 22], we obtain solution of (14) by using STM in the following manner:

$$y(t) = S^{-1}\left[\frac{1}{1 - u^{\alpha-2} - u^{\alpha-1} - u^{\alpha}} \times \left[g(u) - u^{\alpha-1}f(0) - u^{\alpha-2}f(0) + cu^{\alpha}\right]\right]. \tag{17}$$

4. STM Implementations to Homogeneous and Inhomogeneous FODEs

In this Section, we implement the STM to homogeneous and inhomogeneous fractional ordinary differential equations (HFODEs and IHFODEs) in the following three examples.

Example 3. Initially, we consider the inhomogeneous fractional ordinary differential equation [4] as

$$D^{2}y(t) + D^{3/2}y(t) + y(t) = t + 1, \quad \left[{}_{0}D_{t}^{-1/2}y(t)\right]_{t=0} = 0, \quad t > 0, \tag{18}$$

subject to the initial conditions

$$y(0) = y'(0) = 1. \tag{19}$$

In order to find exact solution of (18), we take the Sumudu transform of both sides of (18) as follows:

$$S\left[D^{2}y(t)\right] + S\left[D^{3/2}y(t)\right] + S[y(t)] = S[t] + S[1],$$

$$\frac{1}{u^{2}}\left[F(u) - f(0) - u\left.\frac{df(t)}{dt}\right|_{t=0}\right] + \frac{F(u)}{u^{3/2}} - \left.\frac{D^{1/2}[f(t)]}{u}\right|_{t=0} + F(u) = u + 1,$$

$$\frac{1}{u^{2}}[F(u) - 1 - u] + \frac{F(u)}{u^{3/2}} - \frac{1}{u}\left[\frac{F(u)}{u^{1/2}} - \left.\frac{D^{-1/2}[f(t)]}{u}\right|_{t=0}\right] + F(u) = u + 1,$$

$$\frac{1}{u^{2}}[F(u) - 1 - u] + \left.\frac{D^{-1/2}[f(t)]}{u}\right|_{t=0} + F(u) = u + 1,$$

$$\left(1 + \frac{1}{u^{2}}\right)F(u) = 1 + u + \frac{1}{u} + \frac{1}{u^{2}} \Longrightarrow \left(1 + u^{2}\right)F(u) = 1 + u + u^{2} + u^{3},$$

$$F(u) = \frac{1 + u + u^{2} + u^{3}}{\left(1 + u^{2}\right)} = 1 + u. \tag{20}$$

Upon Sumudu inverting (20) which is linear (also see inverse transform table in [21]), we obtain exact solution as

$$y(t) = 1 + t. \tag{21}$$

Remark 4. The exact solution of (18) found here by using Sumudu transform method agrees with the solution in [4] for corresponding values of tuned parameters.

Example 5. Secondly, we consider homogeneous fractional ordinary differential equation as follows [6]:

$$\frac{dy(t)}{dt} + \frac{d^{1/2}y(t)}{dt^{1/2}} - 2y(t) = 0, \qquad \frac{d^{1/2}y(0^+)}{dt^{1/2}} = c, \quad t > 0, \tag{22}$$

with initial condition

$$y(0^+) = 0. \tag{23}$$

We first apply the Sumudu transform of both sides of (22) to get

$$S\left(\frac{dy(t)}{dt}\right) + S\left(\frac{d^{1/2}y(t)}{dt^{1/2}}\right) - 2S(y(t)) = 0, \tag{24}$$

which yields

$$\begin{aligned}
&\frac{1}{u}\left[F(u) - f(0)\right] + \frac{F(u)}{u^{1/2}} - \left.\frac{D^{-1/2}\left[f(t)\right]}{u}\right|_{t=0} - 2F(u) = 0,\\
&\frac{F(u)}{u} + \frac{F(u)}{u^{1/2}} - \frac{c}{u} - 2F(u) = 0,\\
&\left(1 + u^{1/2} - 2u\right)F(u) = c\\
&\Longrightarrow F(u) = \frac{c}{1 + u^{1/2} - 2u}\\
&\qquad = \frac{2c}{3\left(1 + 2\sqrt{u}\right)} + \frac{c}{3\left(1 - \sqrt{u}\right)},\\
&F(u) = \frac{2c}{3}\left(\frac{1 - 2\sqrt{u}}{1 - 4u}\right) + \frac{c}{3}\left(\frac{1 + \sqrt{u}}{1 - u}\right),\\
&F(u) = \frac{2c}{3}\left(\frac{1}{1 - 4u} - \frac{2\sqrt{u}}{1 - 4u}\right) + \frac{c}{3}\left(\frac{1}{1 - u} - \frac{\sqrt{u}}{1 - u}\right).
\end{aligned} \tag{25}$$

When we take inverse Sumudu transform of (25) by using inverse transform table in [21], we obtain exact solution of (22) by using STM as follows:

$$y(t) = \frac{2c}{3}\left[e^{4t} - e^{4t}\operatorname{erf}\left(2\sqrt{t}\right)\right] + \frac{c}{3}\left[e^{t} + e^{t}\operatorname{erf}\left(\sqrt{t}\right)\right]. \tag{26}$$

Remark 6. To our knowledge, the solution of (22), obtained by using the STM, is new and does not figure in the published literature.

Example 7. Thirdly, we investigate the inhomogeneous fractional ordinary differential equation [6] as

$${}_0D_t^{1/2}y(t) - y(t) = -1, \qquad \left[{}_0D_t^{-1/2}y(t)\right]_{t=0} = 0, \quad t > 0. \tag{27}$$

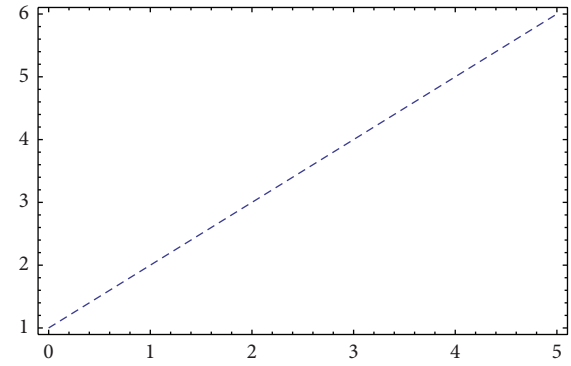


Figure 1: Two-dimensional graphic for exact solution of (18), when $0 < t < 5$, via the STM.

In order to obtain the exact solution of (27) by using the STM, we apply the Sumudu transform to both sides of (27) as follows:

$$\begin{aligned}
&S\left[D^{1/2}y(t)\right] - S\left[y(t)\right] + S[1] = 0,\\
&S\left[D^{1/2}y(t)\right] - F(u) = -S[1],\\
&\frac{F(u)}{u^{1/2}} - \left.\frac{D^{-1/2}\left[f(t)\right]}{u}\right|_{t=0} - F(u) = -S[1],\\
&\frac{F(u)}{u^{1/2}} - F(u) = -1\\
&\Longrightarrow \left(\frac{1}{u^{1/2}} - 1\right)F(u) = -1\\
&\Longrightarrow \left(1 - u^{1/2}\right)F(u) = -u^{1/2},\\
&F(u) = \frac{\sqrt{u}}{\sqrt{u} - 1} = \frac{\sqrt{u}\left(\sqrt{u} + 1\right)}{u - 1} = \frac{u + \sqrt{u}}{u - 1}\\
&\qquad = \frac{u - 1 + 1 + \sqrt{u}}{u - 1} = 1 - \frac{1}{1 - u} - \frac{\sqrt{u}}{1 - u}.
\end{aligned} \tag{28}$$

When we get the inverse Sumudu transform of (28) by using the table in [21], we find exact solution of (27) by the STM as follows:

$$y(t) = 1 - e^{t} - e^{t}\operatorname{erf}\sqrt{t}. \tag{29}$$

Remark 8. Again, the exact solution of (27) exhibited here, found through implementing the Sumudu transform method, is to our knowledge new and not trackable in previous literature.

We plot solution (21) of (18) in Figure 1, solution (26) of (22) in Figure 2, solution (29) of (27) in Figure 3, which shows the dynamics of solutions with suitable parametric choices.

5. Conclusions

In this paper, we considered homogeneous and inhomogeneous fractional ordinary differential equations (HFODEs and IHFODEs) and treated them by applying the Sumudu

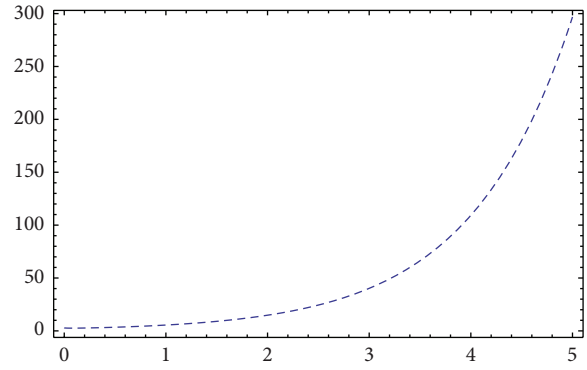


Figure 2: Two-dimensional graphic for exact solution of (22), for $0 < t < 5$ and $c = 3$, via the STM.

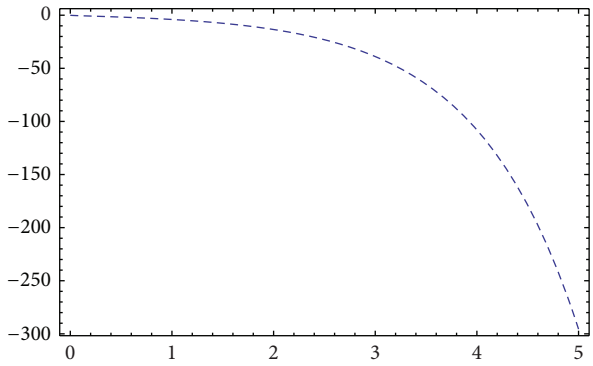


Figure 3: Two-dimensional graphic for exact solution of (27), for $0 < t < 5$, via the STM.

transform method and obtained their exact solutions. To date and to our knowledge, the exact solutions obtained in the last two examples are new. We also exhibited two-dimensional graphics for the obtained solutions by means of programming language Mathematica. Accordingly, we can conclude that not only does the Sumudu transform method play an important role in treating homogeneous and inhomogeneous fractional ordinary differential equations, but also it is highly effective with regard to yielding exact solutions. We expect the STM to be equally successful when treating even more complex applications than presented involving FODEs.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] M. Caputo, "Linear model of dissipation whose Q is almost frequency independent-II," *Geophysical Journal of the Royal Astronomical Society*, vol. 13, no. 5, pp. 529–539, 1967.

[2] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differantial Equations*, John Wiley & Sons, New York, NY, USA, 1993.

[3] I. Podlubny, *Fractional Differantial Equations*, Academic Press, San Diego, Calif, USA, 1999.

[4] K. Diethelm and Y. Luchko, "Numerical solution of linear multi-term initial value problems of fractional order," *Journal of Computational Analysis and Applications*, vol. 6, no. 3, pp. 243–263, 2004.

[5] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier Science, New York, NY, USA, 2006.

[6] R. L. Magin, *Fractional Calculus in Bioengineering*, Begell House, Redding, Conn, USA, 2006.

[7] R. C. Mittal and R. Nigam, "Solution of fractional integro-differential equations by Adomian decomposition method," *The International Journal of Applied Mathematics and Mechanics*, vol. 4, no. 2, pp. 87–94, 2008.

[8] Y. A. Tandogan, Y. Pandir, and Y. Gurefe, "Solutions of the nonlinear differential equations by use of modified Kudryashov method," *Turkish Journal of Mathematics and Computer Science*, vol. 2013, Article ID 20130021, 7 pages, 2013.

[9] S. Tuluce Demiray, Y. Pandir, and H. Bulut, "Generalized kudryashov method for time-fractional differential equations," *Abstract and Applied Analysis*, vol. 2014, Article ID 901540, 13 pages, 2014.

[10] G. Adomian, "A review of the decomposition method and some recent results for nonlinear equations," *Mathematical and Computer Modelling*, vol. 13, no. 7, pp. 17–43, 1990.

[11] A. Atangana, S. Tuluce Demiray, and H. Bulut, "Modelling the nonlinear wave motion within the scope of the fractional calculus," *Abstract and Applied Analysis*, vol. 2014, Article ID 481657, 7 pages, 2014.

[12] G. K. Watugala, "Sumudu transform: a new integral transform to solve differential equations and control engineering problems," *International Journal of Mathematical Education in Science and Technology*, vol. 24, no. 1, pp. 35–43, 1993.

[13] G. K. Watugala, "Sumudu transform—a new integral transform to solve differential equations and control engineering problems," *Mathematical Engineering in Industry*, vol. 6, no. 4, pp. 319–329, 1998.

[14] G. K. Watugala, "The Sumudu transform for functions of two variables," *Mathematical Engineering in Industry*, vol. 8, no. 4, pp. 293–302, 2002.

[15] S. Weerakoon, "Application of Sumudu transform to partial differential equations," *International Journal of Mathematical Education in Science and Technology*, vol. 25, no. 2, pp. 277–283, 1994.

[16] S. Weerakoon, "Complex inversion formula for Sumudu transform," *International Journal of Mathematical Education in Science and Technology*, vol. 29, no. 4, pp. 618–621, 1998.

[17] M. A. Asiru, "Sumudu transform and the solution of integral equations of convolution type," *International Journal of Mathematical Education in Science and Technology*, vol. 32, no. 6, pp. 906–910, 2001.

[18] M. A. Asiru, "Further properties of the Sumudu transform and its applications," *International Journal of Mathematical Education in Science and Technology*, vol. 33, no. 3, pp. 441–449, 2002.

[19] M. A. Asiru, "Application of the sumudu transform to discrete dynamical systems," *International Journal of Mathematical Education in Science and Technology*, vol. 34, no. 6, pp. 944–949, 2003.

[20] F. B. M. Belgacem, A. A. Karaballi, and S. L. Kalla, "Analytical investigations of the Sumudu transform and applications to integral production equations," *Mathematical Problems in Engineering*, vol. 2003, no. 3, pp. 103–118, 2003.

[21] F. B. M. Belgacem and A. A. Karaballi, "Sumudu transform fundamental properties investigations and applications," *Journal of Applied Mathematics and Stochastic Analysis*, vol. 2006, Article ID 91083, 23 pages, 2006.

[22] Q. D. Katatbeh and F. B. M. Belgacem, "Applications of the Sumudu transform to fractional differential equations," *Nonlinear Studies*, vol. 18, no. 1, pp. 99–112, 2011.

[23] H. Bulut, H. M. Baskonus, and F. B. M. Belgacem, "The analytical solution of some fractional ordinary differential equations by the Sumudu transform method," *Abstract and Applied Analysis*, vol. 2013, Article ID 203875, 6 pages, 2013.

[24] H. Bulut, H. M. Baskonus, and S. Tuluce, "The solutions of partial differential equations with variable coefficient by Sumudu transform method," in *Proceedings of the 9th International Conference on Mathematical Problems in Engineering, Aerospace and Sciences (ICNPAA '12)*, vol. 1493 of *AIP Proceedings*, p. 91, Vienna, Austria, July 2012.

[25] H. Bulut, H. M. Baskonus, and S. Tuluce, "Homotopy perturbation Sumudu transform method for heat equations," *Mathematics in Engineering, Science and Aerospace*, vol. 4, no. 1, pp. 49–60, 2013.

[26] A. Atangana and A. Kılıçman, "The use of Sumudu transform for solving certain nonlinear fractional heat-like equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 737481, 12 pages, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 309870, 9 pages
http://dx.doi.org/10.1155/2015/309870

Research Article

Local Fractional Laplace Variational Iteration Method for Solving Diffusion and Wave Equations on Cantor Sets within Local Fractional Operators

Hassan Kamil Jassim,[1] Canan Ünlü,[2] Seithuti Philemon Moshokoa,[3] and Chaudry Masood Khalique[4]

[1]*Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*
[2]*Department of Mathematics, Faculty of Sciences, University of Istanbul, Vezneciler, 34134 Istanbul, Turkey*
[3]*Department of Mathematics and Statistics, Faculty of Science, Tshwane University of Technology, Arcadia Campus, Building 2-117, Nelson Mandela Drive, Pretoria 0001, South Africa*
[4]*Department of Mathematical Sciences, International Institute for Symmetry Analysis and Mathematical Modelling, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa*

Correspondence should be addressed to Seithuti Philemon Moshokoa; moshokoasp@tut.ac.za

Received 14 October 2014; Accepted 26 January 2015

Academic Editor: Samir B. Belhaouari

The local fractional Laplace variational iteration method (LFLVIM) is employed to handle the diffusion and wave equations on Cantor set. The operators are taken in the local sense. The nondifferentiable approximate solutions are obtained by using the local fractional Laplace variational iteration method, which is the coupling method of local fractional variational iteration method and Laplace transform. Illustrative examples are included to demonstrate the high accuracy and fast convergence of this new algorithm.

1. Introduction

Local fractional calculus has played an important role in areas ranging from fundamental science to engineering in the past ten years [1–3] and has been applied to a wide class of complex problems encompassing physics, biology, mechanics, and interdisciplinary areas [4–9]. Various methods, for example, the Adomian decomposition method [10], the Rich-Adomian-Meyers modified decomposition method [11], the variational iteration method [12], the homotopy perturbation method [13, 14], the fractal Laplace and Fourier transforms [15], the homotopy analysis method [16], the heat balance integral method [17–19], the fractional variational iteration method [20–22], and the fractional subequation method [23, 24], have been utilized to solve fractional differential equations [3, 15].

The diffusion equations are important in many processes in science and engineering, for example, the diffusion of a dissolved substance in the solvent liquids and neutrons in a nuclear reactor and Brownian motion, while wave equations characterize the motion of a vibrating string (see [25, 26] and the references therein).

The diffusion equation on Cantor sets (called local fractional diffusion equation) was recently described in [27] as

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} = a^{2\alpha} \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}}, \tag{1}$$

where $a^{2\alpha}$ denotes the fractal diffusion constant which is, in essence, a measure for the efficiency of the spreading of the underlying substance, while local fractional wave equation is written in the following form [28, 29]:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = a^{2\alpha} \frac{\partial^{2\alpha} u(x,y)}{\partial x^{2\alpha}}. \tag{2}$$

The local fractional Laplace operator is given by [28, 29] as follows:

$$\nabla^{2\alpha} = \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial y^{2\alpha}} + \frac{\partial^{2\alpha}}{\partial z^{2\alpha}}. \tag{3}$$

We notice that the local fractional diffusion equation yields

$$\nabla^{2\alpha} u = \frac{1}{a^{2\alpha}} \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}, \tag{4}$$

and the local fractional wave equation has the following form:

$$\nabla^{2\alpha} u = \frac{1}{a^{2\alpha}} \frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}}, \tag{5}$$

where $1/a^{2\alpha}$ is a constant. This equation describes vibrations in a fractal medium.

The quantity $u(x,t)$ is interpreted as the local fractional deviation at the time t from the position at rest of the point with rest position given by x, y, and z. The above fractal derivatives were considered as the local fractional operators [30, 31].

The paper is organized as follows. In Section 2, we introduce the notions of local fractional calculus theory used in this paper. In Section 3, we give the local fractional Laplace variational iteration method. Section 4 presents the solutions for diffusion and wave equations in Cantor set conditions. Section 5 is devoted to our conclusions.

2. Mathematical Fundamentals

2.1. Local Fractional Calculus (See [28, 29, 32, 33])

Definition 1. Suppose that there is the relation

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha}, \quad 0 < \alpha \le 1, \tag{6}$$

with $|x - x_0| < \delta$, for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$; then the function $f(x)$ is called local fractional continuous at $x = x_0$ and it is denoted by $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition 2. Suppose that the function $f(x)$ satisfies condition (6), for $x \in (a,b)$; it is so called local fractional continuous on the interval (a,b), denoted by $f(x) \in C_{\alpha}(a,b)$.

Definition 3. In fractal space, let $f(x) \in C_{\alpha}(a,b)$; local fractional derivative of $f(x)$ of order α at the point $x = x_0$ is given by

$$\begin{aligned} D_x^{\alpha} f(x_0) &= \left. \frac{d^{\alpha}}{dx^{\alpha}} f(x) \right|_{x=x_0} \\ &= f^{(\alpha)}(x_0) = \lim_{x \to x_0} \frac{\Delta^{\alpha}(f(x) - f(x_0))}{(x - x_0)^{\alpha}}, \end{aligned} \tag{7}$$

where $\Delta^{\alpha}(f(x) - f(x_0)) \cong \Gamma(\alpha+1)(f(x) - f(x_0))$.

The formulas of local fractional derivatives of special functions used in the paper are as follows:

$$\begin{aligned} D_x^{(\alpha)} a g(x) &= a D_x^{(\alpha)} g(x), \\ \frac{d^{\alpha}}{dx^{\alpha}} \left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) &= \frac{x^{(n-1)^{\alpha}}}{\Gamma(1+(n-1)\alpha)}, \quad n \in N. \end{aligned} \tag{8}$$

Definition 4. A partition of the interval $[a,b]$ is denoted by (t_j, t_{j+1}), $j = 0, \ldots, N-1$, $t_0 = a$ and $t_N = b$ with $\Delta t_j = t_{j+1} - t_j$ and $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots\}$. Local fractional integral of $f(x)$ in the interval $[a,b]$ is given by

$$\begin{aligned} {}_aI_b^{(\alpha)} f(x) &= \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^{\alpha} \\ &= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^{\alpha}. \end{aligned} \tag{9}$$

The formulas of local fractional integrals of special functions used in the paper are as follows:

$$\begin{aligned} {}_0I_x^{(\alpha)} a g(x) &= a\, {}_0I_x^{(\alpha)} g(x), \\ {}_0I_t^{(\alpha)} \left(\frac{x^{n\alpha}}{\Gamma(1+n\alpha)} \right) &= \frac{x^{(n+1)^{\alpha}}}{\Gamma(1+(n+1)\alpha)}, \quad n \in N. \end{aligned} \tag{10}$$

2.2. Local Fractional Laplace Transform and Its Inverse Formula (See [28, 29, 32])

Definition 5. Let $(1/\Gamma(1+\alpha)) \int_0^{\infty} |f(x)| (dx)^{\alpha} < k < \infty$. The Yang-Laplace transforms of $f(x)$ are given by

$$\begin{aligned} L_{\alpha}\{f(x)\} &= f_s^{L,\alpha}(s) \\ &= \frac{1}{\Gamma(1+\alpha)} \int_0^{\infty} E_{\alpha}(-s^{\alpha} x^{\alpha}) f(x) (dx)^{\alpha}, \\ &\qquad 0 < \alpha \le 1, \end{aligned} \tag{11}$$

where the latter integral converges and $s^{\alpha} \in R^{\alpha}$.

Definition 6. The inverse formula of the Yang-Laplace transforms of $f(x)$ is given by

$$L_{\alpha}^{-1}\left\{ f_s^{L,\alpha}(s) \right\} = f(t) = \frac{1}{(2\pi)^{\alpha}} \int_{\beta - i\omega}^{\beta + i\omega} E_{\alpha}(s^{\alpha} x^{\alpha}) f_s^{L,\alpha}(s) (ds)^{\alpha}, \quad 0 < \alpha \le 1, \tag{12}$$

where $s^{\alpha} = \beta^{\alpha} + i^{\alpha}\omega^{\alpha}$; here i^{α} is fractal imaginary unit os s and $\mathrm{Re}(s) = \beta > 0$.

2.3. Some Basic Properties of Local Fractional Laplace Transform (See [28, 29]). Let $L_\alpha\{f(x)\} = f_s^{L,\alpha}(s)$ and $L_\alpha\{g(x)\} = g_s^{L,\alpha}(s)$; then we have the following formulas:

$$\begin{aligned}
L_\alpha\left\{af(x) + bg(x)\right\} &= af_s^{L,\alpha}(s) + bg_s^{L,\alpha}(s),\\
L_\alpha\left\{E_\alpha\left(c^\alpha x^\alpha\right) f(x)\right\} &= f_s^{L,\alpha}(s-c),\\
L_\alpha\left\{f^{(k\alpha)}(x)\right\} &= s^{k\alpha} f_s^{L,\alpha}(s) - s^{(k-1)\alpha} f(0)\\
&\quad - s^{(k-2)\alpha} f^{(\alpha)}(0) - \cdots - f^{((k-1)\alpha)}(0),\\
L_\alpha\left\{E_\alpha\left(a^\alpha x^\alpha\right)\right\} &= \frac{1}{s^\alpha - a^\alpha},\\
L_\alpha\left\{\sin_\alpha\left(a^\alpha x^\alpha\right)\right\} &= \frac{a^\alpha}{s^{2\alpha} + a^{2\alpha}},\\
L_\alpha\left\{x^{k\alpha}\right\} &= \frac{\Gamma(1+k\alpha)}{s^{(k+1)\alpha}}.
\end{aligned} \tag{13}$$

3. Local Fractional Laplace Variational Iteration Method

Let us consider the following local fractional partial differential equations:

$$L_\alpha u(x,t) + R_\alpha u(x,t) = f(x,t), \tag{14}$$

where L_α is the linear local fractional operator, R_α is the linear local fractional operator of order less than L_α, and $f(x,t)$ is a source term of the nondifferential function.

According to the rule of local fractional variational iteration method, the correction functional for (14) is constructed as follows [34–37]:

$$u_{n+1}(x) = u_n(x) + {}_0I_x^{(\alpha)}\left(\frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha u_n(t) + R_\alpha \widetilde{u}_n(t) - f(t)\right]\right), \tag{15}$$

where $\lambda(x-t)^\alpha/\Gamma(1+\alpha)$ is a fractal Lagrange multiplier and L_α in (14) are $k\alpha$ times local fractional partial differential equations.

For initial value problems of (14), we can start with

$$u_0(x) = u(0) + \frac{x^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(0) + \cdots + \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}u^{((k-1)\alpha)}(0). \tag{16}$$

We now take Yang-Laplace transform of (15); namely,

$$\begin{aligned}
\pounds_\alpha\left\{u_{n+1}(x)\right\} &= \pounds_\alpha\left\{u_n(x)\right\}\\
&\quad + \pounds_\alpha\left\{{}_0I_x^{(\alpha)}\left(\frac{\lambda(x-t)^\alpha}{\Gamma(1+\alpha)}\left[L_\alpha \widetilde{u}_n(t) + R_\alpha u_n(t) - f(t)\right]\right)\right\},
\end{aligned} \tag{17}$$

or

$$\begin{aligned}
\pounds_\alpha\left\{u_{n+1}(x)\right\} &= \pounds_\alpha\left\{u_n(x)\right\}\\
&\quad + \pounds_\alpha\left\{\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}\right\}\pounds_\alpha\left\{L_\alpha u_n(x) + R_\alpha \widetilde{u}_n(x) - f(x)\right\}.
\end{aligned} \tag{18}$$

Take the local fractional variation of (18), which is given by

$$\begin{aligned}
\delta^\alpha\left(\pounds_\alpha\left\{u_{n+1}(x)\right\}\right) &= \delta^\alpha\left(\pounds_\alpha\left\{u_n(x)\right\}\right)\\
&\quad + \delta^\alpha\left(\pounds_\alpha\left\{\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}\right\}\pounds_\alpha\left\{L_\alpha u_n(x) - R_\alpha \widetilde{u}_n(x) - f(x)\right\}\right).
\end{aligned} \tag{19}$$

By using computation of (19), we get

$$\delta^\alpha\left(\pounds_\alpha\left\{u_{n+1}(x)\right\}\right) = \delta^\alpha\left(\pounds_\alpha\left\{u_n(x)\right\}\right) + \pounds_\alpha\left\{\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}\right\}\delta^\alpha\left(\pounds_\alpha\left\{L_\alpha u_n(x)\right\}\right). \tag{20}$$

Hence, from (20), we get

$$1 + \pounds_\alpha\left\{\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}\right\}s^{k\alpha} = 0, \tag{21}$$

where

$$\begin{aligned}
\delta^\alpha\left(\pounds_\alpha\left\{L_\alpha u_n(x)\right\}\right) &= \delta^\alpha\left(s^{k\alpha}\pounds_\alpha\left\{u_n(x)\right\} - s^{(k-1)\alpha}u_n(0) - \cdots - u_n^{((k-1)\alpha)}(0)\right)\\
&= s^{k\alpha}\delta^\alpha\left(\pounds_\alpha\left\{u_n(x)\right\}\right).
\end{aligned} \tag{22}$$

Therefore, we get

$$\pounds_\alpha\left\{\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)}\right\} = -\frac{1}{s^{k\alpha}}. \tag{23}$$

Taking the inverse version of the Yang-Laplace transform, we have

$$\frac{\lambda(x)^\alpha}{\Gamma(1+\alpha)} = \pounds_\alpha^{-1}\left\{-\frac{1}{s^{k\alpha}}\right\} = -\frac{(x)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]}, \quad k \in N. \tag{24}$$

In view of (24), we obtain

$$\begin{aligned}
&\pounds_\alpha \{u_{n+1}(x)\} \\
&= \pounds_\alpha \{u_n(x)\} \\
&\quad - \pounds_\alpha \left\{ {}_0I_x^{(\alpha)} \left(\frac{(x-t)^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \right.\right. \\
&\qquad \left.\left. \cdot \left[L_\alpha \widetilde{u}_n(t) + R_\alpha u_n(t) - f(t) \right] \right) \right\}.
\end{aligned} \tag{25}$$

Therefore, we have the following iteration algorithm:

$$\begin{aligned}
&\pounds_\alpha \{u_{n+1}(x)\} \\
&= \pounds_\alpha \{u_n(x)\} \\
&\quad - \pounds_\alpha \left\{ \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} \right\} \pounds_\alpha \{L_\alpha u_n(x) \\
&\qquad + R_\alpha \widetilde{u}_n(x) - f(x)\},
\end{aligned} \tag{26}$$

or

$$\begin{aligned}
\pounds_\alpha \{u_{n+1}(x)\} &= \pounds_\alpha \{u_n(x)\} \\
&\quad - \frac{1}{s^{k\alpha}} \pounds_\alpha \{L_\alpha u_n(x) + R_\alpha \widetilde{u}_n(x) - f(x)\},
\end{aligned} \tag{27}$$

where the initial value reads as follows:

$$\begin{aligned}
&u_0(x) \\
&= \pounds_\alpha^{-1} \left(\frac{s^{(k-1)\alpha} u(0) + s^{(k-2)\alpha} u^{(\alpha)}(0) + \cdots + u_n^{((k-1)\alpha)}(0)}{s^{k\alpha}} \right) \\
&= u(0) + \frac{x^\alpha}{\Gamma(1+\alpha)} u^{(\alpha)}(0) + \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} u^{(2\alpha)}(0) \\
&\quad + \cdots + \frac{x^{(k-1)\alpha}}{\Gamma[1+(k-1)\alpha]} u^{((k-1)\alpha)}(0).
\end{aligned} \tag{28}$$

Thus, the local fractional series solution of (14) is

$$u(x,t) = \lim_{n\to\infty} \pounds_\alpha^{-1} \left(\pounds_\alpha \{u_n(x,t)\} \right). \tag{29}$$

4. Applications to Diffusion and Wave Equations on Cantor Sets

In this section, four examples for diffusion and wave equations on Cantor sets will demonstrate the efficiency of local fractional Laplace variational iteration method.

Example 1. Let us consider the following diffusion equation on Cantor set:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} = 0, \quad 0 < \alpha \le 1, \tag{30}$$

with the initial value condition

$$u(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{31}$$

Using relation (26), we structure the iterative relation as

$$\begin{aligned}
&\pounds_\alpha \{u_{n+1}(x,t)\} \\
&= \pounds_\alpha \{u_n(x,t)\} - \pounds_\alpha \{1\} \pounds_\alpha \left\{ \frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} \right\} \\
&= \pounds_\alpha \{u_n(x,t)\} \\
&\quad - \frac{1}{s^\alpha} \left(s^\alpha \pounds_\alpha \{u_n(x,t)\} - u_n(x,0) - \frac{\partial^{2\alpha} \pounds_\alpha \{u_n(x,t)\}}{\partial x^{2\alpha}} \right) \\
&= \frac{1}{s^\alpha} u_n(x,0) + \frac{1}{s^\alpha} \frac{\partial^{2\alpha} \pounds_\alpha \{u_n(x,t)\}}{\partial x^{2\alpha}}.
\end{aligned} \tag{32}$$

In view of (28), the initial value reads as follows:

$$u_0(x,t) = u(x,0) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{33}$$

Hence, we get the first approximation; namely,

$$\begin{aligned}
\pounds_\alpha \{u_1(x,t)\} &= \frac{1}{s^\alpha} u_0(x,0) + \frac{1}{s^\alpha} \frac{\partial^{2\alpha} \pounds_\alpha \{u_0(x,t)\}}{\partial x^{2\alpha}} \\
&= \frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{s^\alpha} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \pounds_\alpha \left\{ \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\
&= \frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)}.
\end{aligned} \tag{34}$$

Thus,

$$u_1(x,t) = \pounds_\alpha^{-1} \left(\frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)} \right) = \frac{x^\alpha}{\Gamma(1+\alpha)}. \tag{35}$$

The second approximation reads as follows:

$$\begin{aligned}
\pounds_\alpha \{u_2(x,t)\} &= \frac{1}{s^\alpha} u_1(x,0) + \frac{1}{s^\alpha} \frac{\partial^{2\alpha} \pounds_\alpha \{u_1(x,t)\}}{\partial x^{2\alpha}} \\
&= \frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{1}{s^\alpha} \frac{\partial^{2\alpha}}{\partial x^{2\alpha}} \pounds_\alpha \left\{ \frac{x^\alpha}{\Gamma(1+\alpha)} \right\} \\
&= \frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)}.
\end{aligned} \tag{36}$$

Therefore, we get

$$u_2(x,t) = \pounds_\alpha^{-1} \left(\frac{1}{s^\alpha} \frac{x^\alpha}{\Gamma(1+\alpha)} \right) = \frac{x^\alpha}{\Gamma(1+\alpha)} \cdots. \tag{37}$$

Consequently, the local fractional series solution is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} \pounds_\alpha^{-1}\left(\pounds_\alpha\left\{u_n(x,t)\right\}\right) \\ &= \lim_{n\to\infty} \pounds_\alpha^{-1}\left(\frac{1}{s^\alpha}\frac{x^\alpha}{\Gamma(1+\alpha)}\right) \\ &= \lim_{n\to\infty}\frac{x^\alpha}{\Gamma(1+\alpha)} = \frac{x^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{38}$$

The result is the same as the one which is obtained by the local fractional series expansion method [38].

Example 2. Let us consider the following diffusion equation on Cantor set:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0, \quad 0<\alpha\le 1, \tag{39}$$

with the initial value conditions being as follows:

$$u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{40}$$

Using relation (26), we structure the iterative relation as follows:

$$\begin{aligned} &\pounds_\alpha\left\{u_{n+1}(x,t)\right\} \\ &= \pounds_\alpha\left\{u_n(x,t)\right\} \\ &\quad - \pounds_\alpha\{1\}\pounds_\alpha\left\{\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}\right\} \\ &= \pounds_\alpha\left\{u_n(x,t)\right\} \\ &\quad - \frac{1}{s^\alpha}\left(s^\alpha\pounds_\alpha\left\{u_n(x,t)\right\}\right. \\ &\qquad \left. -u_n(x,0) - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_n(x,t)\right\}}{\partial x^{2\alpha}}\right) \\ &= \frac{1}{s^\alpha}u_n(x,0) + \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_n(x,t)\right\}}{\partial x^{2\alpha}}. \end{aligned} \tag{41}$$

In view of (28), the initial value reads as follows:

$$u_0(x,t) = u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}. \tag{42}$$

Hence, we get the first approximation; namely,

$$\begin{aligned} &\pounds_\alpha\left\{u_1(x,t)\right\} \\ &= \frac{1}{s^\alpha}u_0(x,0) + \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_0(x,t)\right\}}{\partial x^{2\alpha}} \\ &= \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right\} \\ &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^\alpha}+\frac{1}{s^{2\alpha}}\right). \end{aligned} \tag{43}$$

Thus,

$$\begin{aligned} u_1(x,t) &= \pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^\alpha}+\frac{1}{s^{2\alpha}}\right]\right) \\ &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}\right). \end{aligned} \tag{44}$$

The second approximation reads as follows:

$$\begin{aligned} &\pounds_\alpha\left\{u_2(x,t)\right\} \\ &= \frac{1}{s^\alpha}u_1(x,0) + \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_1(x,t)\right\}}{\partial x^{2\alpha}} \\ &= \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \\ &\quad + \frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}\right)\right\} \\ &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^\alpha}+\frac{1}{s^{2\alpha}}+\frac{1}{s^{3\alpha}}\right). \end{aligned} \tag{45}$$

Therefore, we get

$$\begin{aligned} u_2(x,t) &= \pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^\alpha}+\frac{1}{s^{2\alpha}}+\frac{1}{s^{3\alpha}}\right]\right) \\ &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\cdots \end{aligned} \tag{46}$$

Consequently, the local fractional series solution is

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty}\pounds_\alpha^{-1}\left(\pounds_\alpha\left\{u_n(x,t)\right\}\right) \\ &= \lim_{n\to\infty}\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{n}\frac{1}{s^{(n+1)\alpha}}\right) \\ &= \lim_{n\to\infty}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{n}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)} \\ &= \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)} = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}E_\alpha\left(t^\alpha\right). \end{aligned} \tag{47}$$

The result is the same as the one which is obtained by the local fractional series expansion method [38].

Example 3. Let us consider the following wave equation on Cantor set:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}} - \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} = 0, \quad 0<\alpha\le 1, \tag{48}$$

with the initial value conditions being as follows:

$$u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)}, \quad \frac{\partial^\alpha u(x,0)}{\partial t^\alpha} = 0. \tag{49}$$

Using relation (26), we structure the iterative relation as

$$\begin{aligned}
&\pounds_\alpha\left\{u_{n+1}(x,t)\right\}\\
&\quad=\pounds_\alpha\left\{u_n(x,t)\right\}\\
&\qquad-\pounds_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\alpha)}\right\}\pounds_\alpha\left\{\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}\right.\\
&\qquad\qquad\left.-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}\right\}\\
&\quad=\pounds_\alpha\left\{u_n(x,t)\right\}\\
&\qquad-\frac{1}{s^{2\alpha}}\left(s^{2\alpha}\pounds_\alpha\left\{u_n(x,t)\right\}-s^\alpha u_n(x,0)\right.\\
&\qquad\qquad\left.-u_n^{(\alpha)}(x,0)-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_n(x,t)\right\}}{\partial x^{2\alpha}}\right)\\
&\quad=\frac{1}{s^\alpha}u_n(x,0)+\frac{1}{s^{2\alpha}}u_n^{(\alpha)}(x,0)\\
&\qquad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_n(x,t)\right\}}{\partial x^{2\alpha}}.
\end{aligned}\tag{50}$$

In view of (28), the initial value reads as follows:

$$u_0(x,t)=u(x,0)+\frac{t^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(x,0)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}.\tag{51}$$

Hence, we get the first approximation; namely,

$$\begin{aligned}
\pounds_\alpha\left\{u_1(x,t)\right\}&=\frac{1}{s^\alpha}u_0(x,0)+\frac{1}{s^{2\alpha}}u_0^{(\alpha)}(x,0)\\
&\quad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_0(x,t)\right\}}{\partial x^{2\alpha}}\\
&=\frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\quad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right\}\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^\alpha}+\frac{1}{s^{3\alpha}}\right).
\end{aligned}\tag{52}$$

Thus,

$$\begin{aligned}
u_1(x,t)&=\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^\alpha}+\frac{1}{s^{3\alpha}}\right]\right)\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right).
\end{aligned}\tag{53}$$

The second approximation reads as follows:

$$\begin{aligned}
&\pounds_\alpha\left\{u_2(x,t)\right\}\\
&\quad=\frac{1}{s^\alpha}u_1(x,0)+\frac{1}{s^{2\alpha}}u_1^{(\alpha)}(x,0)\\
&\qquad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\left\{u_1(x,t)\right\}}{\partial x^{2\alpha}}\\
&\quad=\frac{1}{s^\alpha}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\qquad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)\right\}\\
&\quad=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^\alpha}+\frac{1}{s^{3\alpha}}+\frac{1}{s^{5\alpha}}\right).
\end{aligned}\tag{54}$$

Therefore, we get

$$\begin{aligned}
u_2(x,t)&=\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^\alpha}+\frac{1}{s^{3\alpha}}+\frac{1}{s^{5\alpha}}\right]\right)\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{4\alpha}}{\Gamma(1+4\alpha)}\right)\cdots.
\end{aligned}\tag{55}$$

Consequently, the local fractional series solution is

$$\begin{aligned}
u(x,t)&=\lim_{n\to\infty}\pounds_\alpha^{-1}\left(\pounds_\alpha\left\{u_n(x,t)\right\}\right)\\
&=\lim_{n\to\infty}\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{n}\frac{1}{s^{(2n+1)\alpha}}\right)\\
&=\lim_{n\to\infty}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{n}\frac{t^{2k\alpha}}{\Gamma(1+2k\alpha)}\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{\infty}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\cosh_\alpha\left(t^\alpha\right).
\end{aligned}\tag{56}$$

The result is the same as the one which is obtained by the local fractional Adomian decomposition method and local fractional variational iteration method in [34].

Example 4. Let us consider the following wave equation on Cantor set:

$$\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}=0,\quad 0<\alpha\le 1,\tag{57}$$

with the initial value conditions being as follows:

$$u(x,0)=0,\quad \frac{\partial^\alpha u(x,0)}{\partial t^\alpha}=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}.\tag{58}$$

Using relation (26), we structure the iterative relation as

$$\begin{aligned}
&\pounds_\alpha\{u_{n+1}(x,t)\}\\
&\quad=\pounds_\alpha\{u_n(x,t)\}\\
&\quad-\pounds_\alpha\left\{\frac{t^\alpha}{\Gamma(1+\alpha)}\right\}\pounds_\alpha\left\{\frac{\partial^{2\alpha}u(x,t)}{\partial t^{2\alpha}}\right.\\
&\qquad\left.-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}}\right\}\\
&\quad=\pounds_\alpha\{u_n(x,t)\}\\
&\quad-\frac{1}{s^{2\alpha}}\left(s^{2\alpha}\pounds_\alpha\{u_n(x,t)\}-s^\alpha u_n(x,0)\right.\\
&\qquad\left.-u_n^{(\alpha)}(x,0)-\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\{u_n(x,t)\}}{\partial x^{2\alpha}}\right)\\
&\quad=\frac{1}{s^\alpha}u_n(x,0)+\frac{1}{s^{2\alpha}}u_n^{(\alpha)}(x,0)\\
&\quad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\{u_n(x,t)\}}{\partial x^{2\alpha}}.
\end{aligned}\tag{59}$$

In view of (28), the initial value reads as follows:

$$\begin{aligned}
&u_0(x,t)\\
&\quad=u(x,0)+\frac{t^\alpha}{\Gamma(1+\alpha)}u^{(\alpha)}(x,0)=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{t^\alpha}{\Gamma(1+\alpha)}.
\end{aligned}\tag{60}$$

Hence, we get the first approximation; namely,

$$\begin{aligned}
\pounds_\alpha\{u_1(x,t)\}&=\frac{1}{s^\alpha}u_0(x,0)+\frac{1}{s^{2\alpha}}u_0^{(\alpha)}(x,0)\\
&+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\{u_0(x,t)\}}{\partial x^{2\alpha}}\\
&=\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\\
&+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right\}\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^{2\alpha}}+\frac{1}{s^{4\alpha}}\right).
\end{aligned}\tag{61}$$

Thus,

$$\begin{aligned}
u_1(x,t)&=\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^{2\alpha}}+\frac{1}{s^{4\alpha}}\right]\right)\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right).
\end{aligned}\tag{62}$$

The second approximation reads as follows:

$$\begin{aligned}
&\pounds_\alpha\{u_2(x,t)\}\\
&\quad=\frac{1}{s^\alpha}u_1(x,0)+\frac{1}{s^{2\alpha}}u_1^{(\alpha)}(x,0)\\
&\qquad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}\pounds_\alpha\{u_1(x,t)\}}{\partial x^{2\alpha}}\\
&\quad=\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\\
&\qquad+\frac{1}{s^{2\alpha}}\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\frac{\partial^{2\alpha}}{\partial x^{2\alpha}}\pounds_\alpha\left\{\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\right.\\
&\qquad\left.\cdot\left(\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right)\right\}\\
&\quad=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{1}{s^{2\alpha}}+\frac{1}{s^{4\alpha}}+\frac{1}{s^{6\alpha}}\right).
\end{aligned}\tag{63}$$

Therefore, we get

$$\begin{aligned}
&u_2(x,t)\\
&\quad=\pounds_\alpha^{-1}\left(\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left[\frac{1}{s^{2\alpha}}+\frac{1}{s^{4\alpha}}+\frac{1}{s^{6\alpha}}\right]\right)\\
&\quad=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\left(\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\frac{t^{5\alpha}}{\Gamma(1+5\alpha)}\right)\cdots.
\end{aligned}\tag{64}$$

Consequently, the local fractional series solution is

$$\begin{aligned}
u(x,t)&=\lim_{n\to\infty}\pounds_\alpha^{-1}\left(\pounds_\alpha\{u_n(x,t)\}\right)\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sum_{k=0}^{\infty}\frac{t^{(2k+1)\alpha}}{\Gamma(1+(2k+1))}\\
&=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)}\sinh_\alpha(t^\alpha).
\end{aligned}\tag{65}$$

5. Conclusions

The local fractional Laplace variational iteration method was applied to the diffusion and wave equations defined on Cantor sets with the fractal conditions. The local fractional Laplace variational iteration method was proved to be effective and very reliable for analytic purposes. Further, the same problems are solved by local fractional expansion series method (LFESM), local fractional variational iteration method (LFVIM), and local fractional Adomian decomposition method (LFADM). The results obtained by the four methods are in agreement and, hence, this technique may be considered an alternative and efficient method for finding approximate solutions of both linear and nonlinear fractional differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.

[2] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*, Imperial College Press, London, UK, 2010.

[3] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[4] R. L. Magin, *Fractional Calculus in Bioengineering*, Begerll House, West Redding, Conn, USA, 2006.

[5] J. Klafter, S. C. Lim, and R. Metzler, *Fractional Dynamics in Physics: Recent Advances*, World Scientific, Singapore, 2011.

[6] G. M. Zaslavsky, *Hamiltonian Chaos and Fractional Dynamics*, Oxford University Press, Oxford, UK, 2008.

[7] B. J. West, M. Bologna, and P. Grigolini, *Physics of Fractal Operators*, Springer, New York, NY, USA, 2003.

[8] V. E. Tarasov, *Fractional Dynamics: Applications of Fractional Calculus to Dynamics of Particles, Fields and Media*, Springer, Berlin, Germany, 2011.

[9] J. A. Tenreiro Machado, A. C. J. Luo, and D. Baleanu, *Nonlinear Dynamics of Complex Systems: Applications in Physical, Biological and Financial Systems*, Springer, New York, NY, USA, 2011.

[10] J.-S. Duan, T. Chaolu, R. Rach, and L. Lu, "The Adomian decomposition method with convergence acceleration techniques for nonlinear fractional differential equations," *Computers and Mathematics with Applications*, vol. 66, no. 5, pp. 728–736, 2013.

[11] J.-S. Duan, T. Chaolu, and R. Rach, "Solutions of the initial value problem for nonlinear fractional ordinary differential equations by the Rach-Adomian-Meyers modified decomposition method," *Applied Mathematics and Computation*, vol. 218, no. 17, pp. 8370–8392, 2012.

[12] L. Chen, Y. Zhao, H. Jafari, J. A. T. Machado, and X.-J. Yang, "Local fractional variational iteration method for local fractional Poisson equations in two independent variables," *Abstract and Applied Analysis*, vol. 2014, Article ID 484323, 7 pages, 2014.

[13] S. Momani and Z. Odibat, "Comparison between the homotopy perturbation method and the variational iteration method for linear fractional partial differential equations," *Computers & Mathematics with Applications*, vol. 54, no. 7-8, pp. 910–919, 2007.

[14] H. Jafari and S. Momani, "Solving fractional diffusion and wave equations by modified homotopy perturbation method," *Physics Letters A*, vol. 370, no. 5-6, pp. 388–396, 2007.

[15] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, Boston, Mass, USA, 2012.

[16] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.

[17] J. Hristov, "Heat-balance integral to fractional (half-time) heat diffusion sub-model," *Thermal Science*, vol. 14, no. 2, pp. 291–316, 2010.

[18] J. Hristov, "Integral-Balance solution to the Stokes' first problem of a viscoelastic generalized second grade fluid," *Thermal Science*, vol. 16, no. 2, pp. 395–410, 2012.

[19] J. Hristov, "Transient flow of a generalized second grade fluid due to a constant surface shear stress: an approximate integral balance solution," *International Review of Chemical Engineering*, vol. 3, no. 6, pp. 802–809, 2011.

[20] G.-C. Wu and E. W. M. Lee, "Fractional variational iteration method and its application," *Physics Letters A*, vol. 374, no. 25, pp. 2506–2509, 2010.

[21] H. Jafari, "A comparison between the variational iteration method and the successive approximations method," *Applied Mathematics Letters*, vol. 32, pp. 1–5, 2014.

[22] H. Jafari, H. Tajadodi, and D. Baleanu, "A modified variational iteration method for solving fractional Riccati differential equation by Adomian polynomials," *Fractional Calculus and Applied Analysis*, vol. 16, no. 1, pp. 109–122, 2013.

[23] H. Jafari, H. Tajadodi, D. Baleanu, A. A. Al-Zahrani, Y. A. Alhamed, and A. H. Zahid, "Fractional sub-equation method for the fractional generalized reaction Duffing model and nonlinear fractional Sharma-Tasso-Olver equation," *Central European Journal of Physics*, vol. 11, no. 10, pp. 1482–1486, 2013.

[24] H. Jafari, H. Tajadodi, N. Kadkhoda, and D. Baleanu, "Fractional subequation method for Cahn-Hilliard and Klein-Gordon equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 587179, 5 pages, 2013.

[25] R. Gorenflo and F. Mainardi, "Some recent advances in theory and simulation of fractional diffusion processes," *Journal of Computational and Applied Mathematics*, vol. 229, no. 2, pp. 400–415, 2009.

[26] R. Metzler and T. F. Nonnenmacher, "Space- and time-fractional diffusion and wave equations, fractional Fokker-Planck equations, and physical motivation," *Chemical Physics*, vol. 284, no. 1-2, pp. 67–90, 2002.

[27] X. J. Yang, "Applications of local fractional calculus to engineering in fractal time-space: local fractional differential equations with local fractional derivative," Mathematical Physics, http://arxiv.org/abs/1106.3010.

[28] X. J. Yang, *Local Fractional Functional Analysis and Its Applications*, Asian Academic Publisher, Hong Kong, 2011.

[29] X. J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science Publisher, New York, NY, USA, 2012.

[30] W. Zhong, X. Yang, and F. Gao, "A Cauchy problem for some local fractional abstract differential equation with fractal conditions," *Journal of Applied Functional Analysis*, vol. 8, no. 1, pp. 92–99, 2013.

[31] M.-S. Hu, R. P. Agarwal, and X.-J. Yang, "Local fractional Fourier series with application to wave equation in fractal vibrating string," *Abstract and Applied Analysis*, vol. 2012, Article ID 567401, 15 pages, 2012.

[32] S.-Q. Wang, Y.-J. Yang, and H. K. Jassim, "Local fractional function decomposition method for solving inhomogeneous wave equations with local fractional derivative," *Abstract and Applied Analysis*, vol. 2014, Article ID 176395, 7 pages, 2014.

[33] S.-P. Yan, H. Jafari, and H. K. Jassim, "Local fractional Adomian decomposition and function decomposition methods for Laplace equation within local fractional operators," *Advances in Mathematical Physics*, vol. 2014, Article ID 161580, 7 pages, 2014.

[34] C. Cattani, D. Baleanu, X.-J. Yang, J. A. T. Machado, and M. C. Baleanu, "Local fractional variational iteration and Decomposition methods for Wave equation on Cantor sets within local

fractional Operators," *Abstract and Applied Analysis*, vol. 2014, Article ID 535048, 6 pages, 2014.

[35] X.-J. Yang and D. Baleanu, "Fractal heat conduction problem solved by local fractional variation iteration method," *Thermal Science*, vol. 17, no. 2, pp. 625–628, 2013.

[36] W.-H. Su, D. Baleanu, X.-J. Yang, and H. Jafari, "Damped wave equation and dissipative wave equation in fractal strings within the local fractional variational iteration method," *Fixed Point Theory and Applications*, vol. 2013, no. 1, article 89, 2013.

[37] J.-H. He and F.-J. Liu, "Local fractional variational iteration method for fractal heat transfer in silk cocoon hierarchy," *Nonlinear Science Letters A*, vol. 4, no. 1, pp. 15–20, 2013.

[38] A.-M. Yang, X.-J. Yang, and Z.-B. Li, "Local fractional series expansion method for solving wave and diffusion equations on Cantor sets," *Abstract and Applied Analysis*, vol. 2013, Article ID 351057, 5 pages, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 780929, 7 pages
http://dx.doi.org/10.1155/2015/780929

Research Article

Analytical Solution of Space-Time Fractional Fokker-Planck Equation by Homotopy Perturbation Sumudu Transform Method

Ravi Shanker Dubey,[1] Badr Saad T. Alkahtani,[2] and Abdon Atangana[3]

[1]*Department of Mathematics, Yagyavalkya Institute of Technology, Jaipur 302022, India*
[2]*Mathematics Department, College of Science, King Saud University, P.O. Box 1142, Riyadh 11989, Saudi Arabia*
[3]*Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa*

Correspondence should be addressed to Ravi Shanker Dubey; ravimath13@gmail.com

Received 27 May 2014; Revised 11 September 2014; Accepted 21 September 2014

Academic Editor: Samir B. Belhaouari

An efficient approach based on homotopy perturbation method by using Sumudu transform is proposed to solve some linear and nonlinear space-time fractional Fokker-Planck equations (FPEs) in closed form. The space and time fractional derivatives are considered in Caputo sense. The homotopy perturbation Sumudu transform method (HPSTM) is a combined form of Sumudu transform, homotopy perturbation method, and He's polynomials. The nonlinear terms can be easily handled by the use of He's polynomials. Some examples show that the HPSTM is an effective tool for solving many space time fractional partial differential equations.

1. Introduction

Fokker-Planck equation (FPE) was introduced by Adriaan Fokker and Max Planck to describe the time evolution of the probability density function of position and velocity of a particle, which is one of the classical widely used equations of statistical physics [1]. FPE arises in a number of different fields in natural sciences; Brownian motion [2] and the diffusion model of chemical reactions [3] are now largely employed, in various generalized forms, in physics, chemistry, engineering, and biology [1]. The FPE arises in kinetic theory [4] where it describes the evolution of the one-particle distribution function of a dilute gas with long-range collisions, such as a Coulomb gas. Some applications of this type of equations can be worked out in the works of He and Wu [5], Jumarie [6], Kamitani and Matsuba [7], Xu et al. [8], and Zak [9].

The general FPE for the motion of a concentration field $v(x,t)$ of one space variable x at time t has the form [1]

$$\frac{\partial v(x,t)}{\partial t} = \left[-\frac{\partial A(x)}{\partial x} + \frac{\partial^2 B(x)}{\partial x^2}\right] v(x,t) \tag{1}$$

with initial condition

$$v(0,x) = f(x), \quad x \in \mathscr{R}, \tag{2}$$

where $A(x)$ and $B(x) > 0$ are called the drift and diffusion coefficients. This equation is also called the forward Kolmogorov equation. The drift and diffusion coefficients may also depend on time as

$$\frac{\partial v(x,t)}{\partial t} = \left[-\frac{\partial A(x,t)}{\partial x} + \frac{\partial^2 B(x,t)}{\partial x^2}\right] v(x,t). \tag{3}$$

There is a more general form of Fokker-Planck equation which is called the nonlinear Fokker-Planck equation. The nonlinear Fokker-Planck equation has important applications in various areas such as plasma physics, surface physics, population dynamics, biophysics, engineering, neurosciences, nonlinear hydrodynamics, polymer physics, laser physics, pattern formation, psychology, and so forth [10]. In the one variable case, the nonlinear FPE can be written as

$$\frac{\partial v(x,t)}{\partial t} = \left[-\frac{\partial A(x,t,v)}{\partial x} + \frac{\partial^2 B(x,t,v)}{\partial x^2}\right] v(x,t). \tag{4}$$

Due to vast range of applications of the FPE, a lot of work has been done to find numerical solution with this equation. In this context, the works of Buet et al. [11], Harrison [12], Palleschi et al. [13], Vanaja [14], and Zorzano et al. [15] are worth mentioning.

For generalization of classical integer order of FPE (4), the following equation can be introduced. The Fokker-Planck equation with fractional space derivative is a particular case of anomalous diffusion and Lévy flights (see[16–19]). This equation is called nonlinear FPE with space-time fractional derivatives [20]:

$$\frac{\partial^{\alpha} v(x,t)}{\partial t^{\alpha}} = \left[-\frac{\partial A(x,t,v)}{\partial x} + \frac{\partial^2 B(x,t,v)}{\partial x^2} \right] v(x,t), \quad (5)$$

where $t > 0$, $x > 0$, and $0 < \alpha \leq 1$. It can be obtained from the general Fokker-Planck equation by replacing the space and time derivatives by fractional derivatives operator. The function $v(x,t)$ is assumed to be a causal function of time and space. Particularly, for $\alpha = 1$, the fractional FPE (5) reduces to the classical nonlinear FPE given by (4) in the case $x > 0$.

In recent years, researchers have studied the fractional partial differential equations and the fractional FPE to investigate various scientific models [21–24]. In the present paper we obtain closed form solutions of a linear-nonlinear time fractional FPE using homotopy perturbation Sumudu transform method (HPSTM); see [25].

2. Some Mathematical Preliminaries and Definitions

2.1. Fundamental Properties of Fractional Calculus. Firstly, we mention some of the fundamental properties of the fractional calculus, that is, fractional derivatives and integrals.

The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f(t) \in C_{\mu}$, $\mu \geq -1$, is defined by implementing the integration operator J^{α} in the following manner:

$$\begin{aligned} J^{\alpha}[f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \quad \alpha > 0, \\ J^{0}[f(t)] &= f(t), \\ J^{\alpha} J^{\beta}[f(t)] &= J^{\alpha+\beta} f(t), \\ J^{\beta} J^{\alpha}[f(t)] &= J^{\alpha} J^{\beta} f(t), \\ J^{\alpha}(t^{\gamma}) &= \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}. \end{aligned} \quad (6)$$

The fundamental properties of fractional integration and fractional differentiation have been introduced to the literature by Podlubny [26].

The fractional derivative of $f(t)$ in the Caputo sense is defined by the following relation [27]:

$$\begin{aligned} D_t^{\alpha} f(t) &= J^{m-\alpha} D^{n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{-\alpha+m-1} f^{(m)}(\tau)\, d\tau, \quad (7) \\ & \qquad m-1 < \alpha \leq m, \end{aligned}$$

where $m \in N$ and $t > 0$.

The relation between Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator is

$$\begin{aligned} J_t^{\alpha} D_t^{\alpha} f(t) &= f(t) - \sum_{k=0}^{m-1} \frac{f^{k}(0+)\, t^{k}}{k!}, \\ D^{-\alpha}[f(t)] &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau, \\ & \qquad 0 < \alpha \leq 1. \end{aligned} \quad (8)$$

Further properties and applications can be found in [26, 28–31].

2.2. The Sumudu Transform. In early 90s, Watugala [32] introduced a new integral transform named the Sumudu transform and applied it to the solution of ordinary differential equation in control engineering problems. The Sumudu transform is defined over the set of functions

$$\begin{aligned} A = \{ f(t) \mid \exists M,\ \tau_1, \tau_2 > 0,\ |f(t)| < M e^{|t|/\tau_j}, \\ \text{if } t \in (-1)^{j} \times [0, \infty) \}; \end{aligned} \quad (9)$$

the Sumudu transform is defined by

$$\widetilde{G}(u) = S[f(t)] = \int_0^{\infty} f(ut)\, e^{-t} dt, \quad u \in (-\tau_1, \tau_2). \quad (10)$$

Sumudu transform has many useful and important properties like linear property, scale properties, shifting properties, duality with Laplace transforms, and so forth. Further detail and properties about this transform can be found in [25, 33–37].

By using the Sumudu transform of multiple differentiation, we obtain

$$\begin{aligned} S[D_t^{\alpha} f(t)] &= u^{m-\alpha} \left[\frac{\widetilde{G}(u)}{u^{m}} - \sum_{k=0}^{m-1} \frac{f^{k}(0)}{u^{m-k}} \right] \\ &= \left[\frac{\widetilde{G}(u)}{u^{\alpha}} - \sum_{k=0}^{m-1} \frac{f^{k}(0+)}{u^{\alpha-k}} \right], \quad (m-1 < \alpha \leq m), \end{aligned} \quad (11)$$

where $\widetilde{G}(u) = S[f(t)]$.

2.3. The Mittag-Leffler Function. The Mittag-Leffler function which is a generalization of exponential function (see [38]) is as follows:

$$E_\alpha(z) = \sum_{n=o}^{\infty} \frac{z_n}{\Gamma(\alpha n + 1)} \quad (\alpha \in \mathscr{C},\ \mathscr{R}(\alpha) > 0). \tag{12}$$

2.4. Adomian Decomposition Method. The Adomian decomposition method (ADM) is a creative and effective method for exactly solving functional equations of various kinds. The method was developed by Adomian [39]. It is important to note that a large amount of research work has been devoted to the application of the ADM in a wide class of linear, nonlinear, ordinary, or partial differential equations. The decomposition method provides the solution as an infinite series in which each term can be determined easily.

3. Solution of Fractional Differential Equations by Homotopy Perturbation Sumudu Transform Method (HPSTM)

The homotopy perturbation method (HPM), introduced by He (see [40–43]) , is a series expansion method used in the solution of nonlinear partial differential equations. The HPM uses a so-called convergence-control parameter to guarantee the convergence of approximation series over a given interval of physical parameters.

We illustrate the basic idea of this method by considering a general fractional nonlinear nonhomogeneous partial differential equation

$$D_t^\alpha U(x,t) = LU(x,t) + NU(x,t) + f(x,t), \quad \alpha > 0, \tag{13}$$

with initial conditions

$$D_0^k U(x,0) = g_k, \quad (k = 0, 1, \ldots, n-1), \qquad D_0^n U(x,0) = 0, \quad n = [\alpha], \tag{14}$$

where $D_t^\alpha U(x,t)$ is the Caputo fractional derivative of the function $U(x,t)$, L is the linear differential operator, N represents the general nonlinear differential operator, and $f(x,t)$ is a known function.

Applying the Sumudu transform on both sides of (13), we get

$$S\left[D_t^\alpha U(x,t)\right] = S\left[LU(x,t)\right] + S\left[NU(x,t)\right] + S\left[f(x,t)\right]. \tag{15}$$

Using the property of the Sumudu transform, we have

$$S\left[U(x,t)\right] = u^\alpha S\left[LU(x,t) + NU(x,t)\right] + g(x,t). \tag{16}$$

Operating with the inverse Sumudu transform on both sides of (16), we get

$$U(x,t) = G(x,t) + S^{-1}\left[u^\alpha S\left[LU(x,t) + NU(x,t)\right]\right], \tag{17}$$

where $G(x,t)$ represents the term arising from the function and the prescribed initial conditions.

Now we use the homotopy perturbation method

$$U(x,t) = \sum_{n=0}^{\infty} p^n U_n(x,t) \tag{18}$$

and the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{n=0}^{\infty} p^n H_n(U) \tag{19}$$

on using He's polynomials $H_n(U)$ (see [10, 44]) which are given by

$$H_n(U_0, U_1, \ldots, U_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i U_i\right)\right]_{p=0}. \tag{20}$$

Substituting (18) and (19) in (17), we get

$$\sum_{n=0}^{\infty} p^n U_n(x,t) = G(x,t) + p\left(S^{-1}\left[u^\alpha S\left[L\sum_{n=0}^{\infty} p^n U_n(x,t) + \sum_{n=0}^{\infty} p^n H_n(U)\right]\right]\right) \tag{21}$$

which shows the coupling of the Sumudu transform and the HPM by using He's polynomials.

