# Dynamics of Delay Differential Equations with Its Applications 2014 

Guest Editors: Chuangxia Huang, Zhiming Guo, Zhichun Yang, Yuming Chen, and Fenghua Wen


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 with Its Applications 2014
## Abstract and Applied Analysis

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## Editorial

# Dynamics of Delay Differential Equations with Its Applications 2014 

Chuangxia Huang, ${ }^{1}$ Zhiming Guo, ${ }^{2}$ Zhichun Yang, ${ }^{3}$ Yuming Chen, ${ }^{4}$ and Fenghua Wen ${ }^{5}$<br>${ }^{1}$ College of Mathematics and Computing Science, Changsha University of Science and Technology, Changsha 410114, China<br>${ }^{2}$ School of Mathematics and Information Science, Guangzhou University, Guangzhou 510006, China<br>${ }^{3}$ College of Mathematics, Chongqing Normal University, Chongqing 400047, China<br>${ }^{4}$ Department of Mathematics, Wilfrid Laurier University, Waterloo, ON, Canada N2L 3C5<br>${ }^{5}$ Business School, Central South University, Changsha 410012, China

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Delay differential equations have attracted a rapidly growing attention in the field of nonlinear dynamics and have become a powerful tool for investigating the complexities of the real-world problems such as infectious diseases, biotic population, neuronal networks, and even economics and finance. When employing delay differential equations to solve practical problems, it is very crucial to be able to completely characterize the dynamical properties of the delay differential equations. In spite of the amount of published results recently focused on such systems, there remain many challenging open questions.

The aim of this special issue is to gather recent research efforts on the development and applications of delay differential equations and to see the latest developments. This special issue contains twenty-five research articles. The original papers explored in this special issue include a wide variety of topics such as the following.

Asymptotic Analysis and Synchronization. Y. Yuan and Z. Guo proposed a fluctuation method to investigate the global asymptotic stability of a very general class of delayed reactiondiffusion equations. L. Zuo and M. Liu investigated the asymptotical stability for an epidemic model with time delay. Z. Zhang et al. investigated a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback and proved the global existence of weak solutions and asymptotic behavior of the energy by using the

Faedo-Galerkin method and the perturbed energy method. J. Zhang et al. used the Lyapunov function method and Lur'e system approach to study the quasisynchronization in a communication system.

Invariant Sets and Attractor. W. Wu used the theory of exponential dichotomy on time scales and fixed point theory based on monotone operator to the global attractivity of the almost periodic solution for a predator-prey system with Beddington-DeAngelis functional response on time scales.

Stability Analysis. P. Wang et al. established a criterion on integral $\varphi_{0}$-stability in terms of two measures for impulsive differential equations with "supremum" by using the conevalued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. C. Liu and Y. Li investigated the global stability of a nonautonomous twospecies competitive system with stage structure and double time delays. L. He and X. Wang proposed a novel method for studying the stability of a macroeconomic system with fractional derivative. H. Peng and Z. Guo proposed a viral infection model with delay and obtained some necessary and sufficient conditions to ensure the global stability of the model.

Bifurcation Analysis. Y. Zhai et al. investigated an avian influenza virus propagation model with nonlinear incidence
rate and delay based on SIR epidemic model and established the global existence of periodic solutions by using a global Hopf bifurcation theory. Y. Dai et al. investigated the local stability of the equilibria and the existence of Hopf bifurcation for a predator-prey system with Michaelis-Menten type functional response and two delays.

Oscillation and Boundary Value Analysis. S. Guo et al. were concerned with oscillation of the first order neutral delay differential equation with constant coefficients and obtained some necessary and sufficient conditions of oscillation for all the solutions in respective cases. Z. Ouyang and H. Liu investigated a class of fractional order three-point boundary value system with resonance. J. Liu and L. Yan used the variational method to investigate the solutions of damped impulsive differential equations with mixed boundary conditions.

Periodic Solutions Analysis. R. Hu used the LyapunovSchmidt reduction method and computations of critical groups to study a higher order difference equation and proved that the equation has four $M$-periodic solutions. G. Lin and Z. Zhou used the critical point theory to obtain a new sufficient condition on the existence of homoclinic solutions of a class of nonperiodic discrete nonlinear systems in infinite lattices. T. Zhang et al. investigated the existence and global attractivity of the "infection-free" periodic solution for a new epidemic disease model governed by system of impulsive delay differential equations.

Numerical Computation Analysis. D. Olvera et al. expanded the application of the enhanced multistage homotopy perturbation method (EMHPM) to solve delay differential equations (DDEs) with constant and variable coefficients.

The response to this special issue was beyond our expectation. We received 46 papers in the interdisciplinary research fields. This special issue includes twenty-five high-quality peer-reviewed papers. These papers contain several new, novel, and innovative techniques and ideas that may stimulate further research in every branch of pure and applied sciences.

## Acknowledgments

The authors would like to express their deepest gratitude to the reviewers, whose professional comments and valuable suggestions guaranteed the high quality of these selected papers. The editors would like to express their gratitude to the authors for their interesting and novel contributions. They would also like to thank the editorial board's members of this journal, for their support and help throughout the preparation of this special issue. The interested readers are advised to explore these interesting and fascinating fields further. The authors hope that problems discussed and investigated in this issue may inspire and motivate discovery of new, innovative, and novel applications in all areas of delay differential equations. Chuangxia Huang would like to express the gratitude to the support of the National Natural Science Foundation of China (nos. 11101053 and 71471020), the Key Project of the Chinese Ministry of Education (no.
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Chuangxia Huang<br>Zhiming Guo<br>Zhichun Yang<br>Yuming Chen<br>Fenghua Wen

# Approximate Solutions of Delay Differential Equations with Constant and Variable Coefficients by the Enhanced Multistage Homotopy Perturbation Method 

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#### Abstract

We expand the application of the enhanced multistage homotopy perturbation method (EMHPM) to solve delay differential equations (DDEs) with constant and variable coefficients. This EMHPM is based on a sequence of subintervals that provide approximate solutions that require less CPU time than those computed from the dde 23 MATLAB numerical integration algorithm solutions. To address the accuracy of our proposed approach, we examine the solutions of several DDEs having constant and variable coefficients, finding predictions with a good match relative to the corresponding numerical integration solutions.


## 1. Introduction

Delayed differential equations (DDEs) are used to describe many physical phenomena of interest in biology, medicine, chemistry, physics, engineering, and economics, among others. Since the introduction of the first delayed models, many publications have appeared as summarizing theorems and homotopy methods of solution that deal with the stability properties of delayed systems (see [1-3] and references cited there in).

For instance, Shakeri and Dehghan introduced an approach to find the solution of delay differential equations by means of the homotopy perturbation technique (HPM) with results that agree well with exact solutions [1]. Wu in [2] used the homotopy analysis method to obtain the approximate solution of a strong nonlinear ENSO delayed oscillator model that provides good agreement when compared to its exact solution under the condition of $B=0$. Alomari and coworkers in [3] developed an algorithm to obtain approximate analytical solutions for DDEs by using the homotopy analysis method (HAM) and the modified homotopy analysis method (MHAM). They used their derived method to obtain
the approximate solution of various linear and nonlinear DDEs with numerical predictions that agree well with the numerical integration solutions, and they also proved that their derived solutions converge to the exact ones. By applying the homotopy perturbation method (HPM), Biazar and Behzad found approximate solutions of neutral differential equations with proportional delays which describe well their corresponding numerical integration solutions [4]. Recently, Anakira and co-workers in [5] extended the applicability of the so called optimal homotopy asymptotic method (OHAM) that does not depend on small or large parameters, to find the approximate analytic solution of DDEs. They used their proposed approach to compare the derived approximate solutions of several DDEs with their exact analytical solutions with predictions that compare well with the exact ones.

On the other hand, Insperger and Stépán in [6] used the semidiscretization method to determine the stability lobes of DDEs that model the dynamics of cutting machine operations. Based on the properties of the Chebyshev polynomials, Butcher and coworkers in [7] developed a methodology to obtain the stability lobes of milling machine operations and
they proved that this technique is faster than that of the full and the semidiscretization methods since these solution techniques approximate the original DDEs by a series of ODEs [8].

Here in this paper, we develop a generalized procedure to solve linear and nonlinear DDEs by introducing some modifications to the multistage homotopy perturbation method (MHPM) derived by Hashim and Chowdhury to obtain approximate solutions of ordinary differential equations [9]. The proposed enhanced multistage homotopy perturbation method (EMHPM) is based on a sequence of subintervals that allow us to find more accurate approximated solutions under a numerical-analytical procedure that requires less CPU time when compared to the numerical integration solutions provided by the MATLAB dde 23 algorithm written by Shampine and Thompson in [10]. The EMHPM is based on a homotopy function that could be divided into a linear operator and a nonlinear operator to satisfy its assumed initial solution. This split of the homotopy function allows us to modify the nonlinear operator to guarantee, by using the enhanced homotopy perturbation method, the stability of the proposed approximate solutions of nonlinear differential equations [11].

To clarify our proposed method, we briefly review in Section 2 some basic concepts of the homotopy perturbation method, and, then in Section 3, we introduce the EMHPM to solve DDEs. The difference between the HPM and the EMHPM is discussed in Section 4 by addressing the approximate solutions of a nonlinear delayed differential equation with variable coefficients. Finally, the general solution of two DDEs that describe the dynamics of two engineering problems, by using the EMHPM, is discussed in Section 5.

## 2. Homotopy Perturbation Method

The homotopy perturbation method (HPM) is a coupling of the traditional perturbation method and homotopy in topology which eliminates the limitation of the small parameter assumed in the perturbation methods [12]. Under this approach, a nonlinear problem can be transformed into an infinite number of simple problems without the restriction of having small nonlinear parameter values. This homotopy perturbation method takes the main advantages of traditional perturbation methods together with homotopy analysis [1315].

To illustrate the basic ideas of the HPM, let us consider the following nonlinear differential equation:

$$
\begin{equation*}
A(u)-f(r)=0, \quad r \in \Omega \tag{1}
\end{equation*}
$$

with boundary conditions

$$
\begin{equation*}
B\left(u, \frac{\partial u}{\partial n}\right)=0, \quad r \in \Gamma \tag{2}
\end{equation*}
$$

where $A$ is a general differential operator, $B$ is a boundary operator, $f(r)$ is a known analytic function, and $\Gamma$ is the boundary of the domain $\Omega$.

The operator $A$ can generally be divided into two parts: $L$ and $N$, where $L$ involves the linear terms and $N$ the nonlinear ones. Equation (1) therefore can be rewritten as follows:

$$
\begin{equation*}
L(u)+N(u)-f(r)=0 . \tag{3}
\end{equation*}
$$

By the homotopy perturbation technique, we construct a homotopy $v(r, p): \Omega \times[0,1] \rightarrow \Re$ that satisfies

$$
\begin{equation*}
H(v, p)=L(v)-L\left(u_{0}\right)+p L\left(u_{0}\right)+p[N(v)-f(r)]=0 \tag{4}
\end{equation*}
$$

where $p \in[0,1]$ is an embedding parameter and $u_{0}$ is an initial approximation of (1) which satisfies the boundary conditions (2). Thus, from (4), we have

$$
\begin{align*}
& H(v, 0)=L(v)-L\left(u_{0}\right)=0 \\
& H(v, 1)=A(v)-f(r)=0 \tag{5}
\end{align*}
$$

The changing process of $p$ from zero to unity is just that of $v(r, p)$ from $u_{0}(r)$ to $u(r)$. In topology, this is called deformation, and $L(v)-L\left(u_{0}\right)$ and $A(v)-f(r)$ are called homotopic.

He in [12] uses the embedding parameter $p$ as the small parameter and assumed that the solution of (4) can be written as a power series of $p$ in the form

$$
\begin{equation*}
v=v_{0}+p v_{1}+p^{2} v_{2}+\cdots \tag{6}
\end{equation*}
$$

By setting $p=1$, He obtained the approximate solution of (1) as

$$
\begin{equation*}
u=\lim _{p \rightarrow 1}=v_{0}+v_{1}+v_{2}+\cdots \tag{7}
\end{equation*}
$$

Then, this method was applied to obtain the approximate solution of some nonlinear ordinary differential equations valid not only for small, but also for large nonlinear parameter values.

We next will introduce an approach based on homotopy methods, to obtain the solution of DDEs with constant and variable coefficients.

## 3. The EMHPM Methodology to Solve DDEs

The HPM is an asymptotic method that depends on the auxiliary linear operator form and the initial guess of the initial conditions. Therefore, the convergence of the approximate solution cannot be guaranteed in some cases [16]. Hashim and Chowdhury showed in [9] that the solutions obtained by the standard HPM were not valid for large time span unless more terms are calculated. Thus, they proposed a multistage homotopy perturbation method (MHPM) which treated the HPM algorithm in a sequence of subintervals in an attempt to improve the accuracy of the approximate solutions of linear and nonlinear ordinary differential equations (ODEs).

However, when the MHPM is applied to obtain the approximate solutions of ODEs which contain coefficients as a function of time, this method cannot provide accurate solutions when $\Delta t \rightarrow 0$. In this work, we introduce some
modifications to the MHPM and focus on the derivation of approximate solutions of DDEs equations with variable coefficient terms. This new approach is based on the enhanced multistage homotopy perturbation method (EMHPM) introduced in [17] to obtain the solution of nonlinear ordinary differential equations.

The EMHPM is an algorithm which approximates the HPM solution by subintervals, utilizing the following transformation rule: $u(t) \rightarrow u_{i}(T)$, where $u_{i}$ satisfies the initial condition $u_{i}(0)=u_{i-1}\left(t_{i-1}\right), T$ is a shifted time scale used to determine the approximate solution in each subinterval, and $u_{i}(T)$ represents the approximate solution in the $i$ th subinterval. In this case, the initial suggested solution in the $i$ th subinterval is given by $u_{i 0}(T)=u_{i-1}\left(t_{i-1}\right)$, where $t_{i-1}$ represents the time at the end of the previous subinterval (i.e., the value of the approximate solution at the end of the previous subinterval represents the initial conditions of the next subinterval under consideration).

To apply the homotopy technique to solve delay differential equations, we also assume the following.
(1) The linear operator $L\left(u_{i}\right)$ represents $L\left(u_{i}\right)=(d / d T) u_{i}$, where the assumed approximate solution $u_{i 0}(T)$ is set equal to the initial condition $u_{i-1}\left(t_{i-1}\right)$; that is, $u_{i 0}=u_{i-1}\left(t_{i-1}\right)$. To simplify the notation, we let $u_{i-1} \equiv$ $u_{i-1}\left(t_{i-1}\right)$.
(2) The transformation $T=t-t_{i-1}$ on $0<T \leq t_{i}-t_{i-1}$ holds in the homotopy $i$-subinterval. Thus, higher order equations are integrated with respect to $T$, while the terms related to the independent variable $t$ are assumed to remain constant.

Therefore, we may conclude that the $m$ order approximate solution, by applying the EMHPM, can be written as

$$
\begin{equation*}
u_{i}\left(T, u_{i-1}\right)=\sum_{k=0}^{m} U_{i k}\left(T, u_{i-1}\right) \tag{8}
\end{equation*}
$$

where the solution $u_{i}\left(T, u_{i-1}\right)$ is valid only in the $i$ th subinterval $\left[t_{i-1}, t_{i}\right]$. Hence, the solution $u(t)$ on the $i$ th subinterval ( $t_{i-1}, t_{i}$ ] can be written as

$$
\begin{equation*}
u(t) \approx u_{i}\left(t-t_{i-1}\right) \tag{9}
\end{equation*}
$$

with initial condition $u_{i-1}\left(t_{i-1}\right)$, and $i=1,2, \ldots, j$. Thus, the approximate solution of $u$ at the time $t_{i}$ is given by

$$
\begin{equation*}
\left.u_{i}\left(t-t_{i-1}\right)\right|_{t=t_{i}}=\left.u_{i+1}\left(t-t_{i}\right)\right|_{t=t_{i}}=u_{i+1}(0)=u_{i} \tag{10}
\end{equation*}
$$

In summary, the solution $u(t)$ for an open-closed interval $\left(t_{0}, t_{1}\right]$ is divided into $j$ subintervals that, in general, are not equally spaced: $\left[t_{0}, t_{1}\right],\left[t_{1}, t_{2}\right], \ldots,\left[t_{j-1}, t_{j}\right]$. Thus, the approximated solution of $u(t)$ for the span time interval is obtained by coupling the $u_{i}(t)$ solutions.

## 4. Approximate Solutions of Some DDEs by Applying the EMHPM

In this section we focus on the solution of DDEs with constant and variable coefficients and examine the applicability of the EMHPM to find the corresponding approximate solutions.
4.1. Delay Differential Equations with Constant Coefficients. First, let us consider the simplest DDE of the form

$$
\begin{equation*}
\dot{x}(t)+x(t-\tau)=0 \tag{11}
\end{equation*}
$$

with initial condition $x(0)=c$. Here, the independent variable $x$ is a scalar $x(t) \in \mathfrak{R}$, the dot stands for differentiation with respect to time $t$, and $\tau$ is the time delay. To evaluate (11) on $a \leq t \leq b$, the term $x(t-\tau)$ must represent a known function $x(t)$ on [ $a-\tau \leq t \leq a$ ]. For instance, if $a=0$, the solution of (11) can be obtained in the interval $(0, \tau]$ by assuming an initial function that satisfies the initial condition. By using this solution, it becomes possible to obtain the solution of (11) in the next $i$ th interval $[(i-1) \tau, i \tau], i=$ $2,3, \ldots, j$, where $j$ is an integer number that can be chosen as $2 \leq j \leq \infty$. With this approach, we can apply the HPM to find the solution of (11) by assuming that the previous delayed function is $x^{\tau_{0}}(T)=c$; thus the solution for the first interval is given by $x^{\tau_{1}}(T)$, valid on $[0, \tau]$. In terms of (4), we now construct the homotopy of (11):

$$
\begin{equation*}
H\left(X^{\tau_{1}}, p\right)=\frac{d}{d T} X^{\tau_{1}}+p x^{\tau_{0}}=0 \tag{12}
\end{equation*}
$$

We next substitute the first order expansion $X^{\tau_{1}}=X_{0}^{\tau_{1}}+p X_{1}^{\tau_{1}}$ in (12) and balance the terms with identical power of $p$ to obtain the following set of linear differential equations:

$$
\begin{align*}
& p^{0}: \frac{d}{d T} X_{0}^{\tau_{1}}=0 \quad X_{0}^{\tau_{1}}(0)=c=X^{\tau_{0}}(\tau) \\
& p^{1}: \frac{d}{d T} X_{1}^{\tau_{1}}=-X^{\tau_{0}} \quad X_{1}^{\tau_{1}}(0)=0 \tag{13}
\end{align*}
$$

Integration of (13) yields

$$
\begin{align*}
& X_{0}^{\tau_{1}}=c \\
& X_{1}^{\tau_{1}}=-c T \tag{14}
\end{align*}
$$

Hence, the first order solution of (12) is given by

$$
\begin{equation*}
x^{\tau_{1}}(T)=c-c T \tag{15}
\end{equation*}
$$

Notice that (15) represents the exact solution of (11) on the first interval. By following the same procedure, it is easy to show that the exact solution of (11), for the second and third intervals, is given, respectively, as

$$
\begin{align*}
& x^{\tau_{2}}(T)=c-c \tau-c T+\frac{1}{2} c T^{2},  \tag{16}\\
& x^{\tau_{3}}(T)=c-2 c \tau+\frac{1}{2} c \tau^{2}-(c-c \tau) T+\frac{1}{2} c T^{2}-\frac{1}{6} c T^{3} .
\end{align*}
$$

Figure 1 shows the exact solution of (11) obtained by coupling at each interval the solution obtained by following HPM procedure for $t=10 \tau$.

It is easy to show that the solution of (11) by the EMHPM coincides with the solution obtained by using the HPM since (11) is a delay differential equation with constant coefficients.


Figure 1: Exact solution of (11) obtained by using the HPM and $\tau=1$.
4.2. Delay Differential Equations with Variable Coefficients. We next show how the EMHPM approach can be applied to obtain the approximate solution of nonlinear delay differential equation with variable coefficients. In this case, we obtain the approximate solutions of a DDE of the form
$\dot{x}+x(t-\tau)-\cos (\pi t) x^{2}=0, \quad \tau=1, \quad x(0)=c=x^{\tau_{0}}(\tau)$
in which the solution $x^{\tau_{0}}(T)=c_{1}$ holds on $(-\tau, 0]$. In order to find the solution $x^{\tau_{1}}$ in the interval $[0, \tau]$, we assume that the homotopy representation of (17) can be given as

$$
\begin{equation*}
H\left(X^{\tau_{1}}, p\right)=\frac{d}{d T} X^{\tau_{1}}+p\left[x^{\tau_{0}}-\cos (\pi t)\left(X^{\tau_{1}}\right)^{2}\right]=0 \tag{18}
\end{equation*}
$$

Notice that the variable $X$ depends on the time $T$ for which $0 \leq T \leq \tau$. If we now substitute the second order expansion $X^{\tau_{1}}=X_{0}^{\tau_{1}}+p X_{1}^{\tau_{1}}+p^{2} X_{2}^{\tau_{1}}$ in (18), and, after balancing the $p$ terms, we get that

$$
\begin{align*}
& p^{0}: \frac{d}{d T} X_{0}^{\tau_{1}}=0, \quad X_{0}(T=0)=c_{1}=X^{\tau_{0}}(T=\tau) \\
& p^{1}: \frac{d}{d T} X_{1}^{\tau_{1}}=-x^{\tau_{0}}+\cos (\pi t)\left(X_{0}^{\tau_{1}}\right)^{2}=0, \quad X_{1}(0)=0 \\
& p^{2}: \frac{d}{d T} X_{2}^{\tau_{1}}=2 \cos (\pi t) X_{0}^{\tau_{1}} X_{1}^{\tau_{1}}, \quad X_{2}(0)=0 \\
& p^{3}: \frac{d}{d T} X_{3}^{\tau_{1}}=\cos (\pi t)\left(2 X_{0}^{\tau_{1}} X_{2}^{\tau_{1}}+\left(X_{1}^{\tau_{1}}\right)^{2}\right)=0, \quad X_{3}(0)=0 \tag{19}
\end{align*}
$$

Equations (19) have the following solutions:

$$
\begin{align*}
& X_{0}^{\tau_{1}}=c_{1} \\
& X_{1}^{\tau_{1}}=-T\left(x^{\tau_{0}}-c_{1}^{2} \cos \pi t\right), \\
& X_{2}^{\tau_{1}}=-c_{1} T^{2}(\cos \pi t)\left(x^{\tau_{0}}-c_{1}^{2} \cos \pi t\right),  \tag{20}\\
& X_{3}^{\tau_{1}}=\frac{1}{3} T^{3}(\cos \pi t)\left[3 c_{1}^{4} \cos ^{2} \pi t-4 c_{1}^{2} x^{\tau_{0}}+\left(x^{\tau_{0}}\right)^{2}\right] .
\end{align*}
$$

Thus, the approximate solution of (17) by using the EMHPM is given by

$$
\begin{equation*}
x^{\tau_{1}}(T) \approx X_{0}^{\tau_{1}}+X_{1}^{\tau_{1}}+X_{2}^{\tau_{1}}+X_{3}^{\tau_{1}} . \tag{21}
\end{equation*}
$$

In this case, the exact solution of $x^{\tau_{1}}(T)$ is unknown. To obtain $x^{\tau_{2}}$, we compute again the approximate solution of $x^{\tau_{1}}(T)$ by applying our EMHPM and the value of the delayed time is assumed to remain constant in each subinterval. To determine $x^{\tau_{2}}$, we next use the homotopy representation of (17) for the interval $(\tau, 2 \tau]$ :

$$
\begin{equation*}
H\left(X^{\tau_{2}}, p\right)=\frac{d}{d T} X^{\tau_{2}}+p\left[x^{\tau_{1}}-\cos (\pi t)\left(X^{\tau_{2}}\right)^{2}\right]=0 \tag{22}
\end{equation*}
$$

Substituting the second order expansion in (22), we get

$$
\begin{align*}
& X_{0}^{\tau_{2}}=c_{2} \\
& X_{1}^{\tau_{2}}=-T\left(x^{\tau_{1}}-c_{2}^{2} \cos \pi t\right) \\
& X_{2}^{\tau_{2}}=-c_{2} T^{2}(\cos \pi t)\left(x^{\tau_{1}}-c_{2}^{2} \cos \pi t\right)  \tag{23}\\
& X_{3}^{\tau_{2}}=\frac{1}{3} T^{3}(\cos \pi t)\left(3 c_{2}^{4} \cos ^{2} \pi t-4 c_{2}^{2} x^{\tau_{1}}+\left(x^{\tau_{1}}\right)^{2}\right) .
\end{align*}
$$

Note that (20) and (23) provide approximate solutions to (17) but evaluated at different interval time delays. To find the third order approximate solution of (17), we can use a homotopy of the form:

$$
\begin{equation*}
H\left(X^{\tau_{i}}, p\right)=\frac{d}{d T} X^{\tau_{i}}+p\left[X^{\tau_{i-1}}-\cos (\pi t)\left(X^{\tau_{i}}\right)^{2}\right]=0 \tag{24}
\end{equation*}
$$

Then, by using our EMPHM approach, we have that

$$
\begin{align*}
& X_{0}^{\tau_{i}}=c \\
& X_{1}^{\tau_{i}}=-T\left(x^{\tau_{i-1}}-c^{2} \cos \pi t\right) \\
& X_{2}^{\tau_{i}}=-c T^{2}(\cos \pi t)\left(x^{\tau_{i-1}}-c^{2} \cos \pi t\right)  \tag{25}\\
& X_{3}^{\tau_{i}}=\frac{1}{3} T^{3}(\cos \pi t)\left(3 c^{4} \cos ^{2} \pi t-4 c^{2} x^{\tau_{i-1}}+\left(x^{\tau_{i-1}}\right)^{2}\right) .
\end{align*}
$$

Notice from (25) that the $k$ th order approximate solution of (17) can be written as

$$
\begin{align*}
& X_{0}^{\tau_{i}}=c \\
& X_{k}^{\tau_{i}}=\frac{T}{k}\left(-x^{\tau_{i-1}} g(k)+\cos \pi t \sum_{n_{1}=0}^{k-1} X_{n_{1}}^{\tau_{i}} X_{k-1-n_{1}}^{\tau_{i}}\right), \tag{26}
\end{align*}
$$

where $k>0, g(k)=1$ when $k=1$ and zero otherwise.
Figure 2 shows the approximate solution of (17) obtained by using the EMHPM approach compared to its numerical integration solution by using the dde 23 MATLAB subroutine program. This case assumes two different initial solutions of the form $x^{\tau_{0}}(T)=\cos (\pi(T+1)), x^{\tau_{0}}(T)=e^{T+1}$, and a time subintervals $\Delta t=0.01$. We can see from Figure 2 that both simulations agree well for the time span showed.

To further assess the applicability of our proposed EMHPM approach to high order delay differential equations, we will next describe a methodology to obtain the approximate solutions of well-known high order delay differential equations by generalizing our EMHPM approach.


Figure 2: EMHPM and dde23 solution of (17).


Figure 3: Schematic of the zeroth order polynomial used to fit the approximate EMHPM solution.

## 5. Generalized Solution of Linear DDEs by the EMHPM Approach

Let us consider an $n$-dimensional delay differential equation of the form

$$
\begin{equation*}
\dot{\mathbf{x}}(t)=\mathbf{A}(t) \mathbf{x}+\mathbf{B}(t) \mathbf{x}(t-\tau) \tag{27}
\end{equation*}
$$

where $\mathbf{A}(t+\tau)=\mathbf{A}(t), \mathbf{B}(t+\tau)=\mathbf{B}(t), \mathbf{x}(t)$ is the state vector, and $\tau$ is the time delay. By following our EMHPM procedure, we can write (27) in equivalent form as

$$
\begin{equation*}
\dot{\mathbf{x}}_{i}(T)-\mathbf{A}_{t} \mathbf{x}_{i}(T) \approx \mathbf{B}_{t} \mathbf{x}_{i}^{\tau}(T) \tag{28}
\end{equation*}
$$

where $\mathbf{x}_{i}(T)$ denotes the $m$ order solution of (27) in the $i$ th subinterval that satisfies the initial conditions $\mathbf{x}_{i}(0)=\mathbf{x}_{i-1}$ and $\mathbf{A}_{t}$ and $\mathbf{B}_{t}$ represent the values of the periodic coefficients at the time $t$. In order to approximate the delayed term $\mathbf{x}_{i}^{\tau}(T)$ in (28), the period [ $t_{0}-\tau, t_{0}$ ] is discretized in $N$ points equally spaced as shown in Figure 3. Here, we assume that the function $\mathbf{x}_{i}^{\tau}(T)$ in the delay subinterval $\left[t_{i-N}, t_{i-N+1}\right]$ is approximated by a constant value

$$
\begin{equation*}
\mathbf{x}_{i}^{\tau}(T)=x_{i-N+1}(T) \approx \mathbf{x}_{i-N} \tag{29}
\end{equation*}
$$

as shown in Figure 3. By following the homotopy perturbation technique, we can write the homotopy representation of (28) as

$$
\begin{equation*}
H\left(\mathbf{X}_{i}, p\right)=L\left(\mathbf{X}_{i}\right)-L\left(\mathbf{x}_{i 0}\right)+p L\left(\mathbf{x}_{i 0}\right)=p\left(\mathbf{A}_{t} \mathbf{X}_{i}+\mathbf{B}_{t} \mathbf{x}_{i-N}\right) \tag{30}
\end{equation*}
$$

Substituting the $m$ order expansion $\mathbf{X}_{i}=\mathbf{X}_{i 0}+p \mathbf{X}_{i 1}+\cdots+$ $p^{m} \mathbf{X}_{i m}$ in (30) and by assuming an initial approximation of the form $\mathbf{x}_{i 0}=\mathbf{x}_{i-1}$, we get, after applying the proposed

EMHPM approach, the following set of first order linear delay differential equations:

$$
\begin{align*}
& p^{0}: \frac{d}{d T} \mathbf{X}_{i 0}+\frac{d}{d T} \mathbf{x}_{i-1}=0, \quad \mathbf{X}_{i}(0)=\mathbf{x}_{i-1}, \\
& p^{1}: \frac{d}{d T} \mathbf{X}_{i 1}=\mathbf{A}_{t} \mathbf{X}_{i 0}+\mathbf{B}_{t} \mathbf{x}_{i-N}, \quad \mathbf{X}_{i 1}(0)=0, \\
& p^{2}: \frac{d}{d T} \mathbf{X}_{i 2}=\mathbf{A}_{t} \mathbf{X}_{i 1}, \quad \mathbf{X}_{i 2}(0)=0, \tag{31}
\end{align*}
$$

$$
p^{m}: \frac{d}{d T} \mathbf{X}_{i m}=\mathbf{A}_{t} \mathbf{X}_{i(m-1)}, \quad \mathbf{X}_{i m}(0)=0
$$

By solving (31), we get

$$
\begin{align*}
& \mathbf{X}_{i 0}=\mathbf{x}_{i-1}, \\
& \mathbf{X}_{i 1}=\mathbf{A}_{t} \mathbf{x}_{i-1} T+\mathbf{B}_{t} \mathbf{x}_{i-N} T \\
& \mathbf{X}_{i 2}=\frac{1}{2} \mathbf{A}_{t}^{2} \mathbf{x}_{i-1} T^{2}+\frac{1}{2} \mathbf{A}_{t} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{2}, \tag{32}
\end{align*}
$$

$$
\mathbf{X}_{i m}=\frac{1}{m!} \mathbf{A}_{t}^{m} \mathbf{x}_{i-1} T^{m}+\frac{1}{m!} \mathbf{A}_{t}^{m-1} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{m}
$$

Equations (32) can be written as

$$
\begin{equation*}
\mathbf{X}_{i k}=\frac{T}{k}\left(\mathbf{A}_{t} \mathbf{X}_{i(k-1)}+g(k) \mathbf{B}_{t} \mathbf{x}_{i-N}\right), \quad k=1,2,3, \ldots, \tag{33}
\end{equation*}
$$

where $\mathbf{X}_{i 0}=\mathbf{x}_{i-1}$ and $g(k)=1$ for $k=1$ and $g(k)=0$, otherwise. Thus, the solution of (27) is obtained by adding the $\mathbf{X}_{i k}$ approximate solutions:

$$
\begin{equation*}
\mathbf{x}_{i}(T) \approx \sum_{k=0}^{m} \mathbf{X}_{i k}(T) \tag{34}
\end{equation*}
$$

Notice, however, that solution (34) may be further improved by using a first order polynomial representation of $\mathbf{x}_{i}^{\tau}(T)$ as shown in Figure 4. Then, the function $\mathbf{x}_{i}^{\tau}(T)$ in the delay subinterval $\left[t_{i-N}, t_{i-N+1}\right]$ takes the form

$$
\begin{equation*}
\mathbf{x}_{i}^{\tau}(T)=\mathbf{x}_{i-N+1}(T) \approx \mathbf{x}_{i-N}+\frac{(N-1)}{\tau}\left(\mathbf{x}_{i-N+1}-\mathbf{x}_{i-N}\right) T . \tag{35}
\end{equation*}
$$

Substituting (35) into (28) gives

$$
\begin{align*}
\dot{\mathbf{x}}_{i}(T) & -\mathbf{A}_{t} \mathbf{x}_{i}(T) \\
& \approx \mathbf{B}_{t} \mathbf{x}_{i-N}-\frac{(N-1)}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N} T+\frac{(N-1)}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N+1} T . \tag{36}
\end{align*}
$$

We next assume that the homotopy representation of (36) is given as

$$
\begin{align*}
H\left(\mathbf{X}_{i}, p\right)= & L\left(\mathbf{X}_{i}\right)-L\left(\mathbf{x}_{i 0}\right)+p L\left(\mathbf{x}_{i 0}\right) \\
& -p\left(\mathbf{A} \mathbf{X}_{i}+\mathbf{B} \mathbf{x}_{i-N}-\frac{(N-1)}{\tau} \mathbf{B} \mathbf{x}_{i-N} T\right.  \tag{37}\\
& \left.+\frac{(N-1)}{\tau} \mathbf{B} \mathbf{x}_{i-N+1} T\right)=0
\end{align*}
$$

Substituting the $m$ order expansion $\mathbf{X}_{i}(T)=\mathbf{X}_{i 0}(T)+$ $p \mathbf{X}_{i 1}(T)+\cdots p^{m} \mathbf{X}_{i m}(T)$ in (37) and assuming that the initial approximation is given by $\mathbf{x}_{i 0}=\mathbf{x}_{i-1}$, we get

$$
\begin{align*}
& \begin{aligned}
& p^{0}: \frac{d}{d T} \mathbf{X}_{i 0}+\frac{d}{d T} \mathbf{x}_{i-1}=0, \quad \mathbf{X}_{i}(0)=\mathbf{x}_{i-1}, \\
& p^{1}: \frac{d}{d T} \mathbf{X}_{i 1}= \mathbf{A}_{t} \mathbf{X}_{i 0}+\mathbf{B}_{t} \mathbf{x}_{i-N}-\frac{N-1}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N} T \\
&+\frac{N-1}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N+1} T, \quad \mathbf{X}_{i_{1}}(0)=0, \\
& p^{2}: \frac{d}{d T} \mathbf{X}_{i 2}= \mathbf{A} \mathbf{X}_{i 1}, \quad \mathbf{X}_{i 2}(0)=0, \\
& \vdots
\end{aligned} \\
& p^{m}: \frac{d}{d T} \mathbf{X}_{i m}=\mathbf{A} \mathbf{X}_{i(m-1)}, \quad \mathbf{X}_{i m}(0)=0 .
\end{align*}
$$

By solving (38) and by following the EMHPM procedure, we get

$$
\begin{align*}
\mathbf{X}_{i 0}= & \mathbf{x}_{i-1}, \\
\mathbf{X}_{i 1}= & \mathbf{A}_{t} \mathbf{x}_{i-1} T+\mathbf{B}_{t} \mathbf{x}_{i-N} T-\frac{1}{2} \frac{N-1}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{2} \\
& +\frac{1}{2} \frac{N-1}{\tau} \mathbf{B}_{t} \mathbf{x}_{i-N+1} T^{2}, \\
\mathbf{X}_{i 2}= & \frac{1}{2} \mathbf{A}_{t}^{2} \mathbf{x}_{i-1} T^{2}+\frac{1}{2} \mathbf{A}_{t} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{2} \\
& -\frac{1}{6} \frac{N-1}{\tau} \mathbf{A}_{t} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{3}+\frac{1}{6} \frac{N-1}{\tau} \mathbf{A}_{t} \mathbf{B}_{t} \mathbf{x}_{i-N+1} T^{3}, \\
& \vdots \\
\mathbf{X}_{i m}= & \frac{1}{m!} \mathbf{A}_{t}^{m} \mathbf{x}_{i-1} T^{m}+\frac{1}{m!} \mathbf{A}_{t}^{m-1} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{m} \\
& -\frac{1}{(m+1)!} \frac{N-1}{\tau} \mathbf{A}_{t}^{m-1} \mathbf{B}_{t} \mathbf{x}_{i-N} T^{m+1}  \tag{39}\\
& +\frac{1}{(m+1)!} \frac{N-1}{\tau} \mathbf{A}_{t}^{m-1} \mathbf{B}_{t} \mathbf{x}_{i-N+1} T^{m+1} .
\end{align*}
$$

Here, the recursive form of $\mathbf{X}_{i_{k}}(T)$ is written as

$$
\begin{equation*}
\mathbf{X}_{i k}=\mathbf{X}_{i k}^{\mathbf{a}}+\mathbf{X}_{i k}^{\mathbf{b}} \quad k=1,2,3, \ldots \tag{40}
\end{equation*}
$$



Figure 4: Schematic EMHPM solution using first polynomial to approximate delay subinterval.
where $\mathbf{X}_{i 0}^{\mathbf{a}}=\mathbf{x}_{i-1}, \mathbf{X}_{i 0}^{\mathbf{b}}=\mathbf{0}$ and

$$
\begin{align*}
\mathbf{X}_{i k}^{\mathbf{a}}= & \frac{T}{k}\left(\mathbf{A}_{t} \mathbf{X}_{i(k-1)}^{\mathbf{a}}+g(k) \mathbf{B}_{t} \mathbf{x}_{i-N}\right), \\
\mathbf{X}_{i k}^{\mathbf{b}}= & \frac{T}{k+1}\left(\mathbf{A}_{t} \mathbf{X}_{i(k-1)}^{\mathbf{b}}\right. \\
& \left.\quad+g(k)\left[\frac{N-1}{\tau} T\left(-\mathbf{B}_{t} \mathbf{x}_{i-N}+\mathbf{B}_{t} \mathbf{x}_{i-N+1}\right)\right]\right) . \tag{41}
\end{align*}
$$

Thus, the approximate solution of (27) by the EMHPM can be obtained by substituting (40) into (34).

In the next section, we will apply our EMHPM procedure to obtain the solution of two second order delay differential equations: (a) the damped Mathieu equation with time delay, and (b) the well-known delay differential equation that describes the dynamics in one degree-of-freedom milling machine operations.
5.1. Solution of the Damped Mathieu Equation with Time Delay. In order to assess the accuracy of our EMHPM approach, we first obtain the solution of the damped Mathieu differential equation with time delay that combines the effect of parametric excitation and damping. This equation is described by the following equation:

$$
\begin{equation*}
\ddot{x}+\kappa \dot{x}+\left(\delta+\varepsilon \cos \left(\frac{2 \pi t}{T}\right)\right) x=b x(t-\tau) \tag{42}
\end{equation*}
$$

where $\kappa, \delta, \varepsilon, \tau$, and $T$ are system parameters whose value depends on the physics of the system. The approximate solution of (42) obtained by using the semidiscretization method is widely discussed in [18, 19]. Here, we focus our attention on applying the EMHPM to find the approximate solution of (42) and we also assess the accuracy of the derived solution by comparing it with the corresponding numerical integration solution of (42).

By following the EMHPM procedure, we first write (42) in the following equivalent form:

$$
\begin{equation*}
\ddot{x}_{i}(T)+\kappa \dot{x}_{i}(T)+\alpha_{t} x_{i}(T) \approx b x_{i-N+1}(T), \tag{43}
\end{equation*}
$$

where $x_{i}(t)$ denotes the $m$ order solution of (43) in the $i$ th subinterval that satisfies the following initial conditions: $x_{i}(0)=x_{i-1}$ and $\dot{x}_{i}(0)=\dot{x}_{i-1}$. The space state form representation of (43) is given by

$$
\begin{equation*}
\dot{\mathbf{x}}_{i}(T)=\mathbf{A}_{t} \mathbf{x}_{i}(T)+\mathbf{B}_{t} \mathbf{x}_{i-N+1}(T), \tag{44}
\end{equation*}
$$



Figure 5: Numerical solutions of the damped Mathieu equation with time delay by dde23, the zeroth EMHPM, and the first EMHPM with $N=50$ and $m=4$.

Table 1: Computer time needed to solve the damped Mathieu equation with time delay. The $m$ order solution of the EMPHM approach is chosen to guarantee the convergence of its approximate solution.

| dde23 [ms] |  | EMHPM |  |  |
| :--- | :---: | :---: | :---: | :---: |
|  | $N$ | $m$ | Zeroth [ms] | First [ms] |
| 19 | 15 | 5 | 4 | 5 |
|  | 20 | 4 | 6 | 6 |
|  | 40 | 3 | 12 | 12 |
|  | 60 | 2 | 17 | 18 |
|  | 60 | 5 | 18 | 19 |
|  | 60 | 10 | 20 | 21 |

where

$$
\mathbf{A}_{t}=\left[\begin{array}{cc}
0 & 1  \tag{45}\\
-\alpha_{t} & -\kappa
\end{array}\right], \quad \mathbf{B}_{t}=\left[\begin{array}{cc}
0 & 0 \\
b_{t} & 0
\end{array}\right]
$$

and $\alpha_{t}=(\delta+\varepsilon \cos (t))$ is a time periodic term. The EMHPM approximate solution of (44) is illustrated in Figure 5 where we have assumed an unstable system behavior for which $\kappa=$ $0.2, \delta=3.0, \varepsilon=1, b=-1$, and $T=\tau=2 \pi$. See [20]. As we can see from Figure 5, our approximate EMHPM solution to (42) is compared with its numerical integration solution obtained from dde 23 MATLAB algorithm for the time interval of $2 T$, by assuming that $N=50$ with the following initial values: $x_{-50}(T)=x_{-49}(T)=\cdots x_{0}(T)=$ 0.001 and $\dot{x}_{-50}(T)=\dot{x}_{-49}(T)=\cdots \dot{x}_{0}(T)=0$.

It can be seen from Figure 5 that in the interval [ $0, T$ ] both the zeroth and the first order solutions are the same since the delay subintervals are constant. See Figure 3. However, in the next interval $[T, 2 T]$ it is clear that the first order EMPHM solution provides a better approximation on the delay subinterval. The computation total time to calculate the solutions in the MATLAB code is listed in Table 1. The order $m$ and the discretized time intervals $N$ in the EMPHM approach are chosen to guarantee the convergence of our approximate solution to the exact one. To provide a full understanding of how the solution is computed by the EMHPM approach, we attached in Algorithm 1 the corresponding MATLAB code.

$$
\begin{aligned}
& \text { - Zeroth order } m=2 \quad \text { First order } m=5 \\
& \text { - First order } m=2 \quad+\quad \text { Zeroth order } m=10 \\
& \text {-- Zeroth order } m=5 \quad \times \quad \text { First order } m=10
\end{aligned}
$$

Figure 6: Estimated relative error values between the numerical solution dde23 and the EMHPM approximate solutions. Here we use for the EMHPM the values of $m=2,5$, and 10 .

Figure 6 shows the relative error between our approximate EMPHM and the dde 23 solution and its relationship with the order $m$ and the discretized time intervals $N$. Notice that the relative error values coincide at values of $N \geq 45$. Also, we can see from Figure 6 that the computed relative error values for approximate solutions of order $m \geq 5$ remain unchanged.

### 5.2. A Practical Application: Cutting Operation on Milling

 Machine. We next use our EMHPM procedure to obtain the solution of the single degree-of-freedom milling operation. We use the simplified form based on [20-22]:$$
\begin{equation*}
\ddot{x}(t)+2 \zeta \omega_{n} \dot{x}(t)+\omega_{n}^{2} x(t)=-\frac{a_{p} K_{s}(t)}{m_{m}}(x(t)-x(t-\tau)), \tag{46}
\end{equation*}
$$

where $\omega_{n}$ is the angular natural frequency of the system, $\zeta$ is the damping ratio, $a_{p}$ is the depth of cut, $m_{m}$ is the modal mass of the tool, $\tau$ represents the time delay which is equal to the tooth passing period, and $K_{s}(t)$ is the specific cutting force coefficient which can be determined from

$$
\begin{align*}
& K_{s}(t) \\
& \qquad=\sum_{j=1}^{z_{n}} g\left(\phi_{j}(t)\right) \sin \left(\phi_{j}(t)\right)\left(K_{t} \cos \phi_{j}(t)+K_{n} \sin \phi_{j}(t)\right), \tag{47}
\end{align*}
$$

where $z_{n}$ is the tool number of teeth, $K_{t}$ and $K_{n}$ are the tangential and the normal linear cutting force coefficients, respectively, $\phi_{j}(t)$ is the angular position of the $j$-tooth defined as

$$
\begin{equation*}
\phi_{j}(t)=\left(\frac{2 \pi n}{60}\right) t+\frac{2 \pi j}{z_{n}} \tag{48}
\end{equation*}
$$

and $n$ is the spindle speed in $\operatorname{rpm}$ [20]. The function $g\left(\varphi_{j}(t)\right)$ is a switching function, which has a unity value when the $j$ tooth is cutting and zero otherwise:

$$
g\left(\phi_{j}(t)\right)= \begin{cases}1 & \phi_{\mathrm{st}}<\phi_{j}(t)<\phi_{\mathrm{ex}}  \tag{49}\\ 0 & \text { otherwise }\end{cases}
$$

```
function dde23_ddeEM_mathieu_paper
disp('Mathieu equation solution')
% Solution by EMHPM with zeroth and first order solution
eps=1; kappa=0.2; T=2*pi; tau=T; b=-1; delta=3; % Mathieu Parameters
pntDelay=1; N=50; m_ord=5; dt=tau/(N-1); ktau=2; tspan=[0,ktau*T]; % EMHPM Parameters
%% Solution
% Solution by dde23
tdde=linspace(0,ktau*tau,ktau*(N-1)+1);
dde=@(t,y,z) mathieu_dde(t,y,z,kappa,delta,eps,b,tau);
te_dde=tic;sol = dde23(dde,tau,@history,tspan); toc(te_dde); xdde=deval(sol,tdde);
% Solution by zeroth EMHPM
dde_emhpm_fun=@(t,c0,tt,zm,zn,tau,N) mathieu_zeroth(t,c0,tt,zm,zn,tau,N,m_ord,
kappa,delta,eps,b,T);
tic;[t0,z0]=ddeEMHPM(dde_emhpm_fun,tspan,@history,tau,dt,pntDelay); toc
% Solution by First EMHPM
dde_emhpm_fun=@(t,c0,tt,zm,zn,tau,N) mathieu_first(t,c0,tt,zm,zn,tau,N,m_ord,kappa,delta,eps,b,T);
tic;[t1,z1]=ddeEMHPM(dde_emhpm_fun,tspan,@history,tau,dt,pntDelay); toc
% Plot results
ind0=1:2:ktau*(N-1); ind1=2:2:ktau*(N-1);
Parent1=figure (1);
axes1 = axes('Parent',Parent1,'FontSize',12,'FontName','Times New Roman');
box(axes1,'on'); hold(axes1,'all');
plot(tdde,xdde(1,:),'Parent',axes1,'LineWidth',2,'Color',[0.502 0.502 0.502],'DisplayName','Numerical
dd23');
plot(t0(ind0),z0(ind0,1),'MarkerSize',5,'Marker','o','LineStyle', 'none','DisplayName', 'Zeroth
EMHPM','Color',[0 0 0]);
plot(t1(ind1),z1(ind1,1),'MarkerSize',7,'Marker','x','LineStyle','none','DisplayName','First
EMHPM','Color',[0 0 0]);
xlabel('\itt','FontSize',12,'FontName','Times New Roman');
ylabel('\itx','FontSize',12,'FontName','Times New Roman');
end
%% Mathieu definitions
function dydt = mathieu_dde(t,y,z,kapa,dlt,eps,b,T)
dydt = [y(2)
    -kapa*y(2)-(dlt+eps*cos(2*pi*t/T))*y(1)+b*z(1)];
end
function Z = mathieu_zeroth(t,c0,tt,zm,zn,tau,N,m,kpa,dlt,eps,b,T)
Z=[cO(1),c0(2)]; alf=dlt+eps*cos(2*pi/T*tt);
for ik=1:m
    Z(ik+1,1)=Z(ik,2)*t/ik;
    Z(ik+1,2)=-kpa*Z(ik,2)-alf*Z(ik,1);
    if ik==1, Z(ik+1,2)=Z(ik+1,2)+b*Zm(1); end
    Z(ik+1,2)=Z(ik+1,2)*t/ik;
end
Z=sum(Z);
end
function Z = mathieu_first(t,c0,tt,zm,zn,tau,N,m,kpa,dlt,eps,b,T)
alf=dlt+eps*cos(2*pi/T*tt); Z=[c0(1),c0(2)]; Z_=[0,0];
for ik=1:m
    Z(ik+1,:)=[Z(ik,2)*t/ik, -kpa*Z(ik,2)-alf*Z(ik,1)];
    Z_(ik+1,:)=[Z_(ik,2)*t/ik, -kpa*Z_(ik,2)-alf*Z_(ik,1)];
    if ik==1,
        Z(ik+1,2)=Z(ik+1,2)+b*zm(1);
        Z_(ik+1,2)=Z_(ik+1,2)+b*(N-1)/tau*(zn(1)-zm(1))*t;
    end
    Z(ik+1,2)=Z(ik+1,2)*t/ik;
    Z_(ik+1,2)=Z_(ik+1,2)*t/(ik+1);
```

```
end
Z=sum(Z)+sum(Z_);
end
function out=history(t)
out=[1E-3+0*t 0+0*t];
end
%% EMHPM algorithm for ODE solutions
function [t,z]= odeEMHPM(nde,tspan,z0,Deltat,pnts)
% ode solver by Enhanced Multistage Homotopy Perturbation Method
tini=tspan(1); tfin=tspan(end); tstart=tini;
tini=tini-tstart; tfin=tfin-tstart; % shifted time set to zero
% Handle errors
if tini == tfin
    error('The ending and starting time values must be different.');
elseif abs(tini)> abs(tfin)
    tspan=flipud(fliplr(tspan)); tini=tspan(1); tfin=tspan(end);
end
tdir=sign(tfin-tini);
if any(tdir*diff(tspan) <= 0)
    error('tspan entries must be strictly sorted.');
end
incT=Deltat/pnts;
if incT<=0
    error ('Increasing must be greater than zero.')
end
z(1,:)=z0'; t(1)=tini; iteT=2; % set initial values
t(iteT)=tini+incT*tdir;
while tdir*t(iteT)<tdir*(tfin+tdir*incT)
    P_act=ceil(.99999*(t(iteT)-tini)*tdir/Deltat); % count sub-intervals
    c=z((Deltat/incT)*(P_act-1)+1,:); % set the corresponding initial condition
    tsub(iteT-1)=t(iteT)-(P_act-1)*Deltat*tdir; % evaluate the solution at the shifted time
    temp=tsub(iteT-1);
    z(iteT,:)=nde(temp,c,tstart+t(iteT));
    if tdir*t(iteT)>=tdir*tfin*0.99999 % repeat for the next sub-interval
        break
    else
        iteT=iteT+1; t(iteT)=tini+(iteT-1)*incT*tdir;
    end
end
t=t' +tstart;
end
%% EMHPM algorithm for DDE solutions
function [t z]=ddeEMHPM(dde,tspan,history,tau,Deltat,pntDelta)
% dde solver by Enhanced Multistage Homotopy Perturbation Method
tini=tspan(1); tfin=tspan(end); % span where the solution is founded
if tau<Deltat
    error('Subtinterval must not be greater than tau');
end
if mod(tau,Deltat)~=0
    pastDlt=Deltat;
    Deltat=tau/round(tau/Deltat);
    warning('Subtinterval was modified from %0.5g to %0.5g.',pastDlt,Deltat);
end
pnt=round(tau/Deltat); % samples-1 [-tau 0]
incT=Deltat/pntDelta; % step for set resolution
t=-tau+tini+(0:pnt*pntDelta)'*incT; % evaluation of the initial solution [-tau 0]
z=history(t); % initial behavior in the inverval [-tau 0]
c=z(end,:); % initial condition
```

```
m=pnt*pntDelta; % samples between tau
itDlt=0;
while (t(end)+eps)<tfin;
    tspan=[tini+itDlt*Deltat;tini+(itDlt+1)*Deltat]; % preparing the span for next Deltat
    zm=z(length(z)-m,:)'; zn=z(1+length(z)-m,:)'; % previous tau solution
    emhpm_fun=@(time,c,T) dde(time,c,T,zm,zn,tau,m); % application of the odeEMHPM
    [t_aux z_aux]=odeEMHPM(emhpm_fun,tspan,z(end,:),Deltat,pntDelta);
    z=[z(1:end-1,:);z_aux]; t=[t(1:end-1,:);t_aux]; % joining the solutions
    itDlt=itDlt+1;
end
z=z(pnt*pntDelta+1:end,:);t=t(pnt*pntDelta+1:end,:);
end
```

Algorithm 1: MATLAB algorithm

Here, $\phi_{\text {st }}$ and $\phi_{\text {ex }}$ are the angles where the teeth enter and exit the workpiece. For upmilling, $\phi_{\text {st }}=0$ and $\phi_{\mathrm{ex}}=\arccos (1-$ $\left.2 a_{d}\right)$, for downmilling, $\phi_{\text {st }}=\arccos \left(2 a_{d}-1\right)$ and $\phi_{\text {ex }}=\pi$, where $a_{d}$ is the radial depth of cut ratio.

By following the EMHPM procedure, we can write (46) in equivalent form as

$$
\begin{align*}
\ddot{x}_{i}(T) & +2 \zeta \omega_{n} \dot{x}_{i}(T)+\omega_{n}^{2} x_{i}(T) \\
& \approx-\frac{a_{p} K_{\mathrm{st}}}{m_{m}}\left(x_{i}(T)-x_{i-N+1}(T)\right), \tag{50}
\end{align*}
$$

where $x_{i}(T)$ denotes the $m$ order solution of (46) on the $i$ th subinterval that satisfies the initial conditions $x_{i}(0)=x_{i-1}$, $\dot{x}_{i}(0)=\dot{x}_{i-1}$, and $h_{t}=h(t)$ and $x_{-\tau}$ is given by (35). Introducing the transformation $\mathbf{x}_{i}=\left[x_{i}, \dot{x}_{i}\right]^{T}$, (50) can be written as a system of first order linear delay differential equations of the form

$$
\begin{equation*}
\dot{\mathbf{x}}_{i}(T)=\mathbf{A}_{t} \mathbf{x}_{i}(T)+\mathbf{B}_{t} \mathbf{x}_{i-N+1}(T), \tag{51}
\end{equation*}
$$

where

$$
\mathbf{A}_{t}=\left[\begin{array}{cc}
0 & 1  \tag{52}\\
-\omega_{n}^{2}-\frac{w}{m_{m}} K_{\mathrm{st}}-2 \zeta \omega_{n}
\end{array}\right] ; \quad \mathbf{B}_{t}=\left[\begin{array}{cc}
0 & 0 \\
\frac{w}{m_{m}} K_{\mathrm{st}} & 0
\end{array}\right] .
$$

We next apply the EMHPM procedure to solve (46) by considering a downmilling operation with the following parameter values: $z_{n}=2, a_{d}=0.1, \omega_{n}=5793 \mathrm{rad} / \mathrm{s}$, $\zeta=0.011, m_{m}=0.03993 \mathrm{~kg}, K_{t}=6 \times 10^{8} \mathrm{~N} / \mathrm{m}^{2}$, and $K_{n}=2 \times 10^{8} \mathrm{~N} / \mathrm{m}^{2}$. As we can see from Figures 7 and 8 and for the depth of cut values of $a_{p}=2 \mathrm{~mm}$ (stable) and $a_{p}=$ 3 mm (unstable), our EMHPM approximate solutions follow closely the numerical integration solutions of (46) obtained by using the dde23 algorithm.

Figure 9 shows the relative error between the EMPHM and the dde 23 numerical solution, while Table 2 shows the CPU time needed for each solution. Here we use $N=75$ since the average step size of the dde 23 algorithm is around $\Delta t \approx \tau / N$. Note that, for $m=7$, the zeroth order EMHPM approximate solution has the fastest CPU time. We can see from Figure 9 that the value of the relative error becomes basically the same for $m=2,7$, and 10 and $N \geq 20$.


Figure 7: EMHPM approximate solutions of (46) with parameter values of $a_{p}=2 \mathrm{~mm}, n=10000 \mathrm{rpm}$. Stable machine operation.


Figure 8: EMHPM approximate solutions of (46) with parameter values of $a_{p}=3 \mathrm{~mm}$ and $n=10000 \mathrm{rpm}$. Unstable (chatter) machine operation.

## 6. Conclusions

We have developed a new algorithm based on the homotopy perturbation method to solve delay differential equations. The proposed EMHPM approach is based on a sequence of subintervals that approximate the solution of delayed differential equations by using the transformation rule $u(t) \rightarrow$ $u_{i}(T)$, where $u_{i}$ satisfies the initial conditions. We have


Figure 9: Estimated relative error values between dde23 and the EMHPM approximate solutions. Here we have used the system parameter values of $a_{p}=2 \mathrm{~mm}$ and $n=1000 \mathrm{rpm}$ and values of $m=2,7$, and 10 .

Table 2: CPU time comparison among the approximate solutions of (46) by using dde23, the zeroth order, and the first order EMHPM solutions.

| Solution method | Time [ms] |
| :--- | :---: |
| dde23 | 139 |
| Zeroth order EMHPM $(m=7)$ | 77 |
| First order EMHPM $(m=7)$ | 87 |

shown that our proposed EMHPM approach can be applied to obtain the approximate solution of a delay differential equation not only with constant, but also with variable coefficients with theoretical predictions that follow well the numerical integration solutions. To further assess the validity of this new approach, we have compared the approximate solutions of two delayed differential equations with respect to their corresponding numerical integration solutions obtained from the MATLAB dde23 algorithm. The test cases were (a) the damped Mathieu differential equation with time delay and (b) the governing equation of motion of downmilling operations. We have found that the EMHPM closely follows the numerical integration solutions of the corresponding equations and that these require less CPU time and have smaller relative errors.

## Conflict of Interests

The authors declare that they have no conflict of interests with any mentioned entities in the paper.

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# Research Article 

# Global Stability Analysis of a Nonautonomous Stage-Structured Competitive System with Toxic Effect and Double Maturation Delays 

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#### Abstract

We investigate a nonautonomous two-species competitive system with stage structure and double time delays due to maturation for two species, where toxic effect of toxin liberating species on nontoxic species is considered and the inhibiting effect is zero in absence of either species. Positivity and boundedness of solutions are analytically studied. By utilizing some comparison arguments, an iterative technique is proposed to discuss permanence of the species within competitive system. Furthermore, existence of positive periodic solutions is investigated based on continuation theorem of coincidence degree theory. By constructing an appropriate Lyapunov functional, sufficient conditions for global stability of the unique positive periodic solution are analyzed. Numerical simulations are carried out to show consistency with theoretical analysis.


## 1. Introduction

In recent years, many research efforts have been made on competitive Lotka-Volterra system with stage structure and time delay. By incorporating a constant time delay into single species model, a stage-structured model is proposed in the pioneering work [1], where time delay reflects a delayed birth of immature population and a reduced survival of immature population to their maturity. The model system takes the following form:

$$
\begin{gather*}
\dot{x}_{i}(t)=\alpha x_{m}(t)-\gamma x_{i}(t)-\alpha e^{-\gamma \tau} x_{m}(t-\tau),  \tag{1}\\
\dot{x}_{m}(t)=\alpha e^{-\gamma \tau} x_{m}(t-\tau)-\beta x_{m}^{2}(t),
\end{gather*}
$$

where $x_{i}(t)$ and $x_{m}(t)$ represent the immature population and mature population density at time $t$, respectively. $\alpha>$ 0 denotes the birth rate of immature population; $\gamma>0$ stands for the death rate of immature population. $\beta>0$ is the death and overcrowding rate of mature population. $\tau$
denotes time of immature population to maturity. The term $\alpha e^{-\gamma \tau} x_{m}(t-\tau)$ represents the immature species which are born at time $t-\tau$ and survive at time $t$ with immature death rate $\gamma$ and therefore represents transformation of immature species to mature species. It is found that all ecologically relevant solutions tend to the positive equilibrium solution as time $t \rightarrow \infty$, and various aspects of the above proposed system including positivity and boundedness of solutions are discussed in [1].

Zeng et al. propose a nonautonomous competitive twospecies model with stage structure in one species in [2], where conditions of permanence are obtained. Furthermore, existence and asymptotic stability of periodic solution are proved under some assumptions if the proposed model turns out to be a periodic system. A two-species Lotka-Volterra type competition model with stage structure for both species is proposed and investigated in [3], where the individuals of each species are classified as immature and mature. By constructing a suitable Lyapunov function, sufficient conditions
are derived for the global stability of nonnegative equilibria of the proposed model in the case of constant coefficients. Furthermore, a set of easily verifiable sufficient conditions are obtained for the existence of positive periodic solution when coefficients are assumed to be positively continuous periodic functions. In [4], there is a time delayed periodic system which describes the competition among mature populations. The evolutionary behavior of model system is analyzed and some sufficient conditions for competitive coexistence and exclusion are obtained.

A nonautonomous competitive Lotka-Volterra system is studied in [5]; it reveals a computable necessary and sufficient condition for the system to be totally permanent when the growth rates have averages and the interaction coefficients are nonnegative constants. Along with this research, permanence for a class of competitive Lotka-Volterra systems is discussed in [6] which extends the work done in [5], and a computable necessary and sufficient condition is found for the permanence of all subsystems of the system and its small perturbation on the interaction matrix. In [7], a twospecies competitive model with stage structure is discussed, and the dynamics of coupled system of semilinear parabolic equations with time delays are investigated, which show that the introduction of diffusion does not affect the permanence and extinction of the species, though the introduction of stage structure brings negative effect on it. In [8], sufficient conditions are obtained for the existence of a unique, globally attractive, strictly positive (componentwise), almost periodic solution of a nonautonomous, almost periodic competitive two-species model with a stage structure in one species. An example together with its numeric simulations shows the feasibility of our main results, which generalize the main results of Zeng et al. [2]. According to two types of wellknown periodic single species population growth models with time delay, two corresponding periodic competitive systems with multiple delays are proposed in [9], and the same criteria for the existence and globally asymptotic stability of positive periodic solutions of the above two competitive systems are derived. In [10], a discrete periodic competitive model with stage structure is established, and some sufficient and realistic conditions are obtained for existence of a positive periodic solution of the proposed system. In [11], a periodic nonautonomous competitive stage-structured system with infinite delay is considered, where the adult members of $n$-species are in competition. For each of the $n$-species the model incorporates a time delay which represents the time from birth to maturity of that species. Infinite delay is introduced which denotes the influential effect of the entire past history of the system on the current competition interactions. By using the comparison principle, if the growth rates are sufficiently large, then the solutions are uniformly permanent. Then, by using Horn's fixed point theorem, the existence of positive periodic solution of the system with finite delay is discussed. Finally, it is proved that even the system with infinite delay admits a positive periodic solution.

In [12], a nonautonomous predator-prey system with discrete time delay is studied, where there is epidemic disease in the predator. By using some techniques of the differential inequalities and delay differential inequalities,
the permanence of system is discussed under some appropriate conditions. When all the coefficients of the system are periodic, the existence and global attractivity of the positive periodic solution are studied by Mawhin's continuation theorem and constructing a suitable Lyapunov functional. Furthermore, when the coefficients of the system are not absolutely periodic but almost periodic, sufficient conditions are also derived for the existence and asymptotic stability of the almost periodic solution. In [13], general nspecies nonautonomous Lotka-Volterra competitive systems with pure-delays and feedback controls are discussed. New sufficient conditions, for which a part of the $n$-species remains permanent, are established by applying the method of multiple Lyapunov functionals and introducing a new analysis technique.

By utilizing Brouwer fixed point theorem and constructing a suitable Lyapunov function, the periodic solution and global stability for a nonautonomous competitive LotkaVolterra diffusion system are investigated in [14]; it can be found that the system has a unique periodic solution which is globally stable under some appropriate conditions. In [15], a delay differential equation model for the interaction between two species is investigated. The maturation delay for each species is modelled as a distribution, to allow for the possibility that individuals may take different amount of time to maturity. Positivity and boundedness of the solutions are studied, and global stability is analyzed for each equilibrium. A Lotka-Volterra competitive system with infinite delay and feedback controls is proposed in [16]. By using Mawhin's continuation theorem of coincidence degree theory, an impulsive nonautonomous Lotka-Volterra predator-prey system with harvesting terms is investigated in [17]. Some sufficient conditions for the existence of multiple positive almost periodic solutions for the system under consideration are discussed. Furthermore, existence of multiple positive almost periodic solutions to other types of population systems can be studied by using the same method obtained in this paper. By using the method of multiple Lyapunov functionals and by developing a new analysis technique, some sufficient conditions are obtained that guarantee that some of the $n$ species are driven to extinction. A three-dimensional nonautonomous competitive Lotka-Volterra system is considered in [18]; it is shown that if the growth rates are positive, bounded, and continuous functions, and the averages of the growth rates satisfy certain inequalities, then any positive solution has the property that one of its components vanishes. In [19], an almost periodic multispecies Lotka-Volterra mutualism system with time delays and impulsive effects is investigated. By using the theory of comparison theorem and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the existence and uniqueness and global asymptotical stability of almost periodic solution of this system are obtained.

It is well known that the effect of toxins on ecological systems is an important issue from mathematical and experimental points of view [20,21]. The first mathematical model to represent the toxic liberating interaction between two competing species is introduced by Maynard Smith [22]. The model is based upon a two-species Lotka-Volterra
competition model with an additional term to take into account the effect of toxic substances released by one species to another, which takes the following form:

$$
\begin{align*}
& \dot{N}_{1}(t)=N_{1}(t)\left[\alpha_{1}-\beta_{1} N_{1}(t)-c_{1} N_{2}(t)-\rho_{1} N_{1}(t) N_{2}(t)\right], \\
& \dot{N}_{2}(t)=N_{2}(t)\left[\alpha_{2}-\beta_{2} N_{2}(t)-c_{2} N_{1}(t)-\rho_{2} N_{1}(t) N_{2}(t)\right], \tag{2}
\end{align*}
$$

where $N_{1}(t)$ and $N_{2}(t)$ represent the density of two competing species at time $t$, respectively. $\alpha_{1}$ and $\alpha_{2}$ denote the birth rate of $N_{1}(t)$ species and $N_{2}(t)$ species, respectively. $\beta_{1}$ and $\beta_{2}$ are the rate of intraspecific competition term for the first and second species, respectively. $c_{1}$ and $c_{2}$ stand for the rate of interspecific competition, respectively. $\rho_{1}$ and $\rho_{2}$ represent the toxic inhibition rate for the first species by the second species and vice versa. By considering that $\rho$ denotes the rate of toxic inhibition for the nontoxic species $N_{1}(t)$ released by the toxin liberating species $N_{2}(t)$ and all other parameters share the same biological interpretations mentioned in model system (2), work done in [22] is extended in [23] and the generalized model system is as follows:

$$
\begin{gather*}
\dot{N}_{1}(t)=N_{1}(t)\left[\alpha_{1}-\beta_{1} N_{1}(t)-c_{1} N_{2}(t)-\rho N_{1}(t) N_{2}^{2}(t)\right] \\
\dot{N}_{2}(t)=N_{2}(t)\left[\alpha_{2}-\beta_{2} N_{2}(t)-c_{2} N_{1}(t)\right] \tag{3}
\end{gather*}
$$

where the toxic substance producing action follows the mathematical term $\rho N_{1}^{2}(t) N_{2}^{2}(t)$ [23].

It should be noted that models of the persistence and extinction of a population or community in a polluted environment have been investigated in [23]. But all of those papers have a basic assumption that the capacity of the environment is so large that the change of toxicant in the environment that comes from uptake and egestation by the organisms can be neglected. This assumption is not made in $[24,25]$, some sufficient conditions on persistence or extinction of a population have been obtained, and the threshold between the two has also been obtained for most situations. In [26, 27], there are modified delay differential equation models of the growth of two species of plankton having competitive and allelopathic effects on each other. By using the continuation theorem of coincidence degree theory, a set of easily verifiable sufficient conditions are obtained for the existence of positive periodic solutions for this model. Recently, some discussions and investigations of the nonautonomous competitive model with toxic effects are made. A periodic competitive stage-structured Lotka-Volterra model with the effects of toxic substances is investigated in [28]. It is shown that toxic substances play an important role in the extinction of species. A set of sufficient conditions guarantee that one of the components is driven to extinction while the other is globally attractive. The dynamical behavior of a two-species competitive system affected by toxic substances is investigated in [21], where each species produces a substance toxic to the other species. Boundedness and local and global stabilities are also addressed. It should be noted that toxic interaction follows the mathematical term
suggested in model system (2) and each mature individual produces a substance toxic to the other mature individuals only when the other mature individual is present, and the immature individual is not affected by the toxicant [21]. However, to the author's best knowledge, dynamical behavior and stability analysis of nonautonomous stage-structured competitive system with toxin liberating species and nontoxic species have not been investigated. Generally speaking, it takes some time for a species to reach maturity to produce the toxicant; then toxin liberating mature individual produces a substance toxic to the nontoxic mature individuals only. The inhibiting effect is zero in absence of either species, and the immature individual of each species is not affected by the toxicant. Furthermore, the species compete each other for the limited life resource within closed environment, but this competition only happens among the mature individuals and does not involve the immature individuals. Consequently, it is necessary to investigate the dynamic effect of stage structure and toxic effect on the population dynamics of two-species competitive system with toxin liberating species and nontoxic species.

The rest section of this paper is organized as follows: a nonautonomous two-species competitive model is established in the second section. Stage structure and maturation delay for each species are introduced, and toxic effect of toxin liberating species on nontoxic species is considered. In the third section, qualitative analyses are performed to investigate the effect of stage structure and toxic substances on the dynamical behavior of two-species competitive model system. The positivity and boundedness of solutions are analytically studied. By utilizing some comparison arguments, an iterative technique is proposed to discuss permanence of the species within competitive system. Furthermore, existence of positive periodic solutions is considered based on continuation theorem of coincidence degree theory. By constructing an appropriate Lyapunov functional, sufficient conditions for global stability of the unique positive periodic solution are analyzed. Numerical simulations are provided to support the theoretical findings obtained in this paper. Finally, this paper ends with a conclusion.

## 2. Model Formulation

In this paper, the effect of stage structure and toxic substances on the dynamical behavior of two-species competitive model system is investigated under the following five hypotheses, which are given as follows.
(H1) Two competing species, that is, nontoxic species and toxin liberating species, are considered in this paper. It is assumed that each species is divided into two-stage groups, and the immature and mature individuals are divided by a fixed period. $x_{1}(t)$ and $y_{1}(t)$ represent immature population density of nontoxic species and toxin liberating species at time $t$, respectively; $x_{2}(t)$ and $y_{2}(t)$ denote mature population density of nontoxic species and toxin liberating species at time $t$, respectively.
(H2) $\omega$-periodic continuous functions $\alpha_{1}(t)>0$ and $\alpha_{2}(t)>0$ denote the birth rate of immature population of nontoxic species and toxin liberating species at time $t$, respectively. $\omega$-periodic continuous functions $\gamma_{1}(t)>0$ and $\gamma_{2}(t)>0$ stand for the death rate of immature population of nontoxic species and toxin liberating species at time $t$, respectively. $\omega$-periodic continuous functions $\beta_{1}(t)>0$ and $\beta_{2}(t)>0$ are the death and overcrowding rate of mature population of nontoxic species and toxin liberating species at time $t$, respectively.
(H3) $\tau_{1}$ denotes time of immature nontoxic species to maturity. The term $\alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} x_{2}\left(t-\tau_{1}\right)$ represents the immature nontoxic species which are born at time $t-\tau_{1}$ and survive at time $t$ with immature death rate. $\tau_{2}$ denotes time of immature toxin liberating species to maturity. The term $\alpha_{2}(t-$ $\left.\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} y_{2}\left(t-\tau_{2}\right)$ represents the immature toxin liberating species which are born at time $t-\tau_{2}$ and survive at time $t$ with immature death rate.
(H4) For toxin liberating species, it takes some time to attain its level of maturity to produce the toxic substances, and toxin liberating mature individual produces a substance toxic to the nontoxic mature individuals only. The inhibiting effect is zero in absence of either species, and the immature individual of each species is not affected by the toxicant. Based on model system (3), the toxic effect released by toxin liberating species on nontoxic species is described by the mathematical term $\rho(t) x_{2}^{2}(t) y_{2}^{2}(t)$, where toxic inhibition rate is represented by an $\omega$ periodic continuous function $\rho(t)>0$.
(H5) Nontoxic species and toxin liberating species compete each other for the common resource within closed environment, but this competition only happens among the mature individuals and does not involve the immature individuals. $\omega$-periodic continuous function $c_{1}(t)>0$ represents interspecific competition rate for the mature nontoxic species by the mature toxin liberating species, and $\omega$-periodic continuous function $c_{2}(t)>0$ represents interspecific competition rate for the mature toxin liberating species by the mature nontoxic species.

Based on hypotheses (H1)-(H5), a nonautonomous stagestructured competitive model with toxic effect and double maturation delays is established as follows:

$$
\begin{aligned}
\dot{x}_{1}(t)= & \alpha_{1}(t) x_{2}(t)-\gamma_{1}(t) x_{1}(t) \\
& -\alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} x_{2}\left(t-\tau_{1}\right), \\
\dot{x}_{2}(t)= & \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} x_{2}\left(t-\tau_{1}\right)-\beta_{1}(t) x_{2}^{2}(t) \\
& -c_{1}(t) x_{2}(t) y_{2}(t)-\rho(t) x_{2}^{2}(t) y_{2}^{2}(t),
\end{aligned}
$$

$$
\begin{align*}
\dot{y}_{1}(t)= & \alpha_{2}(t) y_{2}(t)-\gamma_{2}(t) y_{1}(t) \\
& -\alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} y_{2}\left(t-\tau_{2}\right), \\
\dot{y}_{2}(t)= & \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} y_{2}\left(t-\tau_{2}\right)-\beta_{2}(t) y_{2}^{2}(t) \\
& -c_{2}(t) x_{2}(t) y_{2}(t) . \tag{4}
\end{align*}
$$

In this paper, model system (4) is investigated with the following initial conditions:

$$
\begin{align*}
& x_{i}(\theta)=\phi_{i}(\theta)>0, \quad-\tau_{1} \leq \theta \leq 0, \quad i=1,2, \\
& y_{i}(\theta)=\psi_{i}(\theta)>0, \quad-\tau_{2} \leq \theta \leq 0, \quad i=1,2 . \tag{5}
\end{align*}
$$

For the continuity of the initial conditions, it is required that

$$
\begin{align*}
& x_{1}(0)=\int_{-\tau_{1}}^{0} \alpha_{1}(\theta) \phi_{2}(\theta) e^{\int_{0}^{\theta} \gamma_{1}(s) \mathrm{d} s} \mathrm{~d} \theta  \tag{6}\\
& y_{1}(0)=\int_{-\tau_{2}}^{0} \alpha_{2}(\theta) \psi_{2}(\theta) e^{\int_{0}^{\theta} \gamma_{2}(s) \mathrm{d} s} \mathrm{~d} \theta
\end{align*}
$$

## 3. Qualitative Analysis of Model System

In this section, qualitative analysis of the nonautonomous model system (4) is performed, which is utilized to discuss dynamic effect of toxic effect and maturation delay on population dynamics. The positivity and boundedness of solutions are analytically studied. By utilizing some comparison arguments, an iterative technique is proposed to discuss permanence of the species within competitive system. Furthermore, existence of positive periodic solutions is investigated based on continuation theorem of coincidence degree theory. By constructing an appropriate Lyapunov functional, sufficient conditions for global stability of the unique positive periodic solution are analyzed.

Some mathematical notations are adopted for convenience of the following statement:

$$
\begin{equation*}
f^{L}=\min _{t \in[0, \omega]}|f(t)|, \quad f^{M}=\max _{t \in[0, \omega]}|f(t)|, \tag{7}
\end{equation*}
$$

where $f(t)$ is a $\omega$-periodic continuous function.

### 3.1. Permanence of Solutions

Theorem 1. Solutions of model system (4) with initial conditions (5) and (6) are positive for all $t>0$.

Proof. Firstly, we show that $x_{2}(t)>0$ for all $t>0$. Otherwise, if it is false, since $x_{2}(t)>0$ for all $t \in\left[-\tau_{1}, 0\right]$, then it can be derived that there exists a $t_{1}>0$ such that $x_{2}\left(t_{1}\right)=0$.

Define $t_{0}=\inf \left\{t>0 \mid x_{2}(t)=0\right\}$. According to the definition of $t_{0}$, it can be obtained that

$$
\begin{equation*}
\dot{x}_{2}\left(t_{0}\right) \leq 0 . \tag{8}
\end{equation*}
$$

It follows from the second equation of model system (4) that

$$
\begin{align*}
& \dot{x}_{2}\left(t_{0}\right) \\
& = \begin{cases}\alpha_{1}\left(t_{0}-\tau_{1}\right) e^{-\int_{t_{0}-\tau_{1}}^{t_{0}} \gamma_{1}(s) \mathrm{d} s} \phi_{2}\left(t_{0}-\tau_{1}\right)>0, & 0 \leq t_{0} \leq \tau_{1}, \\
\alpha_{1}\left(t_{0}-\tau_{1}\right) e^{-\int_{t_{0}-\tau_{1}}^{t_{0}} \gamma_{1}(s) \mathrm{d} s} x_{2}\left(t_{0}-\tau_{1}\right)>0, & t>\tau_{1},\end{cases} \tag{9}
\end{align*}
$$

and it is easy to show that $\dot{x}_{2}\left(t_{0}\right)>0$, which is a contradiction to (8). Hence, $x_{2}(t)>0$ for all $t>0$.

By a direct computation, it follows from the first equation of model system (4) that

$$
\begin{equation*}
x_{1}(t)=\int_{t-\tau_{1}}^{t} \alpha_{1}(s) e^{\int_{t}^{s} \gamma_{1}(m) \mathrm{d} s} x_{2}(s) \mathrm{d} s \tag{10}
\end{equation*}
$$

since $x_{2}(t)>0$ for all $t>0$; it is easy to show that $x_{1}(t)>0$ for all $t>0$ based on (10).

By utilizing the similar proof, it can be obtained that $y_{1}(t)>0$ and $y_{2}(t)>0$ for all $t>0$. Consequently, solutions of model system (4) with initial conditions (5) and (6) are positive for all $t>0$.

Theorem 2. Solutions of model system (4) with initial conditions (5) and (6) are ultimately bounded.

Proof. Let $w(t)=x_{1}(t)+x_{2}(t)+y_{1}(t)+y_{2}(t)$, where $\left(x_{1}(t), x_{2}(t), y_{1}(t), y_{2}(t)\right)$ is an arbitrary positive solution of model system (4) with the initial conditions (5) and (6).

Calculating the derivative of $w(t)$ along the solution of model system (4) gives that

$$
\begin{align*}
\dot{w}(t) \leq & \left(\alpha_{1}(t)+\gamma_{1}(t)\right) x_{2}(t)-\gamma_{1}(t)\left(x_{1}(t)+x_{2}(t)\right) \\
& -\beta_{1}(t) x_{2}^{2}(t) \\
& +\left(\alpha_{2}(t)+\gamma_{2}(t)\right) y_{2}(t)-\gamma_{2}(t)\left(y_{1}(t)+y_{2}(t)\right) \\
& -\beta_{2}(t) y_{2}^{2}(t) \\
\leq & \left(\alpha_{1}^{M}+\gamma_{1}^{M}\right) x_{2}(t)-\beta_{1}^{L} x_{2}^{2}(t)-r_{1}^{L}\left(x_{1}(t)+x_{2}(t)\right) \\
& +\left(\alpha_{2}^{M}+\gamma_{2}^{M}\right) y_{2}(t)-\beta_{2}^{L} y_{2}^{2}(t)-r_{2}^{L}\left(y_{1}(t)+y_{2}(t)\right) \\
\leq & -\gamma^{L} w(t)+\left(\alpha_{1}^{M}+\gamma_{1}^{M}\right) x_{2}(t)-\beta_{1}^{L} x_{2}^{2}(t) \\
& +\left(\alpha_{2}^{M}+\gamma_{2}^{M}\right) y_{2}(t)-\beta_{2}^{L} y_{2}^{2}(t) \\
\leq & -\gamma^{L} w(t)+\frac{\left(\alpha_{1}^{M}+\gamma_{1}^{M}\right)^{2}}{4 \beta_{1}^{L}}+\frac{\left(\alpha_{2}^{M}+\gamma_{2}^{M}\right)^{2}}{4 \beta_{2}^{L}} \tag{11}
\end{align*}
$$

where $\gamma^{L}=\min \left\{\gamma_{1}^{L}, \gamma_{2}^{L}\right\}$.
By using the standard comparison principle [20], it follows from (11) that

$$
\begin{equation*}
w(t) \leq \frac{\beta_{2}^{L}\left(\alpha_{1}^{M}+\gamma_{1}^{M}\right)^{2}+\beta_{1}^{L}\left(\alpha_{2}^{M}+\gamma_{2}^{M}\right)^{2}}{4 \gamma^{L} \beta_{1}^{L} \beta_{2}^{L}} \tag{12}
\end{equation*}
$$

which implies that any solution of model system (4) with initial conditions (5) and (6) is ultimately bounded.

Lemma 3 (see [29]). Consider the following differential equation:

$$
\begin{equation*}
\dot{x}(t)=a x(t-\tau)-b x(t)-c x^{2}(t) \tag{13}
\end{equation*}
$$

where $a, b, c, \tau>0$ and $x(t)>0$ for $-\tau \leq t \leq 0$; we have that
(i) if $a>b$, then $\lim _{t \rightarrow+\infty} x(t)=(a-b) / c$;
(ii) if $a<b$, then $\lim _{t \rightarrow+\infty} x(t)=0$.

Lemma 4 (see [29]). Consider the following differential equation:

$$
\begin{equation*}
\dot{x}(t)=d x(t-\sigma)-e x^{2}(t) \tag{14}
\end{equation*}
$$

where $d, e, \sigma>0$ and $x(t)>0$ for $-\sigma \leq t \leq 0$; we have

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} x(t)=\frac{d}{e} \tag{15}
\end{equation*}
$$

Definition 5 (see [30]). Model system

$$
\begin{equation*}
\dot{X}(t)=f\left(t, X_{t}(\theta)\right), \tag{16}
\end{equation*}
$$

where $t \geq 0, \theta \in[-\tau, 0], X \in \mathbb{R}^{n}$. Model system (16) is said to be permanent if, for any solution $X(t, \phi)$, there exists a constant $m>0$ and $T=T(\phi)$ such that $X(t)>m$ for all $t>T$.

Definition 6 (see [30]). The domain $D \in \mathbb{C}^{n}$ is said to be an ultimately bounded domain, if $D$ is a closed, bounded subset of $\mathbb{C}^{n}$, and there exists a constant $T=T(\phi)$ such that $X_{t}(\theta) \in$ $D$ for all $t>T$.

Theorem 7. If $\alpha_{1}^{L} \beta_{2}^{L}>c_{1}^{M} \alpha_{2}^{M}$ and $\alpha_{2}^{L} \beta_{1}^{L}>c_{2}^{M} \alpha_{1}^{M}$, then model system (4) is permanent with initial conditions (5) and (6).

Proof. According to the second equation of model system (4) and Theorem 1, we get that

$$
\begin{equation*}
\dot{x}_{2}(t) \leq \alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}} x_{2}\left(t-\tau_{1}\right)-\beta_{1}^{L} x_{2}^{2}(t) \tag{17}
\end{equation*}
$$

By virtue of Lemma 4 and (17), there exists a positive time $T_{1}$ such that, for sufficiently small $\epsilon>0$ and $t \geq T_{1}$, it yields

$$
\begin{equation*}
x_{2}(t) \leq \frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}}+\epsilon:=M_{2}^{(1)} \tag{18}
\end{equation*}
$$

By rearranging (10), it can be obtained that

$$
\begin{equation*}
x_{1}(t)=e^{-\int_{0}^{t} \gamma_{1}(s) \mathrm{d} s} \int_{t-\tau_{1}}^{t} \alpha_{1}(s) e^{\int_{0}^{s} \gamma_{1}(m) \mathrm{d} m} x_{2}(s) \mathrm{d} s \tag{19}
\end{equation*}
$$

For any $t \geq T_{1}$, it follows from (18) and (19) that

$$
\begin{equation*}
x_{1}(t) \leq \frac{a_{1}^{M} M_{2}^{(1)}\left(1-e^{-\gamma_{1}^{M} \tau_{1}}\right)}{\gamma_{1}^{L}}:=M_{1}^{(1)} . \tag{20}
\end{equation*}
$$

Based on the fourth equation of model system (4) and Theorem 1, it can be obtained that

$$
\begin{equation*}
\dot{y}_{2}(t) \leq \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}} y_{2}\left(t-\tau_{2}\right)-\beta_{2}^{L} y_{2}^{2}(t) \tag{21}
\end{equation*}
$$

holds for $t \geq T_{1}$.
By virtue of Lemma 4 and (21), there exists $T_{2}>T_{1}$ such that, for sufficiently small $\epsilon>0$ and $t \geq T_{2}$, it yields

$$
\begin{equation*}
y_{2}(t) \leq \frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}+\epsilon:=M_{4}^{(1)} \tag{22}
\end{equation*}
$$

By direct computing, it follows from the third equation of model system (4) that

$$
\begin{equation*}
y_{1}(t)=e^{-\int_{0}^{t} \gamma_{2}(s) \mathrm{d} s} \int_{t-\tau_{2}}^{t} \alpha_{2}(s) e^{\int_{0}^{s} \gamma_{2}(m) \mathrm{d} m} y_{2}(s) \mathrm{d} s \tag{23}
\end{equation*}
$$

For any $t \geq T_{2}$, it follows from (22) and (23) that

$$
\begin{equation*}
y_{1}(t) \leq \frac{\alpha_{2}^{M} M_{4}^{(1)}\left(1-e^{-\gamma_{2}^{M} \tau_{2}}\right)}{\gamma_{2}^{L}}:=M_{3}^{(1)} \tag{24}
\end{equation*}
$$

Furthermore, it follows from the second equation of model system (4) that

$$
\begin{align*}
\dot{x}_{2}(t) \geq & \alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}} x_{2}\left(t-\tau_{1}\right) \\
& -\left(\beta_{1}^{M}+\rho_{1}^{M}\left(M_{4}^{(1)}\right)^{2}\right) x_{2}^{2}(t)-c_{1}^{M} M_{4}^{(1)} x_{2}(t) \tag{25}
\end{align*}
$$

Based on Lemma 3 and (25), there exists $T_{3}>T_{2}$ and for any $t \geq T_{3}$ and sufficiently small $\epsilon>0$,

$$
\begin{equation*}
x_{2}(t) \geq \frac{\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}-c_{1}^{M} M_{4}^{(1)}}{\beta_{1}^{M}+\rho_{1}^{M}\left(M_{4}^{(1)}\right)^{2}}-\epsilon:=m_{2}^{(1)} \tag{26}
\end{equation*}
$$

holds provided that

$$
\begin{equation*}
\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}>c_{1}^{M} M_{4}^{(1)} \tag{27}
\end{equation*}
$$

According to (19), for any $t \geq T_{3}$, we get that

$$
\begin{equation*}
x_{1}(t) \geq \frac{\alpha_{1}^{L}\left(1-e^{-\gamma_{1}^{L} \tau_{1}}\right) m_{2}^{(1)}}{\gamma_{1}^{M}}:=m_{1}^{(1)} \tag{28}
\end{equation*}
$$

For any $t \geq T_{3}$, it follows from the fourth equation of model system (4) that

$$
\begin{equation*}
\dot{y}_{2}(t) \geq \alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}} y_{2}\left(t-\tau_{2}\right)-\beta_{2}^{M} y_{2}^{2}(t)-c_{2}^{M} M_{2}^{(1)} y_{2}(t) \tag{29}
\end{equation*}
$$

Based on Lemma 3 and (29), there exists $T_{4}>T_{3}$ and for any $t \geq T_{4}$ and sufficiently small $\epsilon>0$,

$$
\begin{equation*}
y_{2}(t) \geq \frac{\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-c_{2}^{M} M_{2}^{(1)}}{\beta_{2}^{M}}-\epsilon:=m_{4}^{(1)} \tag{30}
\end{equation*}
$$

holds provided that

$$
\begin{equation*}
\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>c_{2}^{M} M_{2}^{(1)} \tag{31}
\end{equation*}
$$

According to (23) and (30), it can be obtained that

$$
\begin{equation*}
y_{1}(t) \geq \frac{\alpha_{2}^{L} m_{4}^{(1)}\left(1-e^{-\gamma_{2}^{L} \tau_{2}}\right)}{\gamma_{2}^{M}}:=m_{3}^{(1)} \tag{32}
\end{equation*}
$$

According to the second equation of model system (4), we get that

$$
\begin{align*}
\dot{x}_{2}(t) \leq & \alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}} x_{2}\left(t-\tau_{1}\right) \\
& -\left(\beta_{1}^{L}+\rho^{L}\left(m_{4}^{(1)}\right)^{2}\right) x_{2}^{2}(t)-c_{1}^{L} m_{4}^{(1)} x_{2}(t) \tag{33}
\end{align*}
$$

By virtue of Lemma 3 and (33), there exists $T_{5}>T_{4}$ such that, for sufficiently small $\epsilon>0$ and $t \geq T_{5}$, it yields

$$
\begin{equation*}
x_{2}(t) \leq \frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}-c_{1}^{L} m_{4}^{(1)}}{\beta_{1}^{L}+\rho^{L}\left(m_{4}^{(1)}\right)^{2}}+\epsilon:=M_{2}^{(2)} \tag{34}
\end{equation*}
$$

which holds provided that

$$
\begin{equation*}
\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}>c_{1}^{L} m_{4}^{(1)} \tag{35}
\end{equation*}
$$

For any $t \geq T_{5}$, it follows from (19) and (34) that

$$
\begin{equation*}
x_{1}(t) \leq \frac{\alpha_{1}^{M} M_{2}^{(2)}\left(1-e^{-\gamma_{1}^{M} \tau_{1}}\right)}{\gamma_{1}^{L}}:=M_{1}^{(2)} \tag{36}
\end{equation*}
$$

Based on the fourth equation of model system (4), it can be obtained that

$$
\begin{equation*}
\dot{y}_{2}(t) \leq \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}} y_{2}\left(t-\tau_{2}\right)-\beta_{2}^{L} y_{2}^{2}(t)-c_{2}^{L} m_{2}^{(1)} y_{2}(t) \tag{37}
\end{equation*}
$$

holds for $t \geq T_{5}$.
By virtue of Lemma 3 and (37), there exists $T_{6}>T_{5}$ such that, for sufficiently small $\epsilon>0$ and $t \geq T_{6}$, it yields

$$
\begin{equation*}
y_{2}(t) \leq \frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}-c_{2}^{L} m_{2}^{(1)}}{\beta_{2}^{L}}+\epsilon:=M_{4}^{(2)} \tag{38}
\end{equation*}
$$

which holds provided that

$$
\begin{equation*}
\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}>c_{2}^{L} m_{2}^{(1)} \tag{39}
\end{equation*}
$$

For any $t \geq T_{6}$, it follows from (23) and (38) that

$$
\begin{equation*}
y_{1}(t) \leq \frac{\alpha_{2}^{M} M_{4}^{(2)}\left(1-e^{-\gamma_{2}^{M} \tau_{2}}\right)}{\gamma_{2}^{L}}:=M_{3}^{(2)} . \tag{40}
\end{equation*}
$$

Furthermore, it follows from the second equation of model system (4) that

$$
\begin{align*}
\dot{x}_{2}(t) \geq & \alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{2}} x_{2}\left(t-\tau_{1}\right) \\
& -\left(\beta_{1}^{M}+\rho^{M}\left(M_{4}^{(2)}\right)^{2}\right) x_{2}^{2}(t)-c_{1}^{M} M_{4}^{(2)} x_{2}(t) \tag{41}
\end{align*}
$$

Based on Lemma 3 and (41), there exists $T_{7}>T_{6}$ and, for any $t \geq T_{7}$ and sufficiently small $\epsilon>0$,

$$
\begin{equation*}
x_{2}(t) \geq \frac{\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}-c_{1}^{M} M_{4}^{(2)}}{\beta_{1}^{M}+\rho^{M}\left(M_{4}^{(2)}\right)^{2}}-\epsilon:=m_{2}^{(2)} \tag{42}
\end{equation*}
$$

holds provided that

$$
\begin{equation*}
\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}>c_{1}^{M} M_{4}^{(2)} \tag{43}
\end{equation*}
$$

According to (19) and (42), for any $t \geq T_{7}$, we get that

$$
\begin{equation*}
x_{1}(t) \geq \frac{\alpha_{1}^{L}\left(1-e^{-\gamma_{1}^{L} \tau_{1}}\right) m_{2}^{(2)}}{\gamma_{1}^{M}}:=m_{1}^{(2)} \tag{44}
\end{equation*}
$$

For any $t \geq T_{7}$, it follows from the fourth equation of model system (4) that

$$
\begin{equation*}
\dot{y}_{2}(t) \geq \alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}} y_{2}\left(t-\tau_{2}\right)-\beta_{2}^{M} y_{2}^{2}(t)-c_{2}^{M} M_{2}^{(2)} y_{2}(t) \tag{45}
\end{equation*}
$$

Based on Lemma 3 and (45), there exists $T_{8}>T_{7}$ and, for any $t \geq T_{8}$ and sufficiently small $\epsilon>0$,

$$
\begin{equation*}
y_{2}(t) \geq \frac{\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-c_{2}^{M} M_{2}^{(2)}}{\beta_{2}^{M}}-\epsilon:=m_{4}^{(2)} \tag{46}
\end{equation*}
$$

holds provided that

$$
\begin{equation*}
\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>c_{2}^{M} M_{2}^{(2)} \tag{47}
\end{equation*}
$$

According to (23) and (46), it can be obtained that

$$
\begin{equation*}
y_{1}(t) \geq \frac{\alpha_{2}^{L} m_{4}^{(2)}\left(1-e^{-\gamma_{2}^{L} \tau_{2}}\right)}{\gamma_{2}^{M}}:=m_{3}^{(2)} \tag{48}
\end{equation*}
$$

By using simple computation, it is easy to show that six inequalities (27), (31), (35), (39), (43), and (47) hold if the following two inequalities $\alpha_{1}^{L} \beta_{2}^{L}>c_{1}^{M} \alpha_{2}^{M}$ and $\alpha_{2}^{L} \beta_{1}^{L}>c_{2}^{M} \alpha_{1}^{M}$ hold.

Furthermore, eight sequences will be obtained by repeating the discussion in this manner, which are given as follows:

$$
\begin{aligned}
& M_{1}^{(n+1)}=\frac{\alpha_{1}^{M} M_{2}^{(n+1)}\left(1-e^{-\gamma_{1}^{M} \tau_{1}}\right)}{\gamma_{1}^{L}}, \\
& M_{2}^{(n+1)}=\frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}-c_{1}^{L} m_{4}^{(n)}}{\beta_{1}^{L}+\rho^{L}\left(m_{4}^{(n)}\right)^{2}}+\epsilon, \\
& M_{3}^{(n+1)}=\frac{\alpha_{2}^{M} M_{4}^{(n+1)}\left(1-e^{-\gamma_{2}^{M} \tau_{2}}\right)}{\gamma_{2}^{L}}, \\
& M_{4}^{(n+1)}=\frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}-c_{2}^{L} m_{2}^{(n)}}{\beta_{2}^{L}}+\epsilon, \\
& m_{1}^{(n+1)}=\frac{\alpha_{1}^{L}\left(1-e^{-\gamma_{1}^{L} \tau_{1}}\right) m_{2}^{(n+1)}}{\gamma_{1}^{M}},
\end{aligned}
$$

$$
\begin{gather*}
m_{2}^{(n+1)}=\frac{\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}-c_{1}^{M} M_{4}^{(n+1)}}{\beta_{1}^{M}+\rho^{M}\left(M_{4}^{(n+1)}\right)^{2}}-\epsilon \\
m_{3}^{(n+1)}=\frac{\alpha_{2}^{L} m_{4}^{(n+1)}\left(1-e^{-\gamma_{2}^{L} \tau_{2}}\right)}{\gamma_{2}^{M}}, \\
m_{4}^{(n+1)}=\frac{\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-c_{2}^{M} M_{2}^{(n+1)}}{\beta_{2}^{M}}-\epsilon \tag{49}
\end{gather*}
$$

It is easy to show that $M_{i}^{(n)}>0$ and the sequences $\left\{M_{i}^{(n)}\right\}$ ( $i=1,2,3,4$ ) are decreasing as $n$ increases, which implies that $\lim _{n \rightarrow \infty} M_{i}^{(n)}=M_{i}^{*}$ exists; furthermore, it is easy to show that $m_{i}^{(n)}<M_{i}^{(n)}$ and the sequences $\left\{m_{i}^{(n)}\right\}(i=1,2,3,4)$ are increasing as $n$ increases, which implies that $\lim _{n \rightarrow \infty} m_{i}^{(n)}=$ $m_{i}^{*}$ exists. Consequently, it follows from (49) that

$$
\begin{array}{ll}
M_{1}^{*}=\frac{\alpha_{1}^{M} M_{2}^{*}\left(1-e^{-\gamma_{1}^{M} \tau_{1}}\right)}{\gamma_{1}^{L}}, & M_{2}^{*}=\frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}-c_{1}^{L} m_{4}^{*}}{\beta_{1}^{L}+\rho^{L} m_{4}^{* 2}}, \\
M_{3}^{*}=\frac{\alpha_{2}^{M} M_{4}^{*}\left(1-e^{-\gamma_{2}^{M} \tau_{2}}\right)}{\gamma_{2}^{L}}, & M_{4}^{*}=\frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}-c_{2}^{L} m_{2}^{*}}{\beta_{2}^{L}}, \\
m_{1}^{*}=\frac{\alpha_{1}^{L}\left(1-e^{-\gamma_{1}^{L} \tau_{1}}\right) m_{2}^{*}}{\gamma_{1}^{M}}, & m_{2}^{*}=\frac{\alpha_{1}^{L} e^{-\gamma_{1}^{M} \tau_{1}}-c_{1}^{M} M_{4}^{*}}{\beta_{1}^{M}+\rho^{M} M_{4}^{* 2}}, \\
m_{3}^{*}=\frac{\alpha_{2}^{L} m_{4}^{*}\left(1-e^{-\gamma_{2}^{L} \tau_{2}}\right)}{\gamma_{2}^{M}}, & m_{4}^{*}=\frac{\alpha_{2}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-c_{2}^{M} M_{2}^{*}}{\beta_{2}^{M}} \tag{50}
\end{array}
$$

Based on Definition 5 and (50), it can be concluded that model system (4) is persistent if $\alpha_{1}^{L} \beta_{2}^{L}>c_{1}^{M} \alpha_{2}^{M}$ and $\alpha_{2}^{L} \beta_{1}^{L}>$ $c_{2}^{M} \alpha_{1}^{M}$ hold.

### 3.2. Existence of Positive Periodic Solutions

Definition 8 (see [31]). Let $L: \operatorname{Dom} L \subset X \rightarrow Y$ be a linear mapping and let $N: X \rightarrow Y$ be a continuous mapping, where $X$ and $Y$ are real Banach spaces. If $\operatorname{dim} \operatorname{Ker} L=$ codimIm $L<+\infty$ and $\operatorname{Im} L$ is closed in $Y$, then $L$ is called a Fredholm mapping of index zero.

If $L$ is Fredholm mapping of index zero and there exist continuous projectors $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ such that

$$
\begin{equation*}
\operatorname{Im} P=\operatorname{Ker} L, \quad \operatorname{Im} L=\operatorname{Ker} Q=\operatorname{Im}(I-Q) \tag{51}
\end{equation*}
$$

then restriction $L_{p}$ of $L$ to $\operatorname{Dom} L \cap \operatorname{Ker} P:(I-P) X \rightarrow \operatorname{Im} L$ is invertible.

Definition 9 (see [31]). Denote the inverse of $L_{p}$ by $K_{p}$. Supposing that $\Omega$ is an open bounded subset of $X$, if $Q N(\bar{\Omega})$ is bounded and $K_{p}(I-Q) N: \Omega \rightarrow X$ is compact, then the mapping $N$ is called $L$-compact on $\bar{\Omega}$. Since $\operatorname{Im} Q$ is
isomorphic to Ker $L$, there exists an isomorphism $J: \operatorname{Im} Q \rightarrow$ $\operatorname{Ker} L$.

Lemma 10 (see [31]). Let $\Omega \subset X$ be an open bounded set, let $L$ be a Fredholm mapping of index zero, and let $N$ be L-compact on $\bar{\Omega}$. If the following three conditions hold:
(i) $L x \neq \lambda N x$ for any $\lambda \in(0,1)$ and $x \in \partial \Omega \cap \operatorname{Dom} L$,
(ii) $Q N x \neq 0$ for any $x \in \partial \Omega \cap \operatorname{Ker} L$,
(iii) $\operatorname{deg}\{J Q N, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$,
then $L x=N x$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.
Theorem 11. If $\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}>\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}, \alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>$ $\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{2}}$, then model system (4) with initial conditions (5) and (6) has at least one positive $\omega$-periodic solution.

Proof. Consider the subsystem of model system (4):

$$
\begin{align*}
\dot{x}_{2}(t)= & \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} x_{2}\left(t-\tau_{1}\right)-\beta_{1}(t) x_{2}^{2}(t) \\
& -c_{1}(t) x_{2}(t) y_{2}(t)-\rho(t) x_{2}^{2}(t) y_{2}^{2}(t) \\
\dot{y}_{2}(t)= & \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} y_{2}\left(t-\tau_{2}\right) \\
& -\beta_{2}(t) y_{2}^{2}(t)-c_{2}(t) x_{2}(t) y_{2}(t) \tag{52}
\end{align*}
$$

Let $u_{1}(t)=\ln \left[x_{2}(t)\right], u_{2}(t)=\ln \left[y_{2}(t)\right]$.
By substituting $u_{1}(t)$ and $u_{2}(t)$ into (52), it can be obtained that

$$
\begin{align*}
\dot{u}_{1}(t)= & \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}\left(t-\tau_{1}\right)-u_{1}(t)} \\
& -\beta_{1}(t) e^{u_{1}(t)}-c_{1}(t) e^{u_{2}(t)}-\rho(t) e^{u_{1}(t)+2 u_{2}(t)} \\
\dot{u}_{2}(t)= & \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{ds}} e^{u_{2}\left(t-\tau_{2}\right)-u_{2}(t)}  \tag{53}\\
& -\beta_{2}(t) e^{u_{2}(t)}-c_{2}(t) e^{u_{1}(t)}
\end{align*}
$$

It should be noted that if model system (53) has one $\omega$-periodic solution $\left(u_{1}^{*}(t), u_{2}^{*}(t)\right)^{T}$, then $\left(x_{2}^{*}(t), y_{2}^{*}(t)\right)^{T}=$ $\left(e^{u_{1}^{*}(t)}, e^{u_{2}^{*}(t)}\right)^{T}$ is a positive $\omega$-periodic solution of model system (52).

In order to utilize Lemma 10 in a straightforward manner, we define

$$
\begin{align*}
& X=Y=\left\{\left(u_{1}(t), u_{2}(t)\right)^{T}\right. \\
& \left.\quad \in C\left(\mathbb{R}, \mathbb{R}^{2}\right): u_{i}(t+\omega)=u_{i}(t), i=1,2\right\}, \\
& \left\|\left(u_{1}(t), u_{2}(t)\right)^{T}\right\|=\max _{t \in[0, \omega]}\left|u_{1}(t)\right|+\max _{t \in[0, \omega]}\left|u_{2}(t)\right|, \tag{54}
\end{align*}
$$

where $|\cdot|$ denotes the Euclidean norm; it is easy to show that both $X$ and $Y$ are Banach spaces with the norm $\|\cdot\|$; then define

$$
\operatorname{Dom} L \cap X \longrightarrow X
$$

$$
\begin{equation*}
L\left(u_{1}(t), u_{2}(t)\right)^{T}=\left(\frac{\mathrm{d} u_{1}(t)}{\mathrm{d} t}, \frac{\mathrm{~d} u_{2}(t)}{\mathrm{d} t}\right)^{T} \tag{55}
\end{equation*}
$$

where $\operatorname{Dom} L=\left\{\left(u_{1}(t), u_{2}(t)\right)^{T} \in C\left(\mathbb{R}, \mathbb{R}^{2}\right)\right\}, N\left[\begin{array}{l}u_{1} \\ u_{2}\end{array}\right]=$ $\left[\begin{array}{l}f_{1}(t) \\ f_{2}(t)\end{array}\right]$, and

$$
\begin{align*}
f_{1}(t)= & \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}\left(t-\tau_{1}\right)-u_{1}(t)} \\
& -\beta_{1}(t) e^{u_{1}(t)}-c_{1}(t) e^{u_{2}(t)}-\rho(t) e^{u_{1}(t)+2 u_{2}(t)},  \tag{56}\\
f_{2}(t)= & \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} e^{u_{2}\left(t-\tau_{2}\right)-u_{2}(t)} \\
& -\beta_{2}(t) e^{u_{2}(t)}-c_{2}(t) e^{u_{1}(t)}
\end{align*}
$$

Furthermore, we define

$$
P\left[\begin{array}{l}
u_{1}  \tag{57}\\
u_{2}
\end{array}\right]=Q\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right]=\left[\begin{array}{l}
\frac{1}{\omega} \int_{0}^{\omega} u_{1}(t) \mathrm{d} t \\
\frac{1}{\omega} \int_{0}^{\omega} u_{2}(t) \mathrm{d} t
\end{array}\right], \quad\left[\begin{array}{l}
u_{1} \\
u_{2}
\end{array}\right] \in X=Y
$$

According to the above definitions, it is not difficult to verify that $\operatorname{Ker} L=\left\{x \mid x \in X, x=h, h \in \mathbb{R}^{2}\right\}$, $\operatorname{Im} L=\left\{y \in Y \mid \int_{0}^{\omega} y(t) \mathrm{d} t=0\right\}$ are closed in $Y, \operatorname{dim} \operatorname{Ker} L=$ codimIm $L=2$, and both $P$ and $Q$ are continuous projectors such that $\operatorname{Im} P=\operatorname{Ker} L$ and $\operatorname{Ker} Q=\operatorname{Im} L=\operatorname{Im}(I-Q)$.

Based on the above analysis, it can be obtained that $L$ is a Fredholm mapping of index zero.

Furthermore, the inverse $K_{p}: \operatorname{Im} L \rightarrow \operatorname{Dom} L \cap \operatorname{Ker} P$ of $L_{p}$ exists and takes the following form:

$$
\begin{equation*}
K_{p}(y)=\int_{0}^{t} y(s) \mathrm{d} s-\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} y(s) \mathrm{d} s \mathrm{~d} t \tag{58}
\end{equation*}
$$

Hence, $Q N: X \rightarrow Y$ and $K_{p}(I-Q) N: X \rightarrow X$ can be defined as follows, respectively,

$$
\begin{gather*}
Q N x=\left[\begin{array}{c}
\frac{1}{\omega} \int_{0}^{\omega} f_{1}(t) \mathrm{d} t \\
\frac{1}{\omega} \int_{0}^{\omega} f_{2}(t) \mathrm{d} t
\end{array}\right] \\
K_{p}(I-Q) N x= \\
\int_{0}^{t} N x(s) \mathrm{d} s-\frac{1}{\omega} \int_{0}^{\omega} \int_{0}^{t} N x(s) \mathrm{d} s \mathrm{~d} t  \tag{59}\\
\\
-\left(\frac{t}{\omega}-\frac{1}{2}\right) \int_{0}^{\omega} N x(s) \mathrm{d} s .
\end{gather*}
$$

It is easy to show that $Q N$ and $K_{p}(I-Q) N$ are continuous. In order to facilitate the proof based on Lemma 10, we also need to find an appropriate open and bounded subset $\Omega$, which can be found by the following two steps.

Step 1. According to the operator equation $L x=\lambda N x$ for $\lambda \in$ $(0,1)$, the upper and lower bound of $u_{1}(t)$ and $u_{2}(t)$ will be estimated as follows:

$$
\begin{align*}
& \frac{\mathrm{d} u_{1}(t)}{\mathrm{d} t}=\lambda f_{1}(t),  \tag{60}\\
& \frac{\mathrm{d} u_{2}(t)}{\mathrm{d} t}=\lambda f_{2}(t),
\end{align*}
$$

where $f_{1}(t)$ and $f_{2}(t)$ have been defined in (56).
Suppose that $\left(u_{1}(t), u_{2}(t)\right)^{T} \in X$ is a solution of model system (60) for some $\lambda \in(0,1)$. By integrating (60) over the interval $[0, \omega]$, it can be obtained that

$$
\begin{align*}
& \int_{0}^{\omega} \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}\left(t-\tau_{1}\right)-u_{1}(t)} \mathrm{d} t \\
& \quad=\int_{0}^{\omega} \beta_{1}(t) e^{u_{1}(t)}+c_{1}(t) e^{u_{2}(t)}+\rho(t) e^{u_{1}(t)+2 u_{2}(t)} \mathrm{d} t  \tag{61}\\
& \int_{0}^{\omega} \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} e^{u_{2}\left(t-\tau_{2}\right)-u_{2}(t)} \mathrm{d} t  \tag{62}\\
& \quad=\int_{0}^{\omega} \beta_{2}(t) e^{u_{2}(t)}+c_{2}(t) e^{u_{1}(t)} \mathrm{d} t
\end{align*}
$$

Based on definition $\left(u_{1}(t), u_{2}(t)\right)^{T} \in X$, there exist $\xi_{i}, \eta_{i} \in$ $[0, \omega]$ such that

$$
\begin{equation*}
u_{i}\left(\xi_{i}\right)=\min _{t \in[0, \omega]} u_{i}(t), \quad u_{i}\left(\eta_{i}\right)=\max _{t \in[0, \omega]} u_{i}(t), \quad i=1,2 \tag{63}
\end{equation*}
$$

Multiplying the first equation of (60) by $e^{u_{1}(t)}$ and integrating it over $[0, \omega]$ give that

$$
\begin{align*}
& \int_{0}^{\omega} \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}\left(t-\tau_{1}\right)} \mathrm{d} t \\
& =\int_{0}^{\omega} \beta_{1}(t) e^{2 u_{1}(t)}+c_{1}(t) e^{u_{1}(t)+u_{2}(t)}  \tag{64}\\
& \quad+\rho(t) e^{2\left(u_{1}(t)+u_{2}(t)\right)} \mathrm{d} t .
\end{align*}
$$

It follows from (64) that

$$
\begin{align*}
& \beta_{1}^{L} \int_{0}^{\omega} e^{2 u_{1}(t)} \mathrm{d} t+c_{1}^{L} \int_{0}^{\omega} e^{u_{1}(t)+u_{2}(t)} \mathrm{d} t  \tag{65}\\
& \quad+\rho^{L} \int_{0}^{\omega} e^{2\left(u_{1}(t)+u_{2}(t)\right)} \mathrm{d} t \leq \alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}} \int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t
\end{align*}
$$

On the other hand, by using the inequality,

$$
\begin{equation*}
\left(\int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t\right)^{2} \leq \omega \int_{0}^{\omega} e^{2 u_{1}(t)} \mathrm{d} t \tag{66}
\end{equation*}
$$

Based on (65) and (66), it can be obtained that

$$
\begin{align*}
\beta_{1}^{L}\left(\int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t\right)^{2} & \leq \omega \beta_{1}^{L} \int_{0}^{\omega} e^{2 u_{1}(t)} \mathrm{d} t  \tag{67}\\
& \leq \omega \alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}} \int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t
\end{align*}
$$

which derives that

$$
\begin{equation*}
\beta_{1}^{L} \int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t \leq \omega \alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}, \quad u_{1}\left(\xi_{1}\right) \leq \ln \frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}} \tag{68}
\end{equation*}
$$

It follows from (60) and (68) that

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| \mathrm{d} t<2 \int_{0}^{\omega} \beta_{1}(t) e^{u_{1}(t)} \leq \frac{2 \omega \alpha_{1}^{M} \beta_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}} . \tag{69}
\end{equation*}
$$

According to (68) and (69), it can be obtained that

$$
\begin{align*}
u_{1}(t) & \leq u_{1}\left(\xi_{1}\right)+\int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| \mathrm{d} t \\
& \leq \ln \frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}}+\frac{2 \omega \alpha_{1}^{M} \beta_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}} . \tag{70}
\end{align*}
$$

Multiplying the second equation of (60) by $e^{u_{2}(t)}$ and integrating it over $[0, \omega]$ give that

$$
\begin{align*}
& \int_{0}^{\omega} \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} e^{u_{2}\left(t-\tau_{2}\right)} \mathrm{d} t  \tag{71}\\
& \quad=\int_{0}^{\omega} \beta_{2}(t) e^{2 u_{2}(t)}+c_{2}(t) e^{u_{1}(t)+u_{2}(t)} \mathrm{d} t .
\end{align*}
$$

It follows from (71) that

$$
\begin{gather*}
\beta_{2}^{L} \int_{0}^{\omega} e^{2 u_{2}(t)} \mathrm{d} t+c_{2}^{L} \int_{0}^{\omega} e^{u_{1}(t)+u_{2}(t)} \mathrm{d} t \\
\leq \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}} \int_{0}^{\omega} e^{u_{2}(t)} \mathrm{d} t \tag{72}
\end{gather*}
$$

Based on (66) and (72), it can be obtained that

$$
\begin{equation*}
\beta_{2}^{L}\left(\int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t\right)^{2} \leq \omega \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}} \int_{0}^{\omega} e^{u_{2}(t)} \mathrm{d} t \tag{73}
\end{equation*}
$$

which derives that

$$
\begin{equation*}
\int_{0}^{\omega} e^{u_{2}(t)} \mathrm{d} t \leq \frac{\omega \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}, \quad u_{2}\left(\xi_{2}\right) \leq \ln \frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}} . \tag{74}
\end{equation*}
$$

It follows from (60) and (74) that

$$
\begin{equation*}
\int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| \mathrm{d} t<2 \int_{0}^{\omega} \beta_{2}(t) e^{u_{2}(t)} \mathrm{d} t \leq \frac{2 \omega \alpha_{2}^{M} \beta_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}} . \tag{75}
\end{equation*}
$$

According to (74) and (75), it can be obtained that

$$
\begin{align*}
u_{2}(t) & \leq u_{2}\left(\xi_{2}\right)+\int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| \mathrm{d} t \\
& \leq \ln \frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}+\frac{2 \omega \alpha_{2}^{M} \beta_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}} . \tag{76}
\end{align*}
$$

It should be noted that

$$
\begin{align*}
& \int_{0}^{\omega} \alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}(t)} \mathrm{d} t  \tag{77}\\
& \quad=\int_{0}^{\omega} \alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(s) \mathrm{d} s} e^{u_{1}\left(t-\tau_{1}\right)} \mathrm{d} t .
\end{align*}
$$

Based on (64), it can be obtained that

$$
\begin{align*}
& \left(\alpha_{1}^{L} e^{-\gamma_{2}^{M} \tau_{1}}-\frac{c_{1}^{M} \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}\right) \int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t \\
& \quad \leq\left(\beta_{1}^{M}+\frac{2 \rho^{M} \alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}\right) \int_{0}^{\omega} e^{2 u_{1}(t)} \mathrm{d} t \tag{78}
\end{align*}
$$

which derives that

$$
\begin{gather*}
\int_{0}^{\omega} e^{u_{1}(t)} \mathrm{d} t \geq \frac{\omega\left(\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}-\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}\right)}{\beta_{1}^{M} \beta_{2}^{L}+2 \alpha_{2}^{M} \rho^{M} e^{-\gamma_{2}^{L} \tau_{2}}},  \tag{79}\\
u_{1}\left(\eta_{1}\right) \geq \ln \frac{\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}-\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{1}^{M} \beta_{2}^{L}+2 \alpha_{2}^{M} \rho^{M} e^{-\gamma_{2}^{L} \tau_{2}}}
\end{gather*}
$$

hold provided that $\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}>\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}$.
According to (69) and (79), it can be obtained that

$$
\begin{align*}
u_{1}(t) & \geq u_{1}\left(\eta_{1}\right)-\int_{0}^{\omega}\left|u_{1}^{\prime}(t)\right| \mathrm{d} t \\
& \geq \ln \frac{\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}-\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{1}^{M} \beta_{2}^{L}+2 \alpha_{2}^{M} \rho^{M} e^{-\gamma_{2}^{L} \tau_{2}}}-\frac{2 \omega \alpha_{1}^{M} \beta_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{1}^{L}} . \tag{80}
\end{align*}
$$

By virtue of (70) and (80), if $\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}>\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}$, then

$$
\begin{aligned}
& \max _{t \in[0, \omega]}\left|u_{1}(t)\right| \\
& <\max \left\{\begin{array}{c}
\left|\ln \frac{\alpha_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}}\right|+\frac{2 \omega \alpha_{1}^{M} \beta_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}}, \\
\left.\ln \frac{\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}-\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{1}^{M} \beta_{2}^{L}+2 \alpha_{2}^{M} \rho^{M} e^{-\gamma_{2}^{L} \tau_{2}}} \right\rvert\,+\frac{2 \omega \alpha_{1}^{M} \beta_{1}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L}}
\end{array}\right\}
\end{aligned}
$$

$$
\begin{equation*}
:=H_{1} . \tag{81}
\end{equation*}
$$

Similarly, it is easy to show that

$$
\begin{align*}
& \int_{0}^{\omega} \alpha_{2}(t) e^{-\int_{t}^{t+\tau_{2}} \gamma_{2}(s) \mathrm{d} s} e^{u_{2}(t)} \mathrm{d} t \\
& \quad=\int_{0}^{\omega} \alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} \gamma_{2}(s) \mathrm{d} s} e^{u_{2}\left(t-\tau_{2}\right)} \mathrm{d} t \tag{82}
\end{align*}
$$

which derives that

$$
\begin{equation*}
u_{2}\left(\eta_{2}\right) \geq \ln \frac{\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L} \beta_{2}^{M}} \tag{83}
\end{equation*}
$$

holds provided that $\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{1}}$.

According to (75) and (83), it is derived that

$$
\begin{align*}
u_{2}(t) & \geq u_{2}\left(\eta_{2}\right)-\int_{0}^{\omega}\left|u_{2}^{\prime}(t)\right| \mathrm{d} t \\
& \geq \ln \frac{\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L} \beta_{2}^{M}}-\frac{2 \omega \alpha_{2}^{M} \beta_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}} . \tag{84}
\end{align*}
$$

By virtue of (76) and (84), if $\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{2}}$, then

$$
\begin{align*}
& \max _{t \in[0, \omega]}\left|u_{2}(t)\right| \\
& <\max \left\{\begin{array}{l}
\left|\ln \frac{\alpha_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}\right|+\frac{2 \omega \alpha_{2}^{M} \beta_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}, \\
\left|\ln \frac{\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}-\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{1}}}{\beta_{1}^{L} \beta_{2}^{M}}\right|+\frac{2 \omega \alpha_{2}^{M} \beta_{2}^{M} e^{-\gamma_{2}^{L} \tau_{2}}}{\beta_{2}^{L}}
\end{array}\right\} \\
& :=H_{2} . \tag{85}
\end{align*}
$$

It is obvious that $H_{1}$ and $H_{2}$ in (81) and (85) are independent of $\lambda$.

Step 2. In order to construct an appropriate open and bounded subset $\Omega$, denote $H=H_{1}+H_{2}+H_{0}$, where $H_{0}$ is sufficiently large such that the unique solution $\left(u^{*}, v^{*}\right)^{T}$ of the following algebraic equations:

$$
\begin{equation*}
\frac{1}{\omega} \int_{0}^{\omega} f_{1}(t) \mathrm{d} t=0, \quad \frac{1}{\omega} \int_{0}^{\omega} f_{2}(t) \mathrm{d} t=0, \tag{86}
\end{equation*}
$$

satisfies $\left\|\left(u^{*}, v^{*}\right)^{T}\right\|=\left|u^{*}\right|+\left|v^{*}\right|<H$.
Select $\Omega=\left\{\left(u_{1}(t), u_{2}(t)\right)^{T} \in X:\left\|\left(u_{1}, u_{2}\right)^{T}\right\|<H\right\}$, which implies that condition (i) of Lemma 10 holds.

When $\left(u_{1}(t), u_{2}(t)\right)^{T} \in \partial \Omega \cap \operatorname{Ker} L=\partial \Omega \cap \mathbb{R}^{2},\left(u_{1}, u_{2}\right)^{T}$ is a constant vector in $\mathbb{R}^{2}$ with $\left|u_{1}\right|+\left|u_{2}\right|=H$. Consequently, it can be concluded that

$$
Q N\left[\begin{array}{l}
u_{1}  \tag{87}\\
u_{2}
\end{array}\right]=\left[\begin{array}{l}
\frac{1}{\omega} \int_{0}^{\omega} f_{1}(t) \mathrm{d} t \\
\frac{1}{\omega} \int_{0}^{\omega} f_{2}(t) \mathrm{d} t
\end{array}\right] \neq\left[\begin{array}{l}
0 \\
0
\end{array}\right]
$$

which implies that condition (ii) of Lemma 10 is satisfied.
Take $J=I: \operatorname{Im} Q \rightarrow \operatorname{Ker} L,\left(u_{1}, u_{2}\right)^{T} \rightarrow\left(u_{1}, u_{2}\right)^{T}$. It follows from straightforward computation that

$$
\begin{equation*}
\operatorname{deg}\left(J Q N\left(u_{1}, u_{2}\right)^{T}, \Omega \cap \operatorname{Ker} L,(0,0)^{T}\right)=1 \tag{88}
\end{equation*}
$$

where $\left(u_{1}^{*}, u_{2}^{*}\right)$ is the unique solution of (86). Hence, the condition (iii) of Lemma 10 holds.

Furthermore, it is easy to see that the set $\left\{K_{p}(I-Q) N x \mid\right.$ $x \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela-Ascoli theorem [31], it can be shown that $K_{p}(I-$ Q) $N: \bar{\Omega} \rightarrow X$ is compact and $N$ is $L$-compact.

Consequently, all conditions (i)-(iii) of Lemma 10 hold for $\Omega$. It follows from Lemma 10 that model system (53) has at least one $\omega$-periodic solution $\left(u_{1}^{*}(t), u_{2}^{*}(t)\right)^{T}$, and model system (52) has at least one $\omega$-periodic solution $\left(x_{2}^{*}(t), y_{2}^{*}(t)\right)^{T}=$ $\left(e^{u_{1}^{*}(t)}, e^{u_{2}^{*}(t)}\right)^{T}$.

Let $\left(x_{2}^{*}(t), y_{2}^{*}(t)\right)^{T}$ be a positive $\omega$-periodic solution of model system (52); it follows from (19) and (23) that

$$
\begin{align*}
& x_{1}^{*}(t)=e^{-\int_{0}^{t} \gamma_{1}(s) \mathrm{d} s} \int_{t-\tau_{1}}^{t} \alpha_{1}(s) e^{\int_{0}^{s} \gamma_{1}(m) \mathrm{d} m} x_{2}^{*}(s) \mathrm{d} s,  \tag{89}\\
& y_{1}^{*}(t)=e^{-\int_{0}^{t} \gamma_{2}(s) \mathrm{d} s} \int_{t-\tau_{2}}^{t} \alpha_{2}(s) e^{\int_{0}^{s} \gamma_{2}(m) \mathrm{d} m} y_{2}^{*}(s) \mathrm{d} s
\end{align*}
$$

are $\omega$-periodic continuous function.
Based on the above analysis, if the following two inequalities hold:

$$
\begin{equation*}
\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}>\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}, \quad \alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{2}} \tag{90}
\end{equation*}
$$

then model system (4) with initial conditions (5) and (6) has at least one positive $\omega$-periodic solution $\left(x_{1}^{*}(t)\right.$, $\left.x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$.

### 3.3. Global Stability Analysis

Theorem 12. If $\liminf _{t \rightarrow+\infty} G_{k}(t)>0, k=1,2$, then model system (4) with initial conditions (5) and (6) has a unique positive $\omega$-periodic globally stable solution $\left(x_{1}^{*}(t)\right.$, $\left.x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$, where

$$
\begin{gather*}
G_{1}=-\alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}} \gamma_{1}(m) \mathrm{d} m}-2 c_{1}(t) M_{4}^{*} \\
+2 \beta_{1}(t) m_{2}^{*}-8 M_{2}^{*} M_{4}^{* 2} \rho(t) \\
G_{2}(t)=-q \alpha_{2}(t) e^{-\int_{t}^{t+\tau_{2}}} \gamma_{2}(m) \mathrm{d} m  \tag{91}\\
+2 q \beta_{2}(t) m_{4}^{*}+2 q c_{1}(t) m_{2}^{*}
\end{gather*}
$$

and $q$ is a positive constant and $m_{i}, M_{i}, i=1,2,3,4$, have been defined in (50).

Proof. Suppose that $\left(x_{1}^{*}(t), x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$ is a positive $\omega$-periodic solution of model system (4) with initial conditions (5) and (6).

Construct a Lyapunov functional as follows:

$$
\begin{align*}
V_{1}(t)=\left|x_{2}(t)-x_{2}^{*}(t)\right|+\int_{t-\tau_{1}}^{t} & \alpha_{1}(s) e^{-\int_{s}^{s+\tau_{1}}} \gamma_{1}(s) \mathrm{d} m  \tag{92}\\
& \times\left|x_{2}(s)-x_{2}^{*}(s)\right| \mathrm{d} s .
\end{align*}
$$

By calculating the upper right derivative of $V_{1}(t)$ along the positive $\omega$-periodic solution of model system (4), it can be obtained that

$$
\left.\begin{array}{rl}
D^{+} V_{1}(t)= & \operatorname{sgn}[
\end{array} x_{2}(t)-x_{2}^{*}(t)\right] .
$$

$$
\begin{align*}
& -\beta_{1}(t) x_{2}^{2}(t)+\beta_{1}(t) x_{2}^{* 2}(t) \\
& -c_{1}(t) x_{2}(t) y_{2}(t)+c_{1}(t) x_{2}^{*}(t) y_{2}^{*}(t) \\
& \left.-\rho(t) x_{2}^{2}(t) y_{2}^{2}(t)+\rho(t) x_{2}^{* 2}(t) y_{2}^{* 2}(t)\right\} \\
& +\alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}}} \gamma_{1}(m) \mathrm{d} m\left|x_{2}(t)-x_{2}^{*}(t)\right| \\
& -\alpha_{1}\left(t-\tau_{1}\right) e^{-\int_{t-\tau_{1}}^{t} \gamma_{1}(m) \mathrm{d} m} \\
& \times\left|x_{2}\left(t-\tau_{1}\right)-x_{2}^{*}\left(t-\tau_{1}\right)\right| \\
& \leq-\beta_{1}(t)\left[x_{2}(t)+x_{2}^{*}(t)\right]\left|x_{2}(t)-x_{2}^{*}(t)\right| \\
& +\alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}} \gamma_{1}(m) \mathrm{d} m}\left|x_{2}(t)-x_{2}^{*}(t)\right| \\
& -c_{1}(t) \operatorname{sgn}\left[x_{2}(t)-x_{2}^{*}(t)\right] \\
& \times\left[x_{2}(t) y_{2}(t)-x_{2}^{*}(t) y_{2}^{*}(t)\right] \\
& -\rho(t) \operatorname{sgn}\left[x_{2}(t)-x_{2}^{*}(t)\right] \\
& \times\left[x_{2}(t) y_{2}(t)+x_{2}^{*}(t) y_{2}^{*}(t)\right] \\
& \times\left[x_{2}(t) y_{2}(t)-x_{2}^{*}(t) y_{2}^{*}(t)\right] \\
& \leq-\left\{\beta_{1}(t)\left[\left(x_{2}(t)+x_{2}^{*}(t)\right)\right]\right. \\
& +c_{1}(t)\left[y_{2}(t)-y_{2}^{*}(t)\right] \\
& +\rho(t)\left(y_{2}(t)-y_{2}^{*}(t)\right) \\
& \times\left[\left(x_{2}(t)-x_{2}^{*}(t)\right) y_{2}(t)\right. \\
& \left.+\left(y_{2}(t)-y_{2}^{*}(t)\right) x_{2}^{*}(t)\right] \\
& \left.-\alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}} \gamma_{1}(m) \mathrm{d} m}\right\}\left|x_{2}(t)-x_{2}^{*}(t)\right| \text {. } \tag{93}
\end{align*}
$$

Similarly, construct another Lyapunov functional as follows:

$$
\begin{align*}
V_{2}(t)= & \left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& +\int_{t-\tau_{2}}^{t} \alpha_{2}(s) e^{-\int_{s}^{s+\tau_{2}} \gamma_{2}(s) \mathrm{d} m}\left|y_{2}(s)-y_{2}^{*}(s)\right| \mathrm{d} s . \tag{94}
\end{align*}
$$

By calculating the upper right derivative of $V_{2}(t)$ along the positive $\omega$-periodic solution of model system (4), it can be obtained that

$$
\left.\begin{array}{rl}
D^{+} V_{2}(t)= & \operatorname{sgn}[
\end{array} y_{2}(t)-y_{2}^{*}(t)\right] .
$$

$$
\begin{align*}
& +\alpha_{2}(t) e^{-\int_{t}^{t+\tau_{2}} \gamma_{2}(m) \mathrm{d} m}\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& -\alpha_{2}\left(t-\tau_{2}\right) e^{-\int_{t-\tau_{2}}^{t} y_{2}(m) \mathrm{d} m} \\
& \times\left|y_{2}\left(t-\tau_{2}\right)-y_{2}^{*}\left(t-\tau_{2}\right)\right| \\
\leq & -\left\{\beta_{2}(t)\left[y_{2}(t)+y_{2}^{*}(t)\right]+c_{1}(t)\left[x_{2}(t)-x_{2}^{*}(t)\right]\right\} \\
& \times\left|y_{2}(t)-y_{2}^{*}(t)\right| \\
& +\alpha_{2}(t) e^{-\int_{t}^{t+\tau_{2}} \gamma_{2}(m) \mathrm{d} m}\left|y_{2}(t)-y_{2}^{*}(t)\right| . \tag{95}
\end{align*}
$$

Let

$$
\begin{equation*}
V(t)=V_{1}(t)+q V_{2}(t), \tag{96}
\end{equation*}
$$

where $q$ is a positive constant.
By calculating the upper right derivative of $V(t)$ along the positive $\omega$-periodic solution of model system (4) based on (93) and (95), it can be obtained as follows:

$$
\begin{align*}
& D^{+} V(t) \leq\left|x_{2}(t)-x_{2}^{*}(t)\right| \\
& \times\left\{\alpha_{1}(t) e^{-\int_{t}^{t+\tau_{1}} \gamma_{1}(m) \mathrm{d} m}-\beta_{1}(t)\left[x_{2}(t)+x_{2}^{*}(t)\right]\right. \\
& +\left[y_{2}(t)-y_{2}^{*}(t)\right]\left[c_{1}(t)+\rho(t)\right. \\
& \\
& \times\left(\left(x_{2}(t)-x_{2}^{*}(t)\right) y_{2}(t)\right. \\
& \\
& +\left(y_{2}(t)-y_{2}^{*}(t)\right) \\
& \left.\left.\left.\times x_{2}^{*}(t)\right)\right]\right\} \\
& +\left|y_{2}(t)-y_{2}^{*}(t)\right|\left\{q \alpha_{2}(t) e^{-\int_{t}^{t+\tau_{2}} \gamma_{2}(m) \mathrm{d} m}-q \beta_{2}(t)\right.  \tag{97}\\
& \\
& \times
\end{align*}
$$

According to Theorem 7, there exists a positive value $T>$ 0 , when $t \geq T$; we get that

$$
\begin{array}{ll}
m_{2}^{*}-\epsilon<x_{2}(t)<M_{2}^{*}+\epsilon, & m_{2}^{*}-\epsilon<x_{2}^{*}(t)<M_{2}^{*}+\epsilon \\
m_{4}^{*}-\epsilon<y_{2}(t)<M_{4}^{*}+\epsilon, & m_{4}^{*}-\epsilon<y_{2}^{*}(t)<M_{4}^{*}+\epsilon \tag{98}
\end{array}
$$

hold for sufficiently small $\epsilon>0$.
Based on (98), when $t>T+\max \left\{\tau_{1}, \tau_{2}\right\}$, it is derived that

$$
\begin{align*}
D^{+} V(t) \leq & -\left(G_{1}(t)-\epsilon\right)\left|x_{2}(t)-x_{2}^{*}(t)\right| \\
& -\left(G_{2}(t)-\epsilon\right)\left|y_{2}(t)-y_{2}^{*}(t)\right| \tag{99}
\end{align*}
$$

where $G_{1}(t)$ and $G_{2}(t)$ have been defined in Theorem 12.
If $\liminf _{t \rightarrow+\infty} G_{k}(t)>0$ for $k=1,2$, then there exist two constants $\delta_{1}>0$ and $\delta_{2}>0$ such that for $t \geq T^{*}:=$ $T+2 \max \left\{\tau_{1}, \tau_{2}\right\}$

$$
\begin{equation*}
G_{1}(t) \geq \delta_{1}>0, \quad G_{2}(t) \geq \delta_{2}>0 \tag{100}
\end{equation*}
$$

Consequently, for $t \geq T^{*}$, we have

$$
\begin{equation*}
D^{+} V(t) \leq-\frac{\delta_{1}}{2}\left|x_{2}(t)-x_{2}^{*}(t)\right|-\frac{\delta_{2}}{2}\left|y_{2}(t)-y_{2}^{*}(t)\right| . \tag{101}
\end{equation*}
$$

By integrating both sides of (101) on the interval [ $T^{*}, t$, it can be obtained that, for $t \geq T^{*}$,

$$
\begin{align*}
V(t) & +\frac{\delta_{1}}{2} \int_{T^{*}}^{t}\left|x_{2}(s)-x_{2}^{*}(s)\right| \mathrm{d} s  \tag{102}\\
& +\frac{\delta_{2}}{2} \int_{T^{*}}^{t}\left|y_{2}(s)-y_{2}^{*}(s)\right| \mathrm{d} s \leq V\left(T^{*}\right)
\end{align*}
$$

Hence, $V(t)$ is bounded on the interval $\left[T^{*},+\infty\right)$ and

$$
\begin{align*}
& \int_{T^{*}}^{t}\left|x_{2}(s)-x_{2}^{*}(s)\right| \mathrm{d} s<+\infty \\
& \int_{T^{*}}^{t}\left|y_{2}(s)-y_{2}^{*}(s)\right| \mathrm{d} s<+\infty \tag{103}
\end{align*}
$$

According to Barbalat's Lemma [31], it can be concluded that

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left|x_{2}(t)-x_{2}^{*}(t)\right|=0, \quad \lim _{t \rightarrow \infty}\left|y_{2}(t)-y_{2}^{*}(t)\right|=0 . \tag{104}
\end{equation*}
$$

It follows from (19) and (23) that

$$
\begin{align*}
\left|x_{1}(t)-x_{1}^{*}(t)\right| & \leq \int_{t-\tau_{1}}^{t} \alpha_{1}(s) e^{\int_{t}^{s} \gamma_{1}(m) \mathrm{d} m}\left|x_{2}(s)-x_{2}^{*}(s)\right| \mathrm{d} s \\
& \leq \int_{t-\tau_{1}}^{t} \alpha_{1}^{M}\left|x_{2}(s)-x_{2}^{*}(s)\right| \mathrm{d} s \\
\left|y_{1}(t)-y_{1}^{*}(t)\right| & \leq \int_{t-\tau_{2}}^{t} \alpha_{2}(s) e^{\int_{t}^{s} \gamma_{2}(m) \mathrm{d} m}\left|y_{2}(s)-y_{2}^{*}(s)\right| \mathrm{d} s \\
& \leq \int_{t-\tau_{2}}^{t} \alpha_{2}^{M}\left|y_{2}(s)-y_{2}^{*}(s)\right| \mathrm{d} s . \tag{105}
\end{align*}
$$

Based on (104) and (105), it can be concluded that

$$
\begin{equation*}
\lim _{t \rightarrow \infty}\left|x_{1}(t)-x_{1}^{*}(t)\right|=0, \quad \lim _{t \rightarrow \infty}\left|y_{1}(t)-y_{1}^{*}(t)\right|=0 . \tag{106}
\end{equation*}
$$

Therefore, it follows from (104) and (106) that model system (4) with initial conditions (5) and (6) has a unique positive $\omega$-periodic globally stable solution $\left(x_{1}^{*}(t)\right.$, $\left.x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$.
3.4. Numerical Simulation. In this subsection, numerical simulations are carried out to substantiate the analytical findings obtained this paper. In order to facilitate the numerical simulations, $\omega$-periodic continuous functions introduced in model system (4) are selected as follows: $\alpha_{1}(t)=2.1+$ $\sin (t) / 10, \gamma_{1}(t)=0.2+\sin (t) / 200, \beta_{1}(t)=1+\sin (t) / 300$, $c_{1}(t)=0.2+\sin (t) / 300, \rho(t)=0.05+\sin (t) / 400, \alpha_{2}(t)=4.1+$ $\sin (t) / 18, \gamma_{2}(t)=0.1+\sin (t) / 580, \beta_{2}(t)=0.3+\sin (t) / 30$, and


Figure 1: Dynamical responses of the unique positive $2 \pi$-periodic globally stable solution of model system (4).
$c_{2}(t)=0.15+\sin (t) / 270$. The maturation delay for nontoxic species and toxin liberating species is given as follows: $\tau_{1}=$ 0.1 and $\tau_{2}=0.2$, respectively. By using straightforward computation, it can be found that $\alpha_{1}^{L} \beta_{2}^{L} e^{-\gamma_{2}^{M} \tau_{1}}>\alpha_{2}^{M} c_{1}^{M} e^{-\gamma_{2}^{L} \tau_{2}}$, $\alpha_{2}^{L} \beta_{1}^{L} e^{-\gamma_{2}^{M} \tau_{2}}>\alpha_{1}^{M} c_{2}^{M} e^{-\gamma_{1}^{L} \tau_{1}}$; then model system (4) has at least one positive $\omega$-periodic solution based on Theorem 11. Further computations show that $G_{1}(t) \geq 0.4281$ and $G_{2}(t) \geq 1.4006$. Consequently, it follows from Theorem 12 that model system (4) has a unique positive $2 \pi$-periodic globally stable solution $\left(x_{1}^{*}(t), x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$, whose dynamical responses are plotted in Figure 1. Furthermore, the unique positive $2 \pi$-periodic globally stable solution $\left(x_{1}^{*}(t), x_{2}^{*}(t), y_{1}^{*}(t), y_{2}^{*}(t)\right)^{T}$ is plotted in the $x_{1}-x_{2}$ plane and $y_{1}-y_{2}$ plane, which can be found in Figures 2 and 3, respectively. It should be noted that two different initial solutions are included to show the attractivity of different solutions.

## 4. Conclusion

In this paper, a nonautonomous dynamical model is proposed to investigate population dynamics of competitive system with toxin liberating species and nontoxic species, where stage structure and maturation delay for two species are considered. It is well known that the effect of toxin ecological systems is an important issue from mathematical and experimental points of view. Generally speaking, it takes some time for a species to reach maturity to produce the toxicant, the toxin liberating mature individual produces a substance toxic to the nontoxic mature individuals only, and the inhibiting effect is zero in absence of either species. Furthermore, the species compete each other for the limited life resource within closed environment, but this competition only happens among the mature individuals and does not


Figure 2: The unique positive $2 \pi$-periodic globally stable solution, which is plotted in the $x_{1}-x_{2}$ plane. Two different initial solutions are included to show the attractivity of different solutions, which are plotted in red and blue color, respectively.


Figure 3: The unique positive $2 \pi$-periodic globally stable solution, which is plotted in the $y_{1}-y_{2}$ plane. Two different initial solutions are included to show the attractivity of different solutions, which are plotted in red and blue color, respectively.
involve the immature individuals. Consequently, it is necessary to investigate the dynamic effect of stage structure and toxic substances on population dynamics of two-species competitive system.

Qualitative analysis of the proposed model system is discussed in the third section of this paper. It follows from Theorems 1 and 2 that solutions of model system (4) with initial conditions are positive and ultimately bounded. By utilizing some comparison arguments, an iterative technique is proposed to discuss permanence of the species; model system (4) is persistent, which can be found in Theorem 7.

Furthermore, existence of positive periodic solutions is considered in Theorem 11 based on continuation theorem of coincidence degree theory, which shows that model system (4) has at least one positive $\omega$-solution. By constructing an appropriate Lyapunov functional, sufficient conditions for global stability of the unique positive periodic solution are analyzed; that is, $\liminf _{t \rightarrow+\infty} G_{k}(t)>0, k=1,2$, which can be found in Theorem 12. Finally, numerical simulations are provided to show dynamical responses of the unique positive $2 \pi$-periodic globally stable solution, which are plotted in Figure 1. Furthermore, the unique positive $2 \pi$-periodic globally stable solution is plotted in the $x_{1}-x_{2}$ plane and $y_{1}-y_{2}$ plane, which can be found in Figures 2 and 3, respectively. Since biological phenomenon associated with stage structure and toxin substances extensively exists within competitive system, theoretical results obtained in this paper are theoretically beneficial to discuss dynamic effect of maturation delay and toxic effect on population dynamics; it makes this work done in this paper has some positive and new features.

## Conflict of Interests

All authors of this paper declare that there is no conflict of interests regarding the publication of this paper. They have no proprietary, financial, professional, or other personal interests of any nature or kind in any product, service, or company that could be construed as influencing the position presented in or the review of this paper.

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## Research Article

# Integral $\varphi_{0}$-Stability in terms of Two Measures for Impulsive Differential Equations with "Supremum" 

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This paper establishes a criterion on integral $\varphi_{0}$-stability in terms of two measures for impulsive differential equations with "supremum" by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Meantime, an example is given to illustrate our result.

## 1. Introduction

In this paper, we discuss the integral $\varphi_{0}$-stability in terms of two measures for impulsive differential equations with "supremum":

$$
\begin{gather*}
x^{\prime}=F\left(t, x(t) \sup _{s \in[t-r, t]} x(s)\right) \quad \text { for } t \geq 0, t \neq \tau_{k} \\
x\left(\tau_{k}+0\right)=I_{k}\left(x\left(\tau_{k}-0\right)\right) \quad \text { for } k=1,2, \ldots  \tag{1}\\
x(t)=\phi(t), \quad t \in\left[t_{0}-r, t_{0}\right]
\end{gather*}
$$

and its perturbed impulsive differential equations with "supremum"

$$
\begin{gather*}
x^{\prime}=F\left(t, x(t), \sup _{s \in[t-r, t]} x(s)\right)+G\left(t, x(t), \sup _{s \in[t-r, t]} x(s)\right) \\
\text { for } t \geq 0, \quad t \neq \tau_{k}, \\
x\left(\tau_{k}+0\right)=I_{k}\left(x\left(\tau_{k}-0\right)\right)+J_{k}\left(x\left(\tau_{k}-0\right)\right) \\
\text { for } k=1,2, \ldots, \\
x(t)=\phi(t), \quad t \in\left[t_{0}-r, t_{0}\right], \tag{2}
\end{gather*}
$$

where $x \in R^{n}, F, G: R^{+} \times R^{n} \times R^{n} \rightarrow R^{n}, F(t, 0,0)=$ $G(t, 0,0) \equiv 0, I_{k}, J_{k}: R^{n} \rightarrow R^{n}, I_{k}(0)=J_{k}(0) \equiv 0, k=$ $1,2, \ldots, r>0, t_{0} \in R^{+}$, and $\phi \in\left(P C\left[t_{0}-r, t_{0}\right], R^{n}\right)$. Let $R^{n}$ be $n$-dimensional Euclidean space with norm $\|x\|, R^{+}=[0, \infty)$, and $\left\{\tau_{k}\right\}_{1}^{\infty}$ a sequence of fixed points in $R^{+}$such that $\tau_{k+1}>\tau_{k}$ and $\lim _{k \rightarrow \infty} \tau_{k}=\infty$. We denote by $x\left(t ; t_{0}, \phi\right)$ the solution of (1). In our further investigation we will assume that solution $x\left(t ; t_{0}, \phi\right)$ is defined on $\left[t_{0}-r, \infty\right)$ for any initial function $\phi \in P C\left(\left[t_{0}-r, t_{0}\right], R^{n}\right)$.

The research on impulsive differential equations with "supremum" problem, Bainov et al. [1] justified the partial averaging for impulsive differential equations, He et al. [2] discussed the periodic boundary value problem for first order impulsive differential equations, Agarwal and Hristova [3] studied the strict stability in terms of two measures for impulsive differential equations, Stamova and Stamov [4] investigated the global stability of models based on impulsive differential equations and variable impulsive perturbations, and Hristova $[5,6]$ obtained the $\varphi_{0}$-stability in terms of two measures for impulsive differential equations.

In recent years, the integral stability theory has been rapid development (see [7-12]). For example, Soliman and Abdalla [10] introduced integral $\varphi_{0}$-stability of perturbed system of ordinary differential equations. Hristova [12] studied the integral stability in terms of two measures for impulsive differential equations with "supremum." However, the corresponding theory of impulsive differential equations with
"supremum" is still at an initial stage of its development, especially for integral $\varphi_{0}$-stability in terms of two measures. Motivated by the idea of [5, 6, 10, 12], in this work, by employing the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method, we extend the notions of $\varphi_{0}$-stability in terms of two measures to integral $\varphi_{0}$-stability in terms of two measures for impulsive differential equations with "supremum."

## 2. Preliminaries

Denote by $P C(X, Y)\left(X \subset R, Y \subset R^{n}\right)$ the set of all functions $u: X \rightarrow Y$ which are piecewise continuous in $X$ with points of discontinuity of the first kind at the points $\tau_{k} \in X$ and which are continuous from the left at the points $\tau_{k} \in$ $X, u\left(\tau_{k}\right)=u\left(\tau_{k}-0\right)$.

We denote by $P C^{1}(X, Y)$ the set of all function $u \in$ $P C(X, Y)$ which are continuously differentiable for $t \in X, t \neq$ $\tau_{k}$.

Let $x, y \in R^{n}$. Denote by $(x \cdot y)$ the dot product of both vectors $x$ and $y$.

Let $\mathscr{K} \subset R^{n}$ be a cone, and $\mathscr{K}^{*}=\left\{\varphi \in R^{n}:(\varphi \cdot x) \geq\right.$ 0 for any $x \in \mathscr{K}\}$ is adjoint cone.

We give the following notations for convenience:

$$
\begin{align*}
K= & \left\{a \in C\left(R^{+}, R^{+}\right):\right. \\
& a(s) \text { is strictly increasing, } a(0)=0\} ; \\
C K= & \left\{b \in C\left[R^{+} \times R^{+}, R^{+}\right]:\right. \\
& b(t, \cdot) \in K \text { for any fixed } t \in[0, \infty)\} ;  \tag{3}\\
\Gamma= & \left\{h \in C\left[[-r, \infty) \times R^{n}, \mathscr{K}\right]:\right. \\
& \left.\inf _{x \in R^{n}} h(t, x)=0 \text { for each } t \in[-r, \infty)\right\} .
\end{align*}
$$

Let $h_{0}, h \in \Gamma, \varphi_{0} \in \mathscr{K}^{*}, t \in R^{+}$, and $\phi \in P C\left(\left[t_{0}-\right.\right.$ $\left.r, t_{0}\right], R^{n}$ ). Define

$$
\begin{equation*}
H_{0}\left(t, \phi, \varphi_{0}\right)=\sup \left\{\left(\varphi_{0} \cdot h_{0}(t+s, \phi(t+s))\right): s \in[-r, 0]\right\} \tag{4}
\end{equation*}
$$

$$
\begin{equation*}
H\left(t, \phi, \varphi_{0}\right)=\sup \left\{\left(\varphi_{0} \cdot h_{0}(t+s, \phi(t+s))\right): s \in[-r, 0]\right\} \tag{5}
\end{equation*}
$$

Let $\rho, t$ and $T>0$ be constants, $\varphi_{0} \in \mathscr{K}^{*}, h \in \Gamma$. Define sets:

$$
\begin{align*}
& S\left(h, \rho, \varphi_{0}\right)=\left\{(t, x) \in R^{+} \times R^{n}:\left(\varphi_{0} \cdot h(t, x)\right)<\rho\right\} ; \\
& S^{c}\left(h, \rho, \varphi_{0}\right)=\left\{(t, x) \in R^{+} \times R^{n}:\left(\varphi_{0} \cdot h(t, x)\right) \geq \rho\right\} ; \\
& \Omega(t, T, \rho)=\left\{(x, y) \in R^{n} \times R^{n}:\right. \\
&\left(\varphi_{0} \cdot h(t, x)\right)<\rho \text { for } s \in[t, t+T] \\
&\left.\left(\varphi_{0} \cdot h(t, y)\right)<\rho \text { for } s \in[t-r, t+T]\right\} \tag{6}
\end{align*}
$$

In our further investigations we use the following comparison scalar impulsive ordinary differential equation:

$$
\begin{gather*}
u^{\prime}=g_{1}(t, u), \quad t \neq \tau_{k}, \\
u\left(\tau_{k}+0\right)=\xi_{k}\left(u\left(\tau_{k}\right)\right), \\
u\left(t_{0}\right)=u_{0}  \tag{7}\\
k=1,2, \ldots
\end{gather*}
$$

the scalar impulsive ordinary differential equation:

$$
\begin{gather*}
w^{\prime}=g_{2}(t, w), \quad t \neq \tau_{k} \\
w\left(\tau_{k}+0\right)=\eta_{k}\left(w\left(\tau_{k}\right)\right)  \tag{8}\\
w\left(t_{0}\right)=w_{0} \\
k=1,2, \ldots
\end{gather*}
$$

and its perturbed scalar impulsive ordinary differential equation:

$$
\begin{gather*}
w^{\prime}=g_{2}(t, w)+q(t), \quad t \neq \tau_{k}, \\
w\left(\tau_{k}+0\right)=\eta_{k}\left(w\left(\tau_{k}\right)\right)+\gamma_{k}\left(w\left(\tau_{k}\right)\right),  \tag{9}\\
w\left(t_{0}\right)=w_{0}, \\
k=1,2, \ldots,
\end{gather*}
$$

where $u, w \in R, g_{1}(t, 0)=g_{2}(t, 0) \equiv 0, \xi_{k}(0)=0, \eta_{k}(0)=$ $0, k=1,2, \ldots$..

Assume that solutions of the scalar impulsive equations (7), (8), and (9) exist on $\left[t_{0}, \infty\right.$ ) for any initial values. Meanwhile, we give some definitions and lemmas. The details can be found in [5].

Definition 1 (see [5]). We say that function $V(t, x):[-r, \infty) \times$ $R^{n} \rightarrow \mathscr{K}, V=\left(V_{1}, V_{2}, \ldots, V_{n}\right)$, belongs to the class $\Lambda$ if
(A1) $V(t, x) \in P C^{1}\left([-r, \infty) \times R^{n}, \mathscr{K}\right)$;
(A2) for each $k=1,2, \ldots$ and $x \in R^{n}$ there exist the finite limits

$$
\begin{equation*}
V\left(\tau_{k}-0, x\right)=\lim _{t \uparrow \tau_{k}} V(t, x), \quad V\left(\tau_{k}+0, x\right)=\lim _{t \downarrow \tau_{k}} V(t, x) \tag{10}
\end{equation*}
$$

(A3) there exist constants $M_{i}>0, i=1,2, \ldots, n$, such that $\left|V_{i}(t, x)-V_{i}(t, y)\right| \leq M_{i}\|x-y\|$ for any $t \in R^{+}, x, y \in$ $R^{n}$.

Definition 2 (see [5]). Let $\varphi_{0} \in \mathscr{K}^{*}, h \in \Gamma$ be given. The function $V(t, x) \in \Lambda$ is said to be $\varphi_{0}$-strongly $h$-decrescent if there exist a constant $\delta>0$ and a function $a \in K$ such that $(t, x) \in[-r, \infty) \times R^{n}:\left(\varphi_{0} \cdot h(t, x)\right)<\delta$ implies that $\left(\varphi_{0} \cdot V(t, x)\right) \leq a\left(\varphi_{0} \cdot h(t, x)\right)$.

