# Recent Advance in Function Spaces and their Applications in Fractional Differential Equations 

Lead Guest Editor: Xinguang Zhang
Guest Editors: Lishan Liu, Yong H. Wu, and Liguang Wang

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## Editorial

# Recent Advance in Function Spaces and Their Applications in Fractional Differential Equations 

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Fractional calculus is a new branch of analytical mathematics which provides useful tools to model many physical and biological phenomena and optimal control of complex processes with memory effects. Therefore, new advancement of fractional calculus theory will greatly promote the development of function space theory, functional theory, and mathematical physics as well as their applications in differential and integral equations.

This special issue mainly focuses on the latest achievements and recent development of fractional calculus in the nonlinear analysis, optimal control, computational methods, space theory, and applications for solving various fractional differential equations; it contains 46 papers selected through a rigorous peer-reviewed process. These papers almost cover most of directions and applications in fractional calculus.

In what follows, we briefly review the highlights and main contributions of each paper.

In the paper titled "Existence Results for a Class of Semilinear Fractional Partial Differential Equations with Delay in Banach Spaces", the authors consider the existence and uniqueness of the mild solutions for a class of nonlinear time fractional partial differential equations with delay by using the theory of solution operator and the general Banach contraction mapping principle. What is significant about this paper is that it does not need extra conditions to ensure the contraction constant $0<k<1$.

In the paper titled "Some Remarks on Estimate of MittagLeffler Function", the authors point out the mistakes in the estimation process of Mittag-Leffler function, provide a
counterexample, and then propose some sufficient conditions to guarantee that part of the estimate for Mittag-Leffler function is correct. Meanwhile, numerical examples are given to illustrate the validity of the two newly established estimates.

In the paper titled "Differential Harnack Estimates for a Semilinear Parabolic System", the author uses the inequalities to construct classical Harnack estimates by integrating along space-time and then proves some differential Harnack inequalities for positive solutions of a semilinear parabolic system on hyperbolic space.

In the paper titled "Toeplitz Operator and Carleson Measure on Weighted Bloch Spaces", the author considers Toeplitz operator acting on weighted Bloch spaces. Meanwhile, the inclusion map from weighted Bloch spaces into tent space is also investigated.

In the paper titled "Solutions for a Class of Hadamard Fractional Boundary Value Problems with Sign-Changing Nonlinearity", by using fixed point index methods, the authors establish some existence theorems of positive (nontrivial) solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity.

In the paper titled "Option Pricing under the Jump Diffusion and Multifactor Stochastic Processes", the authors extend the Heston model to be a hybrid option pricing model driven by multiscale stochastic volatility and jump diffusion process. In this model the correlation effects have been taken into consideration. For the reason that the combination of multiscale volatility processes and jump diffusion process
results in a high dimensional differential equation, an efficient finite element method is proposed and the integral term arising from the jump term is absorbed to simplify the problem. The numerical results show an efficient explanation for volatility smirks when one incorporates jumps into both the stock process and the volatility process.

In the paper titled "Positive Solutions for a Higher-Order Semipositone Nonlocal Fractional Differential Equation with Singularities on Both Time and Space Variable", the author was concerned with a class of higher-order semipositone nonlocal Riemann-Liouville fractional differential equations. The existence results of positive solutions are given by GuoKrasnosel'skii fixed point theorem and Schauder's fixed point theorem.

In the paper titled "Two New Geraghty Type Contractions in Gb-Metric Spaces", two Geraghty type contractions are introduced in $G b$-metric spaces, and some fixed point theorems about the contractions are proved. At the end of this article, a theorem about unique solution of an integral function is proved.

In the paper titled "Solution of Hamilton-Jacobi-Bellman Equation in Optimal Reinsurance Strategy under Dynamic VaR Constraint", the authors analyze the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle. The surplus process of the insurer is described by the diffusion model which is an approximation of the classical Cramér-Lundberg model by assuming the dynamic VaR constraints for proportional reinsurance; the closed form expression of the optimal reinsurance strategy and corresponding survival probability under proportional reinsurance are obtained.

In the paper titled "Parameter Estimation for Fractional Diffusion Process with Discrete Observations", the authors deal with the problem of estimating the parameters for fractional diffusion process from discrete observations when the Hurst parameter $H$ is unknown. With combination of several methods, such as the Donsker type approximate formula of fractional Brownian motion, quadratic variation method, and the maximum likelihood approach, the authors give the parameter estimations of the Hurst index, diffusion coefficients, and volatility and then prove their strong consistency. Finally, an extension for generalized fractional diffusion process and further work are briefly discussed.

In the paper titled "Hermite-Hadamard-Fejér Inequalities for Conformable Fractional Integrals via Preinvex Functions", the authors present a Hermite-Hadamard-Fejér inequality for conformable fractional integrals by using symmetric preinvex functions and also establish an identity associated with the right hand side of Hermite-Hadamard inequality for preinvex functions; then by using this identity and preinvexity of functions and some well-known inequalities, several new Hermite-Hadamard type inequalities for conformal fractional integrals were established.

In the paper titled "Existence of Nontrivial Solutions for Fractional Differential Equations with p-Laplacian", by combining the properties of the Green function with some fixed point theorems, the authors consider the existence of nontrivial solutions for fractional equations with $p$-Laplacian operator.

In the paper titled "Fixed Point Theory and Positive Solutions for a Ratio-Dependent Elliptic System", the authors consider a ratio-dependent predator-prey model under zero Dirichlet boundary condition. By using topological degree theory and fixed index theory, the necessary and sufficient conditions for the existence of positive solutions were studied, and by bifurcation theory and energy estimates, the asymptotic behavior analysis of positive solutions was presented.

In the paper titled "Existence of Nontrivial Solutions for Some Second-Order Multipoint Boundary Value Problems", by using fixed point theorems with lattice structure, the existence of negative solution and sign-changing solution for some second-order multipoint boundary value problems is obtained.

In the paper titled "Existence of Uniqueness and Nonexistence Results of Positive Solution for Fractional Differential Equations Integral Boundary Value Problems", the author considered a class of fractional differential equations with conjugate type integral conditions. Both the existence of uniqueness and nonexistence of positive solution are obtained by means of the iterative technique. The interesting point is that the assumption on nonlinearity is closely associated with the spectral radius corresponding to the relevant linear operator.

In the paper titled "New Fixed Point Theorems and Application of Mixed Monotone Mappings in Partially Ordered Metric Spaces", the authors consider the existence of a coupled fixed point for mixed monotone mapping satisfying a new contractive inequality which involves an altering distance function in partially ordered metric spaces. Some uniqueness results for coupled fixed points, as well as the existence of fixed points of mixed monotone operators, are established.

In the paper titled "Limit Cycles and Invariant Curves in a Class of Switching Systems with Degree Four", a class of switching systems which have an invariant conic is investigated. Half attracting invariant conic is found in switching systems. The coexistence of small-amplitude limit cycles, large amplitude limit cycles, and invariant algebraic curves under perturbations of the coefficients of the systems is proved.

In the paper titled "Positive Solutions for a System of Fractional Differential Equations with Two Parameters", the existence of positive solutions in terms of different values of two parameters for a system of conformable-type fractional differential equations with the p-Laplacian operator is obtained via Guo-Krasnosel'skii fixed point theorem.

In the paper titled "Impulsive Fractional Differential Equations with $p$-Laplacian Operator in Banach Spaces", the authors study a class of boundary value problems with multiple point boundary conditions of impulsive $p$-Laplacian operator fractional differential equations. The sufficient conditions for the existence of solutions in Banach spaces are established by using the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. An example is given to demonstrate the main results.

In the paper titled "Separated Boundary Value Problems of Sequential Caputo and Hadamard Fractional Differential

Equations", the authors discuss the existence and uniqueness of solutions for new classes of separated boundary value problems of Caputo-Hadamard and Hadamard-Caputo sequential fractional differential equations by using standard fixed point theorems. The application of the obtained results with the aid of examples is demonstrated.

In the paper titled "The Existence and Uniqueness of Solutions and Lyapunov-Type Inequality for CFR Fractional Differential Equations", the authors study CFR fractional differential equations with the derivative of order $3<\alpha<$ 4 and prove existence and uniqueness theorems for CFRtype initial value problem. By Green's function and its corresponding maximum value, the Lyapunov-type inequality of corresponding equations is obtained.

The paper titled "Solutions for Integral Boundary Value Problems of Nonlinear Hadamard Fractional Differential Equations" is aimed at establishing some existence theorems of positive (nontrivial) solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations by using fixed point methods.

In the paper titled "The Conjugate Gradient Viscosity Approximation Algorithm for Split Generalized Equilibrium and Variational Inequality Problems", the authors study a kind of conjugate gradient viscosity approximation algorithm for finding a common solution of split generalized equilibrium problem and variational inequality problem. Under mild conditions, the authors prove that the sequence generated by the proposed iterative algorithm converges strongly to the common solution. Some numerical results are illustrated to show the feasibility and efficiency of the proposed algorithm.

In the paper titled "Some Properties for Solutions of Riemann-Liouville Fractional Differential Systems with a Delay", the authors study properties for solutions of Riemann-Liouville fractional differential systems with a delay. Some results on integral inequalities are first presented by Holder inequality. Then by using the obtained inequalities, the properties on solutions for R-L fractional systems with a delay are investigated and upper bound of solutions is obtained. An illustrative example is considered to support the results

In the paper titled "Existence Results for Impulsive Fractional $q$-Difference Equation with Antiperiodic Boundary Conditions", the authors investigate the impulsive fractional $q$-difference equation with antiperiodic conditions. The existence and uniqueness results of solutions are established via the theorem of nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle. Two examples are given to illustrate the main results.

The paper titled "Existence of Generalized Nash Equilibrium in $n$-Person Noncooperative Games under Incomplete Preference" presents a new method to improve the existence of Nash equilibrium. Based on the incomplete preference corresponding to equivalence class set being a partial order set, the author translates the incomplete preference problems into the partial order problems. Using the famous Zorn lemma, the existence theorems of fixed point for noncontinuous operators in incomplete preference sets are obtained. Finally, the existence of generalized Nash equilibrium is
strictly proved in the $n$-person noncooperative games under incomplete preference

In the paper titled "Positive Solutions for Higher Order Nonlocal Fractional Differential Equation with Integral Boundary Conditions", by using the spectral analysis of the relevant linear operator and Gelfand's formula, some properties of the first eigenvalue of a fractional differential equation were obtained; combining fixed point index theorem, sufficient conditions for the existence of positive solutions are established. An example is given to demonstrate the application of the main results.

In the paper titled "Hopf Bifurcation of a Delayed Ecoepidemic Model with Ratio-Dependent Transmission Rate", the authors mainly focus on the effects of the time delay due to the gestation of the predator for a developed delay ecoepidemic model with ratio-dependent transmission rate. Sufficient conditions for local stability and existence of a Hopf bifurcation of the model are derived by regarding the time delay as the bifurcation parameter. Furthermore, properties of the Hopf bifurcation are investigated by using the normal form theory and the center manifold theorem. Finally, numerical simulations are carried out in order to validate the obtained theoretical results

In the paper titled "Uniqueness of Successive Positive Solution for Nonlocal Singular Higher-Order Fractional Differential Equations Involving Arbitrary Derivatives", by means of fixed point theorem on mixed monotone operator, the authors establish the uniqueness of positive solution for some nonlocal singular higher-order fractional differential equations involving arbitrary derivatives. The iterative schemes for approximating unique positive solution are given.

The paper titled "Existence and Nonexistence of Positive Solutions for Mixed Fractional Boundary Value Problem with Parameter and $p$-Laplacian Operator" mainly studies a class of mixed fractional boundary value problem with parameter and p-Laplacian operator. Based on the GuoKrasnosel'skii fixed point theorem, results on the existence and nonexistence of positive solutions for the fractional boundary value problem are established. An example is then presented to illustrate the effectiveness of the results

In the paper titled "Existence Results for Generalized Bagley-Torvik Type Fractional Differential Inclusions with Nonlocal Initial Conditions", the authors prove the existence of solutions for the generalized Bagley-Torvik type fractional-order differential inclusions with nonlocal conditions. It allows one to apply the noncompactness measure of Hausdorff, fractional calculus theory and the nonlinear alternative for Kakutani maps fixed point theorem to obtain the existence results under the assumptions that the nonlocal item is compact continuous and Lipschitz continuous and multifunction is compact and Lipschitz, respectively.

In the paper titled "On the Effective Reducibility of a Class of Quasi-Periodic Linear Hamiltonian Systems Close to Constant Coefficients", the authors consider the effective reducibility of a class of quasi-periodic linear Hamiltonian system. Under nonresonant conditions, it is proved that this system can be reduced to low order system and the change of
variables that perform such a reduction is also quasi-periodic with the same basic frequencies.

In the paper titled "A Note on the Fractional Generalized Higher Order KdV Equation", the author obtains exact solutions to the fractional generalized higher order Korteweg-de Vries equation using the complex method, showing that the applied method is very useful and is practically well suited for the nonlinear differential equations arising in mathematical physics.

In the paper titled "The Eigenvalue Problem for Caputo Type Fractional Differential Equation with Riemann-Stieltjes Integral Boundary Conditions", the authors investigate the eigenvalue problem for Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions. By using the Guo-Krasnoselskii's fixed point theorem on cone and the properties of the Green's function, some new results on the existence and nonexistence of positive solutions for the fractional differential equation are obtained.

The paper titled "Synchronization of Different Uncertain Fractional-Order Chaotic Systems with External Disturbances via T-S Fuzzy Model" presents an adaptive fuzzy synchronization control strategy for a class of different uncertain fractional-order chaotic/hyperchaotic systems with unknown external disturbances via T-S fuzzy systems, where the parallel distributed compensation technology is provided to design adaptive controller with fractional adaptation laws. T-S fuzzy models are employed to approximate the unknown nonlinear systems and tracking error signals are used to update the parametric estimates. The asymptotic stability of the closed-loop system and the boundedness of the states and parameters are guaranteed by fractional Lyapunov theory. This approach is also valid for synchronization of fractional-order chaotic systems with the same system structure. One constructive example is given to verify the feasibility and superiority of the proposed method.

In the paper titled "Positive Solutions for a Fractional Boundary Value Problem with a Perturbation Term", the author obtains some new upper and lower estimates for the Green's function associated with a fractional boundary value problem with a perturbation term. Criteria for the existence of positive solutions of the problem are then obtained based on these theories.

In the paper titled "The Exact Iterative Solution of Fractional Differential Equation with Nonlocal Boundary Value Conditions", the authors deal with a singular nonlocal fractional differential equation with Riemann-Stieltjes integral conditions. The exact iterative solution is established under the iterative technique. The iterative sequences have been proved to converge uniformly to the exact solution, and estimation of the approximation error and the convergence rate have been derived. An example is also given to demonstrate the results

In the paper titled "Fixed-Point Theorems for Systems of Operator Equations and Their Applications to the Fractional Differential Equations", the authors study the existence and uniqueness of positive solution for a class of nonlinear binary operator equations systems by means of the cone theory and monotone iterative technique, under more general conditions. The iterative sequence of the solution and the error
estimation of the system are given. The authors also use this new result to study the existence and uniqueness of the solutions for fractional differential equations systems involving integral boundary value conditions in ordered Banach spaces as an application.

In the paper titled "Generalization of Hermite-Hadamard Type Inequalities via Conformable Fractional Integrals", the authors establish a Hermite-Hadamard type identity and several new Hermite-Hadamard type inequalities for conformable fractional integrals and present their applications to special bivariate means.

In the paper titled "Fixed Point Theorems for Generalized $\alpha s-\psi$-Contractions with Applications", the authors study the sufficient conditions for the existence of a unique common fixed point of generalized $\alpha s-\psi$-Geraghty contractions in an $\alpha s$-complete partial $b$-metric space. An example in support of the findings is given.

In the paper titled "A New Sufficient Condition for Checking the Robust Stabilization of Uncertain Descriptor Fractional-Order Systems", the authors consider the robust asymptotical stabilization of uncertain class of descriptor fractional-order systems. In the state matrix, the authors require that the parameter uncertainties are time-invariant and norm-bounded. A sufficient condition for the system with the fractional-order $\alpha$ satisfying $1 \leqslant \alpha<2$ in terms of linear matrix inequalities is derived. The condition of the proposed stability criterion for fractional-order system is easy to be verified. An illustrative example is given to show that the result is effective.

In the paper titled "Positive Solutions for a System of Semipositone Fractional Difference Boundary Value Problems", by using the fixed point index, the authors establish two existence theorems for positive solutions to a system of semipositone fractional difference boundary value problems. Nonnegative concave functions and nonnegative matrices to characterize the coupling behavior of our nonlinear terms are adopted.

In the paper titled "Non-Nehari Manifold Method for Fractional p-Laplacian Equation with a Sign-Changing Nonlinearity", the authors consider a class of fractional pLaplacian equation. The nonlinear term $f$ has the subcritical growth and may change sign. Under the condition that $V$ is coercive, the existence of ground state solutions for p Laplacian equation is established.

In the paper titled "C*-Algebra-Valued G-Metric Spaces and Related Fixed-Point Theorems", the authors introduce the notion of the $C *$-algebra-valued $G$-metric space. The existence and uniqueness of some fixed-point theorems for self-mappings with contractive or expansive conditions on complete $C *$-algebra-valued $G$-metric spaces are established. As an application, the authors prove the existence and uniqueness of the solution of a type of differential equations.

In the paper titled "The Tensor Padé-Type Approximant with Application in Computing Tensor Exponential Function", tensor Padé-type approximant is defined by introducing a generalized linear functional for the first time. The expression of TPTA is provided with the generating function form. Moreover, by means of formal orthogonal polynomials, the authors propose an efficient algorithm for computing

TPTA. As an application, the TPTA for computing the tensor exponential function is presented and some numerical examples are given to demonstrate the efficiency of the proposed algorithm.

In the paper titled "A New Approach to the Existence of Quasiperiodic Solutions for Second-Order Asymmetric $p$-Laplacian Differential Equations", the authors propose a new estimate approach to study the existence of AubryMather sets and quasiperiodic solutions for the second-order asymmetric $p$-Laplacian differential equations. By using the Aubry-Mather theorem, the existence of Aubry-Mather sets and quasiperiodic solutions under some reasonable conditions are obtained. Particularly, the advantage of the approach is that it not only gives a simpler estimation procedure, but also weakens the smoothness assumption on the function $\psi(t, x)$ in the existing literature.

Through the special issue, we also hope to open the opportunity for the journal readers to make comments on the work presented.

## Conflicts of Interest

The editors declare that they have no conflicts of interest regarding the publication of this special issue.

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# Existence Results for a Class of Semilinear Fractional Partial Differential Equations with Delay in Banach Spaces 

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#### Abstract

In this paper, we consider a class of nonlinear time fractional partial differential equations with delay. We obtain the existence and uniqueness of the mild solutions for the problem by the theory of solution operator and the general Banach contraction mapping principle. We need not extra conditions to ensure the contraction constant $0<k<1$. Therefore, under some general conditions, we obtain our main results.


## 1. Introduction

Fractional derivatives can describe the property of the memory, and they have more advantages than integer-order derivatives. Therefore, fractional differential equations have been successfully applied in many fields, such as engineering and physics. About the fractional differential equations, we refer to these papers [1-9] and the references therein. In [7, 10], the authors studied the existence results of the fractional integrodifferential equations of order $1<\beta \leq 2$. In [11, 12], the authors considered a class of fractional differential equations, where the fractional derivative operator is ${ }_{0}^{A} D_{t}^{\beta}$ with fractional order $\beta$ and $A$ is a closed densely defined operator in a Banach space. Goufo [13] studied the existence results for a class of fractional fragmentation model by theory of strongly continuous solution operators. In [14, 15], the authors investigated a class of space-time fractional diffusion equations, while in [16] the authors studied a class of linear fractional differential equations by the variational iteration method and the Adomian decomposition method. In [17-19], the authors studied the following fractional partial differential equations:

$$
\begin{align*}
& { }^{c} D_{t}^{\beta} u(x, t)-\frac{\partial^{2}}{\partial x^{2}} u(x, t) \\
& \quad=g\left(t, u\left(x, \tau_{1}(t)\right), u\left(x, \tau_{2}(t)\right), \ldots, u\left(x, \tau_{l}(t)\right)\right), \\
& \quad t \in\left[0, T_{0}\right],  \tag{1}\\
& u(x, t)=0, \quad(x, t) \in \partial \Omega \times\left[0, T_{0}\right], \\
& u(x, 0)=\varphi(x), \quad x \in \Omega .
\end{align*}
$$

Ouyang [17] studied the existence of the local mild solutions for such problem by Leray-Schauder's fixed theorem. Zhu et al. [18, 19] also studied the existence of the mild solutions by strict set contraction and Banach contraction mapping theorem of the problem. Li et al. [20] investigated the following fractional differential equations:

$$
\begin{align*}
{ }^{c} D_{t}^{\beta} u(t) & =A u(t)+J_{t}^{1-\beta} f\left(t, u_{t}\right), \quad t \in[0, T]  \tag{2}\\
u(t) & =\varphi(t), \quad t \in[-r, 0]
\end{align*}
$$

where $\beta \in(0,1)$. In [20], the Lipschitz coefficient of the nonlinear function $f$ is a constant.

Inspired by the above said work, we investigate the following nonlinear fractional partial differential equations with delay:

$$
\begin{align*}
& { }^{c} D_{t}^{\beta} u(x, t)-\frac{\partial^{2}}{\partial x^{2}} u(x, t) \\
& =J_{t}^{1-\beta} f\left(t, u\left(x, \tau_{1}(t)\right), \ldots, u\left(x, \tau_{l}(t)\right)\right),  \tag{3}\\
& \quad t \in\left[0, T_{0}\right], x \in \Omega, \\
& u(x, \theta)=\varphi(x, \theta), \quad x \in \Omega, \theta \in[-r, 0]
\end{align*}
$$

where $0<\beta<1,{ }^{c} D_{t}^{\beta}$ is Caputo's fractional derivative of order $\beta, J_{t}^{1-\beta}$ is the Riemann-Liouville fractional integral of order $1-\beta, l$ is a nature number, $\Omega \subset \mathbb{R}^{l}$ is a bounded domain with regular boundary $\partial \Omega, \tau_{i}:\left[0, T_{0}\right] \longrightarrow\left[0, T_{0}\right](i=1,2, \ldots, l)$ are continuous functions, and these functions satisfy $0 \leq$ $\tau_{i}(t) \leq t(i=1,2, \ldots, l), \varphi \in C([-r, 0] ; E), E=C(\bar{\Omega} ; \mathbb{R}), \bar{\Omega}=$ $\Omega \cup \partial \Omega$.

In this paper, we consider the existence results of the mild solutions of problem (3) by general Banach contraction mapping theorem. We need not extra conditions to ensure the contraction constant $0<k<1$. Under some general conditions, we obtain our main results. Therefore, our results presented in this paper improve many classical results.

## 2. Preliminaries

Let $(E,\|\cdot\|)$ be a Banach space and let $C\left(\left[0, T_{0}\right] ; E\right)=\{u$ : $\left[0, T_{0}\right] \longrightarrow E$ is continuous $\}$ be a Banach space with norm $\|u\|_{C}=\max \left\{\|u(t)\|: t \in\left[0, T_{0}\right]\right\}$.

Definition 1 (see [21, 22]). The Riemann-Liouville fractional integral of a function $h:(0, \infty) \longrightarrow \mathbb{R}$ of order $\alpha>0$ is defined as

$$
\begin{equation*}
J_{t}^{\alpha} h(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} h(s) d s \tag{4}
\end{equation*}
$$

where $h(t) \in L^{1}\left(\left(0, T_{0}\right) ; E\right)$.
For convenience, we let

$$
f_{\alpha}(t)= \begin{cases}\frac{t^{\alpha-1}}{\Gamma \alpha}, & t>0  \tag{5}\\ 0, & t \leq 0\end{cases}
$$

and then

$$
\begin{equation*}
J_{t}^{\alpha} h(t)=\left(f_{\alpha} * u\right)(t)=\int_{0}^{t} f_{\alpha}(t-s) h(s) d s \tag{6}
\end{equation*}
$$

Definition 2 (see [21, 22]). The Riemann-Liouville fractional derivative of a function $h:(0, \infty) \longrightarrow \mathbb{R}$ of order $\alpha>0$ is defined as

$$
\begin{equation*}
D_{t}^{\alpha} h(t)=D_{t}^{m} J_{t}^{m-\alpha} h(t), \tag{7}
\end{equation*}
$$

where $D_{t}^{m}=d^{m} / d t^{m}, h(t) \in L^{1}\left(\left(0, T_{0}\right) ; E\right), J_{t}^{m-\alpha} h(t) \in$ $W^{m, 1}\left(\left(0, T_{0}\right) ; E\right)$.

Definition 3 (see [21, 22]). The Caputo fractional derivative of function $h$ of order $\alpha>0$ is defined as

$$
\begin{equation*}
{ }^{c} D_{t}^{\alpha} h(t)=D_{t}^{\alpha}\left(h(t)-\sum_{k=0}^{m-1} \frac{t^{k}}{k!} h^{(k)}(0)\right), \tag{8}
\end{equation*}
$$

where $h(t) \in L^{1}\left(\left(0, T_{0}\right) ; E\right) \cap C^{m-1}\left(\left(0, T_{0}\right) ; E\right)$.
For the Riemann-Liouville fractional integral operator and the Caputo fractional derivative operator, we have

$$
\begin{align*}
& { }^{c} D_{t}^{\alpha}\left(J_{t}^{\alpha} h(t)\right)=h(t), \\
& J_{t}^{\alpha}\left({ }^{c} D_{t}^{\alpha} h(t)\right)=h(t)-\sum_{k=0}^{m-1} \frac{t^{k}}{k!} h^{(k)}(0) . \tag{9}
\end{align*}
$$

Definition 4 (see [23]). Let $A$ be a closed linear operator with dense domain $D(A)$ in a Banach space $E ; \beta>0$. A family $\left\{S_{\beta}(t)\right\}_{t \geq 0} \subset B(E)$ of bounded linear operators in $E$ is called a solution operator for the integral equation

$$
\begin{equation*}
u(t)=x+\frac{1}{\Gamma(\beta)} \int_{0}^{t} \frac{A u(s)}{(t-s)^{1-\beta}} d s, \quad t \geq 0, x \in E \tag{10}
\end{equation*}
$$

if the following conditions are satisfied:
(i) $S_{\beta}(t)$ is strongly continuous on $\mathbb{R}_{+}$and $S_{\beta}(0)=I$.
(ii) $S_{\beta}(t) D(A) \subset D(A)$ and $A S_{\beta}(t) x=S_{\beta}(t) A x$ for all $x \in D(A)$ and $t \geq 0$.
(iii) $S_{\beta}(t) x$ is a solution of

$$
\begin{equation*}
u(t)=x+\frac{1}{\Gamma(\beta)} \int_{0}^{t} \frac{A u(s)}{(t-s)^{1-\beta}} d s \tag{11}
\end{equation*}
$$

for all $x \in D(A), t \geq 0$.
We call $A$ the infinitesimal generator of $S_{\beta}(t)$ or say that $A$ generates $S_{\beta}(t)$.

Let $u(t)=u(\cdot, t)$; then the fractional partial differential equation (3) can be rewritten in the following abstract form:

$$
\begin{align*}
&{ }^{c} D_{t}^{\beta} u(t) \\
& \\
&=A u(t)  \tag{12}\\
&+J_{t}^{1-\beta} f\left(t, u\left(\tau_{1}(t)\right), u\left(\tau_{2}(t)\right), \ldots, u\left(\tau_{l}(t)\right)\right), \\
& t \in\left[0, T_{0}\right]
\end{align*}
$$

$$
u(t)=\varphi(t), \quad t \in[-r, 0],
$$

where

$$
\begin{align*}
D(A) & =\left\{u \in C(\bar{\Omega}, \mathbb{R}) ; u^{\prime \prime} \in C(\bar{\Omega}, \mathbb{R})\right\}, \\
A u & =u^{\prime \prime}  \tag{13}\\
t & \in[-r, 0]
\end{align*}
$$

and $f:\left[0, T_{0}\right] \times C([-r, 0] ; E) \times C([-r, 0] ; E) \cdots \times C([-r$, $0] ; E) \longrightarrow E$ is defined by $f(t, \varphi, \varphi \cdots \varphi)(x)=f(t, \varphi(\cdot, x)$, $\varphi(\cdot, x) \cdots \varphi(\cdot, x))$ for $t \in\left[0, T_{0}\right], \varphi \in C([-r, 0] ; E)$ and $x \in \Omega$.

It is easy to see that $A$ generates a $C_{0}$ semigroup $\left\{T(t)_{t \geq 0}\right\}$ on $E$. Theorem 3.1 in [23] means that $A$ is the infinitesimal generator of a solution operator $S_{\beta}(t)_{t \geq 0}$.

Definition 5 (see [24]). If $u \in C\left(\left[-r, T_{0}\right] ; E\right)$ is a mild solution of problem (12), then $u$ satisfies the following integral equations:

$$
u(t)= \begin{cases}S_{\beta}(t) \varphi(0)+\int_{0}^{t} S_{\beta}(t-s) f\left(s, u\left(\tau_{1}(s)\right), u\left(\tau_{2}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right) d s, & t \in\left[0, T_{0}\right]  \tag{14}\\ \varphi(t), & t \in[-r, 0]\end{cases}
$$

## 3. Main Results

We define the operator $\mathscr{F}: C\left(\left[-r, T_{0}\right] ; E\right) \longrightarrow C\left(\left[-r, T_{0}\right] ; E\right)$ by

$$
\mathscr{F} u(t)= \begin{cases}S_{\beta}(t) \varphi(0)+\int_{0}^{t} S_{\beta}(t-s) f\left(s, u\left(\tau_{1}(s)\right), u\left(\tau_{2}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right) d s, & t \in\left[0, T_{0}\right]  \tag{15}\\ \varphi(t), & t \in[-r, 0]\end{cases}
$$

Theorem 6. Assume that the conditions $\left(H_{1}\right)$ and $\left(H_{2}\right)$ hold.
$\left(H_{1}\right)$ There exists a real number $M>0$ such that $\left\|S_{\beta}(t)\right\| \leq$ $M, t \in\left[0, T_{0}\right] .\left(H_{2}\right)$ The function $f:\left[0, T_{0}\right] \times C([-r, 0] ; E) \times$ $C([-r, 0] ; E) \cdots \times C([-r, 0] ; E) \longrightarrow E$ is continuous in $t$ on $\left[0, T_{0}\right]$, and there exist nonnegative Lebesgue integrable functions $L_{i} \in L^{1}\left(J, \mathbb{R}^{+}\right)(i=1,2, \ldots, l)$ such that

$$
\begin{align*}
& \left\|f\left(t, u_{1}, u_{2}, \ldots, u_{l}\right)-f\left(t, v_{1}, v_{2}, \ldots, v_{l}\right)\right\| \\
& \quad \leq \sum_{i}^{l} L_{i}(t)\left\|u_{i}-v_{i}\right\| \tag{16}
\end{align*}
$$

where $t \in\left[0, T_{0}\right], u_{i}, v_{i} \in C([-r, 0] ; E)$.
Then problem (12) has a unique mild solution $u \in$ $C\left(\left[-r, T_{0}\right] ; E\right)$, which means that (3) has a unique mild solution.

Proof. For any $0<\varepsilon<1$, by the property of the Lebesgue integrable function, there exists a continuous function $\phi(s)$ such that $\int_{0}^{T_{0}}|L(s)-\phi(s)| d s<\varepsilon$, where $L(s)=M \sum_{i}^{l} L_{i}(s)$. From conditions $\left(H_{1}\right)-\left(H_{2}\right)$ and (15), for any $u, v \in C\left(\left[-r, T_{0}\right] ; E\right)$, we obtain

$$
\begin{aligned}
& \|(\mathscr{F} u)(t)-(\mathscr{F} v)(t)\| \leq \int_{0}^{t} S_{\beta}(t-s) \\
& \quad \cdot \| f\left(s, u\left(\tau_{1}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right) \\
& \quad-f\left(s, v\left(\tau_{1}(s)\right), \ldots, v\left(\tau_{l}(s)\right)\right) \| d s \\
& \quad \leq \int_{0}^{t} M \sum_{i}^{l} L_{i}(s)\left\|u\left(\tau_{i}(s)\right)-v\left(\tau_{i}(s)\right)\right\| d s \\
& \quad \leq \int_{0}^{t} L(s) d s\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& \quad \leq\left(\int_{0}^{t}|L(s)-\phi(s)| d s+\int_{0}^{t}|\phi(s)| d s\right) \| u
\end{aligned}
$$

$$
\begin{align*}
& -v\left\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \leq(\varepsilon+N t)\right\| u-v \|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& =\left(C_{1}^{0} \varepsilon^{1}+C_{1}^{1} \frac{(N t)^{1}}{1!}\right)\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \tag{17}
\end{align*}
$$

where $N=\max _{t \in J}|\phi(t)|$. Next, for any nature number $n$, we will prove the following inequality:

$$
\begin{align*}
& \left\|\left(\mathscr{F}^{n} u\right)(t)-\left(\mathscr{F}^{n} v\right)(t)\right\| \\
& \quad \leq\left(C_{n}^{0} \varepsilon^{n}+C_{n}^{1} \varepsilon^{n-1} \frac{(N t)^{1}}{1!}+\cdots+C_{n}^{n} \varepsilon^{n-n} \frac{(N t)^{n}}{n!}\right)  \tag{18}\\
& \quad \cdot\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} .
\end{align*}
$$

Obviously, for $n=1$, (18) holds. Assume that $n=k$ and (18) holds; that is,

$$
\begin{align*}
& \left\|\left(\mathscr{F}^{k} u\right)(t)-\left(\mathscr{F}^{k} v\right)(t)\right\| \\
& \quad \leq\left(C_{k}^{0} \varepsilon^{k}+C_{k}^{1} \varepsilon^{k-1} \frac{(N t)^{1}}{1!}+\cdots+C_{k}^{k} \varepsilon^{k-k} \frac{(N t)^{k}}{k!}\right)  \tag{19}\\
& \quad \cdot\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}
\end{align*}
$$

By (15), $\left(H_{1}\right)-\left(H_{2}\right)$, and formula $C_{k+1}^{m}=C_{k}^{m}+C_{k}^{m-1}$, we have

$$
\begin{aligned}
& \left\|\left(\mathscr{F}^{k+1} u\right)(t)-\left(\mathscr{F}^{k+1} v\right)(t)\right\| \\
& \quad \leq \int_{0}^{t} M \| f\left(s,\left(\mathscr{F}^{k} u\right)\left(\tau_{1}(s)\right), \ldots,\left(\mathscr{F}^{k} u\right)\left(\tau_{l}(s)\right)\right) \\
& \quad-f\left(s,\left(\mathscr{F}^{k} v\right)\left(\tau_{1}(s)\right), \ldots,\left(\mathscr{F}^{k} v\right)\left(\tau_{l}(s)\right)\right) \| d s
\end{aligned}
$$

$$
\begin{align*}
& \leq \int_{0}^{t} L(s)\left\|\left(\mathscr{F}^{k} u\right)\left(\tau_{i}(s)\right)-\left(\mathscr{F}^{k} v\right)\left(\tau_{i}(s)\right)\right\| d s \\
& \leq\left(\int_{0}^{t}|L(s)-\phi(s)|\right. \\
& \left.\cdot\left(C_{k}^{0} \varepsilon^{k}+C_{k}^{1} \varepsilon^{k-1} \frac{(N s)^{1}}{1!}+\cdots+C_{k}^{k} \varepsilon^{k-k} \frac{(N s)^{k}}{k!}\right) d s\right) \\
& \cdot\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}+\left(\int_{0}^{t}|\phi(s)|\right. \\
& \left.\cdot\left(C_{k}^{0} \varepsilon^{k}+C_{k}^{1} \varepsilon^{k-1} \frac{(N s)^{1}}{1!}+\cdots+C_{k}^{k} \varepsilon^{k-k} \frac{(N s)^{k}}{k!}\right) d s\right) \\
& \cdot\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \leq \varepsilon\left(C_{k}^{0} \varepsilon^{k}+C_{k}^{1} \varepsilon^{k-1} \frac{(N t)^{1}}{1!}+\cdots\right. \\
& \left.+C_{k}^{k} \varepsilon^{k-k} \frac{(N t)^{k}}{k!}\right)\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}+N \int_{0}^{t}\left(C_{k}^{0} \varepsilon^{k}\right. \\
& \left.+C_{k}^{1} \varepsilon^{k-1} \frac{(N s)^{1}}{1!}+\cdots+C_{k}^{k} \varepsilon^{k-k} \frac{(N s)^{k}}{k!}\right) d s \| u \\
& -v \|_{C\left(\left[-r, T_{0}\right] ; E\right)} \leq\left\{\varepsilon \left(C_{k}^{0} \varepsilon^{k}+C_{k}^{1} \varepsilon^{k-1} \frac{(N t)^{1}}{1!}+\cdots\right.\right. \\
& \left.+C_{k}^{k} \varepsilon^{k-k} \frac{(N t)^{k}}{k!}\right)+\left(C_{k}^{0} \varepsilon^{k} \frac{(N t)^{1}}{1!}+C_{k}^{1} \varepsilon^{k-1} \frac{(N t)^{2}}{2!}\right. \\
& \left.\left.+\cdots+C_{k}^{k} \varepsilon^{k-k} \frac{(N t)^{k+1}}{k+1!}\right)\right\}\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& \leq\left(C_{k+1}^{0} \varepsilon^{k+1}+C_{k+1}^{1} \varepsilon^{k} \frac{(N t)^{1}}{1!}+\cdots+C_{k+1}^{k+1} \varepsilon^{(k+1)-(k+1)}\right. \\
& \left.\cdot \frac{(N t)^{k+1}}{(k+1)!}\right)\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} . \tag{20}
\end{align*}
$$

By mathematical induction, we obtain that (18) holds for $n=$ $k+1$. Therefore, for any $n$, we have

$$
\begin{align*}
& \left\|\mathscr{F}^{n} u-\mathscr{F}^{n} v\right\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& \quad \leq\left(C_{n}^{0} \varepsilon^{n}+C_{n}^{1} \varepsilon^{n-1} \frac{d^{1}}{1!}+\cdots+C_{n}^{n} \varepsilon^{n-n} \frac{d^{n}}{n!}\right)  \tag{21}\\
& \quad \cdot\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)},
\end{align*}
$$

where $d=N T_{0}$. It is easy to see that

$$
\begin{equation*}
\lim _{k \rightarrow \infty}\left(\varepsilon^{k-1} k\left(\frac{k}{k-1}\right)^{k-1}\right)^{1 / k}=\varepsilon<1 \tag{22}
\end{equation*}
$$

Therefore, we choose sufficiently large nature number $K>2$ such that

$$
\begin{equation*}
\left(\varepsilon^{K-1} K\left(\frac{K}{K-1}\right)^{K-1}\right)^{1 / K} \equiv \alpha<1 \tag{23}
\end{equation*}
$$

For any nature number $n>2 K$, such that $n=K h+p(0 \leq p<$ $K)$. Obviously, $h=[n / K]<[n / 2]$. For any sufficiently large positive integer $n>2 K$; by the Stirling formula

$$
\begin{equation*}
n!=\left(\frac{n}{e}\right)^{n} \sqrt{2 \pi n}\left(1+O\left(\frac{1}{n}\right)\right) \tag{24}
\end{equation*}
$$

and by (23), we have

$$
\begin{align*}
S_{1} & \equiv C_{n}^{0} \varepsilon^{n}+C_{n}^{1} \varepsilon^{n-1} \frac{d^{1}}{1!}+\cdots+C_{n}^{h} \varepsilon^{n-h} \frac{d^{h}}{h!} \\
& \leq \varepsilon^{n-h} C_{n}^{h}\left(1+d+\frac{d^{2}}{2!}+\cdots+\frac{d^{h}}{h!}\right)=\varepsilon^{n-h} C_{n}^{h} O(1) \\
& =\frac{O(1) \varepsilon^{n-h} n^{n} \sqrt{2 \pi n}(1+O(1 / h))}{h^{h} \sqrt{2 \pi h} \sqrt{2 \pi(n-h)}(n-h)^{n-h}} \\
& =O\left(\frac{K^{h}}{\sqrt{h}}\right)\left(\frac{b K}{K-1}\right)^{(n-h) h}  \tag{25}\\
& =O\left(\frac{\left(b^{K-1} K(K /(K-1))^{K-1}\right)^{h}}{\sqrt{h}}\right)=O\left(\frac{\alpha^{K h}}{\sqrt{h}}\right) \\
& =O\left(\frac{\alpha^{n}}{\sqrt{n}}\right) .
\end{align*}
$$

On the other hand, without loss of generality, we assume that $d=N T_{0}>1$. By Stirling formula and $C_{n}^{[n / 2]}=O\left(2^{n} / \sqrt{n}\right)$, we get

$$
\begin{align*}
S_{2} & \equiv C_{n}^{h+1} \varepsilon^{n-h-1} \frac{d^{h+1}}{(h+1)!}+\cdots+C_{n}^{n} \varepsilon^{n-n} \frac{d^{n}}{h!} \\
& \leq \frac{1}{(h+1)!} C_{n}^{[n / 2]}\left(\varepsilon^{n-h-1} d^{h+1}+b^{n-n} d^{n}\right) \\
& =b^{n-h} C_{n}^{h} O(1) \\
& =\frac{O\left(2^{n} / \sqrt{n}\right) e^{h+1}\left(\varepsilon^{n-h-1} d^{h+1}+\varepsilon^{n-n} d^{n}\right)}{\sqrt{2 \pi}(h+1)(h+1)^{h+1}(1+O(1 /(h+1)))}  \tag{26}\\
& \leq \frac{O\left(2^{n} / \sqrt{n}\right) e^{h+1}\left(1+d+d^{2}+\cdots+d^{h+1}+\cdots+d^{n}\right)}{\sqrt{2 \pi(h+1)}(h+1)^{h+1}} \\
& \leq \frac{O(1) 2^{n} e^{h+1} d^{n+1}}{\sqrt{n} \sqrt{h+1}(h+1)^{h+1} \leq \frac{O(1) 2^{n} e^{h+1} d^{n+1}}{h^{h+2}}} \\
& =o\left(\frac{1}{h^{\lambda+1}}\right)=o\left(\frac{1}{n^{\lambda+1}}\right) \quad(n \longrightarrow+\infty),
\end{align*}
$$

where $\lambda>0$. By (21), (25), and (26), we get

$$
\begin{align*}
& \left\|\mathscr{F}^{n} u-\mathscr{F}^{n} v\right\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \leq\left(S_{1}+S_{2}\right)\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& \quad=\left[O\left(\frac{\alpha^{n}}{\sqrt{n}}\right)+o\left(\frac{1}{h^{\lambda+1}}\right)\right]\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}  \tag{27}\\
& \quad=o\left(\frac{1}{n^{\lambda+1}}\right)\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}, \quad(n \longrightarrow+\infty)
\end{align*}
$$

Therefore, for fixed constant $\lambda>0$, there exists a positive integer $n_{0}$ such that, for any $u, v \in C\left(\left[-r, T_{0}\right] ; E\right)$, we have

$$
\begin{align*}
&\left\|\mathscr{F}^{n} u-\mathscr{F}^{n} v\right\|_{C\left(\left[-r, T_{0}\right] ; E\right)} \leq \frac{1}{n^{r+1}}\|u-v\|_{C\left(\left[-r, T_{0}\right] ; E\right)}  \tag{28}\\
& \forall n>n_{0} .
\end{align*}
$$

By general Banach contraction mapping principle, for operator $\mathscr{F}$ there exists a unique fixed point $u \in C\left(\left[-r, T_{0}\right] ; E\right)$, which means that problem (3) has a unique mild solution.

Remark 7. In Theorem 6, we obtain the existence and uniqueness of the global mild solution of problem (3) under the uniform Lipschitz condition of the function $f$. In next Theorem 8, we assume that the function $f$ satisfies the local Lipschitz condition.

Theorem 8. Let $f:\left[0, T_{0}\right] \times C([-r, 0] ; E) \times C([-r, 0] ; E) \times$ $\cdots \times C([-r, 0] ; E) \longrightarrow E$ be continuous in $t$ for $t \in\left[0, T_{0}\right]$, and $f$ satisfies the following local Lipschitz contition: for any
$\varrho>0$, there exist nonnegative Lebesgue integrable functions $L_{\varrho i} \in L^{1}\left(J, \mathbb{R}^{+}\right)(i=1,2, \ldots, l)$ such that

$$
\begin{align*}
& \left\|f\left(t, u_{1}, u_{2}, \ldots, u_{l}\right)-f\left(t, v_{1}, v_{2}, \ldots, v_{l}\right)\right\| \\
& \quad \leq \sum_{i}^{l} L_{\varrho i}(t)\left\|u_{i}-v_{i}\right\| \tag{29}
\end{align*}
$$

where $t \in\left[0, T_{0}\right], u_{i}, v_{i} \in C([-r, 0] ; E)$ with $\left\|u_{i}\right\| \leq \varrho,\left\|v_{i}\right\| \leq \varrho$.
Then there exists a $\tau>0$ such that problem (3) has a unique mild solution on $[-r, \tau)$.

Proof. For all fixed $\xi>0$, there exists $\delta \in\left[0, T_{0}\right]$ such that $\left\|S_{\beta}(t) \varphi(0)-\varphi(0)\right\| \leq \xi / 2$ as $t \in[0, \delta]$. Let $\delta_{1}>0$ such that $M^{*} \varrho \int_{0}^{\delta_{1}} \sum_{i}^{l} L_{\varrho i}(s) d s+\Re \delta_{1} \leq \xi / 2$, where $\Re=$ $\sup _{s \in\left[0, \delta_{1}\right]}\|f(s, 0,0, \ldots, 0)\|,\left\|S_{\beta}(t)\right\| \leq M^{*}, t \in\left[0, \delta_{1}\right]$. Take $\tau=\min \left\{\delta, \delta_{1}\right\}$. Let

$$
\begin{align*}
D_{\xi} & =\{u \in C([-r, \tau] ; E): u(s)=\varphi(s) \text { if } s \\
& \left.\in[-r, 0] \text { and } \sup _{s \in[0, \tau]}\|\mathrm{u}(\mathrm{~s})-\varphi(0)\| \leq \xi\right\} \tag{30}
\end{align*}
$$

Obviously, $D_{\xi}$ is a closed subset of $C([-r, \tau] ; E)$; thus $D_{\xi}$ is a Banach space. Now we consider the mapping

$$
\begin{equation*}
\mathbb{Q}: D_{\xi} \longrightarrow C([-r, \tau] ; E) \tag{31}
\end{equation*}
$$

by

$$
Q u(t)= \begin{cases}S_{\beta}(t) \varphi(0)+\int_{0}^{t} S_{\beta}(t-s) f\left(s, u\left(\tau_{1}(s)\right), u\left(\tau_{2}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right) d s, & t \in[0, \tau]  \tag{32}\\ \varphi(t), & t \in[-r, 0]\end{cases}
$$

Next, we will prove that $\mathbb{Q}$ maps $D_{\xi}$ into $D_{\xi}$.
Let $u \in D_{\xi}$; we have

$$
\begin{align*}
& \|(Q u)(t)-\varphi(0)\| \leq\left\|S_{\beta}(t) \varphi(0)-\varphi(0)\right\|+\int_{0}^{t} S_{\beta}(t \\
& \quad-s)\left\|f\left(s, u\left(\tau_{1}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right)\right\| d s \leq \| S_{\beta}(t) \\
& \quad \cdot \varphi(0)-\varphi(0) \|+\int_{0}^{\tau} S_{\beta}(t-s) \\
& \quad \cdot \| f\left(s, u\left(\tau_{1}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right)  \tag{33}\\
& \quad-f(s, 0, \ldots, 0) \| d s+\int_{0}^{t} S_{\beta}(t-s) \\
& \cdot\|f(s, 0, \ldots, 0)\| d s \leq\left\|S_{\beta}(t) \varphi(0)-\varphi(0)\right\| \\
& \quad+M^{*} \varrho \int_{0}^{\tau} \sum_{i}^{l} L_{\varrho i}(s) d s+M^{*} \Re \tau \leq \xi
\end{align*}
$$

Therefore, $\mathbb{Q}$ maps $D_{\xi}$ into $D_{\xi}$.

Let $u, v \in D_{\xi}$ and $t \in[0, \tau]$; we have

$$
\begin{aligned}
& \|(Q u)(t)-(Q) v)(t) \| \leq \int_{0}^{t} S_{\beta}(t-s) \\
& \quad \cdot \| f\left(s, u\left(\tau_{1}(s)\right), \ldots, u\left(\tau_{l}(s)\right)\right) \\
& \quad-f\left(s, v\left(\tau_{1}(s)\right), \ldots, v\left(\tau_{l}(s)\right)\right) \| d s \\
& \quad \leq M^{*} \int_{0}^{t} \sum_{i}^{l} L_{\rho i}(s)\left\|u\left(\tau_{i}(s)\right)-v\left(\tau_{i}(s)\right)\right\| d s \\
& \quad \leq \int_{0}^{t} L(s) d s\|u-v\|_{C([-r, \tau] ; E)} \\
& \quad \leq\left(\int_{0}^{t}|L(s)-\phi(s)| d s+\int_{0}^{t}|\phi(s)| d s\right) \| u \\
& \quad-v\left\|_{C([-r, \tau] ; E)} \leq(\varepsilon+N t)\right\| u-v \|_{C\left(\left[-r, T_{0}\right] ; E\right)} \\
& \quad=\left(C_{1}^{0} \varepsilon^{1}+C_{1}^{1} \frac{(N t)^{1}}{1!}\right)\|u-v\|_{C([-r, \tau] ; E)}
\end{aligned}
$$

where $L(s)=M^{*} \sum_{i}^{l} L_{\varrho i}(s)$. The following proof of the remainder is similar to the proof of Theorem 6. Therefore, for fixed constant $\lambda>0$, there exists a positive integer $n_{0}$ such that, for any $u, v \in D_{\xi}$, we have

$$
\begin{align*}
\left\|Q^{n} u-Q^{n} v\right\|_{C([-r, \tau] ; E)} \leq \frac{1}{n^{\lambda+1}}\|u-v\|_{C([-r, \tau] ; E)} &  \tag{35}\\
& \forall n>n_{0} .
\end{align*}
$$

By general Banach contraction mapping principle, for operator $\mathbb{Q}$ there exists a unique fixed point $u \in C([-r, \tau] ; E)$, which means that $u \in C([-r, \tau] ; E)$ is the unique mild solution of problem (12). That is, problem (3) has a unique mild solution.

## 4. An Application

Using the main results of this paper, we can solve the following time fractional partial differential equation with delay:

$$
\begin{align*}
{ }^{c} D_{t}^{\beta} u(x, t) & =\frac{\partial^{2}}{\partial x^{2}} u(x, t)+J_{t}^{1-\beta} \frac{t}{\left(1+t^{2}\right)} u(x, \sin t), \\
& t \in\left[0, T_{0}\right], x \in \Omega,  \tag{36}\\
u(x, \theta) & =\varphi(x, \theta), \quad \theta \in[-r, 0], x \in \Omega,
\end{align*}
$$

where $0<\beta \leq 1$. Similar to Section 2 , we can rewrite the time fractional partial differential equation (36) as the following abstract fractional evolution equation:

$$
\begin{align*}
{ }^{c} D_{t}^{\beta} u(t) & =A u(t)+J_{t}^{1-\beta} f(t, u(\tau(t))), \quad t \in\left[0, T_{0}\right]  \tag{37}\\
u(t) & =\varphi(t), \quad t \in[-r, 0] .
\end{align*}
$$

Therefore, all the conditions of Theorems 6 and 8 are satisfied; for problem (36), there exists a unique mild solution.

## 5. Conclusion

This paper considers the existence and uniqueness of the mild solutions for a class of nonlinear fractional partial differential equations with delay by general Banach contraction mapping principle. We know that the Banach contraction mapping principle needs the special conditions to ensure the contraction constant $0<k<1$. In this paper, we successfully overcome this condition. We need not extra conditions to ensure the contraction constant $0<k<1$. Therefore, under some general conditions, we obtain the main results of this paper. Our results generalize and improve many classical results [18-20].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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# Research Article 

# Some Remarks on Estimate of Mittag-Leffler Function 

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#### Abstract

The estimate of Mittag-Leffler function has been widely applied in the dynamic analysis of fractional-order systems in some recently published papers. In this paper, we show that the estimate for Mittag-Leffler function is not correct. First, we point out the mistakes made in the estimation process of Mittag-Leffler function and provide a counterexample. Then, we propose some sufficient conditions to guarantee that part of the estimate for Mittag-Leffler function is correct. Meanwhile, numerical examples are given to illustrate the validity of the two newly established estimates.


## 1. Introduction

Fractional calculus can date back to the seventeenth century, and now it has attracted considerable research interests due to its widespread applications in many fields. There are mainly two types of methods in the dynamic analysis of fractionalorder nonlinear systems, that is, Lyapunov function based method and estimation based method. When estimation based method is employed, the solution of the fractionalorder system being studied is usually expressed in terms of the Mittag-Leffler function. Obviously, the correctness of the estimate of Mittag-Leffler function is crucial to the whole estimation process and plays an important role if the estimation based method is adopted. Recently, estimation based method has been widely applied to the study of finite-time stability and synchronization of fractionalorder memristor-based neural networks [1-7], stability and stabilization of nonlinear fractional-order systems [8-13], finite-time stability of fractional-order neural networks [14, 15], synchronization of fractional-order chaotic systems [16], consensus analysis of fractional-order multiagent systems [17-19], etc. The estimate on Mittag-Leffler function was first proposed in [20]. The definition of Mittag-Leffler function and the estimate of Mittag-Leffler function can be described by Definition 1 and Lemma 2, respectively, as follows.

Definition 1 (see [21]). The Mittag-Leffler function with one parameter is defined as

$$
\begin{equation*}
E_{\alpha}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(k \alpha+1)}, \tag{1}
\end{equation*}
$$

where $\alpha>0$ and $z \in C$.
The Mittag-Leffler function with two parameters is defined as

$$
\begin{equation*}
E_{\alpha, \beta}(z)=\sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma(k \alpha+\beta)} \tag{2}
\end{equation*}
$$

where $\alpha>0$ and $\beta>0$. When $\beta=1$, one has $E_{\alpha, 1}(z)=E_{\alpha}(z)$, and when $\alpha=1$ and $\beta=1$, one further has $E_{1,1}(z)=e^{z}$.

Lemma 2 (see [20]). For Mittag-Leffler function, the following properties hold.
(i) There exist constants $M_{1}, M_{2} \geq 1$ such that, for any $0<$ $\alpha<1$,

$$
\begin{align*}
& \left\|E_{\alpha, 1}\left(A t^{\alpha}\right)\right\| \leq M_{1}\left\|e^{A t}\right\|,  \tag{3}\\
& \left\|E_{\alpha, \alpha}\left(A t^{\alpha}\right)\right\| \leq M_{2}\left\|e^{A t}\right\|, \tag{4}
\end{align*}
$$

where $A$ denotes a matrix and $\|\cdot\|$ denotes any vector or induced matrix norm.
(ii) If $\alpha \geq 1$, then for $\beta=1,2, \alpha$

$$
\begin{equation*}
\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq\left\|e^{A t^{\alpha}}\right\| \tag{5}
\end{equation*}
$$

If $A$ is a diagonal stability matrix, then there exists a constant $N>0$ such that for $t \geq 0$

$$
\begin{align*}
& \left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq N e^{-\omega t}, \quad 0<\alpha<1  \tag{6}\\
& \left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq e^{-\omega t}, \quad 1 \leq \alpha<2 \tag{7}
\end{align*}
$$

where $-\omega(\omega>0)$ is the largest eigenvalue of the diagonal matrix $A$.

However, we have to point out that Lemma 2 is incorrect. In [20], inequalities (3) and (4) are proved as follows:

$$
\begin{align*}
& \left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|=\left\|\sum_{k=0}^{\infty} \frac{\left(A t^{\alpha}\right)^{k}}{\Gamma(k \alpha+\beta)}\right\| \\
& \quad=\left\|\sum_{k=0}^{\infty} \frac{k!}{k^{k(1-\alpha)} \Gamma(k \alpha+\beta)} \frac{(A t)^{k}}{k!}\right\| \\
& \quad \leq\left\|\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{t^{k(1-\alpha)} \Gamma(k \alpha+\beta)}\right) \sum_{k=0}^{\infty} \frac{(A t)^{k}}{k!}\right\|  \tag{8}\\
& \quad=\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{t^{k(1-\alpha)} \Gamma(k \alpha+\beta)}\right)\left\|\sum_{k=0}^{\infty} \frac{(A t)^{k}}{k!}\right\| \\
& \quad \leq \sup _{\tau \in(1, \infty)}\left(\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{\tau^{k(1-\alpha)} \Gamma(k \alpha+\beta)}\right)\right)\left\|e^{A t}\right\| .
\end{align*}
$$

Actually, there are two problems in the above-mentioned proof. First,

$$
\begin{align*}
& \left\|\sum_{k=0}^{\infty} \frac{k!}{t^{k(1-\alpha)} \Gamma(k \alpha+\beta)} \frac{(A t)^{k}}{k!}\right\| \\
& \quad \leq\left\|\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{t^{k(1-\alpha)} \Gamma(k \alpha+\beta)}\right) \sum_{k=0}^{\infty} \frac{(A t)^{k}}{k!}\right\| \tag{9}
\end{align*}
$$

does not necessarily hold. In fact, (9) holds if all the elements $a_{i j}(1 \leq i, j \leq n)$ of matrix $A$ are nonnegative, because matrix norms, such as 1 -norm, 2 -norm, and $\infty$-norm, have the property of weak monotony. In other words, (9) may not hold when there exist negative elements in $A$ or $A^{k}$. Second, $\sup _{k=0,1,2, \ldots, \infty}\left(k!/ t^{k(1-\alpha)} \Gamma(k \alpha+\beta)\right)$ does not exist when $0<$ $\alpha<1$ and $\beta=1, \alpha$. To confirm this point, let $N(k)=$ $k!/ t^{k(1-\alpha)} \Gamma(k \alpha+\beta)$; when $t=2 s$ and $\alpha=0.9$, the behavior of $N(k)$ with $\beta=1$ and $\beta=\alpha$ is shown in Figures 1(a) and 1(b), respectively. It can be obviously observed from Figure 1 that $N(k)$ goes to infinity as $k$ goes to infinity, so $N(k)$ has no supremum when $\beta=1$ or $\beta=\alpha$ as $k$ goes to infinity for a fixed value of $t$.

Next, a counterexample is presented to show that $E_{\alpha, \beta}\left(A t^{\alpha}\right)$ does not satisfy inequality (3) or (4). Assume that
$A=\left[\begin{array}{ccc}1 & 2 & 2 \\ 2 & 3 & 4 \\ 7 & 8 & 1\end{array}\right]$; by simple calculation, it has three different eigenvalues, i.e., $\lambda_{1}=9.6135, \lambda_{2}=-0.3146$, and $\lambda_{3}=-4.2990$. Hence, a nonsingular matrix $P=$ $\left[\begin{array}{ccc}-0.3085 & -0.7235 & -0.1680 \\ -0.5589 \\ -0.7698 & 0.6641 & -0.1885 \\ -0.4879\end{array}\right]$ can be determined to make

$$
\begin{equation*}
P^{-1} A P=\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \lambda_{3}\right) \tag{10}
\end{equation*}
$$

and

$$
\begin{align*}
& E_{\alpha, \beta}\left(A t^{\alpha}\right)=\sum_{k=0}^{\infty} \frac{\left(P \Lambda t^{\alpha} P^{-1}\right)^{k}}{\Gamma(k \alpha+\beta)}=P \sum_{k=0}^{\infty} \frac{\left(\Lambda t^{\alpha}\right)^{k}}{\Gamma(k \alpha+\beta)} P^{-1} \\
& \quad=P\left[\begin{array}{ccc}
E_{\alpha, \beta}\left(\lambda_{1} t^{\alpha}\right) & 0 & 0 \\
0 & E_{\alpha, \beta}\left(\lambda_{2} t^{\alpha}\right) & 0 \\
0 & 0 & E_{\alpha, \beta}\left(\lambda_{3} t^{\alpha}\right)
\end{array}\right] P^{-1} . \tag{11}
\end{align*}
$$

For each fixed value of $t, E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right), i=1,2,3$ can be calculated by means of the OPC algorithm [22]. Thus, $E_{\alpha, \beta}\left(A t^{\alpha}\right)$ can be calculated through (11). When $\alpha=0.9$, the behaviors of $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|_{1} /\left\|e^{A t}\right\|_{1}$ with $\beta=1$ and $\beta=\alpha$ are displayed in Figures 2(a) and 2(b), respectively. It is obvious that $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|_{1} /\left\|e^{A t}\right\|_{1}$ goes to infinity as $t$ goes to infinity, so inequalities (3) and (4) are incorrect.

The conclusion on inequality (6) is straightforward if inequalities (3) and (4) are correct. Because inequalities (3) and (4) are not correct, inequality (6) is not correct either. For example, let $A=\operatorname{diag}(-1,-5,-8)$; the behavior of $\left\|E_{0.9, \beta}\left(A t^{0.9}\right)\right\|_{1} / e^{-t}$ for $\beta=1$ and $\beta=\alpha=0.9$ is shown in Figures 3(a) and 3(b), respectively. It is clear that $\left\|E_{0.9, \beta}\left(A t^{0.9}\right)\right\|_{1} / e^{-t}$ goes to infinity as $t$ goes to infinity for $\beta=1$ and $\beta=\alpha$, so inequality (6) does not hold.

Next, we consider the case that $\alpha \geq 1$. In [20], inequality (5) is proved as follows:

$$
\begin{align*}
\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| & =\left\|\sum_{k=0}^{\infty} \frac{\left(A t^{\alpha}\right)^{k}}{\Gamma(k \alpha+\beta)}\right\| \\
& =\left\|\sum_{k=0}^{\infty} \frac{k!}{\Gamma(k \alpha+\beta)} \frac{\left(A t^{\alpha}\right)^{k}}{k!}\right\| \\
& \leq\left\|\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{\Gamma(k \alpha+\beta)}\right) \sum_{k=0}^{\infty} \frac{\left(A t^{\alpha}\right)^{k}}{k!}\right\|  \tag{12}\\
& =\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{\Gamma(k \alpha+\beta)}\right)\left\|e^{A t^{\alpha}}\right\| \\
& \leq \sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{\Gamma(k+1)}\right)\left\|e^{A t^{\alpha}}\right\| \\
& =\left\|e^{A t^{\alpha}}\right\| .
\end{align*}
$$

With the same argument as stated for (8),

$$
\left\|\sum_{k=0}^{\infty} \frac{k!}{\Gamma(k \alpha+\beta)} \frac{\left(A t^{\alpha}\right)^{k}}{k!}\right\| \text {. }\left\|\sup _{k=0,1,2, \ldots, \infty}\left(\frac{k!}{\Gamma(k \alpha+\beta)}\right) \sum_{k=0}^{\infty} \frac{\left(A t^{\alpha}\right)^{k}}{k!}\right\| .
$$



Figure 1: The behavior of $N(k)$ with (a) $\beta=1$ and (b) $\beta=\alpha$.
does not necessarily hold and it is true if all the elements $a_{i j}(1 \leq i, j \leq n)$ of matrix $A$ are nonnegative. The other problem with (12) is that $\sup _{k=0,1,2, \ldots, \infty}(k!/ \Gamma(k \alpha+\beta)) \leq$ $\sup _{k=0,1,2, \ldots, \infty}(k!/ \Gamma(k+1))$ does not hold for $1<\alpha<2$ and $\beta=$ $\alpha$ as $\sup _{k=0,1,2, \ldots, \infty}(k!/ \Gamma(k \alpha+\beta)) \geq 1 / \Gamma(\beta)=1 / \Gamma(\alpha)>1$. For example, when $\alpha=\beta=1.5, \sup _{k=0,1,2, \ldots, \infty}(k!/ \Gamma(k \alpha+\beta)) \geq$ $1 / \Gamma(1.5) \approx 1.1284>1$. The behavior of Gamma function and its derivative is depicted in Figures 4(a) and 4(b), respectively. From Figure 4, it is clear that $0.8<\Gamma(z)<1$ for $1<z<2$. Hence, we can conclude that $\sup _{k=0,1,2, \ldots, \infty}(k!/ \Gamma(k \alpha+\beta)) \geq$ $1 / \Gamma(\beta)>1$ for $\beta=\alpha$ when $1<\alpha<2$.

From the above discussions, we can infer that inequality (5) holds only under some particular conditions; that is, we have to impose some restrictions on matrix $A$ and $\alpha, \beta$.

Conclusion 3. Suppose all the elements $a_{i j}(1 \leq i, j \leq n)$ of matrix $A$ are nonnegative; if $1 \leq \alpha<2$, then for $\beta=1,2$, inequality (5) holds.

Conclusion 4. Suppose all the elements $a_{i j}(1 \leq i, j \leq n)$ of matrix $A$ are nonnegative; if $\alpha \geq 2$, then for $\beta=1,2, \alpha$, inequality (5) holds.

Now, a counterexample is presented to show that if the conditions in Conclusion 3 or Conclusion 4 are not satisfied, inequality (5) may not be correct. For instance, let $A=\operatorname{diag}(-1,-5,-8)$; then the behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ for $\beta=1, \beta=2$, and $\beta=1.5$ is shown in Figures 5(a)-5(c), respectively. It is obvious that $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ goes to infinity as $t$ goes to infinity, so inequality (5) does not hold.

Similarly, (7) is incorrect because (5) is incorrect. The behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} / e^{-t}$ for $\beta=1, \beta=2$, and $\beta=$ 1.5 is shown in Figures 6(a)-6(c), respectively, which is in contradiction to inequality (7).

## 2. Main Results

In this section, some sufficient conditions are derived to guarantee that $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|$ can be bounded by $F\left\|e^{A t^{\alpha}}\right\|$ for some $F>0$, which can be formulated by the following two theorems.

Theorem 5. If matrix $A$ is diagonalizable, and the largest real part of eigenvalues $\lambda_{i}(i=1,2, \ldots, n)$ of $A$ is positive, then for $1<\alpha<2$ and anyone of the following two conditions:
(i) $\beta>0$,
(ii) $\beta=0$ and $A$ has no zero eigenvalue,

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|}{\left\|e^{A t^{\alpha}}\right\|}=0 \tag{14}
\end{equation*}
$$

and further there exists a positive constant $F$ such that for $t \geq 0$

$$
\begin{equation*}
\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq F\left\|e^{A t^{\alpha}}\right\| \tag{15}
\end{equation*}
$$

where $\|\cdot\|$ denotes 1-norm, 2-norm, or $\infty$-norm of a matrix.
To prove Theorem 5, another two lemmas are presented as follows, which will be used later.

Lemma 6 (see [21]). If $\alpha<2, \beta$ is an arbitrary real number, $\mu$ satisfies $\pi \alpha / 2<\mu<\min \{\pi, \pi \alpha\}$, and $C_{1}$ and $C_{2}$ are real constants, then

$$
\begin{align*}
\left|E_{\alpha, \beta}(z)\right| \leq & C_{1}(1+|z|)^{(1-\beta) / \alpha} \exp \left(\operatorname{Re}\left(z^{1 / \alpha}\right)\right) \\
& +\frac{C_{2}}{1+|z|}, \tag{16}
\end{align*}
$$

where $|\arg (z)| \leq \mu,|z| \geq 0$.

(a)

(b)

Figure 2: The behavior of $\left\|E_{0.9, \beta}\left(A t^{0.9}\right)\right\|_{1} /\left\|e^{A t}\right\|_{1}$ with (a) $\beta=1$ and (b) $\beta=\alpha$.


Figure 3: The behavior of $\left\|E_{0.9, \beta}\left(A t^{0.9}\right)\right\|_{1} / e^{-t}$ with (a) $\beta=1$ and (b) $\beta=\alpha$.


Figure 4: The behavior of (a) $\Gamma(z)$ and (b) $d \Gamma(z) / d z$.


Figure 5: The behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ for (a) $\beta=1$, (b) $\beta=2$, and (c) $\beta=1.5$.


Figure 6: The behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} / e^{-t}$ for (a) $\beta=1$, (b) $\beta=2$, and (c) $\beta=1.5$.

Lemma 7 (see [21]). If $\alpha<2, \beta$ is an arbitrary real number, $\mu$ satisfies $\pi \alpha / 2<\mu<\min \{\pi, \pi \alpha\}$, and $C$ is a real constant, then

$$
\begin{equation*}
\left|E_{\alpha, \beta}(z)\right| \leq \frac{C}{1+|z|} \tag{17}
\end{equation*}
$$

where $\mu \leq|\arg (z)| \leq \pi,|z| \geq 0$.
Now, the proof of Theorem 5 can proceed.
Proof. Because $A$ is diagonalizable, there exists a nonsingular matrix $P$ such that $P^{-1} A P=\Lambda=\operatorname{diag}\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$. Then $E_{\alpha, \beta}\left(A t^{\alpha}\right)=P E_{\alpha, \beta}\left(\Lambda t^{\alpha}\right) P^{-1}$ and $e^{A t^{\alpha}}=P e^{\Lambda t^{\alpha}} P^{-1}$. Since

$$
\begin{align*}
\left|\lambda I-e^{A t^{\alpha}}\right| & =\left|\lambda I-P e^{\Lambda t^{\alpha}} P^{-1}\right|=\left|P \lambda I P^{-1}-P e^{\Lambda t^{\alpha}} P^{-1}\right| \\
& =\left|P\left(\lambda I-e^{\Lambda t^{\alpha}}\right) P^{-1}\right|  \tag{18}\\
& =|P|\left|\lambda I-e^{\Lambda t^{\alpha}}\right|\left|P^{-1}\right|=\left|\lambda I-e^{\Lambda t^{\alpha}}\right|
\end{align*}
$$

$e^{A t^{\alpha}}$ and $e^{\Lambda t^{\alpha}}$ are with the same characteristic polynomial and eigenvalues, so $\left\|e^{A t^{\alpha}}\right\| \geq \rho\left(e^{A t^{\alpha}}\right)=\max _{1 \leq i \leq n}\left\{\left|e^{\lambda_{i} t^{\alpha}}\right|\right\}$. Let $m=$ $\max _{1 \leq i \leq n}\left\{\operatorname{Re}\left(\lambda_{i}\right)\right\}>0$; then $\left\|e^{A t^{\alpha}}\right\| \geq e^{m t^{\alpha}}$.

Let $\theta_{i}$ be the principal value of the argument of $\lambda_{i}$. According to the magnitude of the principal value $\theta_{i}$ of the argument of $\lambda_{i}$, the cases where $\left|\theta_{i}\right| \leq \pi \alpha / 2$ and $\pi \alpha / 2<\left|\theta_{i}\right| \leq$ $\pi$ are considered, separately.

Case 1. If $\left|\theta_{i}\right| \leq \pi \alpha / 2$, it follows from Lemma 6 that

$$
\begin{align*}
& \left|E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)\right| \\
& \quad \leq C_{1}\left(1+\left|\lambda_{i} t^{\alpha}\right|\right)^{(1-\beta) / \alpha} e^{\operatorname{Re}\left(\left(\lambda_{i} t^{\alpha}\right)^{1 / \alpha}\right)}+\frac{C_{2}}{1+\left|\lambda_{i} t^{\alpha}\right|} \\
& = \\
& =C_{1}\left(1+\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\operatorname{Re}\left(\lambda_{i}^{1 / \alpha} t\right)}+\frac{C_{2}}{1+\left|\lambda_{i}\right| t^{\alpha}}  \tag{19}\\
& = \\
& \quad C_{1}\left(1+\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left|\lambda_{i}\right|^{1 / \alpha} \cos \left(\left(\theta_{i}+2 k \pi\right) / \alpha\right) t} \\
& \quad+\frac{C_{2}}{1+\left|\lambda_{i}\right| t^{\alpha}} \\
& \leq \\
& \quad C_{1}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha} t} \\
& \quad+\frac{C_{2}}{1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}},
\end{align*}
$$

where $k=0,1,2, \ldots, q-1 . q$ is the numerator of $\alpha$, where $\alpha=q / p,(p, q)=1$.

Case 2. If $\pi \alpha / 2<\left|\theta_{i}\right| \leq \pi$, it follows from Lemma 7 that

$$
\begin{align*}
\left|E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)\right| & \leq \frac{C}{1+\left|\lambda_{i} t^{\alpha}\right|}=\frac{C}{1+\left|\lambda_{i}\right| t^{\alpha}} \\
& \leq \frac{C}{1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}} \tag{20}
\end{align*}
$$

Then, it follows from (19) and (20) that

$$
\begin{align*}
& \left|E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)\right| \\
& \quad \leq C_{1}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha} t}  \tag{21}\\
& \quad+\frac{C_{\max }}{1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}}
\end{align*}
$$

where $C_{\max }=\max \left\{C_{2}, C\right\}$.
Thus, we have

$$
\begin{aligned}
& \lim _{t \rightarrow+\infty} \frac{\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|}{\left\|e^{A t^{\alpha}}\right\|} \leq \lim _{t \rightarrow+\infty} \frac{\left\|P E_{\alpha, \beta}\left(\Lambda t^{\alpha}\right) P^{-1}\right\|}{e^{m t^{\alpha}}} \\
& \quad \leq \lim _{t \rightarrow+\infty} \frac{\|P\|\left\|E_{\alpha, \beta}\left(\Lambda t^{\alpha}\right)\right\|\left\|P^{-1}\right\|}{e^{m t^{\alpha}}} \\
& \quad=\lim _{t \rightarrow+\infty} \frac{\|P\| \max _{1 \leq i \leq n}\left\{\left|E_{\alpha, \beta}\left(\lambda \lambda_{i} t^{\alpha}\right)\right|\right\}\left\|P^{-1}\right\|}{e^{m t^{\alpha}}}
\end{aligned}
$$

$$
\begin{align*}
& \leq \lim _{t \rightarrow+\infty} C_{1}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} \\
& \cdot e^{\left(\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t} \times\|P\|\left\|P^{-1}\right\| \\
& +\lim _{t \rightarrow+\infty} \frac{C_{\max } e^{-m t^{\alpha}}}{1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}}\|P\|\left\|P^{-1}\right\| . \tag{22}
\end{align*}
$$

When any one of the following two conditions is satisfied:
(1) $\beta \geq 1$,
(2) $0<\beta<1$ and $\exists i \in\{1,2, \ldots, n\}$ such that $\lambda_{i}=0$,
we have

$$
\begin{align*}
& \lim _{t \rightarrow+\infty}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t}  \tag{23}\\
& \quad=0
\end{align*}
$$

When $0 \leq \beta<1$ and $\lambda_{i} \neq 0(i=1,2, \ldots, n)$, the following equality can be derived by L'Hospital's rule,

$$
\begin{equation*}
\lim _{t \rightarrow+\infty}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t}=\lim _{t \rightarrow+\infty} \frac{(1-\beta) \min _{1 \leq i \leq n}\left|\lambda_{i}\right|\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-(\beta+\alpha)) / \alpha} t^{\alpha-1}}{e^{\left(m t^{\alpha-1}-\left(\max _{1 \leq \leq \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}\right) t}\left(\alpha m t^{\alpha-1}-\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}\right)} \tag{24}
\end{equation*}
$$

As

$$
\begin{align*}
& \lim _{t \rightarrow+\infty}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-(\beta+\alpha)) / \alpha} t^{\alpha-1} \\
& \quad=\lim _{t \rightarrow+\infty} \frac{t^{\alpha-1}}{\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(\beta+\alpha-1) / \alpha}} \\
& \quad=\lim _{t \rightarrow+\infty} \frac{t^{-\beta}}{\left(t^{-\alpha}+\min _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{(\beta+\alpha-1) / \alpha}}  \tag{25}\\
& \quad= \begin{cases}0, & \beta \neq 0, \\
\frac{1}{\left(\min _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{(\alpha-1) / \alpha}}, & \beta=0,\end{cases}
\end{align*}
$$

and

$$
\begin{align*}
& \lim _{t \rightarrow+\infty} e^{\left(m t^{\alpha-1}-\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}\right) t} \\
& \quad \cdot\left(\alpha m t^{\alpha-1}-\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}\right)=+\infty \tag{26}
\end{align*}
$$

it can be obtained from (24) that $\lim _{t \rightarrow+\infty}(1+$ $\left.\left.\left.\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\left(\max _{1 \leq i \leq n} n\right.\right.}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t=0$ when $0 \leq \beta<$ 1 and $\lambda_{i} \neq 0(i=1,2, \ldots, n)$. To summarize,

$$
\begin{align*}
& \lim _{t \rightarrow+\infty}\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t}  \tag{27}\\
& \quad=0
\end{align*}
$$

if conditions (i) or (ii) in Theorem 5 are satisfied. It is obvious that $\lim _{t \rightarrow+\infty}\left(C_{\max } e^{-m t^{\alpha}} /\left(1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)\right)=0$. According to (22), we have $\lim _{t \rightarrow+\infty}\left(\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| /\left\|e^{A t^{\alpha}}\right\|\right)=0$. Due to the continuity of matrix norms, there exists a positive constant $F$ such that $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| /\left\|e^{A t^{\alpha}}\right\| \leq F$, which implies that inequality (15) holds. This completes the proof.

Now, an example is presented to verify the correctness of the newly established Theorem 5. Assume that $A=\left[\begin{array}{ccc}1 & 2 & 2 \\ 2 & 3 & 4 \\ 7 & 8 & 1\end{array}\right]$; it is diagonalizable and the eigenvalues are $\lambda_{1}=9.6135, \lambda_{2}=$ $-0.3146, \lambda_{3}=-4.2990$, respectively. Hence, the largest real part of the eigenvalues is $m=9.6135>0$; then the behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ for $\beta=0, \beta=0.3, \beta=0.8, \beta=1$, $\beta=2$, and $\beta=1.5$ is shown in Figures 7(a)-7(f), respectively. It can be seen from Figure 7 that $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ converges to zero, which indicates the validity of Theorem 5.

Remark 8. Note that the condition that $1<\alpha<2$ is needed in Theorem 5. If $0<\alpha<1$, then $(1+$ $\left.\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}\right)^{(1-\beta) / \alpha} e^{\left(\left(\max _{1 \leq i \leq n}\left|\lambda_{i}\right|\right)^{1 / \alpha}-m t^{\alpha-1}\right) t}$ may go to infinity as $t$ goes to infinity, so the process of the proof cannot be carried out and the conclusion in Theorem 5 may not be obtained. To get the similar estimate of Mittag-Leffler function for $0<\alpha<1$, an extra restriction has to be imposed on the eigenvalues of matrix $A$, which is given in the following Theorem 9.


Figure 7: The behavior of $\left\|E_{1.5, \beta}\left(A t^{1.5}\right)\right\|_{1} /\left\|e^{A t^{1.5}}\right\|_{1}$ for (a) $\beta=0$, (b) $\beta=0.3$, (c) $\beta=0.8$, (d) $\beta=1$, (e) $\beta=2$, and (f) $\beta=1.5$.

Theorem 9. If matrix $A$ is diagonalizable and satisfies the following two conditions:
(i) the largest real part of eigenvalues $\lambda_{i}(i=1,2, \ldots, n)$ of $A$ is positive,
(ii) the principal value $\theta_{i}$ of the argument of $\lambda_{i}$ satisfies

$$
\pi \alpha / 2<\left|\theta_{i}\right| \leq \pi, \forall i=1,2, \ldots, n,
$$

then for $0<\alpha<1$ and $\beta \in R$,

$$
\begin{equation*}
\lim _{t \rightarrow+\infty} \frac{\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|}{\left\|e^{A t^{\alpha}}\right\|}=0 \tag{28}
\end{equation*}
$$

and further there exists a positive constant $G$ such that for $t \geq 0$

$$
\begin{equation*}
\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq G\left\|e^{A t^{\alpha}}\right\|, \tag{29}
\end{equation*}
$$

where $\|\cdot\|$ denotes 1-norm, 2-norm, or $\infty$-norm of a matrix.
Proof. According to condition (ii) in Theorem 9, (20) holds for each $\lambda_{i}, i=1,2, \ldots, n$. Thus we have

$$
\begin{aligned}
& \lim _{t \rightarrow+\infty} \frac{\left\|E_{\alpha, \beta}\left(A A^{\alpha}\right)\right\|}{\left\|e^{A t^{\alpha}}\right\|} \leq \lim _{t \rightarrow+\infty} \frac{\left\|P E_{\alpha, \beta}\left(\Lambda t^{\alpha}\right) P^{-1}\right\|}{e^{m t^{\alpha}}} \\
& \quad \leq \lim _{t \rightarrow+\infty} \frac{\|P\|\left\|E_{\alpha, \beta}\left(\Lambda t^{\alpha}\right)\right\|\left\|P^{-1}\right\|}{e^{m t^{\alpha}}}
\end{aligned}
$$

$$
\begin{align*}
& =\lim _{t \rightarrow+\infty} \frac{\|P\| \max _{1 \leq i \leq n}\left\{\left|E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)\right|\right\}\left\|P^{-1}\right\|}{e^{m t^{\alpha}}} \\
& \leq \lim _{t \rightarrow+\infty} \frac{C e^{-m t^{\alpha}}}{1+\min _{1 \leq i \leq n}\left|\lambda_{i}\right| t^{\alpha}}\|P\|\left\|P^{-1}\right\|=0 . \tag{30}
\end{align*}
$$

Similarly, we can prove that there exists a positive constant $G$ such that $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| /\left\|e^{A t^{\alpha}}\right\| \leq G$, which implies that inequality (15) holds. This completes the proof.

Now, an example is presented to verify the correctness of Theorem 9. Assume that $A=\left[\begin{array}{ccc}-3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & -1 & 1\end{array}\right]$; it is diagonalizable and the eigenvalues are $\lambda_{1}=-3, \lambda_{2}=1-j \sqrt{2}, \lambda_{3}=$ $1+j \sqrt{2}$, respectively, where $j=\sqrt{-1}$. It is obvious that the largest real part of the eigenvalues is $m=1>0$. Choose $\alpha=0.5$; then $\left|\theta_{1}\right|=\pi>\pi \alpha / 2=0.25 \pi,\left|\theta_{2}\right|=\left|\theta_{3}\right|=$ $0.9553 \in(0.25 \pi, \pi]$. Hence the two conditions in Theorem 9 are all satisfied, and Figures 8(a)-8(c) depict the behavior of $\left\|E_{0.5, \beta}\left(A t^{0.5}\right)\right\|_{1} /\left\|e^{A t^{0.5}}\right\|_{1}$ for $\beta=0.5, \beta=1$, and $\beta=3$, respectively, which shows that $\left\|E_{0.5, \beta}\left(A t^{0.5}\right)\right\|_{1} /\left\|e^{A t^{0.5}}\right\|_{1}$ converges to zero and indicates the validity of Theorem 9.

Remark 10. Note that the condition that matrix $A$ is diagonalizable is needed in Theorems 5 and 9; otherwise, there exists


Figure 8: The behavior of $\left\|E_{0.5, \beta}\left(A t^{0.5}\right)\right\|_{1} /\left\|e^{A t}{ }^{0.5}\right\|_{1}$ for (a) $\beta=0.5$, (b) $\beta=1$, and (c) $\beta=3$.
a nonsingular matrix $P$ such that $P^{-1} A P=J=\operatorname{diag}\left(J_{1}\right.$, $\left.J_{2}, \ldots, J_{s}\right)$, where $J_{i}=\operatorname{diag}\left(J_{i 1}, J_{i 2}, \ldots, J_{i r_{i}}\right)$ and

$$
J_{i k}=\left[\begin{array}{cccc}
\lambda_{i} & 1 & &  \tag{31}\\
& \lambda_{i} & \ddots & \\
& & \ddots & 1 \\
& & & \lambda_{i}
\end{array}\right]_{u_{i k} \times u_{i k}}
$$

One can obtain that $E_{\alpha, \beta}\left(A t^{\alpha}\right)=P E_{\alpha, \beta}\left(J t^{\alpha}\right) P^{-1}$ and $e^{A t^{\alpha}}=$ $P e^{J t^{\alpha}} P^{-1}$. In the proof of Theorems 5 and 9, the lower bound of $\left\|e^{A t^{\alpha}}\right\|=\left\|P e^{J t^{\alpha}} P^{-1}\right\|$ and the upper bound of $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|=$ $\left\|P E_{\alpha, \beta}\left(J t^{\alpha}\right) P^{-1}\right\|$ are needed. When A is not diagnosable, $\left\|e^{A t^{\alpha}}\right\| \geq e^{m t^{\alpha}}$ still holds but the upper bound of $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|=$ $\left\|P E_{\alpha, \beta}\left(J t^{\alpha}\right) P^{-1}\right\|$ is difficult to obtain. The difficulties are described as follows.

For the Jordan block (31), we have

$$
\left[\begin{array}{cccc}
E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right) \frac{d E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)} & \frac{1}{2!} \frac{d^{2} E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)^{2}} & \cdots & \frac{1}{\left(u_{i k}-1\right)!} \frac{d^{u_{i k}-1} E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)^{u_{i}-1}}  \tag{32}\\
E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right) & \frac{d E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)} & \cdots & \frac{1}{\left(u_{i k}-2\right)!} \frac{d^{u_{i k}-2} E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)^{u_{i k}-2}} \\
& E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right) & \cdots & \frac{1}{\left(u_{i k}-3\right)!} \frac{d^{u_{i k}-3} E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)}{d\left(\lambda_{i} t^{\alpha}\right)^{u_{i k}-3}} \\
& & \ddots & \vdots \\
& & E_{\alpha, \beta}\left(\lambda_{i} t^{\alpha}\right)
\end{array}\right]_{u_{i k} \times u_{i k}} Q_{i k}^{-1},
$$

where $Q_{i k}=\left[\begin{array}{cccccc}1 & 1 & & & & \\ & 1 / t^{\alpha} & 1 / t^{\alpha} & \ldots & & 1 \\ & & 1 / t^{2 \alpha} & \ldots & 1 / t^{\alpha} \\ & & & \ddots & 1 / t^{\alpha} \\ & & & & \\ & & & & 1 / t^{\left.u_{i k}-1\right) \alpha}\end{array}\right]$ is a nonsingular matrix such that $Q_{i k}^{-1}\left(J_{i k} t^{\alpha}\right) Q_{i k}=\left[\begin{array}{ccccc}\lambda_{i} t^{\alpha} & 1 & & \\ & \lambda_{i} t^{\alpha} & \ddots & \\ & & \ddots & \\ & & & \lambda_{i} t^{\alpha}\end{array}\right]_{u_{i k} \times u_{i k}}$. As $J t^{\alpha}$ is composed of $J_{i k} t^{\alpha}$, it is difficult calculate $\left\|E_{\alpha, \beta}\left(J t^{\alpha}\right)\right\|$ via (32), so the upper bound of $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\|$ is difficult to obtain and $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| /\left\|e^{A t^{\alpha}}\right\|$ is difficult to estimate.

To the best of our knowledge, the estimate of MittagLeffler function by the exponential function is still an open problem due to the complexity of Mittag-Leffler function and deserves further research.

## 3. Conclusion

In this paper, several counterexamples are presented to numerically show that the estimate for Mittag-Leffler function used in some recently published papers is not completely correct and the mistakes made in the estimation process are mainly due to the misuse of the properties of matrix norms. Besides, some sufficient conditions are developed to guarantee that the estimate $\left\|E_{\alpha, \beta}\left(A t^{\alpha}\right)\right\| \leq F\left\|e^{A t^{\alpha}}\right\|$ holds for some $F>0$ and numerical examples are given to verify the correctness of the newly developed results.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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## Research Article

# Differential Harnack Estimates for a Semilinear Parabolic System 

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In this paper, we prove differential Harnack inequalities for positive solutions of a semilinear parabolic system on hyperbolic space. We use the inequalities to construct classical Harnack estimates by integrating along space-time.

## 1. Introduction

In this paper, we study the following problem:

$$
\begin{align*}
f_{t} & =\Delta f+\mathrm{e}^{\mu t} g^{p}, \quad \mathbb{R}^{n} \times(0,+\infty), \\
g_{t} & =\Delta g+\mathrm{e}^{\nu t} f^{q}, \quad \mathbb{R}^{n} \times(0,+\infty), \\
f(x, 0) & =f_{0},  \tag{1}\\
g(x, 0) & =g_{0},
\end{align*}
$$

$$
\mathbb{R}^{n}
$$

where $p, q, \mu, \nu$ are positive constants.
P. Li and S.-T.Yau in [1] were first pioneers to the study of differential Harnack inequalities which were brought to general parabolic geometric flows by R. Hamilton (see [2]). Using these inequalities can derive ancient solutions, bounds on gradient Ricci solitons, Holder continuity. Differential Harnack inequalities are important aspects of properties of partial differential equations. Paper [3] described differential Harnack inequalities to the initial value problem of a semilinear parabolic equation when the semilinear term is $e^{\mu t} f^{p}, \mu>0$. There have been numerous interesting results on the properties of solutions of partial differential equations, such as existence of solutions [4-17], nonexistence and blowup of solutions [18-22], and asymptotic behaviors of solutions [23-28].

Let $(f(x, t), g(x, t))$ be positive smooth solutions to (1) and $(u, v):=(\log f, \log g)$. The main object of our study is the following Harnack quantities:

$$
\begin{align*}
& H_{1} \equiv \alpha \Delta u+\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\psi_{1}(t)+\phi_{1}(x)  \tag{2}\\
& H_{2} \equiv \alpha \Delta v+\beta_{2}|\nabla v|^{2}+c \mathrm{e}^{\imath t+q u-v}+\psi_{2}(t)+\phi_{2}(x)
\end{align*}
$$

where $\alpha, \beta_{i}, c \in \mathbb{R}, \alpha>\max \left\{\beta_{1}, \beta_{2}\right\}$ and $\psi_{i}, \phi_{i}$ will be chosen suitably $i=1,2$.

We will derive our differential Harnack estimate.
Theorem 1. Let $(f(x, t), g(x, t))$ be positive classical solutions to (1), and $(u, v):=(\log f, \log g)$. If $\alpha, \beta_{i}, k_{i}$, and $c$ satisfy

$$
\begin{align*}
& \alpha>\max \left\{\beta_{1}, \beta_{2}\right\} \geq 0 \\
& k_{1}-p k_{2} \geq 0 \\
& k_{2}-q k_{1} \geq 0 \\
& k_{i} \geq \frac{n \alpha^{2}}{2\left(\alpha-\beta_{i}\right)}>0, \\
& -\beta_{2}+\alpha p+c(1-p) \geq 0  \tag{3}\\
& -\beta_{1}+\alpha q+c(1-q) \geq 0 \\
& \frac{4}{n \alpha^{2}} \beta_{1} c+1-\frac{p^{2}\left(\alpha-\beta_{1}\right)}{-p \beta_{2}+\alpha p^{2}+c p(1-p)} \geq 0 \\
& \frac{4}{n \alpha^{2}} \beta_{2} c+1-\frac{q^{2}\left(\alpha-\beta_{2}\right)}{-q \beta_{1}+\alpha q^{2}+c q(1-q)} \geq 0,
\end{align*}
$$

then we have

$$
\begin{align*}
& H_{1} \equiv \alpha \Delta u+\beta_{1}|\nabla u|^{2}+c e^{\mu t+p v-u}+\frac{k_{1}}{t} \geq 0 \\
& H_{2} \equiv \alpha \Delta v+\beta_{2}|\nabla v|^{2}+c e^{\mu t+q u-v}+\frac{k_{2}}{t} \geq 0 \tag{4}
\end{align*}
$$

for all $t$.
The paper is organized as follows. In Section 2 we prove Theorem 1 which describes differential Harnack estimate. There are applications of Theorem 1 in Section 3.

## 2. Harnack Estimate

In this section, we shall first obtain our differential Harnack inequalities, relying on the parabolic maximum principle.