On collecting the coefficients of powers of p, we obtain

$$\begin{aligned} p^0 &: U_0(x,t) = G(x,t) \\ p^1 &: U_1(x,t) = S^{-1}\left[u^\alpha S\left\{LU_0(x,t) + H_0(U)\right\}\right] \\ p^2 &: U_2(x,t) = S^{-1}\left[u^\alpha S\left\{LU_1(x,t) + H_1(U)\right\}\right] \end{aligned} \tag{22}$$

and similarly

$$p^n : U_n(x,t) = S^{-1}\left[u^\alpha S\left\{LU_{n-1}(x,t) + H_{n-1}(U)\right\}\right]. \tag{23}$$

Finally, we approximate the analytical solution $U(x,t)$, by truncated series

$$U(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} U_n(x,t). \tag{24}$$

However, as we have seen in many cases, the exact solution in a closed form may be obtained. In addition, the decomposition series solutions generally converge very rapidly. Abbaoui and Cherruault [45] had defined the classical approach of convergence of this type of series.

4. Solution of Fokker-Planck Fractional Differential Equation

Solution of the following nonlinear space-time fractional Fokker-Planck equations (FPE) is

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = \left[-\frac{\partial A(x,t,v)}{\partial x} + \frac{\partial^2 B(x,t,v)}{\partial x^2}\right] v(x,t) \tag{25}$$

with the initial condition

$$v(x,0) = f(x). \tag{26}$$

Applying the Sumudu transform on both sides of (25), subject to initial condition (26), we have

$$S[v(x,t)] = f(x) + u^{\alpha} S \times \left[-\frac{\partial A(x,t,v)}{\partial x} v(x,t) + \frac{\partial^2 B(x,t,v)}{\partial x^2} v(x,t) \right], \tag{27}$$

taking inverse Sumudu transform

$$v(x,t) = f(x) + S^{-1}\left[u^{\alpha} S\left\{ -\frac{\partial}{\partial x}\sum_{n=0}^{\infty} A_n + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} B_n \right\} \right], \tag{28}$$

where

$$\sum_{n=0}^{\infty} A_n = A(x,t,v)\cdot v(x,t), \qquad \sum_{n=0}^{\infty} B_n = B(x,t,v)\cdot v(x,t). \tag{29}$$

Using HPM method, we get

$$\sum_{n=0}^{\infty} p^n v_n(x,t) = f(x) + p \times \left(S^{-1}\left[u^{\alpha} S\left\{ -\sum_{n=0}^{\infty} p^n H_n(x,t,v) + \sum_{n=0}^{\infty} p^n H_n^*(x,t,v) \right\} \right] \right), \tag{30}$$

where $H_n(x,t,v)$ and $H_n^*(x,t,v)$ are He's polynomials that represent the nonlinear terms. The He's polynomials are given by

$$\sum_{n=0}^{\infty} H_n(x,t,v) = \frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} A_n \right), \qquad \sum_{n=0}^{\infty} H_n^*(x,t,v) = \frac{\partial^2}{\partial x^2}\left(\sum_{n=0}^{\infty} B_n \right). \tag{31}$$

Using the above equation we can collect the coefficient of power of p

$$p^0 : v_0(x,t) = f(x), \qquad p^1 : v_1(x,t) = S^{-1}\left[u^{\alpha} S\{[-H_0 + H_0^*]\} \right], \qquad p^2 : v_2(x,t) = S^{-1}\left[u^{\alpha} S\{[-H_1 + H_1^*]\} \right] \tag{32}$$

and similarly

$$p^n : v_n(x,t) = S^{-1}\left[u^{\alpha} S\{[-H_{n-1} + H_{n-1}^*]\} \right]. \tag{33}$$

Finally, we approximate the analytical solution $v_n(x,t)$ by truncated series [25]

$$v(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} v_n(x,t); \tag{34}$$

the series solutions of the above equation converge very rapidly [25, 45].

5. Numerical Examples

In this section we will illustrate the HPSTM techniques by several examples. These examples are somewhat artificial in the sense that the exact answer, for the special case $\alpha = 1$, is known in advance and the initial and boundary conditions are directly taken from this answer. Nonetheless, such an approach is needed to evaluate the accuracy of the analytical techniques and to examine the effect of varying the order of the space- and time-fractional derivatives on the behavior of the solution. All the results are calculated by using the symbolic calculus software MATLAB.

Example 1. Consider the nonlinear time fractional FPE

$$D_t^{\alpha} v(x,t) = \left[-\frac{\partial}{\partial x}\left(3v - \frac{x}{2} \right) + \frac{\partial^2}{\partial x^2}(xv) \right] v(x,t), \tag{35}$$

where $t > 0$, $x > 0$, $0 < \alpha \le 1$, and D_t^{α} is the Caputo fractional derivative defined by (7) and initial condition is

$$v(x,0) = x. \tag{36}$$

Then solution is given by $v(x,t) = xE_{\alpha}(t^{\alpha})$.

Solution 1. On using the method defined in Section 4, we get the coefficient of powers of p

$$p^0 : v_0(x,t) = x, \qquad p^1 : v_1(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)}, \qquad p^2 : v_2(x,t) = \frac{xt^{2\alpha}}{\Gamma(2\alpha+1)}, \qquad p^3 : v_3(x,t) = \frac{xt^{3\alpha}}{\Gamma(3\alpha+1)}; \tag{37}$$

hence, $v(x,t)$ is

$$v(x,t) = x\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots \right) \qquad v(x,t) = xE_{\alpha}(t^{\alpha}). \tag{38}$$

Remark 2. Setting $\alpha = 1$, Example 1 reduces to nonlinear FPE

$$\frac{\partial}{\partial t} v(x,t) = \left[-\frac{\partial}{\partial x}\left(3v - \frac{x}{2} \right) + \frac{\partial^2}{\partial x^2}(xv) \right] v(x,t) \tag{39}$$

with initial condition

$$v(x,0) = x, \tag{40}$$

and a solution as

$$v(x,t) = xe^{t}. \tag{41}$$

Example 3. Consider the nonlinear time fractional FPE

$$D_t^{\alpha} v(x,t) = \left[-\frac{\partial}{\partial x}\left(\frac{4v}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}(v) \right] v(x,t), \quad x > 0,\ t > 0, \tag{42}$$

where $t > 0$, $x > 0$, $0 < \alpha \leq 1$, and D_t^{α} is Caputo fractional derivative defined by (7) and initial condition is

$$v(x,0) = x^2. \tag{43}$$

Then solution is given by $v(x,t) = x^2 E_{\alpha}(t^{\alpha})$.

Solution 2. By using the method defined in Section 4, we get the coefficient of power of p

$$\begin{aligned} p^0 &: v_0(x,t) = x^2, \\ p^1 &: v_1(x,t) = \frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)}, \\ p^2 &: v_2(x,t) = \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ p^3 &: v_3(x,t) = \frac{x^2 t^{3\alpha}}{\Gamma(3\alpha+1)} \cdots; \end{aligned} \tag{44}$$

hence, $v(x,t)$ is

$$v(x,t) = x^2\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$
$$v(x,t) = x^2 E_{\alpha}(t^{\alpha}). \tag{45}$$

Remark 4. Setting $\alpha = 1$, Example 3 reduces to nonlinear FPE

$$\frac{\partial}{\partial t} v(x,t) = \left[-\frac{\partial}{\partial x}\left(\frac{4v}{x} - \frac{x}{3}\right) + \frac{\partial^2}{\partial x^2}(v) \right] v(x,t), \quad x > 0,\ t > 0 \tag{46}$$

with initial condition

$$v(x,0) = x^2 \tag{47}$$

and a solution as

$$v(x,t) = x^2 e^{t}. \tag{48}$$

Example 5. Consider the nonlinear time fractional FPE

$$D_t^{\alpha} v(x,t) = \left[-\frac{\partial}{\partial x}(x) + \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2}\right) \right] v(x,t), \quad x > 0,\ t > 0, \tag{49}$$

where $t > 0$, $x > 0$, $0 < \alpha \leq 1$, and D_t^{α} is Caputo fractional derivative defined by (7) and initial condition is

$$v(x,0) = x. \tag{50}$$

Solution 3. By using the method defined in Section 4, we get the coefficient of p

$$\begin{aligned} p^0 &: v_0(x,t) = x, \\ p^1 &: v_1(x,t) = \frac{xt^{\alpha}}{\Gamma(\alpha+1)}, \\ p^2 &: v_2(x,t) = \frac{xt^{2\alpha}}{\Gamma(2\alpha+1)}, \\ p^3 &: v_3(x,t) = \frac{xt^{3\alpha}}{\Gamma(3\alpha+1)}; \end{aligned} \tag{51}$$

hence, $v(x,t)$ is

$$v(x,t) = x\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots\right)$$
$$v(x,t) = xE_{\alpha}(t^{\alpha}). \tag{52}$$

Remark 6. Setting $\alpha = 1$ in problem, Example 5 reduces to nonlinear FPE

$$\frac{\partial}{\partial t} v(x,t) = \left[-\frac{\partial}{\partial x}(x) + \frac{\partial^2}{\partial x^2}\left(\frac{x^2}{2}\right) \right] v(x,t), \quad x > 0,\ t > 0, \tag{53}$$

with initial condition

$$v(x,0) = x \tag{54}$$

and a solution as

$$v(x,t) = xe^{t}. \tag{55}$$

The approximate solutions for (42) and (49) shown in Tables 1 and 2, respectively, were obtained for different values of x and t by using the homotopy perturbation Sumudu transform method. The exact solutions of (42) and (49) are $v(x,t) = x^2 e^{t}$ and $v(x,t) = xe^{t}$, respectively. The accuracy of our approximate solutions can be improved by computing more terms of the approximate solutions. Both the methods viz. ADM and HPSTM are used adroitly to obtain the exact values and get comparisons. From the numerical results in Tables 1 and 2, it is easy to conclude that the solution continuously depends on the space-fractional derivative and the approximate solutions of (42) and (49) obtained by using

Table 1: Comparison study for solution of (42) for $\alpha = 1$, when $t = 0.2, 0.4, 0.6$ and $x = 0.25, 0.50, 0.75, 1.0$. by various methods.

t	x	HPSTM	ADM (see [20])	EXACT (see [20])
0.2	0.25	0.076333	0.076333	0.076338
0.2	0.50	0.305333	0.305333	0.305351
0.2	0.75	0.687000	0.687000	0.687039
0.2	1.00	1.221333	1.221333	1.221403
0.4	0.25	0.093167	0.093167	0.093239
0.4	0.50	0.372667	0.372667	0.372956
0.4	0.75	0.838500	0.838500	0.839151
0.4	1.00	1.490667	1.490667	1.491825
0.6	0.25	0.113500	0.113500	0.113882
0.6	0.50	0.454000	0.454000	0.455530
0.6	0.75	1.021500	1.021500	1.024942
0.6	1.00	1.816000	1.816000	1.822119

Table 2: Comparison study for solution of (49) for $\alpha = 1$, when $t = 0.2, 0.4, 0.6$ and $x = 0.25, 0.50, 0.75, 1.0$. by various methods.

t	x	HPSTM	ADM (see [20])	EXACT (see [20])
0.2	0.25	0.305333	0.305333	0.305351
0.2	0.50	0.610667	0.610667	0.610701
0.2	0.75	0.916000	0.916000	0.916052
0.2	1.00	1.221333	1.221333	1.221403
0.4	0.25	0.372667	0.372667	0.372956
0.4	0.50	0.745333	0.745333	0.745912
0.4	0.75	1.118000	1.118000	1.118869
0.4	1.00	1.490667	1.490667	1.491825
0.6	0.25	0.454000	0.454000	0.455530
0.6	0.50	0.908000	0.908000	0.911059
0.6	0.75	1.362000	1.362000	1.366589
0.6	1.00	1.816000	1.816000	1.822119

the Adomian decomposition method are the same as those obtained by the homotopy perturbation Sumudu transform method. It is to be noted that only the fourth-order term of the series of the abovesaid methods was used in evaluating the approximate solutions for Tables 1 and 2.

6. Conclusion

Within the scope of the HPSTM, we derive the solution of the nonlinear time fractional Fokker-Planck equations. We made use of the Caputo derivative because it allows usual initial conditions. The numerical result shows that the method used is very simple and straightforward to implement.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are grateful to the anonymous referee for his/her careful reading and many corrections/suggestions to improve this paper. This project was supported by Deanship of Scientific Research, College of Science Research Center, King Saud University

References

[1] H. Risken, *The FokkerPlanck Equation: Method of Solution and Applications*, Springer, Berlin, Germany, 1989.

[2] S. Chandresekhar, "Stochastic problems in physics and astronomy," *Reviews of Modern Physics*, vol. 15, pp. 1–89, 1943.

[3] H. A. Kramers, "Brownian motion in a field of force and the diffusion model of chemical reactions," *Physica*, vol. 7, pp. 284–304, 1940.

[4] A. Fokker, "The median energy of rotating electrical dipoles in radiation fields," *Annalen Der Physik*, vol. 43, pp. 810–820, 1914.

[5] J. H. He and X. H. Wu, "Construction of solitary solution and compaction-like solution by variational iteration method," *Chaos Solitons & Fractals*, vol. 29, pp. 108–113, 2006.

[6] G. Jumarie, "Fractional Brownian motions via random walk in the complex plane and via fractional derivative, comparison and further results on their Fokker-Planck equations," *Chaos Solitons & Fractals*, vol. 22, pp. 907–925, 2004.

[7] Y. Kamitani and I. Matsuba, "Self-similar characteristics of neural networks based on Fokker-Planck equation," *Chaos Solitons & Fractals*, vol. 20, no. 2, pp. 329–335, 2004.

[8] Y. Xu, F. Y. Ren, J. R. Liang, and W. Y. Qiu, "Stretched Gaussian asymptotic behavior for fractional Fokker-Planck equation on fractal structure in external force fields," *Chaos, Solitons & Fractals*, vol. 20, no. 3, pp. 581–586, 2004.

[9] M. Zak, "Expectation-based intelligent control," *Chaos Solitons & Fractals*, vol. 28, pp. 616–626, 2006.

[10] S. T. Mohyud-Din, M. A. Noor, and K. I. Noor, "Traveling wave solutions of seventh-order generalized KdV equations using He's polynomials," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 10, no. 2, pp. 227–233, 2009.

[11] C. Buet, S. Dellacherie, and R. Sentis, "Numerical solution of an ionic Fokker-Planck equation with electronic temperature," *SIAM Journal on Numerical Analysis*, vol. 39, no. 4, pp. 1219–1253, 2001.

[12] G. W. Harrison, "Numerical solution of the Fokker-Planck equation using moving finite elements," *Numerical Methods for Partial Differential Equations*, vol. 4, no. 3, pp. 219–232, 1988.

[13] V. Palleschi, F. Sarri, G. Marcozzi, and M. R. Torquati, "Numerical solution of the Fokker-Planck equation: a fast and accurate algorithm," *Physics Letters A*, vol. 146, no. 7-8, pp. 378–386, 1990.

[14] V. Vanaja, "Numerical solution of a simple Fokker-Planck equation," *Applied Numerical Mathematics*, vol. 9, no. 6, pp. 533–540, 1992.

[15] M. P. Zorzano, H. Mais, and L. Vazquez, "Numerical solution of two-dimensional Fokker-Planck equations," *Applied Mathematics and Computation*, vol. 98, no. 2-3, pp. 109–117, 1999.

[16] A. A. Dubkov, B. Spagnolo, and V. V. Uchaikin, "Lévy flight superdiffusion: an introduction," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 18, no. 9, pp. 2649–2672, 2008.

[17] A. A. Dubkov and B. Spagnolo, "Acceleration of diffusion in randomly switching potential with supersymmetry," *Physical Review E*, vol. 72, no. 4, part 1, Article ID 041104, 8 pages, 2005.

[18] A. Dubkov and B. Spagnolo, "Generalized Wiener process and Kolmogorov's equation for diffusion induced by non-Gaussian noise source," *Fluctuation and Noise Letters*, vol. 5, no. 2, pp. L267–L274, 2005.

[19] A. Dubkov and B. Spagnolo, "Langevin approach to Lévy flights in fixed potentials: exact results for stationary probability distributions," *Acta Physica Polonica B*, vol. 38, no. 5, pp. 1745–1758, 2007.

[20] Z. Odibat and S. Momani, "Numerical solution of Fokker-Planck equation with space- and time-fractional derivatives," *Physics Letters A*, vol. 369, pp. 349–358, 2007.

[21] Q. Yang, F. Liu, and I. Turner, "Computationally efficient numerical methods for time and space-fractional Fokker-Planck equations," *Physica Scripta*, vol. 2009, no. T136, Article ID 014026, 7 pages, 2009.

[22] Q. Yang, F. Liu, and I. Turner, "Stability and convergence of an effective numerical method for the time-space fractional Fokker-Planck equation with a nonlinear source term," *International Journal of Differential Equations*, vol. 2010, Article ID 464321, 22 pages, 2010.

[23] A. Yildirim, "Analytical approach to Fokker-Planck equation with space- and timefractional derivatives by means of the homotopy perturbation method," *Journal of King Saud University (Science)*, vol. 22, pp. 257–264, 2010.

[24] P. Zhuang, F. Liu, V. Anh, and I. Turner, "Numerical treatment for the fractional Fokker-Planck equation," *Anziam Journal*, vol. 48, pp. C759–C774, 2007.

[25] A. Atangana and A. Kılıçman, "The use of Sumudu transform for solving certain nonlinear fractional heat-like equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 737481, 12 pages, 2013.

[26] F. Mainardi, "Fractional calculus: some basic problems in continuum and statistical mechanics," in *Fractals and Fractional Calculus in Continuum Mechanics*, A. Carpinteri and F. Mainardi, Eds., vol. 378, pp. 291–348, Springer, New York, NY, USA, 1997.

[27] M. Caputo, *Elasticitáe Dissipazione*, Zanichelli, Bologna, Italy, 1969, [Italian].

[28] G. Jumarie, "Laplace's transform of fractional order via the Mittag-Leffler function and modified Riemann-Liouville derivative," *Applied Mathematics Letters*, vol. 22, no. 11, pp. 1659–1664, 2009.

[29] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, New York, NY, USA, 1999.

[30] F. Mainardi, "The fundamental solutions for the fractional diffusion-wave equation," *Applied Mathematics Letters*, vol. 9, no. 6, pp. 23–28, 1996.

[31] F. Mainardi, Y. Luchko, and G. Pagnini, "The fundamental solution of the space-time fractional diffusion equation," *Fractional Calculus & Applied Analysis*, vol. 4, no. 2, pp. 153–192, 2001.

[32] G. K. Watugala, "Sumudu transform: a new integral transform to solve differential equations and control engineering problems," *International Journal of Mathematical Education in Science and Technology*, vol. 24, no. 1, pp. 35–43, 1993.

[33] F. B. M. Belgacem, "Applications of the sumudu transform to indefinite periodic parabolic equations," in *Proceedings of the 6th International Conference on Mathematical Problems & Aerospace Sciences (ICNPAA '06)*, chapter 6, pp. 51–60, Cambridge Scientific, Cambridge, UK, 2007.

[34] F. B. Belgacem, A. A. Karaballi, and S. L. Kalla, "Analytical investigations of the Sumudu transform and applications to integral production equations," *Mathematical Problems in Engineering*, vol. 2003, no. 3, pp. 103–118, 2003.

[35] J.-H. He, "A coupling method of a homotopy technique and a perturbation technique for non-linear problems," *International Journal of Non-Linear Mechanics*, vol. 35, no. 1, pp. 37–43, 2000.

[36] V. B. L. Chaurasia, R. S. Dubey, and F. B. M. Belgacem, "Fractional radial diffusion equation analytical solution via Hankel and Sumudu transforms," *Mathematics in Engineering, Science and Aerospace*, vol. 3, no. 2, pp. 1–10, 2012.

[37] Q. D. Katatbeh and F. B. M. Belgacem, "Applications of the Sumudu transform to fractional differential equations," *Nonlinear Studies*, vol. 18, no. 1, pp. 99–112, 2011.

[38] G. M. Mittag-Leffler, *Sur la Nouvelle Fonction Ea(x)*, vol. 137, C. R. Academy of Science, Paris, France, 1903.

[39] G. Adomian, *Solving Frontier Problems of Physics: The Decomp Osition Method*, Kluwer Academic, Boston, Mass, USA, 1994.

[40] J.-H. He, "The homotopy perturbation method nonlinear oscillators with discontinuities," *Applied Mathematics and Computation*, vol. 151, no. 1, pp. 287–292, 2004.

[41] J.-H. He, "Homotopy perturbation technique," *Computer Methods in Applied Mechanics and Engineering*, vol. 178, no. 3-4, pp. 257–262, 1999.

[42] A. A. Hemeda, "Homotopy perturbation method for solving systems of nonlinear coupled equations," *Applied Mathematical Sciences*, vol. 6, no. 93–96, pp. 4787–4800, 2012.

[43] J. Saberi-Nadjafi and A. Ghorbani, "He's homotopy perturbation method: an effective tool for solving nonlinear integral and integro-differential equations," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2379–2390, 2009.

[44] A. Ghorbani, "Beyond Adomian polynomials: he polynomials," *Chaos, Solitons and Fractals*, vol. 39, no. 3, pp. 1486–1492, 2009.

[45] K. Abbaoui and Y. Cherruault, "New ideas for proving convergence of decomposition methods," *Computers & Mathematics with Applications*, vol. 29, no. 7, pp. 103–108, 1995.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 217348, 10 pages
http://dx.doi.org/10.1155/2015/217348

Research Article

Application of Sinc-Galerkin Method for Solving Space-Fractional Boundary Value Problems

Sertan Alkan[1] and Aydin Secer[2]

[1] *Department of Management Information Systems, Bartin University, 74100 Bartin, Turkey*
[2] *Department of Mathematical Engineering, Yildiz Technical University, 34220 Istanbul, Turkey*

Correspondence should be addressed to Aydin Secer; asecer@yildiz.edu.tr

Received 10 June 2014; Accepted 13 July 2014

Academic Editor: Abdon Atangana

We employ the sinc-Galerkin method to obtain approximate solutions of space-fractional order partial differential equations (FPDEs) with variable coefficients. The fractional derivatives are used in the Caputo sense. The method is applied to three different problems and the obtained solutions are compared with the exact solutions of the problems. These comparisons demonstrate that the sinc-Galerkin method is a very efficient tool in solving space-fractional partial differential equations.

1. Introduction

Fractional calculus, which might be considered as an extension of classical calculus, is as old as the classical calculus and fractional differential equations have been often used to describe many scientific phenomena in biomedical engineering, image processing, earthquake engineering, signal processing, physics, statistics, electrochemistry, and control theory.

Because finding the exact or analytical solutions of fractional order differential equations is not an easy task, several different numerical solution techniques have been developed for the approximate solutions of these types of equations. Some of the well-known numerical techniques might be listed as generalized differential transform method [1, 2], finite difference method [3], Adomian decomposition method [4, 5], homotopy perturbation method [6–8], Haar wavelet method [9, 10], differential transform method [11–13], and Adams-Bashforth-Moulton scheme [14]. A detailed and informative study on fractional calculus can be found in [15]. Furthermore a relatively new analytical method was presented in [16] to solve time "The Time-Fractional Coupled-Korteweg-de-Vries Equations" via homotopy decomposition method by the same authors. The sinc methods were introduced in [17] and expanded in [18] by Frank Stenger. Sinc functions were firstly analyzed in [19, 20]. In [21], the sinc-Galerkin method is used to approximate solutions of nonlinear differential equations with homogeneous and nonhomogeneous boundary conditions. In [22], the sinc-Galerkin method is applied to nonlinear fourth-order differential equations with nonhomogeneous and homogeneous boundary conditions. In the paper at [23], the numerical solutions of Troesch's problem are obtained by the sinc-Galerkin method and the results are compared with methods of Laplace, homotopy perturbation, splines, and perturbation. Reference [24] which contains short abstract version of current paper has been presented in an International Conference and Workshop on Mathematical Analysis 2014, Malaysia. In [25], the authors present a comparison between sinc-Galerkin method and sinc-collocation method to obtain approximate solutions of linear and nonlinear boundary value problems. Similarly, the wavelet-Galerkin method and the sinc-Galerkin method for solving nonhomogeneous heat equations are compared in [26]. The paper [27] offers an application of the sinc-Galerkin method for solving second-order singular Dirichlet-type boundary value problems. In [28], the sinc-Galerkin method is used to approximate solutions of fractional order ordinary differential equations in Caputo sense.

In this paper we propose a new solution technique for approximate solution of space-fractional order partial

differential equations (FPDEs) with variable coefficients and boundary conditions by using the sinc-Galerkin method that has almost not been employed for the space-fractional order partial differential equations in the form

$$u_{tt} = a(x)u_{xx} + b(x)\,{}_0^C D_x^\beta u + c(x)\,u + f(x,t), \quad 0 < \beta < 1 \tag{1}$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = u(x,1) = 0, \tag{2}$$

where ${}_0^C D_x$ is Caputo fractional derivative operator.

The paper is organized as follows. Section 2 presents basic theorems of fractional calculus and sinc-Galerkin method. In Section 3, we use the sinc-Galerkin method to obtain an approximate solution of a general space-fractional partial differential equation. In Section 4, we present three examples in order to illustrate the effectiveness and accuracy of the present method. The obtained results are compared with the exact results.

2. Preliminaries

2.1. Fractional Calculus. In this section, we present the definitions of the fractional Riemann-Liouville derivative and the Caputo of fractional derivatives. By using these definitions, we give the definition of the integration by parts of fractional order.

Definition 1 (see [29]). Let $f : [a,b] \times [c,d] \to \mathbb{R}$ be a function; α is a positive real number, and n is the integer. α, n satisfy the inequality $n-1 \le \alpha < n$ and Γ the Euler gamma function. Then,

(i) the left and right Riemann-Liouville fractional derivatives of order α with respect to x of $f(x,t)$ function are given as

$${}_aD_x^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-s)^{n-\alpha-1} f(s,t)\,ds, \tag{3}$$

$${}_xD_b^\alpha f(x,t) = \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (s-x)^{n-\alpha-1} f(s,t)\,ds, \tag{4}$$

respectively;

(ii) the left and right Caputo fractional derivatives of order α with respect to x of $f(x,t)$ function are given as

$${}_a^C D_x^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-s)^{n-\alpha-1}\frac{\partial^n f(s,t)}{\partial s^n}ds, \tag{5}$$

$${}_x^C D_b^\alpha f(x,t) = \frac{1}{\Gamma(n-\alpha)}\int_x^b (-1)^n (s-x)^{n-\alpha-1}\frac{\partial^n f(s,t)}{\partial s^n}ds, \tag{6}$$

respectively.

Now, we can write the definition of integration by parts of fractional order by using the relations given in (3)–(6).

Definition 2. If $0 < \alpha < 1$ and f is a function such that $f(a,t) = f(b,t) = 0$, one can write

$$\int_a^b g(x,t)\,{}_a^C D_x^\alpha f(x,t)\,dx = \int_a^b f(x,t)\,{}_xD_b^\alpha g(x,t)\,dx,$$

$$\int_a^b g(x,t)\,{}_x^C D_b^\alpha f(x,t)\,dx = \int_a^b f(x,t)\,{}_aD_x^\alpha g(x,t)\,dx. \tag{7}$$

2.2. Properties of Sinc Basis Functions and Quadrature Interpolations. In this section, we recall notations and definitions of the sinc function state some known results and derive some useful formulas to be used in the next sections of the present paper.

2.2.1. The Sinc Basis Functions

Definition 3 (see [30]). The function which defined all $z \in \mathbb{C}$ by

$$\operatorname{sinc}(z) = \begin{cases} \dfrac{\sin(\pi z)}{\pi z}, & z \ne 0, \\ 1, & z = 0 \end{cases} \tag{8}$$

is called the *sinc function*.

Definition 4 (see [30]). Let f be a function defined on $\mathbb{R}$ and let $h > 0$. Define the series

$$C(f,h)(x) = \sum_{k=-\infty}^{\infty} f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right), \tag{9}$$

where from (8) we have

$$S(k,h)(x) = \operatorname{sinc}\left(\frac{x-kh}{h}\right) = \begin{cases} \dfrac{\sin(\pi((x-kh)/h))}{\pi((x-kh)/h)} & x \ne kh, \\ 1, & x = kh. \end{cases} \tag{10}$$

If the series in (9) converges, it is called the *Whittaker cardinal function* of f. They are based on the infinite strip D_s in the complex plane

$$D_s \equiv \left\{ w = u + iv : |v| < d \le \frac{\pi}{2} \right\}. \tag{11}$$

Generally, approximations can be constructed for infinite, semi-infinite, and finite intervals. Define the function

$$w = \phi(z) = \ln\left(\frac{z}{1-z}\right) \tag{12}$$

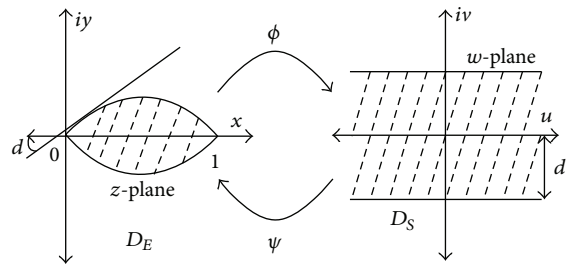


FIGURE 1: The domains D_E and D_S.

which is a conformal mapping from D_E, the eye-shaped domain in the z-plane, onto the infinite strip D_S, where

$$D_E = z = \left\{x + iy : \left|\arg\left(\frac{z}{1-z}\right)\right| < d \le \frac{\pi}{2}\right\}. \quad (13)$$

This is shown in Figure 1. For the sinc-Galerkin method, the bases functions are derived from the composite translated sinc functions

$$S_k(z) = S(k,h)(z) \,o\phi(z) = \operatorname{sinc}\left(\frac{\phi(z) - kh}{h}\right), \quad (14)$$

where $z \in D_E$. The function $z = \phi^{-1}(w) = e^w/(1+e^w)$ is an inverse mapping of $w = \phi(z)$. We may define the range of ϕ^{-1} on the real line as

$$\Gamma = \left\{\phi^{-1}(u) \in D_E : -\infty < u < \infty\right\} \quad (15)$$

evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ on the real line. The image which corresponds to these nodes is denoted by

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1+e^{kh}}. \quad (16)$$

2.2.2. Sinc Function Interpolation and Quadrature

Definition 5 (see [21]). Let D_E be a simply connected domain in the complex plane C and let ∂D_E denote the boundary of D_E. Let a, b be points on ∂D_E and let ϕ be a conformal map D_E onto D_S such that $\phi(a) = -\infty$ and $\phi(b) = \infty$. If the inverse map of ϕ is denoted by φ, define

$$\Gamma = \left\{\phi^{-1}(u) \in D_E : -\infty < u < \infty\right\}, \quad (17)$$

and $z_k = \varphi(kh)$, $k = \pm 1, \pm 2, \ldots$.

Definition 6 (see [21]). Let $B(D_E)$ be the class of functions F that are analytic in D_E and satisfy

$$\int_{\psi(L+u)} |F(z)|\,dz \longrightarrow 0, \quad \text{as } u = \mp\infty, \quad (18)$$

in which

$$L = \left\{iy : |y| < d \le \frac{\pi}{2}\right\}, \quad (19)$$

and those on the boundary of D_E satisfy

$$T(F) = \int_{\partial D_E} |F(z)\,dz| < \infty. \quad (20)$$

Theorem 7 (see [21]). *Let Γ be $(0,1)$, $F \in B(D_E)$; then, for $h > 0$ sufficiently small,*

$$\int_\Gamma F(z)\,dz - h\sum_{j=-\infty}^{\infty}\frac{F(z_j)}{\phi'(z_j)} = \frac{i}{2}\int_{\partial D}\frac{F(z)\,k(\phi,h)(z)}{\sin(\pi\phi(z)/h)}dz \equiv I_F, \quad (21)$$

where

$$\left|k(\phi,h)\right|_{z\in\partial D} = \left|e^{[(i\pi\phi(z)/h)\,\operatorname{sgn}(Im\phi(z))]}\right|_{z\in\partial D} = e^{-\pi d/h}. \quad (22)$$

For the sinc-Galerkin method, the infinite quadrature rule must be truncated to a finite sum. The following theorem indicates the conditions under which an exponential convergence results.

Theorem 8 (see [21]). *If there exist positive constants α, β, and C such that*

$$\left|\frac{F(x)}{\phi'(x)}\right| \le C\begin{cases} e^{-\alpha|\phi(x)|}, & x \in \psi((-\infty,\infty)) \\ e^{-\beta|\phi(x)|}, & x \in \psi((0,\infty)), \end{cases} \quad (23)$$

then the error bound for the quadrature rule (21) *is*

$$\left|\int_\Gamma F(x)\,dx - h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)}\right| \le C\left(\frac{e^{-\alpha Mh}}{\alpha} + \frac{e^{-\beta Nh}}{\beta}\right) + \left|I_F\right|. \quad (24)$$

The infinite sum in (21) *is truncated with the use of* (23) *to arrive at inequality* (24). *Making the selections*

$$h = \sqrt{\frac{\pi d}{\alpha M}}, \qquad N \equiv \left\llbracket \frac{\alpha M}{\beta} + 1 \right\rrbracket, \quad (25)$$

where $\llbracket\cdot\rrbracket$ is an integer part of the statement and M is the integer value which specifies the grid size, then

$$\int_\Gamma F(x)\,dx = h\sum_{j=-M}^{N}\frac{F(x_j)}{\phi'(x_j)} + O\left(e^{-(\pi\alpha dM)^{1/2}}\right). \quad (26)$$

We used these theorems to approximate the integrals that arise in the formulation of the discrete systems corresponding to a second-order boundary value problem.

3. The Sinc-Galerkin Method

Consider fractional boundary value problem

$$u_{tt} = a(x)u_{xx} + b(x)\,{}_0^C D_x^\beta u + c(x)u + f(x,t), \quad 0 < \beta < 1 \quad (27)$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad u(x,0) = u(x,1) = 0, \quad (28)$$

where ${}_0^C D_t$ is Caputo fractional derivative operator. An approximate solution for $u(x,t)$ is represented by the formula

$$u_{m_x,m_t}(x,t) = \sum_{j=-M_t}^{N_t} \sum_{i=-M_x}^{N_x} u_{ij} S_{ij}(x,t), \tag{29}$$

where $m_x = M_x + N_x + 1$ and $m_t = M_t + N_t + 1$. The basis functions $\{S_{ij}(x,t)\}$ are given by

$$S_{ij}(x,t) = S_i(x) S_j(t) = [S(i,h_x) o\phi(x)] [S(j,h_t) o\gamma(t)], \tag{30}$$

where

$$\phi(x) = \ln\left(\frac{x}{1-x}\right), \qquad \gamma(t) = \ln\left(\frac{t}{1-t}\right). \tag{31}$$

The unknown coefficients u_{ij} in (29) are determined by orthogonalizing the residual with respect to the functions $\{S_{kl}(x,t)\}$, $-M_x \le k \le N_x$, $-M_t \le l \le N_t$. This yields the discrete Galerkin system

$$\left\langle L u_{m_x,m_t} - f(x,t), S_{kl} \right\rangle = 0, \quad -M_x \le k \le N_x, \quad -M_t \le l \le N_t, \tag{32}$$

where inner product is defined by

$$\langle f, g \rangle = \iint_0^1 f(x,t)\, g(x,t)\, W(x,t)\, dx\, dt, \tag{33}$$

where $W(x)$ is *weight function* and it is convenient to take

$$W(x,t) = w(x) v(t) = \left[\frac{1}{[\phi'(x)]^{(1/2)}}\right] [\gamma'(t)]^{-1/2} \tag{34}$$

for the problem (27)-(28).

Lemma 9 (see [23]). *Let ϕ be the conformal one-to-one mapping of the simply connected domain D_E onto D_S, given by* (12). *Then*

$$\delta_{jk}^{(0)} = [S(j,h) o\phi(x)]|_{x=x_k} = \begin{cases} 1, & j = k, \\ 0, & j \ne k, \end{cases}$$

$$\delta_{jk}^{(1)} = h \frac{d}{d\phi} [S(j,h) o\phi(x)]\Big|_{x=x_k} = \begin{cases} 0, & j = k, \\ \dfrac{(-1)^{k-j}}{k-j}, & j \ne k, \end{cases}$$

$$\delta_{jk}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(j,h) o\phi(x)]\Big|_{x=x_k} = \begin{cases} -\dfrac{\pi^2}{3}, & j = k, \\ \dfrac{-2(-1)^{k-j}}{(k-j)^2}, & j \ne k. \end{cases} \tag{35}$$

The following theorems which can easily be proven by using Lemma 9 and definitions are used to solve (27).

Theorem 10 (see [31]). *The following relations hold:*

$$\langle u_{tt}, S_k S_l \rangle \cong h_t h_x \frac{w(x_k)}{\phi'(x_k)} \sum_{j=-M_t}^{N_t} \sum_{i=0}^{2} \frac{u(x_k,t_j)}{\gamma'(t_j)} \left[\frac{1}{h_t^i} \delta_{lj}^{(i)} \eta_i\right],$$

$$\langle a(x) u_{xx}, S_k S_l \rangle \cong h_x h_t \frac{v(t_l)}{\gamma'(t_l)} \sum_{i=-M_x}^{N_x} \sum_{j=0}^{2} \frac{u(x_i,t_l)}{\phi'(x_i)} \left[\frac{1}{h_x^j} \delta_{ki}^{(j)} \rho_j\right],$$

$$\langle c(x) u, S_k S_l \rangle \cong h_t h_x \frac{w(x_k) c(x_k) u(x_k,t_l) v(t_l)}{\phi'(x_k) \gamma'(t_l)},$$

$$\langle f(x,t), S_k S_l \rangle \cong h_t h_x \frac{w(x_k) f(x_k,t_l) v(t_l)}{\phi'(x_k) \gamma'(t_l)}, \tag{36}$$

where

$$\eta_2 = (\gamma')^2 v, \quad \eta_1 = \gamma'' v + 2v'\gamma', \quad \eta_0 = v'', \quad \rho_2 = (\phi')^2 aw, \quad \rho_1 = \phi'' aw + 2\phi' (aw' + a'w), \quad \rho_0 = a''w + 2a'w' + aw''. \tag{37}$$

Proof. See [31]. □

Theorem 11. *For $0 < \beta < 1$, the following relations hold:*

$$\left\langle b(x)\, {}_0^C D_x^\beta u, S_k S_l \right\rangle \cong \frac{-h_x h_t}{\Gamma(1-\beta)} \frac{v(t_l)}{\gamma'(t_l)} \times \sum_{i=-M_x}^{N_x} \frac{u(x_i,t_l)}{\phi'(x_i)} \frac{d}{dx} \left[h_P \sum_{r=-P}^{P} \frac{(y_r - x)^{-\beta} R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i}, \tag{38}$$

where $R(x) = b(x) S_k(x) w(x)$, $\mu(s) = \ln((s-x)/(1-s))$, and $h_P = \pi/\sqrt{P}$.

Proof. The inner product with sinc basis elements of $b(x) {}_0^C D_x^\beta u$ is given by

$$\left\langle b(x)\, {}_0^C D_x^\beta u, S_k S_l \right\rangle = \iint_0^1 b(x)\, {}_0^C D_x^\beta u(x,t)\, S_k(x) S_l(t)\, w(x) v(t)\, dx\, dt. \tag{39}$$

Using Definition 2, we can write

$$\begin{aligned}&\left\langle b(x)\ {}_0^C D_x^\beta u, S_k S_l\right\rangle \\ &= \iint_0^1 b(x)\ {}_0^C D_x^\beta u(x,t)\, S_k(x)\, S_l(t)\, w(x)\, v(t)\, dx\, dt \\ &= \iint_0^1 u(x,t)\, S_l(t)\, v(t)\ {}_xD_1^\beta \left(b(x)\, S_k(x)\, w(x)\right) dx\, dt,\end{aligned} \tag{40}$$

where $R(x) = b(x)S_k(x)w(x)$. By the definition of the Riemann-Liouville fractional derivative, we have

$${}_xD_1^\beta R(x) = -\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^1 (s-x)^{-\beta} R(s)\, ds. \tag{41}$$

We will use the sinc quadrature rule given with (26) to compute it because the integral given in last equality is divergent on the interval $[t, 1]$. For this purpose, a conformal map and its inverse image that denotes the sinc grid points are given by

$$\begin{aligned}\mu(s) &= \ln\left(\frac{s-x}{1-s}\right), \\ y_r = \mu^{-1}(rh_P) &= \frac{e^{rh_P}+x}{1+e^{rh_P}},\end{aligned} \tag{42}$$

respectively. Then, according to equality (26) we can write

$$\begin{aligned}&-\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\int_x^1 (s-x)^{-\beta} R(s)\, ds \\ &\cong -\frac{1}{\Gamma(1-\beta)}\frac{d}{dx}\left[h_P \sum_{r=-P}^{P} \frac{(y_r-x)^{-\beta} R(y_r)}{\mu'(y_r)}\right] = N(x),\end{aligned} \tag{43}$$

where $h_P = \pi/\sqrt{P}$. As a result, it can be written in the following way:

$$\left\langle b(x)\, {}_0^C D_x^\beta u, S_k S_l\right\rangle \cong \iint_0^1 u(x,t)\, S_l(t)\, v(t)\, N(x)\, dx\, dt. \tag{44}$$

Now, applying the sinc quadrature rule with respect to x and t in last integral, we obtain

$$\begin{aligned}&\iint_0^1 u(x,t)\, S_l(t)\, v(t)\, N(x)\, dx\, dt \\ &\cong h_x h_t \sum_{i=M_x}^{N_x}\sum_{n=-M_t}^{N_t} \frac{u(x_i,t_n)\, v(t_n)\, S_l(t_n)\, N(x_i)}{\phi'(x_i)\,\gamma'(t_n)}.\end{aligned} \tag{45}$$

Consequently, using $S_l(t)|_{t=t_n} = \delta_{\ln}^{(0)} = \delta_{\ln}$ we obtain

$$\begin{aligned}&\left\langle b(x)\ {}_0^C D_x^\beta u, S_k S_l\right\rangle \\ &\cong \frac{-h_x h_t}{\Gamma(1-\beta)}\frac{v(t_l)}{\gamma'(t_l)} \\ &\quad\times \sum_{i=-M_x}^{N_x} \frac{u(x_i,t_l)}{\phi'(x_i)}\frac{d}{dx}\left[h_P \sum_{r=-P}^{P}\frac{(y_r-x)^{-\beta} R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i}.\end{aligned} \tag{46}$$

This completes the proof. □

Replacing each term of (32) with the approximation defined in Theorems 10 and 11, replacing $u(x_k, t_j)$ by u_{kj}, and dividing by $h_t h_x$ we obtain the following theorem.

Theorem 12. *If the assumed approximate solution of the boundary-value problem* (27)-(28) *is* (29)*, then the discrete sinc-Galerkin system for the determination of the unknown coefficients* $\{u_{kj}, -M_x \le k \le N_x, -M_t \le j \le N_t\}$ *is given by*

$$\begin{aligned}&\frac{w(x_k)}{\phi'(x_k)}\sum_{j=-M_t}^{N_t}\sum_{i=0}^{2}\frac{u_{kj}}{\gamma'(t_j)}\left[\frac{1}{h_t^i}\delta_{lj}^{(i)}\eta_i\right] \\ &= \frac{v(t_l)}{\gamma'(t_l)}\sum_{i=-M_x}^{N_x}\sum_{j=0}^{2}\frac{u_{il}}{\phi'(x_i)}\left[\frac{1}{h_x^j}\delta_{ki}^{(j)}\rho_j\right] \\ &\quad -\frac{1}{\Gamma(1-\beta)}\frac{v(t_l)}{\gamma'(t_l)} \\ &\quad\times \sum_{i=-M_x}^{N_x}\frac{u_{il}}{\phi'(x_i)}\frac{d}{dx}\left[h_P\sum_{r=-P}^{P}\frac{(y_r-x)^{-\beta}R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i} \\ &\quad +\frac{w(x_k)\,c(x_k)\,u_{kl}\,v(t_l)}{\phi'(x_k)\,\gamma'(t_l)} \\ &\quad +\frac{w(x_k)\,f(x_k,t_l)\,v(t_l)}{\phi'(x_k)\,\gamma'(t_l)}.\end{aligned} \tag{47}$$

We introduce the following notations in order the write the system above in a matrix-vector form. Let $I_{m_x}^{(P)}$, $P = 0, 1, 2$ *be the* $m_x \times m_x$ *matrices* $I^{(P)}$*, with jkth entry* $\delta_{jk}^{(P)}$ *as given by Lemma 9. Further,* $D(g_x)$ *is an* $m_x \times m_x$ *diagonal matrix whose diagonal entries are*

$$\left[g(x_{-M_x}), g(x_{-M_x+1}), \ldots, g(x_0), \ldots, g(x_{N_x})\right]^T. \tag{48}$$

The matrices $I_{m_t}^{(P)}$, $P = 0, 1, 2$, and $D(g_t)$ are similarly defined though of size $m_t \times m_t$. Introducing this notation in (47) *leads to the matrix form*

$$\begin{aligned} &D\left(\frac{1}{\phi'}\right) D(w)\, UD(v) \left[\sum_{j=0}^{2} \frac{1}{h_t^j} I_{m_t}^{(j)} D\left(\frac{\eta_j}{\gamma' v}\right)\right]^t \\ &- \left[\sum_{i=0}^{2} \frac{1}{h_x^i} I_{m_x}^{(i)} D\left(\frac{\rho_i}{(\phi')^2 w}\right)\right] D\left(\phi'\right) D(w)\, UD\left(\frac{v}{\gamma'}\right) \\ &- \left[BD\left(\frac{1}{(\phi')^2 w}\right)\right] D\left(\phi'\right) D(w)\, UD\left(\frac{v}{\gamma'}\right) \\ &- D\left(\frac{w}{\phi'}\right) D(c)\, UD\left(\frac{v}{\gamma'}\right) = D\left(\frac{w}{\phi'}\right) FD\left(\frac{v}{\gamma'}\right), \end{aligned} \tag{49}$$

where

$$B = -\frac{1}{\Gamma(1-\beta)} \frac{d}{dx} \left[h_P \sum_{r=-P}^{P} \frac{(y_r - x)^{-\beta} R(y_r)}{\mu'(y_r)}\right]\Bigg|_{x=x_i}. \tag{50}$$

Firstly multiplying this term with $D(\phi')$ and secondly multiplying it with $D(\gamma')$ yield the equivalent system

$$\Phi X + X\Psi = G, \tag{51}$$

where

$$\begin{aligned} \Phi &= -A1 - A2 - C, \\ \Psi &= B1 \end{aligned} \tag{52}$$

for

$$\begin{aligned} A1 &= D\left(\phi'\right) \left[\sum_{i=0}^{2} \frac{1}{h_x^i} I_{m_x}^{(i)} D\left(\frac{\rho_i}{(\phi')^2 w}\right)\right] D\left(\phi'\right), \\ A2 &= D\left(\phi'\right) \left[BD\left(\frac{1}{(\phi')^2 w}\right)\right] D\left(\phi'\right), \\ B1 &= \left[\sum_{j=0}^{2} \frac{1}{h_t^j} I_{m_t}^{(j)} D\left(\frac{\eta_j}{\gamma' v}\right)\right]^t D\left(\gamma'\right), \\ C &= D(c). \end{aligned} \tag{53}$$

Furthermore,

$$\begin{aligned} G &= D(w)\, FD(v), \\ X &= D(w)\, UD(v). \end{aligned} \tag{54}$$

Φ, Ψ, X, and G have dimensions $m_x \times m_x$, $m_t \times m_t$, $m_x \times m_t$, and $m_x \times m_t$, respectively. At last, the $m_x \times m_t$ matrices U and F have klth entries given by u_{kl} and $f(x_k, t_l) = f(e^{kh}/(1 + e^{kh}), e^{lh}/(1 + e^{lh}))$, respectively.

To obtain the approximate solution equation (29)*, we need to solve the system for U which requires solving* (51) *for X. Solution of* (51) *for X is shown in [26].*

4. Examples

In this section, the present method will be tested on three different problems.

Example 1. Consider fractional boundary value problem

$$\begin{aligned} u_{tt} &= u_{xx} + {}_0^C D_x^{0.7} u + f(x,t), \\ u(0,t) &= u(1,t) = 0, \\ u(x,0) &= u(x,1) = 0 \end{aligned} \tag{55}$$

which has the following exact solution:

$$u(x,t) = x^2 (1-x) t^3 (1-t)^2 \tag{56}$$

for

$$\begin{aligned} f(x,t) = {} & -2(1-t)^2 t^3 (1-x) + 4(1-t)^2 t^3 x \\ & + 6(1-t)^2 t (1-x) x^2 - 12 (1-t) t^2 \\ & \times (1-x) x^2 + 2t^3 (1-x) x^2 + 0.334273 \\ & \times (-1+t)^2 t^3 x^{1.3} (-5.12821 + 6.68896x). \end{aligned} \tag{57}$$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 1 and 2 for different values. Also, the graphs of exact and approximate solutions for different values are presented in Figures 2 and 3.

Example 2. Consider fractional boundary value problem

$$\begin{aligned} u_{tt} &= e^x u_{xx} + \left(x^2 + 1\right) {}_0^C D_x^{0.3} u - \frac{1}{x+1} u + f(x,t) \\ u(0,t) &= u(1,t) = 0 \\ u(x,0) &= u(x,1) = 0 \end{aligned} \tag{58}$$

which has the following exact solution:

$$u(x,t) = x^2 (1-x) \sin(\pi t) \tag{59}$$

for

$$\begin{aligned} & f(x,t) \\ & = -\pi^2 (1-x) x^2 \sin(\pi t) + \frac{(1-x) x^2 \sin(\pi t)}{1+x} \\ & \quad - 0.770383 \left(1 + x^2\right) \left(1.68067 x^{1.7} - 1.86741 x^{2.7}\right) \\ & \quad \times \sin(\pi t) - e^x (2(1-x) \sin(\pi t) - 4x \sin(\pi t)). \end{aligned} \tag{60}$$

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 3 and 4. In addition, in Figures 4 and 5, the graphs of exact and approximate solutions for different values are presented.

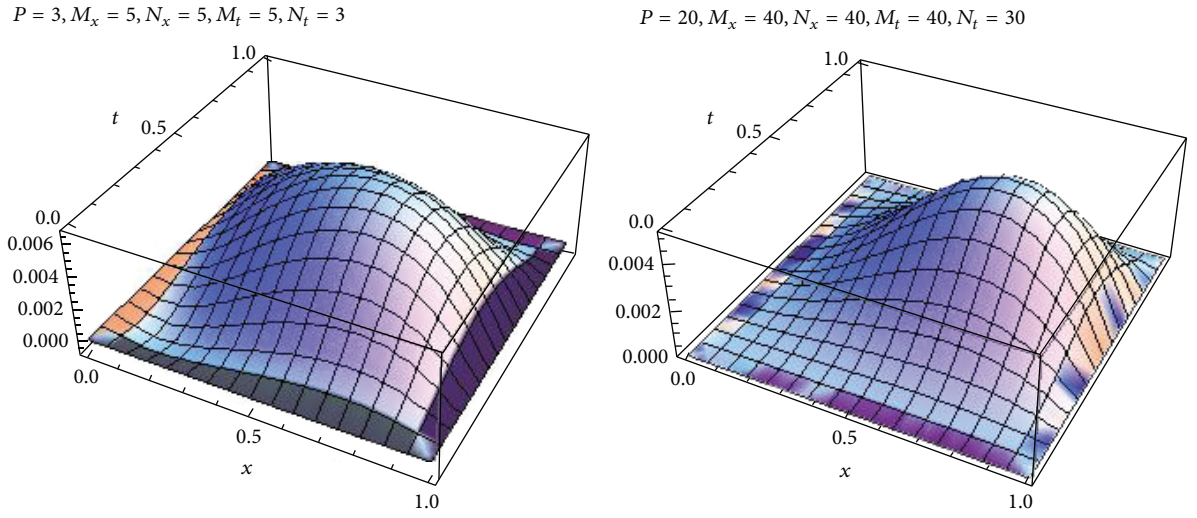


Figure 2: Graphs of approximate solutions for different values.

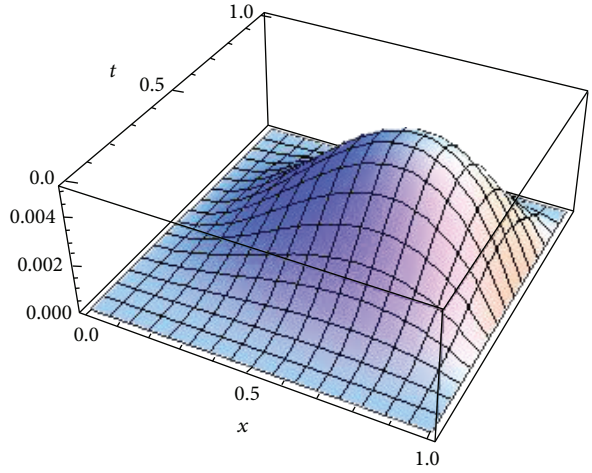


Figure 3: Graph of exact solution.

Example 3 (see [32]). Consider the fractional convection-diffusion equation

$$\frac{\partial^{\gamma} u(x,t)}{\partial t^{\gamma}} = -a(x)\frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} + b(x)\frac{\partial^{\beta} u(x,t)}{\partial x^{\beta}} + f(x,t)$$

$$0 < \gamma \le 2, \quad 1 < \alpha \le 2, \quad 0 < \beta \le 1,$$

$$u(0,t) = u(1,t) = 0,$$

$$u(x,0) = u(x,1) = 0. \tag{61}$$

In particular, if $\gamma = 2$, $\alpha = 2$, $\beta = 0.35$, $a(x) = \Gamma(2.35)\Gamma(2.65)$ $x^{0.35}$, $b(x) = \Gamma(0.7)\Gamma(1)x^2$.

The problem has the following exact solution:

$$u(x,t) = \left(x^{1.7} - x^2\right)\sin(2\pi t) \tag{62}$$

Table 1: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.00000160	0.000069824	0.000068223
	0.6	0.00000365	0.000557026	0.000553368
	0.9	0.00000205	0.000286490	0.000284433
0.06	0.3	0.00001202	−0.000160475	0.000172499
	0.6	0.00002748	0.000688434	0.000660950
	0.9	0.00001545	0.000405155	0.000389695
0.09	0.3	0.00003803	−0.000317899	0.000355931
	0.6	0.00008693	0.000879725	0.000792795
	0.9	0.00004889	0.000577303	0.000528404

Table 2: Numerical results for $P = 20$, $M_x = 40$, $N_x = 40$, $M_t = 40$, $N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.00000160	0.00000376	$2.16753 * 10^{-6}$
	0.6	0.00000365	0.00001237	$8.71244 * 10^{-6}$
	0.9	0.00000205	0.00000667	$4.61338 * 10^{-6}$
0.06	0.3	0.00001202	0.00001645	$4.43271 * 10^{-6}$
	0.6	0.00002748	0.00004476	$1.72861 * 10^{-5}$
	0.9	0.00001545	0.00002455	$9.09322 * 10^{-6}$
0.09	0.3	0.00003803	0.00004497	$6.93846 * 10^{-6}$
	0.6	0.00008693	0.00011262	$2.56904 * 10^{-5}$
	0.9	0.00004889	0.00006228	$1.33882 * 10^{-5}$

for

$$f(x,t) = \left(2\pi x^{1.7} - x^2\right)t^{-1}E_{2,0}\left(-(2\pi t)^2\right) + \left\{\Gamma(2.7)\left(\Gamma(2.65) - \Gamma(1)\right)x^{1.7} + \Gamma(3)\left(\Gamma(0.7) - \Gamma(2.35)\right)x^2\right\}\sin(2\pi t). \tag{63}$$

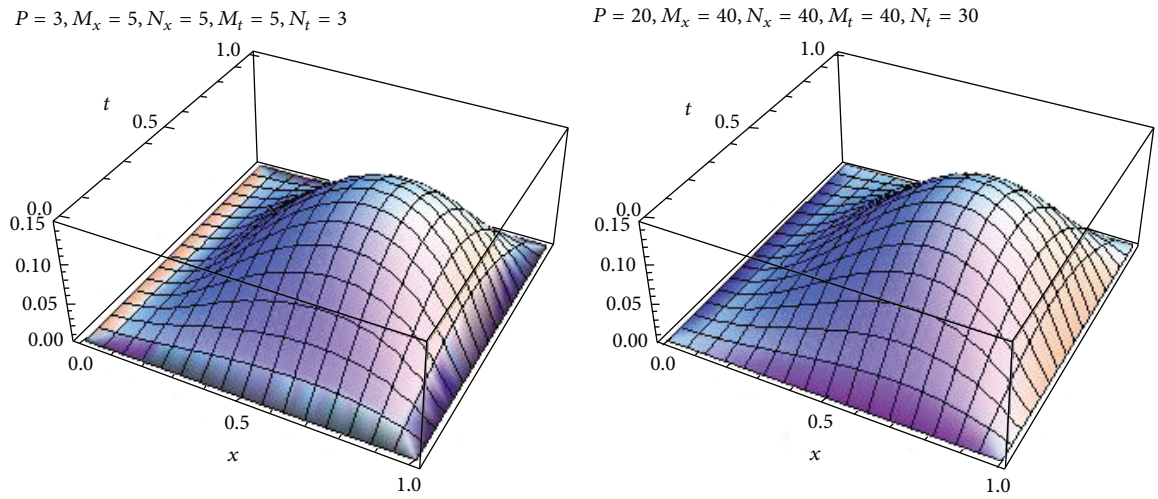


Figure 4: Graphs of approximate solutions for different values.

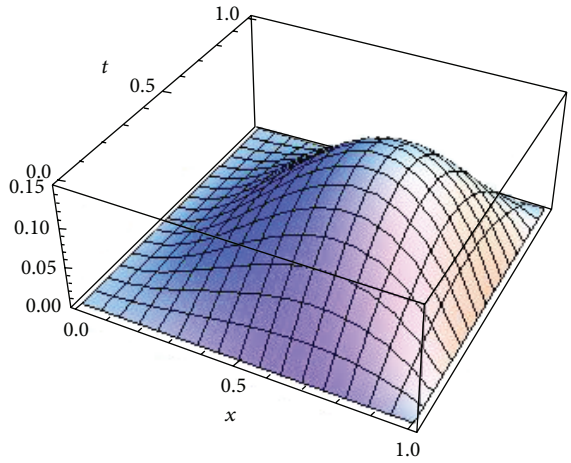


Figure 5: Graphs of exact solution.

Table 3: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0059288	0.0071562	0.00122741
	0.6	0.0135516	0.0149881	0.00143653
	0.9	0.0076227	0.0083725	0.00074978
0.06	0.3	0.0118050	0.0131991	0.00139409
	0.6	0.0269829	0.0254147	0.00156823
	0.9	0.0151779	0.0145261	0.00065176
0.09	0.3	0.0175764	0.0193179	0.00174151
	0.6	0.0401747	0.0362484	0.00392631
	0.9	0.0225983	0.0208906	0.00170767

The numerical solutions which are obtained by using the sinc-Galerkin method (SGM) for this problem are presented in Tables 5 and 6. In addition, in Figures 6 and 7, the graphs of exact and approximate solutions for different values are presented.

Table 4: Numerical results for $P = 20$, $M_x = 40$, $N_x = 40$, $M_t = 40$, $N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0059288	0.0059287	$1.16151 * 10^{-7}$
	0.6	0.0135516	0.0135515	$1.41292 * 10^{-7}$
	0.9	0.0076227	0.0076227	$1.58382 * 10^{-8}$
0.06	0.3	0.0118050	0.0118048	$2.68162 * 10^{-7}$
	0.6	0.0269829	0.0269826	$3.57381 * 10^{-7}$
	0.9	0.0151779	0.0151778	$7.75388 * 10^{-8}$
0.09	0.3	0.0175764	0.0175761	$3.24333 * 10^{-7}$
	0.6	0.0401747	0.0401744	$3.45727 * 10^{-7}$
	0.9	0.0225983	0.0225983	$1.68470 * 10^{-8}$

Table 5: Numerical results for $P = 3$, $M_x = 5$, $N_x = 5$, $M_t = 5$, $N_t = 3$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0073366	0.0043780	0.00295855
	0.6	0.0111718	0.0092590	0.00191284
	0.9	0.0048740	0.0045355	0.00033852
0.06	0.3	0.0144134	0.0107858	0.00362752
	0.6	0.0219479	0.0228301	0.00088218
	0.9	0.0095754	0.0110942	0.00151873
0.09	0.3	0.0209795	0.0167664	0.00421307
	0.6	0.0319465	0.0354700	0.00352348
	0.9	0.0139377	0.0172001	0.00326239

5. Conclusion

In this study, we use the sinc-Galerkin method to obtain approximate solutions of boundary value problems for space-fractional partial differential equations with variable coefficients. In order to illustrate the efficiency and accuracy of the present method, the method is applied to three examples in the literature and the obtained results are compared with exact solutions. As a result, it is shown that sinc-Galerkin

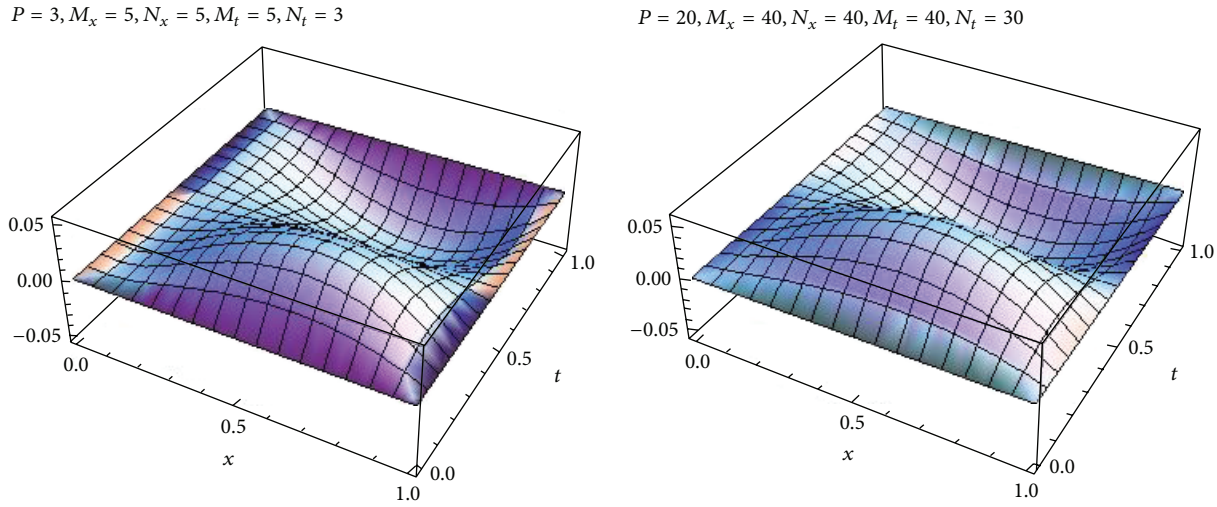


Figure 6: Graphs of approximate solutions for different values.

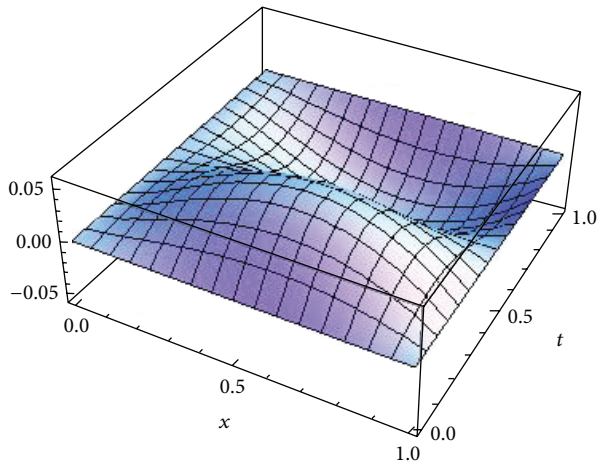


Figure 7: Graphs of exact solution.

Table 6: Numerical results for $P = 20, M_x = 40, N_x = 40, M_t = 40, N_t = 30$.

t	x	Exact sol.	Num. sol.	Error
0.03	0.3	0.0073366	0.0073365	$6.80615 * 10^{-8}$
	0.6	0.0111718	0.0111718	$2.45596 * 10^{-8}$
	0.9	0.0048740	0.0048740	$1.74662 * 10^{-8}$
0.06	0.3	0.0144134	0.0144139	$5.33686 * 10^{-7}$
	0.6	0.0219479	0.0219489	$9.67538 * 10^{-7}$
	0.9	0.0095754	0.0095759	$4.80610 * 10^{-7}$
0.09	0.3	0.0209795	0.0209786	$8.62732 * 10^{-7}$
	0.6	0.0319465	0.0319454	$1.08969 * 10^{-6}$
	0.9	0.0139377	0.0139373	$3.85036 * 10^{-7}$

method is very effective and accurate for obtaining approximate solutions of space-fractional differential equations with variable coefficients. In the future, we plan to extend the present numerical solution technique to nonlinear space-fractional partial differential equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] S. Momani, Z. Odibat, and V. S. Erturk, "Generalized differential transform method for solving a space- and time-fractional diffusion-wave equation," *Physics Letters A*, vol. 370, no. 5-6, pp. 379–387, 2007.