Let $V(t, x) \in \Lambda, t \in \Omega, t \neq \tau_{k}, x \in R^{n}$, and $\phi \in P C([t-$ $\left.r, t], R^{n}\right)$. We define a derivative of the function $V(t, x)$ along the trajectory of solution of (1) as follows:

$$
\begin{align*}
& D_{(1)} V(t, \phi(t)) \\
& \quad=\lim _{\epsilon \rightarrow 0} \sup \frac{1}{\epsilon}\{V(t+\epsilon, \phi(t) \\
& \\
& \left.\quad+\epsilon F\left(t, \phi(t), \sup _{s \in[-r, 0]} \phi(t+s)\right)\right)  \tag{11}\\
& \\
& \quad-V(t, \phi(t))\} .
\end{align*}
$$

Similarly we define a derivative of the function $V(t, x) \in$ $\Lambda$ along the trajectory of solution of the perturbed system (2) for $t \in \Omega, t \neq \tau_{k}, x \in R^{n}$, and $\phi \in P C\left([t-r, t], R^{n}\right)$ as follows:

$$
\begin{align*}
& D_{(2)} V(t, \phi(t)) \\
& \quad=\lim _{\epsilon \rightarrow 0} \sup \frac{1}{\epsilon}\{V(t+\epsilon, \phi(t) \\
& \\
& +\quad \epsilon\left(F\left(t, \phi(t), \sup _{s \in[-r, 0]} \phi(t+s)\right)\right. \\
&  \tag{12}\\
& \left.\left.\quad+G\left(t, \phi(t), \sup _{s \in[-r, 0]} \phi(t+s)\right)\right)\right) \\
& \\
& \\
& \quad-V(t, \phi(t))\} .
\end{align*}
$$

Definition 3 (see [5]). Let $\varphi_{0} \in \mathscr{K}^{*}, h, h_{0} \in \Gamma$ be given. The function $h_{0}$ is $\varphi_{0}$-uniformly finer than $h$ if there exist a constant $\delta>0$ and a function $a \in K$, such that for any point $(t, x) \in[0, \infty) \times R^{n}:\left(\varphi_{0} \cdot h_{0}(t, x)\right)<\delta$ the inequality $\left(\varphi_{0} \cdot h(t, x)\right) \leq a\left(\varphi_{0} \cdot h_{0}(t, x)\right)$ holds.

Lemma 4 (see [5]). Let $h, h_{0} \in \Gamma, \varphi_{0} \in \mathscr{K}^{*}$ be given, and $h_{0}(t, x)$ is $\varphi_{0}$-uniformly finer than $h(t, x)$ with a constant $\delta$ and a function $a \in K$. Then for any $t \in R^{+}$and $\phi \in P C([t-$ $\left.r, t], R^{n}\right)$ inequality $H_{0}\left(t, \phi, \varphi_{0}\right)<\delta$ implies $H\left(t, \phi, \varphi_{0}\right) \leq$ $a\left(H_{0}\left(t, \phi, \varphi_{0}\right)\right)$, where functions $H$ and $H_{0}$ are defined by (4), (5).

In our further investigations we use the following comparison result.

Lemma 5 (see [5]). Let the following conditions be fulfilled.
(B1) The vector $\varphi_{0} \in \mathscr{K}^{*}$ and function $V \in \Lambda$ are such that
(i) for any number $t \geq 0: t \neq \tau_{k}$ and any function $\psi \in P C\left([t-r, t], R^{n}\right)$ such that $\left(\varphi_{0} \cdot V(t, \psi(t))\right) \geq$ $\left(\varphi_{0} \cdot V(t+s, \psi(t+s))\right)$ for $s \in[-r, 0)$ the inequality

$$
\begin{equation*}
\left(\varphi_{0} \cdot D_{(1)} V(t, \psi(t))\right) \leq g_{1}\left(t,\left(\varphi_{0} \cdot V(t, \psi(t))\right)\right) \tag{13}
\end{equation*}
$$

holds, where $g_{1} \in P C\left(R^{+} \times R^{+}, R^{+}\right)$.
(ii) $\left(\varphi_{0} \cdot V\left(\tau_{k}+0, I_{k}(x)\right)\right) \leq \xi_{k}\left(\varphi_{0} \cdot V\left(\tau_{k}, x\right)\right), k=$ $1,2, \ldots, x \in R^{n}$, and $\tau_{k} \in\left[t_{0}, T\right]$, where functions $\xi_{k} \in K$.
(B2) Function $x\left(t ; t_{0}, \phi\right)$ is a solution of (1) that is defined for $t \in\left[t_{0}-r, T\right]$, where $\phi \in P C\left(\left[t_{0}-r, t_{0}\right], R^{n}\right)$.
(B3) Function $u^{*}(t)=u^{*}\left(t ; t_{0}, u_{0}\right)$ is the maximal solution of (7) with initial condition $u^{*}\left(t_{0}\right)=u_{0}$ that is defined for $t \in\left[t_{0}, T\right]$.

Then the inequality $\sup _{s \in[-r, 0]}\left(\varphi_{0} \cdot V\left(t_{0}+s, \phi\left(t_{0}+s\right)\right)\right) \leq u_{0}$ implies the validity of the inequality $\left(\varphi_{0} \cdot V(t, x(t))\right) \leq u^{*}(t)$ for $t \in\left[t_{0}, T\right]$.

Definition 6. Let $h_{0}, h \in \Gamma$. System of impulsive differential equations with "supremum" (1) is said to be
(S1) $\left(H_{0}, h\right)$-equi-integral $\varphi_{0}$-stable if for every $\alpha \geq 0$ and for any $t_{0} \geq 0$ there exists a positive function $\beta=$ $\beta\left(t_{0}, \alpha\right) \in C K$ which is continuous in $t_{0}$ for each $\alpha$ and such that for maximal solution $y^{*}\left(t ; t_{0}, \phi\right)$ of the perturbed system of impulsive differential equations with "supremum" (2) the inequality

$$
\begin{equation*}
\left(\varphi_{0} \cdot h\left(t, y^{*}\left(t ; t_{0}, \phi\right)\right)\right)<\beta, \quad t \geq t_{0} \tag{14}
\end{equation*}
$$

holds, provided that

$$
\begin{equation*}
H_{0}\left(t_{0}, \phi, \varphi_{0}\right) \leq \alpha, \tag{15}
\end{equation*}
$$

and for every $T>0$,

$$
\begin{align*}
& \int_{t_{0}}^{t_{0}+T} \sup _{(x, y) \in \Omega\left(t_{0}, T, \beta\right)}\left\|G\left(s, x, y^{*}\right)\right\| d s  \tag{16}\\
& \quad+\sum_{t_{0} \leq \tau_{k} \leq t_{0}+T^{x}: h\left(\tau_{k}, x\right)<\beta} \sup _{k}\left\|J_{k}(x)\right\| \leq \alpha,
\end{align*}
$$

where $H_{0}\left(t_{0}, \phi, \varphi_{0}\right)$ is defined by (4) and $\phi \in P C\left(\left[t_{0}-\right.\right.$ $\left.\left.r, t_{0}\right], R^{n}\right)$;
(S2) $\left(H_{0}, h\right)$-uniform-integrally $\varphi_{0}$-stable if ( S 1 ) is satisfied, where $\delta$ is independent on $t_{0}$.

Remark 7. We note that in the case when $h_{0}(t, x) \equiv\|x\|$ and $h(t, x) \equiv\|x\|$ the ( $H_{0}, h$ )-equi-integral (uniform-integral) $\varphi_{0}{ }^{-}$ stability reduces to equi-integral (uniform-integral) $\varphi_{0}{ }^{-}$ stability.

## 3. Main Result

## Theorem 8. Let the following conditions be fulfilled.

(H1) Functions $h_{0}, h \in \Gamma ; h_{0}$ is $\varphi_{0}$-uniformly finer than $h$.
(H2) There exists a function $V_{1} \in \Lambda$ that is $\varphi_{0}$-strongly $h_{0}$ decrescent and
(i) for any number $t \geq 0, t \neq \tau_{k}$, and any function $\psi \in P C\left([t-r, t], R^{n}\right)$, such that $\left(\varphi_{0} \cdot V_{1}(t, \psi(t))\right)>$ $\left(\varphi_{0} \cdot V_{1}(t+s, \psi(t+s))\right)$ for $s \in[-r, 0)$ and $(t, \psi(t)) \in S\left(h, \rho, \varphi_{0}\right)$ the inequality

$$
\begin{equation*}
\left(\varphi_{0} \cdot D_{(1)} V_{1}(t, \psi(t))\right) \leq g_{1}\left(t,\left(\varphi_{0} \cdot V_{1}(t, \psi(t))\right)\right) \tag{17}
\end{equation*}
$$

holds, where $\rho>0$ is a constant.
(ii) $\left(\varphi_{0} \cdot V_{1}\left(\tau_{k}+0, I_{k}(x)\right)\right) \leq \xi_{k}\left(\varphi_{0} \cdot V_{1}\left(\tau_{k}, x\right)\right)$, for $\left(\tau_{k}, x\right) \in S\left(h, \rho, \varphi_{0}\right), k=1,2, \ldots$
(H3) For any number $\mu>0$ there exists a function $V_{2}^{(\mu)} \in \Lambda$ such that
(iii) $b\left(\varphi_{0} \cdot h(t, x)\right) \leq\left(\varphi_{0} \cdot V_{2}^{(\mu)}(t, x)\right) \leq a\left(\varphi_{0} \cdot h_{0}(t, x)\right)$ for $(t, x) \in[-r, \infty) \times R^{n}$, where $a, b \in K$ and $\lim _{u \rightarrow \infty} b(u)=\infty$.
(iv) For any number $t \geq 0, t \neq \tau_{k}$, and any function $\psi \in P C\left([t-r, t], R^{n}\right)$, such that $(t, \psi(t)) \in$ $S\left(h, \rho, \varphi_{0}\right) \cap S^{c}\left(h_{0}, \mu, \varphi_{0}\right)$ and $\left(\varphi_{0} \cdot\left(V_{1}(t, \psi(t))+\right.\right.$ $\left.\left.V_{2}^{(\mu)}(t, \psi(t))\right)\right)>\left(\varphi_{0} \cdot\left(V_{1}(t+s, \psi(t+s))+V_{2}^{(\mu)}(t+\right.\right.$ $s, \psi(t+s)))$ ) for $s \in[-r, 0)$ the inequality
$\left(\varphi_{0} \cdot\left(D_{(1)} V_{1}(t, \psi(t))+D_{(2)} V_{2}^{(\mu)}(t, \psi(t))\right)\right)$
$\leq g_{2}\left(t, \varphi_{0} \cdot\left(V_{1}(t, \psi(t))+V_{2}^{(\mu)}(t, \psi(t))\right)\right)$
holds.
(v) $\left(\varphi_{0} \cdot\left(V_{1}\left(\tau_{k}+0, I_{k}(x)\right)+V_{2}^{(\mu)}\left(\tau_{k}+0, I_{k}(x)\right)\right)\right) \leq$ $\eta_{k}\left(\varphi_{0} \cdot\left(V_{1}\left(\tau_{k}, x\right)+V_{2}^{(\mu)}\left(\tau_{k}, x\right)\right)\right)$ for $\left(\tau_{k}, x\right) \in$ $S\left(h, \rho, \varphi_{0}\right) \cap S^{c}\left(h_{0}, \mu, \varphi_{0}\right), k=1,2, \ldots$.
(H4) Zero solution of the scalar impulsive differential equation (7) is equi-stable.
(H5) Zero solution of the scalar impulsive differential equation (8) is uniform-integrally stable.

Then system of impulsive differential equations with "supremum" (1) is $\left(H_{0}, h\right)$-uniform-integrally $\varphi_{0}$-stable.

Proof. Since function $V_{1}(t, x)$ is $\varphi_{0}$-strongly $h_{0}$-decrescent, there exist a constant $\rho_{1} \in(0, \rho)$ and a function $\psi_{1} \in K$ such that $\left(\varphi_{0} \cdot h_{0}(t, x)\right)<\rho_{1}$ implies that

$$
\begin{equation*}
\left(\varphi_{0} \cdot V_{1}(t, x)\right) \leq \psi_{1}\left(\left(\varphi_{0} \cdot h_{0}(t, x)\right)\right) . \tag{19}
\end{equation*}
$$

Since $h_{0}(t, x)$ is $\varphi_{0}$-uniformly finer than $h(t, x)$, there exist a constant $\rho_{0} \in\left(0, \rho_{1}\right)$ and a function $\psi_{2} \in K$ such that $\left(\varphi_{0}\right.$. $\left.h_{0}(t, x)\right)<\rho_{0}$ implies that

$$
\begin{equation*}
\left(\varphi_{0} \cdot h(t, x)\right) \leq \psi_{2}\left(\varphi_{0} \cdot h_{0}(t, x)\right), \tag{20}
\end{equation*}
$$

where $\psi_{2}\left(\rho_{0}\right)<\rho_{1}$.
According to Lemma 4, the inequality $H_{0}\left(t, \phi, \varphi_{0}\right)<\rho_{0}$ implies

$$
\begin{equation*}
H\left(t, \phi, \varphi_{0}\right) \leq \psi_{2}\left(H_{0}\left(t, \phi, \varphi_{0}\right)\right), \quad \phi \in P C\left([t-r, t], R^{n}\right) . \tag{21}
\end{equation*}
$$

Let $t_{0} \geq 0$ be a fixed point. Choose a number $\alpha>0$ such that $\alpha<\rho_{0}$.

According to condition (H3) of Theorem 8, there exists a function $V_{2}^{(\alpha)}(t, x)$ that is Lipshitz with a constant $M_{2}$. Let $M_{1}$ be the Lipshitz constant of function $V(t, x)$.

Denote $\left(M_{1}+M_{2}\right) \alpha=\alpha_{1}$. Without loss of generality we assume $\alpha_{1}<b(\rho)$.

Since the zero solution of the scalar impulsive differential equation (7) is equi-stable, there exists a function $\delta_{1}=$ $\delta_{1}\left(t_{0}, \alpha_{1}\right)>0$ such that the inequality $\left|u_{0}\right|<\delta_{1}$ implies

$$
\begin{equation*}
\left|u\left(t ; t_{0}, u_{0}\right)\right|<\frac{\alpha_{1}}{2}, \quad t \geq t_{0} \tag{22}
\end{equation*}
$$

where $u\left(t ; t_{0}, u_{0}\right)$ is a solution of (7).
Since the function $\psi_{1} \in K$ there exists a $\delta_{2}=\delta_{2}\left(\delta_{1}\right)>$ $0, \delta_{2}<\rho_{1}$, such that for $|u|<\delta_{2}$ the inequality

$$
\begin{equation*}
\psi_{1}(u)<\delta_{1} \tag{23}
\end{equation*}
$$

holds.
Since the zero solution of the scalar impulsive differential equation (8) is uniform-integrally stable, there exists a function $\beta_{1}=\beta_{1}\left(\alpha_{1}\right) \in C K, b(\rho)>\beta_{1} \geq \alpha_{1}$, such that for every solution of the perturbed impulsive equation (9) the inequality

$$
\begin{equation*}
\left|w\left(t ; t_{0}, w_{0}\right)\right|<\beta_{1}, \quad t \geq t_{0} \tag{24}
\end{equation*}
$$

holds, provided that

$$
\begin{equation*}
\left|w_{0}\right|<\alpha_{1} \tag{25}
\end{equation*}
$$

and for every $T>0$,

$$
\begin{equation*}
\int_{t_{0}}^{t_{0}+T}|q(s)| d s+\sum_{t_{0} \leq \tau_{k} \leq t_{0}+T}\left|\gamma_{k}\right| \leq \alpha_{1} . \tag{26}
\end{equation*}
$$

Since the function $b \in K, \lim _{s \rightarrow \infty} b(s)=\infty$, and $\psi_{2}(\alpha)<$ $\psi_{2}\left(\rho_{0}\right)<\rho_{1}<\rho$, we could choose a constant $\beta=\beta\left(\beta_{1}\right)>$ $0, \rho>\beta>\alpha, \beta>\psi_{2}(\alpha)$, such that

$$
\begin{equation*}
b(\beta) \geq \beta_{1} . \tag{27}
\end{equation*}
$$

Since the function $a, \psi_{2} \in K$, and $\beta>\psi_{2}(\alpha)$, we can find a $\delta_{3}=\delta_{3}\left(\alpha_{1}, \beta\right)>0, \alpha<\delta_{3}<\min \left(\delta_{2}, \rho_{0}\right)$, such that the inequalities

$$
\begin{equation*}
a\left(\delta_{3}\right)<\frac{\alpha_{1}}{2}, \quad \psi_{2}\left(\delta_{3}\right)<\beta \tag{28}
\end{equation*}
$$

hold.
From (21) and (28) it follows that $H_{0}\left(t_{0}, \phi, \varphi_{0}\right)<\alpha$ implies

$$
\begin{equation*}
H\left(t_{0}, \phi, \varphi_{0}\right) \leq \psi_{2}\left(H_{0}\left(t_{0}, \phi, \varphi_{0}\right)\right)<\psi_{2}(\alpha)<\psi_{2}\left(\delta_{3}\right)<\beta ; \tag{29}
\end{equation*}
$$

that is, $h\left(t, \phi, \varphi_{0}\right)<\beta$ for $t \in\left[t_{0}-r, t_{0}\right]$.
Now let the initial functions $\phi \in P C\left(\left[t_{0}-r, t_{0}\right], R^{n}\right)$ be such that

$$
\begin{equation*}
H_{0}\left(t_{0}, \phi, \varphi_{0}\right)<\alpha \tag{30}
\end{equation*}
$$

and let the perturbed functions in impulsive equation with "supremum" (2) be such that

$$
\begin{align*}
& \int_{t_{0}}^{t_{0}+T} \sup _{x, y \in \Omega\left(t_{0}, T, \beta\right)}\left\|G\left(s, x, y^{*}\right)\right\| d s  \tag{31}\\
& \quad+\sum_{t_{0} \leq \tau_{k} \leq t_{0}+T}\left\|J_{k}(x)\right\| \leq \alpha
\end{align*}
$$

for every $T>0$.
Let $y^{*}(t)=y^{*}\left(t ; t_{0}, \phi\right)$ be a solution of (2), where the initial function and the perturbed functions satisfy (30) and (31); then

$$
\begin{equation*}
\left(\varphi_{0} \cdot h\left(t, y^{*}\left(t ; t_{0}, \phi\right)\right)\right)<\beta, \quad t \geq t_{0} \tag{32}
\end{equation*}
$$

Suppose it is not true. There exists a point $t^{*}>t_{0}$ such that

$$
\begin{array}{r}
\left(\varphi_{0} \cdot h\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right)=\beta \\
\left(\varphi_{0} \cdot h\left(t, y^{*}\left(t ; t_{0}, \phi\right)\right)\right)<\beta  \tag{33}\\
t \in\left[t_{0}, t^{*}\right)
\end{array}
$$

Case 1. Let $t^{*} \neq \tau_{k}, k=1,2, \ldots$. Then from the continuity of the maximal solution $y^{*}\left(t ; t_{0}, \phi\right)$ at point $t^{*}$ follows that $\left(\varphi_{0} \cdot h\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right)=\beta$.

If we assume that $\left(\varphi_{0} \cdot h_{0}\left(t^{*}, y^{*}\left(t^{*}\right)\right)\right) \leq \delta_{3}$ then from the choice of $\delta_{3}$ and inequality (28) it follows $\left(\varphi_{0}\right.$. $\left.h\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right) \leq\left(\varphi_{0} \cdot h_{0}\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right) \leq \psi_{2}\left(\delta_{3}\right)<\beta$ that contradicts (33).

Therefore

$$
\begin{equation*}
\left(\varphi_{0} \cdot h_{0}\left(t^{*}, y^{*}\left(t^{*}\right)\right)\right) \leq \delta_{3}, \quad H_{0}\left(t_{0}, \phi, \varphi_{0}\right)<\alpha<\delta_{1} . \tag{34}
\end{equation*}
$$

Case 1.1. Let there exist a point $t_{0}^{*} \in\left(t_{0}, t^{*}\right), t_{0}^{*} \neq \tau_{k}, k=$ $1,2, \ldots$, such that $\delta_{3}=\left(\varphi_{0} \cdot h_{0}\left(t^{*}, y^{*}\left(t^{*}\right)\right)\right)$ and $\left(t, y^{*}(t)\right) \in$ $S\left(h, \beta, \varphi_{0}\right) \cap S^{c}\left(h_{0}, \delta_{3}, \varphi_{0}\right)$. Since $\beta<\rho$ and $\delta_{3}>\alpha$ it follows that

$$
\begin{equation*}
\left(t, y^{*}(t)\right) \in S\left(h, \rho, \varphi_{0}\right) \cap S^{c}\left(h_{0}, \alpha, \varphi_{0}\right), \quad t \in\left[t_{0}^{*}, t^{*}\right) . \tag{35}
\end{equation*}
$$

Define a function $\phi^{*}(t)=y^{*}(t)$ for $t \in\left[t_{0}^{*}-r, t_{0}^{*}\right]$ and let $r_{1}\left(t ; t_{0}^{*}, u_{0}\right)$ be the maximal solution of impulsive scalar differential equation (7) where $u_{0}=\sup _{s \in[-r, 0]}\left(\varphi_{0}\right.$. $\left.V_{1}\left(t_{0}^{*}, \phi^{*}\left(t_{0}^{*}\right)\right)\right)$. Let $x^{*}(t) \equiv x^{*}\left(t ; t_{0}^{*}, \phi^{*}\right)$ be the solution of the impulsive equations (1), $t \in\left[t_{0}^{*}-r, t_{0}^{*}\right]$. From conditions (i), (ii) of Theorem 8, according to Lemma 5, it follows that

$$
\begin{equation*}
\left(\varphi_{0} \cdot V_{1}\left(t, x^{*}(t)\right)\right) \leq r_{1}\left(t ; t_{0}^{*}, u_{0}\right), \quad t \in\left[t_{0}^{*}, t^{*}\right] \tag{36}
\end{equation*}
$$

From the choice of the point $t_{0}^{*}$ it follows that $\left(\varphi_{0}\right.$. $\left.h_{0}\left(t_{0}^{*}, \phi^{*}\left(t_{0}^{*}\right)\right)\right)=\left(\varphi_{0} \cdot h_{0}\left(t_{0}^{*}, y^{*}\left(t_{0}^{*}\right)\right)\right)=\delta_{3}<\delta_{2}$.

According to inequalities (19) and (23) we obtain

$$
\begin{align*}
u_{0} & =\left(\varphi_{0} \cdot V_{1}\left(t_{0}^{*}, \phi^{*}\left(t_{0}^{*}\right)\right)\right)  \tag{37}\\
& \leq \psi_{1}\left(\varphi_{0} \cdot h_{0}\left(t_{0}^{*}, \phi^{*}\left(t_{0}^{*}\right)\right)\right)<\delta_{1} .
\end{align*}
$$

From inequalities (22) and (36) it follows that ( $\varphi_{0}$. $\left.V_{1}\left(t, x^{*}(t)\right)\right) \leq r_{1}\left(t ; t_{0}^{*}, u_{0}\right)<\alpha_{1} / 2$ for $t \in\left[t_{0}^{*}, t^{*}\right]$, or

$$
\begin{equation*}
\left(\varphi_{0} \cdot V_{1}\left(t_{0}^{*}, y^{*}\left(t_{0}^{*}\right)\right)\right)<\left(\varphi_{0} \cdot V_{1}\left(t_{0}^{*}, x\left(t_{0}^{*}\right)\right)\right)<\frac{\alpha_{1}}{2} \tag{38}
\end{equation*}
$$

From inequality (28) and condition (iii) of Theorem 8, it follows that

$$
\begin{align*}
\left(\varphi_{0} \cdot V_{2}^{(\alpha)}\left(t_{0}^{*}, y^{*}\left(t_{0}^{*}\right)\right)\right) & <a\left(\varphi_{0} \cdot h_{0}\left(t_{0}^{*}+s, y^{*}\left(t_{0}^{*}+s\right)\right)\right) \\
& =a\left(\delta_{3}\right)<\frac{\alpha_{1}}{2} \tag{39}
\end{align*}
$$

Consider function $V_{2}^{(\alpha)}(t, x)$ that is defined in condition (H7) of Theorem 8, and define the function

$$
\begin{equation*}
V(t, x)=V_{1}(t, x)+V_{2}^{(\alpha)}(t, x) \tag{40}
\end{equation*}
$$

the function $V(t, x)$ satisfies the conditions of Lemma 5. Let point $t \in\left[t_{0}^{*}, t^{*}\right], t \neq t_{k}$, and function $\psi \in P C([t-$ $\left.r, t], R^{n}\right)$ be such that $(t, \psi(t)) \in S\left(h, \beta, \varphi_{0}\right) \cap S^{c}\left(h_{0}, \alpha, \varphi_{0}\right)$, $\left(\psi(t), \sup _{s \in[-r, 0]} \psi(t+s)\right) \in \Omega\left(t_{0}^{*}, T^{*}, \beta\right)$, and $V(t, \psi(t))>$ $V(t+s, \psi(t+s))$ for $s \in[-r, 0)$. Then using the Lipshitz conditions for functions $V_{1}(t, x)$ and $V_{2}^{(\alpha)}(t, x)$, and condition (iv) of Theorem 8, we obtain

$$
\begin{align*}
\left(\varphi_{0}\right. & \left.\cdot D_{(2)} V(t, \psi(t))\right) \\
= & \left(\varphi_{0} \cdot\left(D_{(2)} V_{1}(t, \psi(t))+D_{(2)} V_{2}^{(\alpha)}(t, \psi(t))\right)\right) \\
\leq & \left(\varphi_{0} \cdot D_{(1)} V_{1}(t, \psi(t))+D_{(1)} V_{2}^{(\alpha)}(t, \psi(t))\right) \\
& +\left(M_{1}+M_{2}\right)\left\|G\left(t, \psi(t), \sup _{s \in[-r, 0]} \psi(t+s)\right)\right\|  \tag{41}\\
\leq & g_{2}\left(t,\left(\varphi_{0} \cdot V(t, \psi(t))\right)\right)+\left(M_{1}+M_{2}\right) \\
& \times \sup _{(x, y) \in \Omega\left(t_{0}^{*}, T^{*}, \beta\right)}\left\|G\left(t, x, y^{*}\right)\right\|,
\end{align*}
$$

where $T^{*}=t^{*}-t_{0}^{*}$.
Let $\tau_{k} \in\left(t_{0}^{*}, t^{*}\right), x \in R^{n}$ be such that $\left(\tau_{k}, x\right) \in S\left(h, \beta, \varphi_{0}\right) \cap$ $S^{c}\left(h_{0}, \alpha, \varphi_{0}\right)$. According to condition (v) of Theorem 8, we have

$$
\begin{align*}
\left(\varphi_{0}\right. & \left.\cdot V\left(t_{k}+0, I_{k}(x)+J_{k}(x)\right)\right) \\
= & \left(\varphi_{0} \cdot V\left(t_{k}+0, I_{k}(x)\right)\right) \\
& +\left(\varphi_{0} \cdot\left(V\left(t_{k}+0, I_{k}(x)+J_{k}(x)\right)-V\left(t_{k}+0, I_{k}(x)\right)\right)\right) \\
\leq & \eta_{k}\left(\varphi_{0} \cdot V\left(t_{k}, x\right)\right)+\left(M_{1}+M_{2}\right)\left\|J_{k}(x)\right\| \\
\leq & \eta_{k}\left(\varphi_{0} \cdot V\left(t_{k}, x\right)\right)+\left(M_{1}+M_{2}\right) \\
& \quad \times \sup _{x: h\left(\tau_{k}, x\right)<\beta}\left\|J_{k}(x)\right\| \tag{42}
\end{align*}
$$

According to inequalities (41), (42) and Lemma 5, the inequality

$$
\begin{equation*}
\left(\varphi_{0} \cdot V\left(t, y^{*}(t)\right)\right) \leq r^{*}\left(t ; t_{0}^{*}, w_{0}^{*}\right), \quad t \in\left[t_{0}^{*}, t^{*}\right] \tag{43}
\end{equation*}
$$

holds.

Consider the scalar impulsive differential equation (9), where

$$
\begin{gather*}
q(t)=\left(M_{1}+M_{2}\right) \sup _{(x, y) \in \Omega\left(t_{0}^{*}, T^{*}, \beta\right)}\left\|G\left(t, x, y^{*}\right)\right\|, \\
\gamma_{k}=\left(M_{1}+M_{2}\right) \sup _{x: h\left(\tau_{k}, x\right)<\beta}\left\|J_{k}(x)\right\| . \tag{44}
\end{gather*}
$$

According to above notations and inequality (31) for $T^{*}=$ $t^{*}-t_{0}^{*}$, we obtain

$$
\begin{equation*}
\int_{t_{0}^{*}}^{t^{*}} q(s) d s+\sum_{t_{0}^{*} \leq \tau_{k} \leq t^{*}} \gamma_{k} \leq\left(M_{1}+M_{2}\right) \alpha=\alpha_{1} \tag{45}
\end{equation*}
$$

Let $r^{*}\left(t ; t_{0}^{*}, w_{0}^{*}\right)$ be the maximal solution of (9) through the point $\left(t_{0}^{*}, w_{0}^{*}\right)$, where $w_{0}^{*}=V\left(t_{0}^{*}+s, y^{*}\left(t_{0}^{*}+s\right)\right)$, and perturbations $q(t)$ and $\gamma_{k}$ are defined above and satisfy inequality (45).

Choose a point $T^{*}>t^{*}$ such that

$$
\begin{equation*}
\int_{t_{0}^{*}}^{t^{*}} q(s) d s+\frac{1}{2}\left(T^{*}-t^{*}\right) q\left(t^{*}\right)<\alpha_{1} . \tag{46}
\end{equation*}
$$

Now define the continuous function $q^{*}(t):\left[t_{0}^{*}, \infty\right) \rightarrow$ $R$ :

$$
q^{*}(t)= \begin{cases}q(t) & \text { for } t \in\left[t_{0}^{*}, t^{*}\right]  \tag{47}\\ \frac{q\left(t^{*}\right)}{t^{*}-T^{*}}\left(t-T^{*}\right) & \text { for } t \in\left[t^{*}, T^{*}\right] \\ 0 & \text { for } t \geq T^{*}\end{cases}
$$

and the sequence of numbers $\left\{\gamma_{k}^{*}\right\}_{1}^{\infty}$ :

$$
\gamma_{k}^{*}= \begin{cases}\gamma_{k} & \text { for } k: \tau_{k} \in\left(t_{0}^{*}, t^{*}\right]  \tag{48}\\ 0 & \text { for } k: \tau_{k}>t^{*}\end{cases}
$$

From (45), it follows that for every $T>0$

$$
\begin{align*}
& \int_{t_{0}^{*}}^{t_{0}^{*}+T} q^{*}(s) d s+\sum_{t_{0}^{*} \leq \tau_{k} \leq t_{0}^{*}+T} \gamma_{k}^{*} \\
& \quad \leq \int_{t_{0}^{*}}^{t_{0}^{*}+T} q(s) d s+\sum_{t_{0}^{*} \leq \tau_{k} \leq t_{0}^{*}+T} \gamma_{k} \leq \alpha_{1} . \tag{49}
\end{align*}
$$

Let $R\left(t ; t_{0}^{*}, w_{0}^{*}\right)$ be the maximal solution of the scalar impulsive differential equation (9) through the point $\left(t_{0}^{*}, w_{0}^{*}\right)$, where perturbations of the right parts are defined above function $q^{*}(t)$ and numbers $\gamma_{k}^{*}$. We note that

$$
\begin{equation*}
R\left(t ; t_{0}^{*}, w_{0}^{*}\right)=r^{*}\left(t ; t_{0}^{*}, w_{0}^{*}\right), \quad t \in\left[t_{0}^{*}, t^{*}\right) \tag{50}
\end{equation*}
$$

From inequalities (38) and (39), the definition of point $w_{0}^{*}$, and inequality (49) follows the validity of (24) for the solution $R\left(t ; t_{0}^{*}, w_{0}^{*}\right)$; that is,

$$
\begin{equation*}
R\left(t ; t_{0}^{*}, w_{0}^{*}\right)<\beta_{1}, \quad t \geq t_{0}^{*} . \tag{51}
\end{equation*}
$$

From inequalities (43) and (51), equality (50), the choice of point $t^{*}$, and condition (iii) of Theorem 8, we obtain

$$
\begin{align*}
b(\beta) & \geq \beta_{1}>R\left(t^{*} ; t_{0}^{*}, w_{0}^{*}\right) \\
& \geq\left(\varphi_{0} \cdot V\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right) \\
& \geq\left(\varphi_{0} \cdot V_{2}^{(\alpha)}\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right)  \tag{52}\\
& \geq b\left(\left(\varphi_{0} \cdot h\left(t^{*}, y^{*}\left(t^{*} ; t_{0}, \phi\right)\right)\right)\right) \\
& =b(\beta) .
\end{align*}
$$

The obtained contradiction proves the validity of the inequality (32) for $t \geq t_{0}$.

Case 1.2. Let there exist a point $\tau_{k} \in\left(t_{0}, t^{*}\right)$ such that $\delta_{3}<\left(\varphi_{0}\right.$. $\left.h_{0}\left(\tau_{k}+0, y^{*}\left(\tau_{k}+0 ; t_{0}, x_{0}\right)\right)\right), \delta_{3}>\left(\varphi_{0} \cdot h_{0}\left(\tau_{k}, y^{*}\left(\tau_{k} ; t_{0}, x_{0}\right)\right)\right)$, and (35) is true.

We choose a number $\widetilde{\delta_{3}}: \delta_{3}<\widetilde{\delta_{3}}<\beta$ such that $\widetilde{\delta_{3}}=\left(\varphi_{0}\right.$. $\left.h_{0}\left(t_{0}^{*}, y^{*}\left(t_{0}^{*} ; t_{0}, x_{0}\right)\right)\right)$ and $t_{0}^{*} \in\left(t_{0}, t^{*}\right), t_{0}^{*} \neq \tau_{k}, k=1,2, \ldots$. We repeat the proof of Case 1.1, where instead of $\delta_{3}$ we use $\widetilde{\delta_{3}}$ and obtain a contradiction.

Case 2. Let there exist a natural number $k$ such that $\left(\varphi_{0}\right.$. $\left.h\left(t, y^{*}(t)\right)\right)<\beta$ for $t \in\left[t_{0}, \tau_{k}\right]$ and $\left(\varphi_{0} \cdot h\left(\tau_{k}, y^{*}\left(\tau_{k}+0\right)\right)\right)=$ $\left(\varphi_{0} \cdot h\left(\tau_{k}, I_{k}\left(y^{*}\left(\tau_{k}\right)\right)+J_{k}\left(y^{*}\left(\tau_{k}\right)\right)\right)\right)>\beta$.

We repeat the proof of Case 1 as in this case we choose the constant $\beta=\beta\left(\beta_{1}\right)>0$, such that $b(\beta) \geq \sup _{k}\left\{\eta_{k}\left(\beta_{1}\right)\right\}$.

As in the proof of Case 1.1, we obtain the validity of inequalities (51) and (43). We apply conditions (iii) and (v) of Theorem 8 and obtain

$$
\begin{align*}
b(\beta) \geq & \eta_{k}\left(r^{*}\left(\tau_{k} ; t_{0}^{*}, w_{0}^{*}\right)\right) \\
\geq & \eta_{k}\left(\varphi_{0} \cdot V\left(\tau_{k}, y^{*}\left(\tau_{k}\right)\right)\right) \\
= & \eta_{k}\left(\left(\varphi_{0} \cdot\left(V_{1}\left(\tau_{k}, y^{*}\left(\tau_{k}\right)\right)+V_{2}^{(\alpha)}\left(\tau_{k}, y^{*}\left(\tau_{k}\right)\right)\right)\right)\right) \\
\geq & \left(\varphi_{0} \cdot V_{1}\left(\tau_{k}, I_{k}\left(y^{*}\left(\tau_{k}\right)\right)+J_{k}\left(y^{*}\left(\tau_{k}\right)\right)\right)\right. \\
& \left.\quad+V_{2}^{(\alpha)}\left(\tau_{k}, I_{k}\left(y^{*}\left(\tau_{k}\right)\right)+J_{k}\left(y^{*}\left(\tau_{k}\right)\right)\right)\right) \\
\geq & \left(\varphi_{0} \cdot V_{2}^{(\alpha)}\left(\tau_{k}, I_{k}\left(y^{*}\left(\tau_{k}\right)\right)+J_{k}\left(y^{*}\left(\tau_{k}\right)\right)\right)\right) \\
\geq & b\left(\varphi_{0} \cdot h\left(\left(\tau_{k}, I_{k}\left(y^{*}\left(\tau_{k}\right)\right)+J_{k}\left(y^{*}\left(\tau_{k}\right)\right)\right)\right)\right) \\
> & b(\beta) \tag{53}
\end{align*}
$$

and the obtained contradiction proves the validity of inequality (32) in this case. Inequality (32) proves ( $H_{0}, h$ )-uniformintegral $\varphi_{0}$-stabilities of the considered system of the impulsive differential equations with "supremum."

Next, we will provide an example which satisfies all the hypotheses of Theorem 8.

Example 9. Consider the system of impulsive differential equations with "supremum"

$$
\begin{gather*}
x^{\prime}=-e^{-t} x(t)+2 y(t)+e^{-t} \max _{s \in[t-r, t]} x(s), \quad t \neq k \\
y^{\prime}=-x(t)-e^{-t} y(t)+\frac{1}{2} e^{-t} \max _{s \in[t-r, t]} y(s), \quad t \neq k, \\
x(k+0)=\frac{1}{2^{k / 2}} x(k), \\
y(k+0)=\frac{1}{2^{k / 2}} y(k),  \tag{54}\\
k=1,2, \ldots, \\
x(t)=\phi_{1}\left(t-t_{0}\right), \\
y(t)=\phi_{2}\left(t-t_{0}\right) \\
t \in\left[t_{0}-r, t_{0}\right]
\end{gather*}
$$

and its perturbed impulsive differential equations with "supremum"

$$
\begin{gather*}
x^{\prime}=-e^{-t} x(t)+2 y(t)+e^{-t} \max _{s \in[t-r, t]} x(s)+e^{-t} \max _{s \in[t-r, t]} x^{2}(s) \\
t \neq k, \\
y^{\prime}=-x(t)-e^{-t} y(t)+\frac{1}{2} e^{-t} \max _{s \in[t-r, t]} y(s)+e^{-t} \max _{s \in[t-r, t]} y^{2}(s), \\
t \neq k \\
x(k+0)=\frac{1}{2^{k / 2}} x(k), \\
y(k+0)=\frac{1}{2^{k / 2}} y(k), \\
\quad k=1,2, \ldots, \\
x(t)=\phi_{1}\left(t-t_{0}\right), \quad y(t)=\phi_{2}\left(t-t_{0}\right) \quad t \in\left[t_{0}-r, t_{0}\right] \tag{55}
\end{gather*}
$$

where $x, y \in R, r>0$ is enough small constant, $t \geq t_{0} \geq 0$. Without loss of generality we will assume further that $1 \geq$ $t_{0} \geq 0$.

Let $h_{0}(t, x, y)=(\|x\|,\|y\|), h(t, x, y)=\left(x^{2}, y^{2}\right)$.
Consider function $V: R^{2} \quad \rightarrow \quad \mathscr{K}, V=$ $\left(V_{1}, V_{2}\right), V_{1}(x, y)=(1 / 2) x^{2}, V_{2}(x, y)=(1 / 2) y^{2}$, where $\mathscr{K}=\{(x, y): x \geq 0, y \geq 0\} \subset R^{2}$ is a cone.

Now, let us consider the vector $\varphi_{0}=(1,2)$. It is easy to check that the function $V_{1}(t, x, y)=V(x, y)$ is $\varphi_{0}$-strongly $h_{0}$-decrescent with a function $\psi_{2}=x \in K$ and the condition (iii) is satisfied for the function $V_{2}^{(\mu)}=V(x, y)$, where $b(u)=$ $(1 / 2) u$ and $a(u)=u^{2}$.

Let $t \geq 0, t \neq k, k=1,2 \ldots \psi \in P C\left([t-r, t], R^{2}\right), \psi=$ ( $\psi_{1}, \psi_{2}$ ) be such that the inequality

$$
\begin{align*}
& \left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right) \\
& \quad \geq\left(\varphi_{0} \cdot V\left(\psi_{1}(t+s), \psi_{2}(t+s)\right)\right), \quad s \in[-r, 0] \tag{56}
\end{align*}
$$

or

$$
\begin{equation*}
\frac{1}{2} \psi_{1}^{2}(t)+\psi_{2}^{2}(t) \geq \frac{1}{2} \psi_{1}^{2}(t+s)+\psi_{2}^{2}(t+s), \quad s \in[-r, 0] \tag{57}
\end{equation*}
$$

then

$$
\begin{align*}
& \psi_{1}(t) \max _{s \in[t-r, t]} \psi_{1}(s) \leq 2\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right), \\
& \psi_{2}(t) \max _{s \in[t-r, t]} \psi_{2}(s) \leq\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right) . \tag{58}
\end{align*}
$$

Therefore if inequality (57) is satisfied then

$$
\begin{align*}
& \left(\varphi_{0} \cdot D_{(54)} V\left(\psi_{1}(t), \psi_{2}(t)\right)\right) \\
& \begin{aligned}
&=e^{-t}( \left(-\left(\psi_{1}(t)\right)^{2}-2\left(\psi_{2}(t)\right)^{2}\right. \\
&\left.\quad+\psi_{1}(t) \max _{s \in[t-r, t]} \psi_{1}(s)+\psi_{2}(t) \max _{s \in[t-r, t]} \psi_{2}(s)\right) \\
& \leq e^{-t}\left(-\left(\psi_{1}(t)\right)^{2}-2\left(\psi_{2}(t)\right)^{2}+2\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right)\right. \\
&\left.\quad \quad+\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right)\right) \\
&=e^{-t}\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right)
\end{aligned}
\end{align*}
$$

or

$$
\begin{align*}
& \left(\varphi_{0} \cdot D_{(54)} V\left(\psi_{1}(t), \psi_{2}(t)\right)\right) \\
& \quad \leq e^{-t}\left(\varphi_{0} \cdot V\left(\psi_{1}(t), \psi_{2}(t)\right)\right) \tag{60}
\end{align*}
$$

Inequality (60) proves the validity of condition (i) of Theorem 8 for the function $V_{1}(t, x, y)=V(x, y)$, where $g_{1}(t, u)=u e^{-t}$. Meanwhile, inequality (60) proves the validity of condition (iv) of Theorem 8 for the function $V_{2}^{\mu}(t, x, y)=$ $V(x, y)$, where $g_{2}(t, u)=2 u e^{-t}$.

From jump conditions (54) and the choice of vector $\varphi_{0}$ and function $V$ we obtain the validity of conditions (ii) and (v) of Theorem 8 for the functions $V_{1}(t, x, y)=V(x, y)$ and $V_{2}^{\mu}(t, x, y)=V(x, y)$, where $\xi_{k}(u)=\left(1 / 2^{k}\right) u$ and $\eta_{k}(u)=$ $\left(1 / 2^{k}\right) u$.

Consider following comparison scalar impulsive differential equation:

$$
\begin{gather*}
u^{\prime}=u e^{-t}, \quad t \neq k, \quad u(k+0)=\frac{1}{2^{k}} u(k),  \tag{61}\\
w^{\prime}=2 w e^{-t}, \quad t \neq k, \quad w(k+0)=\frac{1}{2^{k}} w(k) . \tag{62}
\end{gather*}
$$

The solutions of the impulsive differential equation (61) and (62), correspondingly, are equi-stable and uniformintegrally stable. Thus, according to Theorem 8 the system of impulsive differential equations with "supremum" (54) is ( $H_{0}, h$ ) -uniform-integrally $\varphi_{0}$-stable.

## 4. Conclusion

This paper extends the notions of $\varphi_{0}$-stability in terms of two measures to integral $\varphi_{0}$-stability in terms of two measures for impulsive differential equations with "supremum" and establishes a criterion on integral $\varphi_{0}$-stability in terms of two measures for such system by using the cone-valued piecewise continuous Lyapunov functions, Razumikhin method, and comparative method. Finally, an example is given to illustrate our result.

## Conflict of Interests

The authors declare that they have no conflict of interests.

## Authors' Contribution

All authors completed the paper together. All authors read and approved the final paper.

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## Research Article

# Existence and Uniqueness of Globally Attractive Positive Almost Periodic Solution in a Predator-Prey Dynamic System with Beddington-DeAngelis Functional Response 

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#### Abstract

This paper is concerned with a predator-prey system with Beddington-DeAngelis functional response on time scales. By using the theory of exponential dichotomy on time scales and fixed point theory based on monotone operator, some simple conditions are obtained for the existence of at least one positive (almost) periodic solution of the above system. Further, by means of Lyapunov functional, the global attractivity of the almost periodic solution for the above continuous system is also investigated. The main results in this paper extend, complement, and improve the previously known result. And some examples are given to illustrate the feasibility and effectiveness of the main results.


## 1. Introduction

Let

$$
\begin{gather*}
f^{-}=\inf _{s \in \mathbb{T}} f(s), \quad f^{+}=\sup _{s \in \mathbb{U}} f(s), \\
m(f)=\lim _{l \rightarrow \infty} \frac{1}{l} \int_{0}^{l} f(s) \mathrm{d} s, \tag{1}
\end{gather*}
$$

where $f$ is a continuous bounded function defined on $\mathbb{T}$ and $\mathbb{T}$ is a time scale.

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. One significant component of the predator-prey relationship is the functional responses. In general, the functional responses can be either prey dependent or predator dependent. However, the preydependent ones fail to model the interference among predators and have been facing challenges from the biology and physiology communities. The predator-dependent functional responses can provide better descriptions of predator feeding
over a range of predator-prey abundances as is supported by much significant laboratory and field evidence. The Beddington-DeAngelis functional response, first proposed by Beddington [1] and DeAngelis et al. [2], performed even better. So, the dynamics of predator-prey systems with the Beddington-DeAngelis response have been studied extensively in the literature [3-10].

In [8], Cui and Takeuchi considered the following predator-prey system with Beddington-DeAngelis functional response:

$$
\begin{gather*}
x^{\prime}(t)=x(t)[a(t)-b(t) x(t) \\
\left.-\frac{c(t) y(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right]  \tag{2}\\
y^{\prime}(t)=y(t)\left[-d(t)+\frac{f(t) x(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right]
\end{gather*}
$$

where all the coefficients of system (2) are positive $\omega$-periodic functions. Cui and Takeuchi obtained the following.

Theorem 1 (see [8]). System (2) has at least one positive $\omega$ periodic solution provided

$$
\begin{gather*}
(C) \int_{0}^{\omega}\left[-d(t)+\frac{f x_{0}(t)}{\alpha(t)+\beta(t) x_{0}(t)}\right] \mathrm{d} t>0, \quad \text { where } \\
x_{0}(t)=\frac{1-e^{-\int_{0}^{\omega} a(s) \mathrm{d} s}}{\int_{0}^{\omega} b(t-s) e^{-\int_{0}^{s} a(t-u) \mathrm{d} u} \mathrm{~d} s} . \tag{3}
\end{gather*}
$$

In real world phenomenon, the environment varies due to the factors such as seasonal effects of weather, food supplies, mating habits, and harvesting. So it is usual to assume the periodicity of parameters in system (2). However, if the various constituent components of the temporally nonuniform environment are with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since the assumption of almost periodicity is more realistic, more important, and more general when we consider the effects of the environmental factors. However, to the best of the author's knowledge, up to date, there are few works on the existence of positive almost periodic solution of system (2). Therefore, the aim of this paper is to use the fixed point theory based on monotone operator and Lyapunov functional to investigate the positive (almost) periodic solutions of system (2).

In fact, continuous and discrete systems are very important in implementing and applications. It is well known that the theory of time scales has received a lot of attention which was introduced by Hilger [11] in order to unify continuous and discrete analyses. Therefore, it is meaningful to study dynamic systems on time scales which can unify differential and difference systems. Recently, the topic on the dynamics of predator-prey system with Beddington-DeAngelis functional response on time scales has been investigated in some papers (see $[9,10]$ ). Stimulated by the previous reasons, in this paper we will study the following predator-prey system with Beddington-DeAngelis functional response on time scales:

$$
\begin{align*}
& x^{\Delta}(t)=x(t)[a(t)-b(t) x(t) \\
&\left.-\frac{c(t) y(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right]  \tag{4}\\
& y^{\Delta}(t)=y(t)\left[-d(t)+\frac{f(t) x(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right]
\end{align*}
$$

where $t \in \mathbb{T}$ is a periodic time scale; all the coefficients of system (4) are nonnegative almost periodic functions. From the point of view of biology, we focus our discussion on the existence of positive almost periodic solution of system (4) by using the theory of exponential dichotomy on time scales and fixed point theory based on monotone operator. Further, with the help of Lyapunov functional, the global attractivity of a unique positive almost periodic solution of system (2) is considered.

The remainder of this paper is organized in the following ways. In Section 2, we will introduce some necessary notations, definitions, and lemmas which will be used in the paper.

In Section 3, some easy conditions are derived ensuring the existence of at least one positive (almost) periodic solution of system (4) by using the theory of exponential dichotomy on time scales and fixed point theorem of monotone operator. In Section 4, we establish sufficient conditions for the global attractivity of a unique positive (almost) periodic solution of the corresponding continuous system (4) (i.e., system (2)) by means of Lyapunov functional. The main results are illustrated by giving some examples in Section 5.

## 2. Preliminaries

Now, let us state the following definitions and lemmas, which will be useful in proving our main result.

Definition 2 (see [12]). A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real set $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho: \mathbb{T} \rightarrow \mathbb{T}$ and the graininess $\mu, \nu: \mathbb{T} \rightarrow$ $\mathbb{R}^{+}$are defined, respectively, by

$$
\begin{align*}
\sigma(t):=\inf \{s \in \mathbb{T}: s>t\}, & \rho(t):=\sup \{s \in \mathbb{T}: s<t\}, \\
\mu(t):=\sigma(t)-t, & \nu(t):=t-\rho(t) . \tag{5}
\end{align*}
$$

The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, or right-scattered if $\rho(t)=t, \rho(t)<t, \sigma(t)=t$, or $\sigma(t)>t$, respectively. Points that are right-dense and left-dense at the same time are called dense. If $\mathbb{T}$ has a left-scattered maximum $m_{1}$, define $\mathbb{T}^{\kappa}=\mathbb{T}-\left\{m_{1}\right\}$; otherwise, set $\mathbb{T}^{\kappa}=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum $m_{2}$, define $\mathbb{T}_{\kappa}=\mathbb{T}$ - $\left\{m_{2}\right\}$; otherwise, set $\mathbb{T}_{\kappa}=\mathbb{T}$.

Definition 3 (see [12]). A function $p: \mathbb{T} \rightarrow \mathbb{R}$ is said to be regressive provided $1+\mu(t) p(t) \neq 0$ for all $t \in \mathbb{T}^{k}$, where $\mu(t)=$ $\sigma(t)-t$ is the graininess function. The set of all regressive rdcontinuous functions $f: \mathbb{T} \rightarrow \mathbb{R}$ is denoted by $\mathscr{R}$ while the set $\mathscr{R}^{+}$is given by $\{f \in \mathscr{R}: 1+\mu(t) f(t)>0\}$ for all $t \in \mathbb{T}$. Let $p \in \mathscr{R}$. The exponential function is defined by

$$
\begin{equation*}
e_{p}(t, s)=\exp \left(\int_{s}^{t} \xi_{\mu(\tau)}(p(\tau)) \Delta \tau\right) \tag{6}
\end{equation*}
$$

where $\xi_{h(z)}$ is the so-called cylinder transformation.
Lemma 4 (see [12]). Let $p, q \in \mathscr{R}$. Then
(i) $e_{0}(t, s) \equiv 1$ and $e_{p}(t, t) \equiv 1$;
(ii) $1 / e_{p}(t, s)=e_{\ominus p}(t, s)$, where $\Theta p(t)=-p(t) /(1+$ $\mu(t) p(t))$;
(iii) $e_{p}(t, s) e_{p}(s, r)=e_{p}(t, r)$;
(iv) $e_{p}^{\Delta}(\cdot, s)=p e_{p}(\cdot, s)$.

Definition 5 (see [12]). For $f: \mathbb{T} \rightarrow \mathbb{R}$ and $t \in \mathbb{T}^{k}$, the delta derivative of $f$ at $t$, denoted by $f^{\Delta}(t)$, is the number (provided
it exists) with the property that, given any $\epsilon>0$, there is a neighborhood $U \subset \mathbb{T}$ of $t$ such that

$$
\begin{array}{r}
\left|f(\sigma(t))-f(s)-f^{\Delta}(t)[\sigma(t)-t]\right| \leq \epsilon|\sigma(t)-s|  \tag{7}\\
\forall s \in U
\end{array}
$$

Lemma 6 (see [12]). Assume that $p(t) \geq 0$ for $t \geq 0$. Then $e_{p}(t, s) \geq 1$.

Lemma 7 (see [12]). Suppose that $p \in \mathscr{R}^{+}$. Then
(i) $e_{p}(t, s)>0$ for all $t, s \in \mathbb{T}$;
(ii) if $p(t) \leq q(t)$ for all $t \geq s, t, s \in \mathbb{T}$, then $e_{p}(t, s) \leq$ $e_{q}(t, s)$ for all $t \geq s$.

Lemma 8 (see [12]). Suppose that $p \in \mathscr{R}$ and $a, b, c \in \mathbb{T}$; then

$$
\begin{gather*}
{\left[e_{p}(c, \cdot)\right]^{\Delta}=-p\left[e_{p}(c, \cdot)\right]^{\sigma}} \\
\int_{a}^{b} p(t) e_{p}(c, \sigma(t)) \Delta t=e_{p}(c, a)-e_{p}(c, b) . \tag{8}
\end{gather*}
$$

Definition 9 (see [13]). A time scale $\mathbb{T}$ is called a periodic time scale if

$$
\begin{equation*}
\Pi:=\{\tau \in \mathbb{R}: t+\tau \in \mathbb{T}, \forall t \in \mathbb{T}\} \neq\{0\} . \tag{9}
\end{equation*}
$$

Definition 10 (see [14]). Let $\mathbb{T}$ be a periodic time scale. A function $x: \mathbb{T} \rightarrow \mathbb{R}^{n}$ is called almost periodic on $\mathbb{T}$, if, for any $\epsilon>0$, the set

$$
\begin{equation*}
E(\epsilon, x)=\{\tau \in \Pi:|x(t+\tau)-x(t)|<\epsilon, \forall t \in \mathbb{T}\} \tag{10}
\end{equation*}
$$

is relatively dense in $\mathbb{T}$; that is, there exists a constant $l=l(\epsilon)>$ 0 , for any interval with length $l(\epsilon)$; there exists a number $\tau=$ $\tau(\epsilon)$ in this interval such that

$$
\begin{equation*}
\|x(t+\tau)-x(t)\|<\epsilon, \quad \forall t \in \mathbb{T} \tag{11}
\end{equation*}
$$

The set $E(\epsilon, x)$ is called the $\epsilon$-translation set of $x ; \tau$ is called the $\epsilon$-translation number of $x$, and $l(\epsilon)$ is called the inclusion of $E(\epsilon, x)$.

Definition 11 (see [15]). Let $y \in C\left(\mathbb{T}, \mathbb{R}^{n}\right)$ and let $P(t)$ be $n \times n$ continuous matrix defined on $\mathbb{T}$. The linear system

$$
\begin{equation*}
y^{\Delta}(t)=P(t) y(t), \quad t \in \mathbb{T} \tag{12}
\end{equation*}
$$

is said to be an exponential dichotomy on $\mathbb{T}$ if there exist constants $k, \lambda>0$, projection $S$, and the fundamental matrix $Y(t)$ satisfying

$$
\begin{gather*}
\left\|Y(t) S Y^{-1}(s)\right\| \leq k e_{\ominus \lambda}(t, s), \quad \forall t \geq s \\
\left\|Y(t)(I-S) Y^{-1}(s)\right\| \leq k e_{\ominus \lambda}(s, t), \quad \forall t \leq s, t, s \in \mathbb{T} . \tag{13}
\end{gather*}
$$

Lemma 12 (see [16]). If the linear system $y^{\Delta}(t)=P(t) y(t)$ has an exponential dichotomy, then almost periodic system

$$
\begin{equation*}
y^{\Delta}(t)=P(t) y(t)+g(t), \quad t \in \mathbb{T} \tag{14}
\end{equation*}
$$

has a unique almost periodic solution $y(t)$ which can be expressed as follows:

$$
\begin{align*}
y(t)= & \int_{-\infty}^{t} Y(t) S Y^{-1}(\sigma(s)) g(s) \Delta s \\
& -\int_{t}^{\infty} Y(t)(I-S) Y^{-1}(\sigma(s)) g(s) \Delta s \tag{15}
\end{align*}
$$

Lemma 13 (see [15]). If $P(t)=\left(a_{i j}(t)\right)_{n \times n}$ is a uniformly bounded $r$ d-continuous matrix-valued function on $\mathbb{T}$ and there is a $\delta>0$ such that

$$
\begin{array}{r}
\left|a_{i i}(t)\right|-\sum_{j \neq i}\left|a_{i j}(t)\right|-\frac{1}{2} \mu(t)\left[\sum_{j \neq i}\left|a_{i j}(t)\right|\right]^{2}-\delta^{2} \mu(t) \geq 2 \delta \\
t \in \mathbb{T}, \quad i=1,2, \ldots, n \tag{16}
\end{array}
$$

then $y^{\Delta}(t)=P(t) y(t)$ admits an exponential dichotomy on $\mathbb{\mathbb { T }}$.
Lemma 14 (see [12]). Suppose that $r: \mathbb{T} \rightarrow \mathbb{R}$ is regressive. Let $t_{0} \in \mathbb{T}$ and $y_{0} \in \mathbb{R}$. The unique solution of the initial value problem

$$
\begin{equation*}
y^{\Delta}(t)=r(t) y(t)+g(t), \quad y\left(t_{0}\right)=y_{0} \tag{17}
\end{equation*}
$$

is given by

$$
\begin{equation*}
y(t)=e_{r}\left(t, t_{0}\right) y_{0}+\int_{t_{0}}^{t} e_{r}(t, \sigma(s)) g(s) \Delta s \tag{18}
\end{equation*}
$$

Similar to the proof as that in $[14,16]$, we can easily obtain from Lemmas 12-14 the following.

Lemma 15. Assume that $\left(H_{1}\right)-\left(H_{2}\right)$ hold; then system (4) has a unique almost periodic solution $z=(x, y)^{T}$ which can be expressed as follows:

$$
\begin{align*}
x(t)=\int_{t}^{+\infty} & e_{a}(t, \sigma(s)) x(s) \\
& \times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \\
y(t) & =\int_{-\infty}^{t} \frac{e_{-d}(t, \sigma(s)) f(s) x(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)} \Delta s \tag{19}
\end{align*}
$$

In order to obtain the existence of positive almost periodic solution of system (4), we first make the following preparations.

Let $E$ be a Banach space and let $K$ be a cone in $E$. The semiorder induced by the cone $K$ is denoted by " $\leq$ ". That is, $x \leq y$ if and only if $y-x \in K . x<y$ if $x \leq y$ and $x \neq y$. $x \gg y$ if $x-y \in \widehat{K}$, where $\widehat{K}$ is the interior of the cone $K$. A cone $K$ is called minihedral if, for any pair $\{x, y\}, x, y \in$ $E$, bounded above in order that there exists the least upper bound $\sup \{x, y\}$. A cone $K$ is called normal if there exists a constant $N>0$ such that $x \leq y, x, y \in K$ implies $\|x\|_{E} \leq$ $N\|y\|_{E}$.

Definition 16 (see [17]). $\Phi: K \rightarrow K$ is said to be monotone increasing, if, for $\forall x_{1}, x_{2} \in K, x_{1} \leq x_{2}$, one has $\Phi x_{1} \leq \Phi x_{2}$.

The following two lemmas cited from [18] are useful for the proof of our main results in this section.

Lemma 17 (see [18]). Let E be a real Banach space with an order cone $K$ satisfying the following:
(a) K has a nonempty interior,
(b) $K$ is normal and minihedral.

Assume that there are two points in $E, x_{*} \ll x^{*}$, and a monotone increasing complete continuous operator $\Phi:\left[x_{*}, x^{*}\right] \rightarrow$ E. If

$$
\begin{equation*}
\Phi x_{*} \ll x_{*}, \quad x^{*} \ll \Phi x^{*} \tag{20}
\end{equation*}
$$

then $\Phi$ has a fixed point $x \in\left[x_{*}, x^{*}\right]$. Here $\left[x_{*}, x^{*}\right]$ denotes the order interval $\left\{x \in K: x_{*} \leq x \leq x^{*}\right\}$.

Consider the Banach space $E=\operatorname{AP}\left(\mathbb{T}, \mathbb{R}^{n}\right)$ with the norm

$$
\begin{equation*}
\|x\|=\max \left\{x^{+}, y^{+}\right\}, \quad \forall z=(x, y)^{T} \in E . \tag{21}
\end{equation*}
$$

Define the cone $K$ in $E$ by

$$
\begin{equation*}
K=\left\{z=(x, y)^{T} \in E: x \geq 0, y \geq 0\right\} \tag{22}
\end{equation*}
$$

It is not difficult to verify that $K$ is normal and minihedral and has a nonempty interior.

Let the map $L$ be defined by

$$
\begin{equation*}
(L z)(t)=((\Phi z)(t),(\Psi z)(t))^{T} \tag{23}
\end{equation*}
$$

where

$$
\begin{align*}
& (\Phi z)(t) \\
& =\int_{t}^{+\infty} e_{a}(t, \sigma(s)) x(s) \\
& \times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \\
& \quad(\Psi z)(t)=\int_{-\infty}^{t} \frac{e_{-d}(t, \sigma(s)) f(s) x(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)} \Delta s \tag{24}
\end{align*}
$$

where $z \in K, t \in \mathbb{T}$.
By $\left(H_{1}\right)-\left(H_{2}\right)$, one could choose some positive constants $x_{*}<x^{*}$ and $y_{*}<y^{*}$ satisfying

$$
\begin{gathered}
{\left[b^{+} x_{*}+\frac{c^{+} y_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right]<a^{-},} \\
\frac{f^{+} x_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}<d^{-} \\
{\left[b^{-} x^{*}+\frac{c^{-} y^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}}\right]>a^{+}} \\
\frac{f^{-} x^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}}>d^{+} .
\end{gathered}
$$

Lemma 18. $L: D \rightarrow E$ is monotone increasing, where $D=$ $\left[z_{*}, z^{*}\right], z_{*}=\left(x_{*}, y_{*}\right)^{T}$, and $z^{*}=\left(x^{*}, y^{*}\right)^{T}$.

Proof. Let $F_{1}(t, x, y)=x[b(t) x+(c(t) y /(\alpha(t)+\beta(t) x+\gamma(t) y))]$ and $F_{2}(t, x, y)=(f(t) x y /(\alpha(t)+\beta(t) x+\gamma(t) y)), \forall t \in \mathbb{T}$. Then

$$
\begin{align*}
& (\Phi z)(t)=\int_{t}^{+\infty} e_{a}(t, \sigma(s)) F_{1}(s, x, y) \Delta s \\
& (\Psi z)(t)=\int_{-\infty}^{t} e_{-d}(t, \sigma(s)) F_{2}(s, x, y) \Delta s \tag{26}
\end{align*}
$$

Notice that

$$
\begin{gather*}
\frac{\partial F_{1}}{\partial x}=2 b(t) x+\frac{c(t) y[\alpha(t)+\gamma(t) y]}{[\alpha(t)+\beta(t) x+\gamma(t) y]^{2}} \geq 0 \\
\frac{\partial F_{1}}{\partial y}=\frac{c(t) x[\alpha(t)+\beta(t) x]}{[\alpha(t)+\beta(t) x+\gamma(t) y]^{2}} \geq 0  \tag{27}\\
\frac{\partial F_{2}}{\partial x}=\frac{f(t) y[\alpha(t)+\gamma(t) y]}{[\alpha(t)+\beta(t) x+\gamma(t) y]^{2}} \geq 0 \\
\frac{\partial F_{2}}{\partial y}=\frac{f(t) x[\alpha(t)+\beta(t) x]}{[\alpha(t)+\beta(t) x+\gamma(t) y]^{2}} \geq 0
\end{gather*}
$$

which implies that $(L z)(t)=((\Phi z)(t),(\Psi z)(t))^{T}$ is monotone increasing. This completes the proof.

Lemma 19. $\Phi: D \rightarrow E$ is complete continuous.
Proof. First, we show that $L$ maps bounded set into bounded sets. For $\forall z \in D$, we have

$$
\begin{align*}
\sup _{t \in \mathbb{T}}(\Phi z)(t) \leq & x^{*}\left[b^{+} x^{*}+\frac{c^{+} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right] \\
& \times \sup _{t \in \mathbb{T}} \int_{t}^{+\infty} e_{a^{-}}(t, \sigma(s)) \Delta s \\
\leq & \frac{1}{a^{-}} x^{*}\left[b^{+} x^{*}+\frac{c^{+} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right], \\
\sup _{t \in \mathbb{T}}(\Psi z)(t) \leq & \frac{f^{+} x^{*} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}} \sup \int_{-\infty}^{t} e_{-d^{-}}(t, \sigma(s)) \Delta s \\
\leq & \frac{1}{d^{-}} \frac{f^{+} x^{*} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}} . \tag{28}
\end{align*}
$$

That is, $L D$ is uniformly bounded. In addition, for $\forall t_{1}, t_{2} \in \mathbb{T}$ and $t_{1} \leq t_{2}$, notice that

$$
\begin{align*}
& \left|(\Phi z)\left(t_{1}\right)-(\Phi z)\left(t_{2}\right)\right| \\
& =\mid \int_{t_{1}}^{+\infty} e_{a}\left(t_{1}, \sigma(s)\right) x(s) \\
& \times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \\
& -\int_{t_{2}}^{+\infty} e_{a}\left(t_{1}, \sigma(s)\right) x(s) \\
& \times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \\
& +\int_{t_{2}}^{+\infty} e_{a}\left(t_{1}, \sigma(s)\right) x(s) \\
& \times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \\
& -\int_{t_{2}}^{+\infty} e_{a}\left(t_{2}, \sigma(s)\right) x(s) \\
& \left.\times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \right\rvert\, \\
& \leq \mid \int_{t_{1}}^{t_{2}} e_{a}\left(t_{1}, \sigma(s)\right) x(s) \\
& \left.\times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \right\rvert\, \\
& +\mid \int_{t_{2}}^{+\infty}\left[e_{a}\left(t_{2}, \sigma(s)\right)-e_{a}\left(t_{1}, \sigma(s)\right)\right] x(s) \\
& \left.\times\left[b(s) x(s)+\frac{c(s) y(s)}{\alpha(s)+\beta(s) x(s)+\gamma(s) y(s)}\right] \Delta s \right\rvert\, \\
& \leq x^{*}\left[b^{+} x^{*}+\frac{c^{+} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right]\left|t_{2}-t_{1}\right| \\
& +\frac{x^{*}}{a^{-}}\left[b^{+} x^{*}+\frac{c^{+} y^{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right] \\
& \times\left|1-e_{a^{+}}\left(t_{1}, t_{2}\right)\right| \longrightarrow 0, \quad \text { as } t_{1} \longrightarrow t_{2} . \tag{29}
\end{align*}
$$

Similarly, one could easily obtain that

$$
\begin{equation*}
\left|(\Psi z)\left(t_{1}\right)-(\Psi z)\left(t_{2}\right)\right| \longrightarrow 0, \quad \text { as } \quad t_{1} \longrightarrow t_{2} \tag{30}
\end{equation*}
$$

So $L z$ is equicontinuous for any $z \in D$. Using Arzela-Ascoli theorem on time scales [19], we obtain that $L D$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [20], it is easy to prove that $L$ is continuous. Hence, $L$ is complete continuous. The proof of this lemma is complete.

## 3. Almost Periodic Solution

In this section, we will utilize Lemma 17 which is given in the previous section to establish some sufficient criteria for the existence of positive (almost) periodic solutions of system (4).

## Theorem 20. Assume that the following conditions hold:

$$
\begin{aligned}
& \left(H_{1}\right) f^{-}>\beta^{+} d^{+} \\
& \left(H_{2}\right) a^{-}>0, d^{-}>0, \text { and } \alpha^{-}>0 .
\end{aligned}
$$

Then system (4) has at least one positive almost periodic solution.

Proof. Now, we should use Lemma 17 to prove the existence of positive almost periodic solutions of system (4). By Lemmas 18 and 19, we know that $L$ is a monotone increasing complete continuous operator on $D$. It remains to prove that

$$
\begin{equation*}
L z_{*} \ll z_{*}, \quad z^{*} \ll L z^{*} . \tag{31}
\end{equation*}
$$

On the one hand, by the definition of $z_{*}=\left(x_{*}, y_{*}\right)^{T}$, it follows that

$$
\begin{align*}
\Phi z_{*}= & \Phi\left(x_{*}, y_{*}\right)^{T} \\
= & \int_{t}^{\infty} e_{a}(t, \sigma(s)) x_{*} \\
& \times\left[b(s) x_{*}+\frac{c(s) y_{*}}{\alpha(s)+\beta(s) x_{*}+\gamma(s) y_{*}}\right] \Delta s \\
\leq & \frac{1}{a^{-}} x_{*}\left[b^{+} x_{*}+\frac{c^{+} y_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right] \\
< & x_{*},  \tag{32}\\
\Psi z_{*}= & \Psi\left(x_{*}, y_{*}\right)^{T} \\
= & \int_{-\infty}^{t} \frac{e_{-d}(t, \sigma(s)) f(s) x_{*} y_{*}}{\alpha(s)+\beta(s) x_{*}+\gamma(s) y_{*}} \Delta s \\
\leq & \frac{1}{d^{-}} \frac{f^{+} x_{*} y_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}} \\
< & y_{*}
\end{align*}
$$

which implies that

$$
\begin{equation*}
L z_{*}=\left(\Phi z_{*}, \Psi z_{*}\right)^{T}<\left(x_{*}, y_{*}\right)^{T}=z_{*} \Longrightarrow L z_{*} \ll z_{*} . \tag{33}
\end{equation*}
$$

On the other hand, one has from the definition of $z^{*}=$ $\left(x^{*}, y^{*}\right)^{T}$ that

$$
\begin{aligned}
\Phi z^{*} & =\Phi\left(x^{*}, y^{*}\right)^{T} \\
& =\int_{t}^{\infty} e_{a}(t, \sigma(s)) x^{*}
\end{aligned}
$$

$$
\begin{align*}
& \times\left[b(s) x^{*}+\frac{c(s) y^{*}}{\alpha(s)+\beta(s) x^{*}+\gamma(s) y^{*}}\right] \Delta s \\
\geq & \frac{1}{a^{+}} x^{*}\left[b^{-} x^{*}+\frac{c^{-} y^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}}\right] \\
> & x^{*}, \\
\Psi z^{*}= & \Psi\left(x^{*}, y^{*}\right)^{T}  \tag{34}\\
= & \int_{-\infty}^{t} \frac{e_{-d}(t, \sigma(s)) f(s) x^{*} y^{*}}{\alpha(s)+\beta(s) x^{*}+\gamma(s) y^{*}} \Delta s \\
\geq & \frac{1}{d^{+}} \frac{f^{-} x^{*} y^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}} \\
> & y^{*},
\end{align*}
$$

which implies that

$$
\begin{equation*}
L z^{*}=\left(\Phi z^{*}, \Psi z^{*}\right)^{T}>\left(x^{*}, y^{*}\right)^{T}=z^{*} \Longrightarrow L z^{*} \gg z^{*} \tag{35}
\end{equation*}
$$

Applying Lemma 17, we see that $L$ has at least one positive fixed point in $\left[z_{*}, z^{*}\right]$. Therefore, system (4) has at least one positive almost periodic solution. This completes the proof.

From Theorem 20, we can easily obtain the following.
Theorem 21. Assume that $\left(H_{1}\right)-\left(H_{2}\right)$ hold. Suppose further that all the coefficients of system (4) are nonnegative $\omega$-periodic functions; then system (4) has at least one positive $\omega$-periodic solution.

If $\mathbb{T}=\mathbb{R}$ in system (4), then Theorem 21 is changed to the following theorem.

Theorem 22. Assume that $\left(H_{1}\right)-\left(H_{2}\right)$ hold. Suppose further that all the coefficients of system (2) are nonnegative $\omega$-periodic functions; then system (2) has at least one positive $\omega$-periodic solution.

Remark 23. Clearly, the validity of condition (C) in Theorem 1 depends on coefficients $a, b, d, f, \alpha$, and $\beta$ of system (2). But condition $\left(H_{1}\right)$ in Theorem 22 only depends on coefficients $d, f$, and $\beta$. Therefore, compared with Theorem 1, Theorem 22 is easy to verify and then has an extensive application. Sometimes one cannot judge the existence of the periodic solution for some system in the form of (2) by Theorem 1 . However, it can be done by the result in the present theorem. The following example is given to illustrate this point in detail.

Example 24. Let $a(t)=0.1, b(t)=2, c(t)=2, d(t)=(1+$ $1 / 2 \sin t) / 10, f(t)=2, \alpha(t)=3+\sin t, \beta(t)=8+\sin t$, and $\gamma(t)=2+\cos t$; then system (2) becomes

$$
\begin{aligned}
& x^{\prime}(t) \\
& \quad=x(t)[0.1-2 x(t)
\end{aligned}
$$

$$
\left.-\frac{2 y(t)}{3+\sin t+(8+\sin t) x(t)+(2+\cos t) y(t)}\right]
$$

$$
y^{\prime}(t)
$$

$$
\begin{align*}
=y(t)[ & -\frac{1+1 / 2 \sin t}{10} \\
& \left.+\frac{2 x(t)}{3+\sin t+(8+\sin t) x(t)+(2+\cos t) y(t)}\right] . \tag{36}
\end{align*}
$$

We have $f^{-}=2>9 \times 0.15=\beta^{+} d^{+}$, which implies from Theorem 22 that system (36) has at least one positive $2 \pi$ periodic solution.

However, the assumption of Theorem 1 does not hold for system (36) because $x_{0} \equiv 0.05$ and

$$
\begin{align*}
-d(t)+\frac{f x_{0}(t)}{\alpha(t)+\beta(t) x_{0}(t)} & \leq-0.05+\frac{2 \times 0.05}{2+7 \times 0.05}  \tag{37}\\
& \approx-0.0074<0
\end{align*}
$$

Therefore one cannot judge the existence of positive periodic solution of system (36) by Theorem 1.

## 4. Global Attractivity

In this section, we will construct a suitable Lyapunov functional to establish some sufficient criteria for the global attractivity of a unique positive (almost) periodic solution of system (2).

Theorem 25. Assume that $\left(H_{1}\right)-\left(H_{2}\right)$ hold; suppose further that there exists a constant $\rho>0$ such that

$$
\begin{gather*}
\inf _{t \in \mathbb{R}}\left[b(t)-\frac{\alpha(t) f(t)+f(t) \gamma(t) y^{*}+c(t) \beta(t) y^{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y_{*}\right]}\right]>\rho, \\
\inf _{t \in \mathbb{R}}\left[\frac{\gamma(t) f(t) x_{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y^{*}\right]}\right. \\
 \tag{38}\\
\left.-\frac{\alpha(t) c(t)+c(t) \beta(t) x^{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y_{*}\right]}\right]>\rho,
\end{gather*}
$$

where $x_{*}, x^{*}, y_{*}$, and $y^{*}$ are defined as those in Theorem 20. Then system (2) has a unique positive almost periodic solution, which is globally attractive.

Proof. By Theorem 20, system (2) has a unique positive almost periodic solution $(x, y)^{T}$ satisfying

$$
\begin{equation*}
x_{*} \leq x(t) \leq x^{*}, \quad y_{*} \leq y(t) \leq y^{*}, \quad \forall t \in \mathbb{R} \tag{39}
\end{equation*}
$$

Suppose that $(u, v)^{T}$ is another positive solution of system (2). Define

$$
\begin{equation*}
V(t)=|\ln x(t)-\ln u(t)|+|\ln y(t)-\ln v(t)|, \quad \forall t \in \mathbb{R} \tag{40}
\end{equation*}
$$

Calculating the upper right derivatives of $V$ along the solution of system (2), it follows that

$$
\begin{align*}
& D^{+} V(t) \\
&= \operatorname{sgn}[x(t)-u(t)]\left[\frac{x^{\prime}(t)}{x(t)}-\frac{u^{\prime}(t)}{u(t)}\right] \\
&+\operatorname{sgn}[y(t)-v(t)]\left[\frac{y^{\prime}(t)}{y(t)}-\frac{v^{\prime}(t)}{v(t)}\right] \\
&= \operatorname{sgn}[x(t)-u(t)] \\
& \times\left(-b(t)[x(t)-u(t)]-\frac{c(t) y(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right. \\
&\left.+\frac{c(t) v(t)}{\alpha(t)+\beta(t) u(t)+\gamma(t) v(t)}\right) \\
&+\operatorname{sgn}[y(t)-v(t)] \\
& \times\left[\frac{f(t) x(t)}{\alpha(t)+\beta(t) x(t)+\gamma(t) y(t)}\right. \\
&\left.-\frac{f(t) u(t)}{\alpha(t)+\beta(t) u(t)+\gamma(t) v(t)}\right] \\
& \leq-\left(b(t)-\frac{\alpha(t) f(t)+f(t) \gamma(t) y^{*}+c(t) \beta(t) y^{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y_{*}\right]}\right) \\
& \quad \times|x(t)-u(t)| \\
& \leq-\rho[|x(t)-u(t)|+|y(t)-v(t)|] .
\end{align*}
$$

Therefore, $V$ is nonincreasing. Integrating (41) from 0 to $t$ leads to
$V(t)+\rho \int_{0}^{t}[|x(s)-u(s)|+|y(s)-v(s)|] \mathrm{d} s \leq V(0)<+\infty$, $t \in[0, \infty]$.

So

$$
\begin{equation*}
\int_{0}^{\infty}[|x(s)-u(s)|+|y(s)-v(s)|] \mathrm{d} s<+\infty \tag{43}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\lim _{s \rightarrow+\infty}|x(s)-u(s)|=\lim _{s \rightarrow+\infty}|y(s)-v(s)|=0 \tag{44}
\end{equation*}
$$

Thus, the almost periodic solution of system (2) is globally attractive. The global attractivity implies that the almost periodic solution is unique. This completes the proof.

From Theorem 25, we can easily obtain the following.
Theorem 26. Assume that all the conditions of Theorem 25 hold. Suppose further that all the coefficients of system (2) are nonnegative $\omega$-periodic functions; then system (2) has a unique positive $\omega$-periodic solution, which is globally attractive.

## 5. Two Examples

Example 27. Consider the following almost periodic predator-prey system with Beddington-DeAngelis functional response on time scales:

$$
\begin{align*}
& x^{\Delta}(t) \\
&= x(t) \\
& \times[0.1-2 x(t) \\
&\left.\quad-\frac{2 y(t)}{3+\sin t+(8+\sin (\sqrt{2} t)) x(t)+(2+\cos t) y(t)}\right], \\
& y^{\Delta}(t) \\
&= y(t) \\
& \quad \times\left[-\frac{1+1 / 2 \sin (\sqrt{3} t)}{10}\right. \\
& \quad+\frac{\left.+\operatorname{lin}^{2+\sin t+(8+\sin (\sqrt{2} t)) x(t)+(2+\cos t) y(t)}\right] .}{} \tag{45}
\end{align*}
$$

Similar to the argument as that in Example 24, system (45) has at least one positive almost periodic solution by Theorem 20.

Example 28. Consider the following almost periodic predator-prey system with Beddington-DeAngelis functional response:

$$
\begin{aligned}
& x^{\prime}(t) \\
& =x(t) \\
& \quad \times[1-5 x(t) \\
& \left.\quad-\frac{10^{-3} y(t)}{3+\sin (\sqrt{3} t)+0.1 x(t)+(2+\cos (\sqrt{2} t)) y(t)}\right]
\end{aligned}
$$

$$
\begin{align*}
& y^{\prime}(t) \\
& =y(t) \\
& \quad \times\left[-5+\frac{2 x(t)}{3+\sin (\sqrt{3} t)+0.1 x(t)+(2+\cos (\sqrt{2} t)) y(t)}\right] \tag{46}
\end{align*}
$$

Then system (46) has a unique positive almost periodic solution, which is globally attractive.

Proof. Corresponding to system (2), we have $a^{-}=a^{+}=1$, $b^{-}=b^{+}=5, d^{-}=d^{+}=5, c^{-}=c^{+}=10^{-3}, f^{-}=f^{+}=$ $2, \beta^{-}=\beta^{+}=0.1, \alpha^{-}=2, \alpha^{+}=4, \gamma^{-}=1$, and $\gamma^{+}=3$. Obviously, $f^{-}>\beta^{+} d^{+}$. By Theorem 20, system (46) has at least one positive almost periodic solution. Further, we choose $x_{*}=y_{*}=0.1, x^{*}=20$, and $y^{*}=0.3$; then

$$
\begin{gather*}
{\left[b^{+} x_{*}+\frac{c^{+} y_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}\right]<0.8<1=a^{-},} \\
\frac{f^{+} x_{*}}{\alpha^{-}+\beta^{-} x_{*}+\gamma^{-} y_{*}}<0.1<5=d^{-}, \\
{\left[b^{-} x^{*}+\frac{c^{-} y^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}}\right]>5>1=a^{+},}  \tag{47}\\
\frac{f^{-} x^{*}}{\alpha^{+}+\beta^{+} x^{*}+\gamma^{+} y^{*}}>\frac{40}{7}>5=d^{+},
\end{gather*}
$$

which implies that (25) hold. And

$$
\begin{align*}
& \inf _{t \in \mathbb{R}}\left[b(t)-\frac{\alpha(t) f(t)+f(t) \gamma(t) y^{*}+c(t) \beta(t) y^{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y_{*}\right]}\right] \\
& \quad>5-2.5=2.5 \\
& \inf _{t \in \mathbb{R}}\left[\frac{\gamma(t) f(t) x_{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y^{*}\right]}\right.  \tag{48}\\
& \left.\quad-\frac{\alpha(t) c(t)+c(t) \beta(t) x^{*}}{\alpha(t)\left[\alpha(t)+\beta(t) x_{*}+\gamma(t) y_{*}\right]}\right] \\
& \quad>0.0117-0.001=0.0107 .
\end{align*}
$$

Then all conditions of Theorem 25 are satisfied. By Theorem 25, system (46) has a unique positive almost periodic solution, which is globally attractive. This completes the proof.

## 6. Conclusion

In this paper, some sufficient conditions are established for the existence of positive almost periodic solution for a predator-prey system with Beddington-DeAngelis functional response on time scales by using the theory of exponential dichotomy on time scales and fixed point theory based on monotone operator. Further, the global attractivity of the almost periodic solution for the above continuous system is also investigated. The main results obtained in this paper are
completely new even in case of the time scale $\mathbb{T}=\mathbb{R}$ or $\mathbb{Z}$. Besides, the method used in this paper may be used to study the positive almost periodic solution of many other biological models.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Some New Results on the Lotka-Volterra System with Variable Delay 

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#### Abstract

This paper discusses the stochastic Lotka-Volterra system with time-varying delay. The nonexplosion, the boundedness, and the polynomial pathwise growth of the solution are determined once and for all by the same criterion. Moreover, this criterion is constructed by the parameters of the system itself, without any uncertain one. A two-dimensional stochastic delay Lotka-Volterra model is taken as an example to illustrate the effectiveness of our result.


## 1. Introduction

Population systems are often subject to environment noise. In our previous papers [1,2], we considered the following stochastic Lotka-Volterra system:

$$
\begin{align*}
d x(t)=\operatorname{diag}(x(t))\{ & \{a+A x(t)+B y(t)] d t \\
& +[b+D x(t)+E y(t)] d w(t)\} \tag{1}
\end{align*}
$$

and its functional form, where $y(t)=x(t-\delta(t))$ with $\delta(t)$ representing variable delay and $\operatorname{diag}(x)=\operatorname{diag}\left(x_{1}, \ldots, x_{n}\right)$ represents the $n \times n$ matrix with all elements zero except those on the diagonal which are $x_{1}, \ldots, x_{n} . a, b \in \mathbb{R}^{n}$ and matrices $A, B, D$, and $E \in \mathbb{R}^{n \times n}$.

Equation (1) may describe dynamics of $n$ species interaction, in which $x_{i}(t)(1 \leq i \leq n)$ represents the population size of $i$ th species depending both on the current states $x(t)$ and on the past state $x(t-\delta(t))$ of all population. From the point of biological view, the following three properties are very important.
(A) The solution of system (1) is positive and nonexplosive; namely, for any positive initial data $\xi$, (1) has a unique positive global solution $x(t, \xi)$.
(B) The solution of system (1) is ultimately moment bounded and time average moment bounded; that is, this global solution $x(t, \xi)$ of (1) satisfies

$$
\begin{gather*}
\limsup _{t \rightarrow \infty} \mathbb{E}|x(t, \xi)| \leq K  \tag{2}\\
\limsup _{t \rightarrow \infty} \frac{1}{t} \int_{0}^{t} \mathbb{E}|x(t, \xi)|^{2} d s \leq L \tag{3}
\end{gather*}
$$

where $K$ and $L$ are positive constants independent of $\xi$. These two properties show that, in the sense of average, population size is bounded.
(C) The solution of the system (2) grows at most polynomially; namely, this solution $x(t, \xi)$ of (1) satisfies

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{\log |x(t, \xi)|}{\log t} \leq 1, \quad \text { a.s. } \tag{4}
\end{equation*}
$$

There is an extensive literature concerned with these properties of stochastic Lotka-Volterra models. For example, Mao and his coauthors [3-5] discussed the existence and uniqueness of the global positive solution, stochastically ultimate boundedness, and some other asymptotic properties for the stochastic Lotka-Volterra system. References [6, 7] discovered that the presence of the environmental noise may
suppress the potential explosion of the solution in finite time. In our previous work [2], we showed that the environmental noise structure determined whether properties (A)-(C) were affected by the stochastic perturbation parameters or not. In our previous work [1], these three properties were also examined. In this paper, our conclusions will be improved in the following aspects.
(i) In these published works, properties (A)-(C) were given under different conditions, respectively. In this paper, we will give these three properties under the same group of conditions. This is an important improvement since properties (B) and (C) do not imply each other in general.
(ii) In this paper, we will present the conditions, which are easier to be verified, to guarantee properties (A)(C). In these conditions, all parameters are from the models and do not include any uncertain parameters to be determined.

The rest of the paper is arranged as follows. In the next section, we provide some necessary notations and lemmas. Section 3 gives several lemmas to support the main results of this paper. By using Lemmas established in Section 3, Section 4 presents the conditions under which the all desired properties (A)-(C) hold. In Section 5, some simplified cases of model (1) are investigated. Although these models are less general than (1), they have wide applications and satisfy properties (A)-(C) under more simple conditions, which are provided as corollaries of the main theorems. A twodimensional stochastic Lotka-Volterra population model will be examined as an example in Section 6.

## 2. Preliminaries

Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$ satisfying the usual conditions; that is, it is right continuous and increasing while $\mathscr{F}_{0}$ contains all $\mathbb{P}$-null sets. $w(t)(t \geq 0)$ is a one-dimensional Brownian motion defined on $\left(\Omega, \mathscr{F}, \mathscr{F}_{t}, \mathbb{P}\right)$.

For any given $x \in \mathbb{R}^{n}$ and $\mathbb{R}^{n}$-valued function $f$, we always assume that

$$
\begin{gather*}
x=\left(x_{1}, x_{2}, \ldots, x_{n}\right)^{\mathrm{T}}, \quad f=\left(f_{1}, f_{2}, \ldots, f_{n}\right)^{\mathrm{T}}  \tag{5}\\
\operatorname{diag}(x)=\operatorname{diag}\left(x_{i}\right)=\operatorname{diag}\left(x_{1}, x_{2}, \ldots, x_{n}\right) .
\end{gather*}
$$

For matrices $A, B, D$, and $E$ in model (1), we assume that $A=$ $\left[a_{i j}\right], B=\left[b_{i j}\right], D=\left[d_{i j}\right]$, and $E=\left[e_{i j}\right](i, j=1,2, \ldots, n)$. Assume that $A \geq B \Leftrightarrow a_{i j} \geq b_{i j}$ for $i, j=1,2, \ldots, n ; x \gg 0 \Leftrightarrow$ $x_{i}>0$ for $i=1,2, \ldots, n$. Let $\mathbb{R}_{+}=[0, \infty), \mathbb{R}_{+}^{n}=\left(\mathbb{R}_{+}\right)^{n}$, and $\mathbb{R}_{++}^{n}=\left\{x \in \mathbb{R}^{n}: x \gg 0\right\}$. Denote by $|x|$ the Euclidean norm with $x \in \mathbb{R}^{n}$ and $|A|$ is the trace norm of matrix $A$.

Definition 1. Let $A=\left[a_{i j}\right] \in \mathbb{R}^{n \times n}$ satisfy condition

$$
\begin{equation*}
a_{i i}>0 \geq a_{i j} \text { for } i, j=1,2, \ldots, n, i \neq j \tag{6}
\end{equation*}
$$

If all eigenvalues of $A$ have positive real parts, $A$ is called an $M$-matrix.

Lemma 2. Suppose that the matrix $A \in \mathbb{R}^{n \times n}$ satisfies condition (6). Then the following conditions are equivalent (see [8]):
(i) $A$ is an M-matrix;
(ii) there exists $c \in \mathbb{R}_{++}^{n}$ such that $A c \gg 0$;
(iii) all of the leading principal minors of $A$ are positive.

For any given symmetric matrix $Q \in \mathbb{R}^{n \times n}$, define

$$
\begin{equation*}
\lambda_{M}^{+}(Q)=\sup _{x \in \mathbb{R}_{+}^{n},|x|=1} x^{\mathrm{T}} Q x \tag{7}
\end{equation*}
$$

which deduces directly that

$$
\begin{equation*}
\lambda_{M}^{+}(Q) \leq 0 \Longleftrightarrow x^{\mathrm{T}} Q x \leq 0 \quad \text { for any } x \in \mathbb{R}_{+}^{n} \tag{8}
\end{equation*}
$$

Let $\delta(t)$ be the variable delay of system (1). Write $\Delta(t)=t-$ $\delta(t)$ with $\delta(t) \in C^{1}\left(\mathbb{R}_{+}, \mathbb{R}_{+}\right)$and $\delta(t) \leq \delta_{0}<\infty$. Then

$$
\begin{equation*}
\eta=: \inf _{t \geq 0} \Delta^{\prime}(t)>0 \tag{9}
\end{equation*}
$$

implies that $\eta \leq 1$ and $\Delta(t)$ is strictly monotone increasing on $[0, \infty)$. Its inverse function $\Delta^{-1}(s)$ is defined on $[-\delta(0), \infty)$, which satisfies

$$
\begin{equation*}
\left(\Delta^{-1}(s)\right)^{\prime}=\frac{1}{\Delta^{\prime}(t)} \leq \eta^{-1}, \quad(s=\Delta(t), t \geq 0) \tag{10}
\end{equation*}
$$

Assume that $\tau=\delta(0), C=C\left([-\tau, 0], \mathbb{R}^{n}\right)$, and $C_{++}=$ $C\left([-\tau, 0], \mathbb{R}_{++}^{n}\right) . C$ is a Banach space with the supremum norm. For any given initial data $\xi \in C_{++}, x(t, \xi)$ always represents the solution of (2). When $x(t, \xi) \in \mathbb{R}_{++}^{n}$ for all $t$ in the domain, we call it a positive solution; when $x(t, \xi)$ is defined on $-\tau \leq t<\infty$, it is called a global solution.

Denote that

$$
\begin{array}{cl}
f=a+A x+B y, & g=b+D x+E y \\
\bar{f}=\operatorname{diag}(x) f, & \bar{g}=\operatorname{diag}(x) g . \tag{11}
\end{array}
$$

Unless otherwise stated, we assume that $x, y \in \mathbb{R}_{++}^{n}$. For any given $V \in C^{2}\left(\mathbb{R}_{++}^{n}\right)$, define

$$
\begin{equation*}
\mathscr{L} V(x, y)=V_{x}(x) \bar{f}(x, y)+\frac{1}{2}\left[\bar{g}^{\mathrm{T}}(x, y) V_{x x}(x) \bar{g}(x, y)\right] . \tag{12}
\end{equation*}
$$

If $x(t)$ is a positive solution of (1), by the Itô formula and (12), we have that

$$
\begin{align*}
V(x(t))= & V(x(0))+\int_{0}^{t} L V(x(s)) d s \\
& +\int_{0}^{t} V_{x}(x(s)) \bar{g}(x(s), y(s)) d w(s) \tag{13}
\end{align*}
$$

where $L V(x(t))=\mathscr{L} V(x(t), y(t))$ with $y(t)=x(t-\delta(t))$.
Let $p$ and $c_{i}(1 \leq i \leq n)$ be positive constants. Define

$$
\begin{equation*}
V(x)=\sum_{i=1}^{n} c_{i} x_{i}^{p}, \quad U(x)=\sum_{i=1}^{n} c_{i}\left(x_{i}^{p}-p \log x_{i}\right) . \tag{14}
\end{equation*}
$$

Substituting (14) into (12), together with notations in (11), yields that

$$
\begin{gather*}
\mathscr{L} V(x, y)=p \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(f_{i}+\frac{p-1}{2}\left|g_{i}\right|^{2}\right) \\
\mathscr{L} U(x, y)=\mathscr{L} V(x, y)+I  \tag{15}\\
I=p \sum_{i=1}^{n} c_{i}\left(-f_{i}+\frac{1}{2}\left|g_{i}\right|^{2}\right)
\end{gather*}
$$

Particularly, when $V=\sum_{i=1}^{n} x_{i}$ and $U=\sum_{i=1}^{n}\left(x_{i}-\log x_{i}\right)$, we have

$$
\begin{align*}
& \mathscr{L} V(x, y)=x^{\mathrm{T}} f=x^{\mathrm{T}}(a+A x+B y)  \tag{16}\\
& \mathscr{L} U(x, y)=\mathscr{L} V(x, y)-\sum_{i=1}^{n} f_{i}+\frac{1}{2}|g|^{2} \tag{17}
\end{align*}
$$

For the sake of simplicity, let $\Phi_{\varepsilon}$ represent the following function defined on $\mathbb{R}_{++}^{n} \times \mathbb{R}_{++}^{n}$ :

$$
\begin{equation*}
\Phi_{\varepsilon}=\Phi_{\varepsilon}(x, y)=\sum_{l=1}^{L} a_{l}\left[V_{l}(y)-\eta^{-1} e^{\delta_{0} \varepsilon} V_{l}(x)\right] \tag{18}
\end{equation*}
$$

where $V_{l} \in C\left(\mathbb{R}_{+}^{n}, \mathbb{R}_{+}\right), \varepsilon$ and $a_{l}(1 \leq l \leq L)$ are nonnegative constants, and $\eta$ is defined in (9). The following lemma plays a key role in this paper (also see $[1,9,10]$ ).

Lemma 3. Let $\Phi_{\varepsilon}$ be given by (18). Suppose that $x(t)=$ $x(t, \xi)\left(\xi \in C_{++},-\tau \leq t<\sigma\right)$ is a positive solution of (1) with $q \leq \varepsilon$; then

$$
\begin{equation*}
\int_{0}^{t} e^{q s} \Phi_{\varepsilon}(x(s), y(s)) d s \leq \text { const }, \quad(0 \leq t<\sigma) \tag{19}
\end{equation*}
$$

In this paper, const always denotes a positive constant with different values at different places and exact values of these constants are insignificant.

In this paper, we often use the following inequalities:

$$
\begin{gather*}
a^{\alpha} b^{\beta} \leq \frac{\alpha a^{\alpha+\beta}+\beta b^{\alpha+\beta}}{\alpha+\beta} ; \quad(a, b, \alpha, \beta \geq 0, \alpha+\beta>0),  \tag{20}\\
(a+b)^{2} \geq \frac{a^{2}}{\rho}-\frac{b^{2}}{\rho-1} ; \quad(a, b \in \mathbb{R}, \rho>1),  \tag{21}\\
\left(\sum_{i=1}^{n} c_{i} x_{i}\right)^{2} \leq \sum_{i=1}^{n} c_{i} \sum_{i=1}^{n} c_{i} x_{i}^{2} \quad\left(c_{i} \geq 0, x_{i} \in \mathbb{R}\right) . \tag{22}
\end{gather*}
$$

## 3. Main Lemmas

In order to get the desired properties (A)-(C), we need the following three lemmas. Let us first explain that the notation $o\left(|x|^{\alpha}\right): h(x)=o\left(|x|^{\alpha}\right)$ means that $h(x) \in C\left(\mathbb{R}_{+}^{n}\right)$ with

$$
\begin{equation*}
\lim _{|x| \rightarrow \infty}|x|^{-\alpha} h(x)=0 \tag{23}
\end{equation*}
$$

for $x \in \mathbb{R}_{++}^{n}$.
By (12) and (26), we have

$$
\begin{equation*}
\mathscr{L} V(x, y)=J-\frac{p}{2}|Z|^{2}, \quad Z=V_{x} \overline{\mathscr{g}}=\frac{x^{\mathrm{T}} g}{e^{\mathrm{T}} x} \tag{29}
\end{equation*}
$$

Let $h(t)=e^{\varepsilon t} V(x(t))$; then $h(t)=h(0)+I+M(t)$, where

$$
\begin{align*}
M(t) & =\int_{0}^{t} e^{\varepsilon s} V_{x}(x(s)) \bar{g}(x(s), y(s)) d w(s) \\
& =\int_{0}^{t} e^{\varepsilon s} Z(s) d w(s)  \tag{30}\\
I & =\int_{0}^{t} e^{\varepsilon s}[L V(x(s))+\varepsilon V(x(s))] d s \\
& =\int_{0}^{t} e^{\varepsilon s}\left[J-\frac{p}{2}|Z(s)|^{2}+\varepsilon V(x(s))\right] d s \tag{31}
\end{align*}
$$

For any given $\theta>1$ and $k \in \mathbb{N}$, by the exponential martingale inequality, we have that

$$
\begin{gather*}
\mathbb{P}\left\{\sup _{0 \leq t \leq k+1}\left[M(t)-\frac{p}{2 e^{\varepsilon(k+1)}} \int_{0}^{t} e^{2 \varepsilon s}|Z(s)|^{2} d s\right]\right. \\
\left.\quad \geq \frac{e^{\varepsilon(k+1)} \log k^{\theta}}{p}\right\} \leq \frac{1}{k^{\theta}} . \tag{32}
\end{gather*}
$$

Since $\sum_{k=1}^{\infty} k^{-\theta}<\infty$, we can employ the Borel-Cantelli lemma to derive that, almost surely, when $k$ is sufficiently large and $k \leq t \leq k+1$, one can get that

$$
\begin{align*}
M(t) & \leq \frac{e^{\varepsilon(k+1)} \log k^{\theta}}{p}+\frac{p}{2 e^{\varepsilon(k+1)}} \int_{0}^{t} e^{2 \varepsilon s}|Z(s)|^{2} d s  \tag{33}\\
& \leq \frac{\theta e^{\varepsilon}}{p} e^{\varepsilon t} \log t+\frac{p}{2} \int_{0}^{t} e^{\varepsilon s}|Z(s)|^{2} d s
\end{align*}
$$

Note that $-\sum_{i=1}^{n} b_{i} x_{i}^{\sigma}+o\left(|x|^{\sigma}\right)+\varepsilon V(x) \leq$ const. This, together with (31), (33), and (26), gives that in the sense of almost sure, when $t$ is sufficiently large,

$$
\begin{align*}
& h(t)-p^{-1} \theta e^{\varepsilon} e^{\varepsilon t} \log t \\
& \leq \text { const }+\int_{0}^{t} e^{\varepsilon s}[J+\varepsilon V(x(s))] d s \\
& \leq \text { const } \\
& \quad+\int_{0}^{t} e^{\varepsilon s}\left[\Phi_{\varepsilon}-\sum_{i=1}^{n} b_{i} x_{i}^{\sigma}(s)+o\left(|x(s)|^{\sigma}\right)+\varepsilon V(x(s))\right] d s \\
& \leq \text { const }+\operatorname{const} \int_{0}^{t} e^{\varepsilon s} d s \\
& \leq \text { const }\left(1+e^{\varepsilon t}\right), \tag{34}
\end{align*}
$$

where we have used Lemma 3. This implies that in the sense of almost sure

$$
\begin{equation*}
V(x(t)) \leq p^{-1} \theta e^{\varepsilon} \log t+\operatorname{const}\left(1+e^{-\varepsilon t}\right) \tag{35}
\end{equation*}
$$

when $t$ is sufficiently large. Therefore,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{V(x(t))}{\log t} \leq \frac{\theta e^{\varepsilon}}{p}, \quad \text { a.s. } \tag{36}
\end{equation*}
$$

Obviously, $\Phi_{\varepsilon}$ is a monotony decrease function of $\varepsilon$, so $\varepsilon$ can be replaced by any $\varepsilon^{\prime} \in(0, \varepsilon)$ in condition (26). Hence we may assume that $\varepsilon$ is sufficiently small. Letting $\theta \rightarrow 1$ and $\varepsilon \rightarrow 0$, we get that

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} \frac{V(x(t))}{\log t} \leq \frac{1}{p}, \quad \text { a.s. } \tag{37}
\end{equation*}
$$

Note that $V(x) \leq \log |x|$ for $x \in \mathbb{R}_{++}^{n}$. Then (27) follows from (37).

## 4. The Main Results

In this section, let us apply Lemmas 4-6 to establish the main results of this paper. We use the denotations $e=(1,1, \ldots, 1)^{\mathrm{T}}$ and $Q=e e^{T}$.

Theorem 7. Suppose that there exist nonnegative constants $q$, $r, \alpha$, and $\beta$, such that the following conditions are satisfied:

$$
\begin{gather*}
\lambda_{M}^{+}(H) \leq 0, \quad H=\left[\begin{array}{cc}
A+A^{T}+2 q Q & B-r Q \\
B^{T}-r Q & 0
\end{array}\right] ;  \tag{38}\\
\lambda_{M}^{+}(F) \leq 0, \quad F=\left[\begin{array}{cc}
D^{T} D-\alpha Q & D^{T} E \\
E^{T} D & E^{T} E-\beta Q
\end{array}\right] ;  \tag{39}\\
q>r \eta^{-1} \bigvee \frac{r\left(1+\eta^{-1}\right)+\alpha+\beta \eta^{-1}}{2} \tag{40}
\end{gather*}
$$

Then for any given $\xi \in C_{++}$, (1) has a unique positive solution $x(t, \xi)$ and this solution satisfies (2)-(4).

Proof. Let us divide this proof into the following three steps.
Step 1. Let $V=\sum_{i=1}^{n} x_{i}\left(x \in \mathbb{R}_{++}^{n}\right)$. Let us test condition (25). By (8) and condition (38), for any given $x, y \in \mathbb{R}_{++}^{n}$ we have that

$$
\begin{align*}
0 & \geq\left(\begin{array}{ll}
x^{\mathrm{T}} & y^{\mathrm{T}}
\end{array}\right) H\binom{x}{y} \\
& =x^{\mathrm{T}}\left(A+A^{\mathrm{T}}+2 q \mathrm{Q}\right)+2 x^{\mathrm{T}}(B-r \mathrm{Q}) y  \tag{41}\\
& =2 x^{\mathrm{T}}(A x+B y)+2 q\left(e^{\mathrm{T}} x\right)^{2}-2 r\left(e^{\mathrm{T}} x\right)\left(e^{\mathrm{T}} y\right)
\end{align*}
$$

By (16) and (42), we get

$$
\begin{align*}
\mathscr{L} V(x, y) & =x^{\mathrm{T}}(a+A x+B y) \\
& \leq-q\left(e^{\mathrm{T}} x\right)^{2}+r\left(e^{\mathrm{T}} x\right)\left(e^{\mathrm{T}} y\right)+o\left(|x|^{2}\right)  \tag{43}\\
& \leq-\left(q-\frac{r}{2}\right)\left(e^{\mathrm{T}} x\right)^{2}+\frac{r}{2}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right)  \tag{44}\\
& =\Phi_{\varepsilon}-k_{1}\left(e^{\mathrm{T}} x\right)^{2}+o\left(|x|^{2}\right) \tag{45}
\end{align*}
$$

where

$$
\begin{equation*}
\Phi_{\varepsilon}=\frac{r}{2}\left[\left(e^{\mathrm{T}} y\right)^{2}-\eta^{-1} e^{\delta_{0} \varepsilon}\left(e^{\mathrm{T}} x\right)^{2}\right] \tag{46}
\end{equation*}
$$

is a function in the form of (18) with $\varepsilon>0$ sufficiently small and

$$
\begin{equation*}
k_{1}=q-\frac{r}{2}-\frac{r}{2} \eta^{-1} e^{\delta_{0} \varepsilon} \tag{47}
\end{equation*}
$$

By condition (40),

$$
\begin{equation*}
\left.k_{1}\right|_{\varepsilon=0}=q-\frac{r}{2}\left(1+\eta^{-1}\right)>0 . \tag{48}
\end{equation*}
$$

Since $\varepsilon$ is sufficiently small, we may assume that $k_{1}>0$. Obviously, $\left(e^{\mathrm{T}} x\right)^{2} \geq \sum_{i=1}^{n} x_{i}^{2}\left(x \in \mathbb{R}_{++}^{n}\right)$, so (45) implies (25).

Now, we can apply Lemma 5 to obtain that any global positive solution $x(t, \xi)\left(\xi \in C_{++}\right)$of (1) satisfies (2)-(3).

Step 2. Let $U=\sum_{i=1}^{n}\left(x_{i}-\log x_{i}\right)\left(x \in \mathbb{R}_{++}^{n}\right)$. In this step, we will test condition (24). For any given $x, y \in \mathbb{R}_{++}^{n}$, using condition (39) yields

$$
\begin{align*}
0 & \geq\left(\begin{array}{ll}
x^{\mathrm{T}} & y^{\mathrm{T}}
\end{array}\right) F\binom{x}{y} \\
& =x^{\mathrm{T}}\left(D^{\mathrm{T}} D-\alpha Q\right) x+2 x^{\mathrm{T}} D^{\mathrm{T}} E y+y^{\mathrm{T}}\left(E^{\mathrm{T}} E-\beta Q\right) y \\
& =|D x|^{2}+2(D x)^{\mathrm{T}} E y+|E y|^{2}-\alpha x^{\mathrm{T}} Q x-\beta y^{\mathrm{T}} Q y \\
& =|D x+E y|^{2}-\alpha\left(e^{\mathrm{T}} x\right)^{2}-\beta\left(e^{\mathrm{T}} y\right)^{2}, \tag{49}
\end{align*}
$$

which implies

$$
\begin{equation*}
|D x+E y|^{2} \leq \alpha\left(e^{\mathrm{T}} x\right)^{2}+\beta\left(e^{\mathrm{T}} y\right)^{2} \tag{50}
\end{equation*}
$$

By (17), (44), and (50),

$$
\begin{align*}
& \mathscr{L} U(x, y) \\
& \leq-\left(q-\frac{r}{2}\right)\left(e^{\mathrm{T}} x\right)^{2}+\frac{r}{2}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right) \\
&-\sum_{i=1}^{n} f_{i}+\frac{1}{2}|g|^{2} \\
& \leq-\left(q-\frac{r}{2}\right)\left(e^{\mathrm{T}} x\right)^{2}+\frac{r}{2}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right) \\
&+ \text { const }|a+A x+B y|+\frac{1}{2}|b+D x+E y|^{2} \\
& \leq-\left(q-\frac{r}{2}\right)\left(e^{\mathrm{T}} x\right)^{2}+\frac{r}{2}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right)  \tag{51}\\
&+ \text { const }|a+A x+B y|+\frac{1}{2}|b|^{2}+b^{\mathrm{T}}(D x+E y) \\
&+\frac{1}{2}|D x+E y|^{2} \\
& \leq-\left(q-\frac{r+\alpha}{2}\right)\left(e^{\mathrm{T}} x\right)^{2}+\frac{r+\beta}{2}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right) \\
&+\operatorname{const}(1+|x|+|y|) \\
&= \Phi_{\varepsilon}^{\prime}-k_{2}\left(e^{\mathrm{T}} x\right)^{2}+o\left(|x|^{2}\right)
\end{align*}
$$

where

$$
\begin{align*}
\Phi_{\varepsilon}^{\prime}= & \frac{r+\beta}{2}\left[\left(e^{\mathrm{T}} y\right)^{2}-\eta^{-1} e^{\delta_{0} \varepsilon}\left(e^{\mathrm{T}} x\right)^{2}\right]  \tag{52}\\
& +\operatorname{const}\left(|y|-\eta^{-1} e^{\delta_{0} \varepsilon}|x|\right)
\end{align*}
$$

is a function in the form of (18):

$$
\begin{equation*}
k_{2}=q-\frac{r+2}{2}-\frac{r+\beta}{2} \eta^{-1} e^{\delta_{0} \varepsilon} . \tag{53}
\end{equation*}
$$

Condition (40) implies that $\left.k_{2}\right|_{\varepsilon=0}>0$. Since $\varepsilon>0$ can be sufficiently small, we can get $k_{2}>0$. So (51) can imply condition (24) (choose $\sigma=2$ ). Now we can employ Lemma 4 to obtain that, for any given $\xi \in C_{++}$, (1) has a unique positive global solution $x(t, \xi)$.

Step 3. Choose $p=1$. By (26) we have $J=x^{\mathrm{T}} f / e^{\mathrm{T}} x$. Now we test condition (26). Note that $x^{\mathrm{T}} f=\mathscr{L} V(x, y)$, so by (43) we have

$$
\begin{align*}
J & \leq-q e^{\mathrm{T}} x+r e^{\mathrm{T}} y+o(|x|)  \tag{54}\\
& =\Phi_{\varepsilon}^{\prime \prime}-k_{3} e^{\mathrm{T}} x+o(|x|)
\end{align*}
$$

where

$$
\begin{equation*}
\Phi_{\varepsilon}^{\prime \prime}=r\left(e^{\mathrm{T}} y-\eta^{-1} e^{\delta_{0} \varepsilon} e^{\mathrm{T}} x\right) \tag{55}
\end{equation*}
$$

is a function in the form of (18),

$$
\begin{equation*}
k_{3}=q-r \eta^{-1} e^{\delta_{0} \varepsilon} . \tag{56}
\end{equation*}
$$

By condition (40) we have $\left.k_{3}\right|_{\varepsilon=0}=q-r \eta^{-1}>0$, so we may assume that $k_{3}>0$. Then (54) shows that condition (26) is satisfied (choose $\sigma=1$ ).

Applying Lemma 6 yields that any positive solution $x(t, \xi)\left(\xi \in C_{++}\right)$of (1) satisfies (4). This completes the proof.

Theorem 8. Suppose that there exist nonnegative constants $q$ and $r$, such that condition (38) and the following condition are satisfied:

$$
\begin{gather*}
r Q \geq \pm E ;  \tag{57}\\
\lambda_{M}^{+}(R) \leq 0, \quad R=2 q Q-D-D^{T} ;  \tag{58}\\
2 q>r\left(1+\eta^{-1}\right) . \tag{59}
\end{gather*}
$$

Assume that $D \geq 0$,

$$
\begin{equation*}
G=: \operatorname{diag}\left(\eta d_{i i}^{2}\right)-S \quad \text { is an M-matrix, } \tag{60}
\end{equation*}
$$

where $S=\left[s_{i j}\right], s_{i j}=\bar{e}_{i \bullet} \bar{e}_{i j}, \bar{e}_{i j}=\left|e_{i j}\right|$, and $\bar{e}_{i \bullet}=\sum_{j=1}^{n} \bar{e}_{i j}$. Then the conclusion of Theorem 7 holds.

## Proof.

Step 1. By Lemma 2, condition (60) can imply that $G^{T}$ is an $M$-matrix. Thus, there exists $c \in \mathbb{R}_{++}^{n}$ such that $G^{\mathrm{T}} c \gg 0$. Let $U=\sum_{i=1}^{n} c_{i}\left(x_{i}^{p}-p \log x_{i}\right)\left(x \in \mathbb{R}_{++}^{n}\right), \sigma=2+p$, where $p>0$ is sufficiently small. Now we test condition (24). By (15) we have that

$$
\begin{aligned}
& \mathscr{L} U(x, y) \\
& =p \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(f_{i}+\frac{p-1}{2}\left|g_{i}\right|^{2}\right) \\
& \quad+p \sum_{i=1}^{n} c_{i}\left(-f_{i}+\frac{1}{2}\left|g_{i}\right|^{2}\right)
\end{aligned}
$$

$$
\begin{align*}
\leq & \text { const }|x|^{p}|f|-\frac{p(1-p)}{2} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left|g_{i}\right|^{2}+\operatorname{const}\left(|f|+|g|^{2}\right) \\
\leq & -\frac{p(1-p)}{2} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left|g_{i}\right|^{2}+\operatorname{const}\left(|x|^{p}|y|+|y|+|y|^{2}\right) \\
& +o\left(|x|^{\sigma}\right) \\
\leq & -\frac{p(1-p)}{2} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left|g_{i}\right|^{2}+\operatorname{const}\left(|y|^{p+1}+|y|+|y|^{2}\right) \\
& +o\left(|x|^{\sigma}\right) \\
= & \Phi_{\varepsilon}-\frac{p(1-p)}{2} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left|g_{i}\right|^{2}+o\left(|x|^{\sigma}\right), \tag{61}
\end{align*}
$$

where

$$
\begin{equation*}
\Phi_{\varepsilon}=\text { const }\left[|y|^{p+1}+|y|+|y|^{2}-\eta^{-1} e^{\delta_{0} \varepsilon}\left(|x|^{p+1}+|x|+|x|^{2}\right)\right] \tag{62}
\end{equation*}
$$

is a function in the form of (18). Choose $\rho$ sufficiently large; then

$$
\begin{aligned}
& \sum_{i=1}^{n} c_{i} x_{i}^{p}\left|g_{i}\right|^{2} \\
& =\sum_{i=1}^{n} c_{i} x_{i}^{p}\left(b_{i}+\sum_{j=1}^{n} d_{i j} x_{j}+\sum_{j=1}^{n} e_{i j} y_{j}\right)^{2} \\
& \geq \frac{1}{\rho} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(\sum_{j=1}^{n} d_{i j} x_{j}\right)^{2}-\frac{1}{\rho-1} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(b_{i}+\sum_{j=1}^{n} e_{i j} y_{j}\right)^{2} \\
& \geq \\
& \frac{1}{\rho} \sum_{i=1}^{n} c_{i} d_{i i}^{2} x_{i}^{\sigma}-\frac{1}{\rho-1} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(\sum_{j=1}^{n} e_{i j} y_{j}\right)^{2} \\
& \quad-\frac{1}{\rho-1} \sum_{i=1}^{n} c_{i} x_{i}^{p}\left(b_{i}^{2}+2 b_{i} \sum_{j=1}^{n} e_{i j} y_{j}\right) \\
& \geq \\
& \quad \frac{1}{\rho} \sum_{i=1}^{n} c_{i} d_{i i}^{2} x_{i}^{\sigma}-\frac{1}{\rho-1} \sum_{i, j=1}^{n} c_{i} s_{i j} x_{i}^{p} y_{j}^{2} \\
& \quad-\text { const } \sum_{i, j=1}^{n} x_{i}^{p} y_{j}-o\left(|x|^{\sigma}\right) \\
& \geq \\
& \geq \frac{1}{\rho} \sum_{i=1}^{n} c_{i} d_{i i}^{2} x_{i}^{\sigma}-\frac{1}{\rho-1} \sum_{i, j=1}^{n} c_{i} s_{i j} \frac{2 y_{j}^{\sigma}+p x_{i}^{\sigma}}{2+p} \\
& \quad-\text { const } \sum_{i, j=1}^{n} \frac{p x_{i}^{p+1}+y_{j}^{p+1}}{p+1}-o\left(|x|^{\sigma}\right)
\end{aligned}
$$

$$
\begin{align*}
= & \sum_{i=1}^{n} c_{i}\left[\frac{d_{i i}^{2}}{\rho}-\frac{p s_{i}}{(\rho-1)(2+p)}\right] x_{i}^{\sigma} \\
& -\frac{2}{(\rho-1)(2+p)} \sum_{i, j=1}^{n} c_{j} s_{j i} y_{i}^{\sigma}-\mathrm{const} \sum_{i=1}^{n} y_{i}^{p+1}-o\left(|x|^{\sigma}\right) \\
= & -\Phi_{\varepsilon}^{\prime}+\sum_{i=1}^{n} k_{i} x_{i}^{\sigma}+o\left(|x|^{\sigma}\right), \tag{63}
\end{align*}
$$

where we have used inequalities (20)-(22), $s_{i j}=\bar{e}_{i \cdot} \bar{e}_{i j}, s_{i}=$ $\sum_{j=1}^{n} s_{i j}$; consider

$$
\Phi_{\varepsilon}^{\prime}=\frac{2}{(\rho-1)(2+p)} \sum_{i, j=1}^{n} c_{j} s_{j i}\left(y_{i}^{\sigma}-\eta^{-1} e^{\delta_{0} \varepsilon} x_{i}^{\sigma}\right)
$$

$$
+ \text { const } \sum_{i=1}^{n}\left(y_{i}^{p+1}-\eta^{-1} e^{\delta_{0} \varepsilon} x_{i}^{p+1}\right)
$$

is a function in the form of (18):

$$
\begin{equation*}
k_{i}=c_{i}\left(\frac{d_{i i}^{2}}{\rho}-\frac{p s_{i \bullet}}{(\rho-1)(2+p)}\right)-\frac{2 \eta^{-1} e^{\delta_{0} \varepsilon}}{(\rho-1)(2+p)} \sum_{j=1}^{n} c_{j} s_{j i} \tag{65}
\end{equation*}
$$

when $\varepsilon \rightarrow 0, p \rightarrow 0$, and $\rho \rightarrow \infty$,

$$
\begin{equation*}
\rho k_{i} \longrightarrow c_{i} d_{i i}^{2}-\eta^{-1} \sum_{j=1}^{n} c_{j} s_{j i}>0 \tag{66}
\end{equation*}
$$

The last inequality is based on the condition $G^{T} c \gg 0$. Thus we may assume that $\varepsilon$ and $p$ are sufficiently small, while $\rho$ is sufficiently large; then $k_{i}>0(1 \leq i \leq n)$. Substituting (63) into (61) yields that

$$
\begin{equation*}
\mathscr{L U}(x, y) \leq \Phi_{\varepsilon}^{\prime \prime}-\frac{p(1-p)}{2} \sum_{i=1}^{n} k_{i} x_{i}^{\sigma}+o\left(|x|^{\sigma}\right) \tag{67}
\end{equation*}
$$

where

$$
\begin{equation*}
\Phi_{\varepsilon}^{\prime \prime}=\Phi_{\varepsilon}+\frac{p(1-p)}{2} \Phi_{\varepsilon}^{\prime} \tag{68}
\end{equation*}
$$

is a function in the form of (18). Clearly, (67) shows that condition (24) is satisfied.

Now, we can use Lemma 4 to obtain that, for any given $\xi \in C_{++}$, (1) has a unique global positive solution $x(t, \xi)$.

Step 2. Let $V=\sum_{i=1}^{n} x_{i}$. In this step we test condition (25); for that, we only need to show that conditions (38) and (59) hold. The method is similar to the proof of Theorem 7, Step 1.

Step 3. Taking any $p \in(0,1)$, now we test condition (26). We can replace $J$ by $(2 /(1-p)) J$ :

$$
\begin{align*}
& \frac{2}{1-p} J=\frac{2}{1-p} \frac{x^{\mathrm{T}} f}{e^{\mathrm{T}} x}-\left(\frac{x^{\mathrm{T}} g}{e^{\mathrm{T}} x}\right)^{2}=: J_{1}+J_{2}, \\
& J_{1}= \frac{2}{1-p} \frac{x^{\mathrm{T}}(a+A x+B y)}{e^{\mathrm{T}} x} \\
& \leq \text { const }(1+|x|+|y|) \\
&= \text { const }|y|+o\left(|x|^{2}\right) \\
& J_{2}=-\left[\frac{x^{\mathrm{T}} b+x^{\mathrm{T}}(D x+E y)}{e^{\mathrm{T}} x}\right]^{2}  \tag{69}\\
&=-\frac{\left(x^{\mathrm{T}} b\right)^{2}+2\left(x^{\mathrm{T}} b\right) x^{\mathrm{T}}(D x+E y)}{\left(e^{\mathrm{T}} x\right)^{2}} \\
&-\left(\frac{x^{\mathrm{T}} D x+x^{\mathrm{T}} E y}{e^{\mathrm{T}} x}\right)^{2} \\
& \leq \text { const }|D x+E y|-\left(\frac{x^{\mathrm{T}} D x+x^{\mathrm{T}} E y}{e^{\mathrm{T}} x}\right)^{2} \\
&= J_{3}+J_{4} .
\end{align*}
$$

Obviously,

$$
\begin{equation*}
J_{3} \leq \text { const }|y|+o\left(|x|^{2}\right) \tag{70}
\end{equation*}
$$

Letting $\rho$ be sufficiently large, then inequality (21) gives that

$$
\begin{equation*}
J_{4} \leq-\frac{1}{\rho}\left(\frac{x^{\mathrm{T}} D x}{e^{\mathrm{T}} x}\right)^{2}+\frac{1}{\rho-1}\left(\frac{x^{\mathrm{T}} E y}{e^{\mathrm{T}} x}\right)^{2} \tag{71}
\end{equation*}
$$

By condition (58) we have

$$
\begin{equation*}
x^{\mathrm{T}} R x=2 q\left(e^{\mathrm{T}} x\right)^{2}-2 x^{\mathrm{T}} D x \leq 0 \tag{72}
\end{equation*}
$$

thus $x^{\mathrm{T}} D x \geq q\left(e^{\mathrm{T}} x\right)^{2}$, which implies

$$
\begin{equation*}
\left(\frac{x^{\mathrm{T}} D x}{e^{\mathrm{T}} x}\right)^{2} \geq q^{2}\left(e^{\mathrm{T}} x\right)^{2}, \quad\left(x \in \mathbb{R}_{++}^{n}\right) \tag{73}
\end{equation*}
$$

Condition (57) derives that

$$
\begin{equation*}
r\left(e^{\mathrm{T}} x\right)\left(e^{\mathrm{T}} y\right)=r x^{\mathrm{T}} \mathrm{Q} y \geq\left|x^{\mathrm{T}} E y\right|, \quad\left(x, y \in \mathbb{R}_{++}^{n}\right) \tag{74}
\end{equation*}
$$

hence,

$$
\begin{equation*}
\left(\frac{x^{\mathrm{T}} E y}{e^{\mathrm{T}} x}\right)^{2} \leq r^{2}\left(e^{\mathrm{T}} y\right)^{2} \tag{75}
\end{equation*}
$$

So

$$
\begin{equation*}
J_{4} \leq-\frac{q^{2}}{\rho}\left(e^{\mathrm{T}} x\right)^{2}-\frac{r^{2}}{\rho-1}\left(e^{\mathrm{T}} y\right)^{2} \tag{76}
\end{equation*}
$$

Combining (69)-(76) yields

$$
\begin{align*}
\frac{2}{1-p} J & \leq \text { const }|y|-\frac{q^{2}}{\rho}\left(e^{\mathrm{T}} x\right)^{2}+\frac{r^{2}}{\rho-1}\left(e^{\mathrm{T}} y\right)^{2}+o\left(|x|^{2}\right) \\
& =\Phi_{\varepsilon}^{\prime \prime \prime}-k\left(e^{\mathrm{T}} x\right)^{2}+o\left(|x|^{2}\right) \tag{77}
\end{align*}
$$

where

$$
\begin{align*}
\Phi_{\varepsilon}^{\prime \prime \prime}= & \operatorname{const}\left(|y|-\eta^{-1} e^{\delta_{0} \varepsilon}|x|\right) \\
& +\frac{r^{2}}{\rho-1}\left[\left(e^{\mathrm{T}} y\right)^{2}-\eta^{-1} e^{-\delta_{0} \varepsilon}\left(e^{\mathrm{T}} x\right)^{2}\right] \tag{78}
\end{align*}
$$

is a function in the form of (18),

$$
\begin{equation*}
k=\frac{q^{2}}{\rho}-\frac{r^{2}}{\rho-1} \eta^{-1} e^{\delta_{0} \varepsilon} \tag{79}
\end{equation*}
$$

When $\varepsilon \rightarrow 0$ and $\rho \rightarrow \infty$,

$$
\begin{equation*}
\rho k \longrightarrow q^{2}-r^{2} \eta^{-1} \tag{80}
\end{equation*}
$$

By condition (59), we have $q>r\left(1+\eta^{-1}\right) /(2) \geq r / \sqrt{\eta}$; therefore, $q^{2}>r^{2} \eta^{-1}$. Since we may assume that $\varepsilon$ is sufficiently small and $\rho$ is sufficiently large, there must be $k>0$. Thus, condition (77) deduces that condition (26) is satisfied.

Now, we can apply Lemma 6 to obtain that any positive solution $x(t, \xi) \quad\left(\xi \in C_{++}\right)$of (1) satisfies (27). And then we can get that $x(t, \xi)$ satisfies (4) by letting $p \rightarrow 1$. This completes the proof.

Remark 9. Observing and comparing the conditions of Theorems 7 and 8 , the condition they have in common is (38), which only involves parameters from the drift coefficient $f$. Condition (39) in Theorem 7 corresponds to conditions (57), (58), and (60) in Theorem 8 which depend on stochastic disturbances of system (1). Both of them can guarantee the existence and uniqueness of the solution. But it seems that the three conditions of Theorem 8 are more precise than condition (39). Hence, we may expect that Theorem 8 can give more accurate results. However, it needs condition $D \geq$ 0 , which is not requested in Theorem 7. So Theorems 7 and 4.2 have their own strengths and weaknesses.

Remark 10. Theorems 7 and 8 give two classes of conditions under which the desired properties (A)-(C) hold. This is an improvement for our previous results ( $[1,2]$ ), since we only established these three results in different conditions, respectively. Moreover, conditions of the two theorems are directly dependent on the parameters of system, except $q$ and $r$. This implies that these conditions are easier to be verified.

## 5. Some Corollaries

In (1), letting $E=0, D=E=0$, and $B=E=0$, one can get the following "defective" LV systems:

$$
\begin{align*}
& d x(t)=\operatorname{diag}(x(t))\{[a+A x(t)+B y(t)] d t \\
& +[b+D x(t)] d w(t)\} ; \\
& d x(t)=\operatorname{diag}(x(t))\{[a+A x(t)+B y(t)] d t+b d w(t)\} ;  \tag{82}\\
& d x(t)=\operatorname{diag}(x(t))\{[a+A x(t)] d t+[b+D x(t)] d w(t)\}, \tag{83}
\end{align*}
$$

where (83) is equivalent to taking $\delta(t) \equiv 0$ in (1). For (81)(83), we can simplify the conditions of Theorems 7-8 and then obtain corollaries as follows.

Corollary 11. Suppose that there exist nonnegative constants $q$, $r$, and $\alpha$, such that condition (38) and the following conditions are satisfied:

$$
\begin{gather*}
\lambda_{M}^{+}\left(D^{T} D-\alpha Q\right) \leq 0 \\
q>r \eta^{-1} \bigvee \frac{r\left(1+\eta^{-1}\right)+\alpha}{2} \tag{84}
\end{gather*}
$$

Then for any given $\xi \in C_{++}$, (81) has a unique global positive solution $x(t, \xi)$, which satisfies (2)-(4).

Taking $\beta=0$ in Theorem 7, (84) deduces (39)-(40) directly. The following corollary can be found in $[3,4]$.

Corollary 12. Let $D \geq 0, d_{i i}>0(1 \leq i \leq n)$. Then for (81), the conclusion of Corollary 11 holds.

This corollary can be deduced from Theorem 8. First, let $r=0$ such that condition (57) is satisfied. Second, when $q>0$ is sufficiently small, conditions (58)-(59) are satisfied.

Clearly, Theorem 8 cannot be applied on system (82), but employing Theorem 7 we have the following.

Corollary 13. Suppose that there exist nonnegative constants $q$ and $r$, such that (38) and the following condition are satisfied:

$$
\begin{equation*}
q>\left(r \eta^{-1}\right) \bigvee \frac{r\left(1+\eta^{-1}\right)}{2} \tag{85}
\end{equation*}
$$

Then for any given $\xi \in C_{++}$, (82) has a unique global positive solution $x(t, \xi)$, which satisfies (2)-(4).

Note that when $D=E=0$, we should take $\alpha=\beta=0$ such that condition (39) is satisfied.

Applying Theorem 7 on (83) yields the following.
Corollary 14. Suppose that there exist nonnegative constants $q$, $r$, and $\alpha$ such that conditions (84) and the following condition are satisfied:

$$
\begin{equation*}
\lambda_{M}^{+}\left(A+A^{T}+2 q Q\right) \leq 0 \tag{86}
\end{equation*}
$$

Then for any given $\xi \in C_{++}$, (83) has a unique global positive solution $x(t, \xi)$, which satisfies (2)-(4).

## 6. Examples

Consider the following 2-dimensional LV system:

$$
\begin{align*}
\frac{d x_{1}(t)}{x_{1}(t)}= & {\left[-8 x_{1}(t)+x_{2}(t)-y_{1}(t)+y_{2}(t)\right] d t } \\
& +\left[\lambda x_{1}(t)+\lambda x_{2}(t)+\mu y_{1}(t)-\mu y_{2}(t)\right] d w(t), \\
\frac{d x_{2}(t)}{x_{2}(t)}= & {\left[x_{1}(t)-7 x_{2}(t)+y_{1}(t)-y_{2}(t)\right] d t } \\
& +\left[\lambda x_{2}(t)-\mu y_{2}(t)\right] d w(t) \tag{87}
\end{align*}
$$

where $\lambda$ and $\mu$ are nonnegative constants, $y_{i}(t)=x_{i}(t-\tau)(i=$ 1,2 ), and $\tau>0$. Let

$$
\begin{array}{ll}
A=\left(\begin{array}{cc}
-8 & 1 \\
1 & -7
\end{array}\right), & B=\left(\begin{array}{cc}
-1 & 1 \\
1 & -1
\end{array}\right), \\
C=\lambda\left(\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right), & E=\mu\left(\begin{array}{ll}
1 & -1 \\
0 & -1
\end{array}\right) . \tag{88}
\end{array}
$$

By (88), we can compute

$$
\begin{gather*}
A+A^{\mathrm{T}}=\left(\begin{array}{cc}
-16 & 2 \\
2 & -14
\end{array}\right), \quad D^{\mathrm{T}} D=\lambda^{2}\left(\begin{array}{ll}
1 & 1 \\
1 & 2
\end{array}\right) \\
D^{\mathrm{T}} E=\lambda \mu\left(\begin{array}{ll}
1 & -1 \\
1 & -2
\end{array}\right), \quad E^{\mathrm{T}} E=\mu^{2}\left(\begin{array}{cc}
1 & -1 \\
-1 & 2
\end{array}\right)  \tag{89}\\
D+D^{\mathrm{T}}=\lambda\left(\begin{array}{ll}
2 & 1 \\
1 & 2
\end{array}\right), \quad S=\mu^{2}\left(\begin{array}{ll}
2 & 2 \\
0 & 1
\end{array}\right)
\end{gather*}
$$

Then, by (38) and (39) we have

$$
\begin{gather*}
H=\left(\begin{array}{cccc}
2 q-16 & 2 q+2 & -1-r & 1-r \\
2 q+2 & 2 q-14 & 1-r & -1-r \\
-1-r & 1-r & 0 & 0 \\
1-r & -1-r & 0 & 0
\end{array}\right) ; \\
F=\left(\begin{array}{cccc}
\lambda^{2}-\alpha & \lambda^{2}-\alpha & \lambda \mu & -\lambda \mu \\
\lambda^{2}-\alpha & 2 \lambda^{2}-\alpha & \lambda \mu & -2 \lambda \mu \\
\lambda \mu & \lambda \mu & \mu^{2}-\beta & -\mu^{2}-\beta \\
-\lambda \mu & -2 \lambda \mu & -\mu^{2}-\beta & 2 \mu^{2}-\beta
\end{array}\right) ;  \tag{90}\\
G=\left(\begin{array}{cc}
\lambda^{2}-2 \mu^{2} & -2 \mu^{2} \\
0 & \lambda^{2}-\mu^{2}
\end{array}\right), \\
R=\left(\begin{array}{cc}
2 q-2 \lambda & 2 q-\lambda \\
2 q-\lambda & 2 q-2 \lambda
\end{array}\right) .
\end{gather*}
$$

( $1^{\circ}$ ) Apply Theorem 7. For any given $x \in \mathbb{R}_{+}^{4}$, we have

$$
\begin{align*}
x^{\mathrm{T}} H x= & (2 q-16) x_{1}^{2}+2(2 q+2) x_{1} x_{2}-2(1+r) x_{1} x_{3} \\
& +2(1-r) x_{2} x_{4}+(2 q-14) x_{2}^{2}+2(1-r) x_{2} x_{3} \\
& -2(1+r) x_{2} x_{4} \\
\leq & (2 q-16) x_{1}^{2}+(2 q+2)\left(x_{1}^{2}+x_{2}^{2}\right) \\
& +(1-r)^{+}\left(x_{1}^{2}+x_{4}^{2}\right) \\
& +(2 q-14) x_{2}^{2}+(1-r)^{+}\left(x_{2}^{2}+x_{3}^{2}\right) \\
= & x_{1}^{2}\left[4 q-14+(1-r)^{+}\right]+x_{2}^{2}\left[4 q-12+(1-r)^{+}\right] \\
& +(1-r)^{+}\left(x_{3}^{2}+x_{4}^{2}\right) . \tag{91}
\end{align*}
$$

It can be seen that, taking $r \geq 1$ and $q \leq 3$, we have $x^{\mathrm{T}} H x \leq 0$, and then $\lambda_{M}^{+}(H) \leq 0$. Next,

$$
\begin{align*}
x^{\mathrm{T}} F x= & \left(\lambda^{2}-\alpha\right) x_{1}^{2}+2\left(\lambda^{2}-\alpha\right) x_{1} x_{2}+2 \lambda \mu x_{1} x_{3}-2 \lambda \mu x_{1} x_{4} \\
& +\left(2 \lambda^{2}-\alpha\right) x_{2}^{2}+2 \lambda \mu x_{2} x_{3}-4 \lambda \mu x_{2} x_{4} \\
& +\left(\mu^{2}-\beta\right) x_{3}^{2}-2\left(\mu^{2}+\beta\right) x_{3} x_{4}+\left(2 \mu^{2}-\beta\right) x_{4}^{2} \\
\leq & \left(\lambda^{2}-\alpha\right) x_{1}^{2}+\left(\lambda^{2}-\alpha\right)^{+}\left(x_{1}^{2}+x_{2}^{2}\right)+\lambda \mu\left(x_{1}^{2}+x_{3}^{2}\right) \\
& +\left(2 \lambda^{2}-\alpha\right) x_{2}^{2}+\lambda \mu\left(x_{2}^{2}+x_{3}^{2}\right)+\left(\mu^{2}-\beta\right) x_{3}^{2} \\
& +\left(2 \mu^{2}-\beta\right) x_{4}^{2} \\
= & x_{1}^{2}\left[\lambda^{2}-\alpha+\left(\lambda^{2}-\alpha\right)^{+}+\lambda \mu\right] \\
& +x_{2}^{2}\left[\left(\lambda^{2}-\alpha\right)^{+}+2 \lambda^{2}-\alpha+\lambda \mu\right] \\
& +x_{3}^{2}\left(2 \lambda \mu+\mu^{2}-\beta\right)+\left(2 \mu^{2}-\beta\right) x_{4}^{2} . \tag{92}
\end{align*}
$$

Clearly, when $\alpha \geq 2 \lambda^{2}+\lambda \mu$ and $\beta \geq\left(2 \mu^{2}\right) \vee\left(\mu^{2}+2 \lambda \mu\right)$, $\lambda_{M}^{+}(F) \leq 0$. Condition (40) is equivalent to $2 q>2 r+\alpha+\beta$. Combining the above equalities yields

$$
\begin{align*}
6 & \geq 2 q>2 r+\alpha+\beta \\
& \geq 2+2 \lambda^{2}+\lambda \mu+\left(2 \mu^{2}\right) \vee\left(\mu^{2}+2 \lambda \mu\right) \tag{93}
\end{align*}
$$

namely, $\lambda \geq 0$ and $\mu \geq 0$ satisfy

$$
\begin{equation*}
2 \lambda^{2}+\lambda \mu+\left(2 \mu^{2}\right) \vee\left(\mu^{2}+2 \lambda \mu\right)<4 \tag{94}
\end{equation*}
$$

Then we can choose nonnegative constants $q, r, \alpha$, and $\beta$, such that conditions (38)-(40) are satisfied; therefore, Theorem 7


Figure 1: Regions $D_{1}$ and $D_{2}$.
can apply to (87). Through elementary calculation, condition (94) can be expressed as

$$
\lambda< \begin{cases}\frac{\sqrt{32+\mu^{2}}-3 \mu}{4}, & 0 \leq \mu \leq \frac{2}{\sqrt{3}}  \tag{95}\\ \frac{\sqrt{32-15 \mu^{2}}-\mu}{4}, & \frac{2}{\sqrt{3}}<\mu<\sqrt{2}\end{cases}
$$

(2 ${ }^{\circ}$ ) Apply Theorem 8 . Obviously $D \geq 0$. By Lemma 2, $G$ is an $M$-matrix if and only if $\lambda^{2}>2 \mu^{2}$; that is,

$$
\begin{equation*}
\lambda>\sqrt{2} \mu \tag{96}
\end{equation*}
$$

Condition (57) holds $\Leftrightarrow \mu \leq r$, so we may assume $r=\mu$. For any given $x \in \mathbb{R}_{+}^{2}$,

$$
\begin{align*}
x^{\mathrm{T}} R x & =(2 q-2 \lambda) x_{1}^{2}+2(2 q-\lambda) x_{1} x_{2}+(2 q-2 \lambda) x_{2}^{2} \\
& \leq\left[2 q-2 \lambda+(2 q-\lambda)^{+}\right]\left(x_{1}^{2}+x_{2}^{2}\right) \tag{97}
\end{align*}
$$

Obviously, when $q \leq(3 / 4) \lambda, \lambda_{M}^{+}(R) \leq 0$. Let $q=(3 / 4) \lambda$. By condition (96) we have

$$
\begin{equation*}
q=\frac{3}{4} \lambda>\frac{3 \sqrt{2}}{4} \mu \geq \mu=r \tag{98}
\end{equation*}
$$

which shows that condition (40) is satisfied. Thus, when condition (96) holds, Theorem 8 can apply to (87). In Figure 1, regions $D_{1}$ and $D_{2}$ are, respectively, decided by conditions (94) and (96) on the $\lambda \mu$ plane. It is easy to see that $D_{1}$ and $D_{2}$ are partially overlapping. Roughly speaking, $D_{2}$ is much larger than $D_{1}$. This means that applying Theorem 8 on model (87) can get more precise results in some sense. This conclusion is consistent with our expectation in Remark 9.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# The Invertibility, Explicit Determinants, and Inverses of Circulant and Left Circulant and g-Circulant Matrices Involving Any Continuous Fibonacci and Lucas Numbers 

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Circulant matrices play an important role in solving delay differential equations. In this paper, circulant type matrices including the circulant and left circulant and $g$-circulant matrices with any continuous Fibonacci and Lucas numbers are considered. Firstly, the invertibility of the circulant matrix is discussed and the explicit determinant and the inverse matrices by constructing the transformation matrices are presented. Furthermore, the invertibility of the left circulant and $g$-circulant matrices is also studied. We obtain the explicit determinants and the inverse matrices of the left circulant and $g$-circulant matrices by utilizing the relationship between left circulant, $g$-circulant matrices and circulant matrix, respectively.

## 1. Introduction

Circulant matrices have important applications in solving various differential equations [1-3]. The use of circulant preconditioners for solving structured linear systems has been studied extensively since 1986; see [4, 5]. Circulant matrices also play an important role in solving delay differential equations. In [6], Chan et al. proposed a preconditioner called the Strang-type block-circulant preconditioner for solving linear systems from IVPs. The Strang-type preconditioner was also used to solve linear systems from differentialalgebraic equations and delay differential equations; see [714]. In [15], Jin et al. proposed the GMRES method with the Strang-type block-circulant preconditioner for solving singular perturbation delay differential equations.

The $g$-circulant matrices play an important role in various applications as well; please refer to [16, 17] for details. There are discussions about the convergence in probability and in distribution of the spectral norm of $g$-circulant matrices in [18, 19]. Ngondiep et al. showed the singular values of $g$ circulants in [20].

Recently, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices [21, 22]. Unfortunately, the computational complexity of these algorithms is increasing dramatically with the increasing order of matrices. However, some authors gave the explicit determinants and inverse of circulant involving Fibonacci and Lucas numbers. For example, Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [23]. Lind presented the determinants of circulant involving Fibonacci numbers [24]. Lin gave the determinant of the Fibonacci-Lucas quasicyclic matrices in [25]. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses [26]. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [27].

The purpose of this paper is to obtain the explicit determinants, explicit inverses of circulant, left circulant, and $g$-circulant matrices involving any continuous Fibonacci numbers and Lucas numbers. And we generalize the result in [26].

In the following, let $r$ be a nonnegative integer. We adopt the following two conventions $0^{0}=1$, and for any sequence $\left\{a_{n}\right\}, \sum_{k=i}^{n} a_{k}=0$ in the case $i>n$.

The Fibonacci and Lucas sequences are defined by the following recurrence relations [23-26], respectively:

$$
\begin{align*}
& F_{n+1}=F_{n}+F_{n-1} \text { where } F_{0}=0, F_{1}=1, \\
& L_{n+1}=L_{n}+L_{n-1} \text { where } L_{0}=2, L_{1}=1 \tag{1}
\end{align*}
$$

for $n \geq 0$. The first few values of the sequences are given by the following table:

$$
\begin{array}{c|cccccccccc}
n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots  \tag{2}\\
\hline F_{n} & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & 21 & \cdots \\
L_{n} & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & 47 & \cdots
\end{array}
$$

Let $\alpha$ and $\beta$ be the roots of the characteristic equation $x^{2}-$ $x-1=0$; then the Binet formulas of the sequences $\left\{F_{r+n}\right\}$ and $\left\{L_{r+n}\right\}$ have the form

$$
\begin{equation*}
F_{r+n}=\frac{\alpha^{r+n}-\beta^{r+n}}{\alpha-\beta}, \quad L_{r+n}=\alpha^{r+n}+\beta^{r+n} . \tag{3}
\end{equation*}
$$

Definition 1 (see [21, 22]). In a right circulant matrix (or simply, circulant matrix)

$$
\operatorname{Circ}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\left[\begin{array}{cccc}
a_{1} & a_{2} & \cdots & a_{n}  \tag{4}\\
a_{n} & a_{1} & \cdots & a_{n-1} \\
\vdots & \vdots & & \vdots \\
a_{2} & a_{3} & \cdots & a_{1}
\end{array}\right]
$$

each row is a cyclic shift of the row above to the right. Right circulant matrix is a special case of a Toeplitz matrix. It is evidently determined by its first row (or column).

Definition 2 (see [22, 28]). In a left circulant matrix (or reverse circulant matrix )

$$
\operatorname{LCirc}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=\left[\begin{array}{cccc}
a_{1} & a_{2} & \cdots & a_{n}  \tag{5}\\
a_{2} & a_{3} & \cdots & a_{1} \\
\vdots & \vdots & & \vdots \\
a_{n} & a_{1} & \cdots & a_{n-1}
\end{array}\right]
$$

each row is a cyclic shift of the row above to the left. Left circulant matrix is a special Hankel matrix.

Definition 3 (see $[19,29]$ ). A $g$-circulant matrix is an $n \times n$ complex matrix with the following form:

$$
A_{g, n}=\left(\begin{array}{cccc}
a_{1} & a_{2} & \cdots & a_{n}  \tag{6}\\
a_{n-g+1} & a_{n-g+2} & \cdots & a_{n-g} \\
a_{n-2 g+1} & a_{n-2 g+2} & \cdots & a_{n-2 g} \\
\vdots & \vdots & \ddots & \vdots \\
a_{g+1} & a_{g+2} & \cdots & a_{g}
\end{array}\right)
$$

where $g$ is a nonnegative integer and each of the subscripts is understood to be reduced modulo $n$.

The first row of $A_{g, n}$ is $\left(a_{1}, a_{2}, \ldots, a_{n}\right)$; its $(j+1)$ th row is obtained by giving its $j$ th row a right circular shift by $g$ positions (equivalently, $g \bmod n$ positions). Note that $g=1$
or $g=n+1$ yields the standard circulant matrix. If $g=n-1$, then we obtain the so called left circulant matrix.

Lemma 4 (see [26]). Let $A=\operatorname{Circ}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ be circulant matrix; then one has
(i) $A$ is invertible if and only if the eigenvalues of $A$

$$
\begin{equation*}
\lambda_{k}=f\left(\omega^{k}\right) \neq 0, \quad(k=0,1, \ldots, n-1) \tag{7}
\end{equation*}
$$

where $f(x)=\sum_{j=1}^{n} a_{j} x^{j-1}$ and $\omega=\exp (2 \pi i / n)$;
(ii) if $A$ is invertible, then the inverse $A^{-1}$ of $A$ is a circulant matrix.

## Lemma 5. Define

$$
\Delta:=\left(\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 & 0  \tag{8}\\
0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \ddots & & \vdots & \\
0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{array}\right)
$$

the matrix $\Delta$ is an orthogonal cyclic shift matrix (and a left circulant matrix). It holds that $\operatorname{LCirc}\left(a_{1}, a_{2}, \ldots, a_{n}\right)=$ $\Delta \operatorname{Circ}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$.

Lemma 6 (see [29]). The $n \times n$ matrix $\mathbb{Q}_{g}$ is unitary if and only if $(n, g)=1$, where $\mathbb{Q}_{g}$ is a $g$-circulant matrix with first rowe $e^{*}=[1,0, \ldots, 0]$.

Lemma 7 (see [29]). $A_{g, n}$ is a $g$-circulant matrix with first row $\left[a_{1}, a_{2}, \ldots, a_{n}\right]$ if and only if $A_{g, n}=\mathbb{Q}_{g} C$, where $C=$ $\operatorname{Circ}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$.

## 2. Determinant, Invertibility, and Inverse of Circulant Matrix with Any Continuous Fibonacci Numbers

In this section, let $A_{r, n}=\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a circulant matrix. Firstly, we give the determinant equation of the matrix $A_{r, n}$. Afterwards, we prove that $A_{r, n}$ is an invertible matrix for $n>2$, and then we find the inverse of the matrix $A_{r, n}$. Obviously, when $n=2, r \neq 0$, or $n=1, A_{r, n}$ is also an invertible matrix.

Theorem 8. Let $A_{r, n}=\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a circulant matrix. Then one has

$$
\begin{align*}
& \operatorname{det} A_{r, n} \\
& \begin{aligned}
&=F_{r+1} \cdot\left\{\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right. \\
&+\sum_{k=1}^{n-2}\left[\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\right. \\
&\left.\left.\times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\
& \quad \times\left(F_{r+1}-F_{r+n+1}\right)^{n-2} .
\end{aligned}
\end{align*}
$$

where $F_{r+n}$ is the $(r+n)$ th Fibonacci number. Specially, when $r=0$, this result is the same as Theorem 2.1 in [26].
Proof. Obviously, $\operatorname{det} A_{1}=\left(1-F_{n+1}\right)^{n-1}+F_{n}^{n-2} \sum_{k=1}^{n-1} F_{k}((1-$ $\left.\left.F_{n+1}\right) / F_{n}\right)^{k-1}$ satisfies the formula. In the case $n>1$, let

$$
\begin{align*}
& \Pi_{1}=\left(\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{array}\right) \tag{10}
\end{align*}
$$

be two $n \times n$ matrices; then we have

$$
\begin{align*}
& \Gamma A_{r, n} \Pi_{1} \\
& =\left[\begin{array}{ccccc}
F_{r+1} & f_{r, n}^{\prime} & F_{r+n-1} & \cdots & F_{r+2} \\
0 & f_{r, n} & \tau_{n} & \cdots & \tau_{3} \\
0 & 0 & F_{r+1}-F_{r+n+1} & & 0 \\
0 & 0 & F_{r}-F_{r+n} & & 0 \\
\vdots & \vdots & & \ddots & \\
0 & 0 & 0 & & F_{r+1}-F_{r+n+1}
\end{array}\right] \tag{11}
\end{align*}
$$

where

$$
\begin{aligned}
a_{r} & =\frac{F_{r+2}}{F_{r+1}}, f_{r, n}^{\prime} \\
& =\sum_{k=1}^{n-1} F_{r+k+1}\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} \\
\tau_{k} & =F_{r+k}-a_{r} F_{r+k-1}, \quad k=3, \ldots, n
\end{aligned}
$$

$$
\begin{align*}
f_{r, n}=( & \left.F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
+ & \sum_{k=1}^{n-2}\left[\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\right. \\
& \left.\times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right] \tag{12}
\end{align*}
$$

We obtain

$$
\begin{align*}
& \operatorname{det} \Gamma \operatorname{det} A_{r, n} \operatorname{det} \Pi_{1} \\
& \begin{aligned}
&=F_{r+1} \cdot\left\{\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right. \\
&+\sum_{k=1}^{n-2}\left[\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\right. \\
&\left.\left.\times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\
& \quad \times\left(F_{r+1}-F_{r+n+1}\right)^{n-2}
\end{aligned}
\end{align*}
$$

while

$$
\begin{equation*}
\operatorname{det} \Gamma=(-1)^{(n-1)(n-2) / 2}, \operatorname{det} \Pi_{1}=(-1)^{(n-1)(n-2) / 2} \tag{14}
\end{equation*}
$$

we have
$\operatorname{det} A_{r, n}$

$$
=F_{r+1} \cdot\left\{\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right.
$$

$$
\begin{align*}
& +\sum_{k=1}^{n-2}\left[\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\right.  \tag{15}\\
& \left.\left.\times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\
& \times\left(F_{r+1}-F_{r+n+1}\right)^{n-2}
\end{align*}
$$

Theorem 9. Let $A_{r, n}=\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a circulant matrix; if $n>2$, then $A_{r, n}$ is an invertible matrix. Specially, when $r=0$, one gets Theorem 2.2 in [26].

Proof. When $n=3$ in Theorem 8, then we have $\operatorname{det} A_{r, n}=$ $\left(F_{r+1}+F_{r+2}+F_{r+3}\right)\left(F_{r+1}^{2}+F_{r} F_{r+2}\right) \neq 0$; hence $A_{r, n}$ is invertible.

In the case $n>3$, since $F_{r+n}=\left(\alpha^{r+n}-\beta^{r+n}\right) /(\alpha-\beta)$, where $\alpha+\beta=1, \alpha \beta=-1$. We have

$$
\begin{align*}
& f\left(\omega^{k}\right)= \sum_{j=1}^{n} F_{r+j}\left(\omega^{k}\right)^{j-1} \\
&= \frac{1}{\alpha-\beta} \sum_{j=1}^{n}\left(\alpha^{r+j}-\beta^{r+j}\right)\left(\omega^{k}\right)^{j-1} \\
&= \frac{1}{\alpha-\beta}\left[\frac{\alpha^{r+1}\left(1-\alpha^{n}\right)}{1-\alpha \omega^{k}}-\frac{\beta^{r+1}\left(1-\beta^{n}\right)}{1-\beta \omega^{k}}\right] \\
&= \frac{\left(\alpha^{r+1}-\beta^{r+1}\right)-\left(\alpha^{r+n+1}-\beta^{r+n+1}\right)}{(\alpha-\beta)\left(1-(\alpha+\beta) \omega^{k}+\alpha \beta \omega^{2 k}\right)}  \tag{16}\\
&-\frac{\alpha \beta\left(\alpha^{r}-\beta^{r}-\alpha^{r+n}+\beta^{r+n}\right) \omega^{k}}{(\alpha-\beta)\left(1-(\alpha+\beta) \omega^{k}+\alpha \beta \omega^{2 k}\right)} \\
&= \frac{F_{r+1}-F_{r+n+1}+\left(F_{r}-F_{r+n}\right) \omega^{k}}{1-\omega^{k}-\omega^{2 k}} \\
& \quad(k=1,2, \ldots, n-1) .
\end{align*}
$$

If there exists $\omega^{l}(l=1,2, \ldots, n-1)$ such that $f\left(\omega^{l}\right)=0$, we obtain $F_{r+1}-F_{r+n+1}+\left(F_{r}-F_{r+n}\right) \omega^{l}=0$ for $1-\omega^{l}-\omega^{2 l} \neq 0$; thus, $\omega^{l}=\left(F_{r+1}-F_{r+n+1}\right) /\left(F_{r+n}-F_{r}\right)$ is a real number. While

$$
\begin{equation*}
\omega^{l}=\exp \left(\frac{2 l \pi i}{n}\right)=\cos \left(\frac{2 l \pi}{n}\right)+i \sin \left(\frac{2 l \pi}{n}\right) \tag{17}
\end{equation*}
$$

hence, $\sin (2 l \pi / n)=0$; so we have $\omega^{l}=-1$ for $0<2 l \pi / n<2 \pi$. But $x=-1$ is not the root of the equation $F_{r+1}-F_{r+n+1}+\left(F_{r}-\right.$ $\left.F_{r+n}\right) x=0(n>3)$. We obtain $f\left(\omega^{k}\right) \neq 0$ for any $\omega^{k}(k=$ $1,2, \ldots, n-1)$, while $f(1)=\sum_{j=1}^{n} F_{r+j}=-F_{r+1}+F_{r+n+1}-$ $\left(F_{r}-F_{r+n}\right)=F_{r+n+2}-F_{r+2} \neq 0$. By Lemma 4, the proof is completed.