Lemma 2. Suppose $(f(x, t), g(x, t))$ are positive solutions to (1) and $(u, v):=(\log f, \log g)$ and $\left(H_{1}, H_{2}\right)$ are defined as in
(2). Assume that $\alpha, \beta_{1}, \beta_{2}$, and c satisfy

$$
\begin{align*}
& \alpha>\max \left\{\beta_{1}, \beta_{2}\right\} \geq 0 \\
& -\beta_{2}+\alpha p+c(1-p) \geq 0 \\
& -\beta_{1}+\alpha q+c(1-q) \geq 0 \\
& \frac{4}{n \alpha^{2}} \beta_{1} c+1-\frac{p^{2}\left(\alpha-\beta_{1}\right)}{-p \beta_{2}+\alpha p^{2}+c p(1-p)} \geq 0  \tag{5}\\
& \frac{4}{n \alpha^{2}} \beta_{2} c+1-\frac{q^{2}\left(\alpha-\beta_{2}\right)}{-q \beta_{1}+\alpha q^{2}+c q(1-q)} \geq 0
\end{align*}
$$

Then we have

$$
\begin{align*}
\partial_{t} H_{1} \geq & \Delta H_{1}+2 \nabla H_{1} \cdot \nabla u+h_{11} H_{1} \\
& +\mathrm{e}^{\mu t+p v-u}\left(p H_{2}-H_{1}\right)+h_{12}  \tag{6}\\
\partial_{t} H_{2} \geq & \Delta H_{2}+2 \nabla H_{2} \cdot \nabla v+h_{21} H_{2} \\
& +\mathrm{e}^{\imath t+q u-v}\left(q H_{1}-H_{2}\right)+h_{22}
\end{align*}
$$

where

$$
\begin{aligned}
h_{11} & =2\left(\alpha-\beta_{1}\right) \\
& \cdot \frac{1}{n \alpha^{2}}\left\{H_{1}-2\left(\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)\right\} \\
h_{12} & =\partial_{t} \psi_{1}-\Delta \phi_{1}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)^{2} \\
& -\frac{n \alpha^{2}\left|\nabla \phi_{1}\right|^{2}}{4 \beta_{1}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)} \\
& +e^{\mu t+p v-u}\left(c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right)\right) \\
h_{21} & =2\left(\alpha-\beta_{2}\right) \\
& \cdot \frac{1}{n \alpha^{2}}\left\{H_{2}-2\left(\beta_{2}|\nabla v|^{2}+c e^{v t+q u-v}+\psi_{2}+\phi_{2}\right)\right\}
\end{aligned}
$$

$$
\begin{align*}
h_{22} & =\partial_{t} \psi_{2}-\Delta \phi_{2}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)^{2} \\
& -\frac{n \alpha^{2}\left|\nabla \phi_{2}\right|^{2}}{4 \beta_{2}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)} \\
& +\mathrm{e}^{\nu t+q u-v}\left(c v+\psi_{2}+\phi_{2}-q\left(\psi_{1}+\phi_{1}\right)\right) . \tag{7}
\end{align*}
$$

Proof. Substituting $(f, g)=\left(\mathrm{e}^{u}, \mathrm{e}^{v}\right)$ into (1), we have

$$
\begin{align*}
& u_{t}=\Delta u+|\nabla u|^{2}+\mathrm{e}^{\mu t+p v-u} \\
& v_{t}=\Delta v+|\nabla v|^{2}+\mathrm{e}^{v t+q u-v} \tag{8}
\end{align*}
$$

Using the above equations, we have

$$
\begin{align*}
&\left(\partial_{t}-\right.\Delta) \Delta u \\
&= 2|\nabla \nabla u|^{2}+2 \nabla u \cdot \nabla \Delta u \\
&+e^{\mu t+p v-u}\left(p \Delta v-\Delta u+|p \nabla v-\nabla u|^{2}\right), \\
&\left(\partial_{t}-\Delta\right)\left(|\nabla u|^{2}\right)  \tag{9}\\
&= 2 \nabla u \cdot \nabla|\nabla u|^{2}-2|\nabla \nabla u|^{2}+2 e^{\mu t+p v-u} \nabla u \\
& \cdot(p \nabla v-\nabla u) .
\end{align*}
$$

Furthermore, applying (2) and Cauchy-Schwarz inequality $|\nabla \nabla u|^{2} \geq(1 / n)(\Delta u)^{2}$ yields

$$
\begin{align*}
\partial_{t} & H_{1}-\Delta H_{1}-2 \nabla H_{1} \cdot \nabla u=2\left(\alpha-\beta_{1}\right)|\nabla \nabla u|^{2} \\
& +\mathrm{e}^{\mu t+p v-u}\left(p H_{2}-H_{1}\right) \\
& +\mathrm{e}^{\mu t+p v-u}\left\{\left(\alpha p^{2}-p \beta_{2}+c p(1-p)\right)|\nabla v|^{2}\right. \\
& \left.-2 p\left(\alpha-\beta_{1}\right) \nabla u \cdot \nabla v\right\}+\mathrm{e}^{\mu t+p v-u}\left(\alpha-\beta_{1}\right)|\nabla u|^{2} \\
& +\mathrm{e}^{\mu t+p v-u}\left(c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right)\right)+\partial_{t} \psi_{1} \\
& -\Delta \phi_{1}-2 \nabla u \cdot \nabla \phi_{1} \geq 2\left(\alpha-\beta_{1}\right) \frac{1}{n \alpha^{2}} H_{1}\left\{H_{1}\right. \\
& \left.-2\left(\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)\right\}+2\left(\alpha-\beta_{1}\right) \\
& +\frac{1}{n \alpha^{2}} \beta_{1}\left\{\beta_{1}|\nabla u|^{4}+2|\nabla u|^{2}\left(c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)\right\}  \tag{10}\\
& +2\left(\alpha-\beta_{1}\right) \frac{1}{n \alpha^{2}}\left(c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)^{2} \\
& +\mathrm{e}^{\mu t+p v-u}\left(p H_{2}-H_{1}\right) \\
& +\mathrm{e}^{\mu t+p v-u}\left\{p\left(\alpha p-\beta_{2}+c(1-p)\right)|\nabla v|^{2}\right. \\
& \left.-2 p\left(\alpha-\beta_{1}\right) \nabla u \cdot \nabla v\right\}+\mathrm{e}^{\mu t+p v-u}\left(\alpha-\beta_{1}\right)|\nabla u|^{2} \\
& +\mathrm{e}^{\mu t+p v-u}\left(c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right)\right)+\partial_{t} \psi_{1} \\
& -\Delta \phi_{1}-2 \nabla u \cdot \nabla \phi_{1}
\end{align*}
$$

If $\alpha p-\beta_{2}+c(1-p) \geq 0$, the above inequality is

$$
\begin{array}{rl}
\geq 2 & 2\left(\alpha-\beta_{1}\right) \frac{1}{n \alpha^{2}} \\
\cdot & H_{1}\left\{H_{1}-2\left(\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)\right\} \\
+ & \mathrm{e}^{\mu t+p v-u}\left(p H_{2}-H_{1}\right)+\frac{4 \beta_{1}}{n \alpha^{2}}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right) \\
\quad \cdot|\nabla u|^{2}+2\left(\alpha-\beta_{1}\right) \\
\quad \cdot \frac{1}{n \alpha^{2}}\left\{\beta_{1}^{2}|\nabla u|^{4}+\left(c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)^{2}\right\} \\
+\left(\frac{4}{n \alpha^{2}} \beta_{1} c+1-\frac{p\left(\alpha-\beta_{1}\right)}{\alpha p-\beta_{2}+c(1-p)}\right)\left(\alpha-\beta_{1}\right) \\
\quad \cdot \mathrm{e}^{\mu t+p v-u}|\nabla u|^{2} \\
+\mathrm{e}^{\mu t+p v-u}\left(c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right)\right)+\partial_{t} \psi_{1} \\
-\Delta \phi_{1}-2 \nabla u \cdot \nabla \phi_{1}
\end{array}
$$

First note the following inequality:

$$
\begin{align*}
& \frac{4 \beta_{1}}{n \alpha^{2}}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)|\nabla u|^{2}-2 \nabla u \cdot \nabla \phi_{1} \\
& \quad \geq-\frac{n \alpha^{2}\left|\nabla \phi_{1}\right|^{2}}{4 \beta_{1}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)} \tag{12}
\end{align*}
$$

If $\alpha>\beta_{1}$ and $\left(4 / n \alpha^{2}\right) \beta_{1} c+1-p^{2}\left(\alpha-\beta_{1}\right) /\left(-p \beta_{2}+\alpha p^{2}+c p(1-\right.$ $p)) \geq 0$, the above inequality is

$$
\begin{align*}
\geq & 2\left(\alpha-\beta_{1}\right) \frac{1}{n \alpha^{2}} \\
& \cdot H_{1}\left\{H_{1}-2\left(\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\psi_{1}+\phi_{1}\right)\right\} \\
& +\mathrm{e}^{\mu t+p v-u}\left(p H_{2}-H_{1}\right)-\frac{n \alpha^{2}\left|\nabla \phi_{1}\right|^{2}}{4 \beta_{1}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)}  \tag{13}\\
& +\mathrm{e}^{\mu t+p v-u}\left(c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right)\right)+\partial_{t} \psi_{1} \\
& -\Delta \phi_{1}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)^{2}
\end{align*}
$$

For $H_{2}$, we have similar results.

$$
\begin{aligned}
\partial_{t} & H_{2}-\Delta H_{2}-2 \nabla H_{2} \cdot \nabla v \geq 2\left(\alpha-\beta_{2}\right) \frac{1}{n \alpha^{2}} H_{2}\left\{H_{2}\right. \\
& \left.-2\left(\beta_{2}|\nabla v|^{2}+c \mathrm{e}^{v t+q u-v}+\psi_{2}+\phi_{2}\right)\right\}+2\left(\alpha-\beta_{2}\right) \\
& \cdot \frac{1}{n \alpha^{2}} \beta_{2}\left\{\beta_{2}|\nabla v|^{4}+2|\nabla v|^{2}\left(c \mathrm{e}^{v t+q u-v}+\psi_{2}+\phi_{2}\right)\right\} \\
& +2\left(\alpha-\beta_{2}\right) \frac{1}{n \alpha^{2}}\left(c \mathrm{e}^{\imath t+q u-v}+\psi_{2}+\phi_{2}\right)^{2} \\
& +\mathrm{e}^{\imath t+q u-v}\left(q H_{1}-H_{2}\right) \\
& +\mathrm{e}^{v t+q u-v}\left\{q\left(-\beta_{1}+\alpha q+c(1-q)\right)|\nabla u|^{2}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.-2 q\left(\alpha-\beta_{2}\right) \nabla u \cdot \nabla v\right\}+\mathrm{e}^{v t+q u-v}\left(\alpha-\beta_{2}\right)|\nabla v|^{2} \\
& +\mathrm{e}^{v t+q u-v}\left(c v+\psi_{2}+\phi_{2}-q\left(\psi_{1}+\phi_{1}\right)\right)+\partial_{t} \psi_{2} \\
& -\Delta \phi_{2}-2 \nabla v \cdot \nabla \phi_{2} \tag{14}
\end{align*}
$$

If $\alpha q-\beta_{1}+c(1-q) \geq 0$, the above inequality is

$$
\begin{align*}
& \geq 2\left(\alpha-\beta_{2}\right) \frac{1}{n \alpha^{2}} \\
& \cdot H_{2}\left\{H_{2}-2\left(\beta_{2}|\nabla v|^{2}+c \mathrm{e}^{\imath t+q u-v}+\psi_{2}+\phi_{2}\right)\right\} \\
&+\mathrm{e}^{v t+q u-v}\left(q H_{1}-H_{2}\right)+\frac{4 \beta_{2}}{n \alpha^{2}}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right) \\
& \quad \cdot|\nabla v|^{2}+2\left(\alpha-\beta_{2}\right) \\
& \quad \cdot \frac{1}{n \alpha^{2}}\left\{\beta_{2}^{2}|\nabla v|^{4}+\left(c \mathrm{e}^{v t+q u-v}+\psi_{2}+\phi_{2}\right)^{2}\right\}  \tag{15}\\
&+\left(\frac{4}{n \alpha^{2}} \beta_{2} c+1-\frac{q^{2}\left(\alpha-\beta_{2}\right)}{-q \beta_{1}+\alpha q^{2}+c q(1-q)}\right) \\
& \quad \cdot\left(\alpha-\beta_{2}\right) \mathrm{e}^{v t+q u-v}|\nabla v|^{2} \\
&+\mathrm{e}^{v t+q u-v}\left(c v+\psi_{2}+\phi_{2}-q\left(\psi_{1}+\phi_{1}\right)\right)+\partial_{t} \psi_{2} \\
&-\Delta \phi_{2}-2 \nabla v \cdot \nabla \phi_{2} .
\end{align*}
$$

If $\left(4 / n \alpha^{2}\right) \beta_{2} c+1-q^{2}\left(\alpha-\beta_{2}\right) /\left(-q \beta_{1}+\alpha q^{2}+c q(1-q)\right) \geq 0$, the above inequality is

$$
\begin{align*}
\geq & 2\left(\alpha-\beta_{2}\right) \frac{1}{n \alpha^{2}} \\
& \cdot H_{2}\left\{H_{2}-2\left(\beta_{2}|\nabla v|^{2}+c \mathrm{e}^{\imath t+q u-v}+\psi_{2}+\phi_{2}\right)\right\} \\
& +\mathrm{e}^{\imath t+q u-v}\left(q H_{1}-H_{2}\right)-\frac{n \alpha^{2}\left|\nabla \phi_{2}\right|^{2}}{4 \beta_{2}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)}  \tag{16}\\
& +\mathrm{e}^{\imath t+q u-v}\left(c v+\psi_{2}+\phi_{2}-q\left(\psi_{1}+\phi_{1}\right)\right)+\partial_{t} \psi_{2} \\
& -\Delta \phi_{2}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)^{2}
\end{align*}
$$

This completes the proof of Lemma 2.
Next, we need to compute specific $\psi_{i}(t)$ and $\phi_{i}(x)$ that guarantee $h_{i 2}>0$ for the maximum principle to be applicable where $i=1,2$.

Lemma 3. Assume $k_{1}-p k_{2} \geq 0, k_{2}-q k_{1} \geq 0, l_{1}-p l_{2} \geq 0$, $l_{2}-q l_{1} \geq 0$ and

$$
\begin{align*}
k_{i} & \geq \frac{n \alpha^{2}}{2\left(\alpha-\beta_{i}\right)}>0 \\
l_{i} & \geq \frac{n \alpha^{2}}{2\left(\alpha-\beta_{i}\right)}\left(6+\frac{n \alpha^{2}}{\beta_{i}\left(\alpha-\beta_{i}\right)}\right), \quad i=1,2 \tag{17}
\end{align*}
$$

If $\psi_{i}(t)=k_{i} / t, \phi_{i}(x)=\sum_{m=1}^{n}\left(l_{i} /\left(x_{m}-a_{m}\right)^{2}+l_{i} /\left(b_{m}-x_{m}\right)^{2}\right)$, $i=1,2$ for $x \in \Omega=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \times \cdots\left[a_{n}, b_{n}\right]$, then for some $x_{0} \in \Omega,\left.h_{12}\right|_{x=x_{0}}>0$ and $\left.h_{22}\right|_{x=x_{0}}>0$.

Proof. From $k_{1}-p k_{2} \geq 0, k_{2}-q k_{1} \geq 0$ and $l_{1}-p l_{2} \geq 0$, $l_{2}-q l_{1} \geq 0$, we get $c \mu+\psi_{1}+\phi_{1}-p\left(\psi_{2}+\phi_{2}\right) \geq 0$ and $c \nu+\psi_{2}+$ $\phi_{2}-q\left(\psi_{1}+\phi_{1}\right) \geq 0$.

Now applying Lemma 2 yields

$$
\begin{gather*}
h_{12} \geq \partial_{t} \psi_{1}-\Delta \phi_{1}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)^{2} \\
-\frac{n \alpha^{2}\left|\nabla \phi_{1}\right|^{2}}{4 \beta_{1}\left(\alpha-\beta_{1}\right)\left(\psi_{1}+\phi_{1}\right)},  \tag{18}\\
h_{22} \geq \partial_{t} \psi_{2}-\Delta \phi_{2}+\frac{2}{n \alpha^{2}}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)^{2} \\
\\
-\frac{n \alpha^{2}\left|\nabla \phi_{2}\right|^{2}}{4 \beta_{2}\left(\alpha-\beta_{2}\right)\left(\psi_{2}+\phi_{2}\right)} .
\end{gather*}
$$

By the definitions of $\psi_{i}$ and $\phi_{i}$, we obtain

$$
\begin{align*}
\Delta \phi_{i} & =\sum_{m=1}^{n}\left(\frac{6 l_{i}}{\left(x_{m}-a_{m}\right)^{4}}+\frac{6 l_{i}}{\left(b_{m}-x_{m}\right)^{4}}\right) \\
\left|\nabla \phi_{i}\right|^{2} & =\sum_{m=1}^{n}\left(\frac{-2 l_{i}}{\left(x_{m}-a_{m}\right)^{3}}+\frac{2 l_{i}}{\left(b_{m}-x_{m}\right)^{3}}\right)^{2}, \tag{19}
\end{align*}
$$

and

$$
\begin{equation*}
\frac{\left|\nabla \phi_{i}\right|^{2}}{\psi_{i}+\phi_{i}} \leq \sum_{m=1}^{n}\left(\frac{2 \sqrt{l_{i}}}{\left(x_{m}-a_{m}\right)^{2}}+\frac{2 \sqrt{l_{i}}}{\left(b_{m}-x_{m}\right)^{2}}\right)^{2} \tag{20}
\end{equation*}
$$

If

$$
\begin{align*}
k_{i} & \geq \frac{n \alpha^{2}}{2\left(\alpha-\beta_{i}\right)}>0 \\
l_{i} & \geq \frac{n \alpha^{2}}{2\left(\alpha-\beta_{i}\right)}\left(6+\frac{n \alpha^{2}}{\beta_{i}\left(\alpha-\beta_{i}\right)}\right), \quad i=1,2 \tag{21}
\end{align*}
$$

then $h_{12}>0$ and $h_{22}>0$.
This proves Lemma 3.
Proof of Theorem 1. Choose

$$
\begin{align*}
& \psi_{i}(t)=\frac{k_{i}}{t} \\
& \phi_{i}(x)=\sum_{m=1}^{n}\left(\frac{l_{i}}{\left(x_{m}-a_{m}\right)^{2}}+\frac{l_{i}}{\left(b_{m}-x_{m}\right)^{2}}\right), \tag{22}
\end{align*}
$$

$i=1,2$ for $x \in \Omega=\left[a_{1}, b_{1}\right] \times\left[a_{2}, b_{2}\right] \times \cdots\left[a_{n}, b_{n}\right]$. Note that

$$
\begin{equation*}
\lim _{t \rightarrow 0} H_{i}=\infty=\lim _{x_{m} \longrightarrow a_{m}} H_{i}=\lim _{x_{m} \longrightarrow b_{m}} H_{i}, \tag{23}
\end{equation*}
$$

Assume that there exists a first time $t_{0}$ and point $x_{0} \in \Omega$ where $H_{1}\left(x_{0}, t_{0}\right)=0$ and $H_{2}\left(x_{0}, t_{0}\right)>0$. At $\left(x_{0}, t_{0}\right)$; we have

$$
\begin{gather*}
\nabla H_{1}=0 \\
\Delta H_{1} \geq 0  \tag{24}\\
\partial_{t} H_{1} \leq 0
\end{gather*}
$$

Lemma 3 implies that

$$
\begin{align*}
& \partial_{t} H_{1}-\Delta H_{1}-2 \nabla H_{1} \cdot \nabla u-h_{11} H_{1} \\
& \quad-e^{\mu t+p v-u}\left(p H_{2}-H_{1}\right) \geq h_{12}>0 \tag{25}
\end{align*}
$$

This is a contradiction. Assume that there exist a first time $t_{0}$ and point $x_{0} \in \Omega$ where $H_{2}\left(x_{0}, t_{0}\right)=0$ and $H_{1}\left(x_{0}, t_{0}\right)>0$. At $\left(x_{0}, t_{0}\right)$, we have

$$
\begin{align*}
\nabla H_{2} & =0 \\
\Delta H_{2} & \geq 0  \tag{26}\\
\partial_{t} H_{2} & \leq 0
\end{align*}
$$

Lemma 3 implies that

$$
\begin{align*}
\partial_{t} H_{2} & -\Delta H_{2}-2 \nabla H_{2} \cdot \nabla v-h_{21} H_{2} \\
& -\mathrm{e}^{v t+q u-v}\left(q H_{1}-H_{2}\right) \geq h_{22} \tag{27}
\end{align*}
$$

This is a contradiction. Furthermore, $H_{1}\left(x_{0}, t_{0}\right)=0$ and $H_{2}\left(x_{0}, t_{0}\right)=0$ cause the same contradiction as $H_{1}\left(x_{0}, t_{0}\right)=0$ and $H_{2}\left(x_{0}, t_{0}\right)>0$. Thus $H_{1}(x, t)>0$ and $H_{2}(x, t)>0$ for all $x, t>0$.

Taking $\Omega \longrightarrow \mathbb{R}^{n}$ which obtains $\phi_{i} \longrightarrow 0$ then gives the desired result.

## 3. Applications

In this section, we shall give an application of Theorem 1. We integrate along space-time to derive a classical Harnack inequality.
3.1. Classical Harnack Inequality. In this subsection, we integrate our differential Harnack inequality of Theorem 1 along space-time to derive a classical Harnack inequality.

Proposition 4. Let $(f(x, t), g(x, t))$ be positive classical solutions to (1) and $(u(x, t), v(x, t)):=(\log f, \log g)$. Suppose that $x_{1}, x_{2} \in \mathbb{R}^{n}$ and $t_{2}>t_{1}>0$. Assume further that $\alpha \geq$ $2 \max \left\{\beta_{1}, \beta_{2}\right\}, \alpha \geq c$ and $k_{i}=n \alpha^{2} / 2\left(\alpha-\beta_{i}\right), i=1,2$. Then we have

$$
\begin{align*}
& f\left(x_{1}, t_{1}\right) \leq f\left(x_{2}, t_{2}\right)\left(\frac{t_{2}}{t_{1}}\right)^{n} \exp \left(\frac{\left|x_{2}-x_{1}\right|^{2}}{2\left(t_{2}-t_{1}\right)}\right) \\
& g\left(x_{1}, t_{1}\right) \leq g\left(x_{2}, t_{2}\right)\left(\frac{t_{2}}{t_{1}}\right)^{n} \exp \left(\frac{\left|x_{2}-x_{1}\right|^{2}}{2\left(t_{2}-t_{1}\right)}\right) \tag{28}
\end{align*}
$$

Proof. Define the one-variable functions $w_{i}:\left[t_{1}, t_{2}\right] \longrightarrow \mathbb{R}$ as

$$
w_{1}(t):=u(\gamma(t), t)
$$

$$
\begin{equation*}
w_{2}(t):=v(\gamma(t), t) \tag{29}
\end{equation*}
$$

for any $C^{1}$ path $\gamma:\left[t_{1}, t_{2}\right] \longrightarrow \mathbb{R}^{n}$ such that $\gamma\left(t_{1}\right)=x_{1}, \gamma\left(t_{2}\right)=$ $x_{2}$.

Applying Theorem 1, we have

$$
\begin{equation*}
\Delta u \geq-\alpha^{-1}\left(\beta_{1}|\nabla u|^{2}+c e^{\mu t+p v-u}+\frac{k_{1}}{t}\right) . \tag{30}
\end{equation*}
$$

It yields that

$$
\begin{align*}
\frac{d}{d t} w_{1}(t)= & u_{t}+\nabla u \cdot \frac{d \gamma}{d t} \\
= & \Delta u+|\nabla u|^{2}+\mathrm{e}^{\mu t+p v-u}+\nabla u \cdot \frac{d \gamma}{d t} \\
\geq & -\alpha^{-1}\left(\beta_{1}|\nabla u|^{2}+c \mathrm{e}^{\mu t+p v-u}+\frac{k_{1}}{t}\right)+|\nabla u|^{2} \\
& +\mathrm{e}^{\mu t+p v-u}+\nabla u \cdot \frac{d \gamma}{d t}  \tag{31}\\
\geq & \left(\frac{1}{2}-\frac{\beta_{1}}{\alpha}\right)|\nabla u|^{2}+\left(1-\frac{c}{\alpha}\right) \mathrm{e}^{\mu t+p v-u}-\frac{k_{1}}{\alpha t} \\
& -\frac{1}{2}\left|\frac{d \gamma}{d t}\right|^{2} \geq-\frac{1}{2}\left|\frac{d \gamma}{d t}\right|^{2}-\frac{n}{t}
\end{align*}
$$

where $\alpha \geq 0, \alpha \geq 2 \beta_{1}, \alpha \geq c$ and $k_{1}=n \alpha^{2} / 2\left(\alpha-\beta_{1}\right)$. Similarly,

$$
\begin{equation*}
\frac{d}{d t} w_{2}(t) \geq-\frac{1}{2}\left|\frac{d \gamma}{d t}\right|^{2}-\frac{n}{t} \tag{32}
\end{equation*}
$$

for $\alpha \geq 0, \alpha \geq 2 \beta_{2}, \alpha \geq c$ and $k_{2}=n \alpha^{2} / 2\left(\alpha-\beta_{2}\right)$.
By

$$
\begin{align*}
& \int_{t_{1}}^{t_{2}}\left|\frac{d \gamma}{d t}\right|^{2} d t \geq \frac{\left|x_{2}-x_{1}\right|^{2}}{t_{2}-t_{1}} \\
& \int_{t_{1}}^{t_{2}}-\frac{d}{d t} w_{i}(t) \leq \inf _{\gamma(t)} \int_{t_{1}}^{t_{2}}\left(\frac{1}{2}\left|\frac{d \gamma}{d t}\right|^{2}+\frac{n}{t}\right) d t, \quad i=1,2 \tag{33}
\end{align*}
$$

applying $(u, v)=(\log f, \log g)$ gives Proposition 4.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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## Research Article

# Toeplitz Operator and Carleson Measure on Weighted Bloch Spaces 


#### Abstract

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In this paper, we consider Toeplitz operator acting on weighted Bloch spaces. Meanwhile, the inclusion map from weighted Bloch spaces into tent space is also investigated.


## 1. Introduction

Denote the open unit disk of the complex plane $\mathbb{C}$ by $\mathbb{D}$ and the boundary of $\mathbb{D}$ by $\partial \mathbb{D}$. Let $H(\mathbb{D})$ denote the space of all functions analytic in $\mathbb{D}$. For any $a \in \mathbb{D}$,

$$
\begin{equation*}
\varphi_{a}(z)=\frac{a-z}{1-\bar{a} z}, \quad z \in \mathbb{D} \tag{1}
\end{equation*}
$$

is the automorphism of $\mathbb{D}$ which exchanges 0 for $a$. Recall that

$$
\begin{equation*}
\beta(z, a)=\frac{1}{2} \log \frac{1+\left|\varphi_{a}(z)\right|}{1-\left|\varphi_{a}(z)\right|} \tag{2}
\end{equation*}
$$

is the Bergman metric. For any $0<r<\infty, a \in \mathbb{D}$,

$$
\begin{equation*}
D(a, r)=\{z \in D: \beta(z, a)<r\} \tag{3}
\end{equation*}
$$

is the Bergman disk. Let $|D(a, r)|$ denote the normalized area of $D(a, r)$. From [1], we see that $|D(a, r)| \approx\left(1-|a|^{2}\right)^{2}$ when $r$ is fixed.

For $0<p<\infty$ and $\alpha>-1$, the weighted Bergman space $A_{\alpha}^{p}$ is the space of all $f \in H(\mathbb{D})$ such that

$$
\begin{equation*}
\|f\|_{A_{\alpha}^{p}}^{p}:=\int_{\mathbb{D}}|f(z)|^{p}\left(1-|z|^{2}\right)^{\alpha} d A(z)<\infty . \tag{4}
\end{equation*}
$$

When $\alpha=0, A_{\alpha}^{p}$ is the classical Bergman space. We refer the readers to [1,2] for more results on weighted Bergman spaces.

Let $0<\alpha<\infty$. An $f \in H(\mathbb{D})$ is said to belong to the weighted Bloch space, denoted by $\mathscr{B}^{\alpha}$, if

$$
\begin{equation*}
\|f\|_{\mathscr{B}^{\alpha}}=\sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}\left|f^{\prime}(z)\right|<\infty . \tag{5}
\end{equation*}
$$

The space $\mathscr{B}^{\alpha}$ has been studied extensively in [3]. See [1, 4-8] for the study of some operators on weighted Bloch spaces.

Let $\varphi \in L^{\infty}(\mathbb{D})$. The Toeplitz operator $T_{\varphi}$ with symbol $\varphi$ is defined by

$$
\begin{equation*}
T_{\varphi} f(z)=\int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\bar{w} z)^{2+\alpha}} d A_{\alpha}(w) \tag{6}
\end{equation*}
$$

where $d A_{\alpha}(w)=\left(1-|w|^{2}\right)^{\alpha} d A(w)$. There are many results related to $T_{\varphi}$, see [1] and the references therein. Especially, some characterizations for the operator $T_{\varphi}$ on $L_{\alpha}^{2}$ have been obtained by many authors. Since $\mathscr{B}^{\alpha} \subseteq A_{\alpha}^{1}$, it is nature to ask

$$
\begin{equation*}
T_{\varphi} f \in \mathscr{B}^{\alpha} \Longleftrightarrow \text { ?, } \quad f \in \mathscr{B}^{\alpha} \tag{7}
\end{equation*}
$$

The following theorem is the first main result in this paper.

Theorem 1. Let $0<\alpha<\infty$ and $\varphi \in L^{1}(\mathbb{D})$ be harmonic. Then the following statements hold.
(1) $T_{\varphi}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded if and only if $\varphi$ is bounded.
(2) $T_{\varphi}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact if and only if $\varphi=0$.

Given a positive Borel measure $\mu$, the Toeplitz operator with the symbol $\mu$ is defined by
$T_{\mu} f(z)=\int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w} z)^{2+\alpha}} d \mu(w), \quad f \in L^{1}\left(d A_{\alpha}\right)$.
For the Toeplitz operator $T_{\mu}$, we have the following result.

Theorem 2. Let $0<\alpha, r<\infty$ and $\mu$ be a positive Borel measure. Then the following statements hold.
(1) $T_{\mu}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded if and only if

$$
\begin{equation*}
\sup _{a \in \mathbb{D}} \frac{\mu(D(a, r))}{\left(1-|a|^{2}\right)^{2+\alpha}}<\infty \tag{9}
\end{equation*}
$$

(2) $T_{\mu}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact if and only if

$$
\begin{equation*}
\lim _{|a| \rightarrow 0} \frac{\mu(D(a, r))}{\left(1-|a|^{2}\right)^{2+\alpha}}=0 . \tag{10}
\end{equation*}
$$

For $I \subset \partial \mathbb{D},|I|=(1 / 2 \pi) \int_{I}|d \xi|$ is the normalized length of the subarc $I$ and the corresponding Carleson square for $I$ is defined as follows (see [9]).

$$
\begin{equation*}
S(I)=\{r \xi \in \mathbb{D}: r \in[1-|I|, 1), \xi \in I\} . \tag{11}
\end{equation*}
$$

For $0<p<\infty$, a positive Borel measure $\mu$ on $\mathbb{D}$ is said to be a $p$-Carleson measure if

$$
\begin{equation*}
\|\mu\|_{p}=: \sup _{I \subset \partial \mathbb{D}}\left(\frac{\mu(S(I))}{|I|^{p}}\right)^{1 / 2}<\infty . \tag{12}
\end{equation*}
$$

If $p=1, p$-Carleson measure is the classical Carleson measure. From Lemma 3.1.1 of [10], for $p, q \in(0, \infty)$ we know that $\mu$ is a $p$-Carleson measure if and only if

$$
\begin{equation*}
\|\mu\|_{p, q}:=\sup _{z \in \mathbb{D}}\left(\int_{\mathbb{D}} \frac{\left(1-|z|^{2}\right)^{q}}{|1-\bar{z} w|^{p+q}} d \mu(w)\right)^{1 / 2}<\infty . \tag{13}
\end{equation*}
$$

Moreover, $\|\mu\|_{p, q} \approx\|\mu\|_{p}$.
Let $0<p, q<\infty$ and $\mu$ be a positive Borel measure on $\mathbb{D}$. The tent space $T_{\mu}^{p, q}$ is the class of all $f \in H(\mathbb{D})$ which satisfy

$$
\begin{equation*}
\sup _{I \subseteq \partial \mathbb{D}} \frac{1}{|I|^{p}} \int_{S(I)}|f(z)|^{q} d \mu(z)<\infty . \tag{14}
\end{equation*}
$$

The tent space $T_{\mu}^{p, 2}$ was introduced by J. Xiao [11] to studied Carleson measure for $Q_{s}$ space. He proved that $Q_{s}$ space is continuously contained in $T_{\mu}^{p, 2}$ if and only if

$$
\begin{equation*}
\sup _{I \subseteq \partial \mathbb{D}} \frac{\mu(S(I))}{|I|^{s}}\left(\log \frac{2}{|I|}\right)^{2}<\infty . \tag{15}
\end{equation*}
$$

J. Pau and R. Zhao [12] generalized the main results in [11]. In [13], J. Liu and Z.Lou studied Morrey spaces. They proved that an equivalent condition for Morrey spaces $L^{2, s}$ continuously contained in $T_{\mu}^{p, 2}$ is that $\mu$ is a Carleson measure. See [14, 15] for more information of the Morrey space.

We state the last main result in this paper as follows.
Theorem 3. Let $0<\alpha<\infty$ and $\mu$ be a positive Borel measure. Then the following statements hold.
(1) The inclusion map $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$ is bounded if and only if $\mu$ is a $(2+2 \alpha)$-Carleson measure.
(2) The inclusion map $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$ is compact if and only if $\mu$ is a vanishing $(2+2 \alpha)$-Carleson measure.

Throughout this paper, the letter $C$ will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant $C$ such that $A \leq C B$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

## 2. Proofs of Main Results

To prove our main results in this paper, we need some auxiliary results. The following result can be found in [16, Theorem 3.8].

Lemma 4. Let $p \geq 1, \alpha>0,-1+p \alpha<\eta<\infty$, and $c>0$. Then $f \in \mathscr{B}^{\alpha+1}$ if and only if

$$
\begin{equation*}
\sup _{z \in \mathbb{D}} \int_{\mathbb{D}}|f(w)-f(z)|^{p} \frac{\left(1-|z|^{2}\right)^{c+p \alpha}}{|1-\bar{z} w|^{2+c+\eta}} d A_{\eta}(w)<\infty . \tag{16}
\end{equation*}
$$

From Lemma 4, we can easily deduce the following result.
Lemma 5. Let $p \geq 1, \alpha>0,-1+p \alpha<\eta<\infty$, and $c>0$. Then $f \in \mathscr{B}^{\alpha+1}$ if and only if

$$
\begin{equation*}
\sup _{a \in \mathbb{D}} \int_{\mathbb{D}}|f(z)|^{p} \frac{\left(1-|a|^{2}\right)^{c+p \alpha}}{|1-\bar{a} z|^{2+c+\eta}} d A_{\eta}(z)<\infty . \tag{17}
\end{equation*}
$$

Proof. First assume that $f \in \mathscr{B}^{\alpha+1}$. It is clear that

$$
\begin{equation*}
|f(z)| \lesssim \frac{\|f\|_{\mathscr{B}^{\alpha+1}}}{\left(1-|z|^{2}\right)^{\alpha}}, \quad z \in \mathbb{D} . \tag{18}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\int_{\mathbb{D}}|f(z)|^{p} \frac{\left(1-|a|^{2}\right)^{c+p \alpha}}{|1-\bar{a} z|^{2+c+\eta}} d A_{\eta}(z) \\
\quad \leq \int_{\mathbb{D}}|f(z)-f(a)|^{p} \frac{\left(1-|a|^{2}\right)^{c+p \alpha}}{|1-\bar{a} z|^{2+c+\eta}} d A_{\eta}(z) \\
\quad+\int_{\mathbb{D}}|f(a)|^{p} \frac{\left(1-|a|^{2}\right)^{c+p \alpha}}{|1-\bar{a} z|^{2+c+\eta}} d A_{\eta}(z)  \tag{19}\\
\quad \leq\|f\|_{\mathscr{B}^{\alpha+1}}^{p}+\|f\|_{\mathscr{B}^{\alpha+1}}^{p} \int_{\mathbb{D}} \frac{\left(1-|a|^{2}\right)^{c}}{|1-\bar{a} z|^{2+c+\eta}} d A_{\eta}(z) \\
\quad \leq\|f\|_{\mathscr{B}^{\alpha+1}}^{p} .
\end{align*}
$$

The proof of the inverse direction is similar to the above statements we omit the details. The proof is complete.

Proof of Theorem 1. (1) First assume that $\varphi \in L^{\infty}(\mathbb{D})$. For $f \in$ $\mathscr{B}^{\alpha+1}$, since

$$
\begin{equation*}
\|f\|_{\mathscr{B}^{\alpha+1}} \approx \sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}|f(z)| \tag{20}
\end{equation*}
$$

we obtain

$$
\begin{aligned}
& \left\|T_{\varphi} f\right\|_{\mathscr{S}^{\alpha+1}} \approx \sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}\left|T_{\varphi} f(z)\right| \\
& \quad=\sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha}\left|\int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\bar{w} z)^{2+\alpha}} d A_{\alpha}(w)\right| \\
& \quad \leq \sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha} \int_{\mathbb{D}} \frac{|\varphi(w)||f(w)|}{|1-\bar{w} z|^{2+\alpha}} d A_{\alpha}(w) \\
& \quad \leqslant\|\varphi\|_{L^{\infty}(\mathbb{D})}\|f\|_{\mathscr{S}^{\alpha+1}} \\
& \quad \cdot\left(\sup _{z \in \mathbb{D}}\left(1-|z|^{2}\right)^{\alpha} \int_{\mathbb{D}} \frac{1}{|1-\bar{w} z|^{2+\alpha}} d A(w)\right) \\
& \quad \leqslant\| \|_{L^{\infty}(\mathbb{D})}\|f\|_{\mathscr{B}^{\alpha+1}} .
\end{aligned}
$$

Hence $T_{\varphi}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded.
Conversely, assume that $T_{\varphi}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded. For $z \in \mathbb{D}$, set

$$
\begin{equation*}
f_{z}(w)=\frac{\left(1-|z|^{2}\right)^{2}}{(1-\bar{z} w)^{2+\alpha}} \in \mathscr{B}^{\alpha+1} \tag{22}
\end{equation*}
$$

It is easy to check that $\left\|f_{z}\right\|_{\mathscr{B}^{\alpha+1}} \approx 1$. Using Lemma 5 with $p=1, \eta=\alpha$ and $c=2+\alpha$, we get

$$
\begin{align*}
\infty & >\left\|T_{\varphi}\right\| \gtrsim\left\|T_{\varphi} f_{z}\right\|_{\mathscr{B}^{1+\alpha}} \\
& \gtrsim \int_{\mathbb{D}}\left|T_{\varphi} f_{z}(w)\right| \frac{\left(1-|z|^{2}\right)^{2+2 \alpha}}{|1-\bar{z} w|^{2(2+\alpha)}} d A_{\alpha}(w)  \tag{23}\\
& =\left(1-|z|^{2}\right)^{\alpha} \int_{\mathbb{D}}\left|T_{\varphi} f_{z}\left(\varphi_{z}(w)\right)\right| d A_{\alpha}(w) \\
& \gtrsim\left(1-|z|^{2}\right)^{\alpha}\left|T_{\varphi} f_{z}\left(\varphi_{z}(0)\right)\right| \gtrsim|\varphi(z)|
\end{align*}
$$

which implies that $\varphi \in L^{\infty}(\mathbb{D})$, as desired.
(2) Sufficiency. The result is obvious.

Necessity. For any $z_{n} \in \mathbb{D}$, let

$$
\begin{equation*}
f_{z_{n}}(w)=\frac{\left(1-\left|z_{n}\right|^{2}\right)^{2}}{\left(1-\overline{z_{n}} w\right)^{2+\alpha}}, \quad w \in \mathbb{D} . \tag{24}
\end{equation*}
$$

$f_{z_{n}} \in \mathscr{B}^{1+\alpha}$. It is easy to check that $f_{z_{n}} \longrightarrow 0$ uniformly on compact subsets on $\mathbb{D}$ as $\left|z_{n}\right| \longrightarrow 1$. From the fact that $T_{\varphi}$ is compact on $\mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ and the proof of (1), we have

$$
\begin{align*}
0 & \longleftarrow\left\|T_{\varphi} f_{z_{n}}\right\|_{\mathscr{B}^{\alpha+1}} \\
& \gtrsim \int_{\mathbb{D}}\left|T_{\varphi} f_{z_{n}}(w)\right| \frac{\left(1-\left|z_{n}\right|^{2}\right)^{2+2 \alpha}}{\left|1-\overline{z_{n}} w\right|^{2(2+\alpha)}} d A_{\alpha}(w)  \tag{25}\\
& \gtrsim\left(1-|z|^{2}\right)^{\alpha}\left|T_{\varphi} f_{z_{n}}\left(\varphi_{z_{n}}(0)\right)\right| \gtrsim\left|\varphi\left(z_{n}\right)\right| .
\end{align*}
$$

By the arbitrariness of $z_{n}$ and the Maximal Module Principle, we get $\varphi=0$. The proof is complete.

Proof of Theorem 2. (1) First suppose that $T_{\mu}: \mathscr{B}^{\alpha+1} \longrightarrow$ $\mathscr{B}^{\alpha+1}$ is bounded. For any $a \in \mathbb{D}$, from the proof of Theorem 1 , we obtain that $f_{a} \in \mathscr{B}^{\alpha+1}$ and $\left\|f_{a}\right\|_{\mathscr{B}^{\alpha+1}} \leqslant 1$. Thus,

$$
\begin{align*}
\infty & >\left\|T_{\mu}\right\| \gtrsim\left(1-|a|^{2}\right)^{\alpha}\left|T_{\mu} f_{a}(a)\right| \\
& =\left(1-|a|^{2}\right)^{\alpha}\left|\int_{\mathbb{D}} \frac{f_{a}(w)}{(1-\bar{w} a)^{2+\alpha}} d \mu(w)\right| \\
& =\left(1-|a|^{2}\right)^{\alpha} \int_{\mathbb{D}} \frac{\left(1-|a|^{2}\right)^{2}}{|1-\bar{w} a|^{4+2 \alpha}} d \mu(w)  \tag{26}\\
& \geq\left(1-|a|^{2}\right)^{\alpha} \int_{D(a, r)} \frac{\left(1-|a|^{2}\right)^{2}}{|1-\bar{w} a|^{4+2 \alpha}} d \mu(w) \\
& \approx \frac{\mu(D(a, r))}{\left(1-|a|^{2}\right)^{2+\alpha}}
\end{align*}
$$

as desired.
Conversely, suppose that

$$
\begin{equation*}
\sup _{a \in \mathbb{D}} \frac{\mu(D(a, r))}{\left(1-|a|^{2}\right)^{2+\alpha}}<\infty . \tag{27}
\end{equation*}
$$

Then we can get that $\mu$ is a Carleson measure for $L^{1}\left(d A_{\alpha}\right)$. If $g \in L^{1}$ and $f \in \mathscr{B}^{\alpha+1}$, we can easily obtain that $f g \in L^{1}\left(d A_{\alpha}\right)$. Using Fubini's Theorem, we obtain

$$
\begin{align*}
& \left|\int_{\mathbb{D}} g(z) \overline{T_{\mu} f(z)}\left(1-|z|^{2}\right)^{\alpha} d A(z)\right| \\
& =\left|\int_{\mathbb{D}} g(z)\left(1-|z|^{2}\right)^{\alpha} d A(z) \int_{\mathbb{D}} \frac{\overline{f(w)}}{(1-w \bar{z})^{2+\alpha}} d \mu(w)\right| \\
& =\left|\int_{\mathbb{D}} \overline{f(w)} d \mu(w) \int_{\mathbb{D}} \frac{\left(1-|z|^{2}\right)^{\alpha} g(z)}{(1-w \bar{z})^{2+\alpha}} d A(z)\right|  \tag{28}\\
& \leq \int_{\mathbb{D}}|g(w) \overline{f(w)}| d \mu(w) \\
& \quad \leqslant \int_{\mathbb{D}}|g(w)||f(w)| d A_{\alpha}(w) \leqslant\|g\|_{L^{1}}\|f\|_{\mathscr{B}^{\alpha+1}} .
\end{align*}
$$

Hence $T_{\mu}: \mathscr{B}^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is bounded.
(2) Suppose that $T_{\mu}: B^{\alpha+1} \longrightarrow \mathscr{B}^{\alpha+1}$ is compact. Let $a_{n} \in$ D. Set

$$
\begin{equation*}
f_{a_{n}}(z)=\frac{\left(1-\left|a_{n}\right|^{2}\right)^{2}}{\left(1-\overline{a_{n}} z\right)^{2+\alpha}}, \quad z \in \mathbb{D} \tag{29}
\end{equation*}
$$

Then $f_{a_{n}} \in \mathscr{B}^{\alpha+1}$ and $f_{a_{n}} \longrightarrow 0$ uniformly on compact subset on $\mathbb{D}$ as $\left|a_{n}\right| \longrightarrow 1$. Thus,

$$
\begin{aligned}
& \left\|T_{\mu} f_{a_{n}}\right\|_{\mathscr{B}^{\alpha+1}} \geq\left(1-\left|a_{n}\right|^{2}\right)^{\alpha}\left|T_{\mu} f_{a_{n}}\left(a_{n}\right)\right| \\
& \quad=\left(1-\left|a_{n}\right|^{2}\right)^{\alpha}\left|\int_{\mathbb{D}} \frac{f_{a_{n}}(w)}{\left(1-\bar{w} a_{n}\right)^{2+\alpha}} d \mu(w)\right|
\end{aligned}
$$

$$
\begin{align*}
& =\left(1-\left|a_{n}\right|^{2}\right)^{\alpha} \int_{\mathbb{D}} \frac{\left(1-\left|a_{n}\right|^{2}\right)^{2}}{\left|1-\bar{w} a_{n}\right|^{4+2 \alpha}} d \mu(w) \\
& \geq\left(1-\left|a_{n}\right|^{2}\right)^{\alpha} \int_{D\left(a_{n} r\right)} \frac{\left(1-\left|a_{n}\right|^{2}\right)^{2}}{\left|1-\bar{w} a_{n}\right|^{4+2 \alpha}} d \mu(w) \\
& \approx \frac{\mu\left(D\left(a_{n}, r\right)\right)}{\left(1-\left|a_{n}\right|^{2}\right)^{2+\alpha}}, \tag{30}
\end{align*}
$$

which implies the desired result.
Conversely, assume that

$$
\begin{equation*}
\lim _{|a| \rightarrow 1} \frac{\mu(D(a, r))}{\left(1-|a|^{2}\right)^{2+\alpha}}=0 . \tag{31}
\end{equation*}
$$

We know that $\mu$ is a vanishing Carleson measure for $L^{1}\left(d A_{\alpha}\right)$. We want to show that $T_{\mu}$ is compact. Using Fubini's Theorem we have

$$
\begin{equation*}
\int_{\mathbb{D}} \overline{T_{\mu} f(z)} g(z) d A_{\alpha}(z)=\int_{\mathbb{D}} g(z) \overline{f(z)} d \mu(z) \tag{32}
\end{equation*}
$$

If $g \in L^{1}$ and $f \in \mathscr{B}^{\alpha+1}$, we can easily obtain $f g \in$ $L^{1}\left(d A_{\alpha}\right)$. Therefore,

$$
\begin{align*}
& \left|\int_{\mathbb{D}} \overline{T_{\mu} f(z)} g(z) d A_{\alpha}(z)\right| \leq \int_{\mathbb{D}}|g(z) \overline{f(z)}| d \mu(z)  \tag{33}\\
& \quad \leq \int_{\mathbb{D}}|g(z)||f(z)| d A_{\alpha}(z) \leq\|g\|_{L^{1}}\|f\|_{\mathscr{B}^{\alpha+1}}
\end{align*}
$$

That is,

$$
\begin{equation*}
\left\|T_{\mu} f\right\|_{g^{x+1}} \leq\|f\|_{g^{x+1}} . \tag{34}
\end{equation*}
$$

If $f_{n} \longrightarrow 0$ weakly in $\mathscr{B}^{\alpha+1}$, it follows that $\left\|T_{\mu} f_{n}\right\|_{\mathscr{B}^{\alpha+1}} \longrightarrow 0$. The proof of Theorem 2 is complete.

Proof of Theorem 3. (1) Suppose that $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$ is bounded. For $a \in \mathbb{D}$, set

$$
\begin{equation*}
f_{a}(z)=\frac{\left(1-|a|^{2}\right)^{2}}{(1-\bar{a} z)^{2+\alpha}}, \quad z \in \mathbb{D} \tag{35}
\end{equation*}
$$

Then $f_{a} \in \mathscr{B}^{\alpha+1}$. For any $I \subseteq \partial \mathbb{D}$, we get

$$
\begin{equation*}
\frac{\mu(S(I))}{|I|^{2+2 \alpha}} \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)}\left|f_{a}(z)\right| d \mu(z)<\infty, \tag{36}
\end{equation*}
$$

as desired.
Conversely, assume that $\mu$ is a $(2+2 \alpha)$-Carleson measure. Let $f \in \mathscr{B}^{\alpha+1}$. Using the well-known fact

$$
\begin{equation*}
|f(b)| \lesssim \frac{1}{\left(1-|b|^{2}\right)^{\alpha}}, \quad b \in \mathbb{D} \tag{37}
\end{equation*}
$$

we have

$$
\begin{align*}
& \frac{1}{|I|^{2+\alpha}} \int_{S(I)}|f(w)| d \mu(w) \\
& \quad \lesssim \frac{1}{|I|^{2+\alpha}} \int_{S(I)}|f(w)-f(b)| d \mu(w)  \tag{38}\\
& \quad+\frac{1}{|I|^{2+\alpha}} \int_{S(I)}|f(b)| d \mu(w) \\
& \quad \\
& \quad \frac{1}{|I|^{2+\alpha}} \int_{S(I)}|f(w)-f(b)| d \mu(w)+\frac{\mu(S(I))}{|I|^{2+2 \alpha}} .
\end{align*}
$$

Note that

$$
\begin{equation*}
\frac{\mu(S(I))}{|I|^{2+\alpha}} \lesssim \frac{\mu(S(I))}{|I|^{2+2 \alpha}} . \tag{39}
\end{equation*}
$$

Then $\mu$ is a Carleson measure for $L^{1}\left(d A_{\alpha}\right)$. Since $\mathscr{B}^{\alpha+1} \subseteq$ $L^{1}\left(d A_{\alpha}\right)$, combined with Lemma 4, we obtain

$$
\begin{align*}
& \frac{1}{|I|^{2+\alpha}} \int_{S(I)}|f(w)-f(b)| d \mu(w) \\
& \quad \approx\left(1-|b|^{2}\right)^{2+\alpha} \int_{S(I)}\left|\frac{f(w)-f(b)}{(1-\bar{b} w)^{4+2 \alpha}}\right| d \mu(w) \\
& \quad \leq\left(1-|b|^{2}\right)^{2+\alpha} \int_{\mathbb{D}}\left|\frac{f(w)-f(b)}{(1-\bar{b} w)^{4+2 \alpha}}\right| d \mu(w)  \tag{40}\\
& \quad \leq\left(1-|b|^{2}\right)^{2+\alpha} \int_{\mathbb{D}}\left|\frac{f(w)-f(b)}{(1-\bar{b} w)^{4+2 \alpha}}\right| d A_{\alpha}(w) \\
& \quad \leq \int_{\mathbb{D}}|f(w)-f(b)| \frac{\left(1-|b|^{2}\right)^{2+\alpha}}{|1-\bar{b} w|^{4+2 \alpha}} d A_{\alpha}(w) \\
& \quad \leq\|f\|_{\mathscr{B}^{\alpha+1}} .
\end{align*}
$$

Hence $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$ is bounded.
(2) Suppose that $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$ is compact. Let $a_{n} \in$ $\mathbb{D}$ such that $\left|a_{n}\right| \longrightarrow 1$ as $n \longrightarrow \infty$. We know that $f_{a_{n}} \in \mathscr{B}^{\alpha+1}$ and $f_{a_{n}} \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$ as $n \longrightarrow \infty$. By Theorem 5.15 of [1] it follows that $f_{a_{n}} \longrightarrow 0$ weakly as $n \longrightarrow$ $\infty$. Hence for the compact operator $I_{d}: \mathscr{B}^{\alpha+1} \longrightarrow T_{\mu}^{2+\alpha, 1}$, we have $\left\|f_{a_{n}}\right\|_{T_{\mu}^{2+\alpha, 1}} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus,

$$
\begin{align*}
\frac{\mu\left(S\left(I_{n}\right)\right)}{\left|I_{n}\right|^{2+2 \alpha}} & \lesssim \frac{1}{\left(1-\left|a_{n}\right|^{2}\right)^{2+\alpha}} \int_{S\left(I_{n}\right)}\left|f_{a_{n}}(z)\right| d \mu(z)  \tag{41}\\
& \lesssim\left\|f_{a_{n}}\right\|_{T_{\mu}^{2+\alpha, 1}} \longrightarrow 0
\end{align*}
$$

as $n \longrightarrow \infty$. Hence $\mu$ is a vanishing $(2+2 \alpha)$-Carleson measure.
Conversely, assume that $\mu$ is a vanishing $(2+2 \alpha)$-Carleson measure. Let $f_{n} \in \mathscr{B}^{\alpha+1},\left\|f_{n}\right\|_{\mathscr{B}^{\alpha+1}} \leqslant 1$, and $f_{n} \longrightarrow 0$
$(n \longrightarrow \infty)$ uniformly on compact subsets of $\mathbb{D}$. Then it is easy to get that

$$
\begin{align*}
& \frac{1}{|I|^{2+\alpha}} \int_{S(I)}\left|f_{n}(z)\right| d \mu(z) \\
& \quad \\
& \quad \frac{1}{|I|^{2+\alpha}} \int_{S(I)}\left|f_{n}(z)\right| d \mu_{r}(z)  \tag{42}\\
& \quad+\frac{1}{|I|^{2+\alpha}} \int_{S(I)}\left|f_{n}(z)\right| d\left(\mu-\mu_{r}\right)(z) \\
& \quad \\
& \frac{1}{|I|^{2+\alpha}} \int_{S(I)}\left|f_{n}(z)\right| d \mu_{r}(z) \\
& \quad+\left\|\mu-\mu_{r}\right\|^{2}\left\|f_{n}\right\|_{\mathscr{B}^{\alpha+1}} .
\end{align*}
$$

Let $r \longrightarrow 1^{-}$and $n \longrightarrow \infty$; we get the desired result. The proof is complete.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that they have no conflicts of interest.

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# Research Article 

# Solutions for a Class of Hadamard Fractional Boundary Value Problems with Sign-Changing Nonlinearity 

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Using fixed point methods we establish some existence theorems of positive (nontrivial) solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity.

## 1. Introduction

In this paper, using fixed point methods we study the existence of positive (nontrivial) solutions for the Hadamard fractional boundary value problems with sign-changing nonlinearity:

$$
\begin{align*}
-D^{\alpha} u(t) & =f(t, u(t)), \quad t \in[1, e],  \tag{1}\\
u(1) & =\delta u(1)=\delta u(e)=0,
\end{align*}
$$

where $\alpha \in(2,3]$ is a real number, $D^{\alpha}$ is the left-sided Hadamard fractional derivative of order $\alpha, \delta u(t)=t d u(t) / d t$, and $f \in C\left([1, e] \times \mathbb{R}^{+}, \mathbb{R}\right)$ is a sign-changing function; i.e., there exists a constant $M>0$ such that
(H0) $f(t, u)+M \geq 0$ for all $(t, u) \in[1, e] \times \mathbb{R}^{+}$.
As is known, fractional differential equations have been paid special attention by many researchers for the reason that they serve as an excellent tool for wide applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, and control theory; for more details, we refer to books [1-3]. In recent years, there have been a large number of papers dealing with the existence of solutions of nonlinear initial (boundary) value problems of fractional differential equations by using some techniques of nonlinear analysis, such as fixed-point results [4-13], iterative methods [14-23], the topological degree [24-29], the Leray-Schauder alternative [30, 31], and stability [32].

In [4], the authors studied the following abstract evolution of the system for HIV-1 population dynamics, which takes the form in fractional sense:

$$
\begin{align*}
& D_{t}^{\alpha} u(t)+\lambda f\left(t, u(t), D_{t}^{\beta} u(t), v(t)\right)=0 \\
& D_{t}^{\gamma} v(t)+\lambda g(t, u(t))=0, \quad 0<t<1, \\
& D_{t}^{\beta} u(0)=D_{t}^{\beta+1} u(0)=0, \\
& D_{t}^{\beta} u(1)=\int_{0}^{1} D_{t}^{\beta} u(s) d A(s),  \tag{2}\\
& v(0)=v^{\prime}(0)=0 \\
& v(1)=\int_{0}^{1} v(s) d B(s)
\end{align*}
$$

where $f:(0,1) \times[0,+\infty)^{3} \longrightarrow(-\infty,+\infty)$ and $g:(0,1) \times$ $[0,+\infty) \longrightarrow(-\infty,+\infty)$ are two semipositone functions. By using the Guo-Krasnosel'skii fixed point theorem, they not only obtained the existence of positive solutions for (2) but also discussed the effect of parameters $\lambda$ on the existence of solutions.

In [14], the authors adopted generalized $\alpha$-contractive map to study some fractional integro-differential equations with the Caputo-Fabrizio derivation and obtained the
approximate solutions for these problems by using of some appropriate Lipschitz conditions for their nonlinearities.

In [15], Cui used the convergence of Cauchy sequences in complete spaces to obtain the unique solution for the fractional boundary value problems:

$$
\begin{align*}
D_{t}^{p} x(t)+p(t) f(t, x(t))+q(t) & =0, \quad t \in(0,1), \\
x(0) & =x^{\prime}(0)=0,  \tag{3}\\
x(1) & =0,
\end{align*}
$$

where $f$ is a Lipschitz continuous function, with the Lipschitz constant associated with the first eigenvalue for the relevant operator. This method can also be applied in papers [16, 17] and references therein.

However, as a generalization of fractional calculus by Riemann and Liouville, Hadamard fractional equations have seldom been studied in the literature; we only refer to [8-10, 22, 23, 29, 32] and references therein. In [8], the authors used the Guo-Krasnosel'skii fixed point theorem on cones to establish the existence and nonexistence of positive solutions for (1) with nonnegative nonlinearity $\lambda a(t) f(u)$ and considered solvability for the influence of the parameter intervals.

In [32], the authors used Banach and Schauder fixed point theorem to obtain the existence and Hyers-Ulam stability of solutions for Hadamard fractional impulsive Cauchy problems of the form

$$
\begin{align*}
& { }_{H} D_{1^{+}}^{\alpha} u(t)=f(t, u(t)), \alpha \in(0,1), \\
& \\
& \quad t \in(1, e] \backslash\left\{t_{1}, t_{2}, \ldots, t_{m}\right\},  \tag{4}\\
& \Delta u\left(t_{i}\right)={ }_{H} J_{1^{+}}^{1-\alpha} u\left(t_{i}^{+}\right)-{ }_{H} J_{1^{+}}^{1-\alpha} u\left(t_{i}^{-}\right)=p_{i}, \\
& \\
& \quad p_{i} \in \mathbb{R}, i=1,2, \ldots, m, \\
& { }_{H} J_{1^{+}}^{1-\alpha} u\left(1^{+}\right)=u_{0}, \quad u_{0} \in \mathbb{R},
\end{align*}
$$

where $f$ satisfies a Lipschitz condition.
In this paper, motivated by works aforementioned, we used fixed point methods to study the existence of solutions for (1) with sign-changing nonlinearity. We have the main results: (i) when the nonlinear term $f$ grows both superlinearly and sublinearly at $\infty$, we use the fixed point index theory to obtain two existence theorems of positive solutions for (1); (ii) when $f$ satisfies an appropriate Lipschitz condition, we obtain a unique solution for (1) and establish a sequence of iterations uniformly converges to the unique solution.

## 2. Preliminaries

Definition 1 (see [1-3]). The left-sided Hadamard fractional derivative of order $\alpha \in[n-1, n), n \in \mathbb{Z}_{+}$, of a function $f$ is given by

$$
\begin{equation*}
D^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\left(t \frac{d}{d t}\right)^{n} \int_{1}^{t}\left(\ln \frac{t}{s}\right)^{n-\alpha+1} f(s) \frac{d s}{s} \tag{5}
\end{equation*}
$$

$$
1 \leq t \leq e
$$

where $\Gamma(\cdot)$ is the Gamma function.

We now offer Green's function for (1). From Lemma 2.1 of [8], (1) is equivalent to the integral equation

$$
\begin{equation*}
u(t)=\int_{1}^{e} G(t, s) f(s, u(s)) \frac{d s}{s}, \quad t \in[1, e] \tag{6}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, s)=\frac{1}{\Gamma(\alpha)} \\
& \quad \cdot \begin{cases}(1-\ln s)^{\alpha-2}(\ln t)^{\alpha-1}, & 1 \leq t \leq s \leq e \\
(1-\ln s)^{\alpha-2}(\ln t)^{\alpha-1}-\left(\ln \left(\frac{t}{s}\right)\right)^{\alpha-1}, & 1 \leq s \leq t \leq e\end{cases} \tag{7}
\end{align*}
$$

Lemma 2 (see [10, Lemma 2.2]). Let $G$ be defined by (7). Then the following inequalities are satisfied:

$$
\begin{align*}
G(t, s) & \geq 0, \\
(\ln t)^{\alpha-1} G(e, s) \leq G(t, s) \leq G(e, s), &  \tag{8}\\
& \forall(t, s) \in[1, e] \times[1, e],
\end{align*}
$$

where $G(e, s)=(1-\ln s)^{\alpha-2}-(1-\ln s)^{\alpha-1}$ and $\forall s \in[1, e]$.
Lemma 3. Let $\varphi(t)=(1-\ln t)^{\alpha-2}-(1-\ln t)^{\alpha-1}$ and $\forall t \in[1, e]$. Then the following inequalities are satisfied:

$$
\begin{align*}
& \int_{1}^{e}(\ln t)^{\alpha-1} \varphi(t) \frac{d t}{t} \cdot \varphi(s) \leq \int_{1}^{e} G(t, s) \varphi(t) \frac{d t}{t} \\
& \quad \leq \int_{1}^{e} \varphi(t) \frac{d t}{t} \cdot \varphi(s), \quad \forall s \in[1, e] . \tag{9}
\end{align*}
$$

This is a direct result from Lemma 2, so we omit its proof. Moreover, for convenience, let

$$
\begin{align*}
& \kappa_{1}=\int_{1}^{e}(\ln t)^{\alpha-1} \varphi(t) \frac{d t}{t} \\
& \kappa_{2}=\int_{1}^{e} \varphi(t) \frac{d t}{t} \tag{10}
\end{align*}
$$

Let $E:=C[1, e],\|u\|:=\sup _{t \in[1, e]}|u(t)|$, and $P:=\{u \in E:$ $u(t) \geq 0, \forall t \in[1, e]\}$. Then $(E,\|\cdot\|)$ becomes a real Banach space and $P$ is a cone on $E$. Define $B_{\rho}:=\{u \in E:\|u\|<\rho\}$ for $\rho>0$ in the sequel.

Define $P_{0}=\left\{u \in P: u(t) \geq(\ln t)^{\alpha-1}\|u\|\right.$ and $\left.\forall t \in[1, e]\right\}$. Then $P_{0}$ is also a cone on $E$. In what follows, we verify that when $u \in P_{0},\|u\| \geq(M / \Gamma(\alpha)) \int_{1}^{e}(1-\ln s)^{\alpha-2}(d s / s)$; we have $u(t)-w(t) \geq 0, \forall t \in[1, e]$, where $w$ is a solution for the problem

$$
\begin{align*}
-D^{\alpha} u(t) & =M, \quad t \in[1, e]  \tag{11}\\
u(1) & =\delta u(1)=\delta u(e)=0,
\end{align*}
$$

where $M$ is defined by (H0). From (1) and (6), $w$ takes the form as follows:

$$
\begin{equation*}
w(t)=M \int_{1}^{e} G(t, s) \frac{d s}{s}, \quad t \in[1, e] \tag{12}
\end{equation*}
$$

where $G$ is defined by (7). Indeed, when $u \in P_{0}$, from (7) we have

$$
\begin{align*}
& u(t)-w(t) \geq(\ln t)^{\alpha-1}\|u\|-M \int_{1}^{e} G(t, s) \frac{d s}{s} \\
& \quad \geq(\ln t)^{\alpha-1}\|u\|-\frac{M}{\Gamma(\alpha)} \int_{1}^{e}(1-\ln s)^{\alpha-2}(\ln t)^{\alpha-1} \frac{d s}{s}  \tag{13}\\
& \quad=(\ln t)^{\alpha-1}\left(\|u\|-\frac{M}{\Gamma(\alpha)} \int_{1}^{e}(1-\ln s)^{\alpha-2} \frac{d s}{s}\right) \geq 0
\end{align*}
$$

For semipositone condition (H0), we need to construct an appropriate operator with which to study problem (1). Hence, we consider the modified problem

$$
\begin{align*}
-D^{\alpha} u(t)=f(t, \max \{u(t)-w(t), 0\})+M, & \\
& t \in[1, e] \tag{14}
\end{align*}
$$

$$
u(1)=\delta u(1)=\delta u(e)=0
$$

where $w$ is a solution for (11). Clearly, we are easy to show that if $u$ solves (14), $w$ solves (11), and $u(t)-w(t) \geq 0(\equiv \equiv 0)$ and $\forall t \in[1, e]$; then $u(t)-w(t)$ is a positive solution for (1). Consequently, we turn to study the modified problem (14). From (1) and (6), (14) is equivalent to the integral equation
$u(t)$

$$
\begin{array}{r}
=\int_{1}^{e} G(t, s)[f(s, \max \{u(s)-w(s), 0\})+M] \frac{d s}{s}  \tag{15}\\
t \in[1, e], u \in E
\end{array}
$$

Hence, we can define an operator $A: P \longrightarrow P$ as follows:

$$
\begin{array}{r}
(A u)(t) \\
=\int_{1}^{e} G(t, s)[f(s, \max \{u(s)-w(s), 0\})+M] \frac{d s}{s}  \tag{16}\\
\\
t \in[1, e], u \in E .
\end{array}
$$

It is not difficult to prove that if $A u_{0}=u_{0}$, then $u_{0}$ is a solution for (14). Moreover, from Lemma 2 we easily have $A(P) \subset P_{0}$.

Now, we offer some basic theorems for fixed point methods used in our problem.

Lemma 4 (see [33]). Let $E$ be a real Banach space and $P a$ cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set and that $A: \bar{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists $\omega_{0} \in P \backslash\{0\}$ such that

$$
\begin{equation*}
\omega-A \omega \neq \lambda \omega_{0}, \quad \forall \lambda \geq 0, \omega \in \partial \Omega \cap P \tag{17}
\end{equation*}
$$

then $i(A, \Omega \cap P, P)=0$, where $i$ denotes the fixed point index on $P$.

Lemma 5 (see [33]). Let E be a real Banach space and P a cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$
and that $A: \bar{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If

$$
\begin{equation*}
\omega-\lambda A \omega \neq 0, \quad \forall \lambda \in[0,1], \omega \in \partial \Omega \cap P \tag{18}
\end{equation*}
$$

then $i(A, \Omega \cap P, P)=1$.

## 3. Positive Solutions for (1)

Let $\lambda_{1}=\kappa_{1}^{-1}, \lambda_{2}=\kappa_{2}^{-1}$, and $\mathcal{N}=(M / \Gamma(\alpha)) \int_{1}^{e}(1-$ $\ln s)^{\alpha-2}(d s / s)$. Then we give some assumptions for nonlinear term $f$.
(H1) $\liminf _{u \rightarrow+\infty}(f(t, u) / u)>\lambda_{1}$ uniformly on $t \in$ [1,e],
$(\mathrm{H} 2)$ there exist $Q:[1, e] \longrightarrow \mathbb{R}^{+}$such that

$$
\begin{align*}
& f(t, \max \{u(t)-w(t), 0\})+M \leq Q(t), \\
& \forall(t, u) \in[1, e] \times\left[0, \frac{M}{\Gamma(\alpha)} \int_{1}^{e}(1-\ln s)^{\alpha-2} \frac{d s}{s}\right] \tag{19}
\end{align*}
$$

where $\int_{1}^{e} \varphi(t) Q(t)(d t / t)<\mathcal{N}$,
(H3) $\lim \sup _{u \rightarrow+\infty}(f(t, u) / u)<\lambda_{2}$ uniformly on $t \in$ [1,e],
(H4) there exist $l \in(1, e), \widetilde{Q}:[1, e] \longrightarrow \mathbb{R}^{+}$such that

$$
\begin{align*}
& f(t, \max \{u(t)-w(t), 0\})+M \geq \widetilde{Q}(t), \\
& \quad \forall(t, u) \in[1, e] \times\left[0, \frac{M}{\Gamma(\alpha)} \int_{1}^{e}(1-\ln s)^{\alpha-2} \frac{d s}{s}\right], \tag{20}
\end{align*}
$$

where $\int_{1}^{e}(\ln l)^{\alpha-1} \varphi(t) \widetilde{Q}(t)(d t / t)>\mathcal{N}$,
(H5) $f \in C([1, e] \times \mathbb{R}, \mathbb{R})$, and there exists $k \in(0,1)$ such that $|f(t, u)-f(t, v)| \leq k \lambda_{2}|u-v|$ for $t \in[1, e], u, v \in \mathbb{R}$, (H6) $f(t, 0) \not \equiv 0$ for $t \in[1, e]$.
We now state our main results and offer their proofs.
Theorem 6. Suppose that (H0)-(H2) hold. Then (1) has at least a positive solution.

Proof. We first prove that there exist $R>\mathcal{N}$ large enough such that

$$
\begin{equation*}
u-A u \neq \lambda \varphi_{0}^{*}, \quad \forall \lambda \geq 0, u \in \partial B_{R} \cap P, \tag{21}
\end{equation*}
$$

where $\varphi_{0}^{*} \in P_{0}$ is a given element. If false, there exits $u \in$ $\partial B_{R} \cap P, \lambda_{0} \geq 0$ such that $u-A u=\lambda_{0} \varphi_{0}^{*}$. Note that $A(P) \subset P_{0}$; then $A u \in P_{0}$ for $u \in P$, and thus $u \in P_{0}$. This also implies that $u(t) \geq$ for $t \in[1, e]$.

Note that $u \in \partial B_{R} \cap P$; then $\|u\|=R>\mathcal{N}$, and $u(t)-w(t) \geq$ 0 for $u \in P_{0}, t \in[1, e]$. From (H1) we have

$$
\begin{align*}
& \liminf _{u \rightarrow+\infty} \frac{f(t, \max \{u-w, 0\})+M}{u-w} \\
& \quad=\liminf _{u \rightarrow+\infty} \frac{f(t, u-w)+M}{u-w}>\lambda_{1} \tag{22}
\end{align*}
$$

uniformly on $t \in[1, e]$. As a result, there exist $\varepsilon_{1}>0$ and $c_{1}>0$ such that

$$
\begin{align*}
& f(t, u(t)-w(t))+M  \tag{23}\\
& \quad \geq\left(\lambda_{1}+\varepsilon_{1}\right)(u(t)-w(t))-c_{1}, \quad \forall t \in[1, e]
\end{align*}
$$

This implies that

$$
\begin{align*}
u(t) & \geq(A u)(t) \\
& \geq \int_{1}^{e} G(t, s)\left[\left(\lambda_{1}+\varepsilon_{1}\right)(u(s)-w(s))-c_{1}\right] \frac{d s}{s}  \tag{24}\\
& \geq\left(\lambda_{1}+\varepsilon_{1}\right) \int_{1}^{e} G(t, s)(u(s)-w(s)) \frac{d s}{s}-c_{1} \kappa_{2} .
\end{align*}
$$

Note that (12) is multiplied by $\varphi(t)$ on both sides of the above and integrated over $[1, e]$ and use Lemma 3 to obtain

$$
\begin{align*}
& \int_{1}^{e} u(t) \varphi(t) \frac{d t}{t} \geq \int_{1}^{e} \varphi(t) \\
& \cdot\left[\left(\lambda_{1}+\varepsilon_{1}\right) \int_{1}^{e} G(t, s)(u(s)-w(s)) \frac{d s}{s}\right. \\
& \left.-c_{1} \kappa_{2}\right] \frac{d t}{t} \geq \frac{\lambda_{1}+\varepsilon_{1}}{\lambda_{1}} \int_{1}^{e} \varphi(t)(u(t) \\
& -w(t)) \frac{d t}{t}-c_{1} \kappa_{2}^{2}=\left(1+\varepsilon_{1} \kappa_{1}\right) \int_{1}^{e} u(t)  \tag{25}\\
& \cdot \varphi(t) \frac{d t}{t}-\left(1+\varepsilon_{1} \kappa_{1}\right) \int_{1}^{e} \varphi(t) \\
& \cdot M \int_{1}^{e} G(t, s) \frac{d s}{s} \frac{d t}{t}-c_{1} \kappa_{2}^{2} \geq\left(1+\varepsilon_{1} \kappa_{1}\right) \\
& \cdot \int_{1}^{e} u(t) \varphi(t) \frac{d t}{t}-M\left(1+\varepsilon_{1} \kappa_{1}\right) \kappa_{2}^{2}-c_{1} \kappa_{2}^{2}
\end{align*}
$$

Consequently, we have

$$
\begin{equation*}
\int_{1}^{e} u(t) \varphi(t) \frac{d t}{t} \leq\left(\varepsilon_{1} \kappa_{1}\right)^{-1}\left(M\left(1+\varepsilon_{1} \kappa_{1}\right)+c_{1}\right) \kappa_{2}^{2} \tag{26}
\end{equation*}
$$

Noting that $u \in P_{0}$, we have

$$
\begin{align*}
& \|u\| \int_{1}^{e}(\ln t)^{\alpha-1} \varphi(t) \frac{d t}{t} \\
& \quad \leq\left(\varepsilon_{1} \kappa_{1}\right)^{-1}\left(M\left(1+\varepsilon_{1} \kappa_{1}\right)+c_{1}\right) \kappa_{2}^{2} \tag{27}
\end{align*}
$$

and $\|u\| \leq \varepsilon_{1}^{-1} \kappa_{1}^{-2}\left(M\left(1+\varepsilon_{1} \kappa_{1}\right)+c_{1}\right) \kappa_{2}^{2}$.

Therefore, if we choose $R>\max \left\{\mathcal{N}, \varepsilon_{1}^{-1} \kappa_{1}^{-2}\left(M\left(1+\varepsilon_{1} \kappa_{1}\right)+\right.\right.$ $\left.\left.c_{1}\right) \kappa_{2}^{2}\right\}$, then (21) holds true. From Lemma 4 we have

$$
\begin{equation*}
i\left(A, B_{R} \cap P, P\right)=0 \tag{28}
\end{equation*}
$$

On the other hand, we prove that

$$
\begin{equation*}
u \neq \lambda A u, \quad \forall \lambda \in[0,1], u \in \partial B_{\mathcal{N}} \cap P \tag{29}
\end{equation*}
$$

If false, there exist $u \in \partial B_{\mathcal{N}} \cap P, \lambda_{1}^{*} \in[0,1]$ such that $u=$ $\lambda_{1}^{*} A u$; this implies $u(t) \leq(A u)(t), \forall t \in[1, e]$, and $\|u\| \leq$ $\|A u\|$. However, from (H2) we have
$(A u)(t)$

$$
\begin{align*}
& =\int_{1}^{e} G(t, s)[f(s, \max \{u(s)-w(s), 0\})+M] \frac{d s}{s}  \tag{30}\\
& \leq \int_{1}^{e} \varphi(s) Q(s) \frac{d s}{s}<\mathcal{N}=\|u\|,
\end{align*}
$$

for $u \in \partial B_{\mathcal{N}} \cap P$.
This indicates that $\|A u\|<\|u\|$ for $u \in \partial B_{\mathcal{N}} \cap P$. This has a contradiction, and thus (29) holds true. From Lemma 5 we have

$$
\begin{equation*}
i\left(A, B_{\mathcal{N}} \cap P, P\right)=1 \tag{31}
\end{equation*}
$$

From (28) and (31), we obtain

$$
\begin{align*}
i\left(A,\left(B_{R} / \bar{B}_{\mathcal{N}}\right) \cap P, P\right)= & i\left(A, B_{R} \cap P, P\right)  \tag{32}\\
& -i\left(A, B_{\mathcal{N}} \cap P, P\right)=-1 .
\end{align*}
$$

Therefore the operator $A$ has at least one fixed point $u$ in $\left(B_{R} \backslash \bar{B}_{\mathcal{N}}\right) \cap P$ with $\|u\| \geq \mathcal{N}$, and then $u(t)-w(t)$ is a positive solution for (1). This completes the proof.

Theorem 7. Suppose that (H0), (H3), and (H4) hold. Then (1) has at least a positive solution.

Proof. We first prove that there exist $R>\mathcal{N}$ large enough such that

$$
\begin{equation*}
u \neq \lambda A u, \quad \forall \lambda \in[0,1], u \in \partial B_{R} \cap P \tag{33}
\end{equation*}
$$

If false, there exist $u \in \partial B_{R} \cap P, \lambda_{2}^{*} \in[0,1]$ such that $u=\lambda_{2}^{*} A u$, and $u \in P_{0}$ for the fact that $A u \in P_{0}$ when $u \in P$. This also implies $u(t) \leq(A u)(t), t \in[1, e]$.

Note that $u \in \partial B_{R} \cap P$; then $\|u\|=R>\mathcal{N}$, and $u(t)-w(t) \geq$ 0 for $u \in P_{0}, t \in[1, e]$. From (H3) we have

$$
\begin{align*}
& \limsup _{u \rightarrow+\infty} \frac{f(t, \max \{u-w, 0\})+M}{u-w} \\
& \quad=\limsup _{u \rightarrow+\infty} \frac{f(t, u-w)+M}{u-w}<\lambda_{2} \tag{34}
\end{align*}
$$

uniformly on $t \in[1, e]$. As a result, there exist $\varepsilon_{2} \in\left(0, \lambda_{2}\right)$ and $c_{2}>0$ such that

$$
\begin{align*}
& f(t, u(t)-w(t))+M \\
& \quad \leq\left(\lambda_{2}-\varepsilon_{2}\right)(u(t)-w(t))+c_{2}, \quad \forall t \in[1, e] . \tag{35}
\end{align*}
$$

This implies that

$$
\begin{align*}
u(t) & \leq(A u)(t) \\
& \leq \int_{1}^{e} G(t, s)\left[\left(\lambda_{2}-\varepsilon_{2}\right)(u(s)-w(s))+c_{2}\right] \frac{d s}{s}  \tag{36}\\
& \leq\left(\lambda_{2}-\varepsilon_{2}\right) \int_{1}^{e} G(t, s)(u(s)-w(s)) \frac{d s}{s}+c_{2} \kappa_{2} .
\end{align*}
$$

Note that (12) is multiplied by $\varphi(t)$ on both sides of the above and integrated over $[1, e]$ and use Lemma 3 to obtain

$$
\begin{align*}
& \int_{1}^{e} u(t) \varphi(t) \frac{d t}{t} \leq \int_{1}^{e} \varphi(t) \\
& \cdot\left[\left(\lambda_{2}-\varepsilon_{2}\right) \int_{1}^{e} G(t, s)(u(s)-w(s)) \frac{d s}{s}\right. \\
& \left.\quad+c_{2} \kappa_{2}\right] \frac{d t}{t} \leq \frac{\lambda_{2}-\varepsilon_{2}}{\lambda_{2}} \int_{1}^{e} \varphi(t)(u(t) \\
& \quad-w(t)) \frac{d t}{t}+c_{2} \kappa_{2}^{2} \leq\left(1-\varepsilon_{2} \kappa_{2}\right) \int_{1}^{e} u(t)  \tag{37}\\
& \cdot \varphi(t) \frac{d t}{t}+\left(1-\varepsilon_{2} \kappa_{2}\right) \int_{1}^{e} \varphi(t) \\
& \cdot M \int_{1}^{e} G(t, s) \frac{d s}{s} \frac{d t}{t}+c_{2} \kappa_{2}^{2} \leq\left(1-\varepsilon_{2} \kappa_{2}\right) \\
& \cdot \int_{1}^{e} u(t) \varphi(t) \frac{d t}{t}+M\left(1-\varepsilon_{2} \kappa_{2}\right) \kappa_{2}^{2}+c_{2} \kappa_{2}^{2}
\end{align*}
$$

Consequently, we have

$$
\begin{equation*}
\int_{1}^{e} u(t) \varphi(t) \frac{d t}{t} \leq\left(\varepsilon_{2} \kappa_{2}\right)^{-1}\left(M\left(1-\varepsilon_{2} \kappa_{2}\right)+c_{2}\right) \kappa_{2}^{2} \tag{38}
\end{equation*}
$$

Noting that $u \in P_{0}$, we obtain

$$
\begin{align*}
& \|u\| \int_{1}^{e}(\ln t)^{\alpha-1} \varphi(t) \frac{d t}{t} \\
& \quad \leq\left(\varepsilon_{2} \kappa_{2}\right)^{-1}\left(M\left(1-\varepsilon_{2} \kappa_{2}\right)+c_{2}\right) \kappa_{2}^{2} \tag{39}
\end{align*}
$$

$$
\text { and }\|u\| \leq\left(\varepsilon_{2} \kappa_{1} \kappa_{2}\right)^{-1}\left(M\left(1-\varepsilon_{2} \kappa_{2}\right)+c_{2}\right) \kappa_{2}^{2} .
$$

Taking $R>\max \left\{\mathcal{N},\left(\varepsilon_{2} \kappa_{1} \kappa_{2}\right)^{-1}\left(M\left(1-\varepsilon_{2} \kappa_{2}\right)+\mathcal{c}_{2}\right) \kappa_{2}^{2}\right\}$, then (33) is satisfied. From Lemma 5 we have

$$
\begin{equation*}
i\left(A, B_{R} \cap P, P\right)=1 \tag{40}
\end{equation*}
$$

On the other hand, we prove that

$$
\begin{equation*}
u-A u \neq \lambda \varphi_{1}^{*}, \quad \forall \lambda \geq 0, u \in \partial B_{\mathcal{N}} \cap P \tag{41}
\end{equation*}
$$

where $\varphi_{1}^{*} \in P$ is a given element. If false, there exist $u \in$ $\partial B_{\mathcal{N}} \cap P, \lambda_{3} \geq 0$ such that $u-A u=\lambda_{3} \varphi_{1}^{*}$; this implies $u(t) \geq(A u)(t), t \in[1, e]$, and thus $\|u\| \geq\|A u\|$. However, from (H4) we have

$$
\begin{align*}
& \|A u\|=\max _{t \in[1, e]}(A u)(t) \\
& \quad \geq \int_{1}^{e} G(l, s)[f(s, \max \{u(s)-w(s), 0\})+M] \frac{d s}{s}  \tag{42}\\
& \quad \geq \int_{1}^{e}(\ln l)^{\alpha-1} \varphi(s) \widetilde{Q}(s) \frac{d s}{s}>\mathcal{N}=\|u\|
\end{align*}
$$

for $u \in \partial B_{\mathcal{N}} \cap P$. This has a contradiction, and thus (41) holds true. From Lemma 4 we have

$$
\begin{equation*}
i\left(A, B_{\mathcal{N}} \cap P, P\right)=0 \tag{43}
\end{equation*}
$$

From (40) and (43), we obtain

$$
\begin{align*}
i\left(A,\left(B_{R} \backslash \bar{B}_{\mathcal{N}}\right) \cap P, P\right)= & i\left(A, B_{R} \cap P, P\right)  \tag{44}\\
& -i\left(A, B_{\mathcal{N}} \cap P, P\right)=1
\end{align*}
$$

Therefore the operator $A$ has at least one fixed point $u$ in $\left(B_{R} \backslash \bar{B}_{\mathcal{N}}\right) \cap P$ with $\|u\| \geq \mathcal{N}$, and then $u(t)-w(t)$ is a positive solution for (1). This completes the proof.

From (6), we define an operator $T: E \longrightarrow E$ as follows:

$$
\begin{equation*}
(T u)(t)=\int_{1}^{e} G(t, s) f(s, u(s)) \frac{d s}{s}, \quad u \in E \tag{45}
\end{equation*}
$$

Then $T$ is a completely continuous operator, and $u$ is a solution for (1) if and only if $u$ is a fixed point of $T$.

Theorem 8. Suppose that (H5)-(H6) hold. Then (1) has only a nontrivial solution, denoted by $u^{*}$, and for all $u_{0} \in E, u_{0}(t) \not \equiv$ $0, t \in[1, e]$, the sequence $u_{n}=T u_{n-1}(n=1,2, \ldots)$ uniformly converges to $u^{*}$.

Proof. (H6) ensures that 0 is not a solution for (1). Then if (1) has a solution, this solution is nontrivial. For all $n \in \mathbb{N}$, from Lemma 3 we have

$$
\begin{aligned}
& \left|u_{n+1}(t)-u_{n}(t)\right|=\left|\left(T u_{n}\right)(t)-\left(T u_{n-1}\right)(t)\right| \\
& =\left|\int_{1}^{e} G(t, s)\left(f\left(s, u_{n}(s)\right)-f\left(s, u_{n-1}(s)\right)\right) \frac{d s}{s}\right| \\
& \leq \int_{1}^{e} G(t, s)\left|f\left(s, u_{n}(s)\right)-f\left(s, u_{n-1}(s)\right)\right| \frac{d s}{s} \\
& \leq k \lambda_{2} \int_{1}^{e} \varphi(s)\left|u_{n}(s)-u_{n-1}(s)\right| \frac{d s}{s} \\
& =k \lambda_{2} \int_{1}^{e} \varphi(t)\left|\left(T u_{n-1}\right)(t)-\left(T u_{n-2}\right)(t)\right| \frac{d t}{t} \\
& \leq k \lambda_{2} \int_{1}^{e} \varphi(t) \int_{1}^{e} G(t, s) \\
& \quad \cdot\left|f\left(s, u_{n-1}(s)\right)-f\left(s, u_{n-2}(s)\right)\right| \frac{d s}{s} \frac{d t}{t} \\
& \leq k^{2} \lambda_{2} \int_{1}^{e} \varphi(t)\left|u_{n-1}(t)-u_{n-2}(t)\right| \frac{d t}{t} \\
& \vdots \\
& \leq k^{n} \lambda_{2} \int_{1}^{e} \varphi(t)\left|u_{1}(t)-u_{0}(t)\right| \frac{d t}{t} .
\end{aligned}
$$

On the other hand, letting $v=0$ and $f_{0}=$ $\max _{t \in[1, e]}|f(t, 0)|$ in (H5), we have

$$
\begin{equation*}
|f(t, u)| \leq k \lambda_{2}|u|+f_{0}, \quad \forall u \in \mathbb{R}, t \in[1, e] \tag{47}
\end{equation*}
$$

Noting that $u_{1}=T u_{0}$, we obtain

$$
\begin{align*}
& \int_{1}^{e} \varphi(t)\left|u_{1}(t)-u_{0}(t)\right| \frac{d t}{t} \\
& \quad \leq(1+k) \int_{1}^{e} \varphi(t)\left|u_{0}(t)\right| \frac{d t}{t}+\kappa_{2}^{2} f_{0}:=\beta_{1} . \tag{48}
\end{align*}
$$

Therefore, for all $n \in \mathbb{N}$, we have

$$
\begin{equation*}
\left|u_{n+1}(t)-u_{n}(t)\right| \leq k^{n} \beta_{1} \lambda_{2}, \quad \forall t \in[1, e] . \tag{49}
\end{equation*}
$$

Consequently, for all $n, m \in \mathbb{N}$, we have

$$
\begin{align*}
& \left|u_{n+m}(t)-u_{n}(t)\right| \\
& \quad \leq\left|u_{n+m}(t)-u_{n+m-1}(t)\right| \\
& \quad+\left|u_{n+m-1}(t)-u_{n+m-2}(t)\right|+\cdots \\
& \quad+\left|u_{n+1}(t)-u_{n}(t)\right|  \tag{50}\\
& \leq \beta_{1} \lambda_{2}\left(k^{n+m-1}+k^{n+m-2}+\cdots+k^{n}\right) \leq k^{n} \frac{\beta_{1} \lambda_{2}}{1-k} \\
& \quad \longrightarrow 0, \quad \text { when } n \longrightarrow \infty .
\end{align*}
$$

This implies $\left\{u_{n}\right\}$ is a Cauchy sequence, and from E's completeness, there exists $u^{*} \in E$ such that $\lim _{n \rightarrow \infty} u_{n}=u^{*}$. Taking the limits for sequence $u_{n}=T u_{n-1}$ and we have $T u^{*}=u^{*}$; i.e., $u^{*}$ is a nontrivial solution for (1).

Next we prove that (1) has only a solution. If $u, v \in E$ are solutions for (1) and $u \neq v$, then $T^{n} u=u$ and $T^{n} v=v$ for all $n \in \mathbb{N}$. By (H5) we obtain

$$
\begin{aligned}
\mid & u(t)-v(t)\left|=\left|\left(T^{n} u\right)(t)-\left(T^{n} v\right)(t)\right|=\right| T\left(T^{n-1} u\right) \\
& \cdot(t)-T\left(T^{n-1} v\right)(t) \mid \\
\leq & \int_{1}^{e} G(t, s) \mid f\left(s,\left(T^{n-1} u\right)(s)\right) \\
& -f\left(s,\left(T^{n-1} v\right)(s)\right) \left\lvert\, \frac{d s}{s}\right. \\
\leq & k \lambda_{2} \int_{1}^{e} \varphi(s)\left|\left(T^{n-1} u\right)(s)-\left(T^{n-1} v\right)(s)\right| \frac{d s}{s} \\
\leq & k \lambda_{2} \int_{1}^{e} \varphi(t) \int_{1}^{e} G(t, s) \\
& \cdot\left|f\left(s,\left(T^{n-2} u\right)(s)\right)-f\left(s,\left(T^{n-1} v\right)(s)\right)\right| \frac{d s}{s} \frac{d t}{t} \\
\leq & k^{2} \lambda_{2} \int_{1}^{e} \varphi(t)\left|\left(T^{n-2} u\right)(t)-\left(T^{n-1} v\right)(t)\right| \frac{d t}{t} \\
\leq & \quad k^{n-1} \lambda_{2} \int_{1}^{e} \varphi(t)|(T u)(t)-(T v)(t)| \frac{d t}{t} \\
= & k^{n-1} \lambda_{2} \int_{1}^{e} \varphi(t)|u(t)-v(t)| \frac{d t}{t} \\
\leq & k^{n-1}\|u-v\| .
\end{aligned}
$$

This implies that

$$
\begin{equation*}
\|u-v\| \leq k^{n-1}\|u-v\|, \quad \text { for all } n \in \mathbb{N} \tag{52}
\end{equation*}
$$

Noting that $k \in(0,1)$, then there exists $N \in \mathbb{N}$, when $n>N, k^{n-1}<1$, and thus a contradiction for the above inequality. This obtains the uniqueness of solutions for (1). This completes the proof.

In what follows, we offer some examples for our main results. Let $\alpha=2.5, l=\sqrt{e}$. Then $\kappa_{1}=\int_{1}^{e}(\ln t)^{\alpha-1} \varphi(t)(d t / t) \approx$ $0.1227, \kappa_{2}=\int_{1}^{e} \varphi(t)(d t / t)=4 / 15 \approx 0.2667$, and $\mathcal{N}=$ $(M / \Gamma(\alpha)) \int_{1}^{e}(1-\ln s)^{\alpha-2}(d s / s) \approx 0.5015 M$.

Example 9. Let $f(t, u)=\left(1 / M e^{\ln t}\right) u^{2}-M$ for all $t \in[1, e]$, $u \in \mathbb{R}^{+}$, and $Q \equiv 1.8 M$. Then $\lim \inf _{u \rightarrow+\infty}(f(t, u) / u)=$ $\liminf _{u \rightarrow+\infty}\left(\left(\left(1 / M e^{\ln t}\right) u^{2}-M\right) / u\right)=+\infty>\lambda_{1}$ uniformly on $t \in[1, e]$, and when $(t, u) \in[1, e] \times[0, \mathcal{N}], f(t, u)+$ $M \leq(1 / M) 0.2515 M^{2}+M \approx 1.2515 M \leq Q(t)$. Moreover, $\int_{1}^{e} \varphi(t) Q(t)(d t / t) \approx 0.48 M<\mathcal{N}$. Therefore, (H1) and (H2) hold.

Example 10. Let $f(t, u)=\left(6 M / e^{-0.5015 M}\right) e^{-u}-M$ for all $t \in[1, e], u \in \mathbb{R}^{+}$, and $Q \equiv 5.4 M$. Then $\lim \sup _{u \rightarrow+\infty}(f(t$, $u) / u)=\lim \sup _{u \rightarrow+\infty}\left(\left(\left(6 M / e^{-0.5015 M}\right) e^{-u}-M\right) / u\right)=0<\lambda_{2}$ uniformly on $t \in[1, e]$, and when $(t, u) \in[1, e] \times[0, \mathcal{N}]$, $f(t, u)+M \geq\left(6 M / e^{-0.5015 M}\right) e^{-0.5015 M}-M+M=6 M \geq$ $Q(t)$. Moreover, $\int_{1}^{e}(0.5)^{1.5} \varphi(t) 5.4 M(d t / t) \approx 0.5093 M>\mathcal{N}$. Therefore, (H3) and (H4) hold.

Example 11. Let $f(t, u)=3.7 \mathrm{ku}+g(t)$, where $k \in(0,1)$ and $g \in C([1, e], \mathbb{R})$ with $g(t) \not \equiv 0$ for $t \in[1, e]$. Therefore, (H5) and (H6) hold.

## Data Availability

No data were used to support this study

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# Option Pricing under the Jump Diffusion and Multifactor Stochastic Processes 

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#### Abstract

In financial markets, there exists long-observed feature of the implied volatility surface such as volatility smile and skew. Stochastic volatility models are commonly used to model this financial phenomenon more accurately compared with the conventional Black-Scholes pricing models. However, one factor stochastic volatility model is not good enough to capture the term structure phenomenon of volatility smirk. In our paper, we extend the Heston model to be a hybrid option pricing model driven by multiscale stochastic volatility and jump diffusion process. In our model the correlation effects have been taken into consideration. For the reason that the combination of multiscale volatility processes and jump diffusion process results in a high dimensional differential equation (PIDE), an efficient finite element method is proposed and the integral term arising from the jump term is absorbed to simplify the problem. The numerical results show an efficient explanation for volatility smirks when we incorporate jumps into both the stock process and the volatility process.


## 1. Introduction

Due to the well-known phenomenon of volatility 'smile' and 'smirk' exhibited in option pricing processes, many attempts have been made to solve the problem by extending the classical Black-Scholes models [1] and relaxing the assumptions. One of the famous approaches is to replace the constant volatility by a process described by the volatility model such as a local volatility model or a stochastic volatility model, which is widely studied to capture the phenomenon of volatility skew. The local volatility model assumes that the local volatility of the stock is a function of stock price and time $t$. For example, in Dupire's Model [2], the classical Black-Scholes model is modified to include a time dependent local volatility rather than a constant volatility. The constant elasticity of variance model (CEV model) attempts to capture the stochastic nature of volatility and the leverage effect by assuming $\sigma\left(S_{t}\right)=S_{t}^{\gamma}$, which is firstly developed in [3] and then applied to calibrate and estimate the energy commodity market in [4]. Different from the local volatility model, the stochastic volatility model assumes that the volatility process
is related to another stochastic process instead of the stock price process itself. The commonly used model is the Heston model [5], from which Heston generalises the Black-Scholes model to a two-dimensional stochastic model by allowing the volatility to follow a Cox Ingersoll Ross model (CIR) process and derives a semiclosed solution by applying the method of characteristic function. Stein and Stein [6] also promote a stochastic volatility model driven by the OU process. Other stochastic volatility models are proposed for different formation of stochastic volatility. The traditional Heston model [7] assumes that the underlying volatility process is a CIR process with the power of $1 / 2$; the $3 / 2$ model assumes that the diffusion of volatility process is a flipped CIR process, raising the power of $3 / 2$. The $4 / 2$ process is the combination of the CIR process and the flipped CIR process (see [8]).

However, recent empirical study shows that the singlefactor Heston model is overly too restrictive and multifactor stochastic models are required in order to obtain a more accurate result. The multifactor model is based on the modification of the term structure. The term structure of volatility


Figure 1: Variation of SPX price with time.
is much more complicated than that in single factor models. The idea that the mean reversion rate of low frequency data is different to that for the high frequency data has been noticed by [9-11]. Thus, multifactor stochastic processes are introduced considering the different frequencies in the observed data (see [12]). Heston decomposes the stochastic volatility into multifactors by the principle analysis regarding the frequency of the volatility. The author suggests that the use of multifactor stochastic volatility may enhance the option pricing model by a large extent, and at least two factors should be taken into consideration in the study of pathindependent and path-dependent option pricing problems (see [13]). The concept of time-scale is firstly proposed by Fouque to model the volatility process as a combination of fast-scale and slow-scale process (see [14-16]). In 2008, Fouque proposed a numerical algorithm based on asymptotic approximation and asymptotic homogenization to study the effect of the fast and the slow scale of the volatility OU process on option pricing (see [16]). The definition of time-scale is distinguished by the fluctuation frequency of the volatility process. The fast-scale volatility relates to the highly frequent short period fluctuation, while the slow-scale volatility relates to the less frequent and long term variation. The phenomenon of time-scale can easily be observed by stock prices generated by using the 27 years daily SPX data downloaded from the Chicago Board Options Exchange (CBOE) website, as shown in Figure 1. Slow scale volatility can be tracked from the long period variation, and it does not have to be mean reversion, while the fast scale volatility is the smaller but drastic oscillations between the peak and the bottom.

An alternative approach to capture the leptokurtic features and the implied volatility smile is the jump diffusion model. The jump process can be used to sketch the unexpected abrupt change of stock price within a short period. The pioneering work of Merton in [17] assumes that the asset return process follows a Brownian motion plus a jump process, and the jump process is a compound Poisson process with constant jump intensity and normally distributed jumpsize distribution. Different from Merton's Model, the work of Kou in [18] assumes that the distribution of the jumpsize is a double exponential distribution instead of a normal distribution for the simplicity of computation. More recent work proves that combination of the stochastic volatility
model and the traditional jump diffusion model leads to more accurate models. Bates introduces the SVJ (stochastic volatility with jumps) model by allowing both jump diffusion and stochastic volatility in the return process. The SVJ model is then extended in [19] to incorporate the jump term not only in the return process, but also in the stochastic process (see [19]). The SVJ model is also studied in [20], in which the authors assume an affine structure of characteristic function and apply fast Fourier transformation to solve the SVJ problem to obtain a semianalytic solution.

Many numerical algorithms have been proposed to study the option pricing. The traditional Black-scholes equation is a convection-diffusion parabolic equation, and the finite difference scheme for solving the equation is studied in detail by [21]. In [22], Hull and White suggest a modification to the explicit finite difference method (FDM) for valuing derivatives, which ensures a more accurate approximation with small time steps, and the approach has been applied to calculate bond options under two different interest rate processes. A FDM scheme to price the PIDE arising from a jump diffusion model is presented in [23], and an explicitimplicit FDM scheme was proposed for solving the PIDE to price the European and Barrier options with Levy process. Convergence and stability are also considered in [23]. Another important work on the application of the FDM method is studied in [24], in which an alternating direction implicit method (ADI) is applied to solve the PIDE arising from the Possion jump. The ADI approach is shown to be unconditionally stable and efficient when it is combined with the fast Fourier transform (FFT) methods (see [24]). The finite difference method (FDM) is applied in [25] to solve the variance swaps problem using the assumption of constant volatility (see [1]), in which the two-dimensional (2D) problem is tranformed to a system of one-dimensional partial differential equations, and the price of variance swap is calculated as the average of all the solutions. The FEM ensures more flexibility and adaptivity of mesh compared to the FDM. The FEM is suitable for pricing almost all option types. Three simple applications of the FEM approach in option pricing are given in [26], including the standard Black-scholes equation, the stochastic volatility model, and the path-dependent Asian option. The FEM is also applied to study the multiasset American type option in [27]. By adding the penalty term with continuous Jacobian and solving the final ordinary differential equation (ODE) with an adaptive variable order and variable step size solver SUNDIALS, the authors prove that their approach is efficient even for multidimensional PDEs.

In this paper, our contribution includes two aspects. Firstly, we extend the multiscale volatility model in [28] by incorporating both multiscale volatility processes and the jump diffusion process to price European options and the expectation of the realised volatility. The jump term is included both in the stock process and in the volatility process, and the correlation effect is also taken into consideration. Secondly, we develop an efficient FEM method to solve the problem numerically. Inclusion of both of the two factors and the jump results in a high dimensional partial integral
differential equation (PIDE). Our chosen element is eightnodal hexahedron, which can be seen as a tensor product of three one-dimensional non-parametric elements. This largely simplifies the problem by absorbing the integral part in only one tensor (one dimensional problem).

The paper is organised into five sections. In Section 1, we briefly introduce the background of the work and our main contribution. Section 2 describes the model of the underlying asset price, with the volatility following a multiscale stochastic process, which incorporates the jump diffusion term. Section 3 presents the numerical algorithm we use to solve the problem. Numerical results are given in Section 4, followed by a conclusion in Section 5.

## 2. Model Setup

The price of stock is assumed to follow the following stochastic process:

$$
\begin{equation*}
d S=r S d t+f(y, z) S d W_{t}^{(0)}+S d J^{S} \tag{1}
\end{equation*}
$$

where $f(y, z)$ is a function of two factors $y$ and $z$ which denotes fast/slow scale volatility. If $f(y, z)=\sqrt{y}+\sqrt{z}$, the volatility process is formed by a CIR process; if $f(y, z)=$ $\sqrt{y}+1 / \sqrt{z}$, the volatility is a $4 / 2$ process which can be viewed as a combination of the CIR process and the $3 / 2$ process, and the assumption is in line with the idea that the volatility should not remain too close to zero (see [8]). It is assumed that $y$ and $z$ follow the stochastic processes

$$
\begin{align*}
& d y=\frac{1}{\xi} \alpha(y) d t+\frac{1}{\sqrt{\xi}} \beta(y) d W_{t}^{(1)}  \tag{2}\\
& d z=\sigma c(z) d t+\sqrt{\sigma} g(z) d W_{t}^{(2)} \tag{3}
\end{align*}
$$

The concept of fast-scale and slow-scale is distinguished by the frequencies of the observed volatility data, and it is suggested to consider them simultaneously in [12]. Additionally, we assume that the Brownian motion ( $W_{t}^{(0)}, W_{t}^{(1)}, W_{t}^{(2)}$ ) are correlated with the following correlation: $\operatorname{cov}\left(W_{t}^{(0)}, W_{t}^{(1)}\right)=$ $\rho_{1}, \operatorname{cov}\left(W_{t}^{(0)}, W_{t}^{(2)}\right)=\rho_{2}$, and $\operatorname{cov}\left(W_{t}^{(1)}, W_{t}^{(2)}\right)=0$ for simplicity.