[2] Z. Odibat and S. Momani, "A generalized differential transform method for linear partial differential equations of fractional order," *Applied Mathematics Letters*, vol. 21, no. 2, pp. 194–199, 2008.

[3] Y. Zhang, "A finite difference method for fractional partial differential equation," *Applied Mathematics and Computation*, vol. 215, no. 2, pp. 524–529, 2009.

[4] Q. Wang, "Numerical solutions for fractional KdV-Burgers equation by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 182, no. 2, pp. 1048–1055, 2006.

[5] V. Daftardar-Gejji and S. Bhalekar, "Solving multi-term linear and non-linear diffusion-wave equations of fractional order by Adomian decomposition method," *Applied Mathematics and Computation*, vol. 202, no. 1, pp. 113–120, 2008.

[6] Q. Wang, "Homotopy perturbation method for fractional KdV equation," *Applied Mathematics and Computation*, vol. 190, no. 2, pp. 1795–1802, 2007.

[7] Q. Wang, "Homotopy perturbation method for fractional KdV-Burgers equation," *Chaos, Solitons and Fractals*, vol. 35, no. 5, pp. 843–850, 2008.

[8] O. Abdulaziz, I. Hashim, and E. S. Ismail, "Approximate analytical solution to fractional modified KdV equations," *Mathematical and Computer Modelling*, vol. 49, no. 1-2, pp. 136–145, 2009.

[9] M. U. Rehman and R. A. Khan, "Numerical solutions to initial and boundary value problems for linear fractional partial differential equations," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 5233–5244, 2013.

[10] M. A. Akinlar, A. Secer, and M. Bayram, "Numerical solution of fractional Benney equation," *Applied Mathematics and Information Sciences*, vol. 8, no. 4, pp. 1633–1637, 2014.

[11] A. Secer, M. A. Akinlar, and A. Cevikel, "Efficient solutions of systems of fractional PDEs by the differential transform method," *Advances in Difference Equations*, vol. 2012, article 188, 7 pages, 2012.

[12] M. Kurulay and M. Bayram, "Approximate analytical solution for the fractional modified KdV by differential transform method," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 7, pp. 1777–1782, 2010.

[13] M. Kurulay, M. A. Akinlar, and R. Ibragimov, "Computational solution of a fractional integro-differential equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 865952, 4 pages, 2013.

[14] M. A. Akinlar, A. Secer, and M. Bayram, "Stability, synchronization control and numerical solution of fractional Shimizu-Morioka dynamical system," *Applied Mathematics & Information Sciences*, vol. 8, no. 4, pp. 1699–1705, 2014.

[15] A. Atangana and A. Secer, "A note on fractional order derivatives and table of fractional derivatives of some special functions," *Abstract and Applied Analysis*, vol. 2013, Article ID 279681, 8 pages, 2013.

[16] A. Atangana and A. Secer, "The time-fractional coupled-Korteweg-de-Vries equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 947986, 8 pages, 2013.

[17] F. Stenger, "Approximations via Whittaker's cardinal function," *Journal of Approximation Theory*, vol. 17, no. 3, pp. 222–240, 1976.

[18] F. Stenger, "A sinc-Galerkin method of solution of boundary value problems," *Mathematics of Computation*, vol. 33, no. 145, pp. 85–109, 1979.

[19] E. T. Whittaker, "On the functions which are represented by the expansions of the interpolation theory," *Proceedings of the Royal Society of Edinburgh*, vol. 35, pp. 181–194, 1915.

[20] J. M. Whittaker, *Interpolation Function Theory*, vol. 33 of *Cambridge Tracts in Mathematics and Mathematical Physics*, Cambridge University Press, London, UK, 1935.

[21] M. El-Gamel and A. I. Zayed, "Sinc-Galerkin method for solving nonlinear boundary-value problems," *Computers & Mathematics with Applications*, vol. 48, no. 9, pp. 1285–1298, 2004.

[22] M. El-Gamel, S. H. Behiry, and H. Hashish, "Numerical method for the solution of special nonlinear fourth-order boundary value problems," *Applied Mathematics and Computation*, vol. 145, no. 2-3, pp. 717–734, 2003.

[23] M. Zarebnia and M. Sajjadian, "The sinc-Galerkin method for solving Troesch's problem," *Mathematical and Computer Modelling*, vol. 56, no. 9-10, pp. 218–228, 2012.

[24] A. Seçer, S. Alkan, and M. Bayram, "Sinc-galerkin method for approximate solutions of space fractional partial differential equations," in *Proceedings of the International Conference and Workshop on Mathematical Analysis (ICWOMA '14)*, p. 48, 2014.

[25] A. Mohsen and M. El-Gamel, "On the Galerkin and collocation methods for two-point boundary value problems using sinc bases," *Computers & Mathematics with Applications*, vol. 56, no. 4, pp. 930–941, 2008.

[26] M. El-Gamel and A. I. Zayed, "A comparison between the wavelet-GALerkin and the sinc-GALerkin methods in solving nonhomogeneous heat equations," in *Inverse Problem, Image Analysis, and Medical Imaging*, vol. 313, pp. 97–116, 2002.

[27] A. Secer and M. Kurulay, "The sinc-Galerkin method and its applications on singular Dirichlet-type boundary value problems," *Boundary Value Problems*, vol. 2012, article 126, 14 pages, 2012.

[28] A. Secer, S. Alkan, M. A. Akinlar, and M. Bayram, "Sinc-Galerkin method for approximate solutions of fractional order boundary value problems," *Boundary Value Problems*, vol. 2013, article 281, 2013.

[29] R. Almeida and D. F. M. Torres, "Necessary and sufficient conditions for the fractional calculus of variations with Caputo derivatives," *Communications in Nonlinear Science and Numerical Simulation*, vol. 16, no. 3, pp. 1490–1500, 2011.

[30] J. Lund and K. L. Bowers, *Sinc Methods for Quadrature and Differential Equations*, SIAM, Philadelphia, 1992.

[31] M. El-Gamel, "A numerical scheme for solving nonhomogeneous time-dependent problems," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 57, no. 3, pp. 369–383, 2006.

[32] M. ur Rehman and R. A. Khan, "Numerical solutions to initial and boundary value problems for linear fractional partial differential equations," *Applied Mathematical Modelling*, vol. 37, no. 7, pp. 5233–5244, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2015, Article ID 289387, 7 pages
http://dx.doi.org/10.1155/2015/289387

Research Article

On Generalized Fractional Kinetic Equations Involving Generalized Bessel Function of the First Kind

Dinesh Kumar,[1] S. D. Purohit,[2] A. Secer,[3] and A. Atangana[4]

[1] *Department of Mathematics & Statistics, J. N. Vyas University, Jodhpur 342001, India*
[2] *Department of Basic Sciences (Mathematics), College of Technology and Engineering, M. P. University of Agriculture and Technology, Udaipur, Rajasthan 313001, India*
[3] *Department of Mathematical Engineering, Yildiz Technical University, Davutpasa, 34210 Istanbul, Turkey*
[4] *Institute for Groundwater Studies, University of the Free State, Bloemfontein 9300, South Africa*

Correspondence should be addressed to A. Atangana; abdonatangana@yahoo.fr

Received 18 July 2014; Revised 1 September 2014; Accepted 2 September 2014

Academic Editor: Hossein Jafari

We develop a new and further generalized form of the fractional kinetic equation involving generalized Bessel function of the first kind. The manifold generality of the generalized Bessel function of the first kind is discussed in terms of the solution of the fractional kinetic equation in the paper. The results obtained here are quite general in nature and capable of yielding a very large number of known and (presumably) new results.

1. Introduction and Preliminaries

Bessel functions are playing the important role in studying solutions of differential equations, and they are associated with a wide range of problems in important areas of mathematical physics, like problems of acoustics, radiophysics, hydrodynamics, and atomic and nuclear physics. These considerations have led various workers in the field of special functions to explore the possible extensions and applications of the Bessel functions. Among many properties of Bessel functions, they also have investigated some possible extensions of the Bessel functions.

The generalized Bessel function of the first kind $\omega_p(z)$ is defined for $z \in \mathbb{C} \setminus \{0\}$ and $b, c, p \in \mathbb{C}(\Re(p) > -1)$ by the following series [1, page 10, (1.15)] (for recent work, see also [2–6]):

$$\omega_{p,b,c}(z) = \omega_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k c^k}{k!\Gamma(p + (b+1)/2 + k)} \left(\frac{z}{2}\right)^{2k+p}, \tag{1}$$

where $\mathbb{C}$ denotes the set of complex numbers and $\Gamma(a)$ is the familiar Gamma function.

The special cases of series (1) can be obtained as follows.

(i) If we put $b = c = 1$ in (1), then we obtain the familiar Bessel function of the first kind [7] of order p for $z, p \in \mathbb{C}$ with $\Re(p) > -1$ defined and represented by the following expressions (see also [1, 8]):

$$J_p(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}. \tag{2}$$

(ii) Putting $b = 1$ and $c = -1$ in series (1), we get the modified Bessel function of the first kind of order p defined by (see [1, 7])

$$I_p(z) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(p+k+1)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}; \tag{3}$$

the series given by (3) is also a special case of Galué's generalized modified Bessel function [9] depending on parameters $a = 0, 1, 2, \ldots$ and $p > -1$, given as follows:

$${}_aI_p(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+p}}{k!\Gamma(p+ak+1)}, \quad z \in \mathbb{C}. \tag{4}$$

(iii) Letting $b = 2$ and $c = 1$ in series (1), we have the spherical Bessel function of the first kind of order p defined by (see [1])

$$j_p(z) = \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(p+k+3/2)} \left(\frac{z}{2}\right)^{2k+p}, \quad z \in \mathbb{C}. \tag{5}$$

Furthermore, Deniz et al. [10] considered the function $\varphi_{p,b,c}(z)$, defined in terms of the generalized Bessel function $\omega_p(z)$, by the transformation

$$\begin{aligned} \varphi_{p,b,c}(z) &= 2^p \Gamma\left(p + \frac{b+1}{2}\right) z^{1-p/2} \omega_p\left(\sqrt{z}\right) \\ &= z + \sum_{k=1}^{\infty} \frac{(-c)^k}{4^k (\nu)_k} \frac{z^{k+1}}{k!}, \end{aligned} \tag{6}$$

where $\nu = p + (b+1)/2 \notin Z_0^- := \{0, -1, -2, \ldots\}$ and $(a)_k$ is the Pochhammer symbol defined (for $a \in \mathbb{C}$) by

$$\begin{aligned} (a)_k &:= \begin{cases} 1 & (k = 0) \\ a(a+1)\cdots(a+k-1) & (k \in N := \{1, 2, 3, \ldots\}) \end{cases} \\ &= \frac{\Gamma(a+k)}{\Gamma(a)} \quad \left(a \in \mathbb{C} \setminus Z_0^-\right). \end{aligned} \tag{7}$$

Fractional differential equations appear more and more frequently for modeling of relevant systems in several fields of applied sciences. These equations play important roles, not only in mathematics, but also in physics, dynamical systems, control systems, and engineering, to create the mathematical model of many physical phenomena. In particular, the kinetic equations describe the continuity of motion of substance and are the basic equations of mathematical physics and natural science. Therefore, in literature we found several papers that analyze extensions and generalizations of these equations involving various fractional calculus operators. One may, for instance, refer to such type of works by [11–23].

Haubold and Mathai [13] have established a functional differential equation between rate of change of reaction, the destruction rate, and the production rate as follows:

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{8}$$

where $N = N(t)$ is the rate of reaction, $d = d(N)$ is the rate of destruction, $p = p(N)$ is the rate of production, and N_t denotes the function defined by $N_t(t^*) = N(t - t^*)$, $t^* > 0$.

Haubold and Mathai studied a special case of (8), when spatial fluctuations or inhomogeneities in the quantity $N(t)$ are neglected, is given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t), \tag{9}$$

together with the initial condition that $N_i(t = 0) = N_0$, is the number of density of species i at time $t = 0$, $c_i > 0$. If we decline the index i and integrate the standard kinetic equation (9), we have

$$N(t) - N_0 = -c\, {}_0D_t^{-1} N(t), \tag{10}$$

where ${}_0D_t^{-1}$ is the special case of the Riemann-Liouville integral operator ${}_0D_t^{-\nu}$ defined as

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s)\, ds, \quad t > 0,\ \Re(\nu) > 0. \tag{11}$$

Haubold and Mathai [13] have given the fractional generalization of the standard kinetic equation (10) as

$$N(t) - N_0 = -c^{\nu}\, {}_0D_t^{-\nu} N(t) \tag{12}$$

and have provided the solution of (12) as follows:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}. \tag{13}$$

Further, Saxena and Kalla [17] considered the following fractional kinetic equation:

$$N(t) - N_0 f(t) = -c^{\nu} \left({}_0D_t^{-\nu} N\right)(t), \quad (\Re(\nu) > 0), \tag{14}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant, and $f \in L(0, \infty)$.

By applying the Laplace transform to (14), we have

$$\begin{aligned} &L[N(t)](p) \\ &\quad = N_0 \frac{F(p)}{1 + c^{\nu} p^{-\nu}} \\ &\quad = N_0 \left(\sum_{n=0}^{\infty} (-c^{\nu})^n p^{-n\nu}\right) F(p) \quad \left(n \in N_0, \left|\frac{c}{p}\right| < 1\right), \end{aligned} \tag{15}$$

where the Laplace transform [24] is defined by

$$F(p) = L[f(t)] = \int_0^{\infty} e^{-pt} f(t)\, dt, \quad \Re(p) > 0. \tag{16}$$

The aim of this paper is to develop a new and further generalized form of the fractional kinetic equation involving generalized Bessel function of the first kind. The manifold generality of the generalized Bessel function of the first kind is discussed in terms of the solution of the above fractional kinetic equation. Moreover, the results obtained here are quite capable of yielding a very large number of known and (presumably) new results.

2. Solution of Generalized Fractional Kinetic Equations

In this section, we will investigate the solution of the generalized fractional kinetic equations. The results are as follows.

Theorem 1. *If $d > 0$, $\nu > 0$, $c, b, l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 \omega_{l,b,c}(t) = -d^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{17}$$

there holds the formula:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{t}{2}\right)^{2k+l} \times E_{\nu,2k+l+1}\left(-d^{\nu}t^{\nu}\right), \tag{18}$$

where $E_{\nu,2k+l+1}(\cdot)$ is the generalized Mittag-Leffler function [25].

Proof. The Laplace transform of the Riemann-Liouville fractional integral operator is given by [26, 27]

$$L\left\{ {}_0D_t^{-\nu} f(t); p\right\} = p^{-\nu} F(p), \tag{19}$$

where $F(p)$ is defined in (16). Now, applying the Laplace transform to both sides of (17), we get

$$\begin{aligned}
&L\left[N(t); p\right] \\
&\quad = N_0 L\left[\omega_{l,b,c}(t); p\right] - d^{\nu} L\left[{}_0D_t^{-\nu} N(t); p\right], \\
&N(p) \\
&\quad = N_0 \int_0^{\infty} e^{-pt} \sum_{k=0}^{\infty} \frac{(-c)^k}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{t}{2}\right)^{2k+l} \\
&\qquad - d^{\nu} p^{-\nu} N(p), \\
&N(p)\left[1 + d^{\nu} p^{-\nu}\right] \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)}}{k!\Gamma(l+k+(b+1)/2)} \int_0^{\infty} e^{-pt} t^{2k+l} dt \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)}}{k!\Gamma(l+k+(b+1)/2)} \frac{\Gamma(2k+l+1)}{p^{2k+l+1}}, \\
&N(p) \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)} \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \\
&\qquad \times \left\{ p^{-(2k+l+1)} \sum_{r=0}^{\infty} \frac{(1)_r \left[-(p/d)^{-\nu}\right]^r}{(r)!} \right\}.
\end{aligned} \tag{20}$$

Taking Laplace inverse of (20) and using $L^{-1}\{p^{-\nu}\} = t^{\nu-1}/\Gamma(\nu)$, $\Re(\nu) > 0$, we have

$$\begin{aligned}
&L^{-1}\left\{N(p)\right\} \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)} \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \\
&\qquad \times L^{-1}\left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} p^{-(2k+l+\nu r+1)} \right\}, \\
&N(t) \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)} \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \\
&\qquad \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{(2k+l+\nu r)}}{\Gamma(\nu r+2k+l+1)} \right\} \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k 2^{-(2k+l)} \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} t^{2k+l} \\
&\qquad \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{\nu r}}{\Gamma(\nu r+2k+l+1)} \right\} \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{t}{2}\right)^{2k+l} \\
&\qquad \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{\nu r}}{\Gamma(\nu r+2k+l+1)} \right\}, \\
&N(t) \\
&\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k+l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{t}{2}\right)^{2k+l} \\
&\qquad \times E_{\nu,2k+l+1}\left(-d^{\nu}t^{\nu}\right).
\end{aligned} \tag{21}$$

This completes the proof of Theorem 1. □

If we set $b = c = 1$ in (17), then generalized Bessel function $\omega_{l,b,c}(z)$ reduces to Bessel function of the first kind $J_l(z)$ given by (2), and we arrive at the following result.

Corollary 2. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 J_l(t) = -d^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{22}$$

there holds the formula:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+l+1)}{k!\Gamma(l+k+1)} \left(\frac{t}{2}\right)^{2k+l} \times E_{\nu,2k+l+1}\left(-d^{\nu}t^{\nu}\right). \tag{23}$$

Further, taking $b = 1$ and $c = -1$ in (17), then we obtain result of generalized fractional kinetic equation having modified Bessel function of the first kind.

Corollary 3. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 I_l(t) = -d^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{24}$$

there holds the formula:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{\Gamma(2k+l+1)}{k!\Gamma(l+k+1)} \left(\frac{t}{2}\right)^{2k+l} \times E_{\nu,2k+l+1}\left(-d^{\nu}t^{\nu}\right). \tag{25}$$

Letting $b = 2$ and $c = 1$ in (17), then generalized Bessel function $\omega_{l,b,c}(z)$ reduces to the spherical Bessel function of the first kind $j_l(z)$ given by (5), and we obtain the following interesting result.

Corollary 4. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 j_l(t) = -d^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{26}$$

there holds the solution of (22)

$$N(t) = N_0 \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(2k+l+1)}{k!\Gamma(l+k+3/2)} \left(\frac{t}{2}\right)^{2k+l} \times E_{\nu,2k+l+1}\left(-d^{\nu}t^{\nu}\right). \tag{27}$$

Theorem 5. *If $d > 0$, $\nu > 0$, $c, b, l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 \omega_{l,b,c}\left(d^{\nu}t^{\nu}\right) = -d^{\nu}\, {}_0D_t^{-\nu} N(t), \tag{28}$$

there holds the formula:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-d^{\nu}t^{\nu}\right), \tag{29}$$

where $E_{\nu,2k\nu+\nu l+1}(\cdot)$ is the generalized Mittag-Leffler function.

Proof. The Laplace transform of the Riemann-Liouville fractional integral operator is given by [26]

$$L\left\{{}_0D_t^{-\nu} f(t); p\right\} = p^{-\nu} F(p), \tag{30}$$

where $F(p)$ is defined in (16). Now, applying the Laplace transform to both sides of (28), we get

$$\begin{aligned} &L\left[N(t); p\right] \\ &\quad = N_0 L\left[\omega_{l,b,c}\left(d^{\nu}t^{\nu}\right); p\right] - d^{\nu} L\left[{}_0D_t^{-\nu} N(t); p\right], \end{aligned} \tag{31}$$

$$\begin{aligned} &N(p) \\ &\quad = N_0 \int_0^{\infty} e^{-pt} \sum_{k=0}^{\infty} \frac{(-c)^k}{k!\Gamma(l+k+(b+1)/2)} \\ &\qquad \times \left(\frac{d^{\nu}t^{\nu}}{2}\right)^{2k+l} - d^{\nu} p^{-\nu} N(p), \\ &N(p)\left[1 + d^{\nu}p^{-\nu}\right] \\ &\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^{\nu}/2\right)^{2k+l}}{k!\Gamma(l+k+(b+1)/2)} \int_0^{\infty} e^{-pt} t^{2k\nu+\nu l+1-1} dt \\ &\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^{\nu}/2\right)^{2k+l}}{k!\Gamma(l+k+(b+1)/2)} \frac{\Gamma(2k\nu+\nu l+1)}{p^{2k\nu+\nu l+1}}, \\ &N(p) \\ &\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^{\nu}/2\right)^{2k+l} \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \\ &\qquad \times \left\{ p^{-(2k\nu+\nu l+1)} \sum_{r=0}^{\infty} \frac{(1)_r \left[-\left(p/d\right)^{-\nu}\right]^r}{(r)!} \right\}. \end{aligned} \tag{32}$$

Taking Laplace inverse of (32) and using $L^{-1}\{p^{-\nu}\} = t^{\nu-1}/\Gamma(\nu)$, $\Re(\nu) > 0$, we have

$$\begin{aligned} &L^{-1}\left\{N(p)\right\} \\ &\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^{\nu}/2\right)^{2k+l} \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \\ &\qquad \times L^{-1}\left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} p^{-(2k\nu+\nu l+\nu r+1)} \right\}, \\ &N(t) \\ &\quad = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^{\nu}/2\right)^{2k+l} \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \\ &\qquad \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{\nu(2k+l+r)}}{\Gamma(\nu r+2k\nu+\nu l+1)} \right\} \end{aligned}$$

$$= N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^\nu/2\right)^{2k+l} \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + (b+1)/2\right)} t^{\nu(2k+l)} \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{\nu r}}{\Gamma\left(\nu r + 2k\nu + \nu l + 1\right)} \right\}$$

$$= N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + (b+1)/2\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times \left\{ \sum_{r=0}^{\infty} (-1)^r d^{\nu r} \frac{t^{\nu r}}{\Gamma\left(\nu\left(r + 2k + l\right) + 1\right)} \right\},$$

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + (b+1)/2\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-d^\nu t^\nu\right). \tag{33}$$

This completes the proof of Theorem 5. □

If we set $b = c = 1$ in Theorem 5, then generalized Bessel function $\omega_{l,b,c}(z)$ reduces to Bessel function of the first kind $J_l(z)$, and we arrive at the special case of (28).

Corollary 6. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 J_l\left(d^\nu t^\nu\right) = -d^\nu \, {}_0D_t^{-\nu} N(t), \tag{34}$$

the following result holds:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + 1\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-d^\nu t^\nu\right). \tag{35}$$

On taking $b = 1$ and $c = -1$ in (28), then generalized Bessel function $\omega_{l,b,c}(z)$ reduces to Bessel function of the first kind $J_l(z)$, and we get the following result.

Corollary 7. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 I_l\left(d^\nu t^\nu\right) = -d^\nu \, {}_0D_t^{-\nu} N(t), \tag{36}$$

the following result holds:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{\Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + 1\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-d^\nu t^\nu\right). \tag{37}$$

Further, if we put $b = 2$ and $c = 1$ in (28), then we arrive at the following interesting result.

Corollary 8. *If $d > 0$, $\nu > 0$, $l, t \in \mathbb{C}$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 j_l\left(d^\nu t^\nu\right) = -d^\nu \, {}_0D_t^{-\nu} N(t), \tag{38}$$

the following result holds:

$$N(t) = N_0 \sqrt{\frac{\pi}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + 3/2\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-d^\nu t^\nu\right), \tag{39}$$

where $j_l(z)$ is the spherical Bessel function of the first kind.

Theorem 9. *If $a > 0$, $d > 0$, $\nu > 0$, $c, b, l, t \in \mathbb{C}$, $a \neq d$, and $\Re(l) > -1$, then for the solution of the equation*

$$N(t) - N_0 \omega_{l,b,c}\left(d^\nu t^\nu\right) = -a^\nu \, {}_0D_t^{-\nu} N(t), \tag{40}$$

there holds the formula:

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + (b+1)/2\right)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-a^\nu t^\nu\right). \tag{41}$$

Proof. Applying the Laplace transform to both sides of (40), we get

$$L\left[N(t); p\right] = N_0 L\left[\omega_{l,b,c}\left(d^\nu t^\nu\right); p\right] - a^\nu L\left[{}_0D_t^{-\nu} N(t); p\right],$$

$$N(p)\left[1 + a^\nu p^{-\nu}\right] = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^\nu/2\right)^{2k+l}}{k!\Gamma\left(l + k + (b+1)/2\right)} \frac{\Gamma\left(2k\nu + \nu l + 1\right)}{p^{2k\nu+\nu l+1}},$$

$$N(p) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^\nu/2\right)^{2k+l} \Gamma\left(2k\nu + \nu l + 1\right)}{k!\Gamma\left(l + k + (b+1)/2\right)} \times \left\{ p^{-(2k\nu+\nu l+1)} \sum_{r=0}^{\infty} \frac{(1)_r (-1)^r p^{-\nu r} a^{\nu r}}{(r)!} \right\}. \tag{42}$$

Taking Laplace inverse of (42), we arrive at

$$L^{-1}\{N(p)\} = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \left(d^\nu/2\right)^{2k+l} \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \times L^{-1}\left\{\sum_{r=0}^{\infty}(-1)^r a^{\nu r} p^{-(2k\nu+\nu l+\nu r+1)}\right\},$$

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times \left\{\sum_{r=0}^{\infty}(-1)^r a^{\nu r} \frac{t^{\nu r}}{\Gamma(\nu(r+2k+l)+1)}\right\}, \quad (43)$$

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-c)^k \Gamma(2k\nu+\nu l+1)}{k!\Gamma(l+k+(b+1)/2)} \left(\frac{d^\nu t^\nu}{2}\right)^{2k+l} \times E_{\nu,(2k+l)\nu+1}\left(-a^\nu t^\nu\right).$$

This completes the proof of Theorem 9. □

Remark 10. The special cases for Theorem 9 can be developed on similar lines to that of Corollaries 6–8, but we do not state here due to lack of space.

3. Conclusion

In this paper we have studied a new fractional generalization of the standard kinetic equation and derived solutions for it. It is not difficult to obtain several further analogous fractional kinetic equations and their solutions as those exhibited here by main results. Moreover, by the use of close relationships of the generalized Bessel function of the first kind $\omega_p(z)$ with many special functions, we can easily construct various known and new fractional kinetic equations.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors are thankful to the referee for the very careful reading and the valuable suggestions.

References

[1] Á. Baricz, *Generalized Bessel Functions of the First Kind*, vol. 1994 of *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2010.

[2] Á. Baricz, "Geometric properties of generalized Bessel functions," *Publicationes Mathematicae Debrecen*, vol. 73, no. 1-2, pp. 155–178, 2008.

[3] J. Choi, P. Agarwal, S. Mathur, and S. D. Purohit, "Certain new integral formulas involving the generalized Bessel functions," *Bulletin of the Korean Mathematical Society*, vol. 51, no. 4, Article ID 9951003, 2014.

[4] P. Malik, S. R. Mondal, and A. Swaminathan, "Fractional integration of generalized bessel function of the first kind," in *Proceedings of the ASME International Design Engineering Technical Conferences and Computers and Information in Engineering Conference (IDETC/CIE '11)*, pp. 409–418, Washington, DC, USA, August 2011.

[5] S. R. Mondal and K. S. Nisar, "Marichev-Saigo-Maeda fractional integration operators involving generalized Bessel functions," *Mathematical Problems in Engineering*, vol. 2014, Article ID 274093, 11 pages, 2014.

[6] D. Baleanu, P. Agarwal, and S. D. Purohit, "Certain fractional integral formulas involving the product of generalized Bessel functions," *The Scientific World Journal*, vol. 2013, Article ID 567132, 9 pages, 2013.

[7] G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press; The Macmillan, Cambridge, UK, 1944.

[8] J. K. Prajapat, "Certain geometric properties of normalized Bessel functions," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2133–2139, 2011.

[9] L. Galué, "A generalized Bessel function," *Integral Transforms and Special Functions*, vol. 14, no. 5, pp. 395–401, 2003.

[10] E. Deniz, H. Orhan, and H. M. Srivastava, "Some sufficient conditions for univalence of certain families of integral operators involving generalized Bessel functions," *Taiwanese Journal of Mathematics*, vol. 15, no. 2, pp. 883–917, 2011.

[11] G. M. Zaslavsky, "Fractional kinetic equation for Hamiltonian chaos," *Physica D*, vol. 76, no. 1–3, pp. 110–122, 1994.

[12] A. I. Saichev and G. M. Zaslavsky, "Fractional kinetic equations: solutions and applications," *Chaos*, vol. 7, no. 4, pp. 753–764, 1997.

[13] H. J. Haubold and A. M. Mathai, "The fractional kinetic equation and thermonuclear functions," *Astrophysics and Space Science*, vol. 273, no. 1–4, pp. 53–63, 2000.

[14] R. K. Saxena, A. M. Mathai, and H. J. Haubold, "On fractional kinetic equations," *Astrophysics and Space Science*, vol. 282, no. 1, pp. 281–287, 2002.

[15] R. K. Saxena, A. M. Mathai, and H. J. Haubold, "On generalized fractional kinetic equations," *Physica A*, vol. 344, no. 3-4, pp. 657–664, 2004.

[16] R. K. Saxena, A. M. Mathai, and H. J. Haubold, "Solution of generalized fractional reaction-diffusion equations," *Astrophysics and Space Science*, vol. 305, no. 3, pp. 305–313, 2006.

[17] R. K. Saxena and S. L. Kalla, "On the solutions of certain fractional kinetic equations," *Applied Mathematics and Computation*, vol. 199, no. 2, pp. 504–511, 2008.

[18] V. B. L. Chaurasia and S. C. Pandey, "On the new computable solution of the generalized fractional kinetic equations involving the generalized function for the fractional calculus and

related functions," *Astrophysics and Space Science*, vol. 317, no. 3-4, pp. 213–219, 2008.

[19] V. G. Gupta, B. Sharma, and F. B. M. Belgacem, "On the solutions of generalized fractional kinetic equations," *Applied Mathematical Sciences*, vol. 5, no. 19, pp. 899–910, 2011.

[20] A. Chouhan and S. Sarswat, "On solution of generalized Kinetic equation of fractional order," *International journal of Mathematical Sciences and Applications*, vol. 2, no. 2, pp. 813–818, 2012.

[21] A. Chouhan, S. D. Purohit, and S. Saraswat, "An alternative method for solving generalized differential equations of fractional order," *Kragujevac Journal of Mathematics*, vol. 37, no. 2, pp. 299–306, 2013.

[22] A. Gupta and C. L. Parihar, "On solutions of generalized kinetic equations of fractional order," *Boletim da Sociedade Paranaense de Matemática*, vol. 32, no. 1, pp. 181–189, 2014.

[23] R. K. Saxena and D. Kumar, "Solution of fractional kinetic equation associated with aleph function," Submitted.

[24] M. R. Spiegel, *Theory and Problems of Laplace Transforms*, Schaums Outline Series, McGraw-Hill, New York, NY, USA, 1965.

[25] G. Mittag-Leffler, "Sur la représentation analytique d'une branche uniforme d'une fonction monogène," *Acta Mathematica*, vol. 29, no. 1, pp. 101–181, 1905.

[26] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Tables of Integral Transforms*, vol. 1, McGraw-Hill, New York, NY, USA, 1954.

[27] H. M. Srivastava and R. K. Saxena, "Operators of fractional integration and their applications," *Applied Mathematics and Computation*, vol. 118, no. 1, pp. 1–52, 2001.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 146521, 14 pages
http://dx.doi.org/10.1155/2014/146521

Research Article

Newtonian and Non-Newtonian Fluids through Permeable Boundaries

Riëtte Maritz and Emile Franc Doungmo Goufo

Department of Mathematical Sciences, University of South Africa, P.O. Box 392, Pretoria 0003, South Africa

Correspondence should be addressed to Emile Franc Doungmo Goufo; dgoufef@unisa.ac.za

Received 20 August 2014; Accepted 17 September 2014; Published 1 October 2014

Academic Editor: Abdon Atangana

We considered the situation where a container with a permeable boundary is immersed in a larger body of fluid of the same kind. In this paper, we found mathematical expressions at the permeable interface Γ of a domain Ω, where $\Omega \subset \mathfrak{R}^3$. Γ is defined as a smooth two-dimensional (at least class C^2) manifold in Ω. The Sennet-Frenet formulas for curves without torsion were employed to find the expressions on the interface Γ. We modelled the flow of Newtonian as well as non-Newtonian fluids through permeable boundaries which results in nonhomogeneous dynamic and kinematic boundary conditions. The flow is assumed to flow through the boundary only in the direction of the outer normal $\mathbf{n}$, where the tangential components are assumed to be zero. These conditions take into account certain assumptions made on the curvature of the boundary regarding the surface density and the shape of Ω; namely, that the curvature is constrained in a certain way. Stability of the rest state and uniqueness are proved for a special case where a "shear flow" is assumed.

1. Introduction

The flow of incompressible Navier-Stokes fluids and fluids of second grade through permeable boundaries and past porous walls has been studied under various conditions. The equation of motion for incompressible flows in Newtonian fluids (Navier-Stokes equations) under no-slip boundary conditions has been studied extensively from many perspectives. Since the pioneering papers of Leray [1–3] and Hopf [4] questions of the existence, stability [5, 6], and uniqueness of both classical and weak solutions have received more than their fair share of attention.

Recently the same issues have been studied for non-Newtonian fluids of second grade. The studies cover both weak solutions [7–12] and classical solutions for homogeneous Dirichlet boundary data [13] and nonhomogeneous boundary data [6, 14, 15].

Unlike Newtonian fluids, fluids of second grade (and other non-Newtonian species) have the property of developing "normal stresses differences" at boundaries. It was shown, for example, by Berker [16] that if an incompressible flow of a fluid of grade two satisfies the homogeneous Dirichlet boundary condition. The stress at the boundary is given by $\mathbf{t} = (-p + \alpha|\boldsymbol{\omega}|^2)\mathbf{n} + [\mu\boldsymbol{\omega} + 2\alpha\partial_t\boldsymbol{\omega}] \wedge \mathbf{n}$, where $\mathbf{n}$ is the unit exterior normal to the boundary and $\boldsymbol{\omega} = \nabla\wedge\mathbf{v}$ is the vorticity. The wedge denotes a vector product. Thus there is a normal component of stress at the boundary in addition to the pressure. The question becomes *what governs the flow across the boundary*? Possible ways of circumventing this question may be to "prescribe" the normal component of the velocity field at the boundary or to prescribe mass or momentum flux. The prescription of shear stress has also been suggested. ([16, 17]). *Nonlinear* or *non-Newtonian* fluids are fluids like molten metals, multigrade oils, printing inks, paints, suspensions, polymer solutions, molten plastics, blood, protein solutions, and ice [18]. These fluids cannot be described by the above model. The study of these interesting substances has proved to be very important with the growth of the polymer and plastics industry over the last four decades. Consequently, an interest has arisen to study the flow of these nonlinear fluids and, in the case of this model, second-grade fluids, through permeable boundaries. The boundary conditions alone in such circumstances are an interesting topic for study. Works by Berker [16] and Rajagopal and Gupta [19] can be mentioned in this regard.

In this study we shall provide an alternative approach through the formulation of "dynamics at the boundary," the idea being that the normal component of velocity at the boundary is viewed as an unknown function which satisfies a differential equation intricately coupled to the flow in the region "enclosed" by the boundary.

A glimpse of the history of the research on non-Newtonian and Newtonian fluids around porous boundaries is given in Section 2. Notation and definitions precede Sections 4 and 5 which deal with the constitutive equations and the modelling of permeability. In Section 6 the expressions on the interface Γ are given. The alternative model is studied and the stability and uniqueness are proved in Section 7. Section 8 concludes the study and further explorations are discussed.

2. Backgroud

Berker [16] studied the two-dimensional creeping flow of a second-order fluid with nonparallel porous walls. An additional velocity boundary condition was needed. The other conditions they used were due to the usual no-slip conditions. This additional velocity boundary condition was to prescribe the rate of shear at the wall. The problem was then solved numerically by a standard routine.

In 1989 Rajagopal and Kaloni [20] wrote remarks on boundary conditions for flows of fluids of the differential type. Rajagopal [21] discusses a lot of related issues. Rajagopal and Gupta [19] studied the flow of an incompressible fluid of second grade past an infinite porous plate subjected to either suction or blowing at the plate. They studied fluids modelled by

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A} + \alpha_1 \left[D_t\mathbf{A} + \mathbf{A}\left(\nabla\mathbf{v}\right) + \left(\nabla\mathbf{v}\right)^T\mathbf{A} \right] + \alpha_2\mathbf{A}^2. \quad (1)$$

No assumptions were made about the material moduli α_1 and α_2. For the boundary value problem they considered, it was found that the velocity distributions do not depend on the normal stress modulus α_2, but the pressure does. They found that it was possible to produce an exact solution which is asymptotic in nature for both "suction" and "blowing" at the plate if the material modulus $\alpha_1 > 0$. For $\alpha_1 < 0$, they found that such solutions could not exist in the case of blowing, a result which was in keeping with the classical incompressible fluid. Fosdick and Rajagopal [22] have shown that the model (1) whose material modulus $\alpha_1 < 0$ exhibits anomalous behaviour was not to be expected of any fluid of rheological interest (also see [23]). Proudman studied an example of steady laminar flow at a large Reynolds number [24].

Beavers and Joseph [25] studied the flow of a Newtonian fluid over a porous surface in 1967. They found that if the governing differential system was not to be underdetermined, it was necessary to specify some condition on the tangential component of the velocity of the free fluid at the porous interface. It is usual in these analyses to approximate the fluid motion near the true boundary with an adherence condition for the tangential component of velocity of the free fluid at some boundary. Because of a certain ambiguity which is implied by the notion of a "true" boundary for a permeable material, it was found useful to define a nominal boundary. They fixed a nominal boundary by first defining a smooth geometric surface and then assuming that the outermost perimeters of all the surface pores of the permeable material are in this surface. Thus, if the surface pores were filled with solid material to the level of their respective perimeters, a smooth impermeable boundary of the assumed shape would result. This definition is precise when the geometry is simple (planes, spheres, cylinders, etc.) but may not be fully adequate in more complex situations. Beavers and Joseph's [25] experiment was designed to examine the tangential flow in the boundary region of a permeable interface. The results of this experiment indicate that the effects of viscous shear appear to penetrate into the permeable material in a boundary layer region, producing a velocity distribution similar to that depicted in the following figure. The tangential component of the velocity of the free fluid at the porous boundary can be considerably greater than the mean filter velocity within the body of the porous material.

In Figure 1 the plane $y = 0$ defines a nominal surface for the permeable material. The flow through the body of the permeable material, which is homogeneous and isotropic, is assumed to be governed by Darcy's Law. Read more of the status on Darcy's Law in [26]. In the absence of body forces Darcy's Law may be written as $Q = -(k/\mu)(dP/dx)$, where k is the "permeability" of the material and Q is the volume flow rate per unit of the cross-sectional area. As such, Q represents the filter velocity rather than the true velocity of the fluid in the pores. The measured pressure gradient is denoted by dP/dx.

3. Basic Notation

We work in Euclidean 3 space. The following notation will be used throughout:

$$|\mathbf{x}| := \sqrt{\sum_1^3 x_i^2} \text{ denotes the Euclidean norm.}$$

$$\partial_i := \frac{\partial}{\partial x_i}; \quad i = 1, 2, 3.$$

$$\partial_t := \frac{\partial}{\partial t}.$$

$$\left[\nabla p\right]_i := \partial_i p \quad \text{if } p \text{ is a scalar field.}$$

$$\left[\nabla \mathbf{v}\right]_{ij} := \partial_j v_i; \quad i, j = 1, 2, 3, \text{ if } \mathbf{v} \text{ is a vector field.}$$

$$\left[\nabla \mathbf{v}\right]_{ij}^T := \partial_i v_j; \quad i, j = 1, 2, 3, \text{ if } \mathbf{v} \text{ is a vector field.}$$

$$\nabla \cdot \mathbf{v} := \sum_{i=1}^3 \partial_i v_i \quad \text{if } \mathbf{v} \text{ is a vector field.}$$

$$\mathbf{v} \cdot \nabla := \sum_{i=1}^3 v_i \partial_i \quad \text{if } \mathbf{v} \text{ is a vector field.}$$

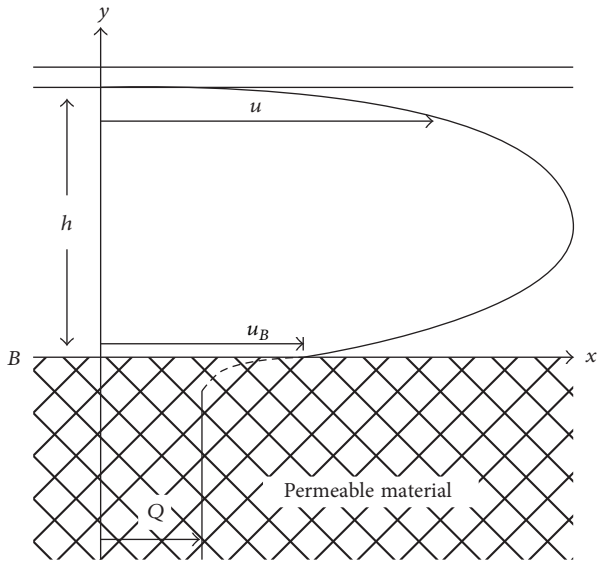


Figure 1: Velocity profile for the rectilinear flow in a horizontal channel formed by a permeable lower wall ($y = 0$) and an impermeable upper wall ($y = h$).

$$
\begin{aligned}
&[\nabla \cdot \mathbf{T}]_j := \sum_{i=1}^{3} \partial_i T_{ij}; \quad j = 1,2,3,\\
&\qquad \text{if } \mathbf{T} \text{ is a matrix (tensor) with}\\
&\qquad \text{Euclidean components } T_{ij}.\\
&[\mathbf{v} \otimes \mathbf{v}]_{ij} := v_i v_j; \quad i,j = 1,2,3, \text{ if } \mathbf{v} \text{ is a vector.}\\
&D_t := \partial_t + \mathbf{v} \cdot \nabla; \quad D_t \text{ is the material time derivative.}\\
&\mathbf{v} \wedge \mathbf{u} := \text{denotes the usual vector}\\
&\qquad \text{product of the vectors } \mathbf{v} \text{ and } \mathbf{u}\\
&\nabla \wedge \mathbf{v} := \text{curl } \mathbf{v}.
\end{aligned}
\tag{2}
$$

If $\mathbf{A}$ and $\mathbf{B}$ are second order tensors we shall use the notations $\mathbf{A} : \mathbf{B} = \sum_{i,j=1}^{3} A_{ij}B_{ij}$ and $|\mathbf{A}|^2 = \mathbf{A} : \mathbf{A}$. Let $\Omega \subset \mathbf{R}^3$ be a bounded domain with a smooth (at least $\mathbf{C}^2$) boundary Γ. Let $\mathbf{n} = \mathbf{n}(x)$ denote the unit exterior normal to Γ at x. We shall be concerned with smooth vector fields $\mathbf{v} = \mathbf{v}(x)$ defined in Ω such that on Γ it has the form $\gamma_o \mathbf{v}(x) = -\eta(x)\mathbf{n}(x)$, where γ_o is the trace operator denoting boundary values and η is a smooth scalar field defined on Γ. Associated with $\nabla\mathbf{v}$ we define the symmetric and skew-symmetric tensors $\mathbf{A}$ and $\mathbf{W}$ as $\mathbf{A} = \mathbf{A}(\mathbf{v}) = \nabla\mathbf{v} + (\nabla\mathbf{v})^T$ and $\mathbf{W} = \mathbf{W}(\mathbf{v}) = \nabla\mathbf{v} - (\nabla\mathbf{v})^T$, where $(\nabla\mathbf{v})^T$ denotes the transpose of the gradient of $\mathbf{v}$. The rate of deformation tensor is related to $\mathbf{A}$ by $\mathbf{D}(\mathbf{v}) = (1/2)\mathbf{A}(\mathbf{v})$. We note that if $\mathbf{v}$ is solenoidal ($\nabla \cdot \mathbf{v} = 0$) then trace $\mathbf{A}(\mathbf{v}) = 2\nabla \cdot \mathbf{v} = 0$ and, for any vector $\mathbf{a}$, $\mathbf{W}(\mathbf{v})\mathbf{a} = \boldsymbol{\omega} \wedge \mathbf{a}$, where $\boldsymbol{\omega} = \nabla \wedge \mathbf{v}$ denotes the vorticity associated with $\mathbf{v}$.

4. The Constitutive Equations

The stress tensor for the linear viscous Newtonian model is $\mathbf{T} = -p\mathbf{I} + \mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$, with p as the pressure, μ as the coefficient of viscosity, and $\mathbf{v}$ as the velocity of the fluid. This model describes the flow of fluids like water and other similar fluids. Lamb [27] and Ladyzhenskaya [28] wrote mathematical theories on viscous incompressible flow.

Fluids of a *differential type* [29–31], of which Rivlin-Ericksen fluids are a subclass, are depicted by a popular nonlinear model. Fluids of *complexity n* form an important subclass of the fluids of a differential type. For incompressible fluids of complexity n the Cauchy stress tensor is of the form $\mathbf{T} = -p\mathbf{I} + \mathbf{F}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$. The pressure p is not a thermodynamic variable and the term $-p\mathbf{I}$ reflects Pascal's law, which is inherent to all fluids. $\mathbf{A}_1, \ldots, \mathbf{A}_n$ are the first n Rivlin-Ericksen tensors [21] defined recursively by

$$
\begin{aligned}
&\mathbf{A}_1 = \nabla\mathbf{v} + (\nabla\mathbf{v})^T = \mathbf{A},\\
&\mathbf{A}_n = D_t\mathbf{A}_{n-1} + \mathbf{A}_{n-1}(\nabla\mathbf{v}) + (\nabla\mathbf{v})^T\mathbf{A}_{n-1}, \quad n \geq 2.
\end{aligned}
\tag{3}
$$

Fluids of *grade n* are examples of fluids of complexity n. The stress tensors for fluids of grades 1 and 2 respectively, are assumed to be of the form

$$
\begin{aligned}
\mathbf{T}^{[1]} &= -p\mathbf{I} + \mu\mathbf{A}_1,\\
\mathbf{T}^{[2]} &= \mathbf{T}^{[1]} + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2,
\end{aligned}
\tag{4}
$$

where μ and α_i are material coefficients (possibly temperature-dependent).

For incompressible fluids of second grade, the stress-deformation relation then becomes

$$
\begin{aligned}
\mathbf{T} = \mathbf{T}^{[2]} &= -p\mathbf{I} + \mu\mathbf{A} + \alpha_1 D_t\mathbf{A}\\
&\quad + \alpha_1\left(\mathbf{A}\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{A}\right) + \alpha_2\mathbf{A}^2,
\end{aligned}
\tag{5}
$$

where p and $\mathbf{v}$ are the pressure and the velocity fields. Here μ is the coefficient of viscosity and α_1 and α_2 are material coefficients or "normal stress moduli." In this case $\mathbf{A} = \mathbf{A}_1$.

To use the relation (5) for the modelling of a fluid, the fluid has to be compatible with thermodynamics in the sense that all flows of the fluid must satisfy the Clausius-Duhem inequality, and the assumption must be made that the specific Helmholtz free energy is at a minimum when the fluid is in equilibrium. Under these assumptions, α_1 and α_2 [32] must satisfy

$$\alpha_1 + \alpha_2 = 0. \tag{6}$$

Considerations of stability of the rest state require the assumptions μ and α_1 to be nonnegative; that is, $\mu > 0$, $\alpha_1 > 0$. See [32]. Under assumption (6), which we shall follow throughout, the form of the stress tensor $\mathbf{T}$ given in (5) reduces to a more compact expression. To obtain this we note that $\nabla\mathbf{v} = (1/2)(\mathbf{A} + \mathbf{W})$ and $(\nabla\mathbf{v})^T = (1/2)(\mathbf{A} - \mathbf{W})$, so that

$$
\begin{aligned}
\alpha_1\left(\mathbf{A}\nabla\mathbf{v} + (\nabla\mathbf{v})^T\mathbf{A}\right) &= \frac{\alpha_1}{2}\left[\mathbf{A}(\mathbf{A} + \mathbf{W}) + (\mathbf{A} - \mathbf{W})\mathbf{A}\right]\\
&= \alpha_1\mathbf{A}^2 + \frac{\alpha_1}{2}(\mathbf{AW} - \mathbf{WA}).
\end{aligned}
\tag{7}
$$

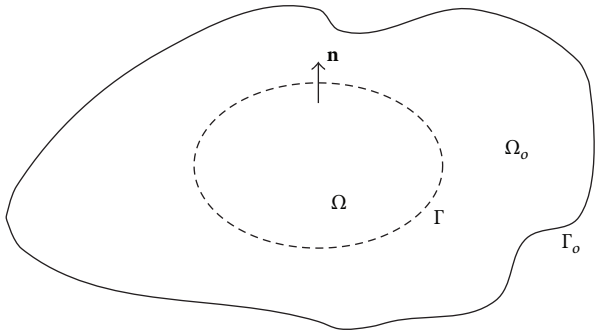


Figure 2: Profile for normal flow through the permeable wall Γ.

Therefore, by (5) and (7)

$$\mathbf{T} = -p\mathbf{I} + \mu\mathbf{A} + \alpha D_t\mathbf{A} + \frac{\alpha}{2}(\mathbf{AW} - \mathbf{WA}), \tag{8}$$

where we have set $\alpha_1 = \alpha$.

Remark 1. Please note that for the Navier-Stokes equations we take $\alpha = 0$ [33].

5. Modelling of Permeability

We study the motion of fluids around and through a fixed porous container filled with the same fluid. The interior of the porous container is an open bounded set $\Omega \subset \mathbf{R}^3$ and the porous boundary, Γ, is smooth. The surrounding fluid domain, Ω_o, is bounded and its outer boundary is denoted by Γ_o. The exterior normal to Ω on Γ is denoted by $\mathbf{n}$. Figure 2 illustrates the situation where the curvature of the boundary Γ of Ω is nonnegative.

Permeability of the walls of the container is described by assuming that at the boundary Γ the flow $\mathbf{v}$ has the direction of the normal:

$$\gamma_o\mathbf{v}(x,t) = -\eta(x,t)\,\mathbf{n}(x). \tag{9}$$

The velocity component η is treated as an unknown and an evolution equation has to be found for it. We model the surface Γ as having an *effective area measure da* which has a density function $\zeta(x)$ with respect to the area measure ds. Thus $da = \zeta(x)ds$. The effective area through which fluid can permeate is not more than the surface area and therefore $0 \le \zeta(x) \le 1$ for any $x \in \Gamma$. If $\zeta(x) \equiv 0$, the wall is impermeable and if $\zeta(x) \equiv 1$, there is no wall.

In order to obtain expressions for mass and momentum in a boundary patch Γ', we let the patch be heuristically represented by a volume G built from copies of Γ' (Figure 3). This is in line with the Beavers-Joseph thinking which was discussed before. For this volume we set up a coordinate system consisting of a "radial part" r, which has the direction of the normal vector $\mathbf{n}$, and a "surface part" made up by vectors tangential to Γ'. For the mass of G we obtain

$$\int_G \rho\,dx = \int_{\Gamma'}\int_0^\delta \rho\,dr\,da = \int_{\Gamma'}\int_0^\delta \rho\,dr\,\zeta\,ds = \int_{\Gamma'} \rho\zeta\delta\,ds, \tag{10}$$

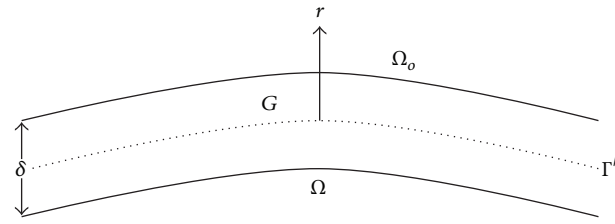


Figure 3: Heuristics of the permeable boundary.

where δ is some measure of thickness. With the aid of these concepts we introduce the *surface density* of the fluid at $x \in \Gamma$ as

$$\sigma(x) = \delta(x)\,\zeta(x)\,\rho, \tag{11}$$

where ρ is the volume density of the fluid.

To obtain the equation of motion for fluid in the boundary, we assume that the rate of change of linear momentum in the boundary is explained by stress forces at both sides of the boundary.

Let $\mathbf{T}$ and $\mathbf{T}'$ denote the stress tensors at the sides of the boundary facing Ω and Ω_o, respectively, and let $\mathbf{P}$ and $\mathbf{P}'$ denote the transfer-of-momentum tensors on the two sides. On an arbitrary boundary patch $\Gamma' \subset \Gamma$ the law of conservation of linear momentum is stated in the following way:

$$\begin{aligned}\partial_t \int_{\Gamma'} \sigma(x)\,\gamma_o\mathbf{v}\,ds = &\int_{\Gamma'} \left[\mathbf{Pn} - \mathbf{P}'\mathbf{n}\right] da \\ &+ \int_{\Gamma'} \left[-\mathbf{T}'(-\mathbf{n}) - \mathbf{Tn}\right] ds\end{aligned} \tag{12}$$

with σ as defined in (11), and it follows that

$$\sigma(x)\,\partial_t\gamma_o\mathbf{v} - \zeta\left[\mathbf{P} - \mathbf{P}'\right]\mathbf{n} = \left[\mathbf{T}' - \mathbf{T}\right]\mathbf{n}. \tag{13}$$

From (13) we have $\sigma(x)\partial_t\eta(x,t) + \zeta\mathbf{n}\cdot[\mathbf{P} - \mathbf{P}']\mathbf{n} = \mathbf{n}\cdot\mathbf{Tn} - \mathbf{n}\cdot\mathbf{T}'\mathbf{n}$. In the domain Ω the momentum flux tensor is given by $\mathbf{P} = \rho\mathbf{v}\otimes\mathbf{v}$. In accordance with this, we shall take $\mathbf{P} = \rho\eta^2\mathbf{n}\otimes\mathbf{n}$ at the boundary. The tensor $\mathbf{P}'$ will be taken as zero.

We take $\mathbf{T}' = \ell\mathbf{I}$ to obtain from (13)

$$\sigma(x)\,\partial_t\eta + \zeta\rho\eta^2 = \mathbf{n}\cdot\mathbf{Tn} - \ell(t). \tag{14}$$

From the incompressibility of the flow in Ω it follows that

$$-\int_\Gamma \eta\,ds = 0. \tag{15}$$

6. Expressions at the Interface

In order to obtain expressions for the stress tensors $\mathbf{T}$ and $\mathbf{T}'$ as well as the acceleration at the boundary through which only normal flow occurs, we obtain a formal expression for the symmetric tensor $\mathbf{A}$ on a surface which is immersed in fluid. We shall eventually use these expressions in postulating

the form of $\mathbf{T}$ and $\mathbf{T}'$ and in formulating a boundary condition which expresses zero tangential acceleration at a wall.

We consider a smooth vector field $\mathbf{v}(\mathbf{x})$ defined on a domain $\Omega \subset \mathbf{R}^3$ and a smooth two-dimensional (at least class C^2) manifold $\Gamma \subset \Omega$ so that $\mathbf{v}$ and $\nabla\mathbf{v}$ are defined on Γ. Let $\mathbf{n}(\mathbf{x})$ be the unit normal to Γ at the point $\mathbf{x} \in \Gamma$.

At any point $\mathbf{x}$ on Γ we consider two orthogonal curves c_1 and c_2 in a neighbourhood of x parametrised by arc lengths s_1 and s_2, respectively. Let $\boldsymbol{\tau_1}$ and $\boldsymbol{\tau_2}$ be the unit tangents to the principal normal curves at a point on the surface. For local coordinates we use the orthogonal system formed by $\boldsymbol{\tau_1}$, $\boldsymbol{\tau_2}$, and $\mathbf{n}$. Under the convention that $\boldsymbol{\tau_1} \wedge \boldsymbol{\tau_2} = \mathbf{n}$ we have $\mathbf{n} \wedge \boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$ and $\mathbf{n} \wedge \boldsymbol{\tau}_2 = -\boldsymbol{\tau}_1$. Let κ_1 and κ_2 represent the principal curvatures at a point on the surface and let $K = \kappa_1 + \kappa_2$ denote the *mean curvature*.

Assumptions

(1) We shall assume throughout that the surface density is bounded and bounded away from zero; that is, there exist constants s and S such that

$$0 < s \leq \sigma(x) \leq S \quad \forall x \in \Gamma. \tag{16}$$

Also, we assume $\sigma \in C^{\infty}(\Gamma)$.

(2) Apart from the smoothness of Γ we make two additional assumptions regarding the shape of Ω; namely, that the curvatures κ_1, κ_2, and K are constrained in the following way:

(a) There exist constants g and G such that

$$0 < g \leq K(x) \leq G \quad \forall x \in \Gamma. \tag{17}$$

(b) There exists a constant H such that

$$0 \leq \kappa_1^2 + \kappa_2^2 \leq H^2 \quad \text{on } \Gamma. \tag{18}$$

Note that these assumptions allow cases where κ_1 and κ_2 can be of opposite signs.

The Frenet-Serret [34–36] formulae in this case, providing that there is no torsion, are then

$$\begin{aligned} \frac{\partial \mathbf{n}}{\partial s_1} &= -\kappa_1 \boldsymbol{\tau}_1 \qquad \frac{\partial \mathbf{n}}{\partial s_2} = -\kappa_2 \boldsymbol{\tau}_2 \\ \frac{\partial \boldsymbol{\tau}_1}{\partial s_1} &= \kappa_1 \mathbf{n} \qquad \frac{\partial \boldsymbol{\tau}_1}{\partial s_2} = 0 \\ \frac{\partial \boldsymbol{\tau}_2}{\partial s_1} &= 0 \qquad \frac{\partial \boldsymbol{\tau}_2}{\partial s_2} = \kappa_2 \mathbf{n}. \end{aligned} \tag{19}$$

The surface gradient ∇_s of a scalar function f may be written as

$$\gamma_o(\nabla f) = \nabla_s(\gamma_o f) + \mathbf{n}\gamma_1 f, \tag{20}$$

where the trace operator γ_1 denotes the normal derivative. Also consider

$$\begin{aligned} \nabla_s f &= \frac{\partial f}{\partial s_1}\boldsymbol{\tau}_1 + \frac{\partial f}{\partial s_2}\boldsymbol{\tau}_2, \\ \Delta_s f &= \nabla_s \cdot (\nabla_s f) = \frac{\partial^2 f}{\partial^2 s_1^2} + \frac{\partial^2 f}{\partial^2 s_2^2}. \end{aligned} \tag{21}$$

If $\mathbf{f}$ is a vector field defined on Γ, the surface gradient ∇_s is defined as the tensor

$$\nabla_s \mathbf{f} = \frac{\partial \mathbf{f}}{\partial s_1} \otimes \boldsymbol{\tau_1} + \frac{\partial \mathbf{f}}{\partial s_2} \otimes \boldsymbol{\tau_2}. \tag{22}$$

Surface divergence and surface curl are defined as

$$\begin{aligned} \nabla_s \cdot \mathbf{f} &= \boldsymbol{\tau_1} \cdot \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau_2} \cdot \frac{\partial \mathbf{f}}{\partial s_2}, \\ \nabla_s \wedge \mathbf{f} &= \boldsymbol{\tau_1} \wedge \frac{\partial \mathbf{f}}{\partial s_1} + \boldsymbol{\tau_2} \wedge \frac{\partial \mathbf{f}}{\partial s_2}. \end{aligned} \tag{23}$$

The relationship between the surface operators and the volume operators for a function defined in Ω is given by

$$\gamma_o(\nabla \mathbf{f}) = \nabla_s \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}] \otimes \mathbf{n}, \tag{24}$$

$$\gamma_o(\nabla \cdot \mathbf{f}) = \nabla_s \cdot \gamma_o \mathbf{f} + \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}] \cdot \mathbf{n}, \tag{25}$$

$$\gamma_o(\nabla \wedge \mathbf{f}) = \nabla_s \wedge \gamma_o \mathbf{f} + \mathbf{n} \wedge \gamma_o[(\mathbf{n} \cdot \nabla)\mathbf{f}]. \tag{26}$$

We use (20)–(25) to prove more important results to make the calculations easier.

Lemma 2. *Let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be two orthogonal unit tangential vectors and let $\mathbf{n}$ be the exterior unit normal vector to Γ. Let α, β, and γ be scalar functions; then*

$$\begin{aligned} &(a)\ \nabla_s \cdot (\alpha \boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1} \\ &(b)\ \nabla_s \cdot (\beta \boldsymbol{\tau}_2) = \frac{\partial \beta}{\partial s_2} \\ &(c)\ \nabla_s \cdot (\gamma \mathbf{n}) = -\gamma K. \end{aligned} \tag{27}$$

Proof. Consider the following:

(a) $\nabla_s \cdot (\alpha \boldsymbol{\tau}_1) = \boldsymbol{\tau}_1 \cdot \frac{\partial}{\partial s_1}(\alpha \boldsymbol{\tau}_1) + \boldsymbol{\tau}_2 \cdot \frac{\partial}{\partial s_2}(\alpha \boldsymbol{\tau}_1) = \frac{\partial \alpha}{\partial s_1}$.

(b) Similar to (a).

(c)
$$\begin{aligned} \nabla_s \cdot (\gamma \mathbf{n}) &= \boldsymbol{\tau}_1 \cdot \left[\frac{\partial \gamma}{\partial s_1}\mathbf{n} - \kappa_1 \gamma \boldsymbol{\tau}_1\right] \\ &\quad + \boldsymbol{\tau}_2 \cdot \left[\frac{\partial \gamma}{\partial s_2}\mathbf{n} - \gamma \kappa_2 \boldsymbol{\tau}_2\right] \\ &= -\gamma[\kappa_1 + \kappa_2] = -\gamma K. \end{aligned} \tag{28}$$

□

We shall apply the expressions above to $\mathbf{v}$. By the Frenet-Serret formulae (torsion is zero) $(\partial/\partial s_1)(\gamma_o \mathbf{v}) = -(\partial\eta/\partial s_1)\mathbf{n} - \eta(\partial \mathbf{n}/\partial s_1) = -(\partial\eta/\partial s_1)\mathbf{n} + \kappa_1 \boldsymbol{\tau_1}\eta$, and, similarly, $\partial(\gamma_0 \mathbf{v})/\partial s_2 = -(\partial\eta/\partial s_2)\mathbf{n} + \kappa_2 \boldsymbol{\tau_2}\eta$. Hence

$$\begin{aligned}\nabla_s \gamma_o \mathbf{v} &= \eta\left[\kappa_1 \boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad - \left[\frac{\partial\eta}{\partial s_1}\mathbf{n}\otimes\boldsymbol{\tau_1} + \frac{\partial\eta}{\partial s_2}\mathbf{n}\otimes\boldsymbol{\tau_2}\right] \\ &= \eta\left[\kappa_1 \boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad - \mathbf{n}\otimes\left[\frac{\partial\eta}{\partial s_1}\boldsymbol{\tau_1} + \frac{\partial\eta}{\partial s_2}\boldsymbol{\tau_2}\right] \\ &= \eta\left[\kappa_1 \boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] - \mathbf{n}\otimes\nabla_s\eta.\end{aligned} \tag{29}$$

The transpose is given by

$$\left(\nabla_s\left(\gamma_o\mathbf{v}\right)\right)^T = \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] - \nabla_s\eta\otimes\mathbf{n}. \tag{30}$$

To find an expression for $\mathbf{A}$ at Γ, we need an expression for $\nabla\mathbf{v}$ on the boundary:

$$\gamma_o\left(\nabla\mathbf{v}\right) = \nabla_s\left(\gamma_o\mathbf{v}\right) + \gamma_o\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\otimes\mathbf{n}. \tag{31}$$

Although we know that the divergence of $\mathbf{v}$ will be zero, it is helpful to observe that $\theta = \gamma_o(\nabla\cdot\mathbf{v}) = \nabla_s\cdot(\gamma_o\mathbf{v}) + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n}$, where

$$\begin{aligned}\nabla_s\cdot\left(\gamma_o\mathbf{v}\right) &= -\boldsymbol{\tau_1}\cdot\frac{\partial}{\partial s_1}\left(\eta\mathbf{n}\right) - \boldsymbol{\tau_2}\cdot\frac{\partial}{\partial s_2}\left(\eta\mathbf{n}\right) \\ &= -\boldsymbol{\tau_1}\cdot\left[\frac{\partial\eta}{\partial s_1}\mathbf{n} - \eta\kappa_1\boldsymbol{\tau_1}\right] - \boldsymbol{\tau_2}\cdot\left(\frac{\partial\eta}{\partial s_2}\mathbf{n} - \eta\kappa_2\boldsymbol{\tau_2}\right) \\ &= \eta\left(\kappa_1+\kappa_2\right) \\ &= \eta K.\end{aligned} \tag{32}$$

Hence $\theta = \eta K + \gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{v}]\cdot\mathbf{n}$.