Lemma 10. Let the entries of the matrix $\mathscr{G}=\left[g_{i, j}\right]_{i, j=1}^{n-2}$ be of the form

$$
g_{i, j}=\left\{\begin{array}{cc}
F_{r+1}-F_{r+n+1}, & i=j  \tag{18}\\
F_{r}-F_{r+n}, & i=j+1 \\
0, & \text { otherwise }
\end{array}\right.
$$

then the entries of the inverse $\mathscr{G}^{-1}=\left[g_{i, j}^{\prime}\right]_{i, j=1}^{n-2}$ of the matrix $\mathscr{G}$ are equal to

$$
g_{i, j}^{\prime}= \begin{cases}\frac{\left(F_{r+n}-F_{r}\right)^{i-j}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j+1}}, & i \geq j  \tag{19}\\ 0, & i<j\end{cases}
$$

In particular, when $r=0$, one gets Lemma 2.1 in [26].
Proof. Let $c_{i, j}=\sum_{k=1}^{n-2} g_{i, k} g_{k, j}^{\prime}$. Obviously, $c_{i, j}=0$ for $i<j$. In the case $i=j$, we obtain

$$
\begin{equation*}
c_{i, i}=g_{i, i} g_{i, i}^{\prime}=\left(F_{r+1}-F_{r+n+1}\right) \cdot \frac{1}{F_{r+1}-F_{r+n+1}}=1 \tag{20}
\end{equation*}
$$

For $i \geq j+1$, we obtain

$$
\begin{align*}
c_{i, j}= & \sum_{k=1}^{n-2} g_{i, k} g_{k, j}^{\prime}=g_{i, i-1} g_{i-1, j}^{\prime}+g_{i, i} g_{i, j}^{\prime} \\
= & \left(F_{r}-F_{r+n}\right) \cdot \frac{\left(F_{r+n}-F_{r}\right)^{i-j-1}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j}}  \tag{21}\\
& +\left(F_{r+1}-F_{r+n+1}\right) \cdot \frac{\left(F_{r+n}-F_{r}\right)^{i-j}}{\left(F_{r+1}-F_{r+n+1}\right)^{i-j+1}} \\
= & 0 .
\end{align*}
$$

Hence, we verify $\mathscr{G} \mathscr{G}^{-1}=I_{n-2}$, where $I_{n-2}$ is $(n-2) \times(n-2)$ identity matrix. Similarly, we can verify $\mathscr{G}^{-1} \mathscr{G}=I_{n-2}$. Thus, the proof is completed.

Theorem 11. Let $A_{r, n}=\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)(n>2) b e$ a circulant matrix.

Then one has

$$
\begin{align*}
& A_{r, n}^{-1} \\
& =\frac{1}{f_{r, n}} \\
& \times \operatorname{Circ}\left(1+\sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n+1-i}\right)\right.\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1},-\frac{F_{r+2}}{F_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(F_{r+n+1-i}-\left(F_{r+2} / F_{r+1}\right) F_{r+n-i}\right)\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right) \text {, } \\
& -\frac{F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)}{\left(F_{r+1}-F_{r+n+1}\right)^{2}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{2}}{\left(F_{r+1}-F_{r+n+1}\right)^{3}}, \ldots, \\
& \left.-\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}\right), \tag{22}
\end{align*}
$$

where

$$
\begin{align*}
f_{r, n}= & \left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} . \tag{23}
\end{align*}
$$

Specially, when $r=0$, this result is the same as Theorem 2.3 in [26].

Proof. Let

$$
\Pi_{2}=\left(\begin{array}{cccccc}
1 & -\frac{f_{r, n}^{\prime}}{F_{r+1}} & x_{3} & x_{4} & \cdots & x_{n}  \tag{24}\\
0 & 1 & y_{3} & y_{4} & \cdots & y_{n} \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right)
$$

where $a_{r}=F_{r+2} / F_{r+1}$,

$$
\begin{gather*}
x_{i}=\frac{f_{r, n}^{\prime}}{f_{r, n}} \frac{F_{r+n+3-i}-\left(F_{r+2} / F_{r+1}\right) F_{r+n+2-i}}{F_{r+1}}-\frac{F_{r+n+2-i}}{F_{r+1}} \\
y_{i}=-\frac{F_{r+n+3-i}-\left(F_{r+2} / F_{r+1}\right) F_{r+n+2-i}}{f_{r, n}} \quad(i=3,4, \ldots, n), \\
f_{r, n}^{\prime}=\sum_{k=1}^{n-1} F_{r+k+1}\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}, \\
f_{r, n}=\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
+\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} .
\end{gather*}
$$

We have

$$
\begin{equation*}
\Gamma A_{r, n} \Pi_{1} \Pi_{2}=\mathscr{D}_{1} \oplus \mathscr{G} \tag{26}
\end{equation*}
$$

where $\mathscr{D}_{1}=\operatorname{diag}\left(F_{r+1}, f_{r, n}\right)$ is a diagonal matrix and $\mathscr{D}_{1} \oplus \mathscr{G}$ is the direct sum of $\mathscr{D}_{1}$ and $\mathscr{G}$. If we denote $\Pi=\Pi_{1} \Pi_{2}$, then we obtain

$$
\begin{equation*}
A_{r, n}^{-1}=\Pi\left(\mathscr{D}_{1}^{-1} \oplus \mathscr{G}^{-1}\right) \Gamma \tag{27}
\end{equation*}
$$

and the last row elements of the matrix $\Pi$ are $0,1, y_{3}, y_{4}$, $\ldots, y_{n-1}, y_{n}$. By Lemma 10, if let $A_{r, n}^{-1}=\operatorname{Circ}\left(u_{1}, u_{2}, \ldots, u_{n}\right)$,
then its last row elements are given by the following equations:

$$
\begin{align*}
& u_{2}=-\frac{1}{f_{r, n}} \frac{F_{r+2}}{F_{r+1}}+\frac{1}{f_{r, n}} C_{n}^{(n-2)}, \\
& u_{3}=-\frac{1}{f_{r, n}} C_{n}^{(1)} \\
& u_{4}=-\frac{1}{f_{r, n}} C_{n}^{(2)}+\frac{1}{f_{r, n}} C_{n}^{(1)}, \\
& u_{5}=-\frac{1}{f_{r, n}} C_{n}^{(3)}+\frac{1}{f_{r, n}} C_{n}^{(2)}+\frac{1}{f_{r, n}} C_{n}^{(1)},  \tag{28}\\
& \vdots \\
& u_{n}=-\frac{1}{f_{r, n}} C_{n}^{(n-2)}+\frac{1}{f_{r, n}} C_{n}^{(n-3)}+\frac{1}{f_{n}} C_{n}^{(n-4)}, \\
& u_{1}=\frac{1}{f_{r, n}}+\frac{1}{f_{r, n}} C_{n}^{(n-2)}+\frac{1}{f_{r, n}} C_{n}^{(n-3)} .
\end{align*}
$$

Let

$$
\begin{align*}
C_{n}^{(j)} & =\sum_{i=1}^{j} \frac{\left(F_{r+3+j-i}-\left(F_{r+2} / F_{r+1}\right) F_{r+2+j-i}\right)\left(F_{r+n}-F_{r}\right)^{i-1}}{\left(F_{r+1}-F_{r+n+1}\right)^{i}} \\
& =\sum_{i=1}^{j} \frac{a_{j, r}^{\prime}}{\left(m_{1, r}\right)^{i}}\left(m_{2, r}\right)^{i-1} \quad(j=1,2, \ldots, n-2) ; \tag{29}
\end{align*}
$$

we have

$$
\begin{aligned}
& C_{n}^{(2)}-C_{n}^{(1)}=\sum_{i=1}^{2} \frac{a_{2, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}-\frac{a_{1, r}^{\prime}}{m_{1, r}} \\
& \quad=\frac{a_{1, r}^{\prime}}{\left(m_{1, r}\right)^{2}} m_{2, r}, \\
& C_{n}^{(n-2)}+C_{n}^{(n-3)}
\end{aligned}
$$

$$
=\sum_{i=1}^{n-2} \frac{a_{n-2, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}+\sum_{i=1}^{n-3} \frac{a_{n-3, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}
$$

$$
=\sum_{i=1}^{n-3} \frac{\left(F_{r+n+2-i}-A_{r, n} F_{r+n+1-i}\right)\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}
$$

$$
+\frac{a_{1, r}^{\prime}}{\left(m_{1, r}\right)^{n-2}}\left(m_{2, r}\right)^{n-3}
$$

$$
=\sum_{i=1}^{n-2} \frac{\left(F_{r+n+2-i}-\left(F_{r+2} / F_{r+1}\right) F_{r+n+1-i}\right)\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}
$$

$$
\begin{align*}
& C_{n}^{(j+2)}-C_{n}^{(j+1)}-C_{n}^{(j)} \\
&= \sum_{i=1}^{j+2} \frac{a_{j+2, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}-\sum_{i=1}^{j+1} \frac{a_{j+1, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}} \\
&-\sum_{i=1}^{j} \frac{a_{j, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}} \\
&= \frac{\left(F_{r+4}-A_{r, n} F_{r+3}\right)\left(m_{2, r}\right)^{j}}{\left(m_{1, r}\right)^{j+1}} \\
&+\frac{\left(F_{r+3}-A_{r, n} F_{r+2}\right)\left(m_{2, r}\right)^{j+1}}{\left(m_{1, r}\right)^{j+2}} \\
&-\frac{\left(F_{r+3}-A_{r, n} F_{r+2}\right)\left(m_{2, r}\right)^{j}}{\left(m_{1, r}\right)^{j+1}} \\
&= \frac{\left(F_{r+3}-A_{r, n} F_{r+2}\right)\left(m_{2, r}\right)^{j+1}}{\left(m_{1, r}\right)^{j+2}} \\
&(j=1,2, \ldots, n-4) . \tag{30}
\end{align*}
$$

We obtain

$$
\begin{aligned}
& A_{r, n}^{-1} \\
&=\operatorname{Circ}( \frac{1+C_{n}^{(n-2)}+C_{n}^{(n-3)}}{f_{r, n}}, \frac{C_{n}^{(n-2)}-F_{r+2} / F_{r+1}}{f_{r, n}}, \\
&-\frac{C_{n}^{(1)}}{f_{r, n}},-\frac{C_{n}^{(2)}-C_{n}^{(1)}}{f_{r, n}}, \\
&-\frac{C_{n}^{(3)}-C_{n}^{(2)}-C_{n}^{(1)}}{f_{r, n}}, \ldots, \\
&=\frac{1}{f_{r, n}} \operatorname{Circ}\left(1+\sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i}-\frac{C_{r+2}^{(n-2)}-C_{n}^{(n-3)}-C_{n}^{(n-4)}}{F_{r+1}}\right)\right.\right. \\
&\left.\quad \times\left(m_{r+n+1-i}\right)^{i-1}\right) \times\left(\left(m_{1, r}\right)^{i}\right)^{-1}, \\
&-\frac{F_{r+2}}{F_{r+1}}+\sum_{i=1}^{n-2} \frac{a_{n-2, r}^{\prime}\left(m_{2, r}\right)^{i-1}}{\left(m_{1, r}\right)^{i}}, \\
&-\frac{a_{1, r}^{\prime}}{m_{1, r}},-\frac{a_{1, r}^{\prime}}{\left(m_{1, r}\right)^{2}} m_{2, r}, \\
&-\frac{a_{1, r}^{\prime}}{\left(m_{1, r}\right)^{3}}\left(m_{2, r}\right)^{2}, \ldots,
\end{aligned}
$$

$$
\begin{align*}
& \left.-\frac{a_{1, r}^{\prime}}{\left(m_{1, r}\right)^{n-2}}\left(m_{2, r}\right)^{n-3}\right) \\
& =\frac{1}{f_{r, n}} \operatorname{Circ}\left(1+\sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n+1-i}\right)\right.\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1},-\frac{F_{r+2}}{F_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(F_{r+n+1-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n-i}\right)\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1} \\
& -\frac{F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)}{\left(F_{r+1}-F_{r+n+1}\right)^{2}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{2}}{\left(F_{r+1}-F_{r+n+1}\right)^{3}}, \ldots, \\
& \left.-\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}\right), \tag{31}
\end{align*}
$$

where

$$
\begin{align*}
f_{r, n}= & \left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} . \tag{32}
\end{align*}
$$

## 3. Determinant, Invertibility, and <br> Inverse of Circulant Matrix with Any Continuous Lucas Numbers

In this section, let $B_{r, n}=\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a circulant matrix. Firstly, we give a determinant formula for the matrix $B_{r, n}$. Afterwards, we prove that $B_{r, n}$ is an invertible matrix for any positive integer $n$, and then we find the inverse of the matrix $B_{r, n}$.

Theorem 12. Let $B_{r, n}=\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a circulant matrix; then one has

$$
\begin{align*}
& \operatorname{det} B_{r, n} \\
& \begin{aligned}
&=L_{r+1} \cdot\left\{\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right)\right. \\
&+\sum_{k=1}^{n-2}\left[\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\right. \\
&\left.\left.\times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\
& \times\left(L_{r+1}-L_{r+n+1}\right)^{n-2}
\end{aligned}
\end{align*}
$$

where $L_{r+n}$ is the $(r+n)$ th Lucas number. In particular, when $r=0$, one gets Theorem 3.1 in [26].
Proof. Obviously, $\operatorname{det} B_{1}=\left(1-L_{n+1}\right)^{n-1}+\left(L_{n}-2\right)^{n-2}$ $\sum_{k=1}^{n-1}\left(L_{k+2}-3 L_{k+1}\right)\left(\left(1-L_{n+1}\right) /\left(L_{n}-2\right)\right)^{k-1}$ satisfies the formula, when $n>1$; let

$$
\begin{align*}
& \Sigma=\left(\begin{array}{cccccccc}
1 & & & & & & \\
-\frac{L_{r+2}}{L_{r+1}} & & & & & & & 1 \\
-1 & & & & & 1 & -1 \\
0 & & 0 & & 1 & -1 & -1 \\
\vdots & & & & \therefore & \therefore & . & \\
0 & & & 1 & \therefore & . & . & \\
0 & & 1 & -1 & . & & 0 &
\end{array}\right), \\
& \Omega_{1}=\left(\begin{array}{cccccc}
1 & 0 & 0 & \cdots & 0 & 0 \\
0 & \left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-2} & 0 & \cdots & 0 & 1 \\
0 & \left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-3} & 0 & \cdots & 1 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}} & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0
\end{array}\right), \tag{34}
\end{align*}
$$

be two $n \times n$ matrices, we have

$$
\begin{align*}
& \Sigma B_{r, n} \Omega_{1} \\
& =\left[\begin{array}{ccccc}
L_{r+1} & l_{r, n}^{\prime} & L_{r+n-1} & \cdots & L_{r+2} \\
0 & l_{r, n} & L_{r+n}-b_{r} L_{r+n-1} & \cdots & L_{r+3}-b_{r} L_{r+2} \\
0 & 0 & L_{r+1}-L_{r+n-1} & & 0 \\
0 & 0 & L_{r}-L_{r+n} & & 0 \\
\vdots & \vdots & & \ddots & \\
0 & 0 & 0 & & L_{r+1}-L_{r+n-1}
\end{array}\right] \tag{35}
\end{align*}
$$

where

$$
\begin{gather*}
b_{r}=\frac{L_{r+2}}{L_{r+1}} \\
l_{r, n}=\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) \\
+\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)  \tag{36}\\
\times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)} \\
l_{r, n}^{\prime}=\sum_{k=1}^{n-1} L_{r+k+1}\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}
\end{gather*}
$$

We obtain

$$
\begin{align*}
& \operatorname{det} \sum \operatorname{det} B_{r, n} \operatorname{det} \Omega_{1} \\
& \begin{aligned}
&=L_{r+1} \cdot\left\{\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right)\right. \\
& \quad+\sum_{k=1}^{n-2}\left[\left(L_{r+k+2}-b_{r} L_{r+k+1}\right)\right. \\
&\left.\left.\times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\right]\right\} \\
& \quad \times\left(L_{r+1}-L_{r+n+1}\right)^{n-2}
\end{aligned}
\end{align*}
$$

while

$$
\begin{equation*}
\operatorname{det} \Sigma=\operatorname{det} \Omega_{1}=(-1)^{(n-1)(n-2) / 2} \tag{38}
\end{equation*}
$$

We have

$$
\begin{align*}
& \operatorname{det} B_{r, n} \\
& \begin{aligned}
&=L_{r+1} \cdot\left\{\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right)\right. \\
& \quad+\sum_{k=1}^{n-2}\left[\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\right.
\end{aligned} \\
& \left.\left.\quad \times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\right]\right\}  \tag{39}\\
& \times\left(L_{r+1}-L_{r+n+1}\right)^{n-2}
\end{align*}
$$

Theorem 13. Let $B_{r, n}=\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a circulant matrix; then $B_{r, n}$ is invertible for any positive integer $n$. Specially, when $r=0$, one gets Theorem 3.2 in [26].

Proof. Since $L_{n+r}=\alpha^{n+r}+\beta^{n+r}$, where $\alpha+\beta=1, \alpha \cdot \beta=-1$. Hence we have

$$
\begin{aligned}
f\left(\omega^{k}\right) & =\sum_{j=1}^{n} L_{n+r}\left(\omega^{k}\right)^{j-1} \\
& =\sum_{j=1}^{n}\left(\alpha^{r+j}+\beta^{r+j}\right)\left(\omega^{k}\right)^{j-1} \\
& =\frac{\alpha^{r+1}\left(1-\alpha^{n}\right)}{1-\alpha \omega^{k}}+\frac{\beta^{r+1}\left(1-\beta^{n}\right)}{1-\beta \omega^{k}}
\end{aligned}
$$

$$
\left(\text { because } 1-\alpha \omega^{k} \neq 0 \text { and } 1-\beta \omega^{k} \neq 0\right)
$$

$$
=\frac{\left(\alpha^{r+1}+\beta^{r+1}\right)-\left(\alpha^{r+n+1}+\beta^{r+n+1}\right)}{1-(\alpha+\beta) \omega^{k}+\alpha \beta \omega^{2 k}}
$$

$$
-\frac{\alpha \beta\left(\alpha^{r}+\beta^{r}-\alpha^{r+n}-\beta^{r+n}\right) \omega^{k}}{1-(\alpha+\beta) \omega^{k}+\alpha \beta \omega^{2 k}}
$$

$$
=\frac{L_{r+1}-L_{r+n+1}+\left(L_{r}-L_{r+n}\right) \omega^{k}}{1-\omega^{k}-\omega^{2 k}}
$$

$$
(k=1,2, \ldots, n-1)
$$

If there exists $\omega^{l}(l=1,2, \ldots, n-1)$ such that $f\left(\omega^{l}\right)=0$, we obtain $L_{r+1}-L_{r+n+1}+\left(L_{r}-L_{r+n}\right) \omega^{l}=0$ for $1-\omega^{l}-\omega^{2 l} \neq 0$; thus, $\omega^{l}=\left(L_{r+1}-L_{r+n+1}\right) /\left(L_{r+n}-L_{r}\right)$ is a real number, while

$$
\begin{equation*}
\omega^{l}=\exp \left(\frac{2 l \pi i}{n}\right)=\cos \frac{2 l \pi}{n}+i \sin \frac{2 l \pi}{n} . \tag{41}
\end{equation*}
$$

Hence, $\sin (2 l \pi / n)=0$; we have $\omega^{l}=-1$ for $0<2 l \pi / n<$ $2 \pi$. But $x=-1$ is not the root of the equation $L_{r+1}-L_{r+n+1}+$ $\left(L_{r}-L_{r+n}\right) x=0$ for any positive integer $n$. We obtain $f\left(\omega^{k}\right) \neq$ 0 for any $\omega^{k}(k=1,2, \ldots, n-1)$, while $f(1)=\sum_{j=1}^{n} L_{r+j}=$ $L_{r+n+2}-L_{r+2} \neq 0$. By Lemma 4, the proof is completed.

Lemma 14. Let the entries of the matrix $\mathscr{H}=\left[h_{i, j} j_{i, j=1}^{n-2}\right.$ be of the form

$$
h_{i, j}= \begin{cases}L_{r+1}-L_{r+n+1}, & i=j  \tag{42}\\ L_{r}-L_{r+n}, & i=j+1 \\ 0, & \text { otherwise }\end{cases}
$$

then the entries of the inverse $\mathscr{H}^{-1}=\left[h_{i, j}^{\prime}\right]_{i, j=1}^{n-2}$ of the matrix $\mathscr{H}$ are equal to

$$
h_{i, j}^{\prime}= \begin{cases}\frac{\left(L_{r+n}-L_{r}\right)^{i-j}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j+1}}, & i \geq j  \tag{43}\\ 0, & i<j\end{cases}
$$

Specially, when $r=0$, one gets Lemma 3.1 in [26].
Proof. Let $r_{i, j}=\sum_{k=1}^{n-2} h_{i, k} h_{k, j}^{\prime}$. Obviously, $r_{i, j}=0$ for $i<j$. In the case $i=j$, we obtain

$$
\begin{equation*}
r_{i, i}=h_{i, i} h_{i, i}^{\prime}=\left(L_{r+1}-L_{r+n+1}\right) \cdot \frac{1}{L_{r+1}-L_{r+n+1}}=1 . \tag{44}
\end{equation*}
$$

For $i \geq j+1$, we obtain

$$
\begin{aligned}
r_{i, j}= & \sum_{k=1}^{n-2} h_{i, k} h_{k, j}^{\prime}=h_{i, i-1} h_{i-1, j}^{\prime}+h_{i, i} h_{i, j}^{\prime} \\
= & \left(L_{r}-L_{r+n}\right) \cdot \frac{\left(L_{r+n}-L_{r}\right)^{i-j-1}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j}} \\
& +\left(L_{r+1}-L_{r+n+1}\right) \cdot \frac{\left(L_{r+n}-L_{r}\right)^{i-j}}{\left(L_{r+1}-L_{r+n+1}\right)^{i-j+1}} \\
= & 0 .
\end{aligned}
$$

Hence, we verify $\mathscr{H} \mathscr{H}^{-1}=I_{n-2}$, where $I_{n-2}$ is $(n-2) \times(n-2)$ identity matrix. Similarly, we can verify $\mathscr{H}^{-1} \mathscr{H}=I_{n-2}$. Thus, the proof is completed.

Theorem 15. Let $B_{r, n}=\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a circulant matrix; then we have

$$
\begin{align*}
& B_{r, n}^{-1} \\
& =\frac{1}{l_{r, n}} \operatorname{Circ}(1 \\
& +\sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i}\right)\right. \\
& \left.\times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1},-\frac{L_{r+2}}{L_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n-i}\right)\right. \\
& \left.\times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}}{L_{r+1}-L_{r+n+1}}, \\
& -\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)}{\left(L_{r+1}-L_{r+n+1}\right)^{2}}, \\
& -\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{2}}{\left(L_{r+1}-L_{r+n+1}\right)^{3}}, \ldots, \\
& \left.-\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}\right), \tag{46}
\end{align*}
$$

where

$$
\begin{align*}
l_{r, n}= & \left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)} \tag{47}
\end{align*}
$$

In particular, when $r=0$, the result is the same as Theorem 3.3 in [26].

Proof. Let $\Omega_{2}$ be the form of

$$
\left(\begin{array}{cccccc}
1 & -\frac{l_{n \prime}}{L_{r+1}} & x_{3}^{\prime} & x_{4}^{\prime} & \cdots & x_{n}^{\prime}  \tag{48}\\
0 & 1 & y_{3}^{\prime} & y_{4}^{\prime} & \cdots & y_{n}^{\prime} \\
0 & 0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1
\end{array}\right)
$$

where

$$
\begin{gather*}
b_{r}=\frac{L_{r+2}}{L_{r+1}}, \\
x_{i}^{\prime}=\frac{l_{r, n}^{\prime}}{l_{r, n}} \frac{L_{r+n+3-i}-\left(L_{r+2} / L_{r+1}\right) L_{r+n+2-i}}{L_{r+1}} \\
-\frac{L_{r+n+2-i}}{L_{r+1}} \quad(i=3,4, \ldots, n), \\
y_{i}^{\prime}=-\frac{L_{r+n+3-i}-\left(L_{r+2} / L_{r+1}\right) L_{r+n+2-i}}{l_{r, n}} \\
(i=3,4, \ldots, n), \\
l_{r, n}^{\prime}=\left(L_{r+1}=\right. \\
\sum_{k=1}^{n-1} L_{r+k+1}\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}, \\
\left.L_{r+1} L_{r+n}\right)  \tag{49}\\
+\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)} .
\end{gather*} .
$$

We have

$$
\begin{equation*}
\Sigma B_{r, n} \Omega_{1} \Omega_{2}=\mathscr{D}_{2} \oplus \mathscr{H} \tag{50}
\end{equation*}
$$

where $\mathscr{D}_{2}=\operatorname{diag}\left(L_{r+1}, l_{r, n}\right)$ is a diagonal matrix and $\mathscr{D}_{2} \oplus \mathscr{H}$ is the direct sum of $\mathscr{D}_{2}$ and $\mathscr{H}$. If we denote $\Omega=\Omega_{1} \Omega_{2}$, we obtain

$$
\begin{equation*}
B_{r, n}^{-1}=\Omega\left(\mathscr{D}_{2}^{-1} \oplus \mathscr{H}^{-1}\right) \Sigma \tag{51}
\end{equation*}
$$

and the last row elements of the matrix $\Omega$ are $0,1, y_{3}^{\prime}, y_{4}^{\prime}, \cdots$, $y_{n}^{\prime}$. By Lemma 14, if let $B_{r, n}^{-1}=\operatorname{Circ}\left(v_{1}, v_{2}, \ldots, v_{n}\right)$, then its last row elements are given by the following equations:

$$
\begin{align*}
v_{2}= & -\frac{1}{l_{r, n}} \frac{L_{r+2}}{L_{r+1}}+\frac{1}{l_{r, n}} D_{n}^{(n-2)}, \\
v_{3} & =-\frac{1}{l_{r, n}} D_{n}^{(1)}, \\
v_{4} & =-\frac{1}{l_{r, n}} D_{n}^{(2)}+\frac{1}{l_{r, n}} C_{n}^{(1)}, \\
v_{5} & =-\frac{1}{l_{r, n}} D_{n}^{(3)}+\frac{1}{l_{r, n}} D_{n}^{(2)}+\frac{1}{l_{r, n}} D_{n}^{(1)},  \tag{52}\\
& \vdots \\
v_{n} & =-\frac{1}{l_{r, n}} D_{n}^{(n-2)}+\frac{1}{l_{r, n}} D_{n}^{(n-3)}+\frac{1}{l_{n}} D_{n}^{(n-4)}, \\
v_{1} & =\frac{1}{l_{r, n}}+\frac{1}{l_{r, n}} D_{n}^{(n-2)}+\frac{1}{l_{r, n}} D_{n}^{(n-3)} .
\end{align*}
$$

Let

$$
\begin{align*}
D_{n}^{(j)}= & \sum_{i=1}^{j}\left(\left(L_{r+3+j-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+2+j-i}\right)\right. \\
& \left.\times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}  \tag{53}\\
= & \sum_{i=1}^{j} \frac{b_{j, r}^{\prime}}{\left(h_{1, r}\right)^{i}}\left(h_{2, r}\right)^{i-1} \quad(j=1,2, \ldots, n-2)
\end{align*}
$$

we have

$$
\begin{aligned}
& D_{n}^{(2)}-D_{n}^{(1)} \\
& \quad=\sum_{i=1}^{2} \frac{b_{2, r}^{\prime}\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}-\frac{b_{1, r}^{\prime}}{h_{1, r}}=\frac{b_{1, r}^{\prime}}{\left(h_{1, r}\right)^{2}} h_{2, r}, \\
& D_{n}^{(n-2)}+D_{n}^{(n-3)} \\
& =\sum_{i=1}^{n-2} \frac{b_{n-2, r}^{\prime}\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}+\sum_{i=1}^{n-3} \frac{\left(b_{n-3, r}^{\prime}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}} \\
& \quad=\sum_{i=1}^{n-3} \frac{\left(L_{r+n+2-i}-B_{r, n} L_{r+n+1-i}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}} \\
& \quad+\frac{b_{1, r}^{\prime}}{\left(h_{1, r}\right)^{n-2}}\left(h_{2, r}\right)^{n-3} \\
& =\sum_{i=1}^{n-2} \frac{\left(L_{r+n+2-i}-B_{r, n} L_{r+n+1-i}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}
\end{aligned}
$$

$$
\begin{align*}
D_{n}^{(j+2)} & -D_{n}^{(j+1)}-D_{n}^{(j)} \\
= & \sum_{i=1}^{j+2} \frac{\left(b_{j+2, r}^{\prime}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}-\sum_{i=1}^{j+1} \frac{\left(b_{j+1, r}^{\prime}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}} \\
& -\sum_{i=1}^{j} \frac{\left(b_{j, r}^{\prime}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}} \\
= & \frac{\left(L_{r+4}-B_{r, n} L_{r+3}\right)\left(h_{2, r}\right)^{j}}{\left(h_{1, r}\right)^{j+1}} \\
& +\frac{\left(L_{r+3}-B_{r, n} L_{r+2}\right)\left(h_{2, r}\right)^{j+1}}{\left(h_{1, r}\right)^{j+2}} \\
& -\frac{\left(L_{r+3}-B_{r, n} L_{r+2}\right)\left(h_{2, r}\right)^{j}}{\left(h_{1, r}\right)^{j+1}} \\
= & \frac{\left(L_{r+3}-B_{r, n} L_{r+2}\right)\left(h_{2, r}\right)^{j+1}}{\left(h_{1, r}\right)^{j+2}} \\
= & \frac{b_{1, r}^{\prime}\left(h_{2, r}\right)^{j+1}}{\left(h_{1, r}\right)^{j+2}} \quad(j=1,2, \ldots, n-4) . \tag{54}
\end{align*}
$$

We obtain

$$
\begin{aligned}
& {B_{r, n}^{-1}}_{=\operatorname{Circ}( } \begin{aligned}
& \frac{D_{n}^{(n-3)}+D_{n}^{(n-2)}}{l_{r, n}}, \frac{D_{n}^{(n-2)}-L_{r+2} / L_{r+1}}{l_{r, n}}, \\
&-\frac{D_{n}^{(1)}}{l_{r, n}}, \frac{D_{n}^{(1)}-D_{n}^{(2)}}{l_{r, n}}, \frac{D_{n}^{(1)}+D_{n}^{(2)}-D_{n}^{(3)}}{l_{r, n}}, \ldots, \\
&\left.\frac{D_{n}^{(n-4)}+D_{n}^{(n-3)}-D_{n}^{(n-2)}}{l_{r, n}}\right) \\
&=\frac{1}{l_{r, n}} \operatorname{Circ}(1 \\
& \quad+\sum_{i=1}^{n-2} \frac{\left(L_{r+n+2-i}-B_{r, n} L_{r+n+1-i}\right)\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}, \\
& \quad-\frac{L_{r+2}}{L_{r+1}}+\sum_{i=1}^{n-2} \frac{b_{n-2, r}^{\prime}\left(h_{2, r}\right)^{i-1}}{\left(h_{1, r}\right)^{i}}, \\
&-\frac{b_{1, r}^{\prime}}{h_{1, r}},-\frac{b_{1, r}^{\prime}}{\left(h_{1, r}\right)^{2}} h_{2, r}, \\
&\left.\quad-\frac{b_{1, r}^{\prime}}{\left(h_{1, r}\right)^{3}}\left(h_{2, r}\right)^{2}, \ldots,-\frac{b_{1, r}^{\prime}}{\left(h_{1, r}\right)^{n-2}}\left(h_{2, r}\right)^{n-3}\right)
\end{aligned}
\end{aligned}
$$

$$
\begin{align*}
=\frac{1}{l_{r, n}} \operatorname{Circ}(1 & \\
& +\sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i}\right)\right. \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \quad \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
- & \frac{L_{r+2}}{L_{r+1}}+\sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n-i}\right)\right. \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
- & \frac{L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}}{L_{r+1}-L_{r+n+1}}, \\
- & \frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)}{\left(L_{r+1}-L_{r+n+1}\right)^{2}}, \\
- & \frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{2}}{\left(L_{r+1}-L_{r+n+1}\right)^{3}}, \ldots, \\
& \left.-\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}\right) . \tag{55}
\end{align*}
$$

## 4. Determinant, Invertibility, and Inverse of Left Circulant Matrix with Any Continuous Fibonacci and Lucas Numbers

In this section, let $A_{r, n}^{\prime}=\operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ and $B_{r, n}^{\prime}=\operatorname{LCirc}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be left circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A_{r, n}^{\prime}$ and $B_{r, n}^{\prime}$. Afterwards, we prove that $A_{r, n}^{\prime}$ is an invertible matrix for $n>2$ and $B_{r, n}^{\prime}$ is an invertible matrix for any positive integer $n$. The inverse of the matrices $A_{r, n}^{\prime}$ and $B_{r, n}^{\prime}$ is also presented.

According to Lemma 5, Theorem 8, Theorem 9, and Theorem 11, we can obtain the following theorems.

Theorem 16. Let $A_{r, n}^{\prime}=\operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a left circulant matrix; then one has

$$
\begin{aligned}
& \operatorname{det} A_{r, n}^{\prime} \\
& =(-1)^{(n-1)(n-2) / 2} \cdot F_{r+1} \\
& \quad \cdot\left[\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right. \\
& \quad \quad+\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)
\end{aligned}
$$

$$
\begin{align*}
& \left.\quad \times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right] \\
& \times\left(F_{r+1}-F_{r+n+1}\right)^{n-2}, \tag{56}
\end{align*}
$$

where $F_{r+n}$ is the $(r+n)$ th Fibonacci number.
Theorem 17. Let $A_{r, n}^{\prime}=\operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a left circulant matrix; if $n>2$, then $\mathscr{A}_{r, n}^{\prime}$ is an invertible matrix.

Theorem 18. Let $A_{r, n}^{\prime}=\operatorname{LCirc}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)(n>2)$ be a left circulant matrix; then one has

$$
\begin{align*}
& A_{r, n}^{\prime-1} \\
& =\frac{1}{f_{r, n}} \\
& \times \operatorname{LCirc}\left(1+\sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n+1-i}\right)\right.\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}, \ldots, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{2}}{\left(F_{r+1}-F_{r+n+1}\right)^{3}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)}{\left(F_{r+1}-F_{r+n+1}\right)^{2}}, \\
& -\frac{F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\
& -\frac{F_{r+2}}{F_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(F_{r+n+1-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n-i}\right)\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \left.\times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}\right) \text {, } \tag{57}
\end{align*}
$$

where

$$
\begin{align*}
f_{r, n}= & \left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} . \tag{58}
\end{align*}
$$

By Lemma 5, Theorem 12, Theorem 13, and Theorem 15, the following conclusions can be attained.

Theorem 19. Let $B_{r, n}^{\prime}=\operatorname{LCirc}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a left circulant matrix; then one has

$$
\begin{align*}
& \operatorname{det} B_{r, n}^{\prime} \\
& \begin{array}{l}
=(-1)^{(n-1)(n-2) / 2} \cdot L_{r+1} \\
\quad \cdot\left[\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right)\right. \\
\quad+\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right) \\
\left.\quad \times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\right] \\
\quad \times\left(L_{r+1}-L_{r+n+1}\right)^{n-2}
\end{array}
\end{align*}
$$

where $L_{r+n}$ is the $(r+n)$ th Lucas number.
Theorem 20. Let $B_{r, n}^{\prime}=\operatorname{LCirc}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a left circulant matrix; then $B_{r, n}^{\prime}$ is invertible for any positive integer $n$.

Theorem 21. Let $B_{r, n}^{\prime}=\operatorname{LCirc}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a left circulant matrix; then one can obtain

$$
\begin{aligned}
& B_{r, n}^{\prime-1} \\
& =\frac{1}{l_{r, n}} \\
& \times \operatorname{LCirc}\left(1+\sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i}\right)\right.\right. \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \quad \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}, \ldots, \\
& - \\
& \quad-\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{2}}{\left(L_{r+1}-L_{r+n+1}\right)^{3}}, \\
& \quad-\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)}{\left(L_{r+1}-L_{r+n+1}\right)^{2}}, \\
& \quad-\frac{L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}}{L_{r+1}-L_{r+n+1}}
\end{aligned}
$$

$$
\begin{align*}
& -\frac{L_{r+2}}{L_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n-i}\right)\right. \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \left.\quad \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}\right) \tag{60}
\end{align*}
$$

where

$$
\begin{align*}
l_{r, n}= & \left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)} . \tag{61}
\end{align*}
$$

## 5. Determinant, Invertibility, and Inverse of $g$-Circulant Matrix with Any Continuous Fibonacci and Lucas Numbers

In this section, let $A_{g, r, n}=g-\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ and $B_{g, r, n}=g$-Circ $\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be $g$-circulant matrices. By using the obtained conclusions, we give a determinant formula for the matrices $A_{g, r, n}$ and $B_{g, r, n}$. Afterwards, we prove that $A_{g, r, n}$ is an invertible matrix for $n>2$ and $B_{g, r, n}$ is an invertible matrix if $(n, g)=1$. The inverse of the matrices $A_{g, r, n}$ and $B_{g, r, n}$ is also presented.

From Lemma 6, Lemma 7, Theorem 8, Theorem 9, and Theorem 11, we deduce the following results.

Theorem 22. Let $A_{g, r, n}=g$ - $\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a $g$-circulant matrix; then one has

$$
\begin{align*}
& \operatorname{det} A_{g, r, n}= \operatorname{det} \\
& \mathbb{Q}_{g} \cdot F_{r+1} \\
& \cdot\left[\left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right)\right.  \tag{62}\\
&+\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right) \\
&\left.\times\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)}\right] \\
& \times\left(F_{r+1}-F_{r+n+1}\right)^{n-2},
\end{align*}
$$

where $F_{r+n}$ is the $(r+n)$ th Fibonacci number.
Theorem 23. Let $A_{g, r, n}=g$ - $\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)$ be a $g$-circulant matrix and $(g, n)=1$; if $n>2$, then $A_{g, r, n}$ is an invertible matrix.

Theorem 24. Let $A_{g, r, n}=g$ - $\operatorname{Circ}\left(F_{r+1}, F_{r+2}, \ldots, F_{r+n}\right)(n>$ 2) be a $g$-circulant matrix and $(g, n)=1$; then

$$
\begin{align*}
& A_{g, r, n}^{-1} \\
& =\left[\frac{1}{f_{r, n}}\right. \\
& \times \operatorname{Circ}\left(1+\sum_{i=1}^{n-2}\left(\left(F_{r+n+2-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n+1-i}\right)\right.\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{F_{r+2}}{F_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(F_{r+n+1-i}-\frac{F_{r+2}}{F_{r+1}} F_{r+n-i}\right)\right. \\
& \left.\times\left(F_{r+n}-F_{r}\right)^{i-1}\right) \\
& \times\left(\left(F_{r+1}-F_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}}{F_{r+1}-F_{r+n+1}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)}{\left(F_{r+1}-F_{r+n+1}\right)^{2}}, \\
& -\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{2}}{\left(F_{r+1}-F_{r+n+1}\right)^{3}}, \ldots, \\
& \left.\left.-\frac{\left(F_{r+3}-\left(F_{r+2} / F_{r+1}\right) F_{r+2}\right)\left(F_{r+n}-F_{r}\right)^{n-3}}{\left(F_{r+1}-F_{r+n+1}\right)^{n-2}}\right)\right] \\
& \times \mathbb{Q}_{g}^{T}, \tag{63}
\end{align*}
$$

where

$$
\begin{align*}
f_{r, n}= & \left(F_{r+1}-\frac{F_{r+2}}{F_{r+1}} F_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(F_{r+k+2}-\frac{F_{r+2}}{F_{r+1}} F_{r+k+1}\right)\left(\frac{F_{r+n}-F_{r}}{F_{r+1}-F_{r+n+1}}\right)^{n-(k+1)} \tag{64}
\end{align*}
$$

Taking Lemma 6, Lemma 7, Theorem 12, Theorem 13, and Theorem 15 into account, one has the following theorems.

Theorem 25. Let $B_{g, r, n}=g-\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a g-circulant matrix; then one has

$$
\begin{align*}
\operatorname{det} B_{g, r, n}= & \operatorname{det} \mathbb{Q}_{g} \cdot L_{r+1} \\
& \cdot\left[\left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right)\right. \\
& +\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)  \tag{65}\\
& \left.\times\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)}\right] \\
& \times\left(L_{r+1}-L_{r+n+1}\right)^{n-2},
\end{align*}
$$

where $L_{r+n}$ is the $(r+n)$ th Lucas number.
Theorem 26. Let $B_{g, r, n}=g$ - $\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)$ be a $g$-circulant matrix and $(g, n)=1$; if $n>2$, then $B_{g, r, n}$ is an invertible matrix.

Theorem 27. Let $B_{g, r, n}=g$ - $\operatorname{Circ}\left(L_{r+1}, L_{r+2}, \ldots, L_{r+n}\right)(n>$ 2) be a $g$-circulant matrix and $(g, n)=1$; then

$$
\begin{aligned}
& B_{g, r, n}^{-1} \\
& =\left[\frac{1}{l_{r, n}}\right. \\
& \quad \times \operatorname{Circ}\left(1+\sum_{i=1}^{n-2}\left(\left(L_{r+n+2-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n+1-i}\right)\right.\right. \\
& \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \\
& \quad \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{L_{r+2}}{L_{r+1}} \\
& +\sum_{i=1}^{n-2}\left(\left(L_{r+n+1-i}-\frac{L_{r+2}}{L_{r+1}} L_{r+n-i}\right)\right. \\
& \\
& \left.\quad \times\left(L_{r+n}-L_{r}\right)^{i-1}\right) \\
& \\
& \times\left(\left(L_{r+1}-L_{r+n+1}\right)^{i}\right)^{-1}, \\
& -\frac{L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}}{L_{r+1}-L_{r+n+1}} \\
& -\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)}{\left(L_{r+1}-L_{r+n+1}\right)^{2}},
\end{aligned}
$$

$$
\begin{align*}
& -\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{2}}{\left(L_{r+1}-L_{r+n+1}\right)^{3}}, \ldots \\
& \left.\left.-\frac{\left(L_{r+3}-\left(L_{r+2} / L_{r+1}\right) L_{r+2}\right)\left(L_{r+n}-L_{r}\right)^{n-3}}{\left(L_{r+1}-L_{r+n+1}\right)^{n-2}}\right)\right] \\
& \times \mathbb{Q}_{g}^{T} \tag{66}
\end{align*}
$$

where

$$
\begin{align*}
l_{r, n}= & \left(L_{r+1}-\frac{L_{r+2}}{L_{r+1}} L_{r+n}\right) \\
& +\sum_{k=1}^{n-2}\left(L_{r+k+2}-\frac{L_{r+2}}{L_{r+1}} L_{r+k+1}\right)\left(\frac{L_{r+n}-L_{r}}{L_{r+1}-L_{r+n+1}}\right)^{n-(k+1)} . \tag{67}
\end{align*}
$$

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Quasisynchronization in Quorum Sensing Systems with Parameter Mismatches 

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#### Abstract

The paper investigates quasisynchronization in a communication system, which consists of cells communicating through quorum sensing. With the help of Lyapunov function method and Lur'e system approach, some sufficient conditions for quasisynchronization are presented, and a bound on the synchronization errors is derived. The obtained theoretical results show that the synchronization quality is influenced by two parameters detrimentally: the error bound depends almost linearly on the mismatches between cells and depends sensitively on the diffusion rates of the signals inward the cell membrane. Numerical experiments are carried out to verify the theoretical results.


## 1. Introduction

In the past decades, increasing interest has been shown to study the dynamics of coupled oscillator networks, which describes many complex systems in the field of nature and science. Due to the couplings among these oscillators, different types of synchronization can be realized in such systems.

In particular, a coupled oscillator network is called to be quasisynchronized, or weak synchronized, if the synchronization errors will be in some neighborhood of zero but will not tend to zero eventually. In other words, quasisynchronization means that the dynamical trajectories of each cell are similar but different from each other [1,2]. By Lyapunov function method and a differential inequality method, two coupled identical oscillators with parameter mismatches were quasisynchronized via periodically intermittent control in [3]. Similar results were also obtained in a discontinuous master-response system with parameter mismatches [4]. It has also been shown that two coupled delayed oscillators with parameter mismatches can be lag quasisynchronized, and the error level is estimated by applying a generalized Halanary inequality and matrix measure [5].

To the best of our knowledge, most of researches on synchronization focused on two oscillators coupled directly [6-8]. And there are very few researches focused on the system composed of several oscillators coupled indirectly. However, many biological systems coupled indirectly can also exhibit synchronization such as a global cellular response. For an example of such mechanisms achieving synchronization, the unicellular bacteria are highly coupled through chemical signaling molecules. This process, termed quorum sensing [ $9-11$ ], allows bacterial populations to exchange intercellular signals with their neighbor cells to coordinate gene expression and integrates cells to realize synchronization effectively. That is to say, each bacterium is connected with all the other bacteria through a mean field and coordinated precisely. As the result, all the bacteria form a "microsociety," behaving synchronously and exhibiting various collective dynamics. In the past decade, synchronization induced by the intercell signaling mechanism has been widely investigated. For instance, synchronization induced by quorum sensing has been studied in networks composed by genetic relaxation oscillators [12], limit-cycle oscillators [13], and synthetic gene oscillators [14]. Another research has shown that a noisy community of such genetic oscillators can self-synchronize
in a robust way and lead to a substantially global rhythmicity [15].

Due to the biological diversity, there are usually some parameter mismatches between the coupled oscillators in biological systems. Therefore, complete synchronization is hard to be achieved. Instead, quasisynchronization, which implies a state of synchronization with an error level, is more common in the biological systems. Motivated by the complexity and similarity of biological organisms and the potential applications, the paper studies quasisynchronization in quorum sensing systems with parameter mismatches. Through a new method different from many previous researches [3-5], the bound on the synchronization errors is estimated with help of Lur'e system, linear matrix inequalities, and Lyapunov function [16-19]. Both the theoretical results and the numerical simulations indicate that the synchronization errors stay in a neighborhood of zero, increase roughly linearly with the mismatches between individual cells, and depend sensitively on the diffusion rate of the signals inward the cell membrane.

The rest of the paper is organized as follows. In Section 2, we give some sufficient conditions for quasisynchronization in quorum sensing systems with parameter mismatches. The bound of the synchronization errors is also estimated. In Section 3, numerical examples are carried out to verify the theoretical results.

## 2. Main Results

2.1. Quorum Sensing Systems with Mismatches. Quorum sensing is a cell concentration dependent phenomenon in bacteria and fungi, which is mediated by small, diffusible signaling molecules that accumulate in the extracellular environment [9]. In such a multicell system, the individual oscillator in each cell is a network with three genes, $a, b$, and $c$, the products of which inhibit the transcription of each other in a cyclic way. The gene $c$ expresses protein $C$, which inhibits transcription of the gene $a$. The product of $a$ inhibits transcription of the gene $b$, the protein product $B$ of which in turn inhibits expression of $c$, completing the cycle. These bacteria exhibit cell-to-cell communication through a mechanism that makes use of two proteins, the first one of which (LuxI) synthesizes a small molecule known as an autoinducer (AI), which can diffuse freely through the cell membrane. The principle of the phenomenon is that when a single bacterium releases autoinducers (AIs) into the environment, their concentration is too low to be detected. However, when sufficient bacteria are present, AI concentrations reach a threshold level that allows the bacteria to sense a critical cell mass and to activate target genes [9]. When a second protein (LuxR) binds to this molecule, the resulting complex activates transcription of various genes, including some coding for light-producing enzymes. The scheme of the network is shown in Figure 1. For further details, one is referred to previous articles [9-11].

Before we carry out the dynamics model for the N cell system described by differential equations, we make the following declaration throughout the paper. Let $R^{n}$ be the $n$ dimensional Euclidean space, $R^{n \times m}$ the set of all $n \times m$ real


Figure 1: Scheme of the repressilator network coupled through signaling molecules, termed quorum sensing. The synchronization scheme of quorum sensing is based on the diffusion of a small molecule (autoinducer AI) to and from the cells.
matrices, $A^{\top}$ the transpose of a square matrix $A$, and $\|\cdot\|$ the usual $L_{2}$ norm of a vector or the usual spectral norm of a square matrix. The notation $M>0(<0)$ is used to define a real symmetric positive (negative) definite matrix. If not explicitly stated, matrices are assumed to have compatible dimensions.

Then the dynamics model for the $N$-cell system is built as follows:

$$
\begin{gather*}
\dot{a}_{i}=-d_{1 i}(t) a_{i}+\frac{\alpha_{6 i}(t)}{\mu_{6}+C_{i}^{m}}, \quad \dot{A}_{i}=-d_{4 i}(t) A_{i}+\beta_{1 i}(t) a_{i}, \\
\dot{b}_{i}=-d_{2 i}(t) b_{i}+\frac{\alpha_{4 i}(t)}{\mu_{4}+A_{i}^{m}}, \quad \dot{B}_{i}=-d_{5 i}(t) B_{i}+\beta_{2 i}(t) b_{i} \\
\dot{c}_{i}=-d_{3 i}(t) c_{i}+\frac{\alpha_{5 i}(t)}{\mu_{5}+B_{i}^{m}}+\frac{\alpha_{7 i}(t) S_{i}}{\mu_{7}+S_{i}} \\
\dot{C}_{i}=-d_{6 i}(t) C_{i}+\beta_{3 i}(t) c_{i} \\
\dot{S}_{i}=-d_{7 i}(t) S_{i}+\beta_{4 i}(t) A_{i}-\eta\left(S_{i}-S_{e}\right) \\
\dot{S}_{e}=-d_{e}(t) S_{e}+\eta_{e}(t) \sum_{j=1}^{N} \frac{S_{j}-S_{e}}{N} \tag{1}
\end{gather*}
$$

where $i=1,2, \ldots, N ; a_{i}, b_{i}$, and $c_{i}$ are the concentrations of $m R N A$ transcribed from genes $a, b$, and $c$ in cell $i$, respectively; $A_{i}, B_{i}$, and $C_{i}$ are the concentrations of the corresponding proteins, respectively; $S_{i}$ and $S_{e}$ are concentrations of AI inside each cell and in the environment, respectively. The parameters $d_{j i}(t)(j=1, \ldots, 7)$ and $d_{e}(t)$ are the dimensionless degradation rates of the chemical molecules in cell $i ; \alpha_{j i}(t)(j=4,5,6)$ are the dimensionless transcription rates in the absence of repressor; $\alpha_{7 i}(t)$ is the maximal contribution to the gene $c$ transcription in the presence of saturating amounts of $\mathrm{AI} ; \beta_{j i}(t)(j=1,2,3)$ are the translation rates of the proteins from the $m R N A s ; \beta_{4 i}(t)$ is the synthesis rate of AI; $m=4$ is the Hill coefficient; $\eta(t)$ and $\eta_{e}(t)$ measure the diffusion rate of AI inward and outward the cell membrane.

We suppose that all the parameters mentioned above are time-varying in the vicinity of certain constants, which indicates that all the cells are similar but different from each other. For convenience, we decompose these parameters into two parts: a constant part that determines the values of the parameters and a time-varying part representing the parameter mismatches. For example, let the parameter $d_{j i}(t)=d_{j}+\delta d_{j i}(t),(j=1, \ldots, 7)$, which implies that the parameter $d_{j i}(t)$ is time-varying around the constant $d_{j}$ with the parameter mismatches $\delta d_{j i}(t),(j=1, \ldots, 7)$.

Denoting $x_{i}=\left(a_{i}, b_{i}, c_{i}, A_{i}, B_{i}, C_{i}, S_{i}\right)^{\top}, i=1, \ldots, N$, and $x_{e}=\left(0,0,0,0,0,0, S_{e}\right)^{\top}$ for convenience, then muiticell system (1) can be rewritten as follows:

$$
\begin{align*}
\dot{x}_{i}= & -\left(d+\delta d_{i}(t)\right) x_{i}+\left(\alpha+\delta \alpha_{i}(t)\right) f\left(x_{i}\right) \\
& -\left(\gamma+\delta \gamma_{i}(t)\right)\left(g\left(x_{i}\right)-e_{n}\right)+\eta\left(x_{e}-I_{n} x_{i}\right), \\
& \dot{x}_{e}=-d_{e}(t) x_{e}+\eta_{e}(t) \sum_{j=1}^{N} \frac{\left(I_{n} x_{j}-x_{e}\right)}{n}, \tag{2}
\end{align*}
$$

where $i=1,2, \ldots, N ; d=\operatorname{diag}\left(d_{1}, d_{2}, d_{3}, d_{4}, d_{5}, d_{6}, d_{7}\right)+\beta$, $\beta \in R^{n \times n}$ with elements equal to zero except $\beta_{41}=\beta_{1}$, $\beta_{52}=\beta_{2}, \beta_{63}=\beta_{3}$, and $\beta_{74}=\beta_{4} ; \alpha \in R^{n \times n}$ with elements equal to zero except $\alpha_{16}=\alpha_{6}, \alpha_{24}=\alpha_{4}$, and $\alpha_{35}=\alpha_{5}$; $\gamma \in R^{n \times n}$ with elements equal to zero except $\gamma_{37}=\alpha_{7}$; $I_{n}=\operatorname{diag}(0,0,0,0,0,0,1) ; e_{n}=(0,0,0,0,0,0,1)^{\top}$; and

$$
\begin{gather*}
g\left(x_{i}\right)=\left(0,0,0,0,0,0, \frac{\mu_{7}}{\mu_{7}+S_{i}}\right)^{\top} \\
f\left(x_{i}\right)=\left(0,0,0, \frac{1}{\mu_{4}+A_{i}^{m}}, \frac{1}{\mu_{5}+B_{i}^{m}}, \frac{1}{\mu_{6}+C_{i}^{m}}, 0\right)^{\top} \tag{3}
\end{gather*}
$$

The matrices $\delta d_{i}(t), \delta \alpha_{i}(t)$, and $\delta \gamma_{i}(t)$ describe the mismatches of the parameters $d_{i}(t), \alpha_{i}(t)$, and $\gamma_{i}(t)$. Note the dimension of individual cells $n=7$ for multicell system (1). In fact, system (2) can describe the general model of quorum sensing mechanism, where the vector components of vector functions $f(\cdot)$ and $g(\cdot)$ are increasing functions and $I_{n}$ describes which components are coupled with the environment. Thus, the results obtained in the paper are also valid for any general multicell system based on quorum sensing.
2.2. Sufficient Conditions for Quasisynchronization. Since the cells in realistic organisms are similar but different from each other, one can suppose that the mismatch matrices $\delta d_{i}(t)$, $\delta \alpha_{i}(t)$, and $\delta \gamma_{i}(t)$ are bounded as follows:

$$
\begin{equation*}
\left\|\delta d_{i}(t)\right\| \leq \varepsilon_{1}, \quad\left\|\delta \alpha_{i}(t)\right\| \leq \varepsilon_{2}, \quad\left\|\delta \gamma_{i}(t)\right\| \leq \varepsilon_{3} \tag{4}
\end{equation*}
$$

Noticing that the concentrations of chemical molecules $x_{i}(t)$ are bounded and $f(\cdot)$ and $g(\cdot)$ are monotonic functions of $x_{i}(t)$, we can conclude that

$$
\begin{equation*}
\left\|x_{i}(t)\right\| \leq \delta_{1}, \quad\left\|f\left(x_{i}(t)\right)\right\| \leq \delta_{2}, \quad\left\|g\left(x_{i}(t)\right)\right\| \leq \delta_{3} \tag{5}
\end{equation*}
$$

Since there exist parameter mismatches between different cells, multicell system (1) cannot be completely synchronized. Instead, we present another type of synchronization, which is defined as follows.

Definition 1. The multicell system (1) is said to reach quasisynchronization or weak synchronization, if, for any initial condition $\left(x_{1}^{\top}(0), \ldots, x_{m}^{\top}(0)\right)^{\top}$, there exists a positive constant $\varepsilon$ such that

$$
\begin{equation*}
\left\|X_{i j}\right\|=\left\|x_{j}-x_{i}\right\| \leq \varepsilon, \quad i, j=1,2, \ldots, N . \tag{6}
\end{equation*}
$$

The regulatory functions $f_{l}(\cdot)$ and $g_{l}(\cdot)$ are both monotonic increasing functions; there exist two diagonal matrices $K_{1}=\operatorname{diag}\left(k_{11}, k_{12}, \ldots, k_{1 n}\right) \geq 0$ and $K_{1}=$ $\operatorname{diag}\left(k_{21}, k_{22}, \ldots, k_{2 n}\right) \geq 0$ such that the following sector conditions are satisfied:

$$
\begin{equation*}
0 \leq \frac{f_{l}(a)-f_{l}(b)}{a-b} \leq k_{1 l}, \quad 0 \leq \frac{g_{l}(a)-g_{l}(b)}{a-b} \leq k_{2 l}, \tag{7}
\end{equation*}
$$

where $a, b \in R, a \neq b$, and $l=1, \ldots, n$. Notice that the Lur'e system consists of a linear system feedback interconnected with a static nonlinearity $f(\cdot)$ that satisfies a sector condition [20]; multicell system (2) can be regarded as a Lur'e system. Consequently, with help of Lur'e system method in control theory and Lyapunov direct method, we obtain the following sufficient conditions for quasisynchronization of system (2).

Theorem 2. If there exist symmetric matrices $P>0$ and $\Lambda_{i}=$ $\operatorname{diag}\left(\lambda_{i 1}, \ldots, \lambda_{\text {in }}\right) \geq 0, i=1,2$, and a constant $\rho>0$ such that the symmetric matrix

$$
M=\left[\begin{array}{ccc}
P d+d^{\top} P-2 \eta I_{n}+\rho E & P \alpha+K_{1} \Lambda_{1} & -P \gamma+K_{2} \Lambda_{2}  \tag{8}\\
\alpha^{\top} P+K_{1} \Lambda_{1} & -2 \Lambda_{1} & 0 \\
-\gamma^{\top} P+K_{2} \Lambda_{2} & 0 & -2 \Lambda_{2}
\end{array}\right] \leq 0,
$$

where $E \in R^{n \times n}$ is the unit matrix, multicell system (1) is quasisynchronized. The bound on the synchronization errors can be estimated by $\delta / \rho$, where

$$
\begin{equation*}
\delta=\frac{2 \lambda_{\max }(P)}{\lambda_{\min }(P)}\left(\varepsilon_{1} \delta_{1}+\varepsilon_{2} \delta_{2}+\varepsilon_{3} \delta_{3}+\varepsilon_{3}\right) \tag{9}
\end{equation*}
$$

Proof. Define a Lyapunov function with respect to multicell system (2) of the following form:

$$
\begin{equation*}
V(x(t))=x^{\top}(t) P x(t) \tag{10}
\end{equation*}
$$

where $P$ is a positive definite matrix. According to [21], the Lyapunov function is equivalent to the following form:

$$
\begin{equation*}
V(x(t))=\sum_{i=1}^{N} \sum_{j=1}^{N} X_{i j}^{\top} P X_{i j} \tag{11}
\end{equation*}
$$

where $X_{i j}=x_{j}-x_{i}$ denotes the synchronization errors.

Based on Lyapunov direct method, if the time derivative of $V(x(t))$ along the trajectories of (2) is negative outside of a neighborhood of the origin $O$, multicell system (2) will achieve quasisynchronization with the errors $X_{i j}$ staying in the neighborhood. Calculating the time derivative of $V(x(t))$ along (2),

$$
\begin{align*}
\dot{X}_{i j}= & \dot{x}_{j}-\dot{x}_{i} \\
= & \left(d+\delta d_{j}(t)\right) x_{j}+\left(\alpha+\delta \alpha_{j}(t)\right) f\left(x_{j}\right) \\
& -\left(\gamma+\delta \gamma_{j}(t)\right)\left(g\left(x_{j}\right)-e_{n}\right)-\eta I_{n} x_{i} \\
& -\left[\left(d+\delta d_{i}(t)\right) x_{i}+\left(\alpha+\delta \alpha_{i}(t)\right) f\left(x_{i}\right)\right. \\
& \left.\quad-\left(\gamma+\delta \gamma_{i}(t)\right)\left(g\left(x_{i}\right)-e_{n}\right)\right]-\eta I_{n} x_{j}  \tag{12}\\
= & \left(d E-\eta I_{n}\right) X_{i j}+\alpha\left(f\left(x_{j}\right)-f\left(x_{i}\right)\right) \\
& -\gamma\left(g\left(x_{j}\right)-g\left(x_{i}\right)\right)+\left(\delta \gamma_{j}(t)-\delta \gamma_{i}(t)\right) I_{n} \\
& +\delta d_{j}(t) x_{j}-\delta d_{i}(t) x_{i}+\delta \alpha_{j}(t) f\left(x_{j}\right) \\
& -\delta \alpha_{i}(t) f\left(x_{i}\right)-\delta \gamma_{j}(t) g\left(x_{j}\right)+\delta \gamma_{i}(t) g\left(x_{i}\right) .
\end{align*}
$$

We have

$$
\begin{align*}
\dot{V}=\sum_{i=1}^{N} \sum_{j=1}^{N} X_{i j}^{\top} P & \\
& \times\left[\left(d E-\eta I_{n}\right) X_{i j}+\alpha\left(f\left(x_{j}\right)-f\left(x_{i}\right)\right)\right. \\
& \quad-\gamma\left(g\left(x_{j}\right)-g\left(x_{i}\right)\right) \\
& +I_{n}\left(\delta \gamma_{j}(t)-\delta \gamma_{i}(t)\right)+\delta d_{j}(t) x_{j} \\
& \quad-\delta d_{i}(t) x_{i}+\delta \alpha_{j}(t) f\left(x_{j}\right)-\delta \alpha_{i}(t) f\left(x_{i}\right) \\
& \left.\quad-\delta \gamma_{j}(t) g\left(x_{j}\right)+\delta \gamma_{i}(t) g\left(x_{i}\right)\right] . \tag{13}
\end{align*}
$$

Noticing the sector conditions (7) of $f(\cdot)$ and $g(\cdot)$, we have

$$
\begin{align*}
& \left(f_{l}\left(x_{j}^{(l)}\right)-f_{l}\left(x_{i}^{(l)}\right)\right)\left(f_{l}\left(x_{j}^{(l)}\right)-f_{l}\left(x_{i}^{(l)}\right)-k_{1 l} X_{i j}^{(l)}\right) \geq 0 \\
& \left(g_{l}\left(x_{j}^{(l)}\right)-g_{l}\left(x_{i}^{(l)}\right)\right)\left(g_{l}\left(x_{j}^{(l)}\right)-g_{l}\left(x_{i}^{(l)}\right)-k_{2 l} X_{i j}^{(l)}\right) \geq 0 \tag{14}
\end{align*}
$$

where $l=1,2, \ldots, n$. Then, for any $\Lambda_{1}=\operatorname{diag}\left(\lambda_{11}, \ldots, \lambda_{1 n}\right) \geq$ 0 and $\Lambda_{2}=\operatorname{diag}\left(\lambda_{21}, \ldots, \lambda_{2 n}\right) \geq 0$ and any constant $\rho>0$, there holds

$$
\begin{align*}
& \dot{V} \leq \sum_{i=1}^{N} \sum_{j=1}^{N} X_{i j}^{\top} P\left[\left(d E-\eta I_{n}+\rho E\right) X_{i j}\right. \\
& \left.+\alpha\left(f\left(x_{j}\right)-f\left(x_{i}\right)\right)\right] \\
& -\gamma \sum_{i=1}^{N} \sum_{j=1}^{N} X_{i j}^{\top} P\left(g\left(x_{j}\right)-g\left(x_{i}\right)\right) \\
& -2 \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_{1}\left(f\left(x_{j}\right)-f\left(x_{i}\right)\right) \\
& \times\left(f\left(x_{j}\right)-f\left(x_{i}\right)-K_{1} X_{i j}\right)  \tag{15}\\
& -2 \sum_{i=1}^{N} \sum_{j=1}^{N} \Lambda_{2}\left(g\left(x_{j}\right)-g\left(x_{i}\right)\right) \\
& \times\left(g\left(x_{j}\right)-g\left(x_{i}\right)-K_{2} X_{i j}\right) \\
& +\lambda_{\text {min }}(P)\left(\delta\left\|X_{i j}\right\|-\rho\left\|X_{i j}\right\|^{2}\right) \\
& =\sum_{i=1}^{N} \sum_{j=1}^{N}\left(\xi_{i j}^{\top} M \xi_{i j}+\lambda_{\min }(P)\left(\delta\left\|X_{i j}\right\|-\rho\left\|X_{i j}\right\|^{2}\right)\right),
\end{align*}
$$

where $M$ is defined in (8),

$$
\begin{gather*}
\xi_{i j}=\left[\left(x_{j}-x_{i}\right)^{\top},\left(f\left(x_{j}\right)-f\left(x_{i}\right)\right)^{\top},\left(g\left(x_{j}\right)-g\left(x_{i}\right)\right)^{\top}\right]^{\top} \\
\delta=\frac{2 \lambda_{\max }(P)}{\lambda_{\min }(P)}\left(\varepsilon_{1} \delta_{1}+\varepsilon_{2} \delta_{2}+\varepsilon_{3} \delta_{3}+\varepsilon_{3}\right) \tag{16}
\end{gather*}
$$

If matrix inequalities (8) and $\left\|x_{j}-x_{i}\right\| \geq \delta / \rho$ hold, one obtains that

$$
\begin{equation*}
\dot{V} \leq \lambda_{\min }(P) \sum_{i=1}^{N} \sum_{j=1}^{N}\left(\delta\left\|X_{i j}\right\|-\rho\left\|X_{i j}\right\|^{2}\right)<0 \tag{17}
\end{equation*}
$$

According to Lyapunov direct method, all the trajectories of cells in system (2) will go closer to each other when $X_{i j}$ stay outside of the neighborhood of the origin $O(0, \delta / \rho)$. Therefore, multicell system (2) realizes quasisynchronization with synchronization errors $X_{i j}$ staying in the neighborhood of the origin $O(0, \delta / \rho)$.

Theorem 2 implies that the estimation of the bound on synchronization error is influenced by two factors, the mismatches between cells $\varepsilon_{i}$ and the parameter $\rho$. On one hand, quasisynchronization can be realized if the mismatches are small and the synchronization errors oscillate in a certain neighborhood of the origin. On the other hand, the parameter $\rho$ is determined by the dynamics of the individual cells and reflects the ability of synchronization of the inherent
dynamics. Furthermore, the larger the parameter $\rho$ is, the smaller the bound on the synchronization errors is.

In fact, the proof of the theorem shows that the mismatches that go against synchronization could be compensated by the linear function $-\rho X_{i j}$. Both the error dynamics caused by the mismatches and $\rho X_{i j}$ could be compensated by the linear function $\Lambda X_{i j}$, which is determined by the diffusion rate of the signals inward the cell membrane $\eta$. Consequently, the parameter $\eta$ measures the quasisynchronization ability of the inherent dynamics of the cells. Therefore, if the two factors mentioned above can be controlled, the synchronization errors can be controlled.

As a special case, if there are no parameter mismatches between different cells, which implies that all the cells are identical, then it is easy to conclude that multicell system (2) realizes complete synchronization. In such a case, multicell systems (1) and (2) could be rewritten as

$$
\begin{gather*}
\dot{a}_{i}=-d_{1 i} a_{i}+\frac{\alpha_{6 i}}{\mu_{6}+C_{i}^{m}}, \quad \dot{A}_{i}=-d_{4 i} A_{i}+\beta_{1 i} a_{i}, \\
\dot{b}_{i}=-d_{2 i} b_{i}+\frac{\alpha_{4 i}}{\mu_{4}+A_{i}^{m}}, \quad \dot{B}_{i}=-d_{5 i} B_{i}+\beta_{2 i} b_{i}, \\
\dot{c}_{i}=-d_{3 i} c_{i}+\frac{\alpha_{5 i}}{\mu_{5}+B_{i}^{m}}+\frac{\alpha_{7 i} S_{i}}{\mu_{7}+S_{i}}  \tag{18}\\
\dot{C}_{i}=-d_{6 i} C_{i}+\beta_{3 i} c_{i} \\
\dot{S}_{i}=-d_{7 i} S_{i}+\beta_{4 i} A_{i}-\eta\left(S_{i}-S_{e}\right), \\
\dot{S}_{e}=-d_{e} S_{e}+\eta_{e} \sum_{j=1}^{N} \frac{S_{j}-S_{e}}{N}, \\
\dot{x}_{i}=-d x_{i}+\alpha f\left(x_{i}\right)-\gamma\left(g\left(x_{i}\right)-e_{n}\right)+\eta\left(x_{e}-I_{n} x_{i}\right), \\
\dot{x}_{e}=-d_{e} x_{e}+\eta_{e} \sum_{j=1}^{N} \frac{\left(I_{n} x_{j}-x_{e}\right)}{n}, \tag{19}
\end{gather*}
$$

where the parameters $d, \gamma, \eta, d_{e}$, and $\eta_{e}$ are defined in multicell system (2). Then we obtain the following corollary.

Corollary 3. If there exist symmetric matrices $P>0$ and $\Lambda_{i}=$ $\operatorname{diag}\left(\lambda_{i 1}, \ldots, \lambda_{i n}\right) \geq 0, i=1,2$, and a constant $\rho>0$ such that the symmetric matrix

$$
M=\left[\begin{array}{ccc}
P d+d^{\top} P-2 \eta I_{n}+\rho E & P \alpha+K_{1} \Lambda_{1} & -P \gamma+K_{2} \Lambda_{2}  \tag{20}\\
\alpha^{\top} P+K_{1} \Lambda_{1} & -2 \Lambda_{1} & 0 \\
-\gamma^{\top} P+K_{2} \Lambda_{2} & 0 & -2 \Lambda_{2}
\end{array}\right] \leq 0,
$$

where $E \in R^{n \times n}$ is the unit matrix, multicell system (18) realizes complete synchronization.

## 3. Numerical Simulations

In order to demonstrate the effectiveness of our theoretical analysis, we give numerical examples based on multicell system (1) consisting of 6 cells. Set the parameters as follows:

$$
\begin{gather*}
d_{1}=d_{2}=d_{3}=0.4, \quad d_{4}=d_{5}=d_{6}=0.5 \\
d_{7}=0.016, \quad d_{e}=0.2, \quad \alpha_{4}=\alpha_{5}=\alpha_{6}=1.96  \tag{21}\\
\alpha_{7}=1, \quad \beta_{1}=\beta_{2}=\beta_{3}=0.13, \quad \beta_{4}=0.018 \\
\eta=0.4, \quad \eta_{e}=0.8, \quad \mu_{4}=\mu_{5}=\mu_{6}=\mu_{7}=0.2
\end{gather*}
$$

Suppose that the initial mismatches $\delta d_{i}, \delta \alpha_{i}$, and $\delta \beta_{i}$ are taken randomly in the open intervals $\left(-\varepsilon d_{i}, \varepsilon d_{i}\right),\left(-\varepsilon \alpha_{i}, \varepsilon \alpha_{i}\right)$, and $\left(-\varepsilon \beta_{i}, \varepsilon \beta_{i}\right)$, respectively. The evolutions of the synchronization errors $X_{i j}$ are shown to converge to a neighborhood of the origin in Figure 2. As can be seen, the smaller the bound on mismatches between the oscillators is, the smaller the synchronization errors $X_{i j}$ are. And complete synchronization can be realized if $\varepsilon=0$.

In order to verify the relationship between the synchronization errors and the mismatches between cells, the figure of the synchronization errors transition for increasing the mismatches $\varepsilon$ is plotted in Figure 3. It is obvious that quasisynchronization is realized and the synchronization errors increase with the mismatches roughly linearly.

Numerical simulations are also carried out to verify the relationship between the synchronization errors and the diffusion rate of the signals inward the cell membrane in the region $\eta \in[0,2]$. Figure 4 shows that the error bounds depend sensitively on the diffusion rates $\eta$, which play the role of the coupling strength. The increase of the coupling strength makes for the decrease of the synchronization errors, but if the parameter $\eta$ is too large, the dynamics of oscillation of multicell systems (1) breaks. Therefore, all the parameters should be taken appropriately to ensure the dynamics of oscillation.

From Figures 3 and 4, one can see that the error concentrations of gene products are much smaller than those of genes; specifically, the concentrations of biosignals AI remain quasisynchronized even when the mismatches are big. During the time course of achieving synchronization, the synchronous states of AI decrease the mismatches between cells through quorum sensing. And the strong synchronizability of AI makes it act as biosignals to synchronize other chemical molecules. On the other hand, the figures also imply that the error concentrations of genes decrease greatly when the mismatches are decreased. The genes' strong sensitivity to regulations (the synchronous states of the concentrations of gene products, especially AI) is the reason why it can be led to synchronization by AI. The two aspects make up of the mechanisms of collective behavior caused by quorum sensing during transcription, translation, translocation, and signaltransduction.

## 4. Conclusions

Many previous researches studied the collective behavior of biological systems by using complete synchronization.


$$
-e=\sum_{i=2}^{6}\left(a_{i}-a_{1}\right) / 5
$$


$-e=\sum_{i=2}^{6}\left(a_{i}-a_{1}\right) / 5$
(a)
(b)

FIgure 2: Time evolutions of the synchronization errors $e=\sum_{i=2}^{6}\left(a_{i}-a_{1}\right) / 5$ in multicell system (1), where the parameter $\varepsilon$ is taken as $10^{-4}$ and $10^{-5}$, respectively.


Figure 3: Synchronization errors transition for increasing the mismatch bound $\varepsilon \in[0,0.002]$.

However, in many real biological systems, individual organisms are similar but different from each other, and their behaviors are not completely identical either. Therefore, it is meaningful to carry out researches on quasisynchronization instead of complete synchronization. Our results on quasisynchronization in multicell systems coupled by quorum sensing indicate that mismatches between cells can lead to quasisynchronization, and the synchronization errors depend heavily on the parameter mismatches. Theoretical analysis shows that the synchronization errors will decrease if the coupling strength increases. All the results agree well with numerical simulations and biological phenomena in practice.


Figure 4: Synchronization errors transition for increasing the parameter $\eta \in[0,2]$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Global Hopf Bifurcation Analysis for an Avian Influenza Virus Propagation Model with Nonlinear Incidence Rate and Delay 

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#### Abstract

The paper investigated an avian influenza virus propagation model with nonlinear incidence rate and delay based on SIR epidemic model. We regard delay as bifurcating parameter to study the dynamical behaviors. At first, local asymptotical stability and existence of Hopf bifurcation are studied; Hopf bifurcation occurs when time delay passes through a sequence of critical values. An explicit algorithm for determining the direction of the Hopf bifurcations and stability of the bifurcation periodic solutions is derived by applying the normal form theory and center manifold theorem. What is more, the global existence of periodic solutions is established by using a global Hopf bifurcation result.


## 1. Introduction

In March 2013, new avian-origin influenza $A(H 7 N 9)$ virus ( $A-O I V$ ) broke out in Shanghai and the surrounding provinces of China [1]. During the first week of April, this virus had been detected in six provinces and municipal cities; this virus has caused global concern as a potential pandemic threat [2]. The virus fast took people's life without timely treatment. Therefore, strong measures should be taken to control the spread of H7N9 viruses.

H7N9 is an infectious disease caused by influenza A virus. Moreover, it is essential to study and to dominate the spread of H7N9. Mathematical models become important instruments in the analysis and control of infectious diseases. The present study evaluates the possible application of SIR model for H7N9 spreading.

Let $S(t), I(t)$, and $R(t)$ be the population densities of susceptible, infective, and recovered, respectively. Recruitment of new individuals is into the susceptible class at a constant rate $B$ [3]. Parameters $\mu_{1}, \mu_{2}$, and $\mu_{3}$ are positive constants which represent the death rate of the classes, respectively. $\tau$ is the length of the infectious period; $1 / \gamma$ is the average time spent in class $I$ before recovery [3].

In 1979, Cooke [4] used mass action incidence $\beta S(t) I(t-$ $\tau)$. In 2009, Xu and Ma [5] developed the model with the force
of infection given by $\beta S(t)(I(t-\tau) /(1+\alpha I(t-\tau)))$, where $\alpha$ determines the level at which the force of infection saturates and $\beta$ is a contract [5]. Then, the avian influenza virus propagation model based on SIR model has the following form:

$$
\begin{gather*}
\dot{S}(t)=B-\mu_{1} S(t)-\frac{\beta S(t) I(t-\tau)}{1+\alpha I(t-\tau)} \\
\dot{I}(t)=\frac{\beta S(t) I(t-\tau)}{1+\alpha I(t-\tau)}-\left(\mu_{2}+\gamma\right) I(t)  \tag{1}\\
\dot{R}(t)=\gamma I(t)-\mu_{3} R(t)
\end{gather*}
$$

Since $R$ does not appear in the first two equations, and avoid excessive use of parentheses in some of the latter calculations, the avian influenza virus propagation model is transformed into the following form

$$
\begin{gather*}
\dot{S}(t)=B-\mu_{1} S(t)-\frac{\beta S(t) I(t-\tau)}{1+\alpha I(t-\tau)} \\
\dot{I}(t)=\frac{\beta S(t) I(t-\tau)}{1+\alpha I(t-\tau)}-\left(\mu_{2}+\gamma\right) I(t)  \tag{2}\\
\dot{R}(t)=\gamma I(t)-\mu_{3} R(t) \tag{3}
\end{gather*}
$$

with the following initial condition:

$$
\begin{gather*}
S(0) \in R^{+}, I(\theta)=\phi(\theta) \quad \text { for } \theta \in[-\tau, 0] \\
\text { where } \phi \in C\left([-\tau, 0], R^{+}\right) \tag{4}
\end{gather*}
$$

which was presented and studied in [3].
The steady state of the model and the stability of epidemic models have been studied in many papers. Zhang and Li [6] studied the global stability of an SIR epidemic model with constant infectious periods. Xu and Ma [5] showed the global stability of the endemic equilibrium for the case of the reproduction number $R_{0}>1$. McCluskey [3] shown that the endemic equilibrium is globally asymptotically stable whenever it exists. In this paper, we investigated the Hopf bifurcation and the global existence of periodic solutions of model (2), which have not been reported yet.

The organization of this paper is as follows. In Section 2, we will investigate the local asymptotical stability and existence of Hopf bifurcation by analyzing the associated characteristic equation. In Section 3, an explicit algorithm for determining the direction of the Hopf bifurcations and stability of the bifurcation periodic solutions will be derived by applying the normal form theory and center manifold theorem. In Section 4, existence of global periodic solutions will be established by using a global Hopf bifurcation result. In Section 5, a brief discussion is offered to conclude this work.

## 2. Local Stability and Hopf Bifurcation

Some results can be directly obtained from [3,5]. The basic reproduction number for the model is $R_{0}=B \beta / \mu_{1}\left(\mu_{2}+\right.$ $\gamma$ ). System (2) always has a disease-free equilibrium $E_{1}=$ $\left(B / \mu_{1}, 0\right)$. If $B \beta>\mu_{1}\left(\mu_{2}+\gamma\right)$, system (2) has a unique endemic equilibrium $E^{*}=\left(S^{*}, I^{*}\right)=\left(\left(B \alpha+\mu_{2}+\gamma\right) /\left(\beta+\alpha \mu_{1}\right),(B \beta-\right.$ $\left.\left.\mu_{1}\left(\mu_{2}+\gamma\right)\right) /\left(\mu_{2}+\gamma\right)\left(\beta+\alpha \mu_{1}\right)\right)$ [3]. The characteristic equation of system (2) at the endemic equilibrium $E^{*}$ is

$$
\begin{equation*}
\lambda^{2}+p_{1} \lambda+p_{0}+\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau}=0 \tag{5}
\end{equation*}
$$

where $p_{0}=\left(\mu_{2}+\gamma\right)\left(\mu_{1}+\beta I^{*} /\left(1+\alpha I^{*}\right)\right)$, $p_{1}=\mu_{1}+\mu_{2}+$ $\gamma+\beta I^{*} /\left(1+\alpha I^{*}\right), q_{0}=-\beta \mu_{1} S^{*} /\left(1+\alpha I^{*}\right)^{2}$, and $q_{1}=-\beta S^{*} /$ $\left(1+\alpha I^{*}\right)^{2}$. If

$$
\begin{equation*}
R_{0}>1 \tag{1}
\end{equation*}
$$

hold, when $\tau=0$, the endemic equilibrium $E^{*}$ of system (2) is locally stable [5].

If $i \omega(\omega>0)$ is a solution of system (2), separating real and imaginary parts, we obtain the following:

$$
\begin{gather*}
p_{1} \omega=q_{0} \sin \omega \tau-q_{1} \omega \cos \omega \tau \\
\omega^{2}-p_{0}=q_{0} \cos \omega \tau+q_{1} \omega \sin \omega \tau \tag{6}
\end{gather*}
$$

Then, we get

$$
\begin{align*}
& \cos \omega \tau=\frac{\left(q_{0}-p_{1} q_{1}\right) \omega^{2}-p_{0} q_{0}}{q_{0}^{2}+q_{1}^{2} \omega^{2}} \\
& \sin \omega \tau=\frac{p_{1} q_{0} \omega+\left(\omega^{2}-p_{0}\right) q_{1} \omega}{q_{0}^{2}+q_{1}^{2} \omega^{2}} \tag{7}
\end{align*}
$$

It follows that

$$
\begin{equation*}
\omega^{4}+\left(p_{1}^{2}-2 p_{0}-q_{1}^{2}\right) \omega^{2}+p_{0}^{2}-q_{0}^{2}=0 \tag{8}
\end{equation*}
$$

Letting $z=\omega^{2}$, we get

$$
\begin{equation*}
z^{2}+\left(p_{1}^{2}-2 p_{0}-q_{1}^{2}\right) z+p_{0}^{2}-q_{0}^{2}=0 \tag{9}
\end{equation*}
$$

It is easy to show that

$$
\begin{align*}
& \begin{aligned}
& p_{1}^{2}-2 p_{0}-q_{1}^{2}=\left(\mu_{1}+\frac{\beta I^{*}}{1+\alpha I^{*}}\right)^{2} \\
&+\left(\mu_{2}+\gamma\right)^{2}-\frac{\left(\mu_{2}+\gamma\right)^{2}}{\left(1+\alpha I^{*}\right)^{2}}>0, \\
& p_{0}^{2}-q_{0}^{2}
\end{aligned} \\
& =\left(\mu_{2}+\gamma\right)\left[\left(\mu_{2}+\gamma\right)\left(\mu_{1}+\frac{\beta I^{*}}{1+\alpha I^{*}}\right)+\frac{\beta \mu_{1} S^{*}}{\left(1+\alpha I^{*}\right)^{2}}\right] \\
& \quad \times\left(\mu_{1}-\frac{\mu_{1}}{1+\alpha I^{*}}+\frac{\beta I^{*}}{1+\alpha I^{*}}\right) . \tag{10}
\end{align*}
$$

The case of

$$
\begin{equation*}
\beta \geq \mu_{1} \alpha \tag{1}
\end{equation*}
$$

has been discussed in [5]. We obtain global asymptotic stability of the endemic equilibrium when $R_{0}>1$. If

$$
\begin{equation*}
\beta<\mu_{1} \alpha \tag{2}
\end{equation*}
$$

hold, that is, $\left(\beta-\mu_{1} \alpha\right) I^{*} /\left(1+\alpha I^{*}\right)<0$, we have $p_{0}^{2}-q_{0}^{2}<0$. Following the theorem given by Ruan [7], there exists critical value

$$
\begin{equation*}
\tau_{k}^{(j)}=\frac{1}{\omega_{k}} \arccos \frac{\left(\omega_{k}^{2}-p_{0}\right) q_{0}-p_{1} q_{1} \omega_{k}^{2}}{q_{0}^{2}+q_{1}^{2} \omega_{k}^{2}}+\frac{2 j \pi}{\omega_{k}} \tag{11}
\end{equation*}
$$

with
$\omega_{j}$

$$
\begin{equation*}
=\left[\frac{2 p_{0}+p_{1}^{2}-q_{1}^{2}+\sqrt{\left(2 p_{0}+p_{1}^{2}-q_{1}^{2}\right)^{2}-4\left(p_{0}^{2}-q_{0}^{2}\right)}}{2}\right]^{1 / 2}, \tag{12}
\end{equation*}
$$

where $k=1,2, \ldots, j=0,1,2, \ldots$. If $\left(P_{1}\right)$ and $\left(H_{2}\right)$ are satisfied, (6) has a pair of purely imaginary roots $\pm \omega_{0} i$ when
$\tau=\tau_{0}$. Additionally, all roots of (6) have negative real parts when $\tau \in\left[0, \tau_{0}\right]$ and when $\tau>\tau_{0}$ (5) has at least a pair of roots with positive real part. In order to give the main results, it is necessary to prove the transversality condition $\operatorname{Re}(d \lambda / d \tau)^{-1}>0$ holds. Denote $\lambda=\alpha(\tau)+i \omega(\tau)$ as the root of (5) with $\alpha(\tau)=0, \omega(\tau)=\omega_{0}$. Differentiating (5) with respect to $\tau$ yields

$$
\begin{equation*}
\left[2 \lambda+p_{1}+\left(q_{1}-\tau\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau}\right)\right] \frac{d \lambda}{d \tau}=\lambda\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau} \tag{13}
\end{equation*}
$$

For the sake of simplicity denoting $\omega_{0}$ and $\tau_{0}$ by $\omega, \tau$, respectively,

$$
\begin{equation*}
\frac{d \lambda}{d \tau}=\frac{\left(q_{1} \lambda+q_{0}\right) \lambda e^{-\lambda \tau}}{2 \lambda+p_{1}+q_{1} e^{-\lambda \tau}-\left(\left(q_{1} \lambda+q_{0}\right) \tau e^{-\lambda \tau}\right)} \tag{14}
\end{equation*}
$$

in the following:

$$
\begin{align*}
& \operatorname{Re}\left(\frac{d \lambda}{d \tau}\right)^{-1} \\
&= \frac{2 \lambda+}{\left(q_{1} \lambda+p_{0}\right) \lambda q_{1} e^{-\lambda \tau}} \\
&=\left(\left(p_{1} \cos \omega \tau-2 \omega \sin \omega \tau+q_{1}\right)\right. \\
&\left.\quad+i\left(2 \omega \cos \omega \tau+p_{1} \sin \omega \tau\right)\right) \\
& \quad \times\left(-q_{1} \omega^{2}+i q_{0} \omega\right)^{-1} \\
&=\left(-p_{1} q_{1} \omega^{2} \cos \omega \tau+2 q_{1} \omega^{3} \sin \omega \tau\right. \\
&\left.\quad \quad-q_{1}^{2} \omega^{2}+2 q_{0} \omega^{2} \cos \omega \tau+p_{1} q_{0} \omega \sin \omega \tau\right) \\
& \quad \times\left(q_{0} \omega^{2}+\omega^{4}\right)^{-1}  \tag{15}\\
&=\left.p_{1} q_{1} \omega^{2}+2 q_{0} \omega^{2}\right)\left(q_{0}-p_{1} q_{1}\right) \omega^{2}-p_{0} q_{0} \\
&+\left(2 q_{1} \omega^{3}+p_{1} q_{0} \omega\right) \\
&\left.\times\left[p_{1} q_{0} \omega+\left(\omega^{2}-p_{0}\right) q_{1} \omega\right]-q_{0}^{2} q_{1}^{2} \omega^{2}-q_{1}^{4} \omega^{2}\right) \\
& \times\left(\left(q_{0}^{2}+q_{1}^{2} \omega^{2}\right)\left(q_{0} \omega^{2}+\omega^{4}\right)\right)^{-1} \\
&=\left(2 q_{1}^{2} \omega^{6}+\left[2 q_{0}^{2}+\left(p_{1}^{2}-2 p_{0}-q_{1}^{2}\right) q_{1}^{2}\right] \omega^{4}\right. \\
&\left.\quad+\left(p_{1}^{2}+2 p_{0}-q_{1}^{2}\right) q_{0}^{2} \omega^{2}\right) \\
& \times\left(\left(q_{0}^{2}+q_{1}^{2} \omega^{2}\right)\left(q_{0} \omega^{2}+\omega^{4}\right)\right)^{-1} .
\end{align*}
$$

From (10), we know $p_{1}^{2}-2 p_{0}-q_{1}^{2}>0$; then, $\operatorname{Re}(d \lambda / d \tau)^{-1}>0$ hold. Under this condition, we have the following theorem.

Theorem 1. (i) If $\left(P_{1}\right)$ and $\left(H_{1}\right)$ holds, the equilibrium ( $S^{*}, I^{*}$ ) of system (2) is asymptotically stable for any $\tau>0$.
(ii) If $\left(P_{1}\right)$ and $\left(H_{2}\right)$ holds, $\left(S^{*}, I^{*}\right)$ is asymptotically stable for $\tau \in\left[0, \tau_{0}\right)$ and unstable for $\tau \in\left(\tau_{0},+\infty\right)$. System (2)
exhibits the Hopf bifurcation at the equilibrium $\left(S^{*}, I^{*}\right)$ for $\tau=\tau_{j}, j=0,1,2, \ldots$.

## 3. Direction and Stability of the Bifurcating Periodic Solutions

In Section 2, we obtain the conditions under which a family of periodic solutions bifurcate from the steady state at the critical value of $\tau$. In this section, we investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solution at critical values $\tau_{0}$, using techniques of the normal form theory and center manifold theorem.

Let $u_{1}=S(t)-S^{*}$ and let $u_{2}=I(t)-I^{*}$. The Taylor expansion of system (2) at $E^{*}$ is

$$
\begin{align*}
\dot{u}_{1}(t)= & a_{1} u_{1}(t)-a_{2} u_{2}(t-\tau) \\
& +a_{6} u_{2}(t)^{2}(t-\tau)+a_{7} u_{1}(t) u_{2}(t-\tau), \\
\dot{u}_{2}(t)= & a_{3} u_{1}(t)+a_{4} u_{2}(t)+a_{5} u_{2}(t-\tau)  \tag{16}\\
& -a_{7} u_{1}(t) u_{2}(t-\tau)-a_{6} u_{2}^{2}(t-\tau),
\end{align*}
$$

where $a_{1}=-\mu_{1}+\beta I^{*} /\left(1+\alpha I^{*}\right), a_{2}=-\beta S^{2} /\left(1+\alpha I^{*}\right)$, $a_{3}=\beta I^{*} /\left(1+\alpha I^{*}\right), a_{4}=-\mu_{2}-\gamma, a_{5}-\beta S^{2} /\left(1+\alpha I^{*}\right)$, $a_{6}=-\alpha \beta S^{*} /\left(1+\alpha I^{*}\right)^{3}, a_{7}=\beta /\left(1+\alpha I^{*}\right)^{2}, \tau=\tau_{0}+\mu$, and $u_{t}=u(t+\theta) \in C_{1}$ for $\theta \in[-1,0]$. System (2) is transformed into FDE as

$$
\begin{equation*}
\dot{u}(t)=L_{\mu}+F\left(u_{t}, \mu\right) \tag{17}
\end{equation*}
$$

with

$$
\begin{gather*}
L_{\mu}(\phi)=\left(\tau_{0}+\mu\right)\left[B_{1} \phi(0)+B_{2} \phi(-1)\right] \\
F(\phi, \mu)=\left(\tau_{0}+\mu\right)\binom{a_{6} \phi_{2}^{2}(-1)+a_{7} \phi_{1}(0) \phi_{2}(-1)}{-a_{6} \phi_{1}(0) \phi_{2}(-1)-a_{7} \phi_{2}^{2}(-1)} \tag{18}
\end{gather*}
$$

where

$$
B_{1}=\left(\begin{array}{cc}
a_{1} & 0  \tag{19}\\
a_{3} & a_{4}
\end{array}\right), \quad B_{2}=\left(\begin{array}{ll}
0 & a_{2} \\
0 & a_{5}
\end{array}\right)
$$

By Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation, for $\theta \in[-1,0]$, such that

$$
\begin{equation*}
L_{\mu} \phi=\int_{-1}^{0} d \eta(\theta, \mu) \phi(\theta) \quad \text { for } \phi \in C \tag{20}
\end{equation*}
$$

In fact, we can choose

$$
\begin{equation*}
\eta(\theta, \mu)=\left(\tau_{0}+\mu\right)\left[B_{1} \delta(\theta)+B_{2} \delta(\theta+1)\right] \tag{21}
\end{equation*}
$$

where $\delta(\theta)$ is a delta function.
For $\phi \in C^{\prime}[-1,0]$, the operators $A$ and $R$ are defined as follows:

$$
\begin{gather*}
A(\mu) \phi(\theta)= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in[-1,0) \\
\int_{-1}^{0} d(\eta(t, \mu) \phi(t)), & \theta=0\end{cases}  \tag{22}\\
R(\mu) \phi(\theta)= \begin{cases}0, & \theta \in[-1,0) \\
f(\mu, \theta), & \theta=0\end{cases} \tag{23}
\end{gather*}
$$

The adjoint operator $A^{*}(0)$ corresponding to $A(0)$ is defined as follows:

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in(0,1]  \tag{24}\\ \int_{-1}^{0} d\left(\eta^{T}(t, 0) \psi(-t)\right), & s=0\end{cases}
$$

and an adjoint bilinear is as follows:

$$
\begin{equation*}
\langle\psi, \phi\rangle=\bar{\psi}(0) \phi(0)-\int_{-1}^{0} \int_{0}^{\theta} \bar{\psi}(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi \tag{25}
\end{equation*}
$$

where $\eta(\theta)=\eta(\theta, 0)$.
From the preceding discussion, we know that $q(\theta)$ and $q^{*}(\theta)$ be the eigenvectors of $A$ and $A^{*}$ corresponding to $i \tau_{0} \omega_{0}$ and $-i \tau_{0} \omega_{0}$, respectively. Next, we calculate $q(\theta)$ and $q^{*}(s)$ to determine the normal form of operator $A$.

Proposition 2. Let $q(\theta)$ and $q^{*}(s)$ be eigenvectors of $A$ and $A^{*}$ corresponding to $i \tau_{0} \omega_{0}$ and $-i \tau_{0} \omega_{0}$, respectively, satisfying $\left\langle q^{*}, q\right\rangle=1$ and $\left\langle q^{*}, \bar{q}\right\rangle=0$.

Then,

$$
\begin{gather*}
q(\theta)=(1, \alpha)^{T} e^{i \omega_{0} \tau_{0} \theta}=\left(1, \frac{\omega_{0} i-a_{1}}{a_{2} e^{-i \omega_{0} \tau_{0}}}\right)^{T} e^{i \omega_{0} \tau_{0} \theta},  \tag{26}\\
q^{*}(s)=D\left(1, \beta^{*}\right) e^{-i \omega_{0} \tau_{0} s}=\left(1, \frac{-\omega_{0} i-a_{1}}{a_{3}}\right)^{T} e^{-i \omega_{0} \tau_{0} s},
\end{gather*}
$$

where

$$
\begin{equation*}
D=\frac{1}{1+\alpha \bar{\beta}^{*}-\tau_{0} \alpha\left(a_{2}+\bar{\beta}^{*} a_{5}\right) e^{-i \omega_{0} \tau_{0}}} . \tag{27}
\end{equation*}
$$

Proof. Without loss of generality, we just consider the eigenvector $q(\theta)$. By the definition of $A$ and $q(\theta)$ with $\theta \in[-1,0)$, we get $q(\theta)=(1, \alpha)^{T} e^{i \omega_{0} \tau_{0}}$ (here, $\alpha$ is a parameter). In what follows, notice that $q(0)=(1, \alpha)^{T}$ and $\operatorname{Aq}(0)=$ $\int_{-1}^{0} d(\eta(t, \mu) \phi(t))=i \omega_{0} \tau_{0} q(0)$; we have $\alpha=\left(\omega_{0} i-\right.$ $\left.a_{1}\right) / a_{2} e^{-i \omega_{0} \tau_{0}}$. Using a proof procedure similar to that in [8], by direct computation, we get $q(\theta)$ and $q^{*}(s)$. Bring $q(\theta)$ and $q^{*}(s)$ into $\left\langle q^{*}, q\right\rangle=1$; it is not hard to obtain the parameter $\bar{D}$. The detailed procedure of proof refers to [9]. The proof is completed.

Then, we construct the coordinates of the center manifold $C_{0}$ at $\mu=0$. Let

$$
\begin{equation*}
z(t)=\left\langle q^{*}, u_{t}\right\rangle, \quad W(t, \theta)=u_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} \tag{28}
\end{equation*}
$$

On the center manifold $C_{0}$, we have

$$
\begin{equation*}
W(t, \theta)=W(z(t), \overline{z(t)}, \theta) \tag{29}
\end{equation*}
$$

where

$$
\begin{equation*}
W(z, \bar{z}, \theta)=W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02} \frac{\bar{z}^{2}}{2}+W_{30} \frac{z^{3}}{6}, \ldots ; \tag{30}
\end{equation*}
$$

and $z$ and $\bar{z}$ are local coordinates for the center manifold $C_{0}$ in the direction of $q$ and $q^{*}$, respectively. Since $\mu=0$, we have

$$
\begin{align*}
z^{\prime}(t) & =i \tau_{0} \omega_{0} z(t)+\left\langle q^{*}(\theta), f(W+2 \operatorname{Re}\{z(t) q(\theta)\})\right\rangle \\
& =i \tau_{0} \omega_{0} z(t)+\overline{q^{*}(0)} f(W(z, \bar{z}, 0)+2 \operatorname{Re}\{z(t) q(0)\}) \\
& \triangleq i \tau_{0} \omega_{0} z(t)+\overline{q^{*}(0)} f_{0}(z, \bar{z}), \tag{31}
\end{align*}
$$

where

$$
\begin{equation*}
f_{0}(z, \bar{z})=f_{z^{2}} \frac{z^{2}}{2}+f_{\bar{z}^{2}} \frac{\bar{z}^{2}}{2}+f_{z \bar{z}} z \bar{z}+\cdots \tag{32}
\end{equation*}
$$

We rewrite this as

$$
\begin{equation*}
z^{\prime}(t)=i \tau_{0} \omega_{0} z+g(z, \bar{z}) \tag{33}
\end{equation*}
$$

with

$$
\begin{align*}
g(z, \bar{z}) & =\overline{q^{*}}(0) f_{0}(z, \bar{z}) \\
& =g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}+g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots, \\
g(z, \bar{z})= & D \tau_{0}\left(1, \overline{\beta^{*}}\right)\binom{a_{6} \phi_{2}^{2}(-1)+a_{7} \phi_{1}(0) \phi_{2}(-1)}{-a_{6} \phi_{1}(0) \phi_{2}(-1)-a_{7} \phi_{2}^{2}(-1)}, \tag{34}
\end{align*}
$$

where

$$
\begin{align*}
\phi_{1}(0)= & z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z} \\
& +W_{02}^{(1)}(0) \frac{\bar{z}}{2}  \tag{35}\\
\phi_{2}(-1)= & z \alpha e^{-i \omega_{0} \tau_{0}}+\bar{z} \bar{\alpha} e^{i \omega_{0} \tau_{0}}+W_{20}^{(2)}(-1) \frac{z^{2}}{2} \\
& +W_{11}^{(2)}(-1) z \bar{z}+W_{02}^{(2)}(-1) \frac{\bar{z}}{2} .
\end{align*}
$$

Comparing the coefficients of the above equation with (22), we obtain
$g_{20}$

$$
=2 D \tau_{0}\left[\left(a_{6}-\overline{\beta^{*}} a_{7}\right) \alpha^{2} e^{-2 i \omega_{0} \tau_{0}}+\left(a_{7}-\overline{\beta^{*}} a_{6}\right) \alpha e^{-i \omega_{0} \tau_{0}}\right],
$$

$g_{11}$

$$
\begin{gathered}
=D \tau_{0}\left[2\left(a_{6}-\overline{\beta^{*}} a_{7}\right) \alpha \bar{\alpha}+\left(a_{7}-\overline{\beta^{*}} a_{6}\right)\right. \\
\left.\times\left(\bar{\alpha} e^{i \omega_{0} \tau_{0}}+\alpha e^{-i \omega_{0} \tau_{0}}\right)\right],
\end{gathered}
$$

$g_{02}$

$$
\begin{aligned}
=2 D \tau_{0}[ & \left(a_{6}-\overline{\beta^{*}} a_{7}\right) \overline{\alpha^{2}} e^{2 i \omega_{0} \tau_{0}} \\
& \left.+\left(a_{7}-\overline{\beta^{*}} a_{6}\right) \bar{\alpha} e^{-i \omega_{0} \tau_{0}}\right]
\end{aligned}
$$

$$
\begin{align*}
& g_{21}=2 D \tau_{0}\left[\left(a_{6}-\overline{\beta^{*}} a_{7}\right)\right. \\
& \times\left[\bar{\alpha} e^{i \omega_{0} \tau_{0}} W_{20}^{(2)}(-1)+2 \alpha e^{-i \omega_{0} \tau_{0}} W_{11}^{(2)}(-1)\right] \\
&+\left(a_{7}-\overline{\beta^{*}} a_{6}\right) \\
& \times\left[\frac{1}{2} \bar{\alpha} e^{i \omega_{0} \tau_{0}} W_{20}^{(1)}(0)+\alpha e^{-i \omega_{0} \tau_{0}} W_{11}^{(1)}(0)\right. \\
&\left.+\frac{1}{2} W_{20}^{(2)}(-1)+W_{11}^{(2)}(-1)\right], \tag{36}
\end{align*}
$$

$$
\begin{align*}
\dot{W} & =\dot{u}_{t}-\dot{z} q-\dot{\bar{z}} \bar{q} \\
& = \begin{cases}A W-2 \operatorname{Re} \overline{q^{*}}(0) f_{0} q(\theta), & \theta \in[-1,0] \\
A W-2 \operatorname{Re} \overline{q^{*}}(0) f_{0} q(\theta)+f_{0}, & \theta=0 .\end{cases}  \tag{37}\\
& \triangleq A W+H(z, \bar{z}, \theta)
\end{align*}
$$

where

$$
\begin{equation*}
H(z, \bar{z}, \theta)=H_{20}(\theta) \frac{z^{2}}{2}+H_{11}(\theta) z \bar{z}+H_{02}(\theta) \frac{\bar{z}^{2}}{2}+\cdots \tag{38}
\end{equation*}
$$

Expanding the above series and comparing the coefficients, we get

$$
\begin{gather*}
\left(A-2 i \omega_{0} \tau_{0} I\right) W_{20}(\theta)=-H_{20}(\theta) \\
A W_{11}(\theta)=-H_{11}(\theta)  \tag{39}\\
\left(A+2 i \omega_{0} \tau_{0} I\right) W_{02}(\theta)=-H_{02}(\theta)
\end{gather*}
$$

Comparing the coefficients with (38), we obtain

$$
\begin{align*}
& H_{20}(\theta)=-g_{20} q(\theta)-\bar{g}_{02} \bar{q}(\theta),  \tag{40}\\
& H_{11}(\theta)=-g_{11} q(\theta)-\bar{g}_{11} \bar{q}(\theta) .
\end{align*}
$$

It follows from (39), (40), and the definition of $A$ that we have

$$
\begin{gather*}
\dot{W}_{20}(\theta)=2 i \tau_{0} \omega_{0} W_{20}(\theta)+g_{20} q(\theta)+\bar{g}_{20} \bar{q}(\theta), \\
\dot{W}_{11}(\theta)=g_{11} q(\theta)+\bar{g}_{11} \bar{q}(\theta) \tag{41}
\end{gather*}
$$

So,

$$
\begin{gather*}
W_{20}(\theta)=\frac{i g_{20}}{\tau_{0} \omega_{0}} q(0) e^{i \tau_{0} \omega_{0} \theta} \\
\quad-\frac{\bar{g}_{02}}{3 i \tau_{0} \omega_{0}} \bar{q}(0) e^{-i \tau_{0} \omega_{0} \theta}+E_{1} e^{2 i \tau_{0} \omega_{0} \theta},  \tag{42}\\
W_{11}(\theta)=-\frac{i g_{11}}{\tau_{0} \omega_{0}} q(0) e^{i \tau_{0} \omega_{0} \theta}+\frac{i \bar{g}_{11}}{\tau_{0} \omega_{0}} \bar{q}(0) e^{-i \tau_{0} \omega_{0} \theta}+E_{2},
\end{gather*}
$$

where

$$
\left.\begin{array}{rl}
E_{1}= & 2\left(\begin{array}{cc}
2 i \omega_{0}-a_{1} & -a_{2} e^{-2 i \omega_{0} \tau_{0}} \\
-a_{3} & 2 i \omega_{0}-a_{4}-a_{5} e^{-2 i \omega_{0} \tau_{0}}
\end{array}\right)^{-1} \\
& \times\binom{ a_{6} \alpha^{2} e^{-2 i \omega_{0} \tau_{0}}+a_{7} \alpha e^{-i \omega_{0} \tau_{0}}}{-a_{7} \alpha^{2} e^{-2 i \omega_{0} \tau_{0}}-a_{6} \alpha e^{-i \omega_{0} \tau_{0}}} \\
E_{2}= & \left(\begin{array}{cc}
-a_{1} & -a_{2} \\
-a_{3} & -a_{4}-a_{5}
\end{array}\right)^{-1}  \tag{43}\\
& \times\binom{ 2 a_{6} \alpha \bar{\alpha}+a_{7}\left(\bar{\alpha} e^{i \omega_{0} \tau_{0}}+\alpha e^{-i \omega_{0} \tau_{0}}\right)}{-2 a_{7} \alpha \bar{\alpha}-a_{6}\left(\bar{\alpha} e^{i \omega_{0} \tau_{0}}+\alpha e^{-i \omega_{0} \tau_{0}}\right.}
\end{array}\right) .
$$

According to the discussion above, we can compute the following parameters:

$$
\begin{gather*}
C_{1}(0)=\frac{i}{2 \tau_{0} \omega_{0}}\left(g_{20} g_{11}-2\left|g_{11}\right|-\frac{1}{3}\left|g_{02}\right|^{2}\right)+\frac{g_{21}}{2} \\
\mu_{2}=-\frac{\operatorname{Re}\left\{C_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}}  \tag{44}\\
\beta_{2}=2 \operatorname{Re}\left\{C_{1}(0)\right\} \\
T_{2}=-\frac{\operatorname{Im}\left\{C_{1}(0)\right\}+\mu_{2}\left(\operatorname{Im}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}\right)}{\omega_{0}}
\end{gather*}
$$

where $\mu_{2}$ determines the directions of the Hopf bifurcations, $\beta_{2}$ determines the stability of the bifurcation periodic solutions, and $T_{2}$ determines the period of the bifurcating periodic solutions [9]. By lemma (5), we know that $\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}>$ 0 ; we have the following theorem.

Theorem 3. If $\operatorname{Re}\left\{C_{1}(0)\right\}<0(>0)$, the direction of the Hopf bifurcation of the system (1) at the equilibrium $(0,0)$ when $\tau=\tau_{0}$ is supercritical (subcritical) and the bifurcating periodic solutions are orbitally asymptotically stable (unstable).

## 4. Global Existence of Periodic Solution

From the above discussion, we know that system (2) undergoes a local Hopf bifurcation at $E^{*}=\left(S^{*}, I^{*}\right)$ when $\tau=\tau_{j}(j=0,1,2, \ldots)$. A natural question is that if the bifurcating periodic solutions of system (2) exist when is $\tau$ far away from critical values? In this section, we will study the global existence of periodic solutions of system (2). Through use of a global Hopf bifurcation theorem given by Wu [10], we obtain the global continuation of periodic solutions bifurcating from the points $\left(E^{*}, \tau_{j}\right)(j=0,1,2, \ldots)$. First of all, we define

$$
\begin{align*}
X= & C([-\tau, 0], R), \\
\Sigma= & C l(x, \tau, l):(x, \tau, l) \in X \times R_{+} \times R_{+}, \\
& \quad x \text { is a } l \text {-periodic solution of system, }  \tag{45}\\
N & =(\widehat{x}, \tau, l): \widehat{x}=0 \quad \text { or } \quad v .
\end{align*}
$$

Let $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ denote the connected component of $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ in $\Sigma$ and $\operatorname{Proj}_{\tau}\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ its projection on $\tau$ component. From theorem (5), we know that
$C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ is nonempty. $\omega_{0}$ and $\tau_{j}$ are defined in (10) and (11).

Lemma 4. All periodic solutions of system (2) are uniformly bounded.

Proof. Let $(S(t), I(t))$ be a nonconstant periodic solution of system (2), and let $S\left(t_{1}\right), S\left(t_{2}\right)$ and $I\left(t_{3}\right), I\left(t_{4}\right)$ be the maximum and minimum of $S(t)$ and $I(t)$, respectively. Using a proof procedure similar to that in [8], we can obtain

$$
\begin{equation*}
0<S(t)<\frac{B}{\mu_{1}}, \quad 0<I(t)<\frac{B}{\mu_{2}+\gamma} . \tag{46}
\end{equation*}
$$

It is shown that all periodic solutions of system (2) are uniformly bounded. This completes the proof.

Lemma 5. System (2) has no nonconstant periodic solution of period $\tau$.

Proof. For a contradiction, if system (5) has a $\tau$-periodic solution, say $(S(t), I(t))$, then it satisfies the ODES as follows:

$$
\begin{gather*}
\dot{S}(t)=B-\mu_{1} S(t)-\frac{\beta S(t) I(t)}{1+\alpha I(t)}=P(S, I) \\
\dot{I}(t)=\frac{\beta S(t) I(t)}{1+\alpha I(t)}-\left(\mu_{2}+\gamma\right) I(t)=Q(S, I) \tag{47}
\end{gather*}
$$

We can get

$$
\begin{equation*}
\frac{\partial P}{\partial S}+\frac{\partial Q}{\partial I}=-\mu_{1}-\frac{\beta I(t)}{1+\alpha I(t)}-\frac{1}{(1+\alpha I(t))^{2}}-\left(\mu_{2}+\gamma\right)<0 . \tag{48}
\end{equation*}
$$

By Bendixson's criterion, we know that system (2) has no nonconstant periodic solutions, which prove the lemma.

Theorem 6. Suppose that the condition $\left(H_{1}\right)$ and $\left(P_{1}\right)$ is satisfied. Then, for each $\tau>\tau_{j}, j=0,1,2, \ldots$, system (2) has at least $j-1$ periodic solutions.

Proof. The characteristic matrix of system (2) at the equilibrium $\bar{z}=\left[\bar{z}^{(1)}, \bar{z}^{(2)}\right] \in R^{2}$ is in the following form:

$$
\begin{equation*}
\Delta(\bar{z}, \tau, l)(\lambda)=\lambda I-D_{\phi} F(\bar{z}, \bar{\tau}, \bar{l})\left(e^{\lambda} I d\right) \tag{49}
\end{equation*}
$$

that is,

$$
\begin{align*}
& \Delta(\bar{z}, \tau, l) \\
& \quad=\left(\begin{array}{cc}
\lambda+\mu_{1}+\frac{\beta \bar{z}^{(2)} e^{-\lambda \tau}}{1+\alpha \bar{z}^{(2)} e^{-\lambda \tau}} & \frac{\beta \bar{z}^{(1)}}{\left(1+\alpha \bar{z}^{(2)} e^{-\lambda \tau}\right)^{2}} \\
-\frac{\beta \bar{z}^{(2)} e^{-\lambda \tau}}{1+\alpha \bar{z}^{(2)} e^{-\lambda \tau}} & \lambda+\frac{\beta \bar{z}^{(1)}}{\left(1+\alpha \bar{z}^{(2)} e^{-\lambda \tau}\right)^{2}}
\end{array}\right) \tag{50}
\end{align*}
$$

Using a proof procedure similar to that in [9], it is easy to obtain that $\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right), j=0,1,2, \ldots$, is an isolated center.

From the proof procedure of Lemmas 4 and 5, it is easy to know that there exist $\varepsilon>0, \delta>0$, smooth curve $\lambda:\left(\tau_{j}-\right.$ $\left.\delta, \tau_{j}+\delta\right) \rightarrow C$ such that

$$
\begin{equation*}
\Delta(\lambda(\tau))=\Delta_{(v, \tau, T)}(\lambda(\tau))=0, \quad\left|\lambda(\tau)-i \omega_{0}\right|<\varepsilon \tag{51}
\end{equation*}
$$

for all $\tau \in\left[\tau_{j}-\delta, \tau_{j}+\delta\right]$, and

$$
\begin{equation*}
\lambda\left(\tau_{j}\right)=i \omega_{0},\left.\quad \frac{d \operatorname{Re}(\lambda(\tau))}{d \tau}\right|_{\tau=\tau_{j}}>0 \tag{52}
\end{equation*}
$$

Define $l_{j}=2 \pi / \omega_{0}$, and let $\Omega_{\varepsilon}=\{(0, l): 0<u<\varepsilon, \mid l-$ $\left.l_{j} \mid<\varepsilon\right\}$. Obviously, if $\left|\tau-\tau_{j}\right| \leq \delta$ and $(u, l) \in \partial \Omega_{\varepsilon}$ such that $\Delta_{\left(E^{*}, \tau, l\right)}(u+2 \pi i / l)=0$, if and only if $\tau=\tau_{j}, u=0, l=l_{j}$, set

$$
\begin{equation*}
H^{ \pm}\left(E^{*}, \tau_{j}, \frac{2 \pi}{\omega_{0}}\right)(u, l)=\Delta_{\left(E^{*}, \tau_{j} \pm \delta, l\right)}\left(u+\frac{2 \pi i}{l}\right) . \tag{53}
\end{equation*}
$$

We obtain the crossing number as follows:

$$
\begin{align*}
\gamma_{1}\left(E^{*}, \tau_{j}, \frac{2 \pi}{\omega_{0}}\right)= & \operatorname{deg}_{B}\left(H^{-}\left(E^{*}, \tau_{j}, \frac{2 \pi}{\omega_{0}}\right), \Omega_{\varepsilon}\right) \\
& -\operatorname{deg}_{B}\left(H^{+}\left(E^{*}, \tau_{j}, \frac{2 \pi}{\omega_{0}}\right), \Omega_{\varepsilon}\right)=-1 . \tag{54}
\end{align*}
$$

We conclude that

$$
\begin{equation*}
\sum_{\left(E^{*}, \tau, l\right) \in C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)} \gamma_{1}\left(E^{*}, \tau, l\right)<0 \tag{55}
\end{equation*}
$$

Since the first crossing number of each center is always -1 , by $\left[10\right.$, Theorem 3.3], we conclude that $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ is unbounded. By the definition of $\tau_{j}$ given in (10), we know that, for $j \geq 1,\left(\tau_{j} /(j+1)\right)<2 \pi / \omega_{0}<\tau_{j}$ automatically hold.

Again, the population densities of susceptible and infective are ultimately uniformly bounded, implying that the projection of $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ onto the $\tau$-space is bounded. Meanwhile, system (2) with $\tau=0$ has no nonconstant periodic solutions; if there exits $\tau^{*}>0$ such that the projection of $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ onto the $\tau$-space ins $\left(0, \tau_{0}\right)$ with $\tau^{*}>\tau_{j}$, then, the projection of $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ onto the $\tau$-space is bounded. Since $\left(2 \pi / \omega_{0}\right)<\tau_{j}$ and from Lemma 5, we can obtain $0<l<\tau^{*}$ for $(E, \tau, T) \in$ $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ with $l<\tau^{*}$; that is to say, $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ onto $l$-space is also bounded. Because $C\left(E^{*}, \tau_{j}, 2 \pi / \omega_{0}\right)$ is unbounded, $\operatorname{Proj}_{\tau}\left(E, \tau_{j}, 2 \pi / \omega_{0}\right)$ must be unbounded. Consequently, $\operatorname{Proj}_{\tau}\left(E, \tau_{j}, 2 \pi / \omega_{0}\right)$ include $\left[\tau_{j}, \infty\right)$ for $j \geq 1$. That is to say, for each $\tau>\tau_{j}(j \geq 1)$, system (2) at least has $j-1$ nonconstant period solutions. The proof is complete.

## 5. Conclusion

In this paper we have analytically studied an avian influenza virus propagation model with nonlinear incidence rate and time delay depending on SIR epidemic model. Some previous efforts in epidemic models have been mainly concerned with the global stability and asymptotical stability. However, it
is a new idea to study the bifurcation periodic solutions and global existence of periodic solutions. The theoretical analysis for the avian influenza virus propagation models is given. Then, Hopf bifurcation occurs when time delay passes through a sequence of critical values. Furthermore, bifurcations and stability of the bifurcation periodic solutions are derived. Finally, global existence of periodic solutions is established.

## Conflict of Interests

The authors declare that they have no financial and personal relationships with other people or organizations that can inappropriately influence their work; there is no professional or other personal interest of any nature or kind in any product, service, and/or company that could be construed as influencing the position presented in, or the review of, in this paper.

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## Research Article

# Effect of Awareness Programs on 

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#### Abstract

An epidemic model with time delay has been proposed and analyzed. In this model the effect of awareness programs driven by media on the prevalence of an infectious disease is studied. It is assumed that pathogens are transmitted via direct contact between the susceptible and the infective populations and further assumed that the growth rate of cumulative density of awareness programs increases at a rate proportional to the infective population. The model is analyzed by using stability theory of differential equations and numerical simulations. Both equilibria have been proved to be globally asymptotically stable. The results we obtained and numerical simulations suggest the increasing of the dissemination rate and implementation rate can reduce the proportion of the infective population.


## 1. Introduction

Infectious diseases that cause mortality, disability, and social and economic disruption are a major threat to mankind, which are responsible for a quarter of all deaths annually in the world $[1-3]$. Once an infectious disease appears and spreads in a region, the Center of Disease Control and Prevention will do its best to stop the propagation of the disease [4]. One of the measures is to tell people the appropriate preventive knowledge of diseases as soon as possible through media and education which make people take precautions to reduce their chances of being infected [5]. As the awareness disseminates, people will change their behaviors to alter their susceptibility. There is much evidence that media coverage can play an important role in the spread and control of infectious diseases [6-11]. In [8] Cui and others established a framework of transmission coefficient with media coverage which is a decreasing function of the number of the infective individuals and they observed a classic threshold-type behavior, with the disease becoming extinct when $R_{0}<1$ and going to a globally asymptotically stable equilibrium when $R_{0}>1$, and they concluded that media coverage is critical in disease eradication.

The models studying the spread of infectious diseases are very useful in evaluating strategies to control the diseases
in populations. Recently some authors studied the impacts of media coverage and education on the spread of infectious diseases in a given region [12-14]. In [12] Misra et al. proposed a nonlinear mathematical model for the effects of awareness programs on the spread of infectious diseases and assumed the growth rate of awareness programs impacting the population is proportional to the number of infective individuals. The model analysis showed that the spread of an infectious disease can be controlled by using awareness programs but the disease remains endemic due to immigration. Yorke and London [15] proposed an SIS type compartmental model for sexually transmitted infections with the assumption that the whole population is aware of risk but only a certain proportion choose to respond by limiting their contacts with the infective population. As a result the spread of infection is controlled, leading to a reduction of the number of individuals becoming infected. A nonlinear mathematical model with delay to capture the dynamics of effect of awareness programs on the prevalence of any epidemic is proposed and analyzed [16], which assumed it increases at a rate proportional to the number of the deaths of infective individuals.

Some scholars focus on the contact rate and most of them assume that media campaigning will aid in modifying the contact rate between susceptible and infective individuals
[17-27]. To prevent the unboundedness of the contact rate, Capasso [17] used a saturated incidence rate of the form $\beta S I /(1+\beta \delta I)$; Liu and coworkers $[23,24]$ used a nonlinear incidence rate given by $k I^{l} S /\left(1+\alpha I^{h}\right)$ to incorporate the effect of behavioral changes. In [25], the authors study how media coverage influences the dynamics of infectious disease by using SIRS model with the contact rate $\left(\beta_{1}-\beta_{2} I /(m+\right.$ I)), where $\beta_{1}, \beta_{2}$, and $m$ are positive constants. Tchuenche and Bauch used an exponentially decreasing function $e^{-M(t)}$ to reveal the force of media [27]. Cui et al. used a similar function as [13] and developed an SIR model using incidence rate $\mu e^{(-m I)} S I$ to investigate the impact of media coverage on the transmission [28]. Stability analysis of the model has shown that Hopf bifurcation can occur which causes oscillatory phenomena when $m$ is sufficiently small. Numerical simulations suggested that the media impact was stronger when the basic reproduction number $R_{0}>1$. Liu et al. [29] described the impact of media coverage using the transmission coefficient $\beta e^{\left(-\alpha_{1} E-\alpha_{2} I-\alpha_{3} H\right)}$, where $H$ is the number of hospitalized individuals. And this impact leads to the change of avoidance and contact patterns at both individual and community levels. Liu and Xiao [30] introduce a segmented function to describe the saturated media impact $e^{-m I_{c}}$ when formulating an epidemic model. A Filippov epidemic model with media coverage is proposed to describe the real characteristics of media impact in the spread of infectious diseases by incorporating a piecewise continuous transmission rate $\beta e^{(-\alpha \epsilon I)} S I$ in [31]. Mathematical and bifurcation analysis with regard to the local, global stability of equilibria and local sliding bifurcations are performed. Bhunu et al. [32] and Tchuenche and Bauch[27] focused on the different types of population in their work.

In order to better describe population mixed conditions, some authors study infectious diseases models on the different networks [33, 34]. Funk et al. have overlaid the model of information spread of a contagious disease on two, not necessarily identical networks, with more informed individuals acting to reduce their susceptibility [35]. Liu et al. took into the random perturbation [36]. In [37], the authors extended the classical SIRS epidemic model from a deterministic framework to a stochastic differential equation, and then they gave the conditions of existence of unique positive solution and the stochastic extinction and discussed the exponential $p$-stability and global stability.

Most of the articles, such as [12], assume that, due to awareness programs, driven by media, some susceptible individuals will avoid their contacts with the infectious individuals resulting in formation of a new class in the population and this newly formed aware class may contract infection only if they lose awareness. But we regard it is unreasonable. In fact sometimes even if persons are conscious of diseases, they will also be infected. Therefore we propose a mathematical model for predicting the future course of any epidemic by considering this newly formed aware class into the model.

The rest of this paper is organized as follows. In the next section, to capture the dynamics of effect of cumulative density of awareness programs on the prevalence of any epidemic
a mathematical model is proposed and analyzed. Then we analyze the local and global stability of the disease-free and the unique endemic equilibrium in Section 3. Furthermore, in Section 4 we perform some numerical examples to validate the analytical findings in Section 3 and introduce the importance of the dissemination rate and implementation rate in disease control. In Section 5 we discuss the above content.

## 2. Mathematical Model and Equilibrium Analysis

In this paper due to the awareness programs about the disease, susceptible individuals are rarely in contact with the infective ones and form a different class, namely, aware susceptible class; thus the total population is divided into three classes, the susceptible population, the aware population, and the infective population, and the proportions of them in the total population are $X(t), Y(t)$, and $X_{m}(t)$. Assuming that at time $t$ the cumulative density of awareness programs driven by media in that region is $M(t)$, which increases at a rate proportional $\mu$ to the infective population and consumes due to causes like inefficiency and psychological barriers at $\mu_{0}$, thus

$$
\begin{equation*}
M^{\prime}(t)=\mu Y(t)-\mu_{0} M(t) \tag{1}
\end{equation*}
$$

In fact people cannot take measures to protect themselves in time after the media reports the disease, so we introduce a time delay $\tau$ that represents the interval between the report time and the time of taking measures. We assume that a proportion of infected individuals recover through treatment and, after recovery, a fraction $q$ of recovered people will become aware and join the aware susceptible class whereas the remaining fraction $p(p+q=1)$ will become unaware susceptible. Keeping the above facts in mind, the dynamics of a model are governed by the following systems of nonlinear ordinary differential equations:

$$
\begin{align*}
X^{\prime}(t)= & b-\beta X(t) Y(t)+\lambda_{0} X_{m}(t) \\
& -\lambda X(t) M(t-\tau)-d X(t)+p \nu Y(t) \\
X_{m}^{\prime}(t)= & \lambda X(t) M(t-\tau)-\left(\lambda_{0}+d\right) X_{m}(t) \\
& -\alpha X_{m}(t) Y(t)+q \nu Y(t)  \tag{2}\\
Y^{\prime}(t)= & \beta X(t) Y(t)-(\nu+d) Y(t) \\
& +\alpha X_{m}(t) Y(t) \\
M^{\prime}(t)= & \mu Y(t)-\mu_{0} M(t)
\end{align*}
$$

Here $X(0)>0, Y(0) \geq 0$, and $X_{m}(0) \geq 0$.
Assume diseases spread due to the direct contact between susceptible and infective individuals only. In the above model, the rate of immigration of susceptible population is $b . \alpha$ is the contact rate of aware susceptible with infective population and $\beta(\alpha<\beta)$ is the contact rate of unaware ones. The constant $\lambda$ represents the dissemination rate of awareness among unaware susceptible due to which they form a different class; then $\lambda_{0}$ denotes the rate of transfer of aware susceptible
to unaware class. The constants $v, d$ represent the recovery rate and the natural death rate, respectively. All the above constants are assumed to be positive. Using the fact that $X(t)+$ $X_{m}(t)+Y(t)=1$, the system (2) becomes as follows:

$$
\begin{align*}
X_{m}^{\prime}(t)= & \lambda\left(1-X_{m}(t)-Y(t)\right) M(t-\tau) \\
& -\left(\lambda_{0}+d\right) X_{m}(t)-\alpha X_{m}(t) Y(t)+q \nu Y(t), \\
Y^{\prime}(t)= & \beta\left(1-X_{m}(t)-Y(t)\right) Y(t)  \tag{3}\\
& -(\nu+d) Y(t)+\alpha X_{m}(t) Y(t), \\
M^{\prime}(t)= & \mu Y(t)-\mu_{0} M(t) .
\end{align*}
$$

Now it is sufficient to study system (3) in detail rather than system (2).

For the analysis of system (3), we need the region of attraction which is given by the set: $\Omega=\left\{\left(X_{m}, Y, M\right) \in \mathfrak{R}_{+}^{3}\right.$ : $\left.0 \leq X_{m}, Y \leq 1,0 \leq M<\mu / \mu_{0}\right\}$, which attracts all solutions initiating in the interior of the positive orthant.

Define the basic reproduction number $R_{0}=\beta /(\nu+$ d). There are two equilibria, the disease-free equilibrium $E_{0}(0,0,0)$ and the endemic equilibrium $E_{*}\left(X_{m}^{*}, Y^{*}, M^{*}\right)$; the existence of $E_{0}$ is trivial; then we prove the existence of $E_{*}$ in detail. In $E_{*}\left(X_{m}^{*}, Y^{*}, M^{*}\right)$ the values of $X_{m}^{*}, M^{*}$ are obtained by solving the following algebraic equations (for $Y \neq 0$ ):

$$
\begin{gather*}
\lambda\left(1-X_{m}-Y\right) M-\left(\lambda_{0}+d\right) X_{m}-\alpha X_{m} Y+q \nu Y=0  \tag{4}\\
\beta\left(1-X_{m}-Y\right) Y-(\nu+d) Y+\alpha X_{m} Y=0  \tag{5}\\
\mu Y-\mu_{0} M=0 . \tag{6}
\end{gather*}
$$

Using (5) and (6), we get

$$
\begin{gather*}
X_{m}^{*}=\frac{\beta\left(1-Y^{*}\right)-(\nu+d)}{\beta-\alpha}  \tag{7}\\
M^{*}=\frac{\mu}{\mu_{0}} Y^{*} \tag{8}
\end{gather*}
$$

$$
J=\left[\begin{array}{ccc}
-\lambda M-\alpha Y-\lambda_{0}-d & -\lambda M-\alpha X_{m}+q \nu & \lambda\left(1-Y-X_{m}\right) e^{-\eta \tau}  \tag{12}\\
(\alpha-\beta) Y & (\alpha-\beta) X_{m}-2 \beta Y+(\beta-\nu-d) & 0 \\
0 & \mu & -\mu_{0}
\end{array}\right]
$$

where $\eta$ is the eigenvalue. Then the characteristic equation is

$$
\begin{equation*}
\left[\eta+\left(\lambda_{0}+d\right)\right][\eta+(\nu+d-\beta)]\left(\eta+\mu_{0}\right)=0 \tag{13}
\end{equation*}
$$

We get $\eta_{1}=-\left(\lambda_{0}+d\right)<0, \eta_{2}=\beta-(\nu+d)$, and $\eta_{3}=-\mu_{0}<0$. So $\eta_{2}<0$ when $R_{0}<1 ; \eta_{2}>0$ when $R_{0}>1$.

Theorem 2. When $\tau \geq 0$, the disease-free equilibrium $E_{0}$ is globally asymptotically stable in $\Omega$ if $R_{0}<1$.

Further, using (7) and (8) in (4), we obtain a quadratic equation in $Y^{*}$ as

$$
\begin{equation*}
P_{1} Y^{* 2}+P_{2} Y^{*}+P_{3}=0, \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
P_{1}= & \alpha\left(\lambda \mu+\mu_{0} \beta\right) \\
P_{2}= & (v+d-\beta)\left(\lambda \mu+\alpha \mu_{0}\right)  \tag{10}\\
& +q v \mu_{0}(\beta-\alpha)+\mu_{0} \beta(\beta+d) \\
P_{3}= & \mu_{0}\left(\lambda_{0}+d\right)(v+d-\beta)
\end{align*}
$$

Solving (9) we get

$$
\begin{equation*}
Y^{*}=\frac{-P_{2} \pm \sqrt{P_{2}^{2}-4 P_{1} P_{3}}}{2 P_{1}} \tag{11}
\end{equation*}
$$

We obtain $P_{1}>0$ and $P_{3}<0$ when $R_{0}>1$ and get $Y^{*}=$ $\left(-P_{2}+\sqrt{P_{2}^{2}-4 P_{1} P_{3}}\right) / 2 P_{1}$ for $Y^{*}>0$.

Remark. From the expression of $Y^{*}$, it is easy to note that $\left(d Y^{*} / d \lambda\right)<0$ and $\left(d Y^{*} / d \mu\right)<0$, which shows that the equilibrium number of infective individuals decreases as the rate of dissemination and the implementation rate of awareness programs increase.

## 3. Stability Analysis

In this section we present the local and global stability of $E_{0}$ and $E_{*}$.

### 3.1. The Stability of the Disease-Free Equilibrium

Theorem 1. When $\tau \geq 0$, the disease-free equilibrium $E_{0}(0,0,0)$ is locally asymptotically stable if $R_{0}<1$ and becomes unstable if $R_{0}>1$.

Proof. The Jacobian matrix corresponding to system (4) is given as follows:

Proof. To establish the global stability of the disease-free equilibrium $E_{0}$, we use Lyapunov's method and consider the following positive definite function without $\tau$ :

$$
\begin{equation*}
V=\frac{1}{2} Y^{2} \tag{14}
\end{equation*}
$$

Now differentiating $V$ with respect to $t$, we get

$$
\begin{equation*}
\frac{d V}{d t}=[\beta-(\nu+d)] Y^{2}-\beta Y^{3}-(\beta-\alpha) X_{m} Y^{2} \tag{15}
\end{equation*}
$$

When $R_{0}<1, d V / d t \leq 0$. The largest compact invariant set in $\left\{\left(X_{m}, Y, M\right) \in \Omega: V^{\prime}=0\right\}$ when $R_{0}<1$ is the singleton $\left\{E_{0}\right\}$. Then LaSalle's invariance principle implies that $E_{0}$ is globally asymptotically stable in $\Omega$.
3.2. The Stability of the Endemic Equilibrium. Linearizing system (4) about $E_{*}$, let $x=X_{m}-X_{m}^{*}, y=Y-Y^{*}$, and $m=M-M^{*}$ and get

$$
\begin{equation*}
\frac{d u}{d t}=M_{1} u(t)+M_{2} u(t-\tau) \tag{16}
\end{equation*}
$$

where

$$
\begin{gather*}
u(t)=[x(t), y(t), m(t)]^{T}, \\
M_{1}=\left[\begin{array}{ccc}
-\left(\lambda M^{*}+\alpha Y^{*}+\lambda_{0}+d\right) & q v-\lambda M^{*}-\alpha X_{m}^{*} & 0 \\
(\alpha-\beta) Y^{*} & -\beta Y^{*} & 0 \\
0 & \mu & -\mu_{0}
\end{array}\right], \\
M_{2}=\left[\begin{array}{ccc}
0 & 0 & \lambda\left(1-Y^{*}-X_{m}^{*}\right) \\
0 & 0 & 0 \\
0 & 0 & 0
\end{array}\right] . \tag{17}
\end{gather*}
$$

We have the disease-free equilibrium $E_{0}^{\prime}(0,0,0)$ and the endemic equilibrium $E_{*}^{\prime}\left(x^{*}, y^{*}, m^{*}\right)$, where the stability of $E_{*}$ about system (3) is corresponding with $E_{0}^{\prime}$. The characteristic equation of the above system at $E_{0}^{\prime}$ is

$$
\begin{equation*}
\eta^{3}+Q_{1} \eta^{2}+Q_{2} \eta+Q_{3}=Q_{4} e^{-\eta \tau} \tag{18}
\end{equation*}
$$

where $\eta$ is the eigenvalue and

$$
\begin{align*}
Q_{1} & =A+\beta Y^{*}+\mu_{0} \\
Q_{2} & =\beta Y^{*} A+\left(A+\beta Y^{*}\right) \mu_{0}+(\alpha-\beta) Y^{*} B \\
Q_{3} & =\beta Y^{*} \mu_{0} A+(\alpha-\beta) \mu_{0} Y^{*} B \\
Q_{4} & =(\beta-\alpha) \mu Y^{*} C  \tag{19}\\
A & =\lambda M^{*}+\alpha Y^{*}+\lambda_{0}+d \\
B & =\lambda M^{*}+\alpha X_{m}^{*}-q \nu \\
C & =\lambda\left(Y^{*}+X_{m}^{*}-1\right)
\end{align*}
$$

The stability of the endemic equilibrium $E_{*}$ of system (3) is stated in the following theorems.

Theorem 3. When $\tau \geq 0$, the endemic equilibrium $E_{*}$ is locally asymptotically stable if $R_{0}>1$.

Proof. When $\tau=0$, the characteristic equation is of the form

$$
\begin{equation*}
\eta^{3}+Q_{1} \eta^{2}+Q_{2} \eta+\left(Q_{3}-Q_{4}\right)=0 \tag{20}
\end{equation*}
$$

It is easy to see that

$$
\begin{align*}
& Q_{1}=A+\beta Y^{*}+\mu_{0} \\
& =\lambda M^{*}+\alpha Y^{*}+\lambda_{0}+d+\beta Y^{*}+\mu_{0}>0, \\
& Q_{2}=\beta Y^{*} A+\left(A+\beta Y^{*}\right) \mu_{0}+(\alpha-\beta) Y^{*} B \\
& =\beta Y^{*} \lambda M^{*}+\beta Y^{*} \alpha Y^{*}+\beta Y^{*}\left(\lambda_{0}+d\right) \\
& +\beta Y^{*} \mu_{0}+\lambda M^{*}+\alpha Y^{*}+\lambda_{0}+d \\
& +(\alpha-\beta) Y^{*} \lambda M^{*}+(\alpha-\beta) Y^{*} \alpha X_{m}^{*} \\
& +(\beta-\alpha) Y^{*} \nu q \\
& =\alpha \lambda Y^{*} M^{*}+\alpha^{2} Y^{*} X_{m}^{*}+\alpha Y^{*}\left(1-X_{m}^{*}\right) \\
& +(\beta-\alpha) v q Y^{*}+\alpha \beta Y^{* 2}+\beta Y^{*}\left(\lambda_{0}+d\right) \\
& +\beta \mu_{0} Y^{*}+\lambda M^{*}+\lambda_{0}+d>0 . \\
& Q_{3}-Q_{4} \\
& =\beta Y^{*} \mu_{0} A+(\alpha-\beta) \mu_{0} Y^{*} B \\
& +(\alpha-\beta) \mu Y^{*} C \\
& =\beta Y^{*} \mu_{0} \lambda M^{*}+\beta Y^{*} \mu_{0} \alpha Y^{*} \\
& +\beta Y^{*} \mu_{0}\left(\lambda_{0}+d\right)+\alpha \mu_{0} Y^{*} \lambda M^{*} \\
& +\alpha \mu_{0} Y^{*} \alpha X_{m}^{*}-\alpha \mu_{0} Y^{*} v q-\beta \mu_{0} Y^{*} \lambda M^{*} \\
& -\beta \mu_{0} Y^{*} \alpha X_{m}^{*}+\beta \mu_{0} Y^{*} \nu q-(\beta-\alpha) \mu Y^{*} C \\
& =(\beta-\alpha) \mu_{0} v q Y^{* 2}+\mu_{0} \alpha \beta X_{m}^{*} Y^{*} x^{*}\left(y^{*}\right)^{-1} \\
& +\beta \mu_{0} \alpha Y^{* 2}+\beta \mu_{0}\left(\lambda_{0}+d\right) Y^{*} \\
& +\alpha \lambda \mu_{0} Y^{*} M^{*}+(\beta-\alpha) \lambda \mu Y^{*}\left(1-Y^{*}-X_{m}^{*}\right)>0, \\
& Q_{1} Q_{2}-\left(Q_{3}-Q_{4}\right) \\
& =\left(A+\beta Y^{*}+\mu_{0}\right)\left[\beta Y^{*} A+\left(A+\beta Y^{*}\right) \mu_{0}+(\alpha-\beta) Y^{*} B\right] \\
& -\left[\beta Y^{*} \mu_{0} A+(\alpha-\beta) \mu_{0} Y^{*} B-(\beta-\alpha) \mu Y^{*} C\right] \\
& =\left(A+\beta Y^{*}+\mu_{0}\right) \beta Y^{*} A+A^{2} \mu_{0} \\
& +\left(\beta Y^{*}+\mu_{0}\right)\left(A+\beta Y^{*}\right) \mu_{0} \\
& +\left(A+\beta Y^{*}\right)(\alpha-\beta) Y^{*} B+(\alpha-\beta) Y^{*} \mu C \\
& =(\beta-\alpha) \mu \lambda Y^{*}\left(Y^{*}+X_{m}^{*}\right)+\alpha \lambda \mu Y^{*} \\
& +\left(A+\beta Y^{*}\right)\left[A \beta Y^{*}+(\alpha-\beta) Y^{*} B\right] \\
& +(\beta-\alpha) \mu \lambda Y^{*} x^{*}\left(y^{*}\right)^{-1}+\mu_{0} \beta Y^{*} A \\
& +A^{2} \mu_{0}+\left(\beta Y^{*}+\mu_{0}\right)\left(A+\beta Y^{*}\right) \mu_{0} . \tag{21}
\end{align*}
$$

From $Q_{3}-Q_{4}>0$, we can get $A \beta Y^{*}+(\alpha-\beta) Y^{*} B>0$; thus $Q_{1} Q_{2}-\left(Q_{3}-Q_{4}\right)>0$. According to Hurwitz criterion, we can know all the $\eta$ 's have negative real parts; then $E_{0}^{\prime}$ is locally asymptotically stable.

When $\tau>0$, notice that (18) does not have nonnegative real roots. If it has roots with nonnegative real parts they must be complex and should have been obtained from a pair of complex conjugate roots which cross the imaginary axis. Consequently, (18) must have a pair of purely imaginary solutions for some $\tau>0$. Assume that $\eta=i \omega(\omega>0)$ is a root of (18) without loss of generality. That is the case if and only if $\omega$ satisfies the equation

$$
\begin{equation*}
-\omega^{3} i-Q_{1} \omega^{2}+Q_{2} \omega i+Q_{3}=Q_{4}(\cos \omega \tau-i \sin \omega \tau) \tag{22}
\end{equation*}
$$

Separating the real and imaginary parts, we have the following system, satisfied by $\omega$ :

$$
\begin{align*}
& Q_{3}-Q_{1} \omega^{2}=Q_{4} \cos \omega \tau \\
& \omega^{3}-Q_{2} \omega=Q_{4} \sin \omega \tau \tag{23}
\end{align*}
$$

To eliminate the trigonometric functions we square both sides of each equation above and we add the squared above equations to obtain the following forth order equation in $\omega$ :

$$
\begin{equation*}
\omega^{6}+\left(Q_{1}^{2}-2 Q_{2}\right) \omega^{4}+\left(Q_{2}^{2}-2 Q_{1} Q_{3}\right) \omega^{2}+\left(Q_{3}^{2}-Q_{4}^{2}\right)=0 \tag{24}
\end{equation*}
$$

To reduce this fourth order equation in $\omega$ to a quadratic equation let $\varphi=\omega^{2}$ and denote the coefficients by

$$
\begin{equation*}
\varphi^{3}+R_{1} \varphi^{2}+R_{2} \varphi+R_{3}=0 \tag{25}
\end{equation*}
$$

where

$$
\begin{aligned}
R_{1}= & Q_{1}^{2}-2 Q_{2} \\
= & \left(A+\beta Y^{*}+\mu_{0}\right)^{2} \\
& -2\left[\beta Y^{*} A+\left(A+\beta Y^{*}\right) \mu_{0}+(\alpha-\beta) Y^{*} B\right] \\
= & A^{2}+\beta^{2} Y^{* 2}+\mu_{0}^{2}+2(\beta-\alpha) Y^{*} B \\
= & A^{2}+\beta^{2} Y^{* 2}+\mu_{0}^{2}+2(\beta-\alpha) Y^{*}\left(\lambda M^{*}+\alpha X_{m}^{*}\right) \\
& +2 v q \beta Y^{*} y^{*}\left(x^{*}\right)^{-1}>0, \\
R_{2}= & Q_{2}^{2}-2 Q_{1} Q_{3} \\
= & {\left[\beta Y^{*} A+(\alpha-\beta) Y^{*} B\right]^{2} } \\
& +\mu_{0}^{2}\left[A^{2}+\beta^{2} Y^{* 2}+(\beta-\alpha) Y^{*} B\right]
\end{aligned}
$$

$$
\begin{align*}
= & {\left[\beta Y^{*} A+(\alpha-\beta) Y^{*} B\right]^{2} } \\
& +\mu_{0}^{2}\left[A^{2}+\beta^{2} Y^{* 2}(\beta-\alpha) Y^{*}\left(\lambda M^{*}+\alpha X_{m}^{*}\right)\right. \\
& \left.+2 v q \beta Y^{*} y^{*}\left(x^{*}\right)^{-1}\right]>0 . \\
R_{3}= & Q_{3}^{2}-Q_{4}^{2}=\left(Q_{3}+Q_{4}\right)\left(Q_{3}-Q_{4}\right), \\
Q_{3}+ & Q_{4} \\
= & \beta Y^{*} \mu_{0} A+(\alpha-\beta) \mu_{0} Y^{*} B+(\beta-\alpha) \mu Y^{*} C \\
= & \beta Y^{*} \mu_{0} A+(\beta-\alpha) \mu_{0} Y^{*} v q \\
& +(\alpha-\beta) \mu_{0} Y^{*}\left(\lambda M^{*}+\alpha X_{m}^{*}\right) \\
+ & (\beta-\alpha) \mu Y^{*} \lambda\left(Y^{*}+X_{m}^{*}\right)+(\alpha-\beta) \mu Y^{*} \lambda \\
= & \beta Y^{*} \mu_{0}\left(\lambda M^{*}+\alpha Y^{*}+\lambda_{0}+b\right)+(\beta-\alpha) \mu_{0} Y^{*} v q \\
+ & \left(\mu_{0} \lambda M^{*}+\mu_{0} \alpha X_{m}^{*}+\mu \lambda\right) \lambda Y^{*} y^{*}\left(x^{*}\right)^{-1} \\
+ & (\beta-\alpha) \mu Y^{*} \lambda\left(Y^{*}+X_{m}^{*}\right)>0 . \tag{26}
\end{align*}
$$

So all the coefficients of (25) are positive numbers. Then according to Lemma 3.3.1 in [38], (25) has no positive real roots; that is, we may not get any positive value of $\omega$, which satisfy the transcendental equation (18). So all the $\eta$ 's have negative real parts for all values of the delay $\tau \geq 0$; then $E_{0}^{\prime}$ is locally asymptotically stable. Thus when $\tau \geq 0, E_{*}$ is locally asymptotically stable if $R_{0}>1$.

Theorem 4. When $\tau \geq 0$, the endemic equilibrium $E_{*}$ is globally asymptotically stable in $\Omega$ if $R_{0}>1$.

Proof. Using Lyapunov's method, we consider the following positive function:

$$
\begin{equation*}
V=\frac{1}{2} y^{2} \tag{27}
\end{equation*}
$$

The derivative of $V$ along the system is given by

$$
\begin{equation*}
\frac{d V}{d t}=(\alpha-\beta) Y^{*} x-\beta Y^{*} y \leq 0 \tag{28}
\end{equation*}
$$

The largest compact invariant set when $V^{\prime}=0$ is the singleton $\left\{E_{0}^{\prime}\right\}$. Then LaSalle's invariance principle implies that $E_{0}^{\prime}$ is globally asymptotically stable; that is, $E_{*}$ is globally asymptotically stable in $\Omega$.

## 4. Numerical Simulations and Results

To check the feasibility of our analysis of $\tau>0$, we present some numerical computations in this section using Matlab by choosing the following set of parameter values: $\beta=0.35$, $\lambda=0.08, \lambda_{0}=0.02, \alpha=0.2, d=0.002, v=0.43, p=0.15$, $q=0.85, \mu=0.002, \mu_{0}=0.02$, and $\tau=1$ when $R_{0}<1$. Let $\beta=0.5$; it may be checked that the condition $R_{0}>1$


Figure 1: The stability of $E_{0}$ and $E_{*}$ with different initial values.


Figure 2: The stability of $X_{m}, Y$ with variational $\lambda$.
of existence of the endemic equilibrium $E_{*}$. The equilibrium values for this data are obtained as

$$
\begin{equation*}
X_{m}^{*}=0.201, \quad Y^{*}=0.014, \quad M^{*}=0.001 \tag{29}
\end{equation*}
$$

The basic reproduction number $R_{0}$, for the above set of parameter values, is found to be 1.157 .

For the above parameter values, we select five sets of different initial starts; then the computer generated graphs of aware population, infective population, and cumulative density of awareness programs, respectively, have been drawn in Figure 1, which shows that all the trajectories initiating inside the region of attraction approach towards $E_{0}$ and $E_{*}$, respectively. Both of the equilibria $E_{0}$ and $E_{*}$ are locally asymptotically stable for given set of parameter values that numerical simulations support the analysis given in Section 3. In fact, they are globally asymptotically stable in $\Omega$ as we have proved.

In the following, we research the relationships of $X_{m}^{*}, Y^{*}$ and the dissemination rate $\lambda$, the implementation rate $\mu$ separately. We make $\lambda, \mu$ change from 0 to 0.4 and get the performances of $X_{m}(t)$ and $Y(t)$, the trajectories of which with respect to time $t$ for different $\lambda$ and $\mu$ are shown in Figures 2 and 3, respectively. And there are no awareness programs when $\lambda$ and $\mu$ are equal to zero. As shown in Figures 2 and

3, $X_{m}^{*}$ both increase and $Y^{*}$ both reduce as the increase of $\lambda$ and $\mu$, which proves the conclusions of the remark. And $\lambda, \mu$ are greater influence on $Y^{*}$ than $X_{m}^{*}$, which state awareness programs have a positive effect on prevention of diseases. In addition the reason why $X_{m}^{*}\left(Y^{*}\right)$ has a similar trend as the variations of $\lambda$ and $\mu$ is that $\lambda$ and $\mu$ have a similar influence on $X_{m}^{*}\left(Y^{*}\right)$. From the figure we obtain that $\mu$ can postpone the time of the balance of equilibrium; thus we can have more time to formulate measures to prevent diseases. There really is an effort here to make it clear that $\lambda$ and $\mu$ (awareness programs) play a key role in the prevention and control of diseases.

## 5. Discussion

The media is widely acknowledged as a key tool for influencing people's behaviors towards the disease to devise proper policies for controlling the epidemic. Awareness programs through media make people be aware about the disease and take various precautions to reduce their chances of being infected. In this paper, we propose and analyze a mathematical model to study the effect of awareness programs driven by media and the delay on the prevalence


Figure 3: The stability of $X_{m}, Y$ with variational $\mu$.
of an infectious disease. It is assumed that pathogens are transmitted via direct contact between the susceptible and the infective populations. Assume further that cumulative density of awareness programs increases at a rate proportional to the infective population. The model exhibits two equilibria; the disease-free equilibrium has been shown to be stable for basic reproduction number $R_{0}<1$. For $R_{0}>1$, it becomes unstable, which leads to the existence of an endemic equilibrium. The endemic equilibriums are globally asymptotically stable. The delay $\tau$ has no effect on the stability of the system. The numerical simulations and results that prove the stability of equilibria suggest that if we want to reduce the proportions of the infective population and increase the aware population, we can increase the dissemination rate $\lambda$ and implementation rate $\mu$. They are conducive to controlling the spread of diseases.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Homoclinic Solutions of a Class of Nonperiodic Discrete Nonlinear Systems in Infinite Higher Dimensional Lattices 

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By using critical point theory, we obtain a new sufficient condition on the existence of homoclinic solutions of a class of nonperiodic discrete nonlinear systems in infinite lattices. The classical Ambrosetti-Rabinowitz superlinear condition is improved by a general superlinear one. Some results in the literature are improved.

## 1. Introduction

Assume that $m$ is a positive integer. Consider the following difference equation in infinite higher dimensional lattices:

$$
\begin{equation*}
L u_{n}-\omega u_{n}=\sigma \gamma_{n} f_{n}\left(u_{n}\right), \quad n \in Z^{m} \tag{1}
\end{equation*}
$$

where $f_{n}(u)$ is continuous in $u, \sigma= \pm 1, n=\left(n_{1}, n_{2}, \ldots, n_{m}\right) \in$ $Z^{m}, \gamma=\left\{\gamma_{n}\right\}$ is a positive real valued sequence, $\omega \in R$, and $L$ is a Jacobi operator [1] given by

$$
\begin{align*}
L u_{n}= & a_{1\left(n_{1}, n_{2}, \ldots, n_{m}\right)} u_{\left(n_{1}+1, n_{2}, \ldots, n_{m}\right)} \\
& +a_{1\left(n_{1}-1, n_{2}, \ldots, n_{m}\right)} u_{\left(n_{1}-1, n_{2}, \ldots, n_{m}\right)} \\
& +a_{2\left(n_{1}, n_{2}, \ldots, n_{m}\right)} u_{\left(n_{1}, n_{2}+1, \ldots, n_{m}\right)} \\
& +a_{2\left(n_{1}, n_{2}-1, \ldots, n_{m}\right)} u_{\left(n_{1}, n_{2}-1, \ldots, n_{m}\right)}  \tag{2}\\
& +\cdots+a_{m\left(n_{1}, n_{2}, \ldots, n_{m}\right)} u_{\left(n_{1}, n_{2}, \ldots, n_{m}+1\right)} \\
& +a_{m\left(n_{1}, n_{2}, \ldots, n_{m}-1\right)} u_{\left(n_{1}, n_{2}, \ldots, n_{m}-1\right)} \\
& +b_{\left(n_{1}, n_{2}, \ldots, n_{m}\right)} u_{\left(n_{1}, n_{2}, \ldots, n_{m}\right)} ;
\end{align*}
$$

here, $\left\{a_{i n}\right\}(i=1,2, \ldots, m)$ and $\left\{b_{n}\right\}$ are real valued bounded sequences.