In this paper, we consider both the European option and the variance swap. For the case of European put option, the pay-off function at the maturity time is

$$
\begin{equation*}
U(T, S, y, z)=\max \{K-S, 0\} \tag{4}
\end{equation*}
$$

Variance and volatility swap are well-known financial derivatives which allow investors to trade the realized volatility against the current implied volatility. Different from European options, variance swap and volatility swap are timedependent. The pay-off function of a variance swap, as shown in (5), is convex in volatility. This phenomenon indicates that the variance swap will boost the gains and discount the losses, which explains why the variance swap is more attractive than the volatility swap. The difference between the realized volatility and the implied volatility is that the realized volatility $\sigma_{R}^{2}$ is calculated by applying the historical data of
option prices, while the implied one is derived from the prices of options.

The realized volatility is commonly approximated by the following two formulas:

$$
\begin{align*}
& \sigma_{R}^{2}=\frac{A F}{N} \sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_{i}}{S_{i}}\right)^{2},  \tag{5}\\
& \sigma_{R}^{2}=\frac{A F}{N} \sum_{i=0}^{N-1}\left(\ln \frac{S_{i+1}}{S_{i}}\right)^{2}
\end{align*}
$$

where $S_{i+1}$ denotes the underlying stock price at the $(i+1)$ th time step. AF is the annualized factor and $A F=12$ if the sampling frequency is every month. In this paper, we let $A F=$ $N / T$ as a simplification. The pay-off of the variance swap is

$$
\begin{equation*}
V(T, x, y, z)=L \cdot E^{Q}\left(\sigma_{R}^{2}-K\right) \tag{6}
\end{equation*}
$$

which is equal to zero under the assumption of zero entry cost. Therefore, the fair strike price can be defined as $K=$ $E^{\mathrm{Q}}\left[\sigma_{R}^{2}\right]$. As a result, the variance swap pricing problem becomes calculating the expected value of the realized variance in the risk neutral world.

We apply the dimensional reduction technique due to [25] by introducing a new variable $I_{t}$ driven by the underlying process

$$
\begin{equation*}
I_{t}=\int_{0}^{t} \delta\left(t_{i-1}-\tau\right) S_{\tau} d \tau \tag{7}
\end{equation*}
$$

where $\delta$ is the Dirac delta function, which means $I_{t}=0$ if $t<$ $t_{i-1}$ and $I_{t}=S_{i-1}$ if $t \geq t_{i-1}$. The terminal condition becomes

$$
\begin{equation*}
U_{i}(T, S, Y, Z, I)=\left(\frac{S_{i}}{I_{i}}-1\right)^{2} \tag{8}
\end{equation*}
$$

For the reason that we are more interested in the relationship between the maturity time and the strike price, we construct a new variable $X=\ln (S / I)$ and then obtain

$$
\begin{equation*}
U_{i}(T, S, Y, Z, I)=\left(e^{X_{i}}-1\right)^{2} \tag{9}
\end{equation*}
$$

According to the Ito formula and (1), we obtain a new process

$$
\begin{equation*}
d x=\mu d t+f(y, z) d W_{t}^{(0)}+d J \tag{10}
\end{equation*}
$$

If the problem in question is an European put option,

$$
\begin{equation*}
\mu=\left(r-\frac{1}{2} f^{2}(y, z)+\lambda\left(1-E\left(e^{z}\right)\right)\right) \tag{11}
\end{equation*}
$$

and the pay-off function is

$$
\begin{equation*}
U(T, S, y, z)=\max \left\{K\left(1-e^{x}\right), 0\right\} \tag{12}
\end{equation*}
$$

If the investigated problem is a variance swap, we have two different situations,

$$
\begin{align*}
& \mu=\mu_{1}=\left(r-\frac{1}{2} f^{2}(y, z)+\lambda\left(1-E\left(e^{z}\right)\right)\right),  \tag{13}\\
& t_{i-1} \leq t \leq t_{i} \\
& \mu=\mu_{2}=\left(r-e^{x}-\frac{1}{2} f^{2}(y, z)+\lambda\left(1-E\left(e^{z}\right)\right)\right),  \tag{14}\\
& 0 \leq t \leq t_{i-1},
\end{align*}
$$

where $E\left(e^{z}\right)=p \eta_{1} /\left(1-\eta_{1}\right)+(1-p) \eta_{2} /\left(\eta_{2}+1\right)$ if the jump rate follows the double exponential distribution as in Kou's model with the density of

$$
\begin{equation*}
p(z)=p \eta_{1} e^{-\eta_{1} z} I_{z \geq 0}+(1-p) \eta_{2} e^{\eta_{2} z} I_{z<0} . \tag{15}
\end{equation*}
$$

In contrast to the model (1) which absorbs the jump in the stock process only, the multidimensional jump process is more interesting. With this motivation, we include the jump process in both the stock price process and the multiscale volatility process, namely,

$$
\begin{align*}
& d y=\frac{1}{\xi} \alpha(y) d t+\frac{1}{\sqrt{\xi}} \beta(y) d W_{t}^{(1)}+d J^{Y}  \tag{16}\\
& d z=\sigma c(z) d t+\sqrt{\sigma} g(z) d W_{t}^{(2)}+d J^{Z} \tag{17}
\end{align*}
$$

However, incorporating more factors makes the model harder to tackle with, and thus development of an efficient numerical method for high dimensional PIDE is of great importance.

## 3. Algorithm of FEM

According to the Feyman-Kac theorem, we obtain the following partial differential equation:

$$
\begin{aligned}
u_{t} & +\mathfrak{D} u+\mathfrak{C} u+\lambda \int_{R}[u(x+\eta)-u(x)] \Gamma(d \eta)-r u \\
& =0
\end{aligned}
$$

with the infinitesimal generator of the three-dimensional Markov process $\left(x_{t}, y_{t}, z_{t}\right)$. Letting $\tau=T-t$, we obtain

$$
\begin{align*}
u_{\tau} & -\mathfrak{D} u-\mathfrak{c} u-\lambda \int_{R}[u(x+\eta)-u(x)] \Gamma(d \eta)+r u  \tag{19}\\
& =0,
\end{align*}
$$

with

$$
\begin{align*}
\mathfrak{D} u(x)= & \frac{1}{2} f^{2}(y, z) U_{x x}+\frac{1}{2 \xi} \beta^{2}(y) U_{y y} \\
& +\frac{1}{2} \sigma g(z) U_{z z} \\
& +\rho_{1} \frac{1}{\sqrt{\xi}} \beta(y) f(y, z) U_{x y}  \tag{20}\\
& +\rho_{2} \sqrt{\sigma} f(y, z) g(z) U_{x z} \\
& +\rho_{12} \sqrt{\frac{\sigma}{\xi}} \beta(z) g(z) U_{y z} \\
\mathfrak{G} u(x)= & \mu U_{x}+\frac{1}{\xi} \alpha(y) U_{y}+\sigma c(z) U_{z} \\
& +\lambda \int_{R} U(x+\eta) \Gamma(d \eta)-r U \tag{21}
\end{align*}
$$

which can be rewritten in vector form by

$$
\begin{align*}
\frac{\partial u}{\partial \tau} & -\nabla \cdot \bar{A} \nabla u-D \cdot \nabla u+(r+\lambda) u \\
& -\lambda \int_{R} u(x+\eta) \Gamma(d \eta)=0 \tag{22}
\end{align*}
$$

where

$$
\begin{align*}
& \bar{A}=\left[\begin{array}{ccc}
\frac{1}{2} f^{2}(y+z) & \frac{1}{2 \sqrt{\xi}} \rho_{1} \beta(y) f(y, z) & \frac{1}{2} \sqrt{\sigma} \rho_{2} g(z) f(y, z) \\
\frac{1}{2 \sqrt{\xi}} \rho_{1} \beta(y) f(y, z) & \frac{1}{2 \xi} \rho_{1} \beta^{2}(y) & \frac{1}{2} \rho_{12} \sqrt{\frac{\sigma}{\xi}} \beta(z) g(z) \\
\frac{1}{2} \sqrt{\sigma} \rho_{2} g(z) f(y, z) & \frac{1}{2} \rho_{12} \sqrt{\frac{\sigma}{\xi}} \beta(z) g(z) & \frac{1}{2} \sigma \rho_{2} g^{2}(z)
\end{array}\right],  \tag{23}\\
& D=\left[\begin{array}{c}
\mu_{i} \\
\frac{1}{\xi} \alpha(y) \\
\sigma c(z)
\end{array}\right], \quad i=1,2 .
\end{align*}
$$

In order to obtain option price, we have to solve the differential equation (22). However, different from $\mu_{1}, \mu_{2}$ is a dynamic process which is related to time. Letting $n=T / \Delta t$, then (22) is divided into $n$ different partial differential equations. We can then solve them one by one and then substitute the solutions back into (6) to obtain the $\sigma_{R}^{2}$.

The weak form of (22) can be written as

$$
\begin{align*}
& \int_{\Omega}\left(\frac{\partial u}{\partial \tau}-\nabla \cdot \bar{A} \nabla u-D \cdot \nabla u+(r+\lambda) u\right. \\
& \left.\quad-\lambda \int_{R} u(x+\eta)\right) v d \Omega=0 . \tag{24}
\end{align*}
$$

Thus, by applying the Green Theorem, we obtain

$$
\begin{align*}
& \left(\frac{\partial u}{\partial \tau}, v\right)+(\bar{A} \nabla u, \nabla v)-(D \cdot \nabla u, v) \\
& \quad-\lambda\left(\int_{R} u(x+\eta) \Gamma(d \eta), v\right)+(r+\gamma)(u, v)  \tag{25}\\
& \quad=0
\end{align*}
$$

which is derived by using the divergence theorem

$$
\begin{equation*}
\int_{\Omega}(\bar{A} \nabla u \cdot \nabla v+\nabla \cdot \bar{A} \nabla u v) d \Omega=\oiint \bar{A} \nabla u v \cdot \vec{n} d S \tag{26}
\end{equation*}
$$

we assume that the test function vanishes on the boundary, and $(a, b)$ denotes inner product.

Letting $u=\sum_{i=1}^{n} u_{i}(\tau) \phi_{i}, v=\sum_{j=1}^{n} u_{j} \phi_{j}$, then we obtain the following ODE system:

$$
\begin{equation*}
M \dot{u}+D u-C u-B u=0 \tag{27}
\end{equation*}
$$

where the mass matrix $M=\sum_{i=1}^{n}\left(\phi_{i}, \phi_{j}\right)$, the matrix of the diffusion part $D=\sum_{i=1}^{n}\left(A \nabla \phi_{i}, \nabla \phi_{j}\right)$, the matrix of the convection part $C=\sum_{i=1}^{n}\left(D \cdot \nabla \phi_{i}, \phi_{j}\right), A=r \sum_{i=1}^{n}\left(\phi_{i}, \phi_{j}\right)$, and $B=\sum_{i=1}^{n}\left(\mathfrak{B} \phi_{i}, \phi_{j}\right)$ denotes the matrix of the integral part:

$$
\begin{align*}
\mathfrak{B} \phi_{i} & =\lambda \int_{R} \phi_{i}(x+\eta, y, z) \Gamma(d \eta) \\
& =\lambda \int_{R} \phi_{i}(x+\eta, y, z) p(\eta) d \eta \tag{28}
\end{align*}
$$

The 8-node hexahedral elements can be seen as the tensor product of three one-dimensional linear elements,

$$
\begin{equation*}
\phi_{i}^{e}(x, y, z)=\phi_{i}^{e}(x) \otimes \phi_{i}^{e}(y) \otimes \phi_{i}^{e}(z) \tag{29}
\end{equation*}
$$

with $\phi_{i}^{e}(x), \phi_{i}^{e}(y)$, and $\phi_{i}^{e}(z)$ denoting the one-dimensional (1D) shape function in each dimension. For example, assume the natural shape functions in one-dimension as

$$
\begin{align*}
& N_{1}^{d}=\frac{1}{2}(1-\epsilon), \\
& N_{2}^{d}=\frac{1}{2}(1+\epsilon) . \tag{30}
\end{align*}
$$

If it is in x -dimension $d=x$; if it is in y -dimension $d=y$; else, $d=z$. Thus, the shape functions of the 8 -node hexahedral element are

$$
\begin{equation*}
\phi_{i}^{e}=\frac{1}{8}\left(1+\epsilon \epsilon_{i}\right)\left(1+\eta \eta_{i}\right)\left(1+\zeta \zeta_{i}\right) \tag{31}
\end{equation*}
$$

with $\epsilon_{i}, \eta_{i}$, and $\zeta_{i}$ denoting the natural coordinates of the ith nodes. To be more specific, the 8 -node hexahedral element can be expanded into form shown in (see Table 1).

Table 1: 8-node hexahedral element.

| $N_{1}=\frac{1}{8}\left(1+\epsilon_{i}\right)\left(1+\eta_{i}\right)\left(1+\zeta_{i}\right)$ | $N_{5}=\frac{1}{8}\left(1-\epsilon_{i}\right)\left(1+\eta_{i}\right)\left(1+\zeta_{i}\right)$ |
| :--- | :--- |
| $N_{2}=\frac{1}{8}\left(1+\epsilon_{i}\right)\left(1+\eta_{i}\right)\left(1-\zeta_{i}\right)$ | $N_{6}=\frac{1}{8}\left(1-\epsilon_{i}\right)\left(1+\eta_{i}\right)\left(1-\zeta_{i}\right)$ |
| $N_{3}=\frac{1}{8}\left(1+\epsilon_{i}\right)\left(1-\eta_{i}\right)\left(1-\zeta_{i}\right)$ | $N_{7}=\frac{1}{8}\left(1-\epsilon_{i}\right)\left(1-\eta_{i}\right)\left(1+\zeta_{i}\right)$ |
| $N_{4}=\frac{1}{8}\left(1+\epsilon_{i}\right)\left(1-\eta_{i}\right)\left(1-\zeta_{i}\right)$ | $N_{8}=\frac{1}{8}\left(1-\epsilon_{i}\right)\left(1-\eta_{i}\right)\left(1-\zeta_{i}\right)$ |

Let $\Phi_{i}$ denote the integral term,

$$
\begin{align*}
\Phi_{i}\left(x_{l}\right)= & \int_{R} \phi_{i}\left(x_{l}+\eta\right) p(\eta) d \eta \\
= & \int_{x_{i}}^{x_{i+1}} \phi_{i}(\bar{x}) p\left(\bar{x}-x_{l}\right) d \eta \\
= & \frac{h}{2} \int_{-1}^{1} \phi_{i}(\xi) p\left(\left(\frac{\xi}{2}+i-l\right) h\right) d \xi  \tag{32}\\
= & \frac{h}{4} \int_{0}^{1} \xi p\left(\left(\frac{\xi}{2}+i-l-\frac{1}{2}\right) h\right) d \xi \\
& +\frac{h}{4} \int_{0}^{1}(1-\xi) p\left(\left(\frac{\xi}{2}+i-l\right) h\right) d \xi
\end{align*}
$$

where $p(\cdot)$ is a double exponential density function and according to (32) and $\Phi_{i}\left(x_{l}\right)$ is determined by the relationship between integers $i$ and $l$. Substituting (32) into (28), the integral term can be rewritten as

$$
\begin{align*}
\mathfrak{B} \phi_{i} & =\lambda \int_{R} \phi_{i}(x+\eta, y, z) p(\eta) d \eta \\
& =\lambda \int_{R} \phi_{i}(x+\eta) p(\eta) d \eta \phi_{i}(y) \phi_{i}(z)  \tag{33}\\
& =\Phi_{i}(x) \otimes \phi_{i}(y) \otimes \phi_{i}(z)
\end{align*}
$$

with function $\Phi_{i}(x)=\lambda \int_{R} \phi_{i}(x+\eta) p(\eta) d \eta$ approximating by the finite element interpolation $\Phi_{i}(x) \approx I_{n} \Phi_{i}(x)=$ $\sum_{l} \Phi_{i}\left(x_{l}\right) \phi_{l}(x)$.

The detail proof is in the Appendix. By simple calculation, we obtain

$$
\begin{gather*}
\frac{p \lambda}{4 \eta_{1} h} e^{-\eta_{1}(i-l-1) h}\left(e^{-\eta_{1} h / 2}-1\right)^{2} \quad i-l \geq 1 \\
\frac{1}{4} \lambda+\frac{p \lambda}{4 \eta_{1} h}\left(e^{-\eta_{1} h / 2}-1\right)+\frac{(1-p) \lambda}{4 \eta_{2} h}\left(e^{-\eta_{2} h / 2}-1\right) \tag{34}
\end{gather*}
$$

$$
\frac{(1-p) \lambda}{4 \eta_{2} h} e^{-\eta_{2}(i-l-1) h}\left(e^{-\eta_{2} h / 2}-1\right)^{2} \quad i-l \leq-1
$$

Table 2: Parameters of model.

| $k_{1}=17.38863$ | $a_{1}=0.04480$ | $b_{1}=1$ | $\rho_{1}=-0.99000$ | $\sigma_{1}=3.70537$ |
| :--- | :--- | :--- | :--- | :--- |
| $k_{2}=16.20866$ | $a_{2}=0.04275$ | $b_{2}=1$ | $\rho_{2}=-0.82897$ | $\sigma_{2}=2.77650$ |

Therefore, $B$ can be seen as a Kronecker product of inner products in three dimensions:

$$
\begin{align*}
B= & B_{x} \otimes B_{y} \otimes B_{z}=\sum_{i=1}^{n}\left(\Phi_{i}(x), \phi_{j}(x)\right)  \tag{35}\\
& \cdot\left(\phi_{i}(y), \phi_{j}(y)\right)\left(\phi_{i}(z), \phi_{j}(z)\right) .
\end{align*}
$$

Moreover,

$$
\left(\phi_{i}, \phi_{j}\right)= \begin{cases}\frac{2}{3} h, & \text { if } j=i  \tag{36}\\ \frac{1}{6} h, & \text { if } j=i \pm 1 \\ 0, & \text { else }\end{cases}
$$

Let $R=D-C+B$, (27) can be written as

$$
\begin{equation*}
M \dot{u}+R u=0 . \tag{37}
\end{equation*}
$$

To solve the ODE system (37), we simply apply the backward Euler method, considering its unconditional stability property

$$
\begin{equation*}
\left(\frac{M}{\Delta t}+R\right) U_{n+1}=U_{n} \tag{38}
\end{equation*}
$$

## 4. Numerical Results and Discussion

In this section, we present our numerical results for European options and variance swap by allowing both multiscale volatility and jump properties. Firstly, we start simulating both the stock price process and the multiscale volatility processes to show the motivation of our study. Then we apply the FEM algorithm to solve the three-dimensional PIDE. The validity of our algorithm is verified by comparing our results with the results of the two factors Heston Model in [13]. Pricing of variance swap is also studied in our paper as an application.
4.1. Validity and Motivation of Our Model. To show the motivation of our model, we firstly apply Monte Carlo simulation to generate a sample path of the stock price. Figure 2 is the stock price generated by the models (1) and (2) by the classic Euler-Maruyama Method [29]. As we can see from the figure that the asset process is a martingale process and upward sloping.

In terms of the algorithm validity, we apply our FEM method to solve the model and compare the result with the semi-analytical result shown in [13]. It is seen from Figure 3(a) that our result is well fitted. Figure 3(b) shows the underlying trajectory of the fast scale volatility process, which


Figure 2: Simulation of stock price process.
is highly oscillated due to the small fast-scale rate $\xi=0.01$. The slow scale volatility is simulated in Figure 3(c) with the slow scale rate $\sigma=0.01$. The incorporation of jump process in both stock price processes has practical significance, as shown in Figures 4(a) and 4(b).

However, analytic solution only exists for some special cases if we can find the characteristic function. For other models, it is not possible to obtain.
4.2. The Effects of Multi-Scale Volatility and Jump Term. Our method is applied to determine the option price of the classical European option model (12) as well as the strike price of variance swap with the payoff function shown in Figure 8(a). To be specific, let $\alpha(y)=k_{1}\left(a_{1}-b_{1} y\right), \beta(y)=$ $\sigma_{1} \sqrt{y}, c(z)=k_{2}\left(a_{2}-b_{2} z\right), g(z)=\sigma_{2} \sqrt{z}$, and $f(y, z)=$ $\sqrt{y}+\sqrt{z}$. Both the fast-scale process and the slow scale process are assumed to be mean reverted process. The parameters we selected are from the calibrated results of [30]. $\lambda=0$, our model reduces to the original multiscale volatility model by [31]. Parameters in (1) and (2) are shown in Table 2.

Figures 5(a) and 5(c) are the surface plot of the option price and strike price of variance swap when the value of slowscale stochastic volatility $z$ is fixed and equivalent to 0.0278 . If both stock price and volatilities are all variables, we obtain the three-dimensional plot shown in Figures 5(b) and 5(d).

To show the validity of our approach, another set of data shown in Table 3 has been applied from [13] with the numerical result shown in Figures 6(a) and 6(b). When $\lambda \neq 0$,


Figure 3: Simulation of volatility process.


Figure 4: Simulation of jump process.


Figure 5: Option price.

Table 3: Parameters of model.

| $k_{1}=8.5$ | $a_{1}=0.03$ | $b_{1}=1$ | $\rho_{1}=-0.84$ | $\sigma_{1}=0.68$ |
| :--- | :--- | :--- | :--- | :---: |
| $k_{2}=0.24$ | $a_{2}=0.02$ | $b_{2}=1$ | $\rho_{2}=-0.77$ | $\sigma_{2}=1.0531$ |

the jump process here is assumed to be a double exponential process with $\eta_{1}=25, \eta_{2}=50$, and $p=0.3$.

It can be seen from Figures 6(c) and 6(d) that the jump intensity has significant effect on option price. Hence, our model is more general compared to the multi-factor Heston model. The option price increases with the jump intensity $\lambda$, mainly because the growth of jump intensity leads to large uncertainty and risk exposure rate, which offers investors more possibilities to be in the money.

Different from jumps, the effect of stochastic volatility is a combination result of fast-scale and slow-scale volatility correction. The effects of fast scale rates and slow scale rates are displayed in Figures 7(a) and 7(b), respectively. As we can see from Figure 7(a), the option price increases with the fastscale rate, while in Figure 7(b), the option price decreases
with the slow scale rate, and the effects of fast-scale rate outweigh the effects of the slow-scale rate in a short period.

Also, the jump terms can also be incorporated into both the fast scale volatility and the slow scale volatility process. The change of the option price, though small, can be seen from Figure 8(a). In Figure 8(a), MSJ denotes the multiscale stochastic volatility model with jumps in the stock price process, MS1J denotes the MSJ model with one jump term in the fast scale volatility, and MSV2J denotes the MSJ model incorporating jump terms in each of the three processes. For the reason that we expand the dimension by Kronecker method, we can avoid the loop and consequently save a lot of time. To make it more understandable, we compare the time (9.3490) that we assemble 3D matrix by using the nested loop with the time of the Kronecker method (5.2345e-04), which is much faster.

We also study the fair strike price of variance swap. Figure 8(b) shows the relationship between the strike price and the maturity time of variance swap, which is anticorrelated due to the introduction of fast and slow scale volatility. The fast scale and slow scale rate we choose in this analysis are $\xi=0.01$ and $\sigma=0.1$ separately. The result verifies that


Figure 6: The effect of jump intensity rate $\lambda$.


Figure 7: Effects of fast-scale rate and slow-scale rate.


Figure 8: The effect of jump intensity rate $\lambda$.
volatility provides a measure of risk exposure. The longer the investors hold the contract, the higher risk they are exposed.

## 5. Conclusions

In this paper, the finite element method and the dimension reduction technique are applied to obtain the approximate solution for the classical European option price and the fair strike price of variance swaps under both multiscale stochastic volatility and jump diffusion process. The time scale rate of stochastic volatility is used to model the short term and the long term perturbation of volatility process. Our numerical results are compared with Monte Carlo simulation and are a good fit. We find that the option price increases with the jump rate and volatility value, which is in line with the reality. In terms of the effects of multiscale volatility, it is a combination result. As assumed in our model, the volatility of the stock process is driven by both the fast scale volatility and the slow scale volatility. The fast scale volatility always relates to the short term volatility with high frequency, while the slow scale volatility relates to the long term volatility and is more smooth. The option price increases with the fast-scale rate and decreases with the slow scale rate, and the effect of slow scale volatility outweighs the effect of fast scale volatility in a long run. Also, the strike price of variance swap is found to be anti-correlated with the maturity time. Volatility is a measure of risk, and the strike price decreases as the maturity time increases.

The significance of this work is in two aspects. Firstly, the exact solution can be obtained only for some specified models. For most partial differential equations, especially the high dimensional ones, closed form solution is impossible to obtain, and hence it becomes necessary to use the numerical approach in order to solve the problem. Secondly, even
though stochastic volatility has already been considered in some work, multifactors in volatility are not to be tackled due to the high dimensional difficulties, and we combine both multiscale rate and jump process to make the result more reliable. In addition, the numerical method and dimensional reduction technique proposed in our paper can be applied to solve some other similar three-dimensional pricing problems.

For the future research, we are more interested in the calibration of this model with high-frequency data and the study of the skew effect. The application of the multiscale model in other financial derivatives can also be a future study area.

## Appendix

In this appendix, we show the detail of approximating the integration term by interpolation method. We have the following approximation:

$$
\begin{align*}
\mathfrak{B}\left(\phi_{i}, \phi_{j}\right)= & \sum_{l=j-1}^{l=j+1} \mathfrak{B} \phi_{i}\left(x_{l}\right)\left(\phi_{l}, \phi_{i}\right) \\
= & \frac{1}{6} j \mathfrak{B} \phi_{i}\left(x_{j-1}\right)+\frac{2}{3} h \mathfrak{B} \phi_{i}\left(x_{j}\right)  \tag{A.1}\\
& +\frac{1}{6} h \mathfrak{B} \phi_{i}\left(x_{j+1}\right),
\end{align*}
$$

with

$$
\begin{align*}
\mathfrak{B} \phi_{j}\left(x_{l}\right) & =\lambda \int_{x_{j-1}}^{x_{j+1}} \phi_{i}\left(x_{l}+z\right) p(z) d z \\
& =\int_{x_{i-1}}^{x_{i+1}} \phi_{i}(x) p\left(x-x_{l}\right) d x \tag{A.2}
\end{align*}
$$

with $x \in\left[x_{j-1}, x_{j+1}\right]$. To be consistent with the natural element, we map it to the interval of $\epsilon \in[-1,1]$ by letting

$$
\begin{equation*}
\epsilon(x)=\frac{1}{2 h}\left(x-x_{j+1}\right)+\frac{1}{2 h}\left(x-x_{j-1}\right) . \tag{A.3}
\end{equation*}
$$

Thus, we can rewrite (A.2) in the form of

$$
\begin{equation*}
\int_{-1}^{1} \phi_{i}(\epsilon) p((\epsilon+i-l) h) h d \epsilon \tag{A.4}
\end{equation*}
$$

$z=x-x_{l}=\epsilon h+(i-l) h$ being only related to the difference of $(i-l)$.

## Data Availability

The data used to support the findings of this study are included within the supplementary information files.

## Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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# Positive Solutions for a Higher-Order Semipositone Nonlocal Fractional Differential Equation with Singularities on Both Time and Space Variable 

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In this paper, we consider the following higher-order semipositone nonlocal Riemann-Liouville fractional differential equation $D_{0+}^{\alpha} x(t)+f\left(t, x(t), D_{0+}^{\beta} x(t)\right)+e(t)=0,0<t<1, D_{0+}^{\beta} x(0)=D_{0+}^{\beta+1} x(0)=\cdots=D_{0+}^{n+\beta-2} x(0)=0$, and $D_{0+}^{\beta} x(1)=\sum_{i=1}^{m-2} \eta_{i} D_{0+}^{\beta} x\left(\xi_{i}\right)$, where $D_{0^{+}}^{\alpha}$ and $D_{0^{+}}^{\beta}$ are the standard Riemann-Liouville fractional derivatives. The existence results of positive solution are given by Guo-krasnosel'skii fixed point theorem and Schauder's fixed point theorem.

## 1. Introduction

In this paper, we devote to the investigation of the following nonlinear fractional differential equation

$$
\begin{align*}
& D_{0+}^{\alpha} x(t)+f\left(t, x(t), D_{0+}^{\beta} x(t)\right)+e(t)=0, \\
& D_{0+}^{\beta} x(0)=D_{0+}^{\beta+1} x(0)=\cdots=D_{0+}^{n+\beta-2} x(0)=0, \\
& D_{0+}^{\beta} x(1)=\sum_{i=1}^{m-2} \eta_{i} D_{0+}^{\beta} x\left(\xi_{i}\right), \tag{1}
\end{align*}
$$

where $D_{0+}^{\alpha}$ and $D_{0+}^{\beta}$ are the standard Riemann-Liouville derivatives, $\alpha \geq 2,1 \leq \alpha-\beta \leq n-1, n-1<\alpha \leq n, 0<$ $\xi_{1}<\xi_{2}<\cdots<\xi_{m-2}<1,0<\sum_{i=1}^{m} \eta_{i} \xi_{i}^{\alpha-\beta-1}<1$. The nonlinear term $f(t, u, v)$ is continuous and may be singular on both $t=0,1$ and $v=0 ; e(t) \in L^{1}([0,1], \mathbb{R})$ permits signchanging.

Differential equation models can describe many nonlinear phenomena in applied mathematics, economics, finance, engineering, and physical and biological processes [1, 2]. In
recent years, there has been a great deal of research on the existence and/or uniqueness of solution in studying FDEs nonlocal problems for their wide applications in modeling some important physical laws (see [3-16], for instance).

In [3], the authors were concerned with the existence of monotone positive solutions to the following fractional-order multipoint boundary value problems

$$
\begin{align*}
D_{0+}^{\alpha} u(t)+q(t) f\left(t, u(t), u^{\prime}(t)\right) & =0, \quad t \in(0,1), \\
u(0) & =u^{\prime}(0)=0  \tag{2}\\
u^{\prime}(1) & =\sum_{i=0}^{m} \eta_{i} u^{\prime}\left(\xi_{i}\right)
\end{align*}
$$

where $0<\xi_{1}<\xi_{2}<\cdots<\xi_{m}<1,2<\alpha<3, \eta_{i} \geq 0$, and $\sum_{i=1}^{m} \eta_{i} \xi_{i}^{\alpha-2}<1$. The authors obtained the existence of monotone positive solutions and establish iterative schemes for approximating the solutions.

In [4], the authors investigated the existence of positive solutions of the following fractional differential equation multipoint boundary value problems with changing sign nonlinearity

$$
\begin{align*}
D_{0+}^{\alpha} u(t)+\lambda f(t, u(t)) & =0, \quad t \in(0,1), \\
u(0) & =u^{\prime}(0)=\cdots=u^{(n-2)}=0,  \tag{3}\\
u^{(i)}(1) & =\sum_{j=0}^{m-2} \eta_{j} u^{\prime}\left(\xi_{j}\right),
\end{align*}
$$

where $\lambda$ is a positive parameter, $\alpha \geq 2, n-1<\alpha \leq n, i \in$ $\mathbb{N}, 1 \leq i \leq n-2, \eta_{j} \geq 0(j=1,2, \cdots, m-2), 0<\xi_{1}<\xi_{2}<$ $\cdots<\xi_{m-2}<1$, and $f$ may change sign and may be singular at $t=0,1$. By employing the cone expansion and compression fixed point theorem, the existence of positive solutions was obtained.

In [5], the authors established the uniqueness of a positive solution to the following higher-order fractional differential equation:

$$
\begin{align*}
& D_{0^{+}}^{\alpha} u(t)+q(t) f\left(t, u(t), D_{0^{+}}^{\mu_{1}} u(t), \cdots, D_{0^{+}}^{\mu_{n-2}} u(t)\right) \\
& \quad=0, \quad t \in(0,1) \\
& u(0)=u^{\mu_{1}}(0)=\cdots=u^{\mu_{n-2}}(0)=0,  \tag{4}\\
& D_{0+}^{\mu} u(1)=\sum_{i=1}^{m-2} \eta_{i} D_{0+}^{\beta} x\left(\xi_{i}\right)
\end{align*}
$$

where $f:[0,1] \times(0,+\infty)^{n-1} \longrightarrow[0,+\infty)$ is continuous and $f\left(t, u_{1}, \ldots, u_{n-1}\right)$ may be singular at $u_{1}=0, \ldots, u_{n-1}=0$, and $q(t):(0,1) \longrightarrow[0,+\infty)$ is continuous and may be singular at $t=0$ and/or $t=1$. By using the fixed point theorem for the mixed monotone operator, the existence of unique positive solutions for above singular nonlocal boundary value problems of fractional differential equations is established. The nonlinear term $f$ in [11] is nonnegative.

In [11], the authors studied the existence of positive solutions for the following nonlocal fractional-order differential equations with sign-changing singular perturbation.

$$
\begin{align*}
& -D_{0^{+}}^{\alpha+2} y(t)+D_{0^{+}}^{\alpha} y(t)=f\left(t, y(t), D_{0^{+}}^{\alpha} y(t)\right)+e(t) \\
& t \in(0,1) \\
& a D_{0^{+}}^{\alpha} y(0)-b D_{0^{+}}^{\alpha+1} y(0)=\sum_{j=1}^{m-2} a_{j} D_{0^{+}}^{\alpha} y\left(\xi_{j}\right)  \tag{5}\\
& c D_{0^{+}}^{\alpha} y(1)-d D_{0^{+}}^{\alpha+1} y(1)=\sum_{j=1}^{m-2} b_{j} D_{0^{+}}^{\alpha} y\left(\xi_{j}\right)
\end{align*}
$$

where $0<\alpha \leq 1, a, c \geq 0, b, d>0,0<\xi_{j}<1, a_{j}, b_{j} \in$ $[0,+\infty), f:(0,1) \times[0, \infty) \times(0, \infty) \longrightarrow(0, \infty)$ is continuous and may be singular near the zero for the third argument, and $e(t) \in L^{1}([0,1], \mathbb{R})$ may be sign-changing. By means of Schauder's fixed point theorem, the conditions for the existence of positive solutions are established, respectively, for the cases where the nonlinearity is positive, negative, and semipositone.

Motivated by the work mentioned above, we consider the fractional-order singular nonlocal BVP (1) and establish the
existence results of positive solutions for (1). The main tools used in this paper are Guo-krasnosel'skii fixed point theorem and Schauder's fixed point theorem. For the concepts and properties about the cone theory and the fixed point theorem, one can refer to [17-21].

The rest of this paper is organized as follows: in Section 2, we present some useful preliminaries and lemmas. The main results are given in Section 3 and Section 4, in which the singular cases with respect to the time variables and space variables are discussed, respectively.

## 2. Preliminaries and Some Lemmas

Definition 1 (see [1, 2]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x(t):(0, \infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
I_{0+}^{\alpha} x(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} x(s) d s \tag{6}
\end{equation*}
$$

provided the right-hand side is pointwisely defined on $(0, \infty)$.
Definition 2 (see [1, 2]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $x(t):(0, \infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
D_{0+}^{\alpha} x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{\alpha-n+1}} d s \tag{7}
\end{equation*}
$$

provided the right-hand side is pointwisely defined on $(0, \infty)$, where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the number $\alpha$.

Lemma 3 (see [1, 2]). The unique solution of the following linear Riemann-Liouville fractional differential equation of order $\alpha>0$

$$
\begin{equation*}
D_{0+}^{\alpha} x(t)=0 \tag{8}
\end{equation*}
$$

is

$$
\begin{equation*}
x(t)=c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{n} t^{\alpha-n} \tag{9}
\end{equation*}
$$

where $c_{i}=1,2, \ldots, n, n=[\alpha]+1,[\alpha]$ denotes the integer part of the number $\alpha$.

Lemma 4 (see $[1,2]$ ). If $x \in L^{1}(0,1)$ and $D_{0^{+}}^{\alpha} x \in L^{1}(0,1)$, then

$$
\begin{equation*}
I_{0^{+}}^{\alpha} D_{0^{+}}^{\alpha} x(t)=x(t)+c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{n} t^{\alpha-n} \tag{10}
\end{equation*}
$$

where $c_{i} \in R, i=1,2, \ldots, n, n=[\alpha]+1$.
The similar proof of the following three lemmas can be traced to [5, 12]; in order to be convenient for readers to read, we now give the detailed process of proof for Lemma 5; the proofs for other two lemmas are omitted here.

Lemma 5. Let $h \in L^{1}(0,1)$ and $0<\sum_{i=1}^{m} \eta_{i} \xi_{i}^{\alpha-\beta-1}<1$, then the unique solution of the problem

$$
\begin{align*}
D_{0+}^{\alpha-\beta} x(t)+h(t) & =0, \quad 0<t<1, \\
x(0) & =x^{\prime}(0)=\cdots=x^{(n-2)}(0)=0,  \tag{11}\\
x(1) & =\sum_{i=1}^{m-2} \eta_{i} x\left(\xi_{i}\right)
\end{align*}
$$

can be expressed uniquely by

$$
\begin{equation*}
x(t)=\int_{0}^{1} G(t, s) h(s) d s \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, s)=\frac{1}{\Gamma(\alpha-\beta) g(0)} \\
& \quad \cdot \begin{cases}g(s)[t(1-s)]^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\
g(s)[t(1-s)]^{\alpha-\beta-1}-(t-s)^{\alpha-\beta-1} g(0), & 0 \leq s \leq t \leq 1,\end{cases}  \tag{13}\\
& g(s)=1-\sum_{s \leq \xi_{i}} \eta_{i}\left(\frac{\xi_{i}-s}{1-s}\right)^{\alpha-\beta-1}, \tag{14}
\end{align*}
$$

Proof. By Lemma 4, the solution of (13) can be written as

$$
\begin{align*}
x(t)= & -\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{t}(t-s)^{\alpha-\beta-1} h(s) d s+C_{1} t^{\alpha-\beta-1}  \tag{15}\\
& +C_{2} t^{\alpha-\beta-2}+\cdots+C_{n} t^{\alpha-\beta-n}
\end{align*}
$$

It follows from $x(0)=x^{\prime}(0)=\cdots=x^{(n-2)}(0)=0$ that $C_{2}=$ $\cdots=C_{n}=0$, i.e.,

$$
\begin{equation*}
x(t)=-\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{t}(t-s)^{\alpha-\beta-1} h(s) d s+C_{1} t^{\alpha-\beta-1} \tag{16}
\end{equation*}
$$

thus

$$
\begin{align*}
x(1)= & -\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{1}(1-s)^{\alpha-\beta-1} h(s) d s+C_{1} \\
x\left(\xi_{i}\right)= & -\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{\xi_{i}}\left(\xi_{i}-s\right)^{\alpha-\beta-1} h(s) d s  \tag{17}\\
& +C_{1} \xi_{i}^{\alpha-\beta-1}
\end{align*}
$$

which, together with the boundary value condition $x(1)=$ $\sum_{i=1}^{m-2} \eta_{i} x\left(\xi_{i}\right)$, implies that

$$
\begin{aligned}
& -\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{1}(1-s)^{\alpha-\beta-1} h(s) d s+C_{1} \\
& \quad=\sum_{i=1}^{m-2} \eta_{i}\left(-\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{\xi_{i}}\left(\xi_{i}-s\right)^{\alpha-\beta-1} h(s) d s\right. \\
& \left.\quad+C_{1} \xi_{i}^{\alpha-\beta-1}\right)
\end{aligned}
$$

$$
\begin{align*}
& C_{1}\left(1-\sum_{i=1}^{m-2} \eta_{i} \xi_{i}^{\alpha-\beta-1}\right)=\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{1}(1-s)^{\alpha-\beta-1} \\
& \cdot h(s) d s-\sum_{i=1}^{m-2} \frac{\eta_{i} \int_{0}^{\xi_{i}}\left(\xi_{i}-s\right)^{\alpha-\beta-1} h(s) d s}{\Gamma(\alpha-\beta)} \\
& C_{1}=\int_{0}^{1} \frac{(1-s)^{\alpha-\beta-1}}{\left(1-\sum_{i=1}^{m-2} \eta_{i} \xi_{i}^{\alpha-\beta-1}\right) \Gamma(\alpha-\beta)} h(s) d s \\
& \quad-\sum_{i=1}^{m-2} \eta_{i} \frac{\int_{0}^{\xi_{i}}\left(\xi_{i}-s\right)^{\alpha-\beta-1} h(s) d s}{\Gamma(\alpha-\beta)\left(1-\sum_{i=1}^{m-2} \eta_{i} \xi_{i}^{\alpha-\beta-1}\right)} . \tag{18}
\end{align*}
$$

i.e.,

$$
\begin{align*}
C_{1} & =\frac{1}{g(0) \Gamma(\alpha-\beta)}\left[\int_{0}^{1}(1-s)^{\alpha-\beta-1} h(s) d s\right. \\
& \left.-\sum_{i=1}^{m-2} \eta_{i} \int_{0}^{\xi_{i}}\left(\xi_{i}-s\right)^{\alpha-\beta-1} h(s) d s\right]  \tag{19}\\
& =\frac{1}{g(0) \Gamma(\alpha-\beta)}\left[\int_{0}^{1}(1-s)^{\alpha-\beta-1} g(s) h(s) d s\right.
\end{align*}
$$

thus

$$
\begin{aligned}
& x(t)=-\frac{1}{\Gamma(\alpha-\beta)} \int_{0}^{t}(t-s)^{\alpha-\beta-1} h(s) d s+C_{1} t^{\alpha-\beta-1} \\
& \quad=\frac{1}{g(0) \Gamma(\alpha-\beta)} \\
& \quad \cdot \int_{0}^{t}\left[t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} g(s)-(t-s)^{\alpha-\beta-1} g(0)\right] \\
& \quad \cdot h(s) d s-\frac{1}{g(0) \Gamma(\alpha-\beta)} \\
& \quad \cdot \int_{t}^{1} t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} h(s) d s=\int_{0}^{1} G(t, s) \\
& \quad \cdot h(s) d s .
\end{aligned}
$$

Lemma 6. If $0<\sum_{i=1}^{m} \eta_{i} \xi_{i}^{\alpha-\beta-1}<1$, then the function $g$ satisfies the following conditions:
(1) $g$ is a nondecreasing function on $[0,1]$;
(2) there exist $M_{1} \geq m_{1} \geq 0$, such that $m_{1} t+g(0) \leq g(t) \leq$ $M_{1} t+g(0)$, for any $t \in[0,1]$, where $M_{1}=\sup _{0<t \leq 1}((g(t)-$ $g(0)) / t), m_{1}=\inf _{0<t \leq 1}((g(t)-g(0)) / t)$.

Remark 7. It is easy to prove that $m_{1}>0$.
Lemma 8. The function $G(t, s)$ defined by (13) has the following properties:
(1) $G(t, s)>0$ for any $(t, s) \in(0,1)$;
(2) $G(t, s) \geq\left(m_{1} / g(0) \Gamma(\alpha-\beta)\right) s(1-s)^{\alpha-\beta-1} t^{\alpha-\beta-1}$ for any $(t, s) \in[0,1]$;
(3) $G(t, s) \leq \quad\left(\left(M_{1}+g(0)[\alpha-\beta]\right) / g(0) \Gamma(\alpha-\right.$ $\beta)) s(1-s)^{\alpha-\beta-1} t^{\alpha-\beta-1}$ for any $(t, s) \in[0,1]$;
(4) $G(t, s) \leq M s(1-s)^{\alpha-\beta-1}$,
where $[\alpha-\beta]$ denotes the integer part of the number $\alpha-\beta$,

$$
\begin{equation*}
M=\frac{M_{1}+g(0)(\alpha-\beta-1)}{g(0) \Gamma(\alpha-\beta)} . \tag{21}
\end{equation*}
$$

Set $v(t)=D_{0^{+}}^{\beta} x(t)$, then (1) can be transformed into the following form:

$$
\begin{align*}
& D_{0+}^{\alpha-\beta} v(t)+f\left(t, I_{0+}^{\beta} v(t), v(t)\right)+e(t)=0, \\
& v(0)=v^{\prime}(0)=\cdots=v^{(n-2)}(0)=0, \\
& v(1)=\sum_{i=1}^{m-2} \eta_{i} v\left(\xi_{i}\right) . \tag{22}
\end{align*}
$$

From Lemma 5, we know that the solution $v(t)$ of (22) satisfies

$$
\begin{equation*}
v(t)=\int_{0}^{1} G(t, s)\left[f\left(s, I_{0+}^{\beta} v(s), v(s)\right)+e(s)\right] d s \tag{23}
\end{equation*}
$$

Lemma 9 (see $[17,18]$ ). Suppose that $E$ is a Banach space and $D \subset E$ is a bounded convex closed set, the operator $A: D \longrightarrow$ $D$ is completely continuous, then $A$ has one fixed point on $D$.

Lemma 10 (see [20] (Guo-krasnosel'skii fixed point theorem)). Let $\Omega_{1}$ and $\Omega_{1}$ be two bounded open sets in Banach space $E$ such that $\theta \in \Omega_{1}$ and $\overline{\Omega_{1}} \subset \Omega_{2}, A: P \bigcap\left(\overline{\Omega_{2}} \backslash \Omega_{1}\right) \longrightarrow P$ a completely continuous operator, where $\theta$ denotes the zero element of $E$ and $P$ a cone of E. Suppose that one of the following conditions
(i) $\|A x\| \geq\|x\|, \forall x \in P \cap \partial \Omega_{1}$ and $\|A x\| \leq\|x\|, \forall x \in$ $P \cap \partial \Omega_{2} ;$
(ii) $\|A x\| \leq\|x\|, \forall x \in P \cap \partial \Omega_{1}$ and $\|A x\| \geq\|x\|, \forall x \in$ $P \cap \partial \Omega_{2}$
holds. Then $A$ has at least one fixed point in $P \cap\left(\overline{\Omega_{2}} \backslash \Omega_{1}\right)$.

## 3. Main Result I: $f$ Is Singular with Respect to the Time Variables

Let $E=C[0,1],\|u\|=\max _{0 \leq t \leq 1}|u(t)|$, then $(E,\|\cdot\|)$ is a Banach space. Set

$$
\begin{align*}
P & =\left\{v \in E: v(t) \geq \frac{m_{1}}{M_{1}+g(0)[\alpha-\beta]} t^{\alpha-\beta-1}\|v\|, t\right. \\
& \in[0,1]\} \tag{24}
\end{align*}
$$

where $[\alpha-\beta]$ denotes the integer part of the number $\alpha-\beta$. Then $P \subset E$ is a positive cone of $E$. For convenience, we list some conditions which will be used in this section.
$\left(H_{1}\right)$ For any $(t, x, y) \in(0,1) \times[0,+\infty) \times[0,+\infty)$,

$$
\begin{equation*}
0 \leq f(t, x, y) \leq \phi(t)(\rho(x)+h(y)) \tag{25}
\end{equation*}
$$

where $\phi \in C(0,1), \phi(t)>0$ on $(0,1), \rho(x)>0$ is continuous and increasing on $[0,+\infty), h(x)>0$ is continuous and decreasing on $[0,+\infty)$.

$$
0<\int_{0}^{1} s(1-s)^{\alpha-\beta-1}\left(\phi(s)+e_{+}(s)+e_{-}(s)\right) d s
$$

$$
<+\infty
$$

where $e_{+}=\max \{e(t), 0\}, e_{-}=-\min \{e(t), 0\}$ are the positive part and negative part of $e(t)$, respectively.
$\left(H_{3}\right)$ There exists $r_{1}>0$, such that

$$
\begin{align*}
& \rho\left(\frac{r_{1}+\|w\|}{\Gamma(\beta)}\right) \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \phi(s) d s \\
& \quad+\int_{0}^{1} s(1-s)^{\alpha-\beta-1}\left[h(0) \phi(s)+e_{+}(s)\right] d s  \tag{27}\\
& \quad<\frac{r_{1}}{M},
\end{align*}
$$

where $w(t)$ is the solution of the following linear equation

$$
\begin{align*}
D_{0+}^{\alpha-\beta} v(t)+e_{-}(t) & =0, \quad 0<t<1 \\
v(0) & =v^{\prime}(0)=\cdots=v^{(n-2)}(0)=0,  \tag{28}\\
v(1) & =\sum_{i=1}^{m-2} \eta_{i} v\left(\xi_{i}\right)
\end{align*}
$$

i.e., $w(t)=\int_{0}^{1} G(t, s) e_{-}(s) d s$.
$\left(H_{4}\right)$ There exists $[a, b] \subset(0,1)$ such that

$$
\begin{equation*}
\lim _{x+y \rightarrow+\infty} \frac{f(t, x, y)}{x+y}>L \tag{29}
\end{equation*}
$$

uniformly holds for $t \in[a, b]$, where

$$
\begin{equation*}
L=\frac{2 \Gamma(\alpha-\beta) g(0)\left(M_{1}+g(0)[\alpha-\beta]\right)}{m_{1}^{2} a^{\alpha-\beta-1} \int_{a}^{b} s^{\alpha-\beta}(1-s)^{\alpha-\beta-1} d s} . \tag{30}
\end{equation*}
$$

For any $v \in P$, let

$$
\begin{equation*}
[v(t)-w(t)]^{*}=\max \{v(t)-w(t), 0\} \tag{31}
\end{equation*}
$$

and define operator

$$
\begin{align*}
& F v(t)=\int_{0}^{1} G(t, s) \\
& \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right.  \tag{32}\\
& \left.\quad+e_{+}(s)\right] d s
\end{align*}
$$

From condition $\left(H_{1}\right)$ and $\left(H_{2}\right)$, it is easy to know that $F$ is well defined.

Lemma 11. $F: P \longrightarrow P$ is a completely continuous operator.
Proof. For any $v \in P$, it follows from Lemma 8 that

$$
\begin{align*}
& F v(t)=\int_{0}^{1} G(t, s) \\
& \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right. \\
& \left.\quad+e_{+}(s)\right] d s \leq \frac{M_{1}+g(0)[\alpha-\beta]}{g(0) \Gamma(\alpha-\beta)} \\
& \quad \cdot t^{\alpha-\beta-1} \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \\
& \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right. \\
& \left.\quad+e_{+}(s)\right] d s \leq \frac{M_{1}+g(0)[\alpha-\beta]}{g(0) \Gamma(\alpha-\beta)} \int_{0}^{1} s(1 \\
& \quad-s)^{\alpha-\beta-1} \\
& \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right.  \tag{33}\\
& \left.\quad+e_{+}(s)\right] d s, \\
& F v(t)=\int_{0}^{1} G(t, s) \\
& \quad \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right. \\
& \left.\quad+e_{+}(s)\right] d s \geq \frac{m_{1}}{g(0) \Gamma(\alpha-\beta)} t^{\alpha-\beta-1} \int_{0}^{1} s(1 \\
& \quad-s)^{\alpha-\beta-1} \\
& \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right. \\
& \left.+e_{+}(s)\right] d s,
\end{align*}
$$

which deduce that $F v(t) \geq\left(m_{1} /\left(M_{1}+g(0)[\alpha-\right.\right.$ $\beta])) t^{\alpha-\beta-1}\|F v\|$, i.e., $F: P \longrightarrow P$.

Let $B \subset P$ be a bounded set, i.e., there exists $L_{1}>0$ such that $\|v\| \leq L_{1}$ for any $v \in B$, then

$$
\begin{aligned}
0 \leq & {[v(t)-w(t)]^{*} \leq L_{1}+\int_{0}^{1} G(t, s) e_{-}(s) d s \leq L_{1} } \\
& +\int_{0}^{1} M s(1-s)^{\alpha-\beta-1} e_{-}(s) d s \doteq \widetilde{L} . \\
F v & (t)=\int_{0}^{1} G(t, s) \\
\cdot & {\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right.} \\
& \left.+e_{+}(s)\right] d s \leq \int_{0}^{1} G(t, s) \\
\cdot & {\left[\phi(s)\left(\rho\left(I_{0+}^{\beta} \widetilde{L}\right)+h(0)\right)+e_{+}(s)\right] d s }
\end{aligned}
$$

$$
\begin{align*}
& \leq M\left(\rho\left(\frac{\widetilde{L}}{\beta \Gamma(\beta)}\right)+h(0)\right) \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \\
& \cdot \phi(s) d s+M \int_{0}^{1} s(1-s)^{\alpha-\beta-1} e_{+}(s) d s \\
& <+\infty \tag{35}
\end{align*}
$$

therefore

$$
\begin{align*}
\|F v\| \leq & M\left(\rho\left(\frac{\widetilde{L}}{\beta \Gamma(\beta)}\right)+h(0)\right) \\
& \cdot \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \phi(s) d s  \tag{36}\\
& +M \int_{0}^{1} s(1-s)^{\alpha-\beta-1} e_{+}(s) d s
\end{align*}
$$

for any $v \in B$, which implies that $F$ is uniformly bounded.
From $\left(\mathrm{H}_{2}\right)$, the absolutely continuity of integral and the uniformly continuity of $G(t, s)$ on $[0,1]$, we know that for any $\varepsilon>0, \exists \delta>0$, such that

$$
\begin{align*}
& \int_{0}^{\delta} s(1-s)^{\alpha-\beta-1} \phi(s) d s \\
& \quad<\frac{\varepsilon}{6 M(\rho(\widetilde{L} / \beta \Gamma(\beta))+h(0))}  \tag{37}\\
& \int_{1-\delta}^{\delta} s(1-s)^{\alpha-\beta-1} \phi(s) d s \\
& \quad<\frac{\varepsilon}{6 M(\rho(\widetilde{L} / \beta \Gamma(\beta))+h(0))} \tag{38}
\end{align*}
$$

and

$$
\begin{equation*}
\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right|<\frac{\varepsilon}{3 l(\rho(\widetilde{L} / \beta \Gamma(\beta))+h(0))} \tag{39}
\end{equation*}
$$

for any $t_{1}, t_{2}, s \in[0,1]$ with $\left|t_{1}-t_{2}\right|<\delta$.
(37)-(39), together with Lemma 8, imply that

$$
\begin{aligned}
& \left|F v\left(t_{1}\right)-F v\left(t_{2}\right)\right| \leq \int_{0}^{1}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right) d s \\
& \quad=\int_{0}^{\delta}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| \\
& \cdot f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right) d s \\
& \quad+\int_{1-\delta}^{1}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right) d s \\
& \quad+\int_{\delta}^{1-\delta}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right) d s
\end{aligned}
$$

$$
\begin{align*}
& \leq 2 M \int_{0}^{\delta} s(1-s)^{\alpha-\beta-1} \phi(s) \\
& \cdot\left(\rho\left(I_{0+}^{\beta} \widetilde{L}\right)+h(0)\right) d s+2 M \int_{1-\delta}^{1} s(1-s)^{\alpha-\beta-1} \\
& \cdot \phi(s)\left(\rho\left(I_{0+}^{\beta} \widetilde{L}\right)+h(0)\right) d s \\
& +\int_{\delta}^{1-\delta}\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right| \\
& \cdot l\left(\rho\left(I_{0+}^{\beta} \widetilde{L}\right)+h(0)\right) d s<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon \tag{40}
\end{align*}
$$

for any $t_{1}, t_{2} \in[0,1]$ with $\left|t_{1}-t_{2}\right|<\delta$ and any $v \in B$, where $l=\max _{\delta \leq t \leq 1-\delta} \phi(t)$, which deduces that $F$ is equicontinuous on $[0,1]$. Thus, according to Ascoli-Arzela theorem, we know that $F B$ is a relatively compact set, and that $F$ is a completely continuous operator.

Theorem 12. Suppose that $\left(H_{1}\right)-\left(H_{4}\right)$ hold, then the FVP (1) has at least one positive solution.

Proof. For any $v \in \partial P_{r_{1}}$, where $P_{r_{1}}=\left\{v \in P \mid\|v\|<r_{1}\right\}$, by (32), Lemma 8 and condition $\left(H_{3}\right)$, one can get that

$$
\begin{aligned}
& F v(t)=\int_{0}^{1} G(t, s) \\
& \cdot {\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right.} \\
&+\left.e_{+}(s)\right] d s \leq \int_{0}^{1} G(t, s) \\
& \cdot {\left[\phi ( s ) \left(\rho\left(I_{0+}^{\beta}(v(s)-w(s))+h(0)\right)\right.\right.} \\
&+\left.e_{+}(s)\right] d s \leq \int_{0}^{1} G(t, s) \\
& \cdot {\left[\phi(s)\left(\rho\left(I_{0+}^{\beta}(\|v\|+\|w\|)\right)+h(0)\right)+e_{+}(s)\right] d s } \\
&=M \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \\
& \cdot {\left[\phi(s)\left(\rho\left(I_{0+}^{\beta}\left(r_{1}+\|w\|\right)\right)+h(0)\right)+e_{+}(s)\right] d s } \\
&=M \int_{0}^{1} s(1-s)^{\alpha-\beta-1} \\
& \cdot {\left[\phi(s)\left(\rho\left(\frac{r_{1}+\|w\|}{\Gamma(\beta)}\right)+h(0)\right)+e_{+}(s)\right] d s } \\
& \quad<r_{1}=\|v\|
\end{aligned}
$$

i.e., $\|F v\| \leq\|v\|, \quad v \in \partial P_{r_{1}}$.

By condition $\left(H_{4}\right), \exists X>0$, such that

$$
\begin{equation*}
f(t, x, y)>L(x+y) \tag{42}
\end{equation*}
$$

for any $x>X$ and any $t \in[a, b]$. Choose $r_{2}$ such that

$$
\begin{equation*}
r_{2}>\max \left\{r_{1}, 2 c_{1}, \frac{2\left(M_{1}+g(0)[\alpha-\beta]\right) X}{m_{1} a^{\alpha-\beta-1}}\right\} \tag{43}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{1}=\frac{\left(M_{1}+g(0)[\alpha-\beta]\right)^{2} \cdot \int_{0}^{1}(1-s)^{\alpha-\beta-1} e_{-}(s) d s}{m_{1} \Gamma(\alpha-\beta) g(0)} . \tag{44}
\end{equation*}
$$

For any $v \in \partial P_{r_{2}}$, where $P_{r_{2}}=\left\{v \in P \mid\|v\|<r_{2}\right\}$. Because

$$
\begin{align*}
w(t) & =\int_{0}^{1} G(t, s) e_{-}(s) d s \\
\leq & \frac{M_{1}+g(0)[\alpha-\beta]}{\Gamma(\alpha-\beta) g(0)} \\
& \cdot t^{\alpha-\beta-1} \int_{0}^{1}(1-s)^{\alpha-\beta-1} e_{-}(s) d s  \tag{45}\\
\leq & \frac{\left(M_{1}+g(0)[\alpha-\beta]\right)^{2} \cdot \int_{0}^{1}(1-s)^{\alpha-\beta-1} e_{-}(s) d s}{m_{1} \Gamma(\alpha-\beta) g(0) r_{2}} \\
& \cdot v(t)=\frac{c_{1}}{r_{2}} v(t),
\end{align*}
$$

so we have

$$
\begin{equation*}
v(t)-w(t) \geq\left(1-\frac{c_{1}}{r_{2}}\right) v(t) \geq \frac{1}{2} v(t), \tag{46}
\end{equation*}
$$

and then for $t \in[a, b]$,

$$
\begin{align*}
v(t)-w(t) & \geq \frac{1}{2} v(t) \geq \frac{1}{2} \cdot \frac{m_{1} a^{\alpha-\beta-1}}{M_{1}+g(0)[\alpha-\beta]} \cdot\|v\|  \tag{47}\\
& =\frac{1}{2} \cdot \frac{m_{1} a^{\alpha-\beta-1}}{M_{1}+g(0)[\alpha-\beta]} \cdot r_{2}>X
\end{align*}
$$

follows from (43) and the definition of cone $P$.
From (42), (43), and (47), one can obtain that

$$
\begin{aligned}
& F v(t)=\int_{0}^{1} G(t, s) \\
& \quad \cdot\left[f\left(s, I_{0+}^{\beta}[v(s)-w(s)]^{*},[v(s)-w(s)]^{*}\right)\right. \\
& \left.\quad+e_{+}(s)\right] d s \geq \int_{a}^{b} G(t, s) L\left(I_{0+}^{\beta}(v(s)-w(s))\right. \\
& \left.\quad+(v(s)-w(s)))+e_{+}(s)\right] d s \geq L \int_{a}^{b} G(t, s) \\
& \quad \cdot \frac{1}{2} v(s) d s \geq \frac{L}{2} \cdot \frac{m_{1}}{\Gamma(\alpha-\beta) g(0)} \cdot \int_{a}^{b} s(1 \\
& \quad-s)^{\alpha-\beta-1} t^{\alpha-\beta-1} v(s) d s \geq \frac{L}{2} \cdot \frac{m_{1} a^{\alpha-\beta-1}}{\Gamma(\alpha-\beta) g(0)}
\end{aligned}
$$

$$
\begin{align*}
& \cdot \frac{m_{1}}{M_{1}+g(0)[\alpha-\beta]} \cdot \int_{a}^{b} s(1-s)^{\alpha-\beta-1} s^{\alpha-\beta-1} d s \\
& \cdot\|v\|=\frac{L}{2} \cdot \frac{m_{1} a^{\alpha-\beta-1}}{\Gamma(\alpha-\beta) g(0)} \cdot \frac{m_{1}}{M_{1}+g(0)[\alpha-\beta]} \\
& \cdot \int_{a}^{b}(1-s)^{\alpha-\beta-1} s^{\alpha-\beta} d s \cdot\|v\| \geq \frac{L}{2} \cdot \frac{m_{1} a^{\alpha-\beta-1}}{\Gamma(\alpha-\beta) g(0)} \\
& \cdot \frac{m_{1}}{M_{1}+g(0)[\alpha-\beta]} \cdot \int_{a}^{b}(1-s)^{\alpha-\beta-1} s^{\alpha-\beta} d s \cdot r_{2} \\
& =r_{2}=\|v\| \tag{48}
\end{align*}
$$

i.e., $\|F v\| \geq\|v\|, \quad v \in \partial P_{r_{2}}$.

It follows from Lemma 10 that $F$ has at least fixed point $v_{1} \in \overline{P_{r_{2}}} \backslash P_{r_{1}}$, i.e., $v_{1}$ satisfies

$$
\begin{align*}
& D_{0+}^{\alpha-\beta} v_{1}(t)+f\left(t, I_{0+}^{\beta}\left(v_{1}(t)-w(t)\right), v_{1}(t)-w(t)\right) \\
& \quad+e(t)=0, \quad 0<t<1, \\
& v_{1}(0)=v_{1}^{\prime}(0)=\cdots=v_{1}^{(n-2)}(0)=0,  \tag{49}\\
& v_{1}(1)=\sum_{i=1}^{m-2} \eta_{i} v_{1}\left(\xi_{i}\right) .
\end{align*}
$$

Set $\overline{v_{1}}(t)=v_{1}(t)-w(t)$, noticing that $v_{1}(t), w(t)$ are the solutions of BVP (32) and (49), respectively; therefore we can conclude that $\overline{v_{1}}(t)$ is a positive solution of (22). Let $x_{1}(t)=$ $I_{0+}^{\beta} \overline{v_{1}}(t)$, then $x_{1}(t)$ is a positive solution of the nonlinear fractional differential equations (1).

## 4. Main Result II: $f$ Is Singular with Respect to Both the Time Variables and the Space Variable

In this section, we always suppose that the following condition holds.
$\left(H_{5}\right) f(t, u, v):(0,1) \times[0, \infty) \times(0, \infty) \longrightarrow[0, \infty)$ is continuous, there exist $\varepsilon \in(0,1)$ and $\mu_{1}, \mu_{2} \in C^{+}[0,1]$, $\mu_{1}(t) \not \equiv 0$ for $t \in[0,1]$ such that

$$
\begin{equation*}
\frac{\mu_{1}(t)}{(x+y)^{\varepsilon}} \leq f(t, x, y) \leq \frac{\mu_{2}(t)}{(x+y)^{\varepsilon}} \tag{50}
\end{equation*}
$$

for any $(x, y) \in[0, \infty) \times(0, \infty), t \in(0,1)$, where

$$
\begin{equation*}
C^{+}[0,1]=\{x(t) \in C[0,1] \mid x(t) \geq 0, t \in[0,1]\} \tag{51}
\end{equation*}
$$

Set

$$
\begin{equation*}
\varphi(t)=\int_{0}^{1} G(t, s) e(s) d s, \quad t \in[0,1] \tag{52}
\end{equation*}
$$

it follows from Lemma 5 that $\varphi(t)$ is the solution of the following linear equation

$$
\begin{align*}
D_{0+}^{\alpha-\beta} v(t)+e(t) & =0, \quad 0<t<1, \\
v(0) & =v^{\prime}(0)=\cdots=v^{(n-2)}(0)=0,  \tag{53}\\
v(1) & =\sum_{i=1}^{m-2} \eta_{i} v\left(\xi_{i}\right) .
\end{align*}
$$

Denote

$$
\begin{align*}
a_{1}(t) & =\int_{0}^{1} G(t, s) \mu_{1}(s) d s, \\
a_{2}(t) & =\int_{0}^{1} G(t, s) \mu_{2}(s) d s, \\
\varphi_{*} & =\inf _{0 \leq t \leq 1} \varphi(t), \\
\varphi^{*} & =\sup _{0 \leq t \leq 1} \varphi(t),  \tag{54}\\
a_{1_{*}} & =\min _{0 \leq t \leq 1} a_{1}(t), \\
a_{1}^{*} & =\max _{0 \leq t \leq 1} a_{1}(t), \\
a_{2_{*}} & =\min _{0 \leq t \leq 1} a_{2}(t), \\
a_{2}^{*} & =\max _{0 \leq t \leq 1} a_{2}(t) .
\end{align*}
$$

Clearly, $a_{j}^{*} \geq a_{j_{*}}>0, j=1,2$.
Theorem 13. Suppose that the condition $\left(H_{5}\right)$ holds and $\varphi_{*} \geq$ 0 . Then the FVP (1) has at least one positive solution.

Proof. Because $\varphi_{*} \geq 0$, so we can choose $R>0$ large enough such that

$$
\begin{align*}
\frac{r}{R^{\varepsilon}}+\varphi_{*} & \geq \frac{1}{R},  \tag{55}\\
R^{\varepsilon}\left(a_{2}^{*}+\varphi^{*}\right) & \leq R,
\end{align*}
$$

where $r=a_{1_{*}} /[1+1 /(1+\beta \Gamma(\beta))]^{\varepsilon}$. In fact, since

$$
\begin{equation*}
\lim _{x \rightarrow+\infty} \frac{x^{\varepsilon}}{x}=0<\min \left\{r, \frac{1}{a_{2}^{*}+\varphi^{*}}\right\}, \tag{56}
\end{equation*}
$$

there exists $X_{1}>0$ such that $x^{\varepsilon} / x<\min \left\{r, 1 /\left(a_{2}^{*}+\varphi^{*}\right)\right\}$ for any $x>X_{1}$, i.e.,

$$
\begin{equation*}
\frac{r}{x^{\varepsilon}} \geq \frac{1}{x}, \quad x^{\varepsilon}\left(a_{2}^{*}+\varphi^{*}\right) \leq x \tag{57}
\end{equation*}
$$

for any $x>X_{1}$. If $\varphi_{*}>0$, then from $\lim _{x \rightarrow+\infty}\left[1 / x-r / x^{\varepsilon}\right]=$ $0<\varphi^{*}$, one can get that there exists $X_{2}>0$ such that $1 / x-$ $r / x^{\varepsilon}<\varphi^{*}$ for any $x>X_{2}$, i.e.,

$$
\begin{equation*}
\frac{r}{x^{\varepsilon}}+\varphi_{*} \geq \frac{1}{x} \tag{58}
\end{equation*}
$$

for any $x>X_{2}$. By (57) (58), we can choose $R>\max \left\{X_{1}, X_{2}\right\}$ such that $R$ satisfies (55). Set

$$
\begin{equation*}
D=\left\{v \in C^{+}[0,1]: \frac{1}{R} \leq v(t) \leq R, t \in[0,1]\right\} . \tag{59}
\end{equation*}
$$

For any $v \in D_{1}$, from (23) we have

$$
\begin{align*}
I_{0+}^{\beta} v(t) & =\frac{1}{\Gamma(\beta)} \int_{0}^{t}(t-s)^{\beta-1} v(s) d s \\
& \leq \frac{R}{\Gamma(\beta)} \int_{0}^{t}(t-s)^{\beta-1} d s=\frac{R t^{\beta}}{\beta \Gamma(\beta)} \leq \frac{R}{\beta \Gamma(\beta)}  \tag{60}\\
I_{0+}^{\beta} v(t) & =\frac{1}{\Gamma(\beta)} \int_{0}^{t}(t-s)^{\beta-1} v(s) d s \\
& \geq \frac{1}{R \Gamma(\beta)} \int_{0}^{t}(t-s)^{\beta-1} d s=\frac{t^{\beta}}{R \beta \Gamma(\beta)} \tag{61}
\end{align*}
$$

It follows from (60), (61), and $\left(\mathrm{H}_{5}\right)$ that

$$
\begin{align*}
\frac{\mu_{1}(t)}{R^{\varepsilon}(1+1 / \beta \Gamma(\beta))^{\varepsilon}} & \leq \frac{\mu_{1}(t)}{\left(v(t)+I_{0+}^{\beta} v(t)\right)^{\varepsilon}} \\
& \leq f\left(t, I_{0+}^{\beta} v(t), v(t)\right), \\
f\left(t, I_{0+}^{\beta} v(t), v(t)\right) & \leq \frac{\mu_{2}(t)}{\left(v(t)+I_{0+}^{\beta} v(t)\right)^{\varepsilon}}  \tag{62}\\
& \leq \frac{\mu_{2}(t)}{\left(1 / R+t^{\beta} / R \beta \Gamma(\beta)\right)^{\varepsilon}} \\
& \leq R^{\varepsilon} \mu_{2}(t),
\end{align*}
$$

i.e.,

$$
\begin{equation*}
\frac{\mu_{1}(t)}{R^{\varepsilon}(1+1 / \beta \Gamma(\beta))^{\varepsilon}} \leq f\left(t, I_{0+}^{\beta} v(t), v(t)\right) \leq R^{\varepsilon} \mu_{2}(t) \tag{63}
\end{equation*}
$$

And then

$$
T v(t)=\int_{0}^{1} G(t, s)\left[f\left(s, I_{0+}^{\beta} v(s), v(s)\right)+e(s)\right] d s
$$

$$
\begin{align*}
& =\int_{0}^{1} G(t, s) f\left(s, I_{0+}^{\beta} v(s), v(s)\right) d s+\varphi(t) \\
& \leq \int_{0}^{1} G(t, s) f\left(s, I_{0+}^{\beta} v(s), v(s)\right) d s+\varphi^{*} \\
& \leq R^{\varepsilon} \int_{0}^{1} G(t, s) \mu_{2}(s) d s+\varphi^{*}<+\infty \tag{64}
\end{align*}
$$

which deduces that the operator $T$ is well defined.
Now, we shall prove that $T: D \longrightarrow D$. For $v \in D$, it is easy to see that $T v(t) \in C^{+}[0,1]$, and by (55) (63) we can obtain that

$$
\begin{align*}
T v(t)= & \int_{0}^{1} G(t, s)\left[f\left(s, I_{0+}^{\beta} v(s), v(s)\right)+e(s)\right] d s \\
= & \int_{0}^{1} G(t, s) f\left(s, I_{0+}^{\beta} v(s), v(s)\right) d s+\varphi(t) \\
\geq & \frac{1}{R^{\varepsilon}(1+1 / \beta \Gamma(\beta))^{\varepsilon}} \int_{0}^{1} G(t, s) \mu_{1}(s) d s  \tag{65}\\
& +\varphi(t) \geq \frac{1}{R^{\varepsilon}(1+1 / \beta \Gamma(\beta))^{\varepsilon}} a_{1_{*}}+\varphi_{*} \geq \frac{1}{R}
\end{align*}
$$

$$
\begin{align*}
T v(t) & =\int_{0}^{1} G(t, s)\left[f\left(s, I_{0+}^{\beta} v(s), v(s)\right)+e(s)\right] d s \\
& =\int_{0}^{1} G(t, s) f\left(s, I_{0+}^{\beta} v(s), v(s)\right) d s+\varphi(t)  \tag{66}\\
& \leq R^{\varepsilon} \int_{0}^{1} G(t, s) \mu_{2}(s) d s+\varphi^{*} \leq R^{\varepsilon} a_{2}^{*}+\varphi^{*} \\
& \leq R^{\varepsilon}\left(a_{2}^{*}+\varphi^{*}\right) \leq R
\end{align*}
$$

i.e., $T: D \longrightarrow D$.

Next, let us prove that $T: D \longrightarrow D$ is completely continuous.

For any $\left\{v_{n}\right\} \subset D, v_{0} \in D$, and $v_{n} \longrightarrow v_{0}$. The continuity of $f$ deduces that

$$
\begin{equation*}
f\left(t, I_{0+}^{\beta} v_{n}(t), v_{n}(t)\right) \longrightarrow f\left(t, I_{0+}^{\beta} v_{0}(t), v_{0}(t)\right) \tag{67}
\end{equation*}
$$

and it follows from (63) that

$$
\begin{align*}
& \left|f\left(t, I_{0+}^{\beta} v_{n}(t), v_{n}(t)\right)-f\left(t, I_{0+}^{\beta} v_{0}(t), v_{0}(t)\right)\right| \\
& \quad \leq 2 R^{\epsilon} \max \left\{\max _{0 \leq t \leq 1} \mu_{1}(t), \max _{0 \leq t \leq 1} \mu_{2}(t)\right\} . \tag{68}
\end{align*}
$$

By using the Lebesgue dominated convergence theorem, we obtain that

$$
\begin{aligned}
\lim _{n \rightarrow \infty}\left\|T v_{n}-T v_{0}\right\| & =\lim _{n \rightarrow \infty} \max _{0 \leq t \leq 1}\left|\int_{0}^{1} G(t, s)\left[f\left(s, I_{0+}^{\beta} v_{n}(s), v_{n}(s)\right)-f\left(s, I_{0+}^{\beta} v_{0}(s), v_{0}(s)\right)\right] d s\right| \\
& \leq \lim _{n \rightarrow \infty} \max _{0 \leq t \leq 1} \int_{0}^{1} G(t, s)\left|f\left(s, I_{0+}^{\beta} v_{n}(s), v_{n}(s)\right)-f\left(s, I_{0+}^{\beta} v_{0}(s), v_{0}(s)\right)\right| d s
\end{aligned}
$$

$$
\begin{equation*}
=\max _{0 \leq t, s \leq 1} G(t, s) \cdot \int_{0}^{1} \lim _{n \longrightarrow \infty}\left|f\left(s, I_{0+}^{\beta} v_{n}(s), v_{n}(s)\right)-f\left(s, I_{0+}^{\beta} v_{0}(s), v_{0}(s)\right)\right| d s=0 \tag{69}
\end{equation*}
$$

and this implies that $T$ is a continuous operator.
Now, we shall prove that $T: D \longrightarrow D$ is compact. For any $v \in D, T v \in D$, which deduces that $1 / R \leq T v(t) \leq R$, for $t \in[0,1]$, i.e., $T$ is uniformly bounded.

Since $G(t, s) \in C([0,1] \times[0,1])$, it is also uniformly continuous on $[0,1] \times[0,1]$, and then for any $\varepsilon>0, \exists \delta>0$, s.t., for any $\left(t_{1}, s\right),\left(t_{2}, s\right) \in[0,1] \times[0,1]$ with $\left|t_{1}-t_{2}\right|<\delta$, we always have

$$
\begin{equation*}
\left|G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right|<\frac{\varepsilon}{R^{\varepsilon} \int_{0}^{1} \mu_{2}(s) d s+\int_{0}^{1}|e(s)| d s} \tag{70}
\end{equation*}
$$

Thus, one can obtain by virtue of (63) (70) that

$$
\begin{align*}
& \left|T v\left(t_{1}\right)-T v\left(t_{2}\right)\right|=\mid \int_{0}^{1}\left[G\left(t_{1}, s\right)-G\left(t_{2}, s\right)\right] \\
& \cdot\left[f\left(s, I^{\beta} v(s), v(s)\right)+e(s)\right] d s\left|\leq \int_{0}^{1}\right| G\left(t_{1}, s\right) \\
& -G\left(t_{2}, s\right) \mid\left[\left|f\left(s, I_{0+}^{\beta} v(s), v(s)\right)\right|+|e(s)|\right] d s  \tag{71}\\
& \quad<\left[R^{\varepsilon} \int_{0}^{1} \mu_{2}(s) d s+\int_{0}^{1}|e(s)| d s\right] \\
& \cdot \frac{\varepsilon}{R^{\varepsilon} \int_{0}^{1} \mu_{2}(s) d s+\int_{0}^{1}|e(s)| d s}=\varepsilon
\end{align*}
$$

for any $t_{1}, t_{2} \in[0,1]$ with $\left|t_{1}-t_{2}\right|<\delta$ and any $v \in D_{1}$, which shows that $T: D \longrightarrow D$ is equicontinuous. Thus, ArzelaAscoli theorem guarantees that $T: D \longrightarrow D$ is completely continuous. Existence of at least one fixed point $v_{1} \in D$ follows from Lemma 6, i.e., $u_{1}(t)=I_{0+}^{\beta} v_{1}(t)$ is a positive solution of differential equation (1), which satisfies

$$
\begin{align*}
\frac{1}{R \Gamma(\alpha)} t^{\alpha} & \leq u_{1}(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} v_{1}(s) d s  \tag{72}\\
& \leq \frac{R}{\Gamma(\alpha)} t^{\alpha}
\end{align*}
$$

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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# Research Article 

# Two New Geraghty Type Contractions in $G_{b}$-Metric Spaces 

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#### Abstract

In this work, two Geraghty type contractions are introduced in $G_{b}$-metric spaces, and some fixed point theorems about the contractions are proved. At the end of this article, a theorem about unique solution of an integral function is proved.


## 1. Introduction

It is well known that fixed point theorem is an important tool for solving many equations in the study of mathematics, such as integral equations [1] and differential equations [2]. It can be applied in several subjects as well, like game theory [3] and economics [4].

The notion of $G$-metric spaces was introduced by Mustafa and Sims [5] as a generalization of metric spaces. Thereafter, $G$-metric spaces have been studied and applied to obtain different kinds of fixed point theorems, see [6-12]. Aghajani et al. [13] introduced the notion of $G_{b}$-metric spaces based on $G$-metric spaces and $b$-metric spaces introduced by Bakhtin in [14]. Also, some further fixed point theorems were studied after $G_{p}$-metric spaces, which related to partial metric spaces and $G$-metric spaces, introduced by Zand and Nezhad in [15]. Ansari et al. [16] proved some common fixed point results in complete $G_{p}$-metric spaces with a new approach. More recently, some Geraghty type contraction results were studied in various metric spaces, see [17-19].

In this work, we introduce two Geraghty type contractions in $G_{b}$-metric spaces and investigate some fixed point theorems about such contractions. In [13], $G_{b}$-metric space was introduced as follows.

Definition 1 (see [13]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. Suppose that $G_{b}: X \times X \times X \longrightarrow[0, \infty)$ is a function satisfying the following properties:
(1) $G_{b}(u, v, w)=0$ if $u=v=w$;
(2) $0<G_{b}(u, u, v)$ for all $u, v \in X$ with $u \neq v$;
(3) $G_{b}(u, u, v) \leq G_{b}(u, v, w)$ for all $u, v, w \in X$ with $v \neq w$;
(4) $G_{b}(u, v, w)=G_{b}(p\{u, v, w\})$, where $p$ is a permutation of $u, v, w$;
(5) $G_{b}(u, v, w) \leq s\left(G_{b}(u, c, c)+G_{b}(c, v, w)\right)$ for all $u, v, w, c \in X$.

Then the function is called a generalized $b$-metric or a $G_{b}{ }^{-}$ metric on $X$. The pair $\left(X, G_{b}\right)$ is called a $G_{b}$-metric space.

It is obvious that $G_{b}$-metric space is effectively larger than that of $G$-metric space. Actually, each $G$-metric space is a $G_{b}{ }^{-}$ metric space with $s=1$.

Definition 2 (see [13]). A $G_{b}$-metric space is said to be symmetric if $G_{b}(u, v, v)=G_{b}(v, u, u)$ for all $u, v \in X$.

Proposition 3 (see [13]). Let $X$ be a $G_{b}$-metric space. Then for each $u, v, w, c \in X$, it satisfies the following properties:
(1) If $G_{b}(u, v, w)=0$, then $u=v=w$;
(2) $G_{b}(u, v, w) \leq s\left(G_{b}(u, u, v)+G_{b}(u, u, w)\right)$;
(3) $G_{b}(u, v, v) \leq 2 s G_{b}(v, u, u)$;
(4) $G_{b}(u, v, w) \leq s\left(G_{b}(u, c, w)+G_{b}(c, v, w)\right)$.

In this paper, we denote $\mathbb{N}$ as the set of all positive integers and $\mathbb{R}$ as the set of all real numbers.

Definition 4 (see [13]). Let $\left(X, G_{b}\right)$ be a $G_{b}$-metric space and $\left\{x_{n}\right\}$ be a sequence in $X$ such that $\lim _{n, m \rightarrow \infty} G_{b}\left(x, x_{n}, x_{m}\right)=$
$0, x \in X$, then $\left\{x_{n}\right\}$ is $G_{b}$-convergent; that is, for any $\varepsilon>0$, there exists $N \in \mathbb{N}$ such that $G_{b}\left(x, x_{n}, x_{m}\right)<\varepsilon$, for all $n, m \geq$ $N$.

Proposition 5 (see [13]). Let $X$ be a $G_{b}$-metric space. The following statements are equivalent:
(1) $\left\{x_{n}\right\}$ is $G_{b}$-convergent to $x$;
(2) $G_{b}\left(x_{n}, x_{n}, x\right) \longrightarrow 0$ as $n \longrightarrow+\infty$;
(3) $G_{b}\left(x_{n}, x, x\right) \longrightarrow 0$ as $n \longrightarrow+\infty$;
(4) $G_{b}\left(x_{n}, x_{m}, x\right) \longrightarrow 0$ as $n, m \longrightarrow+\infty$.

Definition 6 (see [13]). Let $X$ be a $G_{b}$-metric space. A sequence $\left\{x_{n}\right\}$ is called a $G_{b}$-Cauchy sequence if for any $\varepsilon>0$, there is $N \in \mathbb{N}$ such that $G_{b}\left(x_{n}, x_{m}, x_{l}\right)<\varepsilon$ for all $n, m, l \geq N$; that is, $G_{b}\left(x_{n}, x_{m}, x_{l}\right) \longrightarrow 0$ as $n, m, l \longrightarrow \infty$.

Definition 7 (see [13]). A $G_{b}$-metric space $X$ is called $G_{b}{ }^{-}$ complete if every $G_{b}$-Cauchy sequence is $G_{b}$-convergent in X.

Let $\mathscr{A}$ be the family of all functions $\alpha:[0, \infty) \longrightarrow[0,1)$ which satisfies the condition $\lim _{n \rightarrow \infty} \alpha\left(p_{n}\right)=1$ implying $\lim _{n \rightarrow \infty} p_{n}=0$.

Example 8. Let

$$
\begin{align*}
& \alpha_{1}(p)= \begin{cases}\frac{1}{1+p}, & p>0 \\
\frac{1}{2}, & p=0,\end{cases} \\
& \alpha_{2}(p)= \begin{cases}\frac{p^{2}+1}{3 p^{2}+1}, & p>0 \\
\frac{1}{3}, & p=0 .\end{cases} \tag{1}
\end{align*}
$$

Then $\alpha_{1}, \alpha_{2} \in \mathscr{A}$.
Let $\mathscr{B}$ be the family of all functions $\beta:[0, \infty) \longrightarrow[0,1 / s)$ which satisfies the condition $\lim _{n \rightarrow \infty} \beta\left(q_{n}\right)=1 / s$ implying $\lim _{n \rightarrow \infty} q_{n}=0$.

Example 9. Let

$$
\beta(q)= \begin{cases}\frac{1}{s} e^{-q}, & q>0  \tag{2}\\ \frac{1}{s+1}, & q=0\end{cases}
$$

Then, $\beta(q) \in \mathscr{B}$.
In [20], Karapinar et al. proved the following result.
Theorem 10. Let $(X, \sigma)$ be a complete metric-like space and $T: X \longrightarrow X$ be a mapping. Suppose that there exists $\alpha \in \mathscr{A}$ such that

$$
\begin{equation*}
\sigma(T x, T y) \leq \alpha(\sigma(x, y)) \sigma(x, y) \tag{3}
\end{equation*}
$$

for all $x, y \in X$, then $T$ has a unique fixed point $u \in X$ with $\sigma(u, u)=0$.

Recently, Aydi et al. [21] introduced a type of Geraghty contraction in metric-like spaces and proved a fixed point theorem about such contraction as follows:

Theorem 11. Let $(X, \sigma)$ be a complete metric-like space and $T$ : $X \longrightarrow X$ be a mapping. Suppose that there exists $\alpha \in \mathscr{A}$ such that

$$
\begin{equation*}
\sigma(T x, T y) \leq \alpha(F(x, y)) F(x, y) \tag{4}
\end{equation*}
$$

for all $x, y \in X$, where

$$
\begin{equation*}
F(x, y)=\sigma(x, y)+|\sigma(x, T x)-\sigma(y, T y)| \tag{5}
\end{equation*}
$$

Then $T$ has a unique fixed point $u \in X$ with $\sigma(u, u)=0$.
In our work, enlightened by the preceding works, we introduce the Geraghty contraction to the $G_{b}$-metric space and prove fixed point theorems for Geraghty type contractions. At the end, we give an application about a unique solution of an integral function.

## 2. Main Results

Theorem 12. Let $\left(X, G_{b}\right)$ be a complete $G_{b}$-metric space and $T: X \longrightarrow X$ be a given mapping. Suppose there exists $\beta \in \mathscr{B}$ such that

$$
\begin{equation*}
G_{b}(T x, T y, T y) \leq \beta(F(x, y)) F(x, y), \tag{6}
\end{equation*}
$$

for all $x, y \in X$ where

$$
\begin{align*}
F(x, y)= & G_{b}(x, y, y) \\
& +\left|G_{b}(x, T x, T x)-G_{b}(y, T y, T y)\right| . \tag{7}
\end{align*}
$$

Then $T$ has a unique fixed point $u \in X$.
Proof. Let $x_{0} \in X$. Define a sequence $\left\{x_{n}\right\}$ in $X$ by $x_{n+1}=$ $T x_{n}=T^{n+1} x_{0}$ for all $n \in \mathbb{N}$. Assume that $G_{b}\left(x_{n_{0}}\right.$, $\left.x_{n_{0}+1}, x_{n_{0}+1}\right)=0$ for some $n_{0}$, then $x_{0}$ is the fixed point of $T$; the proof is completed. Thus, we assume $G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right) \neq$ 0 for all $n \in \mathbb{N}$. From (6), we have

$$
\begin{align*}
0 & <G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)=G_{b}\left(T x_{n-1}, T x_{n}, T x_{n}\right)  \tag{8}\\
& \leq \beta\left(F\left(x_{n-1}, x_{n}\right)\right) F\left(x_{n-1}, x_{n}\right), \quad n \geq 1,
\end{align*}
$$

where

$$
\begin{align*}
F( & \left.x_{n-1}, x_{n}\right) \\
= & G_{b}\left(x_{n-1}, x_{n}, x_{n}\right) \\
& +\left|G_{b}\left(x_{n-1}, T x_{n-1}, T x_{n-1}\right)-G_{b}\left(x_{n}, T x_{n}, T x_{n}\right)\right|  \tag{9}\\
= & G_{b}\left(x_{n-1}, x_{n}, x_{n}\right) \\
& +\left|G_{b}\left(x_{n-1}, x_{n}, x_{n}\right)-G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)\right| .
\end{align*}
$$

Take

$$
\begin{equation*}
G_{b(n)}=G_{b}\left(x_{n-1}, x_{n}, x_{n}\right) \tag{10}
\end{equation*}
$$

Then, (8) becomes

$$
\begin{align*}
0 & <G_{b(n+1)} \leq \beta\left(G_{b(n)}+\left|G_{b(n)}-G_{b(n+1)}\right|\right)  \tag{11}\\
& \cdot\left(G_{b(n)}+\left|G_{b(n)}-G_{b(n+1)}\right|\right) .
\end{align*}
$$

Suppose that there exists $n_{0}>0$ such that $G_{b\left(n_{0}\right)} \leq G_{b\left(n_{0}+1\right)}$, then from (11), we have

$$
\begin{equation*}
G_{b\left(n_{0}+1\right)} \leq \beta\left(G_{b\left(n_{0}+1\right)}\right) G_{b\left(n_{0}+1\right)}<\frac{1}{s} G_{b\left(n_{0}+1\right)}, \quad s \geq 1 \tag{12}
\end{equation*}
$$

which is a contradiction.
Thus, for all $n>0, G_{b(n)}>G_{b(n+1)}$. From (11), we have

$$
\begin{equation*}
0<G_{b(n+1)} \leq \beta\left(2 G_{b(n)}-G_{b(n+1)}\right)\left(2 G_{b(n)}-G_{b(n+1)}\right) \tag{13}
\end{equation*}
$$

The real sequence $\left\{G_{b(n)}\right\}$ is decreasing; suppose there exists $\gamma \geq 0$ such that $\lim _{n \rightarrow \infty} G_{b(n)}=\gamma$. Assume that $\gamma>0$. Taking $n \longrightarrow \infty$ in (13), we get

$$
\begin{align*}
\gamma & =\lim _{n \rightarrow \infty} G_{b(n+1)} \\
& \leq \lim _{n \rightarrow \infty}\left[\beta\left(2 G_{b(n)}-G_{b(n+1)}\right)\left(2 G_{b(n)}-G_{b(n+1)}\right)\right] . \tag{14}
\end{align*}
$$

There are two situations that need to be discussed.
(A) When $s=1$, then (14) becomes

$$
\begin{align*}
\gamma & =\lim _{n \longrightarrow \infty} G_{b(n+1)} \\
& \leq \lim _{n \longrightarrow \infty}\left[\beta\left(2 G_{b(n)}-G_{b(n+1)}\right)\left(2 G_{b(n)}-G_{b(n+1)}\right)\right]  \tag{15}\\
& \leq \frac{1}{s^{n} \lim _{\rightarrow \infty}\left(2 G_{b(n)}-G_{b(n+1)}\right)=\frac{1}{s} \gamma=\gamma .} .
\end{align*}
$$

We can obtain

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} \beta\left(2 G_{b(n)}-G_{b(n+1)}\right)=1 \tag{16}
\end{equation*}
$$

Since $\beta \in \mathscr{B}$, we get

$$
\begin{equation*}
\gamma=\lim _{n \longrightarrow \infty}\left(2 G_{b(n)}-G_{b(n+1)}\right)=0 \tag{17}
\end{equation*}
$$

which is a contradiction.
(B) When $s>1$, according to (14) we have

$$
\begin{align*}
\gamma & =\lim _{n \longrightarrow \infty} G_{b(\mathrm{n}+1)} \\
& \leq \lim _{n \longrightarrow \infty}\left[\beta\left(2 G_{b(n)}-G_{b(n+1)}\right)\left(2 G_{b(n)}-G_{b(n+1)}\right)\right]  \tag{18}\\
& \leq \frac{1}{s^{n}} \lim _{\longrightarrow \infty}\left(2 G_{b(n)}-G_{b(n+1)}\right)=\frac{1}{s} \gamma, \quad s>1
\end{align*}
$$

which is a contradiction.
In conclusion of the above two conditions, we have $\gamma=0$, that is

$$
\begin{equation*}
\lim _{n \rightarrow \infty} G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)=0 \tag{19}
\end{equation*}
$$

Note that

$$
\begin{equation*}
G_{b}\left(x_{n+1}, x_{n}, x_{n}\right) \leq 2 s G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right) \tag{20}
\end{equation*}
$$

thus, we get

$$
\begin{equation*}
\lim _{n \rightarrow \infty} G_{b}\left(x_{n+1}, x_{n}, x_{n}\right)=0 . \tag{21}
\end{equation*}
$$

We shall prove that $\left\{x_{n}\right\}$ is a Cauchy sequence in $\left(X, G_{b}\right)$. Equation

$$
\begin{equation*}
\lim _{n, m \rightarrow \infty} G_{b}\left(x_{n}, x_{m}, x_{m}\right)=0, \quad m>n \tag{22}
\end{equation*}
$$

will be proved.
Suppose (22) does not hold. Then there exists $\varepsilon>0$ for which we can find subsequences $\left\{x_{m_{i}}\right\}$ and $\left\{x_{n_{i}}\right\}$ of $\left\{x_{n}\right\}$ with $m_{i}>n_{i}>i$ such that for every $i$

$$
\begin{equation*}
G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \geq \varepsilon . \tag{23}
\end{equation*}
$$

Corresponding to $n_{i}$, we can find $m_{i}$ with the smallest index and $m_{i}>n_{i}$ and satisfying (23), then

$$
\begin{equation*}
G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)<\varepsilon \tag{24}
\end{equation*}
$$

By (23) and (24), we have

$$
\begin{align*}
\varepsilon & \leq G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \\
& \leq s\left[G_{b}\left(x_{n_{i}}, x_{n_{i}+1}, x_{n_{i}+1}\right)+G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right)\right] \tag{25}
\end{align*}
$$

Taking $i \longrightarrow \infty$ in (25) and using (19), we obtain

$$
\begin{align*}
\varepsilon & \leq \liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \\
& \leq s \liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right) . \tag{26}
\end{align*}
$$

Back to (6), there is

$$
\begin{align*}
& G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right)=G_{b}\left(T x_{n_{i}}, T x_{m_{i}-1}, T x_{m_{i}-1}\right) \\
& \quad \leq \beta\left(F\left(x_{n_{i}}, x_{m_{i}-1}\right)\right) F\left(x_{n_{i}}, x_{m_{i}-1}\right) \\
& \quad=\beta\left(G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)\right. \\
& \left.\quad+\left|G_{b}\left(x_{n_{i}}, x_{n_{i}+1}, x_{n_{i}+1}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right|\right)  \tag{27}\\
& \quad \cdot\left(G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)\right. \\
& \left.\quad+\left|G_{b}\left(x_{n_{i}}, x_{n_{i}+1}, x_{n_{i}+1}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right|\right) .
\end{align*}
$$

Let

$$
\begin{align*}
\mathscr{M}= & G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right) \\
& +\left|G_{b}\left(x_{n_{i}}, x_{n_{i}+1}, x_{n_{i}+1}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right| . \tag{28}
\end{align*}
$$

Taking $i \longrightarrow \infty$ in the above inequations and by (19), (24), we have

$$
\begin{equation*}
\liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right) \leq \varepsilon \liminf _{i \rightarrow \infty} \beta(\mathscr{M}) . \tag{29}
\end{equation*}
$$

Then we obtain

$$
\begin{equation*}
\frac{1}{\varepsilon} \liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right) \leq \liminf _{i \rightarrow \infty} \beta(\mathscr{M}) . \tag{30}
\end{equation*}
$$

From (26) and (30), we get

$$
\begin{align*}
\frac{1}{s} & =\frac{1}{\varepsilon} \cdot \frac{\varepsilon}{s} \leq \frac{1}{\varepsilon} \liminf _{i \longrightarrow \infty} G_{b}\left(x_{n_{i}+1}, x_{m_{i}}, x_{m_{i}}\right) \\
& \leq \liminf _{i \rightarrow \infty} \beta(\mathscr{M}) \leq \limsup _{i \rightarrow \infty} \beta(\mathscr{M}) \leq \frac{1}{s} . \tag{31}
\end{align*}
$$

We deduce that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \beta(\mathscr{M})=\frac{1}{s} . \tag{32}
\end{equation*}
$$

Since $\beta \in \mathscr{B}$ and by (19), then

$$
\begin{align*}
& \lim _{i \rightarrow \infty} \mathscr{M}=\lim _{i \rightarrow \infty}\left(G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)\right. \\
& \left.\quad+\left|G_{b}\left(x_{n_{i}}, x_{n_{i}+1}, x_{n_{i}+1}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right|\right)  \tag{33}\\
& \quad=\lim _{i \rightarrow \infty} G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)=0 .
\end{align*}
$$

From (19) and (33), we have

$$
\begin{equation*}
\varepsilon \leq G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \leq s\left[G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)+G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right] \longrightarrow 0 \quad \text { as } i \longrightarrow \infty, \tag{34}
\end{equation*}
$$

which is a contradiction. Thus

$$
\begin{equation*}
\lim _{n, m \longrightarrow \infty} G_{b}\left(x_{n}, x_{m}, x_{m}\right)=0 \tag{35}
\end{equation*}
$$

and $\left\{x_{n}\right\}$ is a Cauchy sequence in complete $G_{b}$-metric space. So there exists $u \in X$, such that

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} G_{b}\left(x_{n}, u, u\right)=0 \tag{36}
\end{equation*}
$$

By Proposition 3, we have $G_{b}\left(u, x_{n}, x_{n}\right) \leq 2 s G_{b}\left(x_{n}, u, u\right)$; therefore by (36) we get

$$
\begin{equation*}
\lim _{n \rightarrow \infty} G_{b}\left(u, x_{n}, x_{n}\right)=0 . \tag{37}
\end{equation*}
$$

We shall prove that $u$ is a fixed point of $T$. Assume that $u \neq$ $T u$, then $G_{b}(u, T u, T u)>0$. From (6) and (7), we have

$$
\begin{align*}
G_{b}\left(x_{n+1}, T u, T u\right) & =G_{b}\left(T x_{n}, T u, T u\right) \\
& \leq \beta\left(F\left(x_{n}, u\right)\right) F\left(x_{n}, u\right)  \tag{38}\\
& <\frac{1}{s} F\left(x_{n}, u\right),
\end{align*}
$$

where

$$
\begin{align*}
F\left(x_{n}, u\right)= & G_{b}\left(x_{n}, u, u\right)  \tag{39}\\
& +\left|G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)-G_{b}(u, T u, T u)\right|
\end{align*}
$$

We also have

$$
\begin{aligned}
G_{b} & (u, T u, T u) \\
& \leq s\left(G_{b}\left(u, T x_{n}, T x_{n}\right)+G_{b}\left(T x_{n}, T u, T u\right)\right) \\
& =s G_{b}\left(u, x_{n+1}, x_{n+1}\right)+s G_{b}\left(x_{n+1}, T u, T u\right) \\
& \leq s G_{b}\left(u, x_{n+1}, x_{n+1}\right)+s \beta\left(F\left(x_{n}, u\right)\right) F\left(x_{n}, u\right) \\
& <s G_{b}\left(u, x_{n+1}, x_{n+1}\right)+F\left(x_{n}, u\right) .
\end{aligned}
$$

Taking $n \longrightarrow \infty$ in the above inequation and using (36), (37), and (38) obtains

$$
\begin{align*}
G_{b} & (u, T u, T u) \\
\leq & s_{n \longrightarrow \infty} \lim _{b}\left(u, x_{n+1}, x_{n+1}\right) \\
& +s_{n \longrightarrow \infty}\left[\beta\left(F\left(x_{n}, u\right)\right) F\left(x_{n}, u\right)\right] \\
\leq & \lim _{n \longrightarrow \infty} G_{b}\left(u, x_{n+1}, x_{n+1}\right)+\lim _{n \longrightarrow \infty} F\left(x_{n}, u\right)  \tag{41}\\
\leq & 0+\lim _{n \longrightarrow \infty} G_{b}\left(x_{n}, u, u\right) \\
& +\left|\lim _{n \rightarrow \infty} G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)-\lim _{n \longrightarrow \infty} G_{b}(u, T u, T u)\right| \\
\leq & G_{b}(u, T u, T u),
\end{align*}
$$

which implies

$$
\begin{equation*}
\lim _{n \rightarrow \infty} s \beta\left(F\left(x_{n}, u\right)\right)=1 . \tag{42}
\end{equation*}
$$

Since $\beta \in \mathscr{B}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} F\left(x_{n}, u\right)=0 \tag{43}
\end{equation*}
$$

which is a contradiction. Thus $G_{b}(u, T u, T u)=0$ and so $u=$ $T u$. Consequently, $u$ is a fixed point of $T$.