We proceed to find expressions for $\gamma_o(\mathbf{v}\cdot\nabla)\mathbf{v}$, $\gamma_o(\nabla\mathbf{v})$, and $\gamma_o(\nabla\mathbf{v})^T$.

We know that $\boldsymbol{\omega}\wedge\mathbf{n} = \mathbf{W}(\mathbf{v})\mathbf{n} = (\mathbf{n}\cdot\nabla)\mathbf{v} - (\nabla\mathbf{v})^T\mathbf{n}$, and

$$\begin{aligned}\gamma_o(\nabla\mathbf{v})^T\mathbf{n} &= \left(\nabla_s\left(\gamma_o\mathbf{v}\right)\right)^T\mathbf{n} + \gamma_o\left(\left[\mathbf{n}\cdot\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\mathbf{n}\right) \\ &= -\left(\nabla_s\eta\right) + \left(\theta - K\eta\right)\mathbf{n}.\end{aligned} \tag{33}$$

Therefore,

$$\left(\mathbf{n}\cdot\nabla\right)\gamma_o\mathbf{v} = \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right) - \nabla_s\eta + \left(\theta - K\eta\right)\mathbf{n}. \tag{34}$$

Multiply (34) with $-\eta$ to obtain

$$\gamma_o\left(\mathbf{v}\cdot\nabla\right)\mathbf{v} = -\eta\left(\mathbf{n}\cdot\nabla\right)\gamma_o\mathbf{v} = K\eta^2\mathbf{n} + \eta\left[\nabla_s\eta - \boldsymbol{\gamma_0\omega}\wedge\mathbf{n}\right]. \tag{35}$$

From (31) we now obtain

$$\begin{aligned}\gamma_o\left(\nabla\mathbf{v}\right) &= \nabla_s\mathbf{v} + \left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n} - \nabla_s\eta + \left(\theta - K\eta\right)\mathbf{n}\right]\otimes\mathbf{n} \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right] \\ &\quad - \mathbf{n}\otimes\nabla_s\eta - \nabla_s\eta\otimes\mathbf{n} + \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n} \\ &\quad + \left(\theta - K\eta\right)\mathbf{n}\otimes\mathbf{n} \\ &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2} - K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad - \left[\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad + \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n} + \theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{36}$$

The transpose is

$$\begin{aligned}\gamma_o(\nabla\mathbf{v})^T &= \eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2} - K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad - \left[\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad + \mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right) + \theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{37}$$

Thus we have

$$\begin{aligned}\gamma_o\left(\mathbf{A}\right) &= \gamma_o\left(\nabla\mathbf{v}\right) + \gamma_o(\nabla\mathbf{v})^T \\ &= \nabla_s\gamma_o\mathbf{v} + \left(\nabla_s\gamma_o\mathbf{v}\right)^T + \gamma_o\left(\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\otimes\mathbf{n}\right) \\ &\quad + \gamma_o\left(\mathbf{n}\otimes\left[\left(\mathbf{n}\cdot\nabla\right)\mathbf{v}\right]\right) \\ &= 2\eta\left[\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2} - K\mathbf{n}\otimes\mathbf{n}\right] \\ &\quad - 2\left[\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}\right] \\ &\quad + \mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right) + \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n}. + 2\theta\mathbf{n}\otimes\mathbf{n}.\end{aligned} \tag{38}$$

Let us define the symmetrical tensors $\mathbf{M}$ and $\mathbf{N}$ by

$$\begin{aligned}\mathbf{M} &= \left[K\mathbf{n}\otimes\mathbf{n} - \left(\kappa_1\boldsymbol{\tau_1}\otimes\boldsymbol{\tau_1} + \kappa_2\boldsymbol{\tau_2}\otimes\boldsymbol{\tau_2}\right)\right], \\ \mathbf{N} &= \mathbf{n}\otimes\left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right) + \left(\boldsymbol{\gamma_o\omega}\wedge\mathbf{n}\right)\otimes\mathbf{n} \\ &\quad + 2\theta\mathbf{n}\otimes\mathbf{n} - 2\left(\mathbf{n}\otimes\nabla_s\eta + \nabla_s\eta\otimes\mathbf{n}\right) \\ &= \mathbf{n}\otimes\left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n} - 2\nabla_s\eta\right] \\ &\quad + \left[\boldsymbol{\gamma_o\omega}\wedge\mathbf{n} - 2\nabla_s\eta\right]\otimes\mathbf{n} + 2\theta\mathbf{n}\otimes\mathbf{n} \\ &= \mathbf{n}\otimes\boldsymbol{\psi} + \boldsymbol{\psi}\otimes\mathbf{n} - 2\theta\mathbf{n}\otimes\mathbf{n},\end{aligned} \tag{39}$$

with

$$\boldsymbol{\psi} = \boldsymbol{\gamma_o\omega}\wedge\mathbf{n} - 2\nabla_s\eta \tag{40}$$

a tangential vector. Then, for a vector field of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ, from (38) we have

$$\gamma_o\mathbf{A} = -2\eta\mathbf{M} + \mathbf{N} \quad \text{on } \Gamma. \tag{41}$$

In local coordinates we have the representations

$$\mathbf{M} = \begin{pmatrix} -\kappa_1 & 0 & 0 \\ 0 & -\kappa_2 & 0 \\ 0 & 0 & K \end{pmatrix}, \qquad \mathbf{N} = \begin{pmatrix} 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_1 \\ 0 & 0 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_2 \\ \boldsymbol{\psi}\cdot\boldsymbol{\tau}_1 & \boldsymbol{\psi}\cdot\boldsymbol{\tau}_2 & -2\theta \end{pmatrix}. \tag{42}$$

If $\nabla\cdot\mathbf{v} = 0$, it follows that $\mathrm{tr}\mathbf{A} = 0$, which is in line with incompressibility.

We would further like to obtain expressions for the terms $\mathbf{n}\cdot\Delta\mathbf{v}$, $\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}$, and $\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}$ on the boundary Γ.

Lemma 3. *Let* $\mathbf{n}$ *be the exterior normal to the boundary* Γ, $\mathbf{v}\in\mathscr{D}$, *and* $\mathbf{A} = -2\eta\mathbf{M}+\mathbf{N}$ *with* $\mathbf{M}$ *and* $\mathbf{N}$ *as defined in* (39). *We assume that* $\nabla\cdot\mathbf{v} = 0$ *and* $\boldsymbol{\gamma}_o\boldsymbol{\omega}\wedge\mathbf{n} = 2\nabla_s\eta$, *which implies that* $\mathbf{N} = 0$. *Then*

$$\begin{aligned} &(a)\ \gamma_o(-\mathbf{n}\cdot\Delta\mathbf{v}) = \Delta_s\eta \\ &(b)\ \gamma_o[\mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{A}]\mathbf{n}] = -4\eta^2K_G - 2\eta\Delta_s\eta \\ &(c)\ \gamma_o[\mathbf{n}\cdot[\mathbf{AW}-\mathbf{WA}]\mathbf{n}] = 0, \end{aligned} \tag{43}$$

where K_G *denotes the Gauss-curvature.*

Proof. (a) We have chosen $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$, and $\mathbf{n}$ so that $\boldsymbol{\tau}_1\wedge\boldsymbol{\tau}_2 = \mathbf{n}$. In view of the incompressibility and the fact that there is zero tangential velocity

$$\begin{aligned} \gamma_o(-\Delta\mathbf{v}) &= \nabla\wedge\boldsymbol{\gamma}_o\boldsymbol{\omega} \\ &= \boldsymbol{\tau}_1\wedge\partial_{s_1}[\eta_1\boldsymbol{\tau}_2-\eta_2\boldsymbol{\tau}_1] + \boldsymbol{\tau}_2\wedge\partial_{s_2}[\eta_1\boldsymbol{\tau}_2-\eta_2\boldsymbol{\tau}_1] \\ &\quad + \text{a tangential term} \\ &= \boldsymbol{\tau}_1\wedge\left[\partial_{s_1}\eta_1\boldsymbol{\tau}_2 - \eta_2\kappa_1\mathbf{n}\right] \\ &\quad + \boldsymbol{\tau}_2\wedge\left[\eta_1\kappa_2\mathbf{n} - \partial_{s_1}\eta_2\boldsymbol{\tau}_1\right] + \cdots \\ &= \left(\partial_{s_1}\eta_1 + \partial_{s_2}\eta_2\right)\mathbf{n} + \cdots \\ &= \Delta_s\eta\mathbf{n}. \end{aligned} \tag{44}$$

(b) Consider the tensor $\nabla\cdot\mathbf{A}$ built from "row vectors" with $(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3)$ a basis for $\mathbf{R}^3$. Then

$$\begin{aligned} \nabla\cdot\mathbf{A} &= \begin{pmatrix} \nabla\cdot\mathbf{A}\mathbf{e}_1 \\ \nabla\cdot\mathbf{A}\mathbf{e}_2 \\ \nabla\cdot\mathbf{A}\mathbf{e}_3 \end{pmatrix} \\ &= \begin{pmatrix} \nabla_s\cdot\mathbf{A}\mathbf{e}_1 \\ \nabla_s\cdot\mathbf{A}\mathbf{e}_2 \\ \nabla_s\cdot\mathbf{A}\mathbf{e}_3 \end{pmatrix} + \begin{pmatrix} \mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}\mathbf{e}_1] \\ \mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}\mathbf{e}_2] \\ \mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}\mathbf{e}_3] \end{pmatrix} \\ &= \nabla_s\cdot\mathbf{A} + [(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}. \end{aligned} \tag{45}$$

Hence, $\gamma_o[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n} = \gamma_o[\nabla\cdot\mathbf{A}] - \nabla_s\cdot\gamma_o(\mathbf{A})$.

Furthermore,

$$\nabla_s\cdot\gamma_o[\mathbf{A}] = -2\left(M\nabla_s\eta + \eta\nabla_s\cdot M\right) \tag{46}$$

and now

$$\begin{aligned} \mathbf{n}\cdot[\nabla_s\cdot[\gamma_o\mathbf{A}]] &= -2\left(\mathbf{n}\cdot M\nabla_s\eta + \eta\mathbf{n}\cdot(\nabla_s\cdot M)\right) \\ &= -2\eta\mathbf{n}\cdot(\nabla_s\cdot M). \end{aligned} \tag{47}$$

Here we used the fact that $M\mathbf{n} = -K\mathbf{n}$. Determine $\mathbf{n}\cdot[\nabla_s\cdot M]$ term by term to obtain

$$\begin{aligned} \mathbf{n}\cdot[\nabla_s\cdot[\gamma_o\mathbf{A}]] &= -2\eta\left(\kappa_1^2+\kappa_2^2-K^2\right) \\ &= -2\eta(2\kappa_1\kappa_2) = -4K_G\eta. \end{aligned} \tag{48}$$

K_G denotes the Gauss-curvature and is bounded by assumptions (17) and (18). Hence

$$\gamma_o\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n} = \gamma_o[\mathbf{n}\cdot\Delta\mathbf{v}] - 4\eta K_G = -4\eta K_G - \Delta_s\eta. \tag{49}$$

The term we use in the proof of (57) is therefore

$$-\eta\gamma_o[\mathbf{n}\cdot[(\mathbf{n}\cdot\nabla)\mathbf{A}]\mathbf{n}] = +4\eta^2K_G + \eta\Delta_s\eta. \tag{50}$$

(c) $\mathbf{n}\cdot(\mathbf{AW}-\mathbf{WA})\mathbf{n} = \mathbf{An}\cdot\mathbf{Wn} + \mathbf{Wn}\cdot\mathbf{An} = 2\mathbf{An}\cdot\mathbf{Wn}$.

Here we make use of the additional boundary conditions (52) and (55) and the fact that $\mathbf{Wn} = \boldsymbol{\gamma}_o\boldsymbol{\omega}\wedge\mathbf{n}$ to obtain that

$$\begin{aligned} \mathbf{An}\cdot\mathbf{Wn} &= \mathbf{An}\cdot(\boldsymbol{\gamma}_o\boldsymbol{\omega}\wedge\mathbf{n}) \\ &= (\boldsymbol{\gamma}_o\boldsymbol{\omega}\wedge\mathbf{n})\cdot[-2\eta K\mathbf{n}] \\ &= 0. \end{aligned} \tag{51}$$

□

6.1. Explicit Form of the Dynamic Boundary Condition. It is shown that for a smooth two-dimensional manifold Γ contained in a domain $\Omega\subset\mathbf{R}^3$ the following is true for a vector field $\mathbf{v}$ which is of the form $\mathbf{v} = -\eta\mathbf{n}$ on Γ:

$$\gamma_o[\mathbf{A}] = -2\eta\mathbf{M}+\mathbf{N}, \tag{52}$$

where $\mathbf{M}$ and $\mathbf{N}$ are defined in (39).

If $\mathbf{v}$ is solenoidal, as in the case under consideration, $\theta = 0$. A straightforward application of Stokes' theorem shows that $\boldsymbol{\omega}$ is tangential to the boundary, which implies that $\boldsymbol{\psi}$ is tangential to the boundary. Indeed, let Γ' be any patch of the surface Γ; then

$$\int_{\Gamma'}(\nabla\wedge\mathbf{f})\cdot\mathbf{n}\,ds = \int_{\partial\Gamma'}\mathbf{f}\cdot d\boldsymbol{\tau}, \tag{53}$$

where $d\boldsymbol{\tau}$ is a vector tangential to the boundary. Now if $\mathbf{f} = \gamma_o\mathbf{v} = -\eta\mathbf{n}$, then $\int_{\partial\Gamma'}\mathbf{f}\cdot d\boldsymbol{\tau} = 0$, and that implies that $\int_{\Gamma'}(\nabla\wedge\mathbf{v})\cdot\mathbf{n}ds = 0$ for all $\Gamma'\subset\Gamma$, which in turn implies that $(\nabla\wedge\mathbf{v})\cdot\mathbf{n} = 0$.

In the problem under consideration we shall assume that the "rate of deformation" tensor $\mathbf{A}$ has precisely the form (52) on the boundary Γ with $\mathbf{n}$ the unit exterior normal (the traditional rate of deformation is defined as $\mathbf{D} = (1/2)\mathbf{A}$).

We shall consider a kinematic boundary condition, which has a physical meaning in that there are no tangential components of deformation at the interface boundary. This concerns the form of the tensor $\mathbf{N}$.

Towards this, we observe from (52) that

$$\gamma_0 [\mathbf{An}] = -2K\eta\mathbf{n} + \boldsymbol{\psi}. \tag{54}$$

It follows from (54) that there are no tangential components of deformation at Γ if and only if $\boldsymbol{\psi} = 0$; that is,

$$\boldsymbol{\gamma_o\omega} \wedge \mathbf{n} = 2\nabla_s\eta. \tag{55}$$

This is the kinematic boundary condition.

The various terms in $\mathbf{n} \cdot \mathbf{Tn}$, with $\mathbf{T}$ on a surface Γ, given by (8), may be expressed as follows (see Lemma 3):

$$\begin{aligned} \mathbf{n} \cdot \gamma_o\mathbf{An} &= -2K\eta \\ \mathbf{n} \cdot \gamma_o\partial_t [\mathbf{An}] &= \partial_t [\mathbf{n} \cdot \mathbf{An}] = -2K\eta_t \end{aligned} \tag{56}$$

$$\gamma_o [\mathbf{n} \cdot [(\mathbf{v} \cdot \nabla)\mathbf{A}]\mathbf{n}] = +4K_G\eta^2 + \eta\Delta_s\eta \tag{57}$$

$$\mathbf{n} \cdot [\mathbf{AW} - \mathbf{WA}]\mathbf{n} = 0. \tag{58}$$

Guided by these expressions and (8), we assume that, at Γ,

$$\mathbf{n} \cdot \gamma_o\mathbf{Tn} = -\left[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta\right]. \tag{59}$$

For the stress tensor $\mathbf{T}'$ in the fluid exterior to Ω we assume that $\mathbf{n} \cdot \mathbf{T}'\mathbf{n} = \ell(t)$. This amounts to the situation where the fluid in Ω_o is at rest. As a result we have

$$\begin{aligned} \mathbf{n} \cdot (\delta\mathbf{T})\mathbf{n} &= \mathbf{n} \cdot \left[\mathbf{T} - \mathbf{T}'\right]\mathbf{n} \\ &= -\left[\gamma_o p + 2\mu K\eta + 2\alpha K\eta_t - 4\alpha K_G\eta^2 - \alpha\eta\Delta_s\eta\right] \\ &\quad - \ell(t). \end{aligned} \tag{60}$$

From (13), (14), and (60) we obtain

$$\sigma^{-1/2}(\sigma + 2\alpha K)\eta_t + \sigma^{-1/2}\gamma_0 p = s(\eta) \tag{61}$$

with $s(\eta) = \sigma^{-1/2}[(-k + 4\alpha K_G)\eta^2 - 2\mu\eta K + \alpha\eta\Delta_s\eta - \ell(t)]$, and $k = \zeta\rho$.

Equation (61) is the explicit form of the dynamic boundary condition.

7. An Alternative Model: *Problem* $\mathscr{A}$

Although it was possible to prove stability and uniqueness for the original model (see [33, 37]), we could not find a way to a possible proof of existence for a classical solution. In this chapter we describe an alternative model which displays all the properties of the original problem with respect to stability and uniqueness.

In the alternative model the dynamics at the boundary are formulated by assuming a "shear flow" of the form

$$\mathbf{v}^*(y,t) = -\eta(s_1, s_2, t)\,\mathbf{n}(y) \tag{62}$$

with s_1 and s_2 as the surface parameters (like arc length). It is assumed that the "body force" acting on the shearing fluid at the boundary is proportional to the difference between the pressures $\gamma_o p$ and $\ell(t)$. Under these assumptions the equation governing the evolution of η is

$$\partial_t [\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta + \delta^{-1}\ell(t), \tag{63}$$

where $\gamma_o\mathbf{v} = -\eta\mathbf{n}$, and p is the resulting pressure through the boundary with thickness δ. Δ_s is the Laplace-Beltrami operator ($\Delta_s = \nabla_s \cdot \nabla_s$) and ∇_s denotes the surface gradient. The parameter δ has the physical dimension of length and may be thought of as the "thickness" of the "shear layer" (see [38], Sect 123, p. 506). Equation (63) is derived by calculating the stress tensor for a shear flow and noticing that terms of the form $\mathbf{v}^* \cdot \nabla_s$ vanish. The term $\delta^{-1}\ell(t)$ may be left out since, as before, it disappears when projections are taken. The kinematic boundary condition is still imposed.

7.1. Definitions. All spaces of vector fields are denoted by boldface letters.

(1) Let Ω be a bounded domain in $\mathbf{R}^3$ with a smooth boundary Γ (of class C^∞), $\Omega_T = \Omega \times (0,T)$, and $\Gamma_T = \Gamma \times (0,T)$.

(2) $H^{m,q}(\Omega)$, for m a nonnegative integer and $1 < q < \infty$, is the usual Sobolev space (of real-valued functions) embedded in $L^q(\Omega)$ with norm $\|\cdot\|_{m,q}$. $H^m(\Omega)$, for m a nonnegative integer, denotes the Sobolev space $H^{m,2}(\Omega)$ of order m. By this agreement $\mathbf{H}^m(\Omega)$ is the Sobolev space of three vector fields and the components are elements of $H^m(\Omega)$. In particular, the norm and scalar products in $\mathbf{H}^1(\Omega)$ are defined by $\|\mathbf{u}\|_1^2 = \|\mathbf{u}\|_\Omega^2 + \|\nabla\mathbf{u}\|_\Omega^2$ and $(\mathbf{u},\mathbf{v})_1 = \int_\Omega \mathbf{u}\cdot\mathbf{v}dx + \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}dx$.

(3) With the above notation $\mathbf{H}^0(\Omega)$ denotes the Hilbert space $\mathbf{L}^2(\Omega)$ of vector functions $\mathbf{u}(x) = (u_1(x), u_2(x), u_3(x))$, with $x \in \Omega$, for which $|\mathbf{u}|^2$ is integrable on Ω. The norm and scalar products for $\mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\Omega)$ are defined as $\|\mathbf{u}\|_\Omega^2 = \int_\Omega |\mathbf{u}|^2 dx$ and $(\mathbf{u},\mathbf{v})_\Omega = \int_\Omega \mathbf{u} \cdot \mathbf{v}dx$.

(4) There exists a linear continuous operator $\gamma_o \in \mathscr{L}(\mathbf{H}^1(\Omega), \mathbf{L}^2(\partial\Omega))$, called the *trace operator*, such that $\gamma_o\mathbf{u}$ = the "restriction" of $\mathbf{u}$ to $\partial\Omega$ for every function $\mathbf{u} \in \mathbf{H}^1(\Omega)$ which is continuous in $\overline{\Omega}$. The space $\mathbf{H}_o^1(\Omega)$ is the kernel of γ_o. The image space $\gamma_o(\mathbf{H}^1(\Omega))$ is a dense subspace of $\mathbf{L}^2(\Gamma)$ denoted by $\mathbf{H}^{1/2}(\Gamma)$. The trace operator is bounded. Indeed, there exists a constant $C_1 > 0$ such that

$$\|\gamma_o\mathbf{u}\|_\Gamma \le C_1\|\mathbf{u}\|_1 \quad \forall\mathbf{u} \in \mathbf{H}^1(\Omega). \tag{64}$$

Reference [39, Theorem 9.4, page 41]. We shall refer to this result (64) as the *Trace theorem*.

(5) For the deformation we use the following notation for the norm and scalar products: $(\mathbf{A}(\mathbf{u}), \mathbf{A}(\mathbf{v}))_\Omega = \int_\Omega \mathbf{A}(\mathbf{u}) : \mathbf{A}(\mathbf{v}) dx$ and $\|\mathbf{A}\|_\Omega^2 = \int_\Omega |\mathbf{A}|^2 dx$.

(6) We define the domain $\mathscr{D}$ by

$$\mathscr{D} = \left\{ \mathbf{v} \in \mathbf{H}^2(\Omega) : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \gamma_o \mathbf{v} = -\eta \mathbf{n} \in \mathbf{L}^2(\Gamma), \mathbf{A}(\mathbf{v}) = -2\eta \mathbf{M} \text{ on } \Gamma \right\}. \tag{65}$$

$\mathscr{D}$ is a closed subspace of $\mathbf{H}^2(\Omega)$. Elements of $\mathscr{D}$ satisfy the kinematical boundary conditions (55).

(7) $\mathbf{H}_\varsigma^1(\Omega)$ denotes the closure of $\mathscr{D}$ in $\mathbf{H}^1(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_1$.

(8) The norm of $\gamma_o \mathbf{v} \in \mathbf{L}^2(\Gamma)$ on the boundary Γ is chosen as

$$\|\gamma_o \mathbf{v}\|_\Gamma^2 = \|\eta\|_\Gamma^2 = \int_\Gamma \sigma(x) |\gamma_o \mathbf{v}|^2 ds. \tag{66}$$

The associated scalar product is

$$(\gamma_o \mathbf{u}, \gamma_o \mathbf{v})_\Gamma = \int_\Gamma \sigma(x) \gamma_o \mathbf{u} \cdot \gamma_o \mathbf{v} \, ds. \tag{67}$$

According to assumption (16) this is equivalent to the usual $\mathbf{L}^2$ metric. It is assumed that the function $\sigma \in C^\infty(\Gamma)$.

(9) For the purpose of stability and uniqueness we define the following norm:

$$\int_\Gamma \eta \, ds = \|\eta\|_{0,\Gamma}. \tag{68}$$

(10) We shall deal extensively with the energy associated with fluids of second grade defined for the purpose of this study by

$$\widetilde{E}_v = \frac{\alpha}{2} \|\mathbf{A}(\mathbf{v})\|_\Omega^2 + \rho \|\mathbf{v}\|_\Omega^2 + C_1 \|\eta\|_{0,\Gamma}^2 + C_2 \|\nabla_s \eta\|_{0,\Gamma}^2, \tag{69}$$

with $C_1 = (\delta\rho - 2\alpha K)$ and $C_2 = \alpha\delta$. $\widetilde{E}_v^{1/2}$ is evidently a norm on $\mathbf{H}_\varsigma^1(\Omega)$. We shall refer to the quantity $\widetilde{E}_v^{1/2}$ as the energy norm of $\mathbf{v}$.

(11) The constant C, which appears in inequalities, denotes a generic positive constant. Sometimes it is necessary to indicate the quantities on which a constant depends in brackets or by a subscript.

7.2. Important Identities

Identity 1. For any symmetric tensor $\mathbf{A}$ and any arbitrary tensor $\mathbf{B}$, we have $\mathbf{A} : \mathbf{B} = \mathbf{A} : \mathbf{B}_s$, with $\mathbf{B}_s = (1/2)(\mathbf{B} + \mathbf{B}^T)$.

Proof. Consider the following:

$$\mathbf{A} : \mathbf{B} = \mathbf{A}^T : \mathbf{B}^T = \mathbf{A} : \mathbf{B}^T = \frac{1}{2} \mathbf{A} \left(\mathbf{B} + \mathbf{B}^T \right) = \mathbf{A} : \mathbf{B}_s. \tag{70}$$

□

Expressions are necessary for inner products of the form $(D_t \mathbf{F}, \mathbf{F})_\Omega$, where $\mathbf{F}$ is either a vector or a second order tensor. $D_t = \partial_t + \mathbf{v} \cdot \nabla$ is the material derivative. In order to obtain simple expressions for the scalar product, we notice that if $\circ$ denotes either the usual scalar product or the "colon" product, then

$$\left[\partial_t \mathbf{F} + (\mathbf{v} \cdot \nabla) \mathbf{F} \right] \circ \mathbf{F} = \frac{1}{2} \partial_t |\mathbf{F}|^2 + \frac{1}{2} \nabla \cdot \left(|\mathbf{F}|^2 \mathbf{v} \right), \tag{71}$$

provided $\nabla \cdot \mathbf{v} = 0$. Hence the following identity.

Identity 2. For any smooth vector or tensor quantity $\mathbf{F}(x, t)$ and any $\mathbf{v} \in \mathscr{D}$, we have

$$(D_t \mathbf{F}, \mathbf{F})_\Omega = \frac{1}{2} \partial_t \|\mathbf{F}\|_\Omega^2 - \frac{1}{2} \int_\Gamma |\mathbf{F}|^2 \eta \, ds. \tag{72}$$

Proof. By the divergence theorem

$$\begin{aligned} \left(\partial_t \mathbf{F} + (\mathbf{v} \cdot \nabla) \mathbf{F}, \mathbf{F} \right) &= \frac{1}{2} \partial_t \int_\Omega |\mathbf{F}|^2 dx + \frac{1}{2} \int_\Omega \nabla \cdot \left(|\mathbf{F}|^2 \mathbf{v} \right) dx \\ &= \frac{1}{2} \partial_t \|\mathbf{F}\|_\Omega^2 + \frac{1}{2} \int_\Gamma |\mathbf{F}|^2 \mathbf{v} \cdot \mathbf{n} \, ds \\ &= \frac{1}{2} \partial_t \|\mathbf{F}\|_\Omega^2 - \frac{1}{2} \int_\Gamma |\mathbf{F}|^2 \eta \, ds. \end{aligned} \tag{73}$$

□

Later in this study we shall employ the energy method. It will become necessary to use the various boundary conditions in order to prove stability. The following is important to obtain the required results.

Identity 3. If $f \in H^1(\Omega)$ is a scalar field and $\mathbf{v} \in \mathscr{D}$, then

$$\int_\Omega (\mathbf{v} \cdot \nabla) f \, dx = -\int_\Gamma \eta f \, ds. \tag{74}$$

Proof. Integrating by parts and using the fact that $\mathbf{v}$ is solenoidal

$$\int_\Omega (\mathbf{v} \cdot \nabla) f \, dx = \int_\Gamma f \mathbf{v} \cdot \mathbf{n} \, ds - \int_\Omega f \nabla \cdot \mathbf{v} \, dx = -\int_\Gamma \eta f \, ds. \tag{75}$$

□

We note that, in particular for $\mathbf{v} \in \mathscr{D}$, the imbedding of $\mathbf{H}^2(\Omega)$ in the space of bounded continuous functions makes the choice $f = |\mathbf{v}|^2$ possible, and it follows from Identity 3 that

$$\int_\Omega (\mathbf{v} \cdot \nabla) |\mathbf{v}|^2 ds = -\int_\Gamma |\eta|^3 ds. \tag{76}$$

For $\mathbf{v} \in \mathscr{D}$ we may also choose $f = |\mathbf{A}(\mathbf{v})|^2$, and it follows that

$$\begin{aligned}\int_\Omega (\mathbf{v}\cdot\nabla)\,|\mathbf{A}(\mathbf{v})|^2 dx &= -\int_\Gamma |\mathbf{A}(\mathbf{v})|^2 \eta\, ds \\ &= -\int_\Gamma 4\eta^3 |\mathbf{M}|^2 ds\end{aligned} \tag{77}$$

since $\mathbf{N} = 0$ on $\mathscr{D}$.

The following will be of immediate importance.

Identity 4. For any $\mathbf{v} \in \mathscr{D}$,

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 = 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma K(x)\,\eta^2 ds. \tag{78}$$

Proof. From the definition of $\mathbf{A}$ it is evident that $|\mathbf{A}(\mathbf{v})|^2 = 2|\nabla\mathbf{v}|^2 + 2\nabla\mathbf{v} : \nabla^T\mathbf{v}$. Now $\nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}] = \nabla\mathbf{v} : \nabla^T\mathbf{v} + (\mathbf{v}\cdot\nabla)(\nabla\cdot\mathbf{v})$, and, since $\nabla\cdot\mathbf{v} = 0$, $\nabla\mathbf{v} : \nabla^T\mathbf{v} = \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]$, integration over Ω and Identity 1 yield

$$\begin{aligned}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Omega \nabla\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\,dx \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma \mathbf{n}\cdot[(\mathbf{v}\cdot\nabla)\mathbf{v}]\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 - 2\int_\Gamma \eta\mathbf{n}\otimes\mathbf{n} : [\nabla\mathbf{v}]\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 - \int_\Gamma \eta\mathbf{n}\cdot\mathbf{A}(\mathbf{v})\,\mathbf{n}\,ds \\ &= 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\int_\Gamma K(x)\,\eta^2 ds.\end{aligned} \tag{79}$$

□

Thus, if the curvature K is positive everywhere on Γ, it becomes apparent that $\mathbf{A}(\mathbf{v}) = \mathbf{0}$ if and only if $\mathbf{v} = 0$.

Identity 5. For any bilinear form b on a Hilbert space H, we have, for any $\mathbf{v}, \mathbf{w} \in H$ and with $\mathbf{u} = \mathbf{v} - \mathbf{w}$ that $b(\mathbf{v},\mathbf{v}) - b(\mathbf{w},\mathbf{w}) = b(\mathbf{u},\mathbf{v}) + b(\mathbf{w},\mathbf{u})$.

Identity 6. Let $\mathbf{f}$ and $\mathbf{g}$ be tensor fields of the same order and let $\circ$ denote the "scalar product" in such tensor fields. For $\mathbf{v} \in \mathscr{D}$ it is true that

$$\int_\Omega \left[\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{f}\right] dx = -\int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}\,ds. \tag{80}$$

Proof. Consider the following:

$$\int_\Omega \mathbf{f}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{g}\,dx = \int_\Gamma (\mathbf{v}\cdot\mathbf{n})\,\mathbf{f}\circ\mathbf{g}\,ds - \int_\Omega \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{f}\,dx, \tag{81}$$

thus

$$\int_\Omega \left[\mathbf{f}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{g} + \mathbf{g}\circ(\mathbf{v}\cdot\nabla)\,\mathbf{f}\right] dx = \int_\Gamma \eta_v \mathbf{f}\circ\mathbf{g}\,ds. \tag{82}$$

□

7.3. Inequalities

Lemma 4. *Under the assumptions* (17) *and* (16)*, for* $\mathbf{v} \in \mathscr{D}$, $\mathbf{v} = 0$ *if and only if* $\mathbf{A}(\mathbf{v}) = 0$.

Proof. From (16), (78), and (17) we have

$$\frac{g}{S}\|\eta\|_\Gamma^2 + \|\nabla\mathbf{v}\|_\Omega^2 \le \frac{1}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \le \|\nabla\mathbf{v}\|_\Omega^2 + \frac{G}{s}\|\eta\|_\Gamma^2, \tag{83}$$

and the result follows. □

The following two lemmas are important in establishing a Poincaré inequality.

Lemma 5. *The bilinear forms* $a(\mathbf{u},\mathbf{v}) = (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))_\Omega$ *and* $b(\mathbf{u},\mathbf{v}) = \rho(\mathbf{u},\mathbf{v})_\Omega + C_1(\gamma_o\mathbf{u},\gamma_o\mathbf{v})_\Gamma + C_2(\nabla_s\eta_u,\nabla_s\eta_v)_\Gamma$ *are bounded in the space* $\mathbf{H}_\varsigma^1(\Omega)$. C_1 *and* C_2 *are positive constants.*

Proof. For $\mathbf{u}$ and $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ and by (16), (17), and the Schwartz inequality

$$\begin{aligned}|a(\mathbf{u},\mathbf{v})| &= |(\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))| \\ &= \left|2(\nabla\mathbf{u},\nabla\mathbf{v})_\Omega + 2\int_\Gamma K(x)\,\eta_u\eta_v ds\right| \\ &\le 2\|\mathbf{u}\|_1\|\mathbf{v}\|_1 + \frac{2G}{s^2}\|\gamma_o\mathbf{u}\|_\Gamma\|\gamma_o\mathbf{v}\|_\Gamma.\end{aligned} \tag{84}$$

Hence, by the Trace theorem

$$|a(\mathbf{u},\mathbf{v})| \le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1. \tag{85}$$

Furthermore,

$$\begin{aligned}|b(\mathbf{u},\mathbf{v})| &= \left|\rho(\mathbf{u},\mathbf{v})_\Omega + C_1(\gamma_o\mathbf{u},\gamma_o\mathbf{v})_\Gamma + C_2(\nabla_s\eta_u,\nabla_s\eta_u)_\Gamma\right| \\ &\le \rho\|\mathbf{u}\|_1\|\mathbf{v}\|_1 + C_1\|\gamma_o\mathbf{u}\|_\Gamma\|\gamma_o\mathbf{v}\|_\Gamma \\ &\quad + C_2\|\nabla_s\eta_u\|_\Gamma\|\nabla_s\eta_v\|_\Gamma \\ &\le C\|\mathbf{u}\|_1\|\mathbf{v}\|_1,\end{aligned} \tag{86}$$

by the Trace theorem and C_1 and C_2 as defined and C generic. □

Lemma 6. *The bilinear form* $|a(\mathbf{u},\mathbf{v})| = (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{v}))_\Omega$ *is coercive in the sense that there exist constants* $c_1 > 0$ *and* $c_o \ge 0$ *such that*

$$|a(\mathbf{u},\mathbf{u})| \ge c_1\|\mathbf{u}\|_1^2 - c_o b(\mathbf{u},\mathbf{u}). \tag{87}$$

Proof. From (83) we have

$$\begin{aligned}a(\mathbf{u},\mathbf{u}) &= (\mathbf{A}(\mathbf{u}),\mathbf{A}(\mathbf{u}))_\Omega \\ &\ge 2\|\nabla\mathbf{u}\|_\Omega^2 + \frac{2g}{S}\|\eta\|_\Gamma^2 \\ &= 2\|\mathbf{u}\|_1^2 - 2\|\mathbf{u}\|_\Omega^2 + \frac{2g}{S}\|\eta\|_\Gamma^2\end{aligned}$$

$$\geq 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}\left(\rho\|\mathbf{u}\|_\Omega^2\right) - \frac{2}{\rho}C_1\|\eta\|_\Gamma^2 - \frac{2}{\rho}C_2\|\nabla_s\eta_u\|_\Gamma^2$$
$$= 2\|\mathbf{u}\|_1^2 - \frac{2}{\rho}b(\mathbf{u},\mathbf{u}). \tag{88}$$

□

We can now obtain a generalised Poincaré inequality.

Lemma 7. *There exists a smallest possible constant $\beta > 0$ such that, for every $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$,*

$$\frac{\beta}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \rho\|\mathbf{v}\|_\Omega^2 + C_1\|\gamma_o\mathbf{v}\|_\Gamma^2 + C_2\|\nabla_s\eta_u\|_\Gamma^2. \tag{89}$$

Proof. From the smoothness of Γ (which is always assumed), it follows that the embedding $J : \mathbf{u} \in \mathbf{H}^1(\Omega) \to \langle \mathbf{u}, \gamma_o\mathbf{u}\rangle \in \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Gamma)$ is compact [40]. From the boundedness and coerciveness proved above it follows that there exists a smallest eigenvalue λ and associated eigenfunction $\mathbf{u} \in \mathbf{H}_\varsigma^1(\Omega)$ for which $b(\mathbf{u},\mathbf{u}) = 1$ (see [34]):

$$\lambda = \inf\left\{a(\mathbf{v},\mathbf{v}) : \mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega); b(\mathbf{v},\mathbf{v}) = 1\right\} = a(\mathbf{u},\mathbf{u}) \tag{90}$$

$\lambda > 0$, for if it is zero, it follows that $\mathbf{u} = 0$, which cannot be. It follows from (90) that for any $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ the inequality

$$a(\mathbf{v},\mathbf{v}) \geq \lambda\left[\rho\|\mathbf{v}\|_\Omega^2 + C_1\|\gamma_o\mathbf{v}\|_\Gamma^2 + C_2\|\nabla_s\eta_u\|_\Gamma^2\right] \tag{91}$$

holds and that λ is the largest such constant. Finally, we set $\beta = 2/\lambda$. □

Remark 8. It is now easy to see that $\|\mathbf{A}(\cdot)\|_\Omega$ is a norm on $\mathbf{H}_\varsigma^1(\Omega)$.

In fact, we have the following lemma.

Lemma 9. *For all $\mathbf{v} \in \mathbf{H}_\varsigma^1(\Omega)$ we have*

$$\frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq \widetilde{E}_v \leq \frac{\alpha+\beta+2}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2. \tag{92}$$

Proof. Add $(\alpha/2)\|\mathbf{A}(\mathbf{v})\|_\Omega^2 + C_2\|\nabla_s\eta\|_{0\Gamma}^2$ to both sides of the inequality (89):

$$\widetilde{E}_v = \frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 + \rho\|\mathbf{v}\|_\Omega^2 + C_1\|\eta\|_\Gamma^2 + C_2\|\nabla_s\eta\|_{0,\Gamma}$$
$$\leq \frac{\alpha+\beta+2}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2. \tag{93}$$

From the definition of the energy norm it is clear that

$$\frac{\alpha}{2}\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq \widetilde{E}_v, \tag{94}$$

and the result follows. □

From Lemma 7 it is clear that these are the best estimates of this form.

Lemma 10. *The norms $\|\mathbf{A}(\mathbf{v})\|_\Omega$ and $\widetilde{E}_v^{1/2}$ are equivalent to the norm in the Sobolev space $\mathbf{H}^1(\Omega)$.*

Proof. From (83) and (89) it follows that

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq 2\|\nabla\mathbf{v}\|_\Omega^2,$$
$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \frac{2\rho}{\beta}\|\mathbf{v}\|_\Omega^2. \tag{95}$$

The addition of the two inequalities above yields

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq \|\nabla\mathbf{v}\|_\Omega^2 + \frac{\rho}{\beta}\|\mathbf{v}\|_\Omega^2. \tag{96}$$

Let $k = \min(1, \rho/\beta)$; then

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \geq k\|\mathbf{v}\|_1^2. \tag{97}$$

Equation (83) yields

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq 2\|\nabla\mathbf{v}\|_\Omega^2 + \frac{2G}{s}\|\eta\|_\Gamma^2$$
$$\leq 2\|\nabla\mathbf{v}\|_\Omega^2 + 2\|\mathbf{v}\|_\Omega^2 + \frac{2G}{s}\|\eta\|_\Gamma^2, \tag{98}$$

and from the Trace theorem it follows that

$$\|\mathbf{A}(\mathbf{v})\|_\Omega^2 \leq C\|\mathbf{v}\|_1^2. \tag{99}$$

From (92) it is evident that the energy norm is equivalent to the norm $\|\mathbf{A}(\mathbf{v})\|_\Omega$. □

Remark 11. From the above lemma we may claim from the embedding $\mathbf{H}^1(\Omega) \to \mathbf{L}^3(\Gamma)$, [40], that there exists a constant $\tau > 0$ such that

$$\int_\Gamma |\gamma_o\mathbf{v}|_\Gamma^3 ds \leq \tau\|\mathbf{A}(\mathbf{v})\|_\Omega^3 \quad \text{for every } \mathbf{v} \in \mathbf{H}_\sigma^1(\Omega). \tag{100}$$

7.4. Stability and Uniqueness for the Model Problem $\mathscr{A}$. $\mathbf{v} \in \mathbf{H}^3(\Omega) \cap \mathscr{D}$ satisfies the system of evolution equations

$$D_t[\rho\mathbf{v}(x,t)] = \nabla\cdot\mathbf{T}(\rho,\mathbf{v}) \quad \text{in } \Omega\times(0,\infty)$$
$$\partial_t[\rho\eta - \alpha\Delta_s\eta] + \delta^{-1}\gamma_o p = \mu\Delta_s\eta \quad \text{at } \Gamma\times(0,\infty) \tag{101}$$
$$\gamma_o[\mathbf{A}(\mathbf{v})] = -2\eta M \quad \text{at } \Gamma\times(0,\infty).$$

We now derive an energy identity for the solutions of (101). Take the $\mathbf{L}^2(\Omega)$, scalar product with $\mathbf{v}$ on both sides of $(101)_1$. This produces

$$(D_t(\rho\mathbf{v}),\mathbf{v}) = \frac{\rho}{2}\partial_t\|\mathbf{v}\|_\Omega^2 - \frac{\rho}{2}\int_\Omega \eta^3 ds$$
$$= (\nabla\cdot\mathbf{T},\mathbf{v})$$
$$= \int_\Gamma \gamma_o\mathbf{v}\cdot\mathbf{T}\mathbf{n}\, ds - (\mathbf{T},\nabla\mathbf{v})$$
$$= -\int_\Gamma \eta\mathbf{n}\cdot\mathbf{T}\mathbf{n}\, ds - \frac{1}{2}\int_\Omega \mathbf{T}:\mathbf{A}\, dx. \tag{102}$$

According to the formulation of the original problem on the boundary where $s(\eta) = \mathbf{n} \cdot \mathbf{Tn}$, we obtain

$$\begin{aligned} &- \int_\Gamma \eta \mathbf{n} \cdot \mathbf{Tn}\, ds \\ &= - \int_\Gamma \eta \Big(- \gamma_o p - 2\mu\eta K - 2\alpha K \eta_t \\ &\qquad + 4\alpha K_G \eta^2 - \alpha \eta \Delta_s \eta - \ell(t) \Big) ds. \end{aligned} \tag{103}$$

From $(101)_2$ we obtain

$$\gamma_o p = -\delta \partial_t \left[\rho\eta - \alpha \Delta_s \eta \right] + \delta\mu \Delta_s \eta. \tag{104}$$

Substitute (104) into (103) to obtain

$$\begin{aligned} -\int_\Gamma \eta \mathbf{n} \cdot \mathbf{Tn}\, ds = &-\partial_t \int_\Gamma \frac{\delta\rho}{2} |\eta|^2 ds - \delta\alpha \partial_t \|\nabla_s \eta\|_{0,\Gamma}^2 \\ &- \delta\mu \|\nabla_s \eta\|_{0,\Gamma}^2 \\ &+ 2\mu\delta \int_\Gamma K\eta^2 ds + \partial_t \int_\Gamma \delta\alpha K |\eta|^2 ds \\ &- 4\alpha\delta \int_\Gamma K_G \eta^3 ds - \alpha\delta \int_\Gamma \eta |\nabla_s \eta|^2 ds. \end{aligned} \tag{105}$$

Also

$$-\frac{1}{2}(\mathbf{T}, \mathbf{A})_\Omega = -\frac{1}{2}\mu \|\mathbf{A}\|_\Omega^2 - \frac{\alpha}{4} \partial_t \|\mathbf{A}\|_\Omega^2 + \alpha \int_\Gamma |\mathbf{M}|^2 \eta^3 ds. \tag{106}$$

Therefore the energy identity for *Problem* $\mathscr{A}$ is

$$\begin{aligned} &\partial_t \left[\frac{\rho}{2} \|\mathbf{v}\|_\Omega^2 + \frac{\alpha}{4} \|\mathbf{A}(\mathbf{v})\|_\Omega^2 \right. \\ &\quad \left. + \int_\Gamma \left(\frac{\rho\delta}{2} - \alpha K \right) |\eta|^2 ds + \delta\alpha \|\nabla_s \eta\|_{0,\Gamma}^2 \right] \\ &= -\frac{1}{2}\mu \|\mathbf{A}(\mathbf{v})\|_\Omega^2 + \alpha \int_\Gamma \left(|M|^2 - 4\delta K_G \right) |\eta|^3 ds \\ &\quad - \delta\mu \|\nabla_s \eta\|_\Gamma^2 + 2\mu\delta \int_\Gamma K |\eta|^2 ds - \alpha\delta \int_\Gamma \eta |\nabla_s \eta|^2 ds. \end{aligned} \tag{107}$$

Now we can define an energy norm for *Problem* $\mathscr{A}$ as follows:

$$\begin{aligned} \widetilde{E}_\nu(t) = &\rho \|\mathbf{v}\|_\Omega^2 + \frac{\alpha}{2} \|\mathbf{A}(\mathbf{v})\|_\Omega^2 \\ &+ \int_\Gamma (\delta\rho - 2\alpha K) |\eta|^2 ds + 2\delta\alpha \|\nabla_s \eta\|_{0,\Gamma}^2. \end{aligned} \tag{108}$$

Note that here we have to make the assumption that $\delta\rho - 2\alpha K > 0$, which gives us a restriction on K. We define a parameter

$$p_2 = \frac{\alpha K}{\delta\rho}. \tag{109}$$

It is now clear that stability can only be proved under the assumption that $p_2 \in (0, 1/2)$.

The Poincaré inequality (see [39]) states that there exists a smallest constant c such that $\|\eta\|_{0,\Gamma}^2 \geq c \|\nabla_s \eta\|_{0,\Gamma}^2$. Applying the Schwartz inequality and the above Poincaré inequality, we obtain

$$\begin{aligned} \frac{d}{dt} \widetilde{E}_\nu(t) &\leq -\mu \|\mathbf{A}(\mathbf{v})\|_\Omega^2 + 2\alpha \left(G^2 + H^2 + 4\delta G^2 \right) \|\eta\|_{0,\Gamma}^3 \\ &\quad + 2\delta \|\nabla_s \eta\|_{0,\Gamma}^2 \left(\mu + \alpha \|\eta\|_{0,\Gamma} \right) + 4\delta\mu G \|\eta\|_{0,\Gamma}^2 \\ &\leq -\widetilde{E}_\nu \left[\mu - 2\alpha \left(G^2 + H^2 + 2\delta G^2 - \delta \right) \widetilde{E}_\nu^{1/2} \right. \\ &\quad \left. - 2\delta\mu - 4\delta\mu G \right]. \end{aligned} \tag{110}$$

With $2\alpha(G^2 + H^2 + 4\delta G^2 - \delta) = \epsilon^*$ and $\mu(1 - 2\delta - 4\delta G) = \epsilon^{**}$ we have

$$\frac{d}{dt} \widetilde{E}_\nu \leq -\widetilde{E}_\nu \left[\epsilon^{**} \mu - \epsilon^* \widetilde{E}_\nu^{1/2} \right]. \tag{111}$$

Theorem 12 (stability for problem $\mathscr{A}$). *If* $p_2 \in (0, 1/2)$ *and* $\widetilde{E}_\nu(0) < (\epsilon^{**}\mu/\epsilon^*)^2$, *then the energy* $\widetilde{E}_\nu(t)$ *decreases exponentially to zero as* $t \to \infty$.

The uniqueness of the solution of *Problem* $\mathscr{A}$ is treated in the same way as the uniqueness of the solution of the original problem (see [33]).

8. Conclusion

An extensive study was conducted to find expressions for the stress tensors of Newtonian and non-Newtonian fluids at a permeable surface. We employed the Serret-Frenet formulae exactly for this reason. Stability of the rest state and uniqueness were proven for a special case where a shear flow was taken into account.

These results proved to be valuable in applications for the study of blood flow, where they were applied to model the permeability of special capillaries in the formation of cerebrospinal fluid [41, 42]. Here the authors have presented a mathematical model of the flow of blood through the permeable boundary of a blocked choroidal capillary in which the parameters could be controlled. The blood plasma was modelled as a Newtonian fluid and the nonlinear Stokes equations were supplemented with a boundary condition at the permeable interface of the specialized capillary. The existence of a unique weak solution, which depends on the viscosity and the nature of the curvature of the capillary, was proved. By incorporating in this model all the ultrafiltration parameters, which are presented in [41, 42], the authors have attempted (within the prescribed morphological and physiological properties of the microvascular environment) to adapt the model used by Maritz and Sauer [33] to real-life situations. Further applications could be found in the modelling of other permeable systems in the human body like the lymphatic glands and the urinary system.

With this research the authors have tried to prepare the ground for the applications of these results in the exploring of permeable surfaces in biosciences, engineering, and the natural sciences. The open question regarding the existence

of a classical solution for the system (101) will be addressed in further research.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. Leray, "Etude de diverses équations intégrales nonlinéaires que pose hýdrodynamique," *Journal de Mathématiques Pures et Appliquées*, vol. 12, pp. 1–82, 1933.

[2] J. Leray, "Essai sur les mouvements plans d'un liquide visqueux que limitent des parois," *Journal de Mathématiques Pures et Appliquées*, vol. 13, pp. 331–418, 1934.

[3] J. Leray, "Sur le mouvement d'un liquide visqueux emplissant l'espace," *Acta Mathematica*, vol. 63, no. 1, pp. 193–248, 1934.

[4] E. Hopf, "Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen," *Mathematische Nachrichten*, vol. 4, pp. 213–231, 1951.

[5] G. P. Galdi, M. Padula, and K. R. Rajagopal, "On the conditional stability of the rest state of a fluid of second grade in unbounded domains," *Archive for Rational Mechanics and Analysis*, vol. 109, no. 2, pp. 173–182, 1990.

[6] G. P. Galdi, "Mathematical theory of second-grade fluids," in *Stability and Wave Propagation in Fluids and Solids*, pp. 67–104, Springer, Berlin, Germany, 1995.

[7] S. N. Alekseenko, "Existence and asymptotic representation of weak solutions to the flowing problem under the condition of regular slippage on solid walls," *Siberian Mathematical Journal*, vol. 35, no. 2, pp. 209–230, 1994.

[8] D. Cioranescu and O. El Hacène, "Existence and uniqueness for fluids of second grade," in *Nonlinear Partial Differential Equations and Their Applications*, vol. 109 of *Research Notes in Mathematics*, pp. 178–197, Pitman, Boston, Mass, USA, 1984.

[9] D. Cioranescu and V. Girault, "Solutions variationnelles et classiques d'une famille de fluides de grade deux," *Comptes Rendus de l'Académie des Sciences Series I*, vol. 322, no. 12, pp. 1163–1168, 1996.

[10] D. Cioranescu and V. Girault, "Weak and classical solutions of a family of second grade fluids," *International Journal of Non-Linear Mechanics*, vol. 32, no. 2, pp. 315–335, 1997.

[11] D. Cioranescu and E. H. Quazar, "Existence and uniqueness for fluids of second grade," in *College de France Seminars*, vol. 109 of *Pitman Research Notes in Mathematics*, pp. 178–197, Pitman, Boston, Mass, USA, 1984.

[12] V. Coscia and G. P. Galdi, "Existence, uniqueness and stability of regular steady motions of a second-grade fluid," *International Journal of Non-Linear Mechanics*, vol. 29, no. 4, pp. 493–506, 1994.

[13] G. P. Galdi, M. Grobbelaar-van Dalsen, and N. Sauer, "Existence and uniqueness of classical solutions of the equations of motion for second-grade fluids," *Archive for Rational Mechanics and Analysis*, vol. 124, no. 3, pp. 221–237, 1993.

[14] V. Coscia, A. Sequeira, and J. Videman, "Existence and uniqueness of classical solutions for a class of complexity 2 fluids," *International Journal of Non-Linear Mechanics*, vol. 30, no. 4, pp. 531–551, 1995.

[15] G. P. Galdi, M. Grobbelaar-Van Dalsen, and N. Sauer, "Existence and uniqueness of solutions of the equations of motion for a fluid of second grade with non-homogeneous boundary conditions," *International Journal of Non-Linear Mechanics*, vol. 30, no. 5, pp. 701–709, 1995.

[16] R. Berker, "Contrainte sur une paroi en contact avec un fluide visqueux classique, un fluide de Stokes, un fluide de Coleman-Noll," *Comptes Rendus de l'Académie des Sciences*, vol. 258, pp. 5144–5147, 1964.

[17] P. Bourgin and J. A. Tichy, "The effect of an additional boundary condition on the plane creeping flow of a second-order fluid," *International Journal of Non-Linear Mechanics*, vol. 24, no. 6, pp. 561–569, 1989.

[18] C.-S. Man, "Nonsteady channel flow of ice as a modified second-order fluid with power-law viscosity," *Archive for Rational Mechanics and Analysis*, vol. 119, no. 1, pp. 35–57, 1992.

[19] K. R. Rajagopal and A. S. Gupta, "An exact solution for the flow of a non-Newtonian fluid past an infinite porous plate," *Meccanica*, vol. 19, no. 2, pp. 158–160, 1984.

[20] K. R. Rajagopal and P. N. Kaloni, "Some remarks on boundary conditions for flows of fluids of the differential type," in *Continuum Mechanics and its Applications*, pp. 935–942, Hemisphere Press, 1989.

[21] K. R. Rajagopal, "On boundary conditions for fluids of the differential type," in *Navier-Stokes Equations and Related Non-Linear Problems*, Plenum Press, 1995.

[22] R. L. Fosdick and K. R. Rajagopal, "Anomalous features in the model of 'second order fluids'," *Archive for Rational Mechanics and Analysis*, vol. 70, no. 2, pp. 145–152, 1979.

[23] R. L. Fosdick and K. R. Rajagopal, "Thermodynamics and stability of fluids of third grade," *Proceedings of the Royal Society. London. Series A. Mathematical and Physical Sciences*, vol. 339, no. 1738, pp. 351–377, 1980.

[24] I. Proudman, "An example of steady laminar flow at large Reynolds number," *Journal of Fluid Mechanics*, vol. 9, pp. 593–602, 1960.

[25] G. S. Beavers and D. D. Joseph, "Boundary conditions at a naturally permeable wall," *Journal of Fluid Mechanics*, vol. 30, no. 1, pp. 197–207, 1967.

[26] K. R. Rajagopal and L. Tao, *Mechanics of Mixtures*, vol. 35 of *Advances in Mathematics for Applied Sciences*, World Scientific, River Edge, NJ, USA, 1995.

[27] H. Lamb, *Hydrodynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, UK, 6th edition, 1932.

[28] O. A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York, NY, USA, 1969.

[29] N. Sauer, "On Helmholtz projections," in *Navier-Stokes Equations: Theory and Numerical Methods*, R. Salvi, Ed., vol. 388 of *Pitman Research Notes in Maths Series*, pp. 257–263, Longman, Harlow, UK, 1998.

[30] M. Schwartz, S. Green, and W. A. Rutledge, *Vector Analysis: With Applications to Geometry and Physics*, Harper's Mathematics Series, Harper & Brothers, New York, NY, USA, 1960.

[31] A. J. M. Spencer, *Continuum Mechanics*, Longman Group Limited, London, UK, 1980.

[32] J. E. Dunn and R. L. Fosdick, "Thermodynamics, stability, and boundedness of fluids of complexity 2 and fluids of second grade," *Archive for Rational Mechanics and Analysis*, vol. 56, pp. 191–252, 1974.

[33] R. Maritz and N. Sauer, "On boundary permeation in Navier-Stokes and second grade incompressible fluids," *Mathematical Models & Methods in Applied Sciences*, vol. 16, no. 1, pp. 59–75, 2006.

[34] N. Sauer and A. van der Merwe, "Eigenvalue problems with the spectral parameter also in the boundary condition," *Quaestiones Mathematicae*, vol. 5, no. 1, pp. 1–27, 1982.

[35] C. E. Weatherburn, *Differential Geometry of Three Dimensions*, vol. 1, Cambridge University Press, 1931.

[36] C. E. Weatherburn, *Differential Geometry of Three Dimensions*, vol. 2, Cambridge University Press, New York, NY, USA, 1930.

[37] R. Maritz and N. Sauer, "Stability and uniqueness of second grade fluids in regions with permeable Boundaries," in *Navier Stokes Equations and Related Non linear Problems*, H. Amann, G. P. Galdi, K. Pileckas, and V. A. Solonnikov, Eds., pp. 153–164, VSP, 1998.

[38] C. Truesdell and W. Noll, "The non-linear field theories of mechanics," in *Handbuch der Physik*, S. Flügge, Ed., vol. 3, chapter 1, pp. 1–602, Springer, Berlin, Germany, 1965.

[39] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, vol. 1, Springer, Berlin, Germany, 1972.

[40] R. A. Adams, *Sobolev Spaces*, Academic Press, 1975.

[41] R. Maritz and S. Faleye, "The formation of cerebrospinal fluid in the brain (the linearised model)," *Advances and Applications in Mathematical Sciences*, vol. 8, no. 1, pp. 39–60, 2011.

[42] R. Maritz, J. M. Munganga, and S. Faleye, "On boundary permeation through the walls of blocked cerebral capillaries: a non-linear model," *Journal of Mathematical Fluid Mechanics*, vol. 15, no. 3, pp. 567–581, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 653917, 5 pages
http://dx.doi.org/10.1155/2014/653917

Research Article

Majorization for a Class of Analytic Functions Defined by q-Differentiation

K. A. Selvakumaran,[1] Sunil Dutt Purohit,[2] and Aydin Secer[3]

[1] *Department of Mathematics, R.M.K College of Engineering and Technology, Puduvoyal, Tamil Nadu 601206, India*
[2] *Department of Basic Sciences (Mathematics), College of Technology and Engineering, M.P. University of Agriculture and Technology, Udaipur, Rajasthan 313001, India*
[3] *Department of Mathematical Engineering, Yildiz Technical University, Davutpasa, 34210 Istanbul, Turkey*

Correspondence should be addressed to Aydin Secer; asecer@yildiz.edu.tr

Received 24 May 2014; Accepted 1 July 2014; Published 10 July 2014

Academic Editor: Mustafa Bayram

We introduce a new class of multivalent analytic functions defined by using q-differentiation and fractional q-calculus operators. Further, we investigate majorization properties for functions belonging to this class. Also, we point out some new and known consequences of our main result.

1. Introduction and Preliminaries

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n, \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$. For analytic functions $f(z)$ and $g(z)$ in $\mathcal{U}$, we say that the function $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$ (see [1]) and write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}), \tag{2}$$

if there exists a Schwarz function $w(z)$, analytic in $\mathcal{U}$, such that

$$|w(z)| \leq 1, \quad f(z) = w(z)\, g(z) \quad (z \in \mathcal{U}). \tag{3}$$

For the convenience of the reader, we now give some basic definitions and related details of q-calculus which are used in the sequel.

For any complex number α the q-shifted factorials are defined as

$$(\alpha; q)_0 = 1, \quad (\alpha; q)_n = \prod_{k=0}^{n-1} \left(1 - \alpha q^k\right), \quad n \in \mathbb{N}, \tag{4}$$

and in terms of the basic analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)\,(1-q)^n}{\Gamma_q(\alpha)}, \quad (n > 0), \tag{5}$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q, q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty}, \quad (0 < q < 1). \tag{6}$$

If $|q| < 1$, the definition (4) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} \left(1 - \alpha q^j\right). \tag{7}$$

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n, \tag{8}$$

we observe that the q-shifted factorial (4) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha + 1) \cdots (\alpha + n - 1)$.

It may be noted that the q-Gauss hypergeometric function ${}_2\Phi_1[-]$ (see Gasper and Rahman [2, p.3, eqn. (1.2.14)]) is defined by

$$ {}_2\Phi_1\left[\alpha,\beta;\gamma;q,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n(\beta;q)_n}{(\gamma;q)_n(q;q)_n}z^n,\quad \left(|q|<1,|z|<1\right), \tag{9}$$

and as a special case of the above series for $\gamma=\beta$, we have

$$ {}_1\Phi_0\left[\alpha;-;q,z\right]=\sum_{n=0}^{\infty}\frac{(\alpha;q)_n}{(q;q)_n}z^n,\quad \left(|q|<1,|z|<1\right). \tag{10}$$

Also, the q-derivative and q-integral of a function on a subset of $\mathbb{C}$ are, respectively, given by (see [2, pp. 19–22])

$$ D_qf(z)=\frac{f(z)-f(zq)}{(1-q)z},\quad (z\neq 0,q\neq 0),$$
$$\int_0^z f(t)\,d_qt=z(1-q)\sum_{k=0}^{\infty}q^kf\left(zq^k\right). \tag{11}$$

Therefore, the q-derivative of $f(z)=z^n$, where n is a positive integer, is given by

$$ D_qz^n=\frac{z^n-(zq)^n}{(1-q)z}=[n]_q\,z^{n-1}, \tag{12}$$

where

$$ [n]_q=\frac{1-q^n}{1-q}=q^{n-1}+\cdots+1, \tag{13}$$

and is called the q-analogue of n. As $q\to 1$, we have $[n]_q=q^{n-1}+\cdots+1\to 1+\cdots+1=n$. Here we list some relations satisfied by $[n]_q$:

$$ [m+n]_q=[m]_q+q^m[n]_q=q^n[m]_q+[n]_q,$$
$$ [m-n]_q=q^{-n}[m]_q-q^{-n}[n]_q, \tag{14}$$
$$ [0]_q=0,\qquad [1]_q=1.$$

Recently, many authors have introduced new classes of analytic functions using q-calculus operators. For some recent investigations on the classes of analytic functions defined by using q-calculus operators and related topics, we refer the reader to [3–13] and the references cited therein. In the following, we define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [9].

Definition 1 (fractional q-integral operator). The fractional q-integral operator $I_{q,z}^{\delta}$ of a function $f(z)$ of order δ is defined by

$$ I_{q,z}^{\delta}f(z)\equiv D_{q,z}^{-\delta}f(z)$$
$$=\frac{1}{\Gamma_q(\delta)}\int_0^z(z-tq)_{\delta-1}f(t)\,d_qt,\quad(\delta>0), \tag{15}$$

where $f(z)$ is analytic in a simply connected region of the z-plane containing the origin and the q-binomial function $(z-tq)_{\delta-1}$ is given by

$$ (z-tq)_{\delta-1}=z^{\delta-1}\,{}_1\Phi_0\left[q^{-\delta+1};-;q,\frac{tq^{\delta}}{z}\right]. \tag{16}$$

The series ${}_1\Phi_0[\delta;-;q,z]$ is single valued when $|\arg(z)|<\pi$ and $|z|<1$ (see for details [2], pp. 104–106); therefore, the function $(z-tq)_{\delta-1}$ in (15) is single valued when $|\arg(-tq^{\delta}/z)|<\pi$, $|tq^{\delta}/z|<1$, and $|\arg(z)|<\pi$.

Definition 2 (fractional q-derivative operator). The fractional q-derivative operator $D_{q,z}^{\delta}$ of a function $f(z)$ of order δ is defined by

$$ D_{q,z}^{\delta}f(z)\equiv D_{q,z}I_{q,z}^{1-\delta}f(z)$$
$$=\frac{1}{\Gamma_q(1-\delta)}\,D_{q,z}\int_0^z(z-tq)_{-\delta}\,f(t)\,d_qt, \tag{17}$$
$$(0\le\delta<1),$$

where $f(z)$ is suitably constrained and the multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 1.

Definition 3 (extended fractional q-derivative operator). Under the hypotheses of Definition 2, the fractional q-derivative for a function $f(z)$ of order δ is defined by

$$ D_{q,z}^{\delta}f(z)=D_{q,z}^{m}I_{q,z}^{m-\delta}f(z), \tag{18}$$

where $m-1\le\delta<1, m\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, and $\mathbb{N}$ denotes the set of natural numbers.