Assume that $f_{n}(0)=0$ for $n \in Z^{m}$; then $\left\{u_{n}\right\}=\{0\}$ is a solution of (1), which is called the trivial solution. As usual, we say that $u=\left\{u_{n}\right\}$, a solution of (1), is homoclinic (to 0 ) if

$$
\begin{equation*}
\lim _{|n| \rightarrow \infty} u_{n}=0 \tag{3}
\end{equation*}
$$

where $|n|=\left|n_{1}\right|+\left|n_{2}\right|+\cdots+\left|n_{m}\right|$ is the length of multiindex $n$. In addition, if $\left\{u_{n}\right\} \neq\{0\}$, then $u$ is called a nontrivial homoclinic solution. We are interested in the existence of the nontrivial homoclinic solutions for (1). This problem appears when we seek the discrete solitons of nonperiodic discrete nonlinear Schrödinger (DNLS) equation

$$
\begin{equation*}
i \dot{\psi}_{n}=-\Delta \psi_{n}-\sigma \gamma_{n} f_{n}\left(\psi_{n}\right), \quad n \in Z^{m} \tag{4}
\end{equation*}
$$

where $\sigma= \pm 1$ and

$$
\begin{align*}
\Delta \psi_{n}= & \psi_{\left(n_{1}+1, n_{2}, \ldots, n_{m}\right)} \\
& +\psi_{\left(n_{1}-1, n_{2}, \ldots, n_{m}\right)}+\psi_{\left(n_{1}, n_{2}+1, \ldots, n_{m}\right)}  \tag{5}\\
& +\psi_{\left(n_{1}, n_{2}-1, \ldots, n_{m}\right)}+\cdots+\psi_{\left(n_{1}, n_{2}, \ldots, n_{m}+1\right)} \\
& +\psi_{\left(n_{1}, n_{2}, \ldots, n_{m}-1\right)}-2 m \psi_{\left(n_{1}, n_{2}, \ldots, n_{m}\right)}
\end{align*}
$$

is the discrete Laplacian in $m$ spatial dimension. Typical representatives of power nonlinearities are

$$
\begin{equation*}
f_{n}(u)=l_{n}|u|^{p} u, \quad l_{n}, p>0 \tag{6}
\end{equation*}
$$

Primarily, we are interested in spatially localized, or solitary, standing waves. Such waves are often called breathers or gap solitons. The origin of the last name is that typically such solutions do exist for frequencies in gaps of linear spectrum. Considering (4), we suppose that the nonlinearity is gauge invariant; that is,

$$
\begin{equation*}
f_{n}\left(e^{i \theta} u\right)=e^{i \theta} f_{n}(u), \quad \theta \in R \tag{7}
\end{equation*}
$$

and, in addition, $f_{n}(u) \geq 0$ for $u \geq 0$ for $n \in Z^{m}$.
Making use of the standing wave ansatz,

$$
\begin{align*}
& \psi_{n}=u_{n} e^{-i \omega t}, \\
& \lim _{|n| \rightarrow \infty} \psi_{n}=0, \tag{8}
\end{align*}
$$

where $\left\{u_{n}\right\}$ is a real valued sequence and $\omega \in R$ is the temporal frequency. Then (4) becomes

$$
\begin{equation*}
-\Delta u_{n}-\omega u_{n}=\sigma \gamma_{n} f_{n}\left(u_{n}\right), \quad n \in Z^{m} \tag{9}
\end{equation*}
$$

and (3) holds. This is an equation of the form (1) with $a_{i n}=$ $-1(i=1,2, \ldots, m)$ and $b_{n}=2 m$.

When $f_{n}(u)$ has the form of (6), the homoclinic solutions of (9) were obtained by Karachalios in [2] by assuming that $\gamma \in l^{\rho}, \rho=(q-1) /(q-2)$, for some $q>2$. We note that $\gamma \in l^{\rho}$ implies that $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$. Moreover, (6) satisfies the classical Ambrosetti-Rabinowitz superlinear condition [3], and $f_{n}(u) /|u|$ is nondecreasing with respect to $|u|$, both of which played important roles in the existence of homoclinic solutions of (1.7) in [2].

The aim of this paper is to improve both the monotone condition of $f_{n}(u) /|u|$ and the classical AmbrosettiRabinowitz superlinear condition by general ones; see Remarks 8 and 9 for details. Moreover, in this paper, we only need $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$. Particularly, our results improved the results in [2]; see Remarks 3 and 7 for details.

In the past years, there has been large growth in the study of DNLS equation, which is a nonlinear lattice system that appears in many areas of physics. Discrete solitons which exist in DNLS systems, that is, solitary waves and localized structures in spatially discrete media, are also of particular interest in their own right. Among these, one can mention photorefractive media [4], biomolecular chains [5], and Bose-Einstein condensates [6]. The experimental observations of discrete solitons in nonlinear lattice systems have been reported [7-11]. To mention that, many authors have studied the existence of discrete solitons of the DNLS equations [12-17]. The fruitful methods include centre manifold reduction [16], variational methods [12, 14], the principle of anticontinuity [13, 17], and the Nehari manifold approach [18]. However, most of the existing literature is devoted to the DNLS equations with constant coefficients or periodic coefficients. Results on such DNLS equations have been summarized in [19-23]. And we also want to mention that, in recent years, the existence of homoclinic solutions for difference equations has been studied by many authors, and we refer to [24-36].

Since the operator $L$ is bounded and self-adjoint in the space $l^{2}$ (defined in Section 2), we consider (1) as a nonlinear
equation in $l^{2}$ with (3) being satisfied automatically. The spectrum $\sigma(L)$ of $L$ is closed. Thus, the complement $R \backslash \sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite which are denoted by $(-\infty, \beta)$ and $(\alpha, \infty)$, respectively. In this paper, we consider the homoclinic solutions of (1) in $l^{2}$ for the case where $\omega \in$ $(-\infty, \beta)$ and $\sigma=1$. The case where $\omega \in(\alpha, \infty)$ and $\sigma=-1$ is omitted, since, in this case, we can replace $L$ by $-L$.

The main idea in this paper is as follows. First, we assume that $\left\{\gamma_{n}\right\}$ converges to zero at infinity; that is, $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$. After that, we prove a compact inclusion between ordinary sequence spaces $l^{2}$ and weighted sequence spaces $l_{\gamma}^{2}$ (defined in Section 2), in order to come over lack of compactness for the so-called $(C)_{c}$ condition (defined in Section 2). Finally, by making use of the Mountain Pass Lemma [37], we prove the existence of homoclinic solutions of (1) in $l^{2}$.

## 2. Preliminaries

In this section, we first establish the variational setting associated with (1). Let

$$
\begin{array}{r}
l^{p}=l^{p}\left(Z^{m}\right)=\left\{u=\left\{u_{n}\right\}_{n \in Z^{m}}: \forall n \in Z^{m}, u_{n} \in R,\right. \\
\left.\|u\|_{l^{p}}=\left(\sum_{n \in Z^{m}}\left|u_{n}\right|^{p}\right)^{1 / p}<\infty\right\} . \tag{10}
\end{array}
$$

Then the following embedding between $l^{p}$ spaces holds:

$$
\begin{equation*}
l^{q} \subset l^{p}, \quad\|u\|_{l^{p}} \leq\|u\|_{l^{q}}, \quad 1 \leq q \leq p \leq \infty . \tag{11}
\end{equation*}
$$

For $p=2$, we get the usual Hilbert space of square-summable sequences, with the real scalar product

$$
\begin{equation*}
(u, v)_{l^{2}}=\sum_{n \in Z^{m}} u_{n} v_{n}, \quad u, v \in l^{2} \tag{12}
\end{equation*}
$$

For a positive real valued bounded sequence $\gamma=$ $\left\{\gamma_{n}: 0<\gamma_{n} \leq \bar{\gamma}<\infty\right\}_{n \in Z^{m}}$, we define the weighted sequence spaces $l_{\gamma}^{2}$ :

$$
\begin{align*}
& l_{\gamma}^{2}=\left\{u=\left\{u_{n}\right\}_{n \in Z^{m}}: \forall n \in Z^{m}, u_{n} \in R,\right.  \tag{13}\\
& \left.\quad\|u\|_{l_{\gamma}^{2}}=\left(\sum_{n \in Z^{m}} \gamma_{n}\left|u_{n}\right|^{2}\right)^{1 / 2}<\infty\right\} .
\end{align*}
$$

It is not hard to see that $l_{\gamma}^{2}$ is a Hilbert space, with the scalar product

$$
\begin{equation*}
(u, v)_{l_{\gamma}^{2}}=\sum_{n \in Z^{m}} \gamma_{n} u_{n} v_{n}, \quad u, v \in l_{\gamma}^{2} . \tag{14}
\end{equation*}
$$

For a certain class of weight $\gamma$, we have the following lemmas, which will play a crucial role in our analysis.

Lemma 1. Let $\kappa=\left\{\kappa_{n}:\left|\kappa_{n}\right| \leq \bar{\kappa}<\infty\right\}_{n \in Z^{m}}$ be a multiplication operator from $l_{\gamma}^{2}$ to $l_{\gamma}^{2}$ defined by $\kappa u=\left\{\kappa_{n} u_{n}\right\}_{n \in Z^{m}}$. If $\lim _{|n| \rightarrow \infty} \kappa_{n}=0$, then the operator $\kappa$ is compact.

Proof of Lemma 1. Let

$$
\begin{equation*}
\Lambda=\left\{\kappa u:\|u\|_{l_{\gamma}^{2}} \leq 1\right\} \tag{15}
\end{equation*}
$$

We only need to prove that $\Lambda$ is precompact in $l_{\gamma}^{2}$. By assumption, for any $\varepsilon>0$, there exists $N>0$ such that $\left|\kappa_{n}\right| \leq \varepsilon$ for any $|n|>N$. Define a cutting sequence $\chi=\left\{\chi_{n}\right\}$ by

$$
\chi_{n}= \begin{cases}1, & |n| \leq N  \tag{16}\\ 0, & |n|>N\end{cases}
$$

Denote by $\chi^{c}=1-\chi$ the anticutting sequence. Then for any $\kappa и \in \Lambda$

$$
\begin{align*}
& \left\|\chi^{c} \kappa u\right\|_{l_{\gamma}^{2}}^{2}=\sum_{|n|>N} \gamma_{n}\left|\kappa_{n} u_{n}\right|^{2} \leq \varepsilon^{2}  \tag{17}\\
& \|\chi \kappa u\|_{l_{\gamma}^{2}}^{2}=\sum_{|n| \leq N} \gamma_{n}\left|\kappa_{n} u_{n}\right|^{2} \leq \bar{\kappa}^{2}
\end{align*}
$$

For arbitrary $\varepsilon>0$ and $\Lambda_{\varepsilon}=\left\{\chi \kappa u:\|u\|_{l_{\gamma}^{2}} \leq 1\right\}$ finitedimensional and bounded, we know that $\Lambda$ is precompact. The proof is complete.

Lemma 2. One assumes positive sequence of real numbers $\gamma$ with $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$. Then $l^{2} \hookrightarrow l_{\gamma}^{2}$ with compact inclusion.

Proof of Lemma 2. Note that $\|u\|_{l_{v}} \leq \sqrt{\bar{\gamma}}\|u\|_{l^{2}}$ for any $u \in l^{2}$ and $\gamma^{1 / 2}$ is compact. Thus, $l^{2} \hookrightarrow l_{\gamma}^{2}$ with compact inclusion by Lemma 1 . The proof is complete.

Remark 3. Karachalios [2] proved $l^{2} \hookrightarrow l_{\gamma}^{2}$ with compact inclusion assuming that $\gamma \in l^{\rho}, \rho=(q-1) /(q-2)$, for some $q>2$. Note that $\gamma \in l^{\rho}$ implies that $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$. Thus, we find that Lemma 2 improves Lemma 2.1 in [2].

On the Hilbert space $l^{2}$, we consider the functional

$$
\begin{equation*}
J(u)=\frac{1}{2}((L-\omega) u, u)_{l^{2}}-\sigma \sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(u_{n}\right), \tag{18}
\end{equation*}
$$

where

$$
\begin{equation*}
F_{n}(u)=\int_{0}^{u} f_{n}(s) d s \tag{19}
\end{equation*}
$$

is the primitive function of $f_{n}(u)$. Then $J \in C^{1}\left(l^{2}, R\right)$ and

$$
\begin{align*}
\left\langle J^{\prime}(u), v\right\rangle= & ((L-\omega) u, v)_{l^{2}} \\
& -\sigma \sum_{n \in Z^{m}} \gamma_{n} f_{n}\left(u_{n}\right) v_{n}, \quad u, v \in l^{2} . \tag{20}
\end{align*}
$$

Equation (20) implies that (1) is the corresponding EulerLagrange equation for $J$. Therefore, we have reduced the
problem of finding a nontrivial homoclinic solution of (1) to that of seeking a nonzero critical point of the functional $J$ on $l^{2}$.

Let $\delta$ be the distance from $\omega$ to the spectrum $\sigma(L)$; that is,

$$
\begin{equation*}
\delta=\beta-\omega \tag{21}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
((L-\omega) u, u) \geq \delta\|u\|_{l^{2}}^{2}, \quad u \in l^{2} . \tag{22}
\end{equation*}
$$

We also consider a norm in $l^{2}$ defined by

$$
\begin{equation*}
\|u\|_{l_{\omega}^{2}}=[((L-\omega) u, u)]^{1 / 2}, \quad u \in l^{2} . \tag{23}
\end{equation*}
$$

Since

$$
\begin{equation*}
\delta\|u\|_{l^{2}}^{2} \leq\|u\|_{l_{\omega}^{2}}^{2} \leq\|L-\omega\|\|u\|_{l^{2}}^{2}, \quad u \in l^{2}, \tag{24}
\end{equation*}
$$

norm (23) is an equivalent norm with the usual one of $l^{2}$.
In order to obtain the existence of critical points of $J$ on $l^{2}$, we cite some basic notations and some known results from critical point theory.

Let $H$ be a Hilbert space and $C^{1}(H, R)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on $H$.

Let $J \in C^{1}(H, R)$. A sequence $\left\{u_{j}\right\} \subset H$ is called a $(C)_{c}$ sequence for $J$ if $J\left(u_{j}\right) \rightarrow c$ for some $c \in R$ and $\left(1+\left\|u_{j}\right\|\right)\left\|J^{\prime}\left(u_{j}\right)\right\|$ as $j \rightarrow \infty$. We say $J$ satisfies the $(C)_{c}$ condition if any $(C)_{c}$ sequence for $J$ possesses a convergent subsequence.

Let $B_{r}$ be the open ball in $H$ with radius $r$ and center 0 , and let $\partial B_{r}$ denote its boundary. The following lemma is taken from [37].

Lemma 4 (Mountain Pass Lemma). If $J \in C^{1}(H, R)$ and satisfies the following conditions: there exist $e \in H \backslash\{0\}$ and $r \in(0,\|e\|)$ such that $\max \{J(0), J(e)\}<\inf _{\|u\|=r} J(u)$, then there exists a $(C)_{c}$ sequence $\left\{u_{n}\right\}$ for the mountain pass level $c$ which is defined by

$$
\begin{equation*}
c=\inf _{h \in \Gamma s \in[0,1]} J(h(s)), \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma=\{h \in C([0,1], H): h(0)=0, h(1)=e\} . \tag{26}
\end{equation*}
$$

## 3. Main Results

In this section, we will establish some sufficient conditions on the existence of nontrivial solutions of (1) in $l^{2}$.

Theorem 5. Assume that $\sigma=1, \omega \in(-\infty, \beta)$, and the following conditions hold.
$(H 1) f_{n}(u)$ is continuous in $u, f_{n}(u)=o(u)$ as $u \rightarrow 0$ uniformly for $n \in Z^{m}$.
(H2) There exist $b>0, p>2$ such that

$$
\begin{equation*}
\left|f_{n}(u)\right| \leq b\left(1+|u|^{p-1}\right) \tag{27}
\end{equation*}
$$

uniformly for $n \in Z^{m}$ and $u \in R$.
(H3) There exists some $\theta \geq 1$ such that $\theta G_{n}(u) \geq G_{n}(t u)$, for $n \in Z^{m}, u \in R$, and $t \in[0,1]$, where $G_{n}(u)=$ $(1 / 2) f_{n}(u) u-F_{n}(u)$.
(H4) $F_{n}(u) \geq 0$ for $u \in R$, and $\lim _{|u| \rightarrow \infty}\left(F_{n}(u) / u^{2}\right)=\infty$, uniformly for $n \in Z^{m}$.
(H5) Positive real valued sequence $\gamma=\left\{\gamma_{n}\right\}_{n \in Z^{m}}$ with $\lim _{|n| \rightarrow \infty} \gamma_{n}=0$.

Then (1) has at least a nontrivial solution $u$ in $l^{2}$ and the solution decays exponentially at infinity. That is, there exist two positive constants $C$ and $\tau$ such that

$$
\begin{equation*}
\left|u_{n}\right| \leq C e^{-\tau|n|}, \quad n \in Z^{m} . \tag{28}
\end{equation*}
$$

Theorem 5 gives some sufficient conditions on the existence of nontrivial solutions of (1) in $l^{2}$. However, (1) may have no nontrivial solutions in $l^{2}$. In fact, we have the following proposition.

Proposition 6. Assume that $\sigma=-1, \omega \leq \beta$, and $\gamma_{n} f_{n}(u) u>0$ when $u \neq 0$ for all $n \in Z^{m}$. Then (1) has no nontrivial solutions in $l^{2}$.

Proof of Proposition 6. By way of contradiction, we assume that (1) has a nontrivial solution $u=\left\{u_{n}\right\} \in l^{2}$. Then $u$ is a nonzero critical point of $J$, and

$$
\begin{align*}
\left\langle J^{\prime}(u), u\right\rangle & =((L-\omega) u, u)_{l^{2}}-\sigma \sum_{n \in Z^{m}} \gamma_{n} f_{n}\left(u_{n}\right) u_{n} \\
& \geq \sum_{n \in Z^{m}} \gamma_{n} f_{n}\left(u_{n}\right) u_{n}>0 . \tag{29}
\end{align*}
$$

This is a contradiction as $\left\langle J^{\prime}(u), u\right\rangle=0$, so the conclusion holds.

Remark 7. It is easy to see that the function $f_{n}$ defined by

$$
\begin{equation*}
f_{n}(u)=|u|^{2 \lambda} u \tag{30}
\end{equation*}
$$

where $\lambda>0$ and $\gamma \in l^{\rho}, \rho=(q-1) /(q-2)$, for some $q>2$, satisfies all conditions in Theorem 5. This case was studied by [2], and we find that Theorem 5 improves Theorem 2.3 in [2].

Remark 8. We will introduce another condition ( $\bar{H} 3$ ): $f_{n}(u) /|u|$ is nondecreasing with respect to $|u|$. We want to point out that condition (H3) is equivalent to $(\bar{H} 3)$ when $\theta=1$ and (H3) gives "better monotony" when $\theta>1$, since $(\bar{H} 3)$ implies (H3) (see [38]). Moreover, we can find that $f_{n}(u)=5 u \ln \left(1+u^{2}\right)+9 \sin u$ satisfies (H3) but not $(\bar{H} 3)$ for some $\theta \geq 100$.

Remark 9. As we know, the condition
$(\bar{H} 4) f(u) u>q F(u)>0$ for some $q>2$ and $u \neq 0$
is often called Ambrosetti-Rabinowitz superlinear condition [3]. Clearly, ( $\bar{H} 4$ ) implies (H4). Actually, $(H 4)$ is more general than $(\bar{H} 4)$. Let $\left\{a_{n}\right\}$ be a positive sequence, and $f_{n}(u)=$ $a_{n} u \ln (1+|u|)$. Then $f_{n}$ satisfies (H4). However, $f_{n}$ does not satisfy $(\bar{H} 4)$.

The proof of Theorem 5 is based on a direct application of the following lemmas. The key points read as follows.

Lemma 10. Assume that the conditions of Theorem 5 hold; then one has the following.
$\left(J_{1}\right)$ There exist two constants $a>0$ and $\rho>0$ such that $\left.J\right|_{\partial B_{\rho}} \geq a$.
$\left(J_{2}\right)$ There exists an $e \in l^{2}$ such that $J(t e) \rightarrow-\infty$ as $|t| \rightarrow$ $\infty$.

Proof of Lemma 10. Let $\bar{\gamma}=\max _{n \in Z^{m}}\left\{\gamma_{n}\right\}$ and $\epsilon=\delta / 2 \bar{\gamma}$. By $(H 1)$ and $(H 2)$, there exists $c_{1}>0$, such that

$$
\begin{equation*}
\left|f_{n}(u)\right| \leq \epsilon|u|+c_{1}|u|^{p-1} \tag{31}
\end{equation*}
$$

for all $n \in Z^{m}$ and $u \in R$, and (31) implies that

$$
\begin{equation*}
\left|F_{n}(u)\right| \leq \frac{\epsilon}{2}|u|^{2}+\frac{c_{1}}{p}|u|^{p} . \tag{32}
\end{equation*}
$$

By (32) and the Hölder inequality, we have

$$
\begin{align*}
J(u) & =\frac{1}{2}((L-\omega) u, u)_{l^{2}}-\sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(u_{n}\right) \\
& \geq \frac{\delta}{2}\|u\|_{l^{2}}^{2}-\bar{\gamma} \sum_{n \in Z^{m}}\left(\frac{\epsilon}{2}|u|^{2} \frac{c_{1}}{p}|u|^{p}\right)  \tag{33}\\
& \geq \frac{\delta}{4}\|u\|_{l^{2}}^{2}-\frac{c_{1} \bar{\gamma}}{p}\|u\|_{l^{2}}^{p} .
\end{align*}
$$

Since $p>2$, we have

$$
\begin{equation*}
J(u) \geq \frac{\delta \rho^{2}}{8}=a>0 \quad \text { for }\|u\|_{l^{2}}=\rho \tag{34}
\end{equation*}
$$

where $\rho=\left(\delta \rho / 8 c_{1} \bar{\gamma}\right)^{1 /(p-2)}$.
Let $e=\left\{e_{n}\right\} \in l^{2}$ be the eigenvector of $L$ corresponding to the eigenvalue $\beta$; that is to say, $L e=\beta e$. There exists $N>0$, such that

$$
\begin{equation*}
\sum_{|n| \leq N} e_{n}^{2} \geq \frac{1}{2}\|e\|_{2^{2}}^{2} \tag{35}
\end{equation*}
$$

Let

$$
\begin{equation*}
A^{*}=\left\{n \in Z^{m}:|n| \leq N, e_{n} \neq 0\right\} . \tag{36}
\end{equation*}
$$

By (H4), for any $M>0$, there exists $\eta=\eta(M)>0$ such that

$$
\begin{equation*}
F_{n}(u) \geq M|u|^{2} \quad \text { for } n \in Z^{m},|u| \geq \eta \text {. } \tag{37}
\end{equation*}
$$

Taking $t$ large enough, such that $t e_{n}>\eta$ for all $n \in A^{*}$, then, combining (35), (36), and (37), we have

$$
\begin{align*}
J(t e) & =\frac{1}{2}((L-\omega) t e, t e)_{l^{2}}-\sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(t e_{n}\right) \\
& \leq \frac{\delta}{2} t^{2}\|e\|_{l^{2}}^{2}-\sum_{n \in A^{*}} \gamma_{n} F_{n}\left(t e_{n}\right)  \tag{38}\\
& \leq \frac{\delta}{2} t^{2}\|e\|_{l^{2}}^{2}-\underline{\gamma} M t^{2} \sum_{n \in A^{*}} e_{n}^{2} \\
& \leq \frac{1}{2}(\delta-\underline{\gamma} M) t^{2}\|e\|_{l^{2}}^{2}
\end{align*}
$$

where $\underline{\gamma}=\min _{n \in A^{*}}\left\{\gamma_{n}\right\}>0$. Letting $M$ be large enough, such that $\delta \leq \underline{\leq} M$, we obtain that $J(t e) \rightarrow-\infty$ as $|t| \rightarrow \infty$. The proof is complete.

Lemma 11. Assume that the conditions of Theorem 5 hold; then the functional $J$ satisfies the $(C)_{c}$ condition for any given $c \in R$.

Proof of Lemma 11. Let $\left\{u^{(k)}\right\} \subset l^{2}$ be a $(C)_{c}$ sequence of $J$; that is,

$$
\begin{align*}
J\left(u^{(k)}\right) & \longrightarrow c \\
\left(1+\left\|u^{(k)}\right\|_{l^{2}}\right)\left\|J^{\prime}\left(u^{(k)}\right)\right\|_{l^{2}} & \longrightarrow 0  \tag{39}\\
\text { as } k & \longrightarrow \infty
\end{align*}
$$

First, we prove that $\left\{u^{(k)}\right\}$ is bounded in $l^{2}$. By way of contradiction, assume that $\left\|u^{(k)}\right\|_{l^{2}} \rightarrow \infty$ as $k \rightarrow \infty$. Set $\xi^{(k)}=u^{(k)} /\left\|u^{(k)}\right\|_{l^{2}}$. Up to a sequence, we have

$$
\begin{array}{ll}
\xi^{(k)} \rightharpoonup \xi, & \text { in } l^{2} \\
\xi^{(k)} \longrightarrow \xi, & \text { in } l_{\gamma}^{2} \tag{41}
\end{array}
$$

Case $1(\xi \neq 0)$. By $J\left(u^{(k)}\right)=c+o(1)$, where $o(1) \rightarrow 0$ as $k \rightarrow 0$, we have

$$
\begin{align*}
& \sum_{n \in Z^{m}} \gamma_{n} \frac{F_{n}\left(u_{n}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}=\frac{1}{2} \frac{\left((L-\omega) u^{(k)}, u^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}-\frac{c+o(1)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}  \tag{42}\\
& \quad \leq \frac{\|L-\omega\|}{2}-\frac{c+o(1)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}<\infty .
\end{align*}
$$

Let $B^{*}=\left\{n \in Z^{m}: \xi_{n} \neq 0\right\}$. Obviously, $B^{*}$ is nonempty. Then, for some $n_{0} \in B^{*}$, it follows from (41) that

$$
\begin{equation*}
u_{n_{0}}^{(k)}=\xi_{n_{0}}^{(k)}\left\|u^{(k)}\right\|_{l^{2}} \longrightarrow \infty, \quad \text { as } k \longrightarrow \infty \tag{43}
\end{equation*}
$$

Combining (H4) and $\gamma_{n_{0}}>0$, we have

$$
\begin{equation*}
\gamma_{n_{0}} \frac{F_{n_{0}}\left(u_{n_{0}}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}=\gamma_{n_{0}} \frac{F_{n_{0}}\left(u_{n_{0}}^{(k)}\right)}{\left|u_{n_{0}}^{(k)}\right|^{2}}\left|\xi_{n_{0}}^{(k)}\right|^{2} \longrightarrow \infty, \quad \text { as } k \longrightarrow \infty \tag{44}
\end{equation*}
$$

However,

$$
\begin{align*}
\sum_{n \in Z^{m}} \gamma_{n} \frac{F_{n}\left(u_{n}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}} & =\sum_{n \neq n_{0}} \gamma_{n} \frac{F_{n}\left(u_{n}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}+\gamma_{n_{0}} \frac{F_{n_{0}}\left(u_{n_{0}}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}}  \tag{45}\\
& \geq \gamma_{n_{0}} \frac{F_{n_{0}}\left(u_{n_{0}}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^{2}}^{2}} \longrightarrow \infty,
\end{align*}
$$

as $k \rightarrow \infty$. This contradicts (42).
Case $2(\xi=0)$. Let

$$
\begin{equation*}
J\left(t_{k} u^{(k)}\right)=\max _{t \in[0,1]} J\left(t u^{(k)}\right) \tag{46}
\end{equation*}
$$

For any given $M>\max \{4, \theta c / 2 \delta\}$, let $k$ be large enough such that $\left\|u^{(k)}\right\|_{\mathcal{L}^{2}}>M$ and $\bar{\xi}^{(k)}=2 M^{1 / 2} \xi^{(k)}$. Combining (32), (41), and $\xi=0$, it is easy to see that

$$
\begin{align*}
& \sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(\bar{\xi}_{n}^{(k)}\right) \\
& \leq \frac{\epsilon}{2}\left\|\bar{\xi}^{(k)}\right\|_{l_{\gamma}^{2}}^{2}+\frac{c_{1} \bar{\gamma}^{1 / 2}}{p}\left\|\bar{\xi}^{(k)}\right\|_{l^{(p(p-1)}}^{p-1}\left\|\bar{\xi}^{(k)}\right\|_{l_{\gamma}^{2}} \longrightarrow 0  \tag{47}\\
& \text { as } k \longrightarrow \infty .
\end{align*}
$$

Thus, for $k$ large enough, we have

$$
\begin{align*}
J\left(t_{k} u^{(k)}\right) & \geq J\left(\bar{\xi}^{(k)}\right) \\
& \geq \frac{\delta}{2}\left\|\bar{\xi}^{(k)}\right\|_{l^{2}}^{2}-\sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(\bar{\xi}_{n}^{(k)}\right)  \tag{48}\\
& \geq 2 \delta M-\sum_{n \in Z^{m}} \gamma_{n} F_{n}\left(\bar{\xi}_{n}^{(k)}\right) .
\end{align*}
$$

By (47), (48), and $M>c / 2 \delta$, we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} J\left(t_{k} u^{(k)}\right) \geq 2 \delta M>\theta c \tag{49}
\end{equation*}
$$

Noting that $J(0)=0$ and $J\left(u^{(k)}\right) \rightarrow c$, as $k \rightarrow \infty$, then $0<t_{k}<1$ when $k$ is big enough. Thus, $\left\langle J^{\prime}\left(t_{k} u^{(k)}\right), t_{k} u^{(k)}\right\rangle=0$. In view of (H3), it follows that

$$
\begin{align*}
J\left(t_{k} u^{(k)}\right) & =J\left(t_{k} u^{(k)}\right)-\frac{1}{2}\left\langle J^{\prime}\left(t_{k} u^{(k)}\right), t_{k} u^{(k)}\right\rangle \\
& =\sum_{n \in Z^{m}} \gamma_{n}\left(\frac{1}{2} f_{n}\left(t_{k} u_{n}^{(k)}\right) t_{k} u_{n}^{(k)}-F_{n}\left(t_{k} u_{n}^{(k)}\right)\right)  \tag{50}\\
& \leq \theta \sum_{n \in Z^{m}} \gamma_{n}\left(\frac{1}{2} f_{n}\left(u_{n}^{(k)}\right) u_{n}^{(k)}-F_{n}\left(u_{n}^{(k)}\right)\right) \\
& =\theta\left(J\left(u^{(k)}\right)-\frac{1}{2}\left\langle J^{\prime}\left(u^{(k)}\right), u^{(k)}\right\rangle\right)
\end{align*}
$$

By (50), we have

$$
\begin{equation*}
\lim _{k \rightarrow \infty} J\left(t_{k} u^{(k)}\right) \leq \theta c \tag{51}
\end{equation*}
$$

This contradicts (49), so $\left\{u^{(k)}\right\}$ is bounded in $l^{2}$.
Second, we show that there exists a convergent subsequence of $\left\{u^{(k)}\right\}$. In fact, there exists a subsequence, still denoted by the same notation, such that

$$
\begin{equation*}
u^{(k)} \rightharpoonup u, \quad \text { in } l^{2} \tag{52}
\end{equation*}
$$

By Lemma 2, we have

$$
\begin{equation*}
u^{(k)} \longrightarrow u, \quad \text { in } l_{\gamma}^{2} \tag{53}
\end{equation*}
$$

By direct calculation, we obtain

$$
\begin{align*}
& \left\|u^{(k)}-u\right\|_{l_{\omega}^{2}}^{2} \\
& =\left\langle J^{\prime}\left(u^{(k)}\right)-J^{\prime}(u), u^{(k)}-u\right\rangle \\
& +\sum_{n \in Z^{m}} \gamma_{n}\left(f_{n}\left(u_{n}^{(k)}\right)-f_{n}\left(u_{n}\right)\right)\left(u_{n}^{(k)}-u_{n}\right) \\
& \leq\left\langle J^{\prime}\left(u^{(k)}\right)-J^{\prime}(u), u^{(k)}-u\right\rangle \\
& +\sum_{n \in Z^{m}} \gamma_{n}\left(\epsilon\left(\left|u_{n}^{(k)}\right|+\left|u_{n}\right|\right)+c_{1}\left(\left|u_{n}^{(k)}\right|^{p-1}+\left|u_{n}\right|^{p-1}\right)\right) \\
& \times\left(u_{n}^{(k)}-u_{n}\right) \leq\left\langle J^{\prime}\left(u^{(k)}\right)-J^{\prime}(u), u^{(k)}-u\right\rangle \\
& +\bar{\gamma}^{1 / 2}\left(\epsilon\left(\left\|u^{(k)}\right\|_{l^{2}}+\|u\|_{l^{2}}\right)\right. \\
& \left.+c_{1}\left(\left\|u^{(k)}\right\|_{l^{2(p-1)}}^{p-1}+\|u\|_{l^{2(p-1)}}^{p-1}\right)\right) \\
& \times\left\|u^{(k)}-u\right\|_{l_{\gamma}^{2}} . \tag{54}
\end{align*}
$$

Therefore, combining (11), (24), (52), (53), (54), and the boundedness of $\left\{u^{(k)}\right\}$, it is clear that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left\|u^{(k)}-u\right\|_{l^{2}}=0 \tag{55}
\end{equation*}
$$

and this means $J$ satisfies $(C)_{c}$ condition. The proof is complete.

Now, we are ready to prove Theorem 5.
Proof of Theorem 5. Let $a, \rho$, and $e \in l^{2}$ be obtained in Lemma 10.

Since $J(t e) \rightarrow-\infty$ as $|t| \rightarrow \infty$, there exists a real number $t_{0}$ such that

$$
\begin{equation*}
\left\|t_{0} e\right\|_{l^{2}}>\rho, \quad J\left(t_{0} e\right)<0 \tag{56}
\end{equation*}
$$

Immediately, we obtain

$$
\begin{equation*}
\max \left\{J(0), J\left(t_{0} e\right)\right\}=0<a \leq \inf _{\|u\|=\rho} J(u) . \tag{57}
\end{equation*}
$$

Now that we have verified all assumptions of Lemma 4, we know $J$ possesses a $(C)_{c}$ sequence $\left\{u_{j}\right\} \subset l^{2}$ for the mountain pass level $c \geq a$ with

$$
\begin{equation*}
c=\inf _{h \in \Gamma s \in[0,1]} J(h(s)), \tag{58}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma=\left\{h \in C\left([0,1], l^{2}\right): h(0)=0, h(1)=t_{0} e\right\} . \tag{59}
\end{equation*}
$$

By Lemma 11, $\left\{u_{j}\right\}$ has a convergent subsequence $\left\{u_{j_{m}}\right\}$ such that $u_{j_{m}} \rightarrow u$ as $j_{m} \rightarrow+\infty$ for some bounded $u \in l^{2}$. Since $J \in C^{1}\left(l^{2}, R\right)$, we have

$$
\begin{align*}
J\left(u_{j_{m}}\right) & \longrightarrow J(u), \\
\left(1+\left\|u_{j_{m}}\right\|_{l^{2}}\right) J^{\prime}\left(u_{j_{m}}\right) & \longrightarrow\left(1+\|u\|_{l^{2}}\right) J^{\prime}(u), \tag{60}
\end{align*}
$$

as $j_{m} \rightarrow+\infty$. By the uniqueness of limit and the fact that $u$ is bounded, we obtain that $u$ is a nontrivial critical point of $J$ as the corresponding critical value $c \geq a>0$. Hence, (1) has at least one nontrivial solution $u$ in $l^{2}$.

Finally, we show that $u=\left\{u_{n}\right\}$ satisfies (28). In fact, similar to [39], for $n \in Z$, let

$$
w_{n}= \begin{cases}-\frac{\gamma_{n} f_{n}\left(u_{n}\right)}{u_{n}}, & u_{n} \neq 0  \tag{61}\\ 0, & u_{n}=0\end{cases}
$$

then

$$
\begin{equation*}
\bar{L} u_{n}=\omega u_{n}, \tag{62}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{L} u_{n}=L u_{n}+w_{n} u_{n} \tag{63}
\end{equation*}
$$

Clearly, $\lim _{|n| \rightarrow \infty} w_{n}=0$. Thus, the multiplication by $w_{n}$ is a compact operator in $l^{2}$, which implies that

$$
\begin{equation*}
\sigma_{\mathrm{ess}}(\bar{L})=\sigma_{\mathrm{ess}}(L), \tag{64}
\end{equation*}
$$

where $\sigma_{\text {ess }}$ stands for the essential spectrum. Equation (62) means that $u=\left\{u_{n}\right\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega \notin \sigma_{\text {ess }}(\bar{L})$ of the operator $\bar{L}$. Equation (28) follows from the standard theorem on exponential decay for such eigenfunctions [1]. Now the proof of Theorem 5 is complete.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# The Analysis of Pricing Power of Preponderant Metal Mineral Resources under the Perspective of Intergenerational Equity and Social Preferences: An Analytical Framework Based on Cournot Equilibrium Model 

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#### Abstract

This paper combines intergenerational equity equilibrium and social preferences equilibrium with Cournot equilibrium solving the technological problem of intergenerational equity and strategic value compensation confirmation, achieving the effective combination between sustainable development concept and value evaluation, thinking and expanding the theoretical framework for the lack of pricing power of mineral resources. The conclusion of the theoretical model and the numerical simulation shows that intergenerational equity equilibrium and social preferences equilibrium enhance international trade market power of preponderant metal mineral resources owing to the production of intergenerational equity compensation value and strategic value. However, the impact exerted on Cournot market power by social preferences is inconsistent: that is, changes of altruistic Cournot equilibrium and reciprocal inequity Cournot equilibrium are consistent, while inequity aversion Cournot equilibrium has the characteristic of loss aversion, namely, under the consideration of inequity aversion Cournot competition, Counot-Nash equilibrium transforms monotonically with sympathy and jealousy of inequity aversion.


## 1. Introduction

Pricing power is the ability where related market participants manipulate market equilibrium price away from international trade fair price in its favor by market forces. In recent years, the sustainable growth of China's economy is the important engine driving the growth of the world economy and the increasing demand of staple commodities, such as metal mineral products. The influence exerted on global economy by China is called "China Factor" internationally. However the so-called "China Factor" does not bring corresponding pricing power to China; instead, metal mineral resources international trade price of our country is stuck in the dilemma. The export price of preponderant metal mineral resources, such as rare earth, lithium, and indium, experienced a long-term slump, which not only cause economic loss
but also leave the burden of energy consumption and environmental protection to China, so that it is equivalent to providing hidden subsidies at the cost of ecological environment destruction and mineral resources rapid depletion. Therefore, the reports of the seventeenth and eighteenth congress of the CPC put forward continuously [1, 2], "deepen the resource products price and tax reform, establish compensated use system and eco-compensation system reflected the market supply and demand, resource scarcity and intergenerational compensation." Although intergenerational compensation is stressed in reports, externalities resulting from productive process of metal mineral resources development and utilization are not included in metal mineral resources value system as the form of cost. The proportion of calculated mineral resources compensation fees to sales revenue is approximately $1.18 \%$, far lower than the level of $2 \%-8 \%$ of
foreign premium. The premium rate and compensation fees of rare earth in China are far lower than Australia, the USA, South Africa, and Vietnam. The technological problem of cost confirmation and measurement and serious distortion of tax policy prevent mineral resource development and utilization from reasonable value compensations, causing unfairness in international trade fair price, and above all that is the important reason why China loses pricing power.

Considering that the international trade price of metal mineral resources is also affected by factors such as supply and demand and speculation (expectation), the complete value compensation system including marginal cost of production, marginal user cost, and external cost (ecological value and intergenerational compensation value) is only the static reason to explain pricing power deficiency. Especially to preponderant metal mineral resources, such as rare earth, lithium, and indium, their international trade price is mainly dependent on mutual bargaining, which is affected by psychological preferences of players and thus produce strategic value. While ignoring objectivity of strategic value in policy suggestion making is another reason accounted for pricing power deficiency. Because according to social preference equilibrium analysis, unless the international trade price is fair and it achieves equilibrium among players, it is difficult for metal mineral resources development and utilization to achieve success. Using traditional game equilibrium evaluation method can reflect the economic value connotation accurately; however, a certain mineral resource development and utilization are accepted only when players approve of economic value and ecological value and the fairness of metal mineral resources development compensation price psychologically. So fairness correction on metal mineral resources development compensation basic value is necessary in reality.

Classical literatures discussing pricing fairness from a perspective of mineral resources development value compensation include "Hotelling Rule" [3], raised by Hotelling, namely, mineral resources, as a kind of asset, need depreciation, so the depletion of mineral resources could be compensated by taxation; "Hartwick Rule" [4] raised by Hartwick, pointed out that if mining rent of nonrenewable resources saved as productive investment, the investment across generations is equal to achieve sustainability of economic growth, when the investment is greater than resources value extracted by resources owners. Serafy [5] adds environmental losses to the research work of national income accounting system and raises user cost approach to calculate real income and the value depletion of nonrenewable resources, which is a new national income accounting method in nonrenewable resources field; later, Serafy [6] makes improvement in this method. This approach lays the foundation of depletion cost pricing of nonrenewable mineral resource; therefore, it is used by many scholars to measure user cost of various mineral resources and analyze the reasonability of premium system and resources tax and fee policy, such as Adelman [7] calculates user cost of some large oil and gas companies by user cost approach and compares with premium; Young and Seroa Da Motta [8] count user cost of major minerals in Brazil by this method; Blignaut and Hassan [9] estimate user
costs of underground mineral resources in South Africa; Lin et al. [10] discover the inadaptability of user cost approach in coal resources of China; thus, he uses the modified approach to estimate real cost of coal resources and builds CGE model to determine detailed tax rates; G. P. Li and H. W. Li [11] correct the defects of user cost method and use it to calculate user cost of oil and gas in the United States; Zeng and Li [12] use fixed user cost approach to count user cost of coal, oil, and natural gas in China during 19852010, after taking depletion in resource development and the effects of inflation into account. These researches above solve problems of metal mineral resources compensation scarcity value but ignore environmental costs and intergenerational equity value and lack explanation for influences exerted on market and international trade price by intergenerational equity compensation value. On the other hand, many scholars do researches on pricing power, such as Fattouh [13] who suggests that pricing power is the ability for manufacturers directly affecting the other market participants and market variables, such as price and sales, so market pricing power is a kind of price bonus ability; Kaufmann [14] argues that pricing power is the technical strength which is associated with market power to some extent, that is, enterprises could obtain monopoly pricing power in the market by its unique technology or patents, thus gaining excess profits. Rubinstein [15] explains staple commodity pricing mechanism by using the bargaining model of complete information dynamic game and deems that pricing power advantage between buyers and sellers mainly depends on bargaining patience of two sides when information is complete. Wen et al. [16-19] hold that influences on market structure carried by risk preference and risk premium should be taken into account in bargaining model, for risk preference characteristics will affect pricing power by affecting market power. Some researches specific to rare earth pricing power following the above trend are carried out, such as Zhang [20] who thinks that the rare earth market belongs to a typical oligopoly market, so oligopolistic enterprise must fully consider impacts from competitors before taking any action, which proves game behavior of oligopolists on both sides existed in the pricing process of rare earth; Wang and Zhang [21] analyze the potential impact on China's rare earth export pricing power by the increase of resource tax; Wu and Jiang [22] hold that the formation of pricing power is a result of comprehensive shaping process involving many factors, such as, industry, enterprise, government, and foreign aspects, which all are passed on to the market power.

The analysis above is static interpretation of pricing power, without considering the influence on market power by psychological preferences. Based on remarkable discovery of game experiment, behavioral economics expand and correct the traditional economic theory through integrating behavioral and psychological preferences into it, especially blending social preference in game and decision-making theories. As an effective analytical tool for economic subject of cooperative game, it brings profound impact on the raise of fairness preference and application in motivation theory and industrial organizational theory, such as Rabin [23] who starts original research toward fair game equilibrium and
builds a reciprocal fairness equilibrium game model based on the framework of psychological game raised by Geanakoplos, Pearce, and Stacchetti. This model depicts reciprocal fairness motivation of players as motivation fairness utility function and then discovers a new equilibrium, that is, fairness equilibrium, which meets the Pareto optimality with cooperative equilibrium and provides a reasonable explanation for cooperative results. However, Rabin's model is difficult to predict accurately because it only aimed at games with standard form, not for dynamic game with continuous strategy structure. Dufwenberg and Kirchsteiger [24] improve Rabin's model through expanding it to a dynamic environment with continuous strategy structure, thus obtaining a more extensive application. According to fairness preference based on distribution results revealed by game experiments, Fehr and Schmidt [25] and Bolton and Ockenfels [26] develop inequity aversion model based on distribution results. It could be deduced from research achievements of reciprocal equity equilibrium theory that metal mineral resources development and compensation value are dependent on not only material benefits brought by resource development, but also psychological effect contained by reciprocal fairness belief, if only a reciprocal fairness belief of related subjects is given. Therefore, a correlation consideration is established between reciprocal equity equilibrium analysis and metal mineral resources development and compensation evaluation, consequently revealing the mechanism of pricing power affected by psychological preferences. The breakthrough of theory model in psychological preferences utility assumes the analysis of pricing power affected by market power ignoring psychological preferences, which will affect the bargaining strategy of players, change the market power and influence supply and demand prices of mineral products. For instance, Zhong et al. [27] estimate intergenerational compensation of preponderant high-tech mental mineral resources affected by altruism preference and reciprocal fairness equilibrium with Stackelberg model and points out that the development and utilization of compensation value system should include intergenerational compensation and strategic value besides economic value and ecological value.

Based on the researches above, this paper analyzes market structure of preponderant mental mineral resources such as tungsten, molybdenum, tin, antimony, and rare earth and integrates social preference into Cournot production decision model to analyze the impact exerted on market structure, production decision of developers, price, and profits by social profits, thus discovering the existence of new equilibrium, intergenerational compensation, and strategic value, which clears the agreement pricing mechanism of the metal mineral resources and reveals the pricing power routes affected by intergenerational equity and social preferences.

## 2. The Function Routes of Intergenerational Equity to Preponderant Metal Mineral Resources Pricing Power

2.1. Cournot Market Structure Analysis of Preponderant Metal Mineral Resources. Cournot, French mathematical
economist, first outlined his theory of duopoly market in 1838. In this situation, there exist two enterprises supplying homogeneous products in the market. Each enterprise could choose optimal production to maximize profits by observing others production. He then discovered that a stable equilibrium occurs where each enterprise chooses the production as their rival expected. So the model has a series of strict assumptions: the market is only dominated by two rational suppliers aiming at profit maximization; Oligarch production competition is strategic for supposing each other's output expectation function and price determined by market production; Oligarch determines their own production after prediction and assumes the output of their rival is fixed; the cost of production of oligarchs is zero and marginal cost of production is a certain constant; there is a linear demand function in the market.

Inverse linear demand function in duopoly market is assumed as follows:

$$
\begin{equation*}
p(Q)=a-b Q \tag{1}
\end{equation*}
$$

where $Q$ is the total supply of homogeneous products in duopoly market: $Q=q_{1}+q_{2}$ and $p$ is the market price. The output of oligopolist 1 is $q_{1}$, the output of oligopolist 2 is $q_{2}$, spontaneous demand is $a$, sensitivity coefficient to price of demand is $b$. The profits of oligopolists are

$$
\begin{equation*}
\pi_{i}\left(q_{i}, q_{j}\right)=p(Q) q_{i}-c_{i} q_{i}, \quad i=1,2 ; i \neq j \tag{2}
\end{equation*}
$$

The marginal cost of production of two oligopolists $c_{i}>0$ meets $a>\max \left(c_{1}, c_{2}\right), b>0$. If oligopolist has the same marginal cost of production and chooses optimal output independently, then the profits of each oligopolist are

$$
\begin{align*}
\pi_{1} & =q_{1}(p-c)=q_{1}\left(a-b\left(q_{1}+q_{2}\right)-c\right) \\
& =-b q_{1}^{2}+(a-c) q_{1}-b q_{1} q_{2}  \tag{3}\\
\pi_{2} & =q_{2}(p-c)=q_{2}\left(a-b\left(q_{1}+q_{2}\right)-c\right) \\
& =-b q_{2}^{2}+(a-c) q_{2}-b q_{1} q_{2}
\end{align*}
$$

Since oligopolists are pursuing profits maximization, first order condition is

$$
\begin{align*}
& \frac{\partial \pi_{1}}{\partial q_{1}}=-2 b q_{1}+(a-c)-b q_{2}=0 \\
& \frac{\partial \pi_{1}}{\partial q_{2}}=-2 b q_{2}+(a-c)-b q_{1}=0 \tag{4}
\end{align*}
$$

Combine the above two equations, and we could obtain equilibrium outputs and profits of oligopolists:

$$
\begin{equation*}
q_{1}=q_{2}=\frac{a-c}{3 b}, \quad \pi_{1}=\pi_{2}=\frac{(a-c)^{2}}{9 b} \tag{5}
\end{equation*}
$$

Thus we could obtain Cournot equilibrium under the condition of complete information. Equilibrium outputs of oligopolists $q_{1}$ and $q_{2}$ are optimal output assumed fixed
output of their rival, so Cournot equilibrium is a subset of Nash equilibrium.

According to market concentration $\mathrm{CR}_{2}$ and $\mathrm{CR}_{4}$ of preponderant metal mineral resources in China, these resources are monopolistic and each oligopolist according to its own profit maximization makes decision simultaneously in oligopoly market. Therefore, this paper uses Cournot game model to analyze the influence exerted on development compensation value and pricing mechanism by combinational equilibrium evaluation factors, to analyze the function routes of combinational equilibrium evaluation factors to preponderant metal mineral resources pricing power and follow the classical assumptions of Cournot equilibrium, that is, assuming oligarchs marginal production cost $c$ is equal. According to the market supply and demand situation and national industrial policy, this paper analyzes the relationship between demand and price of preponderant metal mineral resources by using regression analysis, which shows the feasibility to simulate product demand function by linear demand function in oligopoly market. Therefore, it is assumed that linear inverse demand function of metal mineral resources products is $p=a-q_{1}-q_{2}, q_{1}$ is the output of oligopolist 1 , and $q_{2}$ is the output of oligopolist 2 and satisfies the spontaneous demand of the market $a>c$, so oligarchs profits objective functions ignoring psychological preferences of players are as follows:

$$
\begin{align*}
& f_{1}\left(q_{1}, q_{2}\right)=q_{1}\left(a-q_{1}-q_{2}\right)-c q_{1},  \tag{6}\\
& f_{2}\left(q_{1}, q_{2}\right)=q_{2}\left(a-q_{1}-q_{2}\right)-c q_{2}, \tag{7}
\end{align*}
$$

where $a$ in (6) and (7) stands for spontaneous demand of metal minerals; function $f_{1}$ and $f_{2}$ stands for profit function of oligopolists, respectively.

Combine (6) with (7), we could obtain Cournot game equilibrium of each oligopolist:

$$
\begin{equation*}
\left(q_{1}, q_{2}\right)=\left(\frac{a-c}{3}, \frac{a-c}{3}\right) . \tag{8}
\end{equation*}
$$

### 2.2. Intergenerational Compensation Modification of Prepon-

 derant Metal Mineral Resources Development and Compensation. The essence of the preponderant metal mineral resources depletion compensation is value compensation to future losses aiming at excessive mining contemporarily. According to equity theory, externalities of different economic subjects could be solved through negotiations in preponderant metal mineral resources development. If there is reasonable institutional arrangements, the externalities could be internalized to a great extent. However, the externality in mineral resources development for the contemporary is better than the descendant, and because the latter is absent in game negotiation, they could not restrict behavior of the contemporary which cause asymmetry between behaviors. In order to solve the internalization of intergenerational externality problem under the condition of asymmetry, we could build the sustainable development compensation fund in the process of mineral resources development on the basis of the theory of Hotelling mineral resources depletion compensation and Howarth intergenerational property transfertheory. Sustainable development compensation fund is a cash conversion mode; if discount rate is considered, it will keep growing. If intergenerational compensation cost is $F$, time horizon for compensation is $T$, and then the intergenerational compensation fund needed is $s=F /(1+R)^{T}$ and $R$ is social discount rate.

With the development of world economy, the preponderant metal mineral resources are scarcer, and many countries are looking for a new substitute to get rid of the dependence on metal mineral resources. From the perspective of sustainable development, this research input could affect development routes and improve efficiency of preponderant metal mineral resources, to ensure the rights and interests of future generations. In consequence, research input of substitute should be regarded as part of the intergenerational development compensation value. Research input of substitute contributes to lower current consumption of metal mineral resources from the aspect of metal mineral resources recycling and extends the development and utilization period to meet the needs of metal mineral resources for both the contemporary and the descendent from the aspect of substitute researches.
2.3. The Function Routes of Intergenerational Equity Compensation to Preponderant Metal Mineral Resources Strategic Equilibrium Price. Take intergenerational equity value, that is, sustainable development of the compensation fund $s(s>$ 0 ) as intergenerational equity compensation. Metal mineral resources development and utilization cost become the combination of marginal production cost and marginal external cost, namely, $c+s$, and inverse demand function of mineral resources products in international market is $p=a-q_{1}-q_{2}$ and satisfies $a>c+s$; the objective function of each country is

$$
\begin{align*}
& f_{1}\left(q_{1}, q_{2}\right)=q_{1}\left(a-q_{1}-q_{2}-c-s\right)  \tag{9}\\
& f_{2}\left(q_{1}, q_{2}\right)=q_{2}\left(a-q_{1}-q_{2}-c-s\right) \tag{10}
\end{align*}
$$

Combined (9) with (10), we could obtain Cournot equilibrium from the intergenerational compensation perspective:

$$
\begin{equation*}
\left(q_{1}^{*}, q_{2}^{*}\right)=\left(\frac{a-c-s}{3}, \frac{a-c-s}{3}\right) \tag{11}
\end{equation*}
$$

Compare (11) to (8); it could be deduced that international trade price of preponderant metal mineral resources should be included into intergenerational compensation modification so as to show its depletion cost of metal minerals. In this way, the supply of preponderant metal mineral resources will decrease, and international trade price will increase. Besides, the greater the intergenerational equity compensation is, the higher the degree of market monopoly will be, so the initial price of metal mineral prices should be higher.

The intergenerational equity compensation of research inputs mainly considers the effects on metal mineral resources development and utilization by technical progress, which would give rise to the appearance of new substitute and affect price elasticity of demand of the replaced metal
mineral resources products. Given impacts of substitute, the demand equation of new metal mineral resources products is $p_{t}=a^{\prime}-q_{1 t}-q_{2 t}$.

The higher the metal mineral resource price is, the more obvious the substitution will be. Then, the trigger point will appear at a rather low price, that is, $a^{\prime}<a$, reaching the new equilibrium as follows:

$$
\begin{equation*}
\left(q_{1 t}^{*}, q_{2 t}^{*}\right)=\left(\frac{a^{\prime}-c}{3}, \frac{a^{\prime}-c}{3}\right) \tag{12}
\end{equation*}
$$

Compare (12) to (8); we could deduce that total market output of metal mineral resource products is smaller when intergenerational compensation cost of substitutes is considered. Besides, the more the research input of substitutes is, the higher the degree of market monopoly will be, so the initial price of metal mineral prices should be higher. There is no sustainable development compensation fund established to consider intergenerational equity compensation, and no account set up for substitutes research input in accounting system, resulting in underestimation of development compensation costs and deficiency in intrinsic value compensation. In reality, the lower metal premium on mineral resources leads to lower international trade prices. And due to low entry barriers, the development of metal mineral resources exists many problems, such as, small scale, operation chaos and overexploitation, which generate excessive competition and vicious circle to further price reduction.

## 3. The Function Routes of Social Preference to Preponderant Metal Mineral Resources Pricing Power

3.1. The Utility Function Modification in Decision Making of Preponderant Metal Mineral Resources Development. Under the imperfect competition market structure, fair price reflects not only intrinsic value compensation equity, industrial organization trade forces equity, and policies trading forces equity in oligopoly market structure of preponderant metal mineral resources, but also supply and demand of intrinsic value compensation equilibrium fluctuations caused by the above equities. From the perspective of behavioral economics, the influences exerted on shadow price and profits by fair belief of stakeholders should be considered in the trade forces equity. As to measurement of strategic value and equity level, it is advisable to learn from the establishment of social utility function. For example, in strategic production decision, if a resource developer has reciprocal preference hopes that the production of its competitor is more than equity output accepted by players, the oligopolist is willing to reduce profits of competitors by squeezing its own profits; if a resource developer has reciprocal preference hopes that the production of its competitor is less than equity output accepted by players, the oligopolist is willing to increase profits of competitors by squeezing its own profits. The profits variation above is the producer surplus variation; hence, it is possible to measure strategic value produced by psychological preferences by the variation of preponderant metal mineral
resources developers surplus caused by price variation. Under oligopoly market structure, the modification of psychological preferences to developer decision-making utility function should be under the condition of interdependence preferences; metal mineral resources developer output decisionmaking utility function included into psychological effects of social preferences is as follows:

$$
\begin{align*}
U_{i}\left(O\left(s_{i}, s_{i}^{*}\right)\right)= & \pi_{i}\left(O\left(s_{i}, s_{-i}^{*}\right)\right) \\
& +\sum_{j \neq i} w_{i j}\left(s_{i}, s_{-i}^{*}\right) \pi_{j}\left(O\left(s_{i}, s_{-i}^{*}\right)\right) \tag{13}
\end{align*}
$$

In (13), where $O\left(s_{i}, s_{i}^{*}\right)$ is the output decision under interdependence strategy, $s_{i}$ is the output strategy of oligopolist $i, s_{-i}^{*}$ is the output strategy of remaining oligopolists, $\pi_{i}$ is oligopolist i's profits without considering interdependent preferences, $\pi_{j}$ is the profit of other oligopolists without considering interdependence preferences, and $w_{i j}$ is the coefficient of strategic interaction measuring the profit that oligopolist $i$ gives to other oligopolists. Positive values of the coefficient $w_{i j}$ mean that player $i$ is willing to sacrifice his payoff from outcomes in order to increase the payoff of player $j$. Negative values mean that player $i$ is willing to sacrifice his payoff from outcomes in order to lower player $j$ 's payoff. In addition, $w_{i j}\left(s_{i}, s_{-i}^{*}\right) \pi_{j}\left(O\left(s_{i}, s_{-i}^{*}\right)\right)$ can be decided by the different types of social preferences as follows.
(1) If the oligopolist prefers altruism fairness, that is, the oligopolist considers the intertemporal allocation of preponderant mental mineral resources development and the utilization of later generations, then the oligopolists have slight altruistic preferences, and $w_{i j}$ is positive.
(2) For types of inequity averse player, $w_{i j}\left(s_{i}, s_{-i}^{*}\right) \pi_{j}\left(O\left(s_{i}, s_{-i}^{*}\right)\right) \quad$ can be replaced by $w_{i j}\left(q_{i}, Q_{-i}^{*}\right)\left(\pi_{j}-\pi_{i}\right), w_{i j}\left(q_{i}, Q_{-i}^{*}\right)$ is used to measure the deviation profit function of oligopolist $i$ puting weights on oligopolist $j$, and here is

$$
w_{i j}\left(q_{i}, Q_{-i}^{*}\right) \begin{cases}>0, & \pi_{j}<\pi_{i}  \tag{14}\\ =0, & \pi_{j}=\pi_{i} \\ <0, & \pi_{j}>\pi_{i}\end{cases}
$$

The first condition expresses aversion to advantageous inequity, namely, if oligopolist i's profits are greater than those of oligopolist $j$, then oligopolist $i$ is willing to sacrifice its own profits to increase $j$ 's profits. The third condition expresses aversion to disadvantageous inequity. If oligopolist i's profits are lower than those of oligopolist $j$, then oligopolist $i$ is willing to sacrifice its own profits to reduce $j$ 's profits.
(3) If it is the reciprocal fairness preference, the payoff function of the oligopolist $i$ is $U_{i}\left(q_{i}, Q_{-i}\right)=$ $\pi_{i}\left(q_{i}, Q_{-i}\right)+w_{i}\left(Q_{-i}, Q_{-i}^{F}\right) \sum_{j \neq i} \pi_{i}\left(q_{i}, Q_{-i}\right)$. Where $\pi_{i}\left(q_{i}, Q_{-i}\right)$ is oligopolist $i$ 's profits and is the weight that oligopolist $i$ places on its rivals gross profits, that is, $\sum_{j \neq i} \pi_{i}\left(q_{i}, Q_{-i}\right)$, and on the gross output of its rivals $Q_{-i}$, the equation is $\pi_{i}\left(q_{i}, Q_{-i}\right)=R_{i}\left(q_{i}, Q_{-i}\right)-C_{i}\left(q_{i}\right)$,
where $R_{i}\left(q_{i}, Q_{-i}\right)=P(Q) q_{i}$ is revenue. Assuming that oligopolist $i$ is endowed with the weight on its rivals depending on fair gross output $Q_{-i}^{F}$ and output of his rivals. Furthermore, it can be assumed that

$$
w_{i}\left(Q_{-i}, Q_{-i}^{F}\right) \begin{cases}>0, & Q_{-i}<Q_{-i}^{F}  \tag{15}\\ =0, & Q_{-i}=Q_{-i}^{F} \\ <0, & Q_{-i}>Q_{-i}^{F}\end{cases}
$$

That is, when $Q_{-i}<Q_{-i}^{F}$ the oligopolist $i$ has a positive weight on rivals' gross profits; when $Q_{-i}=Q_{-i}^{F}$, the weight is 0 ; and it has a negative weight on its rivals' output when $Q_{-i}>Q_{-i}^{F}$. These conditions reveal the real intention of oligopolist with reciprocal fairness preference to care rivals. The Rabin fairness equilibrium determination method used by reciprocal fairness psychological compensation value modification is that game subjects are willing to sacrifice their material interests to help people who treat them kindly and to punish people who treat them badly; the smaller the sacrifice, the greater motivation to help and punish.
3.2. The Function Routes of Social Preference to Preponderant Metal Mineral Resources Strategic Equilibrium Price. Based on the revised developers' utility function, the developers will play strategic reciprocal game on production when exploiting preponderant mental mineral resources; meanwhile they can tell the industry is oligopoly by judging from the market concentration indicators of $\mathrm{CR}_{2}$ and $\mathrm{CR}_{4}$ of preponderant metal mineral resources. Thus, developers will play oligopolistic reciprocal fairness game, and each oligopolist based on profit maximization principle to make decision simultaneously. Therefore, the Cournot game model is fit to analyze the function routes of social preferences to preponderant metal mineral resources pricing power improvement.
3.2.1. The Function Routes of Altruism Preference to Preponderant Metal Mineral Resources Strategic Equilibrium Price. Preponderant metal mineral resources development requires sustainable development, so it could be assumed that slight altruistic preference is possessed on the consideration of intergenerational equity. According to (13) and Cournot hypothesis, monopoly profit functions under altruism preference of preponderant metal mineral resources development are

$$
\begin{align*}
\pi_{1}\left(q_{1}, q_{2}, \lambda_{1}\right)= & q_{1}\left(a-q_{1}-q_{2}-c-s\right) \\
& +\lambda_{1} q_{2}\left(a-q_{1}-q_{2}-c-s\right),  \tag{16}\\
\pi_{2}\left(q_{1}, q_{2}, \lambda_{2}\right)= & q_{2}\left(a-q_{1}-q_{2}-c-s\right)  \tag{17}\\
& +\lambda_{2} q_{1}\left(a-q_{1}-q_{2}-c-s\right),
\end{align*}
$$

where $\lambda_{i}(i=1,2)$ is oligopolist $i$ 's slight altruism preference coefficient, and $\lambda_{i} \geq 0$. Altruism preference coefficients fall in the interval $(0,1)$ approximately revealed by game experiments according behavior of experimental economics and psychology, such as trust game, gift exchange game, dictator game, and market game with punishment or without
punishment. According to the optimal Cournot equilibrium analysis method, the optimal reaction function of each oligopolist could be gained from (16) and (17):

$$
\begin{align*}
& \frac{\partial \pi_{1}}{q_{1}}=\left(a-2 q_{1}-q_{2}-c-s\right)-\lambda_{1} q_{2}=0  \tag{18}\\
& \frac{\partial \pi_{2}}{q_{2}}=\left(a-2 q_{2}-q_{1}-c-s\right)-\lambda_{2} q_{1}=0 \tag{19}
\end{align*}
$$

Combine (18) with (19), we could obtain Cournot equilibrium under pure altruism preference:

$$
\begin{align*}
& \left(q_{1}^{* *}, q_{2}^{* *}\right) \\
& \quad=\left(\frac{\left(1-\lambda_{1}\right)(a-c-s)}{4-\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)}, \frac{\left(1-\lambda_{2}\right)(a-c-s)}{4-\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)}\right) \tag{20}
\end{align*}
$$

Using Cournot equilibrium output under altruism preference of oligopolist 1 minus that in (11), we could get the equation:

$$
\begin{align*}
& \frac{\left(1+\lambda_{1}\right)(a-c-s)}{4-\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)}-\frac{(a-c-s)}{3} \\
& \quad=\frac{(a-c-s)\left(-2 \lambda_{1}-\lambda_{2}+\lambda_{1} \lambda_{2}\right)}{3\left[4-\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)\right]} . \tag{21}
\end{align*}
$$

Since $\left(\lambda_{1}, \lambda_{2}\right) \rightarrow 0,(a-c-s) / 3\left[4-\left(1-\lambda_{1}\right)\left(1-\lambda_{2}\right)\right]>0$, and $\left(-2 \lambda_{1}-\lambda_{2}+\lambda_{1} \lambda_{2}\right)<0$, the Cournot equilibrium output under pure altruism condition is lower than that under no altruism condition. And the higher the degree of altruism preference is, the smaller the total market output will be, so strategic price of preponderant metal mineral resources development and compensation should be higher.
3.2.2. The Function Routes of Inequity Aversion to Preponderant Metal Mineral Resources Strategic Equilibrium Price. In the development compensation pricing of preponderant mental mineral resources, oligopolists show sympathy preference and jealousy preference on the players' payoffs, say, they will sacrifice their profits to lower those oligopolists who obtain higher profits, but also sacrifice their profits to upgrade those oligopolists who bear lower profits. According to Fehr and Schmidt's definition of inequity aversion, the payoff functions of preponderant mental mineral resources development and utilization are affected by oligopolists' inequity aversion preferences, and thus their payoff functions are

$$
\begin{align*}
\pi_{1}\left(q_{1}, q_{2}, \alpha_{1}, \beta_{1}\right)=f_{1}- & {\left[a_{1} \max \left(f_{2}-f_{1}, 0\right)\right.} \\
& \left.+\beta_{1} \max \left(f_{1}-f_{2}, 0\right)\right]  \tag{22}\\
\pi_{2}\left(q_{1}, q_{2}, \alpha_{2}, \beta_{2}\right)=f_{2}- & {\left[a_{2} \max \left(f_{1}-f_{2}, 0\right)\right.} \\
& \left.+\beta_{1} \max \left(f_{2}-f_{1}, 0\right)\right]
\end{align*}
$$

where $\alpha_{i}(i=1,2)$ is the jealousy preference coefficient under inequity aversion of oligopolist $i$ and $\beta_{i}(i=1,2)$ is the sympathy preference coefficient under inequity aversion;
moreover, $\alpha_{i}>\beta_{i}>0$. And owing to the symmetry form in oligopoly market structure, the assumption of $f_{2}>f_{1}$ will not affect analysis conclusion; thus (22) are as follows:

$$
\begin{align*}
& \pi_{1}\left(q_{1}, q_{2}, \alpha_{1}\right)=q_{1}\left(a-q_{1}-q_{2}-c-s\right) \\
& \\
& -\alpha_{1}\left[q_{2}\left(a-q_{1}-q_{2}-c-s\right)\right. \\
&  \tag{23}\\
& \left.\quad-q_{1}\left(a-q_{1}-q_{2}-c-s\right)\right] \\
& \pi_{2}\left(q_{1}, q_{2}, \alpha_{2}\right)=q_{2}\left(a-q_{1}-q_{2}-c-s\right) \\
& -
\end{aligned} \begin{aligned}
& {\left[q_{2}\left(a-q_{1}-q_{2}-c-s\right)\right.} \\
& \left.-q_{1}\left(a-q_{1}-q_{2}-c-s\right)\right]
\end{align*}
$$

where $\alpha_{i}(i=1,2)$ is the jealousy preference coefficient under inequity aversion of oligopolist $i$ and $\beta_{i}(i=1,2)$ is the sympathy preference coefficient under inequity aversion. The second items on the right side in (23) are disutility produced by oligopolist $i$ 's jealousy preference. According to the optimal Cournot equilibrium analysis, the optimal response function of each oligopolist derived from revenue functions under inequity aversion is as follows:

$$
\begin{align*}
& \frac{\partial \pi_{1}}{q_{1}}=\left(a-2 q_{1}-c-s\right)\left(1+\alpha_{1}\right)-q_{2}=0,  \tag{24}\\
& \frac{\partial \pi_{2}}{q_{2}}=\left(a-2 q_{2}-c-s\right)\left(1-\beta_{2}\right)-q_{1}=0 . \tag{25}
\end{align*}
$$

Combine (24) with (25); the optimal production of each oligopolist preferring inequity aversion can be finally written as

$$
\begin{align*}
\left(q_{1}^{* * *}, q_{2}^{* * *}\right)= & \left(\frac{\left(1+2 \alpha_{1}\right)\left(1-\beta_{2}\right)(a-c-s)}{4\left(1+\alpha_{1}\right)\left(1-\beta_{2}\right)-1}\right.  \tag{26}\\
& \left.\frac{\left(1+\alpha_{1}\right)\left(1-2 \beta_{2}\right)(a-c-s)}{4\left(1+\alpha_{1}\right)\left(1-\beta_{2}\right)-1}\right)
\end{align*}
$$

The fairness equilibrium output function exhibits that under piecewise linear inequity aversion condition, the optimal response function of oligopolist and standard Cournot equilibrium game are both continuous, but the former is no longer monotonous.

Using Cournot equilibrium output in (26) minus that in (11), we could get the equation:

$$
\begin{align*}
& q_{1}^{* * *}-q_{1}^{* *}=\frac{2 a_{1}+\beta_{2}-2 a_{1} \beta_{2}}{12\left(1+\alpha_{1}\right)\left(1-\beta_{2}\right)-3}(a-c-s)  \tag{27}\\
& q_{2}^{* * *}-q_{2}^{* *}=\frac{-a_{1}-2 \beta_{2}-2 a_{1} \beta_{2}}{12\left(1+\alpha_{1}\right)\left(1-\beta_{2}\right)-3}(a-c-s) \tag{28}
\end{align*}
$$

The result of (28) is obviously less than zero. After thousands of game experiments, such as ultimatum game, dictator game, and public good games in different countries, it proves that about $85 \%$ of the people's $\alpha_{1}$ and $\beta_{1}$ fall in the interval $(0.15,0.50)$, so the result of (27) is more than zero. Equation (27) shows that Cournot equilibrium output preferring fairness is more than that when they are only concerned
about their own enterprise profits, while (28) shows that Cournot equilibrium output preferring fairness is less than that when they are only concerned about their own enterprise profits. Such results indicate the effects of sympathy and jealousy preference on the Cournot equilibrium, and weak complementary between degree of oligopolist's sympathy and equilibrium output, that is, when jealousy preferences is greater, the optimal Nash equilibrium market output under inequity aversion in Cournot game will be larger. Under the circumstances, the improvement of jealousy preferences reduces the producer surplus and increases consumer surplus. On the other side, the minimal Nash equilibrium market output under piecewise linear inequity aversion in Cournot game will decrease when sympathy preferences are greater. In this case, the improvement of sympathy preferences increases the producer surplus and reduces consumer surplus. The variation of producer surplus is greater than that of consumer surplus after considering fairness preference, thus producing strategic reciprocal value for oligopolists. Therefore, it is necessary to take compensation of strategic reciprocal value in price system into account in the pricing process of preponderant metal mineral resources.
3.2.3. The Function Routes of Reciprocal Equity Equilibrium to Preponderant Metal Mineral Resources Strategic Equilibrium Price. Under the condition of intergenerational equity equilibrium and according to the definition of reciprocal equity equilibrium, the players' revenue function of preponderant metal mineral resources development is

$$
\begin{align*}
\pi_{1}\left(q_{1}, q_{2}\right) & =q_{1}\left(a-q_{1}-q_{2}-c-s\right) \\
\pi_{2}\left(q_{1}, q_{2}\right) & =q_{2}\left(a-q_{1}-q_{2}-c-s\right)  \tag{29}\\
\pi_{1}^{h}\left(q_{1}\right) & =q_{1}\left(a-q_{1}-c-s\right) \\
\pi_{1}^{e}\left(q_{1}\right) & =\frac{q_{1}\left(a-q_{1}-c-s\right)}{2}  \tag{30}\\
\pi_{2}^{h}\left(q_{2}\right) & =q_{2}\left(a-q_{2}-c-s\right) \\
\pi_{1}^{e}\left(q_{2}\right) & =\frac{q_{2}\left(a-q_{2}-c-s\right)}{2} \tag{31}
\end{align*}
$$

where $\pi_{1}^{l}\left(q_{1}\right)=0, \pi_{1}^{\min }\left(q_{1}\right)=0, \pi_{2}^{l}\left(q_{2}\right)=0, \pi_{2}^{\min }\left(q_{2}\right)=0$. According to the definition of reciprocal equity equilibrium, in the transaction of preponderant metal mineral resources, the friendliness function between oligopolist 1 and oligopolist 2 is

$$
\begin{align*}
& f_{1}\left(q_{1}, q_{2}\right)=\frac{1}{2}-\frac{q_{1}}{a-q_{2}-c-s} \\
& f_{2}\left(q_{1}, q_{2}\right)=\frac{1}{2}-\frac{q_{2}}{a-q_{1}-c-s} \tag{32}
\end{align*}
$$

Friendliness belief between oligopolist 1 and oligopolist 2 is

$$
\begin{align*}
& \tilde{f}_{2}\left(\widetilde{q}_{1}, q_{2}\right)=\frac{1}{2}-\frac{q_{2}}{a-\tilde{q}_{1}-c-s} \\
& \tilde{f}_{1}\left(q_{1}, \tilde{q}_{2}\right)=\frac{1}{2}-\frac{q_{1}}{a-\tilde{q}_{2}-c-s} \tag{33}
\end{align*}
$$

According to the definition of (29) and (33), the utility functions of different players in preponderant metal mineral resources development are

$$
\begin{align*}
U_{1}\left(q_{1}, q_{2}, \tilde{q}_{1}\right)= & \pi_{1}\left(q_{1}, q_{2}\right)+\tilde{f}_{2}\left(\tilde{q}_{1}, q_{2}\right)+\left[1+f_{1}\left(q_{1}, q_{2}\right)\right] \\
= & q_{1}\left(a-q_{1}-q_{2}-c-s\right) \\
& +\left[\frac{1}{2}-\frac{q_{2}}{a-\tilde{q}_{1}-c-s}\right] \\
& \times\left[\frac{3}{2}-\frac{q_{1}}{a-q_{2}-c-s}\right]  \tag{34}\\
U_{1}\left(q_{1}, q_{2}, \widetilde{q}_{1}\right)= & \pi_{2}\left(q_{1}, q_{2}\right)+\tilde{f}_{1}\left(q_{1}, \tilde{q}_{2}\right)\left[1+f_{1}\left(q_{1}, q_{2}\right)\right] \\
= & q_{2}\left(a-q_{1}-q_{2}-c-s\right) \\
& +\left[\frac{1}{2}-\frac{q_{1}}{a-\tilde{q}_{2}-c-s}\right] \\
& \times\left[\frac{3}{2}-\frac{q_{2}}{a-q_{1}-c-s}\right] \tag{35}
\end{align*}
$$

Combine (34) with (35) to get the first-order optimal solution, thus obtaining Cournot equilibrium solution under reciprocal equity equilibrium:

$$
\begin{align*}
q_{1}^{* * * *}= & \frac{1}{2}\left(\frac{4 a-4 c-4 s}{3}-\frac{\sqrt{3+(a-c-s)^{2}}}{3}\right. \\
& \left.-\frac{2(a c+a s) \sqrt{3+(a-c-s)^{2}}}{9}\right) \\
& +\frac{1}{2}\left(\frac{(c+s)^{2} \sqrt{3+(a-c-s)^{2}}}{9}\right)  \tag{36}\\
& \left.+\frac{a^{2} \sqrt{3+(a-c-s)^{2}}}{9}\right) \\
& \left.-\frac{\left[3+(a-c-s)^{2}\right]^{3 / 2}}{9}\right) \\
q_{2}^{* * *}= & \frac{1}{3}\left(2 a-2 c-2 s-\sqrt{3+(a-c-s)^{2}}\right)
\end{align*}
$$

It could be seen in (30) that $q_{2}^{* * * *}<(1 / 3)(2 a-2 c-2 s-$ $\left.\sqrt{(a-c-s)^{2}}\right)=(a-c-s) / 3=q_{2}^{* *}$; similarly, $q_{1}^{* * * *}<q_{1}^{* *}$. It suggests that after considering reciprocal equity preferences, the total market output of preponderant metal mineral resources is reduced and the degree of market monopoly is higher. The demand price elasticity of preponderant metal mineral resources is rather small for the reason that they are industrial raw materials and hard to be replaced. Given reciprocal equity equilibrium, the market capacity is decreased and then caused higher prices. The greater producer surplus is strategic reciprocal value produced by reciprocal equity preference, which should be included in compensation value system. If oligopolist 1 and oligopolist 2 are present two countries, it is observed that two countries having reciprocal equity intention could enhance their metal mineral resources market monopoly status, thereby obtaining pricing power for themselves. It also explains the reason why China should obey reciprocal equality principles in international trade.

## 4. Simulation Analysis

As has been observed from (20), (26), and (36), oligarchs' social preferences in preponderant metal market, that is, altruism preference, inequity aversion, and reciprocal fairness, produce psychological effects, which raise the power of oligarch market and change output decision (reducing production) of each oligarch by being blend in decision function. Therefore, market capacity is decreased and price of supply and demand is increased. Considering the significance of preponderant metals, their demand is rigid, so developers has larger producer surplus, which is strategic reciprocal value produced by reciprocal fairness. However, it can be seen from equilibrium results of the above four equations that forms of strategic value are various. So the method of numerical simulation is used to verify as follows.
4.1. Original Basic Parameter Setting. According to the market supply and demand of lithium, antimony, indium, rare earth, and the national industrial policy as well as the regression analysis of the Cournot linear demand function, the spontaneous demand stabilized at around 2,000 tons, so $a$ in the Cournot model can take the value 2000, namely, $a=2000$, by analyzing the tax subjects of preponderant mental mineral resources development, namely, the property cost (mineral resources compensation, resource tax), mining costs (outlay of exploration, outlay of mining), investment capital (capital investment per ton of mineral resources), production costs (raw materials, power costs, wages and benefits, manufacturing costs, processing fees, finance charges, and operating expenses), security costs (safety training, disinfection equipment, risk assessment costs, occupational funds, and pension), and part of the measurable environmental governance operating costs (water pollution, air pollution, waste pollution, and heavy metal pollution), and environmental restoration costs (mine land reclamation bond, tailings management costs, and mine environmental geology warning inputs). Based on tax subjects above and the statistical analysis of preponderant


Figure 1: The sensitivity analysis of Cournot equilibrium output to altruism preference coefficient.
metal development compensation enterprises, the basic cost c of preponderant mental mineral resources development compensation enterprises is about 800 units; original basic parameter setting in Section 4.1 is obtained by regression estimation in last 10 years, so it conforms to the market situation. Thus the results of numerical simulation could support the application of practical program. The sustainable development fund is used to measure modification of intergenerational compensation, estimates depletion costs of rare earth, lithium, and indium by modified user cost approach and gains the proportion of depletion cost to total cost is $20 \%$. The depletion cost is the reflection of intergenerational compensation modification actually, so the value of sustainable development fund $s$ of intergeneration equity is 160 units.

### 4.2. Impacts Exerted on Cournot Equilibrium by Coefficients

 of Altruism Preference and Inequity Aversion. Reciprocal equity Cournot equilibrium variation under reciprocal equity preference has the consistent results with classic Cournot equilibrium, so numerical simulation is not needed to analyze its character. However, Cournot equilibrium under social preferences produces a new equilibrium bringing about a new character, on account of the variation of altruism preferences and the loss aversion of inequity aversion; thus, numerical simulation is required. Altruism preference coefficients fall in the interval $(0,1)$ revealed by game experiments and modified game experiments of altruism preference and inequity aversion preference, such as, ultimatum game, dictator game, public good games, gift exchange game, and third-party punishment game. While the distribution of inequity aversion coefficients has the following features, as is shown in Table 1.
### 4.2.1. Simulation Results of Impact Exerted on Cournot Equi-

 librium by Altruism Preference. Altruism preference coefficients could be obtained by game experiments and impacts exerted on Cournot equilibrium decision-making by altruism preference could be simulated by (20) and (21). The specific content is shown in Figure 1.As can be seen from Figure 1, the Cournot equilibrium output under altruism preference modified by intergenerational equity is changing as follows. In a certain condition, the altruism degree of oligopolist 2 is in proportion to that of oligopolist 1, namely, the bigger the altruism coefficient of oligopolist 2 is, the smaller the Cournot equilibrium output of oligopolist 1 is, and vice versa. This result is in accordance with the experiment result of pure altruism preference. According to Cournot equilibrium and analysis framework of supply and demand, the reduction of Cournot equilibrium output will raise price in preponderant mental market, while to those preponderant mental resources lacking price elasticity of demand, price increase will lead to producer surplus increase, which is the strategic value of development and compensation of preponderant metal resources.
4.2.2. Simulation Results of Impact Exerted on Cournot Equilibrium by Inequity Aversion. Inequity aversion coefficients could be obtained by game experiments and impacts exerted on Cournot equilibrium decision-making by inequity aversion which could be simulated by (26) and (28). The specific content is shown in Figure 2.

As can be seen from Figure 2, Cournot equilibrium output under intergenerational equity correction when inequity aversion is considered is smaller than that when they are not considered. The oligopolists under inequity aversion lift the degree of market monopoly of preponderant metal mineral resources. Besides, the higher the degree of jealousy preference is, the larger the Cournot equilibrium output will be, while as to sympathy preference, the result is completely opposite. These results are consistent with the results of theoretical model; namely, Cournot equilibrium under inequity aversion has the character of loss aversion.

## 5. Conclusions

This paper analyzes the modification of intergenerational equity and social preferences to fiducial value of preponderant metal mineral resources and qualifies the impacts exerted on Cournot equilibrium by interdependence preferences


Figure 2: The sensitivity analysis of Cournot equilibrium output to inequity aversion preference.

Table 1: The distribution of jealousy and sympathy preferences coefficients $\alpha$ and $\beta$ under inequity aversion.

|  | Value and proportion of $\alpha$ |  | Value and proportion of $\beta$ |
| :--- | :---: | :---: | :---: |
| Value of $\alpha$ | Proportion (\%) | Value of $\beta$ | Proportion (\%) |
| $\alpha=0$ | 20 | $\beta=0$ | 30 |
| $\alpha=0.5$ | 65 | $\beta=0.25$ | 45 |
| $\alpha=1$ | 10 | $\beta=0.6$ | 10 |
| $\alpha=4$ | 5 | $\beta=0.6$ | 5 |

(altruism preference, inequity aversion, and reciprocal equity preference), thus drawing the following conclusions.
(1) Considering the intergenerational equity and social preferences in preponderant metal mineral resources development, Cournot equilibrium market capacity becomes smaller. Because preponderant metal mineral resources are significant to national security and industrial development and hard to replace, the demand of those prices is rigid. Therefore, when Cournot equilibrium decreases, oligopolist's profits increase; that is, oligopolist's producer surplus increases. The variation of producer surplus is intergenerational equity compensation value and strategic value when intergenerational equity and social preferences blend into Cournot game.
(2) However, the impact exerted on Cournot market power by social preferences is inconsistent. Variation of altruism Cournot equilibrium and reciprocal equity Cournot equilibrium are consistent, while Cournot equilibrium under inequity aversion has the characteristic of loss aversion, namely, under the consideration of inequity aversion Cournot competition, Cournot Nash equilibrium transform monotonically with sympathy and jealousy inequity aversion; that is, if the jealousy degree of oligopoly increased, Cournot Nash equilibrium is near the perfect competition output equilibrium; if the sympathy degree of oligopoly increased, Cournot Nash equilibrium is near the optimal collusion equilibrium output.
The results indicate that the essence of price distortion of preponderant metal mineral resources is incomplete value
realization and resource value compensation inequity, failing to realize the goal of mineral resources price reform, namely, two basic conditions for the reform are not satisfied. The production value of mineral resources is achieved through spontaneous effect of the market for its relevance to efficiency. However, the property value, intragenerational value, and intergenerational value of mineral resources are difficult to realize spontaneously on account of their public characteristic under market effect. It is difficult for marketoriented reform of mineral resources price to fully realize the mineral resources value and provide a fundamental guarantee for sufficient and reasonable compensation. Therefore, to achieve the goal of mineral resources price reform, it is necessary to reconstruct value compensation system of metal mineral resources development. According to redefinition of preponderant metal mineral resources development under the principle of multiple equilibrium value evaluation, the actual negotiated pricing mechanism is classified based on mineral resources pricing mechanism of multiple equilibrium evaluation models; that is, pricing should fully reflect the complete elements of mineral resources and coordinate interests between players. As to practical operation scheme of current resources tax reform, value measurement of preponderant metal mineral resources development compensation should analyze from not only development and utilization results but also the perspective of strategic reciprocal psychology. Besides, the value system of preponderant metal mineral resources development compensation contains economic value and ecological value as well as strategic value. Furthermore, since social preferences are added into the value system of preponderant metal mineral resources development compensation, the market monopoly degree
will be strengthened and development compensation price will be higher, which require perfect tax subjects and establish full cost theory system, thus estimating mineral resources value correctly and making the international trade fair prices tend to rationalization.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Solvability for a Fractional Order Three-Point Boundary Value System at Resonance 

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A class of fractional order three-point boundary value system with resonance is investigated in this paper. Using some techniques of inequalities, a completely new method is incorporated. We transform the problem into an integral equation with a pair of undetermined parameters. The topological degree theory is applied to determine the particular value of the parameters so that the system has a solution.

## 1. Introduction

In this paper, we consider the following fractional differential system:

$$
\begin{gather*}
D_{0+}^{\alpha} X(t)+f(t, X(t))=0, \\
0<t<1, \quad X=\left(x_{1}(t), x_{2}(t)\right), \\
X(0)=Y(0)=0, \quad X(1)=\frac{1}{\eta^{\alpha-1}} X(\eta), \quad 0<\eta<1, \tag{1}
\end{gather*}
$$

where $D_{0+}^{\alpha}$ is standard Riemann-Liouville fractional derivative of order $1<\alpha \leq 2,0<\eta<1$ and $f=\left(f_{1}, f_{2}\right)$ is a nonlinear two-dimension continuous vector function.

In the last few decades, many authors have focused on the dynamics of differential equations [1-7]; most of them have investigated fractional differential equations which have been applied in many fields such as physics, mechanics, chemistry, and engineering; see [8-13]. In particular, the positive solutions of the boundary value problem have attracted many authors' attention [14-25].

Recently, the existence of solutions of three-point boundary value problem

$$
\begin{gather*}
D_{0+}^{\alpha} u(t)+f(t, u(t))=0, \quad 0<t<1, \\
u(0)=0, \quad u(1)=\beta u(\eta), \quad 0<\eta<1, \tag{2}
\end{gather*}
$$

where $D_{0+}^{\alpha}$ is standard Riemann-Liouville fractional derivative of order $1<\alpha \leq 2$ has been studied by many authors under the case that $\beta \eta<1$. They obtained some nice results by using some fixed point theorems; see [26-28].

In [29], Ahmad and Nieto considered the existence results for following three-point boundary value problem for a coupled system of nonlinear fractional differential equations given by

$$
\begin{gather*}
D_{0+}^{\alpha} u(t)=f\left(t, v(t), D_{0+}^{p} v(t)\right)=0, \quad 0<t<1, \\
D_{0+}^{\beta} v(t)=g\left(t, u(t), D_{0+}^{q} u(t)\right)=0, \quad 0<t<1,  \tag{3}\\
u(0)=0, \quad u(1)=\gamma u(\eta), \\
v(0)=0, \quad v(1)=\gamma v(\eta),
\end{gather*}
$$

where $1<\alpha, \beta<2, p, q, \gamma>0,0<\eta<1, \alpha-q \geq 1, \beta-p \geq$ 1, $\gamma \eta^{\alpha-1}<1, \gamma \eta^{\beta-1}<1, D_{0+}^{\alpha}$ is standard Riemann-Liouville fractional derivative and $f, g:[0,1] \times R \times R \rightarrow R$ are given continuous functions. An existence result was proved in their paper by applying the Schauder fixed point theorem.

However, few authors have investigated fractional differential boundary value problems with resonance [1, 2, 30-32].

In this paper, we establish some sufficient conditions for the existence of the boundary value system (1) by using intermediate value theorems. To present the main results,
we assume that $f(t, X)=\left(f_{1}(t, X), f_{2}(t, X)\right)$ satisfies the following.
(H) $f(t, X) \in C([0,1] \times R \times R, R \times R), X=\left(x_{1}, x_{2}\right) \in$ $R \times R$. Suppose that there exist nonnegative functions $a_{i}(t), b_{i j}(t)(i, j=1,2)$, with $b_{11}(t)>0, b_{22}(t)>0$, $b_{i j}(t)(i, j=1,2, i \neq j) \leq b_{11}(t), b_{22}(t)$ for any $t \in[0,1]$ such that

$$
\begin{equation*}
\left|f_{i}\left(t, t^{\alpha-1} X\right)\right| \leq a_{i}(t)+b_{i 1}(t)\left|x_{1}\right|^{p_{i}}+b_{i 2}(t)\left|x_{2}\right|^{q_{i}}, \quad i=1,2, \tag{4}
\end{equation*}
$$

where $0 \leq p_{i}, q_{i} \leq 1(i=1,2), q_{1}<p_{1}$, and $p_{2}<q_{2}$. For any real numbers $a$ and $b$, the functions $f_{i}\left(t, t^{\alpha-1}(u, v)\right)(i=1,2)$ satisfy

$$
\begin{gathered}
\lim _{v \rightarrow+\infty} f_{1}\left(t, t^{\alpha-1}(u, v)\right)>-\infty, \\
\lim _{v \rightarrow-\infty} f_{1}\left(t, t^{\alpha-1}(u, v)\right)<+\infty, \\
\quad \text { for any } u \in R, \quad t \in(0,1],
\end{gathered}
$$

$$
\begin{aligned}
& \lim _{v \rightarrow+\infty} f_{2}\left(t, t^{\alpha-1}(u, v)\right)>-\infty, \\
& \lim _{v \rightarrow-\infty} f_{2}\left(t, t^{\alpha-1}(u, v)\right)<+\infty,
\end{aligned}
$$

$$
\text { for any } v \in R, \quad t \in(0,1] .
$$

Furthermore, assume that

$$
\lim _{v \rightarrow+\infty} f_{1}\left(t, t^{\alpha-1}(v, u(v))\right)=+\infty
$$

$$
\text { for any } u(v) \geq-|v|, \quad t \in(0,1],
$$

$$
\lim _{v \rightarrow-\infty} f_{1}\left(t, t^{\alpha-1}(v, u(v))\right)=-\infty
$$

$$
\text { for any } u(v) \leq|v|, \quad t \in(0,1],
$$

$$
\lim _{v \rightarrow+\infty} f_{2}\left(t, t^{\alpha-1}(u(v), v)\right)=+\infty
$$

$$
\text { for any } u(v) \geq-|v|, \quad t \in(0,1]
$$

$$
\lim _{v \rightarrow-\infty} f_{2}\left(t, t^{\alpha-1}(u(v), v)\right)=-\infty
$$

$$
\text { for any } u(v) \leq|v|, \quad t \in(0,1] .
$$

We have the following results.
Theorem 1. Assume that $(H)$ holds. If

$$
\begin{equation*}
\max _{1 \leq i \leq 2}\left\{\int_{0}^{1} G^{*}(s, s)\left(b_{i 1}(s)+b_{i 2}(s)\right) d s\right\}<1, \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
G^{*}(s, s)= & \frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
& \times \begin{cases}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}, & 0 \leq s \leq \eta \\
(1-s)^{\alpha-1}, & \eta \leq s \leq 1,\end{cases} \tag{10}
\end{align*}
$$

then (1) has at least one solution in $[0,1]$.

Also, we consider the following special case of $(H)$ as follows.
$(\widetilde{H}) f(t, X) \in C([0,1] \times R \times R, R \times R), X=\left(x_{1}, x_{2}\right) \in$ $R \times R$. Suppose that there exist nonnegative functions $a_{i}(t), b_{i j}(t)(i, j=1,2)$, with $b_{11}(t)>0, b_{22}(t)>0$, $b_{i j}(t)(i, j=1,2, i \neq j) \leq b_{11}(t), b_{22}(t)$ for any $t \in[0,1]$ such that
$\left|f_{i}\left(t, t^{\alpha-1} X\right)\right| \leq a_{i}(t)+b_{i 1}(t)\left|x_{1}\right|^{p_{i}}+b_{i 2}(t)\left|x_{2}\right|^{q_{i}}, \quad i=1,2$,
where $0 \leq p_{i}, q_{i} \leq 1(i=1,2), q_{1}<p_{1}$, and $p_{2}<q_{2}$. The functions $f_{i}\left(t, t^{\alpha-1}(u, v)\right)(i=1,2)$ satisfy
$\lim _{v \rightarrow \pm \infty}\left|f_{1}\left(t, t^{\alpha-1}(u, v)\right)\right|<\infty \quad$ for any $u \in R, t \in(0,1]$,
$\lim _{u \rightarrow \pm \infty}\left|f_{2}\left(t, t^{\alpha-1}(u, v)\right)\right|<\infty \quad$ for any $v \in R, t \in(0,1]$.

Furthermore, assume that (7) and (8) hold.
From Theorem 1, we have the following corollary.
Corollary 2. Assume that ( $\widetilde{H})$ and (9) hold; then (1) has at least one solution in $[0,1]$.

## 2. Some Lemmas

In this section, we first introduce some definitions and preliminary facts and some lemmas which will be used in this paper.

Definition 3 (see [21]). The fractional integral of order $\alpha>0$ of a function $y:(0, \infty) \rightarrow R$ is given by

$$
\begin{equation*}
I_{0+}^{\alpha} y(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} y(s) d s \tag{14}
\end{equation*}
$$

provided that the right integral converges.
Definition 4 (see [21]). The standard Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $y:(0, \infty) \rightarrow R$ is given by

$$
\begin{equation*}
D_{0+}^{\alpha} y(t)=\frac{1}{\Gamma(n-\alpha)} \frac{d^{n}}{d t^{n}} \int_{0}^{t}(t-s)^{n-\alpha-1} y(s) d s \tag{15}
\end{equation*}
$$

where $n=[\alpha]+1$, provided that the right integral converges.
Lemma 5 (see [21]). Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha>0$ that belongs to $C(0,1) \cap$ $L(0,1)$.Then

$$
\begin{equation*}
I_{0+}^{\alpha} D_{0+}^{\alpha} y(t)=y(t)+C_{1} t^{\alpha-1}+C_{2} t^{\alpha-2}+\cdots+C_{n} t^{\alpha-n}, \tag{16}
\end{equation*}
$$

for some $C_{i} \in R, i=1,2, \ldots, n$, where $n$ is the smallest integer greater than or equal to $\alpha$.

The following lemma is a fixed point theorem in a particular Banach space:

$$
\begin{equation*}
\Omega=\{(x(t), y(t)) \mid x(t), y(t) \in C([0,1], R)\} \tag{17}
\end{equation*}
$$

equipped with the norm

$$
\begin{equation*}
\|(x(t), y(t))\|=\max \left\{\max _{t \in[0,1]}|x(t)|, \max _{t \in[0,1]}|y(t)|\right\} \tag{18}
\end{equation*}
$$

It is easy to show that if $X(t) \in \Omega$, then $t^{\alpha-1} X(t) \in \Omega$.
Lemma 6 (see [33]). Let X be a Banach space with C $\subset X$ closed and convex. Assume that $U$ is a relatively open subset of $C$ with $0 \in U$ and $T: \bar{U} \rightarrow C$ is completely continuous. Then either
(i) T has a fixed point in $\bar{U}$, or
(ii) there exist an $u \in \partial U$ and $\gamma \in(0,1)$ with $u=\gamma T u$.

To use this lemma to prove our main result, we need transfer (1) into an integral operator.

Lemma 7 (see [34]). Problem (1) is equivalent to the following integral equation:

$$
\begin{equation*}
X(t)=\int_{0}^{1} G(t, s) f(s, X(s)) d s+X(1) t^{\alpha-1} \tag{19}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, s) \\
& \quad=\left\{\begin{array}{c}
\left(t^{\alpha-1}(1-s)^{\alpha-1}-t^{\alpha-1}(\eta-s)^{\alpha-1}\right. \\
\left.-\left(1-\eta^{\alpha-1}\right)(t-s)^{\alpha-1}\right) \times\left(\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)\right)^{-1}, \\
0 \leq s \leq \min \{t, \eta\} \leq 1 ; \\
\frac{t^{\alpha-1}(1-s)^{\alpha-1}-t^{\alpha-1}(\eta-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
0 \leq t \leq s \leq \eta \leq 1 ; \\
\frac{t^{\alpha-1}(1-s)^{\alpha-1}-\left(1-\eta^{\alpha-1}\right)(t-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
0 \leq \eta \leq s \leq t \leq 1 ; \\
\frac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, \quad 0 \leq \max \{t, \eta\} \leq s \leq 1 .
\end{array}\right. \tag{20}
\end{align*}
$$

Lemma 8 (see [34]). For any $(t, s) \in[0,1] \times[0,1], G(t, s)$ is continuous, and $G(t, s)>0$ for any $(t, s) \in(0,1) \times(0,1)$.

Let

$$
\begin{equation*}
G(t, s)=t^{\alpha-1} G^{*}(t, s) \tag{21}
\end{equation*}
$$

where

$$
\begin{align*}
& G^{*}(t, s) \\
& \quad=\left\{\begin{array}{l}
\frac{(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}-\left(1-\eta^{\alpha-1}\right)(1-s / t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
0 \leq s \leq \min \{t, \eta\} \leq 1 ; \\
\frac{(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, \quad 0 \leq t \leq s \leq \eta \leq 1 ; \\
\frac{(1-s)^{\alpha-1}-\left(1-\eta^{\alpha-1}\right)(1-s / t)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
0 \leq \eta \leq s \leq t \leq 1 ; \\
\frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}, \quad 0 \leq \max \{t, \eta\} \leq s \leq 1
\end{array}\right. \tag{22}
\end{align*}
$$

Then (1) is equivalent to the following integral equation:

$$
\begin{equation*}
X(t)=\int_{0}^{1} t^{\alpha-1} G^{*}(t, s) f(s, X(s)) d s+X(1) t^{\alpha-1} \tag{23}
\end{equation*}
$$

The new Green's function $G^{*}(t, s)$ has the following properties.

Lemma 9 (see [34]). $G^{*}(t, s)$ is continuous for $(t, s) \in(0,1) \times$ $(0,1)$ and

$$
\begin{align*}
& \lim _{t \rightarrow 0^{+}} G^{*}(t, s) \\
&:=G^{*}(0, s) \\
&=\left\{\begin{array}{cc}
\frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
\times\left\{(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}\right\}, & 0 \leq s \leq \eta ; \\
\frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)}(1-s)^{\alpha-1}, & \eta \leq s \leq 1
\end{array}\right. \tag{24}
\end{align*}
$$

Furthermore, $G^{*}(t, s)>0$ for $(t, s) \in(0,1) \times(0,1)$.
Lemma 10 (see [34]). $G^{*}(t, s)$ is nonincreasing with respect to $t \in[0,1]$ for any $s \in(0,1)$. In particular, for any $s \in[0,1]$, $\partial G^{*}(t, s) / \partial t \leq 0$, and $\partial G^{*}(t, s) / \partial t=0$ for $t \in[0, s]$. That is, $G^{*}(1, s) \leq G^{*}(t, s) \leq G^{*}(s, s)$, where

$$
\begin{align*}
G^{*}(1, s)= & \frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
& \times \begin{cases}\eta^{\alpha-1}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}, & 0 \leq s \leq \eta ; \\
\eta^{\alpha-1}(1-s)^{\alpha-1}, & \eta \leq s \leq 1,\end{cases} \\
G^{*}(s, s)= & \frac{1}{\Gamma(\alpha)\left(1-\eta^{\alpha-1}\right)} \\
& \times \begin{cases}(1-s)^{\alpha-1}-(\eta-s)^{\alpha-1}, & 0 \leq s \leq \eta ; \\
(1-s)^{\alpha-1}, & \eta \leq s \leq 1 .\end{cases} \tag{25}
\end{align*}
$$

Let

$$
\begin{equation*}
X(t)=t^{\alpha-1} Y(t) \tag{26}
\end{equation*}
$$

Then $X(1)=Y(1)$, and (23) gives

$$
\begin{equation*}
Y(t)=\int_{0}^{1} G^{*}(t, s) f\left(s, s^{\alpha-1} Y(s)\right) d s+Y(1) \tag{27}
\end{equation*}
$$

Let

$$
\begin{equation*}
W(t)=Y(t)-Y(1) . \tag{28}
\end{equation*}
$$

Then $Y(t)=W(t)+Y(1)$, and $W(1)=Y(1)-Y(1)=0$. From (27), (28) can be rewritten as

$$
\begin{equation*}
W(t)=\int_{0}^{1} G^{*}(t, s) f\left(s, s^{\alpha-1}(W(s)+Y(1))\right) d s \tag{29}
\end{equation*}
$$

with $W(1)=0$. Now the integral equation (27) is equivalent to (29). It can be seen from (29) that the solution $W(t)$ of (29) is dependent on the value $Y(1)$. Now, instead of (29), we replace $Y(1)$ with a real vector $\kappa=(\mu, \nu)$ and consider

$$
\begin{equation*}
W(t)=\int_{0}^{1} G^{*}(t, s) f\left(s, s^{\alpha-1}(W(s)+\kappa)\right) d s \tag{30}
\end{equation*}
$$

For any $\kappa=(\mu, \nu)$, let

$$
\begin{equation*}
K=\left\{W(t)=\left(w_{1}(t), w_{2}(t)\right) \in \Omega\right\}, \tag{31}
\end{equation*}
$$

equipped with the norm $\|W(t)\|=\max \left\{\max _{t \in[0,1]} w_{1}(t)\right.$, $\left.\max _{t \in[0,1]} w_{2}(t)\right\}$. Define an operator $T$ in $K$ as follows:

$$
\begin{equation*}
T W(t)=\int_{0}^{1} G^{*}(t, s) f\left(s, s^{\alpha-1}(W(s)+\kappa)\right) d s \tag{32}
\end{equation*}
$$

Using a similar method of Lemmas 3.5 and 3.6 in [34], we obtain that $T$ is completely continuous in $K$, and (30) has at least a solution $W(t)$ for any given real constant vector $\kappa$; the solution $W(t)$ is dependent on the given vector $\kappa$. We note the solution $W(t):=W_{\kappa}(t)$.

## 3. The Proof of Theorem 1

From Lemma 10, for any real vector $\kappa$, the integral equation (30) has at least a solution $W(t)$. Therefore, to show that problem (1) has a solution, it remains to show that there exists a $\kappa=(\mu, \nu)$, such that $W(1)=0$, or $Y(1)=\kappa=(\mu, \nu)$.

In what follows, we will use the method of topological degree to prove our main result.

Let $D$ be an open subset of the plane $R^{2}$ with the boundary $\partial D$ being a simple closed curve; $\widetilde{T}$ is a continuous mapping from $\bar{D}=D \cup \partial D$ to $R^{2}$. Let $(c, d) \in R^{2}$. Denote by $A$ a variable point on the boundary $\partial D$. As $A$ traverses the boundary, assume that its image $\widetilde{T}(A)$ traces out a closed curve that does not pass through the point $(c, d)$. As in complex analysis, we can define the winding number of this curve with respect to $(c, d)$, by measuring the total change of the argument of the vector joining $(c, d)$ and the variable point $\widetilde{T}(A)$. For two-dimensional space, this number is equivalent to the topological degree of the mapping $\widetilde{T}$ at $(c, d)$.

We introduce a proposition from [8] as follows.

Proposition 11. If the degree of a continuous mapping $\widetilde{T}$ with respect to a point $(c, d)$ is nonzero, then the equation $\widetilde{T}(\mu, \nu)=$ $(c, d)$ has a solution $(\mu, \nu) \in D$.

From Section 2, for any parameters $\mu, \nu, \kappa=(\mu, \nu)$, there exists a solution $W_{\kappa}(t)$ of (30). At the point $t=1$, we denote $w_{1}(1):=\theta, w_{2}(1):=\mathcal{Y}$. It is obvious that the parameters $\theta$, $\vartheta$ depend on the parameters $\mu, \nu$, so we define a map $\widetilde{T}$ as follows:

$$
\begin{equation*}
\widetilde{T}(\mu, \nu)=(\theta, \vartheta) . \tag{33}
\end{equation*}
$$

Therefore, if we can find a domain $D$ with its boundary as a closed curve $L$, so that its image $\widetilde{T}(L)$ contains the point $(0,0)$ in it, then it is implied by Proposition 11 that there exists a point $\kappa_{0}=\left(\mu_{0}, v_{0}\right)$ in $D$ such that $\widetilde{T}\left(\mu_{0}, v_{0}\right)=(0,0)$. Thus, the function $Y(t)=W_{\kappa_{0}}(t)+\kappa_{0}=W(t)+Y(1)$ is a solution of (29), where

$$
\begin{equation*}
W_{\kappa_{0}}(t)=\int_{0}^{1} G^{*}(t, s) f\left(s, s^{\alpha-1}\left(W_{\kappa_{0}}(s)+\kappa_{0}\right)\right) d s \tag{34}
\end{equation*}
$$

We now proceed to find such $L$. For convenience, we take a curve $L=P Q R S$, where $P=\left(-\mu^{*},-v^{*}\right), Q=\left(-\mu^{*}, \nu^{*}\right)$, $R=\left(\tilde{\mu}^{*}, v^{*}\right), S=\left(\tilde{\mu}^{*},-v^{*}\right)$, and $P Q, Q R, R S$, and $S P$ are a part of line. The image $\widetilde{T}(P Q R S)=P^{\prime} Q^{\prime} R^{\prime} S^{\prime}$. We want to show that the point $(\theta, \vartheta)=(0,0)$ is inside the closed curve $P^{\prime} Q^{\prime} R^{\prime} S^{\prime}$ as the parameters $\mu^{*}, \widetilde{\mu}^{*}$, and $\nu^{*}$ are large enough. In fact, we will prove that the line $P^{\prime} Q^{\prime}\left(R^{\prime} S^{\prime}\right)$ lies in the left (right) side of the $\vartheta$-axis, and the line $Q^{\prime} R^{\prime}\left(S^{\prime} P^{\prime}\right)$ lies above (under) the $\theta$-axis as $\mu^{*}, \widetilde{\mu}^{*}$, and $\nu^{*}$ are large enough.

Let

$$
\begin{gather*}
a_{1}=\int_{0}^{1} G^{*}(s, s) a_{1}(s) d s, \quad b_{11}=\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s \\
b_{12}=\int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
a_{2}=\int_{0}^{1} G^{*}(s, s) a_{2}(s) d s, \quad b_{21}=\int_{0}^{1} G^{*}(s, s) b_{21}(s) d s \\
b_{22}=\int_{0}^{1} G^{*}(s, s) b_{22}(s) d s \tag{35}
\end{gather*}
$$

From (9) and (H), $a_{1}, a_{2} \geq 0$ and $0<b_{11}, b_{12}, b_{21}, b_{22}<1$, and $b_{12}<b_{11}, b_{21}<b_{22}$, we may take $\mu^{*}, \widetilde{\mu}^{*}$, and $\nu^{*}$ large enough satisfying

$$
\begin{align*}
& \mu^{*}=\frac{b_{12}}{b_{11}} v^{*}  \tag{36}\\
& \tilde{\mu}^{*}=\frac{b_{22}}{b_{21}} v^{*} \tag{37}
\end{align*}
$$

Then, the points $P, Q, R$, and $S$ can be expressed as follows:

$$
\begin{align*}
& P=\left(-\frac{b_{12}}{b_{11}} v^{*},-v^{*}\right) \\
& Q=\left(-\frac{b_{12}}{b_{11}} v^{*}, v^{*}\right) \\
& R=\left(\frac{b_{22}}{b_{21}} v^{*}, v^{*}\right)  \tag{38}\\
& S=\left(\frac{b_{22}}{b_{21}} v^{*},-v^{*}\right)
\end{align*}
$$

Now the proof of Theorem 1 is reduced as the following lemmas.

Lemma 12. Suppose that ( $H$ ) and (9) hold. Then, for $v^{*}$ large enough, $P^{\prime}$ lies in the third quadrant.

Proof. From (30), we have

$$
\begin{equation*}
W_{\kappa}(1):=W(1)=\int_{0}^{1} G^{*}(1, s) f\left(s, s^{\alpha-1}(W(s)+\kappa)\right) d s \tag{39}
\end{equation*}
$$

By the definition of $\theta, \vartheta$ in (33), we may rewrite (39) as follows:

$$
\begin{align*}
& \theta_{\kappa}=w_{1}(1)=\int_{0}^{1} G^{*}(1, s) f_{1}\left(s, s^{\alpha-1}(W(s)+\kappa)\right) d s \\
& \vartheta_{\kappa}=w_{2}(1)=\int_{0}^{1} G^{*}(1, s) f_{2}\left(s, s^{\alpha-1}(W(s)+\kappa)\right) d s \tag{40}
\end{align*}
$$

Let $\kappa^{*}=\left(-\mu^{*},-v^{*}\right)=\left(-\left(b_{12} / b_{11}\right) v^{*},-v^{*}\right)$. From (40), we have

$$
\begin{align*}
& \theta_{\kappa^{*}}=w_{1}(1)=\int_{0}^{1} G^{*}(1, s) f_{1}\left(s, s^{\alpha-1}\left(W(s)+\kappa^{*}\right)\right) d s \\
& \vartheta_{\kappa^{*}}=w_{2}(1)=\int_{0}^{1} G^{*}(1, s) f_{2}\left(s, s^{\alpha-1}\left(W(s)+\kappa^{*}\right)\right) d s \tag{41}
\end{align*}
$$

Now we will show that $\theta_{\kappa^{*}}, \vartheta_{\kappa^{*}} \rightarrow-\infty$ as $\nu^{*} \rightarrow \infty$. We only show that $\lim _{v^{*} \rightarrow \infty} \theta_{\kappa^{*}}=-\infty$ and the proof of $\lim _{\nu^{*} \rightarrow \infty} \vartheta_{\kappa^{*}}=$ $-\infty$ is similar. Assume on the contrary that $\varlimsup_{\nu^{*} \rightarrow \infty} \theta_{\kappa^{*}}=$ $l>-\infty$. Thus, there exists a sequence $\left\{\kappa_{n}\right\}=\left\{\left(\mu_{n}, v_{n}\right)\right\}, \mu_{n}=$ $\left(b_{12} / b_{11}\right) v_{n}<0$ such that $\lim _{v_{n} \rightarrow-\infty} \theta_{n}=l>-\infty$.

Recall that

$$
\begin{align*}
& w_{1 \kappa_{n}}(t)=\int_{0}^{1} G^{*}(t, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s  \tag{49}\\
& w_{2 \kappa_{n}}(t)=\int_{0}^{1} G^{*}(t, s) f_{2}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \tag{42}
\end{align*}
$$

Using conditions (6) and (8), we have from (45) that there exists a constant $l$ such that

$$
f_{2}\left(t, t^{\alpha-1}\left(w_{1 \kappa_{n}}(t)+\frac{b_{12}}{b_{11}} v_{n}, w_{2 \kappa_{n}}(t)+v_{n}\right)\right)<l
$$

for $t \in[0,1] \backslash C_{n}$ and any $n$ large enough. From the second formula of (42), (45)-(49), one gets

$$
\begin{aligned}
& w_{2 \kappa_{n}}(t) \leq\left(\int_{[0,1] \backslash C_{n}}+\int_{C_{n} \cap A_{n}}+\int_{C_{n} \cap B_{n}}\right) G^{*}(s, s) \\
& \times f_{2}\left(s, s^{\alpha-1}\left(w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}, w_{2 \kappa_{n}}(s)+v_{n}\right)\right) d s \\
& \leq l \int_{[0,1] \backslash C_{n}} G^{*}(s, s) d s \\
&+\int_{C_{n} \cap B_{n}} G^{*}(s, s) \\
& \quad \times f_{2}\left(s, s^{\alpha-1}\left(w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right.\right. \\
&\left.\left.\quad w_{2 \kappa_{n}}(s)\left(+v_{n}\right)\right)\right) d s
\end{aligned}
$$

$$
=l \int_{[0,1] \backslash C_{n}} G^{*}(s, s) d s
$$

$$
+\int_{B_{n}} G^{*}(s, s)
$$

$$
\times f_{2}\left(s, s^{\alpha-1}\left(w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right.\right.
$$

$$
\left.\left.w_{2 \kappa_{n}}(s)+v_{n}\right)\right) d s
$$

$$
\leq l \int_{[0,1] \backslash C_{n}} G^{*}(s, s) d s+\int_{0}^{1} G^{*}(s, s) a_{2}(s) d s
$$

$$
+\left\|w_{1 \kappa_{n}}(t)+\frac{b_{21}}{b_{11}} v_{n}\right\|_{B_{n}}^{p_{2}} \int_{0}^{1} G^{*}(s, s) b_{21}(s) d s
$$

$$
+\left\|w_{2 \kappa_{n}}(t)+v_{n}\right\|_{B_{n}}^{q_{2}} \int_{0}^{1} G^{*}(s, s) b_{22}(s) d s
$$

$$
\leq l \int_{[0,1] \backslash C_{n}} G^{*}(s, s) d s+a_{2}+\left(b_{21}+b_{22}\right)\left\|w_{2 \kappa_{n}}(t)\right\|_{C_{n}},
$$

$$
\begin{equation*}
t \in C_{n} \tag{50}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left\|w_{2 \kappa_{n}}(t)\right\|_{C_{n}} \leq \frac{l \int_{[0,1] \backslash C_{n}} G^{*}(s, s) d s+a_{2}}{1-b_{21}-b_{22}}<\infty \tag{51}
\end{equation*}
$$

It contradicts (48).
Thus, for any $-v_{n}$ large enough, there exists some $t \in$ $(0,1]$, such that

$$
\begin{equation*}
f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0 \tag{52}
\end{equation*}
$$

Now we define

$$
\begin{align*}
& I_{n}=\left\{t \in[0,1]: f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0\right\}, \\
& I_{n}^{\prime}=\left\{t \in[0,1]: f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0\right\} . \tag{53}
\end{align*}
$$

Then, $I_{n}$ is not empty.

We can further divide the set $I_{n}$ into two sets $\widetilde{I}_{n}$ and $\widehat{I_{n}}$, and divide the set $I_{n}^{\prime}$ into two sets $\widetilde{I_{n}^{\prime}}$ and $\widehat{I_{n}^{\prime}}$ as follows:

$$
\begin{align*}
& \widetilde{I}_{n}=\left\{t \in I_{n} \left\lvert\, w_{1 \kappa_{n}}(t)+\frac{b_{12}}{b_{11}} v_{n} \leq 0\right.\right\} \\
& \widehat{I_{n}}=\left\{t \in I_{n} \left\lvert\, w_{1 \kappa_{n}}(t)+\frac{b_{12}}{b_{11}} v_{n}>0\right.\right\}  \tag{54}\\
& \widetilde{I_{n}^{\prime}}=\left\{t \in I_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n} \leq 0\right\} \\
& \widehat{I_{n}^{\prime}}=\left\{t \in I_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n}>0\right\}
\end{align*}
$$

It is easy to know that $\widetilde{I}_{n} \cap \widehat{I_{n}}=\phi, \widetilde{I_{n}^{\prime}} \cap \widehat{I_{n}^{\prime}}=\phi$ and $I_{n}=\widetilde{I}_{n} \cup \widehat{I_{n}}$, $I_{n}^{\prime}=\widetilde{I_{n}^{\prime}} \cup \widehat{I_{n}^{\prime}}$.

We claim that the set $\widehat{I_{n}}$ is not empty for $-v_{n}$ large enough. Otherwise, the function $f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)$ is bounded from above. In fact, assume that $f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)$ is unbounded from above for $-v_{n}$ large enough; then we have from $(H)$ that there exist a sequence $\left\{t_{i}\right\}$ and a subsequence $\left\{v_{n_{i}}\right\}$ of $\left\{v_{n}\right\}$ such that

$$
\begin{align*}
& \lim _{n_{n_{i}} \rightarrow-\infty} w_{2 \kappa_{n_{i}}}\left(t_{i}\right)=\infty \\
& \lim _{v_{n_{i}} \rightarrow-\infty}\left|w_{1 \kappa_{n_{i}}}\left(t_{i}\right)+\frac{b_{12}}{b_{11}} v_{n_{i}}\right| \leq \lim _{v_{n_{i}} \rightarrow-\infty}\left(w_{2 \kappa_{n_{i}}}\left(t_{i}\right)+v_{n_{i}}\right) \\
&=+\infty . \tag{55}
\end{align*}
$$

Using a similar method of (51), we can derive a contradiction. Therefore, $f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)$ is bounded from above. From (42), $w_{1 \kappa_{n}}(t)$ is bounded from above, which implies that $w_{1 \kappa_{n}}(t)+\left(b_{12} / b_{11}\right) v_{n} \rightarrow-\infty$ as $\nu_{n} \rightarrow-\infty$. If $B_{n}=\phi$ (where $B_{n}$ is defined in (46)), then $\lim _{\nu_{n} \rightarrow-\infty} \theta_{n}=-\infty$, which contradicts our assumption. Thus, $B_{n} \neq \phi$. Using a similar method of getting (51) also gives a contradiction. Therefore, $\widehat{I}_{n}$ is not empty.

Similarly as getting (51) again, we conclude that the function $f_{i}\left(t, t^{\alpha-1} X\right)$ is bounded above by a constant for $t \in$ $[0,1]$ and $x_{i} \in(-\infty, 0](i=1,2)$. From the condition $(H)$, if $w_{1 \kappa_{n}}(t)+v_{n}>0\left(\right.$ or $\left.w_{2 \kappa_{n}}(t)+\mu_{n}>0\right)$ and $f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\right.\right.$ $\left.\left.\kappa_{n}\right)\right)<0\left(\right.$ or $\left.f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0\right)$, then $w_{2 \kappa_{n}}(t)+v_{n}$ (or $w_{1 \kappa_{n}}(t)+\mu_{n}$ ) is also bounded from above by a constant for $t \in[0,1]$. Therefore, from the definition of $\widetilde{I}_{n}, \widetilde{I}_{n}^{\prime}$, there exists a constant $M>1$, independent of $t$ and $v_{n}$ such that

$$
\begin{aligned}
& f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \leq M, \quad \text { for } t \in \widetilde{I}_{n}, \\
& f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \leq M, \quad \text { for } t \in \widetilde{I_{n}^{\prime}}, \\
& w_{2 \kappa_{n}}(t)+v_{n} \leq M \\
& \text { for } f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0, \quad t \in \widehat{I_{n}},
\end{aligned}
$$

$$
\begin{align*}
& w_{1 \kappa_{n}}(t)+\frac{b_{12}}{b_{11}} v_{n} \leq M, \\
& \text { for } f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0, \quad t \in \widehat{I_{n}^{\prime}} . \tag{56}
\end{align*}
$$

Let

$$
\begin{equation*}
\bar{M}_{1}\left(\kappa_{n}\right)=\max _{t \in I_{n}} w_{1 \kappa_{n}}(t) . \tag{57}
\end{equation*}
$$

From the definitions of $\widetilde{I}_{n}$ and $\widehat{I}_{n}$, we have

$$
\begin{equation*}
\bar{M}_{1}\left(\kappa_{n}\right)=\max _{t \in \bar{I}_{n}} w_{1 \kappa_{n}}(t)=\left\|w_{1 \kappa_{n}}(t)\right\|_{I_{n}} . \tag{58}
\end{equation*}
$$

Since $\widehat{I_{n}}$ is not empty, it follows that $\bar{M}_{1}\left(\kappa_{n}\right) \rightarrow \infty$ as $\nu_{n} \rightarrow-\infty$. Recall from (9) and (35) that $b_{i j}<1(i, j=1,2)$. Therefore, we can choose $v_{n_{1}}>0$ large enough so that

$$
\begin{equation*}
\bar{M}_{1}\left(\kappa_{n}\right)>\max \left\{1, P_{1}, P_{2}\right\} \tag{59}
\end{equation*}
$$

for $v_{n}<-v_{n_{1}}$, where

$$
\begin{align*}
P_{1}= & \frac{M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}}{1-b_{11}},  \tag{60}\\
P_{2}= & \left(M \int_{0}^{1} G^{*}(s, s) d s\left(1-b_{22}+b_{12}\right)\right. \\
& +b_{12}\left(1-b_{22}+b_{21}\right)+\left(a_{2}+b_{22}\right) b_{12} \\
& \left.+\left(a_{1}+b_{11}\right)\left(1-b_{22}\right)\right)  \tag{61}\\
& \times\left(\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}\right)^{-1} .
\end{align*}
$$

Now, for later use, for any integral in a domain $A$

$$
\begin{equation*}
\int_{A} G^{*}(s, s) b_{i j}(s) g(s) d s, \quad \text { for } g(s)>0, i, j=1,2, \tag{62}
\end{equation*}
$$

we define a subset $(A)_{1}$ as

$$
\begin{equation*}
(A)_{1}=\{t \in A \mid g(t) \geq 1\} . \tag{63}
\end{equation*}
$$

Thus, the integral in (62) can be rewritten as

$$
\begin{align*}
\int_{A} G^{*}(s, s) b_{i j}(s) g(s) d s= & \int_{(A)_{1}} G^{*}(s, s) b_{i j}(s) g(s) d s \\
& +\int_{A \backslash(A)_{1}} G^{*}(s, s) b_{i j}(s) g(s) d s . \tag{64}
\end{align*}
$$

From (H), (42), and the definitions of $\widetilde{I}_{n}, \widehat{I}_{n}$ and $\widetilde{I_{n}^{\prime}}, \widehat{I_{n}^{\prime}}$, for $v_{n}<-v_{n_{1}}$, we have

$$
\begin{align*}
& w_{1 \kappa_{n}}(t)= \int_{0}^{1} G^{*}(t, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& \leq \int_{I_{n}} G^{*}(t, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& \leq \int_{\tilde{I}_{n}} G^{*}(s, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
&+\int_{\widehat{I}_{n}} G^{*}(s, s) a_{1}(s) d s \\
&+\int_{\widetilde{I}_{n}} G^{*}(s, s)\left(b_{11}(s)\left|w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}}\right. \\
&\left.\quad+b_{12}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{1}}\right) d s \\
& \leq \int_{\tilde{I}_{n}} G^{*}(s, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
&+\int_{\widetilde{I}_{n}} G^{*}(s, s) a_{1}(s) d s \\
&+\int_{\widehat{I}_{n} \backslash\left(\widetilde{I}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
&+\int_{\left(\widetilde{I}_{n}\right)} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
&+\int_{\left(\widetilde{I}_{n} n I_{n}^{\prime}\right)_{1}} G^{*}(s, s) b_{12}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{1}} d s \\
&+\int_{\left.\left.\left(\widetilde{I}_{n} \cap(0,1] I_{n}^{\prime}\right)\right) \cup\left(\widetilde{I}_{n} n I_{n}^{\prime}\right) \backslash\left(\widetilde{I}_{n} \cap I_{n}^{\prime}\right)_{1}\right)} G^{*}(s, s) b_{12}(s) \\
& \times\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{1}} d s, \tag{65}
\end{align*}
$$

which yields from (56) and the definition in (63) that

$$
\begin{align*}
& w_{1 \kappa_{n}}(t) \leq \int_{\tilde{T}_{n}} G^{*}(s, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& +\int_{\widehat{I}_{n}} G^{*}(s, s) a_{1}(s) d s+\int_{\widehat{I}_{n}\left(\widehat{T_{n}}\right)_{1}} G^{*}(s, s) b_{11}(s) d s \\
& +\int_{\left(\widetilde{T}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)+\frac{b_{12}}{b_{11}} v_{n}\right| d s \\
& +\int_{\left(\widetilde{I}_{n} I_{n}^{\prime}\right)_{1}} G^{*}(s, s) b_{12}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right| d s \\
& +M \int_{\left(\widetilde{I}_{n} \cap\left[(0,1] \backslash I_{n}^{\prime}\right)\right) \cup\left(\left(\widetilde{I}_{n} \cap I_{n}^{\prime}\right) \backslash\left(\widehat{I}_{n} \cap I_{n}^{\prime}\right)_{1}\right)} G^{*}(s, s) b_{12}(s) d s . \tag{66}
\end{align*}
$$

Further, one gets from (56) that

$$
\begin{align*}
w_{1 \kappa_{n}}(t) \leq & \int_{0}^{1} G^{*}(s, s) M d s+\int_{0}^{1} G^{*}(s, s) a_{1}(s) d s \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s+M \int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s\left(\bar{M}_{1}\left(\kappa_{n}\right)-\frac{b_{12}}{b_{11}}\left\|v_{n}\right\|\right) \\
& +\int_{0}^{1} G^{*}(s, s) b_{12}(s) d s\left(\left\|w_{2 \kappa_{n}}(t)\right\|_{\widehat{I}_{n} \cap I_{n}^{\prime}}+\left\|v_{n}\right\|\right) \\
= & M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{11} \bar{M}_{1}\left(\kappa_{n}\right) \\
& +b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\widehat{r}_{n} \cap I_{n}^{\prime \prime}}, \tag{67}
\end{align*}
$$

which gives

$$
\begin{align*}
\bar{M}_{1}\left(\kappa_{n}\right)< & M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{11} \bar{M}_{1}\left(\kappa_{n}\right) \\
& +b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\widehat{T}_{n} \cap \cap_{n}^{\prime}} \tag{68}
\end{align*}
$$

That is,

$$
\begin{align*}
& \bar{M}_{1}\left(\kappa_{n}\right) \\
& <\frac{M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\widehat{I}_{n} \cap \cap_{n}^{\prime}}}{1-b_{11}} . \tag{69}
\end{align*}
$$

If $\widehat{I_{n}} \cap I_{n}^{\prime}=\phi$, then we have from (69) that

$$
\begin{equation*}
\bar{M}_{1}\left(\kappa_{n}\right)<\frac{M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}}{1-b_{11}} \tag{70}
\end{equation*}
$$

which contradicts (59).
If $\widehat{I_{n}} \cap I_{n}^{\prime} \neq \phi$, using a similar method of (69), we can estimate $w_{2 \kappa_{n}}(t)$ as

$$
\begin{aligned}
& \left\|w_{2 \kappa_{n}}(t)\right\|_{\widehat{I}_{n} \cap I_{n}^{\prime}} \\
& \quad \leq\left\|w_{2 \kappa_{n}}(t)\right\|_{I_{n}^{\prime}} \\
& \quad<\frac{M\left(\int_{0}^{1} G^{*}(s, s) d s+b_{21}\right)+a_{2}+b_{22}+b_{21} \bar{M}\left(\kappa_{n}\right)}{1-b_{22}} .
\end{aligned}
$$

Substituting this into (69), we obtain

$$
\begin{align*}
& \bar{M}\left(\kappa_{n}\right) \\
& <\left(M\left(\int_{0}^{1} G^{*}(s, s) d s\left(1-b_{22}+b_{12}\right)+b_{12}\left(1-b_{22}+b_{21}\right)\right)\right. \\
& \left.\quad+\left(a_{2}+b_{22}\right) b_{12}+\left(a_{1}+b_{11}\right)\left(1-b_{22}\right)\right) \\
& \quad \times\left(\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}\right)^{-1}, \tag{72}
\end{align*}
$$

which finally contradicts (59). Therefore, our result is proved.
Similarly, we can show that $\lim _{\nu^{*} \rightarrow \infty} \vartheta_{\kappa^{*}}=-\infty$. Thus, the point $\widetilde{T}\left(-\mu^{*},-\nu^{*}\right)$ lies in the third quadrant. The proof is completed.

Lemma 13. Suppose that (H) and (9) hold. Then, for $\nu^{*}>0$ large enough, $Q^{\prime}$ lies in the second quadrant.

Proof. It suffices to show that $\lim _{\nu^{*} \rightarrow \infty} \vartheta^{*}=\infty$ and $\lim _{\nu^{*} \rightarrow \infty} \theta^{*}=-\infty$.

First, we claim that $\lim _{\nu^{*} \rightarrow \infty} \mathcal{V}^{*}=\infty$. On the contrary, we assume that there exists a sequence $\left\{\kappa_{n}\right\}=\left\{\left(\mu_{n}, \nu_{n}\right)\right\}=$ $\left\{\left(-\left(b_{12} / b_{11}\right) v_{n}, v_{n}\right)\right\}$ such that $\lim _{v_{n} \rightarrow \infty} \vartheta_{n}=l<\infty$. By a similar method in Lemma 12, we know it is impossible to have

$$
\begin{equation*}
f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \geq 0, \quad \forall t \in[0,1] \tag{73}
\end{equation*}
$$

as $v_{n}$ is sufficiently large.
Now, for large $v_{n}$, we define

$$
\begin{align*}
& J_{n}=\left\{t \in[0,1]: f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0\right\} \\
& J_{n}^{\prime}=\left\{t \in[0,1]: f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0\right\} . \tag{74}
\end{align*}
$$

Then, $J_{n}^{\prime}$ is not empty.
As in Lemma 12, we can further divide the set $J_{n}$ into two sets $\widetilde{J_{n}}$ and $\widehat{J_{n}}$ and divide the set $J_{n}^{\prime}$ into two sets $\widetilde{J_{n}^{\prime}}$ and $\widehat{J_{n}^{\prime}}$ as follows:

$$
\begin{align*}
& \widetilde{J_{n}}=\left\{t \in J_{n} \left\lvert\, w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n} \geq 0\right.\right\}, \\
& \widehat{J_{n}}=\left\{t \in J_{n} \left\lvert\, w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n}<0\right.\right\},  \tag{75}\\
& \widetilde{J_{n}^{\prime}}=\left\{t \in J_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n} \geq 0\right\}, \\
& \widehat{J_{n}^{\prime}}=\left\{t \in J_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n}<0\right\} .
\end{align*}
$$

Then $\widetilde{J_{n}} \cap \widehat{J_{n}}=\phi, \widetilde{J_{n}^{\prime}} \cap \widehat{J_{n}^{\prime}}=\phi$ and $J_{n}=\widetilde{J_{n}} \cup \widehat{J_{n}}, J_{n}^{\prime}=\widetilde{J_{n}^{\prime}} \cup \widehat{J_{n}^{\prime}}$.
Using a similar method as in the proof of Lemma 12, we can show that the set $\widehat{J_{n}^{\prime}}$ is not empty. Furthermore, the function $f_{i}\left(t, t^{\alpha-1} X\right)$ is bounded below by a constant for $t \in$ $[0,1]$ and $x_{i} \in[0, \infty)(i=1,2)$. If $w_{2 \kappa_{n}}(t)+\mu_{n}<0$ (or $\left.w_{1 \kappa_{n}}(t)+\nu_{n}<0\right)$ and $f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0$ (or $\left.f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0\right)$, then $w_{1 \kappa_{n}}(t)+v_{n}\left(\right.$ or $\left.w_{2 \kappa_{n}}(t)+\mu_{n}\right)$
is also bounded below by a constant for $t \in[0,1]$. From the definition of $\widetilde{J_{n}}, \widetilde{J_{n}^{\prime}}$ and the condition $(H)$, there exists a constant $\widetilde{M}<-1$, independent of $t$ and $v_{n}$ such that

$$
\begin{aligned}
& f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \geq \widetilde{M}, \quad \text { for } t \in \widetilde{J_{n}} \\
& f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \geq \widetilde{M}, \quad \text { for } t \in \widetilde{J_{n}^{\prime}} \\
& w_{2 \kappa_{n}}(t)+v_{n} \geq \widetilde{M} \\
& \text { for } f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0, \quad t \in \widehat{J_{n}} \\
& w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n} \geq \widetilde{M} \\
& \text { for } f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0, \quad t \in \widehat{J_{n}^{\prime}}
\end{aligned}
$$

Let

$$
\begin{equation*}
\bar{m}_{2}\left(\kappa_{n}\right)=\min _{t \in J_{n}^{\prime}} w_{2 \kappa_{n}}(t) \tag{77}
\end{equation*}
$$

From the definitions of $\widetilde{J_{n}^{\prime}}$ and $\widehat{J_{n}^{\prime}}$, we have

$$
\begin{equation*}
\bar{m}_{2}\left(\kappa_{n}\right)=\min _{t \in \widehat{J_{n}^{\prime}}} w_{2 \kappa_{n}}(t)=-\left\|w_{2 \kappa_{n}}(t)\right\|_{J_{n}^{\prime}} \tag{78}
\end{equation*}
$$

and it follows that $\bar{m}_{2}\left(\kappa_{n}\right) \rightarrow-\infty$ as $\nu_{n} \rightarrow \infty$. Therefore, we can choose $v_{n_{1}}$ large enough so that

$$
\begin{equation*}
\bar{m}_{2}\left(\kappa_{n}\right)<\min \left\{-1, Q_{1}, Q_{2}\right\} \tag{79}
\end{equation*}
$$

for $v_{n}>v_{n_{1}}$, where

$$
\begin{align*}
Q_{1}= & \frac{\widetilde{M} \int_{0}^{1} G^{*}(s, s) d s\left(1+b_{21}\right)-a_{2}-b_{22}}{1-b_{22}}, \\
Q_{2}= & \frac{\int_{0}^{1} G^{*}(s, s) d s\left(1-b_{11}-b_{21}\right)+b_{21}\left(1-b_{11}-b_{21}\right)}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}} \widetilde{M} \\
& -\frac{\left(a_{2}+b_{22}\right)\left(1-b_{11}\right)+\left(a_{1}+b_{11}\right) b_{21}}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}} \tag{80}
\end{align*}
$$

Notice that $b_{12}<b_{11}, b_{21}<b_{22}$. From (H), (42), and the definitions of $\widetilde{J_{n}}, \widehat{J_{n}}$ and $\widetilde{J_{n}^{\prime}}, \widehat{J_{n}^{\prime}}$, for $v_{n}>v_{n_{1}}$, we have

$$
\begin{align*}
& w_{2 \kappa_{n}}(t) \geq \int_{J_{n}^{\prime}} G^{*}(s, s) f_{2}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& \geq \int_{J_{n}^{\prime}} G^{*}(s, s) f_{2}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
&-\int_{\widehat{J}_{n}^{\prime}} G^{*}(s, s) a_{2}(s) d s \\
&-\int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s)\left(b_{21}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{2}}\right. \\
&\left.\quad+b_{22}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{2}}\right) d s \\
& \geq \int_{\widetilde{J_{n}^{\prime}}} G^{*}(s, s) f_{2}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
&-\int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s) a_{2}(s) d s \\
&-\int_{J_{J_{n}^{\prime}}^{\prime} \cap J_{n}} G^{*}(s, s) b_{21}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{2}} d s \\
&-\int_{\widehat{J_{n}^{\prime}} \cap\left([0,1] \backslash J_{n}\right)} G^{*}(s, s) b_{21}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{2}} d s \\
&-\int_{\widehat{J}_{n}^{\prime}} G^{*}(s, s) b_{22}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{2}} d s . \tag{81}
\end{align*}
$$

Thus,

$$
\begin{align*}
w_{2 \kappa_{n}} & (t) \\
\geq & \int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s) f_{2}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& -\int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s) a_{2}(s) d s \\
& -\int_{\left(\widehat{\left.J_{n}^{\prime} \cap J_{n}\right)_{1}}\right.} G^{*}(s, s) b_{21}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{2}} d s \\
& -\int_{\left(\widehat{J_{n}^{\prime}} \cap\left([0,1] \backslash J_{n}\right)\right) \cup\left(\left(\widehat{\left.J_{n}^{\prime} \cap J_{n}\right) \backslash\left(\widehat{\left.\left.J_{n}^{\prime} \cap J_{n}\right)_{1}\right)}\right.} G^{*}(s, s) b_{21}(s)\right.\right.} \times\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{2}} d s \\
& -\int_{\left(\widehat{J_{n}^{\prime}}\right)_{1}} G^{*}(s, s) b_{22}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{2}} d s \\
& -\int_{\widehat{J_{n}^{\prime}} \backslash\left(\widehat{J_{n}^{\prime}}\right)_{1}} G^{*}(s, s) b_{22}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{2}} d s,
\end{align*}
$$

which follows from (76) and the definition of (63) that

$$
\begin{aligned}
& w_{2 \kappa_{n}}(t) \geq \widetilde{M} \int_{\widetilde{J}_{n}^{\prime}} G^{*}(s, s) d s-\int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s) a_{2}(s) d s \\
& +\widetilde{M} \int_{\widehat{J_{n}^{\prime}}} G^{*}(s, s) b_{21}(s) d s \\
& -\int_{\left(\widehat{\left.J_{n}^{J} \cap J_{n}\right)_{1}}\right.} G^{*}(s, s) b_{21}(s) d s \\
& \times\left\|w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n}\right\|_{\left(\widehat{\left.J_{n}^{\prime} \cap J_{n}\right)_{1}}\right.} \\
& -\int_{\left(\bar{J}_{n}^{\prime}\right)_{1}} G^{*}(s, s) b_{22}(s) d s\left\|w_{2 \kappa_{n}}(t)+v_{n}\right\|_{\widehat{J_{n}^{\prime}}} \\
& -\int_{\widehat{J_{n}^{\prime}}\left(\left(\widehat{J_{n}^{\prime}}\right)_{1}\right.} G^{*}(s, s) b_{22}(s) d s \\
& \geq \widetilde{M} \int_{0}^{1} G^{*}(s, s) d s-\int_{0}^{1} G^{*}(s, s) a_{2}(s) d s \\
& +\widetilde{M} \int_{0}^{1} G^{*}(s, s) b_{21}(s) d s \\
& -\int_{0}^{1} G^{*}(s, s) b_{21}(s) d s \\
& \times\left(\left\|w_{1 \kappa_{n}}(t)\right\|_{\widehat{J}_{n}^{\prime} \cap J_{n}}+\left\|\frac{b_{12}}{b_{11}} v_{n}\right\|\right) \\
& -\int_{0}^{1} G^{*}(s, s) b_{22}(s) d s \\
& -\int_{0}^{1} G^{*}(s, s) b_{22}(s) d s\left(-\bar{m}_{2}\left(\kappa_{n}\right)-\left\|v_{n}\right\|\right) \\
& -\int_{0}^{1} G^{*}(s, s) b_{22}(s) d s \\
& =\widetilde{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{21}\right)-a_{2}-b_{22} \\
& +\frac{\left\|v_{n}\right\|\left(b_{11} b_{22}-b_{12} b_{21}\right)}{b_{11}} \\
& -b_{21}\left\|w_{1 \kappa_{n}}(t)\right\|_{\widehat{J_{n}^{\prime} \cap J_{n}}}+b_{22} \bar{m}_{2}\left(\kappa_{n}\right) .
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\bar{m}_{2}\left(\kappa_{n}\right)> & \widetilde{M} \\
& \left(\int_{0}^{1} G^{*}(s, s) d s+b_{21}\right)-a_{2}-b_{22} \\
& -b_{21}\left\|w_{1 \kappa_{n}}(t)\right\|_{\widehat{J_{n}^{\prime} \cap J_{n}}}+b_{22} \bar{m}_{2}\left(\kappa_{n}\right),
\end{aligned}
$$

which implies that

$$
\begin{align*}
& \bar{m}_{2}\left(\kappa_{n}\right) \\
& >\frac{\widetilde{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{21}\right)-a_{2}-b_{22}-b_{21}\left\|w_{1 \kappa_{n}}(t)\right\|_{\widehat{J_{n}^{\prime} \cap J_{n}}}}{1-b_{22}} . \tag{85}
\end{align*}
$$

If $\widehat{J_{n}^{\prime}} \cap J_{n}=\phi$, from (85), we have

$$
\begin{equation*}
\bar{m}_{2}\left(\kappa_{n}\right)>\frac{\widetilde{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{21}\right)-a_{2}-b_{22}}{1-b_{22}}, \tag{86}
\end{equation*}
$$

which contradicts (79).
If $\widehat{J_{n}^{\prime}} \cap J_{n} \neq \phi$. Using a similar method of (85), we have

$$
\begin{align*}
& -\left\|w_{1 \kappa_{n}}(t)\right\|_{\widehat{J_{n}^{\prime} \cap J_{n}}} \\
& \quad \geq-\left\|w_{1 \kappa_{n}}(t)\right\|_{J_{n}} \\
& \quad>\frac{\widetilde{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)-a_{1}-b_{11}-b_{12} \bar{m}_{2}\left(\kappa_{n}\right)}{1-b_{11}} . \tag{87}
\end{align*}
$$

Substituting it into (85), we obtain

$$
\begin{align*}
& \bar{m}_{2}\left(\kappa_{n}\right) \\
& \quad>\frac{\int_{0}^{1} G^{*}(s, s) d s\left(1-b_{11}-b_{21}\right)+b_{21}\left(1-b_{11}-b_{21}\right)}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}} \widetilde{M} \\
& \quad-\frac{\left(a_{2}+b_{22}\right)\left(1-b_{11}\right)+\left(a_{1}+b_{11}\right) b_{21}}{\left(1-b_{22}\right)\left(1-b_{11}\right)-b_{21} b_{12}}, \tag{88}
\end{align*}
$$

which also contradicts (79). Therefore, $\lim _{v^{*} \rightarrow \infty} \mathcal{V}=\infty$.
Now, we show that $\lim _{v^{*} \rightarrow \infty} \theta=-\infty$.
On the contrary, assume that there exists a vector sequence $\left\{\kappa_{n}\right\}=\left\{\left(\mu_{n}, \nu_{n}\right)\right\}$ such that $\mu_{n}=-\left(b_{12} / b_{11}\right) \nu_{n}$ and

$$
\begin{equation*}
\lim _{n} \rightarrow \infty \tag{89}
\end{equation*}
$$

Similarly as before, it is impossible to have

$$
\begin{equation*}
f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \leq 0, \quad \forall t \in[0,1] \tag{90}
\end{equation*}
$$

as $\nu_{n}$ is sufficiently large.
Now for large $v_{n}$, we define

$$
\begin{align*}
& \bar{I}_{n}=\left\{t \in[0,1]: f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0\right\}, \\
& \bar{I}_{n}^{\prime}=\left\{t \in[0,1]: f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)>0\right\} . \tag{91}
\end{align*}
$$

Then, $\bar{I}_{n}$ is not empty.

We can further divide the set $\bar{I}_{n}$ into two sets $\tilde{\bar{I}}_{n}$ and $\hat{\bar{I}}_{n}$ and divide the set $\bar{I}_{n}^{\prime}$ into two sets $\tilde{\bar{I}}_{n}^{\prime}$ and $\hat{\bar{I}}_{n}^{\prime}$ as follows:

$$
\begin{align*}
& \tilde{\bar{I}}_{n}=\left\{t \in \bar{I}_{n} \left\lvert\, w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n} \leq 0\right.\right\} \\
& \hat{\bar{I}}_{n}=\left\{t \in I_{n} \left\lvert\, w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n}>0\right.\right\}  \tag{92}\\
& \tilde{\bar{I}}_{n}^{\prime}=\left\{t \in \bar{I}_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n} \leq 0\right\} \\
& \hat{\bar{I}}_{n}^{\prime}=\left\{t \in \bar{I}_{n}^{\prime} \mid w_{2 \kappa_{n}}(t)+v_{n}>0\right\}
\end{align*}
$$

It is easy to know that $\tilde{\bar{I}}_{n} \cap \hat{\bar{I}}_{n}=\phi, \tilde{\bar{I}}_{n}^{\prime} \cap \hat{\bar{I}}_{n}^{\prime}=\phi$ and $\bar{I}_{n}=\tilde{\bar{I}}_{n} \cup \hat{\bar{I}}_{n}$, $\bar{I}_{n}^{\prime}=\tilde{\bar{I}}_{n}^{\prime} \cup \hat{\bar{I}}_{n}^{\prime}$.

Using a similar method of the proof of Lemma 12, we obtain that the set $\hat{\bar{I}}_{n}$ is not empty. Furthermore, there exists a constant $\widehat{M}>1$, independent of $t$ and $v_{n}$ such that

$$
\begin{aligned}
& f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \leq \widehat{M}, \quad \text { for } t \in \tilde{\bar{I}}_{n} \\
& f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right) \leq \widehat{M}, \quad \text { for } t \in \widetilde{\bar{I}}_{n}^{\prime} \\
& w_{2 \kappa_{n}}(t)+v_{n} \leq \widehat{M} \\
& \text { for } f_{2}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0, \quad t \in \widehat{\bar{I}}_{n} \\
& w_{1 \kappa_{n}}(t)-\frac{b_{12}}{b_{11}} v_{n} \leq \widehat{M} \\
& \text { for } f_{1}\left(t, t^{\alpha-1}\left(W_{\kappa_{n}}(t)+\kappa_{n}\right)\right)<0, \quad t \in \hat{\bar{I}}_{n}^{\prime}
\end{aligned}
$$

Let

$$
\begin{equation*}
\widehat{M}_{1}\left(\kappa_{n}\right)=\max _{t \in \bar{I}_{n}} w_{1 \kappa_{n}}(t) \tag{94}
\end{equation*}
$$

From the definitions of $\tilde{\bar{I}}_{n}$ and $\hat{\bar{I}}_{n}$, we have

$$
\begin{equation*}
\widehat{M}_{1}\left(\kappa_{n}\right)=\max _{t \in \overline{\bar{I}}_{n}} w_{1 \kappa_{n}}(t)=\left\|w_{1 \kappa_{n}}(t)\right\|_{\bar{I}_{n}} . \tag{95}
\end{equation*}
$$

Since $\hat{\bar{I}}_{n}$ is not empty, it follows that $\widehat{M}_{1}\left(\kappa_{n}\right) \rightarrow \infty$ as $v_{n} \rightarrow$ $\infty$. Therefore, we can choose $v_{n 1}$ large enough so that

$$
\begin{equation*}
\widehat{M}_{1}\left(\kappa_{n}\right)>\max \left\{1, \widehat{P}_{1}, \widehat{P}_{2}\right\}, \tag{96}
\end{equation*}
$$

for $v_{n}>v_{n_{1}}$, where

$$
\begin{align*}
\widehat{P}_{1}= & \frac{\widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}}{1-b_{11}} \\
\widehat{P}_{2}= & \frac{\int_{0}^{1} G^{*}(s, s) d s\left[\left(1-b_{22}\right)+b_{12}\right]+b_{12}\left(1-b_{22}+b_{21}\right)}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}} \widehat{M} \\
& +\frac{\left(a_{2}+b_{22}\right) b_{12}+\left(a_{1}+b_{11}\right)\left(1-b_{22}\right)}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}} \tag{97}
\end{align*}
$$

From ( $H$ ) and (42), we have

$$
\begin{align*}
& w_{1 \kappa_{n}}(t) \\
& =\int_{0}^{1} G^{*}(t, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& \leq \int_{\tilde{\bar{I}}_{n}} G^{*}(s, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
& +\int_{\hat{\bar{I}}_{n}} G^{*}(s, s) a_{1}(s) d s \\
& +\int_{\left(\overline{\bar{I}}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
& +\int_{\hat{\bar{I}}_{n} \backslash\left(\overline{\bar{I}}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
& +\int_{\left(\overline{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}} G^{*}(s, s) b_{12}(s)\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{1}} d s \\
& +\int_{\left(\overline{\bar{I}}_{n} \cap\left([0,1] \backslash \bar{I}_{n}^{\prime}\right)\right) \cup\left(\left(\overline{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right) \backslash\left(\overline{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}\right)} G^{*}(s, s) b_{12}(s) \\
& \times\left|w_{2 \kappa_{n}}(s)+v_{n}\right|^{q_{1}} d s, \tag{98}
\end{align*}
$$

which follows from (93) and the definition in (63) that

$$
\begin{align*}
w_{1 \kappa_{n}}(t) \leq & \int_{0}^{1} G^{*}(s, s) \widehat{M} d s+\int_{0}^{1} G^{*}(s, s) a_{1}(s) d s \\
& +\widehat{M} \int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& +\int_{\hat{\bar{I}}_{n}} G^{*}(s, s) b_{11}(s) d s\left(\widehat{m}_{1}\left(\kappa_{n}\right)-\left\|\frac{b_{12}}{b_{11}} v_{n}\right\|\right) \\
& +\int_{\left.\hat{\bar{I}}_{n} \backslash \hat{\bar{I}}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s) d s \\
& +\int_{\left(\overline{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)} G^{*}(s, s) b_{12}(s) d s \\
& \times\left(\left\|w_{2 \kappa_{n}}(t)\right\|_{\left(\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}}+\left\|v_{n}\right\|\right) . \tag{99}
\end{align*}
$$

Thus,

$$
\begin{aligned}
w_{1 \kappa_{n}}(t) \leq & \int_{0}^{1} G^{*}(s, s) \widehat{M} d s+\int_{0}^{1} G^{*}(s, s) a_{1}(s) d s \\
& +\widehat{M} \int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s\left(\widehat{m}_{1}\left(\kappa_{n}\right)-\left\|\frac{b_{12}}{b_{11}} v_{n}\right\|\right) \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s
\end{aligned}
$$

$$
\begin{align*}
& +\int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& \times\left(\left\|w_{2 \kappa_{n}}(t)\right\|_{\left.\widehat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}}+\left\|v_{n}\right\|\right) \\
= & \widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11} \\
& +b_{11} \widehat{M}_{1}\left(\kappa_{n}\right)+b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}} \tag{100}
\end{align*}
$$

which implies that

$$
\begin{align*}
\widehat{M}_{1}\left(\kappa_{n}\right)< & \widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{11} \widehat{M}_{1}\left(\kappa_{n}\right) \\
& +b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\bar{I}_{n} \overline{I_{n}^{\prime}}} \tag{101}
\end{align*}
$$

Therefore, we have

$$
\begin{align*}
& \widehat{M}_{1}\left(\kappa_{n}\right) \\
& \quad<\frac{\widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{12}\left\|w_{2 \kappa_{n}}(t)\right\|_{\hat{I}_{n} \cap \bar{I}_{n}^{\prime}}}{1-b_{11}} . \tag{102}
\end{align*}
$$

If $\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}=\phi$, then we have from (76) that

$$
\begin{equation*}
\widehat{M}_{1}\left(\kappa_{n}\right)<\frac{\widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}}{1-b_{11}} \tag{103}
\end{equation*}
$$

which contradicts (102).
If $\widehat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime} \neq \phi$. Using a similar method to that in Lemma 12, we have

$$
\begin{align*}
& \widehat{M}_{1}\left(\kappa_{n}\right) \\
& \quad<\frac{\int_{0}^{1} G^{*}(s, s) d s\left[\left(1-b_{22}\right)+b_{12}\right]+b_{12}\left(1-b_{22}+b_{21}\right)}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}} \widehat{M} \\
& \quad+\frac{\left(a_{2}+b_{22}\right) b_{12}+\left(a_{1}+b_{11}\right)\left(1-b_{22}\right)}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}}, \tag{104}
\end{align*}
$$

which also contradicts (96). Thus, the point $\widetilde{T}\left(-\mu^{*}, \nu^{*}\right)=$ $\left(-\left(b_{12} / b_{11}\right) \nu^{*}, v^{*}\right)$ lies in the second quadrant. The proof is completed.

Lemma 14. Suppose that $(H)$ and (9) hold. Then, for $v^{*}$ large enough, the line $P^{\prime} Q^{\prime}$ lies in the left of 9 -axis.

Proof. For any point $A(\theta, \vartheta)$ in $P^{\prime} Q^{\prime}$, it suffices to show that $\theta \rightarrow-\infty$ as $\nu^{*} \rightarrow \infty$ for any $\nu \in\left[-\nu^{*}, \nu^{*}\right]$.