We shall prove that such $u$ is the unique fixed point of $T$. We argue by contradiction. Assume there exists $v, v \neq u$ such that $v=T v$. We have

$$
\begin{align*}
F(u, v) & =G_{b}(u, v, v)+\left|G_{b}(u, u, u)-G_{b}(v, v, v)\right| \\
& =G_{b}(u, v, v) . \tag{44}
\end{align*}
$$

From (6), we have

$$
\begin{align*}
0 & <G_{b}(u, v, v)=G_{b}(T u, T v, T v) \\
& \leq \beta(F(u, v)) F(u, v)=\beta\left(G_{b}(u, v, v)\right) G_{b}(u, v, v)  \tag{45}\\
& <\frac{1}{s} G_{b}(u, v, v),
\end{align*}
$$

which is a contradiction. Thus, $T$ has a unique fixed point.

We now present the following result.
Theorem 13. Let $\left(X, G_{b}\right)$ be a complete $G_{b}$-metric space and $T: X \longrightarrow X$ be a given mapping. Suppose there exists $\alpha \in \mathscr{A}$ such that

$$
\begin{equation*}
G_{b}(T x, T y, T y) \leq \alpha(F(x, y)) F(x, y) \tag{46}
\end{equation*}
$$

for all $x, y \in X$ where

$$
\begin{align*}
& F(x, y)=\frac{1}{s^{2}}\left(G_{b}(x, y, y)\right.  \tag{47}\\
& \left.\quad+\left|G_{b}(x, T x, T x)-G_{b}(y, T y, T y)\right|\right) .
\end{align*}
$$

Then $T$ has a unique fixed point $u^{*} \in X$.
Proof. Let $x_{0} \in X$. Define a sequence $\left\{x_{n}\right\}$ in $X$ by $x_{n+1}=$ $T x_{n}=T^{n+1} x_{0}$. Assume that $G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)=0$ for $n=n_{0}$, that is, $G_{b}\left(x_{n_{0}}, x_{n_{0}+1}, x_{n_{0}+1}\right)=0$, then $x_{n_{0}}=x_{n_{0}+1}$, i.e., $T x_{n_{0}}=$ $x_{n_{0}}$. Therefore $x_{n_{0}}$ is a fixed point of $T$.

Suppose $G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right) \neq 0$ for all $n \in \mathbb{N}$. From (46) we have

$$
\begin{align*}
0 & <G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)=G_{b}\left(T x_{n-1}, T x_{n}, T x_{n}\right)  \tag{48}\\
& \leq \alpha\left(F\left(x_{n-1}, x_{n}\right)\right) F\left(x_{n-1}, x_{n}\right)
\end{align*}
$$

where

$$
\begin{align*}
F & \left(x_{n-1}, x_{n}\right)=\frac{1}{s^{2}}\left(G_{b}\left(x_{n-1}, x_{n}, x_{n}\right)\right. \\
& \left.+\left|G_{b}\left(x_{n-1}, T x_{n-1}, T x_{n-1}\right)-G_{b}\left(x_{n}, T x_{n}, T x_{n}\right)\right|\right) \\
& =\frac{1}{s^{2}}\left(G_{b}\left(x_{n-1}, x_{n}, x_{n}\right)\right.  \tag{49}\\
& \left.+\left|G_{b}\left(x_{n-1}, x_{n}, x_{n}\right)-G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)\right|\right) .
\end{align*}
$$

Take

$$
\begin{equation*}
G_{b(n)}^{*}=G_{b}\left(x_{n-1}, x_{n}, x_{n}\right) . \tag{50}
\end{equation*}
$$

Then (48) becomes

$$
\begin{align*}
G_{b(n+1)}^{*} \leq & \alpha\left[\frac{1}{s^{2}}\left(G_{b(n)}^{*}+\left|G_{b(n)}^{*}-G_{b(n+1)}^{*}\right|\right)\right]  \tag{51}\\
\cdot & \frac{1}{s^{2}}\left(G_{b(n)}^{*}+\left|G_{b(n)}^{*}-G_{b(n+1)}^{*}\right|\right)
\end{align*}
$$

Suppose there exists $n_{0}>0$ such that $G_{b\left(n_{0}\right)}^{*} \leq G_{b\left(n_{0}+1\right)}^{*}$. From (51), we have

$$
\begin{align*}
G_{b\left(n_{0}+1\right)}^{*} & \leq \alpha\left(\frac{1}{s^{2}} G_{b\left(n_{0}+1\right)}^{*}\right) \cdot\left(\frac{1}{s^{2}} G_{b\left(n_{0}+1\right)}^{*}\right)  \tag{52}\\
& <\frac{1}{s^{2}} G_{b\left(n_{0}+1\right)}^{*}
\end{align*}
$$

which is a contradiction. Thus $G_{b(n)}^{*}>G_{b(n+1)}^{*}$ for all $n>0$, and the real sequence $\left\{G_{b(n)}^{*}\right\}$ is decreasing.

Suppose there exists $\delta \in \mathbb{R}$ such that $\lim _{n \rightarrow \infty} G_{b(n)}^{*}=\delta$. Now we shall prove that

$$
\begin{equation*}
\delta=0 \tag{53}
\end{equation*}
$$

Applying (51), we get

$$
\begin{align*}
G_{b(n+1)}^{*} \leq & \alpha\left[\frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)\right] \\
& \cdot \frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)  \tag{54}\\
<\alpha & \alpha\left[\frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)\right] \\
\cdot & \left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right) \leq 2 G_{b(n)}^{*}-G_{b(n+1)}^{*}
\end{align*}
$$

Take $n \longrightarrow \infty$ in (54) to write

$$
\begin{align*}
\delta= & \lim _{n \rightarrow \infty} G_{b(n+1)}^{*} \\
\leq & \lim _{n \rightarrow \infty} \alpha\left[\frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)\right]  \tag{55}\\
& \cdot \lim _{n \longrightarrow \infty}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right) \\
\leq & \lim _{n \rightarrow \infty}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)=\delta .
\end{align*}
$$

We obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \alpha\left[\frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)\right]=1 \tag{56}
\end{equation*}
$$

Since $\alpha \in \mathscr{A}$, then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{1}{s^{2}}\left(2 G_{b(n)}^{*}-G_{b(n+1)}^{*}\right)=\frac{1}{s^{2}} \cdot \delta=0 . \tag{57}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\delta=0 \tag{58}
\end{equation*}
$$

That is

$$
\begin{equation*}
\lim _{n \rightarrow \infty} G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)=0 \tag{59}
\end{equation*}
$$

Note that

$$
\begin{equation*}
G_{b}\left(x_{n+1}, x_{n}, x_{n}\right) \leq 2 s G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right) \tag{60}
\end{equation*}
$$

then we can get

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} G_{b}\left(x_{n+1}, x_{n}, x_{n}\right)=0 . \tag{61}
\end{equation*}
$$

Now we shall prove that $\left\{x_{n}\right\}$ is a Cauchy sequence in $G_{b}$ metric spaces. We will prove that

$$
\begin{equation*}
\lim _{n, m \longrightarrow \infty} G_{b}\left(x_{n}, x_{m}, x_{m}\right)=0, \quad m>n \tag{62}
\end{equation*}
$$

Suppose (62) does not hold, then there exists $\varepsilon>0$ for which we can find subsequences $\left\{x_{m_{i}}\right\}$ and $\left\{x_{n_{i}}\right\}$ of $\left\{x_{n}\right\}$ with $m_{i}>$ $n_{i}>i$ such that

$$
\begin{equation*}
G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \geq \varepsilon \tag{63}
\end{equation*}
$$

Corresponding to $n_{i}$, we can find $m_{i}$ with the smallest index satisfying (63) and $m_{i}>n_{i}$. That is

$$
\begin{equation*}
G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)<\varepsilon \tag{64}
\end{equation*}
$$

From (46) and (63), we have

$$
\begin{align*}
\varepsilon & \leq G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right)=G_{b}\left(T x_{n_{i}-1}, T x_{m_{i}-1}, T x_{m_{i}-1}\right) \\
& \leq \alpha\left(F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)\right) \cdot F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)  \tag{65}\\
& <F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)
\end{align*}
$$

where

$$
\begin{align*}
& F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)=\frac{1}{s^{2}}\left(G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right)\right.  \tag{66}\\
& \left.\quad+\left|G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right|\right)
\end{align*}
$$

From (65), (66), and applying (59), we have

$$
\begin{align*}
\varepsilon & \leq \liminf _{i \rightarrow \infty} F\left(x_{n_{i}-1}, x_{m_{i}-1}\right) \\
& =\frac{1}{s^{2}} \liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right) . \tag{67}
\end{align*}
$$

On the other hand, applying (64) we have

$$
\begin{align*}
& G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right) \\
& \quad \leq s G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)+s G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)  \tag{68}\\
& \quad<s G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)+s \varepsilon .
\end{align*}
$$

Taking $n \longrightarrow \infty$ in the above inequations, we have

$$
\begin{equation*}
\limsup _{i \rightarrow \infty} G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right) \leq s \varepsilon . \tag{69}
\end{equation*}
$$

From (67) and (69) we obtain

$$
\begin{align*}
\varepsilon & \leq \frac{1}{s^{2}} \liminf _{i \rightarrow \infty} G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right) \\
& \leq \frac{1}{s^{2}} \limsup _{i \rightarrow \infty} G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right) \leq \frac{1}{s^{2}} \cdot s \varepsilon=\frac{\varepsilon}{s}, \tag{70}
\end{align*}
$$

which is a contradiction with $s>1$.
We shall consider the situation which with $s=1$. From (63) and (64) we have

$$
\begin{align*}
\varepsilon & \leq G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right) \\
& \leq G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)+G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)  \tag{71}\\
& <\varepsilon+G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right) .
\end{align*}
$$

Taking $n \longrightarrow \infty$ in (71) and by (59) we have

$$
\begin{equation*}
\lim _{i \rightarrow \infty} G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right)=\varepsilon . \tag{72}
\end{equation*}
$$

In the meantime,

$$
\begin{align*}
& \left|G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right)-G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right)\right| \\
& \quad \leq \mid G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)+G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right) \\
& \quad-\left(G_{b}\left(x_{n_{i}}, x_{m_{i}-1}, x_{m_{i}-1}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right) \mid  \tag{73}\\
& \quad=\left|G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)+G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right| .
\end{align*}
$$

By taking $n \longrightarrow \infty$ and using (59) in the above inequation, we can deduce that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right)=\varepsilon . \tag{74}
\end{equation*}
$$

By (46), we have

$$
\begin{align*}
\varepsilon & \leq G_{b}\left(x_{n_{i}}, x_{m_{i}}, x_{m_{i}}\right)=G_{b}\left(T x_{n_{i}-1}, T x_{m_{i}-1}, T x_{m_{i}-1}\right) \\
& \leq \alpha\left(F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)\right) F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)  \tag{75}\\
& <F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)
\end{align*}
$$

where

$$
\begin{align*}
& F\left(x_{n_{i}-1}, x_{m_{i}-1}\right) \\
& \quad=G_{b}\left(x_{n_{i}-1}, x_{m_{i}-1}, x_{m_{i}-1}\right)  \tag{76}\\
& \quad+\left|G_{b}\left(x_{n_{i}-1}, x_{n_{i}}, x_{n_{i}}\right)-G_{b}\left(x_{m_{i}-1}, x_{m_{i}}, x_{m_{i}}\right)\right| .
\end{align*}
$$

By using (59), we obtain

$$
\begin{equation*}
\lim _{i \rightarrow \infty} F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)=\varepsilon . \tag{77}
\end{equation*}
$$

Taking $i \longrightarrow \infty$ in (75), we can deduce that

$$
\begin{equation*}
\lim _{i \rightarrow \infty} \alpha\left(F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)\right)=1 \tag{78}
\end{equation*}
$$

Since $\alpha \in \mathscr{A}$, we have

$$
\begin{equation*}
\lim _{i \rightarrow \infty} F\left(x_{n_{i}-1}, x_{m_{i}-1}\right)=0, \tag{79}
\end{equation*}
$$

which is a contradiction. In conclusion of two situations, $\left\{x_{n}\right\}$ is a Cauchy sequence in complete $G_{b}$-metric spaces. So there exists $u^{*} \in X$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty} G_{b}\left(x_{n}, u^{*}, u^{*}\right)=0 \tag{80}
\end{equation*}
$$

We can confirm that $G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)=0$. In fact, if $G_{b}\left(u^{*}, T u^{*}, T u^{*}\right) \neq 0$, from (46) and (47) we have

$$
\begin{align*}
G_{b}\left(x_{n+1}, T u^{*}, T u^{*}\right) & =G_{b}\left(T x_{n}, T u^{*}, T u^{*}\right) \\
& \leq \alpha\left(F\left(x_{n}, u^{*}\right)\right) \cdot F\left(x_{n}, u^{*}\right)  \tag{81}\\
& <F\left(x_{n}, u^{*}\right)
\end{align*}
$$

where

$$
\begin{align*}
F & \left(x_{n}, u^{*}\right)=\frac{1}{s^{2}}\left(G_{b}\left(x_{n}, u^{*}, u^{*}\right)\right. \\
& \left.+\left|G_{b}\left(x_{n}, T x_{n}, T x_{n}\right)-G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)\right|\right) \\
& =\frac{1}{s^{2}}\left(G_{b}\left(x_{n}, u^{*}, u^{*}\right)\right.  \tag{82}\\
& \left.+\left|G_{b}\left(x_{n}, x_{n+1}, x_{n+1}\right)-G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)\right|\right) \\
& \rightarrow \frac{1}{s^{2}} G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)
\end{align*}
$$

We also have

$$
\begin{align*}
G_{b}\left(u^{*}, T u^{*}, T u^{*}\right) \leq & s G_{b}\left(u^{*}, T x_{n}, T x_{n}\right) \\
& +s G_{b}\left(T x_{n}, T u^{*}, T u^{*}\right) \\
\leq & s G_{b}\left(u^{*}, x_{n+1}, x_{n+1}\right) \\
& +s \alpha\left(F\left(x_{n}, u^{*}\right)\right) F\left(x_{n}, u^{*}\right)  \tag{83}\\
< & s G_{b}\left(u^{*}, x_{n+1}, x_{n+1}\right) \\
& +s F\left(x_{n}, u^{*}\right)
\end{align*}
$$

Taking $n \longrightarrow \infty$ in the above inequation and using (80), we obtain

$$
\begin{align*}
G_{b} & \left(u^{*}, T u^{*}, T u^{*}\right) \\
\leq & \lim _{n \rightarrow \infty} G_{b}\left(u^{*}, x_{n+1}, \mathrm{x}_{n+1}\right) \\
& +s_{n \rightarrow \infty} \alpha\left(F\left(x_{n}, u^{*}\right)\right) F\left(x_{n}, u^{*}\right)  \tag{84}\\
\leq & s_{n \rightarrow \infty} \lim _{n} F\left(x_{n}, u^{*}\right)=\frac{1}{s} G_{b}\left(u^{*}, T u^{*}, T u^{*}\right) .
\end{align*}
$$

When $s=1$, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \alpha\left(F\left(x_{n}, u^{*}\right)\right)=1 \tag{85}
\end{equation*}
$$

Since $\alpha \in \mathscr{A}$, then

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} F\left(x_{n}, u^{*}\right)=G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)=0 \tag{86}
\end{equation*}
$$

which is a contradiction.
When $s>1$, we get a contradiction from (84).
Thus, in conclusion of two situations, $G_{b}\left(u^{*}, T u^{*}, T u^{*}\right)=$ 0 and so $T u^{*}=u^{*} . u^{*}$ is a fixed point of $T$. We shall prove that such $u^{*}$ is the unique fixed point of $T$. We argue by contradiction. Assume there exists $v^{*}, v^{*} \neq u^{*}$ such that

$$
\begin{equation*}
v^{*}=T v^{*} \tag{87}
\end{equation*}
$$

We have

$$
\begin{align*}
F\left(u^{*}, v^{*}\right)= & G_{b}\left(u^{*}, v^{*}, v^{*}\right) \\
& +\left|G_{b}\left(u^{*}, u^{*}, u^{*}\right)-G_{b}\left(v^{*}, v^{*}, v^{*}\right)\right|  \tag{88}\\
= & G_{b}\left(u^{*}, v^{*}, v^{*}\right) .
\end{align*}
$$

From (46), we have

$$
\begin{align*}
0 & <G_{b}\left(u^{*}, v^{*}, v^{*}\right)=G_{b}\left(T u^{*}, T v^{*}, T v^{*}\right) \\
& \leq \alpha\left(F\left(u^{*}, v^{*}\right)\right) F\left(u^{*}, v^{*}\right) \\
& =\alpha\left(\frac{1}{s^{2}} G_{b}\left(u^{*}, v^{*}, v^{*}\right)\right) \frac{1}{s^{2}} G_{b}\left(u^{*}, v^{*}, v^{*}\right)  \tag{89}\\
& <\frac{1}{s^{2}} G_{b}\left(u^{*}, v^{*}, v^{*}\right)
\end{align*}
$$

which is a contradiction. Thus, there exists a unique fixed point $u^{*} \in X$ such that $u^{*}=T u^{*}$.

## 3. Application

Let $X=C([0,1], \mathbb{R})$ be the set of real continuous functions defined on $[0,1]$. Take the $G_{b}$-metric $G_{b}: X \times X \times X \longrightarrow$ $[0, \infty)$ given by

$$
\begin{align*}
& G_{b}(x, y, z)=\left(\sup _{t \in[0,1]}|x(t)-y(t)|\right. \\
& \left.\quad+\sup _{t \in[0,1]}|x(t)-z(t)|+\sup _{t \in[0,1]}|y(t)-z(t)|\right)^{2} \tag{90}
\end{align*}
$$

for all $x, y \in X$. Then $\left(X, G_{b}\right)$ is $G_{b}$-metric spaces with $s \geq 1$. Consider the following integral equation

$$
\begin{equation*}
x(t)=P(t)+\int_{0}^{1} S(t, u) f(u, x(u)) d u, \quad t \in[0,1] \tag{91}
\end{equation*}
$$

where $f:[0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ and $P:[0,1] \longrightarrow \mathbb{R}$ are two continuous functions and $S:[0,1] \times[0,1] \longrightarrow[0, \infty)$ is a function such that $S(t,.) \in L^{1}([0,1])$ for all $t \in[0,1]$. Consider the operator $T: X \longrightarrow X$ defined by

$$
\begin{align*}
T(x(t))=P(t)+\int_{0}^{1} S(t, u) f(u, x(u)) d u, &  \tag{92}\\
& t \in[0,1]
\end{align*}
$$

Theorem 14. Suppose that the following conditions are satisfied:
(1) there exists $\eta: X \times X \longrightarrow[0, \infty)$ for all $u \in[0,1]$

$$
\begin{align*}
0 & \leq|f(x, x(u))-f(u, y(u))| \\
& \leq \eta(x, y)|x(u)-y(u)| \quad \forall x, y \in X, \tag{93}
\end{align*}
$$

(2) there exists $\beta:[0, \infty) \longrightarrow[0,1 / s)$ such that

$$
\sup _{t \in[0,1]} \int_{0}^{1} S(t, u) \eta(x, y) d u
$$

$$
\begin{equation*}
\leq \sqrt{\beta\left(\left(2 \sup _{t \in[0,1]}|J|\right)^{2}+\left|\left(2 \sup _{t \in[0,1]}|K|\right)^{2}-\left(2 \sup _{t \in[0,1]}|L|\right)^{2}\right|\right)} \tag{94}
\end{equation*}
$$

where

$$
\begin{align*}
J & =x-y \\
K & =x-T x  \tag{95}\\
L & =y-T y .
\end{align*}
$$

Then the integral equation (91) has a unique solution in $X$.
Proof. It is clear that any fixed point of (92) is a solution of (91). By conditions (1) and (2), we get

$$
\begin{align*}
G_{b} & (T(x(t)), T(y(t)), T(y(t))) \\
& =\left(2 \sup _{t \in[0,1]}|T(x(t))-T(y(t))|\right)^{2} \\
& =\left(2 \sup _{t \in[0,1]} \mid \int_{0}^{1} S(t, u) f(u, x(u)) d u-\int_{0}^{1} S(t, u)\right. \\
& \cdot f(u, y(u)) d u \mid)^{2}=\left(2 \sup _{t \in[0,1]} \mid \int_{0}^{1} S(t, u)\right. \\
& \cdot[f(u, x(u))-f(u, y(u))] d u) \mid)^{2} \\
& \leq\left(2 \sup _{t \in[0,1]} \int_{0}^{1} S(t, u) \eta(x, y)|x(u)-y(u)| d u\right)^{2}  \tag{96}\\
& =\left(2 \sup _{t \in[0,1]} \int_{0}^{1} S(t, u) \eta(x, y)\right. \\
& \left.\frac{1}{2}\left((2|x(u)-y(u)|)^{2}\right)^{1 / 2} d u\right)^{2} \leq F(x, y) \\
\cdot & \left(\sup _{t \in[0,1]} \int_{0}^{1} S(t, u) \eta(x, y) d u\right)^{2} \leq F(x, y) \\
\cdot & \beta(F(x, y))
\end{align*}
$$

where

$$
\begin{align*}
F(x, y)= & \left(2 \sup _{t \in[0,1]}|J|\right)^{2}  \tag{97}\\
& +\left|\left(2 \sup _{t \in[0,1]}|K|\right)^{2}-\left(2 \sup _{t \in[0,1]}|L|\right)^{2}\right|
\end{align*}
$$

with

$$
\begin{align*}
J & =x-y \\
K & =x-T x  \tag{98}\\
L & =y-T y
\end{align*}
$$

Then for all $x, y \in X$ we obtain

$$
\begin{equation*}
G_{b}(T(x), T(y), T(y)) \leq \beta(F(x, y)) F(x, y) \tag{99}
\end{equation*}
$$

This implies that Theorem 12 holds. Thus the operator $T$ has a unique fixed point; that is, the integral function has a unique solution in $X$.

## 4. Conclusion

In this paper, we present some fixed point theorems about two new Geraghty type contractions in the setting of $G_{b}$-metric spaces.

In the third section, we study an application about the unique solution of a class of integral functions to illustrate our fixed point theorems.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to the work. All authors read and approved the final manuscript.

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# Solution of Hamilton-Jacobi-Bellman Equation in Optimal Reinsurance Strategy under Dynamic VaR Constraint 

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#### Abstract

This paper analyzes the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle. The surplus process of the insurer is described by the diffusion model which is an approximation of the classical Cramer-Lunderberg model. We assume the dynamic VaR constraints for proportional reinsurance. We obtain the closed form expression of the optimal reinsurance strategy and corresponding survival probability under proportional reinsurance.


## 1. Introduction

In practice, reinsurance is an important way for an insurer to control its risk exposure. In the actuarial literature, the optimal reinsurance problem of minimising ruin probability or equivalently maximising survival probability has been studied extensively in the past two decades. As one type of typical reinsurance strategy, proportional reinsurance has received great attention from both the academics and practitioners. Among others, Choulli et al. (2003), Højgaard and Taksar [1, 2], Schmidli [3, 4], Taksar [5], and Zhang et al. [6] work on the proportional reinsurance.

In the existing literature, the expected value principle is commonly used as the reinsurance premium principle due to its simplicity and popularity in practice. For details, the readers are referred to Bäuerle [7], Bai and Zhang [8], and Liang and Bayraktar [9]. Generally speaking, expected value principle is commonly used in life insurance whose claim frequency and claim sizes are stable and smooth, while the variance premium principle is extensively used in property insurance; see Zhou and Yuen [10] and Sun et al. [11]. Similarly to Zhang et al. [6], in this paper, we focus on a generalized mean-variance premium principle,
which includes the expected value principle and the variance principle as special cases.

More recently, the problem of optimal reinsurance design has been studied by using risk measures such as the Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), and conditional tail expectation (CTE) (to name a few, Cai and Tan [12], Cheung et al. [13], and Cai et al. [14, 15]). Latterly static risk measures have been extended to the dynamic version; see Yiu [16], Alexander and Baptista [17], Cuoco et al. [18], Chen et al. [19], and Zhang et al. [6], all of which investigate the optimal reinsurance problem under dynamic VaR constraint.

In this paper, we investigate an optimal proportional reinsurance problem under dynamic VaR constraint. Assume that an insurer aims to maximize the survival probability. With this assumption, we obtain the closed form expressions. The rest of the paper is organized as follows. In Section 2, we provide a general formulation of the optimal reinsurance problem. Then we investigate the insurance company's maximum survival probability under dynamic VaR constraints, and the corresponding optimal reinsurance strategy is given in proportional reinsurance settings in Section 3.

## 2. Formulation

Let $(\Omega, \mathscr{F}, P)$ be a probability space with a filtration $\left\{\mathscr{F}_{t}\right\}_{t \geq 0}$. Consider a Cramér-Lundberg model with the surplus process of an insurance company being given by

$$
\begin{equation*}
U(t)=u_{0}+c t-\sum_{i=1}^{N(t)} Y_{i}=u_{0}+c t-S(t) \tag{1}
\end{equation*}
$$

where $u_{0}$ is the initial surplus, the claim arrival process $\{N(t)\}_{t \geq 0}$ is a Poisson process with constant intensity $\lambda>0$, and the random variables $Y_{i}, i=1,2, \ldots$, are i.i.d claim sizes independent of $N(t)$. We let $T_{i}$ denote the $i$-th claim occurrence time and $F(y)$ denote the claim size distribution with finite first and second moments $m_{1}, m_{2}$. The premium rate $c$ is assumed to be calculated via the expected value principle; that is,

$$
\begin{equation*}
c=(1+\eta) \lambda m_{1} \tag{2}
\end{equation*}
$$

where $\eta>0$ is the relative loading factor.
In this paper, the insurer can purchase proportional reinsurance to adjust the exposure to insurance risk. The proportional reinsurance level is associated with the risk exposure $q(t)$ at time $t$. We assume $q(t) \in[0,1]$ for all $t$, and it means the insurer purchases proportional reinsurance. In this case, for each claim, the insurer only pays its $q(t) Y$, while the reinsurer pays the rest $(1-q(t)) Y$ for each claim.

For a chosen reinsurance policy $q(t)$, let $\left\{U^{q}(t), t \geq 0\right\}$ denote the associated surplus process; that is, $U^{q}(t)$ is the surplus of insurer at time $t$. This process then evolves as

$$
\begin{equation*}
d U^{q}(t)=\left(c-c^{q}\right) d t-q(t) d \sum_{i=1}^{N(t)} Y_{i} \tag{3}
\end{equation*}
$$

where $c^{q}$ is the net reinsurance rate which the reinsurer receives from the insurer. We assume that the reinsurance premium is calculated by the following generalized meanvariance principle $(1+\theta)[\mathbb{E}(\cdot)+\zeta \mathbb{D}(\cdot)]$, where $\theta, \zeta \geq 0$, and $\mathbb{E}$ and $\mathbb{D}$ denote the expectation and variance, respectively. Thus we have

$$
\begin{equation*}
c^{q} t=(1+\theta) \lambda\left[(1-q(t)) m_{1}+\zeta(1-q(t))^{2} m_{2}\right] t \tag{4}
\end{equation*}
$$

and the premium rate for the insurer is

$$
\begin{align*}
c-c^{q}= & (\eta-\theta) \lambda m_{1}+(1+\theta) \lambda q(t) m_{1} \\
& +\lambda(1+\theta) \zeta(1-q(t))^{2} m_{2} . \tag{5}
\end{align*}
$$

According to Grandell (1991), the surplus process after reinsurance can be approximated by the following diffusion process:

$$
\begin{align*}
& U^{q}(t)=u+\int_{0}^{t}\left[(\eta-\theta) \lambda m_{1}+\theta \lambda q(s) m_{1}\right. \\
& \left.\quad+\lambda(1+\theta) \zeta(1-q(s))^{2} m_{2}\right] d s+\int_{0}^{t} q(s) \tag{6}
\end{align*}
$$

$$
\cdot \sqrt{\lambda m_{2}} d B(s)
$$

where $\{B(t)\}_{t \geq 0}$ is a standard Brownian motion.

We define the ruin time

$$
\begin{equation*}
\tau^{q}=\inf \left\{t>0, U^{q}(t) \leq 0\right\} \tag{7}
\end{equation*}
$$

where the superscript $q$ emphasises that the surplus process and the ruin time are controlled by an admissible policy $q$. Denote the survival probability given the initial surplus $u$ by

$$
\begin{equation*}
v^{q}(u)=\mathbb{P}\left(\tau^{q}=\infty \mid U^{q}(0)=u\right) \tag{8}
\end{equation*}
$$

and the maximum survival probability by

$$
\begin{equation*}
v(u)=\max _{q \in \Pi} v^{q}(u) \tag{9}
\end{equation*}
$$

Our objective is to find the value function $v(u)$ and the optimal policy $q^{*}$ such that

$$
\begin{equation*}
v(u)=v^{q^{*}}(u) . \tag{10}
\end{equation*}
$$

## 3. Maximizing Survival Probability

Under the proportional reinsurance, the insurer could transfer a fraction $1-q(t)$ of the incoming claims to a reinsurer, where $q(t)$ is $\mathscr{F}_{t}$-measurable and satisfies $0 \leq q(t) \leq 1$ for all $t$. The diffusion approximation of insurance company's claim process becomes

$$
\begin{align*}
d S^{q}(t) & =\lambda q(t) m_{1} d t+q(t) \sqrt{\lambda m_{2}} d B(t)  \tag{11}\\
S^{q}(0) & =0
\end{align*}
$$

where $B(t)$ is a standard Brownian motion. The insurer's surplus process satisfies the stochastic differential equation

$$
\begin{align*}
& d U^{q}(t)=\left[(\eta-\theta) \lambda m_{1}+\theta \lambda q(s) m_{1}\right. \\
& \left.\quad+\lambda(1+\theta) \zeta(1-q(s))^{2} m_{2}\right] d t+q(t) \\
& \quad \cdot \sqrt{\lambda m_{2}} d B(t)  \tag{12}\\
& U^{q}(0)=u_{0} .
\end{align*}
$$

Taking $h>0$ is small enough, we assume that risk exposure does not change over the short time period $[t, t+h]$. This means that the risk exposure remains roughly constant in the given time period; that is, $\mathbb{E}[q(s) Y]=\mathbb{E}[q(t) Y]$, $\sqrt{\lambda \mathbb{E}\left[(q(s) Y)^{2}\right]}=\sqrt{\lambda \mathbb{E}\left[(q(t) Y)^{2}\right]}, s \in[t, t+h]$. This setting is reasonable because the insurer can only adjust its reinsurance business at discrete time; and the decision made is based on the holding at time $t$. Thus, we rewrite the claim dynamics as

$$
\begin{align*}
S^{q}(t+h)-S^{q}(t)= & \int_{t}^{t+h} \lambda \mathbb{E}[q(s) Y] d s \\
& +\int_{t}^{t+h} \sqrt{\lambda \mathbb{E}\left[(q(s) Y)^{2}\right]} d B(s)  \tag{13}\\
:= & \lambda q(t) m_{1} h \\
& +q(t) \sqrt{\lambda m_{2}} \int_{t}^{t+h} d B(s) .
\end{align*}
$$

3.1. Dynamic VaR, CVaR, and Worst-Case CVaR. For a given confidence level $1-\alpha \in(0,1)$ and a given horizon $h>0$, the $\operatorname{VaR}$ at time $t$ of a proportional reinsurance policy $q$, denoted by $V a R_{t}^{\alpha, h}$, is defined as

$$
\begin{align*}
& V a R_{t}^{\alpha, h}  \tag{14}\\
& \quad \triangleq \inf \left\{L: \mathbb{P}\left(S^{q}(t+h)-S^{q}(t) \geq L \mid \mathscr{F}_{t}\right)<\alpha\right\} .
\end{align*}
$$

The dynamic Conditional Value-at-Risk $C V a R_{t}^{\alpha, h}$ is given by

$$
\begin{align*}
& C V a R_{t}^{\alpha, h} \triangleq \mathbb{E}\left[S^{q}(t+h)-S^{q}(t) \mid S^{q}(t+h)-S^{q}(t)\right. \\
& \left.\quad \geq V_{t}^{\alpha, h}\right] . \tag{15}
\end{align*}
$$

The dynamic worst-case CVaR is defined as

$$
\begin{align*}
& w c C V a R_{t}^{\alpha, h} \\
& \quad \triangleq \sup _{p(\cdot) \in \mathscr{P}_{1} \in \mathbb{R}} \inf \left\{a+\frac{1}{\alpha} \mathbb{E}_{p}\left[\left(S^{q}(t+h)-S^{q}(t)-a\right)_{+}\right]\right\} \tag{16}
\end{align*}
$$

$$
\begin{align*}
\max _{q \in[0,1]} & \left\{\left[(\eta-\theta) \lambda m_{1}+\theta \lambda q m_{1}-\lambda(1+\theta) \zeta(1-q)^{2} m_{2}\right] v^{\prime}(u)+\frac{1}{2} \lambda q^{2} m_{2} v^{\prime \prime}(u)\right\}=0  \tag{19}\\
\text { s.t. } & q \in[0,1] \\
& V a R_{t}^{\alpha, h} \leq k u  \tag{20}\\
& v(0)=0, \quad v(+\infty)=1
\end{align*}
$$

where

$$
\begin{align*}
\mathscr{P}_{1} & =\left\{p(\cdot): \mathbb{E}_{p}\left[S^{q}(t+h)-S^{q}(t)\right]\right. \\
& =\lambda h m_{1} q(t), \mathbb{E}_{p}\left[\left(S^{q}(t+h)-S^{q}(t)\right)^{2}\right]  \tag{17}\\
& \left.=\lambda h m_{2} q(t)^{2}+\left(\lambda h m_{1} q(t)\right)^{2}\right\}
\end{align*}
$$

Proposition 1 (Zhang et al. [6]).

$$
\begin{align*}
V a R_{t}^{\alpha, h} & =\lambda h q(t) m_{1}-\Phi^{-1}(\alpha) q(t) \sqrt{\lambda h m_{2}}, \\
C V a R_{t}^{\alpha, h} & =\lambda h q(t) m_{1}+\frac{\phi\left(\Phi^{-1}\right)(\alpha)}{\alpha} q(t) \sqrt{\lambda h m_{2}}, \\
w c C V a R_{t}^{\alpha, h} & =\lambda h q(t) m_{1}+\sqrt{\frac{1-\alpha}{\alpha}} q(t) \sqrt{\lambda h m_{2}},  \tag{18}\\
0 & \leq \operatorname{VaR}_{t}^{\alpha, h} \leq C V a R_{t}^{\alpha, h} \leq w c C V a R_{t}^{\alpha, h} \\
& <U^{q}(t),
\end{align*}
$$

where $\phi(x)$ and $\Phi(x)$ denote the probability density function and the cumulative distribution function of a standard normal random variable, respectively. $\Phi^{-1}(x)$ is the inverse function of $\Phi(x)$.
3.2. HJB Equation. Using the dynamic programming technique, we obtain that the value function $v(u)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:
where $k(0<k<+\infty)$ is a constant.
Next we try to construct a solution of the HJB equation (19) with the boundary condition (20). Suppose that
$\eta m_{1} \leq \theta m_{1}+(1+\theta) \zeta m_{2}, v(u)$ with $v^{\prime}(u)>0, v^{\prime \prime}(u) \neq 0$ satisfies (19) and (20).
Theorem 2. (a) If $\theta \geq 2 \eta$, the function

$$
\varphi(u)= \begin{cases}\varphi\left(\frac{A}{k}\right)-\varphi\left(\frac{A}{k}\right) \frac{\int_{u}^{A / k} w^{-\triangle_{2}} e^{-\triangle_{1} / w} e^{\triangle_{3} w} d w}{\int_{0}^{A / k} w^{-\triangle_{2}} e^{-\triangle_{1} / w} e^{\triangle_{3} w} d w}, & \text { if } 0<u \leq \frac{A}{k}  \tag{21}\\ \varphi\left(\frac{A}{k}\right)+\left[1-\varphi\left(\frac{A}{k}\right)\right]\left[1-e^{-\left(2 \eta m_{1} / m_{2}\right)(u-A / k)}\right], & \text { if } u \geq \frac{A}{k}\end{cases}
$$

is a smooth $\left(\mathscr{C}^{2}\right)$ solution to the HJB equation, where

$$
\begin{align*}
A & =\lambda m_{1} h-\Phi^{-1}(\alpha) \sqrt{\lambda m_{2} h} \\
\varphi\left(\frac{A}{k}\right) & =\frac{2 \eta m_{1} / m_{2}}{2 \eta m_{1} / m_{2}+(A / k)^{-\Delta_{2}} e^{-k \Delta_{1} / A} e^{A \triangle_{3} / k} / \int_{0}^{A / k} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w} \tag{22}
\end{align*}
$$

$\Delta_{1}=2 A^{2}\left[(\eta-\theta) m_{1}-(1+\theta) \zeta m_{2}\right] / k^{2} m_{2}, \triangle_{2}=2 A\left[\theta m_{1}+\right.$ $\left.2(1+\theta) \zeta m_{2}\right] / k m_{2}, \triangle_{3}=2(1+\theta) \zeta$. The maximum of the left side of HJB equation is attained at

$$
q^{*}(u)= \begin{cases}\frac{k u}{A}, & \text { if } u<\frac{A}{k}  \tag{23}\\ 1, & \text { if } u \geq \frac{A}{k}\end{cases}
$$

(b) If $\theta<2 \eta$, the function

$$
\begin{align*}
& \varphi(u) \\
& = \begin{cases}\varphi\left(u_{1}\right)-\varphi\left(u_{1}\right) \frac{\int_{u}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\Delta_{3} w} d w}{\int_{0}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\Delta_{3} w} d w}, & \text { if } 0<u \leq u_{1}, \\
\varphi\left(u_{1}\right)+\left[1-\varphi\left(u_{1}\right)\right]\left[1-e^{-\Delta_{4}\left(u-u_{1}\right)}\right], & \text { if } u \geq u_{1},\end{cases} \tag{24}
\end{align*}
$$

is a smooth $\left(\mathscr{C}^{2}\right)$ solution to the HJB equation, where

$$
\begin{align*}
& \varphi\left(u_{1}\right) \\
& \quad=\frac{\triangle_{4}}{\triangle_{4}+u_{1}^{-\triangle_{2}} e^{-\triangle_{1} / u_{1}} e^{\triangle_{3} u_{1}} / \int_{0}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w} \tag{25}
\end{align*}
$$

$\Delta_{4}=\left((1 / 2) \theta^{2} m_{1}^{2}+2 \zeta \eta(1+\theta) m_{1} m_{2}\right) / m_{2}\left[\zeta(1+\theta) m_{2}+(\theta-\right.$ $\left.\eta) m_{1}\right], u_{1}=(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+\right.\right.$ $\left.\left.(1 / 2) \theta m_{1}\right)\right)$. The maximum of the left side of HJB equation is attained at

$$
q^{*}(u)= \begin{cases}\frac{k u}{A}, & \text { if } u<u_{1}  \tag{26}\\ \frac{(\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}}{(1+\theta) \zeta m_{2}+(1 / 2) \theta m_{1}}, & \text { if } u \geq u_{1}\end{cases}
$$

Proof. We solve the HJB equation analytically. First we need to determine the optimal strategy $q^{*}(u)$. Differentiating the terms inside the maximum in (19) with respect to $q(u)$ and setting to 0 yield

$$
\begin{equation*}
q^{0}(u)=\frac{\theta m_{1}+2 \zeta(1+\theta) m_{2}}{2 \zeta(1+\theta) m_{2}-m_{2}\left(\varphi^{\prime \prime} / \varphi^{\prime}\right)} \tag{27}
\end{equation*}
$$

The dynamic VaR constraint implies $q(u) \leq k u / A$, when $A$ is defined by (27). Normally, we take $0<\alpha<1 / 2$; hence, $A$ is always positive.
(1) For $u \geq A / k$, we have $k u / A \geq 1$. Then, from $q(u) \leq$ $k u / A$ obtained from the dynamic VaR constraint and the requirement that the retained proportion of claims $q(u)$ is always within $[0,1]$, we have $q(u) \in[0,1]$.
(a) If $q^{0}(u) \geq 1$, we let $q^{*}(u)=1$, and then the HJB equation becomes

$$
\begin{equation*}
\eta m_{1} \varphi^{\prime}(u)+\frac{1}{2} m_{2} \varphi^{\prime \prime}(u)=0 \tag{28}
\end{equation*}
$$

which implies

$$
\begin{equation*}
\frac{\varphi^{\prime \prime}(u)}{\varphi^{\prime}(u)}=-\frac{2 \eta m_{1}}{m_{2}} \tag{29}
\end{equation*}
$$

Inserting it into (28), we obtain

$$
\begin{equation*}
q^{0}(u)=\frac{\theta m_{1}+2 \zeta(1+\theta) m_{2}}{2 \zeta(1+\theta) m_{2}+2 \eta m_{1}} \tag{30}
\end{equation*}
$$

(i) If $\theta \geq 2 \eta$, we have $q^{0}(u) \geq 1$; consequently $q^{*}(u)=1$, and then the HJB equation becomes (30).
(ii) If $\theta<2 \eta$, we have $q^{0}(u)<1$, where conflict exits.
(b) If $q^{0}(u)<1$, we have $q^{*}(u)=q^{0}(u)$, and then the HJB equation becomes

$$
\begin{equation*}
\frac{\varphi^{\prime \prime}(u)}{\varphi^{\prime}(u)}=-\frac{(1 / 2) \theta^{2} m_{1}^{2}+2 \zeta \eta(1+\theta) m_{1} m_{2}}{m_{2}\left[\zeta(1+\theta) m_{2}+(\theta-\eta) m_{1}\right]} \tag{31}
\end{equation*}
$$

which implies

$$
\begin{equation*}
q^{0}(u)=\frac{\zeta(1+\theta) m_{2}+(\theta-\eta) m_{1}}{(1 / 2) \theta m_{1}+\zeta(1+\theta) m_{2}} \tag{32}
\end{equation*}
$$

(i) If $\theta \geq 2 \eta$, we have $q^{0}(u) \geq 1$, where conflict exits.
(ii) If $\theta<2 \eta$, we have $q^{0}(u)<1$; consequently $q^{*}(u)=$ $q^{0}(u)=\left(\zeta(1+\theta) m_{2}+(\theta-\eta) m_{1}\right) /\left((1 / 2) \theta m_{1}+\zeta(1+\right.$ $\theta) m_{2}$ ), and then the HJB equation becomes (32).
(2) When $0<u<A / k$, we have $k u / A<1$; thus $0 \leq$ $q(u) \leq k u / A$.
(a) If $q^{0}(u) \geq k u / A$, we have $q^{*}(u)=k u / A$, and then the HJB equation becomes

$$
\begin{align*}
& {\left[(\eta-\theta) m_{1}+\theta \frac{k u}{A} m_{1}-\zeta(1+\theta)\left(1-\frac{k u}{A}\right)^{2} m_{2}\right]}  \tag{33}\\
& \cdot \varphi^{\prime}(u)+\frac{1}{2} m_{2}\left(\frac{k u}{A}\right)^{2} \varphi^{\prime \prime}(u)=0 .
\end{align*}
$$

We have

$$
\begin{align*}
& \frac{\varphi^{\prime \prime}(u)}{\varphi^{\prime}(u)} \\
& =-\frac{(\eta-\theta) m_{1}+\theta(k u / A) m_{1}-\zeta(1+\theta)(1-k u / A)^{2} m_{2}}{(1 / 2) m_{2}(k u / A)^{2}} \tag{34}
\end{align*}
$$

which implies

$$
\begin{align*}
q^{0}(u) & =\frac{1}{2}\left(\frac{k u}{A}\right)^{2} \\
\cdot & \frac{1}{k u / A-\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left(\theta m_{1}+2 \zeta(1+\theta) m_{2}\right)} . \tag{35}
\end{align*}
$$

We have $\varphi^{\prime \prime}(u)<0$ when $u>(A / k)(((\theta-$ $\left.\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+(1 / 2)\left(\theta m_{1}+\right.\right.$ $\left.\left.\sqrt{\theta^{2} m_{1}^{2}+4(1+\theta) \zeta \eta m_{1} m_{2}}\right)\right)$ ), and we have $q^{0}(u)>0$ when $u>(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left(2(1+\theta) \zeta m_{2}+\theta m_{1}\right)\right)$. We have $\varphi^{\prime \prime}(u)<0$ and $q^{0}(u)>0$ when $u>$ $(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+(1 / 2)\left(\theta m_{1}+\right.\right.\right.$ $\left.\left.\sqrt{\left.\theta^{2} m_{1}^{2}+4(1+\theta) \zeta \eta m_{1} m_{2}\right)}\right)\right)$.

For $A / k>u>(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /((1+\right.$ $\left.\left.\theta) \zeta m_{2}+(1 / 2)\left(\theta m_{1}+\sqrt{\theta^{2} m_{1}^{2}+4(1+\theta) \zeta \eta m_{1} m_{2}}\right)\right)\right)$, we have the following.
(i) If $\theta \geq 2 \eta$, we have $q^{0}(u) \geq k u / A$; consequently $q^{*}(u)=q^{0}(u)=k u / A$, and then the HJB equation becomes (34).
(ii) If $\theta<2 \eta$, we have $q^{0}(u)<k u / A$, where conflict exits.

For $u \leq(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+\right.\right.$ $\left.\left.(1 / 2)\left(\theta m_{1}+\sqrt{\theta^{2} m_{1}^{2}+4(1+\theta) \zeta \eta m_{1} m_{2}}\right)\right)\right)$, we have $\varphi^{\prime \prime}(u) \geq$ 0 ; therefore $\varphi(u)$ is convex for small $u$. Through the analysis of the HJB equation (19), for $0 \leq u \leq$ $(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+(1 / 2)\left(\theta m_{1}+\right.\right.\right.$ $\left.\left.\left.\sqrt{\theta^{2} m_{1}^{2}+4(1+\theta) \zeta \eta m_{1} m_{2}}\right)\right)\right)$, the maximum of the left side of the HJB is attained at $q^{*}(u)=k u / A$ and the HJB equation becomes (34).
(b) When $q^{0}(u) \leq k u / A$, it is reasonable to let $q^{*}(u)=$ $q^{0}(u)$. Similar to (a), we have the following conclusions.

For $0 \leq u \leq(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+\right.\right.$ $\left.\left.(1 / 2) \theta m_{1}\right)\right)$, the optimal strategy is obtained at $q^{*}(u)=k u / A$.

For $(A / k)\left(\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+\right.\right.$ $\left.\left.(1 / 2) \theta m_{1}\right)\right)<u \leq A / k$, we have the following.
(i) If $\theta \geq 2 \eta$, we have $q^{0}(u) \geq 1>k u / A$, where conflict exits.
(ii) If $\theta<2 \eta$, we have $q^{0}(u)<k u / A$; consequently, $q^{*}(u)=q^{0}(u)=\left((\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+\right.$ $\left.(1 / 2) \theta m_{1}\right)$, and then the HJB equation becomes (32).

From the previous analysis, we have the following conclusions.
(i) If $\theta \geq 2 \eta$, the maximum of the left side of HJB equation is attained at

$$
q^{*}(u)= \begin{cases}\frac{k u}{A}, & \text { if } u<\frac{A}{k}  \tag{36}\\ 1, & \text { if } u \geq \frac{A}{k}\end{cases}
$$

(ii) If $\theta<2 \eta$, the maximum of the left side of HJB equation is attained at

$$
q^{*}(u)= \begin{cases}\frac{k u}{A}, & \text { if } u<u_{1}  \tag{37}\\ \frac{(\theta-\eta) m_{1}+(1+\theta) \zeta m_{2}}{(1+\theta) \zeta m_{2}+(1 / 2) \theta m_{1}}, & \text { if } u \geq u_{1}\end{cases}
$$

$$
\begin{equation*}
\varphi\left(\frac{A}{k}\right)=\frac{2 \eta m_{1} / m_{2}}{2 \eta m_{1} / m_{2}+(A / k)^{-\Delta_{2}} e^{-k \Delta_{1} / A} e^{A \triangle_{3} / k} / \int_{0}^{A / k} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w} \tag{43}
\end{equation*}
$$

Thus, if $\theta \geq 2 \eta$, we have the function

$$
\varphi(u)= \begin{cases}\varphi\left(\frac{A}{k}\right)-\varphi\left(\frac{A}{k}\right) \frac{\int_{u}^{A / k} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w}{\int_{0}^{A / k} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w}, & \text { if } 0<u \leq \frac{A}{k}  \tag{44}\\ \varphi\left(\frac{A}{k}\right)+\left[1-\varphi\left(\frac{A}{k}\right)\right]\left[1-e^{-\left(2 \eta m_{1} / m_{2}\right)(u-A / k)}\right], & \text { if } u \geq \frac{A}{k}\end{cases}
$$

If $\theta<2 \eta$ and $u<u_{1}$, the HJB equation is (34), and the HJB equation is (32) for $\theta<2 \eta$ and $u \geq u_{1}$. From the procedure that is similar to the previous analysis we can get the following function is a $\mathscr{C}^{2}$ solution to HJB; that is,

$$
\begin{align*}
& \varphi(u) \\
& = \begin{cases}\varphi\left(u_{1}\right)-\varphi\left(u_{1}\right) \frac{\int_{u}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\Delta_{3} w} d w}{\int_{0}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\Delta_{3} w} d w}, & \text { if } 0<u \leq u_{1}, \\
\varphi\left(u_{1}\right)+\left[1-\varphi\left(u_{1}\right)\right]\left[1-e^{-\Delta_{4}\left(u-u_{1}\right)}\right], & \text { if } u \geq u_{1},\end{cases} \tag{45}
\end{align*}
$$

where

$$
\begin{aligned}
& \triangle_{4}=\frac{(1 / 2) \theta^{2} m_{1}^{2}+2 \zeta \eta(1+\theta) m_{1} m_{2}}{m_{2}\left[\zeta(1+\theta) m_{2}+(\theta-\eta) m_{1}\right]} \\
& \varphi\left(u_{1}\right)
\end{aligned}
$$

$$
\begin{equation*}
=\frac{\triangle_{4}}{\triangle_{4}+u_{1}^{-\Delta_{2}} e^{-\Delta_{1} / u_{1}} e^{\triangle_{3} u_{1}} / \int_{0}^{u_{1}} w^{-\Delta_{2}} e^{-\Delta_{1} / w} e^{\triangle_{3} w} d w} \tag{46}
\end{equation*}
$$

This ends the proof.
When the value function is twice continuously differentiable, then it is the unique solution of the HJB equation (see, e.g., [20]), and we have the following result.

Proposition 3. The value function $v(x)$ coincides with the smooth function $\varphi(x)$ defined in Theorem 2 and the optimal control, which represents the optimal proportional reinsurance strategy, is described by the $q^{*}(u)$ in Theorem 2, where $\left\{U^{q^{*}}(t), t \geq 0\right\}$ is the corresponding surplus process.

Remark 4. When $\zeta=0$, Theorem 2 coincides with theorem 3.1 in Zhang et al. [6].

Corollary 5. When $\theta=0$, the generalized mean-variance premium principle is mean-variance premium principle, and we have the following:

$$
\varphi(u)= \begin{cases}\varphi\left(u_{1}\right)-\varphi\left(u_{1}\right) \frac{\int_{u}^{u_{1}} w^{-4 A \zeta / k} e^{-2 A^{2}\left(\eta m_{1}-\zeta m_{2}\right) / w k^{2} m_{2}} e^{2 \zeta w} d w}{\int_{0}^{u_{1}} w^{-4 A \zeta / k} e^{-2 A^{2}\left(\eta m_{1}-\zeta m_{2}\right) / w k^{2} m_{2}} e^{2 \zeta w} d w}, & \text { if } 0<u \leq u_{1}  \tag{47}\\ \varphi\left(u_{1}\right)+\left[1-\varphi\left(u_{1}\right)\right]\left[1-e^{-\left(2 \zeta \eta m_{1} / \zeta m_{2}-\eta m_{1}\right)\left(u-u_{1}\right)}\right], & \text { if } u \geq u_{1}\end{cases}
$$

is a smooth $\left(\mathscr{C}^{2}\right)$ solution to the HJB equation, where
$u_{1}=(A / k)\left(1-\eta m_{1} / \zeta m_{2}\right)$. The maximum of the left side of HJB equation is attained at

$$
q^{*}(u)= \begin{cases}\frac{k u}{A}, & \text { if } u<u_{1}  \tag{49}\\ 1-\frac{\eta m_{1}}{\zeta m_{2}}, & \text { if } u \geq u_{1}\end{cases}
$$

Corollary 6. When there is no dynamic VaR, CVaR, or $w c C V a R$ constraints, that is, $k=\infty$, and the model becomes the unconstrained reinsurance problem, we have the following.
(a) If $\theta \geq 2 \eta$, the optimal reinsurance strategy is $q^{*}=1$, and the optimal survival probability is

$$
\begin{equation*}
\varphi(u)=1-e^{-\left(2 \eta m_{1} / m_{2}\right) u} . \tag{50}
\end{equation*}
$$

(b) If $\theta<2 \eta$, the optimal reinsurance strategy is $q^{*}=((\theta-$ $\left.\eta) m_{1}+(1+\theta) \zeta m_{2}\right) /\left((1+\theta) \zeta m_{2}+(1 / 2) \theta m_{1}\right)$, and the optimal survival probability is

$$
\varphi(u)
$$

$$
\begin{equation*}
=1 \tag{51}
\end{equation*}
$$

$$
-e^{-\left(\left((1 / 2) \theta^{2} m_{1}^{2}+2 \zeta \eta(1+\theta) m_{1} m_{2}\right) / m_{2}\left[\zeta(1+\theta) m_{2}+(\theta-\eta) m_{1}\right]\right) u}
$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

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# Parameter Estimation for Fractional Diffusion Process with Discrete Observations 

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#### Abstract

This paper deals with the problem of estimating the parameters for fractional diffusion process from discrete observations when the Hurst parameter $H$ is unknown. With combination of several methods, such as the Donsker type approximate formula of fractional Brownian motion, quadratic variation method, and the maximum likelihood approach, we give the parameter estimations of the Hurst index, diffusion coefficients, and volatility and then prove their strong consistency. Finally, an extension for generalized fractional diffusion process and further work are briefly discussed.


## 1. Introduction

In recent years, many scholars have found that some financial time series data tend to be shown as biased random walk, long memory, and self-similarity, etc., which made the stochastic differential equation model driven by Brownian motion no longer applicable to describe financial data. Perhaps the most popular approach for modeling long memory is the use of fractional Brownian motion (hereafter fBm) that has been verified as a good model to describe the long memory property of some time series.

Compared with the traditional efficient market hypothesis theory, fractional market theory can accurately depict the actual law of financial market, such as the OrnsteinUhlenbeck process driven by fractional Brownian motion, which is more consistent with the characteristics of long-term memory, in place of Vasicek model that is suitable to simulate the short-term interest rate model.

Although the study of fractional Brownian motion has been going on for decades, statistical inference problems related are just in its infancy. Such questions have been recently treated in several papers [1-3]: in general, the techniques used to construct maximum likelihood estimators (MLE) for the drift parameter are based on Girsanov
transforms for fBm and depend on the properties of the deterministic fractional operators related to the fBm. Generally speaking, these papers focused on the problems of estimating the unknown parameters in the continuous-time case. Prakasa Rao [4] gave an extensive review on most of the recent developments related to the parametric and other inference procedures for stochastic models driven by fBm. The latest study can be found in Xiao and Yu [5, 6], who developed the asymptotic theory for least square estimators for two parameters in the drift function in the fractional Vasicek model with a continuous record of observations. Another possibility is to use Euler-type approximations for the solution of the above equation and to construct an MLE estimator based on the density of the observations given "the past", for the case of stochastic equations driven by Brownian motion. "Real-world" data is, however, typically discretely sampled (e.g., stock prices collected once a day or, at best, at every tick). Therefore, statistical inference for discretely observed diffusions is of great interest for practical purposes and at the same time it poses a challenging problem. Some papers are devoted to the parameter estimation for the models with fBm and discrete observations; see, e.g., Hu and Nualart[1], Hu and Song [7], Mishura and Ralchenko [8], Zhang, Xiao, Zhang and Niu [9], and Sun and Shi [10].

In this paper, we shall consider the parameter estimation problem for fractional linear diffusion process (FLDP). Assume that we have the model

$$
\begin{equation*}
d X_{t}=\left(\alpha-\beta X_{t}\right) d t+\sigma d B_{t}^{H} \tag{1}
\end{equation*}
$$

which can describe the intrinsic characteristics of interest rate more accurately in practical problem. The drift parameter $\alpha$, $\beta$ can characterize, respectively, the long-term equilibrium interest rate level and the rate of the short-term interest rates deviate from long-term interest rates. In general, the parameters of long-term equilibrium level of short-term interest rate are unknown. We assume $\beta>0$ throughout the paper so that the process is ergodic (when $\beta<0$ the solution to (1) will diverge), $\sigma$ describes the volatility of interest rates, and $\left(B_{t}^{H}\right)_{t \geq 0}$ is a fBm with Hurst parameter $H \in(0,1)$. In this paper, we suppose the Hurst index $H$, the diffusion coefficients $\alpha, \beta$, and the volatility $\sigma$ are unknown parameters to be estimated. We will furthermore show the strong consistence of these estimators.

In the case of diffusion process driven by Brownian motion, the most important methods are either maximum likelihood estimation or least square estimation. Since fBm is not a Markov process, the Kalman filter method cannot be applied to estimate the parameters of stochastic process driven by fBm. Consequently, it is a convenient way to handle the estimation problem by replacing fBm with its associated disturbed random walk. In this paper, we follow Zhang et. al. [9] to use discrete expressions of fractional Bronwnian motion with Donsker type approximate formula, which can, to some extent, simplify calculation and simulation. Although we do not have martingales in the model, this construction involving random walks allows using martingales arguments to obtain the asymptotic behaviour of the estimators.

Our paper is organized as follows. In Section 2, we propose MLE estimators for FLDP from discrete observations. The almost sure convergence of the estimators is provided in the latter part of this section. In Section 4, an extension for generalized fractional diffusion process is briefly discussed. Finally, Section 5 includes conclusions and directions of further work.

## 2. Estimation Procedure

It is worth emphasizing that the solution of (1) is given by

$$
\begin{equation*}
X_{t}=X_{0}+\int_{0}^{t}\left(\alpha-\beta X_{s}\right) d s+\sigma B_{t}^{H} \tag{2}
\end{equation*}
$$

where the unknown parameters included $\alpha, \beta, \sigma$ and $H$. We now proceed to estimate these parameters based on quadratic variation method and maximum likelihood approach.

Let $\left\{X_{t}, t \in R\right\}$ be the FLDP with $H>1 / 2$ and suppose that $\pi_{n}=\left\{\tau_{k}^{n}, k=0,1, \ldots, i_{n}\right\}, n \geq 1, i_{n} \uparrow \infty$, be a sequence of partitions of the interval $[0, T]$. If partition $\pi_{n}$ is uniform, then $\tau_{k}^{n}=k T / i_{n}$ for all $k \in\left\{0,1, \ldots, i_{n}\right\}$. If $i_{n} \equiv n$, we write $t_{k}^{n}$ instead of $\tau_{k}^{n}$. Assume that process $X_{t}$ is observed at time points $\left(i / m_{n}\right) T, i=1,2, \ldots, m_{n}$, where $m_{n}=n k_{n}$ and $k_{n}$ grows
faster than $n \ln n$, but the growth does not exceed polynomial, e.g., $k_{n}=n n^{\theta} n, \theta>1$ or $k_{n}=n^{2}$.

In applications, the estimation of $H \in(0,1)$ (called the Hurst index) is a fundamental problem. Its solution depends on the theoretical structure of a model under consideration. Therefore particular models usually deserve separate analysis.

According to the notation of Kubilius Skorniakov [11], suppose there are two hypotheses:

$$
\begin{align*}
(\mathrm{C} 1) \Delta X_{\tau_{k}^{n}} & =X_{\tau_{k}^{n}}-X_{\tau_{k-1}^{n}}=O_{\omega}\left(d_{n}^{H-\varepsilon}\right), \\
(\mathrm{C} 2) \Delta^{(2)} X_{\tau_{k}^{n}} & =\Delta X_{\tau_{k}^{n}}-\Delta X_{\tau_{k-1}^{n}}  \tag{3}\\
& =\sigma \Delta^{(2)} B_{\tau_{k-1}^{n}}^{H}+O_{\omega}\left(d_{n}^{2(H-\varepsilon)}\right),
\end{align*}
$$

for all $\varepsilon \in(0, H-1 / 2)$, where $d_{n}=\max _{1 \leq k \leq m_{n}\left(\tau_{k}^{n}-\tau_{k-1}^{n}\right)} Y_{n}=$ $O_{\omega}\left(a_{n}\right)$ means for a sequence of r.v. $Y_{n}$, and $a_{n} \subset(0, \infty)$, and there exists a.s. non-negative r.v. $\varsigma$, such that $\left|Y_{n}\right| \leq \varsigma \cdot a_{n}$. These two conditions are used to prove the strongly consistent and asymptotically normality of the estimator $H$ from discrete observatios.

Denote

$$
\begin{align*}
W_{n, k} & =\sum_{i=-k_{n}+2}^{k_{n}}\left(\Delta^{(2)} X_{s_{i}^{n}+t_{k}^{n}}\right)^{2} \\
& =\sum_{i=-k_{n}+2}^{k_{n}}\left(X_{s_{i}^{n}+t_{k}^{n}}-2 X_{s_{i-1}^{n}+t_{k}^{n}}+X_{s_{i-2}^{n}+t_{k}^{n}}\right)^{2}, \tag{4}
\end{align*}
$$

where $1 \leq k \leq n$ and $s_{i}^{n}=\left(i / m_{n}\right) T$,
Then, the estimator of Hurst parameter $H$ can be written as

$$
\begin{equation*}
\widehat{H}=\frac{1}{2}+\frac{1}{2 \ln k_{n}} \ln \left(\frac{2}{m_{n}} \sum_{k=2}^{m_{n}} \frac{\left(\Delta^{(2)} X_{t_{k}^{n}}\right)^{2}}{W_{n, k-1}}\right) . \tag{5}
\end{equation*}
$$

Next, we turn to the estimation problem of the diffusion parameter $\sigma^{2}$. When $H$ is known, Xiao et al. [12] obtained the estimators based on approximating integrals via Riemann sums with Hurst index $H \in(1 / 2,3 / 4)$. In contrast, we suppose in this paper the Hurst index is unknown. Therefore in the next estimation, the estimator of $H$ will be embedded in the equation. For simplicity, denote $X_{\tau_{i}^{n}} \triangleq X_{i h}, B_{\tau_{k-1}^{n}}^{H}=$ $B_{i}^{H, m_{n}}, i=1, \ldots, m_{n}, h=T / m_{n}$. Thus, the full sequence of $m_{n}$ observations can be written as $\left\{X_{h}, X_{2 h}, \ldots, X_{m_{n} h}\right\}$.

For the diffusion parameter, we easily obtain an estimator for the diffusion parameter by using quadratic variations, such

$$
\begin{equation*}
\widehat{\sigma}^{2}=\frac{\sum_{i=1}^{m_{n}-1}\left(X_{(i+1) h}-X_{i h}\right)^{2}}{\left(m_{n}-1\right) h^{2 \widehat{H}}}, \tag{6}
\end{equation*}
$$

which converges (in $L^{2}$ and almost surely) to $\sigma^{2}$.
Finally, we are in a position to estimate the drift parameter. Note that $B_{t}^{H}-B_{t-1}^{H}$ is not independent and the process $B_{t}^{H}$ is not a semimartingale; therefore the martingale type techniques cannot be used to study this estimator. This
problem will be avoided by the use of the random walks that approximate $B_{t}^{H}$. Based on the results on Sottinen [13], the fractional Brownian motion can be approximated by a "disturbed" random walk, which was called Donsker type approximation for fBm .

Lemma 1. The fBm with Hurst parameter $H>1 / 2$ can be represented by its associated disturbed random walk:

$$
\begin{equation*}
B_{t}^{H, m_{n}}=\sum_{i=1}^{\left\lfloor m_{n} t\right\rfloor} \sqrt{m_{n}}\left(\int_{(i-1) / m_{n}}^{i / m_{n}} K^{H}\left(\frac{\left\lfloor m_{n} t\right\rfloor}{m_{n}}, s\right) d s\right) \varepsilon_{i} \tag{7}
\end{equation*}
$$

with $K^{H}(t, s)=c_{H}(H-1 / 2) s^{1 / 2-H} \int_{s}^{t}(u-s)^{H-1 / 2} u^{H-1 / 2} d u$, which is the kernel function that transforms the standard Brownian motion into a fractional one, $c_{H}$ is the normalizing constant $c_{H}=\sqrt{2 H \Gamma(2 / 3-H) / \Gamma(H+1 / 2) \Gamma(2-2 H)}$, and $\varepsilon_{i}$ are i.i.d. random variables with $E \varepsilon_{i}=0$ and $\operatorname{var} \varepsilon_{i}=1$, and $\lfloor x\rfloor$ denotes the greatest integer not exceeding $x$.

Sottinen (2011) proved that $B_{t}^{H, m_{n}}$ converges weakly in the skorohod topology to the fractional Brownian motion. With the estimators $\widehat{H}, \widehat{\sigma}$ plug-in, the replacing model still kept the main properties of the original process, such as long range dependence and asymptotic self-similar. Therefore, the martingales can be used to treat this replacing model.

In general, numerical approximation of model (1) can be presented by Euler scheme:

$$
\begin{align*}
& X_{(i+1) h}=X_{i h}+\left(\alpha-\beta X_{i h}\right) h+\widehat{\sigma}\left(B_{(i+1) h}^{H, m_{n}}-B_{i h}^{H, m_{n}}\right)  \tag{8}\\
& i=1, \ldots, m_{n}-1 .
\end{align*}
$$

Set

$$
\begin{align*}
& f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right) \\
& =\sqrt{m_{n}} \sum_{j=1}^{i}\left[\int_{(j-1) h}^{j h}\left(K^{H}((i+1) h, s)-K^{H}(i h, s)\right) d s\right] \varepsilon_{j} \tag{9}
\end{align*}
$$

to denote the contribution of the $n-1$ first jumps of the random walk and

$$
\begin{equation*}
F_{i}=\sqrt{m_{n}} \int_{i h}^{(i+1) h} K^{H}((i+1) h, s) d s \tag{10}
\end{equation*}
$$

to denote the contribution of the last jump.

With the approximation of fBm (Lemma 1), we can write

$$
\begin{align*}
B_{(i+1) h}^{H, m_{n}}-B_{i h}^{H, m_{n}}=f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)+ & F_{i} \varepsilon_{i+1}  \tag{11}\\
& i=1, \ldots, m_{n}-1 .
\end{align*}
$$

with which (8) can be written as

$$
\begin{align*}
X_{(i+1) h}= & X_{i h}+\left(\alpha-\beta X_{i h}\right) h \\
& +\widehat{\sigma}\left[f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)+F_{i} \varepsilon_{i+1}\right] \tag{12}
\end{align*}
$$

Hence we have

$$
\begin{align*}
& E\left[X_{(i+1) h}-X_{i h} \mid X_{i h}\right] \\
& \quad=\left(\alpha-\beta X_{i h}\right) h+\widehat{\sigma}\left(f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)\right),  \tag{13}\\
& \operatorname{var}\left[X_{(i+1) h}-X_{i h} \mid X_{i h}\right]=\widehat{\sigma}^{2} F_{i}^{2} .
\end{align*}
$$

We assume that random variables $\varepsilon_{i}$ follow a standard normal law $N(0,1)$. Then, the random variable $X_{(i+1) h}$ is conditionally Gaussian and the conditional density of $X_{(i+1) h}$ given $X_{h}, X_{2 h}, \ldots, X_{i h}$ can be written as

$$
\begin{align*}
& f_{X_{(i+1) h} \mid X_{h}, X_{2 h} \ldots, X_{i h}}\left(x_{(i+1) h} \mid x_{h}, x_{2 h}, \ldots, x_{i h}\right)=\frac{1}{\sqrt{2 \pi \widehat{\sigma}^{2} F_{i}^{2}}} \\
& \quad \cdot \exp \left\{-\frac{1}{2}\right.  \tag{14}\\
& \left.\cdot \frac{\left(x_{(i+1) h}-x_{i h}-\left(\alpha-\beta x_{i h}\right) h-\widehat{\sigma} f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)\right)^{2}}{F_{i}^{2}}\right\} .
\end{align*}
$$

The likelihood function can be expressed as

$$
\begin{align*}
& L(\alpha, \beta)=f_{X_{h}}\left(x_{h}\right) f_{X_{2 h} \mid X_{h}}\left(x_{2 h} \mid x_{h}\right) \\
& \quad \cdots f_{X_{m_{n} h} \mid X_{h}, X_{2 h} \cdots, X_{\left(m_{n}-1\right) h}}\left(x_{m_{n} h} \mid x_{h}, x_{2 h}, \ldots, x_{\left(m_{n}-1\right) h}\right) \\
& \quad=\prod_{i=1}^{m_{n}} \frac{1}{\sqrt{2 \pi \hat{\sigma}^{2} F_{i}^{2}}} \exp \left\{-\frac{1}{2}\right.  \tag{15}\\
& \left.\quad \cdot \frac{\left(x_{(i+1) h}-x_{i h}-\left(\alpha-\beta x_{i h}\right) h-\widehat{\sigma} f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)\right)^{2}}{F_{i}^{2}}\right\} .
\end{align*}
$$

This leads to the MLE of $\alpha$ and $\beta$

$$
\begin{align*}
& \widehat{\alpha}=\frac{\sum_{i=0}^{m_{n}-1}\left(\left(y_{i h}-\widehat{\beta} h x_{i h}\right) / F_{i}^{2}\right)}{\sum_{i=0}^{N-1}\left(h / F_{i}^{2}\right)},  \tag{16}\\
& \widehat{\beta}=\left(\frac{1}{h}\right) \frac{\sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(y_{i h} / F_{i}^{2}\right)-\sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(y_{i h} x_{i h} / F_{i}^{2}\right)}{\sum_{i=0}^{m_{n}-1}\left(x_{i h}^{2} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right)-\left(\sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)\right)^{2}}, \tag{17}
\end{align*}
$$

where $y_{i h}=x_{(i+1) h}-x_{i h}-\widehat{\sigma} f_{i}\left(\varepsilon_{1}, \varepsilon_{2}, \ldots, \varepsilon_{i}\right)=\left(\alpha-\beta X_{i h}\right) h+$ $v_{i}, v_{i}=\widehat{\sigma} F_{i} \varepsilon_{i+1}, i=1,2, \ldots, m_{n}-1$.

Remark 2. Note that the parameter estimators of drift coefficients are related to the volatility $\sigma$, while, in fact, $\sigma^{2}$ can be
(at least theoretically) computed on any finite time interval. Furthermore, fBm is self-similar to stationary increments and it satisfies $E\left|B_{t}^{H}-B_{s}^{H}\right|=|t-s|^{2 H}$ for every $s, t \in[0, T]$. For this reason, we may assume that the diffusion coefficient is equal to 1.

## 3. The Asymptotic Properties

In this section, we turn to study the strong consistency of these estimators by (5), (6), (16), and (17).

Theorem 3. Assume that solution of (1) satisfies hypotheses (C1) and (C2), then estimator $\widehat{H}$ converges to $H$ almost surely as $m_{n}$ goes to infinity.

Detailed proof can be found in Kubilus and Skorniakov [11].

Theorem 4. The estimator $\hat{\sigma}^{2}$ converges to $\sigma^{2}$ almost surely as $m_{n}$ goes to infinity.

Proof. With the strong consistency of $\widehat{H}$ to $H$ and that $\widetilde{\sigma}^{2} \triangleq$ $\left(\sum_{i=1}^{m_{n}-1}\left(X_{(i+1) h}-X_{i h}\right)^{2}\right) /\left(m_{n}-1\right) h^{2 H} \longrightarrow \sigma^{2}$ with probability 1 as $m_{n}$ goes to infinity, it can be easily shown that estimator $\widehat{\sigma}^{2}$ converges to $\sigma^{2}$ almost surely as $m_{n} \longrightarrow \infty$.

Theorem 5. With probability one, $\widehat{\alpha} \longrightarrow \alpha, \widehat{\beta} \longrightarrow \beta$, as $m_{n} \longrightarrow \infty$.

Proof. Clearly, the consistency of $\widehat{\alpha}$ can be inferred combined with (16) and consistency of $\widehat{\beta}$. We just prove that $\widehat{\beta}$ is strong consistent.

A simple calculation shows that

$$
\begin{align*}
\widehat{\beta}-\beta & =\frac{1}{h} \frac{\sum_{i=0}^{m_{n}-1}\left(v_{i} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)-\sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right) \sum_{i=0}^{N-1}\left(v_{i} x_{i h} / F_{i}^{2}\right)}{\sum_{i=0}^{m_{n}-1}\left(x_{i h}^{2} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right)-\left(\sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)\right)^{2}} \\
& =\frac{\left(T / m_{n}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(v_{i} x_{i h} / F_{i}^{2}\right)-\sum_{i=0}^{m_{n}-1}\left(v_{i} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)}{\left(T / m_{n}\right)^{2} \sum_{i=0}^{m_{n}-1}\left(x_{i h}^{2} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right)-\left(\sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)\right)^{2}}=\frac{M_{n}}{\left\langle M>_{n}\right.}, \tag{18}
\end{align*}
$$

where $M_{n}=\left(T / m_{n}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(v_{i h} x_{i h} / F_{i}^{2}\right)-$ $\sum_{i=0}^{m_{n}-1}\left(y_{i h} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)$ is a square-integrable martingle and $\langle M\rangle_{n}=\left(T / m_{n}\right)^{2} \sum_{i=0}^{m_{n}-1}\left(x_{i h}^{2} / F_{i}^{2}\right) \sum_{i=0}^{m_{n}-1}\left(1 / F_{i}^{2}\right)-$ $\left(\sum_{i=0}^{m_{n}-1}\left(x_{i h} / F_{i}^{2}\right)\right)^{2}$ is quadratic characteristic of $M_{n}$.

Using the assumption of $H \in(1 / 2,1)$ and fractional integral, we have the explicit solution of (1) that can be expressed as

$$
\begin{equation*}
X_{t}=\left(1-e^{-\beta t}\right) \frac{\alpha}{\beta}+\sigma \int_{0}^{t} e^{-\beta(t-s)} d B_{s}^{H}, \quad t \geq 0 \tag{19}
\end{equation*}
$$

where the integral can be understood in the Skorohod sense. As a consequence, for any $i$, we have

$$
E\left[X_{i h}^{2}\right]=E\left[\left(1-e^{-\beta i h}\right) \frac{\alpha}{\beta}\right]^{2}
$$

$$
\begin{align*}
& +\left(\sigma \int_{0}^{i h} e^{-\beta(t-s)} d B_{s}^{H}\right)^{2} \\
& \left.+2\left(1-e^{-\beta i h}\right) \frac{\alpha}{\beta} \sigma \int_{0}^{i h} e^{-\beta(t-s)} d B_{s}^{H}\right] \\
& \leq 2\left\{\left[\left(1-e^{-\beta i h}\right) \frac{\alpha}{\beta}\right]^{2}+\sigma^{2} e^{-2 \beta i h} \frac{H \Gamma(2 H)}{\beta^{2} H}\right\} \tag{20}
\end{align*}
$$

Hence, for any $i$, we obtain that $E\left[X_{i h}^{2}\right]$ is bounded. Moreover, by using Cauchy-Schwartz inequality, we show that (see also in [14], with a slight modification below)

$$
\begin{aligned}
E\left|B_{(i+1) h}^{H, m_{n}}-B_{i h}^{H, m_{n}}\right|^{2} & =E\left[\sum_{j=\left\lfloor m_{n} i h\right\rfloor}^{\left\lfloor m_{n}(i+1) h\right\rfloor} \sqrt{m_{n}} \int_{(j-1) / m_{n}}^{j / m_{n}}\left(K^{H}\left(\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}, s\right)-K^{H}\left(\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}, s\right)\right) d s \varepsilon_{j}\right]^{2} \\
& =\left[\sum_{j=\left\lfloor m_{n} i h\right\rfloor}^{\left\lfloor m_{n}(i+1) h\right\rfloor} \sqrt{m_{n}} \int_{(j-1) / m_{n}}^{j / m_{n}}\left(K^{H}\left(\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}, s\right)-K^{H}\left(\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}, s\right)\right) d s\right]^{2} \\
& \leq \sum_{j=\left\lfloor m_{n} i h\right\rfloor}^{\left\lfloor m_{n}(i+1) h\right\rfloor} m_{n} \int_{(j-1) / m_{n}}^{j / m_{n}}\left(\left(K^{H}\left(\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}, s\right)-K^{H}\left(\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}, s\right)\right)\right)^{2} d s \\
& \leq m_{n} \int_{i h}^{(i+1) h}\left(\left(K^{H}\left(\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}, s\right)-K^{H}\left(\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}, s\right)\right)\right)^{2} d s
\end{aligned}
$$

$$
\begin{equation*}
\leq \int_{0}^{T}\left(\left(K^{H}\left(\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}, s\right)-K^{H}\left(\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}, s\right)\right)\right)^{2} d s=\left|\frac{\left\lfloor m_{n}(i+1) h\right\rfloor}{m_{n}}-\frac{\left\lfloor m_{n} i h\right\rfloor}{m_{n}}\right|^{2 H} \leq \frac{1}{m_{n}^{2 H}} \tag{21}
\end{equation*}
$$

By standard calculations, we will have

$$
\begin{equation*}
F_{i}^{2}+E\left[f_{i}^{2}\right]=E\left|B_{(i+1) h}^{H, m_{n}}-B_{i h}^{H, m_{n}}\right|^{2} \leq \frac{1}{m_{n}^{2 H}} \tag{22}
\end{equation*}
$$

and it holds that

$$
\begin{equation*}
F_{j}^{2} \leq \frac{1}{m_{n}^{2 H}} \tag{23}
\end{equation*}
$$

Now, (20) combined with (23) shows that $M_{n} /\langle M\rangle_{n} \longrightarrow$ 0, a.s. as $m_{n} \longrightarrow \infty$.

Remark 6. The asymptotic normality of estimators is not involved in the results of this paper. In fact, Kubilius and Skorniakov [11] proposed the asymptotic normality of the estimators $\widehat{H}$; in view of Remark 2, the asymptotic of $\widehat{\sigma}$ is trivial. For the parameter estimation of fractional diffusion process (1), there are usually two key challenges: the likelihood is intractable and the data is not Markovian. With the Donsker type approximation formula, the statistical inference of fractional diffusion process (FDP) can be simplified to a certain extent. It has proved that the estimator of drift parameter is $L^{p}(p \geq 1)$ - consistent and the asymptotic normality may be obtained with more complex operations by the future studies of this area.

## 4. Extension

Fractional stochastic differential equations have been widely used in the fields of finance, hydrology, information, and stochastic networks. Although model (1) is concerned of simpler linear function, our method can be expected to be applicable for general fractional diffusion processes, such as

$$
\begin{equation*}
d X_{t}=\mu\left(X_{t} ; \theta\right) d t+\sigma d B_{t}^{H} \tag{24}
\end{equation*}
$$

where $\mu(. ; \theta)$ is drift functions representing the conditional mean of the infinitesimal change of $X_{t}$ at time $t ; \sigma d B_{t}^{H}$ is the random perturbation. Here we suppose the diffusion function is constant for simplicity. As far as we know, for a general smooth and elliptic coefficient $\sigma($.$) , only the uniqueness of$ the invariant measure is shown in Haier and Ohashi [15], with an interesting extension to the hypoelliptic case in Haier and Pillai [16]. Nothing is known about the convergence of estimate equation, not to mention rates. Suppose $X_{t}$ is observed at a discrete set of instants $\tau_{i}^{n}=\left(i / m_{n}\right) T, i=$ $1,2, \ldots, m_{n}$. With the Donsker type approximate formula and the above estimation procedure, we use the following global estimation equations to estimate parameter $\theta$ :

$$
\begin{aligned}
& q_{n}(\theta)=\sum_{i=1}^{m_{n}-1}\left\{\dot{g}\left(\theta, X_{i h}\right)\right. \\
& \quad \cdot\left\{X_{(i+1) h}-X_{i h}-g\left(\theta, X_{i h}\right)-\sigma f_{i}\left(\varepsilon_{1}, \ldots, \varepsilon_{i}\right)\right\}=0
\end{aligned}
$$

where $\dot{g}\left(\theta, X_{i h}\right)$ is the derivative of $g\left(\theta, X_{i h}\right)$ on $\theta$. The asymptotic property of the estimators is expected to be studied in the future and how to obtain the asymptotic theory is still an open question.

On the other hand, our method can extend to another self-similar process still with long memory (but not Gaussian), which is called Rosenblatt process $Z_{t}^{H}$. In contrast to the fBm model, the density of Rosenblatt process is not explicitly known any more. However, it can be written as a double integral of a two-variable deterministic with respect to the Wiener process. The method based on random walks approximation offers a solution to the problem of estimating the parameters in fractional diffusion process driven by Rosenblatt process.

## 5. Concluding Remarks

In this paper, we proposed the estimators of FLDP, such as the Hurst index, drift coefficients, and volatility, and provided the strong consistency for these estimators. With the Donsker representation of fractional Brownian motion, the statistical inference of FLDP may be simplified. However, it is important to note that this approximation is satisfied in the sense of weak convergence. This means only when with large number of samples can the simulation be much better. On the other hand, the approximate representation of FLDP is based on the Euler scheme, which is the main source of the error in the computations. There is always a trade-off between the number of Euler steps and the number of simulations, but what is usually computationally costly is the number of Euler steps. The rate of convergence depended on $H$ and the closer the value of $H$ to $1 / 2$. This study also suggests several important directions for further research. How to estimate parameters in FDP from discrete time observations and how to obtain the asymptotic theory are open questions.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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# Research Article 

# Hermite-Hadamard-Fejér Inequalities for Conformable Fractional Integrals via Preinvex Functions 

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#### Abstract

In this paper, we present a Hermite-Hadamard-Fejér inequality for conformable fractional integrals by using symmetric preinvex functions. We also establish an identity associated with the right hand side of Hermite-Hadamard inequality for preinvex functions; then by using this identity and preinvexity of functions and some well-known inequalities, we find several new Hermite-Hadamard type inequalities for conformal fractional integrals.


## 1. Introduction

Let $I \in \mathbb{R}$ be an interval and $h: I \longrightarrow \mathbb{R}$ be a convex function defined on $I$ such that $\kappa_{1}, \kappa_{2} \in I$ with $\kappa_{1}<\kappa_{2}$. Then the wellknown $\mathscr{H} \mathscr{H}$ (Hermite-Hadamard) inequality [1] states that

$$
\begin{equation*}
h\left(\frac{\kappa_{1}+\kappa_{2}}{2}\right) \leq \frac{1}{\kappa_{2}-\kappa_{1}} \int_{\kappa_{1}}^{\kappa_{2}} h(x) d x \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{2}\right)}{2} \tag{1}
\end{equation*}
$$

holds. If the function $h$ is concave on $I$, then both inequalities in (1) hold in the reverse direction.

In the last few years, many researchers have shown their extensive attention on the generalizations, extensions, variations, refinements, and applications of the $\mathscr{H} \mathscr{H}$ inequality (see [2-15]). The most well-known generalization of the $\mathscr{H} \mathscr{H}$ inequality is the Hermite-Hadamard-Fejér inequality [16]. In 1906, Fejér [16] established the following weighted generalization of the Hermite-Hadamard inequality for symmetric functions:

$$
\begin{align*}
& h\left(\frac{\kappa_{1}+\kappa_{2}}{2}\right) \int_{\kappa_{1}}^{\kappa_{2}} g(x) d x \leq \frac{1}{\kappa_{2}-\kappa_{1}} \int_{\kappa_{1}}^{\kappa_{2}} g(x) h(x) d x \\
& \quad \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{2}\right)}{2} \int_{\kappa_{1}}^{\kappa_{2}} g(x) d x \tag{2}
\end{align*}
$$

for all convex functions $h: I \longrightarrow \mathbb{R}, \kappa_{1}, \kappa_{2} \in I$ with $\kappa_{1}<\kappa_{2}$ and $g:\left[\kappa_{1}, \kappa_{2}\right] \longrightarrow \mathbb{R}^{+}$is symmetric with respect to $\left(\kappa_{1}+\right.$ $\left.\kappa_{2}\right) / 2$.

It is well known that the convex sets and convex functions play important roles in the nonlinear programming and optimization theory. Many generalizations and extensions have been considered for the classical convexity in the last few decades. A significant generalization of convex functions is that of invex functions introduced by Hanson in [17]. The basic properties of the preinvex functions and their roles in optimization theory can be found in [18]. The $\mathscr{H} \mathscr{H}$ inequalities for preinvex and log-preinvex functions were established by Noor [19, 20].

Now, we recall some notions and definitions in invexity analysis, which will be used throughout the paper (see [21, 22] and references therein).

Let $\mathfrak{A} \in \mathbb{R}$ be a nonempty set and the functions $h: \mathfrak{A} \longrightarrow$ $\mathbb{R}$ and $\Psi: \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{R}$ be continuous.

Definition 1. The set $\mathfrak{A} \subseteq \mathbb{R}^{n}$ is said to be invex with respect to $\Psi(.,$.$) if$

$$
\begin{equation*}
\mu_{1}+s \Psi\left(\mu_{2}, \mu_{1}\right) \in \mathfrak{A} \tag{3}
\end{equation*}
$$

for all $\mu_{1}, \mu_{2} \in \mathfrak{A}$ and $s \in[0,1]$.

The invex set $\mathfrak{A}$ is also called a $\Psi$-connected set. If $\Psi\left(\mu_{2}, \mu_{1}\right)=\mu_{2}-\mu_{1}$, then the invex set is also a convex set, but some of the invex sets are not convex [21].

Definition 2. The function $h$ is said to be preinvex with respect to $\Psi$ on the invex set $\mathfrak{A}$ if

$$
\begin{equation*}
h\left(\mu_{1}+s \Psi\left(\mu_{2}, \mu_{1}\right)\right) \leq(1-s) h\left(\mu_{1}\right)+\operatorname{sh}\left(\mu_{2}\right) \tag{4}
\end{equation*}
$$

for all $\mu_{1}, \mu_{2} \in \mathfrak{A}$ and $s \in[0,1]$. The function $h$ is called preconcave if $-h$ is preinvex.

The following Condition C was introduced by Mohan and Neogy [23].

Condition C. Suppose $\mathfrak{A}$ is an open invex subset of $\mathbb{R}^{n}$ with respect to $\Psi: \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathbb{R}$ and $\Psi$ satisfies

$$
\begin{align*}
& \Psi\left(\mu_{2}, \mu_{2}+s \Psi\left(\mu_{1}, \mu_{2}\right)\right)=-s \Psi\left(\mu_{1}, \mu_{2}\right) \\
& \Psi\left(\mu_{1}, \mu_{2}+s \Psi\left(\mu_{1}, \mu_{2}\right)\right)=(1-s) \Psi\left(\mu_{1}, \mu_{2}\right) \tag{5}
\end{align*}
$$

for any $\mu_{1}, \mu_{2} \in \mathfrak{A}$ and $s \in[0,1]$.
From Condition C, we clearly see that

$$
\begin{align*}
& \Psi\left(\mu_{2}+s_{2} \Psi\left(\mu_{1}, \mu_{2}\right), \mu_{2}+s_{1} \Psi\left(\mu_{1}, \mu_{2}\right)\right)  \tag{6}\\
& \quad=\left(s_{2}-s_{1}\right) \Psi\left(\mu_{1}, \mu_{2}\right)
\end{align*}
$$

for any $\mu_{1}, \mu_{2} \in \mathfrak{A}$ and $s \in[0,1]$.
The following $\mathscr{H} \mathscr{H}$ inequality for the preinvex functions was proved by Noor [20].

Theorem 3. Let $h: K=\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \longrightarrow(0, \infty)$ be a preinvex function on the interval $K^{\circ}$ (the interior of $K$ ) and $\kappa_{1}$, $\kappa_{2} \in K^{\circ}$ with $\kappa_{1}<\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)$. Then the following inequality holds:

$$
\begin{align*}
& h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \\
& \quad \leq \frac{1}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d x \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{2}\right)}{2} . \tag{7}
\end{align*}
$$

Several important variants of $\mathscr{H} \mathscr{H}$ inequality for preinvex functions have been provided in the literature [24]. Recently, the authors in [25] defined a new well-behaved simple fractional derivative called the "conformable fractional derivative". Namely, the conformable fractional derivative of order $0<\alpha \leq 1$ at $s>0$ for the function $h:[0, \infty) \longrightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
D_{\alpha}(h)(s)=\lim _{\epsilon \longrightarrow 0} \frac{h\left(s+\epsilon s^{1-\alpha}\right)-h(s)}{\epsilon} . \tag{8}
\end{equation*}
$$

If the conformable fractional derivative of $h$ of order $\alpha$ exists, then we say that $h$ is $\alpha$-fractional differentiable. The fractional derivative at 0 is defined as $h^{\alpha}(0)=\lim _{s \rightarrow 0^{+}} h^{\alpha}(s)$.

Next, we present some basic results related to conformable fractional derivative in the following theorem.

Theorem 4 (see [25]). Let $\alpha \in(0,1]$ and $h_{1}, h_{2}$ be $\alpha$-differentiable at a point $s>0$. Then
(i) $\left(d_{\alpha} / d_{\alpha} s\right)\left(s^{n}\right)=n s^{n-\alpha}$ for all $n \in \mathbb{R}$.
(ii) $\left(d_{\alpha} / d_{\alpha} s\right)(c)=0$ for any constant $c \in \mathbb{R}$.
(iii) $\left(d_{\alpha} / d_{\alpha} s\right)\left(\kappa_{1} h_{1}(s)+\kappa_{2} h_{2}(s)\right)=\kappa_{1}\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s)\right)+$ $\kappa_{2}\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)$ for all $\kappa_{1}, \kappa_{2} \in \mathbb{R}$.
(iv) $\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s) h_{2}(s)\right)=h_{1}(s)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)+h_{2}(s)\left(d_{\alpha} /\right.$ $\left.d_{\alpha} s\right)\left(h_{1}(s)\right)$.
(v) $\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s) / h_{2}(s)\right)=\left(h_{2}(s)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s)\right)-h_{1}(s)\left(d_{\alpha} /\right.\right.$ $\left.\left.d_{\alpha} s\right)\left(h_{2}(s)\right)\right) /\left(h_{2}(s)\right)^{2}$.
(vi) $\left(d_{\alpha} / d_{\alpha} s\right)\left(\left(h_{1} \circ h_{2}\right)(s)\right)=h_{1}^{\prime}\left(h_{2}(s)\right)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)$ if $h_{1}$ differentiable at $h_{2}(s)$.

If in addition $h_{1}$ is differentiable, then

$$
\begin{equation*}
\frac{d_{\alpha}}{d_{\alpha} s}\left(h_{1}(s)\right)=s^{1-\alpha} \frac{d}{d s}\left(h_{1}(s)\right) . \tag{9}
\end{equation*}
$$

Definition 5 (see [25] conformable fractional integral). Let $\alpha \in(0,1]$ and $0 \leq \kappa_{1}<\kappa_{2}$. A function $h_{1}:\left[\kappa_{1}, \kappa_{2}\right] \longrightarrow \mathbb{R}$ is $\alpha$-fractional integrable on $\left[\kappa_{1}, \kappa_{2}\right]$ if the integral

$$
\begin{equation*}
\int_{\kappa_{1}}^{\kappa_{2}} h_{1}(x) d_{\alpha} x:=\int_{\kappa_{1}}^{\kappa_{2}} h_{1}(x) x^{\alpha-1} d x \tag{10}
\end{equation*}
$$

exists and is finite. All $\alpha$-fractional integrable functions on $\left[\kappa_{1}, \kappa_{2}\right]$ are indicated by $L_{\alpha}\left(\left[\kappa_{1}, \kappa_{2}\right]\right)$.

Remark 6.

$$
\begin{equation*}
I_{\alpha}^{\kappa_{1}}\left(h_{1}\right)(s)=I_{1}^{\kappa_{1}}\left(s^{\alpha-1} h_{1}\right)=\int_{\kappa_{1}}^{s} \frac{h_{1}(x)}{x^{1-\alpha}} d x \tag{11}
\end{equation*}
$$

where the integral is the usual Riemann improper integral and $\alpha \in(0,1]$.

Recently, the conformable integrals and derivatives have been the subject of intensive research, and many remarkable properties and inequalities involving the conformable integrals and derivatives can be found in the literature [26-38].

In [39], Anderson provided the conformable integral version of $\mathscr{H} \mathscr{H}$ inequality as follows.

Theorem 7 (see [39]). If $\alpha \in(0,1]$ and $h_{1}:\left[\kappa_{1}, \kappa_{2}\right] \longrightarrow$ $\mathbb{R}$ is an $\alpha$-fractional differentiable function such that $D_{\alpha} h$ is increasing, then we have the following inequality:

$$
\begin{equation*}
\frac{\alpha}{\kappa_{2}^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{2}} h(x) d_{\alpha} x \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{2}\right)}{2} . \tag{12}
\end{equation*}
$$

Moreover if the function $h$ is decreasing on $\left[\kappa_{1}, \kappa_{2}\right]$, then we have

$$
\begin{equation*}
h\left(\frac{\kappa_{1}+\kappa_{2}}{2}\right) \leq \frac{\alpha}{\kappa_{2}^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{2}} h(x) d_{\alpha} x . \tag{13}
\end{equation*}
$$

If $\alpha=1$, then this reduces to the classical $\mathscr{H} \mathscr{H}$ inequality.

In this paper, we first establish the Hermite-HadamardFejér inequality for conformable fractional integrals by using symmetric preinvex functions; then we present $\mathscr{H} \mathscr{H}$ inequalities as their special cases (see Corollary 9). Secondly, we give an identity associated with the right side of $\mathscr{H} \mathscr{H}$ inequality for preinvex functions using the conformable fractional integrals; then we establish $\mathscr{H} \mathscr{H}$ inequalities for preinvex functions by use of Hölder inequality, power mean inequality, and preinvexity of functions.