Remark 4. It follows from Definition 2 that

$$ D_{q,z}^{\delta}\,z^n=\frac{\Gamma_q(n+1)}{\Gamma_q(n+1-\delta)}\,z^{n-\delta},\quad(\delta\ge 0,\,n>-1). \tag{19}$$

Using $D_{q,z}^{\delta}$, we define a q-differintegral operator $\Omega_{q,p}^{\delta}:\mathcal{A}_p\to\mathcal{A}_p$, as follows:

$$ \Omega_{q,p}^{\delta}f(z)=\frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)}z^{\delta}D_{q,z}^{\delta}f(z),$$
$$\left(-\infty<\delta<p+1;\ 0<q<1;\ z\in\mathcal{U}\right), \tag{20}$$

where $D_{q,z}^{\delta}f(z)$ in (20) represents, respectively, a fractional q-integral of $f(z)$ of order δ when $-\infty<\delta<0$ and a fractional q-derivative of $f(z)$ of order δ when $0\le\delta<p+1$. It is easy to see from (20) that

$$ \Omega_{q,p}^{\delta}f(z)=z^p+\sum_{n=p+1}^{\infty}\frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)}a_nz^n, \tag{21}$$

$$ q^jzD_q^{j+1}\left(\Omega_{q,p}^{\delta}f(z)\right)=q^{\delta}[p-\delta]_qD_q^j\left(\Omega_{q,p}^{\delta+1}f(z)\right)$$
$$+q^j[\delta-j]_qD_q^j\left(\Omega_{q,p}^{\delta}f(z)\right),$$
$$\left(0\le j\le p;\ -\infty<\delta<p;\ z\in\mathcal{U}\right). \tag{22}$$

Definition 5. A function $f(z) \in \mathcal{A}_p$ is said to be in the class $S_{q,p}^{\delta,j}(b)$ of p-valent functions of complex order $b \neq 0$ in $\mathcal{U}$ if and only if

$$\operatorname{Re}\left\{1+\frac{1}{b}\left(\frac{zD_q^{j+1}\left(\Omega_{q,p}^{\delta}f(z)\right)}{D_q^{j}\left(\Omega_{q,p}^{\delta}f(z)\right)}-[p-j]_q\right)\right\}>0, \quad \left(z\in\mathcal{U};\ p\in\mathbb{N};\ j\in\mathbb{N}_0;\ b\in\mathbb{C}-\{0\};\ \left|2bq^{j-\delta}-[p-\delta]_q\right|\leq[p-\delta]_q\right). \tag{23}$$

It can be seen that, by specializing the parameters, the class $S_{q,p}^{\delta,j}(b)$ reduces to many known subclasses of analytic functions. For instance, if $q \to 1$ then

(1) $\mathcal{S}_{1,1}^{0,0}(b) = \mathcal{S}(b)$, the class of starlike functions of complex order b (see [14]),

(2) $\mathcal{S}_{1,1}^{1,0}(b) = \mathcal{C}(b)$, the class of convex functions of complex order b (see [15]),

(3) $\mathcal{S}_{1,1}^{0,0}(\cos\alpha e^{-i\alpha}) = \mathcal{S}^{\alpha}, (|\alpha| < \pi/2)$, the class of α-spiral-like functions (see [16]),

(4) $\mathcal{S}_{1,1}^{0,0}(1-\alpha) = \mathcal{S}^{*}(\alpha), (0 \leq \alpha < 1)$, the class of starlike functions of order α.

2. Majorization Problem for the Class $S_{q,p}^{\delta,j}(b)$

We start by proving the following q-analogue of the result given by Nehari in [17].

Lemma 6. *If $f(z)$ is analytic and bounded in $\mathcal{U}$, then*

$$\left|D_q\left(f(z)\right)\right| \leq \frac{1-\left|f(z)\right|^2}{1-|z|^2}, \quad (z\in\mathcal{U}). \tag{24}$$

Proof. If $f(z)$ is bounded in $\mathcal{U}$, then

$$g(z) = \frac{f(z)-f(zq)}{1-\overline{f(zq)}f(z)}, \quad (0<q<1) \tag{25}$$

is also bounded in $\mathcal{U}$. Clearly $g(z)$ vanishes when $q = 1$. Therefore, the function

$$h(z) = \frac{g(z)}{\left((z-zq)/(1-z\bar{z}q)\right)} = \left(\frac{f(z)-f(zq)}{z-zq}\right)\left(\frac{1-z\bar{z}q}{1-\overline{f(zq)}f(z)}\right) \tag{26}$$

is regular when $q = 1$ and also at all other points of $|z| < 1$. Furthermore, $h(z)$ is bounded in $|z| < 1$. In fact, $\lim_{|z|\to 1}|g(z)| \leq 1$ and $|(z-zq)/(1-z\bar{z}q)| = 1$ for $|z| = 1$; hence by maximum principle, $|h(z)| \leq 1$ throughout $|z| < 1$. Then from (26) we have

$$\left|D_q\left(f(z)\right)\right|\left(\frac{1-|z|^2}{1-\left|f(z)\right|^2}\right) \leq 1 \quad \text{which implies} \quad \left|D_q\left(f(z)\right)\right| \leq \frac{1-\left|f(z)\right|^2}{1-|z|^2}. \tag{27}$$

□

Theorem 7. *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g(z) \in S_{q,p}^{\delta,j}(b)$. If $D_q^j(\Omega_{q,p}^{\delta}f(z))$ is majorized by $D_q^j(\Omega_{q,p}^{\delta}g(z))$ in $\mathcal{U}$ for $j \in \mathbb{N}_0$, then*

$$\left|D_q^j\left(\Omega_{q,p}^{\delta+1}f(z)\right)\right| \leq \left|D_q^j\left(\Omega_{q,p}^{\delta+1}g(z)\right)\right| \quad \text{for } |z| \leq r_1, \tag{28}$$

where

$$r_1 = r_q(p,j,\delta;b) := \frac{k-\sqrt{k^2-4[p-\delta]_q\left|2bq^{j-\delta}-[p-\delta]_q\right|}}{2\left|2bq^{j-\delta}-[p-\delta]_q\right|} \tag{29}$$

$$\left(k := 2q^{j-\delta}+[p-\delta]_q+\left|2bq^{j-\delta}-[p-\delta]_q\right|;\ p\in\mathbb{N};\ b\in\mathbb{C}-\{0\};\ 0<q<1;\ \delta\geq 0\right).$$

Proof. Let

$$h(z) = 1+\frac{1}{b}\left(\frac{zD_q^{j+1}\left(\Omega_{q,p}^{\delta}g(z)\right)}{D_q^{j}\left(\Omega_{q,p}^{\delta}g(z)\right)}-[p-j]_q\right) \tag{30}$$

$$\left(z\in\mathcal{U};\ b\in\mathbb{C}-\{0\};\ p\in\mathbb{N};\ j\in\mathbb{N}_0 \text{ and } j<p\right).$$

Since $g(z) \in S_{q,p}^{\delta,j}(b)$, we have $\operatorname{Re}(h(z)) > 0$ $(z \in \mathcal{U})$ and

$$h(z) = \frac{1+w(z)}{1-w(z)}, \quad (w\in\mathcal{P}), \tag{31}$$

where $\mathcal{P}$ denotes the well known class of bounded analytic functions in $\mathcal{U}$, which satisfy the conditions (cf. [18])

$$w(0) = 0, \quad |w(z)| \leq |z| \quad (z\in\mathcal{U}). \tag{32}$$

It follows from (30) and (31) that

$$\frac{zD_q^{j+1}\left(\Omega_{q,p}^{\delta}g(z)\right)}{D_q^{j}\left(\Omega_{q,p}^{\delta}g(z)\right)} = \frac{[p-j]_q+\left(2b-[p-j]_q\right)w(z)}{1-w(z)}. \tag{33}$$

In view of the identity (22), we have the following inequality from (33) by making some simple calculations:

$$\left|D_q^j\left(\Omega_{q,p}^{\delta}g(z)\right)\right| \leq \frac{[p-\delta]_q(1+|z|)}{[p-\delta]_q-\left|2bq^{j-\delta}-[p-\delta]_q\right||z|}\times\left|D_q^j\left(\Omega_{q,p}^{\delta+1}g(z)\right)\right|. \tag{34}$$

Since $D_q^j(\Omega_{q,p}^{\delta}f(z))$ is majorized by $D_q^j(\Omega_{q,p}^{\delta}g(z))$ in $\mathcal{U}$, there exists an analytic function $\varphi(z)$ such that

$$D_q^j\left(\Omega_{q,p}^{\delta}f(z)\right) = \varphi(z)\,D_q^j\left(\Omega_{q,p}^{\delta}g(z)\right), \tag{35}$$

and $|\varphi(z)| \leq 1$ $(z \in \mathcal{U})$. Applying q-differentiation with respect to z and multiplying by z, we have from (35)

$$zD_q^{j+1}\left(\Omega_{q,p}^{\delta} f(z)\right) = zD_q(\varphi(z))\, D_q^j\left(\Omega_{q,p}^{\delta} g(z)\right) + z\varphi(z)\, D_q^{j+1}\left(\Omega_{q,p}^{\delta} g(z)\right). \tag{36}$$

Using (22), in the above equation, we get

$$D_q^j\left(\Omega_{q,p}^{\delta+1} f(z)\right) = \frac{zD_q(\varphi(z))}{q^{\delta-j}[p-\delta]_q} D_q^j\left(\Omega_{q,p}^{\delta} g(z)\right) + \varphi(z)\, D_q^j\left(\Omega_{q,p}^{\delta+1} g(z)\right). \tag{37}$$

Noting that $\varphi(z)$ is bounded in $\mathcal{U}$ and using Lemma 6 we obtain

$$\left|D_q(\varphi(z))\right| \leq \frac{1-|\varphi(z)|^2}{1-|z|^2}, \quad (z \in \mathcal{U}). \tag{38}$$

Appling (34) and (38) in (37) we get

$$\begin{aligned}
&\left|D_q^j\left(\Omega_{q,p}^{\delta+1} f(z)\right)\right| \\
&\leq \left\{\varphi(z) + \frac{1-|\varphi(z)|^2}{1-|z|} \cdot \frac{q^{j-\delta}|z|}{[p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right||z|}\right\} \\
&\quad \times \left|D_q^j\left(\Omega_{q,p}^{\delta+1} g(z)\right)\right| \\
&= \frac{-q^{j-\delta}r\rho^2 + (1-r)\left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| r\right)\rho + q^{j-\delta}r}{(1-r)\left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| r\right)} \\
&\quad \times \left|D_q^j\left(\Omega_{q,p}^{\delta+1} g(z)\right)\right| \quad (|z| = r, |\varphi(z)| = \rho, 0 \leq \rho \leq 1) \\
&= \frac{\Theta(\rho)}{(1-r)\left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| r\right)} \\
&\quad \times \left|D_q^j\left(\Omega_{q,p}^{\delta+1} g(z)\right)\right| \quad (z \in \mathcal{U}),
\end{aligned} \tag{39}$$

where the function $\Theta(\rho)$, defined by

$$\Theta(\rho) := -q^{j-\delta}r\rho^2 + (1-r) \times \left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| r\right)\rho + q^{j-\delta}r, \quad (0 \leq \rho \leq 1), \tag{40}$$

takes its maximum value at $\rho = 1$ with $r = r_q(p, j, \delta; b)$ given by (29). Furthermore, if $0 \leq \sigma \leq r_q(p, j, \delta; b)$ where $r_q(p, j, \delta; b)$ given by (29), then the function

$$\Phi(\rho) := -q^{j-\delta}\sigma\rho^2 + (1-\sigma) \times \left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| \sigma\right)\rho + q^{j-\delta}\sigma \tag{41}$$

increases in the interval $0 \leq \rho \leq 1$, so that $\Phi(\rho)$ does not exceed

$$\Phi(1) = (1-\sigma)\left([p-\delta]_q - \left|2bq^{j-\delta} - [p-\delta]_q\right| \sigma\right), \quad \left(0 \leq \sigma \leq r_q(p, j, \delta; b)\right). \tag{42}$$

Therefore, from this fact, (39) gives inequality (28). □

Letting $q \to 1$, $p = 1$, $\delta = 0$, and $j = 0$ in Theorem 7, we have the following.

Corollary 8 (see [19]). *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z) \in \mathcal{S}(b)$, the class of starlike functions of complex order b. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \leq \left|g'(z)\right| \quad \text{for } |z| \leq r_2, \tag{43}$$

where

$$r_2 := \frac{3 + |2b-1| - \sqrt{9 + 2|2b-1| + |2b-1|^2}}{2|2b-1|}. \tag{44}$$

For $b = \cos\alpha e^{-i\alpha}$, Corollary 8 reduces to the following result.

Corollary 9. *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z) \in \mathcal{S}^{\alpha}(|\alpha| < \pi/2)$, the class of α-spiral-like functions. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then $|f'(z)| \leq |g'(z)|$ for $|z| \leq 2-\sqrt{3}$.*

Further setting $b = 1$, in Corollary 8 we get the following.

Corollary 10 (see [1]). *Let the function $f(z) \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z)$ is starlike in $\mathcal{U}$. If $f(z)$ is majorized by $g(z)$ in $\mathcal{U}$, then*

$$\left|f'(z)\right| \leq \left|g'(z)\right| \quad \text{for } |z| \leq 2-\sqrt{3}. \tag{45}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] T. H. MacGregor, "Majorization by univalent functions," *Duke Mathematical Journal*, vol. 34, pp. 95–102, 1967.

[2] G. Gasper and M. Rahman, *Basic Hypergeometric Series*, vol. 35 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, Cambridge, UK, 1990.

[3] H. Aldweby and M. Darus, "A subclass of harmonic univalent functions associated with q-analogue of Dziok-Srivastava operator," *ISRN Mathematical Analysis*, vol. 2013, Article ID 382312, 6 pages, 2013.

[4] H. Aldweby and M. Darus, "On harmonic meromorphic functions associated with basic hypergeometric functions," *The Scientific World Journal*, vol. 2013, Article ID 164287, 7 pages, 2013.

[5] H. Aldweby and M. Darus, "Some subordination results on q-analogue of Ruscheweyh differential operator," *Abstract and Applied Analysis*, vol. 2014, Article ID 958563, 6 pages, 2014.

[6] G. Murugusundaramoorthy, C. Selvaraj, and O. S. Babu, "Subclasses of starlike functions associated with fractional q-calculus operators," *Journal of Complex Analysis*, vol. 2013, Article ID 572718, 8 pages, 2013.

[7] G. Murugusundaramoorthy and T. Janani, "Meromorphic parabolic starlike functions associated with q-hypergeometric series," *ISRN Mathematical Analysis*, vol. 2014, Article ID 923607, 9 pages, 2014.

[8] S. D. Purohit, "A new class of multivalently analytic functions associated with fractional q-calculus operators," *Fractional Differential Calculus*, vol. 2, no. 2, pp. 129–138, 2012.

[9] S. D. Purohit and R. K. Raina, "Certain subclasses of analytic functions associated with fractional q-calculus operators," *Mathematica Scandinavica*, vol. 109, no. 1, pp. 55–70, 2011.

[10] S. D. Purohit and R. K. Raina, "Fractional q-calculus and certain subclass of univalent analytic functions," *Mathematica*, vol. 55, no. 1, pp. 62–74, 2013.

[11] S. D. Purohit and R. K. Raina, "Some classes of analytic and multivalent functions associated with q-derivative operators," *Acta Universitatis Sapientiae, Mathematica*, vol. 6, no. 1, 2014.

[12] S. D. Purohit and R. K. Raina, "On a subclass of p-valent analytic functions involving fractional q-calculus operators," *Kuwait Journal of Science*, vol. 41, no. 3, 2014.

[13] K. A. Selvakumaran, S. D. Purohit, A. Secer, and M. Bayram, "Convexity of certain q-integral operators of p-valent functions," *Abstract and Applied Analysis*, vol. 2014, Article ID 925902, 7 pages, 2014.

[14] M. A. Nasr and M. K. Aouf, "Starlike function of complex order," *The Journal of Natural Sciences and Mathematics*, vol. 25, no. 1, pp. 1–12, 1985.

[15] P. Wiatrowski, "The coefficients of a certain family of holomorphic functions," *Zeszyty Naukowe Uniwersytetu Lodzkiego, Nauki Matematyczno Przyrodnicze II, Zeszyt*, no. 39, pp. 75–85, 1971.

[16] R. J. Libera, "Univalent α-spiral functions," *Canadian Journal of Mathematics*, vol. 19, pp. 449–456, 1967.

[17] Z. Nehari, *Conformal Mapping*, McGraw-Hill, New York, NY, USA, 1952.

[18] A. W. Goodman, *Univalent Functions*, Mariner, Tampa, Fla, USA, 1983.

[19] O. Altintaş, Ö. Özkan, and H. M. Srivastava, "Majorization by starlike functions of complex order," *Complex Variables: Theory and Application*, vol. 46, no. 3, pp. 207–218, 2001.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 789769, 5 pages
http://dx.doi.org/10.1155/2014/789769

Research Article

Exact Solutions of Fragmentation Equations with General Fragmentation Rates and Separable Particles Distribution Kernels

S. C. Oukouomi Noutchie[1] and E. F. Doungmo Goufo[1,2]

[1] *MaSIM Focus Area, North-West University, Mafikeng 2735, South Africa*
[2] *Department of Mathematical Sciences, University of South Africa, Florida 1709, South Africa*

Correspondence should be addressed to S. C. Oukouomi Noutchie; 23238917@nwu.ac.za

Received 27 May 2014; Revised 21 June 2014; Accepted 22 June 2014; Published 8 July 2014

Academic Editor: Abdon Atangana

We make use of Laplace transform techniques and the method of characteristics to solve fragmentation equations explicitly. Our result is a breakthrough in the analysis of pure fragmentation equations as this is the first instance where an exact solution is provided for the fragmentation evolution equation with general fragmentation rates. This paper is the key for resolving most of the open problems in fragmentation theory including "shattering" and the sudden appearance of infinitely many particles in some systems with initial finite particles number.

1. Preliminaries

Fragmentation models occur in a large variety of situations including the study of stellar fragments in astrophysics, rock fracture, polymer degradation, DNA fragmentation, aggregates breakage, breakup of solid drugs in organisms, and liquid droplet decay. The theoretical framework of fragmentation dynamics can be traced back to papers by Melzak [1] (analytically) and Filippov [2] (probabilistically). In the 1980s, a systematic investigation of fragmentation models was undertaken, mainly by Ziff and his students, for example [3, 4]. They provided analytical solutions to a large class of equations of the form

$$\frac{\partial}{\partial t}u(t,x) = -a(x)\,u(t,x) + \int_x^\infty a(y)\,b(x \mid y)\,u(t,y)\,dy, \quad x \geq 0,\ t > 0, \tag{1}$$

with power rates $a(x) = x^\alpha$, $\alpha \in (-\infty, \infty)$ and where $b(x \mid y)$ represented the distribution of particle mass x spawned by the breakage of a particle of mass $y > x$. In their setting, $b(x \mid y)$ was given by a power law

$$b(x \mid y) = (\nu + 2)\,\frac{x^\nu}{y^{\nu+1}}, \tag{2}$$

with $\nu \in (-2, 0]$ (see also [5] for a more insight regarding this case). Note that the density of particles having mass x at time t is denoted by $u(t, x)$. Additionally in absence of any other mechanism, the mass of all daughter particles is equal to the mass of the parent. This "local" conservation mass principle is given by

$$\int_0^y x\,b(x \mid y)\,dx = y. \tag{3}$$

In a similar way, the amount of particles created by a particle of size y is given by

$$n(y) = \int_0^y b(x \mid y)\,dx \tag{4}$$

and $n(y)$ may be infinite.

Formal conservation principles

$$\begin{aligned}\frac{d}{dt}M(t) &= \int_0^\infty \frac{\partial}{\partial t}u(t,x)\,x dx = 0,\\ \frac{d}{dt}N(t) &= \int_0^\infty \frac{\partial}{\partial t}u(t,x)\,dx\\ &= \int_0^\infty a(x)\,(n(x)-1)\,u(t,x)\,dx\end{aligned} \tag{5}$$

can be obtained through the means of local conservation principles (3) and (4) and the integration of (1).

This paper extends the class of power law rates to any positive and continuous function on $(0,\infty)$ and it is assumed that b is separable and can be written as

$$b(x \mid y) = \beta(x)\,\gamma(y), \tag{6}$$

where

$$\gamma(y) = \frac{y}{\int_0^y s\beta(s)\,ds} \tag{7}$$

in order to satisfy the local mass conservation rule. Moreover, it is assumed that β is any continuous function on $(0,\infty)$. A generalization of the power law b described in (2) can be found in (6) which has the advantage of allowing the number of daughter particles,

$$n(y) = \frac{y\int_0^y \beta(s)\,ds}{\int_0^y s\beta(s)\,ds}, \tag{8}$$

to depend on the parent size y [6].

Due to the inability of getting exact solutions in fragmentation models, various authors have used several functional analytic approaches to investigate the dynamics of the system. These methods include semigroup theory [7–10], perturbation theory [11–13], approximation techniques [14], and probabilistic methods [2, 15]. The efficiency of these methods is limited as these problems are reformulated in abstract spaces that are norm dependent and the overall behavior of the dynamics changes radically as different metrics are included in the system. The Laplace transforms as extensively discussed in [16, 17] will play a key role in our investigations. Next we introduce a theorem and a definition that are instrumental in the analysis required in the obtention of exact solutions.

Theorem 1 (see [6]). *Assume that* $\lim_{x\to 0^+} a(x)$ *exists (finite or infinite). Then the fragmentation equation 1 is conservative if and only if there exists* $\delta > 0$ *such that* $b(x \mid x)/a(x) \notin L_1([0,\delta])$.

1.1. Laplace Transforms

Definition 2. The Laplace transform of a piecewise continuous function $f(t), 0 \le t < +\infty$ is the function $F(s) = \mathscr{L}\{f(t)\}$ defined by

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt. \tag{9}$$

The inverse Laplace transform of $F(s)$ is $f(t)$, $f(t) = \mathscr{L}^{-1}(F(s))$.

2. Solvability of the Fragmentation Equation

In this section, we use Laplace transform to solve the fragmentation equation

$$\begin{aligned}\frac{\partial}{\partial t}u(t,x) &= -a(x)\,u(t,x) + \int_x^\infty a(y)\,b(x \mid y)\,u(t,y)\,dy,\\ &\qquad x \ge 0,\ t > 0,\\ u(0,x) &= u_0(x).\end{aligned} \tag{10}$$

Let $\widetilde{u}(s,x) = \mathscr{L}[u(t,x)]$. Clearly, we have that

$$\begin{aligned}\mathscr{L}\left\{\frac{\partial}{\partial t}u(t,x)\right\} &= s\widetilde{u}(s,x) - u_0(x),\\ \mathscr{L}\{a(x)\,u(t,x)\} &= a(x)\,\widetilde{u}(s,x),\\ \mathscr{L}\left\{\int_x^\infty a(y)\,b(x \mid y)\,u(t,y)\,dy\right\} &= \int_x^\infty a(y)\,b(x \mid y)\,\widetilde{u}(s,y)\,dy.\end{aligned} \tag{11}$$

Making use of (10) and (11), we obtain the following equation:

$$\begin{aligned}s\widetilde{u}(s,x) - u_0(x) = &-a(x)\,\widetilde{u}(s,x)\\ &+ \int_x^\infty a(y)\,b(x \mid y)\,\widetilde{u}(s,y)\,dy;\end{aligned} \tag{12}$$

that is,

$$u_0(x) = (s + a(x))\,\widetilde{u}(s,x) - \int_x^\infty a(y)\,b(x \mid y)\,\widetilde{u}(s,y)\,dy. \tag{13}$$

Viewing s as a parameter, this is similar to the resolvent equation solved in 2010 (Banasiak and Noutchie). The solution reads as

$$\begin{aligned}\widetilde{u}(s,x) = &\frac{u_0(x)}{s+a(x)} + \frac{\beta(x)}{s+a(x)}e^{-\xi_s(x)}\\ &\times \int_x^\infty \frac{a(y)\,\gamma(y)}{s+a(y)}e^{\xi_s(y)}u_0(y)\,dy,\end{aligned} \tag{14}$$

where

$$\xi_s(x) = \int_1^x \frac{a(\eta)\,b(\eta \mid \eta)}{s+a(\eta)}d\eta. \tag{15}$$

The solution $u(t,x)$ of (10) is the inverse Laplace transform of $\widetilde{u}(s,x)$. Clearly,

$$\begin{aligned}
&\mathscr{L}^{-1}\left\{\frac{u_0(x)}{s+a(x)}\right\} = u_0(x)\,\mathscr{L}^{-1}\left\{\frac{1}{s+a(x)}\right\} = u_0(x)\,e^{-ta(x)},\\
&\mathscr{L}^{-1}\left\{\frac{\beta(x)}{s+a(x)}e^{-\xi_s(x)}\int_x^\infty \frac{a(y)\,\gamma(y)}{s+a(y)}e^{\xi_s(y)}u_0(y)\,dy\right\}\\
&\quad=\int_x^\infty a(y)\,b(x\mid y)\,u_0(y)\,\mathscr{L}^{-1}\\
&\qquad\times\left\{\frac{1}{s+a(x)}\frac{1}{s+a(y)}e^{\{\xi_s(y)-\xi_s(x)\}}\right\}dy\\
&\quad=\int_x^\infty a(y)\,b(x\mid y)\,u_0(y)\,\mathscr{L}^{-1}\left\{\Theta(s,x,y)\right\}dy,
\end{aligned}\tag{16}$$

where

$$\Theta(s,x,y)=\frac{1}{s+a(x)}\frac{1}{s+a(y)}\exp\left\{\int_x^y\frac{a(\eta)\,b(\eta\mid\eta)}{s+a(\eta)}d\eta\right\}.\tag{17}$$

Therefore, the solution of the fragmentation equation

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) = -a(x)\,u(t,x)+\int_x^\infty a(y)\,b(x\mid y)\,u(t,y)\,dy,\\
&\qquad\qquad x\ge 0,\quad t>0,\\
&u(0,x)=u_0(x)
\end{aligned}\tag{18}$$

is given by

$$\begin{aligned}
u(t,x) &= u_0(x)\,e^{-ta(x)}\\
&\quad+\int_x^\infty a(y)\,b(x\mid y)\,u_0(y)\,\mathscr{L}^{-1}\left\{\Theta(s,x,y)\right\}dy.
\end{aligned}\tag{19}$$

3. Applications

In this section, we assume that

$$\begin{aligned}
a(x)&=x^{\alpha+1},\quad \alpha\in(-\infty,\infty),\\
b(x\mid y)&=(2+\nu)\,\frac{x^\nu}{y^{\nu+1}},
\end{aligned}\tag{20}$$

with $\nu\in(-2,0]$. We have

$$\begin{aligned}
\int_x^y\frac{a(\eta)\,b(\eta\mid\eta)}{s+a(\eta)}d\eta &= (2+\nu)\int_x^y\frac{\eta^\alpha}{s+\eta^{\alpha+1}}d\eta\\
&=\frac{2+\nu}{\alpha+1}\ln\left\{\frac{s+y^{\alpha+1}}{s+x^{\alpha+1}}\right\};
\end{aligned}\tag{21}$$

it follows that

$$\exp\left\{\int_x^y\frac{a(\eta)\,b(\eta\mid\eta)}{s+a(\eta)}d\eta\right\}=\left\{\frac{s+y^{\alpha+1}}{s+x^{\alpha+1}}\right\}^\gamma,\tag{22}$$

where

$$\gamma=\frac{2+\nu}{\alpha+1}.\tag{23}$$

Thus

$$\Theta_{\alpha,\nu}(s,x,y)=\frac{\left(s+y^{\alpha+1}\right)^{\gamma-1}}{\left(s+x^{\alpha+1}\right)^{\gamma+1}}=\left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}.\tag{24}$$

Therefore, the solution $u(t,x)$ is given by

$$\begin{aligned}
u(t,x) &= u_0(x)\,e^{-tx^{\alpha+1}}\\
&\quad+(2+\nu)\int_x^\infty\left\{\frac{x}{y}\right\}^\nu y^\alpha u_0(y)\,\mathscr{L}^{-1}\\
&\qquad\times\left\{\Theta_{\alpha,\nu}(s,x,y)\right\}dy.
\end{aligned}\tag{25}$$

3.1. Case $\alpha=-3$ and $\nu=0$. We want to solve the following equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) = -x^{-2}u(t,x)+2\int_x^\infty y^{-3}u(t,y)\,dy,\\
&\qquad u(0,x)=u_0(x).
\end{aligned}\tag{26}$$

We have $\gamma=-1$; it follows that

$$\Theta_{-3,0}(s,x,y)=\left\{\frac{1}{s+x^{-2}}\right\}^0\left\{s+y^{-2}\right\}^{-2}=\left\{s+y^{-2}\right\}^{-2}.\tag{27}$$

Thus

$$\mathscr{L}^{-1}\left\{\Theta_{-3,0}(s,x,y)\right\}=\mathscr{L}^{-1}\left\{\left(s+y^{-2}\right)^{-2}\right\}=te^{-ty^{-2}}.\tag{28}$$

Therefore,

$$u(t,x)=u_0(x)\,e^{-tx^{-2}}+2t\int_x^\infty y^{-3}e^{-ty^{-2}}u_0(y)\,dy.\tag{29}$$

3.2. Case $\alpha=-2$ and $\nu=0$. We want to solve the following equation:

$$\begin{aligned}
&\frac{\partial}{\partial t}u(t,x) = -x^{-1}u(t,x)+2\int_x^\infty y^{-2}u(t,y)\,dy,\\
&\qquad u(0,x)=u_0(x).
\end{aligned}\tag{30}$$

We have $\gamma=-2$; it follows that

$$\begin{aligned}
\Theta_{-2,0}(s,x,y) &= \left\{\frac{1}{s+x^{-1}}\right\}^{-1}\left\{s+y^{-1}\right\}^{-3}\\
&=\frac{s+y^{-1}-y^{-1}+x^{-1}}{\left(s+y^{-1}\right)^3}\\
&=\frac{1}{\left(s+y^{-1}\right)^2}+\frac{\left(x^{-1}-y^{-1}\right)}{\left(s+y^{-1}\right)^3}.
\end{aligned}\tag{31}$$

Thus

$$\mathscr{L}^{-1}\left\{\Theta_{-2,0}(s,x,y)\right\} = \mathscr{L}^{-1}\left\{\frac{1}{\left(s+y^{-1}\right)^{2}}\right\} + \left(x^{-1}-y^{-1}\right)\mathscr{L}^{-1}\left\{\frac{1}{\left(s+y^{-1}\right)^{3}}\right\} = te^{-ty^{-1}} + \left(x^{-1}-y^{-1}\right)e^{-ty^{-1}}\frac{t^{2}}{2}. \tag{32}$$

Therefore,

$$u(x,t) = e^{-t/x}u_0(x) + 2t\int_x^{\infty}\frac{e^{-t/y}}{y^{2}}e^{-t/y}u_0(y)\,dy + t^{2}\int_x^{\infty}\frac{e^{-t/y}}{y^{2}}\left(\frac{1}{x}-\frac{1}{y}\right)u_0(y)\,dy. \tag{33}$$

3.3. General Case $a(x) = x^{\alpha+1}$ and $b(x \mid y) = (2+\nu)(x^{\nu}/y^{\nu+1})$. We want to solve

$$\frac{\partial}{\partial t}u(t,x) = -x^{\alpha+1}u(t,x) + (2+\nu)\,x^{\nu}\int_x^{\infty}y^{\alpha-\nu}u(t,y)\,dy, \quad x \geq 0, \quad t > 0, \qquad u(0,x) = u_0(x). \tag{34}$$

From the previous section, the solution of this equation is

$$u(t,x) = u_0(x)\,e^{-tx^{\alpha+1}} + (2+\nu) \times \int_x^{\infty}\left\{\frac{x}{y}\right\}^{\nu}y^{\alpha}u_0(y)\,\mathscr{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\}dy. \tag{35}$$

Note that

$$\mathscr{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\} = \mathscr{L}^{-1}\left\{\left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}\right\}$$

$$\mathscr{L}^{-1}\left\{\Theta_{\alpha,\nu}(s,x,y)\right\} = \mathscr{L}^{-1}\left\{\left\{\frac{1}{s+x^{\alpha+1}}\right\}^{\gamma+1}\left\{s+y^{\alpha+1}\right\}^{\gamma-1}\right\} = t\exp\left(-tx^{\alpha+1}\right){}_1F_1 \times \left(1-\gamma;2;t\left(x^{\alpha+1}-y^{\alpha+1}\right)\right). \tag{36}$$

It follows that

$$u(t,x) = u_0(x)\,e^{-tx^{\alpha+1}} + (2+\nu)\,t\exp\left(-tx^{\alpha+1}\right) \times \int_x^{\infty}\left\{\frac{x}{y}\right\}^{\nu}{}_1F_1\left(1-\gamma;2;t\left(x^{\alpha+1}-y^{\alpha+1}\right)\right) \times y^{\alpha}u_0(y)\,dy. \tag{37}$$

We recover the results of Ziff and his students [3, 4, 18].

4. Concluding Remarks

In this paper, we successfully used Laplace transforms and the methods of characteristics to solve an open problem in applied mathematics derived over 60 years ago. We extended the work of Ziff and his students and provided the full solution of fragmentation equations with general explosion rates. This work enables the computation of removal rates and shattering in fragmentation models and provides a general framework for understanding particles distributions in fragmentation processes as time evolves. In particular, it enables a complete classification of shattering regimes.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] Z. A. Melzak, "A scalar transport equation," *Transactions of the American Mathematical Society*, vol. 85, pp. 547–560, 1957.

[2] I. Filippov, "On the distribution of the sizes of particles which undergo splitting," *Theory of Probability and Its Applications*, vol. 6, pp. 275–293, 1961.

[3] R. M. Ziff and E. D. McGrady, "Kinetics of polymer degradation," *Macromolecules*, vol. 19, no. 10, pp. 2513–2519, 1986.

[4] R. M. Ziff and E. D. McGrady, "The kinetics of cluster fragmentation and depolymerisation," *Journal of Physics A: Mathematical and General*, vol. 18, no. 15, pp. 3027–3037, 1985.

[5] J. Banasiak and S. C. Oukouomi Noutchie, "Controlling number of particles in fragmentation equations," *Physica D: Nonlinear Phenomena*, vol. 239, no. 15, pp. 1422–1435, 2010.

[6] J. Banasiak, "Conservative and shattering solutions for some classes of fragmentation models," *Mathematical Models & Methods in Applied Sciences*, vol. 14, no. 4, pp. 483–501, 2004.

[7] J. Banasiak and L. Arlotti, *Perturbations of Positive Semigroups with Applications* , Springer Monographs in Mathematics, Springer, London, UK, 2006.

[8] K. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, vol. 194 of *Graduate Texts in Mathematics*, Springer, 2000.

[9] D. J. McLaughlin, W. Lamb, and A. C. McBride, "A semigroup approach to fragmentation models," *SIAM Journal on Mathematical Analysis*, vol. 28, no. 5, pp. 1158–1172, 1997.

[10] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Applied Mathematical Sciences, 1983.

[11] A. Atangana, "On the singular perturbations for fractional differential equation," *The Scientific World Journal*, vol. 2014, Article ID 752371, 9 pages, 2014.

[12] S. C. Oukouomi Noutchie, "Analysis of the effects of fragmentation-coagulation in planktology," *Comptes Rendus—Biologies*, vol. 333, no. 11-12, pp. 789–792, 2010.

[13] S. C. Oukouomi Noutchie and E. F. Doungmo Goufo, "Global solvability of a continuous model for nonlocal fragmentation dynamics in a moving medium," *Mathematical Problems in Engineering*, vol. 2013, Article ID 320750, 8 pages, 2013.

[14] S. C. Oukouomi Noutchie, "Existence and uniqueness of conservative solutions for nonlocal fragmentation models,"

Mathematical Methods in the Applied Sciences, vol. 33, no. 15, pp. 1871–1881, 2010.

[15] M. H. Ernst and G. Szamel, "Fragmentation kinetics," *Journal of Physics A: Mathematical and General*, vol. 26, no. 22, pp. 6085–6091, 1993.

[16] A. Atangana, "A note on the triple Laplace transform and its applications to some kind of third-order differential equation," *Abstract and Applied Analysis*, vol. 2013, Article ID 769102, 10 pages, 2013.

[17] A. Atangana and S. C. Oukouomi Noutchie, "On multi-laplace transform for solving nonlinear partial differential equations with mixed derivatives," *Mathematical Problems in Engineering*, vol. 2014, Article ID 267843, 9 pages, 2014.

[18] R. M. Ziff and E. D. McGrady, "'Shattering' transition in fragmentation," *Physical Review Letters*, vol. 2, pp. 3027–3037, 1985.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 914527, 9 pages
http://dx.doi.org/10.1155/2014/914527

Research Article

Analysis of Sturm-Liouville Eigenproblem with Interior Singularities and a Perturbation Parameter

Thomas Mbah Acho

Department of Mathematics and Applied Mathematics, University of the Free State, P.O. Box 339, Bloemfontein 9300, South Africa

Correspondence should be addressed to Thomas Mbah Acho; achotm@ufs.ac.za

Received 30 April 2014; Accepted 11 June 2014; Published 8 July 2014

Academic Editor: Abdon Atangana

We devote this work to the discussion underpinning the derivation of eigenvalues and eigenfunction solutions for Sturm-Liouville boundary value problems. The study reveals that the parameter dependent nonstandard Sturm-Liouville boundary value problem with interior singularities may have more than two turning points. The Titchmarsh-Weyl m-function theory is applied here to obtain eigenfunction solutions valid for the whole interval in which pole singularities and two turning points are present. For the first time, with minimal constraints, the validity of the eigenfunction solutions are discussed when there are more than two turning points present. The eigenvalues are subsequently derived.

1. Introduction

The nonstandard Sturm-Liouville (SL) boundary value problems with interior singularities and transition points may require a refinement of some perturbation method or a completely new approach to enable us obtain eigenfunction solutions valid within the entire interval. There have been recent interest in the study of both the standard and nonstandard Sturm-Liouville boundary value problems and other similarly perturbed systems (see [1–9]). Cases where the interior singularities are simple poles or double poles with supplementary one or two turning points (transition points) have been extensively studied (see [10–16]). One method that has been shown to solve some of the SL eigenproblems with interior singularities is the Titchmarsh-Weyl m-function method [10, 11] (we refer the reader to whom the Titchmarsh-Weyl m-function theory is unfamiliar to [17]). However, there has not been any presentation where these interior singularities are associated to more than two turning points which arise naturally from a nonrestricted parameter. The discussion underpinning the turning point analysis will be presented in Section 2.2. The motivation of this study is based on the fact that a typical example of a nonstandard Sturm-Liouville equation with a nonrestricted parameter occurs naturally in the high energy resonance contributing equations, which in this case happens to be the corresponding radial Schrödinger operator in $\mathbb{R}$. The fact that SL eigenproblems with interior singularities (which could either be a simple pole or a double pole singularity) could have a possibility of the presence of more than two turning points requires the introduction of a suitable constraint to enable the use of the Titchmarsh-Weyl m-function technique. We will now give a brief background below of the nature of this problem and in subsequent sections use this example to demonstrate how the Titchmarsh-Weyl m-function method can be applied to obtain eigenfunction solutions for a nonrestricted parameter dependent SL eigenproblem.

The resonance phenomenon appears in most material sciences and partly explains certain cases where structures such as bridges and buildings collapse (see [18]). Resonances are defined as poles of an analytic continuation of the quadratic form of the resolvent to the Riemann sheet through the branch cut along the continuous spectrum (see [19]). There have been studies ([20] and more recently [19]) on the investigation of resonances for the Hamiltonian operator. Asymptotic solutions to resonance equations of the Hamiltonian operator,

$$H = -\left(\frac{1}{2}\right)\Delta - \frac{z}{r} + q(r), \tag{1}$$

acting on $L^2(\mathbb{R}^3)$ for $z > 0$, $r = |x| = (x_1^2 + x_2^2 + x_3^2)^{1/2}$, $x \in \mathbb{R}^3$, with appropriate conditions on a spherical symmetric

support function $q(r)$, were presented in [19]. Eigenfunctions solutions for these operators are valid in the region $0 < r \leq b < \infty$. These solutions are singular at $r = 0$. On considering the operator H acting on $\mathbb{R}$ the desired eigenfunction solutions should satisfy boundary conditions for $r \in [a, b]$, $a < 0 < b$ where a and/or b could be infinite. In this case the eigenfunctions solutions presented in [19] would not be valid for the entire interval because of the pole singularities at $r = 0$ and the behaviour of these functions (which are the Whittaker functions) as $r \to \infty$. In this work we use the following condition on the perturbation term $q(r)$:

$$q(r) = \varepsilon \geq 0, \tag{2}$$

and we examine the case when

$$q(r) = \varepsilon_0 + \varepsilon r. \tag{3}$$

From [19], the radial Schrödinger equation then takes the form

$$\left(-H_l + E\right)\Psi = \left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} + \frac{2z}{r} - \varepsilon + \tau^2\right)\Psi = 0, \tag{4}$$

where τ is the spectral parameter introduced in [19], such that $E = \tau^2$ and the Riemann sheet corresponds to the lower half plane of τ. With specification of $q(r)$ we can maintain the definition of resonances as in Theorem 1.2 of [19] as those points E of the second Riemann sheet that can be represented in the form $E = \tau^2$, where τ belongs to the lower half plane (that is $\operatorname{Im}\tau < 0$). It may be worth mentioning that even in a more simple case like the vibration of a circular membrane [18], the state of resonance is determined by the resulting radial equation. The objective of this work is not to solve the resonance equations as in [19] but to obtain eigenfunctions solutions and eigenvalues valid for the entire interval. We find that (4) has Whittaker functions as the eigenfunction solutions for $r \in \mathbb{R}^3$ as well as for $r \in \mathbb{R}$, where r is finite and different from zero. However, the case for $r \in \mathbb{R}$, with boundary conditions such that $r \in [a, b]$ where $a \leq 0 \leq b$, presents difficulties in obtaining the eigenfunctions solutions as highlighted above. In Section 3 we present these solutions and in Section 4 we derive the eigenvalue relations. The eigenfunction solutions will of course satisfy (4) for $r \in \mathbb{R}^3$ and the restricted interval $0 < r \leq b < \infty$.

2. Preliminaries and Turning Point Analysis

2.1. Preliminaries with a Restricted Perturbation Term $q(r) = \varepsilon \geq 0$. Sequel to our discussion in the previous section we can rewrite (4) with the associated conditions as the following boundary value problem:

$$L\Psi = \left(\frac{d^2}{dr^2} + \frac{\left(\tau^2 - \varepsilon\right)r^2 + 2zr - l}{r^2}\right)\Psi = 0 \tag{5}$$

$$\Psi(a) = \Psi(b) = 0, \quad r \in [a, b] \subset \mathbb{R},\ a < 0 < b, \tag{6}$$

where $l = 0, 1, \ldots; z > 0, \varepsilon > 0$ and $\operatorname{Im}\tau < 0$.

The differential equation (5) has the following turning points:

$$\rho_{1,2} = \frac{-z + \left[z^2 + l(l+1)\left(\tau^2 - \varepsilon\right)\right]^{1/2}}{\tau^2 - \varepsilon}. \tag{7}$$

The value of the angular momentum l is different from zero and $(\tau^2 - \varepsilon)$ is complex; the turning points ρ_1 and ρ_2 are distinct (i.e., do not coincide). Therefore, the eigenfunctions solutions of (5) will be in terms of some asymptotic approximation of Whittaker functions (see [21–24]). Using the transformation

$$x = 2i\left(\tau^2 - \varepsilon\right)^{1/2} r, \qquad \Psi(r) = \Phi(x), \tag{8}$$

equation (5) becomes

$$L\Phi = \left(\frac{d^2}{dx^2} - \left[\frac{1}{4} + \frac{iz\left(\tau^2 - \varepsilon\right)^{-1/2}}{x} + \frac{l(l+1)}{x^2}\right]\right)\Phi = 0. \tag{9}$$

Consider

$$L\Phi = \left(\frac{d^2}{dx^2} + \left[-\frac{1}{4} + \frac{k}{x} + \frac{1/4 - \mu^2}{x^2}\right]\right)\Phi = 0, \tag{10}$$

whose eigenfunctions solution in terms of the $M(\cdot)$ and $W(\cdot)$ Whittaker functions is

$$\Phi(x) = \alpha M_{k,\mu}(x) + \beta W_{k,\mu}(x), \tag{11}$$

where α and β are constants that would be determined. On comparison of (9) and (10), we obtain

$$k = -iz\left(\tau^2 - \varepsilon\right)^{-1/2}, \qquad \mu = l + \frac{1}{2}. \tag{12}$$

From whence we may then write the solution of (5) to be of the form

$$\Psi(r) = \alpha M_{k,\mu}\left(2i\left[\tau^2 - \varepsilon\right]^{1/2} r\right) + \beta W_{k,\mu}\left(2i\left[\tau^2 - \varepsilon\right]^{1/2} r\right), \tag{13}$$

where k and μ are as in (12). It is obvious that for $0 < r \leq b < \infty$ and $\varepsilon = 0$; $M_{k,\mu}(\cdot)$ and $W_{k,\mu}(\cdot)$ as presented in (13) give the two linearly independent solutions of the unperturbed equation as in [19]. However, considering an interval $[a, b]$ that includes the double pole singularity at $r = 0$ and the possibility that a and/or b could be infinite results in the fact that for $|\tau^2 - \varepsilon|$ small, the internal boundary layer in which the asymptotic series for $M_{k,\mu}(\cdot)$ and $W_{k,\mu}(\cdot)$ are inaccurate includes the whole interval $[a, b]$. This is as a result of the prominence in the effect of the double pole for $|\tau^2 - \varepsilon|$ small and the presence of the turning points that lie in the complex plane. Some of the behavioural properties of the $M_{k,\mu}(\cdot)$ and $W_{k,\mu}(\cdot)$ Whittaker functions are presented in [10, 25].

The method of solution we employ requires that we determine the quadrant in the complex plane on which

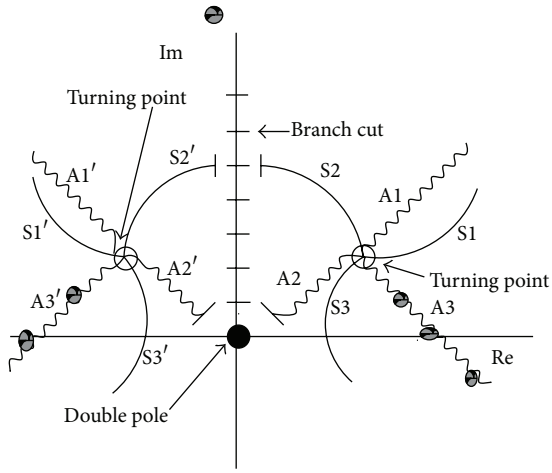


Figure 1: Turning points both lie in the upper half of the complex plane and two of the zeros of the eigenfunctions lie on the real axis. In Figures 1–4, Aj and Aj' and $j = 1, 2, 3$ are antistokes lines; Sj and Sj' and $j = 1, 2, 3$ are stokes lines and zeros of eigenfunctions.

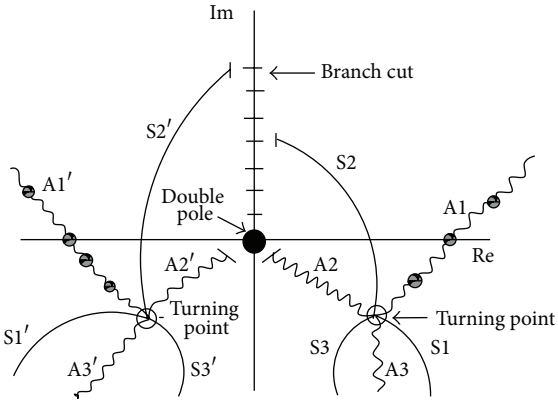


Figure 2: Turning points both lie in the lower half plane and two of the zeros of the eigenfunctions lie on the real axis.

the turning points lie. Considering the resonance condition $\operatorname{Im}\tau < 0$, which implies that either

$$\left(\operatorname{Im}\tau^2 > 0, \operatorname{Re}\tau < 0\right) \quad \text{or} \quad \left(\operatorname{Im}\tau^2 < 0, \operatorname{Re}\tau > 0\right), \tag{14}$$

and the fact that τ^2 appears in both the discriminant and the denominator of (7) means that the turning points could lie in any of the four quadrants. Our method is applicable with the choice of branch cuts as in the theory of atmospheric and ocean waves. The locations of the turning points on the complex plane are as in the Figures 1, 2, 3, and 4. These figures also indicate the stokes lines and antistokes lines and how one of the zeros of the Whitaker function does lie on the positive real axis and the other one on the negative real axis which is essential for the boundary conditions to be satisfied.

Our method would require the use of Titchmarsh-Weyl m-function [10, 17], so we shall only be interested in the turning points that are separated such that one lies in the left half of the complex plane and the other in the right half. The Titchmarsh-Weyl m-function theory for (5) with two singular points a, b requires that we choose a regular point $c \in [a, b]$. We then let

$$\Psi\left(r, \widehat{\tau}; k, \mu\right) = \Theta\left(r, \widehat{\tau}; k, \mu\right) \Phi\left(r, \widehat{\tau}; k, \mu\right). \tag{15}$$

Denote the fundamental solution of (5), defined by

$$\begin{bmatrix} \Theta\left(c, \widehat{\tau}; k, \mu\right) & \Phi\left(c, \widehat{\tau}; k, \mu\right) \\ \Theta'\left(c, \widehat{\tau}; k, \mu\right) & \Phi'\left(c, \widehat{\tau}; k, \mu\right) \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{16}$$

where in this case $\widehat{\tau} = (\tau^2 - \varepsilon)$. The Titchmarsh-Weyl m-function at $r = a$ and $r = b$ is defined, respectively, by

$$\begin{aligned} m^{-}\left(\widehat{\tau}\right) &= \lim_{r \to a} \frac{\Theta\left(r, \widehat{\tau}; k, \mu\right)}{\Phi\left(r, \widehat{\tau}; k, \mu\right)}, \\ m^{+}\left(\widehat{\tau}\right) &= -\lim_{r \to b} \frac{\Theta\left(r, \widehat{\tau}; k, \mu\right)}{\Phi\left(r, \widehat{\tau}; k, \mu\right)} \end{aligned} \tag{17}$$

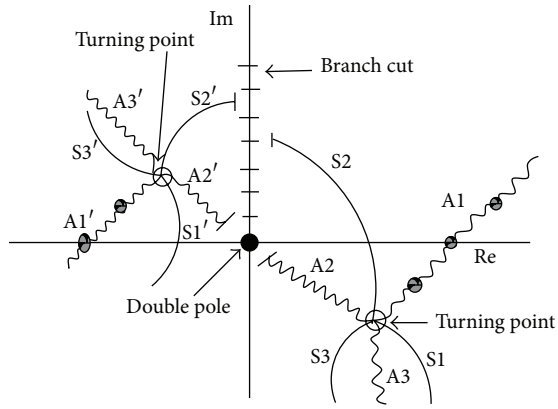


Figure 3: One of the turning points lie in the lower half of the complex plane and the other one in the upper half plane.

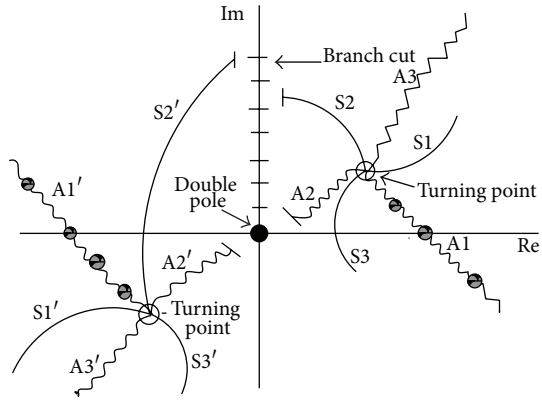


Figure 4: One of the turning points lie in the lower half of the complex plane and the other one in the upper half plane.

for $\operatorname{Im}\tau \neq 0$. Both $m^{-}(\widehat{\tau})$ and $m^{+}(\widehat{\tau})$ are analytic functions in both $\operatorname{Im}(\widehat{\tau}) > 0$ and $\operatorname{Im}(\widehat{\tau}) < 0$ and obey the following relations:

$$\begin{aligned} m^{-}\left(\overline{\widehat{\tau}}\right) &= \overline{m^{-}(\widehat{\tau})}, \qquad \operatorname{Im}\widehat{\tau}\operatorname{Im}m^{-}(\widehat{\tau}) < 0, \\ m^{+}\left(\overline{\widehat{\tau}}\right) &= \overline{m^{+}(\widehat{\tau})}, \qquad \operatorname{Im}\widehat{\tau}\operatorname{Im}m^{+}(\widehat{\tau}) > 0. \end{aligned} \tag{18}$$

Further, if we define $\Psi^{\pm}(r,\widehat{\tau};k,\mu)$ by

$$\Psi^{\pm}(r,\widehat{\tau};k,\mu) = \Theta(r,\widehat{\tau};k,\mu) + m^{\pm}\Phi(r,\widehat{\tau};k,\mu) \tag{19}$$

then it follows that

$$\Psi^{+}(r,\widehat{\tau};k,\mu) \in L^{2}([c,b]), \qquad \Psi^{-}(r,\widehat{\tau};k,\mu) \in L^{2}([a,c]). \tag{20}$$

We use the above to construct our solutions in Section 3. Before doing so we present in the proceeding subsection a turning point analysis for the nonrestricted perturbation parameter.

2.2. Turning Point Analysis for the Nonrestricted Perturbation Parameter $\varepsilon(r) = \varepsilon_0 + \varepsilon_1 r$. The turning point analysis for $\varepsilon(r) = \varepsilon_0 = \varepsilon$ (notation used in the preceding and proceeding sections) would be similar to that presented in [10], so for the sake of brevity it would not be presented here. The corresponding Schrödinger equation for the spectral parameter k and the perturbation parameter $\varepsilon(r) = \varepsilon_0 + \varepsilon_1 r$, $\varepsilon_1 \neq 0$ is given by

$$L\Psi = \left(\frac{d^2}{dr^2} - \frac{l(l+1)}{r^2} - (\varepsilon_0 + \varepsilon_1 r) + k^2\right)\Psi = 0. \tag{21}$$

We rewrite this in the form

$$\left(\frac{d^2}{dr^2} - \frac{\varepsilon_1 r^3 - \left(k^2 - \varepsilon_0\right) r^2 + l(l+1)}{r^2}\right)\Psi = 0. \tag{22}$$

For $r \in \mathbb{R}$, we may consider the generalized interval $r \in (a,b)$, $a < 0 < b$, with no restrictions on a and b. Furthermore, for the purpose of proper analysis of the turning points we will choose the boundary conditions as

$$\Psi(a) = \Psi(b) = 0. \tag{23}$$

The boundary value problems (22) and (23) have a double pole singularity, at $r = 0$, and three turning points which we now determine. The turning points are located at $r = \eta$ for which

$$r^3 - \frac{\left(k^2 - \varepsilon_0\right) r^2}{\varepsilon_1} + \frac{l(l+1)}{\varepsilon_1} = 0. \tag{24}$$

We make the substitution

$$r = y + \frac{\left(k^2 - \varepsilon_0\right)}{3\varepsilon_1} \tag{25}$$

to reduce (24) to the depressed cubic

$$y^3 + \lambda y + \xi = 0, \tag{26}$$

where $\lambda = -(1/3)((k^2 - \varepsilon_0)/\varepsilon_1)^2$ and $\xi = (2/27)((k^2 - \varepsilon_0)/\varepsilon_1)^2 + l(l+1)/\varepsilon_1$.

One of the roots of the depressed cubic is

$$y_1 = \sqrt[3]{\frac{-\xi}{2} + \sqrt{\frac{\xi^2}{4} + \frac{\lambda^3}{27}}} + \sqrt[3]{\frac{-\xi}{2} - \sqrt{\frac{\xi^2}{4} + \frac{\lambda^3}{27}}}. \tag{27}$$

We then obtain the remaining roots as

$$y_{2,3} = \frac{-y_1 + \left(-4\lambda - 3y_1^2\right)^{1/2}}{2}. \tag{28}$$

From (24) to (28) the three turning points are as follows:

$$\begin{aligned} \eta_1 &= y_1 + \frac{k^2 - \varepsilon_0}{3\varepsilon_1}, \\ \eta_{2,3} &= \frac{-y_1 + \left(-4\lambda - 3y_1^2\right)^{1/2}}{2} + \frac{k^2 - \varepsilon_0}{3\varepsilon_1}. \end{aligned} \tag{29}$$

The turning points coalesce into a single turning point when $y_1 = (-2\lambda/3)^{1/2}$ and there are two distinct turning points if $y_1 = (-4\lambda/3)^{1/2}$. For any other values of y_1 there will be three distinct turning points located on the complex plane.

In order to apply the Titchmarsh-Weyl m-function technique for reasons stated in Section 2.1, there should be two distinct turning points; therefore, we must have the restriction that

$$y_1 = \left(-\frac{4\lambda}{3}\right)^{1/2} = \frac{2}{3}\left(\frac{k^2 - \varepsilon_0}{\varepsilon_1}\right). \tag{30}$$

This provides the split intervals required for the application of the Titchmarsh-Weyl m-function theory which will be similar to that when $\varepsilon = \varepsilon_0$, $\varepsilon_1 = 0$; provided of course that one turning point lies in the left half and the other in the right half of the complex plane.

3. Eigenfunction Solutions

The analysis in Section 2 is essential for the application of the method of solutions we now provide here and in the section that follows. We will assume that the turning point ρ_2 lies in the left half plane and ρ_1 lies in the right half plane. We split the interval $[a, b]$ as follows:

$$\begin{aligned} I_1 &= [a, \rho_2^*], \qquad I_2 = [\rho_2^*, 0], \\ I_3 &= [0, \rho_1^*], \qquad I_4 = [\rho_1^*, b], \end{aligned} \tag{31}$$

where $\rho_j^* = \operatorname{Re}\rho_j$, $j = 1, 2$.

We obtain our solution for (5) as a linear combination of eigenfunctions for the split intervals and it turns out that the boundary conditions (6) are not necessary as they are automatically satisfied by our solution. The following theorem summarizes our first results.

Theorem 1. *Let $\Theta(r, \widehat{\tau}; k, \mu)$ and $\Phi(r, \widehat{\tau}; k, \mu)$ be appropriate combinations of the eigenfunctions solutions of* (5) *with* (6) *for $|\widehat{\tau}| = |\tau^2 - \varepsilon|$ small such that*

$$\begin{aligned} m^+(\widehat{\tau}) &= -\lim_{r\to b}\frac{\Theta(r, \widehat{\tau}; k, \mu)}{\Phi(r, \widehat{\tau}; k, \mu)}, \\ m^-(\widehat{\tau}) &= -\lim_{r\to 0}\frac{\Theta(r, \widehat{\tau}; k, \mu)}{\Phi(r, \widehat{\tau}; k, \mu)}, \\ n^+(\widehat{\tau}) &= -\lim_{r\to 0}\frac{\Theta(r, \widehat{\tau}; k, \mu)}{\Phi(r, \widehat{\tau}; k, \mu)}, \\ n^-(\widehat{\tau}) &= -\lim_{r\to a}\frac{\Theta(r, \widehat{\tau}; k, \mu)}{\Phi(r, \widehat{\tau}; k, \mu)}. \end{aligned} \tag{32}$$

Then the solutions to (5) *would be*

$$\begin{aligned} \Psi(r, \widehat{\tau}; k, \mu) &= \Theta(r, \widehat{\tau}; k, \mu) \\ &\quad + \begin{cases} m^+(\widehat{\tau})\,\Phi(r, \widehat{\tau}; k, \mu) \in L^2([\rho_1^*, b]) \\ m^-(\widehat{\tau})\,\Phi(r, \widehat{\tau}; k, \mu) \in L^2([0, \rho_1^*]) \\ n^+(\widehat{\tau})\,\Phi(r, \widehat{\tau}; k, \mu) \in L^2([\rho_2^*, 0]) \\ n^-(\widehat{\tau})\,\Phi(r, \widehat{\tau}; k, \mu) \in L^2([a, \rho_2^*]), \end{cases} \end{aligned} \tag{33}$$

where m^+, m^-, n^+, n^- are as defined in Section 2.1.

Proof. For convenience we suppress the k, μ dependence of Θ and Φ. We start with the regular solutions $M_{k,\mu}(2i\widehat{\tau}^{1/2}r)$ and $W_{k,\mu}(2i\widehat{\tau}^{1/2}r)$ of (5) on $I_3 \cup I_4$, where $\widehat{t} = (\tau^2 - \varepsilon)$, $k = iz(\tau^2 - \varepsilon)^{1/2}$, and $\mu = l + 1/2$.

Thus we let

$$\Theta(r, \widehat{\tau}) = \alpha_0^+ M_{k,\mu}\left(2i\widehat{\tau}^{1/2}r\right) + \alpha_1^+ M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}r\right), \tag{34}$$

$$\Phi(r, \widehat{\tau}) = \beta_0^+ W_{k,\mu}\left(2i\widehat{\tau}^{1/2}r\right) + \beta_1^+ W_{-k,\mu}\left(2i\widehat{\tau}^{1/2}r\right). \tag{35}$$

We now proceed to determine α_0^+, α_1^+, β_0^+, and β_1^+.

Substituting (34) into (16) we obtain

$$\Theta(\rho_1^*, \widehat{\tau}) = \alpha_0^+ M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right) + \alpha_1^+ M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right) = 1,$$

$$\begin{aligned} &\Theta'(\rho_1^*, \widehat{\tau}) \\ &= \alpha_0^+ \left\{ \frac{\left(1 + 2\mu - 4i\widehat{\tau}^{1/2}\rho_1^*\right)}{2i\widehat{\tau}^{1/2}\rho_1^*} M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right) \right. \\ &\qquad \left. + \frac{(2\mu - 2k + 1)}{(2\mu + 1)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)} M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right) \right\} \end{aligned}$$

$$+\alpha_1^+\left\{\frac{\left(1-2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{4i\widehat{\tau}^{1/2}\rho_1^*}M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(1-2\mu-2k)}{(1-2\mu)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}=0. \tag{36}$$

From (36) we obtain the following:

$$\alpha_0^+=\frac{1}{Q_1}\left\{\frac{\left(1-2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{4i\widehat{\tau}^{1/2}\rho_1^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(1-2\mu-2k)}{(1-2\mu)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}M_{k-1/2,-\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\},$$

$$\alpha_1^+=\frac{-1}{Q_1}\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{2i\widehat{\tau}^{1/2}\rho_1^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(2\mu-2k+1)}{(2\mu+1)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}, \tag{37}$$

where

$$\begin{aligned}Q_1=&M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\\&\times\left\{\frac{\left(1-2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{4i\widehat{\tau}^{1/2}\rho_1^*}M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(1-2\mu-2k)}{(1-2\mu)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}M_{k-1/2,-\mu-1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}\\&+\left(-M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right.\\&\times\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{2i\widehat{\tau}^{1/2}\rho_1^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(2\mu-2k+1)}{(2\mu+1)\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}\Bigg).\end{aligned} \tag{38}$$

Similarly, substitution of (35) into (16) yields

$$\beta_0^+=\frac{1}{Q_2}\left\{\frac{\left(1+2\mu+4i\widehat{\tau}^{1/2}\rho_1^*\right)}{4i\widehat{\tau}^{1/2}\rho_1^*}W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)+\left(-\frac{(1+2\mu+2k)}{\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}W_{-k+1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)\right)\right\},$$

$$\beta_1^+=\frac{-1}{Q_2}\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{2i\widehat{\tau}^{1/2}\rho_1^*}W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(2k-2\mu-1)}{\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}W_{k-1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}, \tag{39}$$

where

$$\begin{aligned}Q_2=&W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\\&\times\left\{\frac{\left(1+2\mu+4i\widehat{\tau}^{1/2}\rho_1^*\right)}{4i\widehat{\tau}^{1/2}\rho_1^*}W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)+\left(-\frac{(1+2\mu+2k)}{\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}W_{-k+1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)\right)\right\}\\&+\left(-W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_1^*\right)\right.\\&\times\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_1^*\right)}{2i\widehat{\tau}^{1/2}\rho_1^*}W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)+\frac{(2k-2\mu-1)}{\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)^{1/2}}W_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_1^*\right)\right\}\Bigg).\end{aligned} \tag{40}$$

Using (17) we may then find $m^+(\widehat{\tau})$ and $m^-(\widehat{\tau})$, respectively, at $r=b$ and $r=0$ as follows:

$$m^+(\widehat{\tau})=-\lim_{r\to b}\frac{\Theta(r,\widehat{\tau})}{\Phi(r,\widehat{\tau})},\qquad m^-(\widehat{\tau})=-\lim_{r\to 0}\frac{\Theta(r,\widehat{\tau})}{\Phi(r,\widehat{\tau})}, \tag{41}$$

where Θ and Φ are as given in (34) and (35) with α_j,β_j, $j=0,1$ as obtained in (37) to (40). From (19) we then have

$$\begin{aligned}\Psi_+^+(r,\widehat{\tau})&\equiv\Theta(r,\widehat{\tau})+m^+(\widehat{\tau})\Phi(r,\widehat{\tau})\in L^2\left(\left[\rho_1^*,b\right]\right),\\\Psi_+^-(r,\widehat{\tau})&\equiv\Theta(r,\widehat{\tau})+m^-(\widehat{\tau})\Phi(r,\widehat{\tau})\in L^2\left(\left[0,\rho_1^*\right]\right).\end{aligned} \tag{42}$$

Therefore, the eigenfunctions for $I_3\cup I_4$ are

$$\Psi_+(r,\widehat{\tau})=\begin{cases}\Psi_+^+(r,\widehat{\tau}), & r\in\left[\rho_1^*,b\right]\\\Psi_+^-(r,\widehat{\tau}), & r\in\left[0,\rho_1^*\right],\end{cases} \tag{43}$$

which automatically satisfy the boundary conditions at b.