On the contrary, we assume that there exists a vector sequence $\left\{\kappa_{n}\right\}=\left\{\left(\mu_{n}, v_{n}\right)\right\}$ satisfying $\mu_{n}=-\left(b_{12} / b_{11}\right) \nu_{n}$ and a point $\nu\left(v_{n}\right) \in\left[-v_{n}, v_{n}\right]$ such that $\theta\left(-\left(b_{12} / b_{11}\right) v_{n}, \nu\left(v_{n}\right)\right) \rightarrow$
$l>-\infty$ as $v_{n} \rightarrow \infty$. We define some sets $\bar{I}_{n}, \tilde{\bar{I}}_{n}, \hat{\bar{I}}_{n}$ and $\bar{I}_{n}^{\prime}$, $\tilde{\bar{I}}_{n}^{\prime}, \widehat{\bar{I}}_{n}^{\prime}$, and some numbers $\widehat{M}, \widehat{M}_{1}\left(\kappa_{n}\right)$ as in Lemma 13. Using a similar method of the proof of Lemma 13, we have

$$
\begin{align*}
& w_{1 \kappa_{n}}(t) \\
& \leq \int_{\tilde{\bar{I}}_{n}} G^{*}(s, s) f_{1}\left(s, s^{\alpha-1}\left(W_{\kappa_{n}}(s)+\kappa_{n}\right)\right) d s \\
&+\int_{\overline{\bar{I}}_{n}} G^{*}(s, s) a_{1}(s) d s \\
&+\int_{\left(\hat{\bar{I}}_{n}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
&+\int_{\hat{\bar{I}}_{n} \backslash\left(\overline{\bar{I}_{n}}\right)_{1}} G^{*}(s, s) b_{11}(s)\left|w_{1 \kappa_{n}}(s)-\frac{b_{12}}{b_{11}} v_{n}\right|^{p_{1}} d s \\
&+\int_{\left(\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}} G^{*}(s, s) b_{12}(s)\left|w_{2 \kappa_{n}}(s)+v\left(v_{n}\right)\right|^{q_{1}} d s \\
&+\int_{\left.\left.\left(\overline{\bar{I}}_{n} \cap\left([0,1] \backslash \bar{I}_{n}^{\prime}\right)\right)\right) \cup\left(\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right) \backslash\left(\overline{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}\right)_{1}\right)} G^{*}(s, s) b_{12}(s) \\
& \times\left|w_{2 \kappa_{n}}(s)+v\left(v_{n}\right)\right|^{q_{1}} d s . \tag{105}
\end{align*}
$$

It follows from (93)-(96) that

$$
\begin{align*}
w_{1 \kappa_{n}}(t) \leq & \int_{0}^{1} G^{*}(s, s) \widehat{M} d s+\int_{0}^{1} G^{*}(s, s) a_{1}(s) d s \\
& +\widehat{M} \int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s\left(\widehat{M}_{1}\left(\kappa_{n}\right)-\left\|\frac{b_{12}}{b_{11}} v_{n}\right\|\right) \\
& +\int_{0}^{1} G^{*}(s, s) b_{11}(s) d s \\
& +\int_{0}^{1} G^{*}(s, s) b_{12}(s) d s \\
& \times\left(\left\|w_{2 \kappa_{n}}(t)\right\| \hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}+\left\|v\left(v_{n}\right)\right\|\right) . \tag{106}
\end{align*}
$$

Notice that $-v_{n} \leq \nu\left(v_{n}\right) \leq v_{n}$; from (35), one gets

$$
\begin{align*}
w_{1 \kappa_{n}}(t) \leq & \int_{0}^{1} G^{*}(s, s) \widehat{M} d s+a_{1}+\widehat{M} b_{12}+b_{11} \\
& +b_{11}\left(\widehat{M}_{1}\left(\kappa_{n}\right)-\left\|\frac{b_{12}}{b_{11}} v_{n}\right\|\right) \\
& +b_{12}\left(\left\|w_{2 \kappa_{n}}(t)\right\| \hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}+\left\|v_{n}\right\|\right)  \tag{107}\\
= & \widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11} \\
& +b_{11} \widehat{M}_{1}\left(\kappa_{n}\right)+b_{12}\left\|w_{2 \kappa_{n}}(t)\right\| \hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime},
\end{align*}
$$

which implies that

$$
\begin{align*}
& \widehat{M}_{1}\left(\kappa_{n}\right) \\
& \quad<\frac{\widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}+b_{12}\left\|w_{2 \kappa_{n}}(t)\right\| \|_{\hat{I}_{n} \cap \bar{I}_{n}^{\prime}}}{1-b_{11}} . \tag{108}
\end{align*}
$$

It is easy to show that

$$
\begin{equation*}
\widehat{M}_{1}\left(\kappa_{n}\right)<\frac{\widehat{M}\left(\int_{0}^{1} G^{*}(s, s) d s+b_{12}\right)+a_{1}+b_{11}}{1-b_{11}} \tag{109}
\end{equation*}
$$

for $\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime}=\phi$, which contradicts (102), and

$$
\begin{align*}
& \widehat{M}_{1}\left(\kappa_{n}\right) \\
& <\frac{\int_{0}^{1} G^{*}(s, s) d s\left[\left(1-b_{22}\right)+b_{12}\right]+b_{12}\left(1-b_{22}\right)+b_{12} b_{21}}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}} \widehat{M} \\
& \quad+\frac{\left(a_{2}+b_{22}\right) b_{12}+\left(a_{1}+b_{11}\right)\left(1-b_{22}\right)}{\left(1-b_{11}\right)\left(1-b_{22}\right)-b_{12} b_{21}} \tag{110}
\end{align*}
$$

for $\hat{\bar{I}}_{n} \cap \bar{I}_{n}^{\prime} \neq \phi$, which contradicts (96) also. Thus, the line $P^{\prime} Q^{\prime}$ lies in the left of $\vartheta$-axis. The proof is completed.

Similar to the proof of Lemma 12, we can show that the image point $R^{\prime}$ of the point $R$ lies in the first quadrant. From (37), we have $\tilde{\mu}^{*}=\left(b_{22} / b_{21}\right) \nu^{*}$. Using a similar method of Lemma 13, we can show that the image point $S^{\prime}$ of the point $S$ lies in the fourth quadrant.

Using the conditions (36) and (37), similar to Lemma 14, we can show that the image line $Q^{\prime} R^{\prime}$ of the line $Q R$ lies above the $\theta$-axis, $R^{\prime} S^{\prime}$ lies in the right of the 9 -axis, and $S^{\prime} P^{\prime}$ lies under the $\theta$-axis. Therefore, we have the following lemmas.

Lemma 15. Suppose that $(H)$ and (9) hold. For $v^{*}$ large enough, $Q^{\prime} R^{\prime}$ lies above the $\theta$-axis, $Q^{\prime}$ lies in the second quadrant, and $R^{\prime}$ lies in the first quadrant.

Lemma 16. Suppose that $(H)$ and (9) hold. For $v^{*}$ large enough, $R^{\prime} S^{\prime}$ lies in the right of the 9 -axis and $S^{\prime}$ lies in the fourth quadrant.

Lemma 17. Suppose that $(H)$ and (9) hold. For $v^{*}$ large enough, $S^{\prime} P^{\prime}$ lies below the $\theta$-axis.

Proof of Theorem 1. From Lemmas 12-17, when $\mu^{*}, v^{*}$, and $\tilde{\mu}^{*}$ are large enough and satisfy (36) and (37), then the image $P^{\prime} Q^{\prime} R^{\prime} S^{\prime}$ of the curve $P Q R S$ will contain the zero in it. From Proposition 11, it follows that there exists a vector $\kappa_{0}=\left(\mu_{0}, \nu_{0}\right)$ such that the solution $W_{\kappa_{0}}(t)$ of (30) satisfies $W_{\kappa_{0}}(1)=0$, which implies that the integral equation (27) has a solution $Y(t)$. From (26), it follows that (19) has a solution. Therefore, the problem (1) has at least one solution. The proof is completed.

## 4. Examples

Example 1. Consider the following boundary value system:

$$
\begin{gather*}
D_{0+}^{3 / 2} x(t)+\frac{t^{1 / 3}}{2} x^{1 / 3}(t)+\frac{t^{1 / 3}}{4} \frac{y^{1 / 3}(t)}{1+\left|y^{1 / 3}(t)\right|}+t=0, \\
t \in(0,1), \\
D_{0+}^{3 / 2} y(t)+\frac{t^{1 / 3}}{3} \frac{x^{1 / 3}(t)}{1+\left|x^{1 / 3}(t)\right|}+\frac{t^{1 / 3}}{2} y^{1 / 3}(t)+\frac{t}{4}=0, \\
t \in(0,1), \\
x(0)=y(0)=0, \quad x(1)=2^{1 / 2} x\left(\frac{1}{2}\right), \\
y(1)=2^{1 / 2} y\left(\frac{1}{2}\right), \tag{111}
\end{gather*}
$$

where

$$
\begin{aligned}
f_{1}\left(t, t^{\alpha-1}(x, y)\right) & =f_{1}\left(t, t^{1 / 2}(x, y)\right) \\
& =\frac{t^{1 / 2}}{2} x^{1 / 3}(t)+\frac{t^{1 / 2}}{4} \frac{y^{1 / 3}(t)}{1+t^{1 / 6}\left|y^{1 / 3}(t)\right|}+t \\
f_{2}\left(t, t^{\alpha-1}(x, y)\right) & =f_{2}\left(t, t^{1 / 2}(x, y)\right) \\
& =\frac{t^{1 / 2}}{3} \frac{x^{1 / 3}(t)}{1+t^{1 / 6}\left|x^{1 / 3}(t)\right|}+\frac{t^{1 / 2}}{2} y^{1 / 3}(t)+\frac{t}{4}
\end{aligned}
$$

It is obvious that

$$
\begin{align*}
& \left|f_{1}\left(t, t^{1 / 2}(x, y)\right)\right| \leq \frac{t^{1 / 2}}{2}|x(t)|^{1 / 3}+\frac{t^{1 / 2}}{4}|y(t)|^{1 / 3}+t \\
& \left|f_{2}\left(t, t^{1 / 2}(x, y)\right)\right| \leq \frac{t^{1 / 2}}{3}|x(t)|^{1 / 3}+\frac{t^{1 / 2}}{2}|y(t)|^{1 / 3}+\frac{t}{4} \tag{113}
\end{align*}
$$

where

$$
\begin{gather*}
b_{11}(t)=\frac{t^{1 / 2}}{2}, \quad b_{12}(t)=\frac{t^{1 / 2}}{4} 4, \quad b_{21}(t)=\frac{t^{1 / 2}}{3} \\
b_{22}(t)=\frac{t^{1 / 2}}{2},  \tag{114}\\
a_{1}(t)=t, \quad a_{2}(t)=\frac{t}{4}, \quad \eta=\frac{1}{2} \\
p_{1}=p_{2}=q_{1}=q_{2}=\frac{1}{3}
\end{gather*}
$$

It is easy to check that the function $f$ satisfies (8)-(12). Notice that

$$
\begin{align*}
& G^{*}(s, s)= \frac{1}{\Gamma(1 / 2)\left(1-(1 / 2)^{1 / 2}\right)} \\
& \times \begin{cases}(1-s)^{1 / 2}-\left(\frac{1}{2}-s\right)^{1 / 2}, & 0 \leq s \leq \frac{1}{2}, \\
(1-s)^{1 / 2}, & \frac{1}{2} \leq s \leq 1,\end{cases}  \tag{115}\\
& b_{21}(t), b_{12}(t)<b_{11}(t), b_{22}(t)=\max _{i, j=1,2}\left\{b_{i j}(t)\right\}=\frac{t^{1 / 2}}{2} .
\end{align*}
$$

## Consider

$$
\begin{align*}
\max _{1 \leq i \leq 2} & \int_{0}^{1} G^{*}(s, s)\left(b_{i 1}(s)+b_{i 2}(s)\right) d s \\
& \leq \int_{0}^{1} G^{*}(s, s)\left(\frac{s^{1 / 2}}{2}+\frac{s^{1 / 2}}{2}\right) d s \\
= & \int_{0}^{1} G^{*}(s, s) s^{1 / 2} d s \\
= & \frac{1}{\Gamma(1 / 2)\left(1-(1 / 2)^{1 / 2}\right)}  \tag{116}\\
& \times\left(\int_{0}^{1}[(1-s) s]^{1 / 2} d s-\int_{0}^{1 / 2}\left[\left(\frac{1}{2}-s\right) s\right]^{1 / 2} d s\right) \\
= & \frac{1}{\sqrt{\pi}(1-\sqrt{2} / 2)}\left(\frac{\pi}{8}-\frac{\pi}{32}\right) \approx \frac{0.2945}{1.2533}<1
\end{align*}
$$

which satisfies (9). Therefore, all conditions of Theorem 1 hold and thus the problem (111) has at least a solution.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Boundary Stabilization of a Nonlinear Viscoelastic Equation with Interior Time-Varying Delay and Nonlinear Dissipative Boundary Feedback 

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#### Abstract

We investigate a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback. Under suitable assumptions on the relaxation function and time-varying delay effect together with nonlinear dissipative boundary feedback, we prove the global existence of weak solutions and asymptotic behavior of the energy by using the Faedo-Galerkin method and the perturbed energy method, respectively. This result improves earlier ones in the literature, such as Kirane and SaidHouari (2011) and Ammari et al. (2010). Moreover, we give an positive answer to the open problem given by Kirane and Said-Houari (2011).


## 1. Introduction

In this paper, we consider the global existence and asymptotic behavior of a nonlinear viscoelastic equation with interior time-varying delay and nonlinear dissipative boundary feedback as follows:

$$
\begin{gather*}
u_{t t}-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s+a u_{t}(x, t-\tau(t))=0 \\
x \in \Omega, \quad t>0 \\
u(x, t)=0, \quad \text { on } \Gamma_{0} \times(0, \infty),  \tag{1}\\
\frac{\partial u}{\partial v}+g\left(u_{t}(x, t)\right)=0, \quad \text { on } \Gamma_{1} \times[0, \infty), \\
u(x, 0)=u_{0}(x), \quad u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega \\
u_{t}(x, t-\tau(t))=f_{0}(x, t), \quad x \in \Omega,-\tau(0) \leq t \leq 0
\end{gather*}
$$

where $\Omega$ is a bounded domain of $R^{n}(n \geq 1)$ with a smooth boundary $\partial \Omega$ of $C^{2}, a$ is a positive real constant, $\tau(t)>0$ represents the time-varying delay effect and the initial data
$u_{0}, u_{1}, f_{0}$ are given functions belonging to suitable spaces, $h(t)$ is a positive function that represents the kernel of the memory term, $g\left(u_{t}\right)$ is nonlinear dissipative boundary feedback, and $f_{0}, h, g$ satisfy suitable assumptions (see in Section 2).

This model appears in viscoelasticity (see [1, 2]). In the case of velocity-dependent material density (i.e., $\rho=0$ ) as well as presence of $\mu_{2}=0$ and in the absence of the memory effect (i.e., $g=0$ ), ( 1 ) reduces to the wave equation. There is large literature on the global existence and uniform stabilization of wave equations. We refer the readers to [3-5]. It is worth mentioning that Zhang and Miao [3] considered the nonlinear wave equation with dissipative term and boundary damping

$$
\begin{gather*}
u_{t t}-\Delta u+a(x) u_{t}+f(u)=0, \quad \text { in } \Omega \times[0, \infty), \\
u=0, \quad \text { on } \Gamma_{1} \times[0, \infty) \\
\frac{\partial u}{\partial v}+g\left(u_{t}\right)=0, \text { on } \Gamma_{0} \times[0, \infty)  \tag{2}\\
u(x, 0)=u^{0}(x), \quad u_{t}(x, 0)=u^{1}(x), \quad \text { in } \Omega
\end{gather*}
$$

and they proved the existence and uniform decay of strong and weak solutions by using the Glerkin method and the multiplier technique, respectively. Later on, Zhang et al. [4] improved earlier ones in [3]. More precisely, they investigated the global existence and uniform stabilization of generalized dissipative Klein-Gordon equation with boundary damping

$$
\begin{gather*}
u_{t t}-\Delta u+b(x) u_{t}+f(u)+h(\nabla u)=0, \quad \text { in } \Omega \times[0, \infty), \\
u=0, \quad \text { on } \Gamma_{1} \times[0, \infty), \\
\frac{\partial u}{\partial v}+g\left(u_{t}\right)=0, \quad \text { on } \Gamma_{0} \times[0, \infty), \\
u(x, 0)=u^{0}(x), \quad u_{t}(x, 0)=u^{1}(x), \quad \text { in } \Omega, \tag{3}
\end{gather*}
$$

and they proved the existence and uniform decay of strong and weak solutions by using the nonlinear semigroup method, the perturbed energy method, and the multiplier technique. Quite recently, Cavalcanti et al. [6] considered the following model:

$$
\begin{gather*}
u_{t t}-\Delta_{\mathscr{M}} u+a(x) g\left(u_{t}\right)=0, \quad \text { on } \mathscr{M} \times(0, \infty), \\
u(x, 0)=u^{0}(x), \quad u_{t}(x, 0)=u^{1}(x), \quad \text { for } x \in \mathscr{M} \tag{4}
\end{gather*}
$$

where $\mathscr{M}$ is a smooth oriented embedded compact surface without boundary in $R^{3}$ and $\Delta_{\mathscr{M}}$ is the Laplace-Beltrami operator on manifold $\mathscr{M}$; furthermore, they obtained explicit and optimal decay rates of the energy. Later on, Cavalcanti et al. [7] extended the result for $n$-dimensional compact Riemannian manifolds ( $\mathscr{M}, g$ ) with boundary in two ways: (i) by reducing arbitrarily the region where the dissipative effect lies (this gives us a totally sharp result with respect to the boundary measure and interior measure where the damping is effective) and (ii) by controlling the existence of subsets on the manifold that can be left without any dissipative mechanism, namely, a precise part of radially symmetric subsets. An analogous result holds for compact Riemannian manifolds without boundary.

In the case $\rho=0$ and in the absence of delay (i.e., $\mu_{2}=0$ ), there is large literature on the existence and decay of nonlinear viscoelastic equation during the past decades. In [8], Cavalcanti et al. considered the exponential decay for the solution of viscoelastic wave equation with localized damping

$$
\begin{array}{r}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} g(t-s) \Delta u(s) d s+a(x) u_{t}+u|u|^{r}=0 \\
x \in \Omega, \quad t>0 \tag{5}
\end{array}
$$

Under the condition that $a(x) \geq a_{0}>0$ on $\omega \subset \Omega$, with $\omega$ satisfying some geometry restrictions and

$$
\begin{equation*}
-\xi_{1} g(t) \leq g^{\prime}(t) \leq-\xi_{2} g(t), \quad t \geq 0 \tag{6}
\end{equation*}
$$

they proved an exponential decay result for the energy. Berrimi and Messaoudi [9] improved Cavalcanti's result by introducing a differential functional which allowed to weaken
the conditions on both $a(x)$ and $g$. In [10], Cavalcanti and Oquendo studied

$$
\begin{gather*}
\left|u_{t}\right|^{\rho} u_{t t}-k_{0} \Delta u+\int_{0}^{t} \operatorname{div}[a(x) g(t-s) \nabla u(s)] d s  \tag{7}\\
+b(x) h\left(u_{t}\right)+f(u)=0, \quad x \in \Omega, t>0 .
\end{gather*}
$$

Under some geometric restrictions on $\omega$ and assuming that

$$
\begin{gather*}
a(x) \geq a_{0}>0, \quad \forall x \in \omega \\
-\xi_{1} g(t) \leq g^{\prime}(t) \leq-\xi_{2} g(t), \quad t \geq 0  \tag{8}\\
a(x)+b(x) \geq \rho>0, \quad \forall x \in \Omega
\end{gather*}
$$

they established an exponential stability for the relaxation function $g$ decaying exponentially and $h$ linear and polynomial stability for $g$ decaying polynomially and $h$ nonlinear. It is worth mentioning that Zhang et al. [11] studied the following initial boundary value problem:

$$
\begin{gather*}
u_{t t}+A u+\int_{0}^{t} g(t-s) A u d s=0 \quad \text { in } \Omega \times(0, \infty), \\
u=0 \quad \text { on } \Gamma \times(0, \infty)  \tag{9}\\
u(0)=u_{0}, \quad u_{t}(0)=u_{1}
\end{gather*}
$$

Furthermore, they showed that the solutions of (9) decay uniformly in time, with rates depending on the rate of decay of the kernel $g$. More precisely, the solution decays exponentially to zero provided that $g$ decays exponentially to zero. When $g$ decays polynomially, we show that the corresponding solution also decays polynomially to zero with the same rate of decay. For other related works, we refer the readers to $[12-21]$ and the references therein.

On the other hand, concerning the study of the following nonlinear viscoelastic equation with memory, there are a substantial number of contributions:

$$
\begin{equation*}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} g(t-s) \Delta u(s) d s+F\left(u, u_{t}, u_{t t}\right)=0 \tag{10}
\end{equation*}
$$

Recently, Han and Wang [22] investigated the following problem:

$$
\begin{equation*}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u-\Delta u_{t t}+\int_{0}^{t} g(t-s) \Delta u(s) d s=b|u|^{p-2} u \tag{11}
\end{equation*}
$$

By introducing a new functional and using potential well method, the authors established the global existence and uniform decay if the initial data are in a suitable stable set. Cavalcanti et al. [23] studied a related problem with strong damping as follows:

$$
\begin{equation*}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u-\Delta u_{t t}+\int_{0}^{t} g(t-s) \Delta u(s) d s-\gamma \Delta u_{t}=0 \tag{12}
\end{equation*}
$$

By assuming $0<\rho \leq 2 /(n-2)$, if $n \geq 3$ or $\rho>0$ and if $n=1,2$ and $g(t)$ decays exponentially, they established that the global
existence resulted for $\gamma \geq 0$ and the exponential decay of the energy for $\gamma>0$. This result has been extended to a situation $\gamma=0$ by Messaoudi and Tatar [24] and exponential decay and polynomial decay results have been shown in the absence as well as presence of a source term. Later on, inspired by the ideas of [25-27], Han and Wang [22] investigated the general decay of solutions of energy for the nonlinear viscoelastic equation

$$
\begin{equation*}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u-\Delta u_{t t}+\int_{0}^{t} g(t-s) \Delta u(s) d s+u_{t}\left|u_{t}\right|^{k}=0 \tag{13}
\end{equation*}
$$

In recent years, the control of partial differential equation with time delay effects has become an active area of research; see, for instance, $[28,29]$ and the references therein. The presence of delay may be a source of instability. For instance, it was proved in [30-34] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used. In [32], Nicaise and Pignotti examined (1) with $\rho=0, g \equiv 0, \mu_{1}>0, \mu_{2}>0$, and $\tau(t)=\tau$ being a constant delay in the case of mixed homogeneous Dirichlet-Neumann boundary conditions, under a geometric condition on the Neumann part of the boundary. More precisely, they investigated the following system with linear frictional damping term and internal constant delay:

$$
\begin{gather*}
u_{t t}(x, t)-\Delta u(x, t)+\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau)=0, \\
x \in \Omega, \quad t>0 \\
u(x, t)=0, \quad x \in \Gamma_{0}, t>0  \tag{14}\\
\frac{\partial u}{\partial v}(x, t)=0, \quad x \in \Gamma_{1}, t>0
\end{gather*}
$$

or with boundary constant delay

$$
\begin{align*}
& u_{t t}(x, t)-\Delta u(x, t)=0, \quad x \in \Omega, t>0 \\
& u(x, t)=0, \quad x \in \Gamma_{0}, t>0 \\
& \frac{\partial u}{\partial v}(x, t)+\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau)=0  \tag{15}\\
& x \in \Gamma_{1}, \quad t>0
\end{align*}
$$

In the presence of delay ( $\mu_{2}>0$ ), Nicaise and Pignotti [32] examined systems (14) and (15) and proved under the assumptions $\mu_{2}<\mu_{1}$ that the energy is exponentially stable. Otherwise, they constructed a sequence of delays for which the corresponding solution is instable. The main approach used there is an observability inequality together with a Carleman estimate. See also [35] for treatment to these problems in more general abstract form and [36] for analogous results in the case of boundary time-varying delay. We also recall the result by Nicaise et al. [36], where the researchers proved the same result as in [32] for the one space dimension by applying the spectral analysis approach. Recently, Kirane and Said-Houari [37] considered (1) with $\rho=0, \mu_{1}>0, \mu_{2}>0$, and $\tau(t) \equiv \tau$ being a constant delay in
the case of the initial and Dirichlet boundary wave equation with a linear damping and a delay term as follows:

$$
\begin{gather*}
u_{t t}-\Delta u+\int_{0}^{t} g(t-s) \Delta u(s) d s+\mu_{1} u_{t}(x, t) \\
+\mu_{2} u_{t}(x, t-\tau)=0, \quad x \in \Omega, t>0 \\
u(x, t)=0, \quad x \in \partial \Omega, t>0,  \tag{16}\\
u(x, 0)=u_{0}(x), \quad u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega \\
u_{t}(x, t-\tau)=f_{0}(x, t-\tau), \quad x \in \Omega, t \in(0, \tau)
\end{gather*}
$$

Under an assumption between the weight of the delay term in the feedback and the weight of the term without delay, using the Faedo-Galerkin method combined with some energy estimate, they proved the global existence of (16). Also, they proved exponential decay of (16) via suitable Lyapunov functionals.

Recently, the stability of PDEs with time-varying delays was studied in [38-44]. In [40], Nicaise and Pignotti investigated the stabilization problem by interior damping of the wave equation with internal time-varying delay feedback and established exponential stability estimates by introducing suitable Lyapunov functionals, under the condition $\left|\mu_{2}\right|<$ $\sqrt{1-d} \mu_{1}$ in which the positivity of the coefficient $\mu_{1}$ is not necessary. In [41], Nicaise et al. showed the exponential stability of the heat and wave equations with time-varying boundary delay in 1-D, under the condition $0 \leq \mu_{2}<$ $\sqrt{1-d} \mu_{1}$, where $d$ is a constant such that $\tau^{\prime}(t) \leq d<1$.

The rest of the paper is organized as follows. In Section 2, we show some assumptions and state our main result. In Section 3, we present the proof of our main result. That is, we will prove the global existence by using Faedo-Galerkin method and establish the general decay result (including exponential decay and polynomial decay) by using the perturbed energy method. Finally, in Section 4, we give further remarks on this context.

## 2. Some Assumptions and Main Results

In this section, before proceeding to our analysis, we present some assumptions and state the main result. We use the standard Hilbert space $L^{2}(\Omega)$ and the Sobolev space $H_{0}^{1}(\Omega)$ with their usual scalar products and norms. Throughout this paper, $C_{i}$ is used to denote a generic positive constant from line to line.

For the relaxation function $h$, we assume that
(G1) $h(t):(0, \infty) \rightarrow(0, \infty)$ is a nonincreasing differentiable function such that

$$
\begin{equation*}
1-\int_{0}^{\infty} h(s) d s=l>0 \tag{17}
\end{equation*}
$$

(G2) there exists a nonincreasing differentiable function $\zeta(t)$ such that

$$
\begin{equation*}
h^{\prime}(t) \leq-\zeta(t) h^{p}(t), \quad 1 \leq p<\frac{3}{2}, t \geq 0 \tag{18}
\end{equation*}
$$

We assume that $\rho$ satisfies

$$
\begin{equation*}
0<\rho \leq \frac{2}{n-2}, \quad \text { if } n \geq 3 ; \quad \rho>0, \quad \text { if } n=1,2 . \tag{19}
\end{equation*}
$$

For the time-varying delay, we assume that there exist positive constant $\tau_{0}, \bar{\tau}$ such that

$$
\begin{equation*}
0<\tau_{0} \leq \tau(t) \leq \bar{\tau}, \quad \forall t>0 \tag{20}
\end{equation*}
$$

Furthermore, we assume that the delay satisfies

$$
\begin{equation*}
\tau^{\prime}(t) \leq d<1, \quad \forall t>0 \tag{21}
\end{equation*}
$$

that

$$
\begin{equation*}
\tau(t) \in W^{2, \infty}([0, T]), \quad \forall T>0 \tag{22}
\end{equation*}
$$

and that $\mu_{1}, \mu_{2}$ satisfy

$$
\begin{equation*}
\left|\mu_{2}\right|<\sqrt{1-d} \mu_{1} . \tag{23}
\end{equation*}
$$

Remark 1. We show an example of functions satisfying (G2) as follows:

$$
\begin{gather*}
h(s)=e^{-\sigma s}, \quad p=1 \\
h(s)=\vartheta(1+s)^{-1 /(p-1)}, \quad p>1 \tag{24}
\end{gather*}
$$

for $\sigma, \vartheta>0$ to be chosen properly; see [2].
Remark 2. Condition $p<3 / 2$ is imposed so that $\int_{0}^{\infty} h^{2-p}(s) d s<\infty$.

Now, we are in a position to state our main results.
Theorem 3. Let (20)-(23) be satisfied and $h$ satisfy (G2). Then, given $\left(u_{0}, u_{1}\right) \in H_{0}^{1}(\Omega) \times L^{2}(\Omega), f_{0} \in L^{2}(\Omega \times(0,1))$, and $T>0$, there exists a unique weak solution $u(x, t)$ such that

$$
\begin{align*}
& u \in C\left(0, T ; H_{0}^{1}(\Omega)\right) \cap C^{1}\left(0, T ; L^{2}(\Omega)\right),  \tag{25}\\
& u_{t} \in L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right) \cap L^{2}((0, T) \times \Omega)
\end{align*}
$$

Moreover, if (20)-(23) hold and $h$ satisfies (G1) and (G2), then there exist two positive constants $K, k$ such that for any solution of the problem (1) of the energy satisfies

$$
\begin{gather*}
\mathscr{E}(t) \leq K e^{-k t}, \quad p=1, t \geq t_{0}  \tag{26}\\
\mathscr{E}(t) \leq K(1+t)^{-1 /(p-1)}, \quad p>1, t \geq t_{0} \tag{27}
\end{gather*}
$$

## 3. Proof of the Main Result

In this section, we will divide our proof into two steps. In Step 1, we prove the global existence of weak solutions by using Faedo-Galerkin method benefited from the ideas of [2, 3, 37]. In Step 2, we establish the general decay of energy by introducing the new energy functional and using the perturbed energy method inspired by the contributions; see, for instance, $[2-4,11,39]$.

Step 1 (global existence of weak solutions). Let $\{\omega\}_{j}^{\infty}$ be an orthogonal basis of $H_{0}^{1}(\Omega)$ with $\omega_{j}$ being the eigenfunction of the following problem:

$$
\begin{align*}
-\Delta \omega_{j} & =\lambda_{j} \omega_{j}, \quad x \in \Omega, \\
\omega_{j} & =0, \quad x \in \partial \Omega . \tag{28}
\end{align*}
$$

Denote $W_{n}=\operatorname{Span}\left\{\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right\}$ for subspace generated by the first $n$ vectors of the basis of $\{\omega\}_{j}^{\infty}$. Then, we construct approximation of the solution $(u, z)$ as follows:

$$
\begin{gather*}
u_{n}(x, t)=\sum_{j=1}^{n} g_{j n}(t) \omega_{j}, \\
z_{n}(x, t, \rho)=\sum_{j=1}^{m} h_{j n}(t) \phi_{j}(x, \rho) \tag{29}
\end{gather*}
$$

and we choose two sequences $u_{0 n}$ and $u_{1 n}$ in $W_{n}$ and a sequence $z_{0 n}$ in $V_{n}$ such that $u_{0 n} \rightarrow u_{0}$ strongly in $H_{0}^{1}(\Omega), u_{1 n} \rightarrow u_{1}$ strongly in $L^{2}(\Omega)$, and $z_{0 n} \rightarrow f_{0}$ strongly in $L^{2}(\Omega \times(0,1))$. Define the sequence $\phi_{j}(x, \rho)$ as follows: $\phi_{j}(x, 0)=\phi_{j}(x)$. Then, from [37, pp 1069], we may extend $\phi_{j}(x, 0)$ by $\phi_{j}(x, \rho)$ over $L^{2}(\Omega \times(0,1))$ and denote $V_{n}=$ $\operatorname{Span}\left\{\phi_{1}, \phi_{2}, \ldots, \phi_{n}\right\}$.

To facilitate further our analysis, we introduce as in [32, 36,39 ] the new variable

$$
\begin{equation*}
z(x, \theta, t)=u_{t}(x, t-\tau(t) \theta), \quad x \in \Omega, \theta \in(0,1), t>0 \tag{30}
\end{equation*}
$$

Then, we get

$$
\begin{array}{r}
\tau(t) z(x, \theta, t)+\left(1-\tau^{\prime}(t) \theta\right) z_{\theta}(x, \theta, t)=0,  \tag{31}\\
x \in \Omega, \quad \theta \in(0,1), \quad t>0 .
\end{array}
$$

Therefore, the problem (1) can be rewritten as follows:

$$
\begin{align*}
& \left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s \\
& +\mu_{1} u_{t}(x, t)+\mu_{2} z(x, 1, t)=0, \quad x \in \Omega, t>0 \\
& \tau(t) z(x, \theta, t)+\left(1-\tau^{\prime}(t) \theta\right) z_{\theta}(x, \theta, t)=0 \\
& x \in \Omega, \quad \theta \in(0,1), \quad t>0  \tag{32}\\
& u(x, t)=0, \quad x \in \partial \Omega, t>0, \\
& u(x, 0)=u_{0}(x), \quad u_{t}(x, 0)=u_{1}(x), \quad x \in \Omega \\
& z(x, \theta, 0)=f_{0}(x,-\tau(0)) \\
& x \in \Omega, \quad \theta \in(0,1), \quad-\tau(0) \leq t \leq 0
\end{align*}
$$

Hence, $\left(u_{n}(t), z_{n}(t)\right)$ are solutions to the following Cauchy problem as follows:

$$
\begin{gather*}
\int_{\Omega}\left|u_{t n}\right|^{\rho} u_{t t n} \omega_{j} d x+\int_{\Omega} \nabla u_{n} \nabla \omega_{j} d x \\
-\int_{0}^{t} h(t-s) \nabla u(s) \nabla \omega_{j} d x d s \\
+\int_{\Omega}\left[\mu_{1} u_{t n}(x, t)+\mu_{2} z_{n}(x, 1, t)\right] \omega_{j} d x=0,  \tag{33}\\
z_{n}(x, 0, t)=u_{t n}(x, t), \\
\left(u_{n}(0), u_{t n}(0)\right)=\left(u_{0 n}, u_{1 n}\right), \\
\int_{\Omega}\left[\tau(t) z_{n t}(x, \theta, t)+\left(1-\tau^{\prime}(t) \theta\right) z_{n \theta}(x, \theta, t)\right] \phi_{j} d x=0, \\
z_{n, 0}=z_{0 n} . \tag{34}
\end{gather*}
$$

By standard method of ODE, we know that there exists only one local solution of the Cauchy problem (33) and (34) on some interval $\left[0, t_{n}\right), 0<t_{n}<T$, for arbitrary $T>0$; then, this solution can be extended to the whole interval $[0, T]$ by a priori estimates below.

To facilitate further our analysis, we need some notations and technical Lemmas 4 and 6 . Let us first introduce some notations

$$
\begin{gather*}
(\phi \star \psi)(t)=\int_{0}^{t} \phi(t-s) \psi(s) d s \\
(\phi \diamond \psi)(t)=\int_{0}^{t} \phi(t-s)|\psi(t)-\psi(s)| d s  \tag{35}\\
(\phi \circ \psi)(t)=\int_{0}^{t} \phi(t-s) \int_{\Omega}|\psi(t)-\psi(s)|^{2} d x d s
\end{gather*}
$$

with these notations; we have the following lemma given in [2, 11].

Lemma 4. For $\phi \in C^{1}(\mathscr{R})$ and $\psi \in H^{1}(0, T)$, one has

$$
\begin{align*}
(\phi \star \psi)(t) \cdot \psi(t)= & -\frac{1}{2} \phi(t)\|\psi(t)\|^{2}+\frac{1}{2}\left(\phi^{\prime} \diamond \psi\right)(t) \\
& -\frac{1}{2} \frac{d}{d t}[(\phi \diamond \psi)(t) \\
& \left.-\left(\int_{0}^{t} \phi(s) d s\right)|\phi|^{2} d x\right] . \tag{36}
\end{align*}
$$

Remark 5. In fact, the proof of this lemma follows by differentiating the term $g \diamond \phi$. More details are presented in [2, 11, 37].

Lemma 6. Assuming that $v \in L^{\infty}\left(0, T ; H^{1}(\Omega)\right), h$ is a continuous function such that

$$
\begin{equation*}
\int_{0}^{\infty} h^{1-\alpha}(s) d s<\infty, \quad 0 \leq \alpha \leq 1 \tag{37}
\end{equation*}
$$

Then, we have

$$
\begin{align*}
(h \circ \nabla v)(t) \leq & 2\left[\int_{0}^{t}\|\nabla v(s)\|_{2}^{2} d s+t\|\nabla v(t)\|_{2}^{2}\right]^{(p-1) / p}  \tag{38}\\
& \times((h \circ \nabla v)(t))^{1 / p}
\end{align*}
$$

Proof. It suffices to observe that, for $q>1,0 \leq \alpha \leq 1$,

$$
\begin{align*}
(h \circ \nabla v)(t)= & \int_{0}^{t} h(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2} d s \\
= & \int_{0}^{t} h^{(1-\alpha) / q}(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2 / q} h^{(q-1+\alpha) / q} \\
& \quad \times(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2(q-1) / q} d s \tag{39}
\end{align*}
$$

By applying Hölder inequality, we obtain

$$
\begin{align*}
& (h \circ \nabla v)(t) \\
& \quad \leq\left(\int_{0}^{t} h^{(1-\alpha) / q}(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2} d s\right)^{1 / q} \\
& \quad \times\left(\int_{0}^{t} h^{(q-1+\alpha) /(q-1)}(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2} d s\right)^{(q-1) / q} . \tag{40}
\end{align*}
$$

Taking $q=(p-1+\alpha) /(p-1)$, we get

$$
\begin{align*}
& (h \circ \nabla v)(t) \\
& \leq\left(\int_{0}^{t} h^{(1-\alpha)(p-1) /(p-1+\alpha)}(t-s)\right. \\
& \left.\quad \times\|\nabla v(t)-\nabla v(s)\|_{2}^{2} d s\right)^{(p-1) /(p-1+\alpha)} \\
& \quad \times\left(\int_{0}^{t} h^{p \alpha /(p-1+\alpha)}(t-s)\|\nabla v(t)-\nabla v(s)\|_{2}^{2} d s\right)^{\alpha /(p-1+\alpha)} . \tag{41}
\end{align*}
$$

Finally, taking $\alpha=1$ in the above equality, Lemma 6 is completed.
3.1. A Priori Estimate. Taking $\omega_{j}=u_{t n}$ in (33) and integrating over $(0, t)$, using integration by parts and Lemma 4 , we obtain

$$
\begin{align*}
& \frac{1}{2}\left[\left(1-\int_{0}^{t} h(s) d s\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\frac{2}{\rho+2}\left\|u_{t n}\right\|_{\rho+2}^{\rho+2}+\left(h \circ \nabla u_{n}\right)(t)\right] \\
& \quad+\mu_{1} \int_{0}^{t}\left\|u_{t n}\right\|_{2}^{2} d s+\mu_{2} \int_{0}^{t} \int_{\Omega} z_{n}(x, 1, s) u_{t n}(x, s) d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} h(s)\left\|\nabla u_{n}(s)\right\|_{2}^{2} d s-\frac{1}{2} \int_{0}^{t}\left(h^{\prime} \circ \nabla u_{n}\right)(s) d s \\
& =  \tag{42}\\
& \frac{1}{2}\left(\left\|\nabla u_{0}\right\|_{2}^{2}+\frac{2}{\rho+2}\left\|u_{1}\right\|_{\rho+2}^{\rho+2}\right)
\end{align*}
$$

Taking $\phi_{j}=z_{n}(\xi / \tau(t))$ in (34) and integrating over ( $0, t$ ), we get

$$
\begin{align*}
& \frac{\xi}{2} \int_{\Omega} \quad \int_{0}^{1} z_{n}^{2}(x, \theta, t) d \theta d x \\
& \quad+\xi \int_{0}^{t} \int_{\Omega} \int_{0}^{1} \frac{1-\tau^{\prime}(t) \theta}{\tau(t)} z_{n \theta} z_{n}(x, \theta, s) d \theta d x d s  \tag{43}\\
& \quad=\frac{\xi}{2}\left\|z_{0 n}\right\|_{L^{2}(\Omega \times(0,1))}^{2} .
\end{align*}
$$

Now, integrating by parts, we obtain

$$
\begin{align*}
& \int_{0}^{t} \int_{\Omega} \int_{0}^{1} \frac{1-\tau^{\prime}(t) \theta}{\tau(t)} z_{n \theta} z_{n}(x, \theta, s) d \theta d x d s \\
& =\frac{1}{2} \int_{0}^{t} \int_{\Omega} \int_{0}^{1}\left(\frac{\partial}{\partial \theta} z_{n}^{2}(x, \theta, s) \frac{1-\tau^{\prime}(t) \theta}{\tau(t)}\right) d \theta d x d s \\
& =-\frac{1}{2} \int_{0}^{t} \int_{\Omega} \frac{1-\tau^{\prime}(t) \theta}{\tau(t)} z_{n}^{2}(x, \theta, s) d s d x \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{\Omega}\left[\frac{\left(1-\tau^{\prime}(t) \theta\right) z_{n}^{2}(x, 1, s)-z_{n}^{2}(x, 0, s)}{\tau(t)}\right] d x d s \tag{44}
\end{align*}
$$

It follows from (43) and (44) that

$$
\begin{align*}
& \frac{\xi}{2} \int_{\Omega} \int_{0}^{1} z_{n}^{2}(x, \theta, t) d \theta d x \\
& \quad+\xi \int_{0}^{t} \int_{\Omega} \int_{0}^{1} \frac{\tau^{\prime}(t) \theta-1}{\tau(t)} z_{n}^{2}(x, \theta, s) d x d s \\
& \quad+\frac{1}{2} \int_{0}^{t} \int_{\Omega}\left[\frac{\left(1-\tau^{\prime}(t) \theta\right) z_{n}^{2}(x, 1, s)-z_{n}^{2}(x, 0, s)}{\tau(t)}\right] d x d s \\
& \quad=\frac{\xi}{2}\left\|z_{0 n}\right\|_{L^{2}(\Omega \times(0,1))}^{2} \tag{45}
\end{align*}
$$

Summing up (42) and (45), we conclude that

$$
\begin{align*}
\mathscr{E}_{n}(t) & +\mu_{1} \int_{0}^{t}\left\|u_{t n}\right\|_{2}^{2} d s+\mu_{2} \int_{0}^{t} \int_{\Omega} z_{n}(x, 1, s) u_{t n}(x, s) d x d s \\
& +\frac{1}{2} \int_{0}^{t} h(s)\left\|\nabla u_{n}(s)\right\|_{2}^{2} d s-\frac{1}{2} \int_{0}^{t}\left(h^{\prime} \circ \nabla u_{n}\right)(s) d s \\
& +\frac{\xi}{2} \int_{0}^{t} \int_{\Omega} \frac{\tau^{\prime}(t) \theta-1}{\tau(t)} z_{n}^{2}(x, \theta, s) d x d s \\
& +\frac{1}{2} \int_{0}^{t} \int_{\Omega}\left[\frac{\left(1-\tau^{\prime}(t) \theta\right) z_{n}^{2}(x, 1, s)-z_{n}^{2}(x, 0, s)}{\tau(t)}\right] d x d s \\
= & \frac{1}{2}\left\|\nabla u_{0}\right\|_{2}^{2}+\frac{1}{\rho+2}\left\|u_{t n}\right\|_{\rho+2}^{\rho+2}+\frac{\xi}{2}\left\|z_{0}\right\|_{L^{2}(\Omega \times(0,1))}^{2}=\mathscr{E}_{n}(0), \tag{46}
\end{align*}
$$

where

$$
\begin{align*}
\mathscr{E}_{n}(t)=\frac{1}{2} & {\left[\left(1-\int_{0}^{t} h(s) d s\right)\left\|\nabla u_{n}\right\|_{2}^{2}+\frac{2}{\rho+2}\left\|u_{t n}\right\|_{\rho+2}^{\rho+2}\right.}  \tag{47}\\
& \left.+\left(h \circ \nabla u_{n}\right)(t)\right]+\frac{\xi}{2}\left\|z_{n}\right\|_{L^{2}(\Omega \times(0,1))}^{2}
\end{align*}
$$

Using Young's inequality and noticing (20) and (21), we arrive at

$$
\begin{align*}
& \left(\mu_{1}-\frac{\mu_{2} \xi}{2}\right) \int_{0}^{t}\left\|u_{t n}\right\|_{2}^{2} d s \\
& \quad+\int_{0}^{t} \int_{\Omega}\left[\xi \frac{1-\tau^{\prime}(t)}{2 \tau(t)}-\frac{\mu_{2}}{2 \xi}\right] z_{n}^{2}(x, 1, s)  \tag{48}\\
& \quad+\frac{1}{2} \int_{0}^{t} h(s)\left\|\nabla u_{n}(s)\right\|_{2}^{2} d s-\frac{1}{2} \int_{0}^{t}\left(h^{\prime} \circ \nabla u_{n}\right)(s) d s \\
& \quad+\frac{\xi}{2} \int_{0}^{t} \int_{\Omega} \frac{\tau^{\prime}(t) \theta-1}{\tau(t)} z_{n}^{2}(x, \theta, s) d x d s=\mathscr{E}_{n}(0)
\end{align*}
$$

Choosing some value of $\tau(t)>0$ and $\theta$ and noticing (20) and (21), we have $\left(\tau^{\prime}(t) \theta-1\right) / \tau(t)>0$. Moreover, choosing some value of $\tau(t)>0$ and $\xi$, we obtain

$$
\begin{equation*}
\mu_{1}-\frac{\mu_{2} \xi}{2}>0, \quad \xi \frac{1-\tau^{\prime}(t)}{2 \tau(t)}-\frac{\mu_{2}}{2 \xi}>0 \tag{49}
\end{equation*}
$$

That is,

$$
\begin{equation*}
\sqrt{\frac{\mu_{2} \tau(t)}{1-\tau^{\prime}(t)}}<\xi<\frac{2 \mu_{1}}{\mu_{2}} \tag{50}
\end{equation*}
$$

In fact, by (20) and (21), we get $\sqrt{\mu_{2} \tau(t)_{0} /(1-d)}<\xi<$ $2 \mu_{1} / \mu_{2}$. From (48) and (50), (G1), and (G1) and Lemma 6, we conclude that we can find a positive $C$ independent of $n$, such that

$$
\begin{equation*}
\mathscr{E}_{n}(t) \leq C \tag{51}
\end{equation*}
$$

Hence, using the fact that $1-\int_{0}^{t} h(s) d s \geq l$, the estimate (51), and equality (47), we deduce

$$
\begin{aligned}
& u_{n} \text { is uniformly bounded in } L^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right), \\
& u_{t n} \text { is uniformly bounded in } L^{\infty}\left(0, T ; L^{2}(\Omega)\right)
\end{aligned}
$$

$z_{n}$ is uniformly bounded in $L^{\infty}\left(0, T ; L^{2}(\Omega \times(0,1))\right)$.

By (52), we infer that there exist two subsequences $u_{n}, z_{n}$ (still denoted by $u_{n}, z_{n}$ ) and two functions $u$ and $z$, such that

$$
\begin{gather*}
u_{n} \rightharpoonup u \text { weakly star in } L^{\infty}\left(0, T ; H_{0}^{1}(\Omega)\right), \\
u_{t n} \rightharpoonup u_{t} \text { weakly star in } L^{\infty}\left(0, T ; L^{2}(\Omega)\right)  \tag{53}\\
z_{n} \rightharpoonup z \text { weakly star in } L^{\infty}\left(0, T ; L^{2}(\Omega \times(0,1))\right) .
\end{gather*}
$$

From (52), we have $u_{n}$ is bounded in $L^{2}\left(0, T ; H_{0}^{1}(\Omega)\right)$ and $u_{t n}$ is bounded in $L^{2}\left(0, T ; L^{2}(\Omega)\right)$. Consequently, $u_{n}$ is bounded in $L^{2}\left(0, T ; L^{2}(\Omega)\right)$. More details are present in [37, pp 1072].

Since the Sobolev embedding $H^{1}\left(0, T ; H^{1}(\Omega)\right) \hookrightarrow$ $L^{2}\left(0, T ; L^{2}(\Omega)\right)$ is compact, using Aubin-Lions theorem (see [45]), we can extract a subsequence of $u_{n}$ (still denoted by $\left.u_{n}\right)$, such that

$$
\begin{gather*}
u_{n} \longrightarrow u \text { strongly in } L^{2}\left(0, T ; L^{2}(\Omega)\right) \\
u_{t n} \longrightarrow u_{t} \text { strongly in } L^{2}\left(0, T ; L^{2}(\Omega)\right) \tag{54}
\end{gather*}
$$

which implies $u_{t n} \rightarrow u_{t}$ almost everywhere in $\Omega \times(0, T)$.
Hence,

$$
\begin{equation*}
\left|u_{t n}\right|^{\rho} u_{t n} \longrightarrow\left|u_{t}\right|^{\rho} u_{t} \text { almost everywhere in } \Omega \times(0, T) . \tag{55}
\end{equation*}
$$

On the other hand, by the Sobolev embedding theorem and estimate (51), this yields

$$
\begin{align*}
\left\|\left|u_{t n}\right|^{\rho} u_{t n}\right\|_{L^{2}\left(0, T ; L^{2}(\Omega)\right)} & =\int_{0}^{T} \int_{\Omega}\left|u_{t n}\right|^{2(\rho+1)} d x d t \\
& \leq C_{S}^{2(\rho+1)} \int_{0}^{T}\left\|\nabla u_{t n}\right\|_{2}^{2(\rho+1)} d t  \tag{56}\\
& \leq C_{S}^{2(\rho+1)} C^{\rho+1} T
\end{align*}
$$

where $C_{S}$ is the Sobolev embedding constant. Thus, using (55), (56), and Lions Lemma [46], we get

$$
\begin{equation*}
\left|u_{t n}\right|^{\rho} u_{t n} \rightharpoonup\left|u_{t}\right|^{\rho} u_{t} \text { weakly in } L^{2}\left(0, T ; L^{2}(\Omega)\right) \tag{57}
\end{equation*}
$$

Let $\mathscr{D}(0, T)$ be the space of $C^{\infty}$ functions with compact support in ( $0, T$ ). Multiplying the first equation in (33) by $\Theta(t) \in \mathscr{D}(0, T)$ and integrating over $(0, T)$, we conclude that

$$
\begin{align*}
& -\frac{1}{\rho+1} \int_{0}^{T}\left(\left|u_{t n}\right|^{\rho} u_{t n}, \omega_{j}\right) \Theta_{t}(t) d t \\
& \quad+\int_{0}^{T}\left(\nabla u_{t n}, \nabla \omega_{j}\right) \Theta(t) d t \\
& \quad-\int_{0}^{T} \int_{0}^{t} h(t-s)\left(\nabla u_{t n}, \nabla \omega_{j}\right) \Theta_{t}(t) d s d t  \tag{58}\\
& \quad+\int_{0}^{T}\left(\mu_{1} u_{t n}+\mu_{2} z_{n}, \omega_{j}\right) \Theta(t) d t=0
\end{align*}
$$

Noticing that $\left\{\omega_{j}\right\}_{j}^{\infty}$ is a basis of $H_{0}^{1}(\Omega)$, via convergence (53) and (57), we can pass to the limit in (58) and obtain

$$
\begin{align*}
& \left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s+\mu_{1} u_{t}(x, t)  \tag{59}\\
& \quad+\mu_{2} z(x, 1, t)=0
\end{align*}
$$

Similarly, we get

$$
\begin{equation*}
\tau(t) z_{t}(x, \theta, t)+\left(1-\tau^{\prime}(t) \theta\right) z_{\theta}(x, \theta, t)=0 \tag{60}
\end{equation*}
$$

From (53) and given the label of lemma in [46], we obtain

$$
\begin{align*}
& u_{n}(0) \rightharpoonup u(0) \text { weakly in } H_{0}^{1}(\Omega) \\
& u_{t n}(0) \rightharpoonup u_{t}(0) \text { weakly in } L^{2}(\Omega) \tag{61}
\end{align*}
$$

Therefore, we have $u(0)=u_{0}, u_{t}(0)=u_{1}$. Consequently, the global existence of weak solution is established.

Step 2 (general decay of the energy). First, we introduce the new energy functional $E(t)$ and the perturbed energy $E_{\varepsilon}(t)$; then we apply the perturbed energy method to establish general decay of the energy. More precisely, the method used is based on the construction of suitable Lyapunov functionals $E(t)$ and $E_{\varepsilon}(t)$ satisfying

$$
\begin{equation*}
\frac{d}{d t} E_{\varepsilon}(t) \leq-C_{1} E_{\varepsilon}(t)+C_{2} E(t)^{-r t} \tag{62}
\end{equation*}
$$

for some positive constants $C_{1}, C_{2}, R$. More details are present in [3, pp 1017] or [2, 4, 16].

Now, we introduce the new energy functional as follows:

$$
\begin{align*}
& E(t) \\
& =E(u, z, t) \\
& =\frac{1}{2}\left[\frac{2}{\rho+2}\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\left(1-\int_{0}^{t} h(s) d s\right)\|\nabla u\|_{2}^{2}+(h \circ \nabla u)(t)\right] \\
& \quad+\frac{\xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \tag{63}
\end{align*}
$$

where $\xi, \lambda$ are suitable positive constants.
Next, we will fix $\xi$ such that

$$
\begin{gather*}
2 \mu_{1}-\frac{\left|\mu_{2}\right|}{\sqrt{1-d}}-\xi>0, \quad \xi-\frac{\left|\mu_{2}\right|}{\sqrt{1-d}}>0 \\
\lambda<\frac{1}{\bar{\tau}}\left|\log -\frac{\left|\mu_{2}\right|}{\xi \sqrt{1-d}}\right| . \tag{64}
\end{gather*}
$$

Remark 7. In fact, the existence of such a constant $\xi$ is guaranteed by the assumption (23).

Therefore, we have the following lemma.
Lemma 8. Let (20)-(23) be satisfied and $h$ satisfy (G1). Then, for the solution of problem (1), the energy functional defined by (63) is nonincreasing and satisfies

$$
\begin{align*}
E^{\prime}(t) \leq & \frac{1}{2}(h \circ \nabla u)(t)-\frac{1}{2} h(t) \int_{\Omega}|\nabla u| d x \\
& -C_{1} \int_{\Omega}\left[u_{t}^{2}(x, t)+u_{t}^{2}(x, t-\tau(t))\right] d x  \tag{65}\\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \leq 0
\end{align*}
$$

for some positive constant $C_{1}$.

Proof of Lemma 8. Differentiating (63) and noticing the first equation in (1) together with

$$
\begin{equation*}
(h \circ \nabla u)(t)=\int_{\Omega} \int_{0}^{t} h(t-s)|\nabla u(t)-\nabla u(s)|^{2} d s d x \tag{66}
\end{equation*}
$$

we obtain

$$
\begin{aligned}
& E^{\prime}(t)=\int_{\Omega}\left|u_{t}\right|_{\rho+1} u_{t t}-\frac{1}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x \\
& +\left(1-\int_{0}^{t} h(s) d s\right) \int_{\Omega} \nabla u \cdot \nabla u_{t} d x \\
& +\int_{0}^{t} h(t-s) d s \int_{\Omega}[\nabla u(t)-\nabla u(s)] d s d x \\
& +\frac{1}{2} \int_{0}^{t} h^{\prime}(t-s) \int_{\Omega}|\nabla u(t)-\nabla u(s)|^{2} d s d x \\
& +\frac{\xi}{2} \int_{\Omega} u_{t}^{2}(x, t) d x \\
& -\frac{\xi}{2} \int_{\Omega} e^{-\lambda \tau(t)} u_{t}^{2}(x, t-\tau(t))\left(1-\tau^{\prime}(t)\right) d x \\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \\
& =\int_{\Omega} u_{t}\left[\Delta u-\int_{0}^{t} h(t-s) \Delta u(s) d s-\mu_{1} u_{t}(x, t)\right. \\
& \left.-\mu_{2} u_{t}(x, t-\tau(t))\right] d x \\
& -\frac{1}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x \\
& +\int_{\Omega} \nabla u \cdot \nabla u_{t} d x-\int_{0}^{t} h(s) d s \int_{\Omega} \nabla u \cdot \nabla u_{t} d x \\
& +\int_{0}^{t} h(t-s) d s \int_{\Omega}[\nabla u(t)-\nabla u(s)] d s d x \\
& +\frac{1}{2}\left(h^{\prime} \circ \nabla u\right)(t)+\frac{\xi}{2} \int_{\Omega} u_{t}^{2}(x, t) d x \\
& -\frac{\xi}{2} \int_{\Omega} e^{-\lambda \tau(t)} u_{t}^{2}(x, t-\tau(t))\left(1-\tau^{\prime}(t)\right) d x \\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \\
& =-\mu_{1} \int_{\Omega} u_{t}^{2}(x, t) d x-\mu_{2} \int_{\Omega} u_{t}(x, t) u_{t}(x, t-\tau(t)) d x \\
& -\frac{1}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x+\frac{1}{2}\left(h^{\prime} \circ \nabla u\right)(t)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\xi}{2} \int_{\Omega} u_{t}^{2}(x, t) d x \\
& -\frac{\xi}{2} \int_{\Omega} e^{-\lambda \tau(t)} u_{t}^{2}(x, t-\tau(t))\left(1-\tau^{\prime}(t)\right) d x \\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \tag{67}
\end{align*}
$$

Applying Young's inequality, we obtain

$$
\begin{align*}
& -\mu_{2} \int_{\Omega} u_{t}(x, t) u_{t}(x, t-\tau(t)) d x \\
& \quad \leq \frac{\left|\mu_{2}\right|}{2 \sqrt{1-d}} \int_{\Omega} u_{t}^{2}(x, t) d x  \tag{68}\\
& \quad+\frac{\left|\mu_{2}\right| \sqrt{1-d}}{2} \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x
\end{align*}
$$

Integrating by parts, using the assumption (20), (21) and (67), (68), we arrive at

$$
\begin{align*}
& E^{\prime}(t) \leq-\mu_{1} \int_{\Omega} u_{t}^{2}(x, t) d x \\
& -\mu_{2} \int_{\Omega} u_{t}(x, t) u_{t}(x, t-\tau(t)) d x \\
& -\frac{1}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x+\frac{1}{2}\left(h^{\prime} \circ \nabla u\right)(t) \\
& +\frac{\xi}{2} \int_{\Omega} u_{t}^{2}(x, t) d x-\frac{\xi}{2}(1-d) e^{-\lambda \bar{\tau}} \\
& \times \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x \\
& \left(0<\tau_{0} \leq \tau(t) \leq \bar{\tau}, \tau^{\prime}(t) \leq d<1\right) \\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x  \tag{69}\\
& \leq \frac{1}{2}\left(h^{\prime} \circ \nabla u\right)(t)-\frac{1}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x \\
& -\left(\mu_{1}-\frac{\left|\mu_{2}\right|}{2 \sqrt{1-d}}-\frac{\xi}{2}\right) \int_{\Omega} u_{t}^{2}(x, t) d x \\
& -\left(\frac{\xi}{2}(1-d) e^{-\lambda \bar{\tau}}-\frac{\left|\mu_{2}\right| \sqrt{1-d}}{2}\right) \\
& \times \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x \\
& -\frac{\lambda \xi}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x .
\end{align*}
$$

Combining (64) and (69) and the assumptions (G1) and (G2), (65) is established.

Next, we introduce the following functionals:

$$
\begin{gather*}
\Phi(t)=\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+1} u d x  \tag{70}\\
\Psi(t)=-\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+1} \int_{0}^{t} h(t-s)[u(t)-u(s)] d s d x \tag{71}
\end{gather*}
$$

Set

$$
\begin{equation*}
L(t)=N E(t)+\varepsilon \Phi(t)+\Psi(t) \tag{72}
\end{equation*}
$$

where $N$ and $\varepsilon$ are suitable positive constants to be determined later.

Remark 9. Indeed, we easily see that, for $\varepsilon$ small enough while $N$ large enough, there exist two positive constants $\alpha_{0}, \alpha_{1}$, such that

$$
\begin{equation*}
\alpha_{0} E(t) \leq L(t) \leq \alpha_{1} E(t), \quad \forall t \geq 0 \tag{73}
\end{equation*}
$$

Concerning the estimates of $\Phi(t), \Psi(t)$, we have the following lemmas.

Lemma 10. Under the assumption (G1), the functional $\Phi(t)$ satisfies the estimate

$$
\begin{align*}
& \Phi^{\prime}(t) \\
& \leq-\frac{l}{2} \int_{\Omega}|\nabla u|^{2} d x \\
&+C_{2} \int_{\Omega}\left[u_{t}^{2}(x, t)+u_{t}^{2}(x, t-\tau(t))\right] d x+C_{3}(h \circ \nabla u) \tag{74}
\end{align*}
$$

Proof of Lemma 10. Differentiating (70) and integrating by parts, we get

$$
\begin{aligned}
\Phi^{\prime}(t)= & \int_{\Omega}\left|u_{t}\right|^{\rho+1} u_{t t} u d x+\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+2} d x \\
= & \int_{\Omega} u\left[\Delta u-\int_{0}^{t} h(t-s) \Delta u(s) d s-\mu_{1} u_{t}(x, t)\right. \\
& \left.\quad-\mu_{2} u_{t}(x, t-\tau(t))\right] d x \\
& +\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+2} d x \\
= & -\int_{\Omega}|\nabla u|^{2} d x+\int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s) \nabla u(s) d s d x \\
& -\mu_{1} \int_{\Omega} u u_{t}(x, t) d x \\
& -\mu_{2} \int_{\Omega} u u_{t}(x, t-\tau(t)) d x \frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+2} d x
\end{aligned}
$$

$$
\begin{align*}
= & -l \int_{\Omega}|\nabla u|^{2} d x \\
& +\int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(s)-\nabla u(t)] d s d x \\
& -\mu_{1} \int_{\Omega} u u_{t}(x, t) d x-\mu_{2} \int_{\Omega} u u_{t}(x, t-\tau(t)) d x \\
& +\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho+2} d x \tag{75}
\end{align*}
$$

Using Young's inequality and (G1), we obtain (see [2])

$$
\begin{align*}
& \int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(s)-\nabla u(t)] d s d x \\
& \quad \leq \delta \int_{\Omega}|\nabla u|^{2} d x  \tag{76}\\
& \quad+\frac{1}{4 \delta} \int_{\Omega}\left[\int_{0}^{t} h(t-s)|\nabla u(s)-\nabla u(t)| d s\right]^{2} d x \\
& \leq \delta \int_{\Omega}|\nabla u|^{2} d x+\frac{1-l}{4}(h \circ \nabla u)(t), \quad \forall \delta>0
\end{align*}
$$

Also, applying Young's and Poincaré's inequality yields

$$
\begin{align*}
& -\mu_{1} \int_{\Omega} u u_{t}(x, t) d x \leq \delta \int_{\Omega}|\nabla u|^{2} d x+C(\delta) \int_{\Omega} u_{t}^{2}(x, t) d x \\
& \quad-\mu_{2} \int_{\Omega} u u_{t}(x, t-\tau(t)) d x \\
& \quad \leq \delta \int_{\Omega}|\nabla u|^{2} d x+C(\delta) \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x \tag{77}
\end{align*}
$$

Noticing (75)-(77) and choosing $\delta$ small enough, we obtain estimate (74).

Lemma 11. Under the assumption (G1), the functional $\Psi(t)$ satisfies the estimate

$$
\begin{align*}
\Psi^{\prime}(t) \leq & -\left(\int_{0}^{t} h(s) d s-2 \delta\right) \int_{\Omega} u_{t}^{2} d x+\delta \int_{\Omega}|\nabla u|^{2} d x \\
& +\frac{C_{4}}{\delta}(h \circ \nabla u)(t)-\frac{C_{5}}{\delta}\left(h^{\prime} \circ \nabla u\right)(t)  \tag{78}\\
& +\delta \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x
\end{align*}
$$

Proof of Lemma 11. Differentiating (71), integrating by parts, and noticing the first equation in (1), we have

$$
\begin{align*}
& \Psi^{\prime}(t)=-\int_{\Omega}\left|u_{t}\right|^{\rho} u_{t t} \int_{0}^{t} h(t-s)[u(t)-u(s)] d s d x \\
& -\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} h^{\prime}(t-s)[u(t)-u(s)] d s d x \\
& -\left(\int_{0}^{t} h(s) d s\right) \int_{\Omega} \frac{1}{\rho+1}\left|u_{t}\right|^{\rho+2} d x \\
& =\int_{\Omega}\left[-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s\right. \\
& \left.+\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau(t))\right] \\
& \times \int_{0}^{t} h(t-s)[u(t)-u(s)] d s d x \\
& -\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} h^{\prime}(t-s)[u(t)-u(s)] d s d x \\
& -\left(\int_{0}^{t} h(s) d s\right) \int_{\Omega} \frac{1}{\rho+1}\left|u_{t}\right|^{\rho+2} d x \\
& =\int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s d x \\
& +\int_{\Omega} \int_{0}^{t} h(t-s) \Delta u(s) d s \\
& \times \int_{0}^{t} h(t-s)[u(t)-u(s)] d s d x \\
& +\int_{\Omega}\left[\int_{0}^{t} h(t-s)[u(t)-u(s)] d s\right] \\
& \times\left[\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau(t))\right] d x \\
& -\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} h^{\prime}(t-s)[u(t)-u(s)] d s d x \\
& -\left(\int_{0}^{t} h(s) d s\right) \int_{\Omega} \frac{1}{\rho+1}\left|u_{t}\right|^{\rho+2} d x \text {. } \tag{79}
\end{align*}
$$

Observe that

$$
\begin{aligned}
& \int_{\Omega} \int_{0}^{t} h(t-s) \Delta u(s) d s \int_{0}^{t} h(t-s)[u(t)-u(s)] d s d x \\
&=-\int_{\Omega} {\left[\int_{0}^{t} h(t-s) \nabla u(s) d s\right.} \\
&\left.\quad \times \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s\right] d x
\end{aligned}
$$

$$
\begin{align*}
= & -\int_{\Omega}\left[\int_{0}^{t} h(t-s)[\nabla u(s)-\nabla u(t)+\nabla u(s)] d s\right. \\
& \left.\times \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s\right] d x \\
= & -\int_{\Omega}\left[\int_{0}^{t} h(t-s)[\nabla u(s)-\nabla u(t)] d s\right]^{2} d x \\
& -\left(\int_{0}^{t} h(t) d s\right) \\
& \times \int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s d x . \tag{80}
\end{align*}
$$

It follows from (79) and (80) that

$$
\begin{align*}
\Psi^{\prime}(t)= & \left(1-\int_{0}^{t} h(s) d s\right) \\
& \times \int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s d x \\
& +\int_{\Omega}\left[\int_{0}^{t} h(t-s)[\nabla u(s)-\nabla u(t)] d s\right]^{2} d x \\
& +\int_{\Omega}\left[\int_{0}^{t} h(t-s)[u(t)-u(s)] d s\right] \\
& \times\left[\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau(t))\right] d x \\
& -\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} h^{\prime}(t-s)[u(t)-u(s)] d s d x \\
- & \left(\int_{0}^{t} h(s) d s\right) \int_{\Omega} \frac{1}{\rho+1}\left|u_{t}\right|^{\rho+2} d x . \tag{81}
\end{align*}
$$

Using Young's and Poincaré's inequality, we get (see [2])

$$
\begin{align*}
& \left(1-\int_{0}^{t} h(s) d s\right) \int_{\Omega} \nabla u \cdot \int_{0}^{t} h(t-s)[\nabla u(t)-\nabla u(s)] d s d x \\
& \quad \leq \delta \int_{\Omega}|\nabla u|^{2} d x+\frac{C}{\delta}(h \circ \nabla u)(t) \\
& -\frac{1}{\rho+1} \int_{\Omega}\left|u_{t}\right|^{\rho} u_{t} \int_{0}^{t} h^{\prime}(t-s)[u(t)-u(s)] d s d x \\
& \quad \leq \delta \int_{\Omega} u_{t}^{2} d x-\frac{C}{\delta}\left(h^{\prime} \circ \nabla u\right)(t) . \tag{82}
\end{align*}
$$

From (81) and (82), we derive Lemma 11.
Now, we are ready to finalize our proof of general decay of the energy. Since $h$ is positive, we have

$$
\begin{equation*}
\int_{0}^{t} h(s) d s \geq \int_{0}^{t_{0}} h(s) d s=g_{0}, \quad \forall t \geq t_{0} \tag{83}
\end{equation*}
$$

It follows from (65), (72), (74), and (78) that

$$
\begin{align*}
& L^{\prime}(t)=N E(t)+\varepsilon \Phi^{\prime}(t)+\Psi^{\prime}(t) \\
& \leq \frac{N}{2}\left(h^{\prime} \circ \nabla u\right)(t)-\frac{N}{2} h(t) \int_{\Omega}|\nabla u|^{2} d x \\
& -N C_{1} \int_{\Omega}\left[u_{t}^{2}(x, t)+u_{t}^{2}(x, t-\tau(t))\right] d x \\
& -\frac{\lambda \xi N}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x \\
& +\varepsilon C_{2} \int_{\Omega}\left[u_{t}^{2}(x, t)+u_{t}^{2}(x, t-\tau(t))\right] d x \\
& -\frac{\varepsilon l}{2} \int_{\Omega}|\nabla u|^{2} d x P+\varepsilon C_{3}(h \circ \nabla u)(t) \\
& -\left(\int_{0}^{t} h(s) d s-2 \delta\right) \int_{\Omega} u_{t}^{2} d x+\delta \int_{\Omega}|\nabla u|^{2} d x \\
& +\frac{C_{4}}{\delta}(h \circ \nabla u)(t)-\frac{C_{5}}{\delta}\left(h^{\prime} \circ \nabla u\right)(t)  \tag{84}\\
& +\delta \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) d x  \tag{90}\\
& =-\left[\left(N C_{1}+g_{0}\right)-2 \delta-\varepsilon C_{2}\right] \int_{\Omega} u_{t}^{2}(x, t) d x \\
& +\left(\varepsilon C_{3}+\frac{C_{4}}{\delta}\right)(h \circ \nabla u)(t)  \tag{91}\\
& +\left(\frac{N}{2}-\frac{C_{5}}{\delta}\right)\left(h^{\prime} \circ \nabla u\right)(t) \\
& -\left(\frac{\varepsilon l}{2}-\delta\right) \int_{\Omega}|\nabla u|^{2} d x  \tag{92}\\
& -\left(N C_{1}-\delta-\varepsilon C_{2}\right) \int_{\Omega} u_{t}^{2}(x, t-\tau(t)) \\
& -\frac{\lambda \xi N}{2} \int_{t-\tau(t)}^{t} \int_{\Omega} e^{-\lambda(t-s)} u_{t}^{2}(x, s) d s d x . \tag{93}
\end{align*}
$$

If we choose some constants in the inequality (84), such that

$$
\begin{gather*}
a_{1}=\left(N C_{1}+g_{0}\right)-2 \delta-\varepsilon C_{2}>0 \\
a_{3}=N C_{1}-\delta-\varepsilon C_{2}>0, \quad a_{2}=\frac{\varepsilon l}{2}-\delta>0  \tag{94}\\
a_{4}=\frac{N}{2}-\frac{C_{5}}{\delta}>0, \quad a_{5}=\varepsilon C_{3}+\frac{C_{4}}{\delta}>0  \tag{85}\\
a_{6}=\frac{\lambda \xi N}{2},
\end{gather*}
$$

Observing Remark 9 (i.e., $\alpha_{0} E(t) \leq L(t) \leq \alpha_{1} E(t)$ ) and (89), we derive

$$
\alpha_{0} E(t) \leq L(t) \leq L(0) e^{-\left(\beta_{1} / \alpha_{1}\right) t}, \quad \forall t \geq 0
$$

That is,

$$
E(t) \leq \frac{L(0)}{\alpha_{0}} e^{-\left(\beta_{1} / \alpha_{1}\right) t} \doteq K e^{k t}, \quad p=1, \quad \forall t \geq 0
$$

Assuming $K=L(0) / \alpha_{0}, k=\beta_{1} / \alpha_{1}$, we obtain the exponential decay of the energy. So, (26) is established.

Case $2(1<p<3 / 2)$. Due to (G2), we easily see that

$$
\int_{0}^{\infty} h^{1-r}(s) d s<\infty, \quad 0 \leq r \leq 2-p .
$$

From the sketch of proof of Lemma 6, we observe that

$$
\begin{aligned}
(h \circ \nabla u)(t) \leq & C\left[\int_{0}^{\infty} h^{1-\alpha}(s) d s E(0)\right]^{(p-1+\alpha) /(p-1)} \\
& \times\left[\left(h^{p} \circ \nabla u\right)(t)\right]^{\alpha /(p-1+\alpha)}
\end{aligned}
$$

Thus, for $\sigma>1$, using (63) and (93), we get

$$
\begin{aligned}
E^{\sigma}(t) \leq & C\left[E^{\sigma-1}(0)\left(\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\|\nabla u\|_{2}^{2}+\left\|u_{t}\right\|_{2}^{2}\right)\right. \\
& \left.+(h \circ \nabla u)^{\sigma}(t)\right] \\
\leq & C E^{\sigma-1}(0)\left[\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\|\nabla u\|_{2}^{2}+\left\|u_{t}\right\|_{2}^{2}\right] \\
+ & C\left[\left(\int_{0}^{\infty} h^{1-\alpha}(s) d s\right) E(0)\right]^{(p-1+\alpha) /(p-1)} \\
\times & {\left[\left(h^{p} \circ \nabla u\right)(t)\right]^{\sigma \alpha /(p-1+\alpha)} }
\end{aligned}
$$

Choosing $\alpha=1 / 2, \sigma=2 p-1$ (i.e., $\sigma \alpha /(p-1+\alpha)=1)$ (94) reduces to

$$
\begin{equation*}
E^{\sigma}(t) \leq C\left[\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\|\nabla u\|_{2}^{2}+\left\|u_{t}\right\|_{2}^{2}+\left(h^{p} \circ \nabla u\right)(t)\right] . \tag{95}
\end{equation*}
$$

Combining (86) and (87) with Remark 9, we obtain

$$
\begin{equation*}
L^{\prime}(t) \leq-\frac{\beta_{1}}{C} \alpha_{1}^{\sigma} L^{\sigma}(t), \quad \forall t \geq 0 \tag{96}
\end{equation*}
$$

A simple integration of (96) over $(0, t)$ yields

$$
\begin{equation*}
L^{\prime}(t) \leq C_{6}(1+t)^{-1 /(\sigma-1)}, \quad \forall t \geq 0 . \tag{97}
\end{equation*}
$$

As a consequence of (97), we obtain

$$
\begin{equation*}
\int_{0}^{\infty} L(t) d t+\sup _{t \geq 0} t F(t)<\infty . \tag{98}
\end{equation*}
$$

So, by using Lemma 6, we have

$$
\begin{align*}
& (h \circ \nabla u)(t) \\
& \quad \leq C\left[\int_{0}^{t}\|u(s)\|_{H^{1}(s)} d s+t\|u\|_{H^{1}(\Omega)}\right]^{(p-1) / p} \\
& \quad \times\left(h^{p} \circ \nabla u\right)^{1 / p}(t)  \tag{99}\\
& \quad \leq C\left[\int_{0}^{t} F(s) d s+t F(t)\right]^{(p-1) / p}\left(h^{p} \circ \nabla u\right)^{1 / p}(t) \\
& \quad \leq C\left(h^{p} \circ \nabla u\right)^{1 / p}(t)
\end{align*}
$$

which implies that

$$
\begin{equation*}
\left(h^{p} \circ \nabla u\right)(t) \geq C(h \circ \nabla u)^{p}(t) . \tag{100}
\end{equation*}
$$

Consequently, from (86) and (100), we have

$$
L^{\prime}(t) \leq-C_{7}\left[\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\|\nabla u\|_{2}^{2}+\left\|u_{t}\right\|_{2}^{2}+(h \circ \nabla u)^{p}(t)\right]
$$

$$
\begin{equation*}
\forall t \geq 0 \tag{101}
\end{equation*}
$$

On the other hand, similarly to (95), we

$$
\begin{array}{r}
E^{p}(t) \leq C_{8}\left[\left\|u_{t}\right\|_{\rho+2}^{\rho+2}+\|\nabla u\|_{2}^{2}+\left\|u_{t}\right\|_{2}^{2}+(h \circ \nabla u)^{p}(t)\right] \\
\forall t \geq 0 \tag{102}
\end{array}
$$

Then, it follows from Remark 9, (101), and (102) that

$$
\begin{equation*}
L^{\prime}(t) \leq-C_{9} L^{p}(t), \quad \forall t \geq 0 . \tag{103}
\end{equation*}
$$

A simple integration of (103) over $(0, t)$ gives

$$
\begin{equation*}
L^{\prime}(t) \leq K(1+t)^{-1 /(p-1)}, \quad \forall t \geq 0 . \tag{104}
\end{equation*}
$$

By (104) and Remark 9, we obtain the polynomial decay of the energy. That is,

$$
\begin{equation*}
E(t) \leq K(1+t)^{-1 /(p-1)}, \quad \forall t \geq 0 \tag{105}
\end{equation*}
$$

Thus, our main result is completed.

Remark 12. Our novel contribution is to show that our work improves earlier result in [37] in which only the exponential decay was investigated. More precisely, Kirane and SaidHouari [37] considered the exponential decay of problem (1) with a constant delay (i.e., $\tau(t)=\tau$ ) and velocity-independent material density (i.e., $\rho=0$ ).

Remark 13. By using the fact that energy $E$ is bounded on $\left[0, t_{0}\right]$, we can easily show that estimates (26) and (27) hold for $t \geq 0$. (See, for instance, [2].)

## 4. Further Remarks

In this section, we address some interesting problems of nonlinear viscoelastic equation with time-varying delay effects and velocity-dependent material density. Here, we mention some of them.
(1) An interesting problem is to show the well-posedness and stabilization of the nonlinear viscoelastic equation with boundary feedback with respect to timevarying delay effects. What will happen if the controller with time-varying delay effects is in the equation instead of on the boundary? More precisely, in our forthcoming work, we will investigate the wellposedness and general decay properties of the solutions for the following nonlinear viscoelastic equation with velocity-dependent material density:

$$
\begin{gathered}
\left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s=0, \quad \text { in } \Omega \times[0, \infty), \\
u(x, t)=0, \quad \text { on } \Gamma_{0} \times(0, \infty)
\end{gathered}
$$

$$
\frac{\partial u}{\partial v}+\mu_{1} u_{t}(x, t)+\mu_{2} u_{t}(x, t-\tau(t))=0, \quad \text { on } \Gamma_{1} \times[0, \infty)
$$

$$
u(x, 0)=u_{0}(x), \quad u_{t}(x, 0)=u_{1}(x), \quad \text { in } \Omega
$$

$$
\begin{equation*}
u_{t}(x, t-\tau(t))=f(x, t), \quad \text { on } \Gamma_{1} \times(-\tau(0), 0) \tag{106}
\end{equation*}
$$

where $\Omega$ is bounded domain of $R^{n}$ and $n \geq 1$ with a smooth boundary $\Gamma$ and let $\Gamma_{0}, \Gamma_{1}$ be a partition of $\Gamma$ such that $\overline{\Gamma_{0}} \cap \overline{\Gamma_{1}}=\emptyset, \Gamma_{0} \neq \emptyset, \Gamma_{1} \neq \emptyset, \nu=\left(\nu_{1}, v_{2} \cdots v_{n}\right)$ denotes the unit outward normal to $\Gamma$.
(2) Another interesting problem is to give a positive answer of the open problem given by Kirane and Said-Houari [37]. That is, the linear damping term $\mu_{1} u_{t}$ in the first equation of (16) plays a decisive role in their proofs. Thus, the problem of whether the stability properties they have proved are preserved when $\mu_{1}=0$ is open. In order to overcome the above difficulty, our main idea is to contrast the effects of the time-varying delay by using the dissipative nonlinear
boundary feedback. That is, in our future work, we investigate the following problem:

$$
\begin{align*}
& \left|u_{t}\right|^{\rho} u_{t t}-\Delta u+\int_{0}^{t} h(t-s) \Delta u(s) d s \\
& \quad+\mu_{2} u_{t}(x, t-\tau(t))=0, \quad \text { in } \Omega \times[0, \infty) \\
& \quad u(x, t)=0, \quad \text { on } \Gamma_{0} \times(0, \infty)  \tag{107}\\
& \frac{\partial u}{\partial v}+g\left(u_{t}(x, t)\right)=0, \quad \text { on } \Gamma_{1} \times[0, \infty) \\
& u(x, 0)=u_{0}(x), \quad u_{t}(x, 0)=u_{1}(x), \quad \text { in } \Omega \\
& u_{t}(x, t-\tau(t))=f(x, t), \quad \text { on } \Gamma_{1} \times(-\tau(0), 0)
\end{align*}
$$

where $\mu_{2}$ is constant and $g\left(u_{t}\right)$ is the dissipative nonlinear boundary feedback.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Time-Varying Risk Attitude and Conditional Skewness 

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#### Abstract

Much literature finds that the skewness in the return distribution is negatively correlated with the risk premium coefficient, and speculation is the reason for the skewness in the return distribution. As further research, this paper, first taking up the time-varying property of the risk premium coefficient, proposes a GARCH-M model with a time-varying coefficient of the risk premium for an empirical study of the correlation between the conditional skewness in the return distribution and the time-varying risk attitude. The empirical study indicates that the coefficient of the risk premium varies with the time, and even in a mature market the conditional skewness in the return distribution is negatively correlated with the time-varying coefficient of the risk premium.


## 1. Introduction

The study of the skewness in the return distribution has gradually become the hot topic in Finance. Much literature ([1-3], etc.) finds that there exists more or less skewness in the return distribution.

The negatively skewed return distribution will increase the loss probability, while the positively skewed one will increase the possibility of gaining. Building an optimal asset portfolio allowing for the skewness has been one of the most attractive topics in investment portfolio research. For example, Samuelson [4], Lai [5], Prakash et al. [6], Canela and Collazo [7], and Zakamouline and Koekebakker [8] are all related to this field. Meanwhile, after recognizing the importance of the skewness, it has been given the same importance as the mean and the variance in researching in the problem of security pricing (see, e.g., [9-12]).

More and more scholars have begun to study the properties of the skewness due to the fact that it plays an important role in asset pricing. How the skewness changes with the time under the influence of various factors is one of the most densely studied topics. In specifying most of the models in the past, the distribution assumption did not involve the time variation of the skewness. For example, it was assumed, to the effect, that the skewness does not vary with the time in the GARCH model proposed in Bollerslev [13]. Some
scholars discussed the persistence of the skewness, but their conclusions are not consistent. Singleton and Wingender [14] find that the positive skew is very likely to be negative in the next time period, while the negative skew will very probably turn to be positive in the next period of time; Lau and Wingender [15] believe that the skewness approaches to nothing in the long run, and some researchers hold that there does not exist persistence in the skewness (see, e.g., $[16,17]$ ).

A number of studies have argued for the existence of the time variation of the skewness from various perspectives, among which Harvey and Siddique [18] study is a pioneering study of great significance on the time variation of the skewness. On the basis of a GARCH model, they studied the time variation of the skewness and proposed an autoregressive conditional skewness model with the empirical results showing that the skewness in the return distribution varies with the time. Following their thinking, most of later research studied the time-variation problem in the framework of the GARCH-type models [19]. Higher moments models have difficulties in parameter estimation due to the number of parameters. León et al. [20] estimated the autoregressive conditional variance, the skewness, and the kurtosis using the Gram-Charlier series expansion of the normal density function, as this estimation procedure can incorporate the skewness and the kurtosis in the model as parameters and thus solved the problem of parameter estimation in higher
moments models. Chan [21] modeled the time variation of higher moments using maximum entropy density (MED) and produced relatively better results.

Due to the fact that the reasons for the skewness in the return have not been understood, consensus cannot be achieved on the constraints for the optimization and the pricing kernel process in portfolio selection, although advances have been made in the research and the empirical studies of skewness. Consequently, more and more researchers have switched their research foci to the reasons for the skewness.

Bakshi et al. [22] believed that investors risk aversion will lead to the negative skewness in the return distribution. Ekholm and Pasternack [23] built a theoretic model on the basis of the negative news threshold hypothesis, and with empirical evidence they believe that the different releasing policies for positive and negative information are responsible for the skewness in the return distribution. Wen et al. [24] suggested that, with investors' overconfidence and regret aversion, their reaction to the nonlinear arrival of information will lead to the skewness in the return distribution, which was supported by their simulation. Bae et al. [25] find that the low level of corporate governance may be the cause of the positive skewness in the return distribution. Xu [26] finds that the skewness in the return distribution is positively correlated with the return of the current period but negatively correlated with that of the last period, which indicates that the skewness in the return distribution may be related to investors' reactions to the return. Besides, many scholars have attempted to identify the reasons for the skewness from different perspectives, such as trading volume, and the heterogeneity of the investors [27, 28], but an agreement has yet to be achieved on the conclusion of the reasons.

Wen and Yang [29] suggested that skewness is related to investors' risk attitude in stock market and too much speculative behavior in the market is responsible for the positive skewness in the return distribution. They employed a GARCH-M model to test the composite indices of 33 securities markets, and the empirical evidences showed that the skewness in the return distribution is significantly negatively correlated with the risk premium coefficient. This conclusion was derived from the comparison of different markets, but many factors, such as the differences of the cultural background of market players, and the microstructure of the markets may influence the relationship between the speculation in the market and the skewness in the return distribution, so it is necessary to further study the correlation between them on the dimension of time.

Based on Wen and Yang [29], the autoregressive conditional skewness model in Harvey and Siddique [18] can be introduced to characterize the conditional skewness process in order to further examine the correlation between the skewness in the return distribution and the risk premium coefficient on the dimension of time. GARCH-M can be employed to examine the tradeoff between the risk and the risk premium in investors' investment decision, but in the GARCH-M model, the risk premium coefficient is held constant in a certain period of time and thus cannot generate the time-varying risk premium series corresponding to the conditional skewness process. Anderson et al. [30] proposed an

ANST-GARCH (Asymmetric Nonlinear Smooth-Transition GARCH) model by introducing a smooth-transition specification to extend the GARCH model and studied the time variation of the risk premium using the ANST-GARCH-M model. In Anderson's model, the time variation of the risk premium is essentially related to the error of the last period (the unexpected return), and the reason for the time variation has not been thoroughly discussed.

This paper suggests that the time variation of the risk premium coefficient is related not only to the unexpected return but also to the risk and the speculation of the prior periods. Therefore, we first construct a time-varying risk premium coefficient incorporated GARCH-M model to study the time variation of the risk premium coefficient. On the basis of the above model, we propose a time-varying risk premium coefficient incorporated GARCH-M model with Harvey and Siddique's [18] autoregressive conditional skewness model to introduce the conditional skewness process. Finally, we select 14 most representative stock composite indices as samples to conduct an empirical investigation.

## 2. The Relationship between Risk Attitude and Skewness

2.1. Measure for Risk Attitude: Risk Premium Coefficient. Wen and Yang [29] argued that, "the biggest difference between investors and speculators is their attitude towards risk, most investors are risk averse, while most speculators are risk tolerant," and "the more speculative a market is, the smaller average coefficient of risk premium $\gamma$ is." Then they used the $\gamma$ coefficient in the GARCH-M model as the measure for risk attitude. The $\gamma$ coefficient in the GARCH-M model describes the average risk premium investors demand for a unit risk in the market, that is, risk attitude or risk tolerance, also called the risk premium coefficient. The expression of the GARCHM model is

$$
\begin{align*}
& r_{t}=c+x_{t} \beta+\gamma \sqrt{h_{t}}+\varepsilon_{t} \\
& \varepsilon_{t}=\sqrt{h_{t}} \cdot v_{t}  \tag{1}\\
& h_{t}=\alpha_{0}+\sum_{i=1}^{q} \alpha_{i} \varepsilon_{t-i}^{2}+\sum_{j=1}^{p} \theta_{j} h_{t-j}
\end{align*}
$$

where $v_{t}$ is i.i.d, $E\left(v_{t}\right)=0, D\left(v_{t}\right)=1$, and $\sum_{i=1}^{q} \alpha_{i}+\sum_{j=1}^{p} \theta_{j}<$ 1.

In this model, the return rate is divided into three parts: the average return rate $c+x_{t} \beta$ related to exogenous variables, the risk premium $\gamma \sqrt{h_{t}}$, and the volatility return rate related to exogenous shocks (it is regarded as the gain not expected by investors, i.e., the unexpected gain). Obviously,

$$
\begin{equation*}
\gamma=\frac{\Delta r_{t}}{\Delta \sqrt{h_{t}}} \tag{2}
\end{equation*}
$$

which shows that $\gamma$ is the risk premium coefficient for a unit of risk. So the risk premium coefficient $\gamma$ can be used to measure the magnitude of the compensation for the risk investors take

Table 1: Basic statistics of the daily return series of indices.

|  | Mean | Std. dev. | Skewness | Jarque-Bera | ADF test | ARCH-LM test |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: |
| S\&P 500 | -0.006210 | 1.400254 | -0.114384 | 6587.713 | -38.3026 | 153.6778 |
| Dow Jones | -0.000915 | 1.317075 | 0.047682 | 6299.268 | -37.8115 | 150.5500 |
| NASDAQ | -0.000440 | 1.747697 | 0.175155 | 2298.068 | -37.5163 | 100.7235 |
| NYSE | 0.002528 | 1.397271 | -0.283905 | 9121.484 | -37.44753 | 190.2067 |
| Nikkei225 | -0.011831 | 1.660565 | -0.292542 | 3821.667 | -48.86196 | 325.9931 |
| FTSE 100 | -0.005793 | 1.357928 | -0.106906 | 3927.123 | -21.9967 | 153.3013 |
| SSE | 0.019730 | 1.714406 | -0.105263 | 1629.944 | -27.6192 | 5.505159 |
| DAX | -0.002373 | 1.692539 | 0.070301 | 1798.858 | -49.73979 | 119.7851 |
| CAC 40 | -0.016852 | 1.587710 | 0.052661 | 2746.579 | -23.62872 | 111.7417 |
| GSPTSE | 0.012879 | 1.250443 | -0.679786 | 9174.220 | -51.28554 | 161.5086 |
| MIBTEL | -0.030404 | 1.316993 | -0.158677 | 4999.875 | -20.68796 | 123.2172 |
| IGBM | 0.020879 | 1.364191 | -0.061860 | 4384.783 | -46.61378 | 123.3914 |
| BVSP | 0.067122 | 2.025965 | -0.093354 | 1355.060 | -46.66775 | 131.7443 |
| Hangseng | 0.017219 | 1.676433 | 0.037793 | 7002.364 | -48.77268 | 223.4757 |

Note: the J-B statistic, the ADF test, and the ARCH-LM test are all at $1 \%$ significance level.
in the market. Compared to an investor, a speculator's risk tolerance is higher, that is to say, the risk premium coefficient is lower $\gamma$. The more speculation there is in a market, the smaller the average risk premium coefficient $\gamma$ is.
2.2. Robust Measure for Skewness. For a return series $\left\{R_{t}\right\}$, the most widely used measure for the skewness is

$$
\begin{equation*}
\mathrm{SK}_{1}=E\left(\frac{R_{t}-\mu}{\delta}\right)^{3} \tag{3}
\end{equation*}
$$

where $\mu=E\left(R_{t}\right), \delta^{2}=E\left(R_{t}-\mu\right)^{2}$. But much literature (e.g., [31]) finds that $\mathrm{SK}_{1}$ is very sensitive to outliers and thus not a robust measure. This paper adopted the robust skewness measure proposed by Groeneveld and Meeden [32]:

$$
\begin{align*}
\mathrm{SK}_{2} & =\frac{\int_{0}^{0.5}\left\{F^{-1}(1-\alpha)+F^{-1}(\alpha)-2 \mathrm{Q}_{2}\right\} d \alpha}{\int_{0}^{0.5}\left\{F^{-1}(1-\alpha)-F^{-1}(\alpha)\right\} d \alpha}  \tag{4}\\
& =\frac{\mu-Q_{2}}{E\left|R_{t}-Q_{2}\right|}
\end{align*}
$$

2.3. Samples and Statistics. According to the World Bank Report 2009, the GDP of the top 10 countries accounted for $65.4 \%$ of the world GDP. The 10 countries were USA, Japan, China, Germany, France, UK, Italy, Brazil, Russia, and Spain. This paper would have chosen their representative stock indices from the Yahoo finance as the samples, and the time duration is January 1, 2001, to December 31, 2009. As there are no stock index data of Russia in Yahoo finance, this paper chose Canada which was the eleventh in 2008 world GDP ranking as the substitute for data source consistence. These representative indices are S\&P 500, Dow Jones, Nasdaq, NYSE, Japan's Nikkei 225, China’s SSE and China Hongkong's Hangseng, Germany's DAX, France's CAC 40, UK's FTSE 100, Italy's MIBTEL, Brazil's BVSP, Spain's IGBM, and Canada's GSPTSE, 14 indices all together, which include most of the


Figure 1: Relationship between coefficient $\gamma$ and $\mathrm{SK}_{2}$ measurement.
representative stock markets in North America, Europe, and Asia. The return is calculated with $r_{t}=100 *\left(\ln P_{t}-\ln P_{t-1}\right)$. The basic statistics of the daily return of the above chosen indices are shown in Table 1.

The J-B statistics in Table 1 show that all the return series of the above indices are not normal, but more or less rightor left-skewed (skewness $\neq 0$ ). The ADF test for the return series shows that all the series are stationary. The results of the ARCH-LM test indicate that the accompanying probabilities are all less than those at the $1 \%$ significance level, which implies that there exists error ARCH effect in the error series. Therefore we employ the GARCH type model to capture the volatilities of the indices.
2.4. Empirical Results. Following the thinking of Wen and Yang [29], this paper first compared cross-sectionally the correlation between the risk premium coefficient and the skewness of the 14 stock indices. The results are tabulated in Table 2 and Figure 1. We can see that as the usual skewness measure is very sensitive to the outliers, the values of $\mathrm{SK}_{1}$ for various indices are quite different from each other, while the values of the robust measure $\mathrm{SK}_{2}$ are obviously rather consistent with each other. Further statistical analysis found that the

Table 2: Risk premium coefficients and skewness of indices.

|  | $\mathrm{SK}_{1}$ | $\mathrm{SK}_{2}$ | $\gamma$ | $P$ value |
| :--- | :---: | :---: | :---: | :---: |
| S\&P 500 | -0.114384 | -0.070551 | 0.0026 |  |
| Dow Jones | 0.047682 | -0.047682 | 0.059168 | 0.0046 |
| NASDAQ | 0.175155 | -0.057890 | 0.050783 | 0.0108 |
| NYSE | -0.283905 | -0.074509 | 0.070776 | 0.0004 |
| Nikkei225 | -0.292542 | -0.031569 | 0.036404 | 0.0763 |
| FTSE 100 | -0.106906 | -0.048346 | 0.052449 | 0.0098 |
| SSE | -0.105263 | -0.040100 | 0.046433 | 0.0005 |
| DAX | 0.070301 | -0.066459 | 0.068993 | 0.0193 |
| CAC 40 | 0.052661 | -0.032634 | 0.047085 | 0.0001 |
| GSPTSE | -0.679786 | -0.045354 | 0.074935 | 0.0044 |
| MIBTEL | -0.158677 | -0.065010 | 0.056490 | 0.0000 |
| IGBM | -0.061860 | -0.063446 | 0.095071 | 0.0001 |
| BVSP | -0.093354 | -0.048321 | 0.080769 | 0.0111 |
| Hangseng | 0.037793 | -0.029994 | 0.048903 |  |

correlation between the risk premium coefficient and $\mathrm{SK}_{1}$ for the 14 stock indices is -0.21613 , while the correlation between the robust skewness measure $\mathrm{SK}_{2}$ and the risk premium coefficient is -0.54081 , both being significantly smaller than zero, which shows the skewness in the return distribution and the risk premium coefficient are significantly negatively associated. The above empirical results are consistent with the conclusion in Wen and Yang [29], which has further proved that the skewness in the return distribution is closely related to the degree of speculation in the market.

## 3. Time-Varying Risk Premium Coefficient and Skewness

The above negative correlation between the skewness in the return distribution and the risk premium coefficient was derived from the returns of different markets at the same time period. In the foregoing section, we have pointed out that many factors like different cultural background and different microstructure of different markets may influence the relationship between the degree of speculation and the skewness in the return distribution. If the research is confined in the same market, these factors can be ignored. Therefore it is necessary to further study the correlation between the skewness in the return distribution and the risk premium coefficient in the same market on the dimension of time.
3.1. Time Variation of Risk Premium Coefficient. In fact, the GARCH-M model has an implied assumption that the return compensation investors demand for a unit of risk is invariant in a certain period of time. Wen and Yang [29] conducted a preliminary study of the relationship between the risk premium coefficient and the skewness in the return distribution on the dimension of time in the same markets. They divided the return series of each market into 4 subsamples of 4 different horizons and found that the risk premium coefficients and the skewness for different time periods in
the same market were quite different but were still significantly negatively correlated. The fact that the skewness may vary at different time periods is consistent with many scholars' research conclusions, but there are few studies on the time variation of the risk premium coefficient. The empirical evidences provided by Wen and Yang [29] showed that in the same market the risk premium coefficient may change with the time; that is to say, in reality investors' risk attitude may be different at different time periods under the influence of some factors.

In order to investigate the possible relationship between the risk premium coefficient and the skewness in the return distribution on the time dimension, this paper first studies the time variation of the risk premium coefficient.

Previous studies found that investors' risk aversion varies at different time periods ([33, 34], etc.). The higher risk aversion means higher return compensation investors demand for unit risk they take, which implies that the risk premium coefficient should vary with time and other factors. Anderson et al. [30] proposed the ANST-GARCH-M model, assuming the risk premium coefficient is related to prior unexpected gains and characterizing the premium coefficient as $\delta_{1}+\delta_{2} F\left(\varepsilon_{t-1}\right)$, where $F\left(\varepsilon_{t-1}\right)=\left\{1+\exp \left[-\gamma\left(\varepsilon_{t-1}\right)\right]\right\}^{-1}$, which essentially allows for the time variation of the risk premium coefficient. Their empirical results also indicated that, in reality, investors' risk tolerance will vary with the time under the influence of prior unexpected gains.

Thaler Richard and Johnson Eric's [35] study gave a tentative explanation to the problem of how prior outcomes affect the risk-taking decision of the current period. Their research suggested that as prior gains can cushion the possible loss of the current period, the investor's risk attitude will be enhanced and thus encourage him to take more risk and even engage in speculation. This phenomenon is called the "house money" effect; the investor, in fact, records prior gains in a specific mental account and thinks that is only "house money." On the contrary, prior losses will increase the investor's current risk aversion, since another loss will make
the investor feel much more "painful" than the average loss. Based on the above findings, Barberis et al. [36] introduced the utility function of the prospect theory into their capital asset pricing model and discussed that investors' risk aversion will vary due to their prior behaviors in capital asset pricing. According to the prospect theory, the valuation of the gain and the loss is determined according to the selection of the reference point. This paper, based on Barberis et al. [36], assumes that prior gains confirmed by the investor are only the unexpected gains, that is, the unexpected return in the model.

In addition to allowing for the influence of the last-period unexpected return on the investor's risk attitude, this paper further considers the possible influence of other factors. First, as a market participant facing various risks in the market, the investor must have the capacity to tolerate certain degree of risk; that is to say, every investor has the intrinsically invariant potential to speculate; secondly, human's behavior exhibits a certain measure of continuity; current behavior is more or less influenced by prior behavior, and speculative behavior may, to some extent, find its root in prior behavior; finally, conventional theories hold that, in making their investment decisions, investors will unavoidably consider the gain and the loss, and they will take into account the risk factors in addition to the last-period unexpected return. These factors jointly determine the investor's risk attitude. On the basis of the above discussion, this paper proposes the following GARCH-M model allowing for time-varying risk coefficient to further investigate the possible influence of these factors on the investor's risk attitude:

$$
\begin{align*}
& r_{t}=c+x_{t} \beta+\gamma_{t} f\left(h_{t}\right)+\varepsilon_{t} \\
& \gamma_{t}=\rho_{0}+\rho_{1} \cdot \gamma_{t-1}+\rho_{2} \cdot \frac{\varepsilon_{t-1}}{f\left(h_{t-1}\right)},  \tag{5}\\
& \varepsilon_{t}=\sqrt{h_{t}} \cdot v_{t} \\
& h_{t}=\alpha_{0}+\alpha_{1} \varepsilon_{t-1}^{2}+\alpha_{2} h_{t-1}
\end{align*}
$$

Different from the original GARCH-M model, our model assumes that the risk premium coefficient is time-varying. In the model, $\rho_{0}$ is the basic risk premium investors demand for a unit of risk, which can be interpreted as the invariant risk premium investors demand for certain kinds of risk inherent in the market for a certain period of time; $\varepsilon_{t-1} / f\left(h_{t-1}\right)$ is the risk adjusted unexpected return of the last period, where $f\left(h_{t}\right)$ is a function of the time-varying variance $h_{t}$ (mostly in the forms of $\left.h_{t}, \sqrt{h_{t}}, \ln \left(h_{t}\right)\right), \rho_{1}$ means that the current risk attitude can be more or less jointly influenced by the unexpected return and the risk of the last period; $\rho_{2}$ expresses that the current risk attitude can be influenced, to a certain extent, by the last-period risk attitude, reflecting the average level of the influence of historical behaviors on the current behavior.
3.2. Autoregressive Conditional Skewness Model Allowing for Time-Varying Risk Premium Coefficient. Harvey and Siddique [18], based on a noncentral $t$ distribution, characterized
the time variation of the variance and the skewness simultaneously using a simple autoregressive conditional skewness model, GARCHS ( $1,1,1$ ) (GARCH with skewness):

$$
\begin{align*}
& r_{t}=\varphi^{\prime} Z_{t-1}+\varepsilon_{t}, \quad \varepsilon_{t} \mid \Omega_{t-1} \sim N_{t}\left(v_{t}, \delta_{t}\right) \\
& h_{t}=\alpha_{0}+\alpha_{1} \varepsilon_{t-1}^{2}+\alpha_{2} h_{t-1}  \tag{6}\\
& s_{t}=\beta_{0}+\beta_{1} \varepsilon_{t-1}^{3}+\beta_{2} s_{t-1}
\end{align*}
$$

where $r_{t}$ is a variable to be modeled, for example, the index return in the stock market; $\varphi^{\prime} Z_{t-1}$ is the conditional mean, where $Z_{t}$ is an instrumental variable completely based on the information set $\Omega_{t}$; the error term $\varepsilon_{t}$ follows the conditional noncentral $t$ distribution $N_{t}\left(v_{t}, \delta_{t}\right) ; h_{t}=\operatorname{Var}_{t-1}\left(r_{t}\right)$ is the conditional variance; $s_{t}=\operatorname{Skew}_{t-1}\left(r_{t}\right)$ is the conditional skewness; and $0<\alpha_{1}<1,0<\alpha_{2}<1 ;-1<\beta_{1}<1,-1<$ $\beta_{2}<1 ; \alpha_{1}+\alpha_{2}<1,-1<\beta_{1}+\beta_{2}<1$.

Estimating models (5) and (6) independently can give us the conditional skewness and the time-varying risk premium coefficient, respectively. Then we can investigate the correlation between them on the dimension of time. But estimating the two models independently is likely to separate the various features of the return series undesirably, which are actually embodied in the return series. It is even more likely that the relationship between the various features of the return series will be distorted, especially when the two models are estimated on the basis of different distribution assumptions. On the other hand, in accounting for the skewness, the distribution assumption for the conventional GARCH-type model fails to describe the time-varying process of the skewness, but the noncentral $t$ distribution can be employed to characterize the time variation of the skewness. Therefore, this paper integrated the above two models into the following GARCH-M model allowing for time-varying risk premium coefficient (for consistency, this paper uses $\sqrt{h_{t}}$ for $f\left(h_{t}\right)$ ):

$$
\begin{align*}
& r_{t}=c+x_{t} \delta+\gamma_{t} \sqrt{h_{t}}+\varepsilon_{t}, \quad \varepsilon_{t} \mid \Omega_{t-1} \sim N_{t}\left(v_{t}, \delta_{t}\right) \\
& \gamma_{t}=\rho_{0}+\rho_{1} \cdot \gamma_{t-1}+\rho_{2} \cdot \frac{\varepsilon_{t-1}}{\sqrt{h_{t-1}}}  \tag{7}\\
& h_{t}=\alpha_{0}+\alpha_{1} \varepsilon_{t-1}^{2}+\alpha_{2} h_{t-1} \\
& s_{t}=\beta_{0}+\beta_{1} \varepsilon_{t-1}^{3}+\beta_{2} s_{t-1}
\end{align*}
$$

This model affords a potential tool for further studying the relationship between the risk premium coefficient and the skewness on the dimension of time.
3.3. Model Estimation. The equation of the conditional noncentral $t$ distribution is too complicated, which causes much difficulty in model estimation. León et al. [20] estimated the autoregressive conditional variance, skewness, and kurtosis model using the Gram-Charlier series expansion of the normal density function. Xu [26], on the basis of León et al.'s work, proposed that the parameters of $\operatorname{GARCHS}(1,1,1)$ can be estimated using the Gram-Charlier series expansion of the normal density function and truncating at the third
moment. With the available information set $\Omega_{t-1}$, the approximate expression of the conditional density function of the standardized error $\eta_{t}=\varepsilon_{t} h_{t}^{-1 / 2}$ of the error $\varepsilon_{t}$ can be obtained:

$$
\begin{align*}
g\left(\eta_{t} \mid I_{t-1}\right) & =\frac{1}{\sqrt{2 \pi}} e^{-\eta_{t}^{2} / 2}\left(1+\frac{s_{t}^{*}}{3!}\left(\eta_{t}^{3}-3 \eta_{t}\right)\right)  \tag{8}\\
& =\phi\left(\eta_{t}\right) \psi^{2}\left(\eta_{t}\right)
\end{align*}
$$

where $s_{t}^{*}$ denotes the conditional skewness of $\eta_{t} ; \phi\left(\eta_{t}\right)$ is the probability density function of the standard normal distribution, $(1 / \sqrt{2 \pi}) e^{-\eta_{t}^{2} / 2} ; \psi\left(\eta_{t}\right)$ is the polynomial part of the Gram-Charlier series' third-order expansion, $1+\left(s_{t}^{*} / 3!\right)\left(\eta_{t}^{3}-\right.$ $3 \eta_{t}$ ). Hence we get the likelihood function of the sample:

$$
\begin{align*}
\mathrm{SLF}= & -\frac{1}{2}(T-1) \times \ln (2 \pi)-\frac{1}{2} \sum_{t=2}^{T} \ln h_{t}-\frac{1}{2} \sum_{t=2}^{T} \ln \eta_{t}^{2}  \tag{9}\\
& +\sum_{t=2}^{T} \ln \left(1+\frac{s_{t}^{*}}{3!}\left(\eta_{t}^{3}-3 \eta_{t}\right)\right) .
\end{align*}
$$

However, the likelihood function determined by (8) cannot satisfy the definition of the density function. Based on León et al. [20], this paper modifies it and derives the following probability density function expression:

$$
\begin{align*}
f\left(\eta_{t} \mid I_{t-1}\right) & =\frac{\phi\left(\eta_{t}\right) \psi^{2}\left(\eta_{t}\right)}{\Gamma_{t}} \\
& =\frac{(1 / \sqrt{2 \pi}) e^{-\eta_{t}^{2} / 2}\left(1+\left(s_{t}^{*} / 3!\right)\left(\eta_{t}^{3}-3 \eta_{t}\right)\right)^{2}}{\Gamma_{t}} \tag{10}
\end{align*}
$$

where $\Gamma_{t}=1+s_{t}^{* 2} / 3$ ! is the modified term of the GramCharlier series expansion. Hence we obtain the likelihood function for the sample:

$$
\begin{align*}
\mathrm{SLF}^{\prime}= & -\frac{1}{2}(T-1) \times \ln (2 \pi)-\frac{1}{2} \sum_{t=2}^{T} \ln h_{t}-\frac{1}{2} \sum_{t=2}^{T} \ln \eta_{t}^{2} \\
& +\sum_{t=2}^{T} \ln \left(\psi^{2}\left(\eta_{t}\right)\right)-\sum_{t=2}^{T} \ln \left(\Gamma_{t}\right) \tag{11}
\end{align*}
$$

Solving for the maximum of (11) of the likelihood function of the sample will give us the consistent estimates of the parameters.

As the sample likelihood function is highly nonlinear, the selection of the initial parameters is critical for obtaining the global optimal solution. Harvey and Siddique's [18] strategy of going from the simple model to the complicated model in estimating models has delivered good empirical results and has thus been widely applied. This paper, in line with this strategy, used the following parameter estimation steps:
(1) estimate the equation for the mean and then use the results as the initial values to estimate the GARCH-M model;


Figure 2: The time-varying processes of the risk premium coefficients gamma.
(2) estimate the GARCH-M model and use the parameter estimation results as the initial values for the estimation of the GARCHS-M model and the GARCHM model allowing for time-varying risk premium coefficient;
(3) use the parameter estimation results from the above two steps as the initial values to estimate the GARCHS-M model allowing for time-varying risk premium coefficient.
3.4. Empirical Results. The empirical study in this section was divided into two parts: first the estimation results for model (7) were given and analyzed to investigate the reasonability of the specification of the model; then the correlation between the skewness in the return distribution and the degree of speculation in the market on the basis of the empirical results was further investigated.
3.4.1. Results for Model Estimation. We estimated the model (7) using the data of the 14 samples (we run the estimation on Eviews 5.0.) and the estimation results are shown in Table 3.

Using the estimation results, the time-varying processes of the risk premium coefficients were plotted as shown in Figure 2 (for space, only 2 representative indices, S\&P 500 and Nikkei 225, were given here).

Figure 2 gives the time-varying processes of the risk premium coefficients. From the figure we can see that the risk premium coefficient for each market exhibits an obvious time-varying feature and an average significantly greater than zero. This shows that, in general, the more risk there is in the market, the more return compensation investors demand, which is consistent with the conventional finance theory. But under the influence of the risk attitude, the unexpected return, and the risk of the last period, the current risk premium coefficient may be negative in some cases, which means that investors relax their vigilance for risk under the influence of these factors and exhibit obvious irrationality, which may encourage speculative behavior.

From the examination of the estimation results of the time-varying process of the risk premium coefficients, we can see that, apart from the constant term, most of the estimation

Table 3: Model estimation results.
(a)

| Parameter |  | S\&P 500 | Dow J. | Nasdaq | NYSE | N 225 | FTSE 100 | SSE |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time- varying $\gamma$ | $\rho_{0}$ | $\begin{aligned} & 0.014061 \\ & (0.0668) \end{aligned}$ | $\begin{aligned} & 0.024333 \\ & (0.0960) \end{aligned}$ | $\begin{aligned} & 0.073175 \\ & (0.0220) \end{aligned}$ | $\begin{gathered} 0.015498 \\ (0.1171) \end{gathered}$ | $\begin{gathered} 0.004656 \\ (0.3741) \end{gathered}$ | $\begin{aligned} & 0.022537 \\ & (0.0356) \end{aligned}$ | $\begin{gathered} 0.060807 \\ (0.0767) \end{gathered}$ |
|  | $\rho_{1}$ | $\begin{aligned} & 0.671156 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.548122 \\ & (0.0059) \end{aligned}$ | $\begin{gathered} -0.672522 \\ (0.0011) \end{gathered}$ | $\begin{aligned} & 0.699180 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.864550 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.552242 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} -0.802902 \\ (0.0013) \end{gathered}$ |
|  | $\rho_{2}$ | $\begin{gathered} -0.070729 \\ (0.0002) \\ \hline \end{gathered}$ | $\begin{gathered} -0.056227 \\ (0.0098) \\ \hline \end{gathered}$ | $\begin{aligned} & 0.034182 \\ & (0.0169) \\ & \hline \end{aligned}$ | $\begin{gathered} -0.045118 \\ (0.0142) \\ \hline \end{gathered}$ | $\begin{gathered} -0.017348 \\ (0.1723) \\ \hline \end{gathered}$ | $\begin{gathered} -0.082743 \\ (0.0000) \\ \hline \end{gathered}$ | $\begin{gathered} 0.018254 \\ (0.2415) \\ \hline \end{gathered}$ |
| Variance equation | $\alpha_{0}$ | $\begin{aligned} & 0.011472 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.011674 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.010696 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.014906 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & \hline 0.020784 \\ & (0.0038) \end{aligned}$ | $\begin{aligned} & 0.010016 \\ & (0.0002) \end{aligned}$ | $\begin{aligned} & \hline 0.041504 \\ & (0.0000) \end{aligned}$ |
|  | $\alpha_{1}$ | $\begin{aligned} & 0.075448 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.081726 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.046647 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.078346 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.092420 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.096733 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.099855 \\ & (0.0000) \end{aligned}$ |
|  | $\alpha_{2}$ | $\begin{aligned} & 0.913434 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.908752 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.945602 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.906232 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{gathered} 0.900605 \\ (0.0000) \\ \hline \end{gathered}$ | $\begin{aligned} & 0.894667 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.893033 \\ & (0.0000) \\ & \hline \end{aligned}$ |
| Skewness equation | $\beta_{0}$ | $\begin{gathered} \hline-0.000538 \\ (0.0046) \end{gathered}$ | $\begin{gathered} \hline-0.001420 \\ (0.0311) \end{gathered}$ | $\begin{gathered} \hline-0.045664 \\ (0.0017) \end{gathered}$ | $\begin{gathered} \hline-0.000449 \\ (0.0010) \end{gathered}$ | $\begin{gathered} \hline-0.046385 \\ (0.0385) \end{gathered}$ | $\begin{gathered} -0.008635 \\ (0.1683) \end{gathered}$ | $\begin{aligned} & \hline 0.020711 \\ & (0.0143) \end{aligned}$ |
|  | $\beta_{1}$ | $\begin{aligned} & 0.001963 \\ & (0.0235) \end{aligned}$ | $\begin{gathered} 0.004560 \\ (0.0516) \end{gathered}$ | $\begin{aligned} & 0.032296 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.003019 \\ & (0.0067) \end{aligned}$ | $\begin{gathered} 0.028283 \\ (0.0016) \end{gathered}$ | $\begin{aligned} & 0.011915 \\ & (0.0359) \end{aligned}$ | $\begin{aligned} & 0.006473 \\ & (0.0442) \end{aligned}$ |
|  | $\beta_{2}$ | $\begin{aligned} & 0.987139 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.959255 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.533682 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.983791 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.595621 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{gathered} 0.843206 \\ (0.0000) \\ \hline \end{gathered}$ | $\begin{aligned} & 0.626719 \\ & (0.0003) \\ & \hline \end{aligned}$ |
| Log likelihood |  | -1243.670 | -1154.630 | -1926.634 | -1172.428 | -1845.429 | -1203.899 | -2086.479 |

(b)

| Parameter |  | DAX | CAC 40 | GSPTSE | MIBTEL | IGBM | BVSP | Hangseng |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Time- varying $\gamma$ | $\rho_{0}$ | $\begin{aligned} & 0.013762 \\ & (0.1821) \end{aligned}$ | $\begin{aligned} & 0.012604 \\ & (0.0681) \end{aligned}$ | $\begin{aligned} & 0.012297 \\ & (0.2773) \end{aligned}$ | $\begin{aligned} & 0.035950 \\ & (0.2338) \end{aligned}$ | $\begin{aligned} & 0.021461 \\ & (0.5686) \end{aligned}$ | $\begin{aligned} & 0.014801 \\ & (0.1713) \end{aligned}$ | $\begin{aligned} & 0.007973 \\ & (0.7020) \end{aligned}$ |
|  | $\rho_{1}$ | $\begin{aligned} & 0.771009 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.744133 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} 0.754400 \\ (0.0002) \end{gathered}$ | $\begin{aligned} & 0.082012 \\ & (0.8789) \end{aligned}$ | $\begin{aligned} & 0.761071 \\ & (0.0623) \end{aligned}$ | $\begin{aligned} & 0.728769 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} 0.843089 \\ (0.0367) \end{gathered}$ |
|  | $\rho_{2}$ | $\begin{gathered} -0.031714 \\ (0.0562) \end{gathered}$ | $\begin{gathered} -0.053187 \\ (0.0023) \end{gathered}$ | $\begin{gathered} -0.024171 \\ (0.1279) \end{gathered}$ | $\begin{gathered} -0.036640 \\ (0.0961) \end{gathered}$ | $\begin{gathered} -0.013422 \\ (0.4694) \end{gathered}$ | $\begin{gathered} -0.032091 \\ (0.0483) \end{gathered}$ | $\begin{gathered} -0.007450 \\ (0.6257) \end{gathered}$ |
| Variance equation | $\alpha_{0}$ | $\begin{aligned} & 0.016197 \\ & (0.0002) \end{aligned}$ | $\begin{aligned} & 0.014882 \\ & (0.0001) \end{aligned}$ | $\begin{aligned} & 0.008462 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} 0.009467 \\ (0.0000) \end{gathered}$ | $\begin{aligned} & 0.017090 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.099432 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.015302 \\ & (0.0003) \end{aligned}$ |
|  | $\alpha_{1}$ | $\begin{gathered} 0.089870 \\ (0.0000) \end{gathered}$ | $\begin{gathered} 0.080974 \\ (0.0000) \end{gathered}$ | $\begin{aligned} & 0.072330 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.097325 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.089791 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.072369 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} 0.067647 \\ (0.0000) \end{gathered}$ |
|  | $\alpha_{2}$ | $\begin{aligned} & 0.902930 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.909675 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.918544 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.895624 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.897541 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.894363 \\ & (0.0000) \end{aligned}$ | $\begin{gathered} 0.925600 \\ (0.0000) \end{gathered}$ |
| Skewness equation | $\beta_{0}$ | $\begin{gathered} -0.044249 \\ (0.3287) \end{gathered}$ | $\begin{gathered} -0.012939 \\ (0.0441) \end{gathered}$ | $\begin{gathered} -0.001743 \\ (0.0042) \end{gathered}$ | $\begin{gathered} -0.008658 \\ (0.1932) \end{gathered}$ | $\begin{gathered} -0.008764 \\ (0.2645) \end{gathered}$ | $\begin{gathered} -0.080287 \\ (0.0413) \end{gathered}$ | $\begin{gathered} -0.003197 \\ (0.5593) \end{gathered}$ |
|  | $\beta_{1}$ | $\begin{aligned} & 0.013195 \\ & (0.2559) \end{aligned}$ | $\begin{aligned} & 0.018179 \\ & (0.0018) \end{aligned}$ | $\begin{aligned} & 0.006174 \\ & (0.0001) \end{aligned}$ | $\begin{gathered} 0.007836 \\ (0.0224) \end{gathered}$ | $\begin{aligned} & 0.010878 \\ & (0.1647) \end{aligned}$ | $\begin{gathered} 0.008073 \\ (0.0333) \end{gathered}$ | $\begin{aligned} & 0.001812 \\ & (0.4276) \end{aligned}$ |
|  | $\beta_{2}$ | $\begin{gathered} 0.636849 \\ (0.0780) \end{gathered}$ | $\begin{aligned} & 0.852436 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.964743 \\ & (0.0000) \\ & \hline \end{aligned}$ | $\begin{aligned} & 0.880261 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.750425 \\ & (0.0005) \end{aligned}$ | $\begin{aligned} & 0.909746 \\ & (0.0000) \end{aligned}$ | $\begin{aligned} & 0.945101 \\ & (0.0000) \end{aligned}$ |
| Log likelihood |  | -1815.479 | -1665.186 | -1046.193 | -1071.693 | -1130.580 | -2450.400 | -1753.907 |

results for $\rho_{1}$ and $\rho_{2}$ are significant at $10 \%$; although $\rho_{2}$ for Nikkei 225, SSE, and GSPTSE are not significant at $10 \%$, the corresponding $P$ values are relatively small ( $0.1723,0.2415$, and 0.1279 , resp.). Only the $P$ values for $\rho_{1}$ for MIBTEL and $\rho_{2}$ for IGBM and Hangseng are relatively big ( $0.8789,0.4694$, and 0.6257 , resp.), and the estimation results are obviously not significant. But for all the estimation results there are no cases where the estimation results for the two coefficients of $\rho_{1}$ and $\rho_{2}$, which determine time variation of the risk premium coefficient, are not significant at the same time. This fact indicates that time variation does exist for the risk
premium coefficient, and it is reasonable to assume that the risk premium coefficient for the model is time-varying.

Among the 14 estimation results for the samples, $\rho_{0}$ are all positive and most of them are significant, which imply the risk premium investors demand for unit risk is positive, and the result is consistent with the traditional theory. This can be regarded as the risk premium investors demand for the risk inherent in the market for a certain period of time. Except for the estimation results for NASDAQ and SSE, the other results for $\rho_{1}$ are positive and approximate to 1 (the result for MIBTEL is 0.082012 , rather small), which means
that the current risk attitude is influenced by the persistence of the last-period risk attitude and diminishes a little. The estimation results for $\rho_{2}$ for the samples are all smaller than zero. This indicates that the positive unexpected return after risk adjustment will reduce the risk premium investors demand for unit risk. It can also be viewed as that when the basic risk remains unchanged in the market, the higher the unexpected return is in the last period, the smaller the risk premium coefficient is; this indicates that investors' current risk tolerance is higher and the degree of speculation is also higher; if the unexpected return in the last period is positive, the more riskier it was in the last period, the higher the premium coefficient is, which shows that investors' current risk tolerance is lower than that of the last period, and also lower is the degree of speculation; if the unexpected return is negative in the last period, the more riskier it was in the last period, the smaller the risk premium coefficient is, which shows that investors' current risk tolerance is higher than that of the last period, and also higher is the degree of speculation.

However, the signs of $\rho_{1}$ and $\rho_{2}$ of the estimation results for the samples of NASDAQ and SSE are just the opposite of those for other samples. The fact that $\rho_{1}$ is negative with its absolute value less than 1 indicates that investors' current risk attitude will show a reversal in the nest period in the above 2 markets; that is to say, the higher the risk tolerance for the current period is, the higher the possibility there is for it to become lower for the next period, and the behavior intensity of the risk tolerance will weaken; the sign of $\rho_{2}$ is negative, which implies the risk-adjusted positive unexpected return will drive up investors' unit risk premium. With the risk remaining fundamentally unchanged, the influence of the unexpected return on the degree of speculation is just the opposite to those on other markets. The higher the unexpected return for the last period is, the more likely the risk tolerance is to become higher; that is to say, the degree of speculation is more likely to decrease. This will cause investors to become more cautious in trading, which is the typical psychology to keep the profit.

It is well known that NASDAQ and SSE are widely recognized as two markets with relatively high speculation; the completely different risk tolerance behavior of the two markets from the other markets may be possibly determined by the degrees of speculation on the two markets. Accordingly, we set to investigate the influence of the market risk on the degrees of speculation in the two markets. When the unexpected return is positive and remains basically unchanged, the market is relatively bullish. The fact that the estimation results of $\rho_{2}$ for the two markets are positive indicates that the higher the risk for the last period is, the smaller the risk premium coefficient for the current period is; that is, the higher the risk tolerance is, the higher the speculation degree is; when the unexpected return is negative and remains basically unchanged, the market is relatively bearish. In that case, the positive $\rho_{2}$ implies that the higher the risk is for the last period, the smaller the risk premium coefficient is for the current period; that is to say, the smaller the risk tolerance is, the lower the degree of speculation is. This implies that when it is bullish, investors, more often than not, seek risks in NASDAQ and SSE, and the higher the risk
is, the more likely it is for investors to engage in speculation, while investors tend to evade risk, when the market is bearish, and increased risk will reduce speculation in the market. In a mature market, investors reaction to risk is completely different from investors in the above two markets: investors prefer relatively low risk for stable return in a bullish market, and increased risk will lead to less speculation, while they prefer relatively higher risk, which will prompt speculation in the market.

The above discussion shows that there is striking difference in investors' reaction to the return and the risk in a speculative market and a mature market, which may serve as an indicator for whether the market is mature or not.

Besides, it can be seen from the examination of the estimation results of the coefficient of the skewness process that the conditional skewness process (not including the constant term) of most of the indices is significant at $10 \%$ (DAX, IGBM, and Hangseng are excluded, but part of the coefficients for DAX and IGBM and the corresponding $P$ value are relatively small, being 0.2559 and 0.1647 , resp.). This shows that the time variation of the skewness is well characterized. As the conditional skewness process has been well studied, this paper will not discuss much about the estimation results.
3.4.2. The Relationship between the Risk Premium Coefficient and the Skewness. From the above discussion we can see that the GARCHS-M model with the time-varying risk premium coefficient can well characterize the conditional skewness process and the time variation process of the risk premium coefficient of the 14 stock indices simultaneously. On the basis of the estimation results, this paper will further examine the relationship between the risk premium coefficient and the skewness on the dimension of time.

First, we will plot the conditional skewness and the risk premium coefficient of the 14 indices, as shown in Figure 3 (for S\&P 500 and Nikkei 225), where the line is the regression line. The Pearson correlation can only measure the linear correlation between the variables [37], but the scatter plots will give us the impression that the relationship between the risk premium coefficient and the skewness may be complicated. Consequently, this paper will select Kendall's tau and Spearman's rho to measure whether the 2 variables' variation tendencies are in the same direction. The values for the 2 correlations of the 2 variables are given in Table 4.

It can be seen from Tables 3 and 4, except NASDAQ and SSE, that the conditional skewness and the risk premium coefficient are significantly negatively correlated. The significantly negative correlation further corroborated Wen and Yang's [29] viewpoints: the high degree of speculation in the market will cause the positive skew in the return distribution. It can be shown with further investigation of the causes for the negative correlation using models and their estimation results: in general, the conditional skewness is positively correlated with the cube of the unexpected return, while the risk premium coefficient is negatively correlated with the unexpected return. The different reactions to the unexpected return are possibly the main causes for


Figure 3: The scatter plots of the relationship between the risk premium coefficient gamma and the conditional skewness.

Table 4: The correlation between the risk premium coefficient gamma and the conditional skewness.
(a)

|  | S\&P 500 | Dow J. | NASDAQ | NYSE | N 225 | FTSE 100 | SSE |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pearson correlation | -0.106 | -0.223 | 0.175 | -0.118 | -0.308 | -0.246 | 0.158 |
| Kendall's tau | -0.144 | -0.200 | 0.193 | -0.175 | -0.558 | -0.415 | 0.130 |
| Spearman's rho | -0.211 | -0.295 | 0.272 | -0.255 | -0.756 | -0.591 | 0.183 |

(b)

|  | DAX | CAC 40 | GSPTSE | MIBTEL | IGBM | BVSP | Hangseng |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Pearson correlation | -0.438 | -0.402 | -0.156 | -0.232 | -0.358 | -0.474 | -0.371 |
| Kendall's tau | -0.619 | -0.514 | -0.250 | -0.218 | -0.609 | -0.460 | -0.449 |
| Spearman's rho | -0.818 | -0.714 | -0.368 | -0.321 | -0.808 | -0.644 | -0.632 |

Note: all the values are significantly different from zero at $1 \%$ level.
the negative correlation. Behavioral finance research findings show that the nonnormality of the return distribution is caused by investors' behavioral biases, and the unexpected return series (i.e., the error series) can be viewed as the proxy of the information flow, so we might as well think that the negative correlation is essentially caused by investors' different reactions to information.

On the contrary, the conditional skewness and the risk premium coefficient of the NASDAQ and the SSE are significantly positively correlated, which is also consistent with the findings of Wen and Yang [29]. It can be seen from the scatter plots that none of the correlations between the risk premium coefficient and the conditional skewness for all the markets is linear. Simply judging from the degree of consistency of the tendencies in the changes of the two variables, even in the case of low linear correlation, the directions in the changes are usually of high degree of consistency. This shows that the influence of speculation on
the skewness in the return distribution is not linear and may have a very complex mechanism. Apart from speculation in the market, the skewness in the return distribution may be attributed to other factors, which partly explains the complicated relationship between the speculation and the skewness in the return distribution.

## 4. Conclusions

This paper characterizes the time variation of the risk premium coefficient, that is, how investors' risk attitude varies with the time. By proposing a modified GARCH-M model and using the daily return series of the representative 14 stock indices as samples, this paper empirically investigates the correlation of the skewness in the return distribution and the risk premium coefficient, and the empirical results show the following.

Firstly, the risk premium coefficient is obviously timevarying, and the current risk premium investors demand for unit risk is influenced not only by the last-period unexpected return and the risk but also by the persistence of the risk attitude for the last period. Secondly, there is significant difference in investors' reactions to the return and the risk in a speculative market and a mature market: in a mature market investors prefer low risk for a stable return when the market is bullish, while they seek relatively high risk when it is bearish and may trigger more speculation in the market. Lastly, the skewness in the return distribution and the risk premium coefficient is significantly negatively correlated even in a mature market when examined on the dimension of time.

The above results of this paper further validate the correlation between the skewness in the return distribution and the degree of speculation in the market. Meanwhile, the results also show that the influence of speculation on the skewness in the return distribution is rather complicated. It is necessary and meaningful to further investigate the deep-seated relationship between the skewness in the return distribution and speculation, and the possible influence of other factors on the skewness in the return distribution, such as the cultural background of the market participants, and the microstructure of the market.

## Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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## Research Article

# SVEIRS: A New Epidemic Disease Model with Time Delays and Impulsive Effects 

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#### Abstract

We first propose a new epidemic disease model governed by system of impulsive delay differential equations. Then, based on theories for impulsive delay differential equations, we skillfully solve the difficulty in analyzing the global dynamical behavior of the model with pulse vaccination and impulsive population input effects at two different periodic moments. We prove the existence and global attractivity of the "infection-free" periodic solution and also the permanence of the model. We then carry out numerical simulations to illustrate our theoretical results, showing us that time delay, pulse vaccination, and pulse population input can exert a significant influence on the dynamics of the system which confirms the availability of pulse vaccination strategy for the practical epidemic prevention. Moreover, it is worth pointing out that we obtained an epidemic control strategy for controlling the number of population input.


## 1. Introduction

In epidemic modeling, susceptible-infectious-recovered type of models is well known [1-18] although such models very often ignore the incubation period in the development of mathematical models for some diseases. However, recent research shows for certain diseases, such as smallpox, rabies, BSE, and some skin diseases, the incubation period has significant effect on the epidemic dynamics so that it is nonnegligible. The incubation period varies greatly from a couple of days (e.g., H1N1 outbreaking worldwide has generally an incubation period of one to seven days) to several years (e.g., AIDS virus sometimes can be several years). When taking the incubation period into account in the development of models, we reach SEIR model, which is short for susceptible, exposed, infectious, and recovered [19-30]. And some researchers used time delay to describe the incubation period; for example, Cooke [31], Beretta and Takeuchi [4], Takeuchi et al. [32], and Ma et al. [5] studied a SIR model with time delay and nonlinear incidence rate $\beta S(t) I(t-\tau)$. Liu et al. [33,34] used a nonlinear incidence rate $\beta S^{p}(t) I^{q}(t)$, and Meng et al. [35] and Jiang et al. [30],
respectively, studied an impulsively vaccinating SIR model with nonlinear incidences $\beta S^{q}(t) I(t-\tau)$ and $\beta S^{q}(t-\tau) I(t-\tau)$, which are better to describe the spread process of diseases than linear one.

In order to prevent infectious diseases, $[36,37]$ suggested that vaccination to the susceptible population is an important strategy. The traditional vaccinations are applied to each individual, while impulsive ones are to periodically vaccinate people within certain age groups [7-10, 38]. Some diseases may have a vaccination period after being cured but may cause losing immunity gradually. In this case, people might be infected again. So it is of great significance to investigate epidemic models with time delay and impulsive effects due to the incubation period and vaccination period [26-29]. For some certain regional systems, the immigrations can be periodic impulsive population input because the immigratory population might be susceptible. Certainly two different impulsive effects for periodic vaccination and population input do not usually happen simultaneously. Therefore, motivated by Jiang et al. [30] and Song et al. [19], we built a new mathematical model: susceptible, vaccinated, exposed,
infectious, recovered, and susceptible epidemic model with two time delays and two nonlinear incidences with pulse vaccination and a constant periodic population input at two different moments as follows:

$$
\begin{align*}
& \frac{d S(t)}{d t}=-b S(t)-\beta S^{p}(t) I(t)+\gamma I(t-\omega) e^{-b \omega}, \\
& \frac{d V(t)}{d t}=-\delta \beta V^{q}(t) I(t)-\gamma_{1} V(t)-b V(t), \\
& \frac{d E(t)}{d t}=-b E+\beta S^{p}(t) I(t)+\delta \beta V^{q}(t) I(t) \\
& -\beta e^{-b \tau} S^{p}(t) I(t-\tau)-\delta \beta e^{-b \tau} V^{q}(t) I(t-\tau), \\
& \frac{d I(t)}{d t}=\beta e^{-b \tau} S^{p}(t) I(t-\tau)+\delta \beta e^{-b \tau} V^{q}(t) I(t-\tau) \\
& -(\gamma+b+\alpha) I(t), \\
& \frac{d R(t)}{d t}=\gamma_{1} V(t)+\gamma I(t)-b R(t)-\gamma I(t-\omega) e^{-b \omega}, \\
& t \neq(n+l-1) T, \quad t \neq n T, \\
& \Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t), \quad \Delta E(t)=0, \\
& \Delta I(t)=0, \quad \Delta R(t)=0, \\
& t=(n+l-1) T, \\
& \Delta S(t)=\mu, \quad \Delta V(t)=0, \quad \Delta E(t)=0, \\
& \Delta I(t)=0, \quad \Delta R(t)=0, \\
& t=n T \text {. } \tag{1}
\end{align*}
$$

Here all parameters of system (1) are nonnegative constants. For the significance of parameters in (1), please see literatures Jiang et al. [30] and Song et al. [19]. Terms $\beta S^{p} I$ and $V^{q} I$ are the nonlinear incidence rates, and in our paper we only discuss the case

$$
\begin{equation*}
1 \leq q \leq p . \tag{2}
\end{equation*}
$$

## 2. Preliminaries

Let $N(t)=S(t)+V(t)+E(t)+I(t)+R(t)$, and then it is easy to see that $N(t)$ satisfies the following:

$$
\begin{equation*}
N^{\prime}(t) \leq b(1-N(t)), \quad \lim _{t \rightarrow \infty} \sup N(t) \leq 1 \tag{3}
\end{equation*}
$$

Hence, for time $t$ which is large, we obtain $0 \leq S(t)+$ $V(t)+I(t) \leq 1$. Let $\omega=\max \{\tau, \omega\}$ and $C^{+}=\{\varphi=$ $\left.\left(\varphi_{1}(s), \ldots, \varphi_{5}(s)\right) \in C: \varphi_{i}(0)>0\right\}$; here $\varphi_{i}(s)>0$ is bounded function on interval $[-\widetilde{\omega}, 0]$. Since variable $R(t)$ only appears in the fifth equation, system (1) can be further reduced as

$$
\begin{gathered}
\frac{d S(t)}{d t}=-b S(t)-\beta S^{p}(t) I(t)+\gamma I(t-\omega) e^{-b \omega} \\
\frac{d V(t)}{d t}=-\delta \beta V^{q}(t) I(t)-\gamma_{1} V(t)-b V(t)
\end{gathered}
$$

$$
\begin{gather*}
\frac{d I(t)}{d t}=\beta e^{-b \tau} S^{p}(t) I(t-\tau)+\delta \beta e^{-b \tau} V^{q}(t) I(t-\tau) \\
-(\gamma+b+\alpha) I(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \\
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t), \quad \Delta I(t)=0, \\
t=(n+l-1) T, \\
\Delta S(t)=\mu, \quad \Delta V(t)=0, \quad \Delta I(t)=0, \\
t=n T \tag{4}
\end{gather*}
$$

with the initial conditions

$$
\begin{equation*}
\left(\varphi_{1}(s), \varphi_{2}(s), \varphi_{4}(s)\right) \in C^{+}, \quad \varphi_{i}(0)>0, i=1,2,4 \tag{5}
\end{equation*}
$$

Lemma 1 (see [39, 40]). For the following impulse differential inequalities

$$
\begin{align*}
& s^{\prime}(t) \leq(\geq) q(t) s(t)+r(t), \quad t \neq t_{k}  \tag{6}\\
& s\left(t_{k}^{+}\right) \leq(\geq) b_{k} s\left(t_{k}\right)+p_{k}, \quad t=t_{k}, \quad k \in N
\end{align*}
$$

where $q(t), r(t) \in C\left(R_{+}, R\right), b_{k} \geq 0$, and $p_{k}$ are constants.
Assume the following:
$\left(A_{0}\right)$ the sequence $\left\{t_{k}\right\}$ satisfies $0 \leq t_{0}<t_{1}<t_{2}<\cdots$, with $\lim _{t \rightarrow \infty} t_{k}=\infty$;
$\left(A_{1}\right) w \in P C^{\prime}\left(R_{+}, R\right)$ and $s(t)$ is left-continuous at $t_{k}, k \in N$.

Then

$$
\begin{align*}
& s(t) \leq(\geq) s\left(t_{0}\right) \prod_{t_{0}<t_{k}<t} b_{k} \exp \left(\int_{t_{0}}^{t} q(u) d u\right) \\
&+\sum_{t_{0}<t_{k}<t}\left(\prod_{t_{k}<t_{j}<t} b_{j} \exp \left(\int_{t_{k}}^{t} q(u) d u\right)\right) p_{k}  \tag{7}\\
&+\int_{t_{0}}^{t} \prod_{u<t_{k}<t} b_{k} \exp \left(\int_{u}^{t} q(\theta) d \theta\right) r(u) d u \\
& t \geq t_{0} .
\end{align*}
$$

Lemma 2 (see [41]). For the following delay differential equation

$$
\begin{equation*}
\frac{d z(t)}{d t}=a z(t-\theta)-b z(t) \tag{8}
\end{equation*}
$$

where $a, b$, and $\theta$ are all positive constants and $z(t)>0$ for $t \in[-\theta, 0]$, then we have

$$
\lim _{t \rightarrow \infty} z(t)= \begin{cases}0, & \text { if } a<b  \tag{9}\\ +\infty, & \text { if } a>b\end{cases}
$$

Lemma 3 (see [42]). The following system,

$$
\begin{gather*}
\frac{d x(t)}{d t}=-b x(t), \quad \frac{d y(t)}{d t}=-(a+b) y(t), \\
t \neq n T, \quad t \neq(n+l-1) T, \\
\Delta x(t)=-\theta x(t), \quad \Delta y(t)=\theta x(t), \quad t=(n+l-1) T, \\
\Delta x(t)=\mu, \quad \Delta y(t)=0, \quad t=n T \tag{10}
\end{gather*}
$$

has a unique positive T-periodic solution:

$$
\begin{gather*}
x^{*}(t)=\left\{\begin{array}{l}
\frac{\mu \exp (-b(t-(n-1) T))}{1-(1-\theta) \exp (-b T)}, \\
t \in((n-1) T,(n+l-1) T] \\
\frac{\mu(1-\theta) \exp (-b(t-(n-1) T))}{1-(1-\theta) \exp (-b T)}, \\
t \in((n+l-1) T, \leq n T]
\end{array}\right. \\
y^{*}(t)=\frac{\mu \theta \exp (-b l T) \exp (-(a+b)(t-(n+l-1) T))}{(1-\exp (-(a+b) T))(1-(1-\theta) \exp (-b T))}, \\
t \in((n+l-1) T,(n+l) T] \tag{11}
\end{gather*}
$$

and we further have $x(t) \rightarrow x^{*}(t)$ and $y(t) \rightarrow y^{*}(t)$ as $t \rightarrow$ $+\infty$.

## 3. The Existence and Global Attractivity of "Infection-Free" Periodic Solution

3.1. Existence. In this section, we are committed to investigate the existence of "infection-free" periodic solution. In this case, we have

$$
\begin{equation*}
I(t)=0, \quad t \geq 0 \tag{12}
\end{equation*}
$$

From systems (4) and (12), we obtain

$$
\begin{gathered}
\frac{d S(t)}{d t}=-b S(t), \quad \frac{d V(t)}{d t}=-\left(\gamma_{1}+b\right) V(t) \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N \\
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t)
\end{gathered}
$$

$$
\begin{gather*}
t=(n+l-1) T, \quad n \in N, \\
\Delta S(t)=\mu, \quad \Delta V(t)=0, \quad t=n T, n \in N . \tag{13}
\end{gather*}
$$

By Lemma 3, system (13) has a unique positive $T$-periodic solution:

$$
\begin{gather*}
S^{*}(t)=\left\{\begin{array}{l}
\frac{\mu \exp (-b(t-(n-1) T))}{1-(1-\theta) \exp (-b T)}, \\
t \in((n-1) T,(n+l-1) T] \\
\frac{\mu(1-\theta) \exp (-b(t-(n-1) T))}{1-(1-\theta) \exp (-b T)}, \\
t \in((n+l-1) T, \leq n T]
\end{array}\right. \\
V^{*}(t)=\frac{\mu \theta \exp (-b l T) \exp (-(a+b)(t-(n+l-1) T))}{(1-\exp (-(a+b) T))(1-(1-\theta) \exp (-b T))}, \\
t \in((n+l-1) T,(n+l) T]
\end{gather*}
$$

Furthermore, we can prove that it is the unique globally asymptotically stable positive periodic solution of system (4). We summarize this conclusion in the following lemma.

Lemma 4. The system (4) has an "infection-free" periodic solution $\left(S^{*}(t), V^{*}(t), 0\right)$, for $t \in((n+l-1) T,(n+l) T]$ and $n \in N$; for any solution $(S(t), V(t), I(t))$ of it, the following holds true:

$$
\begin{equation*}
S(t) \longrightarrow S^{*}(t), \quad V(t) \longrightarrow V^{*}(t) \tag{15}
\end{equation*}
$$

as $t \rightarrow \infty$.
This lemma indicates that in between the vaccination the susceptible and vaccinated populations oscillate with period $T$ in synchronization with the periodic pulse vaccination. Next we prove the global attractivity of such solution.
3.2. Global Attractivity. In this section, we will prove our main result on the global attractivity of the infection-free solution. It is stated in the following theorem.

Theorem 5. The system (4) has a unique infection-free periodic solution $\left(S^{*}(t), V^{*}(t), 0\right)$, and when it exists, it is globally attractive if

$$
\begin{equation*}
\mathscr{R}_{1}<1, \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathscr{R}_{1}=\beta e^{-b \tau} \frac{\left(A_{1}^{p}+\delta A_{2}^{q}\right)}{\gamma+b+\alpha} \tag{17}
\end{equation*}
$$

with

$$
\begin{align*}
& A_{1}=\frac{\gamma e^{-b \omega}}{b}+\frac{\mu e^{b T}}{e^{b T}-1} \\
& A_{2}=\frac{\theta e^{-b l T}}{(1-\theta)\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)} \tag{18}
\end{align*}
$$

Proof. Let $(S(t), V(t), I(t))$ be a solution of (4) satisfied initial condition (5). Since $\mathscr{R}_{1}<1$, one can choose an $\varepsilon>0$ small enough such that

$$
\begin{equation*}
\beta e^{-b \tau}\left(\left(\Delta_{1}\right)^{p}+\delta\left(\Delta_{2}\right)^{q}\right)-(\gamma+b+\alpha)<0, \tag{19}
\end{equation*}
$$

where

$$
\begin{align*}
& \Delta_{1}=\frac{\gamma e^{-b \omega}}{b}+\frac{\mu e^{b T}}{e^{b T}-1}+\varepsilon \\
& \Delta_{2}=\frac{\theta e^{-b l T}}{(1-\theta)\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)}+\varepsilon \tag{20}
\end{align*}
$$

For $n>n_{1}$, we have

$$
\begin{gather*}
\frac{d S(t)}{d t} \leq b-b S(t)+\gamma e^{-b \omega}, \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N,  \tag{21}\\
\Delta S(t)=-\theta S(t), \quad t=(n+l-1) T, n \in N, \\
\Delta S(t)=\mu, \quad t=n T, n \in N .
\end{gather*}
$$

By impulsive differential inequality Lemma 1, we have

$$
\begin{align*}
S(t) \leq & S\left(n_{1} T^{+}\right) \prod_{n_{1} T^{+}<n T<t} \exp \left(\int_{n_{1} T}^{t}(-b) d s\right) \\
& +\sum_{n_{1} T<n T<t}\left(\prod_{n T<t_{j}<t} \exp \left(\int_{n T}^{t}(-b) d s\right)\right) \mu  \tag{22}\\
& +\int_{n_{1} T}^{t} \prod_{s<n T<t} \exp \left(\int_{s}^{t}(-b) d \theta\right) \gamma e^{-b \omega} d s \\
= & S_{1}+S_{2}+S_{3}
\end{align*}
$$

where

$$
\begin{aligned}
S_{1} & =S\left(n_{1} T^{+}\right) \prod_{n_{1} T+<n T<t} \exp \left(\int_{n_{1}}^{t / T}(-b) d T \xi\right) \\
& =S\left(n_{1} T^{+}\right) e^{-b\left(t-n_{1} T\right)}, \\
S_{2} & =\sum_{n_{1} T<n T<t}\left(\prod_{n T<t_{j}<t} \exp \left(\int_{n T}^{t}(-b) d s\right)\right) \mu \\
& =\sum_{n_{1} T<n T<t}\left(\mu e^{-b(t-n T)}\right)
\end{aligned}
$$

$$
\begin{align*}
& =\mu \frac{e^{-b\left(t-n_{1}\right) T}-e^{-b\left(t-\left(n+n_{1}\right)\right) T}}{1-e^{b T}}, \\
S_{3} & =\int_{n_{1} T}^{t} \prod_{s<n T<t} \exp \left(\int_{s}^{t}(-b) d \theta\right) \gamma e^{-b \omega} d s \\
& =\gamma e^{-b \omega} \int_{n_{1} T}^{t} \prod_{s<n T<t} \exp \left(\int_{s}^{t}(-b) d \theta\right) d s \\
& =\frac{\gamma e^{-b \omega} e^{-b t}}{b} \int_{n_{1} T}^{t} \prod_{s<n T<t} e^{b s} d(b s) \\
& =\frac{\gamma e^{-b \omega} e^{-b t}}{b}\left(e^{b t}-e^{b n_{1} T}\right) . \tag{23}
\end{align*}
$$

Thus

$$
\begin{align*}
S(t) \leq & S_{1}+S_{2}+S_{3} \\
= & S\left(n_{1} T^{+}\right) e^{-b\left(t-n_{1} T\right)} \\
& +\mu \frac{e^{-b\left(t-n_{1}\right) T}-e^{-b\left(t-\left(n+n_{1}\right)\right) T}}{1-e^{b T}} \\
& +\frac{\gamma e^{-b \omega} e^{-b t}}{b}\left(e^{b t}-e^{b n_{1} T}\right)  \tag{24}\\
\leq & e^{-b t} S\left(n_{1} T^{+}\right) e^{n_{1} b T}+\frac{\mu e^{-b\left(t-n_{1} T\right)}}{1-e^{b T}} \\
& +\frac{\gamma e^{-b \omega}}{b} e^{-b\left(t-n_{1} T\right)}+\frac{\gamma e^{-b \omega}}{b}+\frac{\mu e^{b T}}{e^{b T}-1}
\end{align*}
$$

and then we have

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \sup S(t)<\frac{\gamma e^{-b \omega}}{b}+\frac{\mu e^{b T}}{e^{b T}-1} \tag{25}
\end{equation*}
$$

Thus there exists a positive integer $n_{2}>n_{1}$ and constant $\varepsilon>0$ small enough such that, for all $t>n_{2} T$,

$$
\begin{equation*}
S(t) \leq \frac{\gamma e^{-b \omega}}{b}+\frac{\mu e^{b T}}{e^{b T}-1}+\varepsilon=\Delta_{1} \tag{26}
\end{equation*}
$$

For $n>n_{1}$, system (4) yields

$$
\begin{gather*}
\frac{d V(t)}{d t} \leq-\left(\gamma_{1}+b\right) V(t), \quad t \neq(n+l-1) T, n \in N \\
\Delta V(t)=\theta S(t), \quad t=(n+l-1) T, n \in N \tag{27}
\end{gather*}
$$

We obtain the following comparison impulsive differential system:

$$
\begin{gather*}
\frac{d x(t)}{d t}=-\left(\gamma_{1}+b\right) x(t), \quad t \neq(n+l-1) T, n \in N \\
\Delta x(t)=\theta S(t), \quad t=(n+l-1) T, n \in N \tag{28}
\end{gather*}
$$

By Lemma 3, the system has a periodic solution given by

$$
\begin{align*}
& x^{*}(t)=\frac{\theta e^{-b l T} e^{-\left(\gamma_{1}+b\right)(t-(n+l-1) T)}}{(1-\theta)\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)} \\
& t \in((n+l-1) T,(n+l) T]  \tag{29}\\
& x\left(0^{+}\right)=\frac{\theta e^{-b l T}}{(1-\theta)\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)}
\end{align*}
$$

which is globally asymptotically stable.
Now, assume that $x(t)$ is the solution of system (28) with initial value $x\left(0^{+}\right)=V_{0}$. Then by Lemma 1 , we know there exists a positive integer $n$ such that

$$
\begin{equation*}
V(t)<x(t)<x^{*}(t)+\varepsilon, \quad t \in(n T,(n+1) T] . \tag{30}
\end{equation*}
$$

Hence,

$$
\begin{align*}
V(t) & <x(t)<x^{*}(t)+\varepsilon \\
& <\frac{\theta e^{-b l T}}{(1-\theta)\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)}+\varepsilon_{0}  \tag{31}\\
& =\Delta_{2}
\end{align*}
$$

From (27), (31), and the third equation in (4), for $t>n_{2} T+\tau$ we have

$$
\begin{equation*}
\frac{d I(t)}{d t} \leq \beta e^{-b \tau}\left(\Delta_{1}^{p}+\delta \Delta_{2}^{q}\right) I(t-\tau)-(\gamma+b+\alpha) I(t) \tag{32}
\end{equation*}
$$

Consider the comparison equation:

$$
\begin{equation*}
\frac{d y(t)}{d t} \leq \beta e^{-b \tau}\left(\Delta_{1}^{p}+\delta \Delta_{2}^{q}\right) y(t-\tau)-(\gamma+b+\alpha) y(t) \tag{33}
\end{equation*}
$$

From (19), we have

$$
\begin{equation*}
\beta e^{-b \tau}\left(\Delta_{1}^{p}+\delta \Delta_{2}^{q}\right)-(\gamma+b+\alpha)<0 . \tag{34}
\end{equation*}
$$

According to Lemma 2, we then obtain

$$
\begin{equation*}
\lim _{t \rightarrow \infty} y(t)=0 . \tag{35}
\end{equation*}
$$

Notice the fact that $I(s)=y(s)=\phi_{3}(s)>0$ for all $s \in[-\tau, 0]$ and $I(t) \geq 0$, and the comparison theorem implies $I(t) \rightarrow 0$ as $t \rightarrow \infty$. Without loss of generality, we may assume that $0<I(t)<\varepsilon_{1}$ for all $t \geq 0$. By using the first and second equations in (4), we reach

$$
\begin{gathered}
\frac{d S(t)}{d t} \geq-b S(t)-\beta \varepsilon_{1} S^{p}(t), \\
\frac{d V(t)}{d t} \geq-\delta \beta \varepsilon_{1} V^{q}(t)-\gamma_{1} V(t)-b V(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N
\end{gathered}
$$

$$
\begin{gather*}
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t), \\
t=(n+l-1) T, \quad n \in N \\
\Delta S(t)=\mu, \quad \Delta V(t)=0, \quad t=n T, n \in N . \tag{36}
\end{gather*}
$$

For $1<q<p$, we have

$$
\begin{gather*}
\frac{d S(t)}{d t} \geq-b S(t)-\beta \varepsilon_{1} S(t), \\
\frac{d V(t)}{d t} \geq-\delta \beta \varepsilon_{1} V(t)-\gamma_{1} V(t)-b V(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N,  \tag{37}\\
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t), \\
t=(n+l-1) T, \quad n \in N, \\
\Delta S(t)=\mu, \quad \Delta V(t)=0, \quad t=n T, \quad n \in N
\end{gather*}
$$

considering the following system:

$$
\begin{gather*}
\frac{d f(t)(t)}{d t}=-b f(t)-\beta \varepsilon_{1} f(t), \\
\frac{d g(t)}{d t}=-\delta \beta \varepsilon_{1} g(t)-\gamma_{1} g(t)-b g(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N,  \tag{38}\\
\Delta f(t)=-\theta f(t), \quad \Delta g(t)=\theta f(t), \\
t=(n+l-1) T, \quad n \in N, \\
\Delta f(t)=\mu, \quad \Delta g(t)=0, \quad t=n T, \quad n \in N .
\end{gather*}
$$

We obtain

$$
\tilde{f}(t)=\left\{\begin{array}{c}
\frac{\left(b+\beta \varepsilon_{1}\right) \mu}{\left(b+\beta \varepsilon_{1}\right)\left(1-(1-\theta) e^{-\left(b+\beta \varepsilon_{1}\right) T}\right)} \\
\times e^{-\left(b+\beta \varepsilon_{1}\right)(t-(n-1) T)}, \\
t \in((n-1) T,(n+l-1) T] \\
\frac{\left(b+\beta \varepsilon_{1}\right) \mu(1-\theta) e^{-\left(b+\beta \varepsilon_{1}\right) l T}}{\left(b+\beta \varepsilon_{1}\right)\left(1-(1-\theta) e^{-\left(b+\beta \varepsilon_{1}\right) T}\right)} \\
\times e^{-\left(b+\beta \varepsilon_{1}\right)(t-(n+l-1) T)} \\
t \in((n+l-1) T, n T]
\end{array}\right.
$$

$$
\begin{align*}
& \tilde{g}(t)=\left(\left(\theta\left(b+\beta \varepsilon_{1}\right) \mu e^{-\left(b+\beta \varepsilon_{1}\right) l T}-\beta \varepsilon_{1} \theta\left(1-e^{-\left(b+\beta \varepsilon_{1}\right) T}\right)\right)\right. \\
& \left.\quad \times e^{-\left(\gamma_{1}+b+\delta \beta \varepsilon_{1}\right)(t-(n+l-1) T)}\right) \\
& \times\left(\left(b+\beta \varepsilon_{1}\right)\left(1-(1-\theta) e^{-\left(b+\beta \varepsilon_{1}\right) T}\right)\right. \\
& \left.\quad \times\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)\right)^{-1} \\
& \quad t \in((n+l-1) T,(n+l) T] \tag{39}
\end{align*}
$$

Now by using comparison theorem of impulsive equations, for any $\varepsilon_{2}>0$ there exists a $T_{1}>0$ such that

$$
\begin{align*}
S(t) & >\tilde{f}(t)-\varepsilon_{2}  \tag{40}\\
V(t) & >\tilde{g}(t)-\varepsilon_{2}
\end{align*}
$$

for $t>T_{1}$. On the other side, from the first and second equations of (4), we have

$$
\begin{gather*}
\frac{d S(t)}{d t} \leq-b S(t)+\gamma \varepsilon_{1} e^{-b \omega}, \\
\frac{d V(t)}{d t} \leq-\gamma_{1} V(t)-b V(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N,  \tag{41}\\
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t), \\
t=(n+l-1) T, \quad n \in N, \\
\Delta S(t)=b, \quad \Delta V(t)=0, \quad t=n T, \quad n \in N .
\end{gather*}
$$

Then we have $S(t) \leq \widetilde{h}(t), V(t) \leq \widetilde{g}(t)$ and $\widetilde{h}(t) \rightarrow S^{*}(t)$, $\widetilde{g}(t) \rightarrow V^{*}(t)$, as $\varepsilon_{1} \rightarrow 0$, where $(\widetilde{h}(t), \widetilde{g}(t))$ is a unique positive periodic solution of

$$
\begin{gathered}
\frac{d h(t)}{d t}=-b h(t)+\gamma \varepsilon_{1} e^{-b \omega}, \\
\frac{d g(t)}{d t}=-\gamma_{1} g(t)-b g(t), \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N, \\
\Delta h(t)=-\theta h(t), \quad \Delta g(t)=\theta h(t), \\
t=(n+l-1) T, \quad n \in N \\
\Delta h(t)=\mu, \quad \Delta g(t)=0, \quad t=n T, \quad n \in N
\end{gathered}
$$

from which we have that, for $n T<t \leq(n+1) T$,

$$
\begin{align*}
& \widetilde{h}(t)=\left\{\begin{array}{l}
\frac{\gamma \varepsilon_{1} e^{-b \omega} \theta e^{-b(1-l) T}+b \mu}{b\left(1-(1-\theta) e^{-b T}\right)} \\
\times e^{-b(t-(n-1) T)}-\frac{\gamma \varepsilon_{1} e^{-b \omega}}{b}, \\
(n-1) T<t \leq(n+l-1) T, \\
\frac{b \mu(1-\theta) e^{-b l T}+\gamma \varepsilon_{1} e^{-b \omega} \theta}{b\left(1-(1-\theta) e^{-b T}\right)} \\
\times e^{-b(t-(n+l-1) T)}-\frac{\gamma \varepsilon_{1} e^{-b \omega}}{b}, \\
\quad(n+l-1) T<t \leq n T, \\
\tilde{g}(t)=\frac{\left(\theta b \mu e^{-b l T}-\gamma \varepsilon_{1} e^{-b \omega} \theta\left(1-e^{-b T}\right)\right) e^{-\left(\gamma_{1}+b\right)(t-(n+l-1) T)}}{b\left(1-(1-\theta) e^{-b T}\right)\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)}, \\
(n+l-1) T<t \leq(n+l) T .
\end{array}\right. \\
& \\
& \quad \begin{array}{l}
(n+l
\end{array},
\end{align*}
$$

Applying the comparison theorem again, for any $\varepsilon_{2}>0$, there exists a $T_{2}>0$ such that

$$
\begin{align*}
S(t) & <\widetilde{h}(t)-\varepsilon_{2},  \tag{44}\\
V(t) & <\widetilde{g}(t)-\varepsilon_{2},
\end{align*}
$$

for $t>T_{2}$. Let $\varepsilon_{1} \rightarrow 0$, and then from (40) and (44) we have

$$
\begin{gather*}
S^{*}(t)-\varepsilon_{2}<S(t)<S^{*}(t)-\varepsilon_{2}  \tag{45}\\
V^{*}(t)-\varepsilon_{2}<V(t)<V^{*}(t)-\varepsilon_{2}
\end{gather*}
$$

for $t$ large enough, which implies $S(t) \rightarrow S^{*}(t), V(t) \rightarrow$ $V^{*}(t)$ as $t \rightarrow \infty$. This completes the proof.