## 2. Hermite-Hadamard-Fejér Inequalities for Conformable Fractional Integrals

The preinvex version of Fejer-Hermite-Hadamard inequality can be represented in conformable fractional integrals forms as follows.

Theorem 8. Suppose that $\kappa_{1}, \kappa_{2} \in K$ such that $\Psi\left(\kappa_{2}, \kappa_{1}\right)>0$, $h: K=\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \longrightarrow(0, \infty)$ is a preinvex function and symmetric with respect to $\left(2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right) / 2$, and $g:\left[\kappa_{1}, \kappa_{2}\right] \rightarrow \mathbb{R}$ is a nonnegative integrable function. Also assume that $\Psi$ satisfies Condition $C$; then the inequality

$$
\begin{align*}
& h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) d_{\alpha} x \\
& \quad \leq \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) d_{\alpha} x  \tag{14}\\
& \quad \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) d_{\alpha} x
\end{align*}
$$

holds for any $\alpha \in(0,1]$.
Proof. Since $h: K \longrightarrow \mathbb{R}$ is preinvex function and is symmetric with respect to $\left(2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right) / 2$, then for any $x, y \in K$ and $t=1 / 2$, we have

$$
\begin{equation*}
h\left(x+\frac{\Psi(y, x)}{2}\right) \leq \frac{h(x)+h(y)}{2} \tag{15}
\end{equation*}
$$

i.e., with $x=\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)$ and $y=\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)$, inequality (15) becomes

$$
\begin{aligned}
& h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right. \\
& \left.\quad+\frac{\Psi\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right), \kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}\right) \\
& \quad=h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right. \\
& \left.\quad+\frac{(s-1+s) \Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \text { (using Condition C) } \\
& \quad=h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)+\frac{(2 s-1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right)
\end{aligned}
$$

$$
\begin{align*}
& =h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \\
& \leq \frac{h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)+h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2} \\
& =h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) .(h \text { is symmetric }) \tag{16}
\end{align*}
$$

By change of variables, we have

$$
\begin{align*}
& h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) d_{\alpha} x \\
& \quad=\Psi\left(\kappa_{2}, \kappa_{1}\right) h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \\
& \quad \cdot \int_{0}^{1} g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s  \tag{17}\\
& \quad \leq \Psi\left(\kappa_{2}, \kappa_{1}\right) \int_{0}^{1} h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \quad \cdot g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s \\
& \quad=\int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) d_{\alpha} x .
\end{align*}
$$

So we can write

$$
\begin{align*}
& h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) d_{\alpha} x  \tag{18}\\
& \quad \leq \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) d_{\alpha} x .
\end{align*}
$$

To prove the second inequality in (14), we know that $h$ is preinvex and $\Psi$ satisfies Condition C, so we have

$$
\begin{align*}
& h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \quad=h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)+(1-s) \Psi\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right)  \tag{19}\\
& \quad \leq \operatorname{sh}\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)+(1-s) h\left(\kappa_{1}\right)
\end{align*}
$$

and similarly

$$
\begin{align*}
& h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \quad=h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)+s \Psi\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right)  \tag{20}\\
& \quad \leq \operatorname{sh}\left(\kappa_{1}\right)+(1-s) h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)
\end{align*}
$$

Now with $x=\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)$, we have

$$
\begin{align*}
& \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) x^{\alpha-1} d x \\
& \quad=\int_{0}^{1} h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \cdot\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s \leq \Psi\left(\kappa_{2}, \kappa_{1}\right)  \tag{21}\\
& \quad \cdot \int_{0}^{1}\left[\operatorname{sh}\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)+(1-s) h\left(\kappa_{1}\right)\right] \\
& \quad \times g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s \quad \text { (using (19)) }
\end{align*}
$$

Also

$$
\begin{align*}
& \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) x^{\alpha-1} d x \\
& \quad=\int_{0}^{1} h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \cdot \\
& \quad\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s  \tag{22}\\
& \quad=\int_{0}^{1} h\left(\kappa_{1}+(1-s) \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \\
& \cdot \\
& \quad\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s(h \text { is symmetric }) \\
& \quad \leq \Psi\left(\kappa_{2}, \kappa_{1}\right) \\
& \quad \cdot \int_{0}^{1}\left[s h\left(\kappa_{1}\right)+(1-s) h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right] \\
& \left.\quad \times g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s \text { (using }(20)\right) .
\end{align*}
$$

If we add (21) and (22), we obtain

$$
\begin{align*}
& 2 \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) d_{\alpha} x \leq \Psi\left(\kappa_{2}, \kappa_{1}\right) \\
& \quad \cdot\left(h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right) \\
& \quad \cdot \int_{0}^{1} g\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} d s  \tag{23}\\
& \quad=\left(h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right) \\
& \quad \cdot \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) x^{\alpha-1} d x
\end{align*}
$$

So we can write

$$
\begin{align*}
& \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) g(x) d_{\alpha} x \\
& \quad \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} g(x) d_{\alpha} x . \tag{24}
\end{align*}
$$

From inequalities (18) and (24), we obtain over required result.

Corollary 9. If we put $g(x)=1$ in (14), then we get

$$
\begin{align*}
& h\left(\frac{2 \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2}\right) \\
& \quad \leq \frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x  \tag{25}\\
& \quad \leq \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}
\end{align*}
$$

## 3. $\mathscr{H} \mathscr{H}$ Type Inequalities for Conformable Fractional Integrals

Lemma 10. Let $\kappa_{1}, \kappa_{2} \in K$ with $\Psi\left(\kappa_{2}, \kappa_{1}\right)>0$ and $h: K=$ $\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \rightarrow(0, \infty)$ be an $\alpha$-fractional differentiable
function on $\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)$ for $\alpha \in(0,1]$. If $D_{\alpha}(h) \in$ $L_{\alpha}\left(\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right]\right)$, then the following identity holds:

$$
\begin{align*}
& \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}} \\
& \quad \cdot \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \\
& \quad=\frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right.\right. \\
& \left.-\kappa_{1}^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) \times D_{\alpha}(h)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)  \tag{26}\\
& \quad \cdot s^{1-\alpha} d_{\alpha} s+\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right. \\
& \left.\quad-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) \times D_{\alpha}(h)\left(\kappa_{1}\right. \\
& \left.\left.\quad+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) s^{1-\alpha} d_{\alpha} s\right] .
\end{align*}
$$

Proof. Integrating by parts, we have

$$
\begin{aligned}
& I=\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right. \\
& \left.-\kappa_{1}^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) D_{\alpha}(h)\left(\kappa_{1}\right. \\
& \left.+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s+\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right. \\
& \left.-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) \\
& \times D_{\alpha}(h)\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s \\
& =\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) h^{\prime}\left(\kappa_{1}\right. \\
& \left.+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s+\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right. \\
& \left.-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s \\
& =\left.\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) \frac{h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{\Psi\left(\kappa_{2}, \kappa_{1}\right)}\right|_{0} ^{1} \\
& -\int_{0}^{1} \alpha\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} \Psi\left(\kappa_{2}, \kappa_{1}\right) \\
& \cdot \frac{h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} d s+\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right. \\
& \left.-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)\left.\frac{h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{\Psi\left(\kappa_{2}, \kappa_{1}\right)}\right|_{0} ^{1} \\
& -\int_{0}^{1} \alpha\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1} \Psi\left(\kappa_{2}, \kappa_{1}\right) \\
& \cdot \frac{h\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} d s
\end{aligned}
$$

$$
\begin{align*}
& =\frac{1}{\Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[( ( \kappa _ { 1 } + \Psi ( \kappa _ { 2 } , \kappa _ { 1 } ) ) ^ { \alpha } - \kappa _ { 1 } ^ { \alpha } ) h \left(\kappa_{1}\right.\right. \\
& \left.\left.+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)-\alpha \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x\right] \\
& +\frac{1}{\Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) h\left(\kappa_{1}\right)\right. \\
& \left.-\alpha \int_{\kappa_{1}}^{\kappa_{1} \Psi \Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x\right] \\
& =\frac{\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}{\Psi\left(\kappa_{2}, \kappa_{1}\right)}\left(h\left(\kappa_{1}\right)+h\left(\kappa_{1}\right.\right. \\
& \left.\left.+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right)-\frac{2 \alpha}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x, \tag{27}
\end{align*}
$$

where we have used the change of variable $x=\kappa_{1}+$ $\Psi\left(\kappa_{2}, \kappa_{1}\right)$ and then multiplied both sides by $\Psi\left(\kappa_{2}, \kappa_{1}\right) /\left(2\left(\left(\kappa_{1}+\right.\right.\right.$ $\left.\left.\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)$ ) to get the desired result in (26).

Remark 11. If we set $\alpha=1$ in (26), then we obtain the result which is proved by Barani et al. in [40]

$$
\begin{align*}
& -\frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2} \\
& \quad+\frac{1}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d x  \tag{28}\\
& = \\
& \quad \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2} \int_{0}^{1}(1-2 s) h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s
\end{align*}
$$

Theorem 12. Let $\kappa_{1}, \kappa_{2} \in K$ such that $\Psi\left(\kappa_{2}, \kappa_{1}\right)>0$ and $h$ : $K=\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \longrightarrow(0, \infty)$ be an $\alpha$-differentiable function on $\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)$ for $\alpha \in(0,1]$ such that $D_{\alpha}(h) \in$ $L_{\alpha}\left(\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right]\right)$. If $\left|h^{\prime}\right|$ is preinvex, then we have the following inequality:

$$
\begin{align*}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}\right. \\
& \left.-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \right\rvert\,  \tag{29}\\
& \quad \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{4}\left[\left|h^{\prime}\left(\kappa_{1}\right)\right|+\left|h^{\prime}\left(\kappa_{2}\right)\right|\right] .
\end{align*}
$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $\left|h^{\prime}\right|$, we have

$$
\begin{aligned}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \mid \\
& \quad=\left\lvert\, \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\right.
\end{aligned}
$$

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right. \\
& \left.-\kappa_{1}^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) \times D_{\alpha}(h)\left(\kappa_{1}\right. \\
& \left.+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s+\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2 \alpha-1}\right. \\
& \left.-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha-1}\right) \times D_{\alpha}(h) \\
& \cdot\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) d s \mid \\
& \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right. \\
& \left.-\kappa_{1}^{\alpha}\right)\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s \\
& +\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \\
& \left.\cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s\right] \\
& \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right. \\
& \left.-\kappa_{1}^{\alpha}\right)\left[(1-s)\left|h^{\prime}\left(\kappa_{1}\right)\right|+s\left|h^{\prime}\left(\kappa_{2}\right)\right|\right] d t \\
& +\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \\
& \left.\cdot\left[(1-s)\left|h^{\prime}\left(\kappa_{1}\right)\right|+s\left|h^{\prime}\left(\kappa_{2}\right)\right|\right] d s\right] \\
& =\frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{4}\left[\left|h^{\prime}\left(\kappa_{1}\right)\right|+\left|h^{\prime}\left(\kappa_{2}\right)\right|\right] \tag{30}
\end{align*}
$$

Theorem 13. Let $\kappa_{1}, \kappa_{2} \in K$ such that $\Psi\left(\kappa_{2}, \kappa_{1}\right)>0$ and $h$ : $K=\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \longrightarrow(0, \infty)$ be an $\alpha$-differentiable function on $\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)$ for $\alpha \in(0,1]$ such that $D_{\alpha}(h) \in$ $L_{\alpha}\left(\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right]\right)$. If $\left|h^{\prime}\right|^{q}$ is preinvex for $q>1$ and $q^{-1}+$ $p^{-1}=1$, then we have the following inequality:

$$
\begin{align*}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}\right. \\
& \left.-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \right\rvert\, \\
& \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\left(\mathscr{A}_{1}(\alpha, p)\right)^{1 / p}\right.  \tag{31}\\
& \cdot\left(\frac{\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}}{2}\right)^{1 / q}+\left(\mathscr{A}_{2}(\alpha, p)\right)^{1 / p} \\
&\left.\cdot\left(\frac{\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}}{2}\right)^{1 / q}\right]
\end{align*}
$$

where

$$
\begin{align*}
& \mathscr{A}_{1}(\alpha, p)=\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)^{p} d s \\
& \mathscr{A}_{2}(\alpha, p)  \tag{32}\\
& \quad=\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)^{p} d s .
\end{align*}
$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $\left|h^{\prime}\right|^{9}$, we have

$$
\begin{align*}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \mid \\
& \quad \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right.  \tag{33}\\
& \left.-\kappa_{1}^{\alpha}\right)\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s \\
& \quad+\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \\
& \left.\cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s\right] .
\end{align*}
$$

Now by Hölder's inequality

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s \\
& \quad \leq\left(\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)^{p} d s\right)^{1 / p} \\
& \cdot\left(\int_{0}^{1} \mid h^{\prime}\left(\kappa_{1}+\left.s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right|^{q} d s\right)^{1 / q}\right.  \tag{34}\\
& \quad \leq\left(\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)^{p} d s\right)^{1 / p} \\
& \cdot\left(\int_{0}^{1}(1-s)\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+s\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q} d s\right)^{1 / q} \\
& =\left(\mathscr{A}_{1}(\alpha, p)\right)^{1 / p}\left(\frac{\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}}{2}\right)^{1 / q} .
\end{align*}
$$

Similarly, we have

$$
\begin{aligned}
& \int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \mid h^{\prime}\left(\kappa_{1}\right. \\
& \left.\quad+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \mid d s \leq\left(\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\left.-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)^{p} d s\right)^{1 / p} \\
& \cdot\left(\int_{0}^{1}\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right|^{q} d s\right)^{1 / q} \\
& \leq\left(\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right. \\
& \left.\left.-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)^{p} d s\right)^{1 / p}\left(\int_{0}^{1}(1-s)\right. \\
& \left.\cdot\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+s\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q} d s\right)^{1 / q}=\left(\mathscr{A}_{2}(\alpha, p)\right)^{1 / p} \\
& \cdot\left(\frac{\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}}{2}\right)^{1 / q} \cdot \tag{35}
\end{align*}
$$

Hence, we have the result in (31).
Remark 14. If we set $\alpha=1$ in (31), then we have the following inequality:

$$
\begin{align*}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}\right. \\
& \left.\quad-\frac{1}{\Psi\left(\kappa_{2}, \kappa_{1}\right)} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d x \right\rvert\,  \tag{36}\\
& \quad \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{(1+p)^{1 / p}}\left[\frac{\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}}{2}\right]^{1 / q} .
\end{align*}
$$

Theorem 15. Let $\kappa_{1}, \kappa_{2} \in K$ such that $\Psi\left(\kappa_{2}, \kappa_{1}\right)>0$ and $h$ : $K=\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right] \longrightarrow(0, \infty)$ be an $\alpha$-differentiable function on $\left(\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)$ for $\alpha \in(0,1]$ such that $D_{\alpha}(h) \in$ $L_{\alpha}\left(\left[\kappa_{1}, \kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right]\right)$. If $\left|h^{\prime}\right|^{q}$ is preinvex for $q>1$ and $q^{-1}+$ $p^{-1}=1$, then we have the following inequality:

$$
\begin{align*}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}\right. \\
& \left.-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}} \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \right\rvert\, \\
& \quad \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\left(\mathscr{A}_{1}(\alpha)\right)^{1-1 / q}\right. \tag{37}
\end{align*}
$$

$$
\begin{aligned}
& \cdot\left(\mathscr{A}_{2}(\alpha)\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+\mathscr{A}_{3}(\alpha)\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}\right)^{1 / q} \\
& +\left(\mathscr{B}_{1}(\alpha)\right)^{1-1 / q}\left(\mathscr{B}_{2}(\alpha)\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}\right. \\
& \left.+\mathscr{B}_{3}(\alpha)\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}\right)^{1 / q}
\end{aligned}
$$

## where

$$
\begin{aligned}
& \mathscr{A}_{1}(\alpha)=\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}-\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}-\kappa_{1}^{\alpha}, \\
& \mathscr{B}_{2}(\alpha)=\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left[\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}-\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right] \text {, } \\
& \mathscr{A}_{2}(\alpha)=-\frac{\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)+\kappa_{1}}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right] \\
& +\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}-\frac{\kappa_{1}^{\alpha}}{2}, \\
& \mathscr{A}_{3}(\alpha)=\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}}{2} \\
& -\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right] \\
& -\frac{\kappa_{1}^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}, \\
& \mathscr{B}_{2}(\alpha)=\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}}{2} \\
& +\frac{\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)+\kappa_{1}}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right], \\
& -\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)} \\
& \mathscr{B}_{3}(\alpha)=\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}}{2} \\
& -\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right] \\
& -\frac{\kappa_{1}^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)} .
\end{aligned}
$$

Proof. From Lemma 10, using the property of the modulus and preinvexity of $\left|h^{\prime}\right|^{q}$, we have

$$
\begin{aligned}
& \left\lvert\, \frac{h\left(\kappa_{1}\right)+h\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{2}-\frac{\alpha}{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{\kappa_{1}}^{\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)} h(x) d_{\alpha} x \mid
\end{aligned}
$$

$$
\begin{align*}
& \leq \frac{\Psi\left(\kappa_{2}, \kappa_{1}\right)}{2\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)}\left[\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right. \\
& \left.-\kappa_{1}^{\alpha}\right)\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s \\
& +\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \\
& \left.\cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s\right] . \tag{39}
\end{align*}
$$

Now by the power-mean inequality

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right| d s \\
& \quad \leq\left(\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) d s\right)^{1-1 / q} \\
& \quad \times\left(\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)\right.  \tag{40}\\
& \left.\quad \cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right|^{q} d s\right)^{1 / q}
\end{align*}
$$

and similarly, we have

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \mid h^{\prime}\left(\kappa_{1}\right. \\
& \left.\quad+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right) \mid d s \leq\left(\int _ { 0 } ^ { 1 } \left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right.\right. \\
& \left.\left.\quad-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) d s\right)^{1-1 / q}  \tag{41}\\
& \quad \cdot\left(\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)\right. \\
& \left.\quad \cdot\left|h^{\prime}\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)\right|^{q} d s\right)^{1 / q} \cdot
\end{align*}
$$

Now by the preinvexity of $\left|h^{\prime}\right|^{q}$ from above, we have

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)\left[(1-s)\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+s\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}\right] d s=\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q} \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right)(1-s) d s \\
& \quad+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q} \int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) s d s=\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}\left(-\frac{\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)+\kappa_{1}}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right]\right. \\
& \left.\quad+\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}-\frac{\kappa_{1}^{\alpha}}{2}\right)+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}\left(\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}}{2}\right.  \tag{42}\\
& \left.\quad-\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right]-\frac{\kappa_{1}^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}\right)
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) \\
& \cdot \\
& \cdot\left[(1-s)\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}+s\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q}\right] d s=\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q} \\
& \quad+\mid\left(h^{\prime}\left(\kappa_{2}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{0}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right)(1-s) d s \\
& \left.\quad=\left|h^{\prime}\left(\kappa_{1}\right)\right|^{q}\left(\frac{\left.\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) s d s}{2}, \kappa_{1}\right)\right)^{\alpha} \\
& \quad+\frac{\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)+\kappa_{1}}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right]  \tag{43}\\
& \left.-\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}\right)+\left|h^{\prime}\left(\kappa_{2}\right)\right|^{q} \\
& \quad \cdot\left(\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}}{2}\right. \\
& \\
& -\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\left[\frac{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)-\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)}{(\alpha+2) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right] \\
& \left.-\frac{\kappa_{1}^{\alpha+2}}{(\alpha+1)\left(\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{2}(\alpha+2)}\right),
\end{align*}
$$

where we have the following:

$$
\begin{align*}
& \mathscr{A}_{1}(\alpha)=\int_{0}^{1}\left(\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\kappa_{1}^{\alpha}\right) d s \\
& =\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}-\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}-\kappa_{1}^{\alpha} \\
& \mathscr{B}_{1}(\alpha) \\
& =\int_{0}^{1}\left(\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}-\left(\kappa_{1}+s \Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha}\right) d s  \tag{44}\\
& =\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha} \\
& \quad-\left[\frac{\left(\kappa_{1}+\Psi\left(\kappa_{2}, \kappa_{1}\right)\right)^{\alpha+1}-\kappa_{1}^{\alpha+1}}{(\alpha+1) \Psi\left(\kappa_{2}, \kappa_{1}\right)}\right]
\end{align*}
$$

Hence, we have the result in (37).

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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## Research Article

# Existence of Nontrivial Solutions for Fractional Differential Equations with p-Laplacian 

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Combining the properties of the Green function with some point theorems, we consider the existence of nontrivial solutions for fractional equations with $p$-Laplacian operator $D_{0^{+}}^{\beta} \phi_{p}\left[D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right]+f(t, u(t))=0,0<t<1$, $a u(0)-b p(0) u^{\prime}(0)=0$, and $c u(1)+d p(1) u^{\prime}(1)=0,\left.D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right|_{t=0}=0$, where $a, b, c, d$ are constants and $p(\cdot):[0,1] \longrightarrow(0,+\infty)$ is continuous.

## 1. Introduction

Fractional-order models are better than integer order models to describe the real word, which appear frequently in various fields, such as electrical circuits, biology, material, control theory, and physics (see [1-5]). With the rapid development of the theory of fractional differential equations, during the last two decades, the existence of nontrivial solutions of fractional differential equations has been studied by many researchers in nonsingular case as well as singular case. See [6-19]. Usually, the proof is based on either the method of upper and lower solutions, fixed point theorems, alternative principle of Leray-Schauder, topological degree theory, or critical point theory. To our attention, based on a fixed point theorem in cones, K. Lan and W. Lin [20] obtain some new results on existence of multiple positive solutions of systems of nonlinear Caputo fractional differential equations with some of general separated boundary conditions

$$
\begin{align*}
-^{c} D^{q} z_{i}(t) & =f_{i}(t, z(t)), \quad t \in(0,1), \\
\alpha z_{i}(0)-\beta z_{i}^{\prime}(0) & =0,  \tag{1}\\
\gamma z_{i}(1)+\delta z_{i}^{\prime}(1) & =0,
\end{align*}
$$

where $z(t)=\left(z_{1}(t), \ldots, z_{n}(t)\right), f_{i}:[0,1] \times \mathbb{R}_{+}^{n} \longrightarrow \mathbb{R}_{+}$is continuous on $[0,1] \times \mathbb{R}_{+}^{n}$, and ${ }^{c} D^{q}$ is the Caputo differential operator of order $q \in(1,2), \alpha, \beta, \gamma, \delta$ are positive real
numbers. The relations between the linear Caputo fractional differential equations and the corresponding linear Hammerstein integral equations are studied, which show that suitable Lipschitz type conditions are needed when one studies the nonlinear Caputo fractional differential equations.

Recently, fractional differential equations with $p$ Laplacian operator have gained its popularity and importance due to its distinguished applications in numerous diverse fields of science and engineering, such as viscoelasticity mechanics, non-Newtonian mechanics, electrochemistry, fluid mechanics, combustion theory, and material science. There have appeared some results for the existence of solutions or positive solutions of boundary value problems for fractional differential equations with $p$-Laplacian operator; see [15, 21-26] and the references therein. For example, under different conditions K. Hasib, W. Chen and H. Sun [21] apply some classical fixed-point theorems to study the existence of positive solution for a class of singular fractional differential equations with nonlinear $p$-Laplacian operator in Caputo sense

$$
\begin{aligned}
D^{\beta} \phi_{p}\left(D^{\epsilon} u(t)\right)+\Theta(t) \varphi_{1}(t, u(t))= & 0, \\
\left.\phi_{p}\left(D^{\epsilon} u(t)\right)^{(i)}\right|_{t=0}= & 0, \\
& \quad i=1, \ldots, n-1
\end{aligned}
$$

$$
\begin{align*}
u^{(j)}(0)=0=u^{\prime \prime} & (1) \\
& j=1, \ldots, n, \tag{2}
\end{align*}
$$

where $D^{\beta}, D^{\epsilon}$ is Caputo fractional derivative, $n-1<\beta, \epsilon \leq$ $n, \phi_{p}(r)=|r|^{p-2} r$ is $p$-Laplacian operator, and $\Theta(\cdot)$ is continuous functions. In addition, Hyers-Ulam stability of the proposed problem is also considered.

Inspired by the references, based on some fixed point theorems in cones, under different combinations of local superlinearity and local sublinearity of the function $f$, we will deal with the existence of nontrivial solutions for a certain $p$ Laplacian fractional differential equation

$$
\begin{align*}
D_{0^{+}}^{\beta} \phi_{p}\left(D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right)+f(t, u(t))= & 0, \\
& 0<\alpha, \beta<1, \\
a u(0)-b p(0) u^{\prime}(0)= & 0,  \tag{3}\\
c u(1)+d p(1) u^{\prime}(1)= & 0, \\
\left.\left(D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right)\right|_{t=0}= & 0,
\end{align*}
$$

where $a, b, c, d$ are constants with satisfying $0<a d+b c+$ ac $\int_{0}^{1}(1 / p(s)) d s<+\infty, p(\cdot):[0,1] \longrightarrow \mathbb{R}_{+}$is continuous, and $\phi_{p}(r)=|r|^{p-2} r$ is $p$-Laplacian operator, where $1 / p+1 / q=$ 1 and $\phi_{q}$ denotes inverse of $p$-Laplacian operator. Now we give some notations as follows.

Definition 1 (see [2]). The fractional integral of order $\alpha>0$ of a function $u:(0,+\infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
I_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s \tag{4}
\end{equation*}
$$

provided that the right-hand side integral is pointwise defined on $(0,+\infty)$, where $\Gamma(\alpha)=\int_{0}^{+\infty} e^{-s} s^{\alpha-1} d s$.

Definition 2 (see [2]). The Caputo fractional derivative of order $\alpha>0$ of a continuous function $u:(0,+\infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
D_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-s)^{n-\alpha-1} u^{(n)}(s) d s \tag{5}
\end{equation*}
$$

for $n=[\alpha]+1$, where $[\alpha]$ is used for the integer part of $\alpha$, provided that the right-hand side integral is pointwise defined on $(0,+\infty)$.

Lemma 3 (see [2, Theorem 2.22]). Let $\alpha \in(n-1, n], u \in$ $A C^{n-1}$. Then

$$
\begin{equation*}
I_{0^{+}}^{\alpha} D_{0^{+}}^{\alpha} u(t)=u(t)+c_{0}+c_{1} t+c_{2} t^{2}+\cdots+c_{n-1} t^{n-1} \tag{6}
\end{equation*}
$$

where $c_{i}=-u^{(i)}(0) / i!, i=1, \ldots, n-1$.

The paper is organized as follows. In Section 2, we give some notations and the Green function is examined whether it is increasing or decreasing and positive or negative function. In Section 3, we will give the main results, which are illustrated by some examples.

## 2. Preliminaries

Lemma 4. Let $h(t) \in A C([0,1])$ and $0<\alpha, \beta<1$. Then the solution of the fractional differential equation with $p$-Laplacian operator

$$
\begin{align*}
D_{0^{+}}^{\beta} \phi_{p}\left(D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right)+h(t) & =0, \\
a u(0)-b p(0) u^{\prime}(0) & =0, \\
c u(1)+d p(1) u^{\prime}(1) & =0,  \tag{7}\\
\left.D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right|_{t=0} & =0
\end{align*}
$$

can be expressed by

$$
u(t)
$$

$$
\begin{equation*}
=\int_{0}^{1} G(t, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau \tag{8}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, \tau)=\frac{1}{\rho \Gamma(\alpha)} \begin{cases}\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right)\left(d(1-\tau)^{\alpha-1}+c \int_{t}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right)-H(t, \tau), & 0 \leq \tau \leq t \leq 1 \\
\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right)\left(d(1-\tau)^{\alpha-1}+c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right), & 0 \leq t \leq \tau \leq 1\end{cases} \\
& \rho=a d+b c+a c \int_{0}^{1} \frac{1}{p(s)} d s  \tag{9}\\
& H(t, \tau)=a\left(d+c \int_{t}^{1} \frac{1}{p(s)} d s\right) \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s .
\end{align*}
$$

Proof. Via some computations, from Lemma 3 it follows that

$$
\begin{align*}
p(t) & u^{\prime}(t)+c_{1} \\
& =-I_{0^{+}}^{\alpha}\left(\phi_{q}\left(c_{0}+\frac{1}{\Gamma(\beta)} \int_{0}^{t}(t-s)^{\beta-1} h(s) d s\right)\right) . \tag{10}
\end{align*}
$$

Since $p(t)>0$, we have

$$
\begin{align*}
& u^{\prime}(t)=-\frac{c_{1}}{p(t)}-\frac{1}{\Gamma(\alpha) p(t)} \int_{0}^{t}(t-s)^{\alpha-1} \\
& \quad \cdot \phi_{q}\left(c_{0}+\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s \tag{11}
\end{align*}
$$

Integrating both sides from 0 to $t$, we can obtain

$$
\begin{align*}
& u(t)=u(0)-c_{1} \int_{0}^{t} \frac{1}{p(s)} d s \\
& \quad-\int_{0}^{t} \frac{1}{\Gamma(\alpha) p(s)} \int_{0}^{s}(s-\tau)^{\alpha-1}  \tag{12}\\
& \quad \cdot \phi_{q}\left(c_{0}+\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s
\end{align*}
$$

Due to $\left.D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right|_{t=0}=0$, we get $c_{0}=0$ and $p(0) u^{\prime}(0)=-c_{1}$. According to the boundary conditions $a u(0)-b p(0) u^{\prime}(0)=0$, we have $a u(0)+b c_{1}=0$ and $u(0)=$ $-(b / a) c_{1}$. Since

$$
\begin{aligned}
& c u(1)+d p(1) u^{\prime}(1)=0 \\
& p(1) u^{\prime}(1)=-c_{1}-\frac{1}{\Gamma(\alpha)} \int_{0}^{1}(1-s)^{\alpha-1} \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\quad \cdot \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s \\
& u(1)=u(0)-c_{1} \int_{0}^{1} \frac{1}{p(s)} d s
\end{aligned}
$$

$$
\begin{align*}
& -\int_{0}^{1} \frac{1}{\Gamma(\alpha) p(s)} \int_{0}^{s}(s-\tau)^{\alpha-1} \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s \tag{13}
\end{align*}
$$

it is clear that

$$
\begin{align*}
0= & c\left\{u(0)-c_{1} \int_{0}^{1} \frac{1}{p(s)} d s\right. \\
& -\int_{0}^{1} \frac{1}{\Gamma(\alpha) p(s)} \int_{0}^{s}(s-\tau)^{\alpha-1} \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s\right\}  \tag{14}\\
& +d\left\{-c_{1}-\frac{1}{\Gamma(\alpha)} \int_{0}^{1}(1-s)^{\alpha-1} \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right.\right. \\
& \left.\left.\cdot \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s\right\}
\end{align*}
$$

Since $u(0)=-(b / a) c_{1}$, we have

$$
\begin{align*}
& c_{1}\left(-\frac{b c+a d}{a}-c \int_{0}^{1} \frac{1}{p(s)} d s\right) \\
& \quad=\int_{0}^{1} \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1}\right. \\
& \quad \cdot h(\tau) d \tau) d s  \tag{15}\\
& \quad+\int_{0}^{1} \int_{0}^{s} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\quad \cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s
\end{align*}
$$

which follows that

$$
\begin{align*}
c_{1} & =-\frac{a}{\rho}\left\{\int_{0}^{1} \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s\right. \\
& \left.+\int_{0}^{1} \int_{0}^{s} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s\right\} . \tag{16}
\end{align*}
$$

Then substituting $c_{1}$ and $u(0)$, we obtain

$$
\begin{aligned}
& u(t)=\frac{b}{\rho}\left\{\int_{0}^{1} \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s\right. \\
& \left.\quad+\int_{0}^{1} \int_{0}^{s} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s\right\}+\frac{a}{\rho} \int_{0}^{t} \frac{1}{p(s)} d s
\end{aligned}
$$

$$
\begin{align*}
& \left\{\int_{0}^{1} \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s\right. \\
& \left.+\int_{0}^{1} \int_{0}^{s} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s\right\} \\
& -\int_{0}^{t} \int_{0}^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)^{\prime}} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s \\
& =\frac{b+a \int_{0}^{t}(1 / p(s)) d s}{\rho}\left\{\int_{0}^{1} \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} h(\tau) d \tau\right) d s\right. \\
& \left.+\int_{0}^{1} \int_{0}^{s} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s\right\} \\
& -\int_{0}^{t} \int_{0}^{s} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)^{\prime}} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau d s \\
& =\frac{b+a \int_{0}^{t}(1 / p(s)) d s}{\rho}\left\{\int_{0}^{1} \frac{d(1-\tau)^{\alpha-1}}{\Gamma(\alpha)} \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau\right. \\
& \left.+\int_{0}^{1} \int_{\tau}^{1} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} d s \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau\right\}-\int_{0}^{t} \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)^{2}} d s \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau \\
& =\frac{b+a \int_{0}^{t}(1 / p(s)) d s}{\rho}\left\{\int_{0}^{1}\left(\frac{d(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}+\int_{\tau}^{1} \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)} d s\right) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau\right\} \\
& -\int_{0}^{t} \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha) p(s)^{2}} d s \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} h(\omega) d \omega\right) d \tau, \tag{17}
\end{align*}
$$

which implies the expression of the Green function $G(t, s)$.

Lemma 5. Assume that $a, b, c, d>0$, and $p(t):[0,1] \longrightarrow$ $(0,+\infty)$. The Green function $G(t, \tau)$ has the following properties:
(i) $G(t, \tau)>0$, for $0 \leq t, \tau \leq 1$;
(ii) $G(t, \tau)$ is an increasing function and $\max _{t \in[0,1]} G(t, \tau)=$ $G(1, \tau)$;
(iii) For $0 \leq t, \tau \leq 1$, there exists a $C(t)=(b+$ $\left.a \int_{0}^{t}(1 / p(s)) d s\right) /\left(b+a \int_{0}^{1}(1 / p(s)) d s\right) \in(0,1)$ such that

$$
\begin{equation*}
G(t, \tau) \geq C(t) G(1, \tau) . \tag{18}
\end{equation*}
$$

Proof. (i) For $0 \leq t \leq \tau \leq 1$,

$$
\begin{align*}
G(t, \tau)= & \frac{1}{\rho \Gamma(\alpha)}\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right)  \tag{21}\\
& \cdot\left(d(1-\tau)^{\alpha-1}+c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right) . \tag{19}
\end{align*}
$$

Since $a, b, c, d>0$ and $p(t)>0$, we have $\rho=a d+b c+$ $a c \int_{0}^{1}(1 / p(s)) d s>0$ and $G(t, \tau)>0$.

For $0 \leq \tau \leq t \leq 1$,

$$
\begin{align*}
& G(t, \tau)=\frac{1}{\rho \Gamma(\alpha)}\left\{\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right)\right. \\
& \quad\left(d(1-\tau)^{\alpha-1}+\int_{t}^{1} \frac{c(s-\tau)^{\alpha-1}}{p(s)} d s\right)  \tag{20}\\
& \left.\quad-a\left(d+c \int_{t}^{1} \frac{1}{p(s)} d s\right) \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right\} .
\end{align*}
$$

Let

$$
\begin{aligned}
h(t)= & \left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right) \\
& \cdot\left(d(1-\tau)^{\alpha-1}+\int_{t}^{1} \frac{c(s-\tau)^{\alpha-1}}{p(s)} d s\right) \\
& -a\left(d+c \int_{t}^{1} \frac{1}{p(s)} d s\right) \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s .
\end{aligned}
$$

Via some computations, we get

$$
\begin{align*}
h^{\prime}(t)= & -\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right) \frac{c(t-\tau)^{\alpha-1}}{p(t)} \\
& +\frac{a}{p(t)}\left(d(1-\tau)^{\alpha-1}+\int_{t}^{1} \frac{c(s-\tau)^{\alpha-1}}{p(s)} d s\right) \\
& -\frac{a\left(d+c \int_{t}^{1}(1 / p(s)) d s\right)}{p(t)}(t-\tau)^{\alpha-1}  \tag{22}\\
& +\frac{a c}{p(t)} \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s \\
= & -\frac{\rho(t-\tau)^{\alpha-1}}{p(t)}+\frac{a d(1-\tau)^{\alpha-1}}{p(t)} \\
& +\frac{a c}{p(t)} \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s .
\end{align*}
$$

Let $F(t)=-\rho(t-\tau)^{\alpha-1}+a d(1-\tau)^{\alpha-1}+a c \int_{\tau}^{1}((s-$ $\left.\tau)^{\alpha-1} / p(s)\right) d s$. Since $F^{\prime}(t)=-(\alpha-1)(t-\tau)^{\alpha-2} \rho>0$ and $F(\tau)=a d(1-\tau)^{\alpha-1}+a c \int_{\tau}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s>0$, we obtain $F(t) \geq F(\tau)>0$, which implies that $h^{\prime}(t)=F(t) / p(t)>0$ and $h(t)$ is increasing on $[\tau, t)$. Furthermore, for $0 \leq \tau \leq t \leq 1$, we have

$$
\begin{gather*}
G(t, \tau)=\frac{h(t)}{\rho \Gamma(\alpha)} \geq \frac{h(\tau)}{\rho \Gamma(\alpha)}=\left(b+a \int_{0}^{\tau} \frac{1}{p(s)} d s\right) \\
\cdot\left(d(1-\tau)^{\alpha-1}+\int_{\tau}^{1} \frac{c(s-\tau)^{\alpha-1}}{p(s)} d s\right)>0 \tag{23}
\end{gather*}
$$

(ii) For $0 \leq t \leq \tau \leq 1$, since

$$
\begin{align*}
& \frac{\partial G(t, \tau)}{\partial t} \\
& \quad=\frac{a}{\rho \Gamma(\alpha)}\left(d(1-\tau)^{\alpha-1}+c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right) \frac{1}{p(t)}  \tag{24}\\
& \quad>0
\end{align*}
$$

we obtain that $G(t, \tau)$ is increasing on $t$ in $(0,1]$, which implies that $G(t, \tau) \leq G(1, \tau)$.

For $0 \leq \tau \leq t \leq 1$, from the proof of (i) it follows that $G(t, \tau)$ is increasing on $t$ in $(0,1)$ and $G(t, \tau) \leq G(1, \tau)$.

Then from the above discussion, we can obtain the conclusion

$$
\begin{equation*}
\max _{t \in[0,1]} G(t, \tau)=G(1, \tau) \tag{25}
\end{equation*}
$$

(iii) For $0 \leq t \leq \tau \leq 1$,

$$
\begin{equation*}
\frac{G(t, \tau)}{G(1, \tau)}=\frac{\rho \Gamma(\alpha)\left(b+a \int_{0}^{t}(1 / p(s)) d s\right)\left(d(1-\tau)^{\alpha-1}+\int_{\tau}^{1}\left(c(s-\tau)^{\alpha-1} / p(s)\right) d s\right)}{\rho \Gamma(\alpha)\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)\left(d(1-\tau)^{\alpha-1}+\int_{\tau}^{1}\left(c(s-\tau)^{\alpha-1} / p(s)\right) d s\right)}=\frac{b+a \int_{0}^{t}(1 / p(s)) d s}{b+a \int_{0}^{1}(1 / p(s)) d s}=C(t) \tag{26}
\end{equation*}
$$

For $0 \leq \tau \leq t \leq 1$,

$$
\begin{align*}
\frac{G(t, \tau)}{G(1, \tau)}= & \frac{\rho \Gamma(\alpha)\left\{\left(b+a \int_{0}^{t}(1 / p(s)) d s\right)\left(d(1-\tau)^{\alpha-1}+c \int_{t}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s\right)\right.}{\rho \Gamma(\alpha)\left\{d(1-\tau)^{\alpha-1}\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)-a d \int_{\tau}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s\right\}} \\
& -\frac{\left.a\left(d+c \int_{t}^{1}(1 / p(s)) d s\right) \int_{\tau}^{t}\left((s-\tau)^{\alpha-1} / p(s)\right) d s\right\}}{\rho \Gamma(\alpha)\left\{d(1-\tau)^{\alpha-1}\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)-a d \int_{\tau}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s\right\}} \\
\geq & \frac{d(1-\tau)^{\alpha-1}\left(b+a \int_{0}^{t}(1 / p(s)) d s\right)+c\left(b+a \int_{0}^{t}(1 / p(s)) d s\right) \int_{t}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s}{d(1-\tau)^{\alpha-1}\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)}  \tag{27}\\
& -\frac{a\left(d+c \int_{t}^{1}(1 / p(s)) d s\right) \int_{\tau}^{t}\left((s-\tau)^{\alpha-1} / p(s)\right) d s}{d(1-\tau)^{\alpha-1}\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)} .
\end{align*}
$$

Let

$$
\begin{align*}
K(t)= & c\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right) \int_{t}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s  \tag{28}\\
& -a\left(d+c \int_{t}^{1} \frac{1}{p(s)} d s\right) \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s
\end{align*}
$$

$$
K_{1}(t)=a c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s-(t-\tau)^{\alpha-1} \rho
$$ On one hand, since

$$
\begin{align*}
& K^{\prime}(t)=\frac{a c}{p(t)} \int_{t}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s-c\left(b+a \int_{0}^{t} \frac{1}{p(s)} d s\right) \\
& \quad \cdot \frac{(t-\tau)^{\alpha-1}}{p(t)}+\frac{a c}{p(t)} \int_{\tau}^{t} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s-a(d \\
& \left.\quad+c \int_{t}^{1} \frac{1}{p(s)} d s\right) \frac{(t-\tau)^{\alpha-1}}{p(t)} \\
& \quad=\frac{1}{p(t)}\left\{a c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right.  \tag{29}\\
& \left.\quad-(t-\tau)^{\alpha-1}\left(b c+a d+a c \int_{0}^{1} \frac{1}{p(s)} d s\right)\right\} \\
& \quad=\frac{1}{p(t)}\left\{a c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s-(t-\tau)^{\alpha-1} \rho\right\},
\end{align*}
$$

we have $K^{\prime}(t)=K_{1}(t) / p(t)$. On the other hand, it is not hard to obtain that $K_{1}^{\prime}(t)=-(\alpha-1)(t-\tau)^{\alpha-2} \rho>0$. Then $K_{1}(t)$ is increasing on $t$, which implies that

$$
\begin{equation*}
K_{1}(t) \geq K_{1}(\tau)=a c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s>0 \tag{30}
\end{equation*}
$$

Hence, we get $K^{\prime}(t)>0$. Furthermore, $K(t)$ is increasing on $t$ and

$$
\begin{equation*}
K(t) \geq K(\tau)=c\left(b+a \int_{0}^{\tau} \frac{1}{p(s)} d s\right) \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s \tag{31}
\end{equation*}
$$

$$
>0
$$

Then, we have

$$
\begin{align*}
& \frac{G(t, \tau)}{G(1, \tau)} \\
& \quad \geq \frac{d\left(b+a \int_{0}^{t}(1 / p(s)) d s\right)(1-\tau)^{\alpha-1}+K(t)}{d\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)(1-\tau)^{\alpha-1}} \\
& \quad \geq \frac{d\left(b+a \int_{0}^{t}(1 / p(s)) d s\right)(1-\tau)^{\alpha-1}}{d\left(b+a \int_{0}^{1}(1 / p(s)) d s\right)(1-\tau)^{\alpha-1}}  \tag{32}\\
& \quad=\frac{b+a \int_{0}^{t}(1 / p(s)) d s}{b+a \int_{0}^{1}(1 / p(s)) d s}=C(t) .
\end{align*}
$$

Therefore, from the above discussion, we get the conclusion $G(t, \tau) \geq C(t) G(1, \tau), t \in(0,1)$ and it is clear to see that $0<C(t)<1$.

At the end of this section, we give some notations and crucial lemmas.

Let $E=C[0,1]$ be the Banach space and it endowed with the norm $\|u\|=\max _{t \in[0,1]}\{|u(t)|: u \in E\}$. Define a subcone $K$ as

$$
\begin{equation*}
K=\{u \in E: u(t) \geq C(t)\|u\|, t \in[0,1]\} . \tag{33}
\end{equation*}
$$

For any given $r>0$, let

$$
\begin{align*}
\Omega(r) & =\{u \in K:\|u\|<r\}, \\
\partial \Omega(r) & =\{u \in K:\|u\|=r\} . \tag{34}
\end{align*}
$$

Lemma 6 (see [27]). Let $E$ be a Banach space, $E_{1}$ a closed, convex subset of $E, \Omega$ an open subset of $E_{1}$, and $0 \in \Omega$. Suppose that $T: \bar{\Omega} \longrightarrow E_{1}$ is completely continuous. Then either
(i) $T$ has a fixed point in $\bar{\Omega}$, or
(ii) there are an $u \in \partial \Omega$ (the boundary of $\Omega$ in $E_{1}$ ) and $\lambda \in(0,1)$ with $u=\lambda T u$.

Lemma 7 (see [27]). Let $E$ be a Banach space and $K \subset E$ be a cone in $E$. Assume $\Omega_{1}, \Omega_{2}$ are open subsets of $E$ with $0 \in \Omega_{1}$, $\overline{\Omega_{1}} \subset \Omega_{2}$, and let $T: K \cap\left(\overline{\Omega_{2}} \backslash \Omega_{1}\right) \longrightarrow K$ be a completely continuous operator such that either
(i) $\|T u\| \leq\|u\|, u \in K \cap \partial \Omega_{1}$ and $\|T u\| \geq\|u\|, u \in K \cap \partial \Omega_{2}$; or
(ii) $\|T u\| \geq\|u\|, u \in \partial \Omega_{1}$ and $\|T u\| \leq\|u\|, u \in K \cap \partial \Omega_{2}$.

Then $T$ has a fixed point in $K \cap\left(\overline{\Omega_{2}} \backslash \Omega_{1}\right)$.
Lemma 8 (see [21]). Let $\phi_{p}$ be a $p$-Laplacian operator. Then
(i) If $1<p \leq 2, x_{1} x_{2}>0$ and $\left|x_{1}\right|,\left|x_{2}\right| \geq \lambda>0$, then

$$
\begin{equation*}
\left|\phi_{p}\left(x_{1}\right)-\phi_{p}\left(x_{2}\right)\right| \leq(p-1) \lambda^{p-2}\left|x_{1}-x_{2}\right| \tag{35}
\end{equation*}
$$

(ii) If $p>2$, and $\left|x_{1}\right|,\left|x_{2}\right| \leq \lambda^{*}$, then

$$
\begin{equation*}
\left|\phi_{p}\left(x_{1}\right)-\phi_{p}\left(x_{2}\right)\right| \leq(p-1) \lambda^{*(p-2)}\left|x_{1}-x_{2}\right| \tag{36}
\end{equation*}
$$

## 3. Existence Results

For convenience, the following assumptions hold throughout this paper:
(A1) $f:(0,1) \times(0,+\infty) \longrightarrow \mathbb{R}$ is continuous;
$\left(A 1^{\prime}\right) f:(0,1) \times(0,+\infty) \longrightarrow[0,+\infty)$ is continuous;
(A2) there exists positive constants $\mu_{1}, \mu_{2}$ and $k \in[0,1]$ such that $f$ satisfies

$$
\begin{equation*}
f(t, u(t)) \leq \phi_{p}\left(\mu_{1}|u(t)|^{k}+\mu_{2}\right) \tag{37}
\end{equation*}
$$

(A3) there exists a positive constant $L$ such that for all $u, v \in$ $V$,

$$
\begin{equation*}
|f(t, u)-f(t, v)| \leq L|u(t)-v(t)| . \tag{38}
\end{equation*}
$$

In addition, let

$$
\begin{align*}
& \omega_{0}=\int_{0}^{1} \tau^{\beta} G(1, \tau) d \tau \\
& \bar{\omega}=\int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau  \tag{39}\\
& \max _{0 \leq t \leq 1} f(t, u(t))=\bar{\mu}
\end{align*}
$$

Theorem 9. Suppose that (A1), (A2), and (A3) hold and $p>$ 2. Then problem (3) has a unique solution if

$$
\begin{equation*}
\frac{L(q-1)\left(\lambda^{*}\right)^{q-2}}{\Gamma(\beta+1)} \omega_{0}<1 \tag{40}
\end{equation*}
$$

and $(2-q)(q-1)^{(q-1) /(2-q)} L^{1 /(2-q)} \omega^{1 /(2-q)}+\bar{\mu} \leq 0$.
Proof. By Lemma 4, (3) is equivalent to the following integral equation:

$$
\begin{align*}
& u(t)=\int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \tag{41}
\end{align*}
$$

Define an operator $T: C[0,1] \longrightarrow C[0,1]$ by

$$
\begin{align*}
& T u(t)=\int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \tag{42}
\end{align*}
$$

From (A1) and (A2), the operator $T$ is well defined.
Let $B_{r_{0}}=\left\{u \in C[0,1]:\|u\| \leq r_{0}\right\}$ with $r_{0}=(q-$ 1) ${ }^{(q-1) /(2-q)} L^{(q-1) /(2-q)} \omega^{1 /(2-q)}$. Then we have

$$
\begin{align*}
& \|T u\|=\| \int_{0}^{1} G(t, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \| \\
& \quad \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1}\right. \\
& \cdot(f(\omega, u(\omega))-f(\omega, 0)+f(\omega, 0)) d \omega) d \tau  \tag{43}\\
& \quad \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\left(L r_{0}+\bar{\mu}\right) \frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau=\phi_{q}\left(L r_{0}+\bar{\mu}\right) \\
& \cdot \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau \\
& =\omega\left(L r_{0}+\bar{\mu}\right)^{q-1} \cdot
\end{align*}
$$

Let $g(r)=L r-\omega^{-1 /(q-1)} r^{1 /(q-1)}+\bar{\mu}$, and then we have

$$
\begin{aligned}
& g\left(r_{0}\right)=L^{(q-1)^{(q-1) /(2-q)} L^{(q-1) /(2-q)} \Phi^{1 /(2-q)}} \\
& -\omega^{-1 /(q-1)}\left[(q-1)^{(q-1) /(2-q)}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\cdot L^{(q-1) /(2-q)} \omega^{1 /(2-q)}\right]^{1 /(q-1)}+\bar{\mu}=(q-1)^{(q-1) /(2-q)} \\
& \cdot L^{1 /(2-q)} \omega^{1 /(2-q)}-(q-1)^{1 /(2-q)} L^{1 /(2-q)} \omega^{1 /(2-q)} \\
& +\bar{\mu}=(q-1)^{1 /(2-q)} L^{1 /(2-q)} \omega^{1 /(2-q)}\left[(q-1)^{-1}-1\right] \\
& +\bar{\mu}=(2-q)(q-1)^{(q-1) /(2-q)} L^{1 /(2-q)} \omega^{1 /(2-q)}+\bar{\mu} \\
& \leq 0, \tag{44}
\end{align*}
$$

which implies that $\|T u\|=\omega\left(L r_{0}+\bar{\mu}\right)^{q-1} \leq r_{0}$. Therefore, we proved that $T: B_{r_{0}} \longrightarrow B_{r_{0}}$.

For any $u, v \in E$, by Lemma 8, we have

$$
\begin{align*}
& \|T u-T v\|=\| \int_{0}^{1} G(t, \tau) \\
& \cdot\left\{\phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right)\right. \\
& \left.-\phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, v(\omega)) d \omega\right)\right\} d \tau \| \\
& \quad \leq \int_{0}^{1} G(1, \tau)\left\{(q-1)\left(\lambda^{*}\right)^{q-2} \left\lvert\, \frac{1}{\Gamma(\beta)}\right.\right. \\
& \cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega-\frac{1}{\Gamma(\beta)}  \tag{45}\\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, v(\omega)) d \omega \mid\right\} d \tau \\
& \leq \frac{(q-1)\left(\lambda^{*}\right)^{q-2}}{\Gamma(\beta)} \int_{0}^{1} G(1, \tau) \int_{0}^{\tau}(\tau-\omega)^{\beta-1} \\
& \cdot|f(\omega, u(\omega))-f(\omega, v(\omega))| d \omega d \tau \\
& \leq \frac{L(q-1)\left(\lambda^{*}\right)^{q-2}}{\Gamma(\beta+1)} \int_{0}^{1} \tau^{\beta} G(1, \tau) d \tau\|u-v\| \\
& \leq \frac{L(q-1)\left(\lambda^{*}\right)^{q-2}}{\Gamma(\beta+1)} \omega_{0}\|u-v\| .
\end{align*}
$$

Since $\left(L(q-1)\left(\lambda^{*}\right)^{q-2} / \Gamma(\beta+1)\right) \omega_{0}<1$, from Banach's contraction mapping principle it follows that there exists a unique fixed point for the operator $T$, which corresponds to the unique solution for problem (3).

Lemma 10. Assume that ( $\mathrm{Al}^{\prime}$ ) and (A2) hold. Then the operator $T: K \longrightarrow K$ is completely continuous.

Proof. For any $u \in K$, according to Lemma 5, we can get

$$
\begin{aligned}
& T u(t)=\int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau
\end{aligned}
$$

$$
\begin{align*}
& \leq \int_{0}^{1} G(1, \tau) \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \tag{46}
\end{align*}
$$

and

$$
\begin{align*}
& T u(t)=\int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau  \tag{47}\\
& \quad \geq C(t) \int_{0}^{1} G(1, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau
\end{align*}
$$

So we have

$$
\begin{equation*}
T u(t) \geq C(t)\|T u\|, \quad t \in[0,1] . \tag{48}
\end{equation*}
$$

This implies $T: K \longrightarrow K$.
Given $R>r>0$, now we show that $T$ is completely continuous on $\overline{\Omega(R)} \backslash \Omega(r)$.

Firstly, we will show that $T$ is continuous, and we only need to prove that $\left\|T\left(u_{n}\right)-T(u)\right\| \longrightarrow 0$ for any $u_{n} \longrightarrow u$ as $n \longrightarrow \infty$. It is clear that

$$
\begin{align*}
& \left\|T\left(u_{n}\right)-T(u)\right\|=\max _{0 \leq t \leq 1} \mid \int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f\left(\omega, u_{n}(\omega)\right) d \omega\right) d \tau \\
& \quad-\int_{0}^{1} G(t, \tau) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\,  \tag{49}\\
& \quad \leq \int_{0}^{1} G(1, \tau) \left\lvert\, \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right.\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f\left(\omega, u_{n}(\omega)\right) d \omega\right)-\phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) \mid d \tau .
\end{align*}
$$

From the continuity of $f, \phi_{q}(\cdot)$, we have

$$
\begin{align*}
& \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f\left(\omega, u_{n}(\omega)\right) d \omega\right)  \tag{50}\\
& \quad \longrightarrow \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right)
\end{align*}
$$

as $n \longrightarrow \infty$. Therefore, $\left\|T\left(u_{n}\right)-T(u)\right\| \longrightarrow 0$, as $n \longrightarrow \infty$, and $T$ is continuous.

Next, we show that the operator $T$ is uniformly bounded. By (A2), we get

$$
\begin{align*}
& \|T(u)\|=\max _{0 \leq t \leq 1} \left\lvert\, \int_{0}^{1} G(t, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right.\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \mid \\
& \quad \leq \int_{0}^{1} G(1, \tau) \left\lvert\, \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right.\right.  \tag{51}\\
& \left.\quad \cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} \phi_{p}\left(\mu_{1}|u(t)|^{k}+\mu_{2}\right) d \omega\right) \mid d \tau \\
& \quad=\int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau \\
& \cdot\left(\mu_{1}|u(t)|^{k}+\mu_{2}\right) \leq \bar{\omega}\left(\mu_{1}\|u(t)\|^{k}+\mu_{2}\right),
\end{align*}
$$

which implies that the operator $T$ is uniformly bounded on $\overline{\Omega(R)} \backslash \Omega(r)$.

Finally, we show that the operator $T$ is equicontinuous. Before that, we will proceed with $\partial G(t, \tau) / \partial t$ being bounded.

For $0 \leq t \leq \tau \leq 1$,

$$
\begin{align*}
& \frac{\partial G(t, \tau)}{\partial t} \\
& \quad=\frac{a}{\rho \Gamma(\alpha) p(t)}\left(d(1-\tau)^{\alpha-1}+c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right) . \tag{52}
\end{align*}
$$

Since the function $p(\cdot), d(1-\tau)^{\alpha-1}$ and $\int_{\tau}^{1}\left((s-\tau)^{\alpha-1} / p(s)\right) d s$ are bounded, there exists a positive constant $M_{1}$ such that $|\partial G(t, \tau) / \partial t| \leq M_{1}$. In a similar way, for $0 \leq \tau \leq t \leq 1$, there exists a positive constant $M_{2}$ such that

$$
\begin{align*}
& \left|\frac{\partial G(t, \tau)}{\partial t}\right|=\left\lvert\, \frac{1}{\rho \Gamma(\alpha) p(t)}\left(-\rho(t-\tau)^{\alpha-1}\right.\right. \\
& \left.\quad+a d(1-\tau)^{\alpha-1}+a c \int_{\tau}^{1} \frac{(s-\tau)^{\alpha-1}}{p(s)} d s\right) \mid \leq M_{2} . \tag{53}
\end{align*}
$$

Choosing $\bar{M}=\max \left\{M_{1}, M_{2}\right\}$, we can obtain $|\partial G(t, \tau) / \partial t| \leq$ $\bar{M}$.

Finally, for any $t_{1}, t_{2} \in[0,1]$ with $t_{1}<t_{2}$, there exists a $\xi \in\left(t_{1}, t_{2}\right)$ such that

$$
\begin{aligned}
& \left|T u\left(t_{2}\right)-T u\left(t_{1}\right)\right|=\mid \int_{0}^{1} G\left(t_{2}, \tau\right) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau
\end{aligned}
$$

$$
\begin{align*}
& -\int_{0}^{1} G\left(t_{1}, \tau\right) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& =\mid \int_{0}^{1}\left(G\left(t_{2}, \tau\right)-G\left(t_{1}, \tau\right)\right) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& =\left|\int_{0}^{1} \frac{\partial G(t, \tau)}{\partial t}\right|_{t=\xi}\left(t_{2}-t_{1}\right) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& \leq \left\lvert\, \int_{0}^{1} \frac{\bar{M}\left(t_{2}-t_{1}\right)\left(\mu_{1}|u(t)|^{k}+\mu_{2}\right)}{\left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau \right\rvert\,}\right. \\
& \leq \bar{M}\left(\mu_{1}\|u(t)\|^{k}+\mu_{2}\right) \\
& \cdot \int_{0}^{1}\left|\phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right)\right| d \tau\left|t_{2}-t_{1}\right|
\end{align*}
$$

Thus, the operator $T$ is equicontinuous. According to ArzelaAscoli theorem, $T: \overline{\Omega(R)} \backslash \Omega(r) \longrightarrow E$ is compact.

Theorem 11. Suppose that (A1) and (A2) hold. In addition,
$(\mathrm{C} 1)$ there exists a continuous function $g: \mathbb{R}_{+} \longrightarrow \mathbb{R}_{+}$such that
$f(t, u(t)) \leq \phi_{p}(g(\|u\|))$,
for any $t \in[0,1], u \in \mathbb{R}$;
$(C 2)$ there exists a constant $R$ such that $R / \omega g(R)>1$.
Then problem (3) has at least one solution.
Proof. Now we show the (ii) of Lemma 6 does not hold. If $u$ is a solution of (3), then, for $\lambda \in(0,1)$, we have

$$
\begin{aligned}
& \|u\|=\lambda\|T u\|=\lambda \| \int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \| \\
& \quad \leq \lambda \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} \phi_{p}(g(\|u\|)) d \omega\right) d \tau
\end{aligned}
$$

$$
\begin{align*}
& \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) \\
& \cdot \phi_{p}\left(\phi_{q}(g(\|u\|))\right) d \tau \leq g(\|u\|) \omega . \tag{56}
\end{align*}
$$

Let $B_{R}=\{u \in E:\|u\|<R\}$. From the above inequality and (C2), it yields a contradiction. Therefore, the operator $T$ has a fixed point in $B_{R}$.

For $r>0$, define the following functions:

$$
\begin{align*}
f_{M}(t, r) & =\max \{f(t, u(t)) \mid C(t) r \leq u \leq r\} \\
f_{m}(t, r) & =\min \{f(t, u(t)) \mid C(t) r \leq u \leq r\} \tag{57}
\end{align*}
$$

Theorem 12. Suppose that $\left(A 1^{\prime}\right)$ and (A2) hold. In addition, there exist $r_{0}, R_{0} \in \mathbb{R}^{+}$such that one of the following conditions satisfied:
(B1)

$$
\begin{align*}
r_{0} & \leq \int_{0}^{1} G(1, \tau) \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{m}\left(\omega, r_{0}\right) d \omega\right) d \tau  \tag{58}\\
& <+\infty
\end{align*}
$$

and

$$
\begin{aligned}
& \int_{0}^{1} G(1, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{M}\left(\omega, r_{0}\right) d \omega\right) d \tau
\end{aligned}
$$

$$
\leq R_{0}
$$

or
(B2)

$$
\begin{align*}
& \int_{0}^{1} G(1, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{M}\left(\omega, x_{0}\right) d \omega\right) d \tau  \tag{60}\\
& \quad<r_{0}
\end{align*}
$$

and

$$
\begin{align*}
R_{0} & \leq \int_{0}^{1} G(1, \tau) \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{M}\left(\omega, r_{0}\right) d \omega\right) d \tau \tag{61}
\end{align*}
$$

$<+\infty$.
Then problem (3) has a positive solution $u_{0} \in K$ such that $r_{0} \leq\left\|u_{0}\right\| \leq R_{0}$.

Proof. We only verify the case ( $B 1$ ). On one hand, for any $u \in$ $\partial \Omega\left(r_{0}\right)$, we have $C(t) r_{0} \leq u \leq r_{0}$ and

$$
\begin{align*}
& \|T(u)\|=\max _{0 \leq t \leq 1} \mid \int_{0}^{1} G(t, \tau) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& \quad \geq \max _{0 \leq t \leq 1} \mid \int_{0}^{1} C(t) G(1, \tau) \\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& \quad=\max _{0 \leq t \leq 1} C(t) \mid \int_{0}^{1} G(1, \tau)  \tag{62}\\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& \geq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \\
& \geq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{m}\left(\omega, r_{0}\right) d \omega\right) d \tau \geq r_{0}=\|u\| .
\end{align*}
$$

Thus, $\|T(u)\| \geq\|u\|$, for any $u \in \partial \Omega\left(r_{0}\right)$.
On the other hand, for any $u \in \partial \Omega\left(R_{0}\right)$, we have $C(t) R_{0} \leq$ $u \leq R_{0}, t \in[0,1]$ and

$$
\begin{align*}
& \|T(u)\|=\max _{0 \leq t \leq 1} \mid \int_{0}^{1} G(t, \tau) \\
& \left.\quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \right\rvert\, \\
& \quad \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\quad \cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau  \tag{63}\\
& \quad \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f_{M}\left(\omega, R_{0}\right) d \omega\right) d \tau \leq R_{0}
\end{align*}
$$

$=\|u\|$.
Then, $\|T(u)\| \geq\|u\|$, for any $u \in \partial \Omega\left(R_{0}\right)$.
Therefore, By Lemma 7, the operator $T$ has a fixed point $u_{0} \in \overline{\Omega\left(R_{0}\right)} \backslash \Omega\left(r_{0}\right)$ with $r_{0} \leq\left\|u_{0}\right\| \leq R_{0}$.

Theorem 13. Suppose that $\left(A 1^{\prime}\right)$ holds. In addition
(D1) $\lim _{u \rightarrow 0^{+}}\left(f(t, u) / u^{1 /(q-1)}\right)=0$;
(D2) there exists a constant $K>0$ such that $f(t, u) \leq K$;
(D3) there exists $\bar{R}>0$ and $\theta \in(0,1 / 2)$ such that

$$
\begin{equation*}
\min _{\vartheta \bar{R} \leq u \leq \bar{R}} f(t, u)>(\sigma \bar{R})^{1 /(q-1)}, \tag{64}
\end{equation*}
$$

where

$$
\begin{align*}
0<\vartheta & \min _{\theta \leq t \leq 1-\theta} C(t)<1, \\
\sigma= & \left(\min _{\theta \leq t \leq 1-\theta} C(t) \int_{\theta}^{1-\theta} G(1, \tau)\right.  \tag{65}\\
& \left.\cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau\right)^{-1} .
\end{align*}
$$

Then problem (3) has at least two solutions.
Proof. Since $\lim _{u \rightarrow 0^{+}}\left(f(t, u) / u^{1 /(q-1)}\right)=0$, there exist $\varepsilon>0$ and $r>0$ such that $f(t, u)<\varepsilon u^{1 /(q-1)}$, for $0 \leq u \leq r, t \in[0,1]$, where $\varepsilon$ satisfies $\varepsilon^{q-1} \emptyset<1$. For $u \in \partial \Omega(r)=\{u \in E:\|u\|<$ $r\}$, we have

$$
\begin{align*}
& \|T u\|=\| \int_{0}^{1} G(t, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \| \\
& \quad \leq \| \int_{0}^{1} G(1, \tau)  \tag{66}\\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1}\left(\varepsilon u^{1 /(q-1)}\right) d \omega\right) d \tau \| \\
& \quad \leq \varepsilon^{q-1}\|u\| \int_{0}^{1} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau \leq \varepsilon^{q-1} \omega\|u\|<\|u\| .
\end{align*}
$$

Choosing $R>K^{q-1} \omega$. For $u \in \partial \Omega(R)$, we have

$$
\begin{align*}
& \|T u\|=\| \int_{0}^{1} G(t, \tau) \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \|  \tag{67}\\
& \leq \int_{0}^{1} G(1, \tau) \phi_{q}\left(K \frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) d \tau \\
& \quad=K^{q-1} \omega<R=\|u\| .
\end{align*}
$$

For any $u \in \partial \bar{\Omega}(\bar{R})$, choosing $t_{0} \in(\theta, 1-\theta)$, we have $u\left(t_{0}\right) \in$ $[\vartheta \bar{R}, \bar{R}]$. Furthermore, we have

$$
\begin{align*}
& \left\|T u\left(t_{0}\right)\right\|=\| \int_{0}^{1} G\left(t_{0}, \tau\right) \\
& \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \| \\
& \quad \geq \int_{0}^{1} C\left(t_{0}\right) G(1, \tau) \\
& \quad \cdot \phi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{\tau}(\tau-\omega)^{\beta-1} f(\omega, u(\omega)) d \omega\right) d \tau \| \\
& \quad \geq \int_{\theta}^{1-\theta} C\left(t_{0}\right) G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right.  \tag{68}\\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} \min _{9 \bar{R} \leq u \leq \bar{R}} f(\omega, u(\omega)) d \omega\right) d \tau \\
& \quad \geq\left[\min _{\theta \leq t \leq 1-\theta} C(t)\right] \int_{\theta}^{1-\theta} G(1, \tau) \phi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\cdot \int_{0}^{\tau}(\tau-\omega)^{\beta-1} d \omega\right) \phi_{q}\left((\sigma \bar{R})^{1 /(q-1)}\right) d \tau \\
& \quad=\delta^{-1} \delta \bar{R}=\bar{R}=\|u\| .
\end{align*}
$$

By Lemma 7, problem (3) has at least two positive solution $r \leq\left\|u_{1}(t)\right\| \leq \bar{R}$ and $\bar{R} \leq\left\|u_{2}(t)\right\| \leq R$.

At the end of this section, we give some examples to illustrate our main results.

Example 1. Let us consider the problem

$$
\begin{align*}
& D_{0^{+}}^{\beta} \phi_{p}\left(D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right) \\
& \quad+\phi_{p}\left(\sin t\left(\arctan u^{1 / 3}+\cos u^{1 / 2}+3\right)\right)=0, \\
& a u(0)-b p(0) u^{\prime}(0)=0,  \tag{69}\\
& c u(1)+d p(1) u^{\prime}(1)=0, \\
& \left.D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right|_{t=0}=0 .
\end{align*}
$$

It is clear to see that $|f(t, u)|=\mid \phi_{p}\left(\sin t\left(\arctan u^{1 / 3}+\cos u^{1 / 2}+\right.\right.$ $3)) \mid \leq \phi_{p}\left(\|u\|^{1 / 3}+\|u\|^{1 / 2}+3\right)=\phi_{p}(g(\|u\|))$, for $t \in[0,1]$, $u \in \mathbb{R}$. In addition, there exists a sufficiently large $R>0$, we have $R / \Phi g(R)>1$. Therefore, problem (69) has at least one solution by Theorem 11.

Example 2. Let us consider the problem

$$
\begin{aligned}
& D_{0^{+}}^{\beta} \phi_{p}\left(D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right) \\
& \quad+\frac{(\sigma p)^{p-1}+1}{(p \vartheta)^{p} e^{-p \vartheta}}(\sin t+2) e^{-u} u^{p}=0
\end{aligned}
$$

$$
\begin{gather*}
a u(0)-b p(0) u^{\prime}(0)=0, \\
c u(1)+d p(1) u^{\prime}(1)=0, \\
\left.D_{0^{+}}^{\alpha}\left(p(t) u^{\prime}(t)\right)\right|_{t=0}=0 . \tag{70}
\end{gather*}
$$

Since $f(t, u)=\left(\left((\sigma p)^{p-1}+1\right) /(p \vartheta)^{p} e^{-p} \vartheta\right)(\sin t+2) e^{-u} u^{p}$, we have $f(t, u) / u^{p-1}=\left(\left((\sigma p)^{p-1}+1\right) /(p \vartheta)^{p} e^{-p} \vartheta\right)(\sin t+$ 2) $u e^{-u} \longrightarrow 0$ as $u \longrightarrow 0^{+}$, and (D2) hold. By some calculations, we get $f^{\prime}(t, u)=\left(\left((\sigma p)^{p-1}+1\right) /(p \vartheta)^{p} e^{-(p-1)} \vartheta\right)(\sin t+$ 2) $e^{-u} u^{p-1}(1 / p-u)$, and it is clear to see that $f^{\prime}(t, u)>0$, for $u \in(0,1 / p)$, and $f^{\prime}(t, u)<0$, for $u \in(1 / p,+\infty)$. Let $\bar{R}=1 / p$, and then for any $u \in(\vartheta(1 / p), 1 / p)$, we have $\min _{u \in[\vartheta(1 / p), 1 / p]} f(t, u)=\left(\left((\sigma p)^{p-1}+1\right) /(p \vartheta)^{p} e^{-p} \vartheta\right)(\sin t+$ 2) $(p 9)^{p} e^{-(1 / p) 9} \geq(p \sigma)^{p-1}+1 \geq(\bar{R} 9)^{p-1}$, and (D3) holds. Therefore, problem (70) has at least two positive solutions $u(t)$ by Theorem 13.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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## Research Article

# Fixed Point Theory and Positive Solutions for a Ratio-Dependent Elliptic System 

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#### Abstract

We consider a ratio-dependent predator-prey model under zero Dirichlet boundary condition. By using topological degree theory and fixed index theory, we study the necessary and sufficient conditions for the existence of positive solutions. Then we present the asymptotic behavior analysis of positive solutions, by bifurcation theory and energy estimates.


## 1. Introduction and Preliminaries

There is growing results about elliptic system, which comes from biological and physiological evidence, such as [19]. A more suitable general predator-prey model should be based on the so-called ratio-dependent theory, which asserts that the per capita predator growth rate should be a function of the ratio of prey to predator abundance. In recent years, such model has been studied extensively, and many important phenomena have been observed (see [4-6, 10-12] and references therein).

In this paper, we consider the following semilinear elliptic system with ratio-dependent function response and Dirichlet boundary condition:

$$
\begin{align*}
-\Delta u & =u(\lambda-u)-\frac{b u v}{u+m v}, \quad \text { in } \Omega \\
-\Delta v & =v\left(-k+\frac{u}{u+m v}\right), \quad \text { in } \Omega  \tag{1}\\
u & =v=0, \quad \text { on } \partial \Omega .
\end{align*}
$$

Here $\Omega$ is a bounded domain in $\mathbb{R}^{N}(N \geq 1)$ with smooth boundary $\partial \Omega$. In the biological model, the two unknown functions $u(x), v(x)$ represent the spatial distribution density of the prey and predator. $\lambda$ is termed the birth rate of the prey, while $k$ is the death rate of the predator and $b$ is a positive constant sometimes referred to as the conversion rate. The

Holling-Tanner interaction term $b u v /(u+m v)$ represents the rate at which the prey is consumed by the predator with $m>$ 0 .

We remark that problem (1) with homogeneous Neumann boundary condition was discussed in [6]. We will consider what is more general in this paper, where the parameters $\lambda, b$ are positive and $k \in \mathbb{R}$. What is more, since (1) comes from the biological module, our results and methods are different from those in [13-16] and references therein, which are also the Dirichlet boundary problems.

We say that a solution $(u, v)$ of (1) is positive solution if both $u(x)>0, v(x)>0$ for all $x \in \Omega$ and $(\partial u / \partial n)(x)<$ $0,(\partial v / \partial n)(x)<0$ for all $x \in \partial \Omega$; i.e., $(u, v)$ is also a coexistence state of (1).

Now, we give some notations, definitions and well-known facts which will be used in the sequel.

For each $q \in C(\bar{\Omega})$, let $\lambda_{1}(q)$ be the principal eigenvalue of

$$
\begin{align*}
-\Delta u+q(x) u & =\lambda u, \quad x \in \Omega  \tag{2}\\
u & =0, \quad x \in \partial \Omega .
\end{align*}
$$

As is well known, the principal eigenvalue $\lambda_{1}(q)$ is given by the following variational characterization

$$
\begin{equation*}
\lambda_{1}(q)=\inf _{\phi \in H_{0}^{1}(\Omega),\|\phi\|_{L^{2}(\Omega)}=1} \int_{\Omega}\left(|\nabla \phi|^{2}+q(x) \phi^{2}\right) d x \tag{3}
\end{equation*}
$$

We denote $\lambda_{1}(0)$ by $\lambda_{1}$; some useful properties can be seen [ 6 , Proposition A.1] or [17, Lemma 5.2].

For $q \in C(\bar{\Omega})$, let $p$ be a sufficiently large constant such that $p-q(x)>0$ for all $x \in \bar{\Omega}$. Define a compact linear operator $T: C(\bar{\Omega}) \longrightarrow C(\bar{\Omega})$ by $u=T v=(-\Delta+p)^{-1}(p-$ $q(x)) v$, where $u \in C(\bar{\Omega})$ is the unique solution of the following problem:

$$
\begin{align*}
-\Delta u+p u & =(p-q(x)) v, \quad x \in \Omega \\
u & =0, \quad x \in \partial \Omega . \tag{4}
\end{align*}
$$

Denote $r(T)$ be the spectral radius of $T$. Then the relationship between $\lambda_{1}(q)$ and $r(T)$ can be given as [2, Proposition 1].

Theorem 1 (see [2]). (1) $\lambda_{1}(q)>0 \Longleftrightarrow r(T)<1$;
(2) $\lambda_{1}(q)<0 \Longleftrightarrow r(T)>1$;
(3) $\lambda_{1}(q)=0 \Longleftrightarrow r(T)=1$.

From Theorem 1, we see that it is crucial to determine the eigenvalue $\lambda_{1}(q)$. The following theorem is established by [2] (see also Theorem 2.4 [1]).

Theorem 2 (see [1,2]). Let $q(x) \in L^{\infty}(\Omega)$ and $\phi \geq 0, \phi \not \equiv 0$ in $\Omega$ with $\phi=0$ on $\partial \Omega$. Then we have
(a) If $0 \not \equiv-\triangle \phi+q(x) \phi \leq 0$, then $\lambda_{1}(q)<0$;
(b) If $0 \not \equiv-\triangle \phi+q(x) \phi \geq 0$, then $\lambda_{1}(q)>0$;
(c) If $-\triangle \phi+q(x) \phi \equiv 0$, then $\lambda_{1}(q)=0$.

Some concepts of cone, total wedge, topological degree and fixed point index in a cone can be seen in [18-21] and so on.

Let $E$ be a Banach space and $W \subset E$ be a total wedge. Let $A: W \longrightarrow W$ be a compact operator with a fixed point $y \in W$ and let $D$ be a relatively open subset of $W$ such that $A$ has no fixed point on the boundary of $D$. We denote by $\operatorname{deg}_{W}(I-A, D)$ the degree of $I-A$ in $D$ relative to $W$ and by index $_{W}(A, y)$ the fixed point index of $A$ at $y$ relative to $W$.

Theorem 3 (see [21,22]). Assume that $W$ is a total wedge, and let $A: W \longrightarrow W$ be a compact operator with a fixed point $y \in W$ and it is Frechlet differentiable at $y$. Let $L=A^{\prime}(y)$ be the Frechlet derivative of $A$ at $y$. Then $L$ maps $\bar{W}_{y}$ into itself. Moreover, if $I-L$ is invertible on $\bar{W}_{y}$, then the following results hold:
(i) If $L$ has property $a$ on $\bar{W}_{y}$, then $\operatorname{index}_{W}(A, y)=0$;
(ii) If $L$ does not has property $a$ on $\bar{W}_{y}$, then index $_{W}(A, y)=(-1)^{\sigma}$, where $\sigma$ is the sum of multiplicities of all eigenvalues of $L$, which is greater than 1.

## 2. Existence of Positive Solutions of (1)

In the sequel, for simplicity of notation and more transparent analysis, we redefine

$$
p(u, v)= \begin{cases}\frac{u v}{u+m v}, & (u, v) \neq(0,0)  \tag{5}\\ 0, & (u, v)=(0,0)\end{cases}
$$

It is easy to see that, for $u \neq 0$,

$$
\begin{align*}
& p_{u}^{\prime}(u, v)=\frac{m v^{2}}{(u+m v)^{2}} \Longrightarrow p_{u}^{\prime}(u, 0)=0, \\
& p_{v}^{\prime}(u, v)=\frac{u^{2}}{(u+m v)^{2}} \Longrightarrow p_{v}^{\prime}(u, 0)=1, \\
& p_{u}^{\prime \prime}(u, v)=-\frac{2 m v^{2}}{(u+m v)^{3}} \Longrightarrow p_{u}^{\prime \prime}(u, 0)=0,  \tag{6}\\
& p_{v}^{\prime \prime}(u, v)=-\frac{2 m u^{2}}{(u+m v)^{3}} \Longrightarrow p_{v}^{\prime \prime}(u, 0)=\frac{-2 m}{u}, \\
& p_{u v}^{\prime \prime}(u, v)=p_{v u}^{\prime \prime}=\frac{2 m u v}{(u+m v)^{3}} \Longrightarrow p_{u v}^{\prime \prime}(u, 0)=0
\end{align*}
$$

while

$$
\begin{equation*}
p_{u}^{\prime}(0,0)=p_{v}^{\prime}(0,0)=0 . \tag{7}
\end{equation*}
$$

Under the definition of $p(u, v)$, it is obvious that (1) has a trivial solution $(0,0)$ and only one semitrivial solution $\left(u_{\lambda}, 0\right)$ (if $\lambda>\lambda_{1}, k>-\lambda_{1}$ ), where $u_{\lambda}$ is the unique positive solution of

$$
\begin{align*}
-\Delta u & =\lambda u-u^{2}, \quad x \in \Omega  \tag{8}\\
u & =0, \quad x \in \partial \Omega .
\end{align*}
$$

Lemma 4. If model (1) has a positive solution, then $\lambda>\lambda_{1}$ and $-\lambda_{1}<k<1-\lambda_{1}$.

Proof. Assume $(u, v)$ is a positive solution of (1), and then

$$
\begin{align*}
-\Delta u & <\lambda u, \quad x \in \Omega \\
-\Delta v & <(1-k) v, \quad x \in \Omega  \tag{9}\\
u & =v=0, \quad x \in \partial \Omega .
\end{align*}
$$

It follows from the property of the principal eigenvalue that $\lambda>\lambda_{1}$ and $1-k>\lambda_{1}$, that is, $k<1-\lambda_{1}$.

On the other hand, since $(u, v)$ satisfies

$$
\begin{align*}
-\Delta v & =v\left(-k+\frac{u}{u+m v}\right), \quad x \in \Omega  \tag{10}\\
v & =0, \quad x \in \partial \Omega
\end{align*}
$$

one has

$$
\begin{equation*}
0=\lambda_{1}\left(k-\frac{u}{u+m v}\right)<\lambda_{1}(k)=k+\lambda_{1} . \tag{11}
\end{equation*}
$$

So $k>-\lambda_{1}$.
Remark 5. Above lemma is about the necessary condition for (1) to have positive solutions. Next we show that $\lambda>\lambda_{1}$ and $-\lambda_{1}<k<1-\lambda_{1}$ are also the sufficient conditions for the existence of a positive solution of (1). We will use fixed point index theory.

Lemma 6. Assuming $k>-\lambda_{1}$, then any positive solution ( $u, v$ ) of (1) has a priori bounds

$$
\begin{align*}
& u(x) \leq \lambda, \\
& v(x) \leq \frac{\lambda(\lambda+k)}{b}\left\|(-\Delta+k)^{-1}(1)\right\|_{C(\bar{\Omega})} . \tag{12}
\end{align*}
$$

Proof. Since part 1 is a simple consequence of a standard comparison argument, we omit it. For part 2, by a direct calculation, we get that

$$
\begin{align*}
-\Delta(u+b v) & =u(\lambda-u)-\frac{b u v}{u+m v}+\frac{b u v}{u+m v}-k b v \\
& =-k b v+u(\lambda-u)  \tag{13}\\
& =-k(u+b v)+u(\lambda+k-u)
\end{align*}
$$

and hence

$$
\begin{equation*}
(-\Delta+k)(u+b v)=u(\lambda+k-u) \tag{14}
\end{equation*}
$$

As $\lambda_{1}+k>0$, we have

$$
\begin{equation*}
v=\frac{1}{b}\left[-u+(-\triangle+k)^{-1}(u(\lambda+k-u))\right] . \tag{15}
\end{equation*}
$$

Therefore $\lambda+k>0$. Otherwise, if $\lambda+k \leq 0$, then $(-\triangle$ $+k)(u+b v)<0$, which implies $u+b v<0$, a contradiction. Consequently $\lambda+k>0$ is necessary for the existence of a positive solution; in such case

$$
\begin{align*}
b v & \leq b v+u=(-\Delta+k)^{-1}[u(\lambda+k-u)]  \tag{16}\\
& \leq \lambda(\lambda+k)(-\Delta+k)^{-1}(1)
\end{align*}
$$

then

$$
\begin{equation*}
v \leq \frac{\lambda(\lambda+k)}{b}\left\|(-\Delta+k)^{-1}(1)\right\|_{C(\bar{\Omega})} . \tag{17}
\end{equation*}
$$

Remark 7. Above result shows that the coexistence states of (1) have a priori bounds as soon as $k$ varies in compact subinterval of $\left(-\lambda_{1}, 1-\lambda_{1}\right)$.

Now we introduce the following notations:

$$
\begin{aligned}
E & =C(\bar{\Omega}) \times C(\bar{\Omega}), \\
W & =K \times K,
\end{aligned}
$$

$$
\begin{equation*}
\text { where } K=\{u \in C(\bar{\Omega}): u(x) \geq 0 \text { in } \bar{\Omega}\} \text {, } \tag{18}
\end{equation*}
$$

$$
D=\{(u, v) \in W: u \leq \lambda+1, v \leq Q+1\}
$$

$Q=(\lambda(\lambda+k) / b)\left\|(-\Delta+k)^{-1}(1)\right\|_{C(\bar{\Omega})}$
From Lemma 6, nonnegative solutions of (1) must be in $D$. Define a positive and compact operator $A: E \longrightarrow E$ by

$$
\begin{equation*}
A(u, v)=(-\Delta+p)^{-1}\binom{u(\lambda-u)-\frac{b u v}{u+m v}+p u}{v\left(\frac{u}{u+m v}-k\right)+p v} \tag{19}
\end{equation*}
$$

where $p$ is s sufficiently large number such that $p+\lambda-u-$ $b v /(u+m v)>0$ and $p+u /(u+m v)-k>0$ for $u, v \in$ $[0, \lambda] \times[0, Q]$.

Remark 8. Note that (1) is equivalent to $(u, v)=A(u, v)$ by elliptic regularity, and therefore in order to show the existence of positive solutions of (1), it suffices to prove $A$ has nontrivial fixed points in $D$.

We next compute the fixed point index of the trivial and semitrivial solution of (1). We have shown that (1) has a trivial solution $(u, v)=(0,0)$ and a semitrivial solution $\left(u_{\lambda}, 0\right)$ since $\lambda>\lambda_{1}$ and $k>-\lambda_{1}$. Moreover the following lemma holds.

Lemma 9. Assuming that $\lambda>\lambda_{1}$ and $k>-\lambda_{1}$, then
(i) $\operatorname{deg}_{W}(I-A, D)=1$;
(ii) $\operatorname{index}_{W}(A,(0,0))=0$;
(iii) $\operatorname{index}_{W}\left(A,\left(u_{\lambda}, 0\right)\right)=0$ if $k<1-\lambda_{1}$; and index $_{W}\left(A,\left(u_{\lambda}, 0\right)\right)=1$ ifk $>1-\lambda_{1}$.

Proof. (i) For each $t \in[0,1]$, we define a positive and compact operator $A_{t}: E \longrightarrow E$ by

$$
\begin{align*}
& A_{t}(u, v) \\
& \quad=(-\Delta+p)^{-1}\binom{t u(\lambda-u)-\frac{b u v}{u+m v}+p u}{t v\left(-k+\frac{u}{u+m v}\right)+p v} \tag{20}
\end{align*}
$$

Then $A_{1}=A, A_{t}$ has no fixed point on $\partial D$ and $A_{t}(\bar{D}) \subset W$. Thus $\operatorname{deg}_{W}\left(I-A_{t}, D\right)$ is well defined for all $t \in[0,1]$. By the homotopy invariance of Laray-Schauder degree and $(0,0)$ the only fixed point of $A_{0}$ in $D$, we obtain

$$
\begin{align*}
\operatorname{deg}_{W}(I-A, D) & =\operatorname{deg}_{W}\left(I-A_{0}, D\right) \\
& =\operatorname{index}_{W}\left(A_{0},(0,0)\right) \tag{21}
\end{align*}
$$

Set

$$
L=A_{0}^{\prime}(0,0)=(-\Delta+p)^{-1}\left(\begin{array}{ll}
p & 0  \tag{22}\\
0 & p
\end{array}\right)
$$

Assume that $L\left(\xi_{1}, \xi_{2}\right)=\left(\xi_{1}, \xi_{2}\right)$ for some $\left(\xi_{1}, \xi_{2}\right) \in \bar{W}_{(0,0)}=$ $K \times K$. It is easy to see $\left(\xi_{1}, \xi_{2}\right)=(0,0)$. Thus $I-L$ is invertible on $\bar{W}_{(0,0)}$. Since $\lambda_{1}>0$, we see $r(L)<1$ by Theorem 1 , this implies that $L$ does not have property $a$ on $\bar{W}_{(0,0)}$. Thus by Theorem 3 (ii), index ${ }_{W}\left(A_{0},(0,0)\right)=1$.
(ii) Observe that $A(0,0)=(0,0)$. Let $L=A^{\prime}(0,0)$ and then

$$
L=(-\Delta+p)^{-1}\left(\begin{array}{cc}
\lambda+p & 0  \tag{23}\\
0 & -k+p
\end{array}\right)
$$

Assuming that $L\left(\xi_{1}, \xi_{2}\right)=\left(\xi_{1}, \xi_{2}\right)$ for some $\left(\xi_{1}, \xi_{2}\right) \in \bar{W}_{(0,0)}$, then

$$
\begin{align*}
-\triangle \xi_{1} & =\lambda \xi_{1}, \quad \text { in } \Omega \\
\xi_{1} & =0, \quad \text { on } \partial \Omega  \tag{24}\\
-\triangle \xi_{2} & =-k \xi_{2}, \quad \text { in } \Omega \\
\xi_{2} & =0, \quad \text { on } \partial \Omega
\end{align*}
$$

Since $\lambda>\lambda_{1}$ and $k>-\lambda_{1}$, we see that $\xi_{1}=\xi_{2}=0$. Thus $I-L$ is invertible on $\bar{W}_{(0,0)}$. By Theorem 1, we see $r_{\lambda}:=$ $r\left((-\Delta+p)^{-1}(\lambda+p)\right)>1$ and $r_{\lambda}$ is the principle eigenvalue of the operator $(-\Delta+p)^{-1}(\lambda+p)$ with a corresponding eigenfunction $\phi_{\lambda}>0$ in $\Omega$. Set $t_{\lambda}=1 / r_{\lambda} \in(0,1)$ and then $\left(\phi_{\lambda}, 0\right) \notin S_{(0,0)}=\{(0,0)\}$, but

$$
\begin{equation*}
\left(I-t_{\lambda} L\right)\left(\phi_{\lambda}, 0\right)=(0,0) \in S_{(0,0)} \tag{25}
\end{equation*}
$$

This shows that $L$ has property $a$, and thus index $_{W}(A,(0,0))=0$ by Theorem 3 (i).
(iii) The second part is a straightforward consequence of Lemma 4 and (i)(ii). In fact, from Lemma 4, the nonnegative solutions of $(1)$ are $(0,0)$ and $\left(u_{\lambda}, 0\right)$ if $k>1-\lambda_{1}$, so by the properties of topological degree, we have

$$
\begin{align*}
\operatorname{deg}_{W}(I-A, D)= & \operatorname{index}_{W}(A,(0,0))  \tag{26}\\
& +\operatorname{index}_{W}\left(A,\left(u_{\lambda}, 0\right)\right) .
\end{align*}
$$

Combining (i) and (ii) above and (26), we have $\operatorname{index}_{W}\left(A,\left(u_{\lambda}, 0\right)\right)=1$.

Next, give the proof of the first part. Observe $A\left(u_{\lambda}, 0\right)=$ $\left(u_{\lambda}, 0\right)$. Let $L=A^{\prime}\left(u_{\lambda}, 0\right)$, and then

$$
L=(-\Delta+p)^{-1}\left(\begin{array}{cc}
\lambda-2 u_{\lambda}+p & -b  \tag{27}\\
0 & -k+p+1
\end{array}\right)
$$

Assume that $L(\xi, \eta)=(\xi, \eta)$ for some $(\xi, \eta) \in \bar{W}_{\left(u_{\lambda}, 0\right)}=$ $C(\bar{\Omega}) \times K$; i.e., $(\xi, \eta)$ satisfies

$$
\begin{align*}
-\Delta \xi+\left(2 u_{\lambda}-\lambda\right) \xi & =-b \eta, \quad \text { in } \Omega \\
-\Delta \eta-\eta & =-k \eta, \quad \text { in } \Omega  \tag{28}\\
\xi & =\eta=0, \quad \text { on } \partial \Omega .
\end{align*}
$$

Taking $\eta \in K$, it follows from the second equation of (28) and Theorem 2 that $-k=-1+\lambda_{1}$, if $\eta \neq 0$, which contradicts $k \neq 1-\lambda_{1}$. So $\eta=0$. Then we get from the first equation of (28)

$$
\begin{align*}
-\Delta \xi+\left(2 u_{\lambda}-\lambda\right) \xi & =0, & & \text { in } \Omega  \tag{29}\\
\xi & =0, & & \text { on } \partial \Omega
\end{align*}
$$

If $\xi \neq 0$, then $\lambda_{1}\left(2 u_{\lambda}-\lambda\right)=0$ by Theorem 2 . On the other hand, $\lambda_{1}\left(2 u_{\lambda}-\lambda\right)>\lambda_{1}\left(u_{\lambda}-\lambda\right)=0$, and we get a contradiction. Therefore $(\xi, \eta)=(0,0)$ and $I-L$ is invertible on $\bar{W}_{\left(u_{\lambda}, 0\right)}$.

We claim that $L$ has property $a$ on $\bar{W}_{\left(u_{\lambda}, 0\right)}$. In fact, set

$$
\begin{equation*}
B=(-\Delta+p)^{-1}(-k+1+p) \tag{30}
\end{equation*}
$$

Since $k<1-\lambda_{1}$, by Theorem 1 (ii), $r_{k}:=r(B)>1$ is an eigenvalue of $B$ with a corresponding eigenfunction $\phi_{k}>0$. Since $S_{\left(u_{\lambda}, 0\right)}=C(\bar{\Omega}) \times\{0\}$, we see $\left(0, \phi_{k}\right) \in \bar{W}_{\left(u_{\lambda}, 0\right)} \backslash S_{\left(u_{\lambda}, 0\right)}$. Set $t_{k}=r_{k}^{-1} \in(0,1)$, and then

$$
\begin{align*}
(I & \left.-t_{k} L\right)\binom{0}{\phi_{k}} \\
& =\binom{0}{\phi_{k}}-t_{k}(-\Delta+p)^{-1}\binom{-b \phi_{k}}{(-k+p+1) \phi_{k}}  \tag{31}\\
& =\binom{(-\Delta+p)^{-1}\left(t_{k} b \phi_{k}\right)}{0} \in S_{\left(u_{\lambda}, 0\right)} .
\end{align*}
$$

This proves that $L$ has property $a$. Therefore $\operatorname{index}_{W}(A$, $\left.\left(u_{\lambda}, 0\right)\right)=0$.

Lemma 10. If $\lambda>\lambda_{1}$ and $-\lambda_{1}<k<1-\lambda_{1}$, then (1) has a positive solution.

Proof. By Lemma 9, we have

$$
\begin{gather*}
\operatorname{deg}_{W}(I-A, D)-\operatorname{index}_{W}(A,(0,0))  \tag{32}\\
-\operatorname{index}_{W}\left(A,\left(u_{\lambda}, 0\right)\right)=1 .
\end{gather*}
$$

Hence (1) has at least a positive solution.

From Lemmas 4 and 10, we get the following.
Lemma 11. Model (1) admits a positive solution if and only if $\lambda>\lambda_{1}$ and $-\lambda_{1}<k<1-\lambda_{1}$.

## 3. Structure of Solutions with $k$ as a Bifurcation Parameter

In this section we shall regard $k$ as a bifurcation parameter and suppose that all other constants are fixed. For all values of $k$, we have the branch of zero solutions of (1) $S_{0}=\{(k, 0,0)$ : $k \in \mathbb{R}\}$. Suppose $\lambda>\lambda_{1}$ and $k>-\lambda_{1}$, and then (1) also has the branch of semitrivial solutions $S_{1}=\left\{\left(k, u_{\lambda}, 0\right): k \in \mathbb{R}\right\}$. We next give some results about the bifurcation from $\left(u_{\lambda}, 0\right)$.

Lemma 12. (i) The trivial steady state $(0,0)$ is locally asymptotically stable if $\lambda<\lambda_{1}$ and $k>\lambda_{1}$, while it is unstable if $\lambda>\lambda_{1}$ or $k<-\lambda_{1}$.
(ii) Assume that $\lambda>\lambda_{1}$. Then the semitrivial solution steady state $\left(u_{\lambda}, 0\right)$ is locally asymptotically stable ifk $>1-\lambda_{1}$, while it is unstable if $k<1-\lambda_{1}$.

Proof. From the linearization principle eigenvalue problem

$$
\begin{align*}
- & \Delta \phi-\left[\lambda \phi-2 u \phi-b p_{u}^{\prime}(u, v) \phi-b p_{v}^{\prime}(u, v) \psi\right] \\
& =\eta \phi, \quad \text { in } \Omega \\
- & \Delta \psi-\left[-k \psi+p_{u}^{\prime}(u, v) \phi+p_{v}^{\prime}(u, v) \psi\right]=\eta \psi \tag{33}
\end{align*}
$$

in $\Omega$

$$
\phi=\psi=0, \quad \text { on } \partial \Omega .
$$

The stability of $\left(u_{\lambda}, 0\right)$ is determined by the problem

$$
\begin{align*}
-\triangle \phi-\lambda \phi+2 u_{\lambda} \phi+b \psi & =\eta \phi, \quad \text { in } \Omega \\
-\Delta \psi+k \psi-\psi & =\eta \psi, \quad \text { in } \Omega  \tag{34}\\
\phi & =\psi=0, \quad \text { on } \partial \Omega
\end{align*}
$$

Since (34) is not completely coupled, we only need to consider the following two eigenvalue problems:

$$
\begin{align*}
-\Delta \phi-\lambda \phi+2 u_{\lambda} \phi & =\eta \phi, \quad \text { in } \Omega  \tag{35}\\
\phi & =0, \quad \text { on } \partial \Omega
\end{align*}
$$

and

$$
\begin{align*}
-\Delta \psi+k \psi-\psi & =\eta \psi, \quad \text { in } \Omega  \tag{36}\\
\psi & =0, \quad \text { on } \partial \Omega
\end{align*}
$$

Then an eigenvalues of (34) must be an eigenvalues of (35) or (36) (see [11, 12] and references therein). Denote the principle eigenvalue of (35) and (36) by $\lambda_{*}$ and $\lambda^{*}$, respectively. Then

$$
\begin{align*}
& \lambda_{*}=\lambda_{1}\left(2 u_{\lambda}-\lambda\right)>\lambda_{1}\left(u_{\lambda}-\lambda\right)=0 \\
& \lambda^{*}=\lambda_{1}(k-1)=k-1+\lambda_{1} . \tag{37}
\end{align*}
$$

Combining above results, one can see that if $k>1-\lambda_{1}$, then all eigenvalues of (34) are positive, and thus ( $u_{\lambda}, 0$ ) is locally asymptotically stable. On the other hand, if $k<1-\lambda_{1}$, then (34) has a negative eigenvalue, which implies the instability of $\left(u_{\lambda}, 0\right)$.

Similarly, combining with the definition of $p(u, v)$, we can get (i).