Now, for the interval $I_1\cup I_2$, we let

$$\begin{aligned}\Theta(r,\widehat{\tau})&=\alpha_0^-M_{k,\mu}\left(2i\widehat{\tau}^{1/2}r\right)+\alpha_1^-M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}r\right),\\\Phi(r,\widehat{\tau})&=\beta_0^-W_{k,\mu}\left(2i\widehat{\tau}^{1/2}r\right)+\beta_1^-W_{-k,\mu}\left(2i\widehat{\tau}^{1/2}r\right).\end{aligned} \tag{44}$$

Proceeding as above we obtain

$$\alpha_0^- = \frac{1}{Q_3}\left\{\frac{\left(1-2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{4i\widehat{\tau}^{1/2}\rho_2^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(1-2\mu-2k)}{(1-2\mu)\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)}M_{k-1/2,-\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\},$$

$$\alpha_1^- = \frac{1}{Q_3}\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{2i\widehat{\tau}^{1/2}\rho_2^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(2\mu-2k+1)}{(1+2\mu)\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}} \times M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\}, \tag{45}$$

where

$$\begin{aligned} Q_3 &= M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) \\ &\times\left\{\frac{\left(1-2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{4i\widehat{\tau}^{1/2}\rho_2^*}M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(1-2\mu-2k)}{(1-2\mu)\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}M_{k-1/2,\mu-1/2}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\} \\ &+\left(-M_{k,-\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right. \\ &\times\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{2i\widehat{\tau}^{1/2}\rho_2^*}M_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(2\mu-2k+1)}{(1+2\mu)\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}M_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\}\Bigg). \end{aligned} \tag{46}$$

Using the same process we find that

$$\beta_0^- = \frac{1}{Q_4}\left\{\frac{\left(1+2\mu+4i\widehat{\tau}^{1/2}\rho_2^*\right)}{4i\widehat{\tau}^{1/2}\rho_2^*}W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right) + \left(-\frac{(1+2\mu+2k)}{\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}W_{-k+1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)\right)\right\},$$

$$\beta_1^- = \frac{-1}{Q_4}\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{2i\widehat{\tau}^{1/2}\rho_2^*}W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(2k-2\mu-1)}{\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}W_{k-1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\}, \tag{47}$$

where

$$\begin{aligned} Q_4 &= W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) \\ &\times\left\{\frac{\left(1+2\mu+4i\widehat{\tau}^{1/2}\rho_2^*\right)}{4i\widehat{\tau}^{1/2}\rho_2^*}W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right) + \left(-\frac{(1+2\mu+2k)}{\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}W_{-k+1/2,\mu+1/2}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)\right)\right\} \\ &+\left(-W_{-k,\mu}\left(-2i\widehat{\tau}^{1/2}\rho_2^*\right)\right. \\ &\times\left\{\frac{\left(1+2\mu-4i\widehat{\tau}^{1/2}\rho_2^*\right)}{2i\widehat{\tau}^{1/2}\rho_2^*}W_{k,\mu}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right) + \frac{(2k-2\mu-1)}{\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)^{1/2}}W_{k-1/2,\mu+1/2}\left(2i\widehat{\tau}^{1/2}\rho_2^*\right)\right\}\Bigg). \end{aligned} \tag{48}$$

By using (44) to (48) we obtain

$$\begin{aligned} \Psi_-^-(r,\widehat{\tau}) &\equiv \Theta(r,\widehat{\tau}) + n^+(\widehat{\tau})\,\Phi(r,\widehat{\tau}) \in L^2\left([a,\rho_2^*]\right) \\ \Psi_-^+(r,\widehat{\tau}) &\equiv \Theta(r,\widehat{\tau}) + n^-(\widehat{\tau})\,\Phi(r,\widehat{\tau}) \in L^2\left([\rho_2^*,0]\right), \end{aligned} \tag{49}$$

where

$$n^+(\widehat{\tau}) = -\lim_{r\to 0}\frac{\Theta(r,\widehat{\tau})}{\Phi(r,\widehat{\tau})}, \qquad n^-(\widehat{\tau}) = -\lim_{r\to a}\frac{\Theta(r,\widehat{\tau})}{\Phi(r,\widehat{\tau})}. \tag{50}$$

Therefore, the eigenfunctions for $I_1 \cup I_2$ are

$$\Psi_-(r,\widehat{\tau}) = \begin{cases}\Psi_-^-(r,\widehat{\tau}), & r\in[a,\rho_2^*]\\ \Psi_-^+(r,\widehat{\tau}), & r\in[\rho_2^*,0],\end{cases} \tag{51}$$

and these automatically satisfy the boundary conditions at a. From (43) and (51) we find that

$$\lim_{r\to 0^-}\Psi_-(r,\widehat{\tau}) = \lim_{r\to 0^+}\Psi_+(r,\widehat{\tau}) = 0. \tag{52}$$

We then conclude that the eigenfunctions solutions to (5) with the accompanying boundary conditions would be

$$\Psi(r,\widehat{\tau}) = \begin{cases}\Psi_+(r,\widehat{\tau}), & r\in[a,0]\\ \Psi_-(r,\widehat{\tau}), & r\in[0,b].\end{cases} \tag{53}$$

This completes the proof of the theorem. □

4. Computation of the Eigenvalues

As in [11], the eigenvalues for our problem are $\widehat{\tau}^{(n)}$. To derive the eigenvalue relations for boundary value problems for which the eigenfunctions are obtained using a split interval technique requires a careful examination of the behaviour of the eigenfunctions. To enable us achieve our goal for the interval $r \in [a,b]$, $a < 0 < b$, we will need the following conditions:

(i) that the eigenfunctions solutions are valid for

$$\arg\left(2i\widehat{\tau}^{1/2}rk\right) < 2\pi, \tag{54}$$

(ii) that

$$\begin{aligned}&\sin\left[2\sqrt{2i\widehat{\tau}^{1/2}bk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad = \cos\left[2e^{\pm(1/2)\pi i}2\sqrt{2i\widehat{\tau}^{1/2}bk} - \pi\left(\mu+\frac{1}{4}\right)\right],\end{aligned} \tag{55}$$

(iii) and that

$$\begin{aligned}&\sin\left[2\sqrt{2i\widehat{\tau}^{1/2}ak} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad = \cos\left[2e^{\pm(1/2)\pi i}2\sqrt{2i\widehat{\tau}^{1/2}ak} - \pi\left(\mu+\frac{1}{4}\right)\right].\end{aligned} \tag{56}$$

We summarise our results in this section in the following theorem.

Theorem 2. *The nonstandard SL eigenproblem (or the Schrödinger radial operator in* $\mathbb{R}$*),* (5) *with* (6) *with separated turning points, which have solutions* (53) *and satisfy conditions* (54)–(56) *have its eigenvalues given by*

$$\widehat{\tau}^{(n)} = \frac{n^4\pi^4}{\left[8i\left(b+a\right)+16k\sqrt{ab}\right]^2}. \tag{57}$$

Proof. The asymptotic expansions of the eigenfunctions that meet the requirement (54) are

$$\begin{aligned}&\frac{M_{k,\mu}\left(2i\widehat{\tau}^{1/2}r\right)}{\Gamma\left(1+2\mu\right)}\\ &\sim \left(2i\widehat{\tau}^{1/2}r\right)^{1/4}\pi^{-1/2}k^{-1/4-\mu}\cos\left[2\sqrt{2i\widehat{\tau}^{1/2}rk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad\times\left\{1+O\left(|k|^{-1/2}\right)\right\},\\ &\frac{M_{-k,\mu}\left(2i\widehat{\tau}^{1/2}r\right)}{\Gamma\left(1+2\mu\right)}\\ &\sim \left(2i\widehat{\tau}^{1/2}r\right)^{1/4}\pi^{-1/2}k^{-1/4-\mu}\\ &\quad\times e^{\mp\pi i(\mu+1/4)}\cos\left[2e^{\pm(1/2)\pi i}\sqrt{2i\widehat{\tau}^{1/2}rk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad\cdot\left\{1+O\left(|k|^{-1/2}\right)\right\},\end{aligned} \tag{58}$$

where as usual $k = iz\widehat{\tau}^{-1/2}$ and $\mu = l + 1/2$. □

We then express the asymptotic approximation of the function $\Theta(r,\widehat{\tau})$ as follows:

$$\begin{aligned}&\Theta\left(r,\widehat{\tau}\right)\\ &= \alpha_0\left(2i\widehat{\tau}^{1/2}r\right)^{1/4}\pi^{-1/2}\\ &\quad\times k^{-1/4-\mu}\cos\left[2\sqrt{2i\widehat{\tau}^{1/2}rk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad\cdot\left\{1+O\left(|k|^{-1/2}\right)\right\}\\ &\quad+\alpha_1\left(2i\widehat{\tau}^{1/2}r\right)^{1/4}\pi^{-1/2}k^{-1/4-\mu}\\ &\quad\times e^{\mp\pi i(\mu+1/4)}\cos\left[2e^{\pm(1/2)\pi i}\sqrt{2i\widehat{\tau}^{1/2}rk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad\cdot\left\{1+O\left(|k|^{-1/2}\right)\right\}.\end{aligned} \tag{59}$$

Through a careful limiting process we find that

$$\lim_{r\to a} n^+\left(\widehat{\tau}\right)\Phi\left(r,\widehat{\tau}\right) = \lim_{r\to b} m^+\left(\widehat{\tau}\right)\Phi\left(r,\widehat{\tau}\right) = 0, \tag{60}$$

a result that comes in handy.

Using (60) and (59), we apply the boundary conditions at a and b to obtain

$$\begin{aligned}&\alpha_0\cos\left[2\sqrt{2i\widehat{\tau}^{1/2}ak} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad+\alpha_1 e^{\mp\pi i(\mu+1/4)}\cos\left[2e^{\pm(1/2)\pi i}\sqrt{2i\widehat{\tau}^{1/2}ak} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &= 0,\\ &\alpha_0\cos\left[2\sqrt{2i\widehat{\tau}^{1/2}bk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &\quad+\alpha_1 e^{\mp\pi i(\mu+1/4)}\cos\left[2e^{\pm(1/2)\pi i}\sqrt{2i\widehat{\tau}^{1/2}bk} - \pi\left(\mu+\frac{1}{4}\right)\right]\\ &= 0.\end{aligned} \tag{61}$$

By employing the conditions in (55) and (56), we obtain from (61) that

$$\sin\left[2\sqrt{2i\widehat{\tau}^{1/2}bk} - 2\sqrt{2i\widehat{\tau}^{1/2}ak}\right] = 0. \tag{62}$$

Thus, the eigenvalues $\widehat{\tau}^{(n)}$ may be expressed as

$$\widehat{\tau}^{(n)} = \frac{n^4\pi^4}{\left[8i\left(b+a\right)+16k\sqrt{ab}\right]^2}, \tag{63}$$

where n is the mode number. The eigenvalues are complex, which is consistent with the fact that τ in the radial Schrödinger equation is complex. This result as expected is similar to the result obtained in [11] since the perturbation term ε which is an integral part of $\widehat{\tau}$ is independent of r.

5. Concluding Remarks

The boundary value problem with interior singularities and turning points for which resonance equations result may be represented as nonstandard Sturm-Liouville eigenproblems. With the asymptotic expansions of the eigenfunctions solutions for $r \in \mathbb{R}^3$ being the same for $r \in \mathbb{R}$, we have in this paper presented solutions valid for an unrestricted interval. The three turning points analysis presented here when the perturbation term ε is a function of r demonstrate the applicability of the method in these cases. The beauty of this split interval technique which employs the use of the Titchmarsh-Weyl m-function theory is that eigenfunctions solutions could also be obtained for the nonconstant perturbation term $q(r) = \varepsilon_0 + \varepsilon_1 r$ which happens to have three turning points provided one of these turning points coalesce with another, thus permitting us to use the branch cuts as in the theory of atmospheric and oceanic waves.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

References

[1] D. Eric, *Exact WKB Analysis near a Simple Turning Point. Algebraic Analysis of Differential Equations from Micro-Local Analysis to Exponential Asymtotics*, Springer, Tokyo, Japan, 2008.

[2] A. Fruchard and R. Schäfke, "Composite asymptotic expansions and turning points of singularly perturbed ordinary differential equations," *Comptes Rendus Mathematique*, vol. 348, no. 23-24, pp. 1273–1277, 2010.

[3] H.-R. Sun, Y.-N. Li, J. J. Nieto, and Q. Tang, "Existence of solutions for Sturm-Liouville boundary value problem of impulsive differential equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 707163, 19 pages, 2012.

[4] L. Zhang, X. Huang, and C. Xing, "Solvability of Sturm-Liouville boundary value problems with impulses," *Boundary Value Problems*, vol. 2013, article 192, 2013.

[5] M. Jiaqi, "Singularly perturbed solution of boundary value problem for nonlinear equations of fourth order with two parameters," *Advances in Mathematics*, vol. 39, no. 6, pp. 736–740, 2010.

[6] M. Ni and Z. Wang, "On step-like contrast structure of singularly perturbed systems," *Boundary Value Problems*, vol. 2009, Article ID 634324, 17 pages, 2009.

[7] Y. Tian and W. Ge, "Multiple positive solutions for a second order Sturm-Liouville boundary value problem with a p-Laplacian via variational methods," *The Rocky Mountain Journal of Mathematics*, vol. 39, no. 1, pp. 325–342, 2009.

[8] Y. Tian and W. Ge, "Multiple solutions of impulsive Sturm-Liouville boundary value problem via lower and upper solutions and variational methods," *Journal of Mathematical Analysis and Applications*, vol. 387, no. 2, pp. 475–489, 2012.

[9] Y. Tian and W. Ge, "Variational methods to Sturm-Liouville boundary value problem for impulsive differential equations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 1, pp. 277–287, 2010.

[10] T. M. Acho and D. P. Clemence, "Sturm-Liouville eigenproblems with an interior double pole," *Zeitschrift für Angewandte Mathematik und Physik*, vol. 46, no. 3, pp. 459–474, 1995.

[11] T. M. Acho, "Derivation of eigenrelations for the Sturm-Liouville boundary value problems with interior singularities," *Quarterly of Applied Mathematics*, vol. 65, no. 2, pp. 375–383, 2007.

[12] J. P. Boyd, "Sturm-Liouville eigenproblems with an interior pole," *Journal of Mathematical Physics*, vol. 22, no. 8, pp. 1575–1590, 1981.

[13] F. V. Atkinson, N. N. Everitt, and A. Zettl, "Regularisation of a Sturm-Liouville problem with an interior singularity," *Differential Integral Equations*, vol. 1, no. 2, pp. 213–221, 1988.

[14] J. P. Boyd, "Complex coordinate methods for hydrodynamic instabilities and Sturm-Liouville eigenproblems with an interior singularity," *Journal of Computational Physics*, vol. 57, no. 3, pp. 454–471, 1985.

[15] T. M. Acho, "A matching technique and structure theorems for Sturm-Liouville boundary value problems with interior singularities," *Quaestiones Mathematicae*, vol. 25, no. 3, pp. 275–287, 2002.

[16] T. M. Acho and D. P. Clemence, "The parameter dependent Sturm-Liouville eigenproblem with an interior simple or double pole," *The ANZIAM Journal*, vol. 43, no. 4, pp. 479–491, 2002.

[17] E. A. Coddington and N. Levinson, *Theory of Ordinary Differential Equations*, R.E. Krieger, Malabar, Fla, USA, 1955.

[18] Y. Pinchover and J. Rubinstein, *An Introduction to Partial Differential Equations*, Cambridge University Press, Cambridge, UK, 2007.

[19] M. A. Shubov, "High-energy asymptotics of resonances for three-dimensional Schrödinger operator with screened Coulomb potential," *Journal of Mathematical Physics*, vol. 35, no. 2, pp. 656–679, 1994.

[20] R. G. Newton, *Scattering Theory of Waves and Particles*, Springer, New York, NY, USA, 2nd edition, 1982.

[21] L. J. Slater, *Confluent Hypergeometric Functions*, Cambridge University Press, 1960.

[22] N. Bleistein and R. A. Handelsman, *Asymptotic Expansion of Integrals*, Dover Publications, Ney York, NY, USA, 1986.

[23] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a Complex Variable*, vol. 49 of *Classics in Applied Mathematics*, SIAM, Philadelphia, Pa, USA, 2005.

[24] A. H. Nayfeh, *Perturbation Methods*, John Wiley & Sons, New York, NY, USA, 1973.

[25] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, NY, USA, 1965.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 913202, 7 pages
http://dx.doi.org/10.1155/2014/913202

Research Article

Local Fractional Variational Iteration Method for Inhomogeneous Helmholtz Equation within Local Fractional Derivative Operator

Xian-Jin Wang,[1] Yang Zhao,[2] Carlo Cattani,[3] and Xiao-Jun Yang[4]

[1] *Foundation College, Ningbo Dahongying University, Ningbo 315175, China*
[2] *Electronic and Information Technology Department, Jiangmen Polytechnic, Jiangmen 529090, China*
[3] *Department of Mathematics, University of Salerno, Via Giovanni Paolo II, Fisciano, 84084 Salerno, Italy*
[4] *Department of Mathematics and Mechanics, China University of Mining and Technology, Xuzhou, Jiangsu 221008, China*

Correspondence should be addressed to Yang Zhao; zhaoyang19781023@gmail.com

Received 27 May 2014; Accepted 6 June 2014; Published 26 June 2014

Academic Editor: Abdon Atangana

The inhomogeneous Helmholtz equation within the local fractional derivative operator conditions is investigated in this paper. The local fractional variational iteration method is applied to obtain the nondifferentiable solutions and the graphs of the illustrative examples are also shown.

1. Introduction

Helmholtz equation has played an important role in the partial differential equations arising in mathematical physics [1, 2]. In computing the solution of Helmholtz equation, some analytical and numerical methods were presented. For example, Ihlenburg and Babuška used the finite element method to deal with the Helmholtz equation [3]. Momani and Abuasad suggested the variational iteration method to solve the Helmholtz equation [4]. Rafei and Ganji reported the homotopy perturbation method to report the solution to the Helmholtz equation [5]. Bayliss et al. considered the iterative method to discuss the Helmholtz equation [6]. Benamou and Desprès reported the domain decomposition method for solving the Helmholtz equation [7]. Linton presented Green's function method for the Helmholtz equation [8]. Singer and Turkel proposed the finite difference method to solve the Helmholtz equation [9]. Otto and Larsson applied the second-order method to discuss the Helmholtz equation [10].

Recently, the fractional calculus [11, 12] was developed and applied to present some models in the fields, such as the fractional-order digital control systems [13], the fractional-order viscoelasticity [14], the fractional-order quantum mechanics [15], and fractional-order dynamics [16]. More recently, Samuel and Thomas report the fractional Helmholtz equations [17] and some methods for solving the fractional differential equations were reported in [18–23]. However, we are faced with the problem that there must be some calculus to deal with the nondifferentiable solution for Helmholtz equation, which was structured within the local fractional derivative [24–34]. In this paper, we consider the local fractional inhomogeneous Helmholtz equation in two-dimensional case [31, 32]:

$$\frac{\partial^{2\alpha} M(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M(x,y)}{\partial y^{2\alpha}} + \omega^{2\alpha} M(x,y) = f(x,y), \quad (1)$$

where $f(x,y)$ is a local fractional continuous function and the local fractional partial derivative is defined as follows [24]:

$$\left.\frac{\partial^{\alpha} f(x,y)}{\partial x^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}\left(f(x,y) - f(x_0,y)\right)}{(x-x_0)^{\alpha}}, \quad (2)$$

with

$$\Delta^{\alpha}\left(f\left(x,y\right)-f\left(x_0,y\right)\right) \cong \Gamma\left(1+\alpha\right)\Delta\left(f\left(x,y\right)-f\left(x_0,y\right)\right). \tag{3}$$

The local fractional inhomogeneous Helmholtz equation in three-dimensional case was suggested as follows [29, 30]:

$$\frac{\partial^{2\alpha}M\left(x,y,z\right)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}M\left(x,y,z\right)}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}M\left(x,y,z\right)}{\partial z^{2\alpha}} +\omega^{2\alpha}M\left(x,y,z\right)=f\left(x,y,z\right), \tag{4}$$

where $f(x,y,z)$ is a local fractional continuous function. Here, we use the local fractional variational iteration method [30–34] to solve the local fractional inhomogeneous Helmholtz equation in two-dimensional case. The structure of this paper is as follows. In Section 2, we analyze the local fractional variational iteration method. In Section 3, we present some illustrative examples. Finally, the conclusion is given in Section 4.

2. Analysis of the Local Fractional Variational Iteration Method

Here, we give the analysis of the local fractional variational iteration method as follows. We first consider the local fractional linear partial differential equation:

$$L_{\alpha}u+R_{\alpha}u=g\left(t\right), \tag{5}$$

where L_{α} denotes linear local fractional derivative operator of order 2α, R_{α} denotes a lower-order local fractional derivative operator, and $g(t)$ is the nondifferentiable source term.

Let the local fractional operator be defined as [24, 30–34]

$${}_{a}I_{b}^{(\alpha)}f\left(x\right)=\frac{1}{\Gamma\left(1+\alpha\right)}\int_{a}^{b}f\left(t\right)\left(dt\right)^{\alpha} =\frac{1}{\Gamma\left(1+\alpha\right)}\lim_{\Delta t\to 0}\sum_{j=0}^{j=N-1}f\left(t_j\right)\left(\Delta t_j\right)^{\alpha} \tag{6}$$

with the partitions of the interval $[a,b]$, $\Delta t_j=t_{j+1}-t_j$, $\Delta t=\max\{\Delta t_1,\Delta t_2,\Delta t_j,\ldots\}$, and $j=0,\ldots,N-1$, $t_0=a$, $t_N=b$.

We now structure the correctional local fractional functional in the form

$$u_{n+1}\left(x\right)=u_n\left(x\right)+{}_{0}I_{x}^{(\alpha)} \times\left\{\zeta\left(s\right)\left(L_{\alpha}u_n\left(s\right)+R_{\alpha}u_n\left(s\right)-g\left(s\right)\right)\right\}. \tag{7}$$

Making the local fractional variation, we have

$$\delta^{\alpha}u_{n+1}\left(x\right)=\delta^{\alpha}u_n\left(x\right)+{}_{0}I_{x}^{(\alpha)}\delta^{\alpha} \times\left\{\zeta\left(s\right)\left(L_{\alpha}u_n\left(s\right)+R_{\alpha}\widetilde{u}_n\left(s\right)-\widetilde{g}\left(s\right)\right)\right\}=0 \tag{8}$$

such that the following stationary conditions are given as

$$1-\zeta(s)^{(\alpha)}\Big|_{s=x}=0,\qquad \zeta\left(s\right)\Big|_{s=x}=0,\qquad \zeta(s)^{(2\alpha)}\Big|_{s=x}=0. \tag{9}$$

In view of (9), we obtain the fractal Lagrange multiplier, which is given by

$$\zeta\left(s\right)=\frac{\left(s-x\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}. \tag{10}$$

From (7) and (10), we reach at the local fractional variational iteration algorithm

$$u_{n+1}\left(x\right)=u_n\left(x\right)+{}_{0}I_{x}^{(\alpha)} \times\left\{\frac{\left(s-x\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left(L_{\alpha}u_n\left(s\right)+R_{\alpha}u_n\left(s\right)-g\left(s\right)\right)\right\}, \tag{11}$$

where the nondifferentiable initial value is suggested as

$$u_0\left(x\right)=u\left(0\right)+\frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)}u^{(\alpha)}\left(0\right). \tag{12}$$

Therefore, from (11), we write the solution of (7) as follows:

$$u=\lim_{n\to\infty}u_n. \tag{13}$$

3. Some Illustrative Examples

In this section, we give some illustrative examples for solving the local fractional inhomogeneous Helmholtz equation in two-dimensional case.

We present the following local fractional inhomogeneous Helmholtz equation:

$$\frac{\partial^{2\alpha}M\left(x,y\right)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}M\left(x,y\right)}{\partial y^{2\alpha}}+M\left(x,y\right)=\frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)}E_{\alpha}\left(y^{\alpha}\right), \tag{14}$$

subject to the initial-boundary conditions:

$$\frac{\partial^{\alpha}M\left(0,y\right)}{\partial x^{\alpha}}=E_{\alpha}\left(-y^{\alpha}\right),\qquad M\left(0,y\right)=0. \tag{15}$$

Making use of (11), we structure the local fractional variational iteration algorithm as follows:

$$M_{n+1}\left(x,y\right)=M_n\left(x,y\right)+{}_{0}I_{x}^{(\alpha)}\frac{\left(s-x\right)^{\alpha}}{\Gamma\left(1+\alpha\right)} \times\left\{\frac{\partial^{2\alpha}M_n\left(x,y\right)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}M_n\left(x,y\right)}{\partial y^{2\alpha}} +M_n\left(x,y\right)-\frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)}E_{\alpha}\left(y^{\alpha}\right)\right\}, \tag{16}$$

where the initial value is presented as

$$M_0\left(x,y\right)=\frac{x^{\alpha}}{\Gamma\left(1+\alpha\right)}E_{\alpha}\left(-y^{\alpha}\right), \tag{17}$$

whose plot is shown in Figure 1.

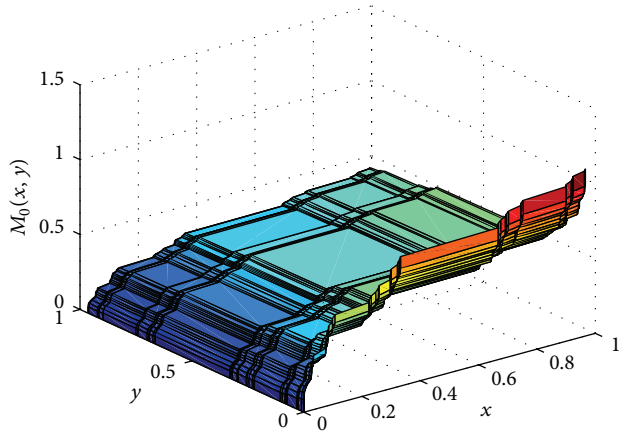


FIGURE 1: The graph of the initial value (17) where $\alpha = \ln 2/\ln 3$.

In view of (16) and (17), we arrive at the first approximation:

$$\begin{aligned}
M_1(x,y) &= M_0(x,y) + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\
&\quad\times\left\{\frac{\partial^{2\alpha}M_0(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_0(x,y)}{\partial y^{2\alpha}} + M_0(x,y)\right. \\
&\quad\left. - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha) \\
&\quad + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)}\left\{\frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(-y^\alpha) \\
&= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right)E_\alpha(-y^\alpha).
\end{aligned} \tag{18}$$

The second approximation is

$$\begin{aligned}
M_2(x,y) &= M_1(x,y) + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\
&\quad\times\left\{\frac{\partial^{2\alpha}M_1(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_1(x,y)}{\partial y^{2\alpha}} + M_1(x,y)\right. \\
&\quad\left. - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\} \\
&= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right)E_\alpha(-y^\alpha) \\
&\quad + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)}\left\{-\frac{2x^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(-y^\alpha)\right\} \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(-y^\alpha) \\
&\quad + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}E_\alpha(-y^\alpha) \\
&= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\
&\quad\times E_\alpha(-y^\alpha).
\end{aligned} \tag{19}$$

Making best of (16) and (19), the third approximation reads as

$$\begin{aligned}
M_3(x,y) &= M_2(x,y) + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\
&\quad\times\left\{\frac{\partial^{2\alpha}M_2(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_2(x,y)}{\partial y^{2\alpha}} + M_2(x,y)\right. \\
&\quad\left. - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\} \\
&= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\
&\quad\times E_\alpha(-y^\alpha) + E_\alpha(-y^\alpha)\,{}_0I_x^{(\alpha)} \\
&\quad\times\frac{(s-x)^\alpha}{\Gamma(1+\alpha)}\left(\frac{4x^{5\alpha}}{\Gamma(1+5\alpha)}\right) \\
&= \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}E_\alpha(-y^\alpha) \\
&\quad + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)}E_\alpha(-y^\alpha) - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)}E_\alpha(-y^\alpha) \\
&= \left(\frac{x^\alpha}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)}\right. \\
&\quad\left. + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)}\right)E_\alpha(-y^\alpha).
\end{aligned} \tag{20}$$

From (16) and (20), we obtain the fourth approximation of (14) given as

$$\begin{aligned}
M_4(x,y) &= M_3(x,y) + {}_0I_x^{(\alpha)}\frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\
&\quad\times\left\{\frac{\partial^{2\alpha}M_3(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha}M_3(x,y)}{\partial y^{2\alpha}} + M_3(x,y)\right. \\
&\quad\left. - \frac{x^\alpha}{\Gamma(1+\alpha)}E_\alpha(-y^\alpha)\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
&\qquad \left. - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} \right) E_{\alpha}(-y^{\alpha}) + E_{\alpha}(-y^{\alpha})\ {}_0I_x^{(\alpha)} \\
&\quad \times \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left(-\frac{8x^{7\alpha}}{\Gamma(1+7\alpha)} \right) \\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} E_{\alpha}(-y^{\alpha}) \\
&\quad + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} E_{\alpha}(-y^{\alpha}) \\
&\quad + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} E_{\alpha}(-y^{\alpha}) \\
&= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
&\quad \left. - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} \right) E_{\alpha}(-y^{\alpha}).
\end{aligned} \tag{21}
$$

As similar manner, from (21), we arrive at the fifth approximate formula:

$$
\begin{aligned}
M_5(x,y) &= M_4(x,y) + {}_0I_x^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \\
&\quad \times \left\{ \frac{\partial^{2\alpha} M_4(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M_4(x,y)}{\partial y^{2\alpha}} + M_4(x,y) \right. \\
&\qquad \left. - \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}) \right\} \\
&= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
&\qquad \left. - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} \right) E_{\alpha}(-y^{\alpha}) \\
&\quad + E_{\alpha}(-y^{\alpha})\ {}_0I_x^{(\alpha)} \frac{(s-x)^{\alpha}}{\Gamma(1+\alpha)} \left(\frac{16x^{9\alpha}}{\Gamma(1+9\alpha)} \right) \\
&= \frac{x^{\alpha}}{\Gamma(1+\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} E_{\alpha}(-y^{\alpha}) \\
&\quad + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} E_{\alpha}(-y^{\alpha}) \\
&\quad + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} E_{\alpha}(-y^{\alpha}) - \frac{16x^{11\alpha}}{\Gamma(1+11\alpha)} \\
&= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
&\qquad \left. - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} - \frac{16x^{11\alpha}}{\Gamma(1+11\alpha)} \right) \\
&\quad \times E_{\alpha}(-y^{\alpha}).
\end{aligned} \tag{22}
$$

Hence, we have the local fractional series solution of (14):

$$
\begin{aligned}
M_n(x,y) &= \left(\frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{x^{3\alpha}}{\Gamma(1+3\alpha)} + \frac{2x^{5\alpha}}{\Gamma(1+5\alpha)} \right. \\
&\qquad \left. - \frac{4x^{7\alpha}}{\Gamma(1+7\alpha)} + \frac{8x^{9\alpha}}{\Gamma(1+9\alpha)} - \frac{16x^{11\alpha}}{\Gamma(1+11\alpha)} + \cdots \right) \\
&\quad \times E_{\alpha}(-y^{\alpha}) \\
&= \left(\frac{1}{2} \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{2^i x^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} \right) \\
&\quad \times E_{\alpha}(-y^{\alpha}).
\end{aligned} \tag{23}
$$

From (13), we get the exact solution of (14) given as

$$
\begin{aligned}
M &= \lim_{n \to \infty} M_n(x,y) \\
&= \lim_{n \to \infty} \left(\frac{1}{2} \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{2^i x^{(2i+1)\alpha}}{\Gamma(1+(2i+1)\alpha)} \right) \\
&\quad \times E_{\alpha}(-y^{\alpha}) \\
&= \left[\frac{1}{2} \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{2} \sin_{\alpha}(2x^{\alpha}) \right] E_{\alpha}(-y^{\alpha})
\end{aligned} \tag{24}
$$

and its plot is illustrated in Figure 2.

Example 1. We suggest the following local fractional inhomogeneous Helmholtz equation:

$$
\begin{aligned}
&\frac{\partial^{2\alpha} M(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M(x,y)}{\partial y^{2\alpha}} + M(x,y) \\
&\qquad = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \frac{y^{\alpha}}{\Gamma(1+\alpha)},
\end{aligned} \tag{25}
$$

and the initial-boundary conditions read as

$$
\begin{aligned}
\frac{\partial^{\alpha} M(0,y)}{\partial x^{\alpha}} &= \frac{y^{\alpha}}{\Gamma(1+\alpha)}, \\
M(0,y) &= 0.
\end{aligned} \tag{26}
$$

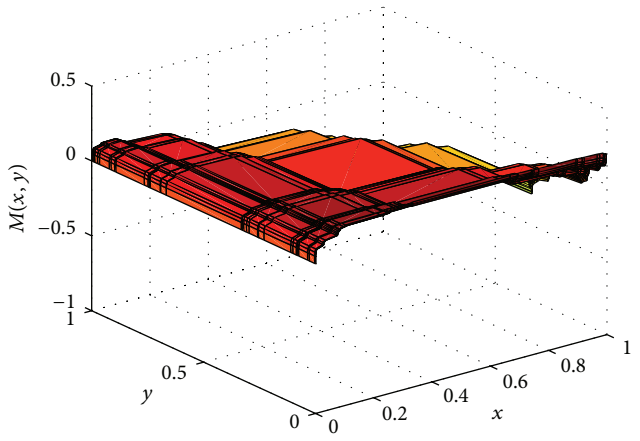


FIGURE 2: The graph of exact solution of (14) where $\alpha = \ln 2/\ln 3$.

From (11), we set up the local fractional variational iteration algorithm as follows:

$$\begin{aligned} M_{n+1}(x,y) &= M_n(x,y) + {}_0I_x^{(\alpha)} \frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\ &\times \left\{ \frac{\partial^{2\alpha} M_n(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M_n(x,y)}{\partial y^{2\alpha}} \right. \\ &\left. + M_n(x,y) - \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \right\}, \end{aligned} \tag{27}$$

where the initial value is suggested as

$$M_0(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \tag{28}$$

Appling (27) and (28) gives the first approximate solution:

$$\begin{aligned} M_1(x,y) &= M_0(x,y) + {}_0I_x^{(\alpha)} \frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\ &\times \left\{ \frac{\partial^{2\alpha} M_0(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M_0(x,y)}{\partial y^{2\alpha}} + M_0(x,y) \right. \\ &\left. - \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{29}$$

Using (27) and (29), we obtain the second approximate term, which is expressed as follows:

$$\begin{aligned} M_2(x,y) &= M_1(x,y) + {}_0I_x^{(\alpha)} \frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\ &\times \left\{ \frac{\partial^{2\alpha} M_1(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M_1(x,y)}{\partial y^{2\alpha}} + M_1(x,y) \right. \\ &\left. - \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{30}$$

In view of (27) and (30), we obtain the third approximation, which reads as follows:

$$\begin{aligned} M_3(x,y) &= M_2(x,y) + {}_0I_x^{(\alpha)} \frac{(s-x)^\alpha}{\Gamma(1+\alpha)} \\ &\times \left\{ \frac{\partial^{2\alpha} M_2(x,y)}{\partial x^{2\alpha}} + \frac{\partial^{2\alpha} M_2(x,y)}{\partial y^{2\alpha}} + M_2(x,y) \right. \\ &\left. - \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \right\} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}. \end{aligned} \tag{31}$$

Therefore, we arrive at the approximate term

$$M_n(x,y) = \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)}, \tag{32}$$

which leads to the exact solution of (25) given as

$$\begin{aligned} M(x,y) &= \lim_{n\to\infty} M_n(x,y) \\ &= \lim_{n\to\infty} \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \\ &= \frac{x^\alpha}{\Gamma(1+\alpha)} \frac{y^\alpha}{\Gamma(1+\alpha)} \end{aligned} \tag{33}$$

and its plot is illustrated in Figure 3.

4. Conclusions

In this work, the boundary value problems for the inhomogeneous Helmholtz equation within local fractional derivative operator were discussed by using the local fractional variational iteration method. Their nondifferentiable solutions are obtained and the graphs of the solutions with fractal dimension $\alpha = \ln 2/\ln 3$ are also given.

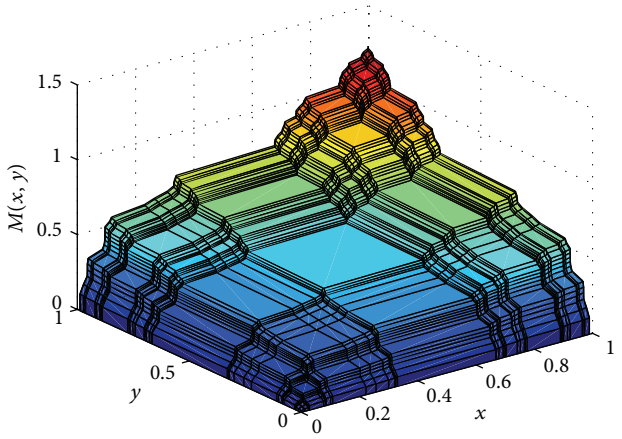


Figure 3: The plot of exact solution of (25) where $\alpha = \ln 2/\ln 3$.

Conflict of Interests

The authors declare that they have no competing interests in this paper.

Acknowledgment

This work was supported by the Zhejiang Provincial Natural Science Foundation (LY13A010007).

References

[1] A. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, New York, NY, USA, 1949.

[2] K. F. Riley, M. P. Hobson, and S. J. Bence, *Mathematical Methods for Physics and Engineering*, Cambridge University Press, Cambridge, UK, 2006.

[3] F. Ihlenburg and I. Babuška, "Finite element solution of the Helmholtz equation with high wave number Part I: the h-version of the FEM," *Computers and Mathematics with Applications*, vol. 30, no. 9, pp. 9–37, 1995.

[4] S. Momani and S. Abuasad, "Application of He's variational iteration method to Helmholtz equation," *Chaos, Solitons & Fractals*, vol. 27, no. 5, pp. 1119–1123, 2006.

[5] M. Rafei and D. D. Ganji, "Explicit solutions of Helmholtz equation and fifth-order KdV equation using homotopy perturbation method," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 7, no. 3, pp. 321–328, 2006.

[6] A. Bayliss, C. I. Goldstein, and E. Turkel, "An iterative method for the Helmholtz equation," *Journal of Computational Physics*, vol. 49, no. 3, pp. 443–457, 1983.

[7] J. Benamou and B. Desprès, "A domain decomposition method for the Helmholtz equation and related optimal control problems," *Journal of Computational Physics*, vol. 136, no. 1, pp. 68–82, 1997.

[8] C. M. Linton, "The Green's function for the two-dimensional Helmholtz equation in periodic domains," *Journal of Engineering Mathematics*, vol. 33, no. 4, pp. 377–402, 1998.

[9] I. Singer and E. Turkel, "High-order finite difference methods for the Helmholtz equation," *Computer Methods in Applied Mechanics and Engineering*, vol. 163, no. 1–4, pp. 343–358, 1998.

[10] K. Otto and E. Larsson, "Iterative solution of the Helmholtz equation by a second-order method," *SIAM Journal on Matrix Analysis and Applications*, vol. 21, no. 1, pp. 209–229, 1999.

[11] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, New York, NY, USA, Elsevier, 2006.

[12] C. Li, Y. Chen, and J. Kurths, "Fractional calculus and its applications," *Philosophical Transactions of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, vol. 371, no. 1990, Article ID 20130037, 3 pages, 2013.

[13] J. A. T. Machado, "Analysis and design of fractional-order digital control systems," *Systems Analysis Modelling Simulation*, vol. 27, no. 2-3, pp. 107–122, 1997.

[14] R. C. Koeller, "Applications of fractional calculus to the theory of viscoelasticity," *Journal of Applied Mechanics*, vol. 51, no. 2, pp. 299–307, 1984.

[15] V. Namias, "The fractional order Fourier transform and its application to quantum mechanics," *Journal of Applied Mathematics*, vol. 25, no. 3, pp. 241–265, 1980.

[16] R. Metzler and J. Klafter, "The random walk's guide to anomalous diffusion: a fractional dynamics approach," *Physics Reports*, vol. 339, no. 1, pp. 1–77, 2000.

[17] M. S. Samuel and A. Thomas, "On fractional Helmholtz equations," *Fractional Calculus & Applied Analysis*, vol. 13, no. 3, pp. 295–308, 2010.

[18] A. Atangana and S. B. Belhaouari, "Solving partial differential equation with space- and time-fractional derivatives via homotopy decomposition method," *Mathematical Problems in Engineering*, vol. 2013, Article ID 318590, 9 pages, 2013.

[19] A. Atangana and A. Kilicman, "A novel integral operator transform and its application to some FODE and FPDE with some kind of singularities," *Mathematical Problems in Engineering*, vol. 2013, Article ID 531984, 7 pages, 2013.

[20] A. H. Bhrawy, A. S. Alofi, and S. S. Ezz-Eldien, "A quadrature tau method for fractional differential equations with variable coefficients," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2146–2152, 2011.

[21] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "A new Jacobi operational matrix: an application for solving fractional differential equations," *Applied Mathematical Modelling*, vol. 36, no. 10, pp. 4931–4943, 2012.

[22] V. Daftardar-Gejji and H. Jafari, "Adomian decomposition: a tool for solving a system of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 2, pp. 508–518, 2005.

[23] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.

[24] X.-J. Yang, *Advanced Local Fractional Calculus and Its Applications*, World Science, New York, NY, USA, 2012.

[25] X. J. Yang, D. Baleanu, and J. A. T. Machado, "Mathematical aspects of the Heisenberg uncertainty principle within local fractional Fourier analysis," *Boundary Value Problems*, vol. 2013, article 131, 2013.

[26] X. Yang, H. M. Srivastava, J. H. He, and D. Baleanu, "Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives," *Physics Letters A*, vol. 377, no. 28–30, pp. 1696–1700, 2013.

[27] W. Zhong, F. Gao, and X. Shen, "Applications of Yang-Fourier transform to local fractional equations with local fractional

derivative and local fractional integral," *Advanced Materials Research*, vol. 461, pp. 306–310, 2012.

[28] C. Long, Y. Zhao, and H. Jafari, "Mathematical models arising in the fractal forest gap via local fractional calculus," *Abstract and Applied Analysis*, vol. 2014, Article ID 782393, 6 pages, 2014.

[29] Y. Hao, H. M. Srivastava, H. Jafari, and X.-J. Yang, "Helmholtz and diffusion equations associated with local fractional derivative operators involving the Cantorian and Cantor-type cylindrical coordinates," *Advances in Mathematical Physics*, vol. 2013, Article ID 754248, 5 pages, 2013.

[30] A. M. Yang, Z. S. Chen, H. M. Srivastava, and X. J. Yang, "Application of the local fractional series expansion method and the variational iteration method to the Helmholtz equation involving local fractional derivative operators," *Abstract and Applied Analysis*, vol. 2013, Article ID 259125, 6 pages, 2013.

[31] X.-J. Yang and D. Baleanu, "Fractal heat conduction problem solved by local fractional variation iteration method," *Thermal Science*, vol. 17, no. 2, pp. 625–628, 2013.

[32] W. Su, D. Baleanu, X. J. Yang, and H. Jafari, "Damped wave equation and dissipative wave equation in fractal strings within the local fractional variational iteration method," *Fixed Point Theory and Applications*, vol. 2013, article 89, 2013.

[33] D. Baleanu, J. A. T. Machado, C. Cattani, M. C. Baleanu, and X. Yang, "Local fractional variational iteration and decomposition methods for wave equation on Cantor sets within local fractional operators," *Abstract and Applied Analysis*, vol. 2014, Article ID 535048, 6 pages, 2014.

[34] J.-H. He, "Local fractional variational iteration method for fractal heat transfer in silk cocoon hierarchy," *Nonlinear Science Letters A*, vol. 4, no. 1, pp. 15–20, 2013.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 392792, 5 pages
http://dx.doi.org/10.1155/2014/392792

Research Article

Mathematical Solvability of a Caputo Fractional Polymer Degradation Model Using Further Generalized Functions

Emile Franc Doungmo Goufo and Stella Mugisha

Department of Mathematical Sciences, University of South Africa, Florida Science Campus, Florida, Gauteng 0003, South Africa

Correspondence should be addressed to Emile Franc Doungmo Goufo; dgoufef@unisa.ac.za

Received 12 May 2014; Accepted 10 June 2014; Published 25 June 2014

Academic Editor: Abdon Atangana

The continuous fission equation with derivative of fractional order α, describing the polymer chain degradation, is solved explicitly. We prove that, whether the breakup rate depends on the size of the chain breaking up or not, the evolution of the polymer sizes distribution is governed by a combination of higher transcendental functions, namely, Mittag-Leffler function, the further generalized G-function, and the Pochhammer polynomial. In particular, this shows the existence of an eigenproperty; that is, the system describing fractional polymer chain degradation contains replicated and partially replicated fractional poles, whose effects are given by these functions.

1. Introduction

Polymer degradation is the process where polymers are converted into monomers or mixtures of monomers. Polymers range from familiar synthetic plastics such as polystyrene (also called styrofoam) to natural biopolymers such as DNA and proteins that are fundamental to biological structure and function. Historically, products arising from the linkage of repeating units by covalent chemical bonds have been the primary focus of polymer science; emerging important areas of the science now focus on noncovalent links. Polyisoprene of latex rubber and the polystyrene of styrofoam are examples of polymeric natural/biological and synthetic polymers, respectively. In biological contexts, essentially, all biological macromolecules, that is, proteins (polyamides), nucleic acids (polynucleotides), and polysaccharides, are purely polymeric and are composed in large part of polymeric components, for instance, isoprenylated/lipid-modified glycoproteins, where small lipidic molecule and oligosaccharide modifications occur on the polyamide backbone of the protein. In the theory of polymers division, one would expect a conservation of mass, especially when polymers are converted into monomers or mixtures of monomers, but [1, 2] an infinite cascade of division events creating a "dust" of monomers of zero size carrying nonzero mass and leading to nonconservativeness (dishonesty) in the model has been observed. Since this remains partially unexplained by classical models of clusters' fission, extending the analysis to the fractional version and expressing the solutions explicitly may bring more light and a broader outlook about this phenomenon which remains a mystery. Thus, we are going to analyze the fractional fission system describing the polymer chain degradation and provide explicit expressions of its solutions. Before that, let us have a look at what is known about the classical kinetics of polymer chain degradation. However, there is a growing interest in extending the normal calculus with integer orders to noninteger orders (real or complex order) [3–6] because its applications, like, for example, the topic of this paper, have caught a great range of consideration in the past decade. For instance, in [3] the authors made use of the homotopy decomposition method (HDM), to solve a system of fractional nonlinear differential equations that arise in the model for HIV infection of CD4+ T cells and attractor one-dimensional Keller-Segel equations. In [4], two methods including Frobenius and Adomian decomposition method were used to generalize the classical Darcy law by regarding the water flow as a function of a noninteger order derivative of the piezometric head.

1.1. The Classic Polymer Degradation Dynamics. The binary fission integrodifferential equation,

$$\frac{\partial}{\partial t}g(x,t) = -g(x,t)\int_0^x H(y,x-y)\,dy + 2\int_x^\infty g(y,t)\,H(x,y-x)\,dy, \quad x,t>0, \tag{1}$$

is the kinetic equation describing the evolution of the sizes distribution. Here, $g(x,t)$ represents the density of x-groups (i.e., groups of size x) at time t, and $H(x,y)$ gives the average fission rate, that is, the average number at which clusters of size $x+y$ undergo splitting to form an x-group and a y-group. This model is applicable in many branches of natural sciences ranging from physics, through chemistry, engineering, and biology, to ecology and in numerous domains of applied sciences, the rock fractures, and break of droplets. Various types of fragmentation equations have been comprehensively analyzed in numerous works (see, e.g., [2, 7–11]). In the domain of polymer science, the fission dynamics have also been of considerable interest, since degradation of bonds or depolymerisation results in fragmentation; see [12–14]. In [13], the authors used the statistical arguments to find and analyse the size distribution of the model. The authors in [12] analysed the model in combination with the inverse process, that is, the coagulation process, and provided a similar result for the size distribution.

2. Fractional Fission Differential Problem

We aim to investigate the evolution of the number density of particles described by the fractional fission integrodifferential equation

$$D_t^\alpha g(x,t) = -g(x,t)\int_0^x H_\alpha(y,x-y)\,dy + 2\int_x^\infty g(y,t)\,H_\alpha(x,y-x)\,dy, \quad 0\le\alpha<1,\ x,t>0, \tag{2}$$

subject to the following initial condition:

$$g(x,0) = f(x), \quad x,t>0, \tag{3}$$

where

$${}_0^C D_t^\alpha g(x,t) = \frac{\partial^\alpha}{\partial t^\alpha}g(x,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-r)^{-\alpha}\frac{\partial}{\partial r}g(x,r)\,dr, \tag{4}$$

where $0\le\alpha<1$ is the fractional derivative of $g(x,t)$ in the sense of Caputo [15], with Γ the Gama function. For reasons of simplicity, we note that ${}_0^C D_t^\alpha = D_t^\alpha$.

3. Mathematical Analysis

Our analysis consists of two distinct cases: the case where the breakup rate depends on the size of the chain breaking up and the case where it does not depend. This will help us compare and analyse the two scenarios.

3.1. The Case $H_\alpha(x,y)=1$. Firstly we assume that the rate of breakup is independent of the length of polymer. Model (2) becomes

$$D_t^\alpha g(x,t) = -xg(x,t) + 2\int_x^\infty g(y,t)\,dy, \quad 0\le\alpha<1; \tag{5}$$

applying the Laplace transform on both sides of the latter equation yields

$$\mathscr{L}\left(D_t^\alpha g(x,t),s\right) = \mathscr{L}\left(-xg(x,t) + 2\int_x^\infty g(y,t)\,dy, s\right). \tag{6}$$

Clearly,

$$\mathscr{L}\left(D_t^\alpha g(x,t),s\right) = s^\alpha\tilde{g}(x,s) - s^{\alpha-1}f(x),$$
$$\mathscr{L}\left(-xg(x,t) + 2\int_x^\infty g(y,t)\,dy\right) = -x\tilde{g}(x,s) + 2\int_x^\infty \tilde{g}(y,s)\,dy, \tag{7}$$

where $\tilde{g}(x,s)$ is the Laplace transform $\mathscr{L}(g(x,t),s)$. We obtain

$$f(x) = \left(\frac{x}{s^{\alpha-1}} + s\right)\tilde{g}(x,s) - \frac{2}{s^{\alpha-1}}\int_x^\infty \tilde{g}(y,s)\,dy. \tag{8}$$

Remark 1. We note that by the differential expression (5) we implicitly require that $y \to g(y,t)$ should be Lebesgue integrable on any $[\epsilon,\infty)$ for $\epsilon>0$ and almost every $x>0$. The same assumption therefore applies to $y\to f(y)$ and $y\to\tilde{g}(y,s)$.

Hence we put $Y(x,s) = (2/s^{\alpha-1})\int_x^\infty \tilde{g}(y,s)dy$ with the confidence that the integrand is integrable over any interval $[\epsilon,\infty)$ so that the integral is absolutely continuous at each $x>0$ and we can thus differentiate so as to convert (8) into the partial differential equation as follows:

$$f(x) = \left(\frac{x}{s^{\alpha-1}} + s\right)\frac{s^{\alpha-1}}{2}\partial_x Y(x,s) - Y(x,s). \tag{9}$$

Choosing the constant in the general solution so as to have solutions converging to zero at ∞, we obtain its solution which is given as

$$Y(x,s) = 2e^{-\xi_{s,\alpha}(x)}\int_x^\infty e^{\xi_{s,\alpha}(\eta)}\frac{f(\eta)}{\eta+s^\alpha}d\eta, \tag{10}$$

where

$$\xi_{s,\alpha}(x) = \int_0^x \frac{2}{\eta+s^\alpha}d\eta = \ln\left(\frac{x+s^\alpha}{s^\alpha}\right)^2. \tag{11}$$

This leads to the solution of (8) as follows:

$$\begin{aligned}\widetilde{g}(x,s) &= \frac{s^{\alpha-1}f(x)}{x+s^{\alpha}} + \frac{2s^{\alpha-1}}{x+s^{\alpha}}e^{-\xi_{s,\alpha}(x)}\int_x^{\infty} e^{\xi_{s,\alpha}(\eta)}\frac{f(\eta)}{\eta+s^{\alpha}}d\eta \\ &= \frac{s^{\alpha-1}f(x)}{x+s^{\alpha}} + 2\frac{s^{\alpha-1}}{(x+s^{\alpha})^3}\int_x^{\infty}(\eta+s^{\alpha})f(\eta)\,d\eta.\end{aligned} \tag{12}$$

Applying the inverse Laplace transform $\mathscr{L}^{-1}(\widetilde{g}(x,s),t) = g(x,t)$ to the latter expression yields

$$\begin{aligned}\mathscr{L}^{-1}\left(\frac{s^{\alpha-1}f(x)}{x+s^{\alpha}},t\right) &= f(x)\,\mathscr{L}^{-1}\left(\frac{s^{\alpha-1}}{x+s^{\alpha}},t\right) \\ &= f(x)\sum_{n=0}^{\infty}\frac{(-x)^n(t)^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= f(x)\,E_{\alpha}\left[-xt^{\alpha}\right],\end{aligned} \tag{13}$$

where E_{α} is the Mittag-Leffler function as follows:

$$E_{\alpha}[x] = \sum_{n=0}^{\infty}\frac{x^n}{\Gamma(n\alpha+1)}. \tag{14}$$

We also have

$$\begin{aligned}&\mathscr{L}^{-1}\left(2\frac{s^{\alpha-1}}{(x+s^{\alpha})^3}\int_x^{\infty}(\eta+s^{\alpha})f(\eta)\,d\eta,t\right) \\ &\quad= 2\int_x^{\infty}f(\eta)\,\mathscr{L}^{-1}\left(\frac{s^{\alpha-1}(\eta+s^{\alpha})}{(x+s^{\alpha})^3},t\right),\end{aligned} \tag{15}$$

and, clearly,

$$\begin{aligned}&\mathscr{L}^{-1}\left(\frac{s^{\alpha-1}(\eta+s^{\alpha})}{(x+s^{\alpha})^3},t\right) \\ &\quad= \mathscr{L}^{-1}\left(\frac{\eta s^{\alpha-1}}{(x+s^{\alpha})^3},t\right) + \mathscr{L}^{-1}\left(\frac{s^{2\alpha-1}}{(x+s^{\alpha})^3},t\right) \\ &\quad= \eta\sum_{j=0}^{\infty}\frac{(-3)(-4)\cdots(-j-2)\,x^j t^{(2+j)\alpha}}{\Gamma(1+j)\,\Gamma(\{2+j\}\alpha+1)} \\ &\qquad+ \sum_{j=0}^{\infty}\frac{(-3)(-4)\cdots(-j-2)\,x^j t^{(1+j)\alpha}}{\Gamma(1+j)\,\Gamma(\{1+j\}\alpha+1)}.\end{aligned} \tag{16}$$

Thus the solution of fractional model (5) is given by

$$\begin{aligned}g(x,t) &= f(x)\,E_{\alpha}\left[-xt^{\alpha}\right] + 2G_{\alpha,\alpha-1,3}(-x,t)\int_x^{\infty}\eta f(\eta)\,d\eta \\ &\quad+ 2G_{\alpha,2\alpha-1,3}(-x,t)\int_x^{\infty}f(\eta)\,d\eta,\end{aligned} \tag{17}$$

where G is the higher transcendental generalized G-function defined by

$$G_{q,\beta,r}(x,t) = \sum_{j=0}^{\infty}\frac{(-r)(-1-r)\cdots(1-j-r)(-x)^j t^{(r+j)q-\beta-1}}{\Gamma(1+j)\,\Gamma(\{r+j\}q-\beta)} \tag{18}$$

and expressed in terms of the Pochhammer polynomial

$$(x-1)_n = (x-1)(x-2)\cdots(x-n), \tag{19}$$

as

$$G_{q,\beta,r}(x,t) = \sum_{j=0}^{\infty}\frac{(r)_j(-x)^j t^{(r+j)q-\beta-1}}{\Gamma(1+j)\,\Gamma(\{r+j\}q-\beta)}. \tag{20}$$

(See the appendix for some properties of the generalized G-function.) We see that putting $\alpha = 1$ in solution (17) reduces to the classic first order derivative and corresponds (as well known; see [14]) to an exponential distribution in x. It also shows that the polymer chains fragment exponentially fast in time. However, in fractional integrodifferential theory, the generalized G-function is of capital importance since it carries increased time domain complexity.

3.2. The Case $H_{\alpha}(x,y) = x+y$. This case represents a process where the rate of fission increases with size. Such a process can occur when the polymers are under tenseness or in a destructive force field such as ultrasound. Model (2) becomes

$$D_t^{\alpha}g(x,t) = -x^2 g(x,t) + 2\int_x^{\infty} yg(y,t)\,dy, \quad 0 \le \alpha < 1. \tag{21}$$

As done previously, we apply the Laplace transform to have

$$f(x) = \left(\frac{x^2}{s^{\alpha-1}} + s\right)\widetilde{g}(x,s) - \frac{2}{s^{\alpha-1}}\int_x^{\infty} y\widetilde{g}(y,s)\,dy. \tag{22}$$

By Remark 1, we can also put $Y(x,s) = (2/s^{\alpha-1})\int_x^{\infty} y\widetilde{g}(y,s)dy$ and then

$$f(x) = \left(\frac{x^2}{s^{\alpha-1}} + s\right)\frac{s^{\alpha-1}}{2x}\partial_x Y(x,s) - Y(x,s), \tag{23}$$

leading to

$$Y(x,s) = 2e^{-\xi_{s,\alpha}(x)}\int_x^{\infty} e^{\xi_{s,\alpha}(\eta)}\frac{\eta f(\eta)}{\eta^2+s^{\alpha}}d\eta, \tag{24}$$

with $\xi_{s,\alpha}(x) = \int_0^x (2\eta/(\eta^2+s^{\alpha}))d\eta = \ln((x^2+s^{\alpha})/s^{\alpha})$. The solution of (22) reads as

$$\widetilde{g}(x,s) = \frac{s^{\alpha-1}f(x)}{x^2+s^{\alpha}} + \frac{2s^{\alpha-1}}{(x^2+s^{\alpha})^2}\int_x^{\infty}(\eta)\,f(\eta)\,d\eta. \tag{25}$$

Applying the inverse Laplace transform and following the same steps as in the previous section finally yield the solution of fractional model (21) which is given by

$$\begin{aligned}g(x,t) &= f(x)\,E_{\alpha}\left[-x^2t^{\alpha}\right] \\ &\quad+ 2G_{\alpha,\alpha-1,2}\left(-x^2,t\right)\int_x^{\infty}\eta f(\eta)\,d\eta,\end{aligned} \tag{26}$$

where E and G are, respectively, defined in (14) and (18).

If we take $f(x) = \delta(x-l)$, then the latter solution becomes

$$g(x,t) = \begin{cases} 0 & \text{for } x > l \\ \delta(x-l)\, E_\alpha\left[-l^2 t^\alpha\right] & \text{for } x = l \\ 2lG_{\alpha,\alpha-1,2}\left(-x^2, t\right) & \text{for } x < l, \end{cases} \tag{27}$$

giving the compact form

$$g(x,t) = \delta(x-l)\, E_\alpha\left[-x^2 t^\alpha\right] + 2l\vartheta(l-x)\, G_{\alpha,\alpha-1,2}\left(-x^2, t\right), \tag{28}$$

where ϑ is the step function.

If we compare this distribution to the previous case where the breakup rate is independent of the length of polymer, we see that the second model shows a much slower production of daughter particles due to fission. This is an expected outcome given the relative behaviour of the two breakup speeds.

4. Concluding Remarks

We have used the model of fractional αth order describing the polymer chain degradation to express the solutions explicitly. We first considered the case where the rate of breakup is independent of the length of the polymer before investigating the case where the rate of fission is a function of the size of the polymer. In both cases, we found that the solutions are given by a combination of higher transcendental functions, the Mittag-Leffler function, the further generalized G-function, and the Pochhammer polynomial, showing the existence in the system of repeated and partially replicated fractional poles, whose effects are given by these functions. Moreover, it is of significant usefulness to obtain here a generalized function which when fractionally differentiated or integrated (differintegrated) by any order returns itself. Like exponential, trigonometric, and hyperbolic functions of integer order calculus, the definitions of such generalized functions are important in fractional calculus, especially to describe real phenomena like the polymer chain degradation. Therefore this work extends the preceding ones, with the inclusion of fractional differentiation which was not considered before, and the results we got here, especially the further generalized G-function which is of capital importance since it carries increased time domain complexity.

Appendix

Relevant Properties of the G-Function: (Partially) Replicated Fractional Poles

It is obvious to see that taking $r = 1$ reduces the G-function into the following generalized function, R-function [16]:

$$R_{q,\beta}(x,t) = \sum_{j=0}^{\infty} \frac{(x)^j t^{(1+j)q-\beta-1}}{\Gamma\left(\{1+j\}q-\beta\right)}, \tag{A.1}$$

subject to some proportional coefficients $1/\Gamma(1+j) = 1/(1+j)!$. Using the fractional derivative D_t^α $(0 < \alpha < 1)$, defined in (4), the R-function is proven to return itself under αth order differentiation. In fact we know (see [17, page 67]) that

$$D_t^\alpha (x)^p = \frac{\Gamma(p+1)(x)^{p-\alpha}}{\Gamma(p-\alpha+1)}, \quad p > -1, \tag{A.2}$$

yields

$$D_t^\alpha R_{\alpha,0}(x,t) = \sum_{j=0}^{\infty} \frac{(x)^j t^{j\alpha-1}}{\Gamma(j\alpha)}. \tag{A.3}$$

Taking $j = k+1$ leads to

$$D_t^\alpha R_{\alpha,0}(x,t) = \sum_{k=-1}^{\infty} \frac{(x)^{k+1} t^{k+1\alpha-1}}{\Gamma(k+1\alpha)}. \tag{A.4}$$

Thus,

$$D_t^\alpha R_{\alpha,0}(x,t) = xR_{\alpha,\beta}(x,t), \quad 1 > \alpha > 0 \tag{A.5}$$

for $t > 0$. Hence for $x = 1$, the function is proven to replicate, showing clearly its eigenproperty. In general, αth order differintegration of the R-function $R_{q,\beta}(x,t)$ returns another R-function, namely, $R_{q,\beta+\alpha}(x,t)$ [18]. Special cases of the G-function also include the exponential function, the sine, cosine, hyperbolic sine, and hyperbolic cosine functions, and the Mittag-Leffler function, Agarwal's function, Erdelyi's function, Hartley's F-function, and Miller and Ross's function.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] G. T. Tsao, "Structures of cellulosic materials and their hydrolysis by enzymes," in *Perspectives in Biotechnology and Applied Microbiology*, D. I. Alani and M. Moo-Young, Eds., pp. 205–212, 1986.

[2] R. M. Ziff and E. D. McGrady, "Shattering transition in fragmentation," *Physical Review Letters*, vol. 58, no. 9, pp. 892–895, 1987.

[3] A. Atangana and E. Alabaraoye, "Solving system of fractional partial differential equations arisen in the model of HIV infection of $CD4^+$ cells and attractor one-dimensional Keller-Segel equation," *Advances in Difference Equations*, vol. 2013, article 94, 2013.

[4] A. Atangana and P. D. Vermeulen, "Analytical solutions of a space-time fractional derivative of groundwater flow equation," *Abstract and Applied Analysis*, vol. 2014, Article ID 381753, 11 pages, 2014.