Corollary 6. If $\tau>\tau^{*}$ or $\mu<\mu^{*}$, then the infection-free periodic solution $\left(S^{*}(t), V^{*}(t), 0\right)$ is globally attractive, where the critical values are given below:

$$
\begin{align*}
\tau^{*}= & \frac{1}{b} \ln \frac{\beta\left(A_{1}^{p}+\delta A_{2}^{q}\right)}{\gamma+b+\alpha}, \\
\mu^{*}= & \left(1-e^{-b T}\right) \\
& \times\left(\sqrt[2]{\frac{\gamma+b+\alpha}{\beta} e^{b \tau}-\delta A_{2}^{q}}-\frac{\gamma e^{-b \omega}}{b}-\frac{\mu e^{b T}}{e^{b T}-1}\right) . \tag{46}
\end{align*}
$$

## 4. Permanence

In this section, we discuss the permanence of the infectious population. First, we introduce the following definition.

Definition 7. System (4) is said to be permanent if there exist positive constants $m_{i}, M_{i}, i=1,2,3$ (independent of initial value), and a finite time $T_{0}$, which may depend on the initial condition, such that every positive solution $(S(t), V(t), I(t))$ with initial condition (5) satisfies $m_{1} \leq S(t) \leq M_{1}, m_{2} \leq$ $I(t) \leq M_{2}, m_{3} \leq V(t) \leq M_{3}$ for all $t>T_{0}$.

Let

$$
\begin{align*}
& S^{*}=\sqrt[p]{\frac{\gamma+b+\alpha}{\beta e^{-b \tau}}}, \\
& V^{*}=\sqrt[q]{\frac{\gamma_{1}+b}{\delta \beta e^{-b \tau}}}, \\
& m^{*}=\frac{(1 / T) \ln \left(\left(\Re_{1}\left(e^{b T}-1+\theta\right)+1-\theta+e^{b T}\right) / 2\right)-b}{\beta}, \\
& \boldsymbol{R}_{1}=\sqrt[p]{\frac{\beta e^{-b \tau}}{\gamma+b+\alpha}}\left(\frac{\mu(1-\theta) e^{-b T}}{1-(1-\theta) e^{-b T}}\right), \\
& \boldsymbol{R}_{2}=\sqrt[q]{\frac{\beta e^{-b \tau}}{\gamma_{1}+b}} \frac{\mu \theta e^{-\left(\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-b T}\right)}, \\
& \mathscr{R}_{2}=\min \left\{\Re_{1}, \Re_{2}\right\} . \tag{47}
\end{align*}
$$

Then we have our main result of this section.
Theorem 8. Let $1 \leq q \leq p$, if $\mathscr{R}_{2}>1$, and then there exists a positive constant $\eta$ small enough such that

$$
\begin{equation*}
I(t) \geq \min \left\{\frac{\eta m^{*}}{2}, \eta m^{*} e^{-(\gamma+b+\alpha) \omega}\right\}=m_{1} \tag{48}
\end{equation*}
$$

with $t$ large enough.
Proof. As before, we suppose that $X(t)=(S(t), V(t), I(t))$ is a positive solution of system (4) with initial condition (5). Then for $t \geq 0$, we construct a function as follows:

$$
\begin{align*}
U(t)= & I(t)+V(t)+\beta e^{-b \tau}\left(S^{*}\right)^{p} \\
& \times \int_{t-\tau}^{t} I(\varrho) d \varrho+\delta \beta e^{-b \tau}\left(V^{*}\right)^{q} \int_{t-\tau}^{t} V(\varrho) d \varrho . \tag{49}
\end{align*}
$$

And then differentiating $U(t)$ along the trajectory of (4) yields

$$
\begin{aligned}
\dot{U}(t)= & \dot{I}(t)+\dot{V}(t)+\beta e^{-b \tau}\left(S^{*}\right)^{p} I(t) \\
& -\beta e^{-b \tau}\left(S^{*}\right)^{p} I(t-\tau)+\delta \beta e^{-b \tau}\left(V^{*}\right)^{q} I(t)
\end{aligned}
$$

$$
\begin{align*}
&-\delta \beta e^{-b \tau}\left(V^{*}\right)^{q} I(t-\tau) \\
&= \beta e^{-b \tau}\left(S^{p}(t)-\left(S^{*}\right)^{p}\right) I(t-\tau) \\
&+\beta e^{-b \tau}\left(V^{q}(t)-\left(V^{*}\right)^{q}\right) I(t-\tau) \\
&+\left(\beta e^{-b \tau}\left(S^{*}\right)^{p}-(\gamma+b+\alpha)\right) I(t) \\
&+\left(\delta \beta e^{-b \tau}\left(V^{*}\right)^{q}-\left(\gamma_{1}+b\right)\right) I(t) \\
&= \beta e^{-b \tau}\left(S^{p}(t)-\left(S^{*}\right)^{p}\right) I(t-\tau) \\
&+\beta e^{-b \tau}\left(V^{q}(t)-\left(V^{*}\right)^{q}\right) I(t-\tau) \\
&= \beta e^{-b \tau}\left(S^{p-1}(t)+S^{p-2}(t) S^{*}+\cdots\right. \\
&\left.\quad+S(t)\left(S^{*}\right)^{p-2}+\left(S^{*}\right)^{p-1}\right) \\
& \times\left(S(t)-S^{*}\right) I(t-\tau) \\
&+\delta \beta e^{-b \tau}\left(V^{q-1}(t)+V^{q-2}(t) V^{*}+\cdots\right. \\
&\left.+V(t)\left(V^{*}\right)^{q-2}+\left(V^{*}\right)^{q-1}\right) \\
& \times\left(V(t)-V^{*}\right) I(t-\tau) \tag{50}
\end{align*}
$$

for $t \geq 0$. Let

$$
\begin{align*}
m^{*} & =\frac{(1 / T) \ln \left(\left(\Re_{1}\left(e^{b T}-1+\theta\right)+1-\theta+e^{b T}\right) / 2\right)-b}{\beta} \\
S^{*} & =\sqrt[p]{\frac{\gamma+b+\alpha}{\beta e^{-b \tau}}} \tag{51}
\end{align*}
$$

Since $\mathscr{R}_{2}>1$, we get $\mathfrak{R}_{1}>1, \mathfrak{R}_{2}>1$. Then we have $m^{*}>0$. And from $\Re_{1}>1$, we can get

$$
\begin{equation*}
\sqrt[p]{\frac{\beta e^{-b \tau}}{\gamma+b+\alpha}}\left(\frac{\mu(1-\theta) e^{-b T}}{1-(1-\theta) e^{-b T}}\right)>1 . \tag{52}
\end{equation*}
$$

Thus, we have

$$
\begin{equation*}
\frac{\mu(1-\theta) e^{-b T}}{1-(1-\theta) e^{-b T}}>\sqrt[p]{\frac{\gamma+b+\alpha}{\beta e^{-b \tau}}}=S^{*} \tag{53}
\end{equation*}
$$

Form $\boldsymbol{R}_{2}>1$, we have

$$
\begin{equation*}
\sqrt[q]{\frac{\beta e^{-b \tau}}{\gamma_{1}+b}} \frac{\mu \theta e^{-\left(\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-b T}\right)}>1 \tag{54}
\end{equation*}
$$

that is,

$$
\begin{equation*}
\frac{\mu \theta e^{-\left(\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-b T}\right)}>\sqrt[q]{\frac{\gamma_{1}+b}{\beta e^{-b \tau}}}=V^{*} \tag{55}
\end{equation*}
$$

We can take $\eta$ small enough such that

$$
\begin{gather*}
\frac{\mu(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}}{1-(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}}>S^{*} \\
\frac{\mu \theta e^{-\left(\beta(\delta+l) \eta m^{*}+\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\delta \beta \eta m^{*}+\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}\right)}>V^{*} \tag{56}
\end{gather*}
$$

Thus we can choose $\varepsilon_{1}, \varepsilon_{2}>0$ to be small enough such that

$$
\begin{align*}
S^{*}< & \frac{\mu(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}}{1-(1-\theta) e^{-\left(\beta \eta m m^{*}+b\right) T}}-\varepsilon_{1} \equiv S_{\Delta} \\
V^{*}< & \frac{\mu \theta e^{-\left(\beta(\delta+l) \eta m^{*}+\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\delta \beta \eta m^{*}+\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}\right)}  \tag{57}\\
& -\varepsilon_{2} \equiv V_{\Delta} .
\end{align*}
$$

Then we claim that there exists an $m_{2}>0$ such that $I(t)>$ $m_{2}$ for $t$ is large enough. We next prove this claim in two steps.

Step $I$. For any positive constant $t_{0}$, that $I(t) \leq \eta m^{*}$ for all $t \geq t_{0}$ is not true.

Otherwise, there is a positive constant $t_{0}$, such that $I(t) \leq$ $\eta m^{*}$ for all $t \geq t_{0}$. First, if $I(t)<\eta m^{*}$ for all $t \geq t_{0}$, it follows from the first, fourth, and fifth equations of (4) that, for $t \geq t_{0}$,

$$
\begin{align*}
\frac{d S(t)}{d t} & \geq-\left(\beta \eta m^{*}+b\right) S(t), \quad t \neq(n+l-1) T, \quad t \neq n T \\
\Delta S(t) & =-\theta S(t), \quad t=(n+l-1) T \\
\Delta S(t) & =\mu, \quad t=n T \tag{58}
\end{align*}
$$

By Lemma 1, there exists $T_{1}>t_{0}+\tau$ so that for $t>T_{1}$

$$
\begin{equation*}
S(t)>\frac{\mu(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}}{1-(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}}-\varepsilon \equiv S_{\Delta} . \tag{59}
\end{equation*}
$$

Similarly, from the second and the fourth equations of (4), we have

$$
\begin{align*}
& \frac{d V(t)}{d t} \geq-\left(\delta \beta \eta m^{*}+\gamma_{1}+b\right) V(t), \quad t \neq(n+l-1) T \\
& \Delta V(t)=\theta S(t), \quad t=(n+l-1) T \tag{60}
\end{align*}
$$

and for $t>T_{1}$,

$$
\begin{align*}
V(t) \geq & \frac{\mu \theta e^{-\left(\beta(\delta+l) \eta m^{*}+\gamma_{1}+b+b l\right) T}}{\left(1-e^{-\left(\delta \beta \eta m^{*}+\gamma_{1}+b\right) T}\right)\left(1-(1-\theta) e^{-\left(\beta \eta m^{*}+b\right) T}\right)} \\
& -\varepsilon \equiv V_{\Delta} \tag{61}
\end{align*}
$$

Then, by (50), for $t \geq T_{1}$,

$$
\begin{align*}
\dot{U}(t)= & \beta e^{-b \tau}\left(S^{p-1}(t)+S^{p-2}(t) S^{*}+\cdots\right. \\
& \left.+S(t)\left(S^{*}\right)^{p-2}+\left(S^{*}\right)^{p-1}\right) \\
& \times\left(S(t)-S^{*}\right) I(t-\tau) \\
& +\delta \beta e^{-b \tau}\left(V^{q-1}(t)+V^{q-2}(t) V^{*}+\cdots\right. \\
& \left.+V(t)\left(V^{*}\right)^{q-2}+\left(V^{*}\right)^{q-1}\right)  \tag{62}\\
& \times\left(V(t)-V^{*}\right) I(t-\tau) \\
> & p \beta e^{-b \tau}\left(S^{*}\right)^{p-1}\left(S_{\Delta}-S^{*}\right) I(t-\tau) \\
& +q \delta \beta e^{-b \tau}\left(V^{*}\right)^{q-1}\left(V_{\Delta}-V^{*}\right) I(t-\tau)
\end{align*}
$$

Let

$$
\begin{equation*}
I_{L}=\min _{t \in\left[T_{1}, T_{1}+\tau\right]} I(t) \tag{63}
\end{equation*}
$$

We can prove that $I(t) \geq I_{L}$ for all $t \geq T_{1}$. Otherwise, there exists a nonnegative constant $T_{2}$ such that $I(t) \geq I_{L}$ for $t \in$ $\left[T_{1}, T_{1}+\tau+T_{2}\right], I\left(T_{1}+\tau+T_{2}\right)=I_{L}$, and $\dot{I}\left(T_{1}+\tau+T_{2}\right) \leq 0$.


Figure 1: The results of numerical simulation on the threshold values $\mathscr{R}_{2}=2.6155>1$, where $p=1.5, q=1.25$.

Then from the second equation of (4) and (37), we easily see that

$$
\begin{aligned}
\dot{I} & \left(T_{1}+\tau+T_{2}\right) \\
& \geq\left(\beta e^{-b \tau} S^{p}(t)+\delta \beta e^{-b \tau} V^{q}(t)-(\gamma+b+\alpha)\right) I_{L} \\
& =(\gamma+b+\alpha)\left(\frac{\beta e^{-b \tau} S^{p}(t)}{\gamma+b+\alpha}+\frac{\delta \beta e^{-b \tau} V^{q}(t)}{\gamma+b+\alpha}-1\right) I_{L} \\
& >(\gamma+b+\alpha)\left(\left(\frac{S_{\Delta}}{S^{*}}\right)^{p}+\frac{\gamma_{1}+b}{\gamma+b+\alpha}\left(\frac{V_{\Delta}}{V^{*}}\right)^{q}-1\right) I_{L}
\end{aligned}
$$

$$
\begin{equation*}
>0 \tag{64}
\end{equation*}
$$

which is a contradiction. Hence $I(t) \geq I_{L}>0$ for all $t \geq T_{1}$. Equation (62) implies

$$
\begin{align*}
\frac{d U(t)}{d t}> & p \beta e^{-b \tau}\left(S^{*}\right)^{p-1}\left(S_{\Delta}-S^{*}\right) I(t-\tau) \\
& +q \delta \beta e^{-b \tau}\left(V^{*}\right)^{q-1}\left(V_{\Delta}-V^{*}\right) I(t-\tau)  \tag{65}\\
> & 0
\end{align*}
$$

It then follows that $U(t) \rightarrow+\infty$ as $t \rightarrow+\infty$. This is a contradiction to $U(t) \leq\left(\alpha+\gamma+\gamma_{1}+2 b\right) \tau+2$. Therefore, for any positive constant $t_{0}$, the inequality $I(t)<\eta m^{*}$ cannot hold for all $t \geq t_{0}$.

Step II. From Step I, we only need to consider the followng: (i) $I(t)>\eta m^{*}$ for all $t$ large enough and (ii) $I(t)$ oscillates


Figure 2: The results of numerical simulation on the threshold values $\mathscr{R}_{1}=0.0339<1$, where $p=1.5, q=1.25$.
about $\eta m^{*}$ for all large $t$. However, Case (i) is obvious in the result of this theorem, so we only need to consider Case (ii), in which we will show that $I(t) \geq m_{1}$ for all large $t$ where

$$
\begin{equation*}
m_{1}=\min \left\{\frac{\eta m^{*}}{2}, \eta m^{*} e^{-(\gamma+b+\alpha) \omega}\right\} \tag{66}
\end{equation*}
$$

First, we notice there exist two positive constants $\bar{t}, \varphi$ such that

$$
\begin{align*}
& I(\bar{t})=I(\bar{t}+\varphi)=I^{*},  \tag{67}\\
& I(t)<\eta m^{*}, \text { for } \bar{t}<t<\bar{t}+\varphi .
\end{align*}
$$

Second, because $I(t)$ is bounded continuous function and $I(t)$ has no pulse, we can get that $I(t)$ is uniformly continuous. Therefore there exists a constant $T_{3}$ (with $0<T_{3}<\omega$ and $T_{3}$ is independent of the choice of $\bar{t})$ such that $I(t)>\eta m^{*} / 2$ for all $\bar{t} \leq t \leq \bar{t}+T_{3}$.

## If $\varphi \leq T_{3}$, our aim is obtained.

If $T_{3}<\varphi \leq \omega$, from the second equation of (4) we have that $\dot{I}(t) \geq-(\gamma+b+\alpha) I(t)$ for $\bar{t}<t \leq \bar{t}+\varphi$. Then we have $I(t) \geq \eta m^{*} e^{-(\gamma+b+\alpha) \omega}$ for $\bar{t}<t \leq \bar{t}+\varphi \leq \bar{t}+\omega$ since $I(\bar{t})=\eta m^{*}$. It is clear that $I(t) \geq m_{1}$ for $\bar{t}<t \leq \bar{t}+\varphi$.

If $\varphi \geq \omega$, then we have $I(t) \geq m_{2}$ for $\bar{t}<t \leq \bar{t}+\omega$. We then can easily prove $I(t) \geq m_{1}$ for $\bar{t}+\omega \leq t \leq \bar{t}+\varphi$. Since the interval $[\bar{t}, \bar{t}+\varphi]$ is arbitrarily chosen, we know that $I(t) \geq m_{1}$ holds for $t$ large enough. Finally, noticing the choice of $m_{1}$ is independent of the positive solution of (4), we completed our proof.

$\qquad$ $-V$
(a) Time series of $S, V$, and $I$. The disease dies out

(c) Phase portrait of $S(t), I(t)$ of the system (4)

(b) Phase portrait of $S(t), V(t)$ of the system (4)

(d) Phase portrait of $S(t), V(t)$, and $I(t)$ of the system (4)

Figure 3: The results of numerical simulation on the threshold values $\mathscr{R}_{1}=0.0449<1$, where $p=1.5, q=1.25$.

Theorem 9. Let $1 \leq q \leq p$, if $\mathscr{R}_{2}>1$, and then system (4) is permanent.

Proof. Suppose that $X(t)=(S(t), V(t), I(t))$ is a positive solution of system (4) with initial conditions (5). Then from system (4), we have

$$
\begin{gathered}
\frac{d S(t)}{d t} \geq-(b+\beta) S(t) \\
\frac{d V(t)}{d t} \geq-\left(\delta \beta+\gamma_{1}+b\right) V(t) \\
t \neq(n+l-1) T, \quad t \neq n T, \quad n \in N \\
\Delta S(t)=-\theta S(t), \quad \Delta V(t)=\theta S(t)
\end{gathered}
$$

$$
\begin{gather*}
t=(n+l-1) T, \quad n \in N, \\
\Delta S(t)=\mu, \quad \Delta V(t)=0, \quad t=n T, n \in N . \tag{68}
\end{gather*}
$$

As what we did in the proof of Theorem 5, we can prove that there exist $t$ large enough and $\varepsilon>0$ small enough such that

$$
\begin{align*}
S(t) \geq & \frac{\mu(1-\theta) e^{-(b+\beta) T}}{1-(1-\theta) e^{-(b+\beta) T}}-\varepsilon=m_{3} \\
V(t) \geq & \frac{\mu \theta e^{-\left(b+\gamma_{1}+\delta \beta+(b+\beta) l\right) T}}{\left(1-(1-\theta) e^{-(b+\beta) T}\right)\left(1-e^{-\left(b+\gamma_{1}+\delta \beta\right) T}\right)}  \tag{69}\\
& -\varepsilon_{2}=m_{4} .
\end{align*}
$$



Figure 4: The results of numerical simulation on the threshold values $\mathscr{R}_{1}=0.0352<1$, where $p=1.5, q=1.25$.

Then for $\mathscr{D}=\left\{(S, V, I) \in R_{+}^{3} \mid S(t)+V(t)+I(t) \leq 1\right\}$, by Theorem 8, we have

$$
\begin{equation*}
m_{3} \leq S(t) \leq 1, \quad m_{1} \leq I(t) \leq 1, \quad m_{4} \leq V(t) \leq 1 \tag{70}
\end{equation*}
$$

for $t$ large enough. Thus the system (4) is uniformly permanent.

## 5. Numerical Simulations and Discussions

Next, we carry out numerical simulations to illustrate the theoretical results obtained in the previous sections. We first set the parameters as follows: $b=0.2, \beta=0.5, \alpha=0.05$, $\gamma=0.04, \delta=0.02, \gamma_{1}=0.06, p=1.5, q=1.25, T=1.5$, $\omega=1, \tau=1, \theta=0.4$, and $\mu=1.4$. Straightforward
calculation shows $\mathscr{R}_{2}=2.6155>1$. Then by Theorem 8 , the disease will be permanent (please see Figures 1(a), 1(b), $1(c)$, and $1(\mathrm{~d})$ ). In order to show the effect of $\tau$, we decrease $\tau$ to 4 , and other parameters are the same with those in Figure 1, and the infection-free periodic solution of system (4) is globally attractive. This phenomenon is also seen from our theoretical analysis as in this case $\mathscr{R}_{1}=0.0339<1$ and then according to Theorem 5 , the disease will be eradicated; please see Figure 2(a).

If we keep $\tau=\omega=1$ and $\mu=1$, as the same with those in Figure 1, but increase vaccination proportion of susceptible persons $\theta$ to 0.9 , then the disease will be eradicated; see Figure 3(a). If we keep $\tau=\omega=1$ and $\theta=0.4$ and decrease $\mu$ to 0.2 , then the disease also will be eradicated; see Figure 4(a).

And if we keep $\tau=\omega=4, \mu=1$ but decrease $\theta$ to 0.1 , then the disease will be permanent; see Figure 5. If we keep $\tau=\omega=$


Figure 5: The results of numerical simulation on the threshold values $\mathscr{R}_{2}=1.1036>1$, where $p=1.5, q=1.25$.

Table 1

| $\omega$ | $\tau$ | $\theta$ | $\mu$ | $\mathscr{R}_{i}$ | Status of the disease |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | 0.4 | 1.4 | $\mathscr{R}_{2}=2.6155>1$ | Permanence |
| 4 | 4 | 0.4 | 1.4 | $\mathscr{R}_{1}=0.0339<1$ | Eradication |
| 1 | 1 | 0.9 | 1.4 | $\mathscr{R}_{1}=0.0449<1$ | Eradication |
| 1 | 1 | 0.4 | 0.2 | $\mathscr{R}_{1}=0.0352<1$ | Eradication |
| 4 | 4 | 0.1 | 1.4 | $\mathscr{R}_{2}=1.1036>1$ | Permanence |
| 4 | 4 | 0.4 | 2 | $\mathscr{R}_{2}=2.3121>1$ | Permanence |

4 and $\theta=0.4$ and increase $\mu$ to 2 , then the disease also will be permanent; see Figure 6. For details please see Table 1.

Lastly, we conclude our paper as follows. In this paper, we proposed an SVEIRS model, which is a new epidemic model with periodic pulse vaccination and pulse population input at two different fixed moments. Our primary result is to investigate the effect of impulsive vaccination, pulse population input, and time delays to the dynamics of population model. With the help of comparison theorems, we proved the existence of the "infection-free" periodic solution and obtained the conditions for global attractivity of the "infection-free" periodic solution and the conditions for the permanence of the system. All the theoretical results show that we believe it might be helpful in disease control: people can select appropriate vaccination rate and population input rate according to our theoretical results to control diseases.


Figure 6: The results of numerical simulation on the threshold values $\mathscr{R}_{2}=2.3121>1$, where $p=1.5, q=1.25$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Research Article 

# Studying Term Structure of SHIBOR with the Two-Factor Vasicek Model 

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#### Abstract

With the development of the Chinese interest rate market, SHIBOR is playing an increasingly important role. Based on principal component analysing SHIBOR, a two-factor Vasicek model is established to portray the change in SHIBOR with different terms. And parameters are estimated by using the Kalman filter. The model is also used to fit and forecast SHIBOR with different terms. The results show that two-factor Vasicek model fits SHIBOR well, especially for SHIBOR in terms of three months or more.


## 1. Introduction

The benchmark interest rate is the core of the formation of market-oriented interest rate system. Without benchmark interest rate, it is difficult to determine the direction of financial derivatives is reasonable. Since Shanghai Interbank Offered Rate (SHIBOR) was launched in 2007, the currency market benchmark interest rates were gradually established, which has the guidance for pricing of stocks, bonds, and financial derivatives. With the improvement of quotation quality and the expanding of application scope of SHIBOR, the system of benchmark interest rate is developing in Chinese financial market. In 2007, based on SHIBOR interest rate swap accounts for about $13 \%$ of the total swaps. In 2008, swaps with SHIBOR as the benchmark interest rate rose by $215 \%$ over the previous year, accounting for $22 \%$ of the change of trading volume. And since 2009, all forward rate agreement was based on SHIBOR. By 2010, swap transactions in the name of the principal proportion linked to SHIBOR of RMB interest rate reached $40.3 \%$. After 2010, the role of SHIBOR in transmission mechanism of monetary policy is more important and the circulation of SHIBOR products is gradually expanding. The reference value to price other financial products of SHIBOR has been increasing [1]. As China's "LIBOR," SHIBOR plays a more and more important role for interest rate marketization in China.

Some researchers studied the term structure of interest rates. Cajueiro and Tabak have studied the long-range
dependence in LIBOR interest rates. Their empirical results show that the degree of long-range dependence of interest rates on most countries decreases with maturity. They also have found interest rates have a multifractal nature [2]. Egorov et al. have modeled the joint term structure of interest rates in the United States and the European Union and have found that a new four-factor model with two common and two local factors captures the joint term structure dynamics in the US and the EU reasonably well [3]. Jagannathan et al. have evaluated the classical CIR model using data on LIBOR, swap rates and caps, and swaptions. And they have found three-factor CIR model is able to fit the term structure of LIBOR and swap rates rather well [4]. Griffiths et al. have examined the robustness of results of Griffiths and Winters [5, 6] and Kotomin et al. [7] using pound sterling and Euro repo rates and have found a year-end preferred habitat for liquidity in the Euro repo rates [8]. Kotomin has studied incorporating year-end and quarter-end preferences for liquidity and other calendar-time effects into the test of the expectations hypothesis in the very short-term LIBOR in seven major world currencies and has found the calendartime effects altering long-term relations between very shortterm rates in these currencies [9]. Wen et al. have proposed a copula-based correlation measure to test the interdependence among stochastic variables in terms of copula function [10, 11]. Because of short SHIBOR launch time, few early launch SHIBOR product category, and small circulation, the study of SHIBOR has few results. Most of the research achievements
are about its term structure, Wang has found that pure expectation hypothesis is rejected by empirical research of Shibor, and term premiums always exist. He also has found that single-factor interest rate models are appropriate in describing overnight and 1W SHIBOR. But if adding GARCH into the diffusion part, the result will be better [12]. X. N. Wang and H. T. Wang have studied the term structure of Shibor and the conclusions show that the expectation theory is valid on the short-term, medium-term, and long-term SHIBOR [13]. Zhang et al. have made an empirical analysis on the term structure of SHBOR under Vasicek and CIR models, respectively, describing the dynamics of SHIBOR. The research presents that Vasicek model does even better in capturing the dynamics of the interest rates [14]. Zhou et al. have taken Vasicek model with jumps or exponential Vasicek model with jumps as the alternative models to describe return series of SHIBOR. And parameters of two models have been estimated by particle filter approach. Comparing goodness-of-fit and forecast effect between the two models, the result shows that Vasicek model with jumps does better [15]. Su has adopted the CIR model, RSCIR model, and no-arbitrage HJM model to study the term structure of SHBOR and the dynamics of its risk premium. The result shows that three-factor HJM model does best to discribe the dynamics characteristic of term structure and volatility structure of SHBOR [16]. Through analyzing the operational mechanism of SHIBOR, Yu and Liu have proposed a practicable pricing model of SHIBOR and tested the model by empirical data [17]. Wen et al. have used the principal component analysis to find the existence of chaotic features of the Chinese finacial market [18]. Wen and Yang have studied the relationship between the skewness and the coefficient of risk premium in financial makets [19]. Huang et al. consider the dynamics of switched cellular neural networks (CNNs) with mixed delays [20]. Liu et al. introduce and investigate some new subclasses of multivalent analytic functions involving the generalized Srivastava-Attiya operator [21]. Based on the modified secant equation, Dai and Wen propose a modified Hestenes-Stiefel (HS) conjugate gradient method which has similar form as the CG-DESCENT method [22]. Under a genal affine data perturbation uncertainty set, Dai and Wen propose a computationally tractable robust optimization method for minimizing the CVaR of a portfolio [23]. Using theories and methods of behavioral finance, Wen et al. take a new look at the characteristics of investors' risk preference, building the D-GARCH-M model, DR-GARCH-M model, and GARCHC-M model to investigate their changes with states of gain and loss and values of return together with other timevarying characteristics of investors' risk preference [24]. The researchers mainly used single factor model to study the term structure of SHIBOR. Among many dynamic equilibrium models describing short-term stochastic interest rates, the most widely used is the Vasicek model [25]. Vasicek model is an equilibrium pricing model about term structure of interest rates, which reflects the risk of debt and investors' expectations of future interest rate changes. The prices of the bonds and interest rate derivatives have a simple analytical expression in Vasicek model. Interest rate derivatives market is a complicated system in real world, so it is difficult to

Table 1: Principal component analysis results.

| Principal <br> component | Eigenvalue | The proportion <br> of explanation | The accumulative <br> proportion of <br> explanation |
| :--- | :---: | :---: | :---: |
| 1 | 6.7432 | 0.8429 | 0.8429 |
| 2 | 0.9823 | 0.1228 | 0.9657 |
| 3 | 0.1395 | 0.0174 | 0.9831 |
| 4 | 0.0736 | 0.0092 | 0.9924 |
| 5 | 0.0362 | 0.0045 | 0.9969 |
| 6 | 0.0221 | 0.0028 | 0.9997 |
| 7 | 0.0027 | 0.0003 | 1.0000 |
| 8 | 0.0003 | 0.0000 | 1 |

describe the term structure of interest rates with single factor. Therefore, the single factor Vasicek model is extended to multiple-factor Vasicek model, and multiple-factor Vasicek model can also be very easy to evaluate the price of bonds and risk. Although there are many more complicated interest rate models later such as Affine model [26], the Libor model [27], and so forth, the Vasicek model is still a very important interest rate model due to the ease in pricing bond prices and the risk. This paper will describe the dynamic characteristic of SHIBOR and study its term structure by two-factor Vasicek model. In the second part, principal component analysis (PCA) will be taken to select two most important factors of SHIBOR for modeling. In the thirtd part, two-factor Vasicek model of SHIBOR will be present and parameters will be estimated by Kalman filter method. In the forth part, the twofactor Vasicek model of SHIBOR will be tested by empirical research. Finally, conclusion will be present.

## 2. Principal Component Analysis of SHIBOR

Different terms of SHIBOR volatility would be influenced by economic cycle, macroeconomic policies, monetary supply, demand, and so on. And there is some correlation between these factors. It is important for modeling dynamically of SHIBOR that irrelated influence factors or components are found in the different term of SHIBOR and less new irrelated compound variables are used to replace the more interdependent variables to build the dynamic model of SHIBOR. This paper uses principal component analysis method to get the principal components affecting the SHIBOR. Then SHIBOR short-term dynamic model is set up with these main components. Although SHIBOR began trial operation from October 2006, its quoted price was a bit chaotic and trading volumes were less in that time. When Launched on January 1,2007, SHIBOR quoted price was improved and trading volumes were also increased. This paper selects $\mathrm{O} / \mathrm{N}, 1$ week, 2 weeks, 1 month, 3 months, 6 months, 9 months, and 1 year of SHIBOR daily data to make principal component analysis from January 4, 2007, to August 21, 2013. Analysis results are shown in Table 1.

In Table 1, the first principal component interpretation for the proportion of SHIBOR volatility reaches $84.29 \%$. The

Table 2: The coefficient of the first two principal components of different terms of SHIBOR.

| Term | The first principal <br> component | The second principal <br> component |
| :--- | :---: | :---: |
| $\mathrm{O} / \mathrm{N}$ | 0.3294 | -0.4278 |
| 1 week | 0.3459 | -0.3992 |
| 2 weeks | 0.3506 | -0.3514 |
| 1 month | 0.3634 | -0.2319 |
| 3 months | 0.3702 | 0.2209 |
| 6 months | 0.3573 | 0.3730 |
| 9 months | 0.3553 | 0.3831 |
| 1 year | 0.3548 | 0.3816 |

cumulative interpretation proportion of the first two principal components reaches $96.57 \%$. The cumulative explain proportion of the first three principal components is above $98 \%$. The interpretation abilities of principal components behind third principal components are weakened observably. Then two irrelated variables can be used to depict the volatility of SHIBOR. By calculating the SHIBOR eigenvectors of covariance matrix, the coefficients of the first two principal components can be gotten.

We get models of the two principal components from the eigenvector as below:

$$
\begin{align*}
F_{1}= & 0.3294 \text { shibor_O/N }+0.3459 \text { shibor_1 } \mathrm{W} \\
& +0.3506 \text { shibor_ } 2 \mathrm{~W}+0.3634 \text { shibor_ } 1 \mathrm{M} \\
& +0.3702 \text { shibor_ } 3 \mathrm{M}+0.3573 \text { shibor_ } 6 \mathrm{M} \\
& +0.3553 \text { shibor_ } 9 \mathrm{M}+0.3548 \text { shibor_ } 1 \mathrm{Y} \\
F_{2}= & -0.4278 \text { shibor_O/ } \mathrm{N}-0.3992 \text { shibor_1 } \mathrm{W}  \tag{1}\\
& -0.3514 \text { shibor_ } 2 \mathrm{~W}-0.2319 \text { shibor_ } 1 \mathrm{M} \\
& +0.2209 \text { shibor_ } 3 \mathrm{M}+0.3730 \text { shibor_ } 6 \mathrm{M} \\
& +0.3831 \text { shibor_ } 9 \mathrm{M}+0.381 \text { shibor_ } 1 \mathrm{Y}
\end{align*}
$$

where $F_{1}, F_{2}$ denote the first principal component and the second principal component, respectively. shibor_O/N, shibor_1W, shibor_2W, shibor_1M, shibor_3M, shibor_6M, shibor_9M, and shibor_1Y denote overnight, 1-week, 2-week, 1-month, 3 -month, 6-month, 9 -month, and 1-year SHIBOR.

## 3. Two-Factor Vasicek Model of SHIBOR

3.1. Two-Factor Vasicek Model. Based on the results of principal component analysis, the term structure of SHIBOR can be described by two-factor model. In this paper, the two-factor Vasicek model is as follows [25]:

$$
\begin{equation*}
R_{t}=\delta_{0}+\delta_{1} F_{1 t}+\delta_{2} F_{2 t}, \tag{2}
\end{equation*}
$$

where $R_{t}$ is short-term interest rate, $\delta_{0}, \delta_{1}$, and $\delta_{2}$ are constants, and $F_{1 t}$ and $F_{2 t}$ are state variables deciding the
value of SHIBOR. Under the risk neutral probability measure, the state variables are subject to the following process:

$$
\begin{align*}
& d F_{1 t}=-\alpha_{1} F_{1 t} d t+\sigma_{1} d W_{1 t}  \tag{3}\\
& d F_{2 t}=-\alpha_{2} F_{2 t} d t+\sigma_{2} d W_{2 t}
\end{align*}
$$

where $\alpha_{1}$ and $\alpha_{2}$ are constants denoting the speed of the mean reversion of state variables, $\sigma_{1}$ and $\sigma_{2}$ are the annual volatility of two state variables, and $W_{1 t}$ and $W_{2 t}$ denote independent standard Brownian motion. Under real probability measure, the state variables are subject to the following process:

$$
\begin{align*}
& d F_{1 t}=k_{1}\left[\theta_{1}-F_{1 t}\right] d t+\sigma_{1} d \omega_{1 t} \\
& d F_{2 t}=k_{2}\left[\theta_{2}-F_{2 t}\right] d t+\sigma_{2} d \omega_{2 t} \tag{4}
\end{align*}
$$

where $k_{1}, k_{2}, \theta_{1}, \theta_{2}, \sigma_{1}$, and $\sigma_{2}$ are constants and $\omega_{1 t}$ and $\omega_{2 t}$ denote independent standard Brownian motion. Under real probability measure, the condition expectation and the condition variance of state variables are as follows:

$$
\begin{gather*}
E\left\{F_{i t} \mid F_{s}\right\}=e^{-k(t-s)} F_{i s}+\theta\left(1-e^{-k(t-s)}\right), \quad i=1,2, \\
\operatorname{Var}\left\{F_{i t} \mid F_{s}\right\}=\frac{\sigma_{i}^{2}}{2 k}\left[1-e^{-2 k(t-s)}\right], \quad i=1,2, \tag{5}
\end{gather*}
$$

where $0 \leq s<t . F_{s}$ is an information set at $s$ time.
3.2. Kalman Filter to Estimate Parameters of Two-Factor Vasicek Model. Many scholars use the generalized moment estimate method (GMM) and maximun likelihood estimate (MLE) to estimate the parameters of Vasicek. However, the parameter estimation of the GMM is not stable. Selecting different moment condition estimates will lead to different parameters. While the parameter estimation of MLE is stable and the effectiveness is better than that of GMM [28]. The Kalman filter estimation methods can build maximum likelihood estimation function of model parameters, and then through maximizing the function to obtain the estimate values of the model parameters. This method is to use state equation and recursive method to estimate, and the obtained solution is given in the form of estimate value (Table 2). Therefore, Kalman filter theory cannot only overcome the disadvantages and limitations of the classical Wiener filter theory but also implement optimal recursive filtering algorithm easily on the computer. These make the Kalman filter theory obtain a wide range of practical applications [29].

In this paper, the kalman filter will be used to estimate parameters of SHIBOR two-factor Vasicek model [30]. Firstly, the two-factor Vasicek model is written in state space system. The observation equation is as follows:

$$
\begin{equation*}
R_{t}=A^{\prime}+\delta^{\prime} \cdot F_{t}+\varepsilon_{t} \tag{6}
\end{equation*}
$$

where the observation vector $R_{t}$ is a $n \times 1$ order matrix, $A$ and $\delta$ are a $1 \times n$ order matrix and a $n \times 2$ order matrix, respectively. The disturbing part $\varepsilon_{t}$ is a $n \times 1$ order matrix. And

$$
E\left(\varepsilon_{t} \varepsilon_{\tau}^{\prime}\right)= \begin{cases}\Omega, & t=\tau  \tag{7}\\ 0, & t \neq \tau\end{cases}
$$

Table 3: The parameter estimation results of the SHIBOR two-factor Vasicek model.

| Parameter | $\delta_{0}$ | $\delta_{1}$ | $\delta_{2}$ | $k_{1}$ | $k_{2}$ | $\theta_{1}$ | $\theta_{2}$ | $\sigma_{1}$ | $\sigma_{2}$ |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| O/N | 0.0318 | 0.5431 | -0.1387 | 1.1691 | 50.5979 | 0.0226 | 0.3002 | -0.0129 | 0.3209 |
| 1 week | 0.0247 | 0.4755 | -0.5196 | 0.6111 | 7.3607 | 0.0126 | 0.044 | -0.0252 | 0.0439 |
| 2 weeks | 0.03136 | 0.4755 | -0.5231 | 0.059 | 3.0249 | 0.014 | 0.0455 | -0.0004 | 0.0017 |
| 1 month | 0.05 | 0.6283 | -0.0164 | 0.1789 | 0.846 | 0.0077 | 0.0292 | -0.0079 | 0.0034 |
| 3 months | 0.0428 | 0.3579 | 0.0002 | 0.0174 | -0.0174 | 0.0016 | 0.0034 | 0.0271 | 0.0239 |
| 6 months | 0.0429 | 0.3282 | -0.0003 | 0.0172 | 0.2084 | 0.0022 | 0.002 | -0.0087 | 0.0069 |
| 9 months | 0.0544 | 0.3034 | -0.0001 | 0.0204 | 0.1201 | 0.001 | 0.001 | -0.0086 | 0.0051 |
| 1 year | 0.0444 | 0.3666 | -0.0002 | 0.0098 | 0.1129 | 0.0015 | 0.0012 | -0.0063 | 0.0033 |

Table 4: The fitting errors of the SHIBOR two-factor Vasicek model.

| Error analysis | Variance | Mean square error | Average relative error | Maximum absolute value error |
| :--- | :---: | :---: | :---: | :---: |
| $\mathrm{O} / \mathrm{N}$ | 0.004204 | 0.012904 | 0.06176 | 1.700516 |
| 1 week | 0.004475 | 0.015309 | 0.068809 | 2.191315 |
| 2 weeks | 0.0047 | 0.015163 | 0.060387 | 3.148987 |
| 1 month | 0.0026 | 0.003363 | 0.028864 | 1.182393 |
| 3 months | $4.94 E-04$ | 0.000162 | 0.003235 | 0.220396 |
| 6 months | $9.36 E-04$ | 0.000883 | 0.016326 | 0.967795 |
| 9 months | $5.89 E-04$ | 0.00054 | 0.014141 | 0.348812 |
| 1 year | 0.0012 | 0.00023 | 0.013917 | 0.215112 |

where $\Omega$ is a $n \times n$ order matrix. The state vector $F_{t}$ is a $2 \times 1$ order matrix and submits to the state equation:

$$
\begin{equation*}
F_{t+1}=H \cdot F_{t}+\mu_{t+1}, \tag{8}
\end{equation*}
$$

where $H$ is a $2 \times 2$ order matrix and $\mu_{t}$ is a $2 \times 1$ order matrix. And

$$
E\left(\mu_{t} \mu_{\tau}^{\prime}\right)= \begin{cases}Q, & t=\tau  \tag{9}\\ 0, & t \neq \tau\end{cases}
$$

where $Q$ is a $2 \times 2$ order matrix. The parameter estimation steps of kalman filtering are as follows.
(1) Setting the initial value, $\widehat{F}_{1 \mid 0}=E\left[\begin{array}{l}F_{11} \\ F_{21}\end{array}\right]$, $\operatorname{vec}\left(P_{0 \mid 1}\right)=$ $\left[I_{\gamma 2}-(H \otimes H)\right]^{-1} \cdot \operatorname{vec}(Q)$, where $P_{t+1 \mid t} \equiv E\left[\left(F_{t+1}-\right.\right.$ $\left.\left.\widehat{F}_{t+1 \mid t}\right)\left(F_{t+1}-\widehat{F}_{t+1 \mid t}\right)^{\prime}\right]$.
(2) Calculating $F_{t+1 \mid t}$ :

$$
\begin{equation*}
\widehat{F}_{t+1 \mid t}=H \widehat{F}_{t \mid t-1}+E_{t}\left(R_{t}-A^{\prime}-\delta^{\prime} F_{t \mid t-1}\right) \tag{10}
\end{equation*}
$$

where $E_{t}=H P_{t \mid t-1} \delta\left(\delta^{\prime} P_{t \mid t-1} \delta+\Omega\right)^{-1}$ is a gain matrix.
(3) Calculating $P_{t+1 \mid t}$ :
$P_{t+1 \mid t}=H\left[P_{t \mid t-1}-P_{t \mid t-1} \delta\left(\delta^{\prime} P_{t \mid t-1} \delta+\Omega\right)^{-1} \delta^{\prime} P_{t \mid t-1}\right] H^{\prime}+Q$.
(4) We can get series of values of $\left\{\widehat{F}_{t \mid t-1}\right\}_{t-1}^{T}$ and $\left\{P_{t \mid t-1}\right\}_{t-1}^{T}$ by calculating steps (2) and (3). Based on these values, the best estimates of parameter matrices $A, \delta, H, \Omega$,
and $Q$ can be gotten by maximizing the following maximum likelihood function:

$$
\begin{align*}
& \operatorname{LnL}=-\frac{1}{2} \ln 2 \pi-\frac{1}{2} \ln \delta^{\prime}\left|\delta^{\prime} P_{t \mid t-1} \delta+\Omega\right| \\
&-\frac{1}{2} \sum_{t=1}^{T}\left[\left(R_{t}-A^{\prime}-\delta^{\prime} \widehat{F}_{t \mid t-1}\right)^{\prime}\left(\delta^{\prime} P_{t \mid t-1} \delta+\Omega\right)^{-1}\right.  \tag{12}\\
&\left.\times\left(R_{t}-A^{\prime}-\delta^{\prime} \widehat{F}_{t \mid t-1}\right)\right]
\end{align*}
$$

## 4. Results and Analysis of the Parameter Estimation

In this paper, overnight, SHIBOR of 1 week, 2 weeks, 1 month, 3 months, 9 months, and 1 year from January 4, 2007, to August 21, 2013 will be adopted as the observed data. The initial values of parameters $A, \delta, H, \Omega$, and $Q$ will be gotten by regression. Then the best values for parameters of SHIBOR two-factor Vasicek model of various terms estimated by Kalman filter are as shown in Table 3.

Accoding to the parameter estimation results in Table 3, we get eight SHIBOR two-factor Vasicek models to fit overnight SHIBOR, 1-week SHIBOR, 2-week SHIBOR, 1month SHIBOR, 3-month SHIBOR, 9-month SHIBOR, and 1-year SHIBOR from January 4, 2007, to August 21, 2013. The goodness of fit of these models is analyzed accoding to the fitting error. We adopt variance, mean square error, the average relative error, and maximum absolute value error to measure the goodness of fit. Their computation formulas are as follows.

Table 5: Analysis results of prediction errors of SHIBOR two-factor Vasicek model.

| Error analysis | Variance | Mean square error | Average relative error | Maximum absolute value error |
| :--- | :---: | :---: | :---: | :---: |
| O/N | 0.004308 | 0.019736 | 0.127703 | 0.232284 |
| 1-week | 0.003746 | 0.008812 | 0.080438 | 0.118666 |
| 2-week | 0.009814 | 0.036827 | 0.14746 | 0.080697 |
| 1-month | 0.008153 | 0.028611 | 0.13772 | 0.301889 |
| 3-month | 0.001746 | 0.001335 | 0.032295 | 0.043602 |
| 6-month | 0.004104 | 0.008984 | 0.088007 | 0.054531 |
| 9-month | 0.004801 | 0.01201 | 0.104805 | 0.074383 |
| 1-year | 0.004243 | 0.08834 | 0.049985 |  |



Figure 1: Overnight SHIBOR forecast figure.

Variance:

$$
\begin{equation*}
\operatorname{MSE}=\frac{\sum_{t=1}^{n}\left(y_{t}^{\prime}-y_{t}\right)^{2}}{n} . \tag{13}
\end{equation*}
$$

Mean square error:

$$
\begin{equation*}
\mathrm{RMSE}=\frac{1}{n} \sum_{t=1}^{n}\left(\frac{y-y^{\prime}}{y}\right)^{2} . \tag{14}
\end{equation*}
$$

Average relative error:

$$
\begin{equation*}
\mathrm{AVGE}=\frac{1}{n} \sum_{t=1}^{n}\left|\frac{y-y^{\prime}}{y}\right| \tag{15}
\end{equation*}
$$

Maximum absolute value error:

$$
\begin{equation*}
\operatorname{MAXE}=\max _{t}\left|\frac{y-y^{\prime}}{y}\right| \tag{16}
\end{equation*}
$$

Results of fitting error analysis of SHIBOR two-factor Vasicek model are shown in Table 4.

Accoding to results of Table 4, the two-factor Vasicek model fitting error is small for SHIBOR of 8 different terms, especially for SHIBOR of more than 3 months. The fitting variance and mean square error of 3-month SHIBOR, 6month SHIBOR, and 9 -month SHIBOR are less than 0.001 . And their average relative error and maximum absolute error are much lower than those of overnight SHIBOR, 1-week


Figure 2: One-week SHIBOR forecast figure.


Figure 3: Two-week SHIBOR forecast figure.

SHIBOR, and 2-week SHIBOR. The fitting variance, the mean square error, and the average relative error of 1-year SHIBOR are less than these of SHIBOR of the former four varieties. Particularly its maximum absolute value error is the smallest. It means that the result of fitting the one-year SHIBOR by using two-factor Vasicek model is robust. Next, in this paper, these SHIBOR two-factor Vasicek models will be used to forecast 8 varieties of SHIBOR from August 22, 2013, to September 18, 2013. The results are shown in Figures 1, 2, 3, $4,5,6,7$, and 8 .

The forecasting precision of SHIBOR two-factor Vasicek model is analyzed. We calculate the variance, difference quotient, the average relative error, and maximum absolute error to compare predicted SHIBOR and real SHIBOR from August 22, 2013, to September 18, 2013. The results are shown in Table 5.


Figure 4: One-month SHIBOR forecast figure.


Figure 5: Three-month SHIBOR forecast figure.


Figure 6: Six-month SHIBOR forecast figure.


Figure 7: Nine-month SHIBOR forecast figure.


FIGURE 8: One-year SHIBOR forecast figure.

The results in Table 5 show that the prediction accuracy of our SHIBOR two-factor Vasicek model is quite high. Accoding to both variance, mean square error, mean relative error, and the maximum absolute error of prediction, prediction accuracy of the 3-month SHIBOR two-factor Vasicek model is superior to other two-factor Vasicek models. The prediction accuracy of SHIBOR two-factor Vasicek model of 1 week, 6 months, and 1 year is slightly higher than it of overnight, 2 weeks, 1 month, and 9 months.

## 5. Conclusion

Through principal component analysis to 8 varieties of SHIBOR, this paper found that two principal components can explain more than $96 \%$ volatility of SHIBOR. Therefore the two-factor Vasicek model can be established to describe the term structure of SHIBOR. Then we use kalman filter to estimate parameters of various terms of SHIBOR, the twofactor Vasicek model, and fit various terms of SHIBOR with this model. The results show that goodness of fit of the twofactor Vasicek model is high, especially for more than 3month SHIBOR. Finally, we test the prediction ability of this model and find that prediction accuracy of 3-month SHIBOR is higher than it of SHIBOR with other terms.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Wealth Share Analysis with "Fundamentalist/Chartist" Heterogeneous Agents 

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#### Abstract

We build a multiassets heterogeneous agents model with fundamentalists and chartists, who make investment decisions by maximizing the constant relative risk aversion utility function. We verify that the model can reproduce the main stylized facts in real markets, such as fat-tailed return distribution and long-term memory in volatility. Based on the calibrated model, we study the impacts of the key strategies' parameters on investors' wealth shares. We find that, as chartists' exponential moving average periods increase, their wealth shares also show an increasing trend. This means that higher memory length can help to improve their wealth shares. This effect saturates when the exponential moving average periods are sufficiently long. On the other hand, the mean reversion parameter has no obvious impacts on wealth shares of either type of traders. It suggests that no matter whether fundamentalists take moderate strategy or aggressive strategy on the mistake of stock prices, it will have no different impact on their wealth shares in the long run.


## 1. Introduction

Compared with traditional economic modeling, agent-based modeling is more flexible in terms of characterizing the individual heterogeneity and population dynamics. This advantage is beneficial in researching on the survivability of different types of investors, namely, market selection.

Previously, the related studies on market selection and wealth share distribution concentrated mainly on the impact of prediction accuracy, risk-aversion level, learning process, and noise trading. Blume and Easley [1] associated market selection with the first-principle of welfare economics and discovered that, in complete market under Pareto optimum allocation, the survival and disappearance of investors depend on the accuracy of their forecasts. Similarly, Fedyk et al. [2] studied the multiasset economy situation and found that, compared with rational investors, unsophisticated investors could suffer severe loss in the long run, and even their predication deviations seem a priori small. Barucci
and Casna [3] also found that, under the mean reverting environment, investors who have inaccuracy predictions cannot survive.

Conversely, Chen and Huang [4, 5] compared the influence of forecasting accuracy and risk preference for investors' long-term survival, based on a multiassets agent-based artificial stock market. They put forward that the risk preference was the determinant and showed that the wealth of the investors who adopted logarithmic utility function could be dominated in the long run. In respect of learning evolution, LeBaron [6] focused on investors' learning on the gain level, which is the weight level for the last step's forecast error in their price forecast rules. This study showed the stylized facts of the market, analyzed the wealth evolution between different strategic investors, and demonstrated their influence on the market instability. Amir et al. [7] identified the adaptive portfolio strategies which could allow investors to survive under the frame of game theory. From the perspective of noise trading, several researchers used the agent-based
modeling method to analyze the survival problem in the long run by comparing the specialists and the noise traders and the noise traders and BSV investors, respectively [8-10]. Zhao [11] studied the survival boundary conditions of different irrational investors by utilizing the market utility maximization rather than the individual utility maximization.

In the field of heterogeneous agents, fundamentalist/ chartist modeling is a very important frame. For instance, Chiarella and He [12], Chiarella et al. [13], Anufriev and Dindo [14], Yuan and Fu [15], and Zou et al. [16] have mainly focused on the price equilibrium and system stability. This paper, however, focuses on the strategies parameters' impacts on investors' long-term wealth share, including fundamentalists' mean reversion parameters and chartists' exponential moving average periods under the fundamentalist/chartist modeling frame.

## 2. Heterogeneous Agent Model

For generality, this paper extends to multiassets case based on the Constant Relative Risk Aversion (CRRA) heterogeneous agent model which was proposed by Chiarella et al. [13]. The setting of the model is as follows.
2.1. The Market. This paper proposes a discrete-time model with $n$ risky assets and one risk-free asset in the financial market. The risk-free interest rate $r$ is constant. There are two types of strategic agents, fundamentalists and chartists, and a Walrasian auctioneer.

Consider risky asset $i$. Under the assumption of traditional financial economics, investors have homogeneous rational expectations to the return of asset $i$, and the fundamental price of asset $i$ can be derived from the "no-arbitrage" equation

$$
\begin{equation*}
E_{t}\left[P_{i, t+1}+D_{i, t+1}\right]=(1+r) P_{i, t} . \tag{1}
\end{equation*}
$$

The fundamental long-run solution is given by

$$
\begin{equation*}
P_{i, t}=P_{i, t}^{*} \equiv \sum_{k=1}^{\infty} \frac{E_{t}\left[D_{i, t+k}\right]}{(1+r)^{k}}, \tag{2}
\end{equation*}
$$

where $P, P^{*}$, and $D$ denote the price, the fundamental value, and the dividend yield, respectively. In particular, if the dividend process is described by

$$
\begin{equation*}
E_{t}\left[D_{i, t+k}\right]=\left(1+\phi_{i}\right)^{k} D_{i, t}, \quad k=1,2, \ldots, 0 \leq \phi_{i} \leq r \tag{3}
\end{equation*}
$$

one can obtain

$$
\begin{equation*}
P_{i, t}^{*}=\frac{\left(1+\phi_{i}\right) D_{i, t}}{r-\phi_{i}} \tag{4}
\end{equation*}
$$

where $\phi_{i}$ denotes the dividend growth rate of asset $i$. Then one can easily obtain that the fundamental values evolve over time according to

$$
\begin{equation*}
E_{t}\left[P_{i, t+1}^{*}\right]=\left(1+\phi_{i}\right) P_{i, t}^{*} \tag{5}
\end{equation*}
$$

and that the capital return is given by

$$
\begin{equation*}
E_{t}\left[\eta_{i, t+1}\right] \equiv E_{t}\left[\frac{P_{i, t+1}^{*}-P_{i, t}^{*}}{P_{i, t}^{*}}\right]=\phi_{i}, \tag{6}
\end{equation*}
$$

where $\eta_{i}$ denotes the dividend growth rate of asset $i$. In the following section we will introduce heterogeneity into the model. We assume agents have heterogeneous, time-varying beliefs about the first and second moment of capital returns, but for simplicity, they are assumed to share the same beliefs about dividend returns.
2.2. The Demand Function. Each agent is assumed to maximize the CRRA (power) utility function to allocate their wealth as follows:

$$
U^{j}(W)= \begin{cases}\frac{1}{1-\lambda^{j}}\left(W^{1-\lambda^{j}}-1\right) & \left(\lambda^{j} \neq 1\right)  \tag{7}\\ \ln (W) & \left(\lambda^{j}=1\right)\end{cases}
$$

where $W>0$ represents the wealth and the parameter $\lambda^{j}>0$ represents the relative risk aversion coefficient. We choose the CRRA utility function because this assumption is quite realistic. The experiment results of Levy et al. [17] support the decreasing absolute risk aversion (DARA). In other words, investor's risk aversion declines with the increase of wealth, which is consistent with the CRRA (power) utility function. In addition, Campbell and Viceira [18] pointed out that relative risk aversion cannot depend strongly on wealth in the long-run behavior of the economy.

This paper extends the solution proposed by Chiarella and He [12] to the multiassets case and derives investors' demand function. At time $t$, the optimal wealth proportion $\pi_{t}$ to be invested in the risky asset is determined by maximizing the expected utility of wealth at $t+1$, as given by

$$
\begin{equation*}
\max _{\pi_{\mathrm{t}}} E_{t}\left[U\left(W_{t+1}\right)\right] \tag{8}
\end{equation*}
$$

To solve this, one needs to work out the evolution of $U(W(t))$.
Assume that the wealth $W(t)$ follows a continuous time stochastic differential equation

$$
\begin{equation*}
d W=\mu(W) d t+\sigma(W) d z(t) \tag{9}
\end{equation*}
$$

where $z(t)$ is a Wiener process. Let $X=U(W)$ be an invertible differentiable function with the inverse function $W=G(X)$. Following Ito's lemma,

$$
\begin{align*}
d X= & {\left[U^{\prime}(W) \mu(W)+\frac{1}{2} \sigma^{2}(W) U^{\prime \prime}(W)\right] d t }  \tag{10}\\
& +\sigma(W) U^{\prime}(W) d z
\end{align*}
$$

which can be written as

$$
\begin{equation*}
d X=\mu(X) d t+\sigma(X) d z \tag{11}
\end{equation*}
$$

where

$$
\begin{gather*}
\mu(X)=U^{\prime}(G(X)) \mu(G(X))+\frac{1}{2} \sigma^{2}(G(X)) U^{\prime \prime}(G(X)),  \tag{12}\\
\sigma(X)=\sigma(G(X)) U^{\prime}(G(X)) . \tag{13}
\end{gather*}
$$

Discretizing (11) using the Euler formula, one obtains the following approximation:

$$
\begin{equation*}
X(t+\Delta t)=X(t)+\mu(X(t)) \Delta t+\sigma(X(t)) \Delta z(t) . \tag{14}
\end{equation*}
$$

It follows that

$$
\begin{gather*}
E_{t}[X(t+\Delta t)]=X(t)+\mu(X(t)) \Delta t  \tag{15}\\
V_{t}[X(t+\Delta t)]=\sigma^{2}(X(t)) \tag{16}
\end{gather*}
$$

Unitizing the time $\Delta t$ in (15), we have

$$
\begin{equation*}
E_{t}\left[X_{t+1}\right]=X_{t}+\mu\left(X_{t}\right) \tag{17}
\end{equation*}
$$

Substituting (12) into (17), one gets

$$
\begin{align*}
E_{t}\left[U\left(W_{t+1}\right)\right] \approx & U\left(W_{t}\right)+\mu_{t}\left(W_{t}\right) U^{\prime}\left(W_{t}\right) \\
& +\frac{1}{2} \sigma_{t}^{2}\left(W_{t}\right) U^{\prime \prime}\left(W_{t}\right) \tag{18}
\end{align*}
$$

This is the evolution of $U(W(t))$.
Let

$$
\begin{equation*}
\varphi_{i, t+1}=E_{t}\left(\varphi_{i, t+1}\right)+\sigma_{i, t+1} \xi_{i, t}, \tag{19}
\end{equation*}
$$

where $\varphi_{i, t+1}$ is the return of asset $i, \sigma_{i, t+1}$ is the standard deviation of the return of asset $i$, and $\xi_{i, t}$ is an $N(0,1)$ process.

Meanwhile, we assume that a trader's wealth change equals to the sum of the return of the risk-free asset and the returns of risky assets; that is,

$$
\begin{equation*}
W_{t+1}-W_{t}=r\left(1-\sum_{i=1}^{n} \pi_{i, t}\right) W_{t}+W_{t} \sum_{i=1}^{n} \pi_{i, t} \varphi_{i, t+1} \tag{20}
\end{equation*}
$$

where $\pi_{i, t}$ is the wealth proportion invested in asset $i$ at time $t$.

Substituting (19) into (20), we have

$$
\begin{align*}
W_{t+1}= & W_{t} \\
= & r\left(1-\sum_{i=1}^{n} \pi_{i, t}\right) W_{t} \\
& +W_{t}\left(\sum_{i=1}^{n} \pi_{i, t} E_{t}\left(\varphi_{i, t+1}\right)+\sum_{i=1}^{n} \pi_{i, t} \sigma_{i, t+1} \xi_{i, t}\right)  \tag{21}\\
= & {\left[r\left(1-\sum_{i=1}^{n} \pi_{i, t}\right)+\sum_{i=1}^{n} \pi_{i, t} E_{t}\left(\varphi_{i, t+1}\right)\right] W_{t} } \\
& +W_{t} \sum_{i=1}^{n} \pi_{i, t} \sigma_{i, t+1} \xi_{i, t}
\end{align*}
$$

which can be written as

$$
\begin{equation*}
W_{t+1}-W_{t}=\mu_{t}(W)+\sigma_{t}(W) \xi_{t} \tag{22}
\end{equation*}
$$

where

$$
\begin{aligned}
\mu_{t}(W) & =\left[r\left(1-\sum_{i=1}^{n} \pi_{i, t}\right)+\sum_{i=1}^{n} \pi_{i, t} E_{t}\left(\varphi_{i, t+1}\right)\right] W_{t} \\
& =\left[r\left(1-\pi_{\mathbf{t}}^{\prime} \mathbf{e}\right)+\pi_{\mathbf{t}}^{\prime} E_{t}\left(\boldsymbol{\varphi}_{\mathbf{t + 1}}\right)\right] W_{t}
\end{aligned}
$$

$$
\begin{align*}
\sigma_{t}^{2}(W)= & D\left(W_{t} \sum_{i=1}^{n} \pi_{i, t} \sigma_{i, t+1} \xi_{i, t}\right) \\
= & W_{t}^{2}\left[\sum_{i=1}^{n} \pi_{i, t}^{2} \sigma_{i, t+1}^{2} D\left(\xi_{i, t}\right)\right. \\
& +\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{i j} \pi_{i, t} \pi_{j, t} \sigma_{i, t+1} \sigma_{j, t+1} \sqrt{D\left(\xi_{i, t}\right)} \\
& \left.\times \sqrt{D\left(\xi_{j, t}\right)}\right] \\
= & W_{t}^{2}\left[\sum_{i=1}^{n} \pi_{i, t}^{2} \sigma_{i, t+1}^{2}+\sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{i j} \pi_{i, t} \pi_{j, t} \sigma_{i, t+1} \sigma_{j, t+1}\right] \\
= & W_{t}^{2}\left[\sum_{i=1}^{n} \pi_{i, t}^{2} \sigma_{i, t+1}^{2}+\sum_{i=1}^{n} \sum_{j=1}^{n} \pi_{i, t} \pi_{j, t} \sigma_{i j, t+1}\right] \\
= & W_{t}^{2} \pi_{\mathbf{t}}^{\prime} \sum_{t} \pi_{\mathbf{t}} \tag{23}
\end{align*}
$$

with

$$
\begin{gather*}
\boldsymbol{\pi}_{\mathbf{t}}=\left[\pi_{1, t}, \pi_{2, t}, \ldots, \pi_{n, t}\right]^{\prime}, \\
\sum_{t}=\left(\begin{array}{ccc}
\sigma_{1, t+1}^{2} & \ldots & \sigma_{1 n, t+1} \\
\vdots & \ddots & \vdots \\
\sigma_{n 1, t+1} & \cdots & \sigma_{n, t+1}^{2}
\end{array}\right) . \tag{24}
\end{gather*}
$$

Substituting (23) into (18), we have

$$
\begin{align*}
E_{t}[ & \left.U\left(W_{t+1}\right)\right] \\
\approx & U\left(W_{t}\right)+\left[r\left(1-\pi_{\mathbf{t}}^{\prime} \mathbf{e}\right)+\pi_{\mathbf{t}}^{\prime} E_{t}\left(\boldsymbol{\varphi}_{\mathbf{t}+\mathbf{1}}\right)\right] W_{t} U^{\prime}\left(W_{t}\right)  \tag{25}\\
& +\frac{1}{2} W_{t}^{2} \boldsymbol{\pi}_{\mathbf{t}}^{\prime} \sum_{t} \boldsymbol{\pi}_{\mathbf{t}} U^{\prime \prime}\left(W_{t}\right) .
\end{align*}
$$

Thus the first order condition of the problem (8) leads to the following optimal solution:

$$
\begin{align*}
\boldsymbol{\pi}_{\mathbf{t}} & =-\frac{U^{\prime}\left(W_{t}\right)}{W_{t} U^{\prime \prime}\left(W_{t}\right)} \sum_{t}^{-1}\left[E_{t}\left(\boldsymbol{\varphi}_{\mathbf{t}+\mathbf{1}}\right)-r \mathbf{e}\right]  \tag{26}\\
& =\frac{1}{\lambda} \sum_{t}^{-1}\left[E_{t}\left(\boldsymbol{\varphi}_{\mathbf{t}+\mathbf{1}}\right)-r \mathbf{e}\right]
\end{align*}
$$

where $E_{t}\left(\boldsymbol{\varphi}_{\mathbf{t}+1}\right)-r e$ represents the vector of the expected excess return on risky assets and $\sum_{t}$ represents the covariance matrix of the expected return on risky assets. Then we can get the investor's position demand for all assets as follows:

$$
\begin{equation*}
\mathbf{z}_{\mathbf{t}}=W_{t-1} \boldsymbol{\pi}_{\mathbf{t}} \cdot / \mathbf{P}_{\mathbf{t}} \tag{27}
\end{equation*}
$$

where $\mathbf{P}_{\mathbf{t}}, \mathbf{z}_{\mathbf{t}}$, and ./ denote the price vector, the vector of the demand for asset position, and the division of the element
as opposed to the vector, respectively. It can be seen that although the optimal investment proportion of an investor's wealth to be invested in the risky asset is independent of wealth, its optimal position demand is proportional to wealth.
2.3. Heterogeneous Expectations. The heterogeneous beliefs of fundamentalists and chartists are reflected in the expected return as well as the expected variance. We assume that the same type of investors predicts all risky assets in the same way.
2.3.1. Fundamentalists. Assume that fundamentalists (denoted by $f$ ) know the fundamental value of assets. These investors believe that the price will move back to the fundamental value when the market price deviates from fundamental value. Therefore their expected price change is

$$
\begin{align*}
E_{t}^{(f)}\left[P_{i, t+1}-P_{i, t}\right] & =E_{t}^{(f)}\left[P_{i, t+1}^{*}-P_{i, t}^{*}\right]+d_{f}\left(P_{i, t}^{*}-P_{i, t}\right) \\
& =\phi P_{i, t}^{*}+d_{f}\left(P_{i, t}^{*}-P_{i, t}\right) \tag{28}
\end{align*}
$$

Fundamentalists' expected price is the sum of the change of the fundamental value and an adjustment item. The adjustment item is proportional to the deviation between current asset price and the fundamental value. The coefficient $d_{f}\left(d_{f}>0\right)$ indicates the speed returning to the fundamental value, and we call it mean reversion coefficient. In addition, to simplify, we assume that the two types of investors have the same expectation to dividend return; that is,

$$
\begin{equation*}
E_{t}\left(D_{i, t+1}\right)=E_{t}^{(f)}\left(D_{i, t+1}\right)=E_{t}^{(c)}\left(D_{i, t+1}\right)=(1+\phi) D_{i, t} \tag{29}
\end{equation*}
$$

Thus, fundamentalist's expected return for asset $i$ is

$$
\begin{align*}
E_{t}^{(f)}\left(\varphi_{i, t+1}\right) & =\frac{E_{t}^{(f)}\left[P_{i, t+1}\right]+E_{t}^{(f)}\left(D_{i, t+1}\right)-P_{i, t}}{P_{i, t}} \\
& =\frac{\phi P_{i, t}^{*}+d_{f}\left(P_{i, t}^{*}-P_{i, t}\right)+(1+\phi) D_{i, t}-P_{i, t}}{P_{i, t}} . \tag{30}
\end{align*}
$$

We further assume that fundamentalists use the exponential moving average of expected return deviation to determine the expected return variance. To certain extent, it reflects the adaptability; that is,

$$
\begin{equation*}
{ }_{t} \sigma_{i, t+1}^{2}=e^{-1 / \tau^{(f)}}{ }_{t-1} \sigma_{i, t}^{2}+\left(1-e^{-1 / \tau^{(f)}}\right)\left(E_{t-1}^{(f)}\left[\varphi_{i, t}\right]-\varphi_{i, t}\right)^{2} \tag{31}
\end{equation*}
$$

where $\tau^{(f)}$ stands for the period length of the exponential moving average. The larger is $\tau^{(f)}$, the smaller is $\left(1-e^{-1 / \tau^{(f)}}\right)$. It shows that the longer is the moving average period, the smaller is the weight of the latest deviation. Thus the latest expected deviation has the smaller influence on the expected variance. Furthermore, this paper assumes that investors' expected correlations between assets are $\rho_{i j}^{(f)}(i, j$ denotes the assets), which vary with different investors, but do not change
with time. Hence, the covariance matrix of the expected return is

$$
\sum_{t}^{(f)}=\left(\begin{array}{ccc}
{ }_{t} \sigma_{1, t+1}^{2} & \cdots & \rho_{1 n}^{(f)}{ }_{t} \sigma_{1, t+1}{ }_{t} \sigma_{n, t+1}  \tag{32}\\
\vdots & \ddots & \vdots \\
\rho_{n 1}^{(f)}{ }_{t} \sigma_{n, t+1 t} \sigma_{1, t+1} & \cdots & { }_{t} \sigma_{n, t+1}^{2}
\end{array}\right)
$$

2.3.2. Chartists. Chartists do not know the fundamental value of the assets. They use the past price series to infer the movement of future prices. Therefore, chartists can be regarded as a kind of adaptive investors. This paper assumes that the first moment and the second moment of the expected return are adaptable for chartists; that is, both the expected return and its variance are obtained by using the exponential moving average. The expected price change of chartists is

$$
\begin{align*}
m_{t}^{(c)} & \equiv E_{t}^{(c)}\left[\eta_{t+1}\right]=E_{t}^{(c)}\left[\frac{P_{t+1}-P_{t}}{P_{t}}\right] \\
& =e^{-1 / \tau^{(c)}} m_{t-1}^{(c)}+\left(1-e^{-1 / \tau^{(c)}}\right)\left(\frac{P_{t}-P_{t-1}}{P_{t-1}}\right) \tag{33}
\end{align*}
$$

where $\tau^{(c)}$ presents the period length of the exponential moving average. Chartists have the same expected dividend yield as fundamentalists, whose expected return for asset $i$ is

$$
\begin{align*}
E_{t}^{(c)}\left(\varphi_{i, t+1}\right)= & \frac{E_{t}^{(c)}\left[P_{i, t+1}\right]+E_{t}^{(c)}\left(D_{i, t+1}\right)-P_{i, t}}{P_{i, t}} \\
= & e^{-1 / \tau^{(c)}} m_{t-1}^{(c)}+\left(1-e^{-1 / \tau^{(c)}}\right)\left(\frac{P_{t}-P_{t-1}}{P_{t-1}}\right)  \tag{34}\\
& +\frac{(1+\phi) D_{i, t}}{P_{i, t}} .
\end{align*}
$$

For chartists, the covariance matrix of the expected return is similar to that of fundamentalists.
2.4. Market Clearing. This model achieves market clearing through the equilibrium between supply and demand as

$$
\begin{equation*}
\sum_{j=1}^{N^{(f)}} \mathbf{Z}^{j}+\sum_{j=N^{(f)}+1}^{N} \mathbf{Z}^{j}=\mathbf{M} \tag{35}
\end{equation*}
$$

where $j$ represents the investors, $N^{(f)}$ represents the number of fundamentalists, $N$ is the total number of investors, $N-$ $N^{(f)}=N^{(c)}$ is the number of chartists, and $\mathbf{M}$ is the number of outstanding stocks in the market. The equation indicates that the sum of risky asset holdings by all traders is equal to $\mathbf{M}$ at any time $t$, which is achieved by adjusting the equilibrium price repeatedly.
2.5. Wealth Shares. After the market price is determined through the clearing mechanism, the wealth of investor $j$ is also determined as follows:

$$
\begin{equation*}
W_{t}^{j}=\left(1-\sum_{i=1}^{n} \pi_{i, t}\right) W_{t-1}(1+r)+W_{t-1} \sum_{i=1}^{n} \pi_{i, t} \frac{P_{i, t}+D_{i, t}}{P_{i, t-1}} \tag{36}
\end{equation*}
$$

At this moment, the total wealth of all investors, the fundamentalists, and the chartists in the market is

$$
\begin{gather*}
W_{t}=\sum_{j=1}^{N} W_{t}^{j} \\
W_{t}^{(f)}=\sum_{j=1}^{N^{(f)}} W_{t}^{j}  \tag{37}\\
W_{t}^{(c)}=\sum_{j=1}^{N^{(c)}} W_{t}^{j}
\end{gather*}
$$

where $N$ is the number of investors, $N^{(f)}$ is the number of fundamentalists, $N^{(c)}$ is the number of chartists, $W_{t}^{(f)}$ is the total wealth of fundamentalists, and $W_{t}^{(c)}$ is the total wealth of chartists. Accordingly, investor $j$ 's wealth share, all fundamentalists' wealth share, and chartists' wealth share in the market are defined as follows:

$$
\begin{align*}
w_{t}^{j} & =\frac{W_{t}^{j}}{W_{t}} \\
w_{t}^{(f)} & =\frac{W_{t}^{(f)}}{W_{t}}  \tag{38}\\
w_{t}^{(c)} & =\frac{W_{t}^{(c)}}{W_{t}}
\end{align*}
$$

By studying the relative wealth share rather than the absolute wealth amount, we can get to the wealth evolution of different types of investors more intuitively.

## 3. Simulation Results

To reflect the randomness of the dividend process, we revised the dividend process in the following agent-based experiments:

$$
\begin{equation*}
D_{i, t+1}=\left(1+\phi_{i}+\sigma_{i, \varepsilon} \varepsilon_{t+1}\right) D_{i, t} . \tag{39}
\end{equation*}
$$

Then we can obtain the fundamental value of the asset as follows:

$$
\begin{equation*}
P_{i, t+1}^{*}=\left(1+\phi_{i}+\sigma_{i, \varepsilon} \varepsilon_{t+1}\right) P_{i, t}^{*}, \tag{40}
\end{equation*}
$$

where $\varepsilon_{t} \sim N(0,1)$ and $\sigma_{i, \varepsilon}>0$ represents standard deviation dividend growth rates.
3.1. Reproducing Stylized Facts. Table 1 lists the parameters for the benchmark model. In this paper, we use three risky stocks as examples to calibrate this model and the parameters of all stocks are setting consistently. The fundamental value of each stock is setting to 10 , each investor's initial wealth is 10 , so that his total initial wealth is 40 . In addition, short selling is permitted in the model. The total number of investors is 40, including 20 fundamentalists and 20 chartists. One step in this model can be seen as one week in reality. Every experiment runs 1000 periods, corresponding to 20 years in reality.

Table 1: The parameters of basic model.

| Parameter | Value |
| :--- | :---: |
| Number of risky assets | 3 |
| Number of agents | 40 |
| Number of fundamentalists | 20 |
| Number of chartists | 20 |
| Initial cash | 10 |
| Initial stock positions | 1 |
| The minimum stock positions | -5 |
| The maximum stock positions | 10 |
| The initial dividend | 0.002 |
| Dividend growth rate | 0.001 |
| The standard deviation of dividend growth rate | 0.01 |
| Random seeds | 0 |
| The risk-free interest rate | 0.0012 |
| The relative risk aversion | 3 |
| The max exponential moving average periods | 80 |
| The min exponential moving average periods | 20 |
| The min mean reversion parameter | 0.5 |
| The max mean reversion parameter | 1 |
| The min expected correlation coefficient | -0.2 |
| The max expected correlation coefficient | 0.8 |
| The max wealth investment proportion | 0.95 |
| The min wealth investment proportion | -0.95 |

The risk-free interest rate is 0.0012 , corresponding to the annual interest rate which is about $6 \%$. The dividend growth rate is 0.001 , corresponding to the annual growth rate which is about $5 \%$. The initial dividend is 0.002 . Many studies suggest that the relative risk aversion is in the range from 2 to 4 . In this paper, we set it to 3 . To reflect the heterogeneity of the investors, the exponential moving average periods, fundamentalists' mean reversion parameter, and the expected correlation coefficients between assets are randomly selected in a certain range by every trader at the beginning of the experiment, which are kept unchanged during the remaining experiment time. For example, the correlation coefficients between stocks are selected randomly in the range $[-0.2,0.8]$, which is consistent with real stock market (Ochiai and Nacher [19] show that the correlations between DJIA and Nikkei 225 roughly fluctuate within $[-0.2,0.8]$ ).

It is easily understood that although stocks have the same parameters, due to the randomness of the dividend process and the different imbalances of supply and demand, the price evolutions of different stocks are not the same, and they can even be opposite.

Considering that our model is a growth model, in which the dividend growth rate is positive, if we compare one day to one time step, then the dividend growth rate will be so small that the model accuracy could be lost. Thus, in this paper, we use weekly closing prices of the S\&P 500 index from December 30, 1991, to March 7, 2011, as the calibration series and compare it with the simulated price series in both the descriptive statistics and the stylized facts.

Table 2: The descriptive statistics.

| Statistic | Asset 1 | Asset 2 | Asset 3 | S\&P 500 |
| :--- | :---: | :---: | :---: | :---: |
| Mean | 0.00090 | 0.0011 | 0.00072 | 0.0014 |
| Median | 0.0016 | 0.00048 | 0.0011 | 0.0024 |
| S. D. | 0.0196 | 0.0235 | 0.0157 | 0.0238 |
| Kurtosis | 7.0397 | 7.9133 | 6.2793 | 8.9414 |
| Skewness | 0.0496 | 0.1523 | 0.1301 | -0.5191 |



Figure 1: Evolution of prices of the three simulated stocks and the S\&P 500 index.

Figure 1 shows the evolution of stock prices. The bottom right plot is for the S\&P 500 index. Table 2 shows the descriptive statistics of the three simulated stock prices and the S\&P 500 prices. We can find that these properties of simulated prices are very similar with those of S\&P 500 index.

Here we test the stylized facts of our model. Figures 2, $3,4,5$, and 6 display the stylized facts of S\&P 500, stock 1 , stock 2 , and stock 3 . Figure 2 shows the probability density in the semilogarithmic axis, where the red line is normal fitted curve. We can find that both S\&P 500 returns and the simulated stocks returns show the fat-tailed distribution. Figures 3-6 compare the autocorrelation functions of the return rate. In each plot, the black line shows the autocorrelation function of the original returns, and the red line is the autocorrelation of the absolute returns. From these autocorrelation plots, we can find that the original returns have no autocorrelations, whereas the absolute returns show significant long-term autocorrelations. In addition, the Hurst indexes for the absolute returns of three simulated stocks are $0.6993,0.7585$, and 0.6817 , respectively, which confirm the
property of long-term autocorrelation as in real markets [20]. We find that all three stocks show similar characteristics as S\&P 500.

We conclude that our model is able to reproduce the main stylized facts of real stocks and stock indexes, including the fat-tailed distribution of returns, the absence of long-memory in the returns, and the strong long-term correlations in the absolute returns. It indicates that our model has captured some key ingredients of the microstructure of real financial markets.
3.2. Wealth Share Analysis. Investors' beliefs play an important role in making investment decisions. Therefore, it is essential to analyze the key parameters that determine investor's beliefs and focus on these parameters' impacts on investors' wealth accumulation. The model has two key parameters, including fundamentalists' mean reversion parameters $d_{f}$ and chartists' exponential moving average periods $\tau^{(c)}$.


Figure 2: The probability density of S\&P 500 returns and the simulated stocks returns. The red line is the normal fitted curve. Both S\&P 500 returns and the simulated stocks returns show the fat-tailed distribution.


Figure 3: Autocorrelation function of the S\&P 500 index. The black line shows the autocorrelation function of the returns. The red line is the autocorrelation of the absolute returns.

In order to analyze the two parameters' impacts on two types of investors' wealth shares, this paper divides the mean reversion parameter values into four intervals: $(0.5,0.6)$, $(0.6,0.7),(0.7,0.8)$, and ( $0.8,0.9$ ). Fundamentalists' mean reversion parameters value is randomly selected in the specific interval. We also take 9 different values for the chartists, that is, $1,2,3,4,5,10,20,50$, and 80 , which correspond to 9 different weights of the latest price information, that is, 0.6321 , $0.3935,0.2835,0.2212,0.1813,0.0952,0.0488,0.0198$, and 0.0124 . This results in 36 different parameters combinations. In order to keep the conclusion robust, this paper selects


Figure 4: Autocorrelation function of the simulated stock 1. The black line shows the autocorrelation function of the returns. The red line is the autocorrelation of the absolute returns.


Figure 5: Autocorrelation function of the simulated stock 2. The black line shows the autocorrelation function of the returns. The red line is the autocorrelation of the absolute returns.
five different random seeds for each combination to conduct five experiments, and then average investors' wealth shares of these five experiments. Thus, we run a total of 180 experiments. We focus on the aggregate wealth of the same type of traders. Figures 7, 8, 9, and 10 show investors' wealth share for 4 different intervals of $d_{f}$. In every figure, each bar corresponds to the average wealth share at the end of experiments when exponential moving average period $\tau^{(c)}$ is given a specific value.

We can see that no matter what kinds of parameters' portfolios are chosen, the two types of investors coexist in the long term. Chartists' wealth shares fall into the range of $0.3-0.5$, and the corresponding fundamentalists' wealth shares locate between 0.7 and 0.5 . Meanwhile, different portfolios $\left(d_{f}, \tau^{(c)}\right)$ do have different distributions of wealth. Obviously, no


Figure 6: Autocorrelation function of the simulated stock 3. The black line shows the autocorrelation function of the returns. The red line is the autocorrelation of the absolute returns.


Figure 7: Wealth shares $\left(d_{f} \in(0.5,0.6)\right)$.
matter $d_{f}$ is at which intervals, with the exponential moving average periods increasing, chartists' wealth shares also show an increasing trend, indicating that a higher memory length will help chartists form more accurate expectations, thus increasing their wealth shares. However, this trend becomes less obvious when exponential moving average periods are high enough, such as $\tau^{(c)}$ is $20,50,80$. This is because chartists' return expectations only have very small weight on the latest price information (namely, $0.0488,0.0198$, and 0.0124 ), and more than $95 \%$ of the weights are given to the past pricing information. Hence, the growth trend of wealth share is no longer obvious when the exponential moving average period is sufficiently long.

In addition, the value of mean reversion coefficient $d_{f}$ has no significant impacts on the wealth share of the two types of investors. It suggests that the aggressiveness of fundamentalists' strategies on the mistake of stock prices, be they moderate (when $d_{f}$ is small) or aggressive (when $d_{f}$ is large), has no impact on their wealth shares in the long run.

## 4. Conclusions

We have built a multiassets heterogeneous-agents model with fundamentalists and chartists. We verified that the model can reproduce the main stylized facts in real markets such as fat tails in the return distribution, absence of


Figure 8: Wealth Shares $\left(d_{f} \in(0.6,0.7)\right)$.


Figure 9: Wealth Shares $\left(d_{f} \in(0.7,0.8)\right)$.
long-memory in returns, and long-term memory in the absolute returns. Based on the calibrated model, we studied the key strategies parameters' impacts on investors' wealth shares. We found that as chartists' exponential moving average periods increase, their wealth shares also show an increasing trend. This means that higher memory length can help to improve their wealth shares. However when the exponential moving average periods are long enough, this trend is no longer obvious and no matter how long the memory is, wealth share of chartists will not be higher than fundamentalists'

That is, chartists' wealth share will not be more than 0.5 . This reflects that chartists can coexist with fundamentalists in stock markets, that is, at least accounting for about $30 \%$ of market wealth, although they cannot equally share the market wealth. On the other hand, the mean reversing parameter has no significant impacts on the wealth share of either type of traders. Therefore, no matter whether fundamentalists take moderate strategy or aggressive strategy on the mistake of stock prices, it has no different impacts on their wealth shares in the long run.


Figure 10: Wealth Shares $\left(d_{f} \in(0.8,0.9)\right)$.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Hopf Bifurcation and Global Periodic Solutions in a Predator-Prey System with Michaelis-Menten Type Functional Response and Two Delays 

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We consider a predator-prey system with Michaelis-Menten type functional response and two delays. We focus on the case with two unequal and non-zero delays present in the model, study the local stability of the equilibria and the existence of Hopf bifurcation, and then obtain explicit formulas to determine the properties of Hopf bifurcation by using the normal form method and center manifold theorem. Special attention is paid to the global continuation of local Hopf bifurcation when the delays $\tau_{1} \neq \tau_{2}$.

## 1. Introduction

In [1], Xu and Chaplain studied the following delayed predator-prey model with Michaelis-Menten type functional response:

$$
\begin{align*}
\frac{d x_{1}}{d t}=x_{1}(t) & {\left[a_{1}-a_{11} x_{1}\left(t-\tau_{11}\right)-\frac{a_{12} x_{2}(t)}{m_{1}+x_{1}(t)}\right] } \\
\frac{d x_{2}}{d t}=x_{2}(t) & {\left[-a_{2}+\frac{a_{21} x_{1}\left(t-\tau_{21}\right)}{m_{1}+x_{1}\left(t-\tau_{21}\right)}\right.} \\
& \left.-a_{22} x_{2}\left(t-\tau_{22}\right)-\frac{a_{23} x_{3}(t)}{m_{2}+x_{2}(t)}\right] \\
\frac{d x_{3}}{d t}= & x_{3}(t)\left[-a_{3}+\frac{a_{32} x_{2}\left(t-\tau_{32}\right)}{m_{2}+x_{2}\left(t-\tau_{32}\right)}-a_{33} x_{3}\left(t-\tau_{33}\right)\right] \tag{1}
\end{align*}
$$

with initial conditions

$$
\begin{equation*}
x_{i}(t)=\phi_{i}(t), \quad t \in[-\tau, 0], \quad \phi_{i}(0)>0, \quad i=1,2,3 \tag{2}
\end{equation*}
$$

where $x_{1}(t), x_{2}(t)$, and $x_{3}(t)$ denote the densities of the prey, predator, and top predator population, respectively.
$a_{i}, a_{i j}(i, j=1,2,3)$ are positive constants. $\tau_{11}, \tau_{21}, \tau_{22}, \tau_{32}$, and $\tau_{33}$ are nonnegative constants. $\tau_{11}, \tau_{22}, \tau_{33}$ denote the delay in the negative feedback of the prey, predator, and top predator crowding, respectively. $\tau_{21}, \tau_{32}$, are constant delays due to gestation; that is, mature adult predators can only contribute to the production of predator biomass. $\tau=$ $\max \left\{\tau_{11}, \tau_{21}, \tau_{22}, \tau_{32}, \tau_{33}\right\} . \phi_{i}(t)(i=1,2,3)$ are continuous bounded functions in the interval $[-\tau, 0]$. The authors proved that the system is uniformly persistent under some appropriate conditions. By means of constructing suitable Lyapunov functional, sufficient conditions are derived for the global asymptotic stability of the positive equilibrium of the system.

Time delays of one type or another have been incorporated into systems by many researchers since a time delay could cause a stable equilibrium to become unstable and fluctuation. In [2-12], authors showed effects of two delays on dynamical behaviors of system.

It is well known that periodic solutions can arise through the Hopf bifurcation in delay differential equations. However, these periodic solutions bifurcating from Hopf bifurcations are generally local. Under some circumstances, periodic solutions exist when the parameter is far away from the critical value. Therefore, global existence of Hopf bifurcation
is a more interesting and difficult topic. A great deal of research has been devoted to the topics [13-21]. In this paper, let $\tau_{11}=\tau_{22}=\tau_{33}=0, \tau_{21}=\tau_{1}, \tau_{32}=\tau_{2}$ in (1); we consider Hopf bifurcation and global periodic solutions of the following system with two unequal and nonzero delays:

$$
\begin{align*}
\frac{d x_{1}}{d t}=x_{1}(t) & {\left[a_{1}-a_{11} x_{1}(t)-\frac{a_{12} x_{2}(t)}{m_{1}+x_{1}(t)}\right] } \\
\frac{d x_{2}}{d t}=x_{2}(t) & {\left[-a_{2}+\frac{a_{21} x_{1}\left(t-\tau_{1}\right)}{m_{1}+x_{1}\left(t-\tau_{1}\right)}\right.} \\
& \left.-a_{22} x_{2}(t)-\frac{a_{23} x_{3}(t)}{m_{2}+x_{2}(t)}\right]  \tag{3}\\
\frac{d x_{3}}{d t}= & x_{3}(t)\left[-a_{3}+\frac{a_{32} x_{2}\left(t-\tau_{2}\right)}{m_{2}+x_{2}\left(t-\tau_{2}\right)}-a_{33} x_{3}(t)\right]
\end{align*}
$$

with initial conditions

$$
\begin{align*}
x_{i}(t)=\phi_{i}(t), \quad & t \in[-\tau, 0], \quad \phi_{i}(0)>0 \\
& i=1,2,3 ; \quad \tau=\max \left\{\tau_{1}, \tau_{2}\right\} . \tag{4}
\end{align*}
$$

Our goal is to investigate the possible stability switches of the positive equilibrium and stability of periodic orbits arising due to a Hopf bifurcation when one of the delays is treated as a bifurcation parameter. Special attention is paid to the global continuation of local Hopf bifurcation when the delays $\tau_{1} \neq \tau_{2}$.

This paper is organized as follows. In Section 2, by analyzing the characteristic equation of the linearized system of system (3) at positive equilibrium, the sufficient conditions ensuring the local stability of the positive equilibrium and the existence of Hopf bifurcation are obtained [22]. Some explicit formulas determining the direction and stability of periodic solutions bifurcating from Hopf bifurcations are demonstrated by applying the normal form method and center manifold theory due to Hassard et al. [23] in Section 3. In Section 4, we consider the global existence of these bifurcating periodic solutions [24] with two different delays. Some numerical simulation results are included in Section 5.

## 2. Stability of the Positive Equilibrium and Local Hopf Bifurcations

In this section, we first study the existence and local stability of the positive equilibrium and then investigate the effect of delay and the conditions for existence of Hopf bifurcations.

There are at most four nonnegative equilibria for system (3):

$$
\begin{align*}
& E_{1}=(0,0,0), \quad E_{2}=\left(\frac{a_{1}}{a_{11}}, 0,0\right),  \tag{5}\\
& E_{3}=\left(\widetilde{x}_{1}, \widetilde{x}_{2}, 0\right), \quad E_{*}=\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right),
\end{align*}
$$

where $\left(\tilde{x}_{1}, \tilde{x}_{2}, 0\right)$ and $\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right)$ satisfy

$$
\begin{gather*}
a_{1}-a_{11} \tilde{x}_{1}-\frac{a_{12} \tilde{x}_{2}}{m_{1}+\widetilde{x}_{1}}=0 \\
-a_{2}+\frac{a_{21} \tilde{x}_{1}}{m_{1}+\widetilde{x}_{1}}-a_{22} \tilde{x}_{2}=0  \tag{6}\\
a_{1}-a_{11} x_{1}^{*}(t)-\frac{a_{12} x_{2}^{*}(t)}{m_{1}+x_{1}^{*}(t)}=0 \\
-a_{2}+\frac{a_{21} x_{1}^{*}(t)}{m_{1}+x_{1}^{*}(t)}-a_{22} x_{2}^{*}(t)-\frac{a_{23} x_{3}^{*}(t)}{m_{2}+x_{2}^{*}(t)}=0  \tag{7}\\
-a_{3}+\frac{a_{32} x_{2}^{*}(t)}{m_{2}+x_{2}^{*}(t)}-a_{33} x_{3}^{*}(t)=0
\end{gather*}
$$

where $E_{3}$ is a nonnegative equilibrium point if there is a positive solution of (6), and $E_{*}$ is a nonnegative equilibrium point if there is a positive solution of (7).

Let

$$
\begin{aligned}
& \left(H_{1}\right) a_{1}\left(a_{21}-a_{2}\right)>m_{1} a_{2} a_{11} ; \\
& \left(H_{2}\right)\left(a_{32}-a_{3}\right)\left[a_{1}\left(a_{21}-a_{2}\right)-m_{1} a_{2}\right]-m_{2} a_{3} a_{22}\left(a_{1}+m_{1} a_{11}\right)>0 ; \\
& \left(H_{3}\right) \widetilde{x}_{2}\left(a_{32}-a_{3}\right)-m_{2} a_{3}>0 ; \\
& \left(H_{4}\right) a_{22}\left(a_{11}-\left(a_{1} / m_{1}\right)\right)-\left(a_{12} a_{21} / m_{1}^{2}\right)>0 .
\end{aligned}
$$

From [1, 25], we know that if $\left(H_{1}\right),\left(H_{2}\right),\left(H_{3}\right)$, and $\left(H_{4}\right)$ hold, $E_{3}$ and $E_{*}$ always exist as nonnegative equilibria.

Let $E=\left(x_{10}, x_{20}, x_{30}\right)$ be the arbitrary equilibrium point, and let $\bar{x}_{1}(t)=x_{1}(t)-x_{10}, \bar{x}_{2}(t)=x_{2}(t)-x_{20}, \bar{x}_{3}(t)=$ $x_{3}(t)-x_{30}$; still denote $\bar{x}_{1}(t), \bar{x}_{2}(t), \bar{x}_{3}(t)$ by $x_{1}(t), x_{2}(t), x_{3}(t)$, respectively; then the linearized system of the corresponding equations at $E$ is as follows:

$$
\begin{equation*}
\dot{u}(t)=B u(t)+C u\left(t-\tau_{1}\right)+D u\left(t-\tau_{2}\right), \tag{8}
\end{equation*}
$$

where

$$
\begin{align*}
& u(t)=\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)^{T}, \\
& B=\left(b_{i j}\right)_{3 \times 3}, \quad C=\left(c_{i j}\right)_{3 \times 3}, \quad D=\left(d_{i j}\right)_{3 \times 3} \\
& b_{11}=a_{1}-2 a_{11} x_{10}-\frac{a_{12} m_{1} x_{20}}{\left(m_{1}+x_{10}\right)^{2}}, \quad b_{12}=-\frac{a_{12} x_{10}}{m_{1}+x_{10}}, \\
& b_{22}=-a_{2}+\frac{a_{21} x_{10}}{m_{1}+x_{10}}-2 a_{22} x_{20}-\frac{a_{23} m_{2} x_{30}}{\left(m_{2}+x_{20}\right)^{2}}, \\
& b_{23}=-\frac{a_{23} x_{20}}{m_{2}+x_{20}}, \quad b_{33}=-a_{3}+\frac{a_{32} x_{20}}{m_{2}+x_{20}}-2 a_{33} x_{30} \\
& c_{21}=\frac{a_{21} m_{1} x_{20}}{\left(m_{1}+x_{10}\right)^{2}}, \quad d_{32}=\frac{a_{32} m_{2} x_{30}}{\left(m_{2}+x_{20}\right)^{2}} \tag{9}
\end{align*}
$$

all the others of $b_{i j}, c_{i j}$, and $d_{i j}$ are 0 .
The characteristic equation for system (8) is

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}+\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau_{1}}+\left(r_{1} \lambda+r_{0}\right) e^{-\lambda \tau_{2}}=0 \tag{10}
\end{equation*}
$$

where

$$
\begin{align*}
& p_{2}=-\left(b_{11}+b_{22}+b_{33}\right), \\
& p_{1}=b_{11} b_{22}+b_{22} b_{33}+b_{11} b_{33}, \\
& p_{0}=-b_{11} b_{22} b_{33} ;  \tag{11}\\
& q_{1}=-b_{12} c_{21}, \quad q_{0}=b_{12} c_{21} b_{33} ; \\
& r_{1}=-b_{23} d_{32}, \quad r_{0}=b_{11} b_{23} d_{32} .
\end{align*}
$$

We consider the following cases.
(1) $E=E_{1}$. The characteristic equation reduces to

$$
\begin{equation*}
\left(\lambda-a_{1}\right)\left(\lambda+a_{2}\right)\left(\lambda+a_{3}\right)=0 . \tag{12}
\end{equation*}
$$

There are always a positive root $a_{1}$ and two negative roots $a_{2}, a_{3}$ of (12); hence $E_{1}$ is a saddle point.
(2) $E=E_{2}$. Equation (10) takes the form

$$
\begin{equation*}
\left(\lambda+a_{1}\right)\left(\lambda+a_{2}-\frac{a_{1} a_{12}}{m_{1} a_{11}+a_{1}}\right)\left(\lambda+a_{3}\right)=0 \tag{13}
\end{equation*}
$$

There is a positive root $\lambda=\left(a_{1} a_{12} /\left(m_{1} a_{11}+a_{1}\right)\right)-a_{2}$ if $a_{1} a_{12} /\left(m_{1} a_{11}+a_{1}\right)>a_{2}$; hence, $E_{2}$ is a saddle point. If $a_{1} a_{12} /\left(m_{1} a_{11}+a_{1}\right)<a_{2}, E_{2}$ is locally asymptotically stable.
(3) $E=E_{3}$. The characteristic equation is

$$
\begin{equation*}
\left(\lambda-b_{33}\right)\left[\lambda^{2}-\left(b_{11}+b_{22}\right) \lambda+b_{11} b_{22}-b_{12} c_{21} e^{-\lambda \tau_{1}}\right]=0 \tag{14}
\end{equation*}
$$

We will analyse the distribution of the characteristic root of (14) from Ruan and Wei [26], which is stated as follows.

Lemma 1. Consider the exponential polynomial

$$
\begin{align*}
P(\lambda, & \left.e^{-\lambda \tau_{1}}, \ldots, e^{-\lambda \tau_{m}}\right) \\
= & \lambda^{n}+p_{1}^{(0)} \lambda^{n-1}+\cdots+p_{n-1}^{(0)} \lambda+p_{n}^{(0)}  \tag{15}\\
& +\left[p_{1}^{(1)} \lambda^{n-1}+\cdots+p_{n-1}^{(1)} \lambda+p_{n}^{(1)}\right] e^{-\lambda \tau_{1}} \\
& +\cdots+\left[p_{1}^{(m)} \lambda^{n-1}+\cdots+p_{n-1}^{(m)} \lambda+p_{n}^{(m)}\right] e^{-\lambda \tau_{m}}
\end{align*}
$$

where $\tau_{i} \geqslant 0(i=1,2, \ldots, m)$ and $p_{j}^{(i)}(i=0,1, \ldots, m ; j=$ $1,2, \ldots, n)$ are constants. As $\left(\tau_{1}, \tau_{2}, \ldots, \tau_{m}\right)$ vary, the sum of the order of the zeros of $P\left(\lambda, e^{-\lambda \tau_{1}}, \ldots, e^{-\lambda \tau_{m}}\right)$ on the open right half plane can change only if a zero appears on or crosses the imaginary axis.

By using Lemma 1, we can easily obtain the following results.

Lemma 2. If $E_{3}$ is a nonnegative equilibrium point, then
(1) $E_{3}$ is unstable if $b_{33}>0$;
(2) $E_{3}$ is locally asymptotically stable if $b_{33}<0, b_{11}+b_{22}<$ $0, b_{11} b_{22}-b_{12} c_{21}>0$ and $b_{11} b_{22}+b_{12} c_{21}>0$.

Proof. (1) $\lambda=b_{33}$ is a root of (14); if $b_{33}>0$, then $E_{3}$ is unstable.
(2) Clearly, $\lambda=0$ is not a root of (14); we should discuss the following equation instead of (14):

$$
\begin{equation*}
\lambda^{2}-\left(b_{11}+b_{22}\right) \lambda+b_{11} b_{22}-b_{12} c_{21} e^{-\lambda \tau_{1}}=0 \tag{16}
\end{equation*}
$$

Assume that $i \omega$ with $\omega>0$ is a solution of (16). Substituting $\lambda=i \omega$ into (16) and separating the real and imaginary parts yield

$$
\begin{align*}
& -\omega^{2}+b_{11} b_{22}=b_{12} c_{21} \cos \omega \tau_{1}  \tag{17}\\
& \omega\left(b_{11}+b_{22}\right)=b_{12} c_{21} \sin \omega \tau_{1}
\end{align*}
$$

which implies

$$
\begin{equation*}
\omega^{4}+\left(b_{11}^{2}+b_{22}^{2}\right) \omega^{2}+b_{11}^{2} b_{22}^{2}-b_{12}^{2} c_{21}^{2}=0 \tag{18}
\end{equation*}
$$

If $b_{11}^{2} b_{22}^{2}-b_{12}^{2} c_{21}^{2}>0$, that is $\left(b_{11} b_{22}+b_{12} c_{21}\right)\left(b_{11} b_{22}-b_{12} c_{21}\right)>$ 0 , there is no real root of (16). Hence there is no purely imaginary root of (18). When $\tau_{1}=0$, (16) reduces to

$$
\begin{equation*}
\lambda^{2}-\left(b_{11}+b_{22}\right) \lambda+b_{11} b_{22}-b_{12} c_{21}=0 \tag{19}
\end{equation*}
$$

If $b_{11}+b_{22}<0$ and $b_{11} b_{22}-b_{12} c_{21}>0$, both roots of (19) have negative real parts. Thus, by using Lemma 1 , when $b_{33}<0$, $b_{11}+b_{22}<0, b_{11} b_{22}-b_{12} c_{21}>0$ and $b_{11} b_{22}+b_{12} c_{21}>0, E_{3}$ is locally asymptotically stable.
(4) $E=E_{*}$. The characteristic equation about $E_{*}$ is (10). In the following, we will analyse the distribution of roots of (10). We consider four cases.

Case a. Consider

$$
\tau_{1}=\tau_{2}=0
$$

The associated characteristic equation of system (3) is

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+\left(p_{1}+q_{1}+r_{1}\right) \lambda+\left(p_{0}+q_{0}+r_{0}\right)=0 \tag{20}
\end{equation*}
$$

Let

$$
\begin{aligned}
& \left(H_{5}\right) p_{2}>0, p_{2}\left(p_{1}+q_{1}+r_{1}\right)-\left(p_{0}+q_{0}+r_{0}\right)>0, p_{0}+q_{0}+r_{0}> \\
& \quad 0 .
\end{aligned}
$$

By Routh-Hurwitz criterion, we have the following.
Theorem 3. For $\tau_{1}=\tau_{2}=0$, assume that $\left(H_{1}\right)-\left(H_{5}\right)$ hold. Then when $\tau_{1}=\tau_{2}=0$, the positive equilibrium $E_{*}\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right)$ of system (3) is locally asymptotically stable.

Case b. Consider

$$
\tau_{1}=0, \tau_{2}>0
$$

The associated characteristic equation of system (3) is

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+\left(p_{1}+q_{1}\right) \lambda+\left(p_{0}+q_{0}\right)+\left(r_{1} \lambda+r_{0}\right) e^{-\lambda \tau_{2}}=0 \tag{21}
\end{equation*}
$$

We want to determine if the real part of some root increases to reach zero and eventually becomes positive as $\tau$ varies. Let $\lambda=i \omega(\omega>0)$ be a root of (21); then we have

$$
\begin{align*}
& -i \omega^{3}-p_{2} \omega^{2}+i\left(p_{1}+q_{1}\right) \omega+\left(p_{0}+q_{0}\right)  \tag{22}\\
& \quad+\left(r_{1} \omega i+r_{0}\right)\left(\cos \omega \tau_{2}-i \sin \omega \tau_{2}\right)=0
\end{align*}
$$

Separating the real and imaginary parts, we have

$$
\begin{align*}
& -\omega^{3}+\left(p_{1}+q_{1}\right) \omega=r_{0} \sin \omega \tau_{2}-r_{1} \omega \cos \omega \tau_{2} \\
& -p_{2} \omega^{2}+\left(p_{0}+q_{0}\right)=-r_{1} \omega \sin \omega \tau_{2}-r_{0} \cos \omega \tau_{2} \tag{23}
\end{align*}
$$

It follows that

$$
\begin{equation*}
\omega^{6}+m_{12} \omega^{4}+m_{11} \omega^{2}+m_{10}=0 \tag{24}
\end{equation*}
$$

where $m_{12}=p_{2}^{2}-2\left(p_{1}+q_{1}\right), m_{11}=\left(p_{1}+q_{1}\right)^{2}-2 p_{2}\left(p_{0}+\right.$ $\left.q_{0}\right)-r_{1}^{2}, m_{10}=\left(p_{0}+q_{0}\right)^{2}-r_{0}^{2}$.

Denoting $z=\omega^{2}$, (24) becomes

$$
\begin{equation*}
z^{3}+m_{12} z^{2}+m_{11} z+m_{10}=0 \tag{25}
\end{equation*}
$$

Let

$$
\begin{equation*}
h_{1}(z)=z^{3}+m_{12} z^{2}+m_{11} z+m_{10} \tag{26}
\end{equation*}
$$

we have

$$
\begin{equation*}
\frac{d h_{1}(z)}{d z}=3 z^{2}+2 m_{12} z+m_{11} \tag{27}
\end{equation*}
$$

If $m_{10}=\left(p_{0}+q_{0}\right)^{2}-r_{0}^{2}<0$, then $h_{1}(0)<$ $0, \lim _{z \rightarrow+\infty} h_{1}(z)=+\infty$. We can know that (25) has at least one positive root.

If $m_{10}=\left(p_{0}+q_{0}\right)^{2}-r_{0}^{2} \geq 0$, we obtain that when $\Delta=$ $m_{12}^{2}-3 m_{11} \leq 0$, (25) has no positive roots for $z \in[0,+\infty)$. On the other hand, when $\Delta=m_{12}^{2}-3 m_{11}>0$, the following equation

$$
\begin{equation*}
3 z^{2}+2 m_{12} z+m_{11}=0 \tag{28}
\end{equation*}
$$

has two real roots: $z_{11}^{*}=\left(-m_{12}+\sqrt{\Delta}\right) / 3, z_{12}^{*}=\left(-m_{12}-\right.$ $\sqrt{\Delta}) / 3$. Because of $h_{1}^{\prime \prime}\left(z_{11}^{*}\right)=2 \sqrt{\Delta}>0, h_{1}^{\prime \prime}\left(z_{12}^{*}\right)=-2 \sqrt{\Delta}<$ $0, z_{11}^{*}$ and $z_{12}^{*}$ are the local minimum and the local maximum of $h_{1}(z)$, respectively. By the above analysis, we immediately obtain the following.

Lemma 4. (1) If $m_{10} \geq 0$ and $\Delta=m_{12}^{2}-3 m_{11} \leq 0$, (25) has no positive root for $z \in[0,+\infty)$.
(2) If $m_{10} \geq 0$ and $\Delta=m_{12}^{2}-3 m_{11}>0$, (25) has at least one positive root if and only if $z_{11}^{*}=\left(-m_{12}+\sqrt{\Delta}\right) / 3>0$ and $h_{1}\left(z_{11}^{*}\right) \leq 0$.
(3) If $m_{10}<0$, (25) has at least one positive root.

Without loss of generality, we assume that (25) has three positive roots, defined by $z_{11}, z_{12}, z_{13}$, respectively. Then (24) has three positive roots:

$$
\begin{equation*}
\omega_{11}=\sqrt{z_{11}}, \quad \omega_{12}=\sqrt{z_{12}}, \quad \omega_{13}=\sqrt{z_{13}} \tag{29}
\end{equation*}
$$

From (23) we have

$$
\begin{align*}
& \cos \omega_{1 k} \tau_{2_{1 k}} \\
& \quad=\frac{r_{1} \omega_{1 k}^{4}+\left[p_{2} r_{0}-\left(q_{1}+p_{1}\right) r_{1}\right] \omega_{1 k}^{2}-r_{0}\left(q_{0}+p_{0}\right)}{r_{0}^{2}+r_{1}^{2} \omega_{1 k}^{2}} \tag{30}
\end{align*}
$$

Thus, if we denote

$$
\begin{align*}
& \tau_{2_{1 k}}^{(j)}=\frac{1}{\omega_{1 k}} \\
&  \tag{31}\\
& \quad \times\left\{\operatorname { a r c c o s } \left(\left(r_{1} \omega_{1 k}^{4}+\left[p_{2} r_{0}-\left(q_{1}+p_{1}\right) r_{1}\right] \omega_{1 k}^{2}\right.\right.\right. \\
& \\
& \left.\quad-r_{0}\left(q_{0}+p_{0}\right)\right) \\
& \\
& \left.\left.\quad \times\left(r_{0}^{2}+r_{1}^{2} \omega_{1 k}^{2}\right)^{-1}\right)+2 j \pi\right\}
\end{align*}
$$

where $k=1,2,3 ; j=0,1,2, \ldots$ then $\pm i \omega_{1 k}$ is a pair of purely imaginary roots of (21) corresponding to $\tau_{2_{1 k}}^{(j)}$. Define

$$
\begin{equation*}
\tau_{2_{10}}=\tau_{2_{1 k_{0}}}^{(0)}=\min _{k=1,2,3}\left\{\tau_{2_{1 k}}^{(0)}\right\}, \quad \omega_{10}=\omega_{1 k_{0}} . \tag{32}
\end{equation*}
$$

Let $\lambda\left(\tau_{2}\right)=\alpha\left(\tau_{2}\right)+i \omega\left(\tau_{2}\right)$ be the root of (21) near $\tau_{2}=\tau_{2_{1 k}}^{(j)}$ satisfying

$$
\begin{equation*}
\alpha\left(\tau_{2_{1 k}}^{(j)}\right)=0, \quad \omega\left(\tau_{2_{1 k}}^{(j)}\right)=\omega_{1 k} . \tag{33}
\end{equation*}
$$

Substituting $\lambda\left(\tau_{2}\right)$ into (21) and taking the derivative with respect to $\tau_{2}$, we have

$$
\begin{align*}
\left\{3 \lambda^{2}\right. & \left.+2 p_{2} \lambda+\left(p_{1}+q_{1}\right)+r_{1} e^{-\lambda \tau_{2}}-\tau_{2}\left(r_{1} \lambda+r_{0}\right) e^{-\lambda \tau_{2}}\right\} \frac{d \lambda}{d \tau_{2}} \\
& =\lambda\left(r_{1} \lambda+r_{0}\right) e^{-\lambda \tau_{2}} \tag{34}
\end{align*}
$$

Therefore,

$$
\begin{align*}
{\left[\frac{d \lambda}{d \tau_{2}}\right]^{-1}=} & \frac{\left[3 \lambda^{2}+2 p_{2} \lambda+\left(p_{1}+q_{1}\right)\right] e^{\lambda \tau_{2}}}{\lambda\left(r_{1} \lambda+r_{0}\right)}  \tag{35}\\
& +\frac{r_{1}}{\lambda\left(r_{1} \lambda+r_{0}\right)}-\frac{\tau_{2}}{\lambda}
\end{align*}
$$

When $\tau_{2}=\tau_{2_{1 k}}^{(j)}, \lambda\left(\tau_{2_{1 k}}^{(j)}\right)=i \omega_{1 k} \quad(k=1,2,3),\left\{\lambda\left(r_{1} \lambda+\right.\right.$ $\left.\left.r_{0}\right)\right\}\left.\right|_{\tau_{2}=\tau_{21 k}^{(j)}}=-r_{1} \omega_{1 k}^{2}+i r_{0} \omega_{1 k},\left\{\left[3 \lambda^{2}+2 p_{2} \lambda+\left(p_{1}+\right.\right.\right.$ $\left.\left.\left.q_{1}\right)\right] e^{\lambda \tau_{2}}\right\}\left.\right|_{\tau_{2}=\tau_{2_{1 k}}^{(j)}}=\left\{\left[-3 \omega_{1 k}^{2}+\left(p_{1}+q_{1}\right)\right] \cos \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)-\right.$ $\left.2 p_{2} \omega_{1 k} \sin \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)\right\}+i\left\{2 p_{2} \omega_{1 k} \cos \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)+\left[-3 \omega_{1 k}^{2}+\left(p_{1}+\right.\right.\right.$ $\left.\left.\left.q_{1}\right)\right] \sin \left(\omega_{1 k} \tau_{2_{k k}}^{(j)}\right)\right\}$.

According to (35), we have

$$
\begin{align*}
& {\left[\begin{array}{rl}
{\left[\frac{\operatorname{Re} d\left(\lambda\left(\tau_{2}\right)\right)}{d \tau_{2}}\right]_{\tau_{2}=\tau_{2_{1 k}}^{(j)}}^{-1}} \\
= & \operatorname{Re}\left[\frac{\left[3 \lambda^{2}+2 p_{2} \lambda+\left(p_{1}+q_{1}\right)\right] e^{\lambda \tau_{2}}}{\lambda\left(r_{1} \lambda+r_{0}\right)}\right]_{\tau_{2}=\tau_{2_{1 k}}^{(j)}} \\
& +\operatorname{Re}\left[\frac{r_{1}}{\lambda\left(r_{1} \lambda+r_{0}\right)}\right]_{\tau_{2}=\tau_{21 k}}^{(j)}
\end{array}\right.} \\
& \begin{aligned}
= & \frac{1}{\Lambda_{1}}\left\{-r_{1} \omega_{1 k}^{2}\left[-3 \omega_{1 k}^{2}+\left(p_{1}+q_{1}\right)\right] \cos \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)\right.
\end{aligned} \\
& \quad+2 r_{1} p_{2} \omega_{1 k}^{3} \sin \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)-r_{1}^{2} \omega_{1 k}^{2} \\
& \\
& \quad+2 r_{0} p_{2} \omega_{1 k}^{2} \cos \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)  \tag{36}\\
& \\
& \left.\quad+r_{0}\left[-3 \omega_{1 k}^{2}+\left(p_{1}+q_{1}\right)\right] \omega_{1 k} \sin \left(\omega_{1 k} \tau_{2_{1 k}}^{(j)}\right)\right\} \\
& =
\end{align*}
$$

where $\Lambda_{1}=r_{1}^{2} \omega_{1 k}^{4}+r_{0}^{2} \omega_{1 k}^{2}>0$. Notice that $\Lambda_{1}>0, z_{1 k}>0$,

$$
\begin{align*}
& \operatorname{sign}\left\{\left[\frac{\operatorname{Re} d\left(\lambda\left(\tau_{2}\right)\right)}{d \tau_{2}}\right]_{\tau_{2}=\tau_{\tau_{1 k}}^{(j)}}\right\} \\
& \quad=\operatorname{sign}\left\{\left[\frac{\operatorname{Re} d\left(\lambda\left(\tau_{2}\right)\right)}{d \tau_{2}}\right]_{\tau_{2}=\tau_{2_{1 k}}^{(j)}}^{-1}\right\} ; \tag{37}
\end{align*}
$$

then we have the following lemma.
Lemma 5. Suppose that $z_{1 k}=\omega_{1 k}^{2}$ and $h_{1}^{\prime}\left(z_{1 k}\right) \neq 0$, where $h_{1}(z)$ is defined by $(26)$; then $d\left(\operatorname{Re} \lambda\left(\tau_{2_{1 k}}^{(j)}\right)\right) / d \tau_{2}$ has the same sign with $h_{1}^{\prime}\left(z_{1 k}\right)$.

From Lemmas 1, 4, and 5 and Theorem 3, we can easily obtain the following theorem.

Theorem 6. For $\tau_{1}=0, \tau_{2}>0$, suppose that $\left(H_{1}\right)-\left(H_{5}\right)$ hold.
(i) If $m_{10} \geq 0$ and $\Delta=m_{12}^{2}-3 m_{11} \leq 0$, then all roots of (10) have negative real parts for all $\tau_{2} \geq 0$, and the positive equilibrium $E_{*}$ is locally asymptotically stable for all $\tau_{2} \geq 0$.
(ii) If either $m_{10}<0$ or $m_{10} \geq 0, \Delta=m_{12}^{2}-3 m_{11}>$ $0, z_{11}^{*}>0$, and $h_{1}\left(z_{11}^{*}\right) \leq 0$, then $h_{1}(z)$ has at least one
positive roots, and all roots of (23) have negative real parts for $\tau_{2} \in\left[0, \tau_{2_{10}}\right)$, and the positive equilibrium $E_{*}$ is locally asymptotically stable for $\tau_{2} \in\left[0, \tau_{2_{10}}\right.$ ).
(iii) If (ii) holds and $h_{1}^{\prime}\left(z_{1 k}\right) \neq 0$, then system (3)undergoes Hopfbifurcations at the positive equilibrium $E_{*}$ for $\tau_{2}=$ $\tau_{2_{1 k}}^{(j)},(k=1,2,3 ; j=0,1,2, \ldots)$.

## Case c. Consider

$\tau_{1}>0, \tau_{2}=0$.
The associated characteristic equation of system (3) is

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+\left(p_{1}+r_{1}\right) \lambda+\left(p_{0}+r_{0}\right)+\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau_{1}}=0 \tag{38}
\end{equation*}
$$

Similar to the analysis of Case $b$, we get the following theorem.

Theorem 7. For $\tau_{1}>0, \tau_{2}=0$, suppose that $\left(H_{1}\right)-\left(H_{5}\right)$ hold.
(i) If $m_{20} \geq 0$ and $\Delta=m_{22}^{2}-3 m_{21} \leq 0$, then all roots of (38) have negative real parts for all $\tau_{1} \geq 0$, and the positive equilibrium $E_{*}$ is locally asymptotically stable for all $\tau_{1} \geq 0$.
(ii) If either $m_{20}<0$ or $m_{20} \geq 0, \Delta=m_{22}^{2}-3 m_{21}>0$, $z_{21}^{*}>0$ and $h_{2}\left(z_{21}^{*}\right) \leq 0$, then $h_{2}(z)$ has at least one positive root $z_{2 k}$, and all roots of (38) have negative real parts for $\tau_{1} \in\left[0, \tau_{1_{20}}\right)$, and the positive equilibrium $E_{*}$ is locally asymptotically stable for $\tau_{1} \in\left[0, \tau_{1_{20}}\right)$.
(iii) If (ii) holds and $h_{2}^{\prime}\left(z_{2 k}\right) \neq 0$, then system (3) undergoes Hopfbifurcations at the positive equilibrium $E_{*}$ for $\tau_{1}=$ $\tau_{1_{2 k}}^{(j)},(k=1,2,3 ; j=0,1,2, \ldots)$,
where

$$
\begin{gather*}
m_{22}=p_{2}^{2}-2\left(p_{1}+r_{1}\right), \\
m_{21}=\left(p_{1}+r_{1}\right)^{2}-2 p_{2}\left(p_{0}+r_{0}\right)-q_{1}^{2}, \\
m_{20}=\left(p_{0}+r_{0}\right)^{2}-q_{0}^{2} ; \\
h_{2}(z)=z^{3}+m_{22} z^{2}+m_{21} z+m_{20}, \quad z_{21}^{*}=\frac{-m_{22}+\sqrt{\Delta}}{3} ; \\
\tau_{1_{2 k}(j)}=\frac{1}{\omega_{2 k}} \\
\times\left\{\operatorname { a r c c o s } \left(\left(q_{1} \omega_{2 k}^{4}+\left[p_{2} q_{0}-\left(r_{1}+p_{1}\right) q_{1}\right] \omega_{2 k}^{2}\right.\right.\right. \\
\left.\quad-q_{0}\left(r_{0}+p_{0}\right)\right) \\
 \tag{39}\\
\left.\left.\times\left(q_{0}^{2}+q_{1}^{2} \omega_{2 k}^{2}\right)^{-1}\right)+2 j \pi\right\},
\end{gather*}
$$

where $k=1,2,3 ; j=0,1,2, \ldots$; then $\pm i \omega_{2 k}$ is a pair of purely imaginary roots of (38) corresponding to $\tau_{1_{2 k}}^{(j)}$. Define

$$
\begin{equation*}
\tau_{1_{20}}=\tau_{1_{2 k_{0}}}^{(0)}=\min _{k=1,2,3}\left\{\tau_{1_{2 k}}^{(0)}\right\}, \quad \omega_{10}=\omega_{1 k_{0}} \tag{40}
\end{equation*}
$$

Case d. Consider
$\tau_{1}>0, \tau_{2}>0, \tau_{1} \neq \tau_{2}$.
The associated characteristic equation of system (3) is

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}+\left(q_{1} \lambda+q_{0}\right) e^{-\lambda \tau_{1}}+\left(r_{1} \lambda+r_{0}\right) e^{-\lambda \tau_{2}}=0 \tag{41}
\end{equation*}
$$

We consider (41) with $\tau_{2}=\tau_{2}^{*}$ in its stable interval $\left[0, \tau_{2_{10}}\right.$ ). Regard $\tau_{1}$ as a parameter.

Let $\lambda=i \omega(\omega>0)$ be a root of (41); then we have

$$
\begin{align*}
& -i \omega^{3}-p_{2} \omega^{2}+i p_{1} \omega+p_{0}+\left(i q_{1} \omega+q_{0}\right)\left(\cos \omega \tau_{1}-i \sin \omega \tau_{1}\right) \\
& +\left(r_{0}+i r_{1} \omega\right)\left(\cos \omega \tau_{2}^{*}-i \sin \omega \tau_{2}^{*}\right)=0 . \tag{42}
\end{align*}
$$

Separating the real and imaginary parts, we have

$$
\begin{align*}
\omega^{3} & -p_{1} \omega-r_{1} \omega \cos \omega \tau_{2}^{*}+r_{0} \sin \omega \tau_{2}^{*} \\
& =q_{1} \omega \cos \omega \tau_{1}-q_{0} \sin \omega \tau_{1}, \\
p_{2} & \omega^{2}-p_{0}-r_{0} \cos \omega \tau_{2}^{*}-r_{1} \omega \sin \omega \tau_{2}^{*}  \tag{43}\\
& =q_{0} \cos \omega \tau_{1}+q_{1} \omega \sin \omega \tau_{1} .
\end{align*}
$$

It follows that

$$
\begin{equation*}
\omega^{6}+m_{33} \omega^{4}+m_{32} \omega^{3}+m_{31} \omega^{2}+m_{30}=0 \tag{44}
\end{equation*}
$$

where

$$
\begin{align*}
& m_{33}=p_{2}^{2}-2 p_{1}-2 r_{1} \cos \omega \tau_{2}^{*}, \\
& m_{32}=2\left(r_{0}-p_{2} r_{1}\right) \sin \omega \tau_{2}^{*}, \\
& m_{31}=p_{1}^{2}-2 p_{0} p_{2}-2\left(p_{2} r_{0}-p_{1} r_{1}\right) \cos \omega \tau_{2}^{*}+r_{1}^{2}-q_{1}^{2},  \tag{45}\\
& m_{30}=p_{0}^{2}+2 p_{0} r_{0} \cos \omega \tau_{2}^{*}+r_{0}^{2}-q_{0}^{2} .
\end{align*}
$$

Denote $F(\omega)=\omega^{6}+m_{33} \omega^{4}+m_{32} \omega^{3}+m_{31} \omega^{2}+m_{30}$. If $m_{30}<0$, then

$$
\begin{equation*}
F(0)<0, \quad \lim _{\omega \rightarrow+\infty} F(\omega)=+\infty . \tag{46}
\end{equation*}
$$

We can obtain that (44) has at most six positive roots $\omega_{1}, \omega_{2}, \ldots, \omega_{6}$. For every fixed $\omega_{k}, k=1,2, \ldots, 6$, there exists a sequence $\left\{\tau_{1 k}^{(j)} \mid j=0,1,2,3, \ldots\right\}$, such that (43) holds.

Let

$$
\begin{equation*}
\tau_{10}=\min \left\{\tau_{1 k}^{(j)} \mid k=1,2, \ldots, 6 ; j=0,1,2,3, \ldots\right\} . \tag{47}
\end{equation*}
$$

When $\tau_{1}=\tau_{1 k}^{(j)}$, (41) has a pair of purely imaginary roots $\pm i \omega_{1 k}^{(j)}$ for $\tau_{2}^{*} \in\left[0, \tau_{2_{10}}\right)$.

In the following, we assume that

$$
\left.\left(H_{6}\right)\left((d \operatorname{Re}(\lambda)) / d \tau_{1}\right)\right|_{\lambda= \pm i \omega_{1 k}^{(j)}} \neq 0
$$

Thus, by the general Hopf bifurcation theorem for FDEs in Hale [22], we have the following result on the stability and Hopf bifurcation in system (3).

Theorem 8. For $\tau_{1}>0, \tau_{2}>0, \tau_{1} \neq \tau_{2}$, suppose that $\left(H_{1}\right)-\left(H_{6}\right)$ is satisfied. If $m_{30}<0$ and $\tau_{2}^{*} \in\left[0, \tau_{2_{10}}\right]$, then the positive equilibrium $E_{*}$ is locally asymptotically stable for $\tau_{1} \in\left[0, \tau_{10}\right.$ ). System (3) undergoes Hopf bifurcations at the positive equilibrium $E_{*}$ for $\tau_{1}=\tau_{1 k}^{(j)}$.

## 3. Direction and Stability of the Hopf Bifurcation

In Section 2, we obtain the conditions under which system (3) undergoes the Hopf bifurcation at the positive equilibrium $E_{*}$. In this section, we consider with $\tau_{2}=\tau_{2}^{*} \in\left[0, \tau_{2_{10}}\right)$ and regard $\tau_{1}$ as a parameter. We will derive the explicit formulas determining the direction, stability, and period of these periodic solutions bifurcating from equilibrium $E_{*}$ at the critical values $\tau_{1}$ by using the normal form and the center manifold theory developed by Hassard et al. [23]. Without loss of generality, denote any one of these critical values $\tau_{1}=$ $\tau_{1 k}^{(j)}(k=1,2, \ldots, 6 ; j=0,1,2, \ldots)$ by $\widetilde{\tau_{1}}$, at which (43) has a pair of purely imaginary roots $\pm i \omega$ and system (3) undergoes Hopf bifurcation from $E_{*}$.

Throughout this section, we always assume that $\tau_{2}^{*}<\tau_{10}$. Let $u_{1}=x_{1}-x_{1}^{*}, u_{2}=x_{1}-x_{2}^{*}, u_{3}=x_{2}-x_{3}^{*}, t=\tau_{1} t$ and $\mu=\tau_{1}-\widetilde{\tau_{1}}, \mu \in \mathscr{R}$. Then $\mu=0$ is the Hopf bifurcation value of system (3). System (3) may be written as a functional differential equation in $\mathscr{C}\left([-1,0], \mathscr{R}^{3}\right)$

$$
\begin{equation*}
\dot{u}(t)=L_{\mu}\left(u_{t}\right)+f\left(\mu, u_{t}\right), \tag{48}
\end{equation*}
$$

where $u=\left(u_{1}, u_{2}, u_{3}\right)^{T} \in \mathscr{R}^{3}$, and

$$
\begin{align*}
& L_{\mu}(\phi)=\left(\widetilde{\tau_{1}}+\mu\right) B\left[\begin{array}{l}
\phi_{1}(0) \\
\phi_{2}(0) \\
\phi_{3}(0)
\end{array}\right]+\left(\widetilde{\tau_{1}}+\mu\right) C\left[\begin{array}{l}
\phi_{1}(-1) \\
\phi_{2}(-1) \\
\phi_{3}(-1)
\end{array}\right] \\
&+\left(\widetilde{\tau_{1}}+\mu\right) D\left[\begin{array}{l}
\phi_{1}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \\
\phi_{2}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \\
\phi_{3}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right)
\end{array}\right] \tag{49}
\end{align*}
$$

$$
f(\mu, \phi)=\left(\widetilde{\tau_{1}}+\mu\right)\left[\begin{array}{l}
f_{1}  \tag{50}\\
f_{2} \\
f_{3}
\end{array}\right]
$$

where $\phi=\left(\phi_{1}, \phi_{2}, \phi_{3}\right)^{T} \in \mathscr{C}\left([-1,0], \mathscr{R}^{3}\right)$, and

$$
\begin{align*}
& B=\left[\begin{array}{ccc}
b_{11} & b_{12} & 0 \\
0 & b_{22} & b_{23} \\
0 & 0 & b_{33}
\end{array}\right], \quad C=\left[\begin{array}{ccc}
0 & 0 & 0 \\
c_{21} & 0 & 0 \\
0 & 0 & 0
\end{array}\right], \\
& D=\left[\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & d_{32} & 0
\end{array}\right] \text {, } \\
& f_{1}=-\left(a_{11}+l_{3}\right) \phi_{1}^{2}(0)-l_{1} \phi_{1}(0) \phi_{2}(0)-l_{2} \phi_{1}^{2}(0) \phi_{2}(0) \\
& -l_{5} \phi_{1}^{3}(0)-l_{4} \phi_{1}^{3}(0) \phi_{2}(0)+\cdots, \\
& f_{2}=-l_{6} \phi_{2}(0) \phi_{3}(0)-\left(l_{7}+a_{22}\right) \phi_{2}^{2}(0)+l_{1} \phi_{1}(-1) \phi_{2}(0) \\
& +l_{3} \phi_{1}^{2}(-1)-l_{8} \phi_{2}^{3}(0)-l_{9} \phi_{2}^{2}(0) \phi_{3}(0) \\
& +l_{2} \phi_{1}^{2}(-1) \phi_{2}(0)+l_{3} \phi_{1}^{3}(-1)+l_{4} \phi_{1}^{3}(-1) \phi_{2}(0) \\
& -l_{10} \phi_{2}^{3}(0) \phi_{3}(0)+\cdots, \\
& f_{3}=l_{6} \phi_{2}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \phi_{3}(0)+l_{7} \phi_{2}^{2}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right)-a_{33} \phi_{3}^{2}(0) \\
& +l_{9} \phi_{2}^{2}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \phi_{3}(0)+l_{8} \phi_{2}^{3}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \\
& +l_{10} \phi_{2}^{3}\left(-\frac{\tau_{2}^{*}}{\tau_{1}}\right) \phi_{3}(0)+\cdots, \\
& p_{1}(x)=\frac{a_{1} x}{1+b_{1} x}, \quad p_{2}(x)=\frac{a_{2} x}{1+b_{2} x}, \quad l_{1}=p_{1}^{\prime}\left(x_{*}\right), \\
& l_{2}=\frac{1}{2!} p_{1}^{\prime \prime}\left(x_{*}\right), \quad l_{3}=\frac{1}{2!} p_{1}^{\prime \prime}\left(x_{*}\right) y_{1 *}, \\
& l_{4}=\frac{1}{3!} p_{1}^{\prime \prime \prime}\left(x_{*}\right), \quad l_{5}=\frac{1}{3!} p_{1}^{\prime \prime \prime}\left(x_{*}\right) y_{1 *}, \\
& l_{6}=p_{2}^{\prime}\left(y_{1 *}\right), \quad l_{7}=\frac{1}{2!} p_{2}^{\prime \prime}\left(y_{1 *}\right) y_{2 *}, \\
& l_{8}=\frac{1}{3!} p_{2}^{\prime \prime \prime}\left(y_{1 *}\right) y_{2 *}, \quad l_{9}=\frac{1}{2!} p_{2}^{\prime \prime}\left(y_{1 *}\right), \\
& l_{10}=\frac{1}{3!} p_{2}^{\prime \prime \prime}\left(y_{1 *}\right) \text {. } \tag{51}
\end{align*}
$$

Obviously, $L_{\mu}(\phi)$ is a continuous linear function mapping $\mathscr{C}\left([-1,0], \mathscr{R}^{3}\right)$ into $\mathscr{R}^{3}$. By the Riesz representation theorem, there exists a $3 \times 3$ matrix function $\eta(\theta, \mu)(-1 \leqslant \theta \leqslant 0)$, whose elements are of bounded variation such that

$$
\begin{equation*}
L_{\mu} \phi=\int_{-1}^{0} d \eta(\theta, \mu) \phi(\theta), \quad \text { for } \phi \in \mathscr{C}\left([-1,0], \mathscr{R}^{3}\right) \tag{52}
\end{equation*}
$$

In fact, we can choose

$$
\begin{equation*}
d \eta(\theta, \mu)=\left(\widetilde{\tau_{1}}+\mu\right)\left[B \delta(\theta)+C \delta(\theta+1)+D \delta\left(\theta+\frac{\tau_{2}^{*}}{\tau_{1}}\right)\right] \tag{53}
\end{equation*}
$$

where $\delta$ is Dirac-delta function. For $\phi \in \mathscr{C}\left([-1,0], \mathscr{R}^{3}\right)$, define

$$
\begin{align*}
& A(\mu) \phi= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & \theta \in[-1,0) \\
\int_{-1}^{0} d \eta(s, \mu) \phi(s), & \theta=0\end{cases}  \tag{54}\\
& R(\mu) \phi= \begin{cases}0, & \theta \in[-1,0) \\
f(\mu, \phi), & \theta=0 .\end{cases}
\end{align*}
$$

Then when $\theta=0$, the system is equivalent to

$$
\begin{equation*}
\dot{x}_{t}=A(\mu) x_{t}+R(\mu) x_{t} \tag{55}
\end{equation*}
$$

where $x_{t}(\theta)=x(t+\theta), \theta \in[-1,0]$. For $\psi \in \mathscr{C}^{1}\left([0,1],\left(\mathscr{R}^{3}\right)^{*}\right)$, define

$$
A^{*} \psi(s)= \begin{cases}-\frac{d \psi(s)}{d s}, & s \in(0,1]  \tag{56}\\ \int_{-1}^{0} d \eta^{T}(t, 0) \psi(-t), & s=0\end{cases}
$$

and a bilinear inner product

$$
\begin{equation*}
\langle\psi(s), \phi(\theta)\rangle=\bar{\psi}(0) \phi(0)-\int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi \tag{57}
\end{equation*}
$$

where $\eta(\theta)=\eta(\theta, 0)$. Let $A=A(0)$; then $A$ and $A^{*}$ are adjoint operators. By the discussion in Section 2, we know that $\pm i \omega \widetilde{\tau_{1}}$ are eigenvalues of $A$. Thus, they are also eigenvalues of $A^{*}$. We first need to compute the eigenvector of $A$ and $A^{*}$ corresponding to $i \omega \widetilde{\tau_{1}}$ and $-i \omega \widetilde{\tau_{1}}$, respectively. Suppose that $q(\theta)=(1, \alpha, \beta)^{T} e^{i \theta \omega \widetilde{\tau}_{1}}$ is the eigenvector of $A$ corresponding to $i \omega \widetilde{\tau_{1}}$. Then $A q(\theta)=i \omega \widetilde{\tau_{1}} q(\theta)$. From the definition of $A, L_{\mu}(\phi)$ and $\eta(\theta, \mu)$, we can easily obtain $q(\theta)=$ $(1, \alpha, \beta)^{T} e^{i \theta \omega \widetilde{\tau}_{1}}$, where

$$
\begin{equation*}
\alpha=\frac{i \omega-b_{11}}{b_{12}}, \quad \beta=\frac{d_{32}\left(i \omega-b_{11}\right)}{b_{12}\left(i \omega-b_{33}\right) e^{i \omega \tau_{2}^{*}}} \tag{58}
\end{equation*}
$$

and $q(0)=(1, \alpha, \beta)^{T}$. Similarly, let $q^{*}(s)=D\left(1, \alpha^{*}, \beta^{*}\right) e^{i s \omega \widetilde{\tau}_{1}}$ be the eigenvector of $A^{*}$ corresponding to $-i \omega \widetilde{\tau_{1}}$. By the definition of $A^{*}$, we can compute

$$
\begin{equation*}
\alpha^{*}=\frac{-i \omega-b_{11}}{c_{21} e^{i \omega \widetilde{\tau}_{1}}}, \quad \beta^{*}=\frac{b_{23}\left(-i \omega-b_{11}\right)}{c_{21}\left(i \omega-b_{33}\right) e^{i \omega \widetilde{\tau}_{1}}} \tag{59}
\end{equation*}
$$

From (57), we have

$$
\begin{align*}
& \left\langle q^{*}(s), q(\theta)\right\rangle \\
& =\bar{D}\left(1, \bar{\alpha}^{*}, \bar{\beta}^{*}\right)(1, \alpha, \beta)^{T} \\
& -\quad \int_{-1}^{0} \int_{\xi=0}^{\theta} \bar{D}\left(1, \bar{\alpha}^{*}, \bar{\beta}^{*}\right) e^{-i \omega \widetilde{\tau}_{1}(\xi-\theta)} d \eta(\theta) \\
& \quad \times(1, \alpha, \beta)^{T} e^{i \omega \widetilde{\omega}_{1} \xi} d \xi \\
& =\bar{D}\left\{1+\alpha \bar{\alpha}^{*}+\beta \bar{\beta}^{*}+c_{21} \bar{\alpha}^{*} \widetilde{\tau}_{1} e^{-i \omega \widetilde{\tau}_{1}}+d_{32} \alpha \bar{\beta}^{*} \tau_{2}^{*} e^{-i \omega \tau_{2}^{*}}\right\} \tag{60}
\end{align*}
$$

Thus, we can choose

$$
\begin{equation*}
\bar{D}=\left\{1+\alpha \bar{\alpha}^{*}+\beta \bar{\beta}^{*}+c_{21} \bar{\alpha}^{*} \widetilde{\tau}_{1} e^{-i \omega \widetilde{\tau}_{1}}+d_{32} \alpha \bar{\beta}^{*} \tau_{2}^{*} e^{-i \omega \tau_{2}^{*}}\right\}^{-1} \tag{61}
\end{equation*}
$$

such that $\left\langle q^{*}(s), q(\theta)\right\rangle=1,\left\langle q^{*}(s), \bar{q}(\theta)\right\rangle=0$.
In the remainder of this section, we follow the ideas in Hassard et al. [23] and use the same notations as there to compute the coordinates describing the center manifold $C_{0}$ at $\mu=0$. Let $x_{t}$ be the solution of (48) when $\mu=0$. Define

$$
\begin{equation*}
z(t)=\left\langle q^{*}, x_{t}\right\rangle, \quad W(t, \theta)=x_{t}(\theta)-2 \operatorname{Re}\{z(t) q(\theta)\} . \tag{62}
\end{equation*}
$$

On the center manifold $C_{0}$, we have

$$
\begin{align*}
W(t, \theta)= & W(z(t), \bar{z}(t), \theta) \\
= & W_{20}(\theta) \frac{z^{2}}{2}+W_{11}(\theta) z \bar{z}+W_{02}(\theta) \frac{\bar{z}^{2}}{2}  \tag{63}\\
& +W_{30}(\theta) \frac{z^{3}}{6}+\cdots
\end{align*}
$$

where $z$ and $\bar{z}$ are local coordinates for center manifold $C_{0}$ in the direction of $q$ and $\bar{q}$. Note that $W$ is real if $x_{t}$ is real. We consider only real solutions. For the solution $x_{t} \in C_{0}$ of (48), since $\mu=0$, we have

$$
\begin{align*}
\dot{z}= & i \omega \widetilde{\tau}_{1} z+\left\langle q^{*}(\theta), f(0, W(z(t), \bar{z}(t), \theta)\right. \\
& +2 \operatorname{Re}\{z(t) q(\theta)\})\rangle \\
= & i \omega \widetilde{\tau}_{1} z+\bar{q}^{*}(0) f(0, W(z(t), \bar{z}(t), 0)+2 \operatorname{Re}\{z(t) q(0)\})  \tag{64}\\
= & i \omega \widetilde{\tau_{1}} z+\bar{q}^{*}(0) f_{0}(z, \bar{z}) \triangleq i \omega \widetilde{\tau_{1}} z+g(z, \bar{z})
\end{align*}
$$

where

$$
\begin{align*}
g(z, \bar{z})= & \bar{q}^{*}(0) f_{0}(z, \bar{z})=g_{20} \frac{z^{2}}{2}+g_{11} z \bar{z}  \tag{65}\\
& +g_{02} \frac{\bar{z}^{2}}{2}+g_{21} \frac{z^{2} \bar{z}}{2}+\cdots .
\end{align*}
$$

By (62), we have $x_{t}(\theta)=\left(x_{1 t}(\theta), x_{2 t}(\theta), x_{3 t}(\theta)\right)^{T}=W(t, \theta)+$ $z q(\theta)+\bar{z} \bar{q}(\theta)$, and then

$$
\begin{aligned}
x_{1 t}(0)= & z+\bar{z}+W_{20}^{(1)}(0) \frac{z^{2}}{2}+W_{11}^{(1)}(0) z \bar{z} \\
& +W_{02}^{(1)}(0) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right) \\
x_{2 t}(0)= & z \alpha+\bar{z} \bar{\alpha}+W_{20}^{(2)}(0) \frac{z^{2}}{2} \\
& +W_{11}^{(2)}(0) z \bar{z}+W_{02}^{(2)}(0) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)
\end{aligned}
$$

$$
x_{3 t}(0)=z \beta+\bar{z} \bar{\beta}+W_{20}^{(3)}(0) \frac{z^{2}}{2}+W_{11}^{(3)}(0) z \bar{z}
$$

$$
+W_{02}^{(3)}(0) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
x_{1 t}(-1)=z e^{-i \omega \widetilde{\tau_{1}}}+\bar{z} e^{i \omega \widetilde{\tau_{1}}}+W_{20}^{(1)}(-1) \frac{z^{2}}{2}
$$

$$
+W_{11}^{(1)}(-1) z \bar{z}+W_{02}^{(1)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
x_{2 t}(-1)=z \alpha e^{-i \omega \widetilde{\tau}_{1}}+\bar{z} \bar{\alpha} e^{i \omega \widetilde{\tau}_{1}}+W_{20}^{(2)}(-1) \frac{z^{2}}{2}
$$

$$
+W_{11}^{(2)}(-1) z \bar{z}+W_{02}^{(2)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
x_{3 t}(-1)=z \beta e^{-i \omega{\widetilde{\tau_{1}}}}+\bar{z} \bar{\beta} e^{\omega \widetilde{\tau_{1}}}+W_{20}^{(3)}(-1) \frac{z^{2}}{2}
$$

$$
+W_{11}^{(3)}(-1) z \bar{z}+W_{02}^{(3)}(-1) \frac{\bar{z}^{2}}{2}+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
x_{1 t}\left(-\frac{\tau_{2}^{*}}{\tau_{10}}\right)=z e^{-i \omega \tau_{2}^{*}}+\bar{z} e^{i \omega \tau_{2}^{*}}+W_{20}^{(1)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{z^{2}}{2}
$$

$$
+W_{11}^{(1)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) z \bar{z}+W_{02}^{(1)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{\bar{z}^{2}}{2}
$$

$$
+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
x_{2 t}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right)=z \alpha e^{-i \omega \tau_{2}^{*}}+\bar{z} \bar{\alpha} e^{i \omega \tau_{2}^{*}}+W_{20}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{z^{2}}{2}
$$

$$
+W_{11}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) z \bar{z}+W_{02}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{\bar{z}^{2}}{2}
$$

$$
+o\left(|(z, \bar{z})|^{3}\right)
$$

$$
\begin{align*}
x_{3 t}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right)= & z \beta e^{-i \omega \tau_{2}^{*}}+\bar{z} \bar{\beta} e^{\omega \tau_{2}^{*}}+W_{20}^{(3)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{z^{2}}{2} \\
& +W_{11}^{(3)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) z \bar{z}+W_{02}^{(3)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) \frac{\bar{z}^{2}}{2} \\
& +o\left(|(z, \bar{z})|^{3}\right) \tag{66}
\end{align*}
$$

It follows together with (50) that

$$
\begin{align*}
& g(z, \bar{z})=\overline{q^{*}}(0) f_{0}(z, \bar{z})=\bar{D} \widetilde{\tau}_{1}\left(1, \bar{\alpha}^{*}, \bar{\beta}^{*}\right)\left(\begin{array}{ll}
\left.f_{1}^{(0)} f_{2}^{(0)} f_{3}^{(0)}\right)^{T} \\
=\bar{D} \widetilde{\tau_{1}}\{ & {\left[-\left(a_{11}+l_{3}\right) \phi_{1}^{2}(0)-l_{1} \phi_{1}(0) \phi_{2}(0)\right.} \\
& -l_{2} \phi_{1}^{2}(0) \phi_{2}(0)-l_{5} \phi_{1}^{3}(0)-l_{4} \phi_{1}^{3}(0) \phi_{2}(0) \\
+\cdots]
\end{array}\right. \\
&+\bar{\alpha}^{*}\left[l_{6} \phi_{2}(0) \phi_{3}(0)\left(l_{7}+a_{22}\right) \phi_{2}^{2}(0)\right. \\
&+l_{1} \phi_{1}(-1) \phi_{2}(0) \\
&+l_{3} \phi_{1}^{2}(-1)-l_{8} \phi_{2}^{3}(0)-l_{9} \phi_{2}^{2}(0) \phi_{3}(0) \\
&\left.+l_{2} \phi_{1}^{2}(-1) \phi_{2}(0)+l_{3} \phi_{1}^{3}(-1)+\cdots\right] \\
&+\bar{\beta}^{*} {\left[l_{6} \phi_{2}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau}_{1}}\right) \phi_{3}(0)+l_{7} \phi_{2}^{2}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau}_{1}}\right)\right.} \\
&\left.\left.\quad-a_{33} \phi_{3}^{2}(0)+l_{8} \phi_{2}^{3}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right)+\cdots\right]\right\}
\end{align*}
$$

Comparing the coefficients with (65), we have

$$
\begin{aligned}
g_{20}=\bar{D} \widetilde{\tau_{1}}\{ & \left\{-2\left(a_{11}+l_{3}\right)-2 \alpha l_{1}\right] \\
& +\bar{\alpha}^{*}\left[2 l_{1} \alpha e^{-i \omega \widetilde{\tau_{1}}}+2 l_{3} e^{-2 i \omega \widetilde{\tau_{1}}}\right. \\
& \left.-2 l_{6} \alpha \beta-2\left(l_{7}+a_{22}\right) \alpha^{2}\right] \\
+ & \bar{\beta}^{*}\left[2 l_{6} \alpha \beta e^{-i \omega \tau_{2}^{*}}+2\left(l_{7}+a_{22}\right) \alpha^{2} e^{-2 i \omega \tau_{2}^{*}}\right. \\
& \left.\left.-2 a_{33} \beta^{2}\right]\right\}
\end{aligned}
$$

$$
g_{11}=\bar{D} \widetilde{\tau}_{1}\left\{\left[-2\left(a_{11}+l_{3}\right)-l_{1}(\alpha+\bar{\alpha})\right]\right.
$$

$$
+\bar{\alpha}^{*}\left[l_{1}\left(\alpha e^{i \omega \widetilde{\tau_{1}}}+\bar{\alpha} e^{-i \omega \widetilde{\tau}_{1}}\right)-l_{6}(\bar{\alpha} \beta+\alpha \bar{\beta})\right.
$$

$$
\left.-2\left(l_{7}+a_{22}\right) \alpha \bar{\alpha}+2 l_{3}\right]
$$

$$
+\bar{\beta}^{*}\left[l_{6}\left(\beta \bar{\alpha} e^{i \omega \tau_{2}^{*}}+\alpha \bar{\beta} e^{-i \omega \tau_{2}^{*}}\right)\right.
$$

$$
\left.\left.+2 l_{7} \alpha \bar{\alpha}-a_{33} \beta \bar{\beta}\right]\right\}
$$

$$
g_{02}=2 \bar{D} \widetilde{\tau}_{1}\left\{\left[-2\left(a_{11}+l_{3}\right)-2 l_{1} \bar{\alpha}\right]\right.
$$

$$
+\bar{\alpha}^{*}\left[-2 l_{6} \bar{\alpha} \bar{\beta}-2\left(l_{7}+a_{22}\right) \bar{\alpha}^{2}+2 l_{1} \bar{\alpha} e^{i \omega \widetilde{\tau_{1}}}\right.
$$

$$
\left.+2 l_{3} e^{2 i \omega \widetilde{\tau_{1}}}\right]
$$

$$
\left.+\bar{\beta}^{*}\left[2 l_{6} \bar{\alpha} \bar{\beta} e^{i \omega \tau_{2}^{*}}+2 l_{7} \bar{\alpha}^{2} e^{2 i \omega \tau_{2}^{*}}-a_{33} \bar{\beta}^{2}\right]\right\}
$$

$$
\begin{align*}
& g_{21}=\bar{D} \widetilde{\tau_{1}}\left\{\left[-\left(a_{11}+l_{3}\right)\left(2 W_{20}^{(1)}(0)+4 W_{11}^{(1)}(0)\right)\right.\right. \\
&-l_{1}\left(2 \alpha W_{11}^{(1)}(0)+\bar{\alpha} W_{20}^{(1)}(0)+W_{20}^{(2)}(0)\right. \\
&\left.\left.+2 W_{11}^{(2)}(0)\right)\right] \\
&+\bar{\alpha}^{*}[ -l_{6}\left(2 \beta W_{11}^{(2)}(0)+\bar{\alpha} W_{20}^{(3)}(0)+\bar{\beta} W_{20}^{(2)}(0)\right. \\
&\left.+2 \alpha W_{11}^{(3)}(0)\right)-\left(l_{7}+a_{22}\right) \\
& \times\left(4 \alpha W_{11}^{(2)}(0)+2 \bar{\alpha} W_{20}^{(2)}(0)\right) \\
&+l_{1}\left(2 \alpha W_{11}^{(1)}(-1)+\bar{\alpha} W_{20}^{(1)}(-1)\right. \\
&\left.+W_{20}^{(2)}(0) e^{i \omega \widetilde{\tau}_{1}}+2 W_{11}^{(2)}(0) e^{-i \omega \widetilde{\tau}_{1}}\right) \\
&\left.+l_{3}\left(4 W_{11}^{(1)}(-1) e^{-i \omega \widetilde{\tau}_{1}}+2 W_{20}^{(1)}(-1) e^{i \omega \widetilde{\tau}_{1}}\right)\right] \\
&+\bar{\beta}^{*} {\left[l _ { 6 } \left(2 \beta W_{11}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right)+\bar{\beta} W_{20}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right)\right.\right.} \\
&\left.+\bar{\alpha} W_{20}^{(3)}(0) e^{i \omega \tau_{2}^{*}}+2 \alpha W_{11}^{(3)}(0) e^{-i \omega \tau_{2}^{*}}\right) \\
&+l_{7}\left(4 \alpha W_{11}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) e^{-i \omega \tau_{2}^{*}}\right. \\
&\left.+2 \bar{\alpha} W_{20}^{(2)}\left(-\frac{\tau_{2}^{*}}{\widetilde{\tau_{1}}}\right) e^{i \omega \tau_{2}^{*}}\right) \\
&\left.\left.\quad-a_{33}\left(4 \beta W_{11}^{(3)}(0)+2 \bar{\beta} W_{20}^{(3)}(0)\right)\right]\right\} \tag{68}
\end{align*}
$$

where

$$
\begin{aligned}
& W_{20}(\theta)=\frac{i g_{20}}{\omega \widetilde{\tau_{1}}} q(0) e^{i \omega \widetilde{\tau}_{1} \theta}+\frac{i \bar{g}_{02}}{3 \omega \widetilde{\tau}_{1}} \bar{q}(0) e^{-i \omega \widetilde{\tau}_{1} \theta}+E_{1} e^{2 i \omega \widetilde{\tau}_{1} \theta} \\
& W_{11}(\theta)=-\frac{i g_{11}}{\omega \widetilde{\tau}_{1}} q(0) e^{i \omega \widetilde{\tau}_{1} \theta}+\frac{i \bar{g}_{11}}{\omega \widetilde{\tau}_{1}} \bar{q}(0) e^{-i \omega \widetilde{\tau}_{1} \theta}+E_{2} \\
& E_{1}=2\left[\begin{array}{ccc}
2 i \omega-b_{11} & -b_{12} & 0 \\
-c_{21} e^{-2 i \omega \widetilde{\tau}_{1}} & 2 i \omega-b_{22} & -b_{23} \\
0 & -d_{32} e^{-2 i \omega \tau_{2}^{*}} & 2 i \omega-b_{33}
\end{array}\right]^{-1} \\
& \quad \times\left[\begin{array}{c}
-2\left(a_{11}+l_{3}\right)-2 \alpha l_{1} \\
2 l_{1} \alpha e^{-i \omega \widetilde{\tau}_{1}}+2 l_{3} e^{-2 i \omega \widetilde{\tau}_{1}}-2 l_{6} \alpha \beta-2\left(l_{7}+a_{22}\right) \alpha^{2} \\
2 l_{6} \alpha \beta e^{-i \omega \tau_{2}^{*}}+2\left(l_{7}+a_{22}\right) \alpha^{2} e^{-2 i \omega \tau_{2}^{*}}-2 a_{33} \beta^{2}
\end{array}\right]
\end{aligned}
$$

$$
\begin{align*}
E_{2}= & 2\left[\begin{array}{ccc}
-b_{11} & -b_{12} & 0 \\
-c_{21} & -b_{22} & -b_{23} \\
0 & -d_{32} & -b_{33}
\end{array}\right]^{-1} \\
& \times\left[\begin{array}{c}
-2\left(a_{11}+l_{3}\right)-l_{1}(\alpha+\bar{\alpha}) \\
l_{1}\left(\alpha e^{i \omega \widetilde{\tau}_{1}}+\bar{\alpha} e^{-i \omega \widetilde{\tau}_{1}}\right)-l_{6}(\bar{\alpha} \beta+\alpha \bar{\beta})-2\left(l_{7}+a_{22}\right) \alpha \bar{\alpha}+2 l_{3} \\
l_{6}\left(\beta \bar{\alpha} e^{i \omega \tau_{2}^{*}}+\alpha \bar{\beta} e^{-i \omega \tau_{2}^{*}}\right)+2 l_{7} \alpha \bar{\alpha}-a_{33} \beta \bar{\beta}
\end{array}\right] . \tag{69}
\end{align*}
$$

Thus, we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$. Furthermore, we can determine each $g_{i j}$ by the parameters and delay in (3). Thus, we can compute the following values:

$$
\begin{align*}
& c_{1}(0)=\frac{i}{2 \omega \widetilde{\tau_{1}}}\left(g_{20} g_{11}-2\left|g_{11}\right|^{2}-\frac{1}{3}\left|g_{02}\right|^{2}\right)+\frac{1}{2} g_{21} \\
& \mu_{2}=-\frac{\operatorname{Re}\left\{c_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\widetilde{\tau_{1}}\right)\right\}} \\
& T_{2}=-\frac{\operatorname{Im}\left\{c_{1}(0)\right\}+\mu_{2} \operatorname{Im}\left\{\lambda^{\prime}\left(\widetilde{\tau_{1}}\right)\right\}}{\omega \widetilde{\tau_{1}}}, \quad \beta_{2}=2 \operatorname{Re}\left\{c_{1}(0)\right\}, \tag{70}
\end{align*}
$$

which determine the quantities of bifurcating periodic solutions in the center manifold at the critical value $\widetilde{\tau_{1}}$. Suppose $\operatorname{Re}\left\{\lambda^{\prime}\left(\widetilde{\tau_{1}}\right)\right\}>0 . \mu_{2}$ determines the directions of the Hopf bifurcation: if $\mu_{2}>0(<0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcation exists for $\tau>\widetilde{\tau_{1}}\left(<\widetilde{\tau_{1}}\right) ; \beta_{2}$ determines the stability of the bifurcation periodic solutions: the bifurcating periodic solutions are stable (unstable) if $\beta_{2}<0(>0)$; and $T_{2}$ determines the period of the bifurcating periodic solutions: the period increases (decreases) if $T_{2}>0(<0)$.