First we shall obtain a local result on bifurcation from $S_{1}$, as the results of [23].

Lemma 13. Let $\lambda>\lambda_{1}$ be fixed. Then $k=1-\lambda_{1}$ is a bifurcation value of (1) where positive solutions bifurcate from the line of semi-trivial solutions $S_{1}$. The set of positive solutions to (1) near $\left(1-\lambda_{1}, u_{\lambda}, 0\right)$ is a smooth curve

$$
\begin{equation*}
\Gamma=\left\{\left(k(s), u_{\lambda}-u(s), v(s)\right), s \in(0, \delta)\right\} \tag{38}
\end{equation*}
$$

such that $k(0)=1-\lambda_{1}, u(s)=s \varphi_{1}(x)+o(|s|)$, and $v(s)=$ $s \varphi_{2}(x)+o(|s|)$. Moreover

$$
\begin{equation*}
k^{\prime}(0)=\frac{-m \int_{\Omega}\left(\varphi_{2}^{3}(x) / u_{\lambda}\right) d x}{\int_{\Omega} \varphi_{2}^{2}(x) d x} \tag{39}
\end{equation*}
$$

and $k=1-\lambda_{1}$ is the unique bifurcation value for which positive solutions bifurcate form $S_{1}$.

Proof. We apply the similar proof of Lemma 12 [4]. By changing the variables $w=u_{\lambda}-u$, define

$$
\begin{align*}
& G(k, w, v) \\
& \quad=\binom{\Delta w+\lambda w-2 u_{\lambda} w+w^{2}+b p\left(u_{\lambda}-w, v\right)}{\Delta v+p\left(u_{\lambda}-w, v\right)-k v} \tag{40}
\end{align*}
$$

A simple calculation implies that

$$
\begin{align*}
G_{k}(k, w, v) & =\binom{0}{-v} \\
G_{k(w, v)}(k, w, v)[\phi, \psi] & =\binom{0}{-\psi} \\
G_{(w, v)}(k, 0,0)[\phi, \psi] & =\binom{\triangle \phi+\lambda \phi-2 u_{\lambda} \phi+b \psi}{\Delta \psi+(1-k) \psi}  \tag{41}\\
G_{(w, v)(w, v)}(k, 0,0)[\phi, \psi]^{2} & =\binom{2 \phi^{2}-\frac{2 m}{u_{\lambda}} \psi^{2}}{-\frac{2 m}{u_{\lambda}} \psi^{2}}
\end{align*}
$$

By letting $(w, v)=(0,0)$, we can find that only when $k=$ $1-\lambda_{1}$, $\operatorname{does} G_{(w, v)}(k, 0,0)[\phi, \psi]=0$ have a solution with $\psi>0$. Thus $k_{1}=1-\lambda_{1}$ is the only bifurcation point along $S_{1}$, where positive solution of (1) bifurcates. At $(k, w, v)=\left(k_{1}, 0,0\right)$, it is easy to verify that the kernel

$$
\begin{equation*}
N\left(G_{(w, v)}\left(k_{1}, 0,0\right)\right)=\operatorname{span}\left\{\left(\varphi_{1}, \varphi_{2}\right)\right\} \tag{42}
\end{equation*}
$$

where $\left(\varphi_{1}, \varphi_{2}\right) \neq(0,0)$ satisfies

$$
\begin{align*}
\Delta \phi+\lambda \phi-2 u_{\lambda} \phi+b \psi & =0, \quad \text { in } \Omega \\
\Delta \psi+\left(1-k_{1}\right) \psi & =0, \quad \text { in } \Omega  \tag{43}\\
\phi & =\psi=0, \quad \text { on } \partial \Omega
\end{align*}
$$

Since $k_{1}=1-\lambda_{1}$, we can choose $\varphi_{2}>0$, and then

$$
\begin{equation*}
\varphi_{1}=\left(-\Delta+2 u_{\lambda}-\lambda\right)^{-1}\left(b \varphi_{2}\right)>0 \tag{44}
\end{equation*}
$$

The range of the operator is given by

$$
\begin{align*}
& R G_{(w, v)}\left(k_{1}, 0,0\right) \\
& \quad=\left\{(f, g) \in Y_{1} \times Y_{2}: \int_{\Omega} g(x) \varphi_{2}(x) d x=0\right\} \tag{45}
\end{align*}
$$

which is of codimension one and

$$
\begin{align*}
& G_{k(w, v)}\left(k_{1}, 0,0\right)\left[\varphi_{1}, \varphi_{2}\right]=\left(0,-\varphi_{2}\right)  \tag{46}\\
& \quad \notin R G_{(w, v)}\left(k_{1}, 0,0\right)
\end{align*}
$$

since $\int_{\Omega} \varphi_{2}^{2}(x) d x>0$. Thus we can conclude that the set of positive solutions to (1) near $\left(k_{1}, u_{\lambda}, 0\right)$ is $\Gamma$. Moreover

$$
\begin{align*}
k^{\prime}(0) & =-\frac{\left\langle G_{(w, v)(w, v)}\left(k_{1}, 0,0\right)\left[\varphi_{1}, \varphi_{2}\right]^{2}, l\right\rangle}{2\left\langle G_{k(w, v)}\left(k_{1}, 0,0\right)\left[\varphi_{1}, \varphi_{2}\right], l\right\rangle} \\
& =\frac{-m \int_{\Omega}\left(\varphi_{2}^{3}(x) / u_{\lambda}\right) d x}{\int_{\Omega} \varphi_{2}^{2}(x) d x}<0 \tag{47}
\end{align*}
$$

where $l$ is the linear functional defined by $\langle[f, g], l\rangle=$ $\int_{\Omega} g(x) \varphi_{2}(x) d x$. Therefore the bifurcation at $\left(k_{1}, u_{\lambda}, 0\right)$ is always subcritical.

Suppose that there is a sequence $\left\{\left(k_{n}, u_{n}, v_{n}\right)\right\}$ of coexistence states of (1) with $\lim _{n \rightarrow \infty}\left(k_{n}, u_{n}, v_{n}\right)=\left(k, u_{\lambda}, 0\right)$ for some $k \in \mathbb{R}$. Then from the $v$-equation of (1), we find for every $n \geq 1$

$$
\begin{equation*}
-\Delta \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}=-k_{n} \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}+\frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}} \frac{u_{n}}{u_{n}+m v_{n}} \tag{48}
\end{equation*}
$$

or equivalently

$$
\begin{align*}
& \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}} \\
& \quad=(-\triangle)^{-1}\left(-k \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}+\frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}} \frac{u_{n}}{u_{n}+m v_{n}}\right)  \tag{49}\\
& \quad+\left(k-k_{n}\right)(-\triangle)^{-1}\left(\frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}\right)
\end{align*}
$$

By the compactness of $(-\triangle)^{-1}$, it is easy to see that along some subsequence, relabeled by $n$, we have

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}=\phi>0 \tag{50}
\end{equation*}
$$

for some $\phi \in C_{0}^{1}(\bar{\Omega})$ with $\|\phi\|_{C(\bar{\Omega})}=1$. Thus passing the limit $n \longrightarrow \infty$ in (49), we find that

$$
\begin{equation*}
\phi=(-\Delta)^{-1}(-k+1) \phi \Longleftrightarrow-\Delta \phi=(-k+1) \phi . \tag{51}
\end{equation*}
$$

Therefore $k=1-\lambda_{1}$, which concludes the proof.
Remark 14. Above results together with Lemma 4 show that the bifurcation of nonnegative solutions from $S_{1}$ at $\left(k_{1}, u_{\lambda}, 0\right)$ must be to the left, while Lemma 4 also shows that the branch of nontrivial solutions cannot extend too far to the left.

We now investigate the global nature of the above curve of nontrivial nonnegative solution in the $k-(u, v)$ plane, i.e., in $\mathbb{R} \times C(\bar{\Omega}) \times C(\bar{\Omega})$, to show that hypotheses of Th3.2 [24] are satisfied.

Writing $U=u_{\lambda}-u$ and $V=v$, it is easy to check that ( $U, V$ ) is a nonnegative solution of (1) if and only if $0 \leq U \leq$ $u_{\lambda}, V \geq 0$ and $(U, V)$ satisfies

$$
\begin{align*}
& -\triangle U=\lambda U-2 u_{\lambda} U+U^{2}+b \frac{\left(u_{\lambda}-U\right) V}{u_{\lambda}-U+m V}  \tag{52}\\
& -\Delta V=-k V+\frac{\left(u_{\lambda}-U\right) V}{u_{\lambda}-U+m V}
\end{align*}
$$

Rewrite (52) as

$$
\begin{align*}
& \binom{U}{V} \\
& \quad=(-\triangle)^{-1}\binom{U\left(\lambda-2 u_{\lambda}+U\right)+b \frac{\left(u_{\lambda}-U\right) V}{u_{\lambda}-U+m V}}{-k V+\frac{\left(u_{\lambda}-U\right) V}{u_{\lambda}-U+m V}} \tag{53}
\end{align*}
$$

Define $T: \mathbb{R} \times C^{1}(\bar{\Omega}) \times C^{1}(\bar{\Omega}) \longrightarrow C^{1}(\bar{\Omega}) \times C^{1}(\bar{\Omega})$ by

$$
\begin{align*}
T(k, u, v)= & (-\Delta)^{-1}\left(\begin{array}{cc}
\lambda-2 u_{\lambda} & b \\
0 & -k+1
\end{array}\right)\binom{u}{v} \\
& +(-\triangle)^{-1}\binom{u^{2}+b \frac{\left(u_{\lambda}-u\right) v}{u_{\lambda}-u+m v}-b v}{-v+\frac{\left(u_{\lambda}-u\right) v}{u_{\lambda}-u+m v}}  \tag{54}\\
= & K(k)(u, v)+R(k, u, v)
\end{align*}
$$

Then $K(k)$ is a linear compact operator and the Frechlet derivative $R_{(u, v)}(k, 0,0)=0$. Let $H=I-T$ and then $H(k, u, v)=0$ with $0 \leq u \leq u_{\lambda}$ and $v \geq 0$ if and only if ( $k, u_{\lambda}-u, v$ ) is a nonnegative solution of (1).

We must calculate the index $i(T(k, \cdot), 0)$ when $k$ is close to $k_{0}:=1-\lambda_{1}$. This index is equal to $(-1)^{\beta}$, where $\beta$ is the sum of the algebraic multiplicities of eigenvalue of $K(k)>1$.

Suppose that $\mu>0$ is an eigenvalue of $K(k)$. Then there exists a nonzero function $v$ such that

$$
\begin{align*}
-\mu \Delta v-v & =-k v \quad \text { in } \Omega \\
v & =0 \quad \text { on } \partial \Omega \tag{55}
\end{align*}
$$

i.e., $-k$ is an eigenvalue of (55). Conversely, if $\mu \geq 1$ and $-k$ is an eigenvalue of (55) with corresponding eigenfunction $v$, then $(u, v)$ is an eigenfunction of $K(k)$ corresponding to the eigenvalue $\mu$, where $u$ is the unique solution of

$$
\begin{align*}
-\mu \triangle u-\lambda u+2 u_{\lambda} u & =b v \quad \text { in } \Omega  \tag{56}\\
u & =0 \quad \text { on } \partial \Omega
\end{align*}
$$

Since all eigenvalues of $-\triangle-\lambda+2 u_{\lambda}$ are positive and $\mu \geq 1$, it follows that $-\Delta-\lambda+2 u_{\lambda}$ is invertible. The eigenvalues of (55) form an increasing sequence $r_{1}(\mu)<r_{2}(\mu) \leq r_{3}(\mu) \leq \cdots$ and $\mu \longrightarrow r_{i}(\mu)$ is a continuous increasing function. Thus $\mu \geq 1$ is an eigenvalue of $K(k)$ if and only if $k=r_{i}(\mu)$ for some $\mu$. Clearly $r_{i}(1)=\lambda_{i}(-1)$.
(i) Suppose $k>k_{0} \Longleftrightarrow-k<-k_{0}$ and $\mu \geq 1$ is an eigenvalue of $K(k)$. Then $-k$ is an eigenvalue of (55) and $-k>$ the least eigenvalue of $-\Delta-1$, i.e., $-k>\lambda_{1}(-1)=\lambda_{1}-1$. But $-k<-k_{0}=\lambda_{1}-1$, a contradiction. Hence $K(k)$ has no eigenvalues $>1$ and so $i(T(k, \cdot), 0)=1$.
(ii) Now suppose $-r_{2}(1)<k<k_{0}$, i.e., $r_{2}(1)>-k>-k_{0}=$ $r_{1}(1)$. Since $\mu \longrightarrow r_{1}(\mu)$ is increasing with $\lim _{\mu \rightarrow \infty} r_{1}(\mu)=$ $\infty$, there exists a unique $\mu>1\left(\mu_{1}\right.$ say $)$ such that $-k=r_{1}\left(\mu_{1}\right)$. Since $-k<r_{2}(1)$, it follows that $-k<r_{i}(\mu)$ for $i=2,3, \cdots$ and $\mu \geq 1$. Thus $\mu_{1}$ is the only eigenvalue of $K(k)$ which is greater than 1 . We now to show that $\mu_{1}$ is a simple eigenvalue of $K(k)$. The discussion above shows that $N\left(K-\mu_{1} I\right)=\operatorname{span}\{(\phi, \psi)\}$, where $\psi$ is the principal eigenfunction corresponding to the eigenvalue $-k$ of

$$
\begin{align*}
-\mu_{1} \triangle v-v & =-k v \quad \text { in } \Omega \\
v & =0 \quad \text { on } \partial \Omega . \tag{57}
\end{align*}
$$

and $\phi=\left(-\mu \Delta-\lambda+2 u_{\lambda}\right)^{-1} \psi$. Thus $\operatorname{dim} N\left(K(k)-\mu_{1} I\right)=1$. Suppose that $(\phi, \psi) \in R\left(K(k)-\mu_{1} I\right)$. Then there exists $v$ such that

$$
\begin{equation*}
-\mu_{1} \Delta v+k v-v=-\Delta \psi=\mu_{1}^{-1}(-k \psi+\psi) \quad \text { on } \Omega \tag{58}
\end{equation*}
$$

Multiplying by $\psi$ and integrating over $\Omega$ shows that $\psi=$ 0 , which is impossible. Hence $R\left(K(k)-\mu_{1} I\right) \cap N(K(k)-$ $\left.\mu_{1} I\right)=\{0\}$ and so $\mu_{1}$ is a simple eigenvalue of $K(k)$. Thus $i(T(k, \cdot), 0)=-1$ whenever $k<k_{0}$.

Therefore Theorem 3.2 [24] can be applied to $T$.
Let

$$
\begin{align*}
P_{1} & =\left\{u \in C(\bar{\Omega}): u(x)>0 \text { for } x \in \Omega \text { and } \frac{\partial u}{\partial n}(x)\right. \\
& <0 \text { for } x \in \partial \Omega\}  \tag{59}\\
P & =\left\{(k, u, v): k \in \mathbb{R} \text { and } u, v \in P_{1}\right\}
\end{align*}
$$

By arguments similar to those used in Section 4 of [24], it can be proved that there exists a continuum $C$ in $\mathbb{R} \times C(\bar{\Omega}) \times C(\bar{\Omega})$ emanating from $\left(k_{0}, u_{\lambda}, 0\right)$ such that
(i) if $(k, u, v) \in C$, then $\left(u_{\lambda}-u, v\right)=T\left(k, u_{\lambda}-u, v\right)$;
(ii) if $(k, u, v) \in C$ and $u, v \geq 0$, then $(k, u, v)$ is a solution of (1).
(iii) close to the bifurcation point $\left(k_{0}, u_{\lambda}, 0\right), C$ consists of the points ( $k, u, v$ ) on the curve given by Lemma 13.

Clearly $C \subset P$ in a neighbourhood of the bifurcation point $\left(k_{0}, u_{\lambda}, 0\right)$.

Theorem 15. Assume $\lambda \neq \lambda_{1}+b / m$
(i) If $(k, u, v) \in C-\left\{\left(k_{0}, u_{\lambda}, 0\right)\right\}$, then $u, v \in P_{1}$, i.e., $C-$ $\left\{\left(k_{0}, u_{\lambda}, 0\right)\right\} \subset P$;
(ii) $C$ is unbounded in $\mathbb{R} \times C(\bar{\Omega}) \times C(\bar{\Omega})$.

Proof. (i) Suppose that $C$ contains a point $(k, u, v) \neq$ $\left(k_{0}, u_{\lambda}, 0\right)$ which lies outside of $P$. Then there exists a point $(\widehat{k}, \widehat{u}, \widehat{v}) \in C-\left\{\left(k_{0}, u_{\lambda}, 0\right)\right\} \cap \partial P$ which is the limit of a sequence of points $\left\{\left(k_{n}, u_{n}, v_{n}\right)\right\}$ in $C \cap P$. As $(\widehat{k}, \widehat{u}, \widehat{v}) \in \partial P$, either $\widehat{u} \in \partial P_{1}$ or $\widehat{v} \in \partial P_{1}$.
(1) Suppose $\hat{v} \in \partial P_{1}$. Then $\hat{v} \geq 0$ for $x \in \Omega$ and either $\widehat{v}(x)=0$ for some $x \in \Omega$ or $(\partial v / \partial n)(x)=0$ for some $x \in \partial \Omega$. It follows that

$$
\begin{equation*}
-\Delta \widehat{v}+\left[M+k-\frac{\widehat{v}}{\widehat{u}+m \widehat{v}}\right] \widehat{v}=\mu \widehat{v} \geq 0 \quad \text { in } \Omega \tag{60}
\end{equation*}
$$

where $M$ is a constant chosen sufficient large so that the term in the square bracket is positive for all $x \in \Omega$. It follows from the strong maximum principle that $\widehat{v} \equiv 0$. A similar argument shows that if $\widehat{u} \in \partial P_{1}$, then $\widehat{u} \equiv 0$. Thus $\widehat{u} \equiv 0$ or $\widehat{v} \equiv 0$.
(2) Suppose $\widehat{u} \equiv 0$ and $\widehat{v} \equiv 0$. Then $(\widehat{k}, \widehat{u}, \widehat{v})=(\widehat{k}, 0,0)$, and so ( $\widehat{k}, \widehat{u}, \widehat{v}$ ) lies on the trivial branch of solutions $S_{0}$. The only trivial nonnegative solutions which are close to $S_{0}$ lie on the semitrivial $S_{1}$, and so there cannot exist a sequence in $C \cap P$
converging to $(\widehat{k}, \widehat{u}, \widehat{v})$. Therefore it is impossible that both $\widehat{u}$ and $\widehat{v}$ are identically zero.
(3) Suppose $u_{n} \longrightarrow \widehat{u} \equiv 0$, then $v_{n} \rightarrow 0$ and we find from the first equation of (1)

$$
\begin{align*}
-\triangle \frac{u_{n}}{\left\|u_{n}\right\|_{C(\bar{\Omega})}}= & \frac{u_{n}}{\left\|u_{n}\right\|_{C(\bar{\Omega})}}\left(\lambda-u_{n}\right)  \tag{61}\\
& -b \frac{u_{n}}{\left\|u_{n}\right\|_{C(\bar{\Omega})}} \frac{v_{n}}{u_{n}+m v_{n}}
\end{align*}
$$

Then by $L^{p}$ - theory and bootstrapping arguments, $\exists \phi>0$ such that

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left(u_{n} /\left\|u_{n}\right\|_{C(\bar{\Omega})}\right) & =\phi \text { and } \\
-\triangle \phi & =\left(\lambda-\frac{b}{m}\right) \phi,  \tag{62}\\
\left.\phi\right|_{\partial \Omega} & =0 .
\end{align*}
$$

which is impossible, since $\lambda-b / m \neq \lambda_{1}$.
(4) Suppose that $\widehat{v} \equiv 0$. Then $\widehat{u} \neq 0,(\widehat{k}, \widehat{u}, \widehat{v}) \in$ $S_{1}$ and there bifurcate from $(\widehat{k}, \widehat{u}, \widehat{v})$ nontrivial, nonnegative solutions. While we have shown that $k_{0}$ is the unique bifurcation value for which positive solutions bifurcate from $S_{1}$, then $\widehat{k}=k_{0}$. That is impossible.

Therefore, if $(k, u, v) \in C-\left(k_{0}, u_{\lambda}, 0\right)$, then $(k, u, v) \in P$.
(ii) $C$ must satisfy one of the three alternatives discussed before Theorem 15. Because of (i) above, $C$ contains no pair of points of the form $\left(k, u_{\lambda}-u, v\right)$ and $\left(k, u_{\lambda}+u,-v\right)$ and $C$ cannot join up with another bifurcation point of the form $\left(k, u_{\lambda}, 0\right)$ on $S_{1}$. Hence $C$ joins ( $k, u_{\lambda}, 0$ ) to $\infty$; i.e., $C$ is unbounded.

Based on above preparations, we have the bifurcation results as follows.

Theorem 16. Assume

$$
\begin{align*}
& k>-\lambda, \\
& \lambda>\lambda_{1} \tag{63}
\end{align*}
$$

$$
\text { and } \lambda \neq \lambda_{1}+\frac{k}{m} .
$$

Then there exists an unbounded component $C_{+}=C-$ $\left\{\left(k_{0}, u_{\lambda}, 0\right)\right\}$ of the set of positive solutions of (1) such that $\left(k_{0}, u_{\lambda}, 0\right) \in \overline{C_{+}}$and $\left(k, u_{\lambda}, 0\right) \notin \overline{C_{+}}$if $k \neq k_{0}$. i.e., $\mathscr{P}_{k} C_{+}=$ $\left(-\lambda_{1}, 1-\lambda_{1}\right)$, where $\mathscr{P}_{k}$ stands for the projection operator into the $k$-component of the term.

## 4. Asymptotic Behavior Analysis of Positive Solutions of (1)

In this section, we give the sketch of asymptotic behavior for positive solution as the parameter $k \longrightarrow-\lambda_{1}$.

Suppose (63) and let ( $u, v$ ) be a coexistence of (1). Then, since $u \leq u_{\lambda} \leq \lambda$, there exists a constant $M$ such that $\|u\|_{C(\bar{\Omega})} \leq M$. Hence owing to Lemma 6, there exists a constant $C(k)$ such that

$$
\begin{equation*}
\|v\|_{C(\bar{\Omega})} \leq\left\|\frac{1}{b}(-\Delta+k)^{-1}(\lambda(\lambda+k))\right\|_{C(\bar{\Omega})}:=C(k) . \tag{64}
\end{equation*}
$$

Note that

$$
\begin{equation*}
\lim _{k \longrightarrow-\lambda_{1}} C(k)=\infty . \tag{65}
\end{equation*}
$$

Theorem 17. Suppose $\lambda-b / m>\lambda_{1}$. Let $\left\{k_{n}\right\}$ be a sequence such that

$$
\begin{align*}
-\lambda_{1} & <k_{n}<1-\lambda_{1} \\
\lim _{n \rightarrow \infty} k_{n} & =-\lambda_{1} . \tag{66}
\end{align*}
$$

For each $n \geq 1$, let $\left(k_{n}, u_{n}, v_{n}\right)$ be a coexistence of (1); then

$$
\begin{equation*}
\lim _{n \rightarrow \infty} v_{n}=\infty \tag{67}
\end{equation*}
$$

uniformly in any compact subset of $\Omega$
and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} u_{n}=u_{0} \quad \text { uniformly on } \bar{\Omega}, \tag{68}
\end{equation*}
$$

where $u_{0}$ is the unique positive solution of

$$
\begin{align*}
-\Delta u & =\left(\lambda-\frac{b}{m}\right) u-u^{2}, \quad \text { in } \Omega  \tag{69}\\
u & =0, \quad \text { on } \partial \Omega
\end{align*}
$$

Proof. By the global bifurcation theory or Theorem 15, we get $\left\|v_{n}\right\|_{C(\bar{\Omega})} \longrightarrow \infty$. From the second equation of (1)

$$
-\frac{\Delta v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}=-k_{n} \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}+\frac{u_{n}}{u_{n}+m v_{n}} \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}}
$$

$$
\begin{equation*}
\text { in } \Omega \tag{70}
\end{equation*}
$$

$$
\left.v_{n}\right|_{\partial \Omega}=0 .
$$

Since $0<u_{n} v_{n} /\left(u_{n}+m v_{n}\right) \leq u_{n} / m \leq \lambda / m$ in $\bar{\Omega}$, by using $L^{p}$ - regularity theory and Schauder bootstrapping technique (see [17] Chapter 5), $\exists \phi>0$ in $\Omega$ such that $v_{n} /\left\|v_{n}\right\|_{C(\bar{\Omega})} \longrightarrow \phi$. In fact, $\phi$ is the principal eigenfunction corresponding to $\lambda_{1}$. This implies that for any compact subset $\Omega_{0}$ of $\Omega, v_{n} \longrightarrow \infty$ uniformly in $\Omega_{0}$.

On the other hand, from the first equation of (1)

$$
\begin{align*}
- & \Delta u_{n} \\
= & u_{n}\left(\lambda-u_{n}\right) \\
& -\frac{b u_{n}}{u_{n} /\left\|v_{n}\right\|_{C(\bar{\Omega})}+m\left(v_{n} /\left\|v_{n}\right\|_{C(\bar{\Omega})}\right)} \frac{v_{n}}{\left\|v_{n}\right\|_{C(\bar{\Omega})}} \tag{71}
\end{align*}
$$

in $\Omega$

$$
\left.u_{n}\right|_{\partial \Omega}=0 .
$$

Similar to arguments above, $\exists u_{0}>0$ in $\Omega$, such that $u_{n} \longrightarrow u_{0}$ on $\bar{\Omega}$ uniformly and $u_{0}$ satisfies

$$
\begin{align*}
-\Delta u_{0} & =u_{0}\left(\lambda-\frac{b}{m}\right)-u_{0}^{2}, \quad \text { in } \Omega  \tag{72}\\
\left.u_{0}\right|_{\partial \Omega} & =0 .
\end{align*}
$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# Existence of Nontrivial Solutions for Some Second-Order Multipoint Boundary Value Problems 

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By using fixed point theorems with lattice structure, the existence of negative solution and sign-changing solution for some secondorder multipoint boundary value problems is obtained.

## 1. Introduction

In this paper, the following second-order ordinary differential equation will be considered:

$$
\begin{equation*}
-x^{\prime \prime}(t)=\varphi(t, x(t)), \quad 0 \leq t \leq 1, \tag{1}
\end{equation*}
$$

subject to the multipoint boundary condition

$$
\begin{align*}
& x(0)=0, \\
& x(1)=\sum_{i=1}^{m-2} \beta_{i} x\left(\alpha_{i}\right), \tag{2}
\end{align*}
$$

where $\beta_{i}>0, i=1,2, \cdots, m-2 ; \sum_{i=1}^{m-2} \beta_{i}<1 ; 0<\alpha_{1}<\alpha_{2}<$ $\cdots<\alpha_{m-2}<1$, and $\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}<1$.

The multipoint boundary value problems of ordinary differential equations arise in different areas of applied mathematics and physics. In 1992, Gupta studied nonlinear second-order three-point boundary value problems (see [1]). Since then, different types of nonlinear multipoint boundary value problems have been studied. Up to now, many great achievements about multipoint boundary value problems have been made. For example, many authors have investigated the existence of nontrivial solutions for nonlinear multipoint boundary value problems. Most of them have used upper and lower solution method, fixed point index theory, Guo-Krasnosel'skii fixed point theorem, bifurcation theory, fixed point theorems on cones, and so on (see [227] and references therein). For instance, in [2], the author
considered the second-order multipoint boundary value problem

$$
\begin{align*}
y^{\prime \prime}(t)+f(y) & =0, \quad 0 \leq t \leq 1, \\
y(0) & =0  \tag{3}\\
y(1) & =\sum_{i=1}^{m-2} \alpha_{i} y\left(\eta_{i}\right)
\end{align*}
$$

By using fixed point index and Leray-Schauder degree methods, the author showed existence of multiple sign-changing solutions for the boundary value problem (3). In [14], the authors have considered the following multipoint boundary value problem:

$$
\begin{align*}
-(L \varphi)(t) & =\lambda f(t, \varphi(t)), \quad 0 \leq t \leq 1 \\
\varphi^{\prime}(0) & =0  \tag{4}\\
\varphi(1) & =\sum_{i=1}^{m-2} \beta_{i} \varphi\left(\eta_{i}\right) .
\end{align*}
$$

The authors have used global bifurcation method to obtain the existence of positive solution of the boundary value problem (4).

In recent years, some authors combine the theory of lattice and the theory of topological degree, so they have obtained some fixed point theorems with lattice structure
for nonlinear operators which are not assumed to be cone mappings (see [28-34]). At present, a few authors have used those fixed point theorems with lattice structure to study boundary value problems (see [6, 17, 28-37]). For example, in [35], by using fixed point theorems with lattice structure, the authors considered the existence of positive solution and sign-changing solution for integral boundary value problem under sublinear condition. In [37], the authors considered the existence of positive solution for fourth-order differential equation with fixed point theorems with lattice structure. In [6], the author considered the following second-order threepoint boundary value problem:

$$
\begin{align*}
-u^{\prime \prime}(t) & =g(t, u(t)), \quad 0 \leq t \leq 1, \\
u(0) & =0,  \tag{5}\\
u(1) & =\alpha u(\beta),
\end{align*}
$$

where $g:[0,1] \times(-\infty,+\infty) \longrightarrow(-\infty,+\infty)$ is continuous, $0<\alpha<1,0<\beta<1$. The author used fixed point theorems with lattice structure to study the existence of sign-changing solutions for the boundary value problem (5) under the unilaterally asymptotically linear condition.

Motivated by [6, 17, 28-37], we shall study the existence of nontrivial solutions for the boundary value problem (1), (2). In this paper, we assume that the nonlinear term satisfies superlinear conditions concerning the first eigenvalue corresponding to the relevant linear operator. The method we use is fixed point theorems with lattice structure. And we obtain the sufficient condition about the existence of negative solution and sign-changing solution for the boundary value problem (1), (2). The method is different from those of [2, 4]. And the main results are different from those of the work $[2,4]$. This paper is arranged as follows. In Section 2, we give some definitions and fixed point theorems with lattice structure. In Section 3, we shall give some lemmas and the main results about the existence of nontrivial solutions (including negative solution and sign-changing solution) for the boundary value problem (1), (2). Finally, in Section 4, some examples are given to illustrate our main results.

## 2. Preliminaries

Let $E$ be an ordered Banach space in which the partial ordering $\leq$ is induced by a cone $P \subset E . P$ is called normal if there exists a positive constant $N>0$ such that $\theta \leq u \leq v$ implies $\|u\| \leq N\|v\| . P$ is called solid if int $\mathrm{P} \neq \theta$, i.e., $P$ has nonempty interior. $P$ is called total if $E=\overline{P-P}$. If $P$ is solid, then $P$ is total. For the concepts and the properties about the cones, we refer to [31, 38, 39].

We call $E$ a lattice under the partial ordering $\leq$, if $\sup \{u, v\}$ and $\inf \{u, v\}$ exist for arbitrary $u, v \in E$.

For $u \in E$, let

$$
\begin{align*}
& u^{+}=\sup \{u, \theta\},  \tag{6}\\
& u^{-}=\sup \{-u, \theta\} .
\end{align*}
$$

$u^{+}$and $u^{-}$are called positive part and negative part of $u$, respectively. Taking $|u|=u^{+}+u^{-}$, then $|u| \in P$. For
the definition and the properties of the lattice, we refer to [40].

For convenience, we use the following notations:

$$
\begin{align*}
& u_{+}=u^{+},  \tag{7}\\
& u_{-}=-u^{-},
\end{align*}
$$

and clearly

$$
\begin{align*}
& u_{+} \in P \\
& u_{-} \in(-P)  \tag{8}\\
& u=u_{+}+u_{-}
\end{align*}
$$

Definition 1 (see [28-31]). Let $D \subset E$ and $F: D \longrightarrow E$ be a nonlinear operator. If there exists $u^{*} \in E$ such that

$$
\begin{equation*}
F u=F u_{+}+F u_{-}+u^{*}, \quad \forall u \in D \tag{9}
\end{equation*}
$$

then $F$ is said to be quasi-additive on lattice.
Let $B: E \longrightarrow E$ be a bounded linear operator. If $B(P) \subset P$, then the operator $B$ is called to be positive.

In this section, we assume that $E$ is a Banach space, $P$ is a total cone, the partial ordering $\leq$ in $E$ is induced by $P$, and $E$ is a lattice in the partial ordering $\leq$.

Let $B: E \longrightarrow E$ be a positive completely continuous linear operator; $B^{*}$ the conjugated operator of $B ; r(B)$ a spectral radius of $B$; and $P^{*}$ the conjugated cone of $P$. Since $P \subset E$ is a total cone, by Krein-Rutman theorem, we can infer that if $r(B) \neq 0$, then there exist $\bar{u} \in P \backslash\{\theta\}$ and $f^{*} \in P^{*} \backslash\{\theta\}$, such that

$$
\begin{align*}
B \bar{u} & =r(B) \bar{u}, \\
B^{*} f^{*} & =r(B) f^{*} . \tag{10}
\end{align*}
$$

For $\delta>0$. Let

$$
\begin{equation*}
P\left(f^{*}, \delta\right)=\left\{u \in P \mid f^{*}(u) \geq \delta\|u\|\right\} . \tag{11}
\end{equation*}
$$

Then $P\left(f^{*}, \delta\right)$ is also a cone in $E$.
Definition 2 (see [30, 31, 41]). If there exist $\bar{u} \in P \backslash\{\theta\}, f^{*} \in$ $P^{*} \backslash\{\theta\}$, and $\delta>0$ such that (10) holds, and $B$ maps $P$ into $P\left(f^{*}, \delta\right)$, then the positive linear operator $B$ is said to satisfy H condition.

Let $P$ be a cone of a Banach space $E$. If $u \in(P \backslash\{\theta\})$ is a fixed point of $A$, then $u$ is said to be a positive fixed point of $A$. If $u \in((-P) \backslash\{\theta\})$ is a fixed point of operator $A$, then $u$ is said to be a positive fixed point of operator $A$. If $u \in(P \backslash\{\theta\})$ is a fixed point of operator $A$, then $u$ is said to be a negative fixed point of operator $A$. If $u \notin(P \cup(-P))$ is a fixed point of operator $A$, then $u$ is said to be a sign-changing fixed point of operator $A$.

In [30], Sun and Liu considered computation for the topological degree about superlinear operators which are not cone mappings and obtained the following results.

Lemma 3. Let the cone $P \subset E$ be solid, and $A: E \longrightarrow E$ be a completely continuous operator, and $A=B F$, where $B$ is
a positive completely continuous linear operator satisfying H condition and $F$ is quasi-additive on lattice. Assume that
(i) there exist $c_{1}>r^{-1}(B)$ and $u_{1} \in P$ such that

$$
\begin{equation*}
F u \geq c_{1} u-u_{1}, \quad \forall u \in P \tag{12}
\end{equation*}
$$

(ii) there exist $0<c_{2}<r^{-1}(B)$ and $u_{2} \in P$ such that

$$
\begin{equation*}
F u \geq c_{2} u-u_{2}, \quad \forall u \in(-P) \tag{13}
\end{equation*}
$$

(iii) $A \theta=\theta$, the Fréchet derivative $A_{\theta}^{\prime}$ of $A$ at $\theta$ exists, and 1 is not an eigenvalue of $A_{\theta}^{\prime}$.

Then the operator A has at least one nonzero fixed point.
In [31], Sun further obtained the following result about the existence of sign-changing fixed points for superlinear operators.

Lemma 4. Let the conditions in Lemma 3 hold, and $\beta$ denote the sum of the algebraic multiplicities for all eigenvalues of $A_{\theta}^{\prime}$ lying in $(1,+\infty)$. In addition, assume that
(iv) $\beta \neq 0, \beta$ is an even number;
(v) $A(P \backslash\{\theta\}) \subset \operatorname{int} P, A((-P) \backslash\{\theta\}) \subset \operatorname{int}(-P)$.

Then the operator $A$ has at least one negative fixed point and one sign-changing fixed point.

## 3. Main Results

For convenience, we list the following conditions.
$\left(\mathrm{C}_{1}\right) \varphi:[0,1] \times R^{1} \longrightarrow R^{1}$ is continuous, $\varphi(t, 0)=0, \forall t \in$ [0, 1].
$\left(\mathrm{C}_{2}\right)$ The sequence of positive solutions of the equation

$$
\begin{equation*}
\sin \sqrt{y}=\sum_{i=1}^{m-2} \beta_{i} \sin \left(\alpha_{i} \sqrt{y}\right) \tag{14}
\end{equation*}
$$

is

$$
\begin{equation*}
0<\lambda_{1}<\lambda_{2}<\cdots<\lambda_{n}<\lambda_{n+1}<\cdots \tag{15}
\end{equation*}
$$

$\left(\mathrm{C}_{3}\right) \lim _{x \rightarrow 0}(\varphi(t, x) / x)=\eta$ uniformly on $t \in[0,1]$.
Let $X=C[0,1]$ with supremum norm $\|x\|=$ $\sup _{0 \leq t \leq 1}|x(t)|$. Set $P=\{x \in X \mid x(t) \geq 0, t \in[0,1]\}$, the $P$ is a solid cone in $X$. And under the partial order $\leq$ which is induced by $P, X$ is a lattice.

In the following, we define some operators $A, B$, and $\Phi$ :

$$
\begin{align*}
& (A x)(t)=\int_{0}^{1} K(t, s) \varphi(s, x(s)) d s, \quad t \in[0,1]  \tag{16}\\
& (B x)(t)=\int_{0}^{1} K(t, s) x(s) d s, \quad t \in[0,1]  \tag{17}\\
& (\Phi x)(t)=\varphi(t, x(t)), \quad t \in[0,1] \tag{18}
\end{align*}
$$

where

$$
\begin{align*}
& K(t, s)=g(t, s)+\frac{t \sum_{i=1}^{m-2} \beta_{i} g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}  \tag{19}\\
& g(t, s)= \begin{cases}t(1-s), & 0 \leq t \leq s \leq 1 \\
s(1-t), & 0 \leq s \leq t \leq 1\end{cases} \tag{20}
\end{align*}
$$

Obviously, $A=B \Phi$, and the nontrivial fixed points of the operator $A$ are nontrivial solutions of the boundary value problem (1), (2) (see [3]).

Lemma 5 (see [2]). Let $\mu$ be a positive number, and the linear operator B be defined by (17). Eigenvalues of the linear operator $\mu B$ are

$$
\begin{equation*}
\frac{\mu}{\lambda_{1}}, \frac{\mu}{\lambda_{2}}, \cdots, \frac{\mu}{\lambda_{n}}, \cdots \tag{21}
\end{equation*}
$$

and algebraic multiplicity of $\mu / \lambda_{n}$ is equal to 1 , where $\lambda_{n}$ is defined by $\left(\mathrm{C}_{2}\right)$.

Lemma 6. The linear operator B satisfies H condition.
Proof. By $\left(\mathrm{C}_{2}\right)$, Lemma 5, and the definition of the spectral radius, we know that

$$
\begin{equation*}
r(B)=\sup _{\lambda \in\left\{1 / \lambda_{n}, n=1,2, \cdots\right\}}|\lambda|=\frac{1}{\lambda_{1}}>0 \tag{22}
\end{equation*}
$$

By (20), we have

$$
\begin{align*}
& g\left(\alpha_{i}, s\right) \geq \alpha_{i}\left(1-\alpha_{i}\right) s(1-s), \quad \forall s \in[0,1]  \tag{23}\\
& g(t, s) \leq s(1-s), \quad \forall t, s \in[0,1] \tag{24}
\end{align*}
$$

By (19) and (23), we have

$$
\begin{align*}
K(t, s) & \geq \frac{t \sum_{i=1}^{m-2} \beta_{i} g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}} \\
& \geq \frac{t \sum_{i=1}^{m-2} \beta_{i} \alpha_{i}\left(1-\alpha_{i}\right) s(1-s)}{1+\sum_{i=1}^{m-2} \beta_{i}\left(1-\alpha_{i}\right)} \tag{25}
\end{align*}
$$

$$
\forall t, s \in[0,1]
$$

From (24) and (25), we have

$$
\begin{equation*}
K(t, s) \geq \frac{t \sum_{i=1}^{m-2} \beta_{i} \alpha_{i}\left(1-\alpha_{i}\right)}{1+\sum_{i=1}^{m-2} \beta_{i}\left(1-\alpha_{i}\right)} g(\tau, s) \tag{26}
\end{equation*}
$$

$$
\forall \tau, t, s \in[0,1]
$$

By (19), we have

$$
\begin{array}{r}
K(t, s) \geq \frac{t \sum_{i=1}^{m-2} \beta_{i} g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}} \geq \frac{t \sum_{i=1}^{m-2} \beta_{i} \tau g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}} \\
\geq \frac{t \sum_{i=1}^{m-2} \beta_{i} \alpha_{i}\left(1-\alpha_{i}\right)}{1+\sum_{i=1}^{m-2} \beta_{i}\left(1-\alpha_{i}\right)} \frac{\sum_{i=1}^{m-2} \beta_{i} \tau g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}  \tag{27}\\
\forall \tau, t, s \in[0,1] .
\end{array}
$$

Hence, by adding (26) to (27), we have

$$
\begin{align*}
& 2 K(t, s) \geq t \frac{\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}\left(1-\alpha_{i}\right)}{1+\sum_{i=1}^{m-2} \beta_{i}\left(1-\alpha_{i}\right)}(g(\tau, s) \\
& \left.\quad+\frac{\sum_{i=1}^{m-2} \beta_{i} \tau g\left(\alpha_{i}, s\right)}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}\right) \tag{28}
\end{align*}
$$

i.e.,

$$
\begin{equation*}
K(t, s) \geq M t K(\tau, s), \quad \forall \tau, t, s \in[0,1] \tag{29}
\end{equation*}
$$

where $M=\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}\left(1-\alpha_{i}\right) / 2\left(1+\sum_{i=1}^{m-2} \beta_{i}\left(1-\alpha_{i}\right)\right)$.
Let

$$
\begin{equation*}
\left(B^{*} x\right)(t)=\int_{0}^{1} K^{*}(t, s) x(s) d s, \quad \forall t \in[0,1] \tag{30}
\end{equation*}
$$

where $K^{*}(t, s)=K(s, t)$. Obviously, $r\left(B^{*}\right)=r(B)=1 / \lambda_{1}>$ 0 . By Krein-Rutman theorem, there exist $x(t) \in P \backslash\{\theta\}$ and $x^{*}(t) \in P \backslash\{\theta\}$ such that

$$
\begin{align*}
(B x)(t) & =r(B) x(t)  \tag{31}\\
\left(B^{*} x^{*}\right)(t) & =r(B) x^{*}(t) . \tag{32}
\end{align*}
$$

By (29) and (32), we obtain

$$
\begin{align*}
x^{*}(s) & =r^{-1}(B)\left(B^{*} x^{*}\right)(s) \\
& =r^{-1}(B) \int_{0}^{1} K^{*}(s, t) x^{*}(t) d t \\
& =r^{-1}(B) \int_{0}^{1} K(t, s) x^{*}(t) d t  \tag{33}\\
& \geq M r^{-1}(B) \int_{0}^{1} t K(\tau, s) x^{*}(t) d t \\
& =\left[M r^{-1}(B) \int_{0}^{1} t x^{*}(t) d t\right] K(\tau, s),
\end{align*}
$$

$$
\forall \tau, s \in[0,1] .
$$

Set

$$
\begin{equation*}
f^{*}(u)=\int_{0}^{1} x^{*}(t) u(t) d t, \quad \forall u \in X, t \in[0,1] . \tag{34}
\end{equation*}
$$

Obviously, $f^{*} \in P^{*} \backslash\{\theta\}$, and by (34), for $u \in X$, we have

$$
\begin{align*}
f^{*}(B u) & =\int_{0}^{1} x^{*}(t)(B u)(t) d t \\
& =\int_{0}^{1} x^{*}(t) d t \int_{0}^{1} K(t, s) u(s) d s \\
& =\int_{0}^{1} \int_{0}^{1} K(t, s) x^{*}(t) u(s) d t d s  \tag{35}\\
& =\int_{0}^{1}\left(\int_{0}^{1} K^{*}(s, t) x^{*}(t) d t\right) u(s) d s \\
& =\int_{0}^{1} r(B) x^{*}(s) u(s) d s=r(B) f^{*}(u) .
\end{align*}
$$

That is,

$$
\begin{equation*}
B^{*} f^{*}=r(B) f^{*} \tag{36}
\end{equation*}
$$

From (33) and (35), we have

$$
\begin{align*}
f^{*}(B u) & =r(B) \int_{0}^{1} x^{*}(s) u(s) d s \\
& \geq M \int_{0}^{1} x^{*}(t) t d t \int_{0}^{1} K(\tau, s) u(s) d s  \tag{37}\\
& =\left(M \int_{0}^{1} t x *(t) d t\right)(B u)(\tau)
\end{align*}
$$

$\forall \tau \in[0,1]$.
By (37), we have

$$
\begin{equation*}
f^{*}(B u) \geq \delta\|B u\| \tag{38}
\end{equation*}
$$

where $\delta=M \int_{0}^{1} t x^{*}(t) d t>0$.
Therefore, from (31), (36), and (38), it is easy to know that the linear operator $B$ satisfies H condition.

Theorem 7. Suppose that $\left(\mathrm{C}_{1}\right),\left(\mathrm{C}_{2}\right)$, and $\left(\mathrm{C}_{3}\right)$ hold. In addition, assume that there exists $\gamma>0$ such that

$$
\begin{align*}
& \liminf _{x \rightarrow+\infty} \frac{\varphi(t, x)}{x} \geq \lambda_{1}+\gamma, \quad \text { uniformly on } t \in[0,1]  \tag{39}\\
& \limsup _{x \rightarrow-\infty} \frac{\varphi(t, x)}{x} \leq \lambda_{1}-\gamma, \quad \text { uniformly on } t \in[0,1] \tag{40}
\end{align*}
$$

If $\eta \neq \lambda_{1}, \lambda_{2}, \cdots, \lambda_{n}, \cdots$, where $\lambda_{i}$ is defined by $\left(\mathrm{C}_{2}\right)$, then the boundary value problem (1), (2) has at least one nontrivial solution.

Proof. By $\left(\mathrm{C}_{1}\right)$, we easily know that $A: X \longrightarrow X$ is a completely continuous operator, and $B: X \longrightarrow X$ is a bounded positive linear completely continuous operator (see [3]). By Lemma 6, we know that the linear operator $B$ satisfies H condition.

For $x \in X$, let

$$
\begin{align*}
& x_{+}(t)= \begin{cases}x(t), & x(t) \geq 0 \\
0, & x(t)<0\end{cases}  \tag{41}\\
& x_{-}(t)= \begin{cases}x(t), & x(t) \leq 0 \\
0, & x(t)>0\end{cases}
\end{align*}
$$

and then $x(t)=x_{+}(t)+x_{-}(t)$.
By $\varphi(t, 0)=0$, we have

$$
\begin{align*}
\Phi(x) & =\varphi(t, x(t))=\varphi\left(t, x_{+}(t)+x_{-}(t)\right) \\
& =\varphi\left(t, x_{+}(t)\right)+\varphi\left(t, x_{-}(t)\right)  \tag{42}\\
& =\Phi\left(x_{+}\right)+\Phi\left(x_{-}\right)
\end{align*}
$$

From (42), we know that $\Phi$ is quasi-additive on lattice.
From (39) and (40), there exists $C>0$ such that

$$
\begin{align*}
& \frac{\varphi(t, x)}{x} \geq \lambda_{1}+\frac{\gamma}{4}, \quad \forall x \geq C, t \in[0,1]  \tag{43}\\
& \frac{\varphi(t, x)}{x} \leq \lambda_{1}-\frac{\gamma}{4}, \quad \forall x \leq-C, t \in[0,1] \tag{44}
\end{align*}
$$

By (43) and (44), we have

$$
\begin{align*}
& \varphi(t, x) \geq\left(\lambda_{1}+\frac{\gamma}{4}\right) x, \quad \forall x \geq C, t \in[0,1]  \tag{45}\\
& \varphi(t, x) \geq\left(\lambda_{1}-\frac{\gamma}{4}\right) x, \quad \forall x \leq-C, t \in[0,1] \tag{46}
\end{align*}
$$

Let $\widetilde{C}=\max _{0 \leq t \leq 1,|x| \leq C}|\varphi(t, x)|$. Then by (45) and (46), we have

$$
\begin{align*}
& \varphi(t, x) \geq\left(\lambda_{1}+\frac{\gamma}{4}\right) x-\widetilde{C}, \quad \forall x \geq 0, t \in[0,1] \\
& \varphi(t, x) \geq\left(\lambda_{1}-\frac{\gamma}{4}\right) x-\widetilde{C}, \quad \forall x \leq 0, t \in[0,1] \tag{47}
\end{align*}
$$

i.e.,

$$
\begin{array}{ll}
\Phi x \geq h_{1} x-\widetilde{C}, & \forall x \in P \\
\Phi x \geq h_{2} x-\widetilde{C}, & \forall x \in(-P) \tag{48}
\end{array}
$$

where $h_{1}=\lambda_{1}+\gamma / 4, h_{2}=\lambda_{1}-\gamma / 4$. Obviously, we have

$$
\begin{align*}
& h_{1}>r^{-1}(B), \\
& h_{2}<r^{-1}(B) . \tag{49}
\end{align*}
$$

In the following, we prove that $A_{\theta}^{\prime}=\eta B$.
In fact, by $\varphi(t, 0)=0, \forall t \in[0,1]$, we have $A \theta=\theta$. From $\left(\mathrm{C}_{3}\right), \forall \epsilon>0, \exists \delta>0$, when $0<|x|<\delta$, we have

$$
\begin{equation*}
\left|\frac{\varphi(t, x)}{x}-\eta\right|<\epsilon \tag{50}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
|\varphi(t, x)-\eta x|<\epsilon|x|, \quad \forall t \in[0,1], \quad 0<|x|<\delta \tag{51}
\end{equation*}
$$

So

$$
\begin{equation*}
\|\Phi x-\eta x\| \leq \epsilon\|x\|, \quad \forall\|x\|<\delta \tag{52}
\end{equation*}
$$

Therefore, by (52), we have

$$
\begin{align*}
&\|A x-A \theta-\eta B x\|=\|B(\Phi x-\eta x)\| \\
& \leq\|B\| \cdot\|(\Phi x-\eta x)\| \leq \epsilon\|B\| \cdot\|x\|  \tag{53}\\
& \forall\|x\|<\delta .
\end{align*}
$$

So

$$
\begin{equation*}
\lim _{\|x\| \longrightarrow 0} \frac{\|A x-A \theta-\eta B x\|}{\|x\|}=0 \tag{54}
\end{equation*}
$$

i.e.,

$$
\begin{equation*}
A_{\theta}^{\prime}=\eta B \tag{55}
\end{equation*}
$$

Since $r\left(A_{\theta}^{\prime}\right)=\eta r(B)$, we know that 1 is not the eigenvalue of $A_{\theta}^{\prime}$ by Lemma 6 and $\left(\mathrm{C}_{2}\right)$.

By the above proof, we know that the conditions of Lemma 3 hold. So by Lemma 3, the boundary value problem (1), (2) has at least one nontrivial solution.

Theorem 8. Assume that $\left(\mathrm{C}_{1}\right)-\left(\mathrm{C}_{3}\right)$, (39), and (40) are satisfied. In addition, suppose that $\varphi(t, x) x>0, \forall t \in[0,1], x \neq 0$, and $\lambda_{2 n_{0}}<\eta<\lambda_{2 n_{0}+1}$, where $n_{0}$ is a natural number. Then the boundary value problem (1), (2) has at least one negative solution and one sign-changing solution.

Proof. By (17), for $\forall x \in P \backslash\{\theta\}$, we have

$$
\begin{align*}
(B x)(t)= & \int_{0}^{1} K(t, s) x(s) d s \\
= & \int_{0}^{1} g(t, s) x(s) d s \\
& +\frac{t \sum_{i=1}^{m-2} \beta_{i} \int_{0}^{1} g\left(\alpha_{i}, s\right) x(s) d s}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}} \\
\leq & t(1-t) \int_{0}^{1} x(s) d s  \tag{56}\\
& +\frac{t \sum_{i=1}^{m-2} \beta_{i}}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}} \int_{0}^{1} x(s) d s \\
= & {\left[t(1-t)+\frac{t \sum_{i=1}^{m-2} \beta_{i}}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}\right] \int_{0}^{1} x(s) d s } \\
(B x)(t)= & \int_{0}^{1} g(t, s) x(s) d s \\
& +\frac{t \sum_{i=1}^{m-2} \beta_{i} \int_{0}^{1} g\left(\alpha_{i}, s\right) x(s) d s}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}  \tag{57}\\
\geq & \frac{t \sum_{i=1}^{m-2} \beta_{i} \int_{0}^{1} g\left(\alpha_{i}, s\right) x(s) d s}{1-\sum_{i=1}^{m-2} \beta_{i} \alpha_{i}}
\end{align*}
$$

From (56) and (57), we obtain that

$$
\begin{equation*}
B(P \backslash\{\theta\}) \subset \operatorname{int} P \tag{58}
\end{equation*}
$$

Similarly, we know that

$$
\begin{equation*}
B(-P \backslash\{\theta\}) \subset \operatorname{int}(-P) \tag{59}
\end{equation*}
$$

Since $\varphi(t, x) x>0, \forall t \in[0,1], x \neq 0$, we have $\varphi(t, x)>$ $0, \forall x>0, t \in[0,1]$, and $\varphi(t, x)<0, \forall x<0, t \in[0,1]$. So we have

$$
\begin{align*}
& (\Phi x) \in P \backslash\{\theta\}, \quad \forall x \in P \backslash\{\theta\}  \tag{60}\\
& (\Phi x) \in(-P) \backslash\{\theta\}, \quad \forall x \in(-P) \backslash\{\theta\} \tag{61}
\end{align*}
$$

By (58)-(61), we have

$$
\begin{gather*}
A(P \backslash\{\theta\}) \subset \operatorname{int} P, \\
A((-P) \backslash\{\theta\}) \subset \operatorname{int}(-P) . \tag{62}
\end{gather*}
$$

Let $\beta$ be the sum of algebraic multiplicities for all the eigenvalues of $A_{\theta}^{\prime}$, lying in the interval $(1, \infty)$. By (55), Lemma 5, and $\lambda_{2 n_{0}}<\eta<\lambda_{2 n_{0}+1}$, we know that

$$
\begin{equation*}
\beta=2 n_{0} \tag{63}
\end{equation*}
$$

By (62) and (63), we know that the conditions (iv) and (v) in Lemma 4 hold. By the proof of Theorem 7, the conditions (i), (ii), and (iii) in Lemma 4 are satisfied. Therefore, by Lemma 4, the boundary value problem (1), (2) has at least one negative solution and one sign-changing solution.

## 4. Examples

We consider second-order four-point boundary value problem

$$
\begin{align*}
-x^{\prime \prime}(t) & =\varphi(t, x(t)), \quad 0 \leq t \leq 1, \\
x(0) & =0,  \tag{64}\\
x(1) & =\frac{1}{3} x\left(\frac{1}{3}\right)+\frac{1}{2} x\left(\frac{1}{2}\right) .
\end{align*}
$$

By simple calculations, $\lambda_{1} \approx 5.602, \lambda_{2} \approx 42.32, \lambda_{3} \approx$ 99.97, and $\lambda_{4} \approx 148.87$ are solutions of the equation

$$
\begin{equation*}
\sin \sqrt{x}=\frac{1}{3} \sin \frac{\sqrt{x}}{3}+\frac{1}{2} \sin \frac{\sqrt{x}}{2} . \tag{65}
\end{equation*}
$$

Example 1. Choose

$$
\begin{align*}
& \varphi(t, x) \\
& = \begin{cases}8 x+(t-1) \sqrt{x}, & t \in[0,1], x \in[4,+\infty) \\
\frac{30+5 t}{3}(x-1)-3 t, & t \in[0,1], x \in(1,4) \\
3 x-3(1+t) x^{2}, & t \in[0,1], x \in[-1,1] \\
\frac{8-t}{7}(x+1)-(6+3 t), & t \in[0,1], x \in(-8,-1) \\
2 x+(t-1) \sqrt[3]{x}, & t \in[0,1], x \in(-\infty,-8]\end{cases} \tag{66}
\end{align*}
$$

By (66), it is easy to know that $\varphi:[0,1] \times(-\infty,+\infty) \longrightarrow$ $(-\infty,+\infty)$ is continuous, and $\varphi(t, 0)=0, \forall t \in[0,1]$. By calculation, $\eta=3<\lambda_{1}$. We can choose $\gamma=2$. Then we have

$$
\begin{align*}
& \liminf _{x \rightarrow+\infty} \frac{\varphi(t, x)}{x}=8 \geq \lambda_{1}+\gamma  \tag{67}\\
& \limsup _{x \rightarrow-\infty} \frac{\varphi(t, x)}{x}=2 \leq \lambda_{1}-\gamma .
\end{align*}
$$

So by Theorem 7, the boundary value problem (64) has at least one nontrivial solution.

Example 2. Choose

$$
\begin{align*}
& \varphi(t, x) \\
& = \begin{cases}10 x+(1-t) \sqrt[3]{x}, & t \in[0,1], x \in[8,+\infty), \\
\frac{31-3 t}{7}(x-1)+(51+t), & t \in[0,1], x \in(1,8), \\
50 x+(1+t) x^{5 / 3}, & t \in[0,1], x \in[-1,1], \\
\frac{-21-4 t}{26}(x+1)-(51+t), & t \in[0,1], x \in(-27,-1), \\
x+(1-t) \sqrt[3]{x}, & t \in[0,1], x \in(-\infty,-27] .\end{cases} \tag{68}
\end{align*}
$$

By (68), we know that $\varphi:[0,1] \times(-\infty,+\infty) \longrightarrow$ $(-\infty,+\infty)$ is continuous, $\varphi(t, 0)=0, \forall t \in[0,1]$, and
$\varphi(t, x) x>0, \forall t \in[0,1], x \neq 0$. By calculation, $\lambda_{2} \leq \eta=$ $50<\lambda_{3}$. We can choose $\gamma=4$. Then we have

$$
\begin{align*}
& \liminf _{x \rightarrow+\infty} \frac{\varphi(t, x)}{x}=10 \geq \lambda_{1}+\gamma \\
& \limsup _{x \rightarrow-\infty} \frac{\varphi(t, x)}{x}=1 \leq \lambda_{1}-\gamma \tag{69}
\end{align*}
$$

So by Theorem 8, the boundary value problem (64) has at least one negative solution and one sign-changing solution.

Example 3. Choose

$$
\begin{align*}
& \varphi(t, x) \\
& = \begin{cases}\frac{45}{32} x^{2}+(1-t) \sqrt[3]{x}, & t \in[0,1], x \in[8,+\infty), \\
\frac{31-3 t}{7}(x-1)+(61+t), & t \in[0,1], x \in(1,8), \\
60 x+(1+t) x^{5 / 3}, & t \in[0,1], x \in[-1,1], \\
\frac{-31-4 t}{26}(x+1)-(61+t), & t \in[0,1], x \in(-27,-1), \\
x+(1-t) \sqrt[3]{x}, & t \in[0,1], x \in(-\infty,-27] .\end{cases} \tag{70}
\end{align*}
$$

By (70), we know that $\varphi:[0,1] \times(-\infty,+\infty) \longrightarrow$ $(-\infty,+\infty)$ is continuous, $\varphi(t, 0)=0, \forall t \in[0,1]$, and $\varphi(t, x) x>0, \forall t \in[0,1], x \neq 0$. By calculation, $\lambda_{2} \leq \eta=$ $60<\lambda_{3}$. We can choose $\gamma=3$. Then we have

$$
\begin{align*}
& \liminf _{x \rightarrow+\infty} \frac{\varphi(t, x)}{x}=+\infty \geq \lambda_{1}+\gamma \\
& \limsup _{x \rightarrow-\infty} \frac{\varphi(t, x)}{x}=1 \leq \lambda_{1}-\gamma \tag{71}
\end{align*}
$$

So by Theorem 8, the boundary value problem (64) has at least one negative solution and one sign-changing solution.

## Data Availability

The date underlying the findings of our manuscript can be obtained by simple calculations. We did not quote other data.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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# Research Article 

# Existence of Uniqueness and Nonexistence Results of Positive Solution for Fractional Differential Equations Integral Boundary Value Problems 

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#### Abstract

In this paper, we consider a class of fractional differential equations with conjugate type integral conditions. Both the existence of uniqueness and nonexistence of positive solution are obtained by means of the iterative technique. The interesting point lies in that the assumption on nonlinearity is closely associated with the spectral radius corresponding to the relevant linear operator.


## 1. Introduction

In this paper, we consider the existence of uniqueness and nonexistence of positive solution for the following fractional differential equations:

$$
\begin{align*}
D_{0+}^{\alpha} u(t)+f(t, u(t)) & =0, \quad 0<t<1, \\
u(0) & =u^{\prime}(0)=\cdots=u^{(n-2)}(0)=0,  \tag{1}\\
D_{0+}^{\beta} u(1) & =\int_{0}^{\eta} a(t) D_{0+}^{\gamma} u(t) d t,
\end{align*}
$$

where $n-1<\alpha \leq n, n \geq 3,0<\beta<1, \Gamma(\alpha-$ $\beta) \int_{0}^{\eta} a(t) t^{\alpha-\gamma-1} d t<\Gamma(\alpha-\gamma), \eta \in(0,1], D_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, $f:(0,1) \times[0,+\infty) \longrightarrow[0$, $+\infty)$ is continuous, and $a(t) \in L^{1}[0,1]$ is nonnegative.

In the recent years, many results were obtained to deal with the existence of solutions for nonlinear differential equations by using nonlinear analysis methods; see [1-16] and references therein. The fractional nonlocal boundary value problems have particularly attracted a great deal of attention (see [17-27]). While there are a lot of works dealing with the existence and multiplicity of solutions for nonlinear fractional differential equations, the results dealing with the uniqueness of solution are relatively scarce (see [28-35]). The main tool used in most of the papers dealing with the uniqueness of solution is the Banach contraction map
principle provided that the nonlinearity $f$ is a Lipschitz continuous function. When $1 \leq \beta<\alpha-1$, and $f$ is continuous on $[0,1] \times(-\infty,+\infty)$, Zhang and Zhong [34] established the uniqueness results of solution to problem (1) by using the Banach contraction map principle. It is worth mentioning that only positive solutions are meaningful in most practical problems. As far as we know, the nonexistence of positive solution has seldom been considered up to now.

Motivated by the above work, the aim of this paper is to establish the existence of uniqueness and nonexistence of positive solution to problem (1). Our analysis relies on the iterative technique on the cone derived from the properties of the Green function. This article provides some new insights. Firstly, the uniqueness results are obtained under some conditions concerning the spectral radius with respect to the relevant linear operator. In addition, the error estimation of the iterative sequences is given. Secondly, we impose weaker positivity conditions on $f$; that is, the Lipschitz constant is generalized to a function and $f(t, x)$ may be singular at $t=$ 0,1 . Finally, the nonexistence results of positive solution are obtained under conditions concerning the spectral radius of the relevant linear operator.

## 2. Preliminaries

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory and lemmas.

Definition 1. The fractional integral of order $\alpha>0$ of a function $u:(0,+\infty) \longrightarrow R$ is given by

$$
\begin{equation*}
I_{0+}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s \tag{2}
\end{equation*}
$$

provided that the right-hand side is point-wise defined on ( $0,+\infty$ ).

Definition 2. The Riemann-Liouville fractional derivative of order $\alpha>0$ of a function $u:(0,+\infty) \longrightarrow R$ is given by

$$
\begin{equation*}
D_{0+}^{\alpha} u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t}(t-s)^{n-\alpha-1} u(s) d s \tag{3}
\end{equation*}
$$

where $n=[\alpha]+1 ;[\alpha]$ denotes the integer part of number $\alpha$, provided that the right-hand side is point-wise defined on $(0,+\infty)$.

For convenience, we here list the assumptions to be used throughout the paper.
$\left(A_{1}\right) a(t) \in L^{1}[0,1]$ is nonnegative, and

$$
\begin{gather*}
\Delta:=\Gamma(\alpha-\gamma)-\Gamma(\alpha-\beta) \int_{0}^{\eta} a(t) t^{\alpha-\gamma-1} d t>0  \tag{4}\\
\left(A_{2}\right) f:(0,1) \times[0,+\infty) \longrightarrow[0,+\infty) \text { is continuous. }
\end{gather*}
$$

Lemma 3 ([34]). For any $y \in L[0,1] \cap C(0,1)$, the unique solution of the boundary value problem

$$
\begin{align*}
D_{0+}^{\alpha} u(t)+y(t) & =0, \quad 0<t<1, \\
u(0) & =u^{\prime}(0)=\cdots=u^{(n-2)}(0)=0,  \tag{5}\\
D_{0+}^{\beta} u(1) & =\int_{0}^{\eta} a(t) D_{0+}^{\gamma} u(t),
\end{align*}
$$

is

$$
\begin{equation*}
u(t)=\int_{0}^{1} G(t, s) y(s) d s \tag{6}
\end{equation*}
$$

where

$$
\begin{aligned}
& G(t, s)=G_{1}(t, s)+h(s) t^{\alpha-1}, \\
& G_{1}(t, s)=\frac{1}{\Gamma(\alpha)} \\
& \quad \cdot \begin{cases}t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\
t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1,\end{cases} \\
& G_{2}(t, s)=\frac{1}{\Gamma(\alpha)} \\
& \qquad \begin{cases}t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1, \\
t^{\alpha-\gamma-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-\gamma-1}, & 0 \leq s \leq t \leq 1,\end{cases} \\
& h(s)=\frac{\Gamma(\alpha-\gamma)}{\Delta} \int_{0}^{\eta} a(t) G_{2}(t, s) d t .
\end{aligned}
$$

Lemma 4. The function $G_{1}(t, s)$ has the following properties:
(1) $G_{1}(t, s)>0, \forall t, s \in(0,1)$;
(2) $\Gamma(\alpha) G_{1}(t, s) \leq t^{\alpha-1}(1-s)^{\alpha-\beta-1}, \forall t, s \in[0,1]$;
(3) $\beta s(1-s)^{\alpha-\beta-1} t^{\alpha-1} \leq \Gamma(\alpha) G_{1}(t, s) \leq s(1-s)^{\alpha-\beta-1}$, $\forall t, s \in[0,1]$.

Proof. It is obvious that (1) and (2) hold. In the following, we will prove (3).

Case (i) ( $0<s \leq t<1$ ). Noticing $\alpha>2$, we have

$$
\begin{align*}
& \frac{\partial}{\partial t}\left[t^{\alpha-1}-\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right] \\
& \quad=(\alpha-1) t^{\alpha-2}\left[1-\left(\frac{t-s}{t(1-s)}\right)^{\alpha-2}\right] \geq 0 \tag{8}
\end{align*}
$$

which implies

$$
\begin{equation*}
t^{\alpha-1}-\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}} \leq 1-(1-s)=s \tag{9}
\end{equation*}
$$

Noticing $0<\beta<1$, we have

$$
\begin{align*}
& t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1} \\
& \quad=(1-s)^{\alpha-\beta-1}\left[t^{\alpha-1}-\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-\beta-1}}\right] \\
& \quad \leq(1-s)^{\alpha-\beta-1}\left[t^{\alpha-1}-\frac{(t-s)^{\alpha-1}}{(1-s)^{\alpha-2}}\right]  \tag{10}\\
& \quad \leq s(1-s)^{\alpha-\beta-1} .
\end{align*}
$$

On the other hand, it follows from

$$
\begin{equation*}
\frac{\partial}{\partial s}\left[\beta s+(1-s)^{\beta}\right] \leq 0, \quad \forall s \in[0,1) \tag{11}
\end{equation*}
$$

that

$$
\begin{equation*}
\beta s+(1-s)^{\beta} \leq 1, \quad \forall s \in[0,1] . \tag{12}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\alpha-1} \\
& \quad \geq t^{\alpha-1}(1-s)^{\alpha-\beta-1}-(t-s)^{\beta}(t-t s)^{\alpha-\beta-1} \\
& \quad=t^{\alpha-1}\left[1-\left(1-\frac{s}{t}\right)^{\beta}\right](1-s)^{\alpha-\beta-1}  \tag{13}\\
& \quad \geq t^{\alpha-1}\left[1-(1-s)^{\beta}\right](1-s)^{\alpha-\beta-1} \\
& \quad \geq \beta s(1-s)^{\alpha-\beta-1} t^{\alpha-1} .
\end{align*}
$$

Case (ii) ( $0 \leq t \leq s \leq 1$ ). It is easy to see that

$$
\begin{equation*}
t^{\alpha-1}(1-s)^{\alpha-\beta-1} \leq s^{\alpha-1}(1-s)^{\alpha-\beta-1} \leq s(1-s)^{\alpha-\beta-1} . \tag{14}
\end{equation*}
$$

On the other hand, we have

$$
\begin{align*}
t^{\alpha-1}(1-s)^{\alpha-\beta-1} & \geq s t^{\alpha-1}(1-s)^{\alpha-\beta-1} \\
& \geq \beta s(1-s)^{\alpha-\beta-1} t^{\alpha-1} \tag{15}
\end{align*}
$$

It follows from (10), (13), (14), and (15) that (3) holds.
Lemma 5. The function $G(t, s)$ has the following properties:
(1) $G(t, s)>0, \forall t, s \in(0,1)$;
(2) $G(t, s) \leq t^{\alpha-1} \Phi_{1}(s), \forall t, s \in[0,1]$;
(3) $\beta t^{\alpha-1} \Phi_{2}(s) \leq G(t, s) \leq \Phi_{2}(s), \forall t, s \in[0,1]$,
where

$$
\begin{align*}
& \Phi_{1}(s)=\frac{(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s) \\
& \Phi_{2}(s)=\frac{s(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha)}+h(s) \tag{16}
\end{align*}
$$

Proof. It can be directly deduced from Lemma 4 and the definition of $G(t, s)$, so we omit the proof.

Let $E=C[0,1]$ be endowed with the maximum norm $\|u\|=\max _{0 \leq t \leq 1}|u(t)|, B_{r}=\{u \in E:\|u\|<r\}$. Define cones $P$, Q by

$$
\begin{align*}
P & =\{u \in E: u(t) \geq 0\}, \\
Q & =\left\{u \in P: \text { there exists } l_{u}>0 \text { such that } \beta\|u\| t^{\alpha-1}\right.  \tag{17}\\
& \left.\leq u(t) \leq l_{u} t^{\alpha-1}\right\} .
\end{align*}
$$

It is clear that $Q$ is nonempty set since $t^{\alpha-1} \in Q$.
For convenience, we list here one more assumption to be used later:
$\left(A_{3}\right)$ There exists $\lambda \in C(0,1) \cap L[0,1]$ such that

$$
\begin{align*}
& |f(t, x)-f(t, y)| \leq \lambda(t)|x-y| \\
& \quad t \in[0,1], x, y \in[0, \infty) \tag{18}
\end{align*}
$$

Moreover,

$$
\begin{align*}
& 0<\int_{0}^{1} \Phi_{1}(s) \lambda(s) d s<+\infty  \tag{19}\\
& 0<\int_{0}^{1} \Phi_{1}(s) f(s, 0) d s<+\infty
\end{align*}
$$

Define operators $A$ and $T$ as follows:

$$
\begin{align*}
A u(t) & =\int_{0}^{1} G(t, s) f(s, u(s)) d s \\
T_{\lambda} u(t) & =\int_{0}^{1} G(t, s) \lambda(s) u(s) d s \tag{20}
\end{align*}
$$

Lemma 6. Assume that $\left(A_{3}\right)$ holds; then $A: P \longrightarrow Q$.

Proof. It is clear that $f(t, x) \leq \lambda(t) x+f(t, 0), \forall x \geq 0$. For any $u \in P$, we have

$$
\begin{align*}
|A u(t)|= & \int_{0}^{1} G(t, s) f(s, u(s)) d s \\
\leq & \int_{0}^{1} \Phi_{1}(s) \lambda(s)\|u\| d s  \tag{21}\\
& +\int_{0}^{1} \Phi_{1}(s) f(s, 0) d s
\end{align*}
$$

Then $A$ is well-defined on $P$. It follows from Lemma 5 that $A: P \longrightarrow Q$.

By virtue of the Krein-Rutmann theorem and Lemma 5, we have the following lemma.

Lemma 7. Assume that $\left(A_{3}\right)$ holds. Then $T_{\lambda}: P \longrightarrow Q$ is a completely continuous linear operator. Moreover, the spectral radius $r\left(T_{\lambda}\right)>0$ and $T_{\lambda}$ has a positive eigenfunction $\varphi_{1}$ corresponding to its first eigenvalue $\left(r\left(T_{\lambda}\right)\right)^{-1}$; that is, $T_{\lambda} \varphi_{1}=$ $r\left(T_{\lambda}\right) \varphi_{1}$.

## 3. Main Results

Theorem 8. Assume that $\left(A_{3}\right)$ holds. Then (1) has a unique positive solution if the spectral radius $r\left(T_{\lambda}\right) \in(0,1)$.

Proof. It follows from $0<\int_{0}^{1} \Phi_{1}(s) f(s, 0) d s<+\infty$ that $\theta$ is not a fixed point of $A$. Then we only need to prove that $A$ has a unique fixed point in $Q$.

Firstly, we will prove $A$ has a fixed point in $Q$.
For any $u \in Q$, let

$$
\begin{equation*}
l_{u}=\int_{0}^{1} \Phi_{1}(s) \lambda(s) u(s) d s \tag{22}
\end{equation*}
$$

Then

$$
\begin{equation*}
\beta\|u\| t^{\alpha-1} \leq\left(T_{\lambda} u\right)(t) \leq l_{u} t^{\alpha-1} \tag{23}
\end{equation*}
$$

By Lemma 7, we have that $T_{\lambda} \varphi_{1}=r\left(T_{\lambda}\right) \varphi_{1}$. It is easy to see that

$$
\begin{equation*}
\frac{\beta\left\|\varphi_{1}\right\|}{r\left(T_{\lambda}\right)} t^{\alpha-1} \leq \varphi_{1}(t) \leq \frac{l_{\varphi_{1}}}{r\left(T_{\lambda}\right)} t^{\alpha-1} \tag{24}
\end{equation*}
$$

For any $u_{0} \in Q$, set

$$
\begin{equation*}
u_{n}=A\left(u_{n-1}\right), \quad n=1,2, \cdots \tag{25}
\end{equation*}
$$

We may suppose that $u_{1}-u_{0} \neq \theta$ (otherwise, the proof is finished). It follows from (23) and (24) that

$$
\begin{equation*}
T_{\lambda}\left(\left|u_{1}-u_{0}\right|\right) \leq \frac{r\left(T_{\lambda}\right) l_{\left|u_{1}-u_{0}\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1} . \tag{26}
\end{equation*}
$$

Then, we have

$$
\begin{aligned}
\mid u_{2} & -u_{1} \mid \\
& =\left|\int_{0}^{1} G(t, s)\left[f\left(s, u_{1}(s)\right)-f\left(s, u_{0}(s)\right)\right] d s\right|
\end{aligned}
$$

$$
\begin{align*}
& \leq \int_{0}^{1} G(t, s) \lambda(s)\left|u_{1}(s)-u_{0}(s)\right| d s \\
& \leq \frac{r\left(T_{\lambda}\right) l_{\left|u_{1}-u_{0}\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1} . \tag{27}
\end{align*}
$$

By induction, we can get

$$
\begin{equation*}
\left|u_{n+1}-u_{n}\right| \leq \frac{\left[r\left(T_{\lambda}\right)\right]^{n} l_{\left|u_{1}-u_{0}\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1}, \quad n=1,2, \cdots . \tag{28}
\end{equation*}
$$

Then, for any $n, m \in \mathbb{N}$, one has

$$
\begin{align*}
& \left|u_{n+m}-u_{n}\right| \leq\left|u_{n+m}-u_{n+m-1}\right|+\cdots+\left|u_{n+1}-u_{n}\right| \\
& \quad \leq\left(\left[r\left(T_{\lambda}\right)\right]^{n+m-1}+\cdots+\left[r\left(T_{\lambda}\right)\right]^{n}\right) \frac{l_{\left|u_{1}-u_{0}\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1}  \tag{29}\\
& \quad \leq \frac{\left[r\left(T_{\lambda}\right)\right]^{n} l_{\left|u_{1}-u_{0}\right|}}{\left[1-r\left(L_{\lambda}\right)\right] \beta\left\|\varphi_{1}\right\|} \varphi_{1} .
\end{align*}
$$

It follows from $r\left(T_{\lambda}\right)<1$ that

$$
\begin{equation*}
\left\|u_{n+m}-u_{m}\right\| \longrightarrow 0 \quad(n \longrightarrow \infty) \tag{30}
\end{equation*}
$$

which implies $\left\{u_{n}\right\}$ is a Cauchy sequence. Therefore, there exists $u^{*} \in Q$ such that $\left\{u_{n}\right\}$ converges to $u^{*}$. Clearly $u^{*}$ is a fixed point of $A$.

In the following, we will prove the fixed point of $A$ is unique.

Suppose $v \neq u^{*}$ is a positive fixed point of $A$. Then there exists $l_{\left|u^{*}-v\right|}>0$ such that

$$
\begin{equation*}
T_{\lambda}\left(\left|u^{*}-v\right|\right) \leq \frac{r\left(T_{\lambda}\right) l_{\left|u^{*}-v\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1} \tag{31}
\end{equation*}
$$

Therefore,

$$
\begin{align*}
& \left|A u^{*}-A v\right| \\
& \quad=\left|\int_{0}^{1} G(t, s)\left[f\left(s, u^{*}(s)\right)-f(s, v(s))\right] d s\right| \\
& \quad \leq \int_{0}^{1} G(t, s) \lambda(s)\left|u^{*}(s)-v(s)\right| d s  \tag{32}\\
& \quad \leq \frac{r\left(T_{\lambda}\right) l_{\left|u^{*}-v\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1} .
\end{align*}
$$

By induction, we can get

$$
\begin{equation*}
\left|A^{n} u^{*}-A^{n} v\right| \leq \frac{\left[r\left(T_{\lambda}\right)\right]^{n} l_{\left|u^{*}-v\right|}}{\beta\left\|\varphi_{1}\right\|} \varphi_{1} . \tag{33}
\end{equation*}
$$

It follows from $r\left(T_{\lambda}\right)<1$ that

$$
\begin{equation*}
\left\|u^{*}-v\right\|=\left\|A^{n} u^{*}-A^{n} v\right\| \longrightarrow 0 \quad(n \longrightarrow \infty) \tag{34}
\end{equation*}
$$

which implies the positive fixed point of $A$ is unique.

Remark 9. The unique positive solution $u^{*}$ of (1) can be approximated by the iterative schemes: for any $u_{0} \in Q$, let

$$
\begin{equation*}
u_{n}=A\left(u_{n-1}\right), \quad n=1,2, \ldots \tag{35}
\end{equation*}
$$

and then $u_{n} \longrightarrow u^{*}$. Furthermore, we have error estimation

$$
\begin{equation*}
\left|u_{n}-u^{*}\right| \leq \frac{\left[r\left(T_{\lambda}\right)\right]^{n} l_{\left|u_{1}-u_{0}\right|}}{\left[1-r\left(T_{\lambda}\right)\right] \beta\left\|\varphi_{1}\right\|} \varphi_{1} \tag{36}
\end{equation*}
$$

and with the rate of convergence

$$
\begin{equation*}
\left\|u_{n}-u^{*}\right\|=O\left(\left[r\left(T_{\lambda}\right)\right]^{n}\right) . \tag{37}
\end{equation*}
$$

Remark 10. The spectral radius satisfies $r\left(T_{\lambda}\right)=$ $\lim _{n \rightarrow \infty}\left\|T_{\lambda}^{n}\right\|^{1 / n}$ and $r\left(T_{\lambda}\right) \leq\left\|T_{\lambda}^{n}\right\|^{1 / n}$. Particularly,

$$
\begin{equation*}
r\left(T_{\lambda}\right) \leq\left\|T_{\lambda}\right\|=\sup _{0 \leq t \leq 1} \int_{0}^{1} G(t, s) \lambda(s) d s . \tag{38}
\end{equation*}
$$

Theorem 11. Assume that the following condition holds:
$\left(A_{4}\right)$ There exists $\lambda_{1} \in C(0,1) \cap L[0,1]$ satisfying

$$
\begin{equation*}
0<\int_{0}^{1} \Phi_{1}(s) \lambda_{1}(s) d s<+\infty \tag{39}
\end{equation*}
$$

such that

$$
\begin{equation*}
f(t, x) \leq \lambda_{1}(t) x, \quad t \in[0,1], x \in[0, \infty) \tag{40}
\end{equation*}
$$

Then (1) has no positive solution if the spectral radius $r\left(T_{\lambda_{1}}\right) \in$ $(0,1)$, where

$$
\begin{equation*}
T_{\lambda_{1}} u(t)=\int_{0}^{1} G(t, s) \lambda_{1}(s) u(s) d s . \tag{41}
\end{equation*}
$$

Proof. We only need to prove that $A$ has no fixed point in $Q \backslash$ $\{\theta\}$. Otherwise, there exists $v \in Q \backslash\{\theta\}$, such that $A v=v$.

By Lemma 7, we have that the spectral radius $r\left(T_{\lambda_{1}}\right)>0$ and $T_{\lambda_{1}}$ has a positive eigenfunction $\psi_{1}$ satisfying

$$
\begin{equation*}
T_{\lambda_{1}} \psi_{1}=r\left(T_{\lambda_{1}}\right) \psi_{1} \tag{42}
\end{equation*}
$$

It is clear that $\psi_{1} \in Q \backslash\{\theta\}$. Therefore, there exists $c_{1}>0$ such that

$$
\begin{equation*}
v \leq c_{1} \psi_{1} . \tag{43}
\end{equation*}
$$

It follows from $f(t, x) \leq b_{1}(t) x$ that $v=A v \leq T_{\lambda_{1}} v$. It is obvious that $T_{\lambda_{1}}$ is increasing on $Q$. By induction, we can get $v \leq T_{\lambda_{1}}^{n} v, \forall n=1,2,3, \cdots$. Thus,

$$
\begin{align*}
v \leq T_{\lambda_{1}}^{n} v \leq T_{\lambda_{1}}^{n} c_{1} \psi_{1}=c_{1}\left[r\left(T_{\lambda_{1}}\right)\right]^{n} \psi_{1}, &  \tag{44}\\
& \forall n=1,2,3, \cdots .
\end{align*}
$$

Noticing $r\left(T_{\lambda_{1}}\right)<1$, we have $v=\theta$, which contradicts with $v \in Q \backslash\{\theta\}$.

Theorem 12. Assume that the following condition holds:
$\left(A_{5}\right)$ There exists $\lambda_{2} \in C(0,1) \cap L[0,1]$ satisfying

$$
\begin{equation*}
0<\int_{0}^{1} \Phi_{1}(s) \lambda_{2}(s) d s<+\infty \tag{45}
\end{equation*}
$$

such that

$$
\begin{equation*}
f(t, x) \geq \lambda_{2}(t) x, \quad t \in[0,1], x \in[0, \infty) \tag{46}
\end{equation*}
$$

Then (1) has no positive solution if the spectral radius $r\left(T_{\lambda_{2}}\right)>$ 1, where

$$
\begin{equation*}
T_{\lambda_{2}} u(t)=\int_{0}^{1} G(t, s) \lambda_{2}(s) u(s) d s \tag{47}
\end{equation*}
$$

Proof. Suppose that there exists $v \in Q \backslash\{\theta\}$, such that $A v=v$.
By Lemma 7, we have that the spectral radius $r\left(T_{\lambda_{2}}\right)>0$ and $T_{\lambda_{2}}$ has a positive eigenfunction $\psi_{2}$ satisfying

$$
\begin{equation*}
T_{\lambda_{2}} \psi_{2}=r\left(T_{\lambda_{2}}\right) \psi_{2} \tag{48}
\end{equation*}
$$

It is clear that $\psi_{2} \in Q \backslash\{\theta\}$. Therefore, there exists $\mathcal{c}_{2}>0$ such that

$$
\begin{equation*}
v \geq c_{2} \psi_{1} \tag{49}
\end{equation*}
$$

It follows from $f(t, x) \geq \lambda_{2}(t) x$ that $v=A v \geq T_{\lambda_{2}} v$. Noticing that $T_{\lambda_{2}}$ is increasing on $Q$, by induction, we have $v \geq T_{\lambda_{2}}^{n} v, \forall n=1,2,3, \cdots$. Thus,

$$
\begin{align*}
v \geq T_{\lambda_{2}}^{n} v \geq T_{\lambda_{2}}^{n} c_{2} \psi_{2}=c_{2}\left[r\left(T_{\lambda_{2}}\right)\right]^{n} \psi_{2} &  \tag{50}\\
& \forall n=1,2,3, \cdots .
\end{align*}
$$

It follows from $r\left(T_{\lambda_{1}}\right)>1$ that $\|v\|=\infty$, which contradicts with $v \in Q$.

## 4. Example

Example 1. Consider the following integral boundary value problem:

$$
\begin{cases}D_{0+}^{5 / 2} u(t)+f(t, u(t))=0, & 0<t<1,  \tag{51}\\ u(0)=u^{\prime}(0)=0, & D_{0+}^{1 / 2} u(1)=\int_{0}^{1} \frac{1}{\sqrt{t}} D_{0+}^{1 / 2} u(t) d t,\end{cases}
$$

with

$$
\begin{equation*}
f(t, x)=\frac{1+x+|\sin x|}{2 C \sqrt{t}} \tag{52}
\end{equation*}
$$

where

$$
\begin{equation*}
C=\frac{8}{3 \sqrt{\pi}}-\frac{\sqrt{\pi}}{2} \tag{53}
\end{equation*}
$$

By direct calculations, we have

$$
\begin{equation*}
\Delta=\Gamma(2)-\Gamma(2) \int_{0}^{1} \frac{t}{\sqrt{t}} d t=\frac{1}{3} \tag{54}
\end{equation*}
$$

It is clear that $\left(A_{1}\right)$ and $\left(A_{2}\right)$ hold. Clearly, we have

$$
\begin{aligned}
& G_{1}(t, s) \\
& \quad=\frac{4}{3 \sqrt{\pi}} \begin{cases}t^{3 / 2}(1-s), & 0 \leq t \leq s \leq 1 \\
t^{3 / 2}(1-s)-(t-s)^{3 / 2}, & 0 \leq s \leq t \leq 1\end{cases}
\end{aligned}
$$

$$
\begin{align*}
& G_{2}(t, s)=\frac{4}{3 \sqrt{\pi}} \begin{cases}t(1-s), & 0 \leq t \leq s \leq 1 \\
s(1-t), & 0 \leq s \leq t \leq 1\end{cases} \\
& h(s)=\frac{16}{3 \sqrt{\pi}} s(1-\sqrt{s}), \\
& \Phi_{1}(s)=\frac{4(1-s)}{3 \sqrt{\pi}}+h(s), \\
& G(t, s)=G_{1}(t, s)+h(s) t^{\alpha-1} . \tag{55}
\end{align*}
$$

Let $\lambda(t)=1 / C \sqrt{t}$; then we have

$$
\begin{align*}
& |f(t, x)-f(t, y)| \leq \lambda(t)|x-y| \\
& \quad t \in[0,1], x, y \in[0, \infty) \tag{56}
\end{align*}
$$

It is easy to get that $\left(A_{3}\right)$ holds.
Denote

$$
\begin{equation*}
e(t) \equiv 1, \quad t \in[0,1] \tag{57}
\end{equation*}
$$

By direct calculations, we have

$$
\begin{align*}
& \left(T_{\lambda} e\right)(t)=\int_{0}^{1} G(t, s) \lambda(s) d s=\frac{8 t^{3 / 2}}{3 \sqrt{\pi} C}-\frac{\sqrt{\pi} t^{2}}{2 C}  \tag{58}\\
& \left(T_{\lambda}^{2} e\right)(t)=\int_{0}^{1} G(t, s) \lambda(s)\left(T_{\lambda} e\right)(s) d s=\left(\frac{4}{3 \sqrt{\pi} C}\right)^{2} \\
& \cdot\left[\left(\frac{5}{7}-\frac{27 \pi}{280}\right) t^{3 / 2}-\frac{8}{35} t^{7 / 2}+\frac{9 \pi^{2}}{1024} t^{4}\right] \\
& \quad \leq\left(\frac{4}{3 \sqrt{\pi} C}\right)^{2}  \tag{59}\\
& \cdot\left[\left(\frac{5}{7}-\frac{27 \pi}{280}\right) t^{3 / 2}-\frac{8}{35} t^{7 / 2}+\frac{9 \pi^{2}}{1024} t^{7 / 2}\right] \\
& \quad \approx\left(\frac{4}{3 \sqrt{\pi} C}\right)^{2}\left(0.41135 t^{3 / 2}-0.14183 t^{7 / 2}\right)
\end{align*}
$$

$\forall t \in[0,1]$, we have

$$
\begin{align*}
& \left(\frac{8 t^{3 / 2}}{3 \sqrt{\pi} C}-\frac{\sqrt{\pi} t^{2}}{2 C}\right)^{\prime}=\frac{4 t^{1 / 2}-\pi t}{\sqrt{\pi} C} \geq 0 \\
& \left(0.41135 t^{3 / 2}-0.14183 t^{7 / 2}\right)^{\prime}  \tag{60}\\
& \quad=0.617025 t^{1 / 2}-0.496405 t^{5 / 2} \geq 0
\end{align*}
$$

Therefore,

$$
\begin{align*}
\max _{0 \leq t \leq 1}\left(T_{\lambda} e\right)(t) & =\left(T_{\lambda} e\right)(1)=1 \\
\max _{0 \leq t \leq 1}\left(T_{\lambda}^{2} e\right)(t) & \leq\left(\frac{4}{3 \sqrt{\pi} C}\right)^{2}(0.41135-0.14183)  \tag{61}\\
& \approx 0.39898
\end{align*}
$$

It is obvious that

$$
\begin{equation*}
\left\|L_{\lambda}\right\|=\max _{0 \leq t \leq 1}\left(T_{\lambda} e\right)(t)=1 \tag{62}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|L_{\lambda}^{2}\right\|=\max _{0 \leq t \leq 1}\left(T_{\lambda}^{2} e\right)(t)<1 \tag{63}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left\|L_{\lambda}^{2}\right\|^{1 / 2}<1 \tag{64}
\end{equation*}
$$

Lemma 7 and Remark 10 can guarantee that

$$
\begin{equation*}
0<r\left(L_{\lambda}\right) \leq\left\|L_{\lambda}^{2}\right\|^{1 / 2}<1<\left\|L_{\lambda}\right\|=1 . \tag{65}
\end{equation*}
$$

So all of the assumptions of Theorem 8 are satisfied. As a result, BVP (51) has a unique positive solution.

Example 2. Consider BVP (51) with

$$
\begin{equation*}
f(t, x)=\frac{x+|\sin x|}{2 C \sqrt{t}} . \tag{66}
\end{equation*}
$$

Clearly, we have

$$
\begin{equation*}
f(t, x) \leq \lambda(t) x, \quad t \in[0,1], x \in[0, \infty) . \tag{67}
\end{equation*}
$$

It is not difficult to check that $\left(A_{4}\right)$ holds.
It follows from Example 1 that

$$
\begin{equation*}
0<r\left(L_{\lambda}\right)<1<\left\|L_{\lambda}\right\|=1 . \tag{68}
\end{equation*}
$$

So all of the assumptions of Theorem 11 are satisfied. As a result, BVP (51) has no positive solution.

## 5. Conclusions

In this paper, we consider the existence of positive solution for fractional differential equations with conjugate type integral conditions. Both the existence of uniqueness and nonexistence of positive solution are established under conditions closely associated with the spectral radius with respect to the relevant linear operator. In addition, the unique positive solution can be approximated by an iterative scheme, and the error estimation of the iterative sequences is also given.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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# New Fixed Point Theorems and Application of Mixed Monotone Mappings in Partially Ordered Metric Spaces 

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#### Abstract

We consider the existence of a coupled fixed point for mixed monotone mapping $F: X \times X \longrightarrow X$ satisfying a new contractive inequality which involves an altering distance function in partially ordered metric spaces. We also establish some uniqueness results for coupled fixed points, as well as the existence of fixed points of mixed monotone operators. The presented results generalize and develop some existing results. In addition to an example as well as an application, we establish some uniqueness results for a system of integral equations.


## 1. Introduction and Preliminaries

In this paper we aim to establish coupled fixed point theorems for a mixed monotone mapping in a metric space endowed with partial order. The concept of the mixed monotone operator was introduced by Guo and Lakshmikantham in [1]. Existence of fixed points in a metric space has been studied for a long time (see [2-10]). The Banach contraction principle, which plays a very important role in nonlinear analysis, is the most famous tool in the study of a fixed point theorem. In the last decade, there has been a trend to weaken the requirement on the contraction by considering metric spaces endowed with partial order. It is of interest to determine if it is still possible to establish the existence of a unique fixed point assuming that the operator considered is monotone in such a setting.

Recently, there have been a lot of generalizations of the Banach contraction-mapping principle in the literature (see [11-28]). Investigation of the existence of fixed points for single-valued mappings in partially ordered metric spaces was initially considered by Ran and Reurings in [29] who proved the existence of a unique fixed point.

Following basically the same approach as the one in [29], Bhaskar and Lakshmikantham in [30] proved a fixed point theorem for a mixed monotone mapping in a metric space
endowed with partial order. Bhaskar and Lakshmikantham extended [29, Theorem 2.1] to mixed monotone operators so that they can enlarge, in a unified manner, the class of problems that can be investigated. The authors in [30] also established some uniqueness results for coupled fixed points, as well as the existence of fixed points of $F$.

Based on the works in [30], Zhao in [31] obtained a more general coupled fixed point theorem for mixed monotone operators. Another generalization of the contraction principle was suggested by Alber and Guerre-Delabriere. In 1997, Alber and Guerre-Delabriere [32] introduced the concept of weak contractions in Hilbert spaces. This concept was extended to metric spaces by Rhoades in [33].

Definition 1. A mapping $T: X \longrightarrow X$, where $(X, d)$ is a metric space, is said to be weakly contractive if

$$
\begin{equation*}
d(T x, T y) \leq d(x, y)-\psi(d(x, y)), \tag{1}
\end{equation*}
$$

where $x, y \in X$ and $\psi$ is an altering distance function.
The notion of altering distance function was introduced by Khan et al. [34]. An altering distance function is a control function that alters distance between two points in a metric space.

It was shown in [32] that, for Hilbert spaces, weakly contractive maps possess a unique fixed point, without any
additional assumptions, and it was noted that the same is true at least for uniformly smooth and uniformly convex Banach spaces. In [33], Rhoades proved that the theorem remains true in arbitrary complete metric spaces, which was improved and extended by Dutta and Choudhury in [35]. Recently, a new version of the context of ordered metric spaces has been proved by Harjani and Sadarangani in [12]. We refer the readers to [36-41] for more related works.

Motivated by the papers mentioned above, we aim to establish coupled fixed point results for mixed monotone operator $F$ which is weakly contractive in partially ordered complete metric spaces. Our main results are Theorems 3, 6 , and 8 . To the best of our knowledge, there are no similar results in the literature on the existence of the coupled fixed point. Compared with the results obtained in the [32, 33, 35], $\psi$ in our results is not necessary to be an altering distance function, which means that $F$ satisfies a more general contractive condition. On the other hand, our result is still valid for $F$ not necessarily continuous if we require that the underlying metric space $X$ has an additional property. Our main results will generalize and develop the results given in [12].

In Section 2, we give the proof of our main results. In Section 3, as an application of our theorems, we consider the existence of a unique solution to a system of integral equations.

## 2. Coupled Fixed Point Theorems

Let $(X, \leq)$ be a partially ordered set and let $d$ be a metric on $X$ such that $(X, d)$ is a complete metric space. Further, we endow the product space $X \times X$ with the following partial order:

$$
\begin{align*}
& \text { for }(x, y),(u, v) \in X \times X \text {, } \\
& (u, v) \leq(x, y) \Longleftrightarrow  \tag{2}\\
& x \geq u, y \leq v .
\end{align*}
$$

Definition 2 (see [34]). A function $\varphi: \quad[0,+\infty) \longrightarrow$ $[0,+\infty)$ is called an altering distance function if the following conditions are satisfied:
(i) $\varphi$ is continuous and nondecreasing
(ii) $\varphi(t)=0 \Longleftrightarrow t=0$

Next we introduce a set of functions

$$
\begin{align*}
\Psi & :=\{\psi \in C([0,+\infty),[0,+\infty)) \mid \psi(0) \\
& =0, \text { and for any } t>0, \psi(t)>0\} . \tag{3}
\end{align*}
$$

It follows from Definition 2 that if $\varphi$ is an altering distance function, $\varphi \in \Psi$.

The first main result is the following coupled fixed point result.