[5] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent-II," *Geophysical Journal of the Royal Astronomical Society*, vol. 13, no. 5, pp. 529–539, 1967.

[6] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, Calif, USA, 1999.

[7] E. F. Doungmo Goufo and S. C. Oukouomi Noutchie, "Honesty in discrete, nonlocal and randomly position structured fragmentation model with unbounded rates," *Comptes Rendus Mathematique*, vol. 351, no. 19-20, pp. 753–759, 2013.

[8] E. F. Doungmo Goufo and S. C. Oukouomi Noutchie, "Global analysis of a discrete nonlocal and nonautonomous fragmentation dynamics occurring in a moving process," *Abstract and Applied Analysis*, vol. 2013, Article ID 484391, 9 pages, 2013.

[9] S. C. Oukouomi Noutchie and E. F. Doungmo Goufo, "Global solvability of a continuous model for nonlocal fragmentation dynamics in a moving medium," *Mathematical Problems in Engineering*, vol. 2013, Article ID 320750, 8 pages, 2013.

[10] S. C. Oukouomi Noutchie and E. F. Doungmo Goufo, "On the honesty in nonlocal and discrete fragmentation dynamics in size and random position," *ISRN Mathematical Analysis*, vol. 2013, Article ID 908753, 7 pages, 2013.

[11] W. Wagner, "Explosion phenomena in stochastic coagulation-fragmentation models," *The Annals of Applied Probability*, vol. 15, no. 3, pp. 2081–2112, 2005.

[12] R. Blatz and J. N. Tobobsky, "Statistical investigation of fragments of polymer molecules," *The Journal of Physical Chemistry*, vol. 49, p. 77, 1945.

[13] H. Mark and R. Simha, "Degradation of long chain molecules," *Transactions of the Faraday Society*, vol. 35, pp. 611–618, 1940.

[14] R. M. Ziff and E. D. McGrady, "The kinetics of cluster fragmentation and depolymerisation," *Journal of Physics A*, vol. 18, no. 15, pp. 3027–3037, 1985.

[15] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent," *Journal of the Royal Australian Historical Society*, vol. 13, part 2, pp. 529–539, 1967.

[16] I. Podlubny, "Fractional-order systems and $PI^{\lambda}D^{\mu}$ controllers," *IEEE Transactions on Automatic Control*, vol. 44, no. 1, pp. 208–214, 1999.

[17] J. Spanier and K. B. Oldham, *An Atlas of Functions*, Hemisphere (Subsidiary of Harper & Row), New York, NY, USA, 1987.

[18] T. T. Hartley and C. F. Lorenzo, "A solution to the fundamental linear fractional order differential equation," Tech. Rep. NASA/TP-1998-208963, 1998.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 728736, 8 pages
http://dx.doi.org/10.1155/2014/728736

Research Article

An Accurate Spectral Galerkin Method for Solving Multiterm Fractional Differential Equations

A. H. Bhrawy[1,2] and A. S. Alofi[1]

[1] *Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia*
[2] *Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef 62511, Egypt*

Correspondence should be addressed to A. H. Bhrawy; alibhrawy@yahoo.co.uk

Received 4 May 2014; Accepted 20 May 2014; Published 12 June 2014

Academic Editor: Abdon Atangana

This paper reports a new formula expressing the Caputo fractional derivatives for any order of shifted generalized Jacobi polynomials of any degree in terms of shifted generalized Jacobi polynomials themselves. A direct solution technique is presented for solving multiterm fractional differential equations (FDEs) subject to nonhomogeneous initial conditions using spectral shifted generalized Jacobi Galerkin method. The homogeneous initial conditions are satisfied exactly by using a class of shifted generalized Jacobi polynomials as a polynomial basis of the truncated expansion for the approximate solution. The approximation of the spatial Caputo fractional order derivatives is expanded in terms of a class of shifted generalized Jacobi polynomials $J_n^{\alpha,-\beta}(x)$ with $x \in (0,1)$, and n is the polynomial degree. Several numerical examples with comparisons with the exact solutions are given to confirm the reliability of the proposed method for multiterm FDEs.

1. Introduction

Fractional differential equations (FDEs), as generalizations of classical integer order differential equations, are increasingly used to model several real phenomena emerging in engineering and science fields. Owing to the increasing applications, there has been important interest in developing analytical and numerical methods for the solution of fractional differential equations (see e.g., [1–7] and the references therein). These methods include variational iteration method [8, 9], Adomian decomposition method [10, 11], generalized differential transform method [12], Laplace decomposition method [13], homotopy analysis method [14], spectral method [15–19], finite difference method [20–22], and wavelet methods [23–25].

Spectral method is one of the principal methods of discretization for the numerical solution of most types of differential equations. The three most widely used spectral versions are the Galerkin, Tau, and collocation methods (see, for instance [26–32]). Recently, spectral method is a class of important tools for obtaining the numerical solutions of fractional differential equations. They have excellent error properties and they offer exponential rates of convergence for smooth problems. In the present paper we intend to extend the application of Galerkin method based on generalized Jacobi polynomials form solving linear problems to solve multiterm FDEs. To the best of our knowledge, there are not so many results on using this technique to solve such problems arising in mathematical physics. This partially motivated our interest in such a method.

Spectral Galerkin method for the numerical solution of fractional differential equations is characterized by expanding the solution by a truncated series of the trial functions. The unknown coefficients of this expansion will be determined by minimizing the error between the exact and numerical solutions in appropriate weighted space. This method provides exponential rates of convergence. An explicit expression for the derivatives of an infinitely differentiable function of any degree and for any fractional order in terms of the function itself is needed. Doha et al. [16] have obtained such a relation in the case of the basis functions of expansion that are shifted Jacobi polynomials. Another formula for shifted Legendre coefficients is obtained by Bhrawy et al. [17]. Moreover, in [33] the authors expressed

explicitly the Caputo fractional derivatives of generalized Laguerre polynomials of any degree in terms of the generalized Laguerre polynomials themselves to solve fractional initial value problems on the half line.

An explicit expression for any Caputo fractional order derivative of the shifted generalized Jacobi polynomials of any degree in terms of the shifted generalized Jacobi polynomials themselves is the first goal of this paper. The fundamental goal of this paper is to develop a direct solution technique based on shifted generalized Jacobi-Galerkin method (SGJG) for solving multiterm FDEs with homogeneous and nonhomogeneous initial conditions. Finally, we present some numerical results exhibiting the accuracy and efficiency of our numerical algorithm.

The next section of this paper is for fractional preliminaries. Section 3 is devoted to proving a formula that expresses the Caputo fractional order derivative of the shifted generalized Jacobi polynomials. In Section 4, we construct and develop algorithms for solving linear FDEs by using shifted generalized Jacobi Galerkin spectral method. In Section 5, several examples are presented. Finally, some concluding remarks are given in the last section.

2. Preliminaries and Notations

In this section, we present some basic knowledge of fractional calculus, orthogonal shifted Jacobi polynomials, and generalized Jacobi polynomials these are most relevant to spectral approximations.

2.1. The Fractional Derivative in the Caputo Sense. In this section, we state the definition and preliminaries of fractional calculus.

Definition 1. For m to be the smallest integer that exceeds ν, Caputo's fractional derivative operator of order $\nu > 0$ is defined as

$$D^{\nu} f(x) = \begin{cases} J^{m-\nu} D^{m} f(x), & \text{if } m-1 < \nu < m, \\ D^{m} f(x), & \text{if } \nu = m,\ m \in N, \end{cases} \tag{1}$$

where

$$J^{\nu} f(x) = \frac{1}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\, dt, \quad \nu > 0,\ x > 0. \tag{2}$$

For the Caputo derivative we have

$$D^{\nu} x^{\beta} = \begin{cases} 0, \\ \quad \text{for } \beta \in N_0,\ \beta < \lceil \nu \rceil, \\ \dfrac{\Gamma(\beta+1)}{\Gamma(\beta+1-\nu)}\, x^{\beta-\nu}, \\ \quad \text{for } \beta \in N_0,\ \beta \geq \lceil \nu \rceil \ \text{ or } \ \beta \notin N,\ \beta > \lfloor \nu \rfloor. \end{cases} \tag{3}$$

Similar to the integer-order differentiation, the Caputo's fractional differentiation is a linear operation; that is,

$$D^{\nu}\left(\lambda f(x) + \mu g(x)\right) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x), \tag{4}$$

where λ and μ are constants.

2.2. Classical Jacobi Polynomials. The Jacobi polynomials with the real parameters ($\alpha > -1, \beta > -1$) are a sequence of polynomials $P_n^{\alpha,\beta}(x)$ ($n = 0, 1, 2, \ldots$), satisfying the orthogonality relation

$$\int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} P_m^{\alpha,\beta}(x)\, P_n^{\alpha,\beta}(x)\, dx = \begin{cases} 0, & m \neq n, \\ h_n^{\alpha,\beta}, & m = n, \end{cases} \tag{5}$$

where

$$h_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\, \Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\, \Gamma(n+\alpha+\beta+1)}. \tag{6}$$

It is convenient to standardize the Jacobi polynomials so that

$$P_n^{\alpha,\beta}(1) = \frac{(\alpha+1)_n}{n!}, \qquad P_n^{\alpha,\beta}(-1) = \frac{(-1)^n\, (\beta+1)_n}{n!}, \tag{7}$$

where $(a)_k = \Gamma(a+k)/\Gamma(a)$. In this form the polynomials may be generated using the standard recurrence relation of Jacobi polynomials starting from $P_0^{\alpha,\beta}(x) = 1$ and $P_1^{\alpha,\beta}(x) = (1/2)[\alpha - \beta + (\lambda+1)x]$, or obtained from Rodrigue's formula

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta} D^n\left[(1-x)^{\alpha+n}(1+x)^{\beta+n}\right], \tag{8}$$

where $\lambda = \alpha + \beta + 1$.

The shifted Jacobi polynomials [34, 35] $R_n^{\alpha,\beta}(x), \alpha, \beta > -1$ are orthogonal polynomials on $[0, 1]$ with respect to the weight function $\omega(x) = (1-x)^{\alpha} x^{\beta}, \alpha, \beta > -1$. Note that the shifted Jacobi polynomials satisfy the orthogonality relation

$$\int_0^1 (1-x)^{\alpha} x^{\beta} R_m^{\alpha,\beta}(x)\, R_n^{\alpha,\beta}(x)\, dx = \begin{cases} 0, & m \neq n, \\ k_n^{\alpha,\beta}, & m = n, \end{cases} \tag{9}$$

where

$$k_n^{\alpha,\beta} = \frac{\Gamma(n+\alpha+1)\, \Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\, n!\, \Gamma(n+\alpha+\beta+1)}. \tag{10}$$

If we denote by $x_{N,j}(x_j^N)$, $0 \leqslant j \leqslant N$, and $\varpi_{N,j}(\varpi_j^N)$, ($0 \leq i \leq N$), to the nodes and Christoffel numbers of the standard (shifted) Legendre-Gauss-Lobatto quadratures on the intervals $(-1, 1), (0, 1)$, respectively, then one can easily show that

$$x_j^N = \frac{1}{2}\left(x_{N,j} + 1\right), \quad \varpi_j^N = \frac{1}{2}\varpi_{N,j}, \quad 0 \leq j \leq N, \tag{11}$$

and if $S_N(0, 1)$ denotes the set of all polynomials of degree at most N, then it follows that for any $\phi \in S_{2N+1}(0, 1)$,

$$\begin{aligned} \int_0^1 w^*(x)\, \phi(x)\, dx &= \frac{1}{2}\int_{-1}^{1} w(x)\, \phi\left(\frac{1}{2}(x+1)\right) dx \\ &= \frac{1}{2}\sum_{j=0}^{N} \varpi_{N,j}\, \phi\left(\frac{1}{2}\left(x_{N,j}+1\right)\right) \\ &= \sum_{j=0}^{N} \varpi_j^N\, \phi\left(x_j^N\right). \end{aligned} \tag{12}$$

According to Legendre-Gauss quadratures

$$x_{N,j} \text{ are the zeros of } L_{N+1}(x),$$
$$\varpi_{N,j} = \frac{2}{\left(1-\left(x_{N,j}\right)^2\right)} \frac{1}{\left(L'_{N+1}\left(x_{N,j}\right)\right)^2}, \quad 0 \le j \le N. \tag{13}$$

We define the discrete inner product and norm as follows:

$$(u,v)_{w^*,N} = \sum_{k=0}^{N} u\left(x_k^N\right) v\left(x_k^N\right) \varpi_k^N,$$
$$\|u\|_{w^*,N} = \sqrt{(u,u)_{w^*,N}}. \tag{14}$$

Obviously,

$$(u,v)_{w^*,N} = (u,v)_{w^*} \quad \forall u, v \in S_{2N-1}. \tag{15}$$

2.3. Generalized Jacobi Polynomials. Recently, Guo et al. [36] presented and developed the generalized Jacobi approximation, in which the parameters α and β considered in the generalized Jacobi polynomials $\widehat{J}_n^{\alpha,\beta}(x)$ might be any real numbers. In this section, we give some properties of such polynomials. Let $\widehat{I} = (-1,1)$ and $\widehat{\omega}^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta$. We denote the set of integers by $\mathbb{Z}$. For any $\alpha, \beta \in \mathbb{Z}$, the generalized Jacobi polynomials are defined by (see [36, 37])

$$\widehat{J}_n^{\alpha,\beta}(x) = \begin{cases} (1-x)^{-\alpha}(1+x)^{-\beta} P_{n+n_0}^{-\alpha,-\beta}(x), \\ \quad \text{if } \alpha,\beta \le 1,\ n_0 = \alpha+\beta, \\ (1-x)^{-\alpha} P_{n+n_0}^{-\alpha,\beta}(x), \\ \quad \text{if } \alpha \le 1, \beta > -1,\ n_0 = \alpha, \\ (1+x)^{-\beta} P_{n+n_0}^{\alpha,-\beta}(x), \\ \quad \text{if } \beta \le 1, \alpha > -1,\ n_0 = \beta, \\ P_{n+n_0}^{\alpha,\beta}(x), \\ \quad \text{if } \alpha,\beta > -1,\ n_0 = 0. \end{cases} \tag{16}$$

For our present purposes it is convenient to use the shifted Jacobi polynomials $R_n^{\alpha,\beta}(x)$; let $I = (0,1)$ and $\omega^{\alpha,\beta}(x) = (1-x)^\alpha x^\beta$. We define the shifted GJPs and separate them into four cases as follows.

Case 1.

$$J_n^{\alpha,\beta}(x) = R_{n+n_0}^{\alpha,\beta}(x), \quad n_0 = 0. \tag{17}$$

Case 2.

$$J_n^{-\alpha,\beta}(x) = (1-x)^\alpha R_{n+n_0}^{\alpha,\beta}(x), \quad n_0 = -\alpha. \tag{18}$$

Case 3.

$$J_n^{\alpha,-\beta}(x) = x^\beta R_{n+n_0}^{\alpha,\beta}(x), \quad n_0 = -\beta. \tag{19}$$

Case 4.

$$J_n^{-\alpha,-\beta}(x) = (1-x)^\alpha x^\beta R_{n+n_0}^{\alpha,\beta}(x), \quad n_0 = -\alpha-\beta, \tag{20}$$

where $\alpha, \beta > -1$, $\alpha, \beta \in \mathbb{Z}$.

Lemma 2. *Each of GJPs $J_n^{\alpha,\beta}(x)$, $J_n^{-\alpha,\beta}(x)$, $J_n^{\alpha,-\beta}(x)$ and $J_n^{-\alpha,-\beta}(x) : n > n_0$ forms a complete orthogonal system in $L^2_{\omega^{\alpha,\beta}}(I)$, $L^2_{\omega^{-\alpha,\beta}}(I)$, $L^2_{\omega^{\alpha,-\beta}}(I)$ and $L^2_{\omega^{-\alpha,-\beta}}(I)$, respectively. And the square of the norm of each of the four GJPs cases is defined as $k_{n+n_0}^{\alpha,\beta}$ where $k_n^{\alpha,\beta}$ is the square of the norm of the classical shifted Jacobi polynomials $R_n^{\alpha,\beta}(x)$ and*

$$k_{n+n_0}^{\alpha,\beta} = \frac{\Gamma(n+n_0+\alpha+1)\Gamma(n+n_0+\beta+1)}{(2n+2n_0+\alpha+\beta+1)(n+n_0)!\,\Gamma(n+n_0+\alpha+\beta+1)}. \tag{21}$$

Proof. Firstly,

$$\left(J_n^{\alpha,\beta}(x), J_m^{\alpha,\beta}(x)\right)_{\omega^{\alpha,\beta}} = \int_0^1 R_n^{\alpha,\beta}(x) R_m^{\alpha,\beta}(x)(1-x)^\alpha x^\beta dx = \delta_{m,n} k_n^{\alpha,\beta}. \tag{22}$$

Secondly,

$$\begin{aligned} &\left(J_n^{-\alpha,\beta}(x), J_m^{-\alpha,\beta}(x)\right)_{\omega^{-\alpha,\beta}} \\ &= \int_0^1 (1-x)^\alpha R_{n-\alpha}^{\alpha,\beta}(x)(1-x)^\alpha R_{m-\alpha}^{\alpha,\beta}(x)(1-x)^{-\alpha} x^\beta dx \\ &= \int_0^1 R_{n-\alpha}^{\alpha,\beta}(x) R_{m-\alpha}^{\alpha,\beta}(x)(1-x)^\alpha x^\beta dx = \delta_{m,n} k_{n-\alpha}^{\alpha,\beta}. \end{aligned} \tag{23}$$

Thirdly,

$$\begin{aligned} &\left(J_n^{\alpha,-\beta}(x), J_m^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}} \\ &= \int_0^1 x^\beta R_{n-\beta}^{\alpha,\beta}(x) x^\beta R_{m-\beta}^{\alpha,\beta}(x)(1-x)^\alpha x^{-\beta} dx \\ &= \int_0^1 R_{n-\beta}^{\alpha,\beta} R_{m-\beta}^{\alpha,\beta}(1-x)^\alpha x^\beta dx = \delta_{m,n} k_{n-\beta}^{\alpha,\beta}. \end{aligned} \tag{24}$$

And lastly,

$$\begin{aligned} &\left(J_n^{-\alpha,-\beta}(x), J_m^{-\alpha,-\beta}(x)\right)_{\omega^{-\alpha,-\beta}} \\ &= \int_0^1 (1-x)^\alpha x^\beta R_{n-\alpha-\beta}^{\alpha,\beta}(x) \\ &\qquad \times (1-x)^\alpha x^\beta R_{m-\alpha-\beta}^{\alpha,\beta}(x)(1-x)^{-\alpha} x^{-\beta} dx \\ &= \int_0^1 R_{n-\alpha-\beta}^{\alpha,\beta}(x) R_{m-\alpha-\beta}^{\alpha,\beta}(x)(1-x)^\alpha x^\beta dx = \delta_{m,n} k_{n-\alpha-\beta}^{\alpha,\beta}. \end{aligned} \tag{25}$$

□

3. The Fractional Derivatives of $J_i^{\alpha,-\beta}(x)$

The main objective of this section is to prove the following theorem for the fractional derivatives of the shifted generalized Jacobi polynomials. The analytic form of the shifted

generalized Jacobi polynomials $J_i^{\alpha,-\beta}(x)$ of degree $i-\beta$ is given by

$$J_i^{\alpha,-\beta}(x) = \sum_{k=0}^{i-\beta}(-1)^{i-\beta-k} \times \frac{\Gamma(i+1)\Gamma(i+k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+1)(i-\beta-k)!k!}x^{k+\beta}. \tag{26}$$

A function $u(x)$, square integrable in $(0,1)$, can be expressed in terms of shifted generalized Jacobi polynomials as

$$u(x) = \sum_{j=0}^{\infty} a_j J_j^{\alpha,-\beta}(x), \tag{27}$$

where the coefficients a_j are given by

$$a_j = \frac{1}{k_{j-\beta}^{\alpha,\beta}} \int_0^1 w^{(\alpha,-\beta)}(x)\, u(x)\, J_j^{\alpha,-\beta}(x)\, dx, \quad j = 0, 1, \ldots. \tag{28}$$

Lemma 3. *Let $J_i^{\alpha,-\beta}(x)$ be a shifted generalized Jacobi polynomial of degree $i-\beta$; then*

$$D^{\nu} J_i^{\alpha,-\beta}(x) = 0, \quad i-\beta = 0, 1, \ldots, \lceil \nu \rceil - 1, \ \nu > 0. \tag{29}$$

Proof. This lemma can be easily proved by making use of relations (3)-(4) with relation (26). □

Theorem 4. *The fractional derivative of order ν in the Caputo sense for the shifted generalized Jacobi polynomials is given by*

$$D^{\nu} J_i^{\alpha,-\beta}(x) = \sum_{j=0}^{\infty} S_{\nu}(i,j,\alpha,\beta)\, J_j^{\alpha,-\beta}(x), \quad i-\beta = \lceil \nu \rceil, \lceil \nu \rceil + 1, \ldots, \tag{30}$$

where

$$\begin{aligned} &S_{\nu}(i,j,\alpha,\beta) \\ &= \sum_{k=\lceil \nu \rceil}^{i-\beta} \Big((-1)^{i-\beta-k} \Gamma(i+1)\Gamma(i+k+\alpha+1) \\ &\qquad \times (2j-\beta+\alpha+1)(j-\beta)! \Big) \\ &\qquad \times \big(\Gamma(i+\alpha+1)\Gamma(k-\nu+\beta+1) \\ &\qquad\quad \times (i-\beta-k)!k!\Gamma(j-\beta+\alpha+1) \big)^{-1} \Big) \\ &\quad \times \sum_{l=0}^{j-\beta} \Big(\Big((-1)^{j-\beta-l} \Gamma(j+l+\alpha+1) \\ &\qquad \times \Gamma(\alpha+1)\Gamma(l+k+\beta-\nu+1) \Big) \\ &\qquad \times \big(\Gamma(l+\beta+1) \\ &\qquad\quad \times (j-\beta-l)!l!\Gamma(l+k+\alpha+\beta-\nu+2) \big)^{-1} \Big). \end{aligned} \tag{31}$$

Proof. The analytic form of the shifted generalized Jacobi polynomials $J_i^{\alpha,-\beta}(x)$ of degree $i-\beta$ is given by (26). Using (3)-(4) and (26), we have

$$\begin{aligned} &D^{\nu} J_i^{\alpha,-\beta}(x) \\ &= \sum_{k=0}^{i-\beta}(-1)^{i-\beta-k} \\ &\quad \times \frac{\Gamma(i+1)\Gamma(i+k+\alpha+1)}{\Gamma(k+\beta+1)\Gamma(i+\alpha+1)(i-\beta-k)!k!} D^{\nu} x^{k+\beta} \\ &= \sum_{k=\lceil \nu \rceil}^{i-\beta}(-1)^{i-\beta-k} \\ &\quad \times \frac{\Gamma(i+1)\Gamma(i+k+\alpha+1)}{\Gamma(i+\alpha+1)(i-\beta-k)!k!\Gamma(k+\beta-\nu+1)} x^{k+\beta-\nu}, \\ &\qquad i-\beta = \lceil \nu \rceil, \lceil \nu \rceil + 1, \ldots. \end{aligned} \tag{32}$$

Now, approximating $x^{k+\beta-\nu}$ by terms of shifted generalized Jacobi series, we have

$$x^{k+\beta-\nu} \simeq \sum_{j=0}^{\infty} b_{k,j} J_j^{\alpha,-\beta}(x), \tag{33}$$

where $b_{k,j}$ is given from (28) with $u(x) = x^{k+\beta-\nu}$, and this immediately gives

$$\begin{aligned} b_{k,j} &= \frac{(2j-\beta+\alpha+1)(j-\beta)!}{\Gamma(j-\beta+\alpha+1)} \\ &\quad \times \sum_{l=0}^{j-\beta} \Big(\big((-1)^{j-\beta-l} \Gamma(j+l+\alpha+1) \\ &\qquad \times \Gamma(\alpha+1)\Gamma(l+k+\beta-\nu+1) \big) \\ &\qquad \times \big(\Gamma(l+\beta+1) \\ &\qquad\quad \times (j-\beta-l)!l!\Gamma(l+k+\alpha+\beta-\nu+2) \big)^{-1} \Big). \end{aligned} \tag{34}$$

Employing (32)–(34), we get

$$D^{\nu} J_i^{\alpha,-\beta}(x) = \sum_{j=0}^{\infty} S_{\nu}(i,j,\alpha,\beta)\, J_j^{\alpha,-\beta}(x), \quad i = \lceil \nu \rceil, \lceil \nu \rceil + 1, \ldots, \tag{35}$$

where $S_{\nu}(i,j,\alpha,\beta)$ is given as in (30), and this proves the theorem. □

4. Shifted Generalized Jacobi Galerkin Method for FDEs

In this section, we are interested in employing the SGJG method to solve the linear multiterm FDE

$$D^{\nu}u(x)+\sum_{\sigma=1}^{r-1}\gamma_{\sigma}D^{\eta_{\sigma}}u(x)+\gamma_{r}u(x)=f(x),\quad x\in I=(0,1),\tag{36}$$

subject to the homogeneous initial conditions

$$u^{(q)}(0)=0,\quad q=0,\ldots,m-1,\tag{37}$$

where γ_{σ} $(\sigma=1,\ldots,r)$ and $0<\eta_1<\eta_2<\cdots<\eta_{r-1}<\nu$, $m-1<\nu\le m$ are constants, $D^{\nu}u(x)\equiv u^{(\nu)}(x)$ denotes the Caputo fractional derivative of order ν for $u(x)$, and $f(x)$ is a given source function. Let us first introduce some basic notation that will be used in the upcoming sections. We set

$$\begin{aligned}S_N&=\operatorname{span}\left\{J_{\beta}^{\alpha,-\beta}(x),J_{\beta+1}^{\alpha,-\beta}(x),\ldots,J_{N}^{\alpha,-\beta}(x)\right\},\\ V_N&=\left\{v\in S_N:v^{(j)}(0)=0,\ j=0,1,\ldots,m-1\right\},\end{aligned}\tag{38}$$

where $v^{(j)}(x)$ denotes jth-order differentiation of $v(x)$ with respect to x. Then the shifted generalized Jacobi-Galerkin approximation to (36) is to find $u_N\in V_N$ such that

$$\begin{aligned}&\left(D^{\nu}u_N,v(x)\right)_{\omega^{\alpha,-\beta}}+\sum_{\sigma=1}^{r-1}\gamma_{\sigma}\left(D^{\eta_{\sigma}}u_N,v(x)\right)_{\omega^{\alpha,-\beta}}\\ &\quad+\gamma_{r}\left(u_N,v(x)\right)_{\omega^{\alpha,-\beta}}=\left(f,v(x)\right)_{\omega^{\alpha,-\beta},N},\\ &\qquad\forall v\in V_N,\end{aligned}\tag{39}$$

where $\omega^{\alpha,-\beta}(x)=(1-x)^{\alpha}x^{-\beta}$ and $(u,v)_{\omega^{\alpha,-\beta}}=\int_I uv\omega^{\alpha,-\beta}dx$ is the inner product in the weighted space $L^2_{\omega^{\alpha,-\beta}}(I)$. The norm in $L^2_{\omega^{\alpha,-\beta}}(I)$ will be denoted by $\|\cdot\|_{\omega^{\alpha,-\beta}}$. Let

$$\begin{aligned}u_N(x)&=\sum_{j=\beta}^{N}a_jJ_j^{\alpha,-\beta}(x),\quad \mathbf{a}=(a_2,a_3,\ldots,a_N)^T,\\ f_k&=\left(f,J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}},\quad k=\beta,\beta+1,\ldots,N,\\ \mathbf{f}&=(f_2,f_3,\ldots,f_N)^T.\end{aligned}\tag{40}$$

Then we can write (39) as follows:

$$\begin{aligned}&\sum_{j=2}^{N}a_j\left[\left(D^{\nu}J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}}\right.\\ &\quad+\sum_{\sigma=1}^{r-1}\gamma_{\sigma}\left(D^{\eta_{\sigma}}J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}}\\ &\quad\left.+\gamma_{r}\left(J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}}\right]\\ &=\left(f,J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta},N},\\ &\qquad k=\beta,\beta+1,\ldots,N.\end{aligned}\tag{41}$$

Let us denote

$$\begin{aligned}A&=\left(a_{kj}\right)_{\beta\le k,j\le N},\qquad B^{\eta_{\sigma}}=\left(b_{kj}^{\eta_{\sigma}}\right)_{\beta\le k,j\le N},\\ C&=\left(c_{kj}\right)_{\beta\le k,j\le N},\\ a_{kj}&=\left(D^{\nu}J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}},\\ &\qquad j=\beta,\beta+1,\ldots,N,\\ b_{kj}^{\eta_{\sigma}}&=\left(D^{\eta_{\sigma}}J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}},\\ &\qquad k,j=\beta,\beta+1,\ldots,N,\\ c_{kj}&=\left(J_j^{\alpha,-\beta}(x),J_k^{\alpha,-\beta}(x)\right)_{\omega^{\alpha,-\beta}},\\ &\qquad k,j=\beta,\beta+1,\ldots,N.\end{aligned}\tag{42}$$

By virtue of (31) and making use of the orthogonality relation of shifted generalized Jacobi polynomials (21), and after some rather lengthy calculation, we get

$$\begin{aligned}a_{kj}&=S_{\nu}(j,k,\alpha,\beta)\,k_{k-\beta}^{\alpha,\beta},\\ &\qquad k,j=\beta,\beta+1,\ldots,N,\\ b_{kj}^{\eta_{\sigma}}&=S_{\eta_{\sigma}}(j,k,\alpha,\beta)\,k_{k-\beta}^{\alpha,\beta},\\ &\qquad k,j=\beta,\beta+1,\ldots,N,\\ c_{kj}&=k_{k-\beta}^{\alpha,\beta},\quad k,j=\beta,\beta+1,\ldots,N.\end{aligned}\tag{43}$$

Thereby, we can write (41) in the following matrix system form

$$\left(A+\sum_{\sigma=1}^{r-1}\gamma_{\sigma}B^{\sigma}+\gamma_{r}C\right)\mathbf{a}=\mathbf{f}.\tag{44}$$

4.1. Treatment of the Nonhomogeneous Initial Conditions. In the following we can always modify the right-hand side to take care of the nonhomogeneous initial conditions. Let us consider for instance the one-dimensional fractional differential equation (36) subject to the nonhomogeneous initial conditions:

$$u^{(j)}(0)=b_j,\quad j=0,1,\ldots,m-1.\tag{45}$$

In such a case we proceed as follows.

Setting

$$V(x)=u(x)+\sum_{i=0}^{m-1}E_ix^i,\tag{46}$$

where

$$E_i=\frac{-b_i}{i!},\quad i=0,1,\ldots,m-1,\tag{47}$$

the transformation (46) turns the nonhomogeneous initial conditions (45) into the homogeneous initial conditions

$$V^{(j)}(0)=0,\quad j=0,1,\ldots,m-1.\tag{48}$$

Table 1: Maximum absolute errors with various choices of ν and N, for Example 1.

N	ν	α	β	SGJG method	ν	SGJG method	ν	SGJG method
8				3.38×10^{-2}		4.33×10^{-2}		6.41×10^{-2}
16	5.1	3	6	9.10×10^{-7}	5.5	3.31×10^{-7}	5.9	1.14×10^{-7}
24				9.10×10^{-7}		3.31×10^{-7}		1.14×10^{-7}

Hence it suffices to solve the following modified multiterm fractional differential equation:

$$D^{\nu}V(x) + \sum_{\sigma=1}^{r-1} \gamma_{\sigma} D^{\beta_{\sigma}} V(x) + \gamma_r V(x) = f^*(x), \quad x \in I, \tag{49}$$

subject to the homogeneous initial conditions (48), where $V(x)$ is given by (46), and

$$f^*(x) = f(x) + \gamma_r \sum_{i=0}^{m-1} E_i x^i + \sum_{\sigma=1}^{r-1} \gamma_{\sigma} D^{\beta_{\sigma}} \left(\sum_{i=0}^{m-1} E_i x^i \right). \tag{50}$$

5. Illustrative Examples

Several test examples are carried out in this section. The results obtained by the present methods reveal that the present method is very effective and convenient for linear FDEs.

Example 1. Consider the linear FDE equation with homogeneous initial conditions

$$\begin{aligned} &D^{\nu}u(x) + 5D^{(11/3)}u(x) + 3D^{(11/5)}u(x) - 4D^{(11/7)}u(x) \\ &\quad - 6D^{(11/13)}u(x) = f(x), \\ &5 < \nu \le 6, \qquad u^{(j)}(0) = 0, \quad j = 0, 1, \ldots, 5, \end{aligned} \tag{51}$$

whose exact solution is given by $u(x) = x^{13}$.

Table 1 lists the maximum absolute errors, using the shifted generalized Jacobi Galerkin (SGJG) method with various choices of ν and N. Accuracy and stability of the SGJG method for all choices of ν are achieved in this table.

Example 2. Consider the equation

$$\begin{aligned} &D^2u(x) - 2Du(x) + D^{(1/2)}u(x) + u(x) = f(x), \quad x \in I, \\ &u(0) = 0, \qquad u'(0) = 0, \end{aligned} \tag{52}$$

whose exact solution is given by $u(x) = x^7 - x^2$.

In Table 2, we present the maximum absolute errors, using SGJG method with various choices of ν and N.

Table 2: Maximum pointwise errors using SGJG method for $N = 8, 12, 16, 20, 24$.

N	α	β	SGJG	α	β	SGJG
8			$1.171 \cdot 10^{-2}$			$3.237 \cdot 10^{-6}$
12			$5.433 \cdot 10^{-7}$			$5.433 \cdot 10^{-7}$
16	0	1	$1.468 \cdot 10^{-7}$	1	2	$1.468 \cdot 10^{-7}$
20			$5.217 \cdot 10^{-8}$			$5.215 \cdot 10^{-8}$
24			$1.979 \cdot 10^{-8}$			$2.421 \cdot 10^{-8}$

Table 3: Maximum pointwise errors using SGJG method for $N = 8, 16, 24$.

N	α	β	SGJG	α	β	SGJG
8			$6.991 \cdot 10^{-3}$			$5.689 \cdot 10^{-10}$
16	0	0	$3.296 \cdot 10^{-4}$	0	2	$5.359 \cdot 10^{-13}$
24			$4.737 \cdot 10^{-5}$			$6.394 \cdot 10^{-14}$
8			$2.715 \cdot 10^{-5}$			$6.091 \cdot 10^{-5}$
16	1	1	$3.009 \cdot 10^{-7}$	2	1	$4.465 \cdot 10^{-7}$
24			$1.848 \cdot 10^{-8}$			$2.407 \cdot 10^{-8}$

Example 3. Consider the equation

$$\begin{aligned} &D^2u(x) - 2Du(x) + 3D^{(1/2)}u(x) + u(x) = f(x), \quad x \in I, \\ &u(0) = 6, \qquad u'(0) = -4, \end{aligned} \tag{53}$$

whose exact solution is given by $u(x) = x^5 - 4x + 6$.

In Table 3, we present the maximum absolute errors, using SGJG method with various choices of ν and N.

Example 4. Consider the equation

$$\begin{aligned} &D^2u(x) + D^{(1/2)}u(x) + u(x) = f(x), \quad x \in I, \\ &u(0) = 0, \qquad u'(0) = \pi, \end{aligned} \tag{54}$$

whose exact solution is given by $u(x) = \sin(\pi x)$.

In Table 4, we exhibit maximum pointwise error using SGJG method with two choices of the shifted generalized Jacobi parameters α, β and $N = 8, 12, 16, 20, 24$. We observe from this table that the suggested algorithm provides accurate and stable numerical results. This numerical experiment demonstrates the utility of the method.

Table 4: Maximum pointwise errors using SGJG method for $N = 8, 12, 16, 20, 24$.

N	α	β	SGJG	α	β	SGJG
8			$2.871 \cdot 10^{-7}$			$5.740 \cdot 10^{-7}$
12			$1.170 \cdot 10^{-8}$			$1.757 \cdot 10^{-8}$
16	0	2	$1.873 \cdot 10^{-9}$	1	3	$2.757 \cdot 10^{-9}$
20			$4.397 \cdot 10^{-10}$			$6.469 \cdot 10^{-9}$
24			$3.004 \cdot 10^{-10}$			$9.909 \cdot 10^{-10}$

6. Conclusion

We have derived a new formula expressing explicitly the Caputo fractional derivatives for any fractional-order of shifted generalized Jacobi polynomials of any degree in terms of shifted generalized Jacobi polynomials themselves. We have derived a Galerkin method, involving a specified class of the shifted generalized Jacobi polynomials, which permits us to numerically solve an important class of FDEs. Indeed, in Section 5, we demonstrated that for all parameter shifted generalized Jacobi considered, the method results in rather small errors with relatively few modes are considered. Since the method is rather robust, it is likely that it may be applied to other types of FDEs. For instance, one- and two-dimensional time-dependent FDEs

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

This paper was funded by the Deanship of Scientific Research DSR, King Abdulaziz University, Jeddah. The authors, therefore, acknowledge with thanks DSR technical and financial support.

References

[1] D. Baleanu, K. Diethelm, E. Scalas, and J. J. Trujillo, *Fractional Calculus*, vol. 3 of *Series on Complexity, Nonlinearity and Chaos*, World Scientific, 2012.

[2] A. H. Bhrawy and A. S. Alofi, "The operational matrix of fractional integration for shifted Chebyshev polynomials," *Applied Mathematics Letters*, vol. 26, no. 1, pp. 25–31, 2013.

[3] A. H. Bhrawy and M. A. Alghamdi, "A shifted Jacobi-Gauss-Lobatto collocation method for solving nonlinear fractional Langevin equation involving two fractional orders in different intervals," *Boundary Value Problems*, vol. 2012, article 62, 2012.

[4] Y. Yang, Y. Chen, and Y. Huang, "Convergence analysis of the Jacobi spectral-collocation method for fractional integro-differential equations," *Acta Mathematica Scientia B*, vol. 34, no. 3, pp. 673–690, 2014.

[5] A. Atangana and N. Bildik, "The use of fractional order derivative to predict the groundwater flow," *Mathematical Problems in Engineering*, vol. 2013, Article ID 543026, 9 pages, 2013.

[6] A. Atangana and S. B. Belhaouari, "Solving partial differential equation with space- and time-fractional derivatives via homotopy decomposition method," *Mathematical Problems in Engineering*, vol. 2013, Article ID 318590, 9 pages, 2013.

[7] X.-J. Yang, D. Baleanu, and J.-H. He, "Transport equations in fractal porous media within fractional complex transform method," *Proceedings of the Romanian Academy A: Mathematics, Physics, Technical Sciences, Information Science*, vol. 14, no. 4, pp. 287–292, 2013.

[8] A. M. Yang, J. Li, H. M. Srivastava, G. Xie, and X. J. Yang, "The local fractional Laplace variational iteration method for solving linear partial differential equations with local fractional derivatives," *Discrete Dynamics in Nature and Society*. In press.

[9] F. Yin, J. Song, and X. Cao, "A general iteration formula of VIM for fractional heat- and wave-like equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 428079, 9 pages, 2013.

[10] J. Song, F. Yin, X. Cao, and F. Lu, "Fractional variational iteration method versus Adomian's decomposition method in some fractional partial differential equations," *Journal of Applied Mathematics*, vol. 2013, Article ID 392567, 10 pages, 2013.

[11] L. Song and W. Wang, "A new improved Adomian decomposition method and its application to fractional differential equations," *Applied Mathematical Modelling*, vol. 37, no. 3, pp. 1590–1598, 2013.

[12] J. Liu and G. Hou, "Numerical solutions of the space- and time-fractional coupled Burgers equations by generalized differential transform method," *Applied Mathematics and Computation*, vol. 217, no. 16, pp. 7001–7008, 2011.

[13] S. Kumar, D. Kumar, S. Abbasbandy, and M. M. Rashidi, "Analytical solution of fractional Navier-Stokes equation by using modified Laplace decomposition method," *Ain Shams Engineering Journal*, vol. 5, no. 2, pp. 569–574, 2014.

[14] S. Kumar, J. Singh, D. Kumar, and S. Kapoor, "New homotopy analysis transform algorithm to solve volterra integral equation," *Ain Shams Engineering Journal*, vol. 5, pp. 243–246, 2014.

[15] A. H. Bhrawy and M. M. Al-Shomrani, "A shifted Legendre spectral method for fractional-order multi-point boundary value problems," *Advances in Difference Equations*, vol. 2012, article 8, 2012.

[16] E. H. Doha, A. H. Bhrawy, and S. S. Ezz-Eldien, "A new Jacobi operational matrix: an application for solving fractional differential equations," *Applied Mathematical Modelling*, vol. 36, no. 10, pp. 4931–4943, 2012.

[17] A. H. Bhrawy, A. S. Alofi, and S. S. Ezz-Eldien, "A quadrature tau method for fractional differential equations with variable coefficients," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2146–2152, 2011.

[18] M. A. Mohamed and M. Sh. Torky, "Approximate solution of fractional nonlinear partial differential equations by the Legendre multiwavelet Galerkin method," *Journal of Applied Mathematics*, vol. 2014, Article ID 192519, 12 pages, 2014.

[19] B. A. Jacobs and C. Harley, "Two hybrid methods for solving two-dimensional linear time-fractional partial differential equations," *Abstract and Applied Analysis*, vol. 2014, Article ID 757204, 10 pages, 2014.

[20] M. Y. Ongun, D. Arslan, and R. Garrappa, "Nonstandard finite difference schemes for a fractional-order Brusselator system," *Advances in Difference Equations*, vol. 2013, article 102, 2013.

[21] L. Su, W. Wang, and Z. Yang, "Finite difference approximations for the fractional advection-diffusion equation," *Physics Letters A: General, Atomic and Solid State Physics*, vol. 373, no. 48, pp. 4405–4408, 2009.

[22] L. Su and P. Cheng, "A weighted average finite difference method for the fractional convection-diffusion equation," *Advances in Mathematical Physics*, vol. 2013, Article ID 129404, 5 pages, 2013.

[23] M. H. Heydari, M. R. Hooshmandasl, C. Cattani, and M. Li, "Legendre wavelets method for solving fractional population growth model in a closed system," *Mathematical Problems in Engineering*, vol. 2013, Article ID 161030, 8 pages, 2013.

[24] M. Mahalakshmi and G. Hariharan, "An efficient wavelet based approximation method to steady state reaction diffusion model arising in mathematical chemistry," *The Journal of Membrane Biology*, vol. 247, pp. 263–271, 2013.

[25] M. H. Heydari, M. R. Hooshmandasl, F. M. Maalek Ghaini, and M. Li, "Chebyshev wavelets method for solution of nonlinear fractional integrodifferential equations in a large interval," *Advances in Mathematical Physics*, vol. 2013, Article ID 482083, 12 pages, 2013.

[26] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral Methods in Fluid Dynamics*, Springer, New York, NY, USA, 1988.

[27] A. H. Bhrawy, E. Tohidi, and F. Soleymani, "A new Bernoulli matrix method for solving high-order linear and nonlinear Fredholm integro-differential equations with piecewise intervals," *Applied Mathematics and Computation*, vol. 219, no. 2, pp. 482–497, 2012.

[28] B. Fornberg, *A Practical Guide to Pseudospectral Methods*, vol. 1, Cambridge University Press, Cambridge, Mass, USA, 1996.

[29] E. H. Doha and A. H. Bhrawy, "An efficient direct solver for multidimensional elliptic Robin boundary value problems using a Legendre spectral-Galerkin method," *Computers & Mathematics with Applications*, vol. 64, no. 4, pp. 558–571, 2012.

[30] R. Peyret, *Spectral Methods for Incompressible Viscous Flow*, vol. 148, Springer, New York, NY, USA, 2002.

[31] A. H. Bhrawy, "A Jacobi-Gauss-Lobatto collocation method for solving generalized Fitzhugh-Nagumo equation with time-dependent coefficients," *Applied Mathematics and Computation*, vol. 222, pp. 255–264, 2013.

[32] L. N. Trefethen, *Spectral Methods in MATLAB*, vol. 10, SIAM, Philadelphia, Pa, USA, 2000.

[33] D. Baleanu, A. H. Bhrawy, and T. M. Taha, "Two efficient generalized Laguerre spectral algorithms for fractional initial value problems," *Abstract and Applied Analysis*, vol. 2013, Article ID 546502, 10 pages, 2013.

[34] A. Rababah, "Jacobi-Bernstein basis transformation," *Computational Methods in Applied Mathematics*, vol. 4, no. 2, pp. 206–214, 2004.

[35] E. H. Doha, A. H. Bhrawy, M. A. Abdelkawy, and R. A. Van Gorder, "Jacobi-Gauss-Lobatto collocation method for the numerical solution of 1 + 1 nonlinear Schrödinger equations," *Journal of Computational Physics*, vol. 261, pp. 244–255, 2014.

[36] B.-Y. Guo, J. Shen, and L.-L. Wang, "Optimal spectral-Galerkin methods using generalized Jacobi polynomials," *Journal of Scientific Computing*, vol. 27, no. 1–3, pp. 305–322, 2006.

[37] B.-Y. Guo and Y.-G. Yi, "Generalized Jacobi rational spectral method and its applications," *Journal of Scientific Computing*, vol. 43, no. 2, pp. 201–238, 2010.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 361234, 8 pages
http://dx.doi.org/10.1155/2014/361234

Research Article

Existence Results for a Michaud Fractional, Nonlocal, and Randomly Position Structured Fragmentation Model

Emile Franc Doungmo Goufo, Riëtte Maritz, and Stella Mugisha

Department of Mathematical Sciences, University of South Africa, Florida Science Campus, Florida 0003, South Africa

Correspondence should be addressed to Emile Franc Doungmo Goufo; dgoufef@unisa.ac.za

Received 12 May 2014; Accepted 28 May 2014; Published 12 June 2014

Academic Editor: Abdon Atangana

Until now, classical models of clusters' fission remain unable to fully explain strange phenomena like the phenomenon of shattering (Ziff and McGrady, 1987) and the sudden appearance of infinitely many particles in some systems having initial finite number of particles. That is why there is a need to extend classical models to models with fractional derivative order and use new and various techniques to analyze them. In this paper, we prove the existence of strongly continuous solution operators for nonlocal fragmentation models with Michaud time derivative of fractional order (Samko et al., 1993). We focus on the case where the splitting rate is dependent on size and position and where new particles generating from fragmentation are distributed in space randomly according to some probability density. In the analysis, we make use of the substochastic semigroup theory, the subordination principle for differential equations of fractional order (Prüss, 1993, Bazhlekova, 2000), the analogy of Hille-Yosida theorem for fractional model (Prüss, 1993), and useful properties of Mittag-Leffler relaxation function (Berberan-Santos, 2005). We are then able to show that the solution operator to the full model is positive and contractive.

1. Model's Motivation and Introduction

The dynamics of clusters' fragmentation occurs in many branches of natural sciences ranging from physics, through chemistry, engineering, biology, ecology, and numerous domains of applied sciences, such as the depolymerization, the rock fractures, and breakage of droplets. There exists a vast literature on classical fragmentation models and many of them have been deeply analyzed in different works (see, e.g., [1–8]) where authors investigated their conservativeness, honesty, and existence of solutions. However, investigations on fractional versions of these models are still yet to come.

The concepts of fractional derivatives and fractional integral started in 1695 when L'Hospital questioned about the meaning of the operator $d^n y/dx^n$ if $n = 1/2$; that is, "what if n is fractional?" Leibniz then replied as "$d^{1/2}x/dx^{1/2}$ will be equal to $x\sqrt{dx : x}$." Then, there is a growing interest in extending the normal calculus with integer orders to noninteger orders (real or complex order) [9–14] because its applications have attracted a great range of attention in the past few years. As an example, fractional evolution equations are used to model many problems in applied sciences, engineering, and technology including applied mathematics, computing, physics, biology, chemistry, and economics. For example, in [13], the authors used the concept of the variational order derivative together with Crank-Nicholson technique to provide a generalization of the groundwater flow equation, solve it, and present a numerical solution. In [14], two methods including Frobenius and Adomian decomposition method were used to generalize the classical Darcy law by regarding the water flow as a function of a noninteger order derivative of the piezometric head. In the process, systems with derivative of fractional order have become a useful tool for describing nonlinear phenomena occurring in epidemiological models and spatial and temporal spreads.

Many processes in real life are described by classical models taking the form of Cauchy problems given by

$$\frac{d}{dt}g(t) = \mathscr{P}g(t) \tag{1}$$

subject to initial condition

$$g(t_0) = 0, \quad t_0 < t, \tag{2}$$

where $t \in \mathbb{R}_+$ is the time and $\mathscr{P}$ is an operator in a Banach space. The aim here is to find the state $g(t)$ of the system at a time $t > t_0$ depending on the initial state. Hence, at an infinitesimal and bounded scale, the rate of accumulation or loss of matter in the system is characterized by the classical derivative d/dt. This leads us to three essential motivations for considering, in this paper, the fractional nonlocal and randomly position structured fragmentation model. The following motivations are the cause of increasing eagerness among scientists to extend well known old models to models with fractional derivative, in order to investigate them with various and different methods. The aim is to try new techniques hoping to establish broader outlooks on the real phenomena these differential evolutions equations describe.

First Motivation. Classical fragmentation processes are difficult to analyze as they involve the evolution of two intertwined quantities: the mass of the ensemble and the number of particles in it. That is why, though linear, they display nonlinear features such as phase transition which, in this case, is called shattering. Indeed, classical models of clusters' fission with normal derivative d/dt cannot fully explain these unusual phenomena, which include not only the phenomenon of shattering, but also the sudden appearance of an infinite number of particles in some systems which contained at the beginning a finite particles number [15]. Shattering is seen as an explosive or dishonest Markov process; see, for example, [16, 17], and it has been associated with an infinite cascade of breakup events creating a "dust" of particles of zero size which, however, carry nonzero mass.

Second Motivation. We mentioned that d/dt can be associated with the representation of the rate of change (accumulation or loss) in the system, considered at infinitesimal bounded space. But, this is not always true since the rate of accumulation or loss can be different from d/dt. Indeed, most of the infinitesimal spaces for complex models can contain hindering or obstacles (of various sizes) where the variable under study is temporarily parked or stuck. In this condition, the classical d/dt will no longer replicate, with certitude, the real picture of accumulation or loss. In the same way, these obstacles could be holes or forbidden zones in the infinitesimal space where the variable (particle, mass, density, flux, etc.) cannot reside. Hence, the fractional differentiation D_t^α, with $\alpha \in \mathbb{R}$ or $\mathbb{C}$, may give the sub- or superrate of accumulation or loss with index α representing the heterogeneity distribution of the infinitesimal space (with hindering or obstacles)! However, substituting d/dt by the fractional derivative D_t^α requires some considerations as we will see later in this paper.

Final Motivation. Most of analysis on fragmentation with first order derivative d/dt has been performed under the assumption that the ensemble of particles is well mixed so that the particle distribution is uniform in space. However, recent approaches, using individual based models [18], yield in a natural way to systems in which splitting particles are distributed in space according to some prescribed probability density, leading, however, to models with explicit space dependence. We note that similar models were also considered earlier in [19] but with emphasis only on well-posedness. In this paper, the fractional version of the same type of model, namely, the fractional, nonlocal, and randomly position structured fragmentation model, is analyzed with full description given in the next section.

2. Model Description and Settings

2.1. Preliminaries: Classical Nonlocal and Randomly Position Structured Fragmentation Model. In the literature review on classical fragmentation models with "normal" time derivative of order one, they have been comprehensively analyzed in numerous works (see, e.g., [1–8]). Transport-type models with convection were investigated in [8] where the author showed that the convection part does not affect the breach of the conservation laws. In [2] the authors studied the nonlocal fragmentation and showed that the process is conservative if at infinity daughter particles tend to go back into the system with a high known probability. Honesty and nonconservative (dishonesty) regimes for fragmentation equations have been thoroughly investigated (see [2, 15]) and, in particular, the breach of the mass conservation law (shattering) is believed to be beyond the model's resolution. The description of the dynamics of the classical nonlocal and randomly position structured fragmentation model is as follows: the state at a given time t is the repartition at that time of all aggregates according to their size m and their position x. In terms of m and x, the state of the system is characterized at any moment t by the particle-mass-position distribution $g = g(t, m, x)$ (g is also called the *density* or *concentration* of particles), where $g : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}_+$. Thus,

$$\int_n^p \int_{\mathbb{R}^3} g(t, m, x)\, dx\, dm \tag{3}$$

is the number of particles having mass between n and p and

$$\int_n^p \int_{\mathbb{R}^3} g(t, m, x)\, m\, dx\, dm \tag{4}$$

is the mass contained in particles in $\mathbb{R}^3$ having mass within this range.

Definition 1. The fragmentation rate $a = a(m, x)$ describes the ability of aggregates of size m and position x to break into smaller particles.

We assume that

$$a \in L_{\infty,\mathrm{loc}}\left(\mathbb{R}_+ \times \mathbb{R}^3\right). \tag{5}$$

When an aggregate of mass s located at x breaks, the expected average number of daughter particles of size m is a nonnegative measurable function $b(m, s, x)$ defined on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^3$. Since a group of size $m \le s$ cannot split to form

a group of size s then $\mathrm{Supp}(b) \subseteq \{(m,s) \in \mathbb{R}_+ \times \mathbb{R}_+ : m < s\} \times \mathbb{R}^3$ which yields

$$b_{s,m} = 0, \quad \forall m \leq s. \tag{6}$$

After the fragmentation of a mass s particle, the sum of masses of all daughter particles should again be s; hence it follows that for any $s > 0$, $x \in \mathbb{R}^3$

$$\int_0^s nb\,(n,s,x)\,dn = s; \tag{7}$$

moreover, the quantity $\int_0^s b(n,s,x)dn$ gives us the expected number of daughter particles produced by fragmentation of a mass s particle (with position x).

After a group's fragmentation, new originating daughter particles have different centers distributed according to a given probabilistic law $\widetilde{b}(\cdot,m,s,y)$. This is the probability density that after a fragmentation of an s-aggregate (with the center at y) the new formed m-group will be located at the position x. Therefore

$$\int_{\mathbb{R}^3} \widetilde{b}\,(x,m,s,y)\,dx = 1. \tag{8}$$

This leads to the nonlocal and randomly position structured fragmentation model given by

$$\begin{aligned}\frac{\partial g}{\partial t}\,(t,m,x) &= -a\,(m,x)\,g\,(t,m,x)\\ &\quad + \int_m^\infty \int_{\mathbb{R}^3} a\,(s,y)\,b\,(m,s,y)\\ &\qquad \times \widetilde{b}\,(x,m,s,y)\,g\,(t,s,y)\,dy\,ds.\end{aligned} \tag{9}$$

The total mass of the ensemble at time t is the quantity

$$\int_0^\infty \int_{\mathbb{R}^3} g\,(t,m,x)\,m\,dx\,dm; \tag{10}$$

thus the natural space for analysis is

$$X = L_1\left(\mathbb{R}_+ \times \mathbb{R}^3, m\,dx\,dm\right). \tag{11}$$

In order to make use of the semigroup theory of linear operators we need to complement (9) with the initial mass-position distribution

$$g\,(0,m,x) = g_0\,(m,x), \quad \text{a.e } (m,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \tag{12}$$

where $g_0 \in X$. In what follows we denote by $\mathscr{A}$ and $\mathscr{B}$ the expressions appearing on the right-hand side of (9); that is,

$$\begin{aligned}\left[\mathscr{A}\psi\right](m,x) &= -a\,(m,x)\,\psi\,(m,x),\\ \left[\mathscr{B}\psi\right](m,x) &= \int_m^\infty \int_{\mathbb{R}^3} a\,(s,y)\,b\,(m,s,y)\\ &\qquad \times \widetilde{b}\,(x,m,s,y)\,\psi\,(s,y)\,dy\,ds,\end{aligned} \tag{13}$$

defined on all measurable and finite almost everywhere functions ψ for which they make pointwise (almost everywhere) sense.

We introduce operators A and B in X defined by

$$[Au]\,(m,x) = \left[\mathscr{A}g\right](m,x), \qquad [Bu]\,(m,x) = \left[\mathscr{B}g\right](m,x) \tag{14}$$

and set $D(A) = \{\psi \in X;\ a\psi \in X\}$.

Lemma 2. *$(A+B, D(A))$ is a well-defined operator.*

Proof. In order to prove the first part of the theorem, we need to show that $\mathscr{B}D(A) \subset X$. Let $g \in D(A)_+$, changing the order of integration by the Fubini theorem; we obtain

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^3} \mathscr{B}g\,(m,x)\,m\,dx\,dm\\ &\quad = \int_{\mathbb{R}^3} \int_0^\infty \left[\int_m^\infty a\,(s,y)\,b\,(m,s,y)\right.\\ &\qquad \times \left(\int_{\mathbb{R}^3} \widetilde{b}\,(x,m,s,y)\,dx\right)\\ &\qquad \left.\times\, g\,(s,y)\,ds\right] m\,dm\,dy\\ &\quad = \int_{\mathbb{R}^3} \left(\int_0^\infty \int_m^\infty ma\,(s,y)\,b\,(m,s,y)\right.\\ &\qquad \left.\times\, g\,(s,y)\,ds\,dm\right) dy\\ &\quad = \int_{\mathbb{R}^3} \left(\int_0^\infty \int_0^s ma\,(s,y)\,b\,(m,s,y)\right.\\ &\qquad \left.\times\, g\,(s,y)\,dm\,ds\right) dy\\ &\quad = \int_0^\infty \int_{\mathbb{R}^3} a\,(s,y)\,g\,(s,y)\,s\,dy\,ds,\end{aligned} \tag{15}$$

where we used (8) and (7), respectively. Because $g \in D(A)_+$ it follows that

$$\int_0^\infty \int_{\mathbb{R}^3} \mathscr{B}g\,(m,x)\,m\,dx\,dm < +\infty. \tag{16}$$

The result follows from the fact that any arbitrary element g of $D(A)$ can be written in the form $g = g_+ - g_-$, where $g_+, g_- \in D(A)_+$. □

Let us define the terms *stochastic* and *substochastic* semigroups.

Definition 3. The strongly continuous semigroup of operators $(G(t))_{t\geq 0}$ on the Banach space X is said to be

(i) substochastic if $S(t) \geq 0$ and $\|G(t)\| \leq 1$ for all $t \geq 0$,

(ii) stochastic if, in addition, it satisfies $\|G(t)\psi\| = \|\psi\|$ for all nonnegative $\psi \in X$.

Usually, to find the semigroup for a given evolution equation, we use the Hille-Yosida theorem, stated as follows.

Theorem 4 (Hille-Yosida theorem). *$\mathscr{P} \in \mathscr{G}(M, \omega)$ if and only if (a) $\mathscr{P}$ is closed and densely defined; (b) there exist $M > 0$, $\omega \in \mathbb{R}$ such that $(\omega, \infty) \in \rho(\mathscr{P})$ and for all $n \geq 1$, $\lambda > \omega$*

$$\left\| (\lambda I - \mathscr{P})^{-n} \right\| \leq \frac{M}{(\lambda - \omega)^n}, \tag{17}$$

where $\rho(\mathscr{P})$ is the resolvent set of the operator $\mathscr{P}$ and is defined as follows:

$$\rho(\mathscr{P}) = \left\{ \lambda \in \mathbb{R}; \lambda I - \mathscr{P} : D(\mathscr{P}) \longrightarrow X \text{ is invertible and } (\lambda I - \mathscr{P})^{-1} \in \mathscr{L}(X) \right\}. \tag{18}$$

Proof. The proof of this classic theorem can be found in any book of functional analysis. Note that $\mathscr{P} \in \mathscr{G}(M, \omega)$ means (see [8]) $\mathscr{P}$ generates a C_0-semigroup $(G_{\mathscr{P}}(t))_{t>0}$ so that there exist $M > 0$ and ω such that

$$\left\| G_{\mathscr{P}}(t) \right\| \leq M e^{\omega t}. \tag{19}$$

□

Then, we can state the following existence theorem for the classical nonlocal model which will help us in the analysis of the fractional model.

Theorem 5. *Consider the model (9) expressed in terms of A and B defined in (14); then there is an extension $\widetilde{K}$ of $A + B$ that generates a substochastic semigroup $(G_{\widetilde{K}}(t))_{t\geq 0}$ on X. Moreover, for each $g_0 \in D(\widetilde{K})$ there is a measurable representation $g(t, m, x)$ of $G_{\widetilde{K}}(t)g_0$ which is absolutely continuous with respect to $t \geq 0$ for almost any (m, x) and such that (9) is satisfied almost everywhere.*

Proof. The proof was done in [3, Theorem 3] or [2, Theorem 3.3] for the discrete case of the same model. □

3. Solvability of the Fractional Nonlocal Model

The fractional and nonlocal differential model of our interest is given by

$$D_t^{\alpha} g(t, m, x) = -a(m, x)\, g(t, m, x) + \int_m^{\infty} \int_{\mathbb{R}^3} a(s, y)\, b(m, s, y) \times \widetilde{b}(x, m, s, y)\, g(t, s, y)\, dy\, ds \tag{20}$$

with $(m, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $0 < \alpha < 1$, and $t > 0$ or expressed in the abstract form as

$$D_t^{\alpha} g(t) = [A + B]\, g(t), \quad 0 < \alpha < 1, \ t > 0, \tag{21}$$

with the assumption that it is subject to the initial condition

$$g(0, m, x) = g_0(m, x), \quad (m, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \tag{22}$$

or simply

$$g(0) = g_0. \tag{23}$$

Here, A and B are defined in (14) and D_t^{α} is defined by

$$D_{\tau}^{\alpha} g(\tau, m, x) = -\lim_{t \to 0} \frac{\mathbf{u}_{\alpha}(t)\, g(\tau, m, x) - g(\tau, m, x)}{t}, \quad x \in \mathbb{R}^3, \quad 0 < \alpha < 1, \quad t > 0 \tag{24}$$

with $\mathbf{u}_{\alpha}(t)$ the fractional time evolution (with index $\alpha, 0 < \alpha < 1$) considered as universal attractor semigroups of coarse grained macroscopic time evolutions [20, 21]. It is shown that [20, 22]

$$D_{\tau}^{\alpha} g(\tau, m, x) = -\frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{g(\tau - r, m, x) - g(\tau, m, x)}{r^{\alpha+1}} dr, \quad 0 < \alpha < 1, \tag{25}$$

which is the fractional derivative of the function g in the sense of Michaud [22]. However, substituting ∂_t by D_t^{α} in (9) is justified in the sense that the presence of ∂_t in (9) reflects a basic symmetry of the time translation invariance and the basic principle of locality. Indeed, from the relation

$$\frac{d}{d\tau} g(\tau) = \lim_{t \to 0} \frac{g(\tau) - g(\tau - t)}{t} = -\lim_{t \to 0} \frac{\mathbf{u}(t)\, g(\tau) - g(\tau)}{t}, \tag{26}$$

we see that $-d/dt$ is identified as the infinitesimal generator of time translation $\mathbf{u}(t)g(\tau) = g(\tau - t)$. Hence, this considers $\mathbf{u}(t)$ as the expression of the general time evolution, which is the same consideration done in Definitions (24) and (25) of D_t^{α}. Thus, the derivative of fractional order D_t^{α}, $0 < \alpha < 1$ was found to be, in general, infinitesimal generator of coarse grained macroscopic time evolution. It is well known [23] that all macroscopic time evolutions have fractional derivatives, with order less than unity, as their infinitesimal generators. Therefore, in Proposition 8, we provide a relation between the generator $D_t^{\alpha}, 0 < \alpha < 1$, of the macroscopic time evolution and d/dt.

Definition 6. Consider an operator $\mathscr{P}$ applied in the fractional model

$$D_t^{\alpha} g(t, m, x) = \mathscr{P} g(t, m, x), \quad 0 < \alpha < 1, \ t > 0, \ x \in \mathbb{R}^3, \tag{27}$$

subject to the initial condition

$$g(x, 0) = f(x), \quad x \in \mathbb{R}^3 \tag{28}$$

and defined in the Banach space X. A family $(G_{\mathscr{P}}(t))_{t>0}$ of bounded operators on X is called a solution operator of the fractional Cauchy problem (27)-(28) if

(1) $G_{\mathscr{P}}(0) = I_X$;

(2) $G_{\mathscr{P}}(t)$ is strongly continuous for every $t \geq 0$;

(3) $\mathscr{P}G_{\mathscr{P}}(t)g = G_{\mathscr{P}}(t)\mathscr{P}g$ for all $g \in D(\mathscr{P})$;

(4) $G_{\mathscr{P}}(t)D(\mathscr{P}) \subset D(\mathscr{P})$;

(5) $G_{\mathscr{P}}(t)g$ is a (classical) solution of the model (27)-(28) for all $g \in D(P)$, $t \geq 0$.