## 4. Numerical Simulation

We consider system (3) by taking the following coefficients: $a_{1}=0.3, a_{11}=5.8889, a_{12}=1, m_{1}=1, a_{2}=$ $0.1, a_{21}=27, a_{22}=12, a_{23}=12, m_{2}=1, a_{3}=$ $0.2, a_{32}=25, a_{33}=12$. We have the unique positive equilibrium $E_{*}=(0.0451,0.0357,0.0551)$.

By computation, we get $m_{10}=-0.0104, \omega_{11}=0.5164$, $z_{11}=0.2666, h_{1}^{\prime}\left(z_{11}\right)=0.3402, \tau_{2_{0}}=3.2348$. From Theorem 6, we know that when $\tau_{1}=0$, the positive equilibrium $E_{*}$ is locally asymptotically stable for $\tau_{2} \in$ $[0,3.2348)$. When $\tau_{2}$ crosses $\tau_{2_{0}}$, the equilibrium $E_{*}$ loses its stability and Hopf bifurcation occurs. From the algorithm in Section 3, we have $\mu_{2}=566.46, \beta_{2}=-315.83, T_{2}=54.45$, which means that the bifurcation is supercritical and periodic solution is stable. The trajectories and the phase graphs are shown in Figures 1 and 2.

Regarding $\tau_{1}$ as a parameter and let $\tau_{2}=2.9 \epsilon$ $[0,3.2348)$, we can observe that with $\tau_{1}$ increasing, the positive equilibrium $E_{*}$ loses its stability and Hopf bifurcation occurs (see Figures 3 and 4).

## 5. Global Continuation of Local Hopf Bifurcations

In this section, we study the global continuation of periodic solutions bifurcating from the positive equilibrium $\left(E_{*}, \tau_{1 k}^{(j)}\right),(k=1,2, \ldots, 6 ; j=0,1, \ldots)$. Throughout this section, we follow closely the notations in [24] and assume that $\tau_{2}=\tau_{2}^{*} \in\left[0, \tau_{2_{10}}\right.$ ) regarding $\tau_{1}$ as a parameter. For simplification of notations, setting $z_{t}(t)=\left(x_{1 t}, x_{2 t}, x_{3 t}\right)^{T}$, we may rewrite system (3) as the following functional differential equation:

$$
\begin{equation*}
\dot{z}(t)=F\left(z_{t}, \tau_{1}, p\right), \tag{71}
\end{equation*}
$$

where $z_{t}(\theta)=\left(x_{1 t}(\theta), x_{2 t}(\theta), x_{3 t}(\theta)\right)^{T}=\left(x_{1}(t+\theta), x_{2}(t+\right.$ $\left.\theta), x_{3}(t+\theta)\right)^{T}$ for $t \geq 0$ and $\theta \in\left[-\tau_{1}, 0\right]$. Since $x_{1}(t), x_{2}(t)$, and $x_{3}(t)$ denote the densities of the prey, the predator, and the top predator, respectively; the positive solution of system (3) is of interest and its periodic solutions only arise in the first quadrant. Thus, we consider system (3) only in the domain $R_{+}^{3}=\left\{\left(x_{1}, x_{2}, x_{3}\right) \in R^{3}, x_{1}>0, x_{2}>0, x_{3}>0\right\}$. It is obvious that (71) has a unique positive equilibrium $E_{*}\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right)$ in $R_{+}^{3}$ under the assumption $\left(H_{1}\right)-\left(H_{4}\right)$. Following the work of [24], we need to define

$$
\begin{align*}
& X=C\left(\left[-\tau_{1}, 0\right], R_{+}^{3}\right) \\
& \Gamma=C l\left\{\left(z, \tau_{1}, p\right) \in \mathbf{X} \times \mathbf{R} \times \mathbf{R}^{+} ;\right. \\
& \quad z \text { is a } p \text {-periodic solution of system (71) }\},  \tag{72}\\
& \mathcal{N}=\left\{\left(\bar{z}, \overline{\tau_{1}}, \bar{p}\right) ; F\left(\bar{z}, \overline{\tau_{1}}, \bar{p}\right)=0\right\} .
\end{align*}
$$

Let $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{1 k}^{(j)}\right)}$ denote the connected component passing through $\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{1 k}^{(j)}\right)$ in $\Gamma$, where $\tau_{1 k}^{(j)}$ is defined by (43). We know that $\ell_{\left(E^{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{1 k}^{(j)}\right)}$ through $\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{1 k}^{(j)}\right)$ is nonempty.

For the benefit of readers, we first state the global Hopf bifurcation theory due to Wu [24] for functional differential equations.

Lemma 9. Assume that $\left(z_{*}, \tau, p\right)$ is an isolated center satisfying the hypotheses (A1)-(A4) in [24]. Denote by $\ell_{\left(z_{*}, \tau, p\right)}$ the connected component of $\left(z_{*}, \tau, p\right)$ in $\Gamma$. Then either
(i) $\ell_{\left(z_{*}, \tau, p\right)}$ is unbounded, or
(ii) $\ell_{\left(z_{*}, \tau, p\right)}$ is bounded, $\ell_{\left(z_{*}, \tau, p\right)} \cap \Gamma$ is finite and

$$
\begin{equation*}
\sum_{\left.(z, \tau, p) \in \ell_{(z *}, \tau, p\right)} \gamma_{m}\left(z_{*}, \tau, p\right)=0 \tag{73}
\end{equation*}
$$

for all $m=1,2, \ldots$, where $\gamma_{m}\left(z_{*}, \tau, p\right)$ is the mth crossing number of $\left(z_{*}, \tau, p\right)$ if $m \in J\left(z_{*}, \tau, p\right)$, or it is zero if otherwise.

Clearly, if (ii) in Lemma 9 is not true, then $\ell_{\left(z_{*}, \tau, p\right)}$ is unbounded. Thus, if the projections of $\ell_{\left(z_{*}, \tau, p\right)}$ onto $z$-space and onto $p$-space are bounded, then the projection of $\ell_{\left(z_{*}, \tau, p\right)}$ onto $\tau$-space is unbounded. Further, if we can show that the


Figure 1: The trajectories and the phase graph with $\tau_{1}=0, \tau_{2}=2.9<\tau_{2_{0}}=3.2348 ; E_{*}$ is locally asymptotically stable.


Figure 2: The trajectories and the phase graph with $\tau_{1}=0, \tau_{2}=3.3>\tau_{2_{0}}=3.2348$; a periodic orbit bifurcate from $E_{*}$.


Figure 3: The trajectories and the phase graph with $\tau_{1}=0.9, \tau_{2}=2.9 ; E_{*}$ is locally asymptotically stable.


Figure 4: The trajectories and the phase graph with $\tau_{1}=1.2, \tau_{2}=2.9$; a periodic orbit bifurcate from $E_{*}$.
projection of $\ell_{\left(z_{*}, \tau, p\right)}$ onto $\tau$-space is away from zero, then the projection of $\ell_{\left(z_{*}, \tau, p\right)}$ onto $\tau$-space must include interval $[\tau, \infty)$. Following this ideal, we can prove our results on the global continuation of local Hopf bifurcation.

Lemma 10. If the conditions $\left(H_{1}\right)-\left(H_{4}\right)$ hold, then all nontrivial periodic solutions of system (71) with initial conditions

$$
\begin{align*}
& x_{1}(\theta)=\varphi(\theta) \geq 0, \quad x_{2}(\theta)=\psi(\theta) \geq 0 \\
& x_{3}(\theta)=\phi(\theta) \geq 0, \quad \theta \in\left[-\tau_{1}, 0\right)  \tag{74}\\
& \varphi(0)>0, \quad \psi(0)>0, \quad \phi(0)>0
\end{align*}
$$

## are uniformly bounded.

Proof. Suppose that $\left(x_{1}(t), x_{2}(t), x_{3}(t)\right)$ are nonconstant periodic solutions of system (3) and define

$$
\begin{array}{ll}
x_{1}\left(\xi_{1}\right)=\min \left\{x_{1}(t)\right\}, & x_{1}\left(\eta_{1}\right)=\max \left\{x_{1}(t)\right\}, \\
x_{2}\left(\xi_{2}\right)=\min \left\{x_{2}(t)\right\}, & x_{2}\left(\eta_{2}\right)=\max \left\{x_{2}(t)\right\},  \tag{75}\\
x_{3}\left(\xi_{3}\right)=\min \left\{x_{3}(t)\right\}, & x_{3}\left(\eta_{3}\right)=\max \left\{x_{3}(t)\right\} .
\end{array}
$$

It follows from system (3) that

$$
\begin{align*}
& x_{1}(t)=x_{1}(0) \exp \left\{\int_{0}^{t}\right. {\left.\left[a_{1}-a_{11} x_{1}(s)-\frac{a_{12} x_{2}(s)}{m_{1}+x_{1}(s)}\right] d s\right\} } \\
& x_{2}(t)=x_{2}(0) \exp \left\{\int_{0}^{t}\right. {\left[-a_{2}+\frac{a_{21} x_{1}\left(s-\tau_{1}\right)}{m_{1}+x_{1}\left(s-\tau_{1}\right)}\right.} \\
&\left.\left.-a_{22} x_{2}(s)-\frac{a_{23} x_{3}(s)}{m_{2}+x_{2}(s)}\right] d s\right\} \\
& \begin{aligned}
x_{3}(t)=x_{3}(0) \exp \left\{\int _ { 0 } ^ { t } \left[-a_{3}+\frac{a_{32} x_{2}\left(s-\tau_{2}^{*}\right)}{m_{2}+x_{2}\left(s-\tau_{2}^{*}\right)}\right.\right.
\end{aligned} \\
&\left.\left.-a_{33} x_{3}(s)\right] d s\right\} \tag{76}
\end{align*}
$$

which implies that the solutions of system (3) cannot cross the $x_{i}$-axis $(i=1,2,3)$. Thus, the nonconstant periodic orbits must be located in the interior of first quadrant. It follows from initial data of system (3) that $x_{1}(t)>0, x_{2}(t)>$ $0, x_{3}(t)>0$ for $t \geq 0$.

From the first equation of system (3), we can get

$$
\begin{equation*}
0=a_{1}-a_{11} x_{1}\left(\eta_{1}\right)-\frac{a_{12} x_{2}\left(\eta_{1}\right)}{m_{1}+x_{1}\left(\eta_{1}\right)} \leq a_{1}-a_{11} x_{1}\left(\eta_{1}\right) \tag{77}
\end{equation*}
$$

thus, we have

$$
\begin{equation*}
x_{1}\left(\eta_{1}\right) \leq \frac{a_{1}}{a_{11}} \tag{78}
\end{equation*}
$$

From the second equation of (3), we obtain

$$
\begin{align*}
0= & -a_{2}+\frac{a_{21} x_{1}\left(\eta_{2}-\tau_{1}\right)}{m_{1}+x_{1}\left(\eta_{2}-\tau_{1}\right)}-a_{22} x_{2}\left(\eta_{2}\right) \\
& -\frac{a_{23} x_{3}\left(\eta_{2}\right)}{m_{2}+x_{2}\left(\eta_{2}\right)} \leq-a_{2}+\frac{a_{21}\left(a_{1} / a_{11}\right)}{m_{1}+\left(a_{1} / a_{11}\right)}-a_{22} x_{2}\left(\eta_{2}\right) \tag{79}
\end{align*}
$$

therefore, one can get

$$
\begin{equation*}
x_{2}\left(\eta_{2}\right) \leq \frac{-a_{2}\left(a_{11} m_{1}+a_{1}\right)+a_{1} a_{21}}{a_{22}\left(a_{11} m_{1}+a_{1}\right)} \triangleq M_{1} . \tag{80}
\end{equation*}
$$

Applying the third equation of system (3), we know

$$
\begin{align*}
0 & =-a_{3}+\frac{a_{32} x_{2}\left(\eta_{3}-\tau_{2}^{*}\right)}{m_{2}+x_{2}\left(\eta_{3}-\tau_{2}^{*}\right)}-a_{33} x_{3}\left(\eta_{3}\right)  \tag{81}\\
& \leq-a_{3}+\frac{a_{32} M_{1}}{m_{2}+M_{1}}-a_{33} x_{3}\left(\eta_{3}\right)
\end{align*}
$$

It follows that

$$
\begin{equation*}
x_{3}\left(\eta_{3}\right) \leq \frac{-a_{3}\left(m_{2}+M_{1}\right)+a_{32} M_{1}}{a_{33}\left(m_{2}+M_{1}\right)} \triangleq M_{2} \tag{82}
\end{equation*}
$$

This shows that the nontrivial periodic solution of system (3) is uniformly bounded and the proof is complete.

Lemma 11. If the conditions $\left(H_{1}\right)-\left(H_{4}\right)$ and

$$
\begin{aligned}
& \left(H_{7}\right) a_{22}-\left(a_{21} / m_{2}\right) \quad>\quad 0,\left[\left(a_{11}-a_{1} / m_{1}\right) a_{22}-\right. \\
& \left.\quad\left(a_{12} a_{21} / m_{1}^{2}\right)\right] a_{33}-\left(a_{11}-\left(a_{1} / m_{1}\right)\right)\left[\left(a_{21} / m_{2}\right) a_{33}+\right. \\
& \left.\quad\left(a_{23} a_{32} / m_{2}^{2}\right)\right]>0
\end{aligned}
$$

hold, then system (3) has no nontrivial $\tau_{1}$-periodic solution.
Proof. Suppose for a contradiction that system (3) has nontrivial periodic solution with period $\tau_{1}$. Then the following system (83) of ordinary differential equations has nontrivial periodic solution:

$$
\begin{align*}
& \frac{d x_{1}}{d t}=x_{1}(t)\left[a_{1}-a_{11} x_{1}(t)-\frac{a_{12} x_{2}(t)}{m_{1}+x_{1}(t)}\right] \\
& \frac{d x_{2}}{d t}=x_{2}(t)\left[-a_{2}+\frac{a_{21} x_{1}(t)}{m_{1}+x_{1}(t)}-a_{22} x_{2}(t)-\frac{a_{23} x_{3}(t)}{m_{2}+x_{2}(t)}\right] \\
& \frac{d x_{3}}{d t}=x_{3}(t)\left[-a_{3}+\frac{a_{32} x_{2}\left(t-\tau_{2}^{*}\right)}{m_{2}+x_{2}\left(t-\tau_{2}^{*}\right)}-a_{33} x_{3}(t)\right] \tag{83}
\end{align*}
$$

which has the same equilibria to system (3); that is,

$$
\begin{array}{ll}
E_{1}=(0,0,0), & E_{2}=\left(\frac{a_{1}}{a_{11}}, 0,0\right)  \tag{84}\\
E_{3}=\left(\tilde{x}_{1}, \tilde{x}_{2}, 0\right), & E_{*}=\left(x_{1}^{*}, x_{2}^{*}, x_{3}^{*}\right) .
\end{array}
$$

Note that $x_{i}$-axis $(i=1,2,3)$ are the invariable manifold of system (83) and the orbits of system (83) do not intersect each
other. Thus, there are no solutions crossing the coordinate axes. On the other hand, note the fact that if system (83) has a periodic solution, then there must be the equilibrium in its interior, and that $E_{1}, E_{2}, E_{3}$ are located on the coordinate axis. Thus, we conclude that the periodic orbit of system (83) must lie in the first quadrant. If $\left(H_{7}\right)$ holds, it is well known that the positive equilibrium $E_{*}$ is globally asymptotically stable in the first quadrant (see [1]). Thus, there is no periodic orbit in the first quadrant too. The above discussion means that (83) does not have any nontrivial periodic solution. It is a contradiction. Therefore, the lemma is confirmed.

Theorem 12. Suppose the conditions of Theorem 8 and $\left(H_{7}\right)$ hold; let $\omega_{k}$ and $\tau_{1 k}^{(j)}$ be defined in Section 2; then when $\tau_{1}>\tau_{1 k}^{(j)}$ system (3) has at least $j-1$ periodic solutions.

Proof. It is sufficient to prove that the projection of $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ onto $\tau_{1}$-space is $\left[\bar{\tau}_{1},+\infty\right)$ for each $j \geq 1$, where $\bar{\tau}_{1} \leq \tau_{1 k}^{(j)}$.

In following we prove that the hypotheses (A1)-(A4) in [24] hold.
(1) From system (3) we know easily that the following conditions hold:
(A1) $\widehat{F} \in C^{2}\left(R_{+}^{3} \times R_{+} \times R_{+}\right)$, where $\widehat{F}=\left.F\right|_{R_{+}^{3} \times R_{+} \times R_{+}} \rightarrow$ $R_{+}^{3}$.
(A3) $F\left(\phi, \tau_{1}, p\right)$ is differential with respect to $\phi$.
(2) It follows from system (3) that

$$
D_{z} \widehat{F}\left(z, \tau_{1}, p\right)=\left[\begin{array}{ccc}
a_{1}-2 a_{11} x_{1}-\frac{a_{12} m_{1} x_{2}}{\left(m_{1}+x_{1}\right)^{2}} & -\frac{a_{12} x_{1}}{m_{1}+x_{1}} & 0  \tag{85}\\
\frac{a_{21} m_{1} x_{2}}{\left(m_{1}+x_{1}\right)^{2}} & -a_{2}+\frac{a_{21} x_{1}}{m_{1}+x_{1}}-2 a_{22} x_{2}-\frac{a_{23} m_{2} x_{3}}{\left(m_{2}+x_{2}\right)^{2}} & -\frac{a_{23} x_{2}}{m_{2}+x_{2}} \\
0 & \frac{a_{32} m_{2} x_{3}}{\left(m_{2}+x_{2}\right)^{2}} & -a_{3}+\frac{a_{32} x_{2}}{m_{2}+x_{2}}-2 a_{33} x_{3}
\end{array}\right] .
$$

Then under the assumption $\left(H_{1}\right)-\left(H_{4}\right)$, we have

$$
\begin{align*}
& \operatorname{det} D_{z} \widehat{F}\left(z^{*}, \tau_{1}, p\right) \\
& =\operatorname{det}\left[\begin{array}{ccc}
-a_{11} x_{1}^{*}+\frac{a_{12} x_{1}^{*} x_{2}^{*}}{\left(m_{1}+x_{1}^{*}\right)^{2}} & \frac{a_{12} x_{1}^{*}}{m_{1}+x_{1}^{*}} & 0 \\
\frac{a_{21} m_{1} x_{2}^{*}}{\left(m_{1}+x_{1}^{*}\right)^{2}} & -a_{22} x_{2}^{*}+\frac{a_{23} x_{2}^{*} x_{3}^{*}}{\left(m_{2}+x_{2}^{*}\right)^{2}} & -\frac{a_{23} x_{2}^{*}}{m_{2}+x_{2}^{*}} \\
0 & \frac{a_{32} m_{2} x_{3}^{*}}{\left(m_{2}+x_{2}^{*}\right)^{2}} & -a_{33} x_{3}^{*}
\end{array}\right] \\
& =-\frac{a_{2}^{2} y_{1 *} y_{2 *}}{\left(1+b_{2} y_{1 *}\right)^{3}}\left[-x_{*}+\frac{a_{1} b_{1} x_{*} y_{1 *}}{\left(1+b_{1} x_{*}\right)^{2}}\right] \neq 0 .
\end{align*}
$$

that is, is satisfied.
(3) The characteristic matrix of (71) at a stationary solution ( $\bar{z}, \tau_{0}, p_{0}$ ) where $\bar{z}=\left(\bar{z}^{(1)}, \bar{z}^{(2)}, \bar{z}^{(3)}\right) \in R^{3}$ takes the following form:

$$
\Delta\left(\bar{z}, \tau_{1}, p\right)(\lambda)=\lambda I d-D_{\phi} F\left(\bar{z}, \bar{\tau}_{1}, \bar{p}\right)\left(e^{\lambda} I\right) ;
$$

$$
\begin{align*}
& \Delta\left(\bar{z}, \tau_{1}, p\right)(\lambda) \\
& \quad=\left[\begin{array}{ccc}
\lambda-a_{1}+2 a_{11} \bar{z}^{(1)}+\frac{a_{12} m_{1} \bar{z}^{(2)}}{\left(m_{1}+\bar{z}^{(1)}\right)^{2}} & \frac{a_{12} \bar{z}^{(1)}}{m_{1}+\bar{z}^{(1)}} & 0 \\
-\frac{a_{21} m_{1} \bar{z}^{(2)}}{\left(m_{1}+\bar{z}^{(1)}\right)^{2}} e^{-\lambda \tau_{1}} & \lambda+a_{2}-\frac{a_{21} \bar{z}^{(1)}}{m_{1}+\bar{z}^{(1)}}+2 a_{22} \bar{z}^{(2)}+\frac{a_{23} m_{2} \bar{z}^{(3)}}{\left(m_{2}+\bar{z}^{(2)}\right)^{2}} & \frac{a_{23} \bar{z}^{(2)}}{m_{2}+\bar{z}^{(2)}} \\
0 & -\frac{a_{32} m_{2} \bar{z}^{(3)}}{\left(m_{2}+\bar{z}^{(2)}\right)^{2}} e^{-\lambda \tau_{2}^{*}} & \lambda+a_{3}-\frac{a_{32} \bar{z}^{(2)}}{m_{2}+\bar{z}^{(2)}+2 a_{33} \bar{z}^{(3)}}
\end{array}\right] . \tag{88}
\end{align*}
$$

From (88), we have

$$
\begin{align*}
& \operatorname{det}\left(\Delta\left(E_{*}, \tau_{1}, p\right)(\lambda)\right) \\
& \qquad=\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}+\left[\left(r_{1}+q_{1}\right) \lambda+\left(q_{0}+r_{0}\right)\right] e^{-\lambda \tau_{1}} . \tag{89}
\end{align*}
$$

Note that (89) is the same as (20); from the discussion in Section 2 about the local Hopf bifurcation, it is easy to verify that $\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)$ is an isolated center, and there exist $\epsilon>$ $0, \delta>0$ and a smooth curve $\lambda:\left(\tau_{1 k}^{(j)}-\delta, \tau_{1 k}^{(j)}+\delta\right) \rightarrow \mathscr{C}$ such that $\operatorname{det}\left(\Delta\left(\lambda\left(\tau_{1}\right)\right)\right)=0,\left|\lambda\left(\tau_{1}\right)-\omega_{k}\right|<\epsilon$ for all $\tau_{1} \in$ $\left[\tau_{1 k}^{(j)}-\delta, \tau_{1 k}^{(j)}+\delta\right]$ and

$$
\begin{equation*}
\lambda\left(\tau_{1 k}^{(j)}\right)=\omega_{k} i,\left.\quad \frac{d \operatorname{Re} \lambda\left(\tau_{1}\right)}{d \tau_{1}}\right|_{\tau_{1}=\tau_{1 k}^{(j)}}>0 . \tag{90}
\end{equation*}
$$

Let

$$
\begin{equation*}
\Omega_{\epsilon, 2 \pi / \omega_{k}}=\left\{(\eta, p) ; 0<\eta<\epsilon,\left|p-\frac{2 \pi}{\omega_{k}}\right|<\epsilon\right\} . \tag{91}
\end{equation*}
$$

It is easy to see that on $\left[\tau_{1 k}^{(j)}-\delta, \tau_{1 k}^{(j)}+\delta\right] \times \partial \Omega_{\epsilon, 2 \pi / \omega_{k}}$, $\operatorname{det}\left(\Delta\left(E_{*}, \tau_{1}, p\right)(\eta+(2 \pi / p) i)\right)=0$ if and only if, $\eta=0$, $\tau_{1}=\tau_{1 k}^{(j)}, p=2 \pi / \omega_{k}, k=1,2,3 ; j=0,1,2, \ldots$.

Therefore, the hypothesis (A4) in [24] is satisfied.
If we define

$$
\begin{align*}
& H^{ \pm}\left(E_{*}, \tau_{1 k}^{(j)}, \frac{2 \pi}{\omega_{k}}\right)(\eta, p)  \tag{92}\\
& \quad=\operatorname{det}\left(\Delta\left(E_{*}, \tau_{1 k}^{(j)} \pm \delta, p\right)\left(\eta+\frac{2 \pi}{p} i\right)\right)
\end{align*}
$$

then we have the crossing number of isolated center $\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)$ as follows:

$$
\begin{align*}
\gamma\left(E_{*}, \tau_{1 k}^{(j)}, \frac{2 \pi}{\omega_{k}}\right)= & \operatorname{deg}_{B}\left(H^{-}\left(E_{*}, \tau_{1 k}^{(j)}, \frac{2 \pi}{\omega_{k}}\right), \Omega_{\epsilon, 2 \pi / \omega_{k}}\right) \\
& -\operatorname{deg}_{B}\left(H^{+}\left(E_{*}, \tau_{1 k}^{(j)}, \frac{2 \pi}{\omega_{k}}\right), \Omega_{\epsilon, 2 \pi / \omega_{k}}\right) \\
= & -1 . \tag{93}
\end{align*}
$$

Thus, we have

$$
\begin{equation*}
\sum_{\left(\bar{z}, \bar{\tau}_{1}, \bar{p}\right) \in \mathscr{C}_{\left(E_{*} * \tau_{1 k}, 2,2 \pi / w_{k}\right)}^{(j)}} \gamma\left(\bar{z}, \bar{\tau}_{1}, \bar{p}\right)<0, \tag{94}
\end{equation*}
$$

where $\left(\bar{z}, \bar{\tau}_{1}, \bar{p}\right)$ has all or parts of the form $\left(E_{*}, \tau_{1 j}^{(k)}, 2 \pi / \omega_{k}\right)(j=0,1, \ldots)$. It follows from Lemma 9 that the connected component $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$
through $\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)$ is unbounded for each center $\left(z_{*}, \tau_{1}, p\right),(j=0,1, \ldots)$. From the discussion in Section 2, we have

$$
\begin{align*}
& \tau_{1 k}^{(j)}=\frac{1}{\omega_{k}} \\
& \qquad \begin{aligned}
& \\
& \quad\left\{\operatorname { a r c c o s } \left(\left(\left(p_{2} \omega_{k}^{2}-p_{0}\right)\left(r_{0}+q_{0}\right)\right.\right.\right. \\
& \left.+\omega^{2}\left(\omega^{2}-p_{1}\right)\left(r_{1}+q_{1}\right)\right) \\
& \left.\left.\times\left(\left(r_{0}+q_{0}\right)^{2}+\left(r_{1}+q_{1}\right)^{2} \omega_{k}^{2}\right)^{-1}\right)+2 j \pi\right\}
\end{aligned}
\end{align*}
$$

where $k=1,2,3 ; \quad j=0,1, \ldots$. Thus, one can get $2 \pi / \omega_{k} \leq \tau_{1 k}^{(j)}$ for $j \geq 1$.

Now we prove that the projection of $\ell_{\left(E_{*} * \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ onto $\tau_{1}$-space is $\left[\bar{\tau}_{1},+\infty\right)$, where $\bar{\tau}_{1} \leq \tau_{1 k}^{(j)}$. Clearly, it follows from the proof of Lemma 11 that system (3) with $\tau_{1}=0$ has no nontrivial periodic solution. Hence, the projection of $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ onto $\tau_{1}$-space is away from zero.

For a contradiction, we suppose that the projection of $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ onto $\tau_{1}$-space is bounded; this means that the projection of $\ell_{\left(E_{*}, \tau_{1 k}, 2 \pi / \omega_{k}\right)}$ onto $\tau_{1}$-space is included in a interval $\left(0, \tau^{*}\right)$. Noticing $2 \pi / \omega_{k}<\tau_{1 k}^{j}$ and applying Lemma 11 we have $p<\tau^{*}$ for $\left(z(t), \tau_{1}, p\right)$ belonging to $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$. This implies that the projection of $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ onto $p$ space is bounded. Then, applying Lemma 10 we get that the connected component $\ell_{\left(E_{*}, \tau_{1 k}^{(j)}, 2 \pi / \omega_{k}\right)}$ is bounded. This contradiction completes the proof.

## 6. Conclusion

In this paper, we take our attention to the stability and Hopf bifurcation analysis of a predator-prey system with MichaelisMenten type functional response and two unequal delays. We obtained some conditions for local stability and Hopf bifurcation occurring. When $\tau_{1} \neq \tau_{2}$, we derived the explicit formulas to determine the properties of periodic solutions by the normal form method and center manifold theorem. Specially, the global existence results of periodic solutions bifurcating from Hopf bifurcations are also established by using a global Hopf bifurcation result due to Wu [24].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Parameters Estimation and Stability Analysis of Nonlinear Fractional-Order Economic System Based on Empirical Data 

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#### Abstract

This paper is devoted to propose a novel method for studying the macroeconomic system with fractional derivative, which can depict the memory property of actual data of economic variables. First of all, we construct a constrained optimal problem to evaluate the coefficients of nonlinear fractional financial system based on empirical data and design the corresponding genetic algorithm. Then, based on the stability criteria of fractional dynamical systems, the methodology of stability analysis is proposed to investigate the stability of the estimated nonlinear fractional dynamic system. Finally, our method is applied to discuss the macroeconomic system of the US, Australia, and UK to demonstrate its effectiveness and applicability.


## 1. Introduction

In a market economy, the macroeconomic stability is an important economic problem which is concerned by all of governments. Macroeconomic instability can take the form of volatility of key macroeconomic variables or of unsustainability in their behavior. Macroeconomic instability refers to phenomena that decrease the predictability of the domestic macroeconomic environment and it is of concern because unpredictability hampers resource-allocation decisions, investment, and growth. Therefore, the analysis of the macroeconomic stability contributes to the making of economic policy decision for governments and controls the macroeconomic instability to improve the predictability.

According to macroeconomic theory, the stability of macroeconomy is measured by the volatility of some key indicators such as consumer price inflation, real GDP growth over business cycles, changes in unemployment, fluctuations in the current of the balance of payments, and volatility of short term policy interest rates and long term interest rates. The existing researches about the stability of macroeconomy concentrate mainly on two directions.

On the one hand, some studies examine the macroeconomic stability by empirical analysis based on the actual data
and econometric models and usually include a few macroeconomic indicators, for example, [1-3]. More precisely, the macroeconomic stability and the properties of the international transmission of business cycles under three exchange rate systems are examined by [1]. In [2], whether monetary policy may have been a source of macroeconomic instability in the 1970s by inducing unstable learning dynamics is investigated. In [3], a conventional New Keynesian model for four countries is used to analyze relationship between monetary policy and macroeconomic stability. Empirical analysis of economic stability obtains scientific conclusion based on actual economic data but pays more attention on the factors which affect the volatility of some key indicators and neglects the stability characteristic which implicit in economic dynamic behavior economic system.

On the other hand, some researches build the simplified economic system model according to economic theory and analyze the stability of economic system based on dynamic economics, for example, [4-6]. What is more, Hopf bifurcation theorem is used to predict the occurrence of a limit cycle bifurcation for the time delay parameter of a new IS-LM business cycle model in [7]. Reference [8] studies the implication of imperfect financial contracting for macroeconomic stability in the context of a stochastic
dynamic general equilibrium model. Reference [9] examines the two most attractive characteristics, memory, and chaos in simulations of fractional financial model. Reference [10] investigates the stability criteria of the bifurcation periodic solutions and then the stability of a business cycle model with discrete delay. Reference [11] analyzes the relationship between interest-rate feedback rules and macroeconomic stability in presence of transaction cost. This method of stability analysis for economic system focuses on dynamic behavior of system evolution, and theory about system stability from dynamic economics is applied extensively in it. However, the considered economic systems are simplified model according to economic theory and neglect the rule of different real economic operation. Thus, there may be some bias between conclusion from it and actual economic situation.

It is well known that the prerequisite for stability analysis of economic system is to construct the reasonable economic model. As an excellent methodology of modeling, fractional calculus is verified to be a powerful tool in modeling most physical processes with memory effect, which cannot be described well by integer-order integral and differential equations (see [12]). With the fractional derivative being applied in economic or financial system, fractional economic models spring up in recent years. For instance, reference [9] proposes the fractional financial system which involves the macroeconomic variables such as investment, interest, and price index. In [13], a delayed fractional-order financial system is proposed. These models are simplified from economic theory. Reference [14] presents macroeconomic modeling based on fractional calculus, but nonlinear structures have not been considered.

The motivation of this paper is to present a new method to model and analyze macroeconomic system based on theory about nonlinear fractional-order system, and this method possesses the merits of the empirical analysis and dynamics economics mentioned above. More precisely, the aim of this paper is to achieve the following: on the one hand, it proposes the method to construct the fractionalorder nonlinear dynamic system which can describe the actual economic data accurately and reveal the rule of operation about real economic system; on the other hand, it introduces stability analysis of fractional-order economic system to investigate the stability of real economic system and proposes a novel method of stability analysis for economic system. In the paper, we will try to design an approach to estimate fractional-order nonlinear economic system based on empirical data and then analyze the stability of estimated system using the stability criteria of fractional-order system.

The remainder of this paper is organized as follows. In Section 2, we review the mathematical preliminaries about fractional calculus and stability of system. In Section 3, we propose a novel methodology to model a nonlinear fractional-order economic system based on nonlinear equation. In Section 4, GA have been used to estimate the macroeconomic system by US, Australia, and UK macroeconomic data. In Section 5, the analysis of dynamic behavior and stability about the macroeconomic system is performed. The conclusions will be shown in Section 6.

## 2. Mathematical Preliminaries

2.1. Definition of Fractional Calculus. In this section, we introduce some preliminaries of fractional calculus. More properties of fractional derivatives could be found in many books and recent papers; for example, see [15, 16].

There exist three most frequently used definitions for the general fractional differintegral which are GrünwaldLetnikov (GL) definition, the Riemann-Liouville (RL), and Caputo definitions. For the GL definition of fractional derivative being convenient to compute numerically, it is applied to solve the fractional-order system in this paper.

Definition 1 (see [17]). The GL definition can be written as

$$
\begin{equation*}
{ }_{a}^{\mathrm{GL}} D_{t}^{\alpha} f(t) \approx h^{-\alpha} \sum_{i=0}^{[(t-a) / h]}(-1)^{k}\binom{\alpha}{k} f\left(t_{k}-i h\right), \tag{1}
\end{equation*}
$$

where $[x]$ means the integer part of $x$.
2.2. Fractional-Order System. In [18], the general form of $n$ dimensional fractional-order system can be expressed as

$$
\begin{align*}
{ }_{0} D_{t}^{\alpha_{i}} x_{i}(t) & =f_{i}\left(x_{1}(t), x_{2}(t), \ldots, x_{n}(t), t\right) \\
x_{i}(0) & =c_{i}, \quad i=1,2, \ldots, n \tag{2}
\end{align*}
$$

where $c_{i}$ are initial conditions. If $\alpha_{1}=\alpha_{2}=\cdots=\alpha_{n}$, then the system is a commensurate fractional-order system otherwise is called as an incommensurate system. If $f_{i}$ is nonlinear function, the system is a nonlinear system.
2.3. Stability of Fractional-Order System. The exponential stability cannot be used to characterize the asymptotic stability of fractional-order system. A new definition was introduced in [18].

Definition 2 (see [18]). The trajectory $x(t)=0$ of the system (2) is $t^{-q}$ asymptotically stable if there is a positive real $q$ such that for all $\|x(t)\|$ with $t \leq t_{0}, \exists N(x(t))$, such that for all $t \geq$ $t_{0},\|x(t)\| \leq N t^{-q}$.

The fact that the components of $x(t)$ slowly decay towards 0 following $t^{-q}$ leads to fractional systems being called long memory systems. The stability theorem in [19] presents that equilibrium points are asymptotically stable for $\alpha_{1}=\alpha_{2}=$ $\cdots=\alpha_{n}=\alpha$ if all the eigenvalues $\lambda_{i}(i=1,2, \ldots, n)$ of the Jacobian matrix $\mathbf{J}=\partial \mathbf{f} / \partial \mathbf{x}$, where $\mathbf{f}=\left[f_{1}, f_{2}, \ldots, f_{n}\right]^{T}$, evaluated at the equilibrium point, satisfy the condition ([18]):

$$
\begin{equation*}
|\arg (\operatorname{eig}(\mathrm{J}))|=\left|\arg \left(\lambda_{i}\right)\right|>\alpha \frac{\pi}{2}, \quad i=1,2, \ldots, n \tag{3}
\end{equation*}
$$

For the incommensurate fractional-order system $\alpha_{1} \neq \alpha_{2} \neq \cdots \neq \alpha_{n}$, suppose the $\alpha_{i}$ is the rational number, which $m$ is the LCM (least common multiple) of the denominators $u_{i}$ of $\alpha_{i}$ 's, where $\alpha_{i}=v_{i} / u_{i}, v_{i} \cdot u_{i} \in Z^{+}$for $i=1,2, \ldots, n$, and set $\gamma=1 / m$. According to [18], the incommensurate fractional-order system is asymptotically stable if

$$
\begin{equation*}
|\arg (\lambda)|>\gamma \frac{\pi}{2} \tag{4}
\end{equation*}
$$

for all roots $\lambda$ of the following equation ([18]):

$$
\begin{equation*}
\operatorname{det}\left(\operatorname{diag}\left(\left[\lambda^{m q_{1}} \lambda^{m q_{2}} \ldots \lambda^{m q_{1}}\right]\right)-\mathbf{J}\right)=0 \tag{5}
\end{equation*}
$$

For fractional-order system (2), the necessary stability condition for it to remain chaotic is keeping at least one eigenvalue $\lambda$ in the unstable region (see [19]). Suppose that the unstable eigenvalues of scroll saddle points are $\lambda_{1,2}=$ $a_{1,2} \pm i b_{1,2}$. The necessary condition to exhibit double-scroll attractor of the system remaining in the unstable region is exhibited in [19]. The condition for commensurate derivatives order is

$$
\begin{equation*}
\alpha>\frac{2}{\pi} \arctan \left(\frac{b_{i}}{a_{i}}\right), \quad i=1,2 . \tag{6}
\end{equation*}
$$

Thus, the instability measure $\pi / 2 m-\min (|\arg (\lambda)|)$ is negative, and the system cannot be chaotic [19].

## 3. Modelling Economic System with Fractional-Order Derivatives

3.1. Nonlinear Fractional-Order System. There exist several financial models reported in recent years. For instance, the study of investment, interest rate, and price index by using a chaotic fractional Chen system is discussed in [9] as

$$
\begin{align*}
& D_{t}^{\alpha_{1}} x_{t}=z_{t}+\left(y_{t}-a\right) x_{t} \\
& D_{t}^{\alpha_{2}} y_{t}=1-b y_{t}-x_{t}^{2}  \tag{7}\\
& D_{t}^{\alpha_{3}} z_{t}=-x_{t}-c z_{t}
\end{align*}
$$

where $x, y, z$ represent the interest rate, investment, and inflation, respectively. The subscript $t$ indicates that the variable depends on $t$. Parameters $a, b$, and $c$ are nonnegative coefficients with economic interpretation. $\alpha_{i} \in(0,1], i=$ $1,2,3$, represents the fractional order of the derivatives. If $\alpha_{1}=\alpha_{2}=\alpha_{3}=1$, (7) reduces to the integer-order Chen system.

Instead of considering the same expressions in fractional chaotic Chen system, we assume a more general form of nonlinear fractional system as

$$
\begin{align*}
D_{t}^{\alpha_{1}} x(t)= & c_{1}+a_{11} x(t)+a_{12} y(t)+a_{13} z(t) \\
& +a_{14} x(t) y(t)+a_{15} y(t) z(t)+a_{16} z(t) x(t) \\
& +a_{17} x^{2}(t)+a_{18} y^{2}(t)+a_{19} z^{2}(t), \\
D_{t}^{\alpha_{2}} y(t)= & c_{2}+a_{21} x(t)+a_{22} y(t)+a_{23} z(t) \\
& +a_{24} x(t) y(t)+a_{25} y(t) z(t)+a_{26} z(t) x(t) \\
& +a_{27} x^{2}(t)+a_{28} y^{2}(t)+a_{29} z^{2}(t), \\
D_{t}^{\alpha_{3}} z(t)= & c_{3}+a_{31} x(t)+a_{32} y(t)+a_{33} z(t) \\
& +a_{34} x(t) y(t)+a_{35} y(t) z(t) a_{36} z(t) x(t) \\
& +a_{37} x^{2}(t)+a_{38} y^{2}(t)+a_{39} z^{2}(t), \tag{8}
\end{align*}
$$

where $x(t), y(t), z(t)$ are the economic variables.

Let

$$
\begin{align*}
& f_{1}(x\left.(t), y(t), z(t), A_{1}\right) \\
&= c_{1}+a_{11} x(t)+a_{12} y(t)+a_{13} z(t) \\
&+a_{14} x(t) y(t)+a_{15} y(t) z(t)+a_{16} z(t) x(t) \\
&+a_{17} x^{2}(t)+a_{18} y^{2}(t)+a_{19} z^{2}(t), \\
& f_{2}\left(x(t), y(t), z(t), A_{2}\right) \\
&= c_{2}+a_{21} x(t)+a_{22} y(t)+a_{23} z(t) \\
&+a_{24} x(t) y(t)+a_{25} y(t) z(t)+a_{26} z(t) x(t) \\
&+a_{27} x^{2}(t)+a_{28} y^{2}(t)+a_{29} z^{2}(t), \\
& f_{3}(x\left.(t), y(t), z(t), A_{3}\right) \\
&= c_{3}+a_{31} x(t)+a_{32} y(t)+a_{33} z(t) \\
&+a_{34} x(t) y(t)+a_{35} y(t) z(t)+a_{36} z(t) x(t) \\
&+a_{37} x^{2}(t)+a_{38} y^{2}(t)+a_{39} z^{2}(t), \\
& A_{i}=\left(c_{i}, a_{i 1}, a_{i 2}, \ldots, a_{i 9}\right), \quad i=1,2,3 . \tag{9}
\end{align*}
$$

Then the model can be rewritten as

$$
\begin{align*}
& D_{t}^{\alpha_{1}} x(t)=f_{1}\left(x(t), y(t), z(t), A_{1}\right) \\
& D_{t}^{\alpha_{2}} y(t)=f_{2}\left(x(t), y(t), z(t), A_{2}\right)  \tag{10}\\
& D_{t}^{\alpha_{3}} z(t)=f_{3}\left(x(t), y(t), z(t), A_{3}\right)
\end{align*}
$$

3.2. Fitting of Fractional-Order Nonlinear System. In order to estimate the parameters of the system, the numerical calculation of the fractional-order derivative will be used. According to the GL definition of fractional-order derivative (1), the explicit numerical approximation has the following form ([17]):

$$
\begin{equation*}
{ }_{\left(k-L_{m} / h\right)} D_{t_{k}}^{\alpha} f(t) \approx h^{-\alpha} \sum_{i=0}^{k} c_{i}^{(\alpha)} f\left(t_{k-i}\right) \tag{11}
\end{equation*}
$$

where $t_{k}=k h, h$ is the time step of discretization, $L_{m}$ is called as "memory length," and $c_{i}^{(\alpha)}(i=0,1, \ldots)$ are binomial coefficients, which can be calculated as

$$
\begin{equation*}
c_{0}^{(\alpha)}=1, \quad c_{i}^{(\alpha)}=\left(1-\frac{1+\alpha}{i}\right) c_{i-1}^{(\alpha)} . \tag{12}
\end{equation*}
$$

Then, the general numerical solution of the fractional-order system (11) can be expressed as

$$
\begin{align*}
& x\left(t_{k}\right)=f_{1}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{1}\right)-\sum_{j=v_{1}}^{k} c_{j}^{\alpha_{1}} x\left(t_{k-j}\right), \\
& y\left(t_{k}\right)=f_{2}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{2}\right)-\sum_{j=v_{2}}^{k} c_{j}^{\alpha_{2}} y\left(t_{k-j}\right), \\
& z\left(t_{k}\right)=f_{3}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{3}\right)-\sum_{j=v_{3}}^{k} c_{j}^{\alpha_{3}} z\left(t_{k-j}\right) . \tag{13}
\end{align*}
$$

The "short memory" principles have been considered in this expression. The lower index of the sums in it will be $v_{i}=1$ for $k<\left(L_{i} / h\right)$ and $v_{i}=k-\left(L_{i} / h\right)$ for $k>\left(L_{i} / h\right)$, or when "short memory" principle is not being considered, we put $v_{i}=1$ for all $k$. Assume that the actual economic data are $(X(i), Y(i), Z(i))(i=1,2, \ldots, n)$. Let $\boldsymbol{\alpha}=\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$, $\mathbf{A}=\left(A_{1}, A_{2}, A_{3}\right)$. In order to ensure that trajectory of the system describes the empirical economic data, we construct constrained optimal problem as follows:

$$
\begin{align*}
\min H(\boldsymbol{\alpha}, \mathbf{A})= & \sum_{i=1}^{n-1}(x(i)-X(i+1))^{2} \\
& +\sum_{i=1}^{n-1}(y(i)-Y(i+1))^{2}  \tag{14}\\
& +\sum_{i=1}^{n-1}(z(i)-Z(i+1))^{2}
\end{align*}
$$

subject to

$$
\begin{align*}
x\left(t_{k}\right)= & f_{1}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{1}\right) \\
& -\sum_{j=1}^{k} c_{j}^{\alpha_{1}} x\left(t_{k-j}\right), \\
y\left(t_{k}\right)= & f_{2}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{2}\right) \\
& -\sum_{j=1}^{k} c_{j}^{\alpha_{2}} y\left(t_{k-j}\right),  \tag{15}\\
z\left(t_{k}\right)= & f_{3}\left(x\left(t_{k}\right), y\left(t_{k}\right), z\left(t_{k}\right), A_{3}\right) \\
& -\sum_{j=1}^{k} c_{j}^{\alpha_{3}} z\left(t_{k-j}\right), \quad 0<\alpha_{i}<2, i=1,2,3,
\end{align*}
$$

where

$$
\begin{equation*}
x\left(t_{0}\right)=X(1), \quad y\left(t_{0}\right)=Y(1), \quad z\left(t_{0}\right)=Z(1) \tag{16}
\end{equation*}
$$

Then, we can estimate the coefficients of fractional-order system (10) by solving the optimal problem (14) when then actual data is known.

Remark 3. About constraints in the optimal problem, $x, y, z$ are obtained by implicit dynamic equations. In the process of solving it, we can calculate it by transferring it to explicit dynamic equations.
3.3. Optimal Parameters Estimations of Fractional-Order Nonlinear System by GA. In this paper, the optimal problem (14) will be solved by genetic algorithms (GA). The solutions of the optimal problem are optimal parameters of fractionalorder nonlinear system which describe accurately empirical economic data. The genetic algorithm is an example of a search procedure that uses random selection for optimization of a function by means of the parameters space coding. It was developed by [20] and the most popular references are [21, 22]. The GA have been proven successful for robust searches in complex spaces. [23] states the validity of the technique in applications of optimization and robust search. The GA have been credited as efficient and effective in the approach for the search.

Comparing with usual constrained optimal problem, the difference of constrained optimal problem (14) includes a process of solving fractional-order system in calculating the fitness of each chromosome in population. The genetic algorithms of solving constrained optimal problem (14) can be designed as the following steps.

Step 1. Generate random population of chromosomes $P(0)$ with population size $N$ in feasible region of problem (14) $\left\{(\boldsymbol{\alpha}, \mathbf{A}) \mid \mathbf{A} \in R^{30}, 0<\alpha_{i}<2, i=1,2,3\right\}$. Set $k=0$.

Step 2. Evaluate the fitness of each chromosomes in the population $P(k)$. Set $x\left(t_{0}\right)=X(1), y\left(t_{0}\right)=Y(1), z\left(t_{0}\right)=$ $Z(1), m=n-1$. Given time step $h$, calculate the $\left(x\left(t_{i}\right), y\left(t_{i}\right), z\left(t_{i}\right)\right)(i=1,2, \ldots, m)$ by recursive formula (13) based on parameters $\left(\boldsymbol{\alpha}_{j}, \mathbf{A}_{j}\right)(j=1,2, \ldots, N)$ which are decoded from chromosomes in $P(k)$. Then, compute the finesses of the chromosomes $f_{j}$ by $H\left(\boldsymbol{\alpha}_{j}, \mathbf{A}_{j}\right)(j=$ $1,2, \ldots, N)$.

Step 3. Create a new population $P(k+1)$ by repeating the following steps until the new population is complete.
(a) Select two parent chromosomes from population $P(k)$ according to their fitness. Calculate selected probability of chromosomes in $P(k)$ by $p_{i}=f_{i} / \sum_{i} f_{i}(i=$ $1,2, \ldots, N)$. The parent chromosomes are selected to be the parent by roulette selection operator.
(b) Cross over the parents to form new offspring with crossover probability $p_{c}$.
(c) Mutate new offspring at each locus with a mutation probability $p_{m}$. Place new offspring in the new population.

Step 4. If stop criterion is satisfied, then stop. Otherwise, replace $P(k)$ by $P(k+1)$, and go to Step 2 .
3.4. Stability Analysis for Nonlinear Fractional-Order System. The stability as an extremely important property of the dynamical systems is investigated in various domains. According to stability criteria in [18], the stability of fractional nonlinear system is investigated by analyzing the characteristic of equilibrium points in system. $\left(x^{*}, y^{*}, z^{*}\right)$ represents an arbitrary equilibrium points of system (10), which is calculated by solving the equations $f_{i}(x(t), y(t), z(t))=$ $0(i=1,2,3)$. The Jacobian matrix is

$$
J=\left(\begin{array}{lll}
j_{11} & j_{12} & j_{13}  \tag{17}\\
j_{21} & j_{22} & j_{23} \\
j_{31} & j_{32} & j_{33}
\end{array}\right)
$$

where

$$
\begin{align*}
& j_{11}=a_{11}+a_{14} x_{2}^{*}+a_{16} x_{3}^{*}+2 a_{17} x_{1}^{*}, \\
& j_{12}=a_{12}+a_{14} x_{1}^{*}+a_{15} x_{1}^{*}+2 a_{18} x_{2}^{*}, \\
& j_{13}=a_{13}+a_{15} x_{1}^{*}+a_{16} x_{2}^{*}+2 a_{19} x_{3}^{*}, \\
& j_{21}=a_{21}+a_{24} x_{2}^{*}+a_{26} x_{3}^{*}+2 a_{27} x_{1}^{*}, \\
& j_{22}=a_{22}+a_{24} x_{1}^{*}+a_{25} x_{1}^{*}+2 a_{28} x_{2}^{*},  \tag{18}\\
& j_{23}=a_{23}+a_{25} x_{1}^{*}+a_{26} x_{2}^{*}+2 a_{29} x_{3}^{*}, \\
& j_{31}=a_{31}+a_{34} x_{2}^{*}+a_{36} x_{3}^{*}+2 a_{37} x_{1}^{*}, \\
& j_{32}=a_{32}+a_{34} x_{1}^{*}+a_{35} x_{1}^{*}+2 a_{38} x_{2}^{*}, \\
& j_{33}=a_{33}+a_{35} x_{1}^{*}+a_{36} x_{2}^{*}+2 a_{39} x_{3}^{*} .
\end{align*}
$$

Let optimal order in fractional nonlinear system (10) be $\left(\widehat{\alpha}_{1}, \widehat{\alpha}_{2}, \widehat{\alpha}_{3}\right) . \widehat{\alpha}_{i}(i=1,2,3)$ can be expressed $v_{i} / u_{i}(i=$ $1,2,3$ ), and $m$ is the LCM (least common multiple) of the denominators $u_{i}(i=1,2,3)$. Let $m_{i}=m \widehat{\alpha}_{i}$. According to (6), the characteristic equation is as follows:

$$
\begin{align*}
& \lambda^{m_{1}+m_{2}+m_{3}}-j_{22} \lambda^{m_{3}+m_{1}}-j_{33} \lambda^{m_{2}+m_{1}}-j_{11} \lambda^{m_{3}+m_{2}} \\
& \quad+b_{1} \lambda^{m_{1}}+b_{2} \lambda^{m_{2}}+b_{3} \lambda^{m_{3}}+b_{4}=0, \tag{19}
\end{align*}
$$

where

$$
\begin{aligned}
b_{1}= & j_{22} j_{33}-j_{23} j_{32}, \\
b_{2}= & j_{11} j_{33}-j_{13} j_{31}, \\
b_{3}= & j_{11} j_{22}-j_{12} j_{21}, \\
b_{4}= & -\left(j_{22} j_{33}-j_{23} j_{32}\right) j_{11} \\
& +j_{12}\left(j_{33} j_{21}-j_{23} j_{31}\right)-j_{13}\left(j_{21} j_{32}-j_{31} j_{22}\right) .
\end{aligned}
$$



Figure 1: The actual data versus trajectories of fractional-order system in each dimension about US.

The roots of (19) are $\omega_{i}\left(i=1,2, \ldots, m_{1}+m_{2}+\right.$ $m_{3}$ ), and the instability measure of equilibrium point is $\pi / 2 m-\min (|\arg (\omega)|)$. If instability measure is negative, the equilibrium point is stable.

## 4. Application to Economic System of US, Australia, and UK

In this section, we will show how to analyze the stability of macroeconomic system based on the actual data by fractional-order system. Firstly, the optimal fractional-order system needs to be estimated based on actual data. This study will take the three nations (i.e., USA, Australia, and UK) for example to show the feasibility and effectiveness of our method.
4.1. Selection of Economic Variables and Data Resource. The GDP, inflation, and unemployment are the key macroeconomic indexes which governments concern. In this paper, the percent change of GDP, average consumer prices percent change rate, and change of unemployment rate percent of total labor force are used to reflect the variables GDP, inflation, and unemployment, respectively. The annual data starts from year 1980 to 2011. The resource of data about percent change of GDP, average consumer prices percent change rate, and unemployment rate percent of total labor force is EconStats which is organized by IMF. Let economic

TABLE 1: Optimal parameters of fractional order economic system for US, Australia, UK.

|  | US |  |  | Australia |  |  | UK |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $x$ | $y$ | $z$ | $x$ | $y$ | $z$ | $x$ | $y$ | $z$ |
| $\hat{q}$ | 0.298 | 0.518 | 1.521 | 0.459 | 0.127 | 1.992 | 0.503 | 0.521 | 0.611 |
| $\widehat{c}$ | 0.141 | 0.845 | -42.503 | 0.936 | 0.128 | 0.957 | 1.271 | 1.618 | 0.959 |
| $\widehat{a}_{1}$ | -7.360 | -26.756 | 19.654 | 9.923 | -2.817 | 3.007 | -10.655 | -70.531 | -13.637 |
| $\widehat{a}_{2}$ | 10.596 | 42.678 | -45.415 | 2.352 | 2.855 | 2.661 | -85.040 | 54.652 | 19.055 |
| $\widehat{a}_{3}$ | -208.649 | -82.896 | -3.255 | 1.696 | -4.907 | -1.889 | -91.272 | -73.995 | 100.756 |
| $\widehat{a}_{4}$ | 106.886 | 134.066 | 7.465 | 1.011 | 2.837 | -1.094 | 37.836 | 98.484 | -31.076 |
| $\widehat{a}_{5}$ | -92.283 | -29.489 | -9.162 | 6.746 | -13.460 | -0.854 | -28.490 | -60.796 | 17.146 |
| $\hat{a}_{6}$ | -46.156 | 0.388 | -9.954 | -10.966 | -8.466 | -6.662 | 0.939 | 37.492 | 2.462 |
| $\hat{a}_{7}$ | -40.568 | 24.615 | 33.139 | 1.639 | -9.793 | -1.183 | -31.064 | -46.859 | 2.666 |
| $\widehat{a}_{8}$ | -28.753 | 98.083 | -19.133 | 0.152 | 14.682 | 2.982 | 65.905 | 28.800 | 32.485 |
| $\widehat{a}_{9}$ | -17.425 | -18.006 | -19.731 | -4.210 | 0.094 | -1.558 | -18.148 | -34.796 | 3.771 |
| MSE |  | $4.8 e-4$ |  |  | $6.6 e-4$ |  |  | $5.5 e-4$ |  |

Notes: $\widehat{c}$ denote the $\left(\widehat{c}_{1}, \widehat{c}_{2}, \widehat{c}_{3}\right) . \widehat{a}_{i}$ denote $\left(\widehat{a}_{1 i}, \widehat{a}_{2 i}, \widehat{a}_{3 i}\right)(i=1,2, \ldots, 9)$. MSE is mean of squared error.

Table 2: Results about stability analysis of economic system about US.

| Point | Jacob Matrix | Unstable measure | Stability |
| :--- | :---: | :---: | :---: |
| $\left[\begin{array}{c}-5.110 \\ -1.548 \\ 7.852\end{array}\right]$ | $\left[\begin{array}{ccc}70.2 & 2023.7 & -844.3 \\ -106.3 & 1373.3 & 943.5 \\ 7.1 & 40.9 & -369.8\end{array}\right]$ | -0.0099 | stable |
| $\left[\begin{array}{c}0.573 \\ -0.664 \\ -0.450\end{array}\right]$ | $\left[\begin{array}{ccc}127.316 & -121.140 & 158.997 \\ 69.603 & -167.274 & 50.804 \\ -41.309 & -35.030 & -17.834\end{array}\right]$ | 0.0314 | unstable |
| $\left[\begin{array}{c}0.726 \\ -0.976 \\ -1.046\end{array}\right]$ | $\left[\begin{array}{ccc}233.291 & -191.453 & 238.417 \\ 113.160 & -275.220 & -36.399 \\ -34.911 & -62.111 & 8.973\end{array}\right]$ | 0.0314 | unstable |
| $\left[\begin{array}{c}0.756 \\ -0.157 \\ -0.194\end{array}\right]$ | $\left[\begin{array}{ccc}-18.986 & -173.095 & 117.048 \\ 20.167 & -123.163 & 110.606 \\ -53.771 & -1.853 & -30.909\end{array}\right]$ | -0.0128 | stable |

Table 3: Results about stability analysis of economic system about Australia.

| Point | Jacob Matrix | Unstable measure |
| :--- | :---: | :---: | Stability

TABLE 4: Results about stability analysis of economic system about UK.

| Point | Jacob matrix | Unstable measure | Stability |
| :--- | :---: | :---: | :---: |
| $\left[\begin{array}{c}-7.396 \\ -3.483 \\ 2.776\end{array}\right]$ | $\left[\begin{array}{ccc}226.2 & 985.8 & 190.3 \\ -464 & 1076.5 & -302.1 \\ -338.8 & -863.7 & 331.4\end{array}\right]$ | 0.0157 | Unstable |
| $\left[\begin{array}{l}2.214 \\ 1.402 \\ 0.570\end{array}\right]$ | $\left[\begin{array}{ccc}-138.774 & -278.268 & 123.998 \\ 29.238 & -309.613 & 220.292 \\ 162.894 & 199.199 & -26.786\end{array}\right]$ | 0.0157 | Unstable |
| $\left[\begin{array}{l}-2.511 \\ 1.331 \\ -1.694\end{array}\right]$ | $\left[\begin{array}{ccc}-76.666 & 71.742 & -351.173 \\ -182.166 & -176.300 & -371.135 \\ 93.656 & -206.896 & -28.261\end{array}\right]$ | -0.0120 | Stable |
| $\left[\begin{array}{l}0.072 \\ 0.074 \\ 0.014\end{array}\right]$ | $\left[\begin{array}{ccc}-5.747 & -21.294 & 7.501 \\ 0.690 & -81.415 & 58.050 \\ 9.010 & -6.423 & -19.114\end{array}\right]$ | -0.0293 | Stable |
| $\left[\begin{array}{l}0.364 \\ 0.381 \\ 0.097\end{array}\right]$ | $\left[\begin{array}{ccc}-35.584 & -63.879 & 20.743 \\ -5.177 & -123.619 & 72.924 \\ 42.803 & 22.055 & -17.532\end{array}\right]$ | -0.0249 | Stable |
| $\left[\begin{array}{l}-4.748 \\ -3.316 \\ -6.044\end{array}\right]$ | $\left[\begin{array}{ccc}467.206 & 400.040 & -876.792 \\ 258.412 & -3.671 & -559.485 \\ -460.158 & -321.884 & -321.041\end{array}\right]$ | 0.0157 | Unstable |



Figure 2: The actual data versus trajectories of fractional-order system in each dimension about Australia.

(c)

Figure 3: The actual data versus trajectories of fractional-order system in each dimension about UK.


Figure 4: Space trajectory and the time responses of the estimated system about US.
variables $x(t), y(t), z(t)$ in (13) be the percent change of GDP, average consumer prices percent change rate, and change of unemployment rate, respectively.

Remark 4. The difference of series may alter the some characteristic of 12 it (such as stationary of series). In order to be consistent with the percent change of GDP, average consumer prices percent change rate, this paper will select change of unemployment rate to be variable which reflects the unemployment of a nation.

### 4.2. Estimation of Fractional-Order Nonlinear System Based on

 Genetic Algorithms. This paper adopts the GA to search the optimal parameters of fractional-order system by using the actual data from US, Australia, and UK. Under the time step $h=0.0005$, (13) have been fit by GA, and optimal parameters are shown in Table 1.In order to show intuitively the accuracy of the optimal system estimated, the comparison of actual data and trajectories of system in each dimension have been shown in

Figures 1-3. From Figure 1, we can find that the accuracy of fit about US economic data is satisfying in economic variables GDP and inflation, and the goodness of fit about unemployment is worse than other variables for which the bias of the last points is obvious. Figure 2 shows the accuracy of fit about Australian economic data in three dimensions. It suggests that the estimated fractional economic system can describe actual data of Australia accurately. Figure 3 shows the goodness of fit about UK data and suggests that trajectory of GDP and inflation are more close to actual data than unemployment.

## 5. Stability Analysis of Nonlinear Fractional-Order Macroeconomic System

In this section, we will analyze the dynamic behavior of the macroeconomic systems with optimal parameters by the simulation technique. The fractional-order economic system estimated by actual economic data from US, Australia, and UK will be analyzed, respectively, as follows.


Figure 5: Space trajectory and the time responses of the estimated system about Australia.
5.1. Fractional-Order Economic System about US. We will analyze the stability of the fractional macroeconomic system for US estimated in Table 1 using the stability criteria in [19]. For number of characteristic root equation (5) depending on the $m$ (LCM of denominates of the fractional order), the optimal order $(0.298,0.518,1.522)$ round to decile $(0.30,0.52,1.52)$ in stability of analysis. Thus, $m=50$ and $\gamma(\pi / 2 m)=0.0314$. The estimated fractional macroeconomic system has four equilibriums at $(-5.110,-1.548,7.852),(0.573,-0.664,-0.450)$, ( $0.726,-0.976,-1.046$ ), and ( $0.756,-0.157,-0.194$ ). We will analyze the stability of each equilibrium point.

In Table 2, the results show that there are two unstable equilibrium points and two stable equilibrium points in the fractional-order macroeconomic system about US. These mean that macroeconomic dynamic systems of US have two saddle points and two stable points. The numerical simulation about fractional economic system is shown in Figure 4. From simulation results about trajectory and time responses of each variable in Figure 4, the variables GDP and inflation
fluctuate periodically and change of unemployment becomes a constant after a given time. This suggests that economic system of US tends to be stable with time changing, and there is nonchaotic behavior in fractional system about US.
5.2. Fractional-Order Economic System about Australia. In Table 1, the optimal order of fractional-order economic system about Australia is $(0.459,0.127,1.991)$ and round to decile $(0.46,0.13,1.99)$ to control the number of characteristic root. Obviously, $m=100$; then $\gamma(\pi / 2 m)=0.0157$. The estimated fractional-order economic system about Australia has four equilibrium points $(-0.246,0.499,0.523),(1.125,0.964,0.509),(-0.053,-0.470$, $-1.272)$ and $(-3.948,-2.578,0.352)$. The stability of the fixed points is analyzed in Table 3. In Table 3, the unstable measures of each equilibrium points show that all of them are unstable and are saddle points. It suggests that the fractional economic system of US is unstable system. From simulation results about trajectory and time responses of each variable in Figure 5, the variables GDP, inflation, and unemployment


Figure 6: Space trajectory and the time responses of the estimated system about UK.
fluctuate irregularly. The simulation of fractional economic system also shows instability of system.

### 5.3. Fractional-Order Economic System about UK. The

 optimal order of fractional-order economic system about Australia is $(0.503,0.521,0.611)$ in Table 1 and analogously round to decile ( $0.50,0.52,0.61$ ). Obviously, $m=100$; then $\gamma(\pi / 2 m)=0.0157$. The estimated fractional-order economic system about Australia has six equilibrium points $(-7.396,-3.483,2.776),(2.214,1.402,0.570),(-2.511,1.331$, $-1.694), \quad(0.072,0.074,0.014), \quad(0.364,0.381,0.097)$, and $(-4.748,-3.316,-6.044)$. The stability of the fixed points is analyzed in Table 4 . Table 4 shows the results about stability analysis of fractional economic at six fixed points. It suggests that there are three stable points and unstable points in fractional economic system about UK. Simulation results for estimated economic system are shown in Figure 6 and show that dynamic behaviors of three variables are unstable and fluctuate with width decreasing and centre on some value. Combining the results about analysis of equilibrium points and simulation for system, the fractional economicsystem for UN is unstable for the initial period and tends to periodical fluctuations.

## 6. Conclusions

In this paper, we propose a novel methodology to construct and analyze macroeconomic system based on fractionalorder derivative that can depict the characteristic of memory in actual data. The methodology to analyze the economic system combine the merit of empirical methods and numerical analysis methods and offset the disadvantage of a single method (empirical analysis or numerical analysis). The application of the methodology is shown by using the US, Australia, and UK macroeconomic data to demonstrate its effectiveness and applicability.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Global Asymptotic Stability in a Class of Reaction-Diffusion Equations with Time Delay 

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#### Abstract

We study a very general class of delayed reaction-diffusion equations in which the reaction term can be nonmonotone and spatially nonlocal. By using a fluctuation method, combined with the careful analysis of the corresponding characteristic equations, we obtain some sufficient conditions for the global asymptotic stability of the trivial solution and the positive steady state to the equations subject to the Neumann boundary condition.


## 1. Introduction

There has been a growing interest in the dynamic behavior of spatial nonlocal and time-delayed population systems since the 1970s [1]. When the death function of such a system is linear many researchers used the theory of monotone semiflows, the comparison arguments, and the fluctuation method to study spreading speeds, traveling waves, and the global stability (see, e.g., [2-10]). However, researches on these problems become relatively rare for nonmonotone delayed reaction-diffusion systems in which the death function is nonlinear (see [11, 12]). The reason lies in the fact that it is difficult to establish an appropriate expression for solutions to study the solution semiflow under this case.

In this paper, we will investigate the global asymptotic stability of the positive steady state for the following timedelayed reaction-diffusion equation:

$$
\begin{aligned}
\frac{\partial w(t, x)}{\partial t}= & d \Delta w(t, x)-f(w(t, x)) \\
& +\int_{\Omega} k(\alpha, x, y) b(w(t-\tau, y)) d y
\end{aligned}
$$

$$
\begin{gather*}
\frac{\partial w(t, x)}{\partial \mathbf{n}}=0, \quad t>0, x \in \partial \Omega \\
w(t, x)=\phi(t, x) \geq 0, \quad t \in[-\tau, 0], x \in \Omega \tag{1}
\end{gather*}
$$

where $d>0, \alpha \geq 0, \tau \geq 0, \Delta$ denotes the Laplacian operator on $\mathbb{R}^{m}, \Omega$ is a bounded and open domain of $\mathbb{R}^{m}$ with a smooth boundary $\partial \Omega, \partial / \partial \mathbf{n}$ is the differentiation in the direction of the outward normal $\mathbf{n}$ to $\partial \Omega$, and the kernel function $k(\alpha, x, y)$ is given by

$$
k(\alpha, x, y)= \begin{cases}\sum_{n=1}^{+\infty} e^{-\lambda_{n} \alpha} \varphi_{n}(x) \varphi_{n}(y), & \text { if } \alpha>0  \tag{2}\\ \delta(x-y), & \text { if } \alpha=0\end{cases}
$$

Here, $0=\lambda_{1}<\lambda_{2} \leq \cdots \leq \lambda_{n} \leq \cdots$ with $\lim _{n \rightarrow \infty} \lambda_{n}=+\infty$ is the eigenvalue of the linear operator $-\Delta$ subject to the homogeneous Neumann boundary condition on $\partial \Omega, \varphi_{n}$ is the eigenvector corresponding to $\lambda_{n},\left\{\varphi_{n}\right\}_{n=1}^{+\infty}$ is a complete orthonormal system in the space $L^{2}(\bar{\Omega}), \varphi_{1}(x)>0$ for all $x \in$ $\Omega$, and $\delta(x)$ is the Dirac function on $\mathbb{R}^{m}[10,13]$. Throughout
this paper, we assume that the functions $f$ and $b$ satisfy the following.
(A1) $b: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$is Lipschitz continuous with $b(0)=0$ and $b^{\prime}(0)>0$, and $b(w) \leq b^{\prime}(0) w$ for all $w \geq 0$.
(A2) $f(w)=w g(w)$ for all $w \geq 0$, where $g: \mathbb{R}^{+} \rightarrow \mathbb{R}^{+}$ is Lipschitz continuous with $g^{\prime}(0) \geq 0, g(w)>0$, and $g^{\prime}(w) \geq 0$ for all $w>0$.
(A3) There exists a positive number $M$ such that, for all $w>$ $M, \bar{b}(w)<f(w)$, where $\bar{b}(w)=\max _{u \in[0, w]} b(u)$.

In the monotone case, where the function $b(w)$ increases with $w>0, \mathrm{Xu}$ and Zhao [12] studied the global dynamics of (1) and obtained some results on the uniqueness and global attractivity of a positive steady state by using the theory of monotone dynamical systems. In the case of $f(w)=$ $\mu w$, Zhao [10] proved the global attractivity of the positive constant equilibrium for (1) by using a fluctuation method of Thieme and Zhao [14], where $\mu$ is a positive constant. In the case where $f(w)=\mu w$ and $\alpha=0$, Yi and Zou [7] proved the global attractivity of the unique positive constant equilibrium for (1) by combining a dynamical systems argument and some subtle inequalities. In the case where $\Omega=[0, L]$ and $f(w)=\mu w$, (1) reduces to the equation derived in [3], where the numerical solutions are considered. A global convergence theorem was obtained in [11] for a special case of (1).

The aim of this paper is to establish some criteria to guarantee the global asymptotic stability of the trivial solution and the positive steady state for (1) by using a fluctuation method, combined with the careful analysis of the corresponding characteristic equations. The interesting thing is that main results obtained in this paper extend the related existing results.

The rest of this paper is organized as follows. We will present some preliminary results in Section 2. Our main results are presented and proved in Sections 3 and 4, where we obtain sufficient conditions to ensure the global asymptotic stability of the trivial solution and the positive steady state for (1) in a nonmonotone case. In Section 5, we provide four examples to illustrate the applicability of the main results.

## 2. Preliminaries

Firstly, we show that the kernel $k(\alpha, x, y)$ in (2) enjoys the following properties.

Lemma 1. For $\alpha>0$, one has
(i) $\left.(\partial / \partial \mathbf{n}) k(\alpha, x, y)\right|_{x \in \partial \Omega}=\left.(\partial / \partial \mathbf{n}) k(\alpha, x, y)\right|_{y \in \partial \Omega}=0$,
(ii) $0<k(\alpha, x, y) \leq C^{*}$, for all $x, y \in \Omega$, where $C^{*}$ is a positive constant depending only on $m$ and $\Omega$,
(iii) $\left|\varphi_{n}(x)\right| \leq \sqrt{C^{*}} \exp \left[(1 / 2) \lambda_{n} \alpha\right]$, for all $x \in \Omega, n=$ $1,2, \ldots$,
(iv) $\int_{\bar{\Omega}} k(\alpha, x, y) d y=1$, for all $x \in \Omega$.

Proof. The verification of (i) is straightforward and is thus omitted. Part (ii) follows from [15, Lemma 3.2.1 and Theorem 4.4.6] since $k(\alpha, x, y)$ is a heat kernel of the heat equation
$(\Delta-(\partial / \partial \alpha)) u(x, \alpha)=0$. Part (iii) follows from $e^{-\lambda_{n} \alpha}\left(\varphi_{n}(x)\right)^{2} \leq k(\alpha, x, x) \leq C^{*}$, for all $x \in \Omega, n=1,2, \ldots$. And part (iv) follows from $\lambda_{1}=0, \varphi_{1}(x) \equiv \sqrt{1 / \operatorname{mes}(\Omega)}$, and $\int_{\bar{\Omega}} \varphi_{n}(y) d y=0$ for all $n=2,3, \ldots$, where $\operatorname{mes}(\Omega)$ is the measure of $\Omega$. The proof is completed.

Let $\mathbb{X}=C(\bar{\Omega}, \mathbb{R})$ and $\mathbb{X}^{+}=\{\phi \in \mathbb{X} \mid \phi(x) \geq 0, \forall x \in \bar{\Omega}\}$. Then $\left(\mathbb{X}, \mathbb{X}^{+}\right)$is a strongly ordered Banach space. It is well known that the differential operator $A=d \Delta$ generates a $C^{0}$-semigroup $T(t)$ on $\mathbb{X}$. Moreover, the standard parabolic maximum principle (see, e.g., [16, Corollary 7.2.3]) implies that the semigroup $T(t): \mathbb{X} \rightarrow \mathbb{X}$ is strongly positive in the sense that $T(t)\left(\mathbb{X}^{+} \backslash\{0\}\right) \subset \operatorname{Int}\left(\mathbb{X}^{+}\right), \forall t>0$.

Let $\mathbb{Y}=C([-\tau, 0], \mathbb{X})$ and $\mathbb{Y}^{+}=C\left([-\tau, 0], \mathbb{X}^{+}\right)$. For the sake of convenience, we will identify an element $\phi \in \mathbb{Y}$ as a function from $[-\tau, 0] \times \bar{\Omega}$ to $\mathbb{R}$ defined by $\phi(s, x)=\phi(s)(x)$, and for each $s \in[-\tau, 0]$, we regard $f(\phi(s))$ as a function on $\bar{\Omega}$ defined by $f(\phi(s))=f(\phi(s, \cdot))$. For any function $w(\cdot)$ : $[-\tau, \sigma) \rightarrow \mathbb{X}$, where $\sigma>0$, we define $w_{t} \in \mathbb{Y}, t \in[0, \sigma)$ by $w_{t}(s)=w(t+s), \forall s \in[-\tau, 0]$. Define $F: \mathbb{Y}^{+} \rightarrow \mathbb{X}$ by

$$
\begin{array}{r}
F(\phi)(x)=-f(\phi(0, x))+\int_{\bar{\Omega}} k(\alpha, x, y) b(\phi(-\tau, y)) d y \\
\forall x \in \bar{\Omega}, \phi \in \mathbb{Y}^{+} . \tag{3}
\end{array}
$$

Then we can rewrite (1) as an abstract functional equation:

$$
\begin{gather*}
\frac{d w(t)}{d t}=A w(t)+F\left(w_{t}\right), \quad t \geq 0  \tag{4}\\
w_{0}=\phi \in \mathbb{Y}^{+}
\end{gather*}
$$

Therefore, we can write (4) as an integral equation:

$$
\begin{gather*}
w(t)=T(t) \phi(0)+\int_{0}^{t} T(t-s) F\left(w_{s}\right) d s, \quad t \geq 0  \tag{5}\\
w_{0}=\phi \in \mathbb{Y}^{+}
\end{gather*}
$$

whose solutions are called mild solutions for (1).
Since $T(t): \mathbb{X} \rightarrow \mathbb{X}$ is strongly positive, we have

$$
\begin{equation*}
\lim _{h \rightarrow 0^{+}} \operatorname{dist}\left(\phi(0)+h F(\phi), \mathbb{X}^{+}\right)=0, \quad \forall \phi \in \mathbb{Y}^{+} \tag{6}
\end{equation*}
$$

By [17, Proposition 3 and Remark 2.4] (or [18, Corollary 8.1.3]), for each $\phi \in \mathbb{Y}^{+}$, (1) has a unique noncontinuable mild solution $w(t, \phi)$ with $w_{0}=\phi$, and $w(t, \phi) \in \mathbb{X}^{+}$for all $t \in\left(0, \sigma_{\phi}\right)$. Moreover, $w(t, \phi)$ is a classical solution of (1) for $t>\tau$ (see [18, Corollary 2.2.5]).

By the same arguments as in the proof of [12, Theorems 2.1 and 3.1], we have the following two lemmas.

Lemma 2. Let (A1)-(A3) hold. Then, for each $\phi \in \mathbb{Y}^{+}$, a unique solution $w(t, \phi)$ of (1) globally exists on $[-\tau, \infty)$, $\lim \sup _{t \rightarrow \infty} w(t, x, \phi) \leq M$ uniformly for $x \in \bar{\Omega}$, and the solution semiflow $\Phi(t)=w_{t}(\cdot): \mathbb{Y}^{+} \rightarrow \mathbb{Y}^{+}, t \geq 0$, admits a connected global attractor.

Lemma 3. Let (A1)-(A3) hold, and let $w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^{+}$. Then the following two statements are valid.
(i) If $b^{\prime}(0)<g(0)$, then for any $\phi \in \mathbb{Y}^{+}$, we have $\lim \sup _{t \rightarrow \infty} w(t, x, \phi)=0$ uniformly for $x \in \bar{\Omega}$.
(ii) If $b^{\prime}(0)>g(0)$, then (1) admits at least one spatially homogeneous steady state $w^{*} \in(0, M]$, and there exists $\eta>0$ such that for any $\phi \in \mathbb{Y}^{+}$with $\phi(0, \cdot) \not \equiv \underline{0}$ we have $\lim \inf _{t \rightarrow \infty} w(t, x, \phi) \geq \eta$ uniformly for $x \in \bar{\Omega}$.

Note that in case (ii) above, the function $S(w)=b(w)-$ $f(w)$ satisfies $S(0)=0, S^{\prime}(0)>0$, and $S(M) \leq 0$. Therefore, there exists at least one positive number $w^{*} \in(0, M]$ such that $S\left(w^{*}\right)=0$, and hence, $w^{*}$ is a spatially homogeneous steady state of (1).

## 3. Global Attractivity

In this section, we establish the global attractivity of the positive and spatially homogeneous steady state $w^{*}$ for (1) by the fluctuation method used in [10, Theorem 3.1].

Motivated by [10, Section 3], we assume further that the functions $f(w)$ and $b(w)$ satisfy the following.
(A4) $b^{\prime}(0)>g(0),(b(w) / f(w))$ is strictly decreasing for $w \in(0, M]$, and $f(w)$ and $b(w)$ have the property ( P ) that, for any $u, v \in(0, M]$ satisfying $u \leq w^{*} \leq v$, $f(u) \geq b(v)$, and $f(v) \leq b(u)$, we have $u=v$.

Note that if $b(w)$ is nondecreasing for $w \in[0, M]$, then $f(w)$ and $b(w)$ have the property $(P)$. Indeed, for any $0<u \leq$ $w^{*} \leq v \leq M$ with $f(u) \geq b(v)$ and $f(v) \leq b(u)$, we have

$$
\begin{equation*}
f\left(w^{*}\right) \leq f(v) \leq b(u) \leq b\left(w^{*}\right) \leq b(v) \leq f(u) \leq f\left(w^{*}\right), \tag{7}
\end{equation*}
$$

which implies that $u=v=w^{*}$. Combining this observation and [10, Lemma 3.1] with $\mu w$ replaced by $f(w)$, where $\mu>0$, we then have the following result.

Lemma 4. Either of the following two conditions is sufficient for the property $(P)$ in condition (A4) to hold.
(P1) $b(w)$ is nondecreasing for $w \in[0, M]$.
(P2) $f(w) b(w)$ is strictly increasing for $w \in(0, M]$.
Now we are in a position to prove our main result in this section.

Theorem 5. Assume that (A1)-(A4) hold, and let $w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^{+}$. Then for any $\phi \in \mathbb{Y}^{+}$with $\phi(0, \cdot) \not \equiv 0$, we have $\lim _{t \rightarrow \infty} w(t, x, \phi)=w^{*}$ uniformly for $x \in \bar{\Omega}$.

In order to prove Theorem 5, we will need the following lemma.

Lemma 6. Assume that (A1)-(A3) hold, and let $w(t, x) \equiv$ $w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^{+}$. Then $w(t, x)$ satisfies

$$
\begin{align*}
& w(t, x) \\
& \begin{aligned}
&=e^{-\gamma t} \int_{\Omega} k(d t, x, y) \phi(0, y) d y \\
&+\int_{0}^{t} e^{-\gamma s} \int_{\Omega} k(d s, x, y) \\
& \times {[\gamma w(t-s, y)-f(w(t-s, y))} \\
&\left.+\int_{\Omega} k(\alpha, y, z) b(w(t-s-\tau, z)) d z\right] d y d s
\end{aligned}
\end{align*}
$$

where $\gamma=\max _{w \in[0, M]} f^{\prime}(w)$ and the kernel function $k$ is given in (2).

## Proof. Let

$$
\begin{align*}
H(t, x) \equiv & \gamma w(t, x)-f(w(t, x)) \\
& +\int_{\Omega} k(\alpha, x, y) b(w(t-\tau, y)) d y \tag{9}
\end{align*}
$$

Since $\mathbb{X} \subset L^{2}(\bar{\Omega})$, for each $t \geq 0$, there exist real numbers $a_{n}(t)$ and $b_{n}(t), n=1,2, \ldots$, such that

$$
\begin{align*}
& w(t, x)=\sum_{n=1}^{+\infty} a_{n}(t) \varphi_{n}(x),  \tag{10}\\
& H(t, x)=\sum_{n=1}^{+\infty} b_{n}(t) \varphi_{n}(x) . \tag{11}
\end{align*}
$$

Therefore, by (10), (11), and (1), we have

$$
\begin{gather*}
a_{n}(0)=\int_{\Omega} \phi(0, y) \varphi_{n}(y) d y \\
b_{n}(s)=\int_{\Omega} H(s, y) \varphi_{n}(y) d y  \tag{12}\\
\frac{d a_{n}(t)}{d t}=-\left(d \lambda_{n}+\gamma\right) a_{n}(t)+b_{n}(t), \quad n=1,2, \ldots \tag{13}
\end{gather*}
$$

By using the variation of constants method, we obtain

$$
\begin{array}{r}
a_{n}(t)=\left[a_{n}(0)+\int_{0}^{t} e^{\left(d \lambda_{n}+\gamma\right) s} b_{n}(s) d s\right] e^{-\left(d \lambda_{n}+\gamma\right) t},  \tag{14}\\
n=1,2, \ldots
\end{array}
$$

Thus, by (10), (12), and (14), we further get

$$
\begin{align*}
& w(t, x) \\
&= \sum_{n=1}^{+\infty}\left[a_{n}(0)+\int_{0}^{t} e^{\left(d \lambda_{n}+\gamma\right) s} b_{n}(s) d s\right] e^{-\left(d \lambda_{n}+\gamma\right) t} \varphi_{n}(x) \\
&= e^{-\gamma t} \sum_{n=1}^{+\infty} \int_{\Omega} \phi(0, y) e^{-d \lambda_{n} t} \varphi_{n}(x) \varphi_{n}(y) d y \\
&+\int_{0}^{t} e^{-\gamma(t-s)} \sum_{n=1}^{+\infty} \int_{\Omega} e^{-d \lambda_{n}(t-s)} H(s, y) \varphi_{n}(x) \varphi_{n}(y) d y d s \\
&= e^{-\gamma t} \int_{\Omega} \phi(0, y) k(d t, x, y) d y \\
&+\int_{0}^{t} e^{-\gamma(t-s)} \int_{\Omega} H(s, y) k(d(t-s), x, y) d y d s \\
&= e^{-\gamma t} \int_{\Omega} \phi(0, y) k(d t, x, y) d y \\
&+\int_{0}^{t} e^{-\gamma s} \int_{\Omega} H(t-s, y) k(d s, x, y) d y d s . \tag{15}
\end{align*}
$$

Therefore, (8) follows immediately from (9) and (15). The proof is completed.

Proof of Theorem 5. For any given $\phi \in \mathbb{Y}^{+}$with $\phi(0, \cdot) \not \equiv 0$, let $\omega(\phi)$ be the omega limit set of the positive orbit through $\phi$ for the solution semiflow $\Phi(t)$. By Lemma 1, we get $\omega(\phi) \subset$ $\mathbb{A} \subseteq \mathbb{Y}_{[0, M]}$, where $\mathbb{A}$ is the global attractor of the solution semiflow $\Phi(t)$ and

$$
\begin{equation*}
\mathbb{Y}_{[0, M]} \equiv\{\phi \in \mathbb{Y} \mid 0 \leq \phi(\theta, x) \leq M, \forall(\theta, x) \in[-\tau, 0] \times \bar{\Omega}\} . \tag{16}
\end{equation*}
$$

Note that $\mathbb{A}$ is a maximal compact invariant set of the solution semiflow $\Phi(t)$. Thus, it is sufficient to prove the global attractivity of $w^{*}$ for all $\phi \in \mathbb{Y}_{[0, M]}$ with $\phi(0, \cdot) \not \equiv 0$.

Let $\phi \in \mathbb{Y}_{[0, M]}$ be given such that $\phi(0, \cdot) \not \equiv 0$. Then it follows from Lemma 6 that

$$
\begin{align*}
& w(t, x) \\
& \begin{aligned}
&=e^{-\gamma t} \int_{\Omega} k(d t, x, y) \phi(0, y) d y \\
&+\int_{0}^{t} e^{-\gamma s} \int_{\Omega} k(d s, x, y) \\
& \times[\gamma w(t-s, y)-f(w(t-s, y)) \\
&\left.+\int_{\Omega} k(\alpha, y, z) b(w(t-s-\tau, z)) d z\right] d y d s
\end{aligned}
\end{align*}
$$

where $w(t, x) \equiv w(t, x, \phi)$ is the solution of (1) starting from the initial function $\phi$. Following [19], we define a function $h$ : $[0, M] \times[0, M] \rightarrow \mathbb{R}$ by

$$
h(u, v)= \begin{cases}\min \{b(w) \mid u \leq w \leq v\}, & \text { if } u \leq v  \tag{18}\\ \max \{b(w) \mid v \leq w \leq u\}, & \text { if } v \leq u\end{cases}
$$

Then $h(u, v)$ is nondecreasing in $u \in[0, M]$ and nonincreasing in $v \in[0, M]$. Moreover, $b(w)=h(w, w), \forall w \in[0, M]$, and $h(u, v)$ is continuous in $(u, v) \in[0, M] \times[0, M]$ (see [20, Section 2]). Therefore, by (17), we have

$$
\begin{align*}
& w(t, x) \\
& \begin{aligned}
&=e^{-\gamma t} \int_{\Omega} k(d t, x, y) \phi(0, y) d y \\
&+\int_{0}^{t} e^{-\gamma s} \int_{\Omega} k(d s, x, y) \\
& \quad \times[\gamma w(t-s, y)-f(w(t-s, y)) \\
& \quad+\int_{\Omega} k(\alpha, y, z) \\
&\quad \times h(w(t-s-\tau, z), w(t-s-\tau, z)) d z] d y d s
\end{aligned}
\end{align*}
$$

Let

$$
w^{\infty}(x) \equiv \limsup _{t \rightarrow \infty} w(t, x), \quad w_{\infty}(x) \equiv \liminf _{t \rightarrow \infty} w(t, x)
$$

$$
\begin{equation*}
\forall x \in \bar{\Omega} \tag{20}
\end{equation*}
$$

Then Lemmas 2 and 3 imply that

$$
\begin{equation*}
M \geq w^{\infty}(x) \geq w_{\infty}(x) \geq \eta>0, \quad \forall x \in \bar{\Omega} \tag{21}
\end{equation*}
$$

On the other hand, note that $\gamma=\max _{w \in[0, M]} f^{\prime}(w)$. Therefore, the function $\gamma w-f(w)$ is nondecreasing in $w \in[0, M]$. Thus, by Fatou's lemma and (19), we further get
$w^{\infty}(x)$

$$
\begin{align*}
\leq \int_{0}^{\infty} e^{-\gamma s} \int_{\Omega} & k(d s, x, y) \\
& \times\left[\gamma w^{\infty}(y)-f\left(w^{\infty}(y)\right)\right. \\
& \left.+\int_{\Omega} k(\alpha, y, z) h\left(w^{\infty}(z), w_{\infty}(z)\right) d z\right] d y d s \tag{22}
\end{align*}
$$

Let

$$
\begin{equation*}
w^{\infty} \equiv \sup _{x \in \bar{\Omega}} w^{\infty}(x), \quad w_{\infty} \equiv \inf _{x \in \bar{\Omega}} w_{\infty}(x) \tag{23}
\end{equation*}
$$

Then

$$
\begin{equation*}
M \geq w^{\infty} \geq w_{\infty} \geq \eta>0 \tag{24}
\end{equation*}
$$

Moreover, it follows from Lemma 1 that

$$
\begin{array}{r}
\int_{\Omega} k(d s, x, y) d y=1, \quad \int_{\Omega} k(\alpha, x, y) d y=1  \tag{25}\\
\forall s \geq 0, \quad x \in \bar{\Omega}
\end{array}
$$

Therefore, by (22), we have

$$
\begin{align*}
w^{\infty} & \leq\left[\gamma w^{\infty}-f\left(w^{\infty}\right)+h\left(w^{\infty}, w_{\infty}\right)\right] \int_{0}^{\infty} e^{-\gamma s} d s \\
& =\frac{1}{\gamma}\left[\gamma w^{\infty}-f\left(w^{\infty}\right)+h\left(w^{\infty}, w_{\infty}\right)\right] \tag{26}
\end{align*}
$$

Thus,

$$
\begin{equation*}
f\left(w^{\infty}\right) \leq h\left(w^{\infty}, w_{\infty}\right) \tag{27}
\end{equation*}
$$

Similarly, we have

$$
\begin{equation*}
f\left(w_{\infty}\right) \geq h\left(w_{\infty}, w^{\infty}\right) \tag{28}
\end{equation*}
$$

By (18), we may find $u, v \in\left[w_{\infty}, w^{\infty}\right] \subset(0, M]$ such that

$$
\begin{equation*}
h\left(w^{\infty}, w_{\infty}\right)=b(u), \quad h\left(w_{\infty}, w^{\infty}\right)=b(v) \tag{29}
\end{equation*}
$$

It then follows from (27) and (28) that

$$
\begin{equation*}
b(u) \geq f\left(w^{\infty}\right) \geq f(u), \quad b(v) \leq f\left(w_{\infty}\right) \leq f(v) \tag{30}
\end{equation*}
$$

and hence,

$$
\begin{equation*}
\frac{b(v)}{f(v)} \leq 1=\frac{b\left(w^{*}\right)}{f\left(w^{*}\right)} \leq \frac{b(u)}{f(u)} \tag{31}
\end{equation*}
$$

This, together with the strict monotonicity of $b(w) / f(w)$ for $w \in(0, M]$, implies that $u \leq w^{*} \leq v$. Moreover, by (27) and (28), we also have

$$
\begin{equation*}
b(u) \geq f\left(w^{\infty}\right) \geq f(v), \quad b(v) \leq f\left(w_{\infty}\right) \leq f(u) \tag{32}
\end{equation*}
$$

Therefore, the property $(\mathrm{P})$ implies that

$$
\begin{equation*}
u=v=w^{*} \tag{33}
\end{equation*}
$$

Thus, by (30), we obtain

$$
\begin{equation*}
w^{\infty}=w_{\infty}=w^{*} \tag{34}
\end{equation*}
$$

Since

$$
\begin{equation*}
w^{\infty} \geq w^{\infty}(x) \geq w^{\infty}(x) \geq w^{\infty}, \quad x \in \bar{\Omega} \tag{35}
\end{equation*}
$$

we further get

$$
\begin{equation*}
w^{\infty}(x)=w_{\infty}(x)=w^{*}, \quad x \in \bar{\Omega} \tag{36}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\lim _{t \rightarrow \infty} w(t, x)=w^{*}, \quad x \in \bar{\Omega} \tag{37}
\end{equation*}
$$

It remains to prove that $\lim _{t \rightarrow \infty} w(t, x)=w^{*}$ uniformly for $x \in \bar{\Omega}$. For any $\psi \in \omega(\phi)$, there exists a sequence $t_{n} \rightarrow \infty$ such that $\Phi\left(t_{n}\right) \phi \rightarrow \psi$ in $\mathbb{Y}$ as $n \rightarrow \infty$. Therefore, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} w\left(t_{n}+\theta, x, \phi\right)=\psi(\theta, x) \tag{38}
\end{equation*}
$$

uniformly for $(\theta, x) \in[-\tau, 0] \times \bar{\Omega}$. By (37), we further get

$$
\begin{equation*}
\psi(\theta, x)=w^{*}, \quad \forall(\theta, x) \in[-\tau, 0] \times \bar{\Omega} . \tag{39}
\end{equation*}
$$

Thus, we obtain $\omega(\phi)=\left\{w^{*}\right\}$, which implies that $w(t, \cdot, \phi)$ converges to $w^{*}$ in $\mathbb{X}$ as $t \rightarrow \infty$. The proof is completed.

## 4. Global Asymptotic Stability

In this section, we establish the global asymptotic stability of the trivial solution and the positive and spatially homogeneous steady state $w^{*}$ for (1) by the careful analysis of the corresponding characteristic equations. To this end, we first give the following formal definitions of stability (see, e.g., [18, Remark 2.1.3]).

Definition 7. Let $w=\widehat{w}$ be a steady state of the abstract equation (4). It is called stable if for any $\varepsilon>0$ there exists $\delta>0$ such that the solution $w(t, \phi)$ of (4) with $\|\phi-\widehat{w}\|_{\bigvee}<$ $\delta$ satisfies $\|w(t, \phi)-\widehat{w}\|_{\mathbb{X}}<\varepsilon$, for all $t \geq 0$. It is called unstable if it is not stable. It is asymptotically stable if it is stable and there exists $\delta_{0}>0$ such that the solution $w(t, \phi)$ of (4) with $\|\phi-\widehat{w}\|_{\mathbb{Y}}<\delta_{0}$ satisfies $\lim _{t \rightarrow+\infty}\|w(t, \phi)-\widehat{w}\|_{\mathbb{X}}=$ 0 . It is globally asymptotically stable if it is stable and any solution $w(t, \phi)$ of (4) with arbitrary $\phi \in \mathbb{Y}$ satisfies $\lim _{t \rightarrow+\infty}\|w(t, \phi)-\widehat{w}\|_{\mathbb{X}}=0$.

Let $\widehat{w}$ be a spatially homogeneous steady state for (1) (e.g., the trivial solution and $\left.w^{*}\right)$. Define $G: \mathbb{Y}^{+} \rightarrow \mathbb{X}$ by

$$
\begin{align*}
& G(\phi)(x)=-f^{\prime}(\widehat{w}) \phi(0, x) \\
& \quad+b^{\prime}(\widehat{w}) \int_{\bar{\Omega}} k(\alpha, x, y) \phi(-\tau, y) d y  \tag{40}\\
& \forall x \in \bar{\Omega}, \quad \phi \in \mathbb{Y}^{+}
\end{align*}
$$

where $f^{\prime}(\widehat{w})=\left.(d f(w) / d w)\right|_{w=\widehat{w}}$ and $b^{\prime}(\widehat{w})=(d b(w) /$ $d w)\left.\right|_{w=\widehat{w}}$. Note that $k(\alpha, x, y)$ is given in (2). Then we can write the linearized equation of (1) at $w=\widehat{w}$ as the following abstract functional equation

$$
\begin{gather*}
\frac{d w(t)}{d t}=A w(t)+G\left(w_{t}\right), \quad t \geq 0  \tag{41}\\
w_{0}=\phi \in \mathbb{Y}^{+}
\end{gather*}
$$

where $A$ can be referred to Section 2 .
For each complex number $\lambda$ we define the $\mathbb{X}$-valued linear operator $\Theta(\lambda)$ by

$$
\begin{equation*}
\Theta(\lambda) u=A u-\lambda u+G\left(e^{\lambda} u\right), \quad u \in \operatorname{Dom}(A) \tag{42}
\end{equation*}
$$

where $e^{\lambda .} u \in \mathbb{Y}$ is defined by (note that we use $\mathbb{Y}$ to denote its complexification here)

$$
\begin{equation*}
\left(e^{\lambda \cdot} u\right)(\theta)=e^{\lambda \theta} u, \quad \theta \in[-\tau, 0] \tag{43}
\end{equation*}
$$

We will call $\lambda$ a characteristic value of (41) if there exists $u \in$ $\operatorname{Dom}(A) \backslash\{0\}$ solving the characteristic equation $\Theta(\lambda) u=0$ (see, e.g., [18]). Since $\operatorname{Dom}(A) \subset \mathbb{X} \subset L^{2}(\bar{\Omega})$, for any $u \in$ $\operatorname{Dom}(A) \backslash\{0\}$, there exist complex numbers $a_{n}, n=1,2, \ldots$, such that

$$
\begin{equation*}
u(x)=\sum_{n=1}^{+\infty} a_{n} \varphi_{n}(x) \tag{44}
\end{equation*}
$$

Therefore, by (2), (42), and (44), we have

$$
\begin{align*}
& \Theta(\lambda) u(x) \\
& =d \Delta u(x)-\lambda u(x)-f^{\prime}(\widehat{w}) u(x) \\
& \quad+b^{\prime}(\widehat{w}) \int_{\bar{\Omega}} e^{-\tau \lambda} u(y) k(\alpha, x, y) d y \\
& =\sum_{n=1}^{+\infty} a_{n}\left[-d \lambda_{n}-\lambda-f^{\prime}(\widehat{w})+b^{\prime}(\widehat{w}) e^{-\tau \lambda} e^{-\lambda_{n} \alpha}\right] \varphi_{n}(x) . \tag{45}
\end{align*}
$$

Thus, the characteristic value $\lambda$ of (41) satisfies at least one of the following equations:

$$
\begin{equation*}
\lambda=-d \lambda_{n}-f^{\prime}(\widehat{w})+b^{\prime}(\widehat{w}) e^{-\lambda_{n} \alpha} e^{-\tau \lambda}, \quad n=1,2, \ldots \tag{46}
\end{equation*}
$$

Lemma 8. Assume that (A1)-(A3) hold, and let $\beta$ be the smallest real number such that if $\lambda$ is a characteristic value of (41), then $\operatorname{Re} \lambda \leq \beta$. One has the following:
(i) if $b^{\prime}(\widehat{w})>f^{\prime}(\widehat{w})$, then $\beta>0$,
(ii) if $-f^{\prime}(\widehat{w}) \leq b^{\prime}(\widehat{w})<f^{\prime}(\widehat{w})$, then $\beta<0$,
(iii) if $b^{\prime}(\widehat{w})=f^{\prime}(\widehat{w})$, then $\beta=0$.

Proof. (i) If $b^{\prime}(\widehat{w})>f^{\prime}(\widehat{w})$, then, by (46) and [21, Proposition 4.6], there exists at least one characteristic value $\lambda$ of (41) such that $\operatorname{Re} \lambda>0$. Therefore, $\beta>0$.
(ii) If $-f^{\prime}(\widehat{w}) \leq b^{\prime}(\widehat{w})<f^{\prime}(\widehat{w})$, then since $0=\lambda_{1}<\lambda_{2} \leq$ $\cdots \leq \lambda_{n} \leq \cdots$, we have

$$
\begin{align*}
-\left[d \lambda_{n}+f^{\prime}(\widehat{w})\right] e^{\lambda_{n} \alpha} & \leq b^{\prime}(\widehat{w}) \\
& <\left[d \lambda_{n}+f^{\prime}(\widehat{w})\right] e^{\lambda_{n} \alpha}, \quad n=1,2, \ldots \tag{47}
\end{align*}
$$

Therefore, by (46) and [21, Proposition 4.6], all the characteristic values of (41) have negative real parts. Thus, it follows from [18, Theorem 3.1.10] that $\beta<0$.
(iii) If $b^{\prime}(\widehat{w})=f^{\prime}(\widehat{w})$, then $\lambda=0$ is a characteristic value of (41). Therefore, $\beta \geq 0$. If $\beta>0$, then there exists at least one characteristic value of (41) $\lambda^{(0)}$ and a positive number $n$ such that $\operatorname{Re} \lambda^{(0)}>0$ and

$$
\begin{equation*}
\lambda^{(0)}=-d \lambda_{n}-f^{\prime}(\widehat{w})+b^{\prime}(\widehat{w}) e^{-\lambda_{n} \alpha} e^{-\tau \lambda^{(0)}} \tag{48}
\end{equation*}
$$

Let $\lambda^{(0)}=x^{(0)}+i y^{(0)}$, where $x^{(0)}$ and $y^{(0)}$ both are real numbers. Then $x^{(0)}>0$. By (48), we have

$$
\begin{equation*}
x^{(0)}=-d \lambda_{n}-f^{\prime}(\widehat{w})+f^{\prime}(\widehat{w}) e^{-\lambda_{n} \alpha} e^{-\tau x^{(0)}} \cos \left(\tau y^{(0)}\right) \tag{49}
\end{equation*}
$$

and hence, $\cos \left(\tau y^{(0)}\right)>0$. This implies that

$$
\begin{align*}
x^{(0)} & \leq-f^{\prime}(\widehat{w})+f^{\prime}(\widehat{w}) e^{-\tau x^{(0)}} \cos \left(\tau y^{(0)}\right) \\
& =f^{\prime}(\widehat{w})\left[e^{-\tau x^{(0)}} \cos \left(\tau y^{(0)}\right)-1\right] \tag{50}
\end{align*}
$$

But, since $x^{(0)}>0$, we have $e^{-\tau x^{(0)}} \cos \left(\tau y^{(0)}\right)<1$. Therefore,

$$
\begin{equation*}
x^{(0)} \leq f^{\prime}(\widehat{w})\left[e^{-\tau x^{(0)}} \cos \left(\tau y^{(0)}\right)-1\right]<0 \tag{51}
\end{equation*}
$$

contradicting $x^{(0)}>0$. This contradiction proves $\beta=0$. The proof is completed.

Now we are ready to summarize our main results on the global stability. By Definition 7, Lemmas 3 and 8, Theorem 5, [18, Corollary 3.1.11], and the principle of linearized stability (see, e.g., [21]), we obtain the following.

Theorem 9. Assume that (A1)-(A3) hold. Then the following two statements are valid.
(i) If $b^{\prime}(0)<g(0)$, then the zero solution of (1) is globally asymptotically stable in $\mathbb{Y}^{+}$.
(ii) $I f b^{\prime}(0)>g(0)$, then the zero solution of (1) is unstable, and (1) admits at least one spatially homogeneous steady state $w^{*} \in(0, M]$.

Theorem 10. Assume that (A1)-(A3) hold, and $b^{\prime}(0)>g(0)$. Then the following two statements for the positive and spatially homogeneous steady state $w^{*}$ of (1) are valid.
(i) If $b^{\prime}\left(w^{*}\right)>f^{\prime}\left(w^{*}\right)$, then $w^{*}$ is unstable.
(ii) If $-f^{\prime}\left(w^{*}\right) \leq b^{\prime}\left(w^{*}\right)<f^{\prime}\left(w^{*}\right)$ and (A4) hold, then $w^{*}$ is globally asymptotically stable in $\mathbb{Y}^{+} \backslash\{0\}$.

## 5. Examples

In this section, we present four examples to illustrate the feasibility of our main results.

Example 1. Consider the equation resulting from letting $f(w)=\mu w$ in (1); that is,

$$
\begin{align*}
& \begin{array}{l}
\frac{\partial w(t, x)}{\partial t}= \\
\quad d \Delta w(t, x)-\mu w(t, x) \\
\quad \\
\quad \int_{\Omega} k(\alpha, x, y) b(w(t-\tau, y)) d y, \\
\frac{\partial w(t, x)}{\partial \mathbf{n}}=0, \quad t>0, x \in \partial \Omega, \\
w(t, x)=
\end{array} \quad \phi(t, x) \geq 0, \quad t \in[-\tau, 0], x \in \Omega,
\end{align*}
$$

where $\mu$ is a positive constant.
In this case, we now formulate the following assumptions to replace (A2)-(A4):
(A2 ${ }^{\prime}$ ) There exists a positive number $M$ such that, for all $w>$ $M, \bar{b}(w)<\mu w$, where $\bar{b}(w)=\max _{u \in[0, w]} b(u)$.
$\left(\mathrm{A}^{\prime}\right) b^{\prime}(0)>\mu,(b(w) / w)$ is strictly decreasing for $w \in$ $(0, M]$, and $b(w)$ has the property $\left(P^{\prime}\right)$ that, for any $u, v \in(0, M]$ satisfying $u \leq w^{*} \leq v, \mu u \geq b(v)$, and $\mu v \leq b(u)$, we have $u=v$.

By applying Theorems 9 and 10, we then obtain the following results for (52).

Theorem 11. Assume that (A1) and (A2') hold. Then the following two statements are valid.
(i) If $b^{\prime}(0)<\mu$, then the zero solution of (52) is globally asymptotically stable in $\mathbb{Y}^{+}$.
(ii) If $b^{\prime}(0)>\mu$, then the zero solution of (52) is unstable, and (52) admits at least one spatially homogeneous steady state $w^{*} \in(0, M]$.

Theorem 12. Assume that (A1) and $\left(A 2^{\prime}\right)$ hold, and $b^{\prime}(0)>\mu$. Then the following two statements for the positive and spatially homogeneous steady state $w^{*}$ of (52) are valid.
(i) If $b^{\prime}\left(w^{*}\right)>\mu$, then $w^{*}$ is unstable.
(ii) If $-\mu \leq b^{\prime}\left(w^{*}\right)<\mu$ and $\left(A 3^{\prime}\right)$ hold, then $w^{*}$ is globally asymptotically stable in $\mathbb{Y}^{+} \backslash\{0\}$.

Remark 13. It is easy to see that (52) is discussed in [10] and some partial results of Theorems 11 and 12 have been obtained [10].

Example 2. Consider the following Nicholson's blowfly equation resulting from letting $f(w)=\mu w$ and $b(w)=p w e^{-q w}$ in (1):

$$
\begin{align*}
& \frac{\partial w(t, x)}{\partial t}= d \Delta w(t, x)-\mu w(t, x) \\
&+\int_{\Omega} p w(t-\tau, y) e^{-q w(t-\tau, y)} k(\alpha, x, y) d y \\
& \frac{\partial w(t, x)}{\partial \mathbf{n}}=0, \quad t>0, x \in \partial \Omega \\
& w(t, x)=\phi(t, x) \geq 0, \quad t \in[-\tau, 0], x \in \Omega \tag{53}
\end{align*}
$$

where $p$ and $q$ are two positive constants.
By the same arguments as in [10, Section 4], together with Theorems 11 and 12, we have the following results for (53).

Theorem 14. (i) If $p<\mu$, then the zero solution of (53) is globally asymptotically stable in $\mathbb{Y}^{+}$.
(ii) If $p>\mu$, then the zero solution of (53) is unstable, and (53) admits the unique positive constant equilibrium $w^{*}=$ $(\ln (p / \mu)) / q$.

Theorem 15. If $\mu<p \leq e^{2} \mu$, the unique positive constant equilibrium $w^{*}=(\ln (p / \mu)) / q$ of (53) is globally asymptotically stable in $\mathbb{Y}^{+} \backslash\{0\}$.

Remark 16. It is easy to see that the equation in [7] is a special case of $\alpha=0$ of (53). Hence all main results of [7] are special cases of our Theorems 14 and 15.

Example 3. Consider the following Mackey-Glass equation resulting from letting $f(w)=\mu w^{l+1}$ and $b(w)=p w /\left(q+w^{l}\right)$ in (1):

$$
\begin{align*}
& \begin{array}{l}
\frac{\partial w(t, x)}{\partial t}= \\
\quad d \Delta w(t, x)-\mu w^{l+1}(t, x) \\
\\
\quad+\int_{\Omega} \frac{p w(t-\tau, y)}{q+w^{l}(t-\tau, y)} k(\alpha, x, y) d y \\
\frac{\partial w(t, x)}{\partial \mathbf{n}}=0, \quad t>0, x \in \partial \Omega \\
w(t, x)=
\end{array} \quad \phi(t, x) \geq 0, \quad t \in[-\tau, 0], x \in \Omega
\end{align*}
$$

where $l$ is a positive constant.
By the same arguments as in [10, Section 4], together with Theorems 9 and 10, we have the following results for (54).

Theorem 17. The zero solution of (54) is always unstable, and (54) must admit the unique positive constant equilibrium $w^{*}=$ $\left(Z_{0}\right)^{1 / l}$, where

$$
\begin{equation*}
Z_{0}=\frac{1}{2}\left(-q+\sqrt{q^{2}+\frac{4 p}{\mu}}\right) \tag{55}
\end{equation*}
$$

Theorem 18. If $\left(\left(p q+p(1-l) Z_{0}\right) /\left(q+Z_{0}\right)^{2}\right)>\mu(1+l) Z_{0}$, the unique positive constant equilibrium $w^{*}=\left(Z_{0}\right)^{1 / l}$ is unstable, and if $-\mu(1+l) Z_{0} \leq\left(\left(p q+p(1-l) Z_{0}\right) /\left(q+Z_{0}\right)^{2}\right)<\mu(1+l) Z_{0}$ it is globally asymptotically stable in $\mathbb{Y}^{+} \backslash\{0\}$.

Example 4. Consider the equation resulting from letting $f(w)=\mu w^{2}$ and $b(w)=p w(1-(w / r))$ in (1); that is,

$$
\begin{align*}
& \frac{\partial w(t, x)}{\partial t} \\
& =d \Delta w(t, x)-\mu w^{2}(t, x) \\
& \quad+\int_{\Omega} p w(t-\tau, y)\left(1-\frac{w(t-\tau, y)}{r}\right) k(\alpha, x, y) d y, \\
& \quad \frac{\partial w(t, x)}{\partial \mathbf{n}}=0, \quad t>0, x \in \partial \Omega, \\
& \quad w(t, x)=\phi(t, x) \geq 0, \quad t \in[-\tau, 0], x \in \Omega \tag{56}
\end{align*}
$$

where $0<r \leq+\infty$.
Clearly, $w^{*}=\left(p /\left(\mu+p r^{-1}\right)\right), \max _{w \geq 0} b(w)=b(r / 2)$, and (A1)-(A3) hold, where $r^{-1}=0$ if $r=+\infty$. Moreover, $0<$ $w^{*} \leq r / 2$ if $\mu \geq p r^{-1}$. Therefore, (A4) is satisfied if $\mu \geq p r^{-1}$. Thus, Theorems 9 and 10 imply the following results.

Theorem 19. The zero solution of (56) is always unstable, and (56) must admit the unique positive constant equilibrium $w^{*}=$ $\left(p /\left(\mu+p r^{-1}\right)\right)$.

Theorem 20. If $\mu \geq p r^{-1}$, then the unique positive constant equilibrium $w^{*}=\left(p /\left(\mu+p r^{-1}\right)\right)$ is globally asymptotically stable in $\mathbb{Y}^{+} \backslash\{0\}$.

Remark 21. It is easy to see that the equation in [11] is a special case of $r=+\infty$ of (56) and hence some partial results of Theorems 19 and 20 have been obtained [11].

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Necessary and Sufficient Conditions of Oscillation in First Order Neutral Delay Differential Equations 

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#### Abstract

We are concerned with oscillation of the first order neutral delay differential equation $[x(t)-p x(t-\tau)]^{\prime}+q x(t-\sigma)=0$ with constant coefficients, and we obtain some necessary and sufficient conditions of oscillation for all the solutions in respective cases $0<p<1$ and $p>1$.


## 1. Introduction

Delay differential equations (DDEs) arose widely in many fields, like oscillation theory [1-9], stability theory [10-12], dynamical behavior of delayed network systems [13-15], and so on. Theoretical studies on oscillation of solutions for DDEs have fundamental significance (see [16, 17]). For this reason, DDEs have been attracting great interest of many mathematicians during the last few decades.

In this paper, we consider a class of neutral DDEs

$$
\begin{equation*}
[x(t)-p x(t-\tau)]^{\prime}+q x(t-\sigma)=0, \quad t \geqslant t_{0} \tag{1}
\end{equation*}
$$

where $t_{0}$ is a positive number and $p, q, \tau$, and $\sigma$ are positive constants. Generally, a solution of (1) is called oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is nonoscillatory. It can be seen in the literature that the oscillation theory regarding solutions of (1) has been extensively developed in the recent years.

In [18], Zhang came to the following conclusion.
Theorem I. Assume that $p \in(0,1)$ and $q \sigma e>1-p$; then all solutions of (1) are oscillatory.

This result in Theorem I improves the corresponding result in [19]. Afterward, many authors have been devoted to studying this problem and have obtained many better
results. For details, Gopalsamy and Zhang [20] obtained the improved result shown in Theorem II.

Theorem II. If $p \in(0,1)$ and $q \sigma e>1-p[1+q \tau /(1-p)]$, then all solutions of (1) are oscillatory.

Further, Zhou and Yu [21] proved the following theorem.
Theorem III. Suppose that $p \in(0,1)$ and $q \sigma e>1-p[1+$ $\left.q \tau /(1-p)+(q \tau)^{2} / 2(1-p)^{2}\right]$; then all solutions of (1) are oscillatory.

Continuing to improve the research work, Xiao and Li [22] obtained the following.

Theorem IV. Let $p \in(0,1)$ and $q \sigma e>1-p e^{q \tau /(1-p)}$; then all solutions of (1) are oscillatory.

Finally, Lin [23] obtained the result shown in Theorem $V$.
Theorem V. Assume that $p \in(0,1)$ and qбe $>1$ $p e^{q \tau /(1-p-q \sigma)}$; then all solutions of (1) are oscillatory.

However, all the conclusions mentioned above are limited to sufficient conditions in the case $0<p<1$. The aim of this paper is to establish systematically the necessary and sufficient conditions of oscillation for all solutions of (1) for the cases $0<$ $p<1$ and $p>1$.

## 2. Main Results

It is well known [24] that all solutions of (1) are oscillatory if and only if the characteristic equation of (1)

$$
\begin{equation*}
f(\lambda) \equiv \lambda-p \lambda e^{-\lambda \tau}+q e^{-\lambda \sigma}=0 \tag{2}
\end{equation*}
$$

has no real roots.
Theorem 1. Assume that $p \in(0,1)$ and let

$$
\begin{align*}
& \varphi(\mu):=q(\sigma \mu-1)+p \tau \mu^{2} e^{(\tau-\sigma) \mu}  \tag{3}\\
& h(\mu):=q e^{\mu \sigma}[(\tau-\sigma) \mu+1]-\tau \mu^{2} \tag{4}
\end{align*}
$$

Then all solutions of (1) are oscillatory if and only if

$$
\begin{equation*}
h(\theta)=q e^{\theta \sigma}[(\tau-\sigma) \theta+1]-\tau \theta^{2}>0 \tag{5}
\end{equation*}
$$

where $\theta$ is a unique zero of $\varphi(\mu)$ in $(0,1 / \sigma)$.
Proof. It is easy to see that, for $\lambda \geqslant 0$, we have

$$
\begin{equation*}
f(\lambda)=\lambda\left(1-p e^{-\lambda \tau}\right)+q e^{-\lambda \sigma} \geq q e^{-\lambda \sigma}>0 \tag{6}
\end{equation*}
$$

Thus any real root of (2) must be negative.
Next, let

$$
\begin{equation*}
g(\mu)=\frac{q}{\mu} e^{\mu \sigma}+p e^{\mu \tau}-1=0 \tag{7}
\end{equation*}
$$

We consider the monotonicity of the function $g(\mu)$ := $f(-\mu) / \mu$. Differentiation yields

$$
\begin{equation*}
g^{\prime}(\mu)=\frac{e^{\mu \sigma} \varphi(\mu)}{\mu^{2}} \tag{8}
\end{equation*}
$$

where $\varphi(\mu)$ satisfies the following properties:
(1) $\varphi(\mu)>0$ for $\mu \in(1 / \sigma,+\infty)$;
(2) $\varphi(\mu)$ is strictly increasing on $(0,1 / \sigma)$ since the function $\mu^{2} e^{(\tau-\sigma) \mu}$ is strictly increasing on $(0,1 / \sigma)$.

In addition,

$$
\begin{equation*}
\varphi(0)=-q<0, \quad \varphi\left(\frac{1}{\sigma}\right)=p \tau \frac{1}{\sigma^{2}} e^{(\tau-\sigma) / \sigma}>0 \tag{9}
\end{equation*}
$$

Thus, we get that function $\varphi(\mu)$ has a unique zero $\theta$ in $(0,1 / \sigma)$. Hence $g^{\prime}(\mu)<0$ for $\mu \in(0, \theta)$ and $g^{\prime}(\mu)>0$ for $\mu \in(\theta,+\infty)$, which imply that $g(\mu)$ is decreasing on $(0, \theta)$ and increasing on $(\theta,+\infty)$. Therefore, $g(\mu)>0$ for $\mu \in(0,+\infty)$ if and only if (7) has no real roots in $\mu \in(0,1 / \sigma)$. It is easy to see that $g(\theta)$ is the minimum value of $g(\mu)$ in $(0,1 / \sigma)$. Consequently, $g(\mu)=0$ has no real roots in $(0,1 / \sigma)$ if and only if $g(\theta)>0$. Since

$$
\begin{equation*}
g(\theta)=\frac{q}{\theta} e^{\theta \sigma}+p e^{\theta \tau}-1=\frac{h(\theta)}{\tau \theta^{2}} \tag{10}
\end{equation*}
$$

we obtain the result immediately.
From Theorem 1, we obtain immediately the following.

Corollary 2. If $p \in(0,1)$ and $\tau=\sigma$, then all solutions of (1) are oscillatory if and only if $q e^{\theta \sigma}>\sigma \theta^{2}$ holds, where $\theta=$ $(\sqrt{q \sigma(q \sigma+4 p)}-q \sigma) / 2 p \sigma$.

Theorem 3. Suppose that $p \in(0,1)$; then all solutions of (1) are oscillatory if and only if one of the following conditions holds:

$$
\begin{aligned}
& \left(H_{1}\right) q \sigma e \geqslant 1 ; \\
& \left(H_{2}\right) \bar{\theta}>\theta,
\end{aligned}
$$

where $\theta$ and $\bar{\theta}$ are the unique zeros of $\varphi(\mu)$ and $h(\mu)$ (see (3) and (4)) in $(0,1 / \sigma)$, respectively.

Proof. Let $y(\mu)=h(\mu) / \mu^{2}=q e^{\mu \sigma}\left((\tau-\sigma) / \mu+1 / \mu^{2}\right)-\tau$; then

$$
\begin{equation*}
y^{\prime}(\mu)=\frac{q e^{\mu \sigma} z(\mu)}{\mu^{3}} \tag{11}
\end{equation*}
$$

where $z(\mu)=(\tau-\sigma) \sigma \mu^{2}+(2 \sigma-\tau) \mu-2$, which satisfies

$$
\begin{equation*}
z(0)=-2<0, \quad z\left(\frac{1}{\sigma}\right)=-1<0 \tag{12}
\end{equation*}
$$

If $\tau \geqslant \sigma$, we get obviously that $z(\mu)<0$ for all $\mu \in(0,1 / \sigma]$. If $\tau<\sigma$, we also get $z(\mu)<0$ for all $\mu \in(0,1 / \sigma]$ since $z^{\prime}(1 / \sigma)=$ $\tau>0$. Thus, $z(\mu)<0$ for all $\mu \in(0,1 / \sigma]$. From this and (11) we get that $y^{\prime}(\mu)<0$ for all $\mu \in(0,1 / \sigma]$. Consequently, $y(\mu)$ is strictly decreasing on $(0,1 / \sigma]$. Further,

$$
\begin{equation*}
\lim _{\mu \rightarrow 0^{+}} y(\mu)=+\infty, \quad y\left(\frac{1}{\sigma}\right)=(q e \sigma-1) \tau \tag{13}
\end{equation*}
$$

Therefore, if $q \sigma e \geqslant 1$, we have $y(\theta)>0$. Hence $h(\theta)>0$. If $q \sigma e<1$, we have $y(1 / \sigma)<0$. Hence, it is easy to find that both functions $y(\mu)$ and $h(\mu)$ have an equal and unique zero $\bar{\theta} \in$ $(0,1 / \sigma)$. Consequently, $h(\theta)>0$ is equivalent to $\bar{\theta}>\theta$.

From Theorem 1, all solutions of (1) are oscillatory if and only if one of $\left(\mathrm{H}_{1}\right)$ or $\left(\mathrm{H}_{2}\right)$ holds.

Theorem 4. Assume that $p \in(0,1)$; then all solutions of (1) are oscillatory if one of the following conditions holds:

$$
\begin{aligned}
& \left(H_{1}\right) q / \theta+q \sigma \geqslant 1-p \\
& \left(H_{2}\right) q \sigma e \geqslant 1-p e^{q \tau /(1-p-q \sigma)}
\end{aligned}
$$

where $\theta$ is a unique zero of $\varphi(\mu)$ in $(0,1 / \sigma)$.
Proof. If $q / \theta+q \sigma \geqslant 1-p$, we have that

$$
\begin{align*}
g(\mu) & =\frac{q}{\mu} e^{\mu \sigma}+p e^{\mu \tau}-1>\frac{q}{\mu}(1+\mu \sigma)+p-1 \\
& =\frac{q}{\mu}+q \sigma+p-1 . \tag{14}
\end{align*}
$$

From the proof of Theorem 1, all solutions of (1) are oscillatory.

If $q \sigma e \geqslant 1-p e^{q \tau /(1-p-q \sigma)}$, we suppose furthermore that $q / \theta+q \sigma<1-p$ (otherwise, all solutions of (1) are oscillatory by the above conclusion); that is, $\theta>q /(1-p-q \sigma)$. Since $q \sigma e$ is a minimum value of the function $(q / \mu) e^{\mu \sigma}$ at $\mu=1 / \sigma$, we have that

$$
\begin{equation*}
g(\theta)=\frac{q}{\theta} e^{\theta \sigma}+p e^{\theta \tau}-1>q \sigma e+p e^{q \tau /(1-p-q \sigma)}-1 \geqslant 0 \tag{15}
\end{equation*}
$$

and the result follows.
So far, for $p \in(0,1)$ we have discussed the necessary and sufficient conditions of oscillation for all solutions of (1). Our results have perfected the results in [23] (see Theorem 4). Next, we will discuss the behavior of oscillation of solutions of (1) in the case $p>1$.

Lemma 5. Let $p>1$; then all solutions of (1) are oscillatory if and only if the equation

$$
\begin{equation*}
g(\mu)=\frac{q}{\mu} e^{\mu \sigma}+p e^{\mu \tau}-1=0 \tag{16}
\end{equation*}
$$

has no real roots in $(-\ln p / \tau, 0)$.
Proof. By (14), we know that $g(\mu)>0$ for $\mu \in(0, \infty)$. It is not difficult to see that $e^{\mu \sigma} / \mu$ is strictly decreasing on $(-\infty, 0)$ while $e^{\mu \tau}$ is strictly increasing on $(-\infty, 0)$. Notice that $p e^{\mu \tau}-$ $1=0$ at $\mu=-\ln p / \tau$; we find that

$$
\begin{equation*}
g(\mu)<0 \quad \text { for } u \in\left(-\infty, \frac{-\ln p}{\tau}\right] . \tag{17}
\end{equation*}
$$

Hence, $f(\lambda)$ has no real roots which is equivalent to $g(\mu)$ that has no real roots in $(-\ln p / \tau, 0)$.

Theorem 6. Suppose that $p>1$ and $\tau=\sigma$; then all solutions of (1) are oscillatory if and only if

$$
\begin{equation*}
q e^{\theta \sigma}<\sigma \theta^{2} \tag{18}
\end{equation*}
$$

where $\theta=(-\sqrt{q \sigma(q \sigma+4 p)}-q \sigma) / 2 p \sigma$.
Proof. It is similar to the proof of Theorem 1; $g(\theta)$ is the maximum value of $g(\mu)$ for $\mu \in(-\infty, 0)$. This and Lemma 5 imply the result.

Theorem 7. Assume that $p>1$ and $\tau<\sigma$; then all solutions of (1) are oscillatory if and only if

$$
\begin{equation*}
h(\theta)=q e^{\theta \sigma}[(\tau-\sigma) \theta+1]-\tau \theta^{2}<0 \tag{19}
\end{equation*}
$$

where $\theta$ is a unique zero of (3) in $(-\infty, 0)$.
Proof. Firstly, we prove that $\varphi(\mu)$ has a unique zero $\theta$ in $(-\infty, 0)$. In fact,

$$
\begin{equation*}
\varphi^{\prime}(\mu)=p \tau e^{(\tau-\sigma) \mu}\left[(\tau-\sigma) \mu^{2}+2 \mu\right]+q \sigma \tag{20}
\end{equation*}
$$

It is easy to verify that $\varphi^{\prime}(\mu)$ is strictly increasing on $(-\infty, 0)$. In addition,

$$
\begin{equation*}
\varphi^{\prime}(0)=q \sigma>0, \quad \varphi^{\prime}(\mu) \rightarrow-\infty(\mu \rightarrow-\infty) \tag{21}
\end{equation*}
$$

Therefore, $\varphi^{\prime}(\mu)$ has a unique zero $\omega_{0}$ in ( $-\infty, 0$ ). Hence, $\varphi(\mu)$ is strictly decreasing on $\left(-\infty, \omega_{0}\right)$ and strictly increasing on $\left(\omega_{0}, 0\right)$, so that $\varphi(\mu)$ has a unique zero $\theta$ in $(-\infty, 0)$ as $\varphi(0)=$ $-q<0$ and $\varphi(\mu) \rightarrow+\infty(\mu \rightarrow-\infty)$.

Now, from (8), it follows that $g(\theta)$ is the maximum value of $g(\mu)$ in $(-\infty, 0)$. By (10), we know that (19) is equivalent to $g(\mu)<0$ for $\mu \in(-\infty, 0)$.

From Theorem 7, we obtain the following corollary that extends Theorem 1 in [25] for $\tau<\sigma$.

Corollary 8. If $p>1, \tau<\sigma$, and $\tau q e^{-(\sigma / \tau) \ln p} \geqslant$ $\tau \ln ^{2} p /(\sigma \ln p+\tau)$, then all solutions of (1) are oscillatory.

Proof. The inequality $\tau q e^{-(\sigma / \tau) \ln p} \geqslant \tau \ln ^{2} p /(\sigma \ln p+\tau)$ is equivalent to $\varphi(-\ln p / \tau) \leqslant 0$. From the proof of Theorem 7, we get that $\theta \leqslant-\ln p / \tau$. This and (17) imply $g(\theta)<0$; therefore, $h(\theta)<0$.

Theorem 9. Suppose that $p>1$ and $\tau>\sigma$; then all solutions of (1) are oscillatory if and only if one of the following conditions holds:

$$
\begin{aligned}
& \left(H_{1}\right) q \sigma e^{2-\sqrt{2}} \geqslant(2 \sqrt{2}-2) p \tau /(\tau-\sigma) \\
& \left(H_{2}\right) \sigma(\tau-\sigma) \omega_{1}^{2}+(2 \sigma-\tau) \omega_{1} \leqslant 2 \\
& \left(H_{3}\right) h\left(\theta_{2}\right)=q e^{\theta_{2} \sigma}\left[(\tau-\sigma) \theta_{2}+1\right]-\tau \theta_{2}^{2}<0
\end{aligned}
$$

where $\omega_{1}$ is a unique zero of $\varphi^{\prime}(\mu)$ in $(-2 /(\tau-\sigma),(\sqrt{2}-2) /(\tau-$ $\sigma)$ ) and $\theta_{2}$ is the maximum negative zero of $\varphi(\mu)$.

Proof. By Lemma 5, all solutions of (1) are oscillatory if and only if

$$
\begin{equation*}
g(\mu)<0, \quad \text { for } \mu \in(-\infty, 0) \tag{22}
\end{equation*}
$$

From (20), we have that

$$
\begin{equation*}
\varphi^{\prime}(\mu)>0 \quad \text { for } \mu \in\left(-\infty, \frac{-2}{\tau-\sigma}\right] \tag{23}
\end{equation*}
$$

and $\varphi^{\prime}(\mu)$ is strictly decreasing on $(-2 /(\tau-\sigma),(\sqrt{2}-2) /(\tau-\sigma))$ and strictly increasing on $((\sqrt{2}-2) /(\tau-\sigma), 0)$. Thus, $\varphi^{\prime}((\sqrt{2}-$ $2) /(\tau-\sigma))$ is the minimum value of $\varphi^{\prime}(\mu)$ in $(-2 /(\tau-\sigma), 0)$.
(1) If $\varphi^{\prime}((\sqrt{2}-2) /(\tau-\sigma)) \geqslant 0$, which is the case of $\left(\mathrm{H}_{1}\right)$, we have that

$$
\begin{equation*}
\varphi^{\prime}(\mu) \geqslant 0, \quad \mu \in\left(\frac{-2}{\tau-\sigma}, 0\right) \tag{24}
\end{equation*}
$$

Combining (23) and (24), we obtain that

$$
\begin{equation*}
\varphi(\mu) \leqslant \varphi(0)=-q<0, \quad \mu \in(-\infty, 0) \tag{25}
\end{equation*}
$$

This means that $g(\mu)$ is strictly decreasing on $(-\infty, 0)$ and, consequently,

$$
\begin{equation*}
g(\mu)<\lim _{\mu \rightarrow-\infty} g(\mu)=-1 \tag{26}
\end{equation*}
$$

(2) If $\varphi^{\prime}((\sqrt{2}-2) /(\tau-\sigma))<0, \varphi^{\prime}(\mu)$ has a unique zero $\omega_{1}$ in $(-2 /(\tau-\sigma),(\sqrt{2}-2) /(\tau-\sigma))$ and a unique zero $\omega_{2}$ in
$((\sqrt{2}-2) /(\tau-\sigma), 0)$ since $\varphi^{\prime}(-2 /(\tau-\sigma))=q \sigma>0$. Hence $\varphi(\mu)$ is strictly increasing on $\left(-\infty, \omega_{1}\right)$, strictly decreasing on ( $\omega_{1}, \omega_{2}$ ), and strictly increasing on ( $\left.\omega_{2}, 0\right)$. Consequently, $\varphi\left(\omega_{1}\right)$ is the maximum value of $\varphi(\mu)$ in $\left(-\infty, \omega_{2}\right)$. Now, it is easy to find that (22) holds if $\varphi\left(\omega_{1}\right) \leqslant 0$.

On the other hand, applying $\varphi^{\prime}\left(\omega_{1}\right)=0$, we can get

$$
\begin{equation*}
\varphi\left(\omega_{1}\right)=\frac{q\left[\sigma(\tau-\sigma) \omega_{1}^{2}+(2 \sigma-\tau) \omega_{1}-2\right]}{(\tau-\sigma) \omega_{1}+2} \tag{27}
\end{equation*}
$$

So $\varphi\left(\omega_{1}\right) \leqslant 0$ is equivalent to $\sigma(\tau-\sigma) \omega_{1}^{2}+(2 \sigma-\tau) \omega_{1} \leqslant 2$. This is the case of $\left(\mathrm{H}_{2}\right)$.

If $\varphi\left(\omega_{1}\right)>0$, we obtain that $\varphi(\mu)$ has a unique zero $\theta_{1}$ in $\left(-\infty, \omega_{1}\right)$ and a unique zero $\theta_{2}$ in $\left(\omega_{1}, \omega_{2}\right)$. Therefore, $g(\mu)$ is strictly decreasing on $\left(-\infty, \theta_{1}\right)$, strictly increasing on $\left(\theta_{1}, \theta_{2}\right)$, and strictly decreasing on $\left(\theta_{2}, 0\right)$. Therefore, it is not difficult to find that (22) holds if and only if $g\left(\theta_{2}\right)<0$ and it is the case of $\left(\mathrm{H}_{3}\right)$.

From Theorem 9, we obtain the following corollary immediately.

Corollary 10. If $p>1, \tau>\sigma$, and $q \sigma \geqslant p \tau /(\tau-\sigma)$, then all solutions of (1) are oscillatory.

Example 11. Consider the following neutral delay differential equation:

$$
\begin{equation*}
[x(t)-20 x(t-12)]^{\prime}+10.5 x(t-2)=0 . \tag{28}
\end{equation*}
$$

It is not difficult to see that $p=20, q=10.5, \tau=12$, and $\sigma=2$. Consequently, $\tau>\sigma$, and

$$
\begin{align*}
& q \sigma e^{2-\sqrt{2}}-\frac{(2 \sqrt{2}-2) p \tau}{\tau-\sigma}>21(3-\sqrt{2})-24(2 \sqrt{2}-2) \\
& \quad=3(37-23 \sqrt{2})>0 \tag{29}
\end{align*}
$$

so that all the solutions of (28) are oscillatory from Theorem 9.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Multiplicity of Periodic Solutions for a Higher Order Difference Equation 

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We study a higher order difference equation. By Lyapunov-Schmidt reduction methods and computations of critical groups, we prove that the equation has four $M$-periodic solutions.

## 1. Introduction

Considering the following higher order difference equation

$$
\begin{equation*}
\sum_{i=0}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f\left(n, x_{n}\right)=0, \quad n \in \mathbf{Z} \tag{1}
\end{equation*}
$$

where $k \in \mathbf{N}, \mathbf{N}$ and $\mathbf{Z}$ are the sets of all positive integers and integers, respectively, $f \in C^{1}(\mathbf{R} \times \mathbf{R}, \mathbf{R}), \mathbf{R}$ is the set of all real numbers, and there exists a positive integer $M$ such that, for any $(t, z) \in(\mathbf{R} \times \mathbf{R}), f(t+M, Z)=f(t, Z), F(t, z)=$ $\int_{0}^{z} f(t, s) \mathrm{d} s$.

Throughout this paper, for $a, b \in \mathbf{Z}$, we define $\mathbf{Z}(a)$ := $\{a, a+1, \ldots\}, \mathbf{Z}(a, b):=\{a, a+1, \ldots, b\}$ when $a \leq b$.

When $k=1, a_{0}=-1, a_{1}=1$, (1) can be reduced to the following second order difference equation:

$$
\begin{equation*}
\Delta^{2} x_{n-1}+f\left(n, x_{n}\right)=0, \quad n \in \mathbf{Z} \tag{2}
\end{equation*}
$$

Equation (2) can be seen as an analogue discrete form of the following second order differential equation:

$$
\begin{equation*}
\frac{\mathrm{d}^{2} x}{\mathrm{~d} t^{2}}+f(t, x)=0 \tag{3}
\end{equation*}
$$

In recent years, much attention has been given to second order Hamiltonian systems and elliptic boundary value problems by a number of authors; see [1-3] and references therein. On one hand, there have been many approaches to study periodic solutions of differential equations or difference
equations, such as critical point theory (which includes the minimax theory, the Kaplan-Yorke method, and Morse theory), fixed point theory, and coincidence theory; see, for example, [4-20].

Among these approaches, Morse theory is an important tool to deal with such problems. However, there are, at present, only a few papers dealing with higher order difference equation except [21-23]. On the other hand, under some assumptions, the functional $f$ may not satisfy the PalasisSmale condition. Thus, we cannot apply the Morse theory to $f$ directly. To go around this difficulty, Tang and Wu [24] and Liu [25] obtain many interesting results of elliptic boundary value problems by combining Morse theory with LyapunovSchmidt reduction method or minimax principle. Inspired by this, we study the existence of periodic solutions of a higher order difference equation (1) by combining computations of critical groups with Lyapunov-Schmidt reduction method, and an existence theorem on multiple periodic solutions for such an equation is obtained.

For a given integer $M>0$, let

$$
\begin{equation*}
\lambda_{j}=-2 \sum_{s=0}^{k} a_{s} \cos \frac{2 s \pi}{M} j, \quad j=1, \ldots, M \tag{4}
\end{equation*}
$$

We denote $p_{1}=M / 2$ when $M$ is even, or $p_{1}=(M+$ 1)/ 2 when $M$ is odd. Because of $\lambda_{M-j}=\lambda_{j}, j \in \mathbf{Z}(\mathbf{1}, \mathbf{M})$, then, $\lambda_{j}, j \in \mathbf{Z}(\mathbf{1}, \mathbf{M})$ has $p_{1}$ different values. Therefore, we can write these numbers in such a way:

$$
\begin{equation*}
\lambda_{1}<\lambda_{2}<\cdots<\lambda_{p_{1}} \tag{5}
\end{equation*}
$$

Assume $\lambda_{\text {min }}=\min \left\{\lambda_{j}, \lambda_{j} \neq 0, j=1, \ldots, p_{1}\right\}, \lambda_{\max }=\max$ $\left\{\lambda_{j}, \lambda_{j} \neq 0, j=1, \ldots, p_{1}\right\}$.

Combing Morse theory with Lyapunov-Schmidt reduction method, we have the following results.

Theorem 1. Suppose that $M \geq 2 k+1, a_{0}+\sum_{s=1}^{k}\left|a_{s}\right|<0$, and $f(t, z)=f(z)$; we assume that
$\left(f_{1}\right) f(z) \in C^{1}(\mathbf{R}, \mathbf{R}), f(0)=0, f^{\prime}(0)<\lambda_{\text {min }}<$ $f_{\infty}=\lambda_{m} \leq \lambda_{\text {max }}, m \in \mathbf{N}\left(1, p_{1}\right)$, where $f_{\infty}=$ $\lim _{|z| \rightarrow \infty} f(z) / z ;$
$\left(f_{2}\right)$ there exists a constant $\gamma \geq \lambda_{1}$ such that $f^{\prime}(z) \leq \gamma<$ $\lambda_{m+1}$;
$\left(f_{3}\right)$ for any $t \in \mathbf{Z}$,

$$
\begin{equation*}
F(z)-\frac{1}{2} \lambda_{m}|z|^{2} \longrightarrow+\infty, \quad \text { as }|z| \longrightarrow \infty \tag{6}
\end{equation*}
$$

Then (1) possesses at least four nontrivial M-periodic solutions.
This paper is divided into four parts. Section 2 presents variational structure. In Section 3, we present some propositions. The proof of Theorem 1 is given in Section 4.

## 2. Preliminaries

To apply Morse theory to study the existence of periodic solutions of (1), we will construct suitable variational structure.

Let $\mathbf{S}$ be the set of sequences $x=\left\{x_{n}\right\}_{n=-\infty}^{+\infty}$, where $x_{n} \in \mathbf{R}$. For any $x, y \in \mathbf{S}$ and $a, b \in \mathbf{R}, a x+b y$ is defined by

$$
\begin{equation*}
a x+b x:=\left\{a x_{n}+b x_{n}\right\} . \tag{7}
\end{equation*}
$$

Then $\mathbf{S}$ is a vector space.
For any given positive integer $M, E_{M}$ is defined as a subspace of $\mathbf{S}$ by

$$
\begin{equation*}
E_{M}=\left\{x=\left\{x_{n}\right\} \in \mathbf{S} \mid x_{n+M}=x_{n}, n \in \mathbf{Z}\right\} . \tag{8}
\end{equation*}
$$

$E_{M}$ can be equipped with inner product $\langle\cdot, \cdot\rangle_{E_{M}}$ and norm $\|\cdot\|_{E_{M}}$ as follows:

$$
\begin{align*}
& \langle x, y\rangle_{M}=\sum_{j=1}^{M} x_{j} \cdot y_{j}, \quad \forall x, y \in E_{M} \\
& \|x\|_{E_{M}}=\left(\sum_{j=1}^{M} x_{j}^{2}\right)^{1 / 2}, \quad \forall x \in E_{M} \tag{9}
\end{align*}
$$

where $|\cdot|$ denotes the Euclidean Norm in $\mathbf{R}^{M}$, and $x_{n} \cdot y_{n}$ denotes the usual scalar product in $\mathbf{R}$.

Define a linear map $L: E_{M} \rightarrow \mathbf{R}^{M}$ by

$$
\begin{equation*}
L x=\left(x_{1}, \ldots, x_{M}\right)^{T} \tag{10}
\end{equation*}
$$

It is easy to see that the map $L$ defined in (10) is a linear homeomorphism with $\|x\|_{E_{M}}=|L x|$ and $\left(E_{M},\langle\ldots, \ldots\rangle\right)_{E_{M}}$ is a finite dimensional Hilbert space, which can be identified with $\mathbf{R}^{M}$.

For (1), we consider the functional $I$ defined on $E_{M}$ by

$$
\begin{align*}
I(x)= & -\frac{1}{2} \sum_{n=1}^{M} \sum_{i=0}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right) x_{n} \\
& -\sum_{n=1}^{M} F\left(n, x_{n}\right), \quad \forall x \in E_{M} \tag{11}
\end{align*}
$$

where $x_{n+M}=x_{n}, \forall x \in E_{M}, F(t, z)=\int_{0}^{z} f(t, s) \mathrm{d} s$.
Since $E_{M}$ is linearly homeomorphic to $\mathbf{R}^{M}$, by the continuity of $f(t, z)$, I can be viewed as continuously differentiable functional defined on a finite dimensional Hilbert space. That is, $I \in C^{1}\left(E_{M}, \mathbf{R}\right)$. If we define $x_{0}:=x_{M}$, then

$$
\begin{equation*}
\frac{\partial I(x)}{\partial x_{n}}=-\left[\sum_{i=0}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f\left(n, x_{n}\right)\right], \tag{12}
\end{equation*}
$$

where $n \in \mathbf{Z}(1, M)$. Therefore, $x \in E_{M}$ is a critical point of $I$; that is, $I^{\prime}(x)=0$ if and only if

$$
\begin{equation*}
\sum_{i=0}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f\left(n, x_{n}\right)=0, \quad n \in \mathbf{Z}(1, M) \tag{13}
\end{equation*}
$$

On the other hand, $\left\{x_{n}\right\} \in E_{M}$ is $M$-periodic in $n$, and $f(t, z)$ is $M$-periodic in $t$; hence, $x \in E_{M}$ is a critical point of $I$ if and only if $\sum_{i=0}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f\left(n, x_{n}\right)=0$ for any $n \in \mathbf{Z}$, and $x=\left\{x_{n}\right\}$ is a $M$-periodic solution of (1). Thus, we reduce the problem of finding $M$-periodic solutions of (1) to that of seeking critical points of the functional $I$ in $E_{M}$.

Apparently, $I(x) \in C^{2}\left(E_{M}, \mathbf{R}\right)$. Consider

$$
\begin{align*}
& \left(I^{\prime}(x), v\right) \\
& \quad=-\frac{1}{2} \sum_{n=1}^{M} \sum_{i=1}^{k} a_{i}\left[\left(x_{n-i}+x_{n+i}\right) v_{n}+\left(v_{n-i}+v_{n+i}\right) x_{n}\right] \\
& \quad-\sum_{n=1}^{M} f\left(n, x_{n}\right) v_{n}, \\
& \left(I^{\prime \prime}(x) v, w\right)  \tag{14}\\
& = \\
& \quad-\frac{1}{2} \sum_{n=1}^{M} \sum_{i=1}^{k} a_{i}\left[\left(w_{n-i}+w_{n+i}\right) v_{n}+\left(v_{n-i}+v_{n+i}\right) w_{n}\right] \\
& \quad-\sum_{n=1}^{M} f^{\prime}\left(n, x_{n}\right) v_{n} w_{n},
\end{align*}
$$

for all $x, v, w \in E_{M}$. For convenience, we write $x \in E_{M}$ as $x=\left(x_{1}, x_{2}, \ldots, x_{M}\right)^{T}$.

In view of $x_{n+M}=x_{n}, \forall x=\left(x_{1}, x_{2}, \ldots, x_{M}\right)^{T} \in E_{M}$, $n \in \mathbf{Z}$, when $M \geq 2 k+1, I$ can be rewritten as

$$
\begin{equation*}
I(x)=\frac{1}{2} x^{T} A x-\sum_{n=1}^{M} F\left(n, x_{n}\right) \tag{15}
\end{equation*}
$$

where

$$
-A=\left(\begin{array}{ccccccccccccccc}
2 a_{0} & a_{1} & a_{2} & \cdots & a_{k-1} & a_{k} & 0 & 0 & \cdots & 0 & a_{k} & a_{k-1} & \cdots & a_{2} & a_{1}  \tag{16}\\
a_{1} & 2 a_{0} & a_{1} & \cdots & a_{k-2} & a_{k-1} & a_{k} & 0 & \cdots & 0 & 0 & a_{k} & \cdots & a_{3} & a_{2} \\
a_{2} & a_{1} & 2 a_{0} & \cdots & a_{k-3} & a_{k-2} & a_{k-1} & a_{k} & \cdots & 0 & 0 & 0 & \cdots & a_{4} & a_{3} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \ddots & \vdots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots & \cdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
a_{2} & a_{3} & a_{4} & \cdots & 0 & 0 & 0 & 0 & \cdots & a_{k} & a_{k-1} & a_{k-2} & \cdots & 2 a_{0} & a_{1} \\
a_{1} & a_{2} & a_{3} & \cdots & a_{k} & 0 & 0 & 0 & \cdots & a_{k-1} & a_{k-2} & a_{k-3} & \cdots & a_{1} & 2 a_{0}
\end{array}\right)_{M \times M}
$$

Let the eigenvalues of $A$ be $\lambda_{1}^{\prime}, \lambda_{1}^{\prime}, \ldots, \lambda_{M}^{\prime}$, and let $A$ be a circulant matrix [18] denoted by

$$
\begin{array}{r}
A \stackrel{\text { def }}{=} \operatorname{Circ}\left\{-2 a_{0},-a_{1},-a_{2}, \ldots,-a_{k}, 0, \ldots,\right.  \tag{17}\\
\left.0,-a_{k},-a_{k-1}, \ldots,-a_{2},-a_{1}\right\}
\end{array}
$$

By [18], the eigenvalues of $A$ are

$$
\begin{align*}
\lambda_{j}^{\prime} & =-2 a_{0}-\sum_{s=1}^{k} a_{s}\left\{\exp i \frac{2 j \pi}{M}\right\}^{s}-\sum_{s=1}^{k} a_{s}\left\{\exp i \frac{2 j \pi}{M}\right\}^{M-s} \\
& =-2 \sum_{s=0}^{k} a_{s} \cos \left(\frac{2 j s \pi}{M}\right) \tag{18}
\end{align*}
$$

where $j=1, \ldots, M$.
According to (18), for any positive integer $M$ with $M \geq$ $2 k+1$, we know that.

If $a_{0}+\sum_{s=1}^{k}\left|a_{s}\right|<0$, then $\lambda_{j}^{\prime}>0(j=1,2, \ldots, M)$. That is, the matrix $A$ is positive definite.

Comparing (18) with (4), we know that $\lambda_{j}^{\prime}=\lambda_{j}(j=$ $1, \ldots, M)$, then, the matrix $A$ has $p_{1}$ different eigenvalues denoted in such a way:

$$
\begin{equation*}
\lambda_{1}<\lambda_{2}<\cdots<\lambda_{p_{1}} \tag{19}
\end{equation*}
$$

## 3. Main Propositions

In order to prove our main results, we will give several propositions and notations as follows.

Definition 2 (see [4]). Let $X$ be a Banach space, let $J \in$ $C^{1}(X, \mathbf{R})$, and let $H_{q}(A, B)$ be the $q$ th singular relative homology group of the topological pair $(A, B)$ with coefficients in
an Abelian group G. $\beta_{q}=\operatorname{rank} H_{q}(A, B)$ is called the $q$ dimension Betti number. Let $u$ be an isolated critical point of $J$ with $J(u)=c, c \in \mathbf{R}$, and let $U$ be a neighborhood of $u_{0}$ in which $J$ has no critical points except $u_{0}$. Then the group

$$
\begin{equation*}
C_{q}\left(J, u_{0}\right):=H_{q}\left(J_{c} \bigcap U, J_{c} \bigcap U \backslash\left\{u_{0}\right\}\right), \quad q=0,1,2, \ldots \tag{20}
\end{equation*}
$$

is called the $q$ th critical group of $J$ at $u$, here $J_{c}=J^{-1}(-\infty, c]$. Assume that $J$ satisfies PS condition; $J$ has no critical value less than $\alpha \in \mathbf{R}$; then the $q$ th critical group at infinity of $J$ is defined as

$$
\begin{equation*}
C_{q}(J, \infty):=H_{q}\left(X, J_{a}\right), \quad q=0,1,2, \ldots \tag{21}
\end{equation*}
$$

If $J^{\prime \prime}\left(u_{0}\right)=0$, then the Morse index of $J$ at $u_{0}$ is defined as the dimension of the maximal subspace of $X$ on which the quadratic form $\left(J^{\prime \prime}\left(u_{0}\right) v, v\right)$ is negative definite. Define $K_{c}=\left\{u \in X: J^{\prime}(u)=0, J(u)=c\right\}$. We need the following condition.
(A) Suppose that $a<b$ are two regular values of $J$; $J$ has at most finitely many critical points on $J^{-1}[a, b]$ and the rank of the critical group for every critical point is finite.