## Theorem 3. Assume

$\left(H_{1}\right) \psi \in \Psi(\psi$ is not necessary to be an altering distance function)
$\left(H_{2}\right) F: X \times X \longrightarrow X$ being a mixed monotone mapping, there exist a constant $k \in(0,1)$ such that

$$
\begin{align*}
\varphi(d & (F(u, v), F(x, y))+d(F(v, u), F(y, x))) \\
\leq & k \varphi(d(u, x)+d(v, y))  \tag{4}\\
& -\psi(k[d(u, x)+d(v, y)])
\end{align*}
$$

and, for each $u \geq x, v \leq y, \varphi$ is an altering distance function which satisfies

$$
\begin{equation*}
\varphi(t+s) \leq \varphi(t)+\varphi(s), \quad \forall t, s \in[0,+\infty) \tag{5}
\end{equation*}
$$

$\left(H_{3}\right)$ there exist $\left(u_{0}, v_{0}\right) \in X \times X$ such that $u_{0} \leq F\left(u_{0}, v_{0}\right)$ and $v_{0} \geq F\left(v_{0}, u_{0}\right)$
$\left(H_{4}\right)$
(a) $F$ is continuous or
(b) $X$ has the following properties:
(i) If a nondecreasing sequence $\left\{u_{n}\right\} \longrightarrow u$, then $u_{n} \leq u, \forall n$
(ii) If a nonincreasing sequence $\left\{v_{n}\right\} \longrightarrow v$, then $v_{n} \geq v, \forall n$

Then there exist $u, v \in X$, such that

$$
\begin{align*}
u & =F(u, v) \\
\text { and } v & =F(v, u) . \tag{6}
\end{align*}
$$

Proof. Since

$$
\begin{align*}
u_{0} & \leq F\left(u_{0}, v_{0}\right) \\
\text { and } v_{0} & \geq F\left(v_{0}, u_{0}\right), \tag{7}
\end{align*}
$$

let $F\left(u_{0}, v_{0}\right)=u_{1}, F\left(v_{0}, u_{0}\right)=v_{1}, F\left(u_{1}, v_{1}\right)=u_{2}$, and $F\left(v_{1}\right.$, $\left.u_{1}\right)=v_{2}$; we have $u_{0} \leq u_{1} v_{0} \geq v_{1}$. Denote

$$
\begin{align*}
F^{2}\left(u_{0}, v_{0}\right) & =F\left(F\left(u_{0}, v_{0}\right), F\left(v_{0}, u_{0}\right)\right)=F\left(u_{1}, v_{1}\right) \\
& =u_{2}  \tag{8}\\
F^{2}\left(v_{0}, u_{0}\right) & =F\left(F\left(v_{0}, u_{0}\right), F\left(u_{0}, v_{0}\right)\right)=F\left(v_{1}, u_{1}\right) \\
& =v_{2}
\end{align*}
$$

Note that $u_{0} \leq u_{1}, v_{0} \geq v_{1}$; it follows from the mixed monotone property of $F$ that

$$
\begin{align*}
& F\left(u_{1}, v_{1}\right) \geq F\left(u_{0}, v_{1}\right) \geq F\left(u_{0}, v_{0}\right) \\
& F\left(v_{1}, u_{1}\right) \leq F\left(v_{0}, u_{1}\right) \leq F\left(v_{0}, u_{0}\right) \tag{9}
\end{align*}
$$

which implies

$$
\begin{align*}
& u_{2}=F^{2}\left(u_{0}, v_{0}\right)=F\left(u_{1}, v_{1}\right) \geq F\left(u_{0}, v_{0}\right)=u_{1}, \\
& v_{2}=F^{2}\left(v_{0}, u_{0}\right)=F\left(v_{1}, u_{1}\right) \leq F\left(v_{0}, u_{0}\right)=v_{1} . \tag{10}
\end{align*}
$$

For $n=1,2, \cdots$, let

$$
\begin{align*}
& u_{n+1}=F^{n+1}\left(u_{0}, v_{0}\right)=F\left(F^{n}\left(u_{0}, v_{0}\right), F^{n}\left(v_{0}, u_{0}\right)\right), \\
& v_{n+1}=F^{n+1}\left(v_{0}, u_{0}\right)=F\left(F^{n}\left(v_{0}, u_{0}\right), F^{n}\left(u_{0}, v_{0}\right)\right) . \tag{11}
\end{align*}
$$

i.e.,

$$
\begin{align*}
& u_{n+1}=F\left(u_{n}, v_{n}\right),  \tag{12}\\
& v_{n+1}=F\left(v_{n}, u_{n}\right) .
\end{align*}
$$

It is easy to see that

$$
\begin{aligned}
u_{0} & \leq F\left(u_{0}, v_{0}\right)=u_{1} \leq F^{2}\left(u_{0}, v_{0}\right)=u_{2} \leq \cdots \\
& \leq F^{n+1}\left(u_{0}, v_{0}\right) \leq \cdots, \\
v_{0} & \geq F\left(v_{0}, u_{0}\right)=v_{1} \geq F^{2}\left(v_{0}, u_{0}\right)=v_{2} \geq \cdots \\
& \geq F^{n+1}\left(v_{0}, u_{0}\right) \geq \cdots,
\end{aligned}
$$

i.e.,

$$
\begin{align*}
& u_{0} \leq u_{1} \leq \cdots \leq u_{n} \leq \cdots \\
& v_{0} \geq v_{1} \geq \cdots \geq v_{n} \geq \cdots \tag{14}
\end{align*}
$$

$$
n=1,2, \cdots .
$$

By (4) (12), we have

$$
\begin{align*}
\varphi & \left(d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)\right) \\
& =\varphi\left(d\left(F\left(u_{n}, v_{n}\right), F\left(u_{n-1}, v_{n-1}\right)\right)\right.  \tag{15}\\
& \left.+d\left(F\left(v_{n}, u_{n}\right), F\left(v_{n-1}, u_{n-1}\right)\right)\right) \leq k \varphi\left(d\left(u_{n}, u_{n-1}\right)\right. \\
& \left.+d\left(v_{n}, v_{n-1}\right)\right)-\psi\left(k\left[d\left(u_{n}, u_{n-1}\right)+d\left(v_{n}, v_{n-1}\right)\right]\right),
\end{align*}
$$

and thus

$$
\begin{align*}
& \varphi\left(d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)\right) \\
& \quad \leq k \varphi\left(d\left(u_{n}, u_{n-1}\right)+d\left(v_{n}, v_{n-1}\right)\right) \tag{16}
\end{align*}
$$

From $k \in[0,1)$, we have

$$
\begin{align*}
& \varphi\left(d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)\right) \\
& \quad \leq \varphi\left(d\left(u_{n}, v_{n-1}\right)+d\left(v_{n}, v_{n-1}\right)\right) \tag{17}
\end{align*}
$$

Since $\varphi$ is continuous and nondecreasing,

$$
\begin{align*}
& d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)  \tag{18}\\
& \quad \leq d\left(u_{n}, u_{n-1}\right)+d\left(v_{n}, v_{n-1}\right) .
\end{align*}
$$

Let $\xi_{n}=d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)$. From $0 \leq \xi_{n+1} \leq \xi_{n}$, $\left\{\xi_{n}\right\}$ is a Cauchy sequence, and thus there exist $\xi \geq 0$ such that

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left[d\left(u_{n+1}, u_{n}\right)+d\left(\left(v_{n+1}, v_{n}\right)\right]=\xi\right. \tag{19}
\end{equation*}
$$

Now we claim $\xi=0$. In fact, by (14) and $\left(H_{1}\right)$, one has

$$
\begin{align*}
\varphi(\xi)= & \lim _{n \longrightarrow \infty} \varphi\left(\xi_{n}\right) \\
= & \lim _{n \longrightarrow \infty} \varphi\left(d\left(u_{n+1}, u_{n}\right)+d\left(v_{n+1}, v_{n}\right)\right) \\
\leq & k_{n \longrightarrow \infty} \varphi\left(d\left(u_{n}, u_{n-1}\right)+d\left(v_{n}, v_{n-1}\right)\right)  \tag{20}\\
& -\lim _{n \longrightarrow \infty} \psi\left(k\left[d\left(u_{n}, u_{n-1}\right)+d\left(v_{n}, v_{n-1}\right)\right]\right) \\
= & k \varphi(\xi)-\lim _{n \longrightarrow \infty} \psi\left(k \xi_{n-1}\right)=k \varphi(\xi)-\lim _{t \rightarrow k \xi} \psi(t) \\
\leq & k \varphi(\xi)
\end{align*}
$$

which implies $\varphi(\xi)=0$, so is $\xi$.
Next we will show $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are Cauchy sequences.
Arguing indirectly we suppose $\left\{u_{n}\right\}$ is not a Cauchy sequence. Thus, there exists a positive constant $\epsilon$ such that, for any $K>0$, there exist $n_{k}>m_{k}>K$ such that

$$
\begin{equation*}
d\left(u_{n_{k}}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{m_{k}}\right) \geq \epsilon \tag{21}
\end{equation*}
$$

For $m_{k}$, let $n_{k}$ be the smallest integer satisfying $n_{k} \geq m_{k}$ and (21). Thus, one has

$$
\begin{equation*}
d\left(u_{n_{k}-1}, u_{m_{k}}\right)+d\left(v_{n_{k}-1}, v_{m_{k}}\right)<\epsilon, \tag{22}
\end{equation*}
$$

which implies

$$
\begin{align*}
& d\left(u_{n_{k}}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{m_{k}}\right) \\
& \quad \leq d\left(u_{n_{k}}, u_{n_{k}-1}\right)+d\left(u_{n_{k}-1}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{n_{k}-1}\right)  \tag{23}\\
& \quad+d\left(v_{n_{k}-1}, v_{m_{k}}\right) \\
& \quad \leq d\left(u_{n_{k}}, u_{n_{k}-1}\right)+d\left(v_{n_{k}}, v_{n_{k}-1}\right)+\epsilon=\xi_{n_{k}-1}+\epsilon
\end{align*}
$$

and, by (21) and $\xi_{n} \longrightarrow 0(n \longrightarrow \infty)$, we have

$$
\begin{equation*}
s_{k}:=d\left(u_{n_{k}}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{m_{k}}\right) \longrightarrow \epsilon, \quad k \longrightarrow \infty \tag{24}
\end{equation*}
$$

The triangular inequality gives us

$$
\begin{align*}
s_{k} \leq & d\left(u_{n_{k}}, u_{n_{k}+1}\right)+d\left(u_{n_{k}+1}, u_{m_{k}+1}\right) \\
& +d\left(u_{m_{k}+1}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{n_{k}+1}\right) \\
& +d\left(v_{n_{k}+1}, v_{m_{k}+1}\right)+d\left(v_{m_{k}+1}, v_{m_{k}}\right)  \tag{25}\\
= & \xi_{n_{k}}+\xi_{m_{k}}+d\left(u_{n_{k}+1}, u_{m_{k}+1}\right)+d\left(v_{n_{k}+1}, v_{m_{k}+1}\right)
\end{align*}
$$

and thus

$$
\begin{align*}
& \varphi\left(s_{k}\right) \\
&= \varphi\left(\xi_{n_{k}}+\xi_{m_{k}}+d\left(u_{n_{k}+1}, u_{m_{k}+1}\right)+d\left(v_{n_{k}+1}, v_{m_{k}+1}\right)\right) \\
& \leq \varphi\left(\xi_{n_{k}}+\xi_{m_{k}}\right)+\varphi\left(d\left(u_{n_{k}+1}, u_{m_{k}+1}\right)\right)  \tag{26}\\
& \quad+\varphi\left(d\left(v_{n_{k}+1}, v_{m_{k}+1}\right)\right) .
\end{align*}
$$

By (14),

$$
\begin{align*}
& \varphi\left(d\left(u_{n_{k}+1}, u_{m_{k}+1}\right)\right)+\varphi\left(d\left(v_{n_{k}+1}, v_{m_{k}+1}\right)\right) \\
& \quad=\varphi\left(d\left(F\left(u_{n_{k}}, v_{n_{k}}\right), F\left(u_{m_{k}}, v_{m_{k}}\right)\right)\right. \\
& \left.\quad+d\left(F\left(v_{n_{k}}, u_{n_{k}}\right), F\left(v_{m_{k}}, u_{m_{k}}\right)\right)\right)  \tag{27}\\
& \quad \leq k \varphi\left(d\left(u_{n_{k}}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{m_{k}}\right)\right) \\
& \quad-\psi\left(k\left[d\left(u_{n_{k}}, u_{m_{k}}\right)+d\left(v_{n_{k}}, v_{m_{k}}\right)\right]\right)=k \varphi\left(s_{k}\right) \\
& \quad-\psi\left(k s_{k}\right)
\end{align*}
$$

From (26) (27), one has

$$
\begin{align*}
\varphi\left(s_{k}\right) & \leq \varphi\left(\xi_{n_{k}}+\xi_{m_{k}}\right)+k \varphi\left(s_{k}\right)-\psi\left(s_{k}\right)  \tag{28}\\
& \leq \varphi\left(\xi_{n_{k}}+\xi_{m_{k}}\right)+\varphi\left(s_{k}\right)-\psi\left(s_{k}\right)
\end{align*}
$$

and, letting $k \longrightarrow \infty$ in the above inequality,

$$
\begin{align*}
\lim _{k \rightarrow \infty} \varphi\left(s_{k}\right) \leq & \lim _{k \rightarrow \infty}\left[\varphi\left(\xi_{n_{k}}+\xi_{m_{k}}\right)+\varphi\left(s_{k}\right)\right]  \tag{29}\\
& -\lim _{k \rightarrow \infty} \psi\left(s_{k}\right) .
\end{align*}
$$

From $\left(H_{1}\right)$, note that $\xi_{n} \longrightarrow 0, s_{k} \longrightarrow \epsilon$; we have

$$
\begin{align*}
\varphi(\epsilon) & \leq \varphi(0)+\varphi(\epsilon)-\lim _{k \rightarrow \infty} \psi\left(s_{k}\right) \\
& =\varphi(\epsilon)-\lim _{k \rightarrow \infty} \psi\left(s_{k}\right)<\varphi(\epsilon), \tag{30}
\end{align*}
$$

which is a contradiction. This shows that $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are Cauchy sequences. Since $X$ is a complete metric space, there exist $u, v \in X$ such that

$$
\begin{gather*}
u_{n} \longrightarrow u \\
v_{n} \longrightarrow v  \tag{31}\\
n \longrightarrow \infty
\end{gather*}
$$

Case (a). Assume $F$ is continuous. Then

$$
\begin{align*}
u & =\lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} F\left(u_{n-1}, v_{n-1}\right) \\
& =F\left(\lim _{n \longrightarrow \infty} u_{n-1}, \lim _{n \rightarrow \infty} v_{n-1}\right)=F(u, v), \\
v & =\lim _{n \rightarrow \infty} v_{n}=\lim _{n \rightarrow \infty} F\left(v_{n-1}, u_{n-1}\right)  \tag{32}\\
& =F\left(\lim _{n \longrightarrow \infty} v_{n-1}, \lim _{n \rightarrow \infty} u_{n-1}\right)=F(v, u),
\end{align*}
$$

and, thus, $(u, v) \in X \times X$ is a coupled fixed point of $F$.
Case (b). Assume $X$ has the following properties:
(i) If a nondecreasing sequence $\left\{u_{n}\right\} \longrightarrow u$, then $u_{n} \leq$ $u, \forall n$
(ii) If a nonincreasing sequence $\left\{v_{n}\right\} \longrightarrow v$, then $v_{n} \geq$ $v, \forall n$

By (14) (31), we have

$$
\begin{align*}
& u_{n} \geq u \\
& v_{n} \leq v \tag{33}
\end{align*}
$$

$$
n=1,2, \cdots
$$

Again, the triangular inequality gives us

$$
\begin{align*}
d(u, F(u, v)) & \leq d\left(u, u_{n+1}\right)+d\left(u_{n+1}, F(u, v)\right) \\
& =d\left(u, u_{n+1}\right)+d\left(F\left(u_{n}, v_{n}\right), F(u, v)\right) . \tag{34}
\end{align*}
$$

From $\left(H_{1}\right)\left(H_{2}\right)$, observing that $\varphi$ is nondecreasing and using $u_{n} \leq u, v_{n} \geq v$, we get

$$
\begin{align*}
& \varphi(d(u, F(u, v))) \leq \varphi\left(d\left(u, u_{n+1}\right)+d\left(u_{n+1}, F(u, v)\right)\right) \\
&\left.=\varphi\left(d\left(u, u_{n+1}\right)\right)+d\left(F\left(u_{n}, v_{n}\right), F(u, v)\right)\right) \\
& \quad \leq \varphi\left(d\left(u, u_{n+1}\right)\right)+\varphi\left(d\left(F\left(u_{n}, v_{n}\right), F(u, v)\right)\right) \\
& \quad \leq \varphi\left(d\left(u, u_{n+1}\right)\right)+\varphi\left(d\left(F\left(u_{n}, v_{n}\right), F(u, v)\right)\right. \\
&\left.\quad+d\left(F\left(v_{n}, u_{n}\right), F(v, u)\right)\right) \leq \varphi\left(d\left(u, u_{n+1}\right)\right)  \tag{35}\\
& \quad+k \varphi\left(d\left(u_{n}, u\right)+d\left(v_{n}, v\right)\right) \\
& \quad-\psi\left(k\left(d\left(u_{n}, u\right)+d\left(v_{n}, v\right)\right)\right) \longrightarrow 0
\end{align*}
$$

$$
(\text { as } n \longrightarrow \infty)
$$

and we get $\varphi(d(u, F(u, v)))=0$. Then $u=F(u, v)$. Similarly, we get $v=F(v, u)$. The proof is completed.

Corollary 4. Let $(X, \leq)$ be a partially ordered set and let d be a metric on $X$ such that $(X, d)$ is a complete metric space. Assume $F: X \times X \longrightarrow X$ is a fixed monotone operator and
$\left(\mathrm{f}_{1}\right)$ there exist $\left(u_{0}, v_{0}\right) \in X \times X$ such that $u_{0} \leq F\left(u_{0}, v_{0}\right)$ and $v_{0} \geq F\left(v_{0}, u_{0}\right)$
$\left(\mathrm{f}_{2}\right)$ there exist $\psi \in \Psi, k \in(0,1)$, for $u, v, x, y \in X$ with $u \geq x, v \leq y$,

$$
\begin{align*}
& d(F(u, v), F(x, y))+d(F(v, u), F(y, x)) \\
& \quad \leq k(d(u, x)+d(v, y))  \tag{36}\\
& \quad-\psi(k(d(u, x)+d(v, y)))
\end{align*}
$$

$\left(f_{3}\right)$
(a) $F$ is continuous or
(b) $X$ has the following properties:
(i) If a nondecreasing sequence $\left\{u_{n}\right\} \longrightarrow u$, then $u_{n} \leq u, \forall n$
(ii) If a nonincreasing sequence $\left\{v_{n}\right\} \longrightarrow v$, then $v_{n} \geq v, \forall n$

Then there exist $u, v \in X$ such that

$$
\begin{align*}
u & =F(u, v) \\
\text { and } v & =F(v, u) . \tag{37}
\end{align*}
$$

Proof. Let $\varphi(t)=t, t \in[0,+\infty)$. The proof is finished by Theorem 3.

Note that, in (36), $\psi$ is not necessary to be an altering distance function.

Corollary 5. Let $(X, \leq)$ be a partially ordered set and let d be a metric on $X$ such that $(X, d)$ is a complete metric space. Assume $F: X \times X \longrightarrow X$ is a fixed monotone operator and
( $\mathrm{f}_{1}$ ) there exist $\left(u_{0}, v_{0}\right) \in X \times X$ such that $u_{0} \leq F\left(u_{0}, v_{0}\right)$ and $v_{0} \geq F\left(v_{0}, u_{0}\right)$
$\left(\mathrm{f}_{2}\right)$ there exist $\psi \in \Psi, k \in(0,1)$, for $u, v, x, y \in X$ with $u \geq x, v \leq y$,

$$
\begin{align*}
& d(F(u, v), F(x, y))+d(F(v, u), F(y, x)) \\
& \quad \leq \frac{k}{2}[d(u, x)+d(v, y)] \tag{38}
\end{align*}
$$

$\left(f_{3}\right)$
(a) $F$ is continuous or
(b) $X$ has the following property:
(i) If a nondecreasing sequence $\left\{u_{n}\right\} \longrightarrow u$, then
$u_{n} \leq u, \forall n$
(ii) If a nonincreasing sequence $\left\{v_{n}\right\} \longrightarrow v$, then $v_{n} \geq v, \forall n$

Then there exist $u, v \in X$ such that

$$
\begin{align*}
u & =F(u, v) \\
\text { and } v & =F(v, u) . \tag{39}
\end{align*}
$$

Proof. Let $\psi(t)=t / 2, t \in[0,+\infty)$. The proof is finished by Corollary 4.

Next, we discuss the uniqueness of the coupled fixed point. Since the contractivity assumption is made only on comparable elements in $X \times X$, Theorem 3 cannot guarantee the uniqueness of the coupled fixed point. Before stating our uniqueness results, we require that the product space $X \times X$ endowed with the partial order mentioned earlier have the following property:
$\left(H_{5}\right)$ For every $(u, v),(s, w) \in X \times X$, there exists $(x, y) \in$ $X \times X$ which is comparable to $(u, v)$ and $(s, w)$.

Note that $\left(\mathrm{H}_{5}\right)$ is equivalent to (see [42]) the following:
$\left(H_{5}^{\prime}\right)$ Every pair of elements in $X \times X$ has either a lower bound or an upper bound.

Next, we state our second main result.
Theorem 6. Assume $\left(H_{1}\right)-\left(H_{4}\right)$ and $\left(H_{5}\right)\left(\right.$ or $\left.\left(H_{5}^{\prime}\right)\right)$ hold. Then the coupled fixed point of $F$ is unique.

Proof. By $\left(H_{1}\right)-\left(H_{4}\right)$, applying Theorem 3, $F$ has a coupled fixed point. Let $(u, v),(w, s)$ be the coupled fixed points of $F$, i.e.,

$$
\begin{align*}
u & =F(u, v), \\
v & =F(v, u), \\
w & =F(w, s),  \tag{40}\\
s & =F(s, w)
\end{align*}
$$

To show the uniqueness, we need to prove $u=w, v=s$.
By $\left(H_{5}\right)$, there exists $(x, y) \in X \times X$ which is comparable to $(u, v)$ and $(s, w)$. Let

$$
\begin{align*}
x_{0} & =x, \\
y_{0} & =y, \\
x_{n+1} & =F\left(x_{n}, y_{n}\right),  \tag{41}\\
y_{n+1} & =F\left(x_{n}, y_{n}\right) .
\end{align*}
$$

Since $(x, y)$ is comparable to $(u, v)$ with respect to the ordering in $X \times X$, we suppose

$$
\begin{equation*}
\left(x_{0}, y_{0}\right)=(x, y) \leq(u, v) \tag{42}
\end{equation*}
$$

Then, for $n=1,2, \ldots$,

$$
\begin{equation*}
\left(x_{n}, y_{n}\right) \leq(u, v) \tag{43}
\end{equation*}
$$

In fact, (42) implies that

$$
\begin{align*}
& x_{0}=x \leq u  \tag{44}\\
& y_{0}=y \geq v
\end{align*}
$$

For $n=1$, by the mixed monotone property, we have

$$
\begin{align*}
x_{1}=F\left(x_{0}, y_{0}\right) \leq F\left(u, y_{0}\right) \leq F(u, v) & =u \\
y_{1}=F\left(y_{0}, x_{0}\right) \geq F\left(v, x_{0}\right) \geq F(v, u) & =v,  \tag{45}\\
& \quad \text { i.e., }\left(x_{1}, y_{1}\right) \leq(u, v) .
\end{align*}
$$

For $n=k$, we suppose $\left(x_{k}, y_{k}\right) \leq(u, v)$. Therefore, for $n=$ $k+1$, we have

$$
\begin{align*}
& x_{k+1}=F\left(x_{k}, y_{k}\right) \leq F\left(u, y_{k}\right) \leq F(u, v)=u \\
& y_{k+1}=F\left(y_{k}, x_{k}\right) \geq F\left(v, x_{k}\right) \geq F(v, u)=v,  \tag{46}\\
& \text { i.e., }\left(x_{k+1}, y_{k+1}\right) \leq(u, v),
\end{align*}
$$

which implies (43) holds.
From (41) $\left(H_{2}\right)$,

$$
\begin{align*}
& \varphi( d \\
&\left.\left(u, x_{n+1}\right)+d\left(y_{n+1}, v\right)\right) \\
&= \varphi\left(d\left(F(u, v), F\left(x_{n}, y_{n}\right)\right)\right. \\
&+d\left(F\left(y_{n}, x_{n}\right), F(v, u)\right)  \tag{47}\\
& \leq k \varphi\left(d\left(u, x_{n}\right)+d\left(v, y_{n}\right)\right) \\
&-\psi\left(k\left[d\left(u, x_{n}\right)+d\left(v, y_{n}\right)\right]\right) \\
& \leq \varphi\left(d\left(u, x_{n}\right)+d\left(v, y_{n}\right)\right)
\end{align*}
$$

which implies

$$
\begin{equation*}
d\left(u, x_{n+1}\right)+d\left(y_{n+1}, v\right) \leq d\left(u, x_{n}\right)+d\left(v, y_{n}\right) . \tag{48}
\end{equation*}
$$

Set $\xi_{n}:=d\left(u, x_{n}\right)+d\left(y_{n}, v\right), n \in \mathbb{N}$. By (48), we have $0 \leq$ $\xi_{n+1} \leq \xi_{n}$. Then, there exists $\xi \in \mathbb{R}^{+}=[0,+\infty)$ such that

$$
\begin{equation*}
\xi=\lim _{n \longrightarrow \infty} \xi_{n}=\lim _{n \longrightarrow \infty}\left[d\left(u, x_{n}\right)+d\left(y_{n}, v\right)\right] . \tag{49}
\end{equation*}
$$

If $\xi>0$, by $\left(\mathrm{H}_{2}\right)$,

$$
\begin{align*}
\varphi(\xi) & =\lim _{n \rightarrow \infty} \varphi\left(\xi_{n}\right) \leq \lim _{n \rightarrow \infty}\left(k \varphi\left(\xi_{n}\right)\right)-\lim _{n \rightarrow \infty} \psi\left(k \xi_{n}\right) \\
& =k \varphi(\xi)-\lim _{n \rightarrow \infty} \psi\left(k \xi_{n}\right)<k \varphi(\xi)<\varphi(\xi) \tag{50}
\end{align*}
$$

which is a contradiction. So $\xi=0$. We have

$$
\begin{equation*}
0=\lim _{n} \xi_{n}=\lim _{n \rightarrow \infty}\left[d\left(u, x_{n}\right)+d\left(y_{n}, v\right)\right]=0 \tag{51}
\end{equation*}
$$

and then

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} d\left(u, x_{n}\right)=\lim _{n \longrightarrow \infty} d\left(y_{n}, v\right)=0 \tag{52}
\end{equation*}
$$

Similarly, we also have

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} d\left(w, x_{n}\right)=\lim _{n \longrightarrow \infty} d\left(y_{n}, s\right)=0 \tag{53}
\end{equation*}
$$

It follows from (52) (53) that

$$
\begin{equation*}
(u, v)=(w, s) . \tag{54}
\end{equation*}
$$

Corollary 7. In addition to the hypothesis of Corollary 4, suppose $\left(H_{5}\right)\left(\operatorname{or}\left(H_{5}^{\prime}\right)\right)$ holds. Then $F$ has a unique coupled fixed point.

Next, we state our third main result.
Theorem 8. In addition to the hypothesis of Theorem 6, suppose one of the followings holds:
(i) Elements of the coupled fixed point $(u, v)$ are comparable
(ii) $u_{0}$ and $v_{0}$ are comparable
(iii) Every pair of elements of $X$ has an upper bound or a lower bound in X

Then, we have $u=v$; that is, $F$ has a unique fixed point:

$$
\begin{equation*}
F(u, u)=u . \tag{55}
\end{equation*}
$$

Proof. From Theorem 6, $F$ has a unique coupled fixed point (u,v).

Case (i). Since $u$ and $v$ are comparable, we have $u \geq v$ or $u \leq v$. Suppose $u \geq v$; then $u=F(u, v)$ and $v=F(v, u)$ are comparable. By (4) in $\left(\mathrm{H}_{2}\right)$, one obtains

$$
\begin{align*}
\varphi( & 2 d(u, v))=\varphi(d(u, v)+d(v, u)) \\
= & \varphi(d(F(u, v), F(v, u))+d(F(v, u), F(u, v))) \\
\leq & k \varphi(d(u, v)+d(v, u))  \tag{56}\\
& \quad-\psi(k[d(u, v)+d(v, u)]) \leq k \varphi(2 d(u, v)) .
\end{align*}
$$

Noting $k \in(0,1)$, it follows from (56) that $d(u, v)=0$, i.e., $u=v$.

Case (ii). Since $u_{0}$ and $v_{0}$ are comparable, we have $u_{0} \geq v_{0}$ or $u_{0} \leq v_{0}$. Suppose we are in the first case, $u_{0} \geq v_{0}$. Let

$$
\begin{align*}
& u_{n+1}=F\left(u_{n}, v_{n}\right), \\
& v_{n+1}=F\left(v_{n}, u_{n}\right), \tag{57}
\end{align*}
$$

$$
n=0,1,2 \cdots
$$

From Theorem 3, the coupled fixed point $(u, v)$ satisfies

$$
\begin{align*}
& u=\lim _{n \rightarrow \infty} u_{n}=\lim _{n \rightarrow \infty} F\left(u_{n}, v_{n}\right), \\
& v=\lim _{n \rightarrow \infty} v_{n}=\lim _{n \rightarrow \infty} F\left(v_{n}, u_{n}\right) . \tag{58}
\end{align*}
$$

By $u_{0} \geq v_{0}$ and the mixed monotone property of $F$, one has

$$
\begin{equation*}
u_{1}=F\left(u_{0}, v_{0}\right) \geq F\left(v_{0}, u_{0}\right)=v_{1}, \tag{59}
\end{equation*}
$$

and, hence, by induction one obtains

$$
\begin{equation*}
u_{n} \geq v_{n}, \quad n=0,1,2 \cdots \tag{60}
\end{equation*}
$$

On the one hand, it follows from the continuity of the distance $d$ that

$$
\begin{align*}
d(u, v) & =d\left(\lim _{n \longrightarrow \infty} F\left(u_{n}, v_{n}\right), \lim _{n \longrightarrow \infty} F\left(v_{n}, u_{n}\right)\right) \\
& =\lim _{n \longrightarrow \infty} d\left(F\left(u_{n}, v_{n}\right), F\left(v_{n}, u_{n}\right)\right)  \tag{61}\\
& =\lim _{n \longrightarrow \infty} d\left(u_{n+1}, v_{n+1}\right) .
\end{align*}
$$

On the other hand, by $(4)$ in $\left(H_{2}\right)$, one gets

$$
\begin{align*}
\varphi & \left(2 d\left(u_{n+1}, v_{n+1}\right)\right)=\varphi\left(d\left(u_{n+1}, v_{n+1}\right)\right. \\
& \left.+d\left(v_{n+1}, u_{n+1}\right)\right)=\varphi\left(d\left(F\left(u_{n}, v_{n}\right), F\left(v_{n}, u_{n}\right)\right)\right. \\
& \left.+d\left(F\left(v_{n}, u_{n}\right), F\left(u_{n}, v_{n}\right)\right)\right) \leq k \varphi\left(d\left(u_{n}, v_{n}\right)\right.  \tag{62}\\
& \left.+d\left(v_{n}, u_{n}\right)\right)-\psi\left(k\left(d\left(u_{n}, v_{n}\right)+d\left(v_{n}, u_{n}\right)\right)\right) \\
& \leq k \varphi\left(2 d\left(u_{n}, v_{n}\right)\right) \leq k \cdot k \varphi\left(2 d\left(u_{n-1}, v_{n-1}\right)\right) \\
& \leq k^{n} \varphi\left(2 d\left(u_{1}, v_{1}\right)\right)
\end{align*}
$$

which implies

$$
\begin{equation*}
0=\lim _{n \longrightarrow \infty} \varphi\left(2 d\left(u_{n+1}, v_{n+1}\right)\right)=\varphi(2 d(u, v)) \tag{63}
\end{equation*}
$$

and therefore $d(u, v)=0$, which finishes the proof.

Case (iii). If $u, v$ are comparable, see Case (i). If $u, v$ are not comparable, then there exists $w \in X$ comparable to $u$ and $v$. Without loss of generality, we suppose $u \leq w, v \leq w$ (the other case is similar). In view of the order of $X \times X$ ( or $X^{2}$ for short), one has

$$
\begin{align*}
(u, v) & \geq(u, w) \\
(u, w) & \leq(w, u)  \tag{64}\\
(w, u) & \geq(v, u)
\end{align*}
$$

that is, $(u, v),(u, w),(u, w),(w, u),(w, u),(v, u)$ are comparable in $X^{2}$.

Inspired by [31], we consider the functional $d_{2}: X^{2} \times$ $X^{2} \longrightarrow \mathbb{R}^{+}=[0,+\infty)$ defined by

$$
\begin{align*}
& d_{2}(Y, V)=\frac{1}{2}[d(x, u)+d(y, v)],  \tag{65}\\
& \qquad \text { for } Y=(x, y), V=(u, v) \in X^{2} .
\end{align*}
$$

$d_{2}$ is a metric on $X^{2}$ and, moreover, if $(X, d)$ is complete, then $\left(X^{2}, d_{2}\right)$ is a complete metric space, too. Define another operator $T: X^{2} \longrightarrow X^{2}$ as follows:

$$
\begin{equation*}
T(V)=(F(u, v), F(v, u)) \quad \text { for } V=(u, v) \in X^{2} \tag{66}
\end{equation*}
$$

For $Y=(x, y), V=(u, v) \in X^{2}$, let $\bar{Y}=(y, x), \bar{V}=(v, u)$; one has

$$
\begin{align*}
& d_{2}(Y, V)=\frac{d(x, u)+d(y, v)}{2} \\
& d_{2}(T(Y), T(V)) \\
& \quad=\frac{d(F(x, y), F(u, v))+d(F(y, x), F(v, u))}{2}  \tag{67}\\
& \quad=\frac{d(F(Y), F(V))+d(F(\bar{Y}), F(\bar{V}))}{2} .
\end{align*}
$$

By contractive condition (4) in ( $H_{2}$ ), we get a Banach type contraction condition:

$$
\begin{align*}
\varphi( & 2 d_{2}(T(Y), T(V)) \\
& =\varphi(d(F(x, y), F(u, v))+d(F(y, x), F(v, u)))  \tag{68}\\
& \leq k \varphi\left(2 d_{2}(Y, V)\right)
\end{align*}
$$

and thus

$$
\begin{align*}
& \varphi\left(2 d_{2}(T(T(Y)), T(T(V)))\right. \\
& \quad \leq k \varphi\left(2 d_{2}(T(Y), T(V))\right) \leq k^{2} \varphi\left(2 d_{2}(Y, V)\right) \tag{69}
\end{align*}
$$

and, hence, by induction we have

$$
\begin{align*}
\varphi\left(2 d_{2}\left(T^{n}(Y), T^{n}(V)\right) \leq k^{n} \varphi\left(2 d_{2}(Y, V)\right)\right.  \tag{70}\\
n=1,2,3, \cdots .
\end{align*}
$$

Now, applying (70) to the comparable pairs $Y=$ $(u, v), V=(u, w), Y=(u, w), V=(w, u), Y=(w, u), V=$ $(v, u)$, one obtains

$$
\begin{align*}
& \varphi\left(2 d_{2}\left(T^{n}((u, v)), T^{n}((u, w))\right)\right. \\
& \quad \leq k^{n} \varphi\left(2 d_{2}((u, v),(u, w))\right)=k^{n} \varphi(d(v, w)),  \tag{71}\\
& \varphi\left(2 d_{2}\left(T^{n}((u, w)), T^{n}((w, u))\right)\right.  \tag{72}\\
& \quad \leq k^{n} \varphi\left(2 d_{2}((u, w),(w, u))\right)=k^{n} \varphi(2 d(u, w)), \\
& \varphi\left(2 d_{2}\left(T^{n}((w, u)), T^{n}((v, u))\right)\right.  \tag{73}\\
& \quad \leq k^{n} \varphi\left(2 d_{2}((w, u),(v, u))\right)=k^{n} \varphi(d(w, v)) .
\end{align*}
$$

Now, for $Y=(u, v), V=(v, u)$, note that $u=F(u, v)=$ $F(Y), v=F(v, u)=F(V)$; we get

$$
\begin{align*}
T(Y) & =(F(u, v), F(v, u))=(u, v)=Y \\
d(u, v) & =d(F(Y), F(V))=d(F(u, v), F(v, u))  \tag{74}\\
& =d(F(\bar{V}), F(\bar{Y}))
\end{align*}
$$

So, using the triangle inequality and (4) (71)-(74), we have

$$
\begin{align*}
& \varphi(2 d(u, v)) \\
&= \varphi\left(2 \times \frac{d(F(Y), F(V))+d(F(\bar{V}), F(\bar{Y}))}{2}\right) \\
&= \varphi\left(2 d_{2}\left(T^{n}((u, v)), T^{n}((v, u))\right)\right) \\
& \leq \varphi\left(2 d_{2}\left(T^{n}((u, v)), T^{n}((u, w))\right)\right)  \tag{75}\\
&+\varphi\left(2 d_{2}\left(T^{n}((u, w)), T^{n}((w, u))\right)\right) \\
&+\varphi\left(2 d_{2}\left(T^{n}((w, u)), T^{n}((v, u))\right)\right) \\
& \leq k^{n}[\varphi(d(v, w))+\varphi(2 d(u, w))+\varphi(d(w, v))] \\
& \longrightarrow 0 \quad(\text { as } n \longrightarrow \infty),
\end{align*}
$$

which implies $d(u, v)=0$, so $u=v$.
Corollary 9. In addition to the hypothesis of Corollary 7, suppose one of the followings holds:
(i) Elements of the coupled fixed point $(u, v)$ are comparable
(ii) $u_{0}$ and $v_{0}$ are comparable
(iii) Every pair of elements of $X$ has an upper bound or a lower bound in $X$

Then, we have $u=v$; that is, $F$ has a unique fixed point:

$$
\begin{equation*}
F(u, u)=u . \tag{76}
\end{equation*}
$$

Remark 10. Corollary 9 includes theorems in [31].
Example 11. Let $X=\mathbb{R}, d(x, y)=|x-y|$ and $F: X \times X \longrightarrow X$ be defined by

$$
\begin{equation*}
F(x, y)=\frac{1}{6}(x-2 y), \quad(x, y) \in X^{2}=X \times X \tag{77}
\end{equation*}
$$

Then,
(i) $F$ has a coupled fixed point
(ii) $F$ has a unique coupled fixed point
(iii) $F$ has a unique fixed point

Proof. Let $\varphi(t)=4 t, \psi(t)=t, t \in[0,+\infty)$. Since

$$
\begin{equation*}
F(x, y)=\frac{1}{6}(x-2 y), \quad(x, y) \in X^{2} \tag{78}
\end{equation*}
$$

one gets, for $(u, v),(m, n) \in X^{2}$,

$$
\begin{align*}
d(F(u, v), F(m, n)) & =\frac{1}{6}|u-2 v-m+2 n| \\
& =\frac{1}{6}|(u-m)+2(n-v)|  \tag{79}\\
& \leq \frac{1}{6}|u-m|+\frac{1}{3}|n-v|
\end{align*}
$$

$$
\begin{align*}
d(F(v, u), F(n, m)) & =\frac{1}{6}|v-2 u-n+2 m| \\
& =\frac{1}{6}|(v-n)+2(m-u)|  \tag{80}\\
& \leq \frac{1}{6}|n-v|+\frac{1}{3}|u-m|
\end{align*}
$$

and

$$
\begin{align*}
& \varphi(d(F(u, v), F(m, n))+d(F(v, u), F(n, m))) \\
& \quad \leq 2|(u-m)|+2|(n-v)| . \tag{81}
\end{align*}
$$

Note that

$$
\begin{equation*}
d(u, m)+d(v, n)=|u-m|+|n-v|, \tag{82}
\end{equation*}
$$

and, for $k \in[2 / 3,1)$,

$$
\begin{align*}
& k \varphi(d(u, m)+d(v, n))-\psi(k[d(u, m)+d(v, n)])  \tag{83}\\
& \quad=3 k|u-m|+3 k|n-v|,
\end{align*}
$$

and, hence, (81) (83) imply that

$$
\begin{align*}
\varphi(d & (F(u, v), F(m, n))+d(F(v, u), F(n, m))) \\
\leq & k \varphi(d(u, m)+d(v, n))  \tag{84}\\
& -\psi(k[d(u, m)+d(v, n)]) .
\end{align*}
$$

On the other hand, $F$ is continuous and $u_{0}=-1, v_{0}=1$ satisfy

$$
\begin{align*}
& F\left(u_{0}, v_{0}\right)=\frac{1}{6}\left(u_{0}-2 v_{0}\right)=-\frac{1}{2}>-1=u_{0}, \\
& F\left(v_{0}, u_{0}\right)=\frac{1}{6}\left(v_{0}-2 u_{0}\right)=\frac{1}{2}<1=v_{0} . \tag{85}
\end{align*}
$$

(i) As mentioned above, by Theorem 3, F has a coupled fixed point
(ii) Since $X^{2}$ has property $\left(H_{5}\right)$ or $\left(H_{5}^{\prime}\right)$, by Theorem 6 , the uniqueness of the coupled fixed point is obtained
(iii) Noting that $u_{0}$ and $v_{0}$ are comparable, by Theorem 8 (ii), $F$ has a unique fixed point

## 3. Applications to Integral Equations

As an application to the fixed point theorem proved in Section 2, we shall study a class of integral equation

$$
\begin{align*}
& u(x)=\int_{m}^{n}\left(q_{1}(x, y)+q_{2}(x, y)\right)  \tag{86}\\
& \quad \cdot(f(y, u(y))+g(y, u(y))) d y+p(x)
\end{align*}
$$

where $x \in I=[m, n]$. We will establish existence and uniqueness results. It is well known that some boundary value
problems are equivalent to an integral equation or a system of integral equations.

Let $X=C(I, \mathbb{R})$ be a partially ordered set such that, for $u, v \in X$,

$$
\begin{gather*}
u \leq v \Longleftrightarrow \\
u(x) \leq v(x), \tag{87}
\end{gather*}
$$

$$
x \in I .
$$

$X$ is endowed with the sup metric:

$$
\begin{equation*}
d(u, v)=\sup _{x \in I}|u(x)-v(x)|, \quad u, v \in X \tag{88}
\end{equation*}
$$

so ( $X, d$ ) is a complete metric space. The corresponding metric $d_{1}$ on $X^{2}$ is defined by

$$
\begin{align*}
d_{1} & \left(\left(u_{1}, v_{1}\right),\left(u_{2}, v_{2}\right)\right) \\
& =\frac{1}{2}\left[\sup _{x \in I}\left|u_{1}(x)-u_{2}(x)\right|+\sup _{x \in I}\left|v_{1}(x)-v_{2}(x)\right|\right], \tag{89}
\end{align*}
$$

and then consider on $X^{2}$ the partial order relation:

$$
\begin{aligned}
\quad\left(u_{1}, v_{1}\right) & \leq\left(u_{2}, v_{2}\right) \Longleftrightarrow \\
u_{1}(x) & \leq u_{2}(x) \\
\text { and } v_{1}(x) & \geq v_{2}(x) \text {, }
\end{aligned}
$$

$$
x \in I .
$$

A pair $(\alpha, \beta) \in X^{2}$ is called a coupled lower - upper solution of (86) if

$$
\alpha(x)
$$

$$
\begin{aligned}
\leq & \int_{m}^{n} q_{1}(x, y)(f(y, \alpha(y))+g(y, \beta(y))) d y \\
& +\int_{m}^{n} q_{2}(x, y)(f(y, \beta(y))+g(y, \alpha(y))) d y \\
& +p(x)
\end{aligned}
$$

$$
\beta(x)
$$

$$
\geq \int_{m}^{n} q_{1}(x, y)(f(y, \beta(y))+g(y, \alpha(y))) d y
$$

$$
+\int_{m}^{n} q_{2}(x, y)(f(y, \alpha(y))+g(y, \beta(y))) d y
$$

$$
+p(x)
$$

If $\alpha=\beta, \alpha$ will be a solution of (86).

Let $\theta:[0,+\infty) \longrightarrow[0,+\infty)$ be a nondecreasing function such that

$$
\begin{equation*}
\theta(u)=\frac{u}{2}-\psi\left(\frac{u}{2}\right), \quad u \in[0,+\infty) \tag{92}
\end{equation*}
$$

where $\psi \in \Psi$ satisfies $\psi\left(t_{1}+t_{2}\right) \leq \psi\left(t_{1}\right)+\psi\left(t_{2}\right), t_{1}, t_{2} \in$ $[0,+\infty)$.

We make the following assumptions:
(i) $q_{1} \in C(I \times I,[0,+\infty)), q_{2} \in C(I \times I,(-\infty, 0])$
(ii) $p \in C(I, \mathbb{R})$
(iii) $f, g \in C(I \times \mathbb{R}, \mathbb{R})$
(iv) there exist constants $\lambda, \mu>0$ such that, for $u, v \in$ $\mathbb{R}, u \geq$

$$
\begin{align*}
& 0 \leq f(x, u)-f(x, v) \leq \lambda \theta(u-v) \\
& -\mu \theta(u-v) \leq g(x, u)-g(x, v) \leq 0,  \tag{93}\\
& \max \{\lambda, \mu\} \sup _{x \in I} \int_{m}^{n}\left(q_{1}(x, y)-q_{2}(x, y)\right) d y \leq \frac{1}{4},
\end{align*}
$$

(v) $(\alpha, \beta) \in X^{2}$ is a coupled lower-upper solution of (86)

Theorem 12. Suppose (i) - (v) hold. Then (86) has a unique solution.

Proof. In order to obtain the (unique) solution of (86), define for $x \in[m, n]$ the operator $F: X \times X \longrightarrow X$ by

$$
\begin{aligned}
F(u, v) & (x) \\
= & \int_{m}^{n} q_{1}(x, y)(f(y, u(y))+g(y, v(y))) d y \\
& +\int_{m}^{n} q_{2}(x, y)(f(y, v(y))+g(y, u(y))) d y \\
& +p(x), \quad \forall x \in[m, n] .
\end{aligned}
$$

If $(u, v) \in X^{2}$ is a coupled fixed point of $F$, then we have

$$
\begin{align*}
& u(x)=F(u, v)(x), \\
& v(x)=F(v, u)(x) . \tag{95}
\end{align*}
$$

It is obvious that the fixed point of $F$ is the solution of (86). In what follows, we will show that $F$ has a unique fixed point.
(1) We will show that the operator $F$ has a unique coupled fixed point in $X^{2}$

Firstly, by (v), we have

$$
\begin{align*}
\alpha & \leq F(\alpha, \beta)  \tag{96}\\
F(\beta, \alpha) & \leq \beta .
\end{align*}
$$

Secondly, we check $F$ is mixed monotone for $u_{1}, u_{2} \in X$ such that $u_{1} \leq u_{2}$. From (i) and (iv), we have

$$
\begin{align*}
& F\left(u_{1}, v\right)(x)-F\left(u_{2}, v\right)(x) \\
& \quad=\int_{m}^{n} q_{1}(x, y)\left(f\left(y, u_{1}(y)\right)-\left(f\left(y, u_{2}(y)\right)\right) d y\right.  \tag{97}\\
& \quad+\int_{m}^{n} q_{2}(x, y)\left(g\left(y, u_{1}(y)\right)-g\left(y, u_{2}(y)\right)\right) d y
\end{align*}
$$

$$
\leq 0
$$

Similarly, for $v_{1}, v_{2} \in X$ such that $v_{1} \leq v_{2}$, we have

$$
\begin{align*}
& F\left(u, v_{1}\right)(x)-F\left(u, v_{2}\right)(x) \\
& \quad=\int_{m}^{n} q_{1}(x, y)\left(g\left(y, v_{1}(y)\right)-\left(g\left(y, v_{2}(y)\right)\right) d y\right.  \tag{98}\\
& \quad+\int_{m}^{n} q_{2}(x, y)\left(f\left(y, v_{1}(y)\right)-f\left(y, v_{2}(y)\right)\right) d y \\
& \quad \geq 0,
\end{align*}
$$

which yields that $F$ is mixed monotone.
Thirdly, we show that $F$ verifies the contraction condition. Let us consider $u, v, a, b \in X$ with $u \geq a, v \leq b$; we have

$$
\begin{align*}
& d(F(u, v), F(a, b))=\sup _{x \in I} \mid \int_{m}^{n} q_{1}(x, y)[(f(y, u(y))-f(y, a(y))-(g(y, b(y))-g(y, v(y))] d y \\
& \quad-\int_{m}^{n} q_{2}(x, y)[(f(y, b(y))-f(y, v(y))-(g(y, u(y))-g(y, a(y))] d y \mid \\
& \quad \leq \sup _{x \in I}\left\{\int_{m}^{n} q_{1}(x, y)[\lambda \theta(u(y)-a(y))+\mu \theta(b(y)-v(y))] d y\right.  \tag{99}\\
& \left.\quad-\int_{m}^{n} q_{2}(x, y)[\lambda \theta(b(y)-v(y))+\mu \theta(u(y)-a(y))] d y\right\} \leq \max \{\lambda, \mu\} \\
& \quad \cdot \sup _{x \in I} \int_{m}^{n}\left(q_{1}(x, y)-q_{2}(x, y)\right)[\theta(u(y)-a(y))+\theta(b(y)-v(y))] d y .
\end{align*}
$$

Similarly, one obtains

$$
\begin{align*}
& d(F(v, u), F(b, a)) \leq \max \{\lambda, \mu\} \\
& \quad \cdot \sup _{x \in I} \int_{m}^{n}\left(q_{1}(x, y)-q_{2}(x, y)\right)  \tag{100}\\
& \quad \cdot[\theta(u(y)-a(y))+\theta(b(y)-v(y))] d y .
\end{align*}
$$

Since

$$
\begin{align*}
& \theta(u(y)-a(y)) \leq \theta(d(u, a)), \\
& \theta(b(y)-v(y)) \leq \theta(d(b, v)), \tag{101}
\end{align*}
$$

which together with (iv) imply that

$$
\begin{align*}
& d(F(u, v), F(a, b))+d(F(v, u), F(b, a))=2 \\
& \quad \cdot \max \{\lambda, \mu\} \sup _{x \in I} \int_{m}^{n}\left(q_{1}(x, y)-q_{2}(x, y)\right) \\
& \quad \cdot[\theta(u(y)-a(y))+\theta(b(y)-v(y))] d y \\
& \quad \leq \theta(d(u, a))+\theta(d(b, v))=\frac{1}{2}[d(u, a)+d(b, v)]  \tag{102}\\
& \quad-\left[\psi\left(\frac{1}{2} d(u, a)\right)+\psi\left(\frac{1}{2} d(b, v)\right)\right] \\
& \quad \leq \frac{d(u, a)+d(b, v)}{2}-\psi\left(\frac{d(u, a)+d(b, v)}{2}\right)
\end{align*}
$$

which proves that $F$ verifies contraction condition (36) in Corollary 4.

Next, we consider a monotone nondecreasing sequence $\left\{u_{n}\right\} \subset X$ converging to $u \in X$. Then, for every $x \in I$, the sequence of real numbers

$$
\begin{equation*}
u_{1}(x) \leq u_{2}(x) \leq \cdots \leq u_{n}(x) \leq \cdots, \tag{103}
\end{equation*}
$$

converges to $u(x)$. So, for all $x \in I, n \in \mathbb{N}, u_{n}(x) \leq u(x)$ Hence, $u_{n} \leq u$, for all $n$. Similarly, we can verify that the limit $v(x)$ of monotone nonincreasing sequence $\left\{v_{n}\right\}$ in $X$ is a lower bound for all the elements in the sequence. That is, $v \leq v_{n}$ for all $n$.

Therefore, it follows from Corollary 4 that $F$ has a coupled fixed point $\left(u_{0}, v_{0}\right) \in X^{2}$, i.e.,

$$
\begin{align*}
u_{0} & =F\left(u_{0}, v_{0}\right) \\
\text { and } v_{0} & =F\left(v_{0}, u_{0}\right) . \tag{104}
\end{align*}
$$

On the other hand, $X$ has property $\left(H_{5}^{\prime}\right)$ since, for any $g, h \in X, M(x)=\max (g(x), h(x)), m(x)=\min (g(x), h(x))$, for each $x \in I$, are in $X$ and are the upper and lower bounds of $g, h$, respectively. Then, by Corollary $7, F$ has a unique coupled fixed point.
(2) Noting that $(\alpha, \beta)$ is a coupled lower - upper solution of (86), one obtains $\alpha(x) \leq \beta(x)$ for all $x \in I$. So, $\alpha$ and $\beta$ are comparable. By Corollary $9, F$ has a unique fixed point

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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# Research Article 

# Limit Cycles and Invariant Curves in a Class of Switching Systems with Degree Four 

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In this paper, a class of switching systems which have an invariant conic $x^{2}+c y^{2}=1, c \in R$, is investigated. Half attracting invariant conic $x^{2}+c y^{2}=1, c \in R$, is found in switching systems. The coexistence of small-amplitude limit cycles, large amplitude limit cycles, and invariant algebraic curves under perturbations of the coefficients of the systems is proved.

## 1. Introduction

It is well known that the 16th problem stated in 1900 by D. Hilbert is considered to be the most difficult problem in the 23 problems; it is far from being solved. Over past three decades, there have been many good results about this problem. As far as the maximal number of small-amplitude limit cycles which are bifurcated from an elementary center or focus is concerned, the best known result obtained by Bautin in 1952 [1] is $M(2)=3$, where $M(n)$ denotes the maximal number of small-amplitude limit cycles around a singular point with $n$ being the degree of polynomials in the system. For cubicdegree system, many good results have also been obtained. For example, a cubic system was constructed by Lloyd and Pearson [2] to show 9 limit cycles with the aid of purely symbolic computation. Moreover, Yu and Tian [3] proved that there can exist 12 limit cycles around an elementary center in a planar cubic-degree polynomial system. As far as we know this is the best result obtained so far for cubic-degree polynomial systems with all limit cycles around a single singular point. For $n \geq 4$, because of the difficulty of computation of focal values, there are very few results. An example of a quartic system with 8 limit cycles bifurcating from a fine focus [4] was given by Huang et al. Theory of rotated equations and applications to a population model can be found in [5]; they gave a new method to solve the center problem.

As far as the maximal number of limit cycles of polynomial systems is concerned, the best results published were given as follows. Articles [6, 7] proved that $H(2) \geqslant 4$, then [8-10] gave $H(3) \geqslant 12$ and [11, 12] obtained $H(4) \geqslant 16$ etc. Here, $H(n)$ denotes the maximal number of limit cycles of polynomial systems. Furthermore, 13 limit cycles bifurcated from $Z_{2}$-equivariant systems with degree 3 were proved in [13-15], respectively. An improvement on the number of limit cycles bifurcating from a nondegenerate center of homogeneous polynomial systems was given in [16].

Center and the coexistence of large and small-amplitude limit cycles problems are two closely related questions of the 16th problem. Algebraic trajectories play an important role in the dynamical behavior of polynomial systems, so it has been an interesting problem in planar polynomial systems. Over the past twenty years, many interesting results were got for quadratic systems; the authors in $[17,18]$ proved that quadratic systems with a pair of straight lines or an invariant hyperbola, ellipse, can have no limit cycles other than the possible ellipse itself. Furthermore, if there is an invariant line, there can be no more than one limit cycle. The case of parabola was considered in [19]. For cubic systems, there exist different classes of cubic systems in which there may coexist an invariant hyperbola or straight lines with limit cycles (see [20-28]). For a given family of real planar polynomial systems of ordinary differential equations depending on parameters,
the problem of how to find the systems in the family which become time-reversible was solved in [29].

In modelling many practical problems in science and engineering, switching systems have been widely used recently. The richness of dynamical behavior found in switching systems covers almost all the phenomena discussed in general continuous systems. For example, the maximum number of limit cycles bifurcating from the periodic orbits of the quadratic isochronous centers of switching system was studied in [30]. In [31], limit cycles in a class of continuous and switching cubic polynomial systems were investigated. Bifurcation of limit cycles in switching quadratic systems with two zones was considered in [30]. In [32, 33], the authors considered nonsmooth Hopf bifurcation in switching systems. Limit cycles bifurcating from centers of discontinuous quadratic systems were studied by Chen and Du [34]. Switching Bautin system was also investigated in [35]; they got 10 limit cycles for this system. $Z_{2}$-equivariant cubic systems were also investigated, and 14 limit cycles were obtained in [36]. Bifurcation of periodic orbits by perturbing high-dimensional piecewise smooth integrable systems was investigated in [37]. Bifurcation theory for finitely smooth planar autonomous differential systems was considered in [38]. All results obtained show that the dynamical behavior of switching systems is more complex than continuous system.

About algebraic invariant curves, as far as we know, there are few papers to consider switching system with algebraic invariant curves. In this paper we are concerned with the limit cycle problem and the center problem for a class of degree four polynomial differential systems

$$
\begin{align*}
\frac{d x}{d t} & =\lambda x+y\left(-1-b x-e y+(1+c d) x^{2}\right. \\
& \left.+c(1+2 c d) y^{2}+b x^{3}+e x^{2} y+b c x y^{2}+c e y^{3}\right) \\
& -\left(x^{3}+c x y^{2}\right) \lambda, \\
\frac{d y}{d t} & =\lambda y+x\left(1-c y-(1+d) x^{2}-c(1+2 d) y^{2}\right. \\
& \left.+c x^{2} y+c^{2} y^{3}\right)-\lambda\left(x^{2} y+c y^{3}\right), \\
\frac{d x}{d t} & =\lambda_{1} x+y\left(-1-b_{1} x-e_{1} y+\left(1+c d_{1}\right) x^{2}\right.  \tag{1}\\
& \left.+c\left(1+2 c d_{1}\right) y^{2}+b_{1} x^{3}+e_{1} x^{2} y+b_{1} c x y^{2}+c e_{1} y^{3}\right) \\
& -\left(x^{3}+c_{1} x y^{2}\right) \lambda_{1}, \\
\frac{d y}{d t} & =\lambda_{1} x+x\left(1-c y-\left(1+d_{1}\right) x^{2}-c\left(1+2 d_{1}\right) y^{2}\right. \\
& \left.+c x^{2} y+c^{2} y^{3}\right)-\lambda_{1}\left(x^{2} y+c_{1} y^{3}\right),
\end{align*}
$$

which have an invariant conic $x^{2}+c y^{2}=1, c \in R$, and we prove the coexistence of large elliptic limit cycle that contains at least four small-amplitude limit cycles generated by Hopf bifurcations.

The rest of the paper is organized as follows. In the next section, we prove that the switching system (1) has an
invariant conic $x^{2}+c y^{2}=1, c \in R$, and there exists a large limit cycle in switching system (1); half attracting invariant conic $x^{2}+c y^{2}=1, c \in R$, is found in switching systems. In Section 3, the first eight Lyapunov constants will be computed; bifurcation of limit cycles and center conditions of (1) are investigated. Section 4 is devoted to discuss the number of limit cycles with different parameter $c$ of (1). At last, coexistence of invariant curve and limit cycles of (1) is drawn in Section 5.

## 2. Invariant Curve and Large Limit Cycle of (1)

In this section, we will prove that the switching system (1) has an invariant conic $x^{2}+c y^{2}=1, c \in R$, and there exists a large limit cycle in switching system (1).

Lemma 1. The conic $h(x, y)=x^{2}+c y^{2}-1=0, c \in R$, is an invariant algebraic curve of system (1). In particular, if $c>$ 0 and $d d_{1} \neq 0$, this conic is an elliptic hyperbolic limit cycle, attracting if $\lambda>0, \lambda_{1}>0$, a repelling if $\lambda \leq 0, \lambda_{1} \leq 0$, and half attracting if $\lambda \lambda_{1}<0$.

Proof. It is easy to know that the conic $h(x, y)=x^{2}+c y^{2}-1$, $c \in R$, is an invariant algebraic curve of systems

$$
\begin{align*}
& \frac{d x}{d t}=\lambda x+y\left(-1-b x-e y+(1+c d) x^{2}\right. \\
& \left.\quad+c(1+2 c d) y^{2}+b x^{3}+e x^{2} y+b c x y^{2}+c e y^{3}\right) \\
& \quad-\left(x^{3}+c x y^{2}\right) \lambda,  \tag{2}\\
& \frac{d y}{d t}=\lambda y+x\left(1-c y-(1+d) x^{2}-c(1+2 d) y^{2}\right. \\
& \left.\quad+c x^{2} y+c^{2} y^{3}\right)-\lambda\left(x^{2} y+c y^{3}\right),
\end{align*}
$$

and

$$
\begin{align*}
& \frac{d x}{d t}=\lambda_{1} x+y\left(-1-b_{1} x-e_{1} y+\left(1+c d_{1}\right) x^{2}\right. \\
& \left.+c\left(1+2 c d_{1}\right) y^{2}+b_{1} x^{3}+e_{1} x^{2} y+b_{1} c x y^{2}+c e_{1} y^{3}\right) \\
& -\left(x^{3}+c_{1} x y^{2}\right) \lambda_{1},  \tag{3}\\
& \frac{d y}{d t}=\lambda_{1} x+x\left(1-c y-\left(1+d_{1}\right) x^{2}-c\left(1+2 d_{1}\right) y^{2}\right. \\
& \left.+c x^{2} y+c^{2} y^{3}\right)-\lambda_{1}\left(x^{2} y+c_{1} y^{3}\right),
\end{align*}
$$

because

$$
\begin{equation*}
\frac{d h(x, y)}{d t}=h(x, y) k_{i}(x, y), \quad i=1,2 \tag{4}
\end{equation*}
$$



Figure 1: The half attracting conic when $\lambda>0, \lambda_{1}<0$ or $\lambda<0, \lambda_{1}>0$.
respectively, where

$$
\begin{align*}
k_{1}(x, y)= & -\lambda x^{2}+(1-c) x y-c \lambda y^{2}+b x^{2} y \\
& +\left(c^{2}+e\right) x y^{2},  \tag{5}\\
k_{2}(x, y)= & -\lambda_{1} x^{2}+(1-c) x y-c \lambda_{1} y^{2}+b_{1} x^{2} y \\
& +\left(c^{2}+e_{1}\right) x y^{2} .
\end{align*}
$$

In particular, according to Lemma 1 in [39], if $c>0$ and $d \neq 0$, this conic is an elliptic hyperbolic limit cycle of system (2), attracting if $\lambda>0$ and a repelling if $\lambda<0$. Similarly, if $c>0$ and $d_{1} \neq 0$, this conic is an elliptic hyperbolic limit cycle of system (3), attracting if $\lambda_{1}>0$ and a repelling if $\lambda_{1}<0$. Especially, if $c>0$ and $d d_{1} \neq 0$ and $\lambda \lambda_{1}<0$, the stability of the conic $x^{2}+c y^{2}=1, c \in R$, is contradict for the upper half system and lower half system.

So, for switching system (1), the conic $x^{2}+c y^{2}=1, c \in$ $R$, is an invariant algebraic curve. Furthermore, if $c>0$ and $d d_{1} \neq 0$, this conic is an elliptic hyperbolic limit cycle, and

$$
\begin{array}{cl}
\text { attracting } & \lambda>0, \lambda_{1}>0 \\
\text { repelling } & \lambda<0, \lambda_{1}<0 \tag{6}
\end{array}
$$

halfattracting $\quad \lambda \lambda_{1}<0$.

Remark 2. For planar continuous system, if $c>0$ and $d \neq 0$, the conic $x^{2}+c y^{2}=1, c \in R$, is an elliptic hyperbolic limit cycle, attracting if $\lambda>0$, a repelling if $\lambda \leq 0$. For switching system, half attracting cases which are different from continuous systems appear. Namely, for the conic $x^{2}+$ $c y^{2}=1, c \in R$, it is attracting (repelling) for $y>0$ and repelling (attracting) for $y<0$. It is an interesting phenomena; see Figure 1.

## 3. Bifurcation of Limit Cycle and Center Conditions of (1)

First of all, it is easy to know that the origin of upper half system and lower half system is a fine focus if $\lambda \lambda_{1} \neq 0$, so we let $\lambda=\lambda_{1}=0$ in order to consider the center conditions
and the number of small limit cycles. With the aid of symbolic computation, we obtain the following result.

Theorem 3. For system (1), the first eight Lyapunov constants at the origin are given by

$$
\begin{align*}
\mu_{1} & =-\frac{2}{3}\left(b_{1}-b\right) \\
\mu_{2} & =\frac{b \pi}{8}\left(e_{1}+e\right),  \tag{7}\\
\mu_{3} & =-\frac{2 b}{45}\left(6 d+3 c d+12 c^{2} d-6 d_{1}-3 c d_{1}-12 c^{2} d_{1}\right. \\
& +4 c e),
\end{align*}
$$

with two cases: $(I) c= \pm \sqrt{2} / 2$.

$$
\begin{align*}
\mu_{4} & =\frac{b c(64+21 c) e^{2} \pi}{288(4+c)}, \\
\mu_{5} & =0 \\
\mu_{6} & =\frac{b c d_{1} \pi}{96}\left(-9+b^{2}+7 c-6 c d_{1}\right), \\
\mu_{7} & =\frac{b c d_{1} \pi}{829440 c}\left(11763+2434 b^{2}-66 b^{4}-16486 c\right. \\
& \left.-3792 b^{2} c+544 b^{4} c\right), \\
\mu_{8} & =\frac{b \pi}{978447237120 c^{3}}(-632874756819  \tag{8}\\
& -1564815308390 b^{2}-48058429290 b^{4} \\
& +31851335688 b^{6}-1734059496 b^{8} \\
& +903172527570 c+2206928649816 b^{2} c \\
& +78231277128 b^{4} c-42656920272 b^{6} c \\
& \left.+1359726496 b^{8} c\right)
\end{align*}
$$

$$
\begin{align*}
& \text { (II) } c \neq \pm \sqrt{2} / 2 \text {. } \\
& \mu_{4}=\frac{b c \pi}{288\left(2+c+4 c^{2}\right)}\left(-48 d_{1}-24 c d_{1}+48 c^{3} d_{1}\right. \\
& \left.+192 c^{4} d_{1}+16 c e-32 c^{3} e+64 e^{2}+21 c e^{2}\right), \\
& \mu_{5}=-\frac{b c e}{14175\left(-1+2 c^{2}\right)\left(2+c+4 c^{2}\right)^{2}}(14400 \\
& -1440 b^{2}+12240 c-720 b^{2} c+8824 c^{2}+304 c^{3} \\
& +1440 b^{2} c^{3}-54816 c^{4}+5760 b^{2} c^{4}-25440 c^{5} \\
& -36768 c^{6}-48256 c^{7}-8192 c^{8}-1152 e^{2} \\
& -10764 c e^{2}-128130 c^{2} e^{2}-74937 c^{3} e^{2} \\
& -43644 c^{4} e^{2} \text { ), } \\
& \mu_{6}=-\frac{b c e^{2} \pi}{1658880\left(-1+2 c^{2}\right)^{2}\left(2+c+4 c^{2}\right)^{2}}(-938880  \tag{9}\\
& -740160 c+1065264 c^{2}-483240 c^{3}+2351920 c^{4} \\
& -1092224 c^{5}-3670368 c^{6}+8556192 c^{7} \\
& +4228224 c^{8}+2964992 c^{9}+409600 c^{10}-58752 e^{2} \\
& +1558128 c e^{2}+8882094 c^{2} e^{2}+6387798 c^{3} e^{2} \\
& \left.+3861933 c^{4} e^{2}+2842044 c^{5} e^{2}+2182200 c^{6} e^{2}\right), \\
& \mu_{7}=-\frac{8 b c e}{8037225\left(2+c+4 c^{2}\right)^{3} f(c)^{2}} f_{1}(c) \text {, } \\
& \mu_{8}=-\frac{8 b c e \pi}{2799360\left(2+c+4 c^{2}\right)^{3} f(c)^{3}} f_{2}(c) \text {. }
\end{align*}
$$

where

$$
\begin{aligned}
& f(c)=-19584+519376 c+2960698 c^{2}+2129266 c^{3} \\
& \quad+1287311 c^{4}+947348 c^{5}+727400 c^{6}, \\
& f_{1}(c)=(-16642999723622400 \\
& \quad-533641875920732160 c \\
& \quad-1422483722108868096 c^{2} \\
& \quad-6179682878361868032 c^{3} \\
& \quad-16468630764742211040 c^{4} \\
& \quad-20952718578375299760 c^{5} \\
& \quad-47133134356437147504 c^{6} \\
& -20260317246600069740 c^{7}
\end{aligned}
$$

$$
\begin{aligned}
& +1872045962529690518 c^{8} \\
& +124503308483978457972 c^{9} \\
& +384905600471956478466 c^{10} \\
& +639925619448235723897 c^{11} \\
& +995949036122547224578 c^{12} \\
& +1051854664680123230362 c^{13} \\
& +989202162935234549752 c^{14} \\
& +722398370839930377872 c^{15} \\
& +477074650742338886960 c^{16} \\
& +266605013954075833728 c^{17} \\
& +102005297657011481216 c^{18} \\
& +26101293898978550016 c^{19} \\
& +1081597406307225600 c^{20} \\
& +215716197433324981017 c^{12} \\
& +1064766586075843968365 c^{13} \\
& +1162211975363604480 c^{21} \\
& +812488329237341411240 c^{7} \\
& +516471221184905286240 c^{6} \\
& \left.+2379375230976000 c^{22}\right) \\
& + \\
& +178634371590216048576 c^{5} \\
& +2718583098083868672 c \\
& +
\end{aligned}
$$

$$
\begin{align*}
& +4034487317277427888208 c^{14} \\
& +5640449144757272315336 c^{15} \\
& +6613022295691417350356 c^{16} \\
& +5376059971880816237612 c^{17} \\
& +4585410563561734908832 c^{18} \\
& +3103125495812061611648 c^{19} \\
& +1361862580731165716224 c^{20} \\
& +378280087173029995008 c^{21} \\
& +12407010796058419200 c^{22} \\
& -18830025122086993920 c^{23} \\
& \left.+39259691311104000 c^{24}\right) . \tag{10}
\end{align*}
$$

Note that in computing the above expressions $\mu_{k}, k=2, \ldots, 8$, and $\mu_{1}=\mu_{2}=\cdots=\mu_{k-1}=0$ have been used.

The following proposition follows directly from Theorem 3.

Proposition 4. The first eight Lyapunov constants at the origin of system (1) become zero if and only if one of the following conditions is satisfied:

$$
\begin{align*}
& \lambda_{1}=\lambda=0, \\
& b_{1}=b=0,  \tag{11}\\
& \lambda_{1}=\lambda=0, \\
& b_{1}=b, \\
& e_{1}=-e,  \tag{12}\\
& d_{1}=d, \\
& c=0 ; \\
& \lambda_{1}=\lambda=0, \\
& b_{1}=b, \\
& e_{1}=-e,  \tag{13}\\
& d_{1}=d, \\
& e=0 .
\end{align*}
$$

They are also the center conditions of system (1).
Proof. When the conditions in (11) hold, system (1) can be brought to

$$
\begin{align*}
& \frac{d x}{d t}=y\left(-1-e y+(1+c d) x^{2}+c(1+2 c d) y^{2}\right. \\
& \left.\quad+e x^{2} y+c e y^{3}\right), \\
& \frac{d y}{d t}=x\left(1-c y-(1+d) x^{2}-c(1+2 d) y^{2}+c x^{2} y\right. \\
& \left.\quad+c^{2} y^{3}\right) \\
& \quad(y>0),  \tag{14}\\
& \frac{d x}{d t}=y\left(-1-e_{1} y+\left(1+c d_{1}\right) x^{2}+c\left(1+2 c d_{1}\right) y^{2}\right. \\
& \left.\quad+e_{1} x^{2} y+c e_{1} y^{3}\right), \\
& \frac{d y}{d t}=x\left(1-c y-\left(1+d_{1}\right) x^{2}-c\left(1+2 d_{1}\right) y^{2}+c x^{2} y\right. \\
& \left.\quad+c^{2} y^{3}\right)
\end{align*}
$$

Obviously, the system is symmetric with the $y$-axis, and so the origin is a center of system (14).

When the conditions in (12) hold, system (1) can be rewritten as

$$
\begin{align*}
& \frac{d x}{d t}=y\left(-1-b x-e y+x^{2}+b x^{3}+e x^{2} y\right) \\
& \frac{d y}{d t}=x\left(1-(1+d) x^{2}\right) \\
& \frac{d x}{d t}=y\left(-1-b x+e y+x^{2}+b x^{3}-e x^{2} y\right)  \tag{15}\\
& \frac{d y}{d t}=x\left(1-(1+d) x^{2}\right)
\end{align*}
$$

$$
(y<0),
$$

which is symmetric with the $x$-axis, and so the origin is a center of system (15).

When the conditions in (13) hold, system (1) becomes a continuous system

$$
\begin{align*}
& \frac{d x}{d t}=y(1+b x)\left(-1+x^{2}+c y^{2}\right) \\
& \frac{d y}{d t}=x(-1+c y)\left(-1+x^{2}+c y^{2}\right) \tag{16}
\end{align*}
$$

By elementary integration, the above system in $\Omega=(x, y) \mid$ $x^{2}+c y^{2}<1$ is topologically equivalent to the system

$$
\begin{align*}
& \frac{d x}{d t}=y(1+b x)  \tag{17}\\
& \frac{d y}{d t}=x(-1+c y)
\end{align*}
$$

which has the first analytic integral

$$
\begin{align*}
H(x, y)= & b c(b y-c x)+b^{2} \ln (1-c y)  \tag{18}\\
& +c^{2} \ln (1+b x)
\end{align*}
$$

Remark 5. The phase plane of system (16) can be drawn by Maple; see Figure 2.

As far as limit cycles are concerned, it follows from Theorem 3 that at most 8 limit cycles can bifurcate from the origin of system (1). We have the following theorem.

Theorem 6. If the origin of system (1) is a 8th-order weak focus, then for $0<\delta_{1}, \delta_{2} \ll 1,8$ small-amplitude limit cycles can bifurcate from the origin of the perturbed system (1).

Proof. When the origin of system (1) is a 8th-order weak focus, the conditions

$$
\begin{aligned}
& b_{1}= \\
& e_{1}=-e \\
& d= \frac{6 d_{1}+3 c d_{1}+12 c^{2} d_{1}-4 c e}{3\left(2+c+4 c^{2}\right)}, \\
& d_{1}=-\frac{e_{1}\left(-16 c+32 c^{3}+64 e_{1}+21 c e_{1}\right)}{24\left(-1+2 c^{2}\right)\left(2+c+4 c^{2}\right)}, \\
& b^{2}= \frac{1}{720\left(-1+2 c^{2}\right)\left(2+c+4 c^{2}\right)}(-14400-12240 c \\
&-8824 c^{2}-304 c^{3}+54816 c^{4}+25440 c^{5}+36768 c^{6} \\
&+48256 c^{7}+8192 c^{8}+1152 e^{2}+10764 c e^{2} \\
&\left.+128130 c^{2} e^{2}+74937 c^{3} e^{2}+43644 c^{4} e^{2}\right), \\
& e^{2}=-\frac{8}{3 f(c)}\left(-117360-92520 c+133158 c^{2}\right. \\
&-60405 c^{3}+293990 c^{4}-136528 c^{5}-458796 c^{6} \\
&\left.+1069524 c^{7}+528528 c^{8}+370624 c^{9}+51200 c^{10}\right), \\
& c=-2.09067,-1.9427,-0.763201,0.581824
\end{aligned}
$$

should be satisfied. Furthermore, one has the following: When $c=-2.09067$,

$$
\begin{equation*}
\frac{\partial\left(\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}, \mu_{6}, \mu_{7}\right)}{\partial\left(b_{1}, e_{1}, d, d_{1}, b, e, c\right)}=-2.23342 \times 10^{6} \neq 0 \tag{20}
\end{equation*}
$$

When $c=-1.9427$,

$$
\begin{equation*}
\frac{\partial\left(\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}, \mu_{6}, \mu_{7}\right)}{\partial\left(b_{1}, e_{1}, d, d_{1}, b, e, c\right)}=1.31166 \times 10^{6} \neq 0 \tag{21}
\end{equation*}
$$



Figure 2: The phase plane of system (16) when $b=1, c=1$ or $b=1, c=-1$.

When $c=-0.763201$,

$$
\begin{equation*}
\frac{\partial\left(\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}, \mu_{6}, \mu_{7}\right)}{\partial\left(b_{1}, e_{1}, d, d_{1}, b, e, c\right)}=-0.23494 \neq 0 \tag{22}
\end{equation*}
$$

When $c=0.581824$,

$$
\begin{equation*}
\frac{\partial\left(\mu_{1}, \mu_{2}, \mu_{3}, \mu_{4}, \mu_{5}, \mu_{6}, \mu_{7}\right)}{\partial\left(b_{1}, e_{1}, d, d_{1}, b, e, c\right)}=-0.000330085 \neq 0 \tag{23}
\end{equation*}
$$

So it implies that 8 small-amplitude limit cycles can bifurcate from the origin of the perturbed system (1).

## 4. Number of Limit Cycles with Different Parameter $c$ of (1)

In this section, we devote to discuss the number of limit cycles with different parameter $c$ of (1). The following theorem could be concluded from Theorem 3.

Theorem 7. The number of limit cycles with different parameter $c$ of (1) can be shown in the Table 1.

Proof. Let $\mu_{1}=\mu_{2}=\mu_{3}=0$, it is easy to obtain that

$$
\begin{align*}
b_{1} & =b \\
e_{1} & =-e  \tag{24}\\
d & =\frac{6 d_{1}+3 c d_{1}+12 c^{2} d_{1}-4 c e}{3\left(2+c+4 c^{2}\right)}
\end{align*}
$$

If $c=0$, it is easy to check that the origin is a three-order weak focus. Furthermore, if $c \neq 0$, when $c=\sqrt{2} / 2$, the Lyapunov constants in case 1 yield that the origin is a seventh order weak focus.

When $c \neq \sqrt{2} / 2$, the Lyapunov constants $\mu_{4}=\mu_{5}=\mu_{6}=$ $\mu_{7}=0, \mu_{8} \neq 0$ yield that the origin is an eighth-order weak focus if

$$
\begin{align*}
& d_{1}=-\frac{e_{1}\left(-16 c+32 c^{3}+64 e_{1}+21 c e_{1}\right)}{24\left(-1+2 c^{2}\right)\left(2+c+4 c^{2}\right)}, \\
& b^{2}=\frac{f_{3}(c)}{720\left(-1+2 c^{2}\right)\left(2+c+4 c^{2}\right)}=F_{3}(c),  \tag{25}\\
& e^{2}=-\frac{8 f_{4}(c)}{3 f(c)}=F_{4}(c),
\end{align*}
$$

Table 1: The maximum number of small limit cycles around the origin for different parameter c .

| Parameter c | The maximum number of <br> small limit cycle around <br> the origin |
| :--- | :---: |
| $(-\infty,-6.14829)$ | 6 |
| $(-6.14829,-2.09067)$ | 7 |
| -2.09067 | 8 |
| $(-2.09067,-1.94278)$ | 7 |
| -1.94278 | 8 |
| $(-1.94278,-0.7632018)$ | 7 |
| -0.7632018 | 8 |
| $(-0.7632018,-0.707107]$ | 7 |
| $(-0.707107,-0.588861]$ | 6 |
| $(-0.588861,-0.44638)$ | 7 |
| $[-0.44638,-0.031805]$ | 6 |
| $(-0.031805,0)$ | 7 |
| 0 | 3 |
| $(0,0.031805)$ | 7 |
| $[0.031805,0.129745]$ | 6 |
| $(0.129745,0.581824)$ | 7 |
| $(0.581824$ | $881824,0.707107]$ |
| $(0.707107,+\infty)$ | 7 |

where

$$
\begin{align*}
f_{3}(c)= & -14400-12240 c-8824 c^{2}-304 c^{3} \\
& +54816 c^{4}+25440 c^{5}+36768 c^{6} \\
& +48256 c^{7}+8192 c^{8}+1152 e^{2}+10764 c e^{2} \\
& +128130 c^{2} e^{2}+74937 c^{3} e^{2}+43644 c^{4} e^{2}  \tag{26}\\
f_{4}(c)= & -117360-92520 c+133158 c^{2}-60405 c^{3} \\
& +293990 c^{4}-136528 c^{5}-458796 c^{6} \\
& +1069524 c^{7}+528528 c^{8}+370624 c^{9} \\
& +51200 c^{10} .
\end{align*}
$$

and $c$ satisfy that

$$
\begin{align*}
f(c)= & -19584+519376 c+2960698 c^{2}+2129266 c^{3}  \tag{27}\\
& +1287311 c^{4}+947348 c^{5}+727400 c^{6} \neq 0 .
\end{align*}
$$

It is easy to conclude that if $F_{3}(c)>0, F_{4}(c)>0$ and $f_{1}(c)=$ $0, f_{2}(c) \neq 0$, there exist 8 limit cycles; namely,

$$
\begin{equation*}
c \approx-2.09067,-1.9427,-0.763201,0.581824 . \tag{28}
\end{equation*}
$$

If $F_{3}(c)>0, F_{4}(c)>0$, and $f_{1}(c) \neq 0$, there exist 7 small limit cycles.


Figure 3: When $c=-2.09067$, an invariant algebraic curve $x^{2}-$ $2.09067 y^{2}=1$ and eight small limit cycles.


Figure 4: When $c=-\sqrt{2} / 2$, an invariant algebraic curve $x^{2}-$ $(\sqrt{2} / 2) y^{2}=1$ and seven small limit cycles.

If $F_{3}(c) F_{4}(c) \leq 0$, there exist 6 limit cycles.
When $c \neq \sqrt{2} / 2, f(c)=0$, the Lyapunov constants $\mu_{4}=$ $\mu_{5}=0, \mu_{6} \neq 0$ yield that the origin is a sixth-order weak focus. The conclusion can be given in Table 1 for simplify.

## 5. Coexistence of Invariant Curve and Limit Cycles of (1)

From above discussion, we study the coexistence of invariant curve and limit cycles of (1), by perturbation method of small parameters, the following conclusions could be got easily; for example, when $c=-2.09067$, there exist eight small limit cycles at least and $x^{2}-2.09067 y^{2}=1$ is an invariant algebraic curve. The distribution of limit cycle can be drawn in Figure 3.

When $c=-\sqrt{2} / 2$, there exist seven small limit cycles at least and $x^{2}-(\sqrt{2} / 2) y^{2}=1$ is an invariant algebraic curve. The distribution of limit cycle can be drawn in Figure 4.

When $c=0$, there exist three small limit cycles at least and $x=1$ and $x=-1$ are two invariant lines. The distribution of limit cycle can be drawn in Figure 5.

When $c=0.581824$, there exist eight small limit cycles and a large limit cycle $x^{2}+0.581824 y^{2}=1$ at the same time, namely, nine limit cycles in total for this system. The distribution of limit cycle can be drawn in Figure 6.

When $c=\sqrt{2} / 2$, there exist seven small limit cycles at least and there is a large limit cycle $x^{2}+(\sqrt{2} / 2) y^{2}=1$ at the


Figure 5: When $c=0$, two invariant lines $x= \pm 1$ and three small limit cycles.


Figure 6: When $c=0.581824$, an invariant algebraic curve $x^{2}+$ $0.581824 y^{2}=1$ and eight small limit cycles.


Figure 7: When $c=\sqrt{2} / 2$, an invariant algebraic curve $x^{2}+$ $(\sqrt{2} / 2) y^{2}=1$ and seven small limit cycles.


Figure 8: When $c=1$, an invariant algebraic curve $x^{2}+y^{2}=1$ and six small limit cycles.
same time. The distribution of limit cycles can be drawn in Figure 7.

When $c=1$, there exist six small limit cycles at least and there is a large limit cycle $x^{2}+y^{2}=1$ at the same time. The distribution of limit cycles can be drawn in Figure 8.

## 6. Conclusion

In this paper, a class of switching systems is investigated; the coexistence of small limit cycles and algebraic an invariant curve is proves. An interesting phenomenon that the algebraic invariant curve $x^{2}+c y^{2}=1, c>0$, can be half attracting is found.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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## Research Article

# Positive Solutions for a System of Fractional Differential Equations with Two Parameters 

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In this paper, the existence of positive solutions in terms of different values of two parameters for a system of conformable-type fractional differential equations with the p-Laplacian operator is obtained via Guo-Krasnosel'skii fixed point theorem.

## 1. Introduction

In this paper, we study the existence of positive solutions for the following system of fractional differential equations:

$$
\begin{array}{ll}
D^{\alpha_{1}}\left(\phi_{p_{1}}\left(D^{\alpha_{1}} x(t)\right)\right)=\lambda g(t, x(t), y(t)), & 0<t<1, \\
D^{\alpha_{2}}\left(\phi_{p_{2}}\left(D^{\alpha_{2}} y(t)\right)\right)=\mu f(t, x(t), y(t)), & 0<t<1, \tag{1}
\end{array}
$$

subject to the following boundary condition:

$$
\begin{align*}
x(0) & =x(1)=0, \\
D^{\alpha_{1}} x(0) & =D^{\alpha_{1}} x(1)=0, \\
y(0) & =y(1)=0,  \tag{2}\\
D^{\alpha_{2}} y(0) & =D^{\alpha_{2}} y(1)=0,
\end{align*}
$$

where $\alpha_{1}, \alpha_{2} \in(1,2]$ are real numbers; $D^{\alpha_{1}}$ and $D^{\alpha_{2}}$ are the conformable fractional derivative; $\phi_{p_{i}}(s)=|s|^{p_{i}-2} s, p_{i}>$ $1, \phi_{q_{i}}=\phi_{p_{i}}^{-1}, 1 / p_{i}+1 / q_{i}=1, i=1,2 ; g, f:[0,1] \times$ $[0,+\infty) \longrightarrow[0,+\infty)$ are continuous; $\lambda$ and $\mu$ are positive parameters.

Fractional differential equations have many applications in various fields such as biological science, chemistry, physics, and engineering. Many authors have made large achievements about the study of fractional differential equations boundary value problems. Most results have adopted the Riemann-Liouville and Caputo-type fractional derivatives;
we can see [1-28] and the references therein; for example, in [28], by using Guo-Krasnosel'skii fixed point theorem, the authors obtained the various existence results for positive solutions about a system of Riemann-Liouville type fractional boundary value problems with two parameters and the p-Laplacian operator. As we know, there is another kind of fractional derivative which is conformable fractional derivative. Recently, in [29], the authors Khalil R. et al. first introduced a new simple well-behaved definition of the fractional derivative called conformable fractional derivative. They first presented the definition of conformable fractional derivative of order $\alpha \in(0,1]$ and generalized the definition to include order $\alpha \in(n, n+1], n \in \mathbf{N}$. In [30], Abdeljawad proceeded on to develop the definitions and set the basic concepts in this new simple interesting fractional calculus. Since then, there are a few authors to study the boundary value problems for conformable-type fractional differential equations; for example, we can see [31-33] and the reference therein. In [33], the authors applied approximation method and fixed point theorems on cone to consider the existence and multiplicity of positive solution about the following fractional differential equation with the p-Laplacian operator:

$$
\begin{align*}
D^{\alpha}\left(\phi_{p}\left(D^{\alpha} u(t)\right)\right) & =\lambda f(t, u(t)), \quad 0<t<1, \\
u(0) & =u(1)=0,  \tag{3}\\
D^{\alpha} u(0) & =D^{\alpha} u(1)=0,
\end{align*}
$$

where $\alpha \in(1,2]$ is a real number, $D^{\alpha}$ is the conformable fractional derivative, $\phi_{p}(s)=|s|^{p-2} s, p>1$, and $f:[0,1] \times$ $[0,+\infty) \longrightarrow[0,+\infty)$ is continuous.

There are few papers about the system of fractional differential equations concerning conformable fractional derivative. System (1), (2) is a new type of conformable fractional differential equations. Motivated by the recent papers [28, 33, 34], we consider the existence of positive solutions of the system for conformable fractional differential equations (1), (2). By using Guo-Krasnosel'skii fixed point theorem, we establish some sufficient conditions on $g, f, \lambda, \mu$ for the existence of at least one positive solutions of system (1), (2) for appropriately chosen parameters.

The organization of this paper is as follows. In Section 2, we recall some concepts about the conformable fractional derivative and give some lemmas with respect to the corresponding Green's function. In Section 3, we give some results about the existence of positive solutions of system (1), (2). In Section 4, we summarize the main results of the third section.

## 2. Preliminaries

For the convenience of the reader, we give the following concepts and lemmas of conformable fractional calculus, and some auxiliary results that will be used to prove our main theorems (see [29-33]).

Definition 1. Let $\alpha \in(n, n+1]$ and $f$ be a $n$-differential function at $t>0$, then the fractional conformable derivative of order $\alpha$ at $t>0$ is given by

$$
\begin{align*}
D^{\alpha} f(t) & =D^{\alpha-n} f^{(n)}(t) \\
& =\lim _{\epsilon \rightarrow 0} \frac{f^{(n)}\left(t+\epsilon t^{n+1-\alpha}\right)-f^{(n)}(t)}{\epsilon} \tag{4}
\end{align*}
$$

provided the limit of the right hand side exists. If $f$ is $\alpha$-differentiable in some $(0, a)$, where $a>0$, and $\lim _{t \rightarrow 0^{+}} D^{\alpha} f(t)$ exists, then define $D^{\alpha} f(0)=$ $\lim _{t \rightarrow 0^{+}} D^{\alpha} f(t)$.

Definition 2. Let $\alpha \in(n, n+1]$. The fractional integral of order $\alpha>0$ at $t>0$ of a function $f:(0,+\infty) \longrightarrow(0,+\infty)$ is given by

$$
\begin{align*}
I^{\alpha} f(t) & =I^{n+1}\left(t^{\alpha-n-1} f(t)\right) \\
& =\frac{1}{n!} \int_{0}^{t}(t-s)^{n} s^{\alpha-n-1} f(s) d s \tag{5}
\end{align*}
$$

where $I^{n+1}$ denotes the integration operator of order $n+1$.
Lemma 3. Let $\alpha \in(n, n+1]$ and $f, g$ be $\alpha$-differentiable at a point $t>0$. Then
(i) $D^{\alpha}[a f+b g]=a D^{\alpha}(f)+b D^{\alpha}(g), \forall a, b \in R^{1}$.
(ii) $D^{\alpha} t^{k}=0$, where $k=0,1, \ldots, n$.
(iii) $D^{\alpha}(C)=0$, for all constant functions $f(t)=C$.
(iv) $D^{\alpha}(f g)=f D^{\alpha}(g)+g D^{\alpha}(f)$.
(v) $D^{\alpha}(f / g)=\left(g D^{\alpha}(f)-f D^{\alpha}(g)\right) / g^{2}$.
(vi) If, in addition, $f$ is differentiable, then $D^{\alpha} f(t)=$ $t^{n+1-\alpha} f^{(n+1)}(t)$
(vii) If, in addition, $f$ is differentiable at $g(t)$, then $D^{\alpha}(f \circ$ $g)(t)=f^{(n)}(g(t)) D^{\alpha} g(t)$.

Lemma 4. Given $\alpha \in(n, n+1]$ and $f$ a continuous function defined in the domain of $I^{\alpha}$, one has that $D^{\alpha} I^{\alpha} f(t)=f(t)$ for $t>0$.

Lemma 5 (mean value theorem). Let $a>0$ and $f:[a, b] \longrightarrow$ $R^{1}$ be a given function that satisfies
(i) $f$ is continuous on $[a, b]$,
(ii) $f$ is $\alpha$-differentiable for some $\alpha \in(0,1)$.

Then, there exists $c \in(a, b)$ such that

$$
\begin{equation*}
D^{\alpha} f(c)=\frac{f(b)-f(a)}{(1 / \alpha) b^{\alpha}-(1 / \alpha) a^{\alpha}} \tag{6}
\end{equation*}
$$

Lemma 6. Given $\alpha \in(n, n+1]$ and $f:[0,+\infty) \longrightarrow R^{1}$ an $\alpha$-differentiable function, one has that $D^{\alpha} f(t)=0$ if and only if $f(t)=a_{0}+a_{1} t+\cdots+a_{n} t^{n}$, where $a_{1}, a_{2}, \ldots, a_{n} \in R^{1}$.

Lemma 7. Given $a \in(n, n+1]$ and $x \in C(0,+\infty)$ an $\alpha$ differentiable function that belongs to $C(0,1) \cap L(0,1)$, one has that $I^{\alpha} D^{\alpha} x(t)=x(t)+c_{0}+c_{1} t+\cdots+c_{n} t^{n}$, for some $c_{k} \in R^{1}$, $k=0,1, \ldots, n$.

By Lemma 2.7 in [33], we can obtain the following lemmas.

Lemma 8 (see [33]). Let $u \in C[0,1]$ and $\alpha_{1}, \alpha_{2} \in(1,2]$. Then the conformable fractional differential equation

$$
\begin{align*}
D^{\alpha_{i}}\left(\phi_{p}\left(D^{\alpha_{i}} x(t)\right)\right. & =u(t), \quad 0<t<1, \\
x(0) & =x(1)=0,  \tag{7}\\
D^{\alpha_{i}} x(0) & =D^{\alpha_{i}} x(1)=0
\end{align*}
$$

has a unique solution

$$
\begin{equation*}
x(t)=\int_{0}^{1} K_{i}(t, s) \phi_{q_{i}}\left(\int_{0}^{1} K_{i}(s, \tau) u(\tau)\right) d s \tag{8}
\end{equation*}
$$

where

$$
k_{i}(t, s)= \begin{cases}(1-t) s^{\alpha_{i}-1}, & 0 \leq s \leq t \leq 1  \tag{9}\\ t s^{\alpha_{i}-2}(1-s), & 0 \leq t \leq s \leq 1\end{cases}
$$

and $i=1,2$.
Lemma 9 (see [33]). The function $K_{i}(t, s)(i=1,2)$ defined by (9) has the following properties:
(i) $K_{i}(t, s)>0$, for all $t, s \in(0,1)$;
(ii) $\min _{1 / 4 \leq t \leq 3 / 4} K_{i}(t, s) \geq(1 / 4) \max _{0 \leq t \leq 1} K_{i}(t, s)=$ $(1 / 4) K_{i}(s, s)$, for $s \in(0,1)$.

Lemma 10 (see [35]). Let $E$ be a Banach space, $P \subset E$ be a cone, $\Omega_{1}$ and $\Omega_{2}$ be bounded open subsets of $E, \theta \in \Omega_{1}$, and $\bar{\Omega}_{1} \subset \Omega_{2}$. Assume that $T: P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \longrightarrow P$ is a completely continuous operator such that
(i) $\|T u\| \leq\|u\|, \forall u \in P \cap \partial \Omega_{1},\|T u\| \geq\|u\|, \forall u \in P \cap$ $\partial \Omega_{2}$
or
(ii) $\|T u\| \geq\|u\|, \forall u \in P \cap \partial \Omega_{1},\|T u\| \leq\|u\|, \forall u \in P \cap$ $\partial \Omega_{2}$.

Then the operator $T$ has at least one fixed point in $P \cap$ $\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

## 3. Main Results

Let $X=C[0,1]$ with supremum norm $\|x\|=\max _{0 \leq t \leq 1}|x(t)|$. Let $E=X \times X$ with the norm $\|(x, y)\|_{E}=\|x\|+\|y\|$. Then $E$ is a Banach space. We define the cone

$$
\begin{align*}
P= & \{(x, y) \in E \mid x(t) \geq 0, y(t) \geq 0, \forall t  \tag{10}\\
& \left.\in[0,1], \min _{1 / 4 \leq t \leq 3 / 4}(x(t)+y(t)) \geq \frac{1}{4}\|(x, y)\|_{E}\right\} .
\end{align*}
$$

In the following, we define the operators $A, B: E \longrightarrow X$ and $T: E \longrightarrow E$ :

$$
\begin{aligned}
& A(x, y)(t)=\int_{0}^{1} K_{1}(t, s) \\
& \cdot \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s \\
& t \in[0,1], \\
& B(x, y)(t)=\int_{0}^{1} K_{2}(t, s) \\
& \quad \cdot \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s, \\
& \\
& T(x, y)=(A(x, y), B(x, y)), \quad(x, y) \in E,
\end{aligned}
$$

where $K_{i}(t, s)(i=1,2)$ is defined by (9).
Obviously, the nontrivial fixed points of the operator $T$ in P are positive solutions of system (1), (2).

Lemma 11. The operator $T: P \longrightarrow P$ is a completely continuous operator.