Recall that the operator $\widetilde{\mathscr{P}} \in \mathscr{G}(M, \omega)$ means $\widetilde{\mathscr{P}}$ generates a C_0-semigroup $(G_{\widetilde{\mathscr{P}}}(t))_{t>0}$ so that there exist $M > 0$ and ω such that

$$\left\|G_{\widetilde{\mathscr{P}}}(t)\right\| \leq Me^{\omega t}. \tag{29}$$

Whence, by analogy if the fractional Cauchy problem (27)-(28) has a solution operator $(G_{\mathscr{P}}(t))_{t>0}$ verifying (29), then we say that $\mathscr{P} \in \mathscr{G}^{\alpha}(M, \omega)$. The solution operator $(G_{\mathscr{P}}(t))_{t>0}$ is contractive if

$$\left\|G_{\mathscr{P}}(t)\right\|_X \leq 1, \tag{30}$$

and we say $\mathscr{P} \in \mathscr{G}^{\alpha}(1, 0)$.

Note that if we have a contraction solution operator, we can use Definition 6 to identify the fractional Cauchy problem of which it is a solution. Usually, however, we are interested in the reverse question, that is, in finding the solution operator that is contractive, for a given fractional model. The answer is given by the following theorem (seen as an analogue of Hille-Yosida Theorem 4).

Theorem 7. *An operator $A_\alpha \in \mathscr{G}^{\alpha}(1, 0)$ for $0 < \alpha \leq 1$ if and only if*

(a)

$$(0, \infty) \subset \rho\left(A_\alpha\right), \tag{31}$$

(b)

$$\lambda^{\alpha-1}R\left(\lambda^{\alpha}, A_\alpha\right) f = \int_0^{\infty} e^{-\lambda r}G_\alpha(r)\, f\, dr, \tag{32}$$

with $\lambda > 0$, $f \in X$, where $(G_\alpha(t))_{t>0}$ is a family of strongly continuous operators satisfying (30) *and $\rho(A_\alpha)$ is the resolvent set of the operator A_α:*

$$\rho\left(A_\alpha\right) = \left\{\lambda \in \mathbb{R}; \lambda I - A_\alpha : D\left(A_\alpha\right) \longrightarrow Y_1 \right. \\ \left. \text{is invertible and } \left(\lambda I - A_\alpha\right)^{-1} \text{ bounded and linear}\right\}. \tag{33}$$

Proof. This theorem is a particular version of [24, Theorem 1.3] and the proof follows the same steps. □

Consider the following problem:

$$D_t^{\alpha} g(t, m, x) = -a(m, x)\, g(t, m, x) \\ \left(= Ag(t, m, x)\right), \quad 0 < \alpha < 1, \tag{34}$$

$$g(x, 0) = f(x), \quad x \in \mathbb{R}^3, \quad t > 0, \tag{35}$$

with

$$f \in D(A). \tag{36}$$

The full operator in the problem (21) is seen as a perturbation of operator A in (34), which represents the loss part of the fractional fragmentation dynamics. Then, from the original semigroup $(G(t))_{t\geq 0}$ (the multiplication semigroup) generated by the differentiation operator $-d/dt$ and given after a simple integration by

$$\left(G(t)\, g\right)(t, m, x) = e^{-a(m,x)t} g(t, m, x), \tag{37}$$

one can exploit the theory of C_0-semigroup to state that the operator A is the infinitesimal generator of a C_0-semigroup $(G_A(t))_{t\geq 0}$. The existence of $(G(t))_{t\geq 0}$ comes from assumption that the fission rate satisfies (5). Therefore the infinitesimal generators of $(G_A(t))_{t\geq 0}$ may be interpreted [25] as the distributions, evaluated on the right time translation group, which leads to the following proposition.

Proposition 8. *(a) $(G(t))_{t\geq 0}$ is an equibounded semigroup and the norms of all operators $G(t)$, $t \in \mathbb{R}$ are bounded above by 1.*

(b) Let $g \in D(A)$. The infinitesimal generators A of C_0-semigroups $(G_A(t))_{t\geq 0}$ are related to $(S(t))_{t\geq 0}$ by the representations

$$-\left(-\frac{d}{dt}\right)^{\alpha} g(x) = -D_t^{\alpha}\left(g(x)\right) \\ = \lim_{\varepsilon \to 0+} \frac{1}{\Gamma(-\alpha)} \int_{\varepsilon}^{\infty} r^{-\alpha-1} \\ \times \left[1 - e^{-a(m,x)r}\right] g(x)\, dr, \tag{38}$$

$$\left\|G(t)\, g - g\right\| \leq Mt^{\alpha} \|Au\| \quad \text{for some } M > 0. \tag{39}$$

(c) For every $g \in D(A)$ one has $G_A(t)Au = AG_A(t)g$.

Proof. (a) By condition (5) we know (see [1, 2, 8]) that A is the infinitesimal generator of a positive semigroup of contractions and the assertion follows.

To prove (b) we use Marchaud type representation [26] of fractional powers of infinitesimal generators. Let δ be the Dirac measure; then if we take $C_0^1(\mathbb{R})$ as the set of test functions on $\mathbb{R}$, we know that (see [27]) the fractional derivative distribution δ^{α}, $0 < \alpha < 1$, of δ can be expressed by

$$\left\langle \varphi, \delta^{\alpha} \right\rangle = \frac{1}{-\Gamma(-\alpha)} \int_0^{\infty} r^{-\alpha-1} \left[\varphi(x) - \varphi(x + r)\right] dr \tag{40}$$

and approximated by the family $(\mathscr{M}_{\varepsilon}^{\alpha})_{\varepsilon>0}$ of finite Borel measure on $\mathbb{R}_+$ given as

$$\left\langle \varphi, \mathscr{M}_{\varepsilon}^{\alpha} \right\rangle \\ = \frac{1}{-\Gamma(-\alpha)} \int_{\varepsilon}^{\infty} r^{-\alpha-1} \left[\varphi(x) - \varphi(x + r)\right] dr, \tag{41}$$

$$\forall \varphi \in C_0^1(\mathbb{R}).$$

The family $(\mathscr{M}_\varepsilon^\alpha)_{\varepsilon>0}$ generates the bounded linear operators $l_{\mathscr{M}_\varepsilon^\alpha}$ defined by

$$l_{\mathscr{M}_\varepsilon^\alpha} g(x) = \frac{1}{-\Gamma(-\alpha)} \int_\varepsilon^\infty r^{-\alpha-1} \left[g(x) - G_A(r)\, g(x)\right] dr, \quad g \in X_1. \tag{42}$$

Using Laplace transform $\mathscr{L}$, we can show (see [28]) that

$$\mathscr{L}\left(l_{\mathscr{M}_\varepsilon^\alpha}\right)(x) = x^\alpha \cdot \mathscr{L}\left[\frac{1}{-\Gamma(-\alpha)\,\Gamma(1+\alpha)} \times \left(\left(\frac{\cdot}{\varepsilon}\right)_+^\alpha - \left(\frac{\cdot}{\varepsilon}+1\right)_+^\alpha\right)\right](x), \quad x > 0, \tag{43}$$

which yields, by uniqueness theorems for Laplace transformation and well known properties of convolution operator $*$,

$$\mathscr{M}_\varepsilon^\alpha = \delta^\alpha * \left[\frac{1}{-\Gamma(-\alpha)\,\Gamma(1+\alpha)} \left(\left(\frac{\cdot}{\varepsilon}\right)_+^\alpha - \left(\frac{\cdot}{\varepsilon}+1\right)_+^\alpha\right)\right] \tag{44}$$

and then

$$A \int_0^\infty \frac{1}{-\Gamma(-\alpha)\,\Gamma(1+\alpha)} \left(\left(\frac{r}{\varepsilon}\right)_+^\alpha - \left(\frac{r}{\varepsilon}+1\right)_+^\alpha\right) G_A(r)\, g(x)\, dr = \frac{1}{-\Gamma(-\alpha)} \int_\varepsilon^\alpha r^{-\alpha-1} \left[g(x) - G_A(r)\, g(x)\right] dr. \tag{45}$$

Now taking $\varepsilon \to 0$, the assertion (b) follows by using the fact that A_α is a closed operator [29] and

$$\lim_{\varepsilon\to 0} \int_0^\infty \frac{1}{-\Gamma(-\alpha)\ \Gamma(1+\alpha)} \left(\left(\frac{r}{\varepsilon}\right)_+^\alpha - \left(\frac{r}{\varepsilon}+1\right)_+^\alpha\right) G_A(r) \times g(x)\, dr = g(x). \tag{46}$$

To prove (39) and (c), we make use of [30, Theorem 6.13 p.74]. We just need to show that $0 \in \rho(A)$. In fact since $a \in L_{\infty,\mathrm{loc}}(0,\infty)$ then $0 \notin \overline{a((0,\infty))}$; hence, a has abounded inverse $1/a$ and so does A. Thus $0 \in \rho(A)$, which completes the proof. □

Let us now consider the Mittag-Leffler relaxation function $E_\alpha(-x)$, where x is a positive real number (time variable for example) and used to describe complex relaxation processes. We have that [31, 32]

$$E_\alpha[z] = \sum_{n=0}^\alpha \frac{z^n}{\Gamma(n\alpha+1)}, \tag{47}$$

where Γ is the Gamma function. Using the Laplace transform $\mathscr{L}$ and its inverse, it can be proved (see [33–35]) that

$$\begin{aligned} E_\alpha(-x) &= \frac{2x}{\pi} \int_0^\infty \frac{E_{2\alpha}\left(-s^2\right)}{x^2+s^2} ds \\ &= \frac{\sin(\alpha\pi)}{\pi} \int_0^\infty \frac{s^{\alpha-1} e^{-x^{1/\alpha} s} ds}{s^{2\alpha} + 2s^\alpha \cos(\alpha\pi) + 1} \\ &= \int_0^\infty \sum_{n=0}^\infty \frac{(-s)^n e^{-xs} ds}{n!\Gamma(1-\alpha-\alpha n)} \\ &= \frac{1}{2\pi i} \int_\Upsilon e^{(s-xs^\alpha)} s^{\alpha-1} e^{-xs} ds, \quad 0 < \alpha < 1, \end{aligned} \tag{48}$$

where Υ is a contour domain which encircles the origin counterclockwise, going from $-\infty$ to $-\infty$.

Theorem 9. *There is a positive and contractive solution operator $(G_K(t))_{t\geq 0}$ to the fractional model (21), where $K \in \mathscr{G}^\alpha(1,0)$ is an extension of the operator sum $A + B$ defined in (14).*

Proof. Consider the classical model (9) given as

$$\frac{d}{dt} g(t,m,x) = [A+B]\, g(t,m,x), \quad g(0) = g_0, \tag{49}$$

where A and B are defined in (14). This model is the same as

$$D_t^1 g(t,m,x) = [A+B]\, g(t,m,x), \quad g(0) = g_0. \tag{50}$$

Then, by Theorem 5, there is an extension $\widetilde{K}$ of the operator $A + B$ which generates a positive semigroup of contractions $(G_{\widetilde{K}}(t))_{t\geq 0}$ and, then, satisfies

$$\left\|G_{\widetilde{K}}(t)\right\|_X \leq 1. \tag{51}$$

Now, using Definition 6, $\widetilde{K} \in \mathscr{G}(1,0)$. Exploiting the relaxation relation (48), we set

$$G_K(t)\, f = \int_0^\infty \Psi_\alpha(t,s)\, G_{\widetilde{K}}(s)\, f\, ds, \tag{52}$$

where

$$\Psi_\alpha(t,s) = \frac{1}{t^\alpha} \sum_{n=0}^\infty \frac{(-s)^n t^{-\alpha n}}{n!\Gamma(1-\alpha-\alpha n)}. \tag{53}$$

We aim to show that $G_K(t)$ is the solution operator for the fractional model (21) and is positive and contractive. The second last relation of (48) and monotonicity of $E_\alpha(-x)$ imply that $(G_K(t))_{t>0}$ is positive. By Hille-Yosida Theorem 4, we have $(0,\infty) \subset \rho(K)$ and $\|G_K(s)\|_X \leq 1, s \geq 0$ since $G_K(t)$ is substochastic. Then using (48) and the fact (see [34, 35]) that

$$\int_0^\infty \left(\sum_{n=0}^\infty \frac{(-s)^n}{n!\Gamma(1-\alpha-\alpha n)}\right) ds = 1 \tag{54}$$

yield $\|G(s)\|_X \leq \int_0^\infty \Psi_\alpha(t,s)\|G_K(s)\|_X ds \leq 1$, $t \geq 0$; hence $(G(t))_{t>0}$ is contractive. To conclude the theorem, we make use of the subordination principle [24, 36] to state that $\widetilde{K} \in \mathcal{G}^\alpha(1,0)$ since $\widetilde{K} \in \mathcal{G}(1,0)$. Thus, by (52) and making use of Theorem 7, we have (see also [36, Theorem 3.1]) that $G_K(t)$ verifies $\|G_K(t)\|_X \leq 1$ and is the solution operator for the fractional model (21), that is, positive and contractive. □

4. Conclusion and Possible Future Analysis

In this paper, we proved that there is a strongly continuous solution operator, positive and contractive, to nonlocal fragmentation models with Michaud time derivative of fractional order. To obtain this existence result, we made use of the subordination principle for differential equations of fractional order together with the theory of substochastic semigroup, the analogue of Hille-Yosida theorem for fractional model, and properties of Mittag-Leffler relaxation function. This existence result extends the preceding ones with the inclusion of the position dependence on the fragmentation rate and the derivative of order less than unity. Thus, more analysis can be done on the existence of the solution operator to the model and the study of its characteristics (strong continuity, positivity, and contractivity) may lead to a better understanding of strange observed phenomena like the sudden appearance of an infinite number of particles in some systems with initial finite particles number and the shattering phenomenon. The results in the present paper may also lead to the full characterization of the operator under which the fractional model admits a solution operator.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] E. F. Doungmo Goufo and S. C. Oukouomi Noutchie, "Global analysis of a discrete non-local and non-autonomous fragmentation dynamics occurring in a moving process," *Abstract and Applied Analysis*, vol. 2013, Article ID 484391, 9 pages, 2013.

[2] E. F. Doungmo Goufo and S. C. Oukouomi Noutchie, "Honesty in discrete, nonlocal and randomly position structured fragmentation model with unbounded rates," *Comptes Rendus Mathematique: Comptes Rendus de l'Académie des Sciences I*, vol. 351, no. 19-20, pp. 753–759, 2013.

[3] S. C. Oukouomi Noutchie and E. F. Doungmo Goufo, "On the honesty in nonlocal and discrete fragmentation dynamics in size and random position," *ISRN Mathematical Analysis*, vol. 2013, Article ID 908753, 7 pages, 2013.

[4] R. M. Ziff and E. D. McGrady, "The kinetics of cluster fragmentation and depolymerisation," *Journal of Physics A*, vol. 18, no. 15, pp. 3027–3037, 1985.

[5] C. R. Garibotti and G. Spiga, "Boltzmann equation for inelastic scattering," *Journal of Physics A*, vol. 27, no. 8, pp. 2709–2717, 1994.

[6] A. Majorana and C. Milazzo, "Space homogeneous solutions of the linear semiconductor Boltzmann equation," *Journal of Mathematical Analysis and Applications*, vol. 259, no. 2, pp. 609–629, 2001.

[7] W. Wagner, "Explosion phenomena in stochastic coagulation-fragmentation models," *Annals of Applied Probability*, vol. 15, no. 3, pp. 2081–2112, 2005.

[8] S. C. Oukouomi Noutchie and E. F. Doungmo Goufo, "Global solvability of a continuous model for nonlocal fragmentation dynamics in a moving medium," *Mathematical Problems in Engineering*, vol. 2013, Article ID 320750, 8 pages, 2013.

[9] M. Caputo, "Linear models of dissipation whose Q is almost frequency independent: part II," *Geophysical Journal International*, vol. 13, no. 5, pp. 529–539, 1967.

[10] K. B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, NY, USA, 1999.

[11] I. Podlubny, *Fractional Differential Equations*, vol. 198, Academic Press, San Diego, Calif, USA, 1999.

[12] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley & Sons, New York, NY, USA, 2003.

[13] A. Atangana and J. F. Botha, "A generalized groundwater flow equation using the concept of variable order derivative," *Boundary Value Problems*, vol. 2013, article 53, 2013.

[14] A. Atangana and P. D. Vermeulen, "Analytical solutions of a space-time fractional derivative of groundwater flow equation," *Abstract and Applied Analysis*, vol. 2014, Article ID 381753, 11 pages, 2014.

[15] R. M. Ziff and E. D. McGrady, "'Shattering' transition in fragmentation," *Physical Review Letters*, vol. 58, no. 9, pp. 892–895, 1987.

[16] W. J. Anderson, *Continuous-Time Markov Chains. An Applications-Oriented Approach*, Springer, New York, NY, USA, 1991.

[17] J. R. Norris, *Markov Chains*, Cambridge University Press, Cambridge, UK, 1998.

[18] R. Rudnicki and R. Wieczorek, "Phytoplankton dynamics: from the behaviour of cells to a transport equation," *Mathematical Modelling of Natural Phenomena*, vol. 1, no. 1, pp. 83–100, 2006.

[19] M. Lachowicz and D. Wrzosek, "A nonlocal coagulation-fragmentation model," *Applicationes Mathematicae*, vol. 27, no. 1, pp. 45–66, 2000.

[20] A. V. Balakrishnan, "Fractional powers of closed operators and the semigroups generated by them," *Pacific Journal of Mathematics*, vol. 10, pp. 419–437, 1960.

[21] K. Yosida, *Functional Analysis*, Springer, New York, NY, USA, 6th edition, 1980.

[22] S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Franctional Integrals and Derivatives. Theory and Application*, Gordon and Breach Science, Amsterdam, The Netherlands, 1993.

[23] R. Hilfer, *On New Class of Phase Transitions, Random Magnetism and High Temprature Superconductivity*, World Scientific, Singapore, 1994.

[24] J. Prüss, *Evolutionary Integral Equations and Applications*, Birkhäuser, Basel, Switzerland, 1993.

[25] I. M. Gel'fand and G. Shilov, *Generalized Functions*, vol. 1, Academic Press, New York, NY, USA, 1964.

[26] J. L. Lions and J. Peetre, "Sur une classe d'espaces d'interpolation," *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, vol. 19, no. 1, pp. 5–68, 1964.

[27] B. Rubin, *Fractional Integrals and Potentials*, Addison-Wesley, Longman, Harlow, UK, 1996.

[28] U. Westphal, "Ein Kalkül für gebrochene Potenzen infinitesimaler Erzeuger von Halbgruppen und Gruppen von Operatoren, Teil I: Halbgruppen-erzeuger," *Compositio Mathematica*, vol. 22, pp. 67–103, 1970.

[29] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics (Book 194), Springer, New York, NY, USA, 2000.

[30] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of *Applied Mathematical Sciences*, Springer, New York, NY, USA, 1983.

[31] G. M. Mittag-Leffler, "Sur la nouvelle fonction E$\alpha(x)$," *Comptes Rendus Mathematique: Comptes Rendus de l'Académie des Sciences II*, vol. 137, pp. 554–558, 1903.

[32] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, vol. 3, McGraw-Hill, New York, NY, USA, 1955.

[33] M. N. Berberan-Santos, "Properties of the Mittag-Leffler relaxation function," *Journal of Mathematical Chemistry*, vol. 38, no. 4, pp. 629–635, 2005.

[34] R. Gorenflo, Y. Luchko, and F. Mainardi, "Analytical properties and applications of the Wright function," *Fractional Calculus & Applied Analysis*, vol. 2, no. 4, pp. 383–414, 1999.

[35] E. M. Wright, "The generalized Bessel function of order greater than one," *The Quarterly Journal of Mathematics*, vol. 11, pp. 36–48, 1940.

[36] E. G. Bazhlekova, "Subordination principle for fractional evolution equations," *Fractional Calculus & Applied Analysis*, vol. 3, no. 3, pp. 213–230, 2000.

Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2014, Article ID 267843, 9 pages
http://dx.doi.org/10.1155/2014/267843

Research Article

On Multi-Laplace Transform for Solving Nonlinear Partial Differential Equations with Mixed Derivatives

Abdon Atangana[1] and Suares Clovis Oukouomi Noutchie[2]

[1] *Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, University of the Free State, Bloemfontein 9300, South Africa*
[2] *Department of Mathematical Sciences, North-West University, Mafikeng Campus, Mmabatho 2735, South Africa*

Correspondence should be addressed to Abdon Atangana; abdonatangana@yahoo.fr

Received 28 April 2014; Accepted 21 May 2014; Published 11 June 2014

Academic Editor: Hossein Jafari

A novel approach is proposed to deal with a class of nonlinear partial equations including integer and noninteger order derivative. This class of equations cannot be handled with any other commonly used analytical technique. The proposed method is based on the multi-Laplace transform. We solved as an example some complicated equations. Three illustrative examples are presented to confirm the applicability of the proposed method. We have presented in detail the stability, the convergence and the uniqueness analysis of some examples.

1. Introduction

In order to investigate the present and future behaviors of a physical problem, many scholars always convert the observed fact into mathematical formula. In the recent decade it was observed that many physical problems were described with partial differential equation with either integer order derivative or fractional order derivative. Some of these equations can be classified as ordinary (linear or nonlinear) partial (linear or nonlinear) differential equations. The partial differential equations, both fractional and integer-order, have been documented as an overriding modeling technique particularly in the last few decades [1–5]. To accurately replicate the nonlocal, frequency-, and history-dependent properties of power law phenomena, some different modeling tools based on fractional operators have to be introduced.

However, as soon as this conversion is done, the next challenge is to find the solutions to these equations. Many scholars have developed method to show the existence and the uniqueness of the solution of these equations [6–8]. But when we are dealing with real world fact, one needs to find approximate, numerical, or exact solution of these equations in order to predict and analyze the solution as function of time and space and this renders the existence concept useless [9–11]. It is perhaps important to recall that finding the exact solution implies the existence of a solution and this is more convenient than just to provide a proof of the existence without presenting the solution. In order to be more practical, several methods have been proposed to find solutions to these equations. In the case of linear equations, some techniques using integral transform such as Mellin transform, Laplace transform, the Fourier transform, and the Sumudu transform, as well as other recent techniques, were proposed. In the case of nonlinear partial differential equations, asymptotic methods are dealing with equations with small parameters, perturbation methods are dealing with multilayers problems, and for some strong linearity, iterations methods such as homotopy perturbation, Adomian's decomposition, homotopy decomposition, variational iteration, and many others have been documented and proven efficient with limitations [12–17]. In 2011, Khan showed that it was possible to make use of the Laplace transform to actually derive solution of nonlinear equations. In his method, he coupled the Laplace series with the Poincare

series [16]. Others adapted this method using the Sumudu transform and the idea of the Lagrange multiplier.

The question that remains is what will happen if a nonlinear partial differential is made up with mixed derivative only? Can these commonly used methods be suitable in finding the approximate or exact solution? The answer is perhaps no. One of the purposes of this paper is to present a novel or extended method that will be used to handle this class of partial derivative. The method makes use of the double Laplace transform and the Poincare series. Without loss of generality, the general form of this class of equation is given as follows:

$$\begin{aligned}\partial_{x^n}^n \partial_{y^m}^m \cdots \partial_{t^i}^i \left[U\left(x, y, \ldots, t\right)\right] &= L\left[U\left(x, y, \ldots, t\right)\right] + N\left[U\left(x, y, \ldots, t\right)\right] \\ &\quad + f\left(x, y, \ldots, t\right),\end{aligned} \tag{1}$$

where $m, n, \ldots, i$ are integer and noninteger numbers, L and N are linear and nonlinear operators with only mixed derivative, respectively, and f is a known function. However, to be more practical we will only consider the dimension to be two or three. The rest of the paper is presented as follows. The methodology of the technique is presented in Section 2. Application of the method with some examples is presented in Section 3. The stability, convergence, and uniqueness analysis will be presented in Section 4, and finally a conclusion is reached in Section 5.

2. Methodology

We devote this section to the discussion underpinning the methodology of the technique for solving (1). Nevertheless, before we present this methodology we will first present some properties of Laplace transform for integer and noninteger order.

Definition 1. The *Laplace transform* is a widely used integral transform with many applications in physics and engineering. The Laplace transform of the function f is defined as follows:

$$\mathscr{L}\left(f\left(x\right)\right)\left(s\right) = \int_0^\infty e^{-sx} f\left(x\right) dx. \tag{2}$$

2.1. Properties of Fractional Calculus. Two properties of the Laplace transform can be used to define the fractional integral operator as follows [18, 19]:

$$\begin{aligned}\mathscr{L}\left(I\left(f\right)\right) &= \mathscr{L}\left(\int_0^x f\left(\tau\right) d\tau\right)\left(s\right) = \frac{1}{s}\mathscr{L}\left(f\left(x\right)\right)\left(s\right), \\ \mathscr{L}\left(I^2\left(f\right)\right) &= \mathscr{L}\left(\int_0^x \int_0^{x_1} f\left(\tau\right) d\tau\, dx_1\right)\left(s\right) \\ &= \frac{1}{s^2}\mathscr{L}\left(f\left(x\right)\right)\left(s\right).\end{aligned} \tag{3}$$

Now using the recursive method, we arrive at the following:

$$\mathscr{L}\left(I^n\left(f\right)\right) = \frac{1}{s^n}\mathscr{L}\left(f\left(x\right)\right)\left(s\right). \tag{4}$$

From the above equation we can obtain

$$\begin{aligned}\mathscr{L}\left(I^n\left(f\right)\right) &= \frac{1}{s^n}\mathscr{L}\left(f\left(x\right)\right)\left(s\right) \\ &\Longrightarrow I^n\left(f\right) = \mathscr{L}^{-1}\left(\frac{1}{s^n}\mathscr{L}\left(f\left(x\right)\right)\left(s\right)\right).\end{aligned} \tag{5}$$

It is well known from the convolution theorem of Laplace transform that

$$\mathscr{L}\left(f * g\left(x\right)\right)\left(s\right) = \mathscr{L}\left(f\left(x\right)\right)\left(s\right) \cdot \mathscr{L}\left(g\left(x\right)\right)\left(s\right). \tag{6}$$

Now from the above formula if we chose $g(x) = x^{\alpha-1}$, with the information of (4), the fractional integral operator can be defined as follows:

$$\begin{aligned}{}_aD_t^\alpha f\left(x\right) &= \frac{1}{\Gamma\left(\alpha\right)}\mathscr{L}^{-1}\left(\mathscr{L}\left(f\left(x\right)\right)\left(s\right) \cdot \mathscr{L}\left(g\left(x\right)\right)\left(s\right)\right), \\ {}_aD_t^\alpha f\left(x\right) &= \frac{1}{\Gamma\left(\alpha\right)}\left(f * g\left(x\right)\right) \\ &= \frac{1}{\Gamma\left(\alpha\right)}\int_a^x \left(x-t\right)^{\alpha-1} f\left(t\right) dt.\end{aligned} \tag{7}$$

Let us observe that Laplace transform of the fractional derivative with both Riemann-Liouville and Caputo is as follows:

$$\begin{aligned}&\mathscr{L}\left({}_0^C D_t^\alpha f\left(x\right)\right)\left(s\right) \\ &\quad = s^\alpha F\left(s\right) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}\left(0\right), \quad \left(n-1 < \alpha \le n\right).\end{aligned} \tag{8}$$

Caputo use the usual initial conditions or values of the functions. On the other hand, we have the Riemann-Liouville; that is,

$$\begin{aligned}\mathscr{L}\left({}_0D_t^\alpha f\left(x\right)\right)\left(s\right) &= s^\alpha F\left(s\right) - \sum_{k=0}^{n-1} s^{\alpha-k-1}\, {}_aD_t^{p-k-1} f\left(0\right), \\ &\quad \left(n-1 \le \alpha < n\right).\end{aligned} \tag{9}$$

The above make use of the unusual initial value or conditions of the function; therefore, it is not suitable for the real world problems [19].

With the above references we will present the methodology of the technique. The first step in this technique is to apply on both sides of (1) the multi-Laplace transform to obtain

$$\begin{aligned}s_1^m s_2^n \cdots s_i^i u\left(s_1, s_2, \ldots, s_i\right) - F\left(s_1, s_2, \ldots, s_i, x, y, \ldots, t\right) &= \mathscr{L}_{x,y,\ldots,t}\left[L\left[U\left(x, y, \ldots, t\right)\right] + N\left[U\left(x, y, \ldots, t\right)\right]\right. \\ &\quad \left. + f\left(x, y, \ldots, t\right)\right]\left(s_1, s_2, \ldots, s_i\right).\end{aligned} \tag{10}$$

We can now divide both sides by $s_1^m s_2^n \cdots s_i^i$ and apply the inverse Laplace transform on both sides to obtain

$$\begin{aligned} U(x,y,\ldots,t) &= g(x,y,\ldots,t) \\ &= \mathscr{L}^{-1}_{x,y,\ldots,t}\left[\frac{1}{s_1^m s_2^n \cdots s_i^i} \times \left(\mathscr{L}_{x,y,\ldots,t}\left[L\left[U(x,y,\ldots,t)\right] + N\left[U(x,y,\ldots,t)\right] + f(x,y,\ldots,t)\right] \times (s_1,s_2,\ldots,s_i)\right)\right](x,y,\ldots,t). \end{aligned} \tag{11}$$

At this stage two iteration formulae can be developed, the first one using the idea of Lagrange multiplier and the second one using the homotopy idea. If we use the idea of Lagrange multiplier, we have that the Lagrange multiplier in Laplace space is given as follows:

$$\lambda(s_1,s_2,\ldots,s_i) = \frac{1}{s_1^m s_2^n \cdots s_i^i}. \tag{12}$$

With the above Lagrange multiplier in hand, we proposed the general integration to be in the form of

$$\begin{aligned} U_0(x,y,\ldots,t) &= g(x,y,\ldots,t), \\ U_{n+1}(x,y,\ldots,t) &= U_n(x,y,\ldots,t) + \mathscr{L}^{-1}_{x,y,\ldots,t}\left[\frac{1}{s_1^m s_2^n \cdots s_i^i} \times \left(\mathscr{L}_{x,y,\ldots,t}\left[L\left[U_n(x,y,\ldots,t)\right] + N\left[U_n(x,y,\ldots,t)\right] + f(x,y,\ldots,t)\right] \times (s_1,s_2,\ldots,s_i)\right)\right](x,y,\ldots,t) \end{aligned} \tag{13}$$

and the approximate or special solution can be obtained as

$$U(x,y,\ldots,t) = \lim_{n\to\infty} U_{n+1}(x,y,\ldots,t). \tag{14}$$

If we use the idea of homotopy, we will assume that the solution can be in the form of series as follows:

$$U(x,y,\ldots,t) = \sum_{n=0}^{\infty} p^n U_n(x,y,\ldots,t). \tag{15}$$

With replacing this expression in (11) and after comparing terms of the same power of p, as well as using the polynomial proposed in [15], we obtain

$$\begin{aligned} U_0(x,y,\ldots,t) &= g(x,y,\ldots,t), \\ U_1(x,y,\ldots,t) &= \mathscr{L}^{-1}_{x,y,\ldots,t}\left[\frac{1}{s_1^m s_2^n \cdots s_i^i} \times \left(\mathscr{L}_{x,y,\ldots,t}\left[L\left[U_0(x,y,\ldots,t)\right] + N\left[U_0(x,y,\ldots,t)\right] + f(x,y,\ldots,t)\right] \times (s_1,s_2,\ldots,s_i)\right)\right](x,y,\ldots,t) \end{aligned} \tag{16}$$

and for $n \geq 2$, we have

$$U_{n+1}(x,y,\ldots,t) = \mathscr{L}^{-1}_{x,y,\ldots,t}\left[\frac{1}{s_1^m s_2^n \cdots s_i^i} \times \left(\mathscr{L}_{x,y,\ldots,t}\left[L\left[U_0(x,y,\ldots,t)\right] + \mathscr{H}_n\left[(x,y,\ldots,t)\right]\right] \times (s_1,s_2,\ldots,s_i)\right)\right](x,y,\ldots,t) \tag{17}$$

with of course $\mathscr{H}_n[(x,y,\ldots,t)]$ the polynomial proposed in [15].

We will illustrate this method with some examples and this is done in the next section

3. Application

We present in this section the application of this extension by solving some nonlinear and linear partial differential equations with mixed derivative only.

Example 2. To illustrate these methods, let us consider the following simple linear equation:

$$\partial_{xt} u(x,t) + u = 0. \tag{18}$$

Making use of methodology 2 presented is Section 2, we have

$$u(x,t) = u(x,0) + u(0,t) - u(0,0) - \mathscr{L}^{-1}_{x,t}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[u(x,t)\right](s,v)\right)\right](x,t) \tag{19}$$

and the general iteration formula for this is given by

$$\begin{aligned} u_0(x,t) &= u(x,0) + u(0,t) - u(0,0), \\ u_{n+1}(x,t) &= -\mathscr{L}^{-1}_{x,t}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[u_n(x,t)\right](s,v)\right)\right](x,t). \end{aligned} \tag{20}$$

Therefore, using the iteration formula, we obtain

$$
\begin{aligned}
u_0(x,t) &= \operatorname{Cosh}[x] + \operatorname{Cosh}[t] - 1,\\
u_1(x,t) &= -\frac{1}{2}e^{-t}x + \frac{e^t x}{2} + \frac{1}{2}e^{-x}t\left(-1 + e^{2x} - 2e^x x\right),\\
u_2(x,t) &= -\frac{x^2}{2} + \frac{1}{4}e^{-t}x^2 + \frac{e^t x^2}{4}\\
&\quad + \frac{1}{4}e^{-x}t^2\left(1 - 2e^x + e^{2x} - e^x x^2\right),\\
u_3(x,t) &= -\frac{1}{12}e^{-t}x^3 + \frac{e^t x^3}{12} - \frac{tx^3}{6}\\
&\quad + \frac{1}{36}e^{-x}t^3\left(-3 + 3e^{2x} - 6e^x x - e^x x^3\right),\\
u_4(x,t) &= -\frac{x^4}{24} + \frac{1}{48}e^{-t}x^4 + \frac{e^t x^4}{48} - \frac{t^2x^4}{48} + \frac{1}{576}e^{-x}t^4\\
&\quad \times\left(12 - 24e^x + 12e^{2x} - 12e^x x^2 - e^x x^4\right),\\
u_5(x,t)&\\
&= -\frac{1}{240}e^{-t}x^5 + \frac{e^t x^5}{240} - \frac{tx^5}{120} - \frac{t^3x^5}{720}\\
&\quad + \frac{e^{-x}t^5\left(-60 + 60e^{2x} - 120e^x x - 20e^x x^3 - e^x x^5\right)}{14400},\\
u_6(x,t) &= -\frac{x^6}{720} + \frac{e^{-t}x^6}{1440} + \frac{e^t x^6}{1440} - \frac{t^2x^6}{1440} - \frac{t^4x^6}{17280}\\
&\quad + \left(e^{-x}t^6\left(360 - 720e^x + 360e^{2x} - 360e^x x^2\right.\right.\\
&\qquad \left.\left. - 30e^x x^4 - e^x x^6\right) \times (518400)^{-1}\right),\\
u_7(x,t) &= -\frac{e^{-t}x^7}{10080} + \frac{e^t x^7}{10080} - \frac{tx^7}{5040} - \frac{t^3x^7}{30240} - \frac{t^5x^7}{604800}\\
&\quad + \left(e^{-x}t^7\left(-2520 + 2520e^{2x} - 5040e^x x\right.\right.\\
&\qquad \left. - 840e^x x^3 - 42e^x x^5 - e^x x^7\right)\\
&\qquad \left.\times (25401600)^{-1}\right).
\end{aligned}
\tag{21}
$$

And then, the summation of the first 11 terms is given as

$$
\begin{aligned}
u[x,t] = \sum_{n=0}^{10} u[x,t,n] &= -1 - \frac{e^{-t}x}{2} + \frac{e^t x}{2} - \frac{x^2}{2} + \frac{1}{4}e^{-t}x^2\\
&\quad + \frac{e^t x^2}{4} - \frac{1}{12}e^{-t}x^3 + \frac{e^t x^3}{12} - \frac{tx^3}{6} - \frac{x^4}{24}\\
&\quad + \frac{1}{48}e^{-t}x^4 + \frac{e^t x^4}{48} - \frac{t^2x^4}{48} - \frac{1}{240}e^{-t}x^5 + \frac{e^t x^5}{240}\\
&\quad - \frac{tx^5}{120} - \frac{t^3x^5}{720} - \frac{x^6}{720} + \frac{e^{-t}x^6}{1440} + \frac{e^t x^6}{1440} - \frac{t^2x^6}{1440}\\
&\quad - \frac{t^4x^6}{17280} - \frac{e^{-t}x^7}{10080} + \frac{e^t x^7}{10080} - \frac{tx^7}{5040} - \frac{t^3x^7}{30240}\\
&\quad - \frac{t^5x^7}{604800} - \frac{x^8}{40320} + \frac{e^{-t}x^8}{80640} + \frac{e^t x^8}{80640} - \frac{t^2x^8}{80640}\\
&\quad - \frac{t^4x^8}{967680} - \frac{t^6x^8}{29030400} - \frac{e^{-t}x^9}{725760} + \frac{e^t x^9}{725760}\\
&\quad - \frac{tx^9}{362880} - \frac{t^3x^9}{2177280} - \frac{t^5x^9}{43545600}\\
&\quad - \frac{t^7x^9}{1828915200} - \frac{x^{10}}{3628800} + \frac{e^{-t}x^{10}}{7257600}\\
&\quad + \frac{e^t x^{10}}{7257600} - \frac{t^2x^{10}}{7257600} - \frac{t^4x^{10}}{87091200}\\
&\quad - \frac{t^6x^{10}}{2612736000} - \frac{t^8x^{10}}{146313216000}\\
&\quad + \frac{1}{2}e^{-x}t\left(-1 + e^{2x} - 2e^x x\right)\\
&\quad + \frac{1}{4}e^{-x}t^2\left(1 - 2e^x + e^{2x} - e^x x^2\right)\\
&\quad + \frac{1}{36}e^{-x}t^3\left(-3 + 3e^{2x} - 6e^x x - e^x x^3\right)\\
&\quad + \frac{1}{576}e^{-x}t^4\left(12 - 24e^x + 12e^{2x} - 12e^x x^2 - e^x x^4\right)\\
&\quad + \left(e^{-x}t^5\left(-60 + 60e^{2x} - 120e^x x\right.\right.\\
&\qquad \left.\left. - 20e^x x^3 - e^x x^5\right) \times (14400)^{-1}\right)\\
&\quad + \left(e^{-x}t^6\left(360 - 720e^x + 360e^{2x} - 360e^x x^2\right.\right.\\
&\qquad \left.\left. - 30e^x x^4 - e^x x^6\right) \times (518400)^{-1}\right)\\
&\quad + \left(e^{-x}t^7\left(-2520 + 2520e^{2x} - 5040e^x x - 840e^x x^3\right.\right.\\
&\qquad \left.\left. - 42e^x x^5 - e^x x^7\right) \times (25401600)^{-1}\right)\\
&\quad + \left(e^{-x}t^8\left(20160 - 40320e^x + 20160e^{2x}\right.\right.\\
&\qquad - 20160e^x x^2 - 1680e^x x^4 - 56e^x x^6\\
&\qquad \left.\left. - e^x x^8\right) \times (1625702400)^{-1}\right)\\
&\quad + \left(e^{-x}t^9\left(-181440 + 181440e^{2x} - 362880e^x x\right.\right.\\
&\qquad - 60480e^x x^3 - 3024e^x x^5 - 72e^x x^7\\
&\qquad \left.\left. - e^x x^9\right) \times (131681894400)^{-1}\right)\\
&\quad + \left(e^{-x}t^{10}\left(1814400 - 3628800e^x + 1814400e^{2x}\right.\right.\\
&\qquad - 1814400e^x x^2 - 151200e^x x^4\\
&\qquad \left.\left. - 5040e^x x^6 - 90e^x x^8 - e^x x^{10}\right)\right.
\end{aligned}
$$

$$\times (13168189440000)^{-1}\Big) + \text{Cosh}[t] + \text{Cosh}[x]. \tag{22}$$

Realize that if n is very large, then the solution of this equation is

$$u[x,t] = \sum_{n=0}^{\infty} u_n(x,t) = \text{Cosh}[x-t]. \tag{23}$$

This is the exact solution of our equation. We will examine the solution for the fractional version.

Example 3. Let us consider the following linear fractional differential equation:

$$\partial^{\alpha+\beta}_{x^\alpha t^\beta} u(x,t) + u = 0. \tag{24}$$

Making use of methodology 2 presented in Section 2, we have

$$u(x,t) = u(x,0) + u(0,t) - u(0,0) - \mathscr{L}^{-1}_{x,t}\left[\frac{1}{s^\alpha v^\beta}\left(\mathscr{L}_{x,t}[u(x,t)](s,v)\right)\right](x,t) \tag{25}$$

and the general iteration formula for this is given by

$$u_0(x,t) = u(x,0) + u(0,t) - u(0,0),$$
$$u_{n+1}(x,t) = -\mathscr{L}^{-1}_{x,t}\left[\frac{1}{s^\alpha v^\beta}\left(\mathscr{L}_{x,t}[u_n(x,t)](s,v)\right)\right](x,t). \tag{26}$$

Therefore, using the iteration formula we obtain

$$\begin{aligned}
u_0(x,t) &= u(x,0) + u(0,t) - u(0,0) = 1,\\
u_1(x,t) &= \frac{t^\beta x^\alpha}{\Gamma[1+\alpha]\Gamma[1+\beta]},\\
u_2(x,t) &= \frac{t^{2\beta} x^{2\alpha}}{\Gamma[1+2\alpha]\Gamma[1+2\beta]},\\
u_3(x,t) &= \frac{t^{3\beta} x^{3\alpha}}{\Gamma[1+3\alpha]\Gamma[1+3\beta]},\\
u_4(x,t) &= \frac{t^{4\beta} x^{4\alpha}}{\Gamma[1+4\alpha]\Gamma[1+4\beta]},\\
u_n(x,t) &= \frac{t^{n\beta} x^{n\alpha}}{\Gamma[1+n\alpha]\Gamma[1+n\beta]}.
\end{aligned} \tag{27}$$

It is very important to realize that if n is very large, then the solution of this equation is

$$u[x,t] = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \frac{t^{n\beta} x^{n\alpha}}{\Gamma[1+n\alpha]\Gamma[1+n\beta]}. \tag{28}$$

This is the exact solution of our equation.

Example 4. Let us consider the following nonlinear partial differential equation :

$$\begin{aligned}
\partial^2_{xt} u = {} & 2u\partial^4_{x^2t^2} u + 4\partial_x u \partial^3_{xt^2} u + 4\partial^3_{x^2t} u \partial_t u\\
& + 4\left(\partial^2_{xt} u\right)^2 + \partial^2_{t^2} u \partial^2_{x^2} u + u^2 + \partial^3_{xt^2} u^3.
\end{aligned} \tag{29}$$

The above equation is very complicated due to the strong nonlinearity; we will therefore present a special solution to it by applying methodology two presented in Section 2.

$$\begin{aligned}
u(x,t) = {} & u(x,0) + u(0,t) - u(0,0)\\
& - \mathscr{L}^{-1}_{x,t}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[2u\partial^4_{x^2t^2} u + 4\partial_x u \partial^3_{xt^2} u\right.\right.\right.\\
& \qquad + 4\partial^3_{x^2t} u \partial_t u + 4\left(\partial^2_{xt} u\right)^2\\
& \qquad + \partial^2_{t^2} u \partial^2_{x^2} u + u^2\\
& \qquad \left.\left.\left. + \partial^3_{xt^2} u^3\right](s,v)\right)\right](x,t).
\end{aligned} \tag{30}$$

The second and the third steps produce

$$\begin{aligned}
u_0(x,t) = {} & u(x,0) + u(0,t) - u(0,0),\\
u_1(x,t) = {} & -\mathscr{L}^{-1}_{x,t}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[2u_0\partial^4_{x^2t^2} u_0 + 4\partial_x u_0 \partial^3_{xt^2} u_0\right.\right.\right.\\
& \qquad + 4\partial^3_{x^2t} u_0 \partial_t u_0 + 4\left(\partial^2_{xt} u_0\right)^2\\
& \qquad + \partial^2_{t^2} u_0 \partial^2_{x^2} u_0 + u_0^2\\
& \qquad \left.\left.\left. + \partial^3_{xt^2} u_0^3\right](s,v)\right)\right](x,t).
\end{aligned} \tag{31}$$

And for any natural number greater than one we have

$$\begin{aligned}
u_n(x,t) = {} & -\mathscr{L}^{-1}_{x,t}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[2H^1_n(x,t) + 4H^2_n(x,t)\right.\right.\right.\\
& \qquad + 4H^3_n(x,t) + 4H^4_n(x,t)\\
& \qquad + H^5_n(x,t) + H^6_n(x,t)\\
& \qquad \left.\left.\left. + H^7_n(x,t)\right](s,v)\right)\right](x,t),
\end{aligned}$$

$$H^1_n(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\, \partial_{x^2t^2} u_{n-j}(x,t),$$

$$H_n^2(x,t) = \sum_{j=0}^{n-1} \partial_x u_j(x,t)\, \partial_{xt^2} u_{n-j}(x,t),$$

$$H_n^3(x,t) = \sum_{j=0}^{n-1} \partial_t u_j(x,t)\, \partial_{x^2t} u_{n-j}(x,t),$$

$$H_n^4(x,t) = \sum_{j=0}^{n-1} \partial_{xt} u_j(x,t)\, \partial_{xt} u_{n-j}(x,t),$$

$$H_n^5(x,t) = \sum_{j=0}^{n-1} \partial_{t^2} u_j(x,t)\, \partial_{x^2} u_{n-j}(x,t),$$

$$H_n^6(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\, u_{n-j}(x,t),$$

$$H_n^7(x,t) = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \partial_{xt^2} u_j(x,t)\, \partial_{xt^2} u_{j-k}(x,t)\, \partial_{xt^2} \times u_{n-j-1}(x,t). \tag{32}$$

The above can be resumed in the following algorithm.

Algorithm 5. (i) Input: $J(x,t)$ as initial guest,
(ii) j—number terms in the rough calculation
(iii) Output: $u_{\text{approx}}(x,t)$, the approximate solution.

Step 1. Put $u_0(x,t) = u(x,0) + u(0,t) - u(0,0)$ and $u_{\text{approx}}(x,t) = u_0(x,t)$.

Step 2. For $j = 0$ to $n-1$ do Step 3, Step 4, and Step 5.

Step 3. Compute

$$H_n^1(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\, \partial_{x^2t^2} u_{n-j}(x,t),$$

$$H_n^2(x,t) = \sum_{j=0}^{n-1} \partial_x u_j(x,t)\, \partial_{xt^2} u_{n-j}(x,t),$$

$$H_n^3(x,t) = \sum_{j=0}^{n-1} \partial_t u_j(x,t)\, \partial_{x^2t} u_{n-j}(x,t),$$

$$H_n^4(x,t) = \sum_{j=0}^{n-1} \partial_{xt} u_j(x,t)\, \partial_{xt} u_{n-j}(x,t),$$

$$H_n^5(x,t) = \sum_{j=0}^{n-1} \partial_{t^2} u_j(x,t)\, \partial_{x^2} u_{n-j}(x,t),$$

$$H_n^6(x,t) = \sum_{j=0}^{n-1} u_j(x,t)\, u_{n-j}(x,t),$$

$$H_n^7(x,t) = \sum_{j=0}^{n-1} \sum_{k=0}^{j} \partial_{xt^2} u_j(x,t)\, \partial_{xt^2} u_{j-k}(x,t)\, \partial_{xt^2} \times u_{n-j-1}(x,t). \tag{33}$$

Step 4. Compute

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) \\ &= -\mathscr{L}_{x,t}^{-1}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[2H_n^1(x,t) + 4H_n^2(x,t) + 4H_n^3(x,t) + 4H_n^4(x,t) + H_n^5(x,t) + H_n^6(x,t) + H_n^7(x,t)\right](s,v)\right)\right](x,t). \end{aligned} \tag{34}$$

Step 5. Compute $u_{\text{approx}}(x,t) = u_{\text{approx}}(x,t) + u_{n+1}(x,t)$

Stop.

We will present in the next section the analysis of convergence and uniqueness of the especial solution of (29) for using method 2.

4. Convergence and Uniqueness Analysis

The reason of this part is to demonstrate in detail the convergence and the uniqueness of the nonlinear equation while using the proposed iteration method; we will therefore consider the following equation:

$$\begin{aligned} \partial_{xt}^2 u &= 2u\partial_{x^2t^2}^4 u + 4\partial_x u \partial_{xt^2}^3 u + 4\partial_{x^2t}^3 u \partial_t u \\ &\quad + 4\left(\partial_{xt}^2 u\right)^2 + \partial_{t^2}^2 u \partial_{x^2}^2 u + u^2 + \partial_{xt^2}^3 u^3. \end{aligned} \tag{35}$$

Think about the Hilbert space $\mathscr{H} = L^2((\eta,\lambda)\times[0,T])$ defined as

$$\mathscr{H} = \left\{(u,v) : (\eta,\lambda)\times[0,T] \text{ with, } \mathscr{L}_{x,t}^{-1}\left[\frac{1}{sv}\left(\mathscr{L}_{x,t}\left[u(x,t)\right](s,v)\right)\right](x,t) < \infty\right\}. \tag{36}$$

Then the operator is of the form

$$\begin{aligned} K(u) &= \partial_{xt}^2 u = 2u\partial_{x^2t^2}^4 u + 4\partial_x u \partial_{xt^2}^3 u + 4\partial_{x^2t}^3 u \partial_t u \\ &\quad + 4\left(\partial_{xt}^2 u\right)^2 + \partial_{t^2}^2 u \partial_{x^2}^2 u + u^2 + \partial_{xt^2}^3 u^3. \end{aligned} \tag{37}$$

The proposed analytical technique is convergent if the subsequent necessities are met.

Hypothesis 1. It is likely for us to establish a positive constant, say P, such that the inner product holds in $\mathscr{H}$:

$$\left(K(u) - K(v), u - v\right) \le P\|u - v\|, \quad \forall v, u \in \mathscr{H}. \tag{38}$$

Hypothesis 2. As far as for all $v, u \in H$are bounded, this implies, we can find a positive constant say D such that $\|u\|, \|v\| \leq D$; then, we can find $\Phi(D) > 0$ such that

$$\left(K(u) - K(v), g\right) \leq \Phi(D) \|u - v\| \left\|g\right\|, \quad \forall g \in H. \tag{39}$$

We can accordingly declare the consequential theorem for the sufficient condition for the convergence of (41).

Theorem 6. *Let us think about*

$$\begin{aligned} K(u) = {} & 2u\partial^4_{x^2t^2}u + 4\partial_x u\partial^3_{xt^2}u + 4\partial^3_{x^2t}u\partial_t u \\ & + 4\left(\partial^2_{xt}u\right)^2 + \partial^2_{t^2}u\partial^2_{x^2}u + u^2 + \partial^3_{xt^2}u^3 \end{aligned} \tag{40}$$

and think about the initial and boundary condition for (41)*; then, the proposed technique shows the way to a particular solution of* (41).

We will present the proof of this theorem by just verifying Hypotheses 1 and 2.

Proof. Using the definition of our operator K, we have the following:

$$\begin{aligned} K(u) - K(v) = {} & 2u\partial^4_{x^2t^2}u + 4\partial_x u\partial^3_{xt^2}u + 4\partial^3_{x^2t}u\partial_t u \\ & + 4\left(\partial^2_{xt}u\right)^2 + \partial^2_{t^2}u\partial^2_{x^2}u + u^2 - v^2 \\ & + \partial^3_{xt^2}\left(u^3 - v^3\right) - 2v\partial^4_{x^2t^2}v - 4\partial_x v\partial^3_{xt^2}v \\ & - 4\partial^3_{x^2t}v\partial_t v - 4\left(\partial^2_{xt}v\right)^2 - \partial^2_{t^2}v\partial^2_{x^2}v, \\ K(u) - K(v) = {} & \partial^2_{xt}(u - v) + u^2 - v^2 \\ & + \partial^3_{xt^2}\left(u^3 - v^3\right) \\ & + 2\partial_x\left(u\partial^3_{xt^2}u + \partial^2_{xt}u\partial_t u\right. \\ & \qquad \left. + \partial^2_{tx}u\partial_t u + \partial_x u\partial^2_{t^2}u\right) \\ & - 2\partial_x\left(v\partial^3_{xt^2}v + \partial^2_{xt}v\partial_t v\right. \\ & \qquad \left. + \partial^2_{tx}v\partial_t v + \partial_x v\partial^2_{t^2}v\right), \\ K(u) - K(v) = {} & \partial^2_{xt}(u - v) + \left(u^2 - v^2\right) \\ & + \partial^3_{xt^2}\left(u^3 - v^3\right) + \partial^4_{x^2t^2}u - \partial^4_{x^2t^2}v, \\ K(u) - K(v) = {} & \partial^2_{xt}(u - v) + \left(u^2 - v^2\right) \\ & + \partial^3_{xt^2}\left(u^3 - v^3\right) + \partial^4_{x^2t^2}\left(u^2 - v^2\right). \end{aligned} \tag{41}$$

With the above reduction in hand, it is therefore possible for us to evaluate the following inner product:

$$\begin{aligned} & \left(K(u) - K(v), (u - v)\right) \\ & \quad = \left(\left(u^2 - v^2\right), u - v\right) + \left(\partial^3_{xt^2}\left(u^3 - v^3\right), u - v\right) \\ & \qquad + \left(\partial^4_{x^2t^2}\left(u^2 - v^2\right), u - v\right). \end{aligned} \tag{42}$$

We will examine case after case and start with the high nonlinear part.

Take for granted that u, v are bounded; consequently, we can find a positive constant M such that $(u, u), (v, v) < M^2$. It follows by the use of Schwartz inequality that

$$\left(\partial^3_{xt^2}\left(u^3 - v^3\right), u - v\right) \leq \left\|\partial^3_{xt^2}\left(u^3 - v^3\right)\right\| \|u - v\|. \tag{43}$$

However, we can find a positive constant $\theta_1$such that $\|\partial^2_{t^2}(u^3 - v^3)\| \leq \theta_1\|\partial^3_{t^2}(u^3 - v^3)\|$ it follows from equation (43) that

$$\left(\partial^2_{xt^2}\left(u^3 - v^3\right), u - v\right) \leq \theta_1\theta_2\theta_3\left\|u^3 - v^3\right\| \|u - v\|. \tag{44}$$

We will recall that

$$u^3 - v^3 = (u - v)\left(u^2 + uv + v^2\right). \tag{45}$$

Thus

$$\begin{aligned} \left\|u^3 - v^3\right\| &= \left\|(u - v)\left(u^2 + uv + v^2\right)\right\| \\ &= \|(u - v)\|\left\|u^2 + uv + v^2\right\|. \end{aligned} \tag{46}$$

Now making use of the triangular inequality together with the fact that u and v are bounded, we arrive at the following result:

$$\left\|u^3 - v^3\right\| \leq 6M^3. \tag{47}$$

It follows that

$$\left(\partial^3_{xt^2}\left(u^3 - v^3\right), u - v\right) \leq 6\theta_1\theta_2\theta_3 M^3 \|u - v\|. \tag{48}$$

Also we have the following inequality:

$$\begin{aligned} \left(u^2 - v^2, u - v\right) &\leq \left\|u^2 - v^2\right\| \|u - v\| \\ &\leq 4M^2 \|u - v\|. \end{aligned} \tag{49}$$

We moreover have also that the Cauchy-Schwarz-Bunyakovsky inequality yields

$$\begin{aligned} & \partial^4_{x^2t^2}\left(\left(u^2 - v^2\right), u - v\right) \\ & \quad \leq \theta_4\theta_5\theta_6\theta_7\left\|u^2 - v^2\right\| \|u - v\|. \end{aligned} \tag{50}$$

Obviously due to the fact that it is possible for us to find two positive constants θ_3, θ_4 such that

$$\begin{aligned} & \partial^4_{x^2t^2}\left(\left(u^2 - v^2\right), u - v\right) \\ & \quad \leq \theta_4\theta_5\left\|\left(u^2 - v^2\right)_{xt}\right\| \|u - v\|, \end{aligned} \tag{51}$$

we can find another set of positive constants $\theta_6\theta_7$ in respect of the following inequality:

$$\left\|\left(u^2 - v^2\right)_{xt}\right\| \leq \theta_6\theta_7\left\|u^2 - v^2\right\| \tag{52}$$

and finally using the fact that u and v are bounded we obtain

$$\partial^4_{x^2t^2}\left(\left(u^2 - v^2\right), u - v\right) \leq 4M^2\theta_4\theta_5\theta_6\theta_7 \|u - v\|. \tag{53}$$

Now substituting (51), (44), and (49) into (42) we arrive at the following:

$$\begin{aligned}&(K(u)-K(v),(u-v))\\&\quad\le\left(6\theta_1\theta_2\theta_3M^3+4M^2+4M^2\theta_4\theta_5\theta_6\theta_7\right)\|u-v\|.\end{aligned}\tag{54}$$

Take here

$$P=6\theta_1\theta_2\theta_3M^3+4M^2+4M^2\theta_4\theta_5\theta_6\theta_7.\tag{55}$$

And then, Hypothesis 1 is verified. We will now verify Hypothesis 2, and to do this we quickly compute the relation as follows:

$$\begin{aligned}(K(u)-K(v),g)=&\left(\left(u^2-v^2\right),g\right)+\left(\partial^3_{xt^2}\left(u^3-v^3\right),g\right)\\&+\left(\partial^4_{x^2t^2}\left(u^2-v^2\right),g\right).\end{aligned}\tag{56}$$

Now using the Schwartz inequality and the fact that u and v are bounded, we obtain the following:

$$\begin{aligned}&(K(u)-K(v),g)\\&\quad\le\left(2D+3D^2\theta_1\theta_2\theta_3+2D\theta_4\theta_5\theta_6\theta_7\right)\|u-v\|\|g\|.\end{aligned}\tag{57}$$

We will then consider $\Phi(D)=(2D+3D^2\theta_1\theta_2\theta_3+2D\theta_4\theta_5\theta_6\theta_7)$ and Hypothesis 2 is also verified. We can therefore conclude that the proposed method works for (23). Our next concern is to establish the uniqueness of the special solution. This can be resumed in the following theorem. □

Theorem 7. *Taking into account the initial conditions for* (23) *then the special solution of* (23) u_{esp} *to whom u convergence is unique.*

Proof. Assuming that we can find another special solution, say v_{esp}, then by making use of the inner product, we have

$$\left(u_{\text{esp}}-v_{\text{esp}},u_{\text{esp}}-v_{\text{esp}}\right)\le 2D\left\|u_{\text{esp}}-v_{\text{esp}}\right\|.\tag{58}$$

Now if u is the exact solution of (23), then according to Theorem 6 we can find suitable natural number n and m such that $\|u_{\text{esp}}-u\|<\varepsilon/4D$ and $\|u-v_{\text{esp}}\|<\varepsilon/4D$ with ε a very small number closest to zero. Now taking the $\max(m,n)$, then

$$\left(u_{\text{esp}}-v_{\text{esp}},u_{\text{esp}}-v_{\text{esp}}\right)\le\varepsilon.\tag{59}$$

However according to the properties of the inner product,

$$\begin{aligned}&\left(u_{\text{esp}}-v_{\text{esp}},u_{\text{esp}}-v_{\text{esp}}\right)=0\\&\text{implies } u_{\text{esp}}-v_{\text{esp}}=0\end{aligned}\tag{60}$$

and Theorem 7 is verified. □

5. Conclusion

Although many methods have been developed to deal with the existence and the uniqueness of some class of differential equations, there are still a large number of equations for which these methods cannot be used and even it is not possible to prove the existence of their solution, for instance, the Navier-Stock equation. However, it is important to point out that solving an equation analytically and finding the exact solution are more than proving the existence of this solution. The question that arises at this level is that should we perhaps focus on developing analytical methods that can be used to solve these complicated equations? Or should we perhaps focus on developing methods to prove their existence? We are afraid to reveal that for those dealing with real world problems the answer will be as follows: let us instead focus on developing methods to find analytical solutions of these equations since we need them to predict the future behaviour of these physical phenomena. Numerical methods can also be used for this purpose [20]. The main aim of this paper was to propose a method that can be used to solve a class of partial differential equations that other commonly used methods, such as the normal Laplace transform method, the Fourier method, the Sumudu method, the Green function method, and the Mellin transform method, as well as the recent developed iteration methods, cannot handle. We therefore presented some examples. We have with great success presented the stability, the convergence, and the uniqueness analysis. We presented this for both fractional and ordinary partial differential equations. The method proposed here makes use of the Laplace transform.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

[1] J. A. Barker, "A generalized radial flow model for hydraulic tests in fractured rock," *Water Resources Research*, vol. 24, no. 10, pp. 1796–1804, 1988.

[2] J. Bear, *Dynamics of Fluids in Porous Media*, American Elsevier Environmental Science Series, Elsevier, New York, NY, USA, 1972.

[3] I. Podlubny, *Fractional Differential Equations*, vol. 198 of *Mathematics in Science and Engineering*, Academic Press, San Diego, Calif, USA, 1999.

[4] J. F. Botha and A. Cloot, "A generalised groundwater flow equation using the concept of non-integer order derivatives," *Water SA*, vol. 32, no. 1, pp. 1–7, 2006.

[5] A. Atangana and P. D. Vermeulen, "Analytical solutions of a space-time fractional derivative of groundwater flow equation," *Abstract and Applied Analysis*, vol. 2014, Article ID 381753, 11 pages, 2014.

[6] R. P. Agarwal, Y. Zhou, and Y. He, "Existence of fractional neutral functional differential equations," *Computers & Mathematics with Applications*, vol. 59, no. 3, pp. 1095–1100, 2010.

[7] D. Baleanu, R. P. Agarwal, H. Mohammadi, and S. Rezapour, "Some existence results for a nonlinear fractional differential equation on partially ordered Banach spaces," *Boundary Value Problems*, vol. 2013, article 112, 2013.

[8] Y.-K. Chang and J. J. Nieto, "Some new existence results for fractional differential inclusions with boundary conditions,"

Mathematical and Computer Modelling, vol. 49, no. 3-4, pp. 605–609, 2009.

[9] A. Atangana and S. C. Oukouomi Noutchie, "A modified groundwater flow model using the space time Riemann-Liouville fractional derivatives approximation," *Abstract and Applied Analysis*, vol. 2014, Article ID 498381, 7 pages, 2014.

[10] A. Atangana, "Drawdown in prolate spheroidal-spherical coordinates obtained via Green's function and perturbation methods," *Communications in Nonlinear Science and Numerical Simulation*, vol. 19, no. 5, pp. 1259–1269, 2014.

[11] J.-H. Moon and B.-G. Lee, "Modeling and sensitivity analysis of a pneumatic vibration isolation system with two air chambers," *Mechanism and Machine Theory*, vol. 45, no. 12, pp. 1828–1850, 2010.

[12] J. H. He and F. J. Liu, "Local fractional variational iteration method for fractal heat transfer in silk cocoon hierarchy," *Nonlinear Science Letters A*, vol. 4, no. 1, pp. 15–20, 2013.

[13] H. Jafari and S. Seifi, "Homotopy analysis method for solving linear and nonlinear fractional diffusion-wave equation," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 5, pp. 2006–2012, 2009.

[14] V. Daftardar-Gejji and H. Jafari, "Adomian decomposition: a tool for solving a system of fractional differential equations," *Journal of Mathematical Analysis and Applications*, vol. 301, no. 2, pp. 508–518, 2005.

[15] A. Atangana and A. Secer, "The time-fractional coupled-Korteweg-de-Vries equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 947986, 8 pages, 2013.

[16] Y. Khan and Q. Wu, "Homotopy perturbation transform method for nonlinear equations using He's polynomials," *Computers & Mathematics with Applications*, vol. 61, no. 8, pp. 1963–1967, 2011.

[17] Y. Khan, "A method for solving nonlinear time-dependent drainage model," *Neural Computing and Applications*, vol. 23, no. 2, pp. 411–415, 2013.

[18] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, vol. 204 of *North-Holland Mathematics Studies*, Elsevier, Amsterdam, The Netherlands, 2006.

[19] A. Atangana and A. Secer, "A note on fractional order derivatives and table of fractional derivatives of some special functions," *Abstract and Applied Analysis*, vol. 2013, Article ID 279681, 8 pages, 2013.

[20] A. H. Bhrawy and M. M. Al-Shomrani, "A Jacobi dual-Petrov Galerkin-Jacobi collocation method for solving Korteweg-de Vries equations," *Abstract and Applied Analysis*, vol. 2012, Article ID 418943, 16 pages, 2012.