Definition 3 (see [4]). Assume that $J$ satisfies condition (A); $c_{1}<c_{2}<\cdots<c_{m}$ are all critical values of $J$ in $[a, b]$ and
$K_{c_{i}}=\left\{z_{1}^{i}, z_{2}^{i}, \ldots, z_{n_{i}}^{i}\right\}, i=1,2, \ldots, m$. Choose $0<\epsilon<$ $\min \left\{c_{1}-a, c_{2}-c_{1}, \ldots, c_{m}-c_{m-1}, b-c_{m}\right\}$. Define

$$
\begin{align*}
M_{q} & =M_{q}(a, b) \\
& =\sum_{i=1}^{m} \operatorname{rank} H_{q}\left(J_{c_{i}+\epsilon}, J_{c_{i}-\epsilon}\right)  \tag{22}\\
& =\sum_{i=1}^{m} \sum_{j=1}^{n_{i}} \operatorname{rank} C_{q}\left(J, z_{j}^{i}\right), \quad q=0,1, \ldots
\end{align*}
$$

Then $M_{q}$ is called the $q$ th Morse-type number of $J$ about the interval $[a, b]$.

Here the critical groups of $J$ at an isolated critical point $u$ describe the local behavior of $J$ near $u$, while the critical groups of $J$ at infinity describe the global property of $J$. The Morse inequality gives the relation between them.

Proposition 4 (see [4]). Suppose that $J \in C^{1}(X, \mathbf{R})$ satisfies the PS condition and has only isolated critical points, and the critical values of $f$ are bounded below. Then we have

$$
\begin{equation*}
\sum_{q=0}^{\infty} M_{q} t^{q}=\sum_{q=0}^{\infty} \beta_{q} t^{q}+(1+t) Q(t) \tag{23}
\end{equation*}
$$

where $M_{q}=\sum_{J^{\prime}(u)=0} \operatorname{rank} C_{q}(J, u), \beta_{q}=\operatorname{rank} C_{q}(J, \infty) ; Q$ is a formal series with nonnegative integer coefficients.

Now we recall the Lyapunov-Schmidt reduction method.
Proposition 5 (see [5]). Let X be a separable Hilbert space with inner product $\langle u, v\rangle$ and norm $\|u\|$ and let $X^{-}$and $X^{+}$be closed subspaces of $X$ such that $X=X^{-} \oplus X^{+}$. Let $J \in C^{1}(X, \mathbf{R})$. If there is a real number $\beta>0$ such that, for all $v \in X^{-}$, $w_{1}, w_{2} \in X^{+}$, there holds

$$
\begin{equation*}
\left\langle\nabla f\left(v+w_{1}\right)-\nabla J\left(v+w_{2}\right), w_{1}-w_{2}\right\rangle \geq \beta\left\|w_{1}-w_{2}\right\|^{2} \tag{24}
\end{equation*}
$$

then we have the following:
(i) there exists a continuous function $\psi: X^{-} \rightarrow X^{+}$ such that

$$
\begin{equation*}
J(v+\psi(v))=\min _{w \in X^{+}} J(v+w), \tag{25}
\end{equation*}
$$

and $\psi(v)$ is the unique member of $X^{+}$such that

$$
\begin{equation*}
\langle\nabla J(v+\psi(v)), w\rangle=0, \quad \forall w \in X^{+} \tag{26}
\end{equation*}
$$

(ii) the functional $\varphi \in C^{1}\left(X^{-}, \mathbf{R}\right)$ defined by $\varphi(v)=J(v+$ $\psi(v))$ and

$$
\begin{equation*}
\left\langle\nabla \varphi(v), v_{1}\right\rangle=\langle\nabla J(v+\psi(v)), v\rangle, \quad \forall v, v_{1} \in X^{-} \tag{27}
\end{equation*}
$$

(iii) an element $v \in X^{-}$is a critical point of $\varphi$ if and only if $v+\psi(v)$ is a critical point of $J$.

Proposition 6 (see [25]). Assume that the assumptions of Proposition 5 hold, then at any isolated critical point $v$ of $\varphi$ we have

$$
\begin{equation*}
C_{q}(\varphi, v) \cong C_{q}(f, \psi(v)), \quad q=0,1,2, \ldots . \tag{28}
\end{equation*}
$$

Proposition 7 (see [25]). Assume that the assumptions of Proposition 5 hold, if there exists a compact mapping $T: X \rightarrow$ $X$ such that, for any $u \in X$, we have $\nabla J(u)=u-T(u)$, then we have $\varphi$ :

$$
\begin{equation*}
\operatorname{ind}(\nabla \varphi, v)=\operatorname{ind}(\nabla J, v+\psi(v)) \tag{29}
\end{equation*}
$$

at any isolated critical point $v$ of $\varphi$.

## 4. Proof of Theorem

Consider the following $C^{1}$ functional:

$$
\begin{equation*}
I(x)=\frac{1}{2} x^{T} A x-\sum_{n=1}^{M} F\left(n, x_{n}\right) \tag{30}
\end{equation*}
$$

As we know, the PS condition is an important part of critical point theory. However, under our assumptions $\left(f_{1}\right)-\left(f_{3}\right)$, the functional I may not satisfy PS condition. Thus, we cannot apply the Morse theory directly. But the truncated functional $I_{ \pm}$does satisfy the PS condition. So we can obtain two critical points of $I$ via mountain pass lemma; then we can obtain other critical points by combing Morse theory with Lyapunov-Schmidt reduction method.

At first, we consider the truncated problem

$$
\begin{equation*}
\sum_{i=1}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f_{+}\left(x_{n}\right)=0, \quad n \in \mathbf{Z}, \mathbf{k} \in \mathbf{N} \tag{31}
\end{equation*}
$$

where

$$
\begin{gather*}
f_{+}(z)= \begin{cases}f(n, z), & z \geq 0, \\
0, & z<0,\end{cases}  \tag{32}\\
\sum_{i=1}^{k} a_{i}\left(x_{n-i}+x_{n+i}\right)+f_{-}\left(x_{n}\right)=0, \quad n \in \mathbf{Z}, \mathbf{k} \in \mathbf{N}, \tag{31}
\end{gather*}
$$

where

$$
f_{-}(z)= \begin{cases}f(n, z), & z \leq 0  \tag{33}\\ 0, & z>0\end{cases}
$$

Then the functional $I_{+}: \mathbf{Z} \times \mathbf{R} \rightarrow \mathbf{R}$ corresponding to (31) can be written as

$$
\begin{equation*}
I_{+}(x)=\frac{1}{2} x^{T} A x-\sum_{n=1}^{M} F_{+}\left(x_{n}\right) \tag{34}
\end{equation*}
$$

where $F_{+}(n, z)=\int_{0}^{z} f_{+}(n, s) \mathrm{d} s$. Apparently, $I_{+} \in C^{1}$.
The functional $I_{-}: \mathbf{Z} \times \mathbf{R} \rightarrow \mathbf{R}$ corresponding to (31)' can be written as

$$
\begin{equation*}
I_{-}(x)=\frac{1}{2} x^{T} A x-\sum_{n=1}^{M} F_{-}\left(x_{n}\right) \tag{35}
\end{equation*}
$$

where $F_{-}(n, z)=\int_{0}^{z} f_{-}(n, s) \mathrm{d} s$. Apparently, $I_{-} \in C^{1}$.

We only consider the case of $I_{+}$; the case of $I_{-}$is similar and omitted.

By $\left(f_{1}\right)$, we know that

$$
\begin{equation*}
\lim _{z \rightarrow-\infty} \frac{f_{+}(z)}{z}=0, \quad \lim _{z \rightarrow+\infty} \frac{f_{+}(z)}{z}=f_{\infty} \tag{36}
\end{equation*}
$$

Then there exist real number $\epsilon>0$ (small enough) and $C_{\epsilon}>0$ such that

$$
\begin{equation*}
f_{+}(z)=f_{\infty} z+C_{\epsilon}, \quad \text { if } z \longrightarrow+\infty . \tag{37}
\end{equation*}
$$

Lemma 8. Under the conditions of Theorem 1, the functional $I_{+}(x)$ satisfies the PS condition.

Proof. Let $\left\{x^{q}\right\} \in E_{M}$ be such a sequence; that is, there exists a positive constant $M_{1}$ such that $\left|I_{+}\left(x^{q}\right)\right| \leq M_{1}, \forall q \in \mathbf{N}$, and that $\left|\left(I_{+}^{\prime}\left(x^{q}\right), v\right)\right| \rightarrow 0$ as $q \rightarrow+\infty, \forall v \in E_{M}$.

Therefore,

$$
\begin{align*}
2 M_{1} & \geq 2 I_{+}\left(x^{q}\right)-\left(I_{+}^{\prime}\left(x^{q}\right), x^{q}\right) \\
& =\sum_{n=1}^{M}\left[f_{+}\left(x_{n}^{q}\right) x_{n}^{q}-F_{+}\left(x_{n}^{q}\right)\right] \\
& =\sum_{n=1}^{M}\left[\left(f_{\infty}\left(x_{n}^{q}\right)^{2}+C_{\epsilon} x_{n}^{q}\right)-\left(\frac{1}{2} f_{\infty}\left(x_{n}^{q}\right)^{2}+C_{\epsilon} x_{n}^{q}+C_{\epsilon}\right)\right] \\
& =\frac{1}{2} f_{\infty}\left\|x^{q}\right\|^{2}-C_{\epsilon} M . \tag{38}
\end{align*}
$$

That is, $\left\{x^{q}\right\} \in E_{M}$ is a bounded sequence in the finite dimensional space $E_{M}$. Consequently, it has a convergent subsequence. Thus, we obtain Lemma 8.

Let $z^{+}=\max (z, 0), z^{-}=\max (-z, 0)$, and $z=z^{+}-z^{-}$.
Lemma 9. If $x \in E_{M}$ is a local minimizer of $I_{+}$, then $x$ must be a local minimizer of $I$.

Proof. Let $x>0$ be a local minimizer of $I_{+}$; then for any sequence $\left\{x^{q}\right\} \subset E_{M}, x^{q} \rightarrow x(q \rightarrow \infty)$, for big enough $q$, we have $I\left(x^{q}\right) \geq I(x)$.

In fact,

$$
\begin{align*}
I\left(x^{q}\right)-I(x) & =I\left(x^{q}\right)-I_{+}(x) \\
& \geq I\left(x^{q}\right)-I_{+}\left(x^{q}\right) \\
& =\sum_{n=1}^{M}\left[F_{+}\left(x_{n}^{q}\right)-F\left(x_{n}^{q}\right)\right]  \tag{39}\\
& =-\sum_{n \in \mathbf{Z}(1, M), x_{n}^{q}<0} F\left(x_{n}^{q}\right) .
\end{align*}
$$

Because $x^{q} \rightarrow x, x_{n}^{q}=\left(x_{n}^{q}\right)^{+}-\left(x_{n}^{q}\right)^{-}$, and $x_{n}=\left(x_{n}\right)^{+}-$ $\left(x_{n}\right)^{-}$, so $\left(x_{n}^{q}\right)^{+} \rightarrow\left(x_{n}\right)^{+}=x_{n},-\left(x_{n}^{q}\right)^{-} \rightarrow 0^{-}$.

For any $n \in \mathbf{Z}(1, M)$, if $\left(x_{n}^{q}\right)^{-}=0$, then $I\left(x^{q}\right)=I(x)$.
If $-\left(x_{n}^{q}\right)^{-} \rightarrow 0^{-}$, by $\left(f_{1}\right), f(0)=0$, and $0<$ $f^{\prime}(z)<\gamma$, then $f(z)<0$ for $z \rightarrow 0^{-}$. Therefore, $-\sum_{n \in \mathbf{Z}(1, M), x_{n}^{q}<0} F\left(x_{n}^{q}\right)>0$; that is, $I\left(x^{q}\right)>I(x)$.

The proof of Lemma 9 is complete.

It is easy to see that the zero function 0 is a local minimizer of $I_{+}$, and $I_{+}\left(s \phi_{1}\right) \rightarrow-\infty$ as $s \rightarrow+\infty$, where $\phi_{1}$ is a first eigenfunction corresponding to the first nonzero eigenvalue of $A$. Thus, by the mountain pass lemma we obtain a critical point $x_{+}$of $I_{+}$. However, it is true that $x_{+}$is a critical point of $I$ if $x_{+}$is a critical point of $I_{+}$; then we deduce that $x_{+}$is a critical point of $I$ with

$$
\begin{equation*}
C_{q}\left(I, x_{+}\right) \cong \delta_{q, 1} G, \quad x_{+}>0 \text { in } E_{M} . \tag{40}
\end{equation*}
$$

Similarly, we obtain another critical point $x_{-}$of $I$ and

$$
\begin{equation*}
C_{q}\left(I, x_{-}\right) \cong \delta_{q, 1} G, \quad x_{-}<0 \text { in } E_{M} . \tag{41}
\end{equation*}
$$

Next we will prove that $I$ has two more nonzero critical points. We decompose $E_{M}=X^{-} \oplus X^{+}$according to $f_{\infty}=\lambda_{m}$. We set

$$
\begin{align*}
& X^{-}=\bigoplus_{i=1}^{m} \operatorname{Ker}\left(A-\lambda_{i} I\right), \\
& X^{+}=\bigoplus_{i=m+1}^{M} \operatorname{Ker}\left(A-\lambda_{i} I\right),  \tag{42}\\
& E_{M}=X^{-} \bigoplus X^{+} .
\end{align*}
$$

Since $f^{\prime}(z) \leq \gamma<\lambda_{m+1}$, for any $v \in X^{-}$and $w_{1}, w_{2} \in X^{+}$, we have

$$
\begin{align*}
& \left\langle\nabla I\left(n, v+w_{1}\right)-\nabla I\left(n, v+w_{2}\right), w_{1}-w_{2}\right\rangle  \tag{43}\\
& \quad \geq \beta\left\|w_{1}-w_{2}\right\|^{2},
\end{align*}
$$

where $\beta=1-\gamma \lambda_{m+1}^{-1}$. Then, by Proposition 5, there exist a continuous map $\psi: X^{-} \rightarrow X^{+}$and a $C^{1}$-functional $\varphi$ : $X^{-} \rightarrow \mathbf{R}$ such that

$$
\begin{equation*}
\varphi(v)=I(v+\psi(v))=\min _{w \in X^{+}} \tag{44}
\end{equation*}
$$

We need to show that $\varphi$ has at least five critical points. Hence, we assume that $\varphi$ has no critical value less than some $\alpha \in \mathbf{R}$.

Lemma 10. Suppose that $f \in C^{1}(\mathbf{R}, \mathbf{R})$ satisfies $\left(f_{1}\right)-\left(f_{3}\right)$, then the functional $\varphi$ is anticoercive.

Proof. According to $\left(f_{3}\right)$, there exists $R>0$ such that

$$
\begin{equation*}
\frac{1}{2} \lambda_{m} z^{2}-F(z) \leq 0, \quad|z| \geq R . \tag{45}
\end{equation*}
$$

Then, for any $z \in \mathbf{R}$, we have

$$
\begin{equation*}
\frac{1}{2} \lambda_{m} z^{2}-F(z) \leq T=\max _{|z| \leq R}\left|\frac{1}{2} \lambda_{m} z^{2}-F(z)\right| . \tag{46}
\end{equation*}
$$

Assume that $\left\{v^{t}\right\}_{t=1}^{\infty}$ is a sequence in $X^{-}$such that $\left\|v^{t}\right\| \rightarrow \infty$. Let $\xi^{t}=v^{t} /\left\|v^{t}\right\|$, then $\left\|\xi^{t}\right\|=1$. Because of $\operatorname{dim} X^{-}<\infty$, there exist some $\xi \in X^{-}$such that, up to subsequence $\left\|\xi^{t}-\xi\right\| \rightarrow 0$, $\|\xi\|=1$.

In particular, $\xi \neq 0$, meas $\Theta=\operatorname{meas}\{n \in \mathbf{Z}[1, M]$ : $\left.\xi_{n} \neq 0\right\}>0$. For $n \in \Theta,\left|v_{n}^{t}\right| \rightarrow \infty$. Hence, by $\left(f_{3}\right)$,

$$
\begin{equation*}
\sum_{n \in \Theta}\left(\frac{1}{2} \lambda_{m}\left\|v^{t}\right\|^{2}-F\left(v_{n}^{t}\right)\right) \longrightarrow-\infty, \quad \text { as } t \longrightarrow \infty \tag{47}
\end{equation*}
$$

By the above discussion, we have

$$
\begin{align*}
\varphi\left(v^{t}\right) \leq & I\left(v^{t}\right) \\
= & -\frac{1}{2} \sum_{n=1}^{M} \sum_{i=1}^{k} a_{i}\left(v_{n-i}^{t}+v_{n+i}^{t}\right) v_{n}^{t}-\sum_{n=1}^{M} F\left(v_{n}^{t}\right) \\
\leq & \frac{1}{2} \lambda_{m}\left\|v_{t}\right\|^{2}-\sum_{n=1}^{M} F\left(v_{n}^{t}\right)  \tag{49}\\
= & \sum_{n \in \Theta}\left[\frac{1}{2} \lambda_{m}\left\|v_{t}\right\|^{2}-F\left(v_{n}^{t}\right)\right] \\
& +\sum_{n \in[1, M] \backslash \Theta}\left[\frac{1}{2} \lambda_{m}\left\|v_{t}\right\|^{2}-F\left(v_{n}^{t}\right)\right]
\end{align*}
$$

$$
\begin{align*}
& \leq \sum_{n \in \Theta}\left[\frac{1}{2} \lambda_{m}\left\|v^{t}\right\|^{2}-F\left(v_{n}^{t}\right)\right]+M T \\
& \longrightarrow-\infty \tag{48}
\end{align*}
$$

This concludes the proof.

Because $\varphi$ is anticoercive, we choose $a<b<\alpha$ and $\rho>$ $r>0$ such that

$$
A_{\rho} \subset \varphi_{a} \subset A_{r} \subset \varphi_{b}
$$

where $A_{\rho}=\left\{v \in X^{-}:\|v\| \geq \rho\right\}$. Since $\varphi$ has no critical value in $[a, b], H_{*}\left(\varphi_{b}, \varphi_{a}\right)=0$.

Thus, we have the following commutative diagram with exact rows:
where all the homomorphisms except $\partial_{*}$ are induced by inclusions. The exactness of rows implies that $i_{*}, k_{*}$ are isomorphisms. Hence $l_{*}: H_{q}\left(X^{-}, \varphi_{a}\right) \rightarrow H_{q}\left(X^{-}, A_{r}\right)$ is also an isomorphism, and we get

$$
\begin{equation*}
C_{q}(\varphi, \infty)=H_{q}\left(X^{-}, \varphi_{a}\right) \cong H_{q}\left(X^{-}, A_{r}\right)=\delta_{q, m} G \tag{51}
\end{equation*}
$$

Because the anticoercive functional $\varphi$ is defined on the $m$-dimensional $X^{-}$, it has a critical point $v$, with

$$
\begin{equation*}
C_{q}(\varphi, v) \cong \delta_{q, m} G \tag{52}
\end{equation*}
$$

Let $0, v_{+}, v_{-}$be the projection of $0, x_{+}, x_{-}$in $X^{-}$, respectively. Then they are all critical points of $\varphi$. By (11), (14), and Proposition 6, and 0 is a local minimizer of $I$, we have

$$
\begin{align*}
C_{q}\left(\varphi, v_{ \pm}\right) & \cong C_{q}\left(I, x_{ \pm}\right) \cong \delta_{q, 1} Q  \tag{53}\\
C_{q}(\varphi, 0) & \cong C_{q}(I, 0) \cong \delta_{q, 0} Q
\end{align*}
$$

If $0, v_{+}, v_{-}, v$ are the only critical points of $\varphi$, then by Proposition 4 with $t=-1$,

$$
\begin{equation*}
(-1)^{0}+2 \times(-1)^{1}+(-1)^{m}=(-1)^{m} . \tag{54}
\end{equation*}
$$

This is impossible. Thus $\varphi$ has at least five critical points. So $I$ also has five critical points, four of which are nonzero. Therefore, (1) has at least four nontrivial solutions. This completes the proof of Theorem 1.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Multiple Solutions of Second-Order Damped Impulsive Differential Equations with Mixed Boundary Conditions 

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We use variational methods to investigate the solutions of damped impulsive differential equations with mixed boundary conditions. The conditions for the multiplicity of solutions are established. The main results are also demonstrated with examples.

## 1. Introduction

Impulsive effect exists widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians [1-6]. Applications of impulsive differential equations with or without delays occur in biology, medicine, mechanics, engineering, chaos theory, and so on [7-11].

In this paper, we consider the following second-order damped impulsive differential equations with mixed boundary conditions:

$$
\begin{gather*}
-u^{\prime \prime}(t)+g(t) u^{\prime}(t)-\lambda u(t)=f(t, u(t)), \\
t \neq t_{j}, \quad \text { a.e. } t \in[0, T], \\
-\Delta u^{\prime}\left(t_{j}\right)=I_{j}\left(u\left(t_{j}\right)\right), \quad j=1,2, \ldots, n,  \tag{1}\\
u^{\prime}(0)=0, \quad u(T)=0,
\end{gather*}
$$

where $0=t_{0}<t_{1}<t_{2}<\cdots<t_{n}<t_{n+1}=T, g \in C[0, T]$, $f:[0, T] \times R \rightarrow R$ is continuous, $I_{j}: R \rightarrow R, j=1,2, \ldots, n$ are continuous, and $\Delta u^{\prime}\left(t_{j}\right)=u^{\prime}\left(t_{j}^{+}\right)-u^{\prime}\left(t_{j}^{-}\right)$for $u^{\prime}\left(t_{j}^{ \pm}\right)=$ $\lim _{t \rightarrow t_{j}^{ \pm}} u^{\prime}(t), j=1,2, \ldots, n$.

The characteristic of (1) is the presence of the damped term $g(t) u^{\prime}$. Most of the results concerning the existence of solutions of these equations are obtained using upper and lower solutions methods, coincidence degree theory,
and fixed point theorems [12-15]. On the other hand, when there is no presence of the damped term, some researchers have used variational methods to study the existence of solutions for these problems [16-21]. However, to the best of our knowledge, there are few papers concerned with the existence of solutions for impulsive boundary value problems like problem (1) by using variational methods.

For this nonlinear damped mixed boundary problem (1), the variational structure due to the presence of the damped term $g(t) u^{\prime}$ is not apparent. However, inspired by the work [22, 23], we will be able to transform it into a variational formulation. In this paper, our aim is to study the existence of $n$ distinct pairs of nontrivial solutions of problem (1). Our main results extend the study made in [22, 23], in the sense that we deal with a class of problems that is not considered in those papers.

## 2. Preliminaries and Statements

Let $m=\min _{t \in[0, T]} e^{G(t)}, M=\max _{t \in[0, T]} e^{G(t)}, G(t)=$ $-\int_{0}^{t} g(s) d s, t \in[0, T]$. We transform (1) into the following equivalent form:

$$
\begin{array}{r}
-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}-\lambda e^{G(t)} u(t)=e^{G(t)} f(t, u(t)) \\
t \neq t_{j}, \quad \text { a.e. } t \in[0, T]
\end{array}
$$

$$
\begin{gather*}
-\Delta u^{\prime}\left(t_{j}\right)=I_{j}\left(u\left(t_{j}\right)\right), \quad j=1,2, \ldots, n \\
u^{\prime}(0)=0, u(T)=0 \tag{2}
\end{gather*}
$$

Obviously, the solutions of (2) are solutions of (1).
Define the space $X=\{u(t) \quad \mid u(t)$ is absolutely continuous on $\left.[0, T], u^{\prime}(\cdot) \in L^{2}[0, T], u(T)=0\right\}$. It is easy to see that $H_{0}^{1}(0, T) \subset X \subset H^{1}(0, T)$ and $X$ is a closed subset of $H^{1}(0, T)$. So $X$ is a Hilbert space with the usual inner product in $H^{1}(0, T)$.

Consider the Hilbert spaces $X$ with the inner product

$$
\begin{equation*}
(u, v)=\int_{0}^{T} e^{G(t)} u^{\prime}(t) v^{\prime}(t) d t \tag{3}
\end{equation*}
$$

inducing the norm

$$
\begin{equation*}
\|u\|=\left(\int_{0}^{T} e^{G(t)}\left|u^{\prime}(t)\right|^{2} d t\right)^{1 / 2} \tag{4}
\end{equation*}
$$

We also consider the inner product

$$
\begin{equation*}
(u, v)=\int_{0}^{T} u^{\prime}(t) v^{\prime}(t) d t \tag{5}
\end{equation*}
$$

inducing the norm

$$
\begin{equation*}
\|u\|_{X}=\left(\int_{0}^{T}\left|u^{\prime}(t)\right|^{2} d t\right)^{1 / 2} \tag{6}
\end{equation*}
$$

Consider the problem

$$
\begin{gather*}
-u^{\prime \prime}(t)=\lambda u(t), \quad t \in[0, T], \\
u^{\prime}(0)=0, \quad u(T)=0 . \tag{7}
\end{gather*}
$$

As is well known, (7) possesses a sequence of eigenvalues $\left(\lambda_{i}\right)\left(\lambda_{i}=[(2 i-1) \pi / 2 T]^{2}\right)$ with

$$
\begin{equation*}
0<\lambda_{1}<\lambda_{2}<\cdots<\lambda_{j}<\cdots \tag{8}
\end{equation*}
$$

The corresponding eigenfunctions are normalized so that $\left\|\varphi_{j}\right\|_{X}=1=\lambda_{j} \int_{0}^{T}\left|\varphi_{j}(t)\right|^{2} d t$; here

$$
\begin{equation*}
\varphi_{j}(t)=\sqrt{\frac{2}{T \lambda_{j}}} \cos \left(\sqrt{\lambda_{j}} t\right), \quad j=1,2, \ldots . \tag{9}
\end{equation*}
$$

Now multiply (2) by $v \in X$ and integrate on the interval $[0, T]$ :

$$
\begin{align*}
& \int_{0}^{T} e^{G(t)} u^{\prime}(t) v^{\prime}(t) d t-\lambda \int_{0}^{T} e^{G(t)} u(t) v(t) d t \\
& \quad=\sum_{j=1}^{n} e^{G\left(t_{j}\right)} I_{j}\left(u\left(t_{j}\right)\right) v\left(t_{j}\right)+\int_{0}^{T} e^{G(t)} f(t, u(t)) v(t) d t . \tag{10}
\end{align*}
$$

Then, a weak solution of (2) is a critical point of the following functional:

$$
\begin{align*}
E(u)= & \frac{1}{2} \int_{0}^{T} e^{G(t)}\left|u^{\prime}(t)\right|^{2} d t-\frac{\lambda}{2} \int_{0}^{T} e^{G(t)}|u(t)|^{2} d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t-\int_{0}^{T} e^{G(t)} F(t, u(t)) d t \tag{11}
\end{align*}
$$

where $F(t, u)=\int_{0}^{u} f(t, \xi) d \xi$.
We say that $u \in C[0, T]$ is a classical solution of IBVP (1) if it satisfies the following conditions: $u$ satisfies the first equation of (1) a.e. on [0,T]; the limits $u^{\prime}\left(t_{j}^{+}\right), u^{\prime}\left(t_{j}^{-}\right)$, $j=1,2, \ldots, n$, exist and impulsive condition of (1) holds; $u$ satisfies the boundary condition of (1).

Lemma 1. If $u \in X$ is a weak solution of (1), then $u$ is a classical solution of (1).

Proof. If $u \in X$ is a weak solution of (1), then $u$ is a weak solution of (2), so $\left(E^{\prime}(u), v\right)=0$ holds for all $v \in X$; that is,

$$
\begin{align*}
& \int_{0}^{T}\left[e^{G(t)} u^{\prime}(t) v^{\prime}(t)+\lambda e^{G(t)} u(t) v(t)\right] d t \\
& \quad-\sum_{j=1}^{n} e^{G\left(t_{j}\right)} I_{j}\left(u\left(t_{j}\right)\right) v\left(t_{j}\right)  \tag{12}\\
& \quad-\int_{0}^{T} e^{G(t)} f(t, u(t)) v(t) d t=0 .
\end{align*}
$$

By integrating by part, we have

$$
\begin{align*}
& \int_{0}^{T}\left[e^{G(t)} u^{\prime}(t) v^{\prime}(t)+\lambda e^{G(t)} u(t) v(t)\right] d t \\
&-\sum_{j=1}^{n} e^{G\left(t_{j}\right)} I_{j}\left(u\left(t_{j}\right)\right) v\left(t_{j}\right) \\
&-\int_{0}^{T} e^{G(t)} f(t, u(t)) v(t) d t \\
&= \int_{0}^{T}\left[-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}+\lambda e^{G(t)} u(t)-e^{G(t)} f(t, u(t))\right] v(t) d t \\
&-\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right) \\
& \quad-e^{G(0)} u^{\prime}(0) v(0)+e^{G(T)} u^{\prime}(T) v(T) \\
&= \int_{0}^{T}\left[-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}+\lambda e^{G(t)} u(t)-e^{G(t)} f(t, u(t))\right] v(t) d t \\
&-\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right)-u^{\prime}(0) v(0) . \tag{13}
\end{align*}
$$

Thus

$$
\begin{align*}
& \int_{0}^{T}\left[-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}+\lambda e^{G(t)} u(t)-e^{G(t)} f(t, u(t))\right] v(t) d t \\
& \quad-\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right) \\
& \quad-u^{\prime}(0) v(0)=0 \tag{14}
\end{align*}
$$

holds for all $v \in X$. Without loss of generality, for any $j=$ $\{1,2, \ldots, n\}$ and $v \in X$ with $v(t) \equiv 0$, for every $t \in\left[0, t_{j}\right] \cup$ [ $\left.t_{j+1}, T\right]$, then substituting $v$ into (14), we get

$$
\begin{align*}
&-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}+\lambda e^{G(t)} u(t)-e^{G(t)} f(t, u(t))=0 \\
& t \in\left(t_{j}, t_{j+1}\right) \tag{15}
\end{align*}
$$

Hence $u$ satisfies the first equation of (2). Therefore, by (14) we have

$$
\begin{equation*}
-\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right)-u^{\prime}(0) v(0)=0 \tag{16}
\end{equation*}
$$

Next we will show that $u$ satisfies the impulsive and the boundary condition in (2). If the impulsive condition in (2) does not hold, without loss of generality, we assume that there exists $j \in\{1,2, \ldots, n\}$ such that

$$
\begin{equation*}
\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right) \neq 0 \tag{17}
\end{equation*}
$$

Let $v(t)=\prod_{i=0, i \neq j}^{n+1}\left(t-t_{i}\right)$; then

$$
\begin{gather*}
-\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right)-u^{\prime}(0) v(0)  \tag{18}\\
=-e^{G\left(t_{j}\right)}\left[\Delta u^{\prime}\left(t_{j}\right)+I_{j}\left(u\left(t_{j}\right)\right)\right] v\left(t_{j}\right) \neq 0,
\end{gather*}
$$

which contradicts (16). So $u$ satisfies the impulsive condition in (2) and (16) implies

$$
\begin{equation*}
u^{\prime}(0) v(0)=0 \tag{19}
\end{equation*}
$$

If $u^{\prime}(0) \neq 0$, pick $v(t)=\prod_{i=1}^{n+1}\left(t-t_{i}\right)$; one has

$$
\begin{equation*}
u^{\prime}(0) \prod_{i=1}^{n+1}\left(t_{0}-t_{i}\right) \neq 0 \tag{20}
\end{equation*}
$$

which contradicts (19), so $u$ satisfies the boundary condition. Therefore, $u$ is a solution of (1).

Lemma 2. Let $u \in X$. Then there exists a constant $\sigma>0$, such that

$$
\begin{equation*}
\|u\|_{\infty} \leq \sigma\|u\| \tag{21}
\end{equation*}
$$

where $\|u\|_{\infty}=\max _{t \in[0, T]}|u(t)|$.

Proof. By Hölder inequality, for $u \in X$,

$$
\begin{align*}
|u(t)| & =\left|u(T)-\int_{t}^{T} u^{\prime}(s) d s\right| \\
& \leq\left(\int_{0}^{T} \frac{1}{e^{G(s)}} d s\right)^{1 / 2}\left(\int_{0}^{T} e^{G(s)}\left|u^{\prime}(s)\right|^{2} d s\right)^{1 / 2}  \tag{22}\\
& \leq \sqrt{\frac{T}{m}}\|u\|=\sigma\|u\|
\end{align*}
$$

Lemma 3 (see [24, Theorem 9.1]). Let E be a real Banach space, $I \in C^{1}(E, R)$ with $I$ even, bounded from below, and satisfying P.S. condition. Suppose $I(0)=0$; there is a set $K \subset E$ such that $K$ is homeomorphic to $S^{j-1}$ by an odd map and $\sup _{K} I<0$. Then I possesses at least $j$ distinct pairs of critical points.

## 3. Main Results

Theorem 4. Suppose that the following conditions hold.
(H1) There exist $u_{1}>0, r>M \lambda_{k} / m, \lambda_{k}$ which is the $k t h$ eigenvalue of (7) such that

$$
r M u_{1}+e^{G(t)} f\left(t, u_{1}\right)=0, \quad r M u_{1}+e^{G(t)} f(t, u)>0
$$

$$
\begin{equation*}
\text { for every } u \in\left(0, u_{1}\right) \tag{23}
\end{equation*}
$$

(H2) There exist $a_{j}, b_{j}>0$ and $r_{j} \in[0,1)(j=1,2, \ldots, n)$ such that

$$
\begin{equation*}
\left|I_{j}(u)\right| \leq a_{j}+b_{j}|u|^{r_{j}} \quad \text { for any } u \in R . \tag{24}
\end{equation*}
$$

(H3) $f(t, u)$ and $I_{j}(u)(j=1,2, \ldots, n)$ are odd about $u$.
(H4) $f(t, u)=o(|u|), I_{j}(u)=o(|u|)$, as $|u| \rightarrow 0, j=$ $1,2, \ldots, n$.

Then, for $\lambda \in\left(M \lambda_{k} / m, r\right]$, problem (1) has at least $k$ distinct pairs of solutions.

Proof. Set

$$
h_{1}(\lambda, t, u)= \begin{cases}\lambda e^{G(t)} u+e^{G(t)} f(t, u) & u \in\left[-u_{1}, u_{1}\right]  \tag{25}\\ \lambda e^{G(t)} u_{1}+e^{G(t)} f\left(t, u_{1}\right), & u \in\left[u_{1},+\infty\right), \\ -\lambda e^{G(t)} u_{1}-e^{G(t)} f\left(t,-u_{1}\right), & u \in\left(-\infty,-u_{1}\right]\end{cases}
$$

Consider

$$
\begin{gather*}
-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}=h_{1}(\lambda, t, u(t)), \\
t \neq t_{j}, \quad \text { a.e. } t \in[0, T],  \tag{26}\\
-\Delta u^{\prime}\left(t_{j}\right)=I_{j}\left(u\left(t_{j}\right)\right), \quad j=1,2, \ldots, n, \\
u^{\prime}(0)=0, \quad u(T)=0 .
\end{gather*}
$$

Next, we will verify that the solutions of problem (26) are solutions of problem (1).

In fact, let $u_{0}(t)$ be the solution of problem (26). If $\max _{0 \leq t \leq T} u_{0}(t)>u_{1}$, then there exists an interval $[a, b] \subset$ $[0, T]$ such that

$$
\begin{equation*}
u_{0}(a)=u_{0}(b)=u_{1}, \quad u_{0}(t)>u_{1} \quad \text { for any } t \in(a, b) \tag{27}
\end{equation*}
$$

When $t \in[a, b]$, by (H1), we have

$$
\begin{align*}
-\left(e^{G(t)} u_{0}^{\prime}(t)\right)^{\prime}= & h_{1}(\lambda, t, u) \\
= & \lambda e^{G(t)} u_{1}+e^{G(t)} f\left(t, u_{1}\right) \leq r M u_{1}  \tag{28}\\
& +e^{G(t)} f\left(t, u_{1}\right)=0
\end{align*}
$$

That is, $e^{G(t)} u_{0}^{\prime}(t)$ is nondecreasing in $[a, b]$. By $u_{0}^{\prime}(a) \geq 0$ and $u_{0}^{\prime}(b) \leq 0$, we have

$$
\begin{array}{r}
0 \leq e^{G(t)} u_{0}^{\prime}(a) \leq e^{G(t)} u_{0}^{\prime}(t) \leq e^{G(t)} u_{0}^{\prime}(b) \leq 0  \tag{29}\\
\text { for every } t \in[a, b]
\end{array}
$$

That is, $e^{G(t)} u_{0}^{\prime}(t) \equiv 0$ for any $t \in[a, b]$. Since $e^{G(t)} \neq 0$, then $u_{0}^{\prime}(t) \equiv 0$. So, there exists a constant $\epsilon$ such that $u_{0}(t) \equiv \epsilon$, which contradicts (27). Then $\max _{0 \leq t \leq T} u_{0}(t) \leq u_{1}$. Similarly, we can prove that $\min _{0 \leq t \leq T} u_{0}(t)>-u_{1}$.

Therefore, any solution of (26) is a solution of (1). Hence to prove Theorem 4, it suffices to produce at least $k$ distinct pairs of critical points of

$$
\begin{align*}
E_{1}(u)= & \frac{1}{2} \int_{0}^{T} e^{G(t)}\left|u^{\prime}(t)\right|^{2} d t-\int_{0}^{T} H_{1}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \tag{30}
\end{align*}
$$

where $H_{1}(\lambda, t, u(t))=\int_{0}^{u} h_{1}(\lambda, t, s) d s$.
We will apply Lemma 3 to finish the proof.
By (30) and (H3), $E_{1} \in C^{\prime}(X, R)$ is even and $E_{1}(0)=0$.
Next, we will show that $E_{1}$ is bounded from below.
Let $C_{1}=\max \left\{a_{1}, a_{2}, \ldots, a_{n}\right\}, C_{2}=\max \left\{b_{1}, b_{2}, \ldots, b_{n}\right\}$. By (H1) and (H3), we have $u h_{1}(\lambda, t, u(t)) \leq 0$ for $|u| \geq u_{1}$; thus

$$
\begin{align*}
\int_{0}^{T} H_{1}(\lambda, t, u(t)) d t & =\int_{0}^{T} \int_{0}^{u(t)} h_{1}(\lambda, t, s) d s d t \\
& \leq \int_{0}^{T} \int_{0}^{u_{1}} h_{1}(\lambda, t, s) d s d t \\
& \leq \int_{0}^{T} \int_{0}^{u_{1}}[r M s+f(t, s)] d s d t=\rho>0 . \tag{31}
\end{align*}
$$

So, we have

$$
\begin{align*}
E_{1}(u)= & \frac{1}{2}\|u\|^{2}-\int_{0}^{T} H_{1}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \\
\geq & \frac{1}{2}\|u\|^{2}-\rho-n \sigma C_{1} M\|u\|-C_{2} M \sum_{j=1}^{n} \sigma^{r_{j}+1}\|u\|^{r_{j}+1} \\
> & -\infty, \tag{32}
\end{align*}
$$

for any $u \in X$. Therefore, $E_{1}$ is bounded from below.
In the following we will show that $E_{1}$ satisfies the P.S. condition. Let $\left\{u_{k}\right\} \subset X$ such that $\left\{E_{1}\left(u_{k}\right)\right\}$ is a bounded sequence and $\lim _{k \rightarrow \infty} E_{1}^{\prime}\left(u_{k}\right)=0$; then there exists $C_{3}>0$ such that

$$
\begin{equation*}
\left|E_{1}\left(u_{k}\right)\right| \leq C_{3} . \tag{33}
\end{equation*}
$$

By (32), we have

$$
\begin{align*}
\frac{1}{2}\left\|u_{k}\right\|^{2} \leq & C_{3}+\rho+n \sigma C_{1} M\left\|u_{k}\right\| \\
& +C_{2} M \sum_{j=1}^{n} \sigma^{r_{j}+1}\|u\|^{r_{j}+1} \tag{34}
\end{align*}
$$

So $\left\{u_{k}\right\}$ is bounded in $X$. From the reflexivity of $X$, we may extract a weakly convergent subsequence that, for simplicity, we call $\left\{u_{k}\right\}, u_{k} \rightharpoonup u$ in $X$. In the following we will verify that $\left\{u_{k}\right\}$ strongly converges to $u$ :

$$
\begin{align*}
& \left(E_{1}^{\prime}\left(u_{k}\right)-E_{1}^{\prime}(u)\right)\left(u_{k}-u\right) \\
& =\left\|u_{k}-u\right\|^{2}-\int_{0}^{T}\left[h_{1}\left(\lambda, t, u_{k}(t)\right)-h_{1}(\lambda, t, u(t))\right] \\
& \quad \times\left(u_{k}(t)-u(t)\right) d t  \tag{35}\\
& +\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[I_{j}\left(u_{k}\left(t_{j}\right)\right)-I_{j}\left(u\left(t_{j}\right)\right)\right] \\
& \quad \times\left(u_{k}\left(t_{j}\right)-u\left(t_{j}\right)\right) .
\end{align*}
$$

By $u_{k} \rightharpoonup u$ in $X$, we see that $\left\{u_{k}\right\}$ uniformly converges to $u$ in $C[0, T]$. So

$$
\begin{gather*}
\int_{0}^{T}\left[h_{1}\left(\lambda, t, u_{k}(t)\right)-h_{1}(\lambda, t, u(t))\right] \\
\times\left(u_{k}(t)-u(t)\right) d t \longrightarrow 0, \\
\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[I_{j}\left(u_{k}\left(t_{j}\right)\right)-I_{j}\left(u\left(t_{j}\right)\right)\right]  \tag{36}\\
\times\left(u_{k}\left(t_{j}\right)-u\left(t_{j}\right)\right) \longrightarrow 0, \\
\left(E_{1}^{\prime}\left(u_{k}\right)-E_{1}^{\prime}(u)\right)\left(u_{k}-u\right) \longrightarrow 0, \quad \text { as } k \longrightarrow+\infty
\end{gather*}
$$

So we obtain $\left\|u_{k}-u\right\| \rightarrow 0$, as $k \rightarrow+\infty$. That is, $\left\{u_{k}\right\}$ strongly converges to $u$ in $X$, which means that $E_{1}$ satisfies the P.S. condition.

Now set $K=\left\{\sum_{i=1}^{k} c_{i} \varphi_{i}: \sum_{i=1}^{k} c_{i}^{2}=c^{2}\right\}$, where $\varphi_{i}$ is defined in (9). It is clear that $K$ is homeomorphic to $S^{k-1}$ by an odd map for any $c>0$. In the following we verify that $\left.E_{1}\right|_{K}<0$ if $c$ is sufficiently small.

For any $u \in K, u=\sum_{i=1}^{k} c_{i} \varphi_{i}$. By (H4) and (30), we have

$$
\begin{align*}
& E_{1}(u)=\frac{1}{2} \int_{0}^{T} e^{G(t)}\left[\left(\sum_{i=1}^{k} c_{i} \varphi_{i}(t)\right)^{\prime}\right]^{2} d t \\
& -\int_{0}^{T} H_{1}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \\
& =\frac{1}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T} e^{\mathrm{G}(t)}\left[\varphi_{i}^{\prime}(t)\right]^{2} d t \\
& -\frac{\lambda}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T} e^{G(t)}\left[\varphi_{i}(t)\right]^{2} d t \\
& -\int_{0}^{T} e^{G(t)} F(t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \\
& \leq \frac{M}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T}\left[\varphi_{i}^{\prime}(t)\right]^{2} d t \\
& -\frac{m \lambda}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T}\left[\varphi_{i}(t)\right]^{2} d t \\
& -\int_{0}^{T} e^{G(t)} F(t, u(t)) d t-\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \\
& =\frac{1}{2} \sum_{i=1}^{k} c_{i}^{2}\left(M-\frac{m \lambda}{\lambda_{i}}\right)-\int_{0}^{T} e^{G(t)} F(t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} I_{j}(t) d t \\
& \leq \frac{1}{2}\left(M-\frac{m \lambda}{\lambda_{k}}\right) c^{2}+o\left(c^{2}\right)+o\left(c^{2}\right), \tag{37}
\end{align*}
$$

for small $c>0$. Since $\lambda \in\left(M \lambda_{k} / m, r\right], E_{1}(u)<0$ and the proof is complete.

Theorem 5. Suppose that the following conditions hold.
(H1) There exist $u_{1}>0, r>M \lambda_{k} / m, \lambda_{k}$ which is the $k t h$ eigenvalue of (7) such that

$$
\begin{equation*}
r M u_{1}+e^{G(t)} f\left(t, u_{1}\right)=0, \quad r M u_{1}+e^{G(t)} f(t, u)>0 \tag{38}
\end{equation*}
$$ for every $u \in\left(0, u_{1}\right)$.

(H2) $\int_{0}^{u} I_{j}(s) d s \leq 0$ for any $u \in R(j=1,2, \ldots, n)$.
(H3) $f(t, u)$ and $I_{j}(u)(j=1,2, \ldots, n)$ are odd about $u$.
(H4) $f(t, u)=o(|u|), I_{j}(u)=o(|u|)$, as $|u| \rightarrow 0, j=$ $1,2, \ldots, n$.
Then, for $\lambda \in\left(M \lambda_{k} / m, r\right]$, problem (1) has at least $k$ distinct pairs of solutions.

Proof. The proof is similar to the proof of Theorem 4, and therefore we omit it.

Theorem 6. Suppose that the following conditions hold.
(H1) There exist $u_{2}>0, r>M \lambda_{k} / m, \lambda_{k}$ which is the $k t h$ eigenvalue of (7) such that

$$
\begin{array}{r}
r M u_{2}+e^{G(t)} f\left(t, u_{2}\right) \leq 0, \quad I_{j}\left(u_{2}\right) \leq 0  \tag{39}\\
j=1,2, \ldots, n
\end{array}
$$

(H2) $f(t, u)$ and $I_{j}(u)(j=1,2, \ldots, n)$ are odd about $u$.
(H3) $f(t, u)=o(|u|), I_{j}(u)=o(|u|)$, as $|u| \rightarrow 0, j=$ $1,2, \ldots, n$.

Then, for $\lambda \in\left(M \lambda_{k} / m, r\right]$, problem (1) has at least $k$ distinct pairs of solutions.

Proof. Set

$$
\begin{gather*}
h_{2}(\lambda, t, u)= \begin{cases}\lambda e^{G(t)} u+e^{G(t)} f(t, u), & u \in\left[-u_{2}, u_{2}\right] \\
\lambda e^{G(t)} u_{2}+e^{G(t)} f\left(t, u_{2}\right), & u \in\left[u_{2},+\infty\right), \\
-\lambda e^{G(t)} u_{2}-e^{G(t)} f\left(t,-u_{2}\right), & u \in\left(-\infty,-u_{2}\right],\end{cases} \\
T_{j}(u)= \begin{cases}I_{j}(u), & u \in\left[-u_{2}, u_{2}\right], \\
I_{j}\left(u_{2}\right), & u \in\left[u_{2},+\infty\right), \\
I_{j}\left(-u_{2}\right), & u \in\left(-\infty,-u_{2}\right]\end{cases} \tag{40}
\end{gather*}
$$

Consider

$$
\begin{gather*}
-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}=h_{2}(\lambda, t, u(t)), \\
t \neq t_{j}, \quad \text { a.e. } t \in[0, T], \\
-\Delta u^{\prime}\left(t_{j}\right)=T_{j}\left(u\left(t_{j}\right)\right), \quad j=1,2, \ldots, n,  \tag{41}\\
u^{\prime}(0)=0, u(T)=0 .
\end{gather*}
$$

Next, we will verify that the solutions of problem (41) are solutions of problem (1).

In fact, let $\omega_{1}=\left\{t \in\left(a_{1}, b_{1}\right) \subseteq[0, T]: u(t)>u_{2}\right\}$. By the definitions of $h_{2}(\lambda, t, u)$ and $T_{j}(u)$, (41) is reduced to

$$
\begin{gather*}
-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}=h_{2}\left(\lambda, t, u_{2}\right)=\lambda e^{G(t)} u_{2}+e^{G(t)} f\left(t, u_{2}\right) \\
\leq r M u_{2}+e^{G(t)} f\left(t, u_{2}\right) \leq 0, \quad t \neq t_{j}, \quad \text { a.e. } t \in\left(a_{1}, b_{1}\right), \\
-\Delta u^{\prime}\left(t_{j}\right)=T_{j}\left(u\left(t_{j}\right)\right)=I_{j}\left(u_{2}\right) \leq 0, \quad j=1,2, \ldots, n \\
u\left(a_{1}\right)=u\left(b_{1}\right)=u_{2} . \tag{42}
\end{gather*}
$$

The solution $u(t)$ of (42) satisfies $u(t) \leq u_{2}, t \in\left(a_{1}, b_{1}\right)$. So $\omega_{1}=\emptyset$ and $u(t) \leq u_{2}$.

Let $\omega_{2}=\left\{t \in\left(a_{2}, b_{2}\right) \subseteq[0, T]: u(t)<-u_{2}\right\}$. By the definitions of $h_{2}(\lambda, t, u)$ and $T_{j}(u),(41)$ is reduced to

$$
\begin{gather*}
-\left(e^{G(t)} u^{\prime}(t)\right)^{\prime}=h_{2}\left(\lambda, t,-u_{2}\right)=-\lambda e^{G(t)} u_{2}+e^{G(t)} f\left(t,-u_{2}\right) \\
\geq-r M u_{2}-e^{G(t)} f\left(t, u_{2}\right) \geq 0, \\
t \neq t_{j}, \quad \text { a.e. } t \in\left(a_{2}, b_{2}\right), \\
-\Delta u^{\prime}\left(t_{j}\right)=T_{j}\left(u\left(t_{j}\right)\right)=-I_{j}\left(u_{2}\right) \geq 0, \quad j=1,2, \ldots, n, \\
u\left(a_{2}\right)=u\left(b_{2}\right)=-u_{2} . \tag{43}
\end{gather*}
$$

The solution $u(t)$ of (43) satisfies $u(t) \geq-u_{2}, t \in\left(a_{2}, b_{2}\right)$. So $\omega_{2}=\emptyset$ and $u(t) \geq-u_{2}$.

Therefore, the solutions of (41) are solutions of (1). Hence to prove Theorem 6, it suffices to produce at least $k$ distinct pairs of critical points of

$$
\begin{align*}
E_{2}(u)= & \frac{1}{2} \int_{0}^{T} e^{G(t)}\left|u^{\prime}(t)\right|^{2} d t-\int_{0}^{T} H_{2}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t \tag{44}
\end{align*}
$$

where $H_{2}(\lambda, t, u(t))=\int_{0}^{u} h_{2}(\lambda, t, s) d s$.
We will apply Lemma 3 to finish the proof.
By (44) and (H2), $E_{2} \in C^{\prime}(X, R)$ is even and $E_{2}(0)=0$.
Next, we will show that $E_{2}$ is bounded from below.
By (H1) and (H2), we have $u h_{2}(\lambda, t, u(t)) \leq 0$ and $u T_{j}(u) \leq 0$ for $|u| \geq u_{2}$; thus

$$
\begin{align*}
& \int_{0}^{T} H_{2}(\lambda, t, u(t)) d t=\int_{0}^{T} \int_{0}^{u(t)} h_{2}(\lambda, t, s) d s d t \\
& \leq \int_{0}^{T} \int_{0}^{u_{2}} h_{2}(\lambda, t, s) d s d t \\
& \leq \int_{0}^{T} \int_{0}^{u_{2}}[r M s+f(t, s)] d s d t=\rho>0, \\
& \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t \leq \int_{0}^{u_{2}} T_{j}(t) d t=\delta>0 . \tag{45}
\end{align*}
$$

So, we have

$$
\begin{align*}
E_{2}(u)= & \frac{1}{2}\|u\|^{2}-\int_{0}^{T} H_{2}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t  \tag{46}\\
\geq & \frac{1}{2}\|u\|^{2}-\rho-n M \delta \\
> & -\infty
\end{align*}
$$

for any $u \in X$. Therefore, $E_{2}$ is bounded from below.
In the following we will show that $E_{2}$ satisfies the P.S. condition. Let $\left\{u_{k}\right\} \subset X$ such that $\left\{E_{2}\left(u_{k}\right)\right\}$ is a bounded sequence and $\lim _{k \rightarrow \infty} E_{2}^{\prime}\left(u_{k}\right)=0$; then there exists $C_{4}>0$ such that

$$
\begin{equation*}
\left|E_{2}\left(u_{k}\right)\right| \leq C_{4} . \tag{47}
\end{equation*}
$$

By (46), we have

$$
\begin{equation*}
\frac{1}{2}\left\|u_{k}\right\|^{2} \leq C_{4}+\rho+n M \delta \tag{48}
\end{equation*}
$$

So $\left\{u_{k}\right\}$ is bounded in $X$. From the reflexivity of $X$, we may extract a weakly convergent subsequence that, for simplicity, we call $\left\{u_{k}\right\}, u_{k} \rightharpoonup u$ in $X$. In the following we will verify that $\left\{u_{k}\right\}$ strongly converges to $u$ :

$$
\begin{align*}
& \left(E_{2}^{\prime}\left(u_{k}\right)-E_{2}^{\prime}(u)\right)\left(u_{k}-u\right) \\
& =\left\|u_{k}-u\right\|^{2}-\int_{0}^{T}\left[h_{2}\left(\lambda, t, u_{k}(t)\right)-h_{2}(\lambda, t, u(t))\right] \\
& \quad \times\left(u_{k}(t)-u(t)\right) d t  \tag{49}\\
& +\sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[T_{j}\left(u_{k}\left(t_{j}\right)\right)-T_{j}\left(u\left(t_{j}\right)\right)\right] \\
& \quad \times\left(u_{k}\left(t_{j}\right)-u\left(t_{j}\right)\right) .
\end{align*}
$$

By $u_{k} \rightharpoonup u$ in $X$, we see that $\left\{u_{k}\right\}$ uniformly converges to $u$ in $C[0, T]$. So

$$
\begin{align*}
& \int_{0}^{T}\left[h_{2}\left(\lambda, t, u_{k}(t)\right)-h_{2}(\lambda, t, u(t))\right]\left(u_{k}(t)-u(t)\right) d t \longrightarrow 0 \\
& \sum_{j=1}^{n} e^{G\left(t_{j}\right)}\left[T_{j}\left(u_{k}\left(t_{j}\right)\right)-T_{j}\left(u\left(t_{j}\right)\right)\right]\left(u_{k}\left(t_{j}\right)-u\left(t_{j}\right)\right) \longrightarrow 0 \\
& \quad\left(E_{2}^{\prime}\left(u_{k}\right)-E_{2}^{\prime}(u)\right)\left(u_{k}-u\right) \rightarrow 0, \quad \text { as } k \longrightarrow+\infty \tag{50}
\end{align*}
$$

So we obtain $\left\|u_{k}-u\right\| \rightarrow 0$, as $k \rightarrow+\infty$. That is, $\left\{u_{k}\right\}$ strongly converges to $u$ in $X$, which means $E_{2}$ satisfies the P.S. condition.

Now set $K=\left\{\sum_{i=1}^{k} c_{i} \varphi_{i}: \sum_{i=1}^{k} c_{i}^{2}=c^{2}\right\}$, where $\varphi_{i}$ is defined in (9). It is clear that $K$ is homeomorphic to $S^{k-1}$ by an odd
map for any $c>0$. In the following we verify that $\left.E_{2}\right|_{K}<0$ if $c$ is sufficiently small.

For any $u \in K, u=\sum_{i=1}^{k} c_{i} \varphi_{i}$. By (H3) and (44), we have

$$
\left.\begin{array}{rl}
E_{2}(u)= & \frac{1}{2} \int_{0}^{T} e^{G(t)}\left[\left(\sum_{i=1}^{k} c_{i} \varphi_{i}(t)\right)^{\prime}\right]^{2} d t \\
& -\int_{0}^{T} H_{2}(\lambda, t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t \\
= & \frac{1}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T} e^{G(t)}\left[\varphi_{i}^{\prime}(t)\right]^{2} d t \\
& -\frac{\lambda}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T} e^{G(t)}\left[\varphi_{i}(t)\right]^{2} d t \\
& -\int_{0}^{T} e^{G(t)} F(t, u(t)) d t \\
& -\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t \\
\leq & \frac{M}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T}\left[\varphi_{i}^{\prime}(t)\right]^{2} d t-\frac{m \lambda}{2} \sum_{i=1}^{k} c_{i}^{2} \int_{0}^{T}\left[\varphi_{i}(t)\right]^{2} d t \\
= & -\int_{0}^{T} e^{G(t)} F(t, u(t)) d t-\sum_{j=1}^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t \\
= & \frac{1}{2} \sum_{i=1}^{k} c_{i}^{2}\left(M-\frac{1}{2} e^{n} e^{G\left(t_{j}\right)} \int_{0}^{u\left(t_{j}\right)} T_{j}(t) d t\right. \\
\lambda_{i}
\end{array}\right)-\int_{0}^{T} e^{G(t)} F(t, u(t)) d t
$$

for small $c>0$. Since $\lambda \in\left(M \lambda_{k} / m, r\right], E_{2}(u)<0$ and the proof is complete.

## 4. Example

To illustrate how our main results can be used in practice we present the following example.

Example 1. Let $T=\pi / 4, g(t)=-2 t$, and consider the following problem:

$$
\begin{aligned}
& -u^{\prime \prime}(t)-2 t u^{\prime}(t)-\lambda u(t) \\
& =(1+t)\left(u-u^{2}\right)-1000 e^{\left(\pi^{2} / 16\right)-t^{2}}, \quad t \in\left[0, \frac{\pi}{4}\right], t \neq t_{j},
\end{aligned}
$$

$$
\begin{gather*}
-\Delta u^{\prime}\left(t_{j}\right)=2-\sqrt[3]{u\left(t_{j}\right)}, \quad j=1,2, \ldots, n \\
u^{\prime}(0)=0, \quad u\left(\frac{\pi}{4}\right)=0 \tag{52}
\end{gather*}
$$

Compared with (1), $f(t, u)=(1+t)\left(u-u^{2}\right)-$ $1000 e^{\left(\pi^{2} / 16\right)-t^{2}}, I_{j}(u)=2-\sqrt[3]{u(t)}$. Obviously (H2), (H3), and (H4) are satisfied. Let $u_{1}=1, r=1000$; then (H1) is satisfied. By Theorem 4 , for $\left(M \lambda_{k} / m, 1000\right]=\left(4 e^{\pi^{2} / 16}(2 k-\right.$ $\left.1)^{2} \pi^{2}, 1000\right], k=1,2$, problem (1) has at least $k$ distinct pairs of solutions.

Example 2. Let $T=\pi / 2, g(t)=-t / 4$, and consider the following problem:

$$
\begin{align*}
& -u^{\prime \prime}(t)-\frac{t}{4} u^{\prime}(t)-\lambda u(t) \\
& =\left(1+t^{2}\right)\left(2 u-u^{2}\right)-1000 e^{\left(\pi^{2} / 32\right)-t^{2}} \\
& t \in\left[0, \frac{\pi}{2}\right], t \neq t_{j}  \tag{53}\\
& -\Delta u^{\prime}\left(t_{j}\right)=-u\left(t_{j}\right), \quad j=1,2, \ldots, n \\
& u^{\prime}(0)=0, \quad u\left(\frac{\pi}{2}\right)=0
\end{align*}
$$

Compared with (1), $f(t, u)=\left(1+t^{2}\right)\left(2 u-u^{2}\right)-$ $1000 e^{\left(\pi^{2} / 32\right)-t^{2}}, I_{j}(u)=-u\left(t_{j}\right)$. Obviously (H2), (H3), and (H4) are satisfied. Let $u_{1}=2, r=500$; then (H1) is satisfied. By Theorem 5 , for $\left(M \lambda_{k} / m, 500\right]=\left(e^{\pi^{2} / 32}(2 k-1)^{2} \pi^{2}, 500\right]$, $k=1,2,3,4$, problem (53) has at least $k$ distinct pairs of solutions.

Example 3. Let $T=\pi / 2, g(t)=-t / 2$, and consider the following problem:

$$
\begin{align*}
& -u^{\prime \prime}(t)-\frac{t}{2} u^{\prime}(t)-\lambda u(t) \\
& =-e^{\pi^{2} / 16}\left(1+t^{2}\right) u^{3}(t), \quad t \in\left[0, \frac{\pi}{2}\right], t \neq t_{j}  \tag{54}\\
& -\Delta u^{\prime}\left(t_{j}\right)=-3 u^{3}\left(t_{j}\right), \quad j=1,2, \ldots, n, \\
& u^{\prime}(0)=0, \quad u\left(\frac{\pi}{2}\right)=0 .
\end{align*}
$$

Compared with (1), $f(t, u)=-e^{\pi^{2} / 16}\left(1+t^{2}\right) u^{3}(t), I_{j}(u)=$ $-3 u^{3}\left(t_{j}\right)$. Obviously (H2) and (H3) are satisfied. Let $u_{2}=$ 25, $r=625$; then (H1) is satisfied. By Theorem 6, for $\left(M \lambda_{k} / m, 625\right]=\left(e^{\pi^{2} / 16}(2 k-1)^{2} \pi^{2}, 625\right], k=1,2,3$, problem (54) has at least $k$ distinct pairs of solutions.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# On Ground States of Discrete $p(k)$-Laplacian Systems in Generalized Orlicz Sequence Spaces 

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Using the critical point theory, we establish sufficient conditions on the existence of ground states for discrete $p(k)$-Laplacian systems. Our results considerably generalize some existing ones.

## 1. Introduction and Main Results

The aim of this paper is to study the existence of ground state for discrete $p(k)$-Laplacian system

$$
\begin{align*}
& \Delta\left[a(k)|\Delta u(k-1)|^{p(k)-2} \Delta u(k-1)\right] \\
& \quad-q(k)|u(k)|^{p(k)-2} u(k)+f(k, u(k))=0 \tag{1}
\end{align*}
$$

where $p(k)>1$, for all $k \in Z, a(k)$ and $q(k)$ are real valued on $Z$. $f: Z \times R \rightarrow R$ is continuous in the second variable. Moreover, $\Delta$ is the forward difference operator defined by $\Delta u(k)=u(k+1)-u(k)$.

We may think of (1) as being a discrete analogue of the following differential system:

$$
\begin{align*}
& \frac{d}{d t}\left(a(t)|\dot{u}(t)|^{p(t)-2} \dot{u}(t)\right)  \tag{2}\\
& \quad-q(t)|u(t)|^{p(t)-2} u(t)+f(t, u(t))=0 .
\end{align*}
$$

For the case $p(t) \equiv p$, system (2) is a $p$-Laplacian system, which has been widely studied; to mention a few, see $[1,2]$. Even for the special case $p=2$, system (2) can be regarded as the more general form of Emden-Fowler equation appearing in the study of astrophysics, fluid mechanics, gas dynamics, nuclear physics, relativistic mechanics, and chemically reacting system in terms of various special forms
of $f(t, u(t))$ (see, e.g., [3]). The more general differential operator (2), namely, the so-called $p(t)$-Laplacian, has been studied by Fan et al. [4-7]. The $p(t)$-Laplacian operator can be used to describe the physical phenomena with "pointwise different properties." The $p(t)$-Laplacian operator has more complicated properties than that of the $p$-Laplacian; for example, it is not homogeneous, and this makes some classic theories and methods, such as the theory of Sobolev spaces, not applicable.

With the theory of nonlinear discrete dynamical systems being widely used to study discrete models appearing in many fields such as economics, ecology, computer science, neural networks, and cybernetics [8], the existence of solutions of discrete dynamical systems has become a hot topic; to mention a few, see [9-15]. For the case $p(k) \equiv p$, Iannizzotto and Tersian [16] obtained multiple homoclinic solutions for system (1) by using the critical point theorem, and for the special case $p=2, \mathrm{Ma}$ and Guo [13, 14] provided some sufficient conditions on the existence of homoclinic solutions for system (1). For the more general case- $p(k)$ Laplacian system (1)-Chen et al. [17] established some existence criteria to guarantee that the system has at least one or infinitely many homoclinic orbits. Motivated by Liu [2], which discussed the existence of ground state for $p$-Laplacian system, in this paper, we will consider the existence of ground state for the $p(k)$-Laplacian system (1).

Now we are in a position to state our main results.

Theorem 1. Assume the following conditions hold:
(a) $a(k)>0$ for all $k \in Z$,
(q) $q(k)>0$ for all $k \in Z$ and $q(k) \rightarrow+\infty$ as $|k| \rightarrow+\infty$,
(p) $1<p^{-}=\inf _{k \in Z} p(k) \leq \sup _{k \in Z} p(k)=p^{+}<+\infty$ for all $k \in Z$,
(f) $f(k, u)=f_{1}(k, u)-f_{2}(k, u)$ is continuous in $u$ for all $k \in Z$, and $F(k, u)=\int_{0}^{u} f(k, s) d s$ for $u \in R$. Moreover,

$$
\begin{equation*}
\frac{|f(k, u)|}{q(k)|u|^{p(k)-1}} \longrightarrow 0 \quad \text { as } u \longrightarrow 0 \text { uniformly in } k \in Z \tag{3}
\end{equation*}
$$

$\left(f_{1}\right)$ there exists a constant $\beta>p^{+}$such that

$$
\begin{equation*}
u f_{1}(k, u) \geq \beta F_{1}(k, u)>0 \quad \forall k \in Z, u \in R \backslash\{0\}, \tag{4}
\end{equation*}
$$

where $F_{1}(k, u)=\int_{0}^{u} f_{1}(k, s) d s$ for $u \in R$;
$\left(f_{2}\right)$ there exists a constant $\tau \in(0, \beta)$ such that

$$
\begin{array}{r}
F_{2}(k, u) \geq 0, \quad u f_{2}(k, u) \leq \tau F_{2}(k, u)  \tag{5}\\
\forall k \in Z, \quad u \in R,
\end{array}
$$

where $F_{2}(k, u)=\int_{0}^{u} f_{2}(k, s) d s$ for $u \in R$.
Then (1) has a ground state solution.
Remark 2. (q) implies that there exists $q_{*}>0$ such that $q(k) \geq$ $q_{*}$ for all $k \in Z$.

Remark 3. We extend Theorem 3.1 in [14] to the more general case- $p(k)$-Laplacian system. Furthermore, we obtain the existence of the ground state.

The rest of this paper is organized as follows. In Section 2, we establish the variational structure associated with (1). Some preliminary results are also provided in this section. In Section 3, we give the proof of the main result.

## 2. Variational Structure and Some Preliminary Results

In this section, we establish a variational structure which enables us to reduce the existence of solutions for (1) to the existence of critical points of the corresponding functional.

Let $S$ be the set of all two-sided sequences; that is,

$$
\begin{equation*}
S=\{u=\{u(k)\}: u(k) \in R, k \in Z\} . \tag{6}
\end{equation*}
$$

Then $S$ is a vector space with $a u+b v=\{a u(k)+b v(k)\}$ for $u, v \in S, a, b \in R$.

We define $l^{p(k)}$ as the set of all functions $u \in S$ such that

$$
\begin{equation*}
l^{p(k)}=\left\{u \in S: \sum_{k \in Z}|u(k)|^{p(k)}<+\infty\right\} \tag{7}
\end{equation*}
$$

with the norm

$$
\begin{equation*}
\|u\|_{p(k)}=\inf \left\{r>0: \sum_{k \in Z}\left|\frac{u(k)}{r}\right|^{p(k)} \leq 1\right\} . \tag{8}
\end{equation*}
$$

We also define

$$
\begin{align*}
E=\left\{u \in S: \sum_{k \in Z}\right. & {\left[a(k)|\Delta u(k-1)|^{p(k)}\right.}  \tag{9}\\
& \left.\left.+q(k)|u(k)|^{p(k)}\right]<+\infty\right\}
\end{align*}
$$

with the norm

$$
\begin{align*}
\|u\|=\inf \left\{r>0: \sum_{k \in Z}\right. & {\left[a(k)\left|\frac{\Delta u(k-1)}{r}\right|^{p(k)}\right.}  \tag{10}\\
& \left.\left.+q(k)\left|\frac{u(k)}{r}\right|^{p(k)}\right] \leq 1\right\} .
\end{align*}
$$

We call the space $E$ a sequence space; it is a special kind of generalized Orlicz sequence space. For the general theory of generalized Orlicz spaces, see [18, 19].

Consider the functional $I$ on $E$ defined by

$$
\begin{align*}
I(u)=\sum_{k \in Z}[ & \frac{a(k)}{p(k)}|\Delta u(k-1)|^{p(k)}+\frac{q(k)}{p(k)}|u(k)|^{p(k)}  \tag{11}\\
& -F(k, u(k))] .
\end{align*}
$$

Using the similar arguments as [17], we have the following lemmas.

Lemma 4. $\left(l^{p(k)},\|\cdot\|_{p(k)}\right)$ is a reflexive Banach space. Let $u \in$ $l^{p(k)}$ and

$$
\begin{equation*}
\phi(u)=\sum_{k \in Z}|u(k)|^{p(k)} ; \tag{12}
\end{equation*}
$$

one has
(1) if $\|u\|_{p(k)}>1$, then $\|u\|_{p(k)}^{p^{-}} \leq \phi(u) \leq\|u\|_{p(k)}^{p^{+}} ;$
(2) if $\|u\|_{p(k)}<1$, then $\|u\|_{p(k)}^{p^{+}} \leq \phi(u) \leq\|u\|_{p(k)}^{p^{-}}$.

Lemma 5. $(E,\|\cdot\|)$ is a reflexive Banach space. Let $u \in E$ and

$$
\begin{equation*}
\varphi(u)=\sum_{k \in Z}\left[a(k)|\Delta u(k-1)|^{p(k)}+q(k)|u(k)|^{p(k)}\right] ; \tag{13}
\end{equation*}
$$

one has
(1) if $\|u\|>1$, then $\|u\|^{p^{-}} \leq \varphi(u) \leq\|u\|^{p^{+}}$;
(2) if $\|u\|<1$, then $\|u\|^{p^{+}} \leq \varphi(u) \leq\|u\|^{p^{-}}$.

Lemma 6. $I \in C^{1}(E, R)$ and the Fréchet derivative is given by

$$
\begin{align*}
&\left\langle I^{\prime}(u), v\right\rangle=\sum_{k \in Z}[ a(k)|\Delta u(k-1)|^{p(k)-2} \\
& \times \Delta u(k-1) \Delta v(k-1) \\
&\left.+q(k)|u(k)|^{p(k)-2} u(k) v(k)\right]  \tag{14}\\
& \quad-\sum_{k \in Z} f(k, u(k)) v(k)
\end{align*}
$$

for all $u, v \in E$. Moreover, the nonzero critical points of the functional I on E are the nontrivial solutions of (1).

Lemma 7. $\left(f_{1}\right)$ and $\left(f_{2}\right)$ imply that

$$
\begin{equation*}
u f(k, u) \geq \beta F(k, u) \quad \forall k \in Z, u \in R . \tag{15}
\end{equation*}
$$

Moreover, for every $k \in Z$ and $u \in R, s^{-\beta} F_{1}(k, s u)$ is nondecreasing on $(0,+\infty)$ and $s^{-\tau} F_{2}(k, s u)$ is nonincreasing on ( $0,+\infty$ ).

Let

$$
\begin{equation*}
c_{\min }=\inf \left\{I(u): I^{\prime}(u)=0, u \in E \backslash\{0\}\right\} . \tag{16}
\end{equation*}
$$

Then $u_{0} \neq 0$ with $I\left(u_{0}\right)=c_{\text {min }}$ is said to be a ground state solution of (1).

As usual, we make use of the following basic notations. Let $H$ be a Hilbert space and $C^{1}(H, R)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on $H$.

Definition 8. Let $I \in C^{1}(H, R)$. A sequence $\left\{x_{j}\right\} \subset H$ is called a Palais-Smale sequence (P.S. sequence) for $I$ if $\left\{I\left(x_{j}\right)\right\}$ is bounded and $I^{\prime}\left(x_{j}\right) \rightarrow 0$ as $j \rightarrow+\infty$. We say that $I$ satisfies the Palais-Smale condition (P.S. condition) if any P.S. sequence for $I$ possesses a convergent subsequence.

Let $B_{r}$ be the open ball in $H$ with radius $r$ and center 0 , and let $\partial B_{r}$ denote its boundary.

Lemma 9 (mountain pass lemma). Let $H$ be a real Hilbert space and $I \in C^{1}(H, R)$ satisfies the P.S. condition. Assume that $I(0) \leq 0$ and the following two conditions hold.
$\left(\mathrm{J}_{1}\right)$ There exist constants $a>0$ and $\rho>0$ such that $I_{\partial B_{\rho}} \geq$ a.
$\left(\mathrm{J}_{2}\right)$ There exists an $e \in H \backslash B_{\rho}$ such that $I(e) \leq 0$.
Then I possesses a critical value $c \geq a$. Moreover, $c$ can be characterized as

$$
\begin{equation*}
c=\inf _{h \in \Gamma} \max _{s \in[0,1]} I(h(s)), \tag{17}
\end{equation*}
$$

where

$$
\begin{equation*}
\Gamma=\{h \in C([0,1], H): h(0)=0, h(1)=e\} . \tag{18}
\end{equation*}
$$

## 3. Proof of Main Result

In order to prove Theorem 1, we first prove the following lemmas.

Lemma 10. The embedding $E \hookrightarrow l^{p(k)}$ is compact.
Proof. Let $\left\{u_{j}\right\}$ be a bounded sequence in $E$; that is, there exists $M>0$ such that $\left\|u_{j}\right\|<M$ for all $j \in Z^{+}$. By reflexivity, passing to a subsequence we have $u_{j} \rightharpoonup u$ in $E$ for some $u \in E$. We may assume $u=0$, in particular $u_{j}(k) \rightarrow 0$ as $j \rightarrow+\infty$ for all $k \in Z$. For all $\epsilon>0$, we can find $h \in Z^{+}$such that

$$
\begin{equation*}
q(k)>\frac{1+M}{\epsilon} \quad \forall|k|>h . \tag{19}
\end{equation*}
$$

By continuity of the finite sum, there exists $\nu_{0} \in Z^{+}$such that

$$
\begin{equation*}
\sum_{|k| \leq h}\left|u_{j}(k)\right|^{p(k)} \leq \frac{\epsilon}{1+M} \quad \forall j>v_{0} \tag{20}
\end{equation*}
$$

So for all $j \geq v_{0}$ we have

$$
\begin{align*}
\sum_{k \in Z}\left|u_{j}(k)\right|^{p(k)} & \leq \frac{\epsilon}{1+M}+\frac{\epsilon}{1+M} \sum_{|k|>h} q(k)\left|u_{j}(k)\right|^{p(k)} \\
& \leq \frac{\epsilon}{1+M}+\frac{\epsilon}{1+M} \sum_{k \in Z} q(k)\left|u_{j}(k)\right|^{p(k)} \tag{21}
\end{align*}
$$

Since

$$
\sum_{k \in Z} q(k)\left|u_{j}(k)\right|^{p(k)} \leq \varphi\left(u_{j}\right) \leq \begin{cases}\left\|u_{j}\right\|^{p-}, & \text { if }\left\|u_{j}\right\|<1  \tag{22}\\ 1, & \text { if }\left\|u_{j}\right\|=1 \\ \left\|u_{j}\right\|^{p+}, & \text { if }\left\|u_{j}\right\|>1\end{cases}
$$

letting $M_{0}=\max \left\{M^{p-}, 1, M^{p+}\right\}$, we have

$$
\begin{equation*}
\sum_{k \in Z}\left|u_{j}(k)\right|^{p(k)} \leq \frac{1+M_{0}}{1+M} \epsilon \quad \forall j \geq v_{0} . \tag{23}
\end{equation*}
$$

Thus, $u_{j}(k) \rightarrow 0$ in $l^{p(k)}$, and the proof is completed.
Lemma 11. Assume that $u \in l^{p(k)}$ and $v \in l^{r(k)}$. Moreover, $p(k)$ satisfies condition $(p)$ and $1 / r(k)+1 / p(k)=1$ for all $k \in Z$. Then

$$
\begin{equation*}
\sum_{k \in Z} u(k) v(k) \leq\left(\frac{1}{p^{-}}+\frac{1}{r^{-}}\right)\|u\|_{p(k)}\|v\|_{r(k)}, \tag{24}
\end{equation*}
$$

where $r^{-}=\inf \{r(k): k \in Z\}$ and $r^{-}=p^{+} /\left(p^{+}-1\right)$.
Proof. Let

$$
\begin{equation*}
r_{1}=\|u\|_{p(k)}, \quad r_{2}=\|v\|_{r(k)} \tag{25}
\end{equation*}
$$

then

$$
\begin{align*}
\sum_{k \in Z} \frac{u(k) v(k)}{r_{1} r_{2}} & \leq \sum_{k \in Z}\left[\frac{1}{p(k)}\left(\frac{u(k)}{r_{1}}\right)^{p(k)}+\frac{1}{r(k)}\left(\frac{v(k)}{r_{2}}\right)^{r(k)}\right] \\
& \leq \frac{1}{p^{-}}+\frac{1}{r^{-}} \tag{26}
\end{align*}
$$

The proof is completed.
Lemma 12. Assume that all the conditions of Theorem 1 hold. Then the functional I satisfies the P.S. condition.

Proof. Assume that $\left\{u_{j}\right\}_{j \in N} \subset E$ is a sequence such that $\left\{I\left(u_{j}\right)\right\}$ is a bounded and $I^{\prime}\left(u_{j}\right) \rightarrow 0$ as $j \rightarrow+\infty$. Then there exists a positive constant $M^{\prime}$ such that $\left|I\left(u_{j}\right)\right| \leq M^{\prime}$ for all $j \in Z^{+}$.

First, we show that $\left\|u_{j}\right\|$ is bounded. Now we may assume that $\left\|u_{j}\right\|>1$; otherwise, $\left\|u_{j}\right\|$ is bounded obviously. When $j$ is large enough, we have

$$
\begin{align*}
M^{\prime}+\frac{1}{\beta}\left\|u_{j}\right\| \geq & I\left(u_{j}\right)-\frac{1}{\beta}\left\langle I^{\prime}\left(u_{j}\right), u_{j}\right\rangle \\
= & \sum_{k \in Z}\left(\frac{1}{p(k)}-\frac{1}{\beta}\right)\left[a(k)\left|\Delta u_{j}(k-1)\right|^{p(k)}\right. \\
& \left.+q(k)\left|u_{j}(k)\right|^{p(k)}\right] \\
& +\sum_{k \in Z}\left[\frac{1}{\beta} f\left(k, u_{j}\right) u_{j}(k)-F\left(k, u_{j}(k)\right)\right] \\
\geq & \left(\frac{1}{p^{+}}-\frac{1}{\beta}\right)\left\|u_{j}\right\|^{p^{-}} . \tag{27}
\end{align*}
$$

It follows from $\beta>p^{+}$and $p^{-}>1$ that there exists a constant $M_{*}>0$ such that

$$
\begin{equation*}
\left\|u_{j}\right\| \leq M_{*}, \quad \forall j \in Z^{+} . \tag{28}
\end{equation*}
$$

By Lemma 10, we can choose a subsequence, still denoted by $\left\{u_{j}\right\}$, such that

$$
\begin{gather*}
u_{j} \rightarrow u_{*} \quad \text { in } E,  \tag{29}\\
u_{j} \longrightarrow u_{*} \quad \text { in } l^{p(k)} \tag{30}
\end{gather*}
$$

for some $u_{*} \in E$.
Next, we prove that

$$
\begin{align*}
\lim _{j \rightarrow+\infty} \sum_{k \in Z}( & \left.f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right)  \tag{31}\\
\times & \left(u_{j}(k)-u_{*}(k)\right)=0
\end{align*}
$$

By (f), for any $0<\epsilon<\min \left\{1 / 2,1 / 2 p^{+}\right\}$, there exists a positive constant $\rho<1$ with $q_{*}^{\left(1 / p^{-}\right)} \rho^{\left(p^{+} / p^{-}\right)}<1$ such that

$$
\begin{equation*}
f(k, u) \leq \epsilon q(k)|u|^{p(k)-1} \quad \forall k \in Z,|u| \leq \rho, \tag{32}
\end{equation*}
$$

Since $u_{*} \in E$, there exists a positive integer $T$ such that

$$
\begin{equation*}
\left|u_{*}(k)\right| \leq \frac{\rho}{2} \quad \forall k>T ; \tag{33}
\end{equation*}
$$

we have

$$
\begin{equation*}
\left|f\left(k, u_{*}(k)\right)\right| \leq \epsilon q(k)\left|u_{*}(k)\right|^{p(x)-1} \quad \forall k>T . \tag{34}
\end{equation*}
$$

By (30), there exists $v_{1} \in Z^{+}$such that

$$
\begin{equation*}
\left|u_{j}(k)-u_{*}(k)\right| \leq \frac{\rho}{2} \quad \forall j>v_{1}, k \in Z . \tag{35}
\end{equation*}
$$

This, combined with (33) and (32), gives us

$$
\begin{gather*}
\left|u_{j}(k)\right| \leq\left|u_{j}(k)-u_{*}(k)\right|+\left|u_{*}(k)\right| \leq \rho \quad \forall j>v_{1}, k>T \\
\left|f\left(k, u_{j}(k)\right)\right| \leq \epsilon q(k)\left|u_{j}(k)\right|^{p(k)-1} \quad \forall j>v_{1}, k>T . \tag{36}
\end{gather*}
$$

Then for $j>\nu_{1}$,

$$
\begin{align*}
& \sum_{|k|>T}\left|f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right|\left|u_{j}(k)-u_{*}(k)\right| \\
& \leq \epsilon \sum_{|k|>T}\left(q(k)\left|u_{j}(k)\right|^{p(k)-1}+q(k)\left|u_{*}(k)\right|^{p(k)-1}\right) \\
& \times\left(\left|u_{j}(k)\right|+\left|u_{*}(k)\right|\right) \\
& \leq \epsilon \sum_{k \in Z} q(k)^{1 / r(k)}\left|u_{j}(k)\right|^{p(k)-1} q(k)^{1 / p(k)}\left|u_{j}(k)\right|  \tag{37}\\
& +\epsilon \sum_{k \in Z} q(k)^{1 / r(k)}\left|u_{j}(k)\right|^{p(k)-1} q(k)^{1 / p(k)}\left|u_{*}(k)\right| \\
& +\epsilon \sum_{k \in Z} q(k)^{1 / r(k)}\left|u_{*}(k)\right|^{p(k)-1} q(k)^{1 / p(k)}\left|u_{j}(k)\right| \\
& +\epsilon \sum_{k \in Z} q(k)^{1 / r(k)}\left|u_{*}(k)\right|^{p(k)-1} q(k)^{1 / p(k)}\left|u_{*}(k)\right|,
\end{align*}
$$

where $1 / r(k)+1 / p(k)=1$ for all $k \in Z$.
Let $v_{1}=\left\{v_{1}(k)\right\}$ and $v_{1}(k)=q(k)^{1 / r(k)}\left|u_{j}(k)\right|^{p(k)-1}$, $v_{2}=\left\{v_{2}(k)\right\}$ and $v_{2}(k)=q(k)^{1 / p(k)}\left|u_{j}(k)\right|, h_{1}=\left\{h_{1}(k)\right\}$ and $h_{1}(k)=q(k)^{1 / r(k)}\left|u_{*}(k)\right|^{p(k)-1}$, and $h_{2}=\left\{h_{2}(k)\right\}$ and $h_{2}(k)=q(k)^{1 / p(k)}\left|u_{*}(k)\right|$.

It is easy to check that $v_{1}, h_{1} \in l^{r(k)}$ and $v_{2}, h_{2} \in l^{p(k)}$. Then using Lemma 11, for $j>\nu_{1}$, we have

$$
\begin{align*}
& \sum_{|k|>T}\left|f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right|\left|u_{j}(k)-u_{*}(k)\right| \\
& \leq\left(\frac{\epsilon}{p^{-}}+\frac{\epsilon}{r^{-}}\right)  \tag{38}\\
& \quad \times\left(\left\|v_{1}\right\|_{r(k)}\left\|v_{2}\right\|_{p(k)}+\left\|v_{1}\right\|_{r(k)}\left\|h_{2}\right\|_{p(k)}\right. \\
& \left.\quad+\left\|h_{1}\right\|_{r(k)}\left\|v_{2}\right\|_{p(k)}+\left\|h_{1}\right\|_{r(k)}\left\|h_{2}\right\|_{p(k)}\right) .
\end{align*}
$$

Now we show that $\left\|v_{1}\right\|_{r(k)}$ is bounded. We may assume that $\left\|v_{1}\right\|_{r(k)}>1$; otherwise, $\left\|v_{1}\right\|_{r(k)}$ is bounded obviously. By Lemmas 4 and 5, we have

$$
\begin{equation*}
\left\|v_{1}\right\|_{r(k)} \leq\left[\sum_{k \in Z} q(k)\left|u_{j}(k)\right|^{p^{(k)}}\right]^{1 / r^{-}} \leq\left\|u_{j}\right\|^{p^{+} / r^{-}} \leq M_{*}^{p^{+} / r^{-}} \tag{39}
\end{equation*}
$$

Let $M_{1}=\left\{1, M_{*}^{p^{+} / r^{-}}\right\}$; then $\left\|v_{1}\right\|_{r(k)} \leq M_{1}$; that is, $\left\|v_{1}\right\|_{r(k)}$ is bounded. Using the similar arguments as above, we obtain that $\left\|v_{2}\right\|_{p(k)},\left\|h_{1}\right\|_{r(k)}$, and $\left\|h_{2}\right\|_{p(k)}$ are bounded; that is, there exist three positive constants $M_{2}, M_{3}$, and $M_{4}$ such that

$$
\begin{equation*}
\left\|v_{2}\right\|_{p(k)} \leq M_{2}, \quad\left\|h_{1}\right\|_{r(k)} \leq M_{3}, \quad\left\|h_{2}\right\|_{p(k)} \leq M_{4} \tag{40}
\end{equation*}
$$

This, combined with (38), gives us

$$
\begin{align*}
& \sum_{|k|>T}\left|f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right|\left|u_{j}(k)-u_{*}(k)\right| \\
& \quad \leq \epsilon\left(\frac{1}{p^{-}}+\frac{1}{r^{-}}\right) \\
& \quad \times\left(M_{1} M_{2}+M_{1} M_{4}+M_{3} M_{2}+M_{3} M_{4}\right) \quad \forall j>v_{1} . \tag{41}
\end{align*}
$$

By continuity of the finite sum and (30), there exists $\nu_{2} \in Z^{+}$ such that

$$
\begin{align*}
& \sum_{|k| \leq T}\left|f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right|  \tag{42}\\
& \quad \times\left|u_{j}(k)-u_{*}(k)\right| \leq \epsilon \quad \forall j \geq v_{2}
\end{align*}
$$

Let $v=\max \left\{v_{1}, v_{2}\right\}$. Combining (41) and (42) together, we have

$$
\begin{align*}
& \sum_{k \in Z}\left|f\left(k, u_{j}(k)\right)-f\left(k, u_{*}(k)\right)\right|\left|u_{j}(k)-u_{*}(k)\right| \\
& \quad \leq \epsilon\left(\frac{1}{p^{-}}+\frac{1}{r^{-}}\right) \\
& \times\left(M_{1} M_{2}+M_{1} M_{4}+M_{3} M_{2}+M_{3} M_{4}\right)+\epsilon \quad \forall j>\nu . \tag{43}
\end{align*}
$$

Thus, (31) holds.
Finally, we show that $\left\{u_{j}\right\}$ possesses a convergent subsequence. Since $I^{\prime}\left(u_{j}\right) \rightarrow 0$ and $u_{j} \rightharpoonup u_{*}$, it follows at once that

$$
\begin{equation*}
\lim _{j \rightarrow+\infty}\left\langle I^{\prime}\left(u_{j}\right)-I^{\prime}\left(u_{*}\right), u_{j}-u_{*}\right\rangle=0 \tag{44}
\end{equation*}
$$

This, combined with (31), gives us

$$
\begin{gather*}
\lim _{j \rightarrow+\infty}\left\{\sum _ { k \in Z } a ( k ) \left[\left|\Delta u_{j}(k-1)\right|^{p(k)-2} \Delta u_{j}(k-1)\right.\right. \\
\left.-\left|\Delta u_{*}(k-1)\right|^{p(k)-2} \Delta u_{*}(k-1)\right] \\
\times\left(\Delta u_{j}(k-1)-\Delta u_{*}(k-1)\right) \\
+\sum_{k \in Z} q(k)\left[\left|u_{j}(k)\right|^{p(k)-2} u_{j}(k)\right.  \tag{45}\\
\left.-\left|u_{*}(k)\right|^{p(k)-2} u_{*}(k)\right] \\
\left.\times\left(u_{j}(k)-u_{*}(k)\right)\right\}=0
\end{gather*}
$$

The following two inequalities are taken from [20] and will play an important role in the proof of our main result:

$$
\begin{align*}
& \quad\left[\left(|\xi|^{p-2} \xi-|\eta|^{p-2} \eta\right)(\xi-\eta)\right] \\
& \quad \times(|\xi|+|\eta|)^{2-p} \geq|\xi-\eta|^{2}, \quad 1<p<2  \tag{46}\\
& 2^{p}\left[\left(|\xi|^{p-2} \xi-|\eta|^{p-2} \eta\right)(\xi-\eta)\right] \geq|\xi-\eta|^{p}, \quad p \geq 2
\end{align*}
$$

for every $\xi$ and $\eta$ in $R$. We define

$$
\begin{align*}
R_{j}(k)= & a(k)\left[\left|\Delta u_{j}(k-1)\right|^{p(k)-2} \Delta u_{j}(k-1)\right. \\
& \left.-\left|\Delta u_{*}(k-1)\right|^{p(k)-2} \Delta u_{*}(k-1)\right] \\
& \times\left(\Delta u_{j}(k-1)-\Delta u_{*}(k-1)\right) \quad \forall k \in Z . \\
Q_{j}(k)= & q(k)\left[\left|u_{j}(k)\right|^{p(k)-2} u_{j}(k)-\left|u_{*}(k)\right|^{p(k)-2} u_{*}(k)\right] \\
& \times\left(u_{j}(k)-u_{*}(k)\right) \quad \forall k \in Z . \tag{47}
\end{align*}
$$

This, combined with (45), produces at once

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \sum_{k \in Z} R_{j}(k)=0, \quad \lim _{j \rightarrow+\infty} \sum_{k \in Z} Q_{j}(k)=0 \tag{48}
\end{equation*}
$$

Now we show that $\varphi\left(u_{j}-u_{*}\right) \rightarrow 0$ as $j \rightarrow+\infty$. That is,

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \sum_{k \in Z} a(k)\left|\Delta u_{j}(k-1)-\Delta u_{*}(k-1)\right|^{p(k)}=0 \tag{49}
\end{equation*}
$$

$$
\begin{equation*}
\lim _{j \rightarrow+\infty} \sum_{k \in Z} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)}=0 \tag{50}
\end{equation*}
$$

Let us first prove (50). Since $\left\{u_{j}\right\}$ and $u_{*}$ are bounded in $E$, there exists a constant $M^{*}>1$ such that $\varphi\left(u_{j}\right) \leq M^{*}$ and $\varphi\left(u_{*}\right) \leq M^{*}$ for all $j \in Z^{+}$. We denote

$$
\begin{align*}
& W_{1}=\{k \in Z: 1<p(k)<2\}, \\
& W_{2}=\{k \in Z: p(k) \geq 2\} . \tag{51}
\end{align*}
$$

By (46), we have

$$
\begin{aligned}
& \sum_{k \in W_{2}} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)} \\
& \quad \leq \sum_{k \in W_{2}} 2^{p^{+}} q(k)\left(\left|u_{j}(k)\right|^{p(k)-2} u_{j}(k)\right. \\
& \left.\quad-\left|u_{*}(k)\right|^{p(k)-2} u_{*}(k)\right) \\
& \quad \times\left(u_{j}(k)-u_{*}(k)\right) \leq 2^{p^{+}} \sum_{k \in Z} Q_{j}(k),
\end{aligned}
$$

$$
\sum_{k \in W_{1}} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)}
$$

$$
=\sum_{k \in W_{1}} q(k)\left[\left|u_{j}(k)-u_{*}(k)\right|^{2}\right]^{p(k) / 2}
$$

$$
\leq \sum_{k \in W_{1}}\left(Q_{j}(k)\right)^{p(k) / 2}
$$

$$
\times\left\{q(k)\left[\left|u_{j}(k)\right|+\left|u_{*}(k)\right|\right]^{p(k)}\right\}^{(2-p(k)) / 2}
$$

$$
\leq \sum_{k \in W_{1}}\left(Q_{j}(k)\right)^{p(k) / 2}
$$

$$
\times\left[2^{p^{+}} q(k)\left(\left|u_{j}(k)\right|^{p(k)}+\left|u_{*}(k)\right|^{p(k)}\right)\right]^{(2-p(k)) / 2}
$$

$$
\leq 2^{p^{+}\left(2-p^{-}\right) / 2}
$$

$$
\times \sum_{k \in Z}\left(Q_{j}(k)\right)^{p(k) / 2}
$$

$$
\begin{equation*}
\times\left[q(k)\left|u_{j}(k)\right|^{p(k)}+q(k)\left|u_{*}(k)\right|^{p(k)}\right]^{(2-p(k)) / 2} . \tag{53}
\end{equation*}
$$

Let

$$
\begin{align*}
& p_{1}(k)=\frac{2}{p(k)}, \quad r_{1}(k)=\frac{2}{2-p(k)}, \\
& v_{j *}(k)=\left(Q_{j}(k)\right)^{p(k) / 2}, \\
& w_{j *}(k)=\left[q(k)\left|u_{j}(k)\right|^{p(k)}+q(k)\left|u_{*}(k)\right|^{p(k)}\right]^{(2-p(k)) / 2} . \tag{54}
\end{align*}
$$

Then it is easy to check that $v_{j *}=\left\{v_{j *}(k)\right\} \in l_{p_{1}(k)}$ and $w_{j *}=$ $\left\{w_{j *}(k)\right\} \in l_{r_{1}(k)}$. By Lemma 11 and (53), we have

$$
\begin{align*}
& \sum_{k \in W_{1}} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)} \\
& \quad \leq 2^{p^{+}\left(2-p^{-}\right) / 2}\left(1+\frac{p^{+}}{2}-\frac{p^{-}}{2}\right)\left\|v_{j *}\right\|_{p_{1}(k)}\left\|w_{j *}\right\|_{r_{1}(k)} . \tag{55}
\end{align*}
$$

Since

$$
\begin{gather*}
\lim _{j \rightarrow+\infty} \sum_{k \in Z} Q_{j}(k)=0, \\
\sum_{k \in Z}\left[q(k)\left|u_{j}(k)\right|^{p(k)}+q(k)\left|u_{*}(k)\right|^{p(k)}\right]  \tag{56}\\
\leq \varphi\left(u_{j}\right)+\varphi\left(u_{*}\right) \leq 2 M^{*} .
\end{gather*}
$$

It is easy to see that $\left\|v_{j *}\right\|_{p_{1}(k)} \rightarrow 0$ as $j \rightarrow+\infty$ and $\left\|w_{j *}\right\|_{r_{1}(k)}$ is bounded for all $j \in Z^{+}$. This, combined with (52) and (55), gives us

$$
\begin{align*}
\lim _{j \rightarrow+\infty} & \sum_{k \in Z} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)} \\
= & \lim _{j \rightarrow+\infty} \sum_{k \in W_{1}} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)}  \tag{57}\\
& +\lim _{j \rightarrow+\infty} \sum_{k \in W_{2}} q(k)\left|u_{j}(k)-u_{*}(k)\right|^{p(k)}=0 .
\end{align*}
$$

Using the similar arguments, we have (49). So $\varphi\left(u_{j}-u_{*}\right) \rightarrow 0$ as $j \rightarrow+\infty$. By Lemma 5 , it follows that $\left\|u_{j}-u_{*}\right\| \rightarrow 0$ as $j \rightarrow+\infty$, and the proof is completed.

Proof of Theorem 1. The proof consists of two steps.
Step 1. We use Lemma 9 to show that (1) has a nontrivial solution in $E$.

First we prove that $I$ satisfies $\left(\mathrm{J}_{1}\right)$ of Lemma 9. It follows from (32) that

$$
\begin{equation*}
|F(k, u(k))| \leq \frac{1}{2 p^{+}} q(k)|u(k)|^{p(k)} \tag{58}
\end{equation*}
$$

for $|u(k)| \leq \rho$ and $k \in Z$. Then, let $\delta=q_{*}^{\left(1 / p^{-}\right)} \rho^{\left(p^{+} / p^{-}\right)}<1$, for all $u \in \partial B_{\delta} \cap E$, we have $|u(k)| \leq \rho$ and

$$
\begin{align*}
I(u)= & \sum_{k \in Z}\left[\frac{a(k)}{p(k)}|\Delta u(k-1)|^{p(k)}+\frac{q(k)}{p(k)}|u(k)|^{p(k)}\right] \\
& -\sum_{k \in Z} F(k, u(k)) \\
\geq & \sum_{k \in Z} \frac{1}{p^{+}}\left[a(k)|\Delta u(k-1)|^{p(k)}+q(k)|u(k)|^{p(k)}\right] \\
& -\sum_{k \in Z} \frac{1}{2 p^{+}} q(k)|u(k)|^{p(k)} \\
\geq & \sum_{k \in Z} \frac{1}{2 p^{+}}\left[a(k)|\Delta u(k-1)|^{p(k)}+q(k)|u(k)|^{p(k)}\right] \\
\geq & \frac{1}{2 p^{+}}\|u\|^{p^{+}}=\frac{1}{2 p^{+}} \delta^{p^{+}}>0, \tag{59}
\end{align*}
$$

and hence $I$ satisfies $\left(\mathrm{J}_{1}\right)$ of Lemma 9.

Next, we prove that $I$ satisfies $\left(\mathrm{J}_{2}\right)$ of Lemma 9. Let $e=$ $\{e(k)\} \in E$ and

$$
e(k)= \begin{cases}0, & \text { if } k \neq 0  \tag{60}\\ 1, & \text { if } k=0\end{cases}
$$

Then

$$
F(k, e(k))= \begin{cases}0, & \text { if } k \neq 0  \tag{61}\\ F(0,1), & \text { if } k=0\end{cases}
$$

By Lemma 7, for $s>1$, we have

$$
\begin{array}{ll}
\sum_{k \in Z} F_{1}(k, s e(k))=F_{1}(0, s) \geq s^{\beta} F_{1}(0,1), & F_{1}(0,1)>0 \\
\sum_{k \in Z} F_{2}(k, s e(k))=F_{2}(0, s) \leq s^{\tau} F_{2}(0,1), & F_{2}(0,1) \geq 0 \tag{62}
\end{array}
$$

Then

$$
\begin{align*}
I(s e)= & \sum_{k \in Z}\left[\frac{a(k)}{p(k)}|s \Delta e(k-1)|^{p(k)}+\frac{q(k)}{p(k)}|s e(k)|^{p(k)}\right] \\
& -\sum_{k \in Z}\left[F_{1}(k, s e(k))-F_{2}(k, s e(k))\right]  \tag{63}\\
\leq & \frac{a(0)}{p(0)} s^{p(0)}+\frac{a(1)}{p(1)} s^{p(1)}+\frac{q(0)}{p(0)} s^{p(0)} \\
& -s^{\beta} F_{1}(0,1)+s^{\tau} F_{2}(0,1)
\end{align*}
$$

Since $F_{1}(0,1)>0$ and $p(0), p(1), \tau$ are smaller than $\beta$, we can choose $s^{*}$ large enough such that $I\left(s^{*} e\right)<0$. So we have verified all assumptions of Lemma 9; we know that $I$ possesses a critical value $\alpha \geq\left(1 / 2 p^{+}\right) \delta^{p^{+}}>0$, where

$$
\begin{align*}
\alpha & =\inf _{h \in \Gamma} \max _{s \in[0,1]} I(h(s))  \tag{64}\\
\Gamma & =\left\{h \in C([0,1], E): h(0)=0, h(1)=s^{*} e\right\}
\end{align*}
$$

A critical point $u^{*}$ of $I$ corresponding to $\alpha$ is nonzero as $\alpha>0$.
Step 2. We prove that (1) has a ground state in $E$.
Let

$$
\begin{equation*}
K=\left\{u \in E: I^{\prime}(u)=0, u \neq 0\right\} \tag{65}
\end{equation*}
$$

be the critical set of $I$. Obviously, $K$ is a nonempty set. Denote

$$
\begin{equation*}
\tilde{c}=\inf \{I(u): u \in K\} \tag{66}
\end{equation*}
$$

Since $u \in K$, we have

$$
\begin{equation*}
I(u)=I(u)-\frac{1}{\beta}\left\langle I^{\prime}(u), u\right\rangle \geq\left(\frac{1}{p^{+}}-\frac{1}{\beta}\right) \varphi(u) \geq 0 . \tag{67}
\end{equation*}
$$

Then $0 \leq \tilde{c} \leq I(u)$.

Suppose that $\left\{u_{j}\right\} \subset K$ such that $I\left(u_{j}\right) \rightarrow \widetilde{c}$. Obviously, $\left\{u_{j}\right\}$ is a P.S. sequence. By Lemma 12, we can choose a subsequence, still denoted by $\left\{u_{j}\right\}$, such that

$$
\begin{equation*}
\lim _{j \rightarrow \infty}\left\|u_{j}-u_{0}\right\|=0, \quad u_{0} \in E \tag{68}
\end{equation*}
$$

Then $I\left(u_{j}\right) \rightarrow I\left(u_{0}\right)$ and $I\left(u_{0}\right)=\widetilde{c}$. Now we prove that $u_{0}$ is nonzero. If $u_{0}=0$, then there exists a positive integer $W$ such that for all $j>W$, we have

$$
\begin{equation*}
\left\|u_{j}\right\| \leq q_{*}^{\left(1 / p^{-}\right)} \rho^{\left(p^{+} / p^{-}\right)}<1 \tag{69}
\end{equation*}
$$

By (32), it follows that

$$
\begin{align*}
\left\langle I^{\prime}\left(u_{j}\right), u_{j}\right\rangle= & \sum_{k \in Z}\left[a(k)\left|\Delta u_{j}(k-1)\right|^{p(k)}+q(k)\left|u_{j}(k)\right|^{p(k)}\right. \\
& \left.-f\left(k, u_{j}(k)\right) u_{j}(k)\right] \\
\geq \sum_{k \in Z}[ & {\left[a(k)\left|\Delta u_{j}(k-1)\right|^{p(k)}\right.} \\
& \left.+q(k)\left|u_{j}(k)\right|^{p(k)}\right] \\
& \quad-\frac{1}{2} \sum_{k \in Z} q(k)\left|u_{j}(k)\right|^{p(k)} \\
\geq & \frac{1}{2} \varphi\left(u_{j}\right) \geq \frac{1}{2}\left\|u_{j}\right\|^{p^{+}}>0 \tag{70}
\end{align*}
$$

which is in contradiction to $I^{\prime}\left(u_{j}\right)=0$. Thus, $u_{0}$ is the ground state solution of (1). The proof is completed.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Global Stability for a Viral Infection Model with Saturated Incidence Rate 

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#### Abstract

A viral infection model with saturated incidence rate and viral infection with delay is derived and analyzed; the incidence rate is assumed to be a specific nonlinear form $\beta x v /(1+\alpha v)$. The existence and uniqueness of equilibrium are proved. The basic reproductive number $R_{0}$ is given. The model is divided into two cases: with or without delay. In each case, by constructing Lyapunov functionals, necessary and sufficient conditions are given to ensure the global stability of the models.


## 1. Introduction

In recent years, study of infectious disease model has been a hot issue; the main cause of infectious disease is the virus invasion. As we know, viral cytopathicity within target cells is very common. A number of mathematical models have been used to study virus dynamics. In 1996, Nowak et al. [1] designed a simple but natural mathematical model based on ordinary differential equation. The model is as follows:

$$
\begin{align*}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\beta x(t) v(t) \\
& \frac{d y(t)}{d t}=\beta x(t) v(t)-a y(t)  \tag{1}\\
& \frac{d v(t)}{d t}=\kappa y(t)-\gamma v(t)
\end{align*}
$$

where $x(t)$ denotes the number of uninfected cells, $y(t)$ the numbers of infected cells, and $v(t)$ the numbers of free viral particles at time $t$, respectively. In model (1), uninfected target cells are assumed to be produced at a constant rate $\lambda$ and died at rate $d x$. Infection of target cells by in-host free viruses is assumed to occur at a bilinear rate $\beta x v$; infected cells are lost at a rate $a y$. Free viruses are produced by infected cells at a rate $\kappa y$, in which $\kappa$ is the average number of viral particles
produced over the lifetime of a single infected cell. Free viral particles die at a rate $\gamma v$. For model (1), Korobeinikov [2] established the condition of global stability in 2004. Some other viral dynamical models were proposed by later researchers; see for example [3-8].

In [7], Wodarz and Levy pointed out that the term $a y(t)$ in model (1) should consist of two parts: one is the natural death of infected cells, the other is viral cytopathicity. In 2012, Li et al. [4] assumed that infected cells burst and then release viral particles (i.e., viral cytopathicity occurs) after uninfected cells were infected by a constant period of time $\tau$; that is, the time period of viral cytopathicity within target cells is $\tau$. They incorporated the delay of viral cytopathicity within target cells and built a new model:

$$
\begin{align*}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\beta x(t) v(t), \\
& \frac{d y(t)}{d t}=\beta x(t) v(t)-\beta e^{-d \tau} x(t-\tau) v(t-\tau)-d y(t),  \tag{2}\\
& \frac{d v(t)}{d t}=\kappa \beta e^{-d \tau} x(t-\tau) v(t-\tau)-\gamma v(t) .
\end{align*}
$$

By constructing Lyapunov functionals, necessary and sufficient conditions were obtained ensuring the global stability of the model.

In models (1) and (2), the researcher studied the viral dynamics with bilinear incidence rate $\beta x v$. As we know, as the viral particles diffuse in the body, the person often takes some actions when $v$ gets large. In order to describe the inhibitory effect from the uninfected cells when the number of viral cytopathicity is large enough, following the idea of [9], we propose an incidence rate $\beta x v /(1+\alpha v)$, where $\beta v$ measures the infection force of the viral, and $\alpha$ reflects the level of inhibitory action.

Similar to the discussions in [4], we assume that the viral cytopathicity has time delay. When the delay of viral cytopathicity within target cells is $\tau$ and the natural death rate of per target cell is $d$, the number of infected cells at time $t(t>\tau)$ can be represented by

$$
\begin{equation*}
y(t)=\int_{t-\tau}^{t} \frac{\beta x(\theta) v(\theta)}{1+\alpha v(\theta)} e^{-d(t-\theta)} d \theta, \quad \text { for } t>\tau \tag{3}
\end{equation*}
$$

where $e^{-d(t-\theta)}$ is the probability that target cells survive from time $\theta$ to time $t$, and $(\beta x(\theta) v(\theta) /(1+\alpha v(\theta))) e^{-d(t-\theta)}$ is the number of target cells being infected at time $\theta$ and still surviving at time $t$.

Differentiating $y(t)$ of (3), we get

$$
\begin{align*}
\frac{d y(t)}{d t}= & -d \int_{t-\tau}^{t} \frac{\beta x(\theta) v(\theta)}{1+\alpha v(\theta)} e^{-d(t-\theta)} d \theta+\frac{\beta x(t) v(t)}{1+\alpha v(t)} \\
& -e^{-d \tau} \frac{\beta x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}  \tag{4}\\
= & \frac{\beta x(t) v(t)}{1+\alpha v(t)}-\beta e^{-d \tau} \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-d y(t)
\end{align*}
$$

where $\beta e^{-d \tau}(x(t-\tau) v(t-\tau) /(1+\alpha v(t-\tau)))$ is the transfer rate of the infected cells being used to produce free viruses at time $t$; the recruitment rate of free virus at time $t$ is $\kappa \beta e^{-d \tau}(x(t-$ $\tau) v(t-\tau) /(1+\alpha v(t-\tau)))$, in which $\kappa$ is the average number of viral particles produced by an infected target cell when viral cytopathicity occurs, which implies that the recruitment of virus at time $t$ depends on the number of target cells that were newly infected at time $t-\tau$ and still alive at time $t$. Therefore following the model (2), we obtain a basic viral dynamical model of delay differential equations:

$$
\begin{aligned}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)} \\
& \frac{d y(t)}{d t}=\frac{\beta x(t) v(t)}{1+\alpha v(t)}-\beta e^{-d \tau} \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-d y(t) \\
& \frac{d v(t)}{d t}=\kappa \beta e^{-d \tau} \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t)
\end{aligned}
$$

Since the variable $y$ does not appear in the first and the third equations of (5), we only focus on the following equations:

$$
\begin{align*}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)} \\
& \frac{d v(t)}{d t}=\kappa \beta e^{-d \tau} \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t) \tag{6}
\end{align*}
$$

which has the same dynamics with system (5).

Let $b=\kappa e^{-d \tau}$, by (6), we have

$$
\begin{align*}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)} \\
& \frac{d v(t)}{d t}=\beta b \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t) \tag{7}
\end{align*}
$$

where all the parameters are assumed to be positive.
The rest of this paper is organized as follows. In the next section we will derive the infection-free equilibrium and the infection equilibrium. In Section 3, we carry out a qualitative analysis of the model, and stability conditions for the infection-free equilibrium and the infection equilibrium are derived, respectively. A brief conclusion will be given in Section 4.

## 2. Positive Solutions and Equilibria

Due to the biological meaning of the components $(x(t), v(t))$, we consider system (7) with the following initial conditions:

$$
\begin{align*}
\frac{d x(t)}{d t} & =\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)} \\
\frac{d v(t)}{d t} & =\beta b \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t)  \tag{8}\\
x(\theta) & =\varphi_{1}(\theta) \geq 0, \varphi_{1}(0)>0, \quad \theta \in[-\tau, 0] \\
v(\theta) & =\varphi_{2}(\theta) \geq 0, \varphi_{2}(0)>0, \quad \theta \in[-\tau, 0]
\end{align*}
$$

Equation (8) is a system of retarded differential equations in $C=C\left([-\tau, 0], \mathbb{R}^{2}\right) . C$ is a Banach space of continuous mappings from $[-\tau, 0]$ into $\mathbb{R}^{2}$ with norm $\|\psi\|=\sup _{-\tau \leq \theta \leq 0}|\psi(\theta)|$ for $\psi \in C$. We denote

$$
\begin{align*}
& C^{+}=\{ \left(\varphi_{1}, \varphi_{2}\right) \in C \mid \varphi_{1}(0)>0, \varphi_{2}(0)>0 \\
&\left.\varphi_{1}(\theta) \geq 0, \varphi_{2}(\theta) \geq 0, \theta \in[-\tau, 0]\right\} \tag{9}
\end{align*}
$$

As usual, for any continuous function $x \in C([-\tau,+\infty), \mathbb{R})$ and any given $t \geq 0, x_{t}$ is defined as $x_{t} \in C([-\tau, 0] \mathbb{R}), x_{t}(\theta)=$ $x(t+\theta)$, for any $\theta \in[-\tau, 0]$.

Theorem 1. All the solutions $(x(t), v(t))^{T}$ of (8) under the initial conditions are positive on $[0, \infty)$.

Proof. Assume that there is a $t_{1}\left(t_{1}>0\right)$ such that $x\left(t_{1}\right)=0$; then by $x(0)>0$ and the continuity of $x$, there is $t^{*}=\inf \{t>$ $0, x(t)=0\}>0$ such that $x(t)>0$ for $t \in\left[0, t^{*}\right)$. Then we have $x^{\prime}\left(t^{*}\right) \leq 0$. However, $x^{\prime}\left(t^{*}\right)=\lambda>0$ by the first equation of (8); this is a contradiction. Therefore $x(t)>0$ for all $t>0$.

From the second equation of (8)

$$
\begin{equation*}
\frac{d v(t)}{d t}=\beta b \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t) \tag{10}
\end{equation*}
$$

Multiplying $e^{\gamma t}$ in both sides of the above equation and integrating it from 0 to $t$, we have

$$
\begin{gather*}
\int_{0}^{t} e^{\gamma \theta} d v(\theta)+\int_{0}^{t} e^{\gamma \theta} \gamma v(\theta) d \theta \\
=\int_{0}^{t} e^{\gamma \theta} \beta b \frac{x(\theta-\tau) v(\theta-\tau)}{1+\alpha v(\theta-\tau)} d \theta \\
e^{\gamma t} v(t)-v(0)=\int_{0}^{t} e^{\gamma \theta} \beta b \frac{x(\theta-\tau) v(\theta-\tau)}{1+\alpha v(\theta-\tau)} d \theta  \tag{11}\\
v(t)=\left[v(0)+\int_{0}^{t} e^{\gamma \theta} \beta b \frac{x(\theta-\tau) v(\theta-\tau)}{1+\alpha v(\theta-\tau)} d \theta\right] e^{-\gamma t}
\end{gather*}
$$

Let $\eta=\theta-\tau$

$$
\begin{equation*}
v(t)=\left[v(0)+\int_{-\tau}^{t-\tau} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta\right] e^{-\gamma t} \tag{12}
\end{equation*}
$$

Since $x(t) \geq 0, v(t) \geq 0$, and $v(0)>0$ for $-\tau \leq t \leq 0$, then $v(t)>0$ for $0 \leq t<\tau$.

Further, when $\tau \leq t<2 \tau$, we have

$$
\begin{align*}
& v(t)=\left\{v(0)+\int_{-\tau}^{0} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta\right. \\
&\left.+\int_{0}^{t-\tau} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta\right\} e^{-\gamma t} \tag{13}
\end{align*}
$$

By the fact that $x(\eta) \geq 0$ and $v(\eta) \geq 0$ for $-\tau \leq \eta<0$, then

$$
\begin{equation*}
\int_{-\tau}^{0} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta \geq 0 \tag{14}
\end{equation*}
$$

Also $x(\eta)>0$ and $v(\eta)>0$ for $0 \leq \eta<\tau$; then

$$
\begin{equation*}
\int_{0}^{t-\tau} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta>0 \tag{15}
\end{equation*}
$$

Consequently, $v(t)>0$ for $\tau \leq t<2 \tau$, which implies that $v(t)>0$ holds true for $0 \leq t<2 \tau$.

We assume that for a positive integer $k, v(t)>0$ for $0 \leq$ $t<k \tau$. When $k \tau \leq t<(k+1) \tau$, we have

$$
\begin{align*}
v(t)=\{ & v(0)+\int_{-\tau}^{(k-1) \tau} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta \\
& \left.+\int_{(k-1) \tau}^{t-\tau} e^{\gamma(\eta+\tau)} \beta b \frac{x(\eta) v(\eta)}{1+\alpha v(\eta)} d \eta\right\} e^{-\gamma t} . \tag{16}
\end{align*}
$$

Then similar discussions show that $v(t)>0$ for $k \tau \leq t<$ $(k+1) \tau$. Hence, $v(t)>0$ for all $t>0$.

Theorem 2. All the solutions $(x(t), v(t))^{T}$ of (8) under the initial conditions are ultimately bounded.

Proof. For any solution $(x(t), v(t))^{T}$ of (8), define a function $f(t)=b x(t-\tau)+v(t)$. Then the derivative of $f(t)$ is

$$
\begin{align*}
\frac{d f(t)}{d t}= & b \frac{d x(t-\tau)}{d t}+\frac{d v(t)}{d t} \\
= & b \lambda-b d x(t-\tau)-\beta b \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)} \\
& +\beta b \frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}-\gamma v(t)  \tag{17}\\
= & b \lambda-b d x(t-\tau)-\gamma v(t) \\
\leq & b \lambda-\rho f(t),
\end{align*}
$$

where $\rho=\min \{d, \gamma\}$.
Integrating both sides of inequality above from 0 to $t$, we have

$$
\begin{equation*}
f(t) \leq \frac{b \lambda}{\rho}+\left(f(0)-\frac{b \lambda}{\rho}\right) e^{-\rho t} \tag{18}
\end{equation*}
$$

It means that $b x_{t}(-\tau)+v_{t}(0) \leq b \lambda / \rho$ for any $t \geq 0$ as long as $b \varphi_{1}(-\tau)+\varphi_{2}(0) \leq b \lambda / \rho$. Also,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} f(t) \leq \frac{b \lambda}{\rho} \tag{19}
\end{equation*}
$$

From the first equation of (8), we have

$$
\begin{equation*}
\frac{d x(t)}{d t} \leq \lambda-d x(t) \tag{20}
\end{equation*}
$$

Similar discussion shows that

$$
\begin{equation*}
x(t) \leq \frac{\lambda}{d}+\left(x(0)-\frac{\lambda}{d}\right) e^{-d t} . \tag{21}
\end{equation*}
$$

Then $x_{t}(0) \leq \lambda / d$ for any $t \geq 0$ as long as $\varphi_{1}(0) \leq \lambda / d$. Moreover,

$$
\begin{equation*}
\limsup _{t \rightarrow \infty} x(t) \leq \frac{\lambda}{d} \tag{22}
\end{equation*}
$$

Thus, the region

$$
\begin{equation*}
\Omega=\left\{\left(\varphi_{1}, \varphi_{2}\right) \in C^{+}: \varphi_{1}(0) \leq \frac{\lambda}{d}, b \varphi_{1}(-\tau)+\varphi_{2}(0) \leq \frac{b \lambda}{\rho}\right\} \tag{23}
\end{equation*}
$$

is an invariant set and an attractor of system (8) with initial condition $\left(\varphi_{1}, \varphi_{2}\right) \in C^{+}$.

In what follows, we study the existence of equilibria.
We consider algebraic equations

$$
\begin{align*}
& \lambda-d x-\frac{\beta x v}{1+\alpha v}=0  \tag{24}\\
& \beta b \frac{x v}{1+\alpha v}-\gamma v=0
\end{align*}
$$

It is easy to see that system (24) always has an infectionfree equilibrium $E_{1}(\lambda / d, 0)$. To find the other equilibrium, we assume $v \neq 0$. By the second equation of (24), we have

$$
\begin{equation*}
\frac{\beta b x}{1+\alpha v}=\gamma \tag{25}
\end{equation*}
$$

Then

$$
\begin{equation*}
x=\frac{\gamma(1+\alpha v)}{\beta b} \tag{26}
\end{equation*}
$$

We put $x$ into the first equation of (24); then

$$
\begin{gather*}
\lambda-\frac{d \gamma(1+\alpha v)}{\beta b}-\frac{\beta \gamma(1+\alpha v) v}{\beta b(1+\alpha v)}=0  \tag{27}\\
\beta b \lambda-d \gamma=(d \gamma \alpha+\beta \gamma) v .
\end{gather*}
$$

Thus the positive root $E_{2}$ exists if and only if $\beta b \lambda-d \gamma>0$. For system (8), define the basic reproduction number [10] as follows

$$
\begin{equation*}
R_{0}=\frac{\beta b \lambda}{d \gamma} \tag{28}
\end{equation*}
$$

It is easy to see that
(i) If $R_{0} \leq 1$, then system (8) has a unique equilibrium $E_{1}(\lambda / d, 0)$, which corresponds to the case that viruses die out, and it is called infection-free equilibrium.
(ii) If $R_{0}>1$, then system (8) has two equilibria, one is the infection-free equilibrium $E_{1}(\lambda / d, 0)$ and the other is a positive equilibrium $E_{2}\left((\gamma+\alpha b \lambda) / b(d \alpha+\beta), d\left(R_{0}-\right.\right.$ 1) $/(d \alpha+\beta))$.

## 3. Stability of the Equilibrium

In this section, we consider the stability of the equilibrium. There are two cases, $\tau=0$ and $\tau>0$.
3.1. Local Stability of Equilibria. First we consider the case of $\tau=0$. In this case system (8) is reduced to a system of ordinary differential equations. In order to examine local stability of an equilibrium, we should compute the eigenvalues of the linearized operator for system (8) at the equilibrium.

By a direct computation, the Jacobian matrix is as follows:

$$
\left(\begin{array}{cc}
-d-\frac{\beta v}{1+\alpha v}, & -\frac{\beta x}{(1+\alpha v)^{2}}  \tag{29}\\
\beta b \frac{v}{1+\alpha v}, & \beta b \frac{x}{(1+\alpha v)^{2}}-\gamma
\end{array}\right)
$$

Consider infection-free equilibrium $E_{1}(\lambda / d, 0)$. The characteristic equation is obtained by the standard method as follows.

It is obvious that $\mu_{1}=-d<0$, and $\mu_{2}=\beta b \lambda / d-\gamma=$ $\gamma\left(R_{0}-1\right)$ are the characteristic roots of the characteristic equation. Therefore, we have the following theorem.

Theorem 3. (i) If $R_{0}<1$, then infection-free equilibrium $E_{1}(\lambda / d, 0)$ is locally asymptotically stable.
(ii) If $R_{0}>1$, then infection-free equilibrium $E_{1}(\lambda / d, 0)$ is unstable.
(iii) If $R_{0}=1$, then infection-free equilibrium $E_{1}(\lambda / d, 0)$ is degenerated.

Now, local stability of the infection equilibrium $E_{2}((\gamma+$ $\left.\alpha b \lambda) / b(d \alpha+\beta), d\left(R_{0}-1\right) /(d \alpha+\beta)\right)$ is considered. As we know, infection equilibrium $E_{2}\left((\gamma+\alpha b \lambda) / b(d \alpha+\beta), d\left(R_{0}-1\right) /(d \alpha+\right.$ $\beta$ ) exists if and only if $R_{0}>1$.

Theorem 4. If $R_{0}>1$, then the infection equilibrium $E_{2}$ is locally asymptotically stable.

Proof. Set $f(x, v)=\beta x v /(1+\alpha v)$. Then system (8) at the equilibrium $E_{2}\left(x^{*}, v^{*}\right)$ has Jacobian matrix

$$
A=\left(\begin{array}{cc}
-d-\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right), & -\frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)  \tag{30}\\
b \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right), & b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)-\gamma
\end{array}\right)
$$

A direct computation shows that the characteristic equation is

$$
\begin{align*}
h(\mu)= & \mu^{2}+\left[d+\gamma+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)-b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right] \mu \\
& +d \gamma-b d \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)+\gamma \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)=0 . \tag{31}
\end{align*}
$$

By Hurwitz criterion, all of the eigenvalues of characteristic equation have negative real parts if and only if

$$
\begin{align*}
& d+\gamma+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)-b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)>0 \\
& d \gamma-b d \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)+\gamma \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)>0 \tag{32}
\end{align*}
$$

Indeed,

$$
\begin{align*}
& d+\gamma+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)-b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right) \\
& =\frac{(d+\gamma)\left(1+\alpha v^{*}\right)^{2}+\beta v^{*}\left(1+\alpha v^{*}\right)-b \beta x^{*}}{\left(1+\alpha v^{*}\right)^{2}} \\
& =\frac{\left(\alpha^{2} d+\alpha^{2} \gamma+\alpha \beta\right) v^{* 2}+(2 \alpha d+2 \gamma \alpha+\beta) v^{*}-b \beta x^{*}+d+\gamma}{\left(1+\alpha v^{*}\right)^{2}} \\
& =\frac{\left(\alpha^{2} d+\alpha^{2} \gamma+\alpha \beta\right) v^{* 2}+(2 \alpha d+\gamma \alpha+\beta) v^{*}+d}{\left(1+\alpha v^{*}\right)^{2}}>0, \\
& d \gamma-b d \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)+\gamma \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right) \\
& =\frac{d \gamma\left(1+\alpha v^{*}\right)^{2}-b d \beta x^{*}+\gamma \beta v^{*}+\gamma \beta \alpha v^{* 2}}{\left(1+\alpha v^{*}\right)^{2}} \\
& =\frac{\left(\alpha^{2} d \gamma+\alpha \gamma \beta\right) v^{* 2}+(d \alpha \gamma+\gamma \beta) v^{*}}{\left(1+\alpha v^{*}\right)^{2}}>0 . \tag{33}
\end{align*}
$$

This implies that all the eigenvalues of characteristic equation have negative real parts. Then the infection equilibrium $E_{2}$
is locally asymptotically stable. This completes the proof of theorem.

Now we consider the case $\tau>0$. By linearizing system (8) at the infection-free equilibrium $E_{1}(\lambda / d, 0)$, we obtain the characteristic equation as follows:

$$
\begin{equation*}
(\mu+d)\left(\mu+\gamma-e^{-\mu \tau} \beta b \frac{\lambda}{d}\right)=0 \tag{34}
\end{equation*}
$$

It is easy to see that $\mu_{1}=-d<0$; hence we only need to discuss the roots of the following equation:

$$
\begin{equation*}
h(\mu, \tau)=\mu+\gamma-e^{-\mu \tau} \beta b \frac{\lambda}{d}=0 \tag{35}
\end{equation*}
$$

Theorem 5. When $\tau>0$, then
(i) If $R_{0}<1$, then the infection-free equilibrium $E_{1}(\lambda / d, 0)$ is locally asymptotically stable.
(ii) If $R_{0}>1$, then the infection-free equilibrium $E_{1}(\lambda / d, 0)$ is unstable.
(iii) If $R_{0}=1$, then the infection-free equilibrium $E_{1}(\lambda / d, 0)$ is degenerated.

Proof. (i) By implicit function theorem for complex variables, we know that the roots of (35) are continuous on the parameter $\tau$.

If $R_{0}<1$, then 0 is not a root of (35) for all $\tau>0$. Note that all complex roots of (35) must come in conjugate pairs and the root of (35) is negative for $\tau=0$. Thus, all roots of (35) have negative real parts for small $\tau$; that is, $0<\tau=1$. Suppose that there exists a positive number $\tau=\tau_{0}$ such that (35) has a pair of purely imaginary roots $\lambda= \pm \omega i$; here $\omega$ is a positive number. We have

$$
\begin{equation*}
\omega i+\gamma-e^{-\omega i \tau_{0}} \beta b \frac{\lambda}{d}=0 \tag{36}
\end{equation*}
$$

Then

$$
\begin{align*}
& \beta b \frac{\lambda}{d} \cos \omega \tau_{0}=\gamma  \tag{37}\\
& \beta b \frac{\lambda}{d} \sin \omega \tau_{0}=-\omega
\end{align*}
$$

Summing up the square of both equations in (37) we obtain

$$
\begin{align*}
\omega^{2} & =\beta^{2} b^{2} \frac{\lambda^{2}}{d^{2}}-\gamma^{2}=\frac{\beta^{2} b^{2} \lambda^{2}-\gamma^{2} d^{2}}{d^{2}}=\frac{R_{0}^{2} d^{2} \gamma^{2}-\gamma^{2} d^{2}}{d^{2}}  \tag{38}\\
& =\gamma^{2}\left(R_{0}^{2}-1\right)
\end{align*}
$$

When $R_{0}<1$, then $\omega^{2}<0$. It is a contradiction with $\omega^{2}>$ 0 which leads to the nonexistence of $\tau_{0}$. This contradiction proves the result.
(ii) When $\mu=0$, and $R_{0}>1$, then

$$
\begin{align*}
h(0, \tau)= & \gamma-\beta b \frac{\lambda}{d}=\gamma\left(1-R_{0}\right)<0  \tag{39}\\
& \lim _{\mu \rightarrow \infty} h(\mu, \tau)=+\infty
\end{align*}
$$

Therefore equation must have a positive real root for all $\tau>0$.
(iii) If $R_{0}=1$, it is easy to know that $\mu=0$ is a root of (35) for all $\tau>0$, which leads to conclusion. (iii) This completes the proof of theorem.

Now we consider the local stability of the infection equilibrium $E_{2}\left((\gamma+\alpha b \lambda) / b(d \alpha+\beta), d\left(R_{0}-1\right) /(d \alpha+\beta)\right)$. As we know, the infection equilibrium $E_{2}\left((\gamma+\alpha b \lambda) / b(d \alpha+\beta), d\left(R_{0}-\right.\right.$ $1) /(d \alpha+\beta))$ exists if and only if $R_{0}>1$. By computation, the associated transcendental characteristic equation of (8) at $E_{2}$ becomes

$$
\begin{equation*}
\mu^{2}+A \mu+B-(C \mu+D) e^{-\mu \tau}=0 \tag{40}
\end{equation*}
$$

where

$$
\begin{align*}
A=d+\gamma+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right), & B & =\left(d+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)\right) \gamma, \\
C=b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right), & D & =d b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right) . \tag{41}
\end{align*}
$$

Theorem 6. When $\tau>0$, if $R_{0}>1$, then the infection equilibrium $E_{2}$ is locally asymptotically stable.

Proof. By implicit function theorem for complex variables, we know that the root of (40) is continuous on the parameter $\tau$. If $R_{0}>1$, then all roots of (40) have negative real parts as $\tau=0$ and (40) has no zero root for all $\tau>0$. Thus, all roots of (40) have negative real parts for very small $\tau$; that is, $0<\tau \ll 1$. Assume that there exists a positive $\tau_{0}$ such that (40) has a pair of purely imaginary roots $\pm \omega i, \omega>0$. Then $\omega>0$ must satisfy

$$
\begin{equation*}
-\omega^{2}+A \omega i+B-(C \omega i+D)\left(\cos \omega \tau_{0}-i \sin \omega \tau_{0}\right)=0 \tag{42}
\end{equation*}
$$

Separating the real and imaginary parts, we have

$$
\begin{gather*}
C \omega \sin \omega \tau_{0}+D \cos \omega \tau_{0}=B-\omega^{2}  \tag{43}\\
D \sin \omega \tau_{0}-C \omega \cos \omega \tau_{0}=-A \omega
\end{gather*}
$$

which implies that

$$
\begin{equation*}
\omega^{4}+\left(A^{2}-2 B-C^{2}\right) \omega^{2}+B^{2}-D^{2}=0 \tag{44}
\end{equation*}
$$

Direct computation shows that

$$
\begin{aligned}
B^{2}-D^{2}= & {\left[\gamma\left(d+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)\right)\right]^{2}-\left[d b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right]^{2} } \\
= & {\left[\gamma \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)+\gamma d-d b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right] } \\
& \times\left[\gamma \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)+\gamma d+d b \frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right]>0,
\end{aligned}
$$

$$
\begin{align*}
A^{2}-2 B-C^{2}= & d^{2}+\gamma^{2}+\left(\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)\right)^{2}+2 d \frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right) \\
& -b^{2}\left(\frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right)^{2} \\
= & \left(d+\frac{\partial f}{\partial x}\left(x^{*}, v^{*}\right)\right)^{2} \\
& -b^{2}\left(\frac{\partial f}{\partial v}\left(x^{*}, v^{*}\right)\right)^{2}+\gamma^{2} \\
= & \left(\left(d^{2}+\gamma^{2}\right)\left(1+\alpha v^{*}\right)^{4}+\beta^{2}\left(v^{*}\right)^{2}\left(1+\alpha v^{*}\right)^{2}\right. \\
& \left.+2 d \beta v^{*}\left(1+\alpha v^{*}\right)^{3}-b^{2} \beta^{2}\left(x^{*}\right)^{2}\right) \\
& \times\left(\left(1+\alpha v^{*}\right)^{4}\right)^{-1} . \tag{45}
\end{align*}
$$

Let

$$
\begin{align*}
H\left(x^{*}, v^{*}\right)= & \left(d^{2}+\gamma^{2}\right)\left(1+\alpha v^{*}\right)^{4}+\beta^{2}\left(v^{*}\right)^{2}\left(1+\alpha v^{*}\right)^{2} \\
& +2 d \beta v^{*}\left(1+\alpha v^{*}\right)^{3}-b^{2} \beta^{2}\left(x^{*}\right)^{2} . \tag{46}
\end{align*}
$$

Note that $1+\alpha v^{*}=(\beta b / \gamma) x^{*}$,

$$
\begin{align*}
H & \left(x^{*}, v^{*}\right) \\
= & \left(d^{2}+\gamma^{2}\right) \frac{\beta^{4} b^{4}}{\gamma^{4}}\left(x^{*}\right)^{4}+\beta^{2}\left(v^{*}\right)^{2} \frac{\beta^{2} b^{2}}{\gamma^{2}}\left(x^{*}\right)^{2} \\
& +2 d \beta v^{*} \frac{\beta^{3} b^{3}}{\gamma^{3}}\left(x^{*}\right)^{3}-\beta^{2} b^{2}\left(x^{*}\right)^{2} \\
= & \left(x^{*}\right)^{2}\left[\frac{\beta^{4} b^{2}}{\gamma^{4}}\left(b^{2} d^{2}\left(x^{*}\right)^{2}+\gamma^{2}\left(v^{*}\right)^{2}+2 \gamma d b v^{*} x^{*}\right)\right. \\
& \left.\quad+\frac{\beta^{4} b^{4}}{\gamma^{2}}\left(x^{*}\right)^{2}-b^{2} \beta^{2}\right] \\
= & \left(x^{*}\right)^{2}\left[\frac{\beta^{4} b^{2}}{\gamma^{4}}\left(b d x^{*}+\gamma v^{*}\right)^{2}+\frac{\beta^{2} b^{2}}{\gamma^{2}}\left(\beta^{2} b^{2}\left(x^{*}\right)^{2}-\gamma^{2}\right)\right] \\
= & \left(x^{*}\right)^{2}\left[\frac{\beta^{4} b^{2}}{\gamma^{4}}\left(b d x^{*}+\gamma v^{*}\right)^{2}+\beta^{2} b^{2}\left(\alpha^{2}\left(v^{*}\right)^{2}+2 \alpha v^{*}\right)\right] \\
> & 0 . \tag{47}
\end{align*}
$$

By Hurwitz criterion, (44) has no positive roots, which implies the nonexistence of $\tau_{0}$. Thus all roots of (40) have negative real parts for $\tau>0$.
3.2. Global Stability of Equilibria. In the section, we study the global stability of equilibria; we first consider the infectionfree equilibrium $E_{1}$.

Theorem 7. When $\tau=0$,
(i) If $R_{0} \leq 1$, then infection-free equilibrium $E_{1}(\lambda / d, 0)$ is globally asymptotically stable.
(ii) If $R_{0}>1$, then infection equilibrium $E_{2}\left(x^{*}, v^{*}\right)$ is globally asymptotically stable.

Proof. (i) Define a Lyapunov function as what follows

$$
\begin{gather*}
V(x, v)=x-\frac{\lambda}{d} \ln x+\frac{v}{b}+\frac{\lambda}{d} \ln \frac{\lambda}{d}-\frac{\lambda}{d} \\
\left.\frac{d V(x, v)}{d t}\right|_{(8)} \\
=\frac{x-\lambda / d}{x}\left(\lambda-d x-\frac{\beta x v}{1+\alpha v}\right)+\frac{1}{b}\left(\frac{\beta b x v}{1+\alpha v}-\gamma v\right) \\
=-\frac{(d x-\lambda)^{2}}{d x}+\frac{\beta x v}{1+\alpha v}\left(1-\frac{d x-\lambda}{d x}\right)-\frac{\gamma}{b} v \\
=-\frac{(d x-\lambda)^{2}}{d x}+\left(\frac{\beta \lambda}{d(1+\alpha v)}-\frac{\gamma}{b}\right) v \\
=-\frac{(d x-\lambda)^{2}}{d x}+\frac{\gamma}{b}\left(R_{0}-1\right) v-\frac{\beta \gamma}{d} \frac{\alpha v^{2}}{1+\alpha v} . \tag{48}
\end{gather*}
$$

It means that $d V(x, v) /\left.d t\right|_{(8)}$ is negative semidefinite as $R_{0} \leq$ 1. Moreover, the last equality of the above equation shows that the largest invariant set of system (8) on the region $\left\{(x, v)^{T} \in \mathbb{R}_{+}^{2}: d V / d t=0\right\}$ is the singleton $\left\{E_{1}\right\}$. Therefore, the infection-free equilibrium $E_{1}$ is global asymptotically stability.
(ii) We rewrite the system (8)

$$
\begin{align*}
& \frac{d x(t)}{d t}=\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)}=Q(x, v) \\
& \frac{d v(t)}{d t}=\beta b \frac{x(t) v(t)}{1+\alpha v(t)}-\gamma v(t)=P(x, v) \tag{49}
\end{align*}
$$

Choose a Dulac function

$$
\begin{equation*}
D(x, v)=\frac{1+\alpha v}{\beta v} \tag{50}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{\partial(D P)}{\partial v}+\frac{\partial(D Q)}{\partial x}=-\frac{\gamma \alpha}{\beta}-d \frac{1+\alpha v}{\beta v}-1<0 \tag{51}
\end{equation*}
$$

Thus system (49) does not have nontrivial periodic orbits in $\Omega$. The conclusion follows.

Theorem 8. When $\tau>0$, if $R_{0} \leq 1$, then infection-free equilibrium $E_{1}(\lambda / d, 0)$ is globally asymptotically stable.

Proof. Define a functional $g: C^{+} \rightarrow \mathbb{R}$ as follows:

$$
\begin{align*}
g\left(\varphi_{1}, \varphi_{2}\right)= & \frac{1}{2}\left[\varphi_{1}(0)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda}{d}\left[\varphi_{2}(0)+\beta b \int_{-\tau}^{0} \frac{\varphi_{1}(\theta) \varphi_{2}(\theta)}{1+\alpha \varphi_{2}(\theta)} d \theta\right] \tag{52}
\end{align*}
$$

For any $t \geq 0, x_{t}, v_{t} \in C$, then

$$
\begin{align*}
& g\left(x_{t}, v_{t}\right)=\frac{b}{2}\left[x(t)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda}{d}\left[v(t)+\beta b \int_{t-\tau}^{t} \frac{x(\theta) v(\theta)}{1+\alpha v(\theta)} d \theta\right], \\
& \frac{d g\left(x_{t}, v_{t}\right)}{d t}=b\left(x(t)-\frac{\lambda}{d}\right) \frac{d x(t)}{d t}+\frac{\lambda}{d} \frac{d v(t)}{d t} \\
& +\frac{\beta b \lambda}{d}\left[\frac{x(t) v(t)}{1+\alpha v(t)}-\frac{x(t-\tau) v(t-\tau)}{1+\alpha v(t-\tau)}\right] \\
& =b\left[x(t)-\frac{\lambda}{d}\right]\left[\lambda-d x(t)-\frac{\beta x(t) v(t)}{1+\alpha v(t)}\right] \\
& +\frac{\lambda}{d}\left[\beta b \frac{x(t) v(t)}{1+\alpha v(t)}-\gamma v(t)\right] \\
& =-b d\left[x(t)-\frac{\lambda}{d}\right]^{2}-\beta b \frac{x(t) v(t)}{1+\alpha v(t)}\left[x(t)-\frac{\lambda}{d}\right] \\
& +\frac{\lambda}{d}\left[\beta b \frac{x(t) v(t)}{1+\alpha v(t)}-\gamma v(t)\right] \\
& =-b\left(d+\frac{\beta v(t)}{1+\alpha v(t)}\right)\left[x(t)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda v(t)}{d}\left(\frac{\beta b \lambda}{d(1+\alpha v(t))}-\gamma\right) \\
& =-b\left(d+\frac{\beta v(t)}{1+\alpha v(t)}\right)\left[x(t)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda v(t) \gamma}{d}\left(\frac{R_{0}}{1+\alpha v(t)}-1\right) \\
& =-b\left(d+\frac{\beta v(t)}{1+\alpha v(t)}\right)\left[x(t)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda v(t) \gamma}{d}\left(R_{0}-1\right)-\frac{\alpha v(t) R_{0}}{1+\alpha v(t)} \\
& \leq-b\left(d+\frac{\beta v(t)}{1+\alpha v(t)}\right)\left[x(t)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda v(t) \gamma}{d}\left(R_{0}-1\right) . \tag{53}
\end{align*}
$$

Then,

$$
\begin{align*}
\left.\frac{d g\left(\varphi_{1}, \varphi_{2}\right)}{d t}\right|_{(8)}= & -b\left(d+\frac{\beta \varphi_{2}(0)}{1+\alpha \varphi_{2}(0)}\right)\left[\varphi_{1}(0)-\frac{\lambda}{d}\right]^{2} \\
& +\frac{\lambda \varphi_{2}(0) \gamma}{d}\left(R_{0}-1\right)-\frac{\alpha \varphi_{2}(0) R_{0}}{1+\alpha \varphi_{2}(0)} \tag{54}
\end{align*}
$$

When $R_{0} \leq 1$, we have $d g\left(x_{t}, v_{t}\right) / d t \leq 0$. It is easy to know, when $R_{0} \leq 1$ the largest invariant set of system (8) on the region $\left\{\left(\varphi_{1}, \varphi_{2}\right) \in C^{+}: d g\left(\varphi_{1}, \varphi_{2}\right) / d t=0\right\}$ is the singleton $\left\{E_{1}\right\}$. By Lassalle invariant principle for autonomous retarded differential equations [11], infection-free equilibrium $E_{1}$ is globally asymptotically stable. This completes the proof.

Theorem 9. When $\tau>0$, if $R_{0}>1$, then the infection equilibrium $E_{2}\left(x^{*}, v^{*}\right)$ is globally asymptotically stable.

Proof. Let $V(t)=v(t+\tau)$ and $X(t)=x(t)$; system (8) becomes

$$
\begin{align*}
& \frac{d X(t)}{d t}=\lambda-d X(t)-\frac{\beta X(t) V(t-\tau)}{1+\alpha V(t-\tau)}  \tag{55}\\
& \frac{d V(t)}{d t}=\beta b \frac{X(t) V(t-\tau)}{1+\alpha V(t-\tau)}-\gamma V(t)
\end{align*}
$$

Denote $u(x)=x /(1+\alpha x)$. Evaluating both sides of (55) at $E_{2}$, we obtain

$$
\begin{equation*}
\lambda=d x^{*}+\beta x^{*} u\left(v^{*}\right), \quad \gamma v^{*}=\beta b x^{*} u\left(v^{*}\right) . \tag{56}
\end{equation*}
$$

Define a Lyapunov functional $L: C^{+} \rightarrow \mathbb{R}$ as follows:

$$
\begin{align*}
L\left(\varphi_{1}, \varphi_{2}\right)= & \frac{1}{\beta u\left(v^{*}\right)} L_{1}\left(\varphi_{1}, \varphi_{2}\right)+\frac{v^{*}}{\beta b x^{*} u\left(v^{*}\right)} L_{2}\left(\varphi_{1}, \varphi_{2}\right) \\
& +L_{3}\left(\varphi_{1}, \varphi_{2}\right) \tag{57}
\end{align*}
$$

where

$$
\begin{align*}
& L_{1}\left(\varphi_{1}, \varphi_{2}\right)=\frac{\varphi_{1}(0)}{x^{*}}-1-\ln \frac{\varphi_{1}(0)}{x^{*}}, \\
& L_{2}\left(\varphi_{1}, \varphi_{2}\right)=\frac{\varphi_{2}(0)}{v^{*}}-1-\ln \frac{\varphi_{2}(0)}{v^{*}},  \tag{58}\\
& L_{3}\left(\varphi_{1}, \varphi_{2}\right)=\int_{-\tau}^{0}\left(\frac{\varphi_{2}(\theta)}{v^{*}}-1-\ln \frac{\varphi_{2}(\theta)}{v^{*}}\right) d \theta .
\end{align*}
$$

Thus, $L\left(\varphi_{1}, \varphi_{2}\right) \geq 0$ with equality if and only if $\varphi_{1}(0) / x^{*}=$ $\varphi_{2}(0) / v^{*}=1$.

For any $t \geq 0, X_{t}, V_{t} \in C$, then

$$
\begin{gather*}
L_{1}\left(X_{t}, V_{t}\right)=\frac{X(t)}{x^{*}}-1-\ln \frac{X(t)}{x^{*}}, \\
L_{2}\left(X_{t}, V_{t}\right)=\frac{V(t)}{v^{*}}-1-\ln \frac{V(t)}{v^{*}},  \tag{59}\\
L_{3}\left(X_{t}, V_{t}\right)=\int_{-\tau}^{0}\left(\frac{V(t+\theta)}{v^{*}}-1-\ln \frac{V(t+\theta)}{v^{*}}\right) d \theta .
\end{gather*}
$$

We calculate derivatives of $L_{1}\left(X_{t}, V_{t}\right), L_{2}\left(X_{t}, V_{t}\right)$, and $L_{3}\left(X_{t}, V_{t}\right)$ with respect to the system (55):

$$
\begin{align*}
& \frac{d L_{1}\left(X_{t}, V_{t}\right)}{d t} \\
& =\frac{1}{x^{*}}\left(1-\frac{x^{*}}{X}\right)(\lambda-d X-\beta X u(V(t-\tau))) \\
& =\frac{1}{x^{*}}\left(1-\frac{x^{*}}{X}\right)\left(d x^{*}+\beta x^{*} u\left(v^{*}\right)-d X-\beta X u(V(t-\tau))\right) \\
& =-d \frac{\left(X-x^{*}\right)^{2}}{X x^{*}}+\beta u\left(v^{*}\right)\left(1-\frac{x^{*}}{X}\right)\left(1-\frac{X u(V(t-\tau))}{x^{*} u\left(v^{*}\right)}\right) \\
& =-d \frac{\left(X-x^{*}\right)^{2}}{X x^{*}} \\
& +\beta u\left(v^{*}\right)\left(1-\frac{x^{*}}{X}+\frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{X u(V(t-\tau))}{x^{*} u\left(v^{*}\right)}\right), \\
& \frac{d L_{2}\left(X_{t}, V_{t}\right)}{d t} \\
& =\frac{1}{v^{*}}\left(1-\frac{v^{*}}{V}\right)(\beta b X u(V(t-\tau))-\gamma V) \\
& =\frac{1}{v^{*}}\left(1-\frac{v^{*}}{V}\right)\left(\beta b x^{*} u\left(v^{*}\right) \frac{X u(V(t-\tau))}{x^{*} u\left(v^{*}\right)}-\gamma \frac{V v^{*}}{v^{*}}\right) \\
& =\beta b \frac{x^{*}}{v^{*}} u\left(v^{*}\right)\left(1-\frac{v^{*}}{V}\right)\left(\frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{V}{v^{*}}\right) \\
& =\beta b \frac{x^{*}}{v^{*}} u\left(v^{*}\right) \\
& \times\left(1-\frac{v^{*}}{V} \frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}+\frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{V}{v^{*}}\right), \\
& \frac{d L_{3}\left(X_{t}, V_{t}\right)}{d t} \\
& =\int_{-\tau}^{0} \frac{d}{d t}\left(\frac{V(t+\theta)}{v^{*}}-1-\ln \frac{V(t+\theta)}{v^{*}}\right) d \theta \\
& =\frac{V(t)}{v^{*}}-\ln \frac{V(t)}{v^{*}}-\frac{V(t-\tau)}{v^{*}}+\ln \frac{V(t-\tau)}{v^{*}} \text {. } \tag{60}
\end{align*}
$$

## We obtain

$$
\begin{aligned}
& \frac{d L\left(X_{t}, V_{t}\right)}{d t} \\
&=-\frac{d}{\beta u\left(v^{*}\right)} \frac{\left(X-x^{*}\right)^{2}}{X x^{*}}+1-\frac{x^{*}}{X} \\
&+\frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{X u(V(t-\tau))}{x^{*} u\left(v^{*}\right)}+1-\frac{v^{*}}{V} \frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)} \\
&+\frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{V}{v^{*}}+\frac{V}{v^{*}}-\ln \frac{V}{v^{*}}-\frac{V(t-\tau)}{v^{*}}
\end{aligned}
$$

$$
\begin{align*}
& +\ln \frac{V(t-\tau)}{v^{*}} \\
= & -\frac{d}{\beta u\left(v^{*}\right)} \frac{\left(X-x^{*}\right)^{2}}{X x^{*}}+2-\frac{x^{*}}{X}+\frac{u(V(t-\tau))}{u\left(v^{*}\right)} \\
& -\frac{v^{*}}{V} \frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)} \\
& -\ln \frac{V}{v^{*}}-\frac{V(t-\tau)}{v^{*}}+\ln \frac{V(t-\tau)}{v^{*}} . \tag{61}
\end{align*}
$$

By adding and subtracting the quantity $\ln \left(\left(X / x^{*}\right)(u(V(t-\right.$ $\left.\tau)) / u\left(v^{*}\right)\right)$ ), we have

$$
\begin{align*}
& \frac{d L\left(X_{t}, V_{t}\right)}{d t} \\
&=-\frac{d}{\beta u\left(v^{*}\right)} \frac{\left(X-x^{*}\right)^{2}}{X x^{*}}+2-\frac{x^{*}}{X} \\
&+\frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{v^{*}}{V} \frac{X}{x^{*}} \frac{u(V(t-\tau))}{u\left(v^{*}\right)} \\
&-\frac{V(t-\tau)}{v^{*}}+\ln \frac{V(t-\tau)}{v^{*}}+\ln \left(\frac{X}{x^{*}} \frac{v^{*}}{V} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}\right) \\
&-\ln \left(\frac{X}{x^{*}}\right)-\ln \left(\frac{u(V(t-\tau))}{u\left(v^{*}\right)}\right) \\
&=-\frac{d}{\beta u\left(v^{*}\right)} \frac{\left(X-x^{*}\right)^{2}}{X x^{*}}-\left(\frac{x^{*}}{X}-1+\ln \frac{X}{x^{*}}\right) \\
&-\left(\frac{X}{x^{*}} \frac{v^{*}}{V} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}-1\right. \\
&\left.-\ln \left(\frac{X}{x^{*}} \frac{v^{*}}{V} \frac{u(V(t-\tau))}{u\left(v^{*}\right)}\right)\right) \\
&+\frac{u(V(t-\tau))}{u\left(v^{*}\right)}-\frac{V(t-\tau)}{v^{*}} \\
&+\ln \frac{V(t-\tau)}{v^{*}}-\ln \left(\frac{u(V(t-\tau))}{u\left(v^{*}\right)}\right) . \tag{62}
\end{align*}
$$

Then,

$$
\begin{aligned}
\frac{d L\left(\varphi_{1}, \varphi_{2}\right)}{d t}= & -\frac{d}{\beta u\left(v^{*}\right)} \frac{\left(\varphi_{1}(0)-x^{*}\right)^{2}}{\varphi_{1}(0) x^{*}} \\
& -\left(\frac{x^{*}}{\varphi_{1}(0)}-1+\ln \frac{\varphi_{1}(0)}{x^{*}}\right) \\
& -\left(\frac{\varphi_{1}(0)}{x^{*}} \frac{v^{*}}{\varphi_{2}(0)} \frac{u\left(\varphi_{2}(-\tau)\right)}{u\left(v^{*}\right)}-1\right. \\
& \left.-\ln \left(\frac{\varphi_{1}(0)}{x^{*}} \frac{v^{*}}{\varphi_{2}(0)} \frac{u\left(\varphi_{2}(-\tau)\right)}{u\left(v^{*}\right)}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{u\left(\varphi_{2}(-\tau)\right)}{u\left(v^{*}\right)}-\frac{\varphi_{2}(-\tau)}{v^{*}} \\
& +\ln \frac{\varphi_{2}(-\tau)}{v^{*}}-\ln \left(\frac{u\left(\varphi_{2}(-\tau)\right)}{u\left(v^{*}\right)}\right) \tag{63}
\end{align*}
$$

It is easy to know that $u\left(\varphi_{2}(-\tau)\right) / u\left(v^{*}\right)-\varphi_{2}(-\tau) / v^{*}+$ $\ln \left(\varphi_{2}(-\tau) / v^{*}\right)-\ln \left(u\left(\varphi_{2}(-\tau)\right) / u\left(v^{*}\right)\right) \leq 0$, and $u\left(\varphi_{2}(-\tau)\right) /$ $u\left(v^{*}\right)-\varphi_{2}(-\tau) / v^{*}+\ln \left(\varphi_{2}(-\tau) / v^{*}\right)-\ln \left(u\left(\varphi_{2}(-\tau)\right) / u\left(v^{*}\right)\right)=0$ if and only if $\varphi_{2}(-\tau) / v^{*}=1$. It follows that $d L\left(\varphi_{1}, \varphi_{2}\right) / d t \leq 0$, and $d L\left(\varphi_{1}, \varphi_{2}\right) / d t=0$ if and only if $\varphi_{1}(0) / x^{*}=\varphi_{2}(0) / v^{*}=$ $\varphi_{2}(-\tau) / v^{*}=1$. By classical stability theory for functional differential equations, $E_{2}$ is globally asymptotically stable. This completes the proof.

## 4. Conclusion

The viral infection model addressed in this paper has saturated incidence rate and viral infection with delay. The basic reproductive number $R_{0}$ is given. When $R_{0}<1$, for the model with or without delay time, the infection-free equilibrium is globally asymptotically stable, which implies that the viral infection goes extinct eventually; when $R_{0}>1$, the infection equilibrium is globally asymptotically stable, which implies that the viral infection persists in the body of the host.

## Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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