Proof. It is obvious that $A(x, y)(t) \geq 0, B(x, y)(t) \geq 0$ for $(x, y) \in P, t \in[0,1]$. By Lemma 8, we have

$$
\begin{align*}
& A(x, y)(t)=\int_{0}^{1} K_{1}(t, s) \\
& \quad \cdot \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{1}}(\lambda) \int_{0}^{1} s(1-s)^{\alpha_{1}-1}  \tag{12}\\
& \quad \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s
\end{align*}
$$

and

$$
\begin{align*}
& B(x, y)(t)=\int_{0}^{1} K_{2}(t, s) \\
& \quad \cdot \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s  \tag{13}\\
& \quad \leq \phi_{q_{2}}(\mu) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \quad \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s
\end{align*}
$$

Then

$$
\begin{align*}
& \|T(x, y)\|_{E}=\|A(x, y)\|+\|B(x, y)\| \leq \phi_{q_{1}}(\lambda) \\
& \quad \cdot \int_{0}^{1} s(1-s)^{\alpha_{1}-1} \\
& \quad \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s  \tag{14}\\
& \quad+\phi_{q_{2}}(\mu) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \quad \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s
\end{align*}
$$

On the other hand, by Lemma 9, we have

$$
\begin{align*}
& \min _{1 / 4 \leq t \leq 3 / 4} A(x, y)(t) \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \int_{0}^{1} K_{1}(s, s) \\
& \quad \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s  \tag{15}\\
& \quad \geq \frac{1}{4}\|A(x, y)\|,
\end{align*}
$$

and

$$
\begin{aligned}
& \min _{1 / 4 \leq t \leq 3 / 4} B(x, y)(t) \geq \frac{1}{4} \phi_{q_{2}}(\mu) \int_{0}^{1} K_{2}(s, s) \\
& \quad \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s \\
& \quad \geq \frac{1}{4}\|B(x, y)\| .
\end{aligned}
$$

So we have

$$
\begin{align*}
& \min _{1 / 4 \leq t \leq 3 / 4}(A(x, y)(t)+B(x, y)(t)) \\
& \quad \geq \min _{1 / 4 \leq t \leq 3 / 4} A(x, y)(t)+\min _{1 / 4 \leq t \leq 3 / 4} B(x, y)(t)  \tag{17}\\
& \quad \geq \frac{1}{4}\|(x, y)\|_{E}
\end{align*}
$$

i.e., $T(P) \subset P$.

By the paper [33], we know that $A$ and $B$ are completely continuous operator. It is obvious that $T$ is completely continuous. The proof is completed.

## Denote

$$
\begin{align*}
& g_{0}=\lim _{x+y \rightarrow 0^{+}} \max _{t \in[0,1]} \frac{g(t, x, y)}{\phi_{p_{1}}(x+y)} . \\
& f_{0}=\lim _{x+y \rightarrow 0^{+}} \max _{t \in[0,1]} \frac{f(t, x, y)}{\phi_{p_{2}}(x+y)} . \\
& g_{\infty}=\lim _{x+y \rightarrow+\infty} \min _{1 / 4 \leq t \leq 3 / 4} \frac{g(t, x, y)}{\phi_{p_{1}}(x+y)} . \\
& f_{\infty}=\lim _{x+y \rightarrow+\infty} \min _{1 / 4 \leq t \leq 3 / 4} \frac{f(t, x, y)}{\phi_{p_{2}}(x+y)} .  \tag{18}\\
& C_{1}=\int_{0}^{1} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) d \tau\right) d s, \\
& C_{2}=\int_{0}^{1} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) d \tau\right) d s . \\
& C_{3}=\int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau) d \tau\right) d s, \\
& C_{4}=\int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau) d \tau\right) d s .
\end{align*}
$$

Theorem 12. Assume that $g_{0}, f_{0}, g_{\infty}, f_{\infty} \in(0,+\infty), M_{1}<$ $M_{2}$ and $M_{3}<M_{4}$, then for each $\lambda \in\left(M_{1}, M_{2}\right)$ and $\mu \in$ $\left(M_{3}, M_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t))$, $t \in[0,1]$, where

$$
\begin{aligned}
& M_{1}=\phi_{p_{1}}\left(\frac{8}{C_{3}}\right) \frac{1}{g_{\infty}} \\
& M_{2}=\phi_{p_{1}}\left(\frac{1}{2 C_{1}}\right) \frac{1}{g_{0}} \\
& M_{3}=\phi_{p_{2}}\left(\frac{8}{C_{4}}\right) \frac{1}{f_{\infty}} \\
& M_{4}=\phi_{p_{2}}\left(\frac{1}{2 C_{2}}\right) \frac{1}{f_{0}}
\end{aligned}
$$

Proof. Let $\lambda \in\left(M_{1}, M_{2}\right)$ and $\mu \in\left(M_{3}, M_{4}\right)$. Then there exists a number $\epsilon>0$ such that $\epsilon<\min \left\{g_{\infty}, f_{\infty}\right\}$, and

$$
\begin{align*}
& \phi_{p_{1}}\left(\frac{8}{C_{3}}\right) \frac{1}{g_{\infty}-\epsilon} \leq \lambda \leq \phi_{p_{1}}\left(\frac{1}{2 C_{1}}\right) \frac{1}{g_{0}+\epsilon},  \tag{20}\\
& \phi_{p_{2}}\left(\frac{8}{C_{4}}\right) \frac{1}{f_{\infty}-\epsilon} \leq \mu \leq \phi_{p_{2}}\left(\frac{1}{2 C_{2}}\right) \frac{1}{f_{0}+\epsilon} .
\end{align*}
$$

For the above $\epsilon>0$, we know that there exists $R_{1}>0$ such that

$$
\begin{align*}
& g(t, x, y)<\left(g_{0}+\epsilon\right) \phi_{p_{1}}(x+y), \\
& \quad 0 \leq x+y \leq R_{1}, t \in[0,1] . \\
& f(t, x, y)<\left(f_{0}+\epsilon\right) \phi_{p_{2}}(x+y),  \tag{21}\\
& \quad 0 \leq x+y \leq R_{1}, t \in[0,1] .
\end{align*}
$$

Let $\Omega_{1}=\left\{(x, y) \in E \mid\|(x, y)\|_{E}<R_{1}\right\}$. For any $(x, y) \in$ $P \cap \partial \Omega_{1}$, we have

$$
\begin{align*}
& A(x, y)(t)=\int_{0}^{1} K_{1}(t, s) \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau)\right. \\
& \quad \cdot g(\tau, x(\tau), y(\tau)) d \tau) d s \leq \phi_{q_{1}}(\lambda) \\
& \quad \cdot \int_{0}^{1} K_{1}(t, s) \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau)\right. \\
& \cdot\left[\left(g_{0}+\epsilon\right) \phi_{p_{1}}(x(\tau)+y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{1}}(\lambda) \phi_{q_{1}}\left(g_{0}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{1}-1}  \tag{22}\\
& \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) \phi_{p_{1}}(x(\tau)+y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{1}}(\lambda) \phi_{q_{1}}\left(g_{0}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{1}-1} \\
& \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) \phi_{p_{1}}(\|x\|+\|y\|) d \tau\right) d s \\
& \quad=\phi_{q_{1}}\left(\lambda\left(g_{0}+\epsilon\right)\right) C_{1}(\|x\|+\|y\|) \leq \frac{1}{2}\|(x, y)\|_{E}
\end{align*}
$$

So

$$
\begin{equation*}
\|A(x, y)\| \leq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{1} . \tag{23}
\end{equation*}
$$

Similarly, we have

$$
\begin{aligned}
& B(x, y)(t)=\int_{0}^{1} K_{2}(t, s) \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau)\right. \\
& \cdot f(\tau, x(\tau), y(\tau)) d \tau) d s \leq \phi_{q_{2}}(\mu) \\
& \cdot \int_{0}^{1} K_{2}(t, s) \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau)\right. \\
& \cdot\left[\left(f_{0}+\epsilon\right) \phi_{p_{2}}(x(\tau)+y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{2}}(\mu) \phi_{q_{2}}\left(f_{0}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) \phi_{p_{2}}(x(\tau)+y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{2}}(\mu) \phi_{q_{2}}\left(f_{0}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) \phi_{p_{2}}(\|x\|+\|y\|) d \tau\right) d s \\
& \quad=\phi_{q_{2}}\left(\lambda\left(f_{0}+\epsilon\right)\right) C_{2}(\|x\|+\|y\|) \leq \frac{1}{2}\|(x, y)\|_{E} .
\end{aligned}
$$

So

$$
\begin{equation*}
\|B(x, y)\| \leq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{1} . \tag{25}
\end{equation*}
$$

Hence

$$
\begin{array}{r}
\|T(x, y)\|_{E}=\|A(x, y)\|+\|b(x, y)\| \leq\|(x, y)\|_{E}  \tag{26}\\
\forall(x, y) \in P \cap \partial \Omega_{1}
\end{array}
$$

On the other hand, for the above $\epsilon>0$, there exists $\widetilde{R}_{2}>0$ such that

$$
\begin{align*}
& g(t, x, y) \geq\left(g_{\infty}-\epsilon\right) \phi_{p_{1}}(x+y) \\
& \qquad x+y \geq \widetilde{R}_{2}>0, t \in\left[\frac{1}{4}, \frac{3}{4}\right], \tag{27}
\end{align*}
$$

$f(t, x, y) \geq\left(f_{\infty}-\epsilon\right) \phi_{p_{2}}(x+y)$,

$$
x+y \geq \widetilde{R}_{2}>0, t \in\left[\frac{1}{4}, \frac{3}{4}\right]
$$

Let $R_{2}=\max \left\{2 R_{1}, 4 \widetilde{R}_{2}\right\}$. Let $\Omega_{2}=\{(x, y) \in$ $\left.E \mid\|(x, y)\|_{E}<R_{2}\right\}$. For any $(x, y) \in P \cap \partial \Omega_{2}$, we have $\min _{1 / 4 \leq t \leq 3 / 4}\{x(t)+y(t)\} \geq(1 / 4)(\|x\|+\|y\|)=$ $(1 / 4)\|(x, y)\|_{E}=R_{2} / 4 \geq \widetilde{R}_{2}$. So by Lemma 9 , we have

$$
\begin{align*}
& A(x, y)\left(\frac{1}{4}\right)=\int_{0}^{1} K_{1}\left(\frac{1}{4}, s\right) \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau)\right. \\
& \cdot g(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \quad \cdot \int_{0}^{1} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\right. \\
& \quad \cdot g(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\left(g_{\infty}-\epsilon\right)\right.  \tag{28}\\
& \left.\cdot \phi_{p_{1}}(x(\tau)+y(\tau)) d \tau\right) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \cdot \phi_{q_{1}}\left(g_{\infty}-\epsilon\right) \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\right. \\
& \left.\cdot \phi_{p_{1}}\left(\frac{1}{4}(\|x\|+\|y\|)\right) d \tau\right) d s=\frac{1}{16} \\
& \cdot \phi_{q_{1}}\left(\lambda\left(g_{\infty}-\epsilon\right)\right) C_{3}\left(\|(x, y)\|_{E}\right) \geq \frac{1}{2}\|(x, y)\|_{E}
\end{align*}
$$

So
$\|A(x, y)\| \geq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{2}$.
Similarly, we have

$$
\begin{aligned}
& B(x, y)\left(\frac{3}{4}\right)=\int_{0}^{1} K_{2}\left(\frac{3}{4}, s\right) \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau)\right. \\
& \cdot f(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\right. \\
& \cdot f(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\left(f_{\infty}-\epsilon\right)\right. \\
& \left.\cdot \phi_{p_{2}}(x(\tau)+y(\tau)) d \tau\right) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \phi_{q_{2}}\left(f_{\infty}-\epsilon\right) \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.\phi_{p_{2}}\left(\frac{1}{4}(\|x\|+\|y\|)\right) d \tau\right) d s=\frac{1}{16} \\
& \cdot \phi_{q_{2}}\left(\lambda\left(f_{\infty}-\epsilon\right)\right) C_{4}\left(\|(x, y)\|_{E}\right) \geq \frac{1}{2}\|(x, y)\|_{E} \tag{30}
\end{align*}
$$

So

$$
\begin{equation*}
\|B(x, y)\| \geq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{2} \tag{31}
\end{equation*}
$$

Hence

$$
\begin{array}{r}
\|T(x, y)\|_{E}=\|A(x, y)\|+\|B(x, y)\| \geq\|(x, y)\|_{E} \\
\forall(x, y) \in P \cap \partial \Omega_{2} \tag{32}
\end{array}
$$

By Lemma 10 and (26) (32), the operator $T$ has one fixed point $(x, y) \in P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$. That is, $(x, y)$ is a positive solution of system (1), (2).

Similar to the proof of Theorem 12, we can easily obtain the following results.

Theorem 13. Assume that $g_{0}=0, g_{\infty}, f_{0}, f_{\infty} \in(0,+\infty)$ and $M_{3}<M_{4}$, then for each $\lambda \in\left(M_{1},+\infty\right)$ and $\mu \in\left(M_{3}, M_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 14. Assume that $f_{0}=0, g_{\infty}, g_{0}, f_{\infty} \in(0,+\infty)$ and $M_{1}<M_{2}$, then for each $\lambda \in\left(M_{1}, M_{2}\right)$ and $\mu \in\left(M_{3},+\infty\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 15. Assume that $g_{0}=0, f_{0}=0, g_{\infty}, f_{\infty} \in$ $(0,+\infty)$, then for each $\lambda \in\left(M_{1},+\infty\right)$ and $\mu \in\left(M_{3},+\infty\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 16. Assume that $g_{0}, f_{0} \in(0,+\infty), g_{\infty}=+\infty$ or $g_{0}, f_{0} \in(0,+\infty), f_{\infty}=+\infty$, then for each $\lambda \in\left(0, M_{2}\right)$ and $\mu \in\left(0, M_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 17. Assume that $g_{0} \in(0,+\infty), f_{0}=0, f_{\infty}=+\infty$ or $g_{0} \in(0,+\infty), f_{0}=0, g_{\infty}=+\infty$, then for each $\lambda \in$ ( $0, M_{2}$ ) and $\mu \in(0,+\infty)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 18. Assume that $g_{0}=f_{0}=0, f_{\infty}=+\infty$ or $g_{0}=f_{0}=0, g_{\infty}=+\infty$, then for each $\lambda \in(0,+\infty)$ and $\mu \in$ $(0,+\infty)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in$ [0, 1].

Denote

$$
\begin{aligned}
& \bar{g}_{0}=\lim _{x+y \rightarrow 0^{+}} \min _{t \in[1 / 4,3 / 4]} \frac{g(t, x, y)}{\phi_{p_{1}}(x+y)} \\
& \bar{f}_{0}=\lim _{x+y \rightarrow 0^{+}} \min _{t \in[1 / 4,3 / 4]} \frac{f(t, x, y)}{\phi_{p_{2}}(x+y)}
\end{aligned}
$$

$$
\begin{align*}
& \bar{g}_{\infty}=\lim _{x+y \rightarrow+\infty} \max _{t \in[0,1]} \frac{g(t, x, y)}{\phi_{p_{1}}(x+y)} \\
& \bar{f}_{\infty}=\lim _{x+y \rightarrow+\infty} \max _{t \in[0,1]} \frac{f(t, x, y)}{\phi_{p_{2}}(x+y)} \tag{33}
\end{align*}
$$

Theorem 19. Assume that $\bar{g}_{0}, \bar{f}_{0}, \bar{g}_{\infty}, \bar{f}_{\infty} \in(0,+\infty), \bar{M}_{1}<$ $\bar{M}_{2}$ and $\bar{M}_{3}<\bar{M}_{4}$, then for each $\lambda \in\left(\bar{M}_{1}, \bar{M}_{2}\right)$ and $\mu \in$ $\left(\bar{M}_{3}, \bar{M}_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t))$, $t \in[0,1]$, where

$$
\begin{align*}
& \bar{M}_{1}=\phi_{p_{1}}\left(\frac{8}{C_{3}}\right) \frac{1}{\bar{g}_{0}} \\
& \bar{M}_{2}=\phi_{p_{1}}\left(\frac{1}{2 C_{1}}\right) \frac{1}{\bar{g}_{\infty}} \\
& \bar{M}_{3}=\phi_{p_{2}}\left(\frac{8}{C_{4}}\right) \frac{1}{\bar{f}_{0}}  \tag{34}\\
& \bar{M}_{4}=\phi_{p_{2}}\left(\frac{1}{2 C_{2}}\right) \frac{1}{\bar{f}_{\infty}}
\end{align*}
$$

Proof. Let $\lambda \in\left(\bar{M}_{1}, \bar{M}_{2}\right)$ and $\mu \in\left(\bar{M}_{3}, \bar{M}_{4}\right)$. Then there exists a number $\epsilon>0$ such that $\epsilon<\min \left\{\bar{g}_{0}, \bar{f}_{0}\right\}$, and

$$
\begin{align*}
& \phi_{p_{1}}\left(\frac{8}{C_{3}}\right) \frac{1}{\bar{g}_{0}-\epsilon} \leq \lambda \leq \phi_{p_{1}}\left(\frac{1}{2 C_{1}}\right) \frac{1}{\bar{g}_{\infty}+\epsilon}, \\
& \phi_{p_{2}}\left(\frac{8}{C_{4}}\right) \frac{1}{\bar{f}_{0}-\epsilon} \leq \mu \leq \phi_{p_{2}}\left(\frac{1}{2 C_{2}}\right) \frac{1}{\bar{f}_{\infty}+\epsilon} \tag{35}
\end{align*}
$$

For the above $\epsilon>0$, we know that there exists $R_{3}>0$ such that

$$
\begin{align*}
& g(t, x, y)>\left(\bar{g}_{0}-\epsilon\right) \phi_{p_{1}}(x+y), \\
& \\
& \quad x \geq 0, y \geq 0, x+y \leq R_{3}, t \in\left[\frac{1}{4}, \frac{3}{4}\right],  \tag{36}\\
& f(t, x, y)>\left(\bar{f}_{0}-\epsilon\right) \phi_{p_{2}}(x+y), \\
& \\
& \quad x \geq 0, y \geq 0, x+y \leq R_{3}, t \in\left[\frac{1}{4}, \frac{3}{4}\right] .
\end{align*}
$$

Let $\Omega_{3}=\left\{(x, y) \in E \mid\|(x, y)\|_{E}<R_{3}\right\}$. For any $(x, y) \in$ $P \cap \partial \Omega_{3}$, we have $\min _{1 / 4 \leq t \leq 3 / 4}\{x(t)+y(t)\} \geq(1 / 4)(\|x\|+$ $\|y\|)=(1 / 4)\|(x, y)\|_{E}$. So by Lemma 9, we obtain

$$
\begin{align*}
& A(x, y)\left(\frac{1}{4}\right)=\int_{0}^{1} K_{1}\left(\frac{1}{4}, s\right) \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau)\right. \\
& \cdot g(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\right. \\
& \quad \cdot g(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \quad \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\left(\bar{g}_{0}-\epsilon\right)\right.  \tag{37}\\
& \left.\cdot \phi_{p_{1}}(x(\tau)+y(\tau)) d \tau\right) d s \geq \frac{1}{4} \phi_{q_{1}}(\lambda) \\
& \cdot \phi_{q_{1}}\left(\bar{g}_{0}-\epsilon\right) \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{1}-1} \phi_{q_{1}}\left(\int_{1 / 4}^{3 / 4} K_{1}(s, \tau)\right. \\
& \left.\cdot \phi_{p_{1}}\left(\frac{1}{4}(\|x\|+\|y\|)\right) d \tau\right) d s=\frac{1}{16} \\
& \cdot \phi_{q_{1}}\left(\lambda\left(\bar{g}_{0}-\epsilon\right)\right) C_{3}\left(\|(x, y)\|_{E}\right) \geq \frac{1}{2}\|(x, y)\|_{E} .
\end{align*}
$$

So

$$
\begin{equation*}
\|A(x, y)\| \geq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{3} . \tag{38}
\end{equation*}
$$

Similarly, we get

$$
\begin{aligned}
& B(x, y)\left(\frac{3}{4}\right)=\int_{0}^{1} K_{2}\left(\frac{3}{4}, s\right) \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau)\right. \\
& \cdot f(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\right. \\
& \cdot f(\tau, x(\tau), y(\tau)) d \tau) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\left(\bar{f}_{0}-\epsilon\right)\right. \\
& \left.\cdot \phi_{p_{2}}(x(\tau)+y(\tau)) d \tau\right) d s \geq \frac{1}{4} \phi_{q_{2}}(\mu) \\
& \cdot \phi_{q_{2}}\left(\bar{f}_{0}-\epsilon\right) \int_{1 / 4}^{3 / 4} s(1-s)^{\alpha_{2}-1} \phi_{q_{2}}\left(\int_{1 / 4}^{3 / 4} K_{2}(s, \tau)\right. \\
& \left.\cdot \phi_{p_{2}}\left(\frac{1}{4}(\|x\|+\|y\|)\right) d \tau\right) d s=\frac{1}{16} \\
& \cdot \phi_{q_{2}}\left(\mu\left(\bar{f}_{0}-\epsilon\right)\right) C_{3}\left(\|(x, y)\|_{E}\right) \geq \frac{1}{2}\|(x, y)\|_{E} .
\end{aligned}
$$

So

$$
\begin{equation*}
\|B(x, y)\| \geq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{3} . \tag{40}
\end{equation*}
$$

Hence

$$
\begin{array}{r}
\|T(x, y)\|_{E}=\|A(x, y)\|+\|B(x, y)\| \geq\|(x, y)\|_{E} \\
\forall(x, y) \in P \cap \partial \Omega_{3} . \tag{41}
\end{array}
$$

We define the functions $\tilde{g}, \tilde{f}:[0,1] \times[0,+\infty) \longrightarrow$ $[0,+\infty), \widetilde{g}(t, u)=\max _{0 \leq x+y \leq u} g(t, x, y)$, and $\tilde{f}(t, u)=$ $\max _{0 \leq x+y \leq u} f(t, x, y)$. So it is obvious that $\widetilde{g}(t, u)$ and $\widetilde{f}(t, u)$ are nondecreasing on $u$ for every $t \in[0,1]$; and $g(t, x, y) \leq$ $\widetilde{g}(t, u), f(t, x, y) \leq \tilde{f}(t, u), x \geq 0, y \geq 0, x+y \leq u, t \in$ $[0,1]$; and they satisfy the conditions

$$
\begin{align*}
& \limsup _{u \rightarrow+\infty} \max _{t \in[0,1]} \frac{\widetilde{g}(t, u)}{\phi_{p_{1}}(u)} \leq \bar{g}_{\infty} \\
& \limsup _{u \rightarrow+\infty} \max _{t \in[0,1]} \frac{\widetilde{f}(t, u)}{\phi_{p_{2}}(u)} \leq \bar{f}_{\infty} \tag{42}
\end{align*}
$$

For the above $\epsilon>0$, there exists $\widetilde{R}_{4}>0$ such that

$$
\begin{align*}
& \frac{\tilde{\mathcal{g}}(t, u)}{\phi_{p_{1}}(u)} \leq \limsup _{u \rightarrow+\infty} \max _{t \in[0,1]} \frac{\tilde{\mathcal{g}}(t, u)}{\phi_{p_{1}}(u)}+\epsilon \leq \bar{g}_{\infty}+\epsilon, \\
& t \in[0,1], u \geq \widetilde{R}_{4} \tag{43}
\end{align*}
$$

$\frac{\tilde{f}(t, u)}{\phi_{p_{2}}(u)} \leq \limsup _{u \rightarrow+\infty} \max _{t \in[0,1]} \frac{\tilde{f}(t, u)}{\phi_{p_{2}}(u)}+\epsilon \leq \bar{f}_{\infty}+\epsilon$,

$$
t \in[0,1], u \geq \widetilde{R}_{4} .
$$

So $\widetilde{g}(t, u) \leq\left(\bar{g}_{\infty}+\epsilon\right) \phi_{p_{1}}(u), \tilde{f}(t, u) \leq\left(\bar{f}_{\infty}+\epsilon\right) \phi_{p_{2}}(u)$, and $t \in[0,1], u \geq \widetilde{R}_{4}$.

By the definition of $\tilde{g}, \tilde{f}$, we get $g(t, x, y) \leq$ $\widetilde{g}\left(t,\|(x, y)\|_{E}\right), \quad f(t, x, y) \leq \tilde{f}\left(t,\|(x, y)\|_{E}\right)$. Let $R_{4}=\max \left\{2 R_{3}, \widetilde{R}_{4}\right\}$. Let $\Omega_{4}=\left\{(x, y) \in E \mid\|(x, y)\|_{E}<R_{4}\right\}$. For any $(x, y) \in P \cap \partial \Omega_{4}$, we have

$$
\begin{align*}
& A(x, y)(t)=\int_{0}^{1} K_{1}(t, s) \\
& \quad \cdot \phi_{q_{1}}\left(\lambda \int_{0}^{1} K_{1}(s, \tau) g(\tau, x(\tau), y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{1}}(\lambda) \int_{0}^{1} s(1-s)^{\alpha_{1}-1} \\
& \quad \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) \widetilde{g}\left(\tau,\|(x, y)\|_{E}\right) d \tau\right) d s  \tag{44}\\
& \quad \leq \phi_{q_{1}}(\lambda) \phi_{q_{1}}\left(\bar{g}_{\infty}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{1}-1} \\
& \left.\quad \cdot \phi_{q_{1}}\left(\int_{0}^{1} K_{1}(s, \tau) \phi_{p_{1}}\left(\|(x, y)\|_{E}\right)\right) d \tau\right) d s \\
& \quad=\phi_{q_{1}}\left(\lambda\left(\bar{g}_{\infty}+\epsilon\right)\right) C_{1}\left(\|(x, y)\|_{E}\right) \leq \frac{1}{2}\|(x, y)\|_{E}
\end{align*}
$$

So

$$
\begin{equation*}
\|A(x, y)\| \leq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{4} \tag{45}
\end{equation*}
$$

Similarly, we have

$$
\begin{align*}
& B(x, y)(t)=\int_{0}^{1} K_{2}(t, s) \\
& \quad \cdot \phi_{q_{2}}\left(\mu \int_{0}^{1} K_{2}(s, \tau) f(\tau, x(\tau), y(\tau)) d \tau\right) d s \\
& \quad \leq \phi_{q_{2}}(\mu) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \quad \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) \widetilde{f}\left(\tau,\|(x, y)\|_{E}\right) d \tau\right) d s  \tag{46}\\
& \quad \leq \phi_{q_{2}}(\mu) \phi_{q_{2}}\left(\bar{f}_{\infty}+\epsilon\right) \int_{0}^{1} s(1-s)^{\alpha_{2}-1} \\
& \left.\quad \cdot \phi_{q_{2}}\left(\int_{0}^{1} K_{2}(s, \tau) \phi_{p_{2}}\left(\|(x, y)\|_{E}\right)\right) d \tau\right) d s \\
& \quad=\phi_{q_{2}}\left(\mu\left(\bar{f}_{\infty}+\epsilon\right)\right) C_{2}\left(\|(x, y)\|_{E}\right) \leq \frac{1}{2}\|(x, y)\|_{E}
\end{align*}
$$

So

$$
\begin{equation*}
\|B(x, y)\| \leq \frac{1}{2}\|(x, y)\|_{E}, \quad \forall(x, y) \in P \cap \partial \Omega_{4} \tag{47}
\end{equation*}
$$

Hence

$$
\begin{array}{r}
\|T(x, y)\|_{E}=\|A(x, y)\|+\|B(x, y)\| \leq\|(x, y)\|_{E}  \tag{48}\\
\forall(x, y) \in P \cap \partial \Omega_{4} .
\end{array}
$$

By Lemma 10 and (41) (48), the operator $T$ has one fixed point $(x, y) \in P \cap\left(\bar{\Omega}_{4} \backslash \Omega_{3}\right)$. So $(x, y)$ is a positive solution of system (1), (2).

Similar to the proof of Theorem 19, we can easily obtain the following results.

Theorem 20. Assume that $\bar{g}_{0}, \bar{g}_{\infty}, \bar{f}_{0} \in(0,+\infty), \bar{f}_{\infty}=$ 0 and $\bar{M}_{1}<\bar{M}_{2}$, then for each $\lambda \in\left(\bar{M}_{1}, \bar{M}_{2}\right)$ and $\mu \in\left(\bar{M}_{3},+\infty\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 21. Assume that $\bar{g}_{0}, \bar{f}_{\infty}, \bar{f}_{0} \in(0,+\infty), \bar{g}_{\infty}=$ 0 and $\bar{M}_{3} \leq \bar{M}_{4}$, then for each $\lambda \in\left(\bar{M}_{1},+\infty\right)$ and $\mu \in\left(\bar{M}_{3}, \bar{M}_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 22. Assume that $\bar{g}_{0}, \bar{f}_{0} \in(0,+\infty), \bar{g}_{\infty}=0, \bar{f}_{\infty}=$ 0 , then for each $\lambda \in\left(\bar{M}_{1},+\infty\right)$ and $\mu \in\left(\bar{M}_{3},+\infty\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 23. Assume that $\bar{g}_{\infty}, \bar{f}_{\infty} \in(0,+\infty), \bar{g}_{0}=+\infty$ or $\bar{f}_{0}=+\infty, \bar{g}_{\infty}, \bar{f}_{\infty} \in(0,+\infty)$, then for each $\lambda \in\left(0, \bar{M}_{2}\right)$ and $\mu \in\left(0, \bar{M}_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 24. Assume that $\bar{g}_{0}=+\infty, \bar{g}_{\infty} \in(0,+\infty), \bar{f}_{\infty}=$ 0 or $\bar{f}_{0}=+\infty, \bar{g}_{\infty} \in(0,+\infty), \bar{f}_{\infty}=0$, then for each $\lambda \in$ $\left(0, \bar{M}_{2}\right)$ and $\mu \in(0,+\infty)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 25. Assume that $\bar{f}_{\infty} \in(0,+\infty), \bar{g}_{\infty}=0, \bar{g}_{0}=$ $+\infty$ or $\bar{f}_{\infty} \in(0,+\infty), \bar{f}_{0}=+\infty, \bar{g}_{\infty}=0$, then for each $\lambda \in(0,+\infty)$ and $\mu \in\left(0, \bar{M}_{4}\right)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

Theorem 26. Assume that $\bar{g}_{0}=+\infty, \bar{g}_{\infty}=0, \bar{f}_{\infty}=0$ or $\bar{g}_{\infty}=0, \bar{f}_{\infty}=0, \bar{f}_{0}=+\infty$, then for each $\lambda \in(0,+\infty)$ and $\mu \in(0,+\infty)$, system (1), (2) has a positive solution $(x(t), y(t)), t \in[0,1]$.

## 4. Conclusion

In this paper, we investigate the existence of positive solutions for a system of conformable-type fractional differential equations with two parameters and the p-Laplacian operator. By employing Guo-Krasnosel'skii fixed point theorem, under different combinations of sublinearity and superlinearity of the nonlinearities $f, g$, various sufficient conditions for the existence of at least one positive solutions of system (1), (2) are derived in terms of appropriately chosen parameters $\lambda, \mu$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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# Impulsive Fractional Differential Equations with p-Laplacian Operator in Banach Spaces 

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In this paper, we study a class of boundary value problem (BVP) with multiple point boundary conditions of impulsive p-Laplacian operator fractional differential equations. We establish the sufficient conditions for the existence of solutions in Banach spaces. Our analysis relies on the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. An example is given to demonstrate the main results.

## 1. Introduction

With the development of the theory of fractional order calculus, fractional differential equations are getting more and more extensively used (see[1-15]). For instance, the impulsive fractional differential equations are widely used in various scientific fields, such as the problem of dynamics of populations subject to abrupt changes, harvesting, diseases, and so on. Lakshmikantham et al. [16], Bainov and Simeonov [17], and Benchohra et al. [18] have done in-depth studies on this issue. Moreover, the p-Laplacian operator differential equation was first proposed by Leibenson [19] in order to study the problem of turbulent flow in a porous medium. He converted this problem into the existence of solution of the following differential equation:

$$
\begin{equation*}
\left(\varphi_{p}\left(u^{\prime}(t)\right)\right)^{\prime}=f(t, u(t)), \quad t \in(0,1) . \tag{1}
\end{equation*}
$$

$\varphi_{p}$ is the p-Laplacian operator, $\varphi_{q}$ is the inverse function of $\varphi_{p}$ with $\varphi_{q}(s)=|s|^{q-2} s$, and $p, q$ satisfy $1 / p+1 / q=1$. In recent years, many results about the solutions of the pLaplacian operator fractional differential equation BVP have been obtained (see [20-26]). The research of the solutions of the BVP with p-Laplacian operator and with impulsive has been attracting increasing interest.

Zhao and Gong [27] study the solution of the following impulsive fractional differential equations with generalized periodic boundary conditions:

$$
\begin{align*}
C^{C} \mathscr{D}^{q} u(t) & =f(t, u(t)), \\
\left.\Delta u(t)\right|_{t=t_{k}} & =I_{k}\left(u\left(t_{k}\right)\right), \quad k=1,2, \cdots, m, \\
\left.\Delta u^{\prime}(t)\right|_{t=t_{k}} & =J_{k}\left(u\left(t_{k}\right)\right), \quad k=1,2, \cdots, m,  \tag{2}\\
a u(0)-b u(1) & =0, \\
a u^{\prime}(0)-b u^{\prime}(1) & =0 .
\end{align*}
$$

By means of the Schauder and Guo- Krasnosel'skii fixed point theorem, they get the existence of single and multiple positive solutions of the above BVP. By the technique of the GuoKrasnosel'skii fixed point theorem and the Leggett-Williams theorem, Wang et al. [28] obtained the results of the following BVP.

$$
\begin{aligned}
\mathscr{D}_{0_{+}}^{\eta}\left(\phi_{p}\left(\mathscr{D}_{0_{+}}^{\alpha} u(t)\right)\right) & =f(t, u(t)), \\
u(0) & =0, \\
u(1) & =a u(\xi),
\end{aligned}
$$

$$
\begin{align*}
& \mathscr{D}_{0_{+}}^{\alpha} u(0)=0, \\
& \quad 0<\xi<1,0 \leq a \leq 1 \tag{3}
\end{align*}
$$

The results about the BVP with multiple point boundary conditions of impulsive p-Laplacian operator fractional differential equations are few, especially in Banach space.

In this paper, we study the following BVP with multiple point boundary conditions of impulsive p-Laplacian operator fractional differential equations in Banach space $E$ :

$$
\begin{align*}
& \mathscr{D}_{0+}^{\beta}\left(\varphi_{p}\left(\mathscr{D}_{0+}^{\alpha} x\right)\right)(t)=f\left(t, x(t), x^{\prime}(t)\right), \\
& 1<\alpha \leq 2,0<\beta \leq 1, \\
&\left.\Delta x(t)\right|_{t=t_{k}}=I_{k}\left(x\left(t_{k}\right)\right), \\
&\left.\Delta x^{\prime}(t)\right|_{t=t_{k}}=J_{k}\left(x\left(t_{k}\right)\right),  \tag{4}\\
& x(0)=x^{\prime}(0)=\int_{0}^{1} a_{1}(x(s)) d s \\
& x(1)=x^{\prime}(1)=\int_{0}^{1} a_{2}(x(s)) d s \\
& \mathscr{D}_{0+}^{\alpha} x(0)=\theta
\end{align*}
$$

where $\theta$ is the zero element of $E, \mathscr{D}_{0+}^{\alpha}$ and $\mathscr{D}_{0+}^{\beta}$ are the fractional derivatives of order $\alpha(\alpha>0), I=[0,1], t \in I^{\prime}=$ $I \backslash\left\{t_{1}, t_{2}, \cdots, t_{m}\right\}, \varphi_{p}(s)=\|s\|^{p-2} s$ with $p>1 . f: I \times E \times E \longrightarrow$ $E, I_{k}, J_{k}: E \longrightarrow E$, and $a_{1}, a_{2}: E \longrightarrow E$ are continuous. The impulsive point set $\left\{t_{k}\right\}_{k=1}^{m}$ satisfies $0=t_{0}<t_{1}<\cdots<t_{m}<$ $t_{m+1}=1$; we denote $T_{0}=\left[t_{0}, t_{1}\right], T_{k}=\left(t_{k}, t_{k+1}\right], 1 \leq k \leq m$. We denote by $x\left(t_{k}^{+}\right)$and $x\left(t_{k}^{-}\right)$the right and left limits of $x(t)$ at the point $t=t_{k}$, i.e., $x\left(t_{k}^{+}\right)=\lim _{h \rightarrow 0^{+}} x\left(t_{k}^{+}+h\right)$ and $x\left(t_{k}^{-}\right)=$ $\lim _{h \rightarrow 0^{-}} x\left(t_{k}^{+}+h\right)$. We establish the existence of solution to BVP (4). by the technique of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem. The main innovations of this paper are as follows. Firstly, we study the impulsive fractional differential equations with the pLaplacian Operator. Secondly, we study the BVP in Banach space. Thirdly, the nonlinear term contains the derivatives $x^{\prime}(t)$.

The paper is organized as follows. In Section 2, we recall some definitions and lemmas. In Section 3, the main results of this paper are discussed. Finally, one example is given in Section 4.

## 2. Preliminaries

First, we recall some definitions and preliminary.
Definition 1 (see [29]). Let $\alpha>0, f \in C([0, \infty), R)$.
(1) The fractional integral of order $\alpha(\alpha>0)$ of $f$ is given by

$$
\begin{equation*}
\mathscr{J}_{0+}^{\alpha} f(x)=\frac{1}{\Gamma(\alpha)} \int_{0}^{x} \frac{f(t)}{(x-t)^{1-\alpha}} d t \tag{5}
\end{equation*}
$$

(2) the fractional derivative of order $\alpha(\alpha>0)$ of $f$ is given by

$$
\begin{equation*}
\mathscr{D}_{0+}^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{f(s)}{(t-s)^{\alpha-n+1}} d s \tag{6}
\end{equation*}
$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of $\alpha$ and the right side integral is pointwise defined on $[0, \infty)$.

Lemma 2 (see[30, 31]). Let $f(x)$ be integrable, $\alpha>0, \beta>\alpha>$ 0 . Then
(1)

$$
\begin{align*}
\mathscr{F}_{0+}^{\alpha} \mathscr{D}_{0+}^{\alpha} f(x)= & f(x)+c_{1} x^{\alpha-1}+c_{2} x^{\alpha-2}+\cdots  \tag{7}\\
& +c_{n} x^{\alpha-n},
\end{align*}
$$

where $c_{i} \in R, i=1,2,3, \cdots, N, N=\lceil\alpha\rceil$.
(2)

$$
\begin{align*}
& \mathscr{D}_{0+}^{\alpha} \mathscr{I}_{0+}^{\beta} f(x)=\mathscr{J}_{0+}^{\beta-\alpha} f(x), \\
& \mathscr{D}_{0+}^{\alpha} \mathscr{J}_{0+}^{\alpha} f(x)=f(x) \tag{8}
\end{align*}
$$

Definition 3 (see [32]). Let $S$ be a bounded set in a real Banach space $E$; the Kuratowski noncompactness measure of $S$ is given by

$$
\begin{align*}
\alpha(S) & =\inf \left\{\delta>0: S=\bigcup_{i=1}^{m} S_{i}, \operatorname{diam}\left(\mathrm{~S}_{\mathrm{i}}\right)<\delta, \mathrm{i}\right.  \tag{9}\\
& =1,2, \cdots, \mathrm{~m}\}
\end{align*}
$$

where $\operatorname{diam}\left(\mathrm{S}_{\mathrm{i}}\right)$ denote the diameters of $S_{i}$.
Remark 4. From the definition, it is obvious that $0 \leq \alpha(S)<$ $\infty$.

Lemma 5 (see [33]). If $H \subset C(I)$ is bounded and equicontinuous, then $\alpha_{E}(H(t))$ is continuous on $I$ and $\alpha_{C}(H)=$ $\max _{t \in I} \alpha_{E}(H(t)), \alpha_{E}\left(\int_{I} x(t) d t: x \in H\right) \leq \int_{I} \alpha_{E}(H(t)) d t$, where $H(t)=\{x(t): x \in H\}$ for each $t \in I$.

Definition 6 (see [32]). Let $E_{1}$ and $E_{2}$ be real Banach spaces, $D \subset E_{1}$, and $A: D \longrightarrow E_{2}$ be a continuous and bounded operator; $A$ is called a k-set contraction operator if there exists a constant $k \geq 0$, for any bounded set $S$ in $D$, such that $\alpha(A(S)) \leq k \alpha(S)$.

Remark 7 (see [34]). A is called a strict set contraction operator if $k<1$. It is clear that a strict set contraction operator is a condensing operator.

In the following, we define the basic space of this paper. Denote

$$
\begin{aligned}
& P C(I)=\left\{x \in C(I) \mid x \in C\left(I^{\prime}\right) \cap C^{\prime}\left(I^{\prime}\right), x\left(t_{k}^{-}\right),\right. \\
& x^{\prime}\left(t_{k}^{-}\right), x\left(t_{k}^{+}\right) \text {and } x^{\prime}\left(t_{k}^{+}\right) \text {exist with } x\left(t_{k}^{-}\right) \\
&\left.=x\left(t_{k}\right) \text { and } x^{\prime}\left(t_{k}^{-}\right)=x^{\prime}\left(t_{k}\right), 1 \leq k \leq m\right\}
\end{aligned}
$$

$$
\begin{align*}
& Q C(I)=\left\{x \in P C(I): \sup _{t \in I} \frac{\|x(t)\|}{1+t}<+\infty\right\}, \\
& D C(I)=\left\{x \in P C(I): \sup _{t \in I} \frac{\|x(t)\|}{1+t}\right. \\
& \left.\quad<+\infty \text { and } \sup _{t \in I}\left\|x^{\prime}(t)\right\|<+\infty\right\}, \tag{10}
\end{align*}
$$

and it is easy to see that $D C(I)$ is Banach space with the norm

$$
\begin{equation*}
\|x\|_{D}=\max \left\{\sup _{t \in I} \frac{\|x(t)\|}{1+t},\left\|x^{\prime}\right\|_{C}\right\} \tag{11}
\end{equation*}
$$

The basic space used in this paper is $D C(I)$. The Kuratowski noncompactness measures in $E, C(I)$, and $\mathrm{DC}(\mathrm{I})$ are denoted by $\alpha_{E}(\cdot), \alpha_{C}(\cdot)$, and $\alpha_{D}(\cdot)$, respectively.

The following Sadovskii fixed point theorem is needed for the proof of our main results.

Lemma 8 ((Sadovskii)(see [32])). Let D be a bounded, closed, and convex subset of the Banach space E. If the operator A: $D \longrightarrow D$ is condensing, then $A$ has a fixed point in $D$.

## 3. Existence of Solutions

Before proceeding further, let us give some denotations as follows:

$$
\begin{align*}
M & =\frac{10 m+4}{\Gamma(\alpha) \Gamma(\beta)^{q-1}} \\
l & =M \int_{0}^{1}\left[(1+t) l_{1}(t)+l_{2}(t)\right] d t \\
K_{\rho} & =\left\{x \in D C(I):\|x\|_{D} \leq \rho\right\} \\
K_{r} & =\{x \in E:\|x\| \leq r\} \\
F(s) & =\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} f\left(\tau, x(\tau), x^{\prime}(\tau)\right) d \tau  \tag{12}\\
L & =\max \left\{L_{1}, L_{2}, L_{3}, L_{4}\right\} \\
M_{1} & =\max \left\{\max _{t \in I} a(t), \max _{t \in I} b(t), \max _{t \in I} c(t)\right\} \\
M_{2} & =\max _{i=1, \cdots, m}\left\{\left\|a_{1}(x)\right\|,\left\|a_{2}(x)\right\|,\left\|I_{i}(x)\right\|,\left\|J_{i}(x)\right\|\right\}
\end{align*}
$$

For simplicity of presentation, we list some conditions. $\left(\mathrm{H}_{1}\right) a, b, c \in C(I)$ are nonnegative functions and satisfy

$$
\int_{0}^{t}\|f(s, x, y)\| d s \leq \varphi_{p}[a(t)\|x\|+b(t)\|y\|+c(t)]
$$

$\forall x \in E$,

$$
\begin{align*}
& \int_{0}^{1}[(1+t) a(t)+b(t)] d t<M^{-1} \\
& \int_{0}^{1} c(t) d t<+\infty \tag{13}
\end{align*}
$$

$\left(\mathrm{H}_{2}\right)$ For any $r>0,[\alpha, \beta] \subset I, f(t, x, y)$ is uniformly continuous on $[\alpha, \beta] \times K_{r} \times K_{r}$.
$\left(\mathrm{H}_{3}\right)$ For any $x, y \in E,\left\|a_{k}(x)-a_{k}(y)\right\| \leq L_{k}\|x-y\|(k=$ $1,2),\left\|I_{i}(x)-I_{i}(y)\right\| \leq L_{3}\|x-y\|$, and $\left\|J_{i}(x)-J_{i}(y)\right\| \leq L_{4} \| x-$ $y \|, i=1,2, \cdots, m$.
$\left(\mathrm{H}_{4}\right)$ For all $t \in I$ and all bounded subsets $D_{1}, D_{2} \subset E$, there exist $l_{1}, l_{2} \in L[0, \infty)$ such that

$$
\begin{align*}
& \int_{0}^{t} \alpha_{E}\left(f\left(s, D_{1}, D_{2}\right)\right) d s  \tag{14}\\
& \quad \leq \varphi_{p}\left(\left[l_{1}(t) \alpha_{E}\left(D_{1}\right)+l_{2}(t) \alpha_{E}\left(D_{2}\right)\right]\right)
\end{align*}
$$

with $l<1$.
Lemma 9. Given $y \in C(I)$, the following $B V P$,

$$
\begin{align*}
\mathscr{D}_{0+}^{\beta}\left(\varphi_{p}\left(\mathscr{D}_{0+}^{\alpha} x(t)\right)\right) & =y(t), \\
\left.\Delta x(t)\right|_{t=t_{k}} & =I_{k}\left(x\left(t_{k}\right)\right), \\
\left.\Delta x^{\prime}(t)\right|_{t=t_{k}} & =J_{k}\left(x\left(t_{k}\right)\right) \\
x(0) & =x^{\prime}(0)=\int_{0}^{1} a_{1}(x(s)) d s  \tag{15}\\
x(1) & =x^{\prime}(1)=\int_{0}^{1} a_{2}(x(s)) d s \\
\mathscr{D}_{0+}^{\alpha} x(0) & =\theta
\end{align*}
$$

has a unique solution satisfying the following.

$$
\begin{aligned}
& x(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s \\
& \quad+\sum_{i=1}^{m+1} \frac{\alpha(2-t)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s \\
& \quad+\frac{1}{\Gamma(\alpha)} \int_{t_{k}}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s+(2-t) \\
& \quad \cdot \int_{0}^{1}\left[a_{1}(x(s))-a_{2}(x(s))\right] d s+\sum_{i=1}^{m}(2-t) \\
& \quad \cdot\left[\frac{\left(1-t_{m}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.\quad+\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right]+\sum_{i=1}^{m-1}(2-t) \\
& \quad \cdot\left[\frac{\left(t_{m}-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right] \\
& +\sum_{i=1}^{k}\left[\frac{\left(t-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}\left(\mathcal{F}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.+\frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}\left(\mathcal{J}_{0+}^{\beta} y(s)\right) d s\right] \\
& +\sum_{i=1}^{k}\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(t-t_{k}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]+\sum_{i=1}^{k-1}\left(t_{k}-t_{i}\right) \\
& \cdot J_{i}\left(x\left(t_{i}\right)\right) \tag{16}
\end{align*}
$$

Proof. From (15) and Lemma 2, we know

$$
\begin{equation*}
\varphi_{p}\left(\mathscr{D}_{0+}^{\alpha} x(t)\right)=\mathscr{J}_{0+}^{\beta} y(t)+c_{0} . \tag{17}
\end{equation*}
$$

Because of $\mathscr{D}_{0+}^{\alpha} x(0)=\theta$, we can obtain that $c_{0}=0$. According to the definition of the p-Laplacian operator it follows that

$$
\begin{equation*}
\mathscr{D}_{0+}^{\alpha} x(t)=\varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(t)\right) . \tag{18}
\end{equation*}
$$

Therefore

$$
\begin{align*}
x(t)= & \frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\mathscr{f}_{0+}^{\beta} y(s)\right) d s-c_{1}  \tag{19}\\
& -c_{2} t \\
x^{\prime}(t)= & \frac{1}{\Gamma(\alpha-1)} \int_{0}^{t}(t-s)^{\alpha-2} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s-c_{2} \tag{20}
\end{align*}
$$

If $t \in T_{1}$, then

$$
\begin{align*}
x(t)= & \frac{1}{\Gamma(\alpha)} \int_{t_{1}}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\mathcal{F}_{0+}^{\beta} y(s)\right) d s-d_{1} \\
& -d_{2}\left(t-t_{1}\right)  \tag{21}\\
x^{\prime}(t)= & \frac{1}{\Gamma(\alpha-1)} \int_{t_{1}}^{t}(t-s)^{\alpha-2} \varphi_{q}\left(\mathscr{F}_{0+}^{\beta} y(s)\right) d s-d_{2} .
\end{align*}
$$

For $d_{1}$ and $d_{2} \in R$, we have

$$
\begin{align*}
x\left(t_{1}^{-}\right)= & \frac{1}{\Gamma(\alpha)} \int_{0}^{t_{1}}\left(t_{1}-s\right)^{\alpha-1} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s-c_{1} \\
& -c_{2} t_{1}, \\
x\left(t_{1}^{+}\right)= & -d_{1}, \\
x^{\prime}\left(t_{1}^{-}\right)= & \frac{1}{\Gamma(\alpha-1)} \int_{0}^{t}(t-s)^{\alpha-2} \varphi_{q}\left(\mathscr{F}_{0+}^{\beta} y(s)\right) d s  \tag{22}\\
& -c_{2}, \\
x^{\prime}\left(t_{1}^{+}\right)= & -d_{2} .
\end{align*}
$$

When $t \in T_{k}$, we can also obtain

$$
\begin{align*}
x(t) & =\frac{1}{\Gamma(\alpha)} \int_{t_{k}}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s \\
& +\sum_{i=1}^{k-1}\left[\int_{t_{i-1}}^{t_{i}} \frac{t_{i}\left(t_{k}-t_{i}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right. \\
& \left.+\left(t_{k}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]  \tag{23}\\
& +\sum_{i=1}^{k}\left\{\int_{t_{i-1}}^{t_{i}}\left[\frac{\left(t_{i}-s\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left(t-t_{k}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\right. \\
& \left.+I_{i}\left(x\left(t_{i}\right)\right)+\left(t-t_{k}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}-c_{1}-c_{2} t
\end{align*}
$$

and taking the boundary condition of (15) into consideration, some tedious manipulation yields the following.

$$
\begin{align*}
& c_{1}=-\sum_{i=1}^{m+1}\left\{\int_{t_{i-1}}^{t_{i}}\left[\frac{\left(t_{i}-s\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\right. \\
& \text { - } \left.\varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s\right\} \\
& -\sum_{i=1}^{m}\left\{\int_{t_{i-1}}^{t_{i}} \frac{\left(1-t_{m}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_{q}\left(\mathcal{F}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.+I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}  \tag{24}\\
& -\sum_{i=1}^{m-1}\left\{\int_{t_{i-1}}^{t_{i}} \frac{\left(t_{m}-t_{i}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}-\int_{0}^{1}\left[2 a_{1}(x(s))\right. \\
& \left.-a_{2}(x(s))\right] d s \\
& c_{2}=\sum_{i=1}^{m+1}\left\{\int_{t_{i-1}}^{t_{i}}\left[\frac{\left(t_{i}-s\right)^{\alpha-1}}{\Gamma(\alpha)}+\frac{\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)}\right]\right. \\
& \text { - } \left.\varphi_{q}\left(\mathscr{\mathscr { G }}_{0+}^{\beta} y(s)\right) d s\right\} \\
& +\sum_{i=1}^{m}\left\{\int_{t_{i-1}}^{t_{i}} \frac{\left(1-t_{m}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_{q}\left(\mathscr{J}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.+I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}  \tag{25}\\
& +\sum_{i=1}^{m-1}\left\{\int_{t_{i-1}}^{t_{i}} \frac{\left(t_{m}-t_{i}\right)\left(t_{i}-s\right)^{\alpha-2}}{\Gamma(\alpha-1)} \varphi_{q}\left(\mathscr{F}_{0+}^{\beta} y(s)\right) d s\right. \\
& \left.+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}+\int_{0}^{1} a_{1}(x(s)) d s
\end{align*}
$$

Substituting (24) and (25) into (19), we can get (16), which implies that the solution of BVP (15) is given by (16).

From Lemma 9, we can establish the following conclusion.

Lemma 10. If $\left(\mathrm{H}_{1}\right)$ is satisfied, then BVP (4) has a unique solution satisfying

$$
\begin{aligned}
x & (t) \\
& =\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s \\
& +\sum_{i=1}^{m+1} \frac{\alpha(2-t)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s+\frac{1}{\Gamma(\alpha)} \\
& \cdot \int_{t_{k}}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s+(2-t) \\
& \cdot \int_{0}^{1}\left[a_{1}(x(s))-a_{2}(x(s))\right] d s+\sum_{i=1}^{m}(2-t) \\
& \cdot\left\{\frac{\left(1-t_{m}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right\}+\sum_{i=1}^{m-1}(2-t) \\
& \cdot\left\{\frac{\left(t_{m}-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\} \\
& +\sum_{i=1}^{k}\left\{\frac{\left(t-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s\right\} \\
& +\sum_{i=1}^{k}\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(t-t_{k}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]+\sum_{i=1}^{k-1}\left(t_{k}-t_{i}\right) \\
& \cdot J_{i}\left(x\left(t_{i}\right)\right) \cdot
\end{aligned}
$$

Proof. The proof is almost identical to Lemma 9, with the major change being the substitution of $y(t)$ for $f(t, x(t)$, $\left.x^{\prime}(t)\right)$.

Remark 11. From Lemma 10, we can get the conclusion that the solutions to the BVP (4) are equivalent to the fixed point of the following operator:

$$
\begin{aligned}
& (T x)(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s \\
& \quad+\sum_{i=1}^{m+1} \frac{\alpha(2-t)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s+\frac{1}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{align*}
& \cdot \int_{t_{k}}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s+(2-t) \\
& \cdot \int_{0}^{1}\left[a_{1}(x(s))-a_{2}(x(s))\right] d s+\sum_{i=1}^{m}(2-t) \\
& \cdot\left\{\frac{\left(1-t_{m}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right\}+\sum_{i=1}^{m-1}(2-t) \\
& \cdot\left\{\frac{\left(t_{m}-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\} \\
& +\sum_{i=1}^{k}\left\{\frac{\left(t-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+\frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s\right\} \\
& +\sum_{i=1}^{k}\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(t-t_{k}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]+\sum_{i=1}^{k-1}\left(t_{k}-t_{i}\right) \\
& \cdot J_{i}\left(x\left(t_{i}\right)\right), \forall x \in D C(I) \tag{27}
\end{align*}
$$

Taking derivative to both sides of (27), we have

$$
\begin{aligned}
& \left(T^{\prime} x\right)(t)=\frac{1}{\Gamma(\alpha-1)} \int_{0}^{t}(t-s)^{\alpha-2} \varphi_{q}(F(s)) d s \\
& \quad+\frac{1}{\Gamma(\alpha-1)} \int_{t_{k}}^{t}(t-s)^{\alpha-2} \varphi_{q}(F(s)) d s \\
& \quad-\int_{0}^{1}\left[a_{1}(x(s))-a_{2}(x(s))\right] d s \\
& \quad-\sum_{i=1}^{m+1} \frac{\alpha}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s \\
& \quad-\sum_{i=1}^{m}\left\{\frac{\left(1-t_{m}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.\quad+\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right\} \\
& \quad-\sum_{i=1}^{m-1}\left\{\frac{\left(t_{m}-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.\quad+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}
\end{aligned}
$$

$$
\begin{align*}
& +\sum_{i=1}^{k}\left\{\frac{1}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s\right. \\
& \left.+J_{i}\left(x\left(t_{i}\right)\right)\right\}, \quad \forall x \in D C(I) . \tag{28}
\end{align*}
$$

Lemma 12. If $\left(\mathrm{H}_{1}\right),\left(\mathrm{H}_{2}\right)$, and $\left(\mathrm{H}_{3}\right)$ are satisfied, then operator $T: D C(I) \longrightarrow D C(I)$ is continuous and bounded.

## Proof.

Step 1. For any $x \in D C(I)$, we prove that $T$ is well defined and $(T x)(t) \in D C(I)$. From condition $\left(\mathrm{H}_{1}\right)$, we have

$$
\begin{align*}
& \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s=\int_{0}^{1} \| \varphi_{q}\left(\frac{1}{\Gamma(\beta)}\right. \\
& \left.\quad \cdot \int_{0}^{s}(s-\tau)^{\beta-1} f\left(\tau, x(\tau), x^{\prime}(\tau)\right) d \tau\right) \| d s \\
& \quad \leq \int_{0}^{1} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}\left\|f\left(\tau, x(\tau), x^{\prime}(\tau)\right)\right\| d \tau\right)  \tag{29}\\
& \quad \leq \varphi_{q} \int_{0}^{1}\left(\frac { 1 } { \Gamma ( \beta ) } \varphi _ { p } \left[a(s)\|x\|+b(s)\left\|x^{\prime}\right\|\right.\right. \\
& \quad+c(s)]) d s \leq \Gamma(\beta)^{1-q} M_{1}\left(3\|x\|_{D}+1\right) .
\end{align*}
$$

Together with the definition of operator $T$, we have

$$
\begin{align*}
\left\|\frac{(T x)(t)}{1+t}\right\| \leq & \frac{1}{1+t}\left\{\left\|\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s\right\|+\left\|\sum_{i=1}^{m+1} \frac{\alpha(2-t)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s\right\|\right. \\
& +\left\|\frac{1}{\Gamma(\alpha)} \int_{t_{k}}^{t}(t-s)^{\alpha-1} \varphi_{q}(F(s)) d s\right\|+\left\|(2-t) \int_{0}^{1}\left[a_{1}(x(s))-a_{2}(x(s))\right] d s\right\| \\
& +\left\|\sum_{i=1}^{m}(2-t)\left\{\frac{\left(1-t_{m}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s+\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(2-t_{m}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right\}\right\| \\
& +\| \sum_{i=1}^{m-1}(2-t)\left\{\frac{\left(t_{m}-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s+\left(t_{m}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\}  \tag{30}\\
& +\left\|\sum_{i=1}^{k}\left\{\frac{\left(t-t_{i}\right)}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}(F(s)) d s+\frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}(F(s)) d s\right\}\right\| \\
& +\left\|\sum_{i=1}^{k}\left[I_{i}\left(x\left(t_{i}\right)\right)+\left(t-t_{k}\right) J_{i}\left(x\left(t_{i}\right)\right)\right]\right\|+\left\|\sum_{i=1}^{k-1}\left(t_{k}-t_{i}\right) J_{i}\left(x\left(t_{i}\right)\right)\right\| \\
\leq & \frac{10 m+4}{\Gamma(\alpha)} \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s+(11 m+1) M_{2}<+\infty .
\end{align*}
$$

As above, from (28), a tedious calculation gives

$$
\begin{align*}
\left\|(T x)^{\prime}(t)\right\| \leq & \frac{5 m+3}{\Gamma(\alpha)} \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s  \tag{31}\\
& +(5 m+1) M_{2}<+\infty
\end{align*}
$$

From (30) and (31), we have that $(T x)(t)$ is well defined and $(T x)(t) \in D C(I)$ for any $x \in D C(I)$.

Step 2. We prove that $T$ is bounded. For any $x \in K_{\rho}$, from (29)-(31), we get

$$
\begin{align*}
& \left\|\frac{(T x)(t)}{1+t}\right\| \leq(3 \rho+1) M M_{1}+(11 m+1) M_{2}  \tag{32}\\
& \left\|(T x)^{\prime}(t)\right\| \leq(3 \rho+1) M M_{1}+(5 m+1) M_{2}
\end{align*}
$$

So $T$ is bounded operator.

Step 3. It is time to prove that $T$ is continuous. Let $x_{n}, x \in$ $D C(I)$, and for any $s \in I, \varepsilon>0$, there exists $N_{1}>0$, when $n \geq N_{1}$, satisfying that $\left\|x_{n}-x\right\|_{D}<\varepsilon /(22 m+2) L$. So $\left\{x_{n}\right\}$ is a bounded subset of $D C(I)$; let $\eta>0$ such that $\left\|x_{n}\right\|_{D} \leq \eta$; taking limit, we obtain $\|x\|_{D} \leq \eta$. From $\left(\mathrm{H}_{2}\right)$, we know that, for any $\varepsilon>0$ and $s \in I$, there exists $N_{2}>0$; when $n \geq N_{2}$, we have

$$
\begin{align*}
& \left\|f\left(s, x_{n}(s), x_{n}^{\prime}(s)\right)-f\left(s, x(s), x^{\prime}(s)\right)\right\| \\
& \quad \leq M^{1 /(1-q)}\left(\frac{\varepsilon}{2}\right)^{1 /(q-1)} \tag{33}
\end{align*}
$$

Taking $N=\max \left\{N_{1}, N_{2}\right\}$, for any $\varepsilon>0, n \geq N$, and $t \in I$, according to (27), (28), we have

$$
\left\|\frac{\left(T x_{n}\right)(t)}{1+t}-\frac{(T x)(t)}{1+t}\right\| \leq \frac{10 m+4}{\Gamma(\alpha)(1+t)}\left\{\int _ { 0 } ^ { 1 } \varphi _ { q } \left(\frac{1}{\Gamma(\beta)}\right.\right.
$$

$$
\begin{align*}
& \left.\left.\cdot \int_{0}^{s}\left\|f\left(\tau, x_{n}(\tau), x_{n}^{\prime}(\tau)\right)-f\left(\tau, x(\tau), x^{\prime}(\tau)\right)\right\| d \tau\right) d s\right\} \\
& +\frac{2-t}{1+t} \int_{0}^{1}\left[\left\|a_{1}\left(x_{n}(s)\right)-a_{1}(x(s))\right\|+\| a_{1}\left(x_{n}(s)\right)\right. \\
& \left.-a_{1}(x(s)) \|\right] d s+\frac{1}{1+t}\left[3 m L_{3}+(8 m-3) L_{4}\right] \| x_{n}(s) \\
& \quad-x(s)\left\|\leq \frac{\varepsilon}{2(1+t)}+\frac{(11 m+1)}{1+t} L\right\| x_{n}(s)-x(s) \| \\
& \quad \leq \frac{\varepsilon}{2(1+t)}+\frac{\varepsilon}{2(1+t)}<\varepsilon  \tag{34}\\
& \left\|\left(T x_{n}\right)^{\prime}(t)-(T x)^{\prime}(t)\right\| \leq \frac{5 m}{\Gamma(\alpha)}\left\{\int _ { 0 } ^ { 1 } \varphi _ { q } \left(\frac{1}{\Gamma(\beta)}\right.\right. \\
& \left.\left.\quad \cdot \int_{0}^{s}\left\|f\left(\tau, x_{n}(\tau), x^{\prime}(\tau)\right)-f\left(\tau, x(\tau), x^{\prime}(\tau)\right)\right\| d \tau\right) d s\right\} \\
& \quad+\int_{0}^{1}\left[\left\|a_{1}\left(x_{n}(s)\right)-a_{1}(x(s))\right\|+\| a_{1}\left(x_{n}(s)\right)\right.  \tag{35}\\
& \left.\quad-a_{1}(x(s)) \|\right] d s+\left[m L_{3}+(4 m-1) L_{4}\right] \| x_{n}(s) \\
& \quad-x(s) \| \leq \frac{(5 m+3) \varepsilon}{10 m+4}+\frac{(5 m+1) \varepsilon}{22 m+2}<\varepsilon
\end{align*}
$$

Thus $\left\|\left(T x_{n}\right)(t) /(1+t)-(T x)(t) /(1+t)\right\|_{D}<\varepsilon$. So $T: D C(I) \longrightarrow$ $D C(I)$ is continuous.

Lemma 13. If $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{3}\right)$ are satisfied, $U$ is a bounded subset of $D C(I)$. Then $(T U)(t) /(1+t)$ and $(T U)^{\prime}(t)$ are equicontinuous on I.

Proof. In fact, from the boundedness of $U$, that is, for any $x \in U$, there exists $\eta>0$ such that $\|x\|_{D} \leq \eta$. Suppose that $t^{\prime}, t^{\prime \prime} \in I$ with $t^{\prime}<t^{\prime \prime}$; by the mean value theorem, we have the following.

$$
\begin{aligned}
& \left\|\frac{(T x)\left(t^{\prime \prime}\right)}{1+t^{\prime \prime}}-\frac{(T x)\left(t^{\prime}\right)}{1+t^{\prime}}\right\| \\
& \quad \leq \frac{1}{\Gamma(\alpha)} \| \int_{0}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& \quad-\int_{0}^{t^{\prime}} \frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}} \varphi_{q}(F(s)) d s \| \\
& \quad+\frac{1}{\Gamma(\alpha)} \| \int_{t_{k}}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& \quad-\int_{t_{k}}^{t^{\prime}} \frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}} \varphi_{q}(F(s)) d s \|+\left[\frac{3 m+2}{\Gamma(\alpha-1)}\right. \\
& \left.\quad \cdot \int_{0}^{1} \varphi_{q}(F(s)) d s+3 m M_{1}\right]\left|\frac{2-t^{\prime \prime}}{1+t^{\prime \prime}}-\frac{2-t^{\prime}}{1+t^{\prime}}\right|
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{m}{\Gamma(\alpha-1)}\left|\frac{t^{\prime \prime}}{1+t^{\prime \prime}}-\frac{t^{\prime}}{1+t^{\prime}}\right| \int_{0}^{1} \varphi_{q}(F(s)) d s \\
& +m M_{1}\left(t^{\prime \prime}-t^{\prime}\right) \\
& \leq \frac{1}{\Gamma(\alpha)} \| \int_{0}^{t^{\prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& +\int_{t^{\prime}}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& -\int_{0}^{t^{\prime}} \frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}} \varphi_{q}(F(s)) d s \| \\
& +\frac{1}{\Gamma(\alpha)} \| \int_{t_{k}}^{t^{\prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& +\int_{t^{\prime}}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s \\
& -\int_{t_{k}}^{t^{\prime}} \frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}} \varphi_{q}(F(s)) d s \|+3\left[\frac{3 m+2}{\Gamma(\alpha-1)}\right. \\
& \left.\cdot \int_{0}^{1} \varphi_{q}(F(s)) d s+3 m M_{1}\right]\left(t^{\prime \prime}-t^{\prime}\right) \\
& +\frac{m}{\Gamma(\alpha-1)} \int_{0}^{1} \varphi_{q}(F(s)) d s\left(t^{\prime \prime}-t^{\prime}\right)+m M_{1}\left(t^{\prime \prime}\right. \\
& \left.-t^{\prime}\right) \leq \frac{1}{\Gamma(\alpha)} \| \int_{0}^{t^{\prime}}\left[\frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}}-\frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}}\right] \\
& \cdot \varphi_{q}(F(s)) d s \| \\
& +\frac{1}{\Gamma(\alpha)}\left\|\int_{t^{\prime}}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s\right\| \\
& +\frac{1}{\Gamma(\alpha)} \| \int_{t_{k}}^{t^{\prime}}\left[\frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}}-\frac{\left(t^{\prime}-s\right)^{\alpha-1}}{1+t^{\prime}}\right] \\
& \varphi_{q}(F(s)) d s \| \\
& +\frac{1}{\Gamma(\alpha)}\left\|\int_{t^{\prime}}^{t^{\prime \prime}} \frac{\left(t^{\prime \prime}-s\right)^{\alpha-1}}{1+t^{\prime \prime}} \varphi_{q}(F(s)) d s\right\| \\
& +3\left[\frac{3 m+2}{\Gamma(\alpha-1)} \int_{0}^{1} \varphi_{q}(F(s)) d s+3 m M_{1}\right]\left(t^{\prime \prime}-t^{\prime}\right)
\end{aligned}
$$

$$
\begin{align*}
& +\frac{m}{\Gamma(\alpha-1)} \int_{0}^{1} \varphi_{q}(F(s)) d s\left(t^{\prime \prime}-t^{\prime}\right)+m M_{1}\left(t^{\prime \prime}\right. \\
& \left.-t^{\prime}\right) \leq \frac{2}{\Gamma(\alpha)}\left|\frac{\left(t^{\prime \prime}\right)^{\alpha}}{1+t^{\prime \prime}}-\frac{\left(t^{\prime}\right)^{\alpha}}{1+t^{\prime}}-\frac{\left(t^{\prime \prime}-t^{\prime}\right)^{\alpha}}{1+t^{\prime \prime}}\right| \\
& \cdot \int_{0}^{t^{\prime}}\left\|\varphi_{q}(F(s))\right\| d s+\frac{2}{\Gamma(\alpha)} \\
& \cdot \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s\left(t^{\prime \prime}-t^{\prime}\right) \\
& +3\left[\frac{3 m+2}{\Gamma(\alpha-1)} \int_{0}^{1} \varphi_{q}(F(s)) d s+3 m M_{1}\right]\left(t^{\prime \prime}-t^{\prime}\right) \\
& +\frac{m}{\Gamma(\alpha-1)} \int_{0}^{1} \varphi_{q}(F(s)) d s\left(t^{\prime \prime}-t^{\prime}\right)+m M_{1}\left(t^{\prime \prime}\right. \\
& \left.-t^{\prime}\right) \leq\left[\frac{10 m+18}{\Gamma(\alpha)} \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s+10 m M_{1}\right] \\
& \cdot\left(t^{\prime \prime}-t^{\prime}\right) \tag{36}
\end{align*}
$$

Let

$$
\begin{equation*}
\delta=\left[\frac{(10 m+18)(3 \eta+1) M_{1}}{\Gamma(\alpha) \Gamma(\beta)^{q-1}}+10 m M_{1}\right]^{-1} \cdot \frac{\varepsilon}{2} \tag{37}
\end{equation*}
$$

From (37),

$$
\begin{equation*}
\left\|\frac{(T x)\left(t^{\prime \prime}\right)}{1+t^{\prime \prime}}-\frac{(T x)\left(t^{\prime}\right)}{1+t^{\prime}}\right\|<\varepsilon \tag{38}
\end{equation*}
$$

For the case of $t^{\prime} \geq t^{\prime \prime}$, using the same methods we can also get (38). So $(T U)(t) /(1+t)$ is equicontinuous on $I$. An argument similar to the one used in $(T U)(t) /(1+t)$ shows that

$$
\begin{equation*}
\left\|(T x)^{\prime}\left(t^{\prime \prime}\right)-(T x)^{\prime}\left(t^{\prime}\right)\right\|<\varepsilon \tag{39}
\end{equation*}
$$

So $(T U)^{\prime}(t)$ is equicontinuous on $I$.
Lemma 14 (see [35]). Let condition $\left(\mathrm{H}_{1}\right)$ be satisfied and $U$ be a bounded subset of $D C(I)$. Then

$$
\begin{align*}
& \alpha_{D}(T U) \\
& \quad=\max \left\{\sup _{t \in I} \alpha_{E}\left(\frac{(T U)(t)}{1+t}\right), \sup _{t \in I} \alpha_{E}(T U)^{\prime}(t)\right\} . \tag{40}
\end{align*}
$$

Theorem 15. Let conditions $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$ be satisfied. Then BVP (4) has at least one solution belonging to $D C(I)$.

Proof. By Remark 11, we need only to show that the operator $T$ has a fixed point in $D C(I)$. Let

$$
\begin{equation*}
K_{R}=\left\{x \in D C(I):\|x\|_{D} \leq R\right\} \tag{41}
\end{equation*}
$$

where

$$
\begin{align*}
R> & {\left[(11 m+1) M^{-1} M_{2}+\int_{0}^{1} c(s) d s\right] } \\
& \cdot\left[M^{-1}-\int_{0}^{1}[(1+t) a(t)+b(t)] d t\right]^{-1} . \tag{42}
\end{align*}
$$

Step 1. we prove that $T K_{R} \subset K_{R}$. In fact, for any $x \in K_{R}, t \in I$, by (27), we have

$$
\begin{align*}
& \left\|\frac{(T x)(t)}{1+t}\right\| \leq \frac{10 m+4}{\Gamma(\alpha)} \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s+(11 m+1) \\
& \quad \cdot M_{2} \leq M\left[\int_{0}^{1}[(1+t) a(t)+b(t)] d t\|x\|_{D}\right. \\
& \left.\quad+\int_{0}^{1} c(t) d t\right]+(11 m+1) M_{2}  \tag{43}\\
& \quad \leq M\left\{\int_{0}^{1}[(1+t) a(t)+b(t)] d t R+R\left[M^{-1}\right.\right. \\
& \left.\quad-\int_{0}^{1}[(1+t) a(t)+b(t)] d t\right]-(11 m+1) \\
& \left.\quad \cdot M^{-1} M_{2}\right\}+(11 m+1) M_{2}<R, \\
& \left\|(T x)^{\prime}(t)\right\| \leq \frac{5 m+3}{\Gamma(\alpha)} \int_{0}^{1}\left\|\varphi_{q}(F(s))\right\| d s+(5 m+1) \\
& \quad \cdot M_{2}<M\left[\int_{0}^{1}[(1+t) a(t)+b(t)] d t\|x\|_{D}\right. \\
& \left.\quad+\int_{0}^{1} c(t) d t\right]+(5 m+1) M_{2}  \tag{44}\\
& \quad \leq M\left\{\int_{0}^{1}[(1+t) a(\mathrm{t})+b(t)] d t R+R\left[M^{-1}\right.\right. \\
& \quad-\int_{0}^{1}[(1+t) a(t)+b(t)] d t \\
& \left.\left.\quad-(11 m+1) M^{-1} M_{2}\right]\right\}+(5 m+1) M_{2}<R .
\end{align*}
$$

From (43) and (44), we know $\|T x\|_{D}<R$. Thus, from Lemma 12, $T K_{R} \subset K_{R}$ follows.

Step 2. It is time to prove that $T$ is a strict set contraction operator. Let $D=\overline{c o}_{D}\left(T K_{R}\right)$. It is easy to see that $D$ is nonempty, bounded, convex, and closed subset of $K_{R}$. According to Lemma 13, $\left(T K_{R}\right)(t) /(1+t)$ and $\left(T K_{R}\right)^{\prime}(t)$ are equicontinuous on $I$, so $(D)(t) /(1+t)$ and $(D)^{\prime}(t)$ are equicontinuous on $I$. From the definition of $D$, we get $D \subset K_{R}$ and $T K_{R} \subset D$, and by Lemma 12, $T: D \longrightarrow D$ is bounded and continuous. From $\left(\mathrm{H}_{2}\right)$, it follows that $\left\{f\left(s, x(s), x^{\prime}(s)\right)\right.$ : $x \in D\}$ are equicontinuous on $I$. For any $t \in I$ and $U \subset D$, by $\left(\mathrm{H}_{3}\right)$ and Lemma 2, we obtain

$$
\begin{aligned}
& \alpha_{E}\left(\frac{(T U)(t)}{1+t}\right) \leq \frac{1}{1+t}\left[\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s\right. \\
& \quad+\sum_{i=1}^{m+1} \frac{\alpha(2-t)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s+\frac{1}{\Gamma(\alpha)} \int_{t_{k}}^{t}(t-s)^{\alpha-1} \\
& \quad \cdot \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s \\
& \quad+\sum_{i=1}^{m} \frac{(2-t)\left(1-t_{m}\right)}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s \\
& \quad+\sum_{i=1}^{k} \frac{t-t_{i}}{\Gamma(\alpha-1)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-2} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s \\
& \left.\quad+\sum_{i=1}^{k} \frac{1}{\Gamma(\alpha)} \int_{t_{i-1}}^{t_{i}}\left(t_{i}-s\right)^{\alpha-1} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \int_{0}^{s}(s-\tau)^{\beta-1} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s\right] \leq \frac{10 m+4}{1+t} \\
& \quad \cdot \int_{0}^{1} \varphi_{q}\left(\frac{1}{\Gamma(\beta)} \alpha_{E}\left(\left\{f\left(\tau, x(\tau), x^{\prime}(\tau)\right): x \in U\right\}\right) d \tau\right) d s \leq \frac{M}{1+t} \int_{0}^{1}\left[l_{1}(s) \alpha_{E}(U(s))+l_{2}(s) \alpha_{E}\left(U^{\prime}(s)\right)\right] d s \\
& \quad \leq \frac{M}{1+t} \int_{0}^{1}\left[(1+s) l_{1}(s)+l_{2}(s)\right] d s \alpha_{D}(U) .
\end{aligned}
$$

Since $t$ is arbitrary, we obtain

$$
\begin{equation*}
\sup _{t \in I} \alpha_{E}\left(\frac{(T U)(t)}{1+t}\right) \leq l \alpha_{D}(U) . \tag{46}
\end{equation*}
$$

Using similar methods, we can also show that

$$
\begin{equation*}
\sup _{t \in I} \alpha_{E}\left((T U)^{\prime}(t)\right) \leq l \alpha_{D}(U) . \tag{47}
\end{equation*}
$$

Form (46) and (47), using Lemma 14 we get

$$
\begin{equation*}
\alpha_{D}(T U) \leq l \alpha_{D}(U) \tag{48}
\end{equation*}
$$

Obviously, $0 \leq l<1$, From Remark 7, $T$ is a condensing operator too. It follows from the Sadovskii fixed point theorem that $T$ has at least one fixed point in $D$; therefore, $\operatorname{BVP}(4)$ has at least one solution in $D C(I)$.

Remark 16. If $E=[0, \infty)$, we can get the following result.
Corollary 17. Let $f \in C[I \times[0, \infty) \times[0, \infty),[0, \infty)]$ and $\left(\mathrm{H}_{1}\right)-\left(\mathrm{H}_{4}\right)$ be satisfied. Then BVP (4) has at least one solution in $D C[I \times[0, \infty) \times[0, \infty),[0, \infty)]$.

Proof. Letting $E=[0, \infty)$ in Theorem 15, we can prove the desired result.

Corollary 18. Suppose the following assumptions holds:
$\left(\mathrm{H}_{1}\right) a, b, c \in C(I)$ are nonnegative functions and satisfy

$$
\int_{0}^{t}\|f(s, x)\| d s \leq \varphi_{p}[a(t)\|x\|+b(t)], \quad \forall x \in E
$$

$$
\begin{gather*}
\int_{0}^{1}(1+t) a(t) d t<M^{-1} \\
\int_{0}^{1} b(t) d t<+\infty \tag{49}
\end{gather*}
$$

$\left(\mathrm{H}_{2}\right)$ for any $r>0,[\alpha, \beta] \subset I, f(t, x)$ is uniformly continuous on $[\alpha, \beta] \times K_{r}$;
$\left(\mathrm{H}_{3}\right)$ for any $x, y \in E,\left\|a_{k}(x)-a_{k}(y)\right\| \leq L_{k}\|x-y\|(k=$ $1,2),\left\|I_{i}(x)-I_{i}(y)\right\| \leq L_{3}\|x-y\|$, and $\left\|J_{i}(x)-J_{i}(y)\right\| \leq L_{4} \| x-$ $y \|, i=1,2, \cdots, m$;
$\left(\mathrm{H}_{4}\right)$ for all $t \in I$ and all bounded subsets $D \subset E$, there exists $l_{1} \in L[0, \infty)$ such that

$$
\begin{equation*}
\int_{0}^{t} \alpha_{E}(f(s, D)) d s \leq \varphi_{p}\left(l_{1}(t) \alpha_{E}(D)\right) \tag{50}
\end{equation*}
$$

with $l=M \int_{0}^{1} l_{1}(t) d t<1$. Then the following $B V P$,

$$
\begin{align*}
\mathscr{D}_{0+}^{\beta}\left(\varphi_{p}\left(\mathscr{D}_{0+}^{\alpha} x\right)\right)(t) & =f(t, x(t)), \\
\left.\Delta x(t)\right|_{t=t_{k}} & =I_{k}\left(x\left(t_{k}\right)\right), \\
\left.\Delta x^{\prime}(t)\right|_{t=t_{k}} & =J_{k}\left(x\left(t_{k}\right)\right), \\
x(0) & =x^{\prime}(0)=\int_{0}^{1} a_{1}(x(s)) d s  \tag{51}\\
x(1) & =x^{\prime}(1)=\int_{0}^{1} a_{2}(x(s)) d s \\
\mathscr{D}_{0+}^{\alpha} x(0) & =\theta
\end{align*}
$$

has at least one solution belonging to $Q C(I)$.

Proof. The proofs are similar to Theorem 15, except the need for substitution of $f(t, x(t))$ for nonlinear term $f\left(t, x(t), x^{\prime}(t)\right)$.

## 4. Illustrative Example

Let $E=l_{\infty}=\left\{x=\left(x_{1}, x_{2}, \cdots, x_{n}, \cdots\right), \sup _{n}\left|x_{n}\right|<+\infty, t \in\right.$ $I\}$ and equipped with the norm $\|x\|=\sup _{n}\left|x_{n}\right|$. The following BVP

$$
\begin{align*}
& \mathscr{D}_{0+}^{1 / 2}\left(\varphi_{3 / 2}\left(\mathscr{D}_{0+}^{3 / 2} x_{n}\right)\right)(t)=\left[\frac{x_{n}(t)}{8(1+t)\left(1+t^{2}\right)}\right. \\
& \left.+\frac{\sqrt{\mid \sin \left(x_{2 n}(t)\right)^{2}}+\sqrt{\left|\sin \left(x_{n}^{\prime}(t)\right)\right|^{2}}}{36 n^{3} e^{\sqrt{t}}}\right]^{1 / 2} \\
& \left.\Delta x_{n}(t)\right|_{t=1 / 2}=\frac{\left|x_{n}\left((1 / 2)^{-}\right)\right|}{2+\left|x_{n}\left((1 / 2)^{-}\right)\right|},  \tag{52}\\
& \left.\Delta x_{n}^{\prime}(t)\right|_{t=1 / 2}=\frac{\left|x_{n}^{\prime}\left((1 / 2)^{-}\right)\right|}{2+\left|x_{n}^{\prime}\left((1 / 2)^{-}\right)\right|} \\
& x_{n}(0)=x_{n}^{\prime}(0)=\int_{0}^{1} \frac{\left\|x_{n}(s)\right\|}{4+\left\|x_{n}(s)\right\|} d s \\
& x_{n}(1)=x_{n}^{\prime}(1)=\int_{0}^{1} \frac{\left\|x_{n}(s)\right\|}{6+\left\|x_{n}(s)\right\|} d s \\
& \mathscr{D}_{0+}^{3 / 2} x_{n}(0)=0
\end{align*}
$$

can be regarded as a problem with the form of BVP (4), where

$$
\begin{align*}
& f(t, x, y) \\
& \quad=\left(f_{1}(t, x, y), f_{2}(t, x, y), \cdots, f_{n}(t, x, y), \cdots\right), \\
& f_{n}(t, x, y)=\left[\frac{x_{n}(t)}{8(1+t)\left(1+t^{2}\right)}\right.  \tag{53}\\
& \left.\quad+\frac{\sqrt{\left|\sin \left(x_{2 n}(t)\right)\right|^{2}}+\sqrt{\left|\sin \left(y_{n}(t)\right)\right|^{2}}}{36 n^{3} e^{\sqrt{t}}}\right]^{1 / 2} .
\end{align*}
$$

Clearly $L_{1}=1 / 4, L_{2}=1 / 6, L_{3}=L_{4}=1 / 2$, and

$$
\begin{aligned}
& \int_{0}^{t}\left\|f_{n}(s, x, y)\right\| d s \\
& \quad \leq \varphi_{3 / 2}\left(\left[\frac{1}{8(1+t)\left(1+t^{2}\right)}+\frac{1}{36 e^{\sqrt{t}}}\right]\|x\|\right. \\
& \left.\quad+\frac{1}{36 e^{\sqrt{t}}}\|y\|\right) .
\end{aligned}
$$

Let $a(t)=1 / 8(1+t)\left(1+t^{2}\right)+1 / 36 e^{\sqrt{t}}, b(t)=1 / 36 e^{\sqrt{t}}$, and $c(t)=0$. According to $\Gamma(1 / 2) \approx 1.772$ and $\Gamma(3 / 2) \approx 0.8862$, we have

$$
\begin{align*}
& M=\frac{10 m+4}{\Gamma(\alpha) \Gamma(\beta)^{q-1}}=6.6979 \\
& \int_{0}^{1}[(1+t) a(t)+b(t)] d t=0.0242<M^{-1} \\
& \int_{0}^{1} c(t) d t=0<+\infty  \tag{55}\\
& l=M \int_{0}^{1}\left[(1+t) l_{1}(t)+l_{2}(t)\right] d t=0.162<1
\end{align*}
$$

where $l_{1}(t)=1 / 8(1+t)\left(1+t^{2}\right)+1 / 36 e^{\sqrt{t}}$ and $l_{2}(t)=1 / 36 e^{\sqrt{t}}$. So, all the conditions of Theorem 15 are satisfied. Therefore, BVP (52) has at least one solution.

## 5. Conclusion

In this paper, we present some sufficient conditions which ensure the existence of solution to BVP (4) in Banach spaces. Through construction space $D C(I)$, using the technique of the Kuratowski noncompactness measure and the Sadovskii fixed point theorem, we obtain some new existence criteria for BVP (4). As far as we know, only a few papers have dealt with the boundary value problem for impulsive p-Laplacian fractional differential equations, especially in Banach space.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors wrote, read, and approved the final manuscript.

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# Separated Boundary Value Problems of Sequential Caputo and Hadamard Fractional Differential Equations 

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In this paper, we discuss the existence and uniqueness of solutions for new classes of separated boundary value problems of Caputo-Hadamard and Hadamard-Caputo sequential fractional differential equations by using standard fixed point theorems. We demonstrate the application of the obtained results with the aid of examples.

## 1. Introduction

Fractional differential equations have been of increasing importance for the past decades due to their diverse applications in science and engineering such as biophysics, bioengineering, virology, control theory, signal and image processing, blood flow phenomena, etc.; see [1-6]. Many interesting results of the existence of solutions of various classes of fractional differential equations have been obtained; see [7-15] and the references therein.

Sequential fractional differential equations are also found to be of much interest $[16,17]$. In fact, the concept of sequential fractional derivative is closely related to the nonsequential Riemann-Liouville derivatives, for details, see [3]. For some recent results on boundary value problems for sequential fractional differential equations; see [18-22] and references cited therein.

In this paper, we discuss existence and uniqueness of solutions for two sequential Caputo-Hadamard and HadamardCaputo fractional differential equations subject to separated boundary conditions as

$$
{ }^{C} D^{p}\left({ }^{H} D^{q} x\right)(t)=f(t, x(t)), \quad t \in(a, b)
$$

$$
\begin{align*}
& \alpha_{1} x(a)+\alpha_{2}\left({ }^{H} D^{q} x\right)(a)=0, \\
& \beta_{1} x(b)+\beta_{2}\left({ }^{H} D^{q} x\right)(b)=0, \tag{1}
\end{align*}
$$

and

$$
\begin{align*}
{ }^{H} D^{q}\left({ }^{C} D^{p} x\right)(t) & =f(t, x(t)), \quad t \in(a, b), \\
\alpha_{1} x(a)+\alpha_{2}\left({ }^{C} D^{p} x\right)(a) & =0,  \tag{2}\\
\beta_{1} x(b)+\beta_{2}\left({ }^{C} D^{p} x\right)(b) & =0,
\end{align*}
$$

where ${ }^{C} D^{p}$ and ${ }^{H} D^{q}$ are the Caputo and Hadamard fractional derivatives of orders $p$ and $q$, respectively, $0<p, q \leq 1, f$ : $[a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $a>0$ and $\alpha_{i}, \beta_{i} \in \mathbb{R}$, $i=1,2$.

It can be observed that the sequential Caputo-Hadamard and Hadamard-Caputo fractional differential equations in (1) and (2) are different type when $p=1$ and $q=1$, since

$$
\begin{equation*}
\frac{d}{d t}\left(t \frac{d}{d t} x(t)\right)=t \frac{d^{2} x(t)}{d t^{2}}+\frac{d x(t)}{d t}=f(t, x(t)) \tag{3}
\end{equation*}
$$

and

$$
\begin{equation*}
t \frac{d}{d t}\left(\frac{d}{d t} x(t)\right)=t \frac{d^{2} x(t)}{d t^{2}}=f(t, x(t)) \tag{4}
\end{equation*}
$$

for $t \in(a, b)$, respectively.
The rest of the paper is arranged as follows. In Section 2, we establish basic results that lay the foundation for defining a fixed point problem equivalent to the given problems (1) and (2). The main results, based on Banach's contraction mapping principle, Krasnoselskii's fixed point theorem, and nonlinear alternative of Leray-Schauder type, are obtained in Section 3. Illustrating examples are discussed in Section 4.

## 2. Preliminaries

In this section, we introduce some notations and definitions of fractional calculus [4,5] and present preliminary results needed in our proofs later.

Definition 1 (see [5]). For an at least $n$-times differentiable function $g:[a, \infty) \longrightarrow \mathbb{R}$, the Caputo derivative of fractional order $q$ is defined as

$$
\begin{align*}
&{ }^{C} D^{q} g(t)=\frac{1}{\Gamma(n-q)} \int_{a}^{t}(t-s)^{n-q-1} g^{(n)}(s) d s  \tag{5}\\
& n-1<q<n, n=[q]+1
\end{align*}
$$

where $[q]$ denotes the integer part of the real number $q$.
Definition 2 (see [5]). The Riemann-Liouville fractional integral of order $q$ of a function $g:[a, \infty) \longrightarrow \mathbb{R}$ is defined as

$$
\begin{equation*}
{ }^{R L} I^{q} g(t)=\frac{1}{\Gamma(q)} \int_{a}^{t} \frac{g(s)}{(t-s)^{1-q}} d s, \quad q>0 \tag{6}
\end{equation*}
$$

provided the integral exists.
Definition 3 (see [5]). For an at least $n$-times differentiable function $g:[a, \infty) \longrightarrow \mathbb{R}$, the Caputo-type Hadamard derivative of fractional order $q$ is defined as

$$
\begin{align*}
&{ }^{H} D^{q} g(t)=\frac{1}{\Gamma(n-q)} \int_{a}^{t}\left(\log \frac{t}{s}\right)^{n-q-1} \delta^{n} g(s) \frac{d s}{s}  \tag{7}\\
& n-1<q<n, n=[q]+1
\end{align*}
$$

where $\delta=t(d / d t), \log (\cdot)=\log _{e}(\cdot)$.
Definition 4 (see [5]). The Hadamard fractional integral of order $q$ is defined as

$$
\begin{equation*}
{ }^{H} I^{q} g(t)=\frac{1}{\Gamma(q)} \int_{a}^{t}\left(\log \frac{t}{s}\right)^{q-1} g(s) \frac{d s}{s}, \quad q>0 \tag{8}
\end{equation*}
$$

provided the integral exists.
Lemma 5 (see [5]). For $q>0$, the general solution of the fractional differential equation ${ }^{C} D^{q} u(t)=0$ is given by

$$
\begin{equation*}
u(t)=c_{0}+c_{1}(t-a)+\cdots+c_{n-1}(t-a)^{n-1} \tag{9}
\end{equation*}
$$

where $c_{i} \in \mathbb{R}, i=0,1,2, \ldots, n-1(n=[q]+1)$.

In view of Lemma 5, it follows that

$$
\begin{align*}
{ }^{R L} I^{q}\left({ }^{C} D^{q} u\right)(t)= & u(t)+c_{0}+c_{1}(t-a)+\cdots  \tag{10}\\
& +c_{n-1}(t-a)^{n-1}
\end{align*}
$$

for some $c_{i} \in \mathbb{R}, i=0,1,2, \ldots, n-1(n=[q]+1)$.
Lemma 6 (see [23]). Let $u \in A C_{\delta}^{n}[a, b]$ or $C_{\delta}^{n}[a, b]$ and $q \in \mathbb{C}$, where $X_{\delta}^{n}[a, b]=\left\{g:[a, b] \longrightarrow \mathbb{C}: \delta^{n-1} g(t) \in X[a, b]\right\}$. Then, we have

$$
\begin{equation*}
{ }^{H} I^{q}\left({ }^{H} D^{q}\right) u(t)=u(t)-\sum_{k=0}^{n-1} c_{k}\left(\log \left(\frac{t}{a}\right)\right)^{k} \tag{11}
\end{equation*}
$$

where $c_{i} \in \mathbb{R}, i=0,1,2, \ldots, n-1(n=[q]+1)$.
In order to define the solution of the boundary value problem (1), we consider the linear variant

$$
\begin{align*}
{ }^{C} D^{p}\left({ }^{H} D^{q} x\right)(t) & =y(t), \quad t \in(a, b), \\
\alpha_{1} x(a)+\alpha_{2}\left({ }^{H} D^{q} x\right)(a) & =0,  \tag{12}\\
\beta_{1} x(b)+\beta_{2}\left({ }^{H} D^{q} x\right)(b) & =0,
\end{align*}
$$

where $y \in C([a, b], \mathbb{R})$.
Lemma 7. Let

$$
\begin{equation*}
\Omega:=\beta_{1} \alpha_{2}-\alpha_{1}\left(\beta_{1} \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\beta_{2}\right) \neq 0 \tag{13}
\end{equation*}
$$

Then, the unique solution of the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (12) is given by the integral equation

$$
\begin{align*}
x(t)= & \frac{\beta_{1}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{H} I^{q}\left({ }^{R L} I^{p} y\right)(b) \\
& +\frac{\beta_{2}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{R L} I^{p} y(b)  \tag{14}\\
& +{ }^{H} I^{q}\left({ }^{R L} I^{p} y\right)(t), \quad t \in[a, b]
\end{align*}
$$

Proof. Taking the Riemann-Liouville fractional integral of order $p$ to the first equation of (12), we get

$$
\begin{equation*}
\left({ }^{H} D^{q} x\right)(t)=c_{1}+{ }^{R L} I^{p} y(t), \quad c_{1} \in \mathbb{R} . \tag{15}
\end{equation*}
$$

Again taking the Hadamard fractional integral of order $q$ to the above equation, we obtain

$$
\begin{equation*}
x(t)=c_{2}+c_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}+{ }^{H} I^{q}\left({ }^{R L} I^{p} y\right)(t) \tag{16}
\end{equation*}
$$

$$
c_{2} \in \mathbb{R}
$$

Substituting $t=a$ in (15)-(16) and applying the first boundary condition of (12), it follows that

$$
\begin{equation*}
\alpha_{2} c_{1}+\alpha_{1} c_{2}=0 \tag{17}
\end{equation*}
$$

For $t=b$ in equations (15)-(16) and using the second boundary condition of (12), it yields

$$
\begin{align*}
& c_{1}\left(\beta_{1} \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\beta_{2}\right)+\beta_{1} c_{2}  \tag{18}\\
& \quad=-\beta_{1}{ }^{H} I^{q}\left({ }^{R L} I^{p} y\right)(b)-\beta_{2}{ }^{R L} I^{p} y(b)
\end{align*}
$$

Solving the linear system of (17) and (18) for finding two constants $c_{1}, c_{2}$, we get

$$
\begin{equation*}
c_{1}=\frac{\alpha_{1} \beta_{1}}{\Omega} I^{q}\left({ }^{R L} I^{p} y\right)(b)+\frac{\alpha_{1} \beta_{2}}{\Omega}{ }^{R L} I^{p} y(b) \tag{19}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{2}=-\frac{\beta_{1} \alpha_{2}}{\Omega} I^{q}\left({ }^{R L} I^{p} y\right)(b)-\frac{\alpha_{2} \beta_{2}}{\Omega}{ }^{R L} I^{p} y(b) \tag{20}
\end{equation*}
$$

Substituting constants $c_{1}$ and $c_{2}$ in (16), we get the integral equation (14). The converse follows by direct computation. The proof is completed.

In the same way, we can prove the following lemma, which concerns a linear variant of problem (2):

$$
\begin{align*}
{ }^{H} D^{q}\left({ }^{C} D^{p} x\right)(t) & =z(t), \quad t \in(a, b), \\
\alpha_{1} x(a)+\alpha_{2}\left({ }^{C} D^{p} x\right)(a) & =0,  \tag{21}\\
\beta_{1} x(b)+\beta_{2}\left({ }^{C} D^{p} x\right)(b) & =0,
\end{align*}
$$

where $z \in C([a, b], \mathbb{R})$.
Lemma 8. Let

$$
\begin{equation*}
\Omega^{*}:=\beta_{1} \alpha_{2}-\alpha_{1}\left(\beta_{1} \frac{(b-a)^{p}}{\Gamma(p+1)}+\beta_{2}\right) \neq 0 \tag{22}
\end{equation*}
$$

Then, the unique solution of the separated boundary value problem of sequential Caputo and Hadamard fractional differential equation (21) is given by the integral equation

$$
\begin{align*}
x(t)= & \frac{\beta_{1}}{\Omega^{*}}\left(\alpha_{1} \frac{(t-a)^{p}}{\Gamma(p+1)}-\alpha_{2}\right){ }^{R L} I^{p}\left({ }^{H} I^{q} z\right)(b) \\
& +\frac{\beta_{2}}{\Omega^{*}}\left(\alpha_{1} \frac{(t-a)^{p}}{\Gamma(p+1)}-\alpha_{2}\right){ }^{H} I^{q} z(b)  \tag{23}\\
& +{ }^{R L} I^{p}\left({ }^{H} I^{q} z\right)(t), \quad t \in[a, b] .
\end{align*}
$$

## 3. Main Results

We set some abbreviate notations for sequential RiemannLiouville and Hadamard fractional integrals of a function with two variables as

$$
\begin{align*}
& { }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(\phi)=\frac{1}{\Gamma(q) \Gamma(p)} \\
& \cdot \int_{a}^{\phi} \int_{a}^{s}\left(\log \frac{\phi}{s}\right)^{q-1}(s-r)^{p-1} f(r, x(r)) d r \frac{d s}{s} \tag{24}
\end{align*}
$$

and

$$
\begin{align*}
& { }^{R L} I^{p}\left({ }^{H} I^{q}\left(f_{x}\right)\right)(\phi)=\frac{1}{\Gamma(p) \Gamma(q)}  \tag{25}\\
& \quad \cdot \int_{a}^{\phi} \int_{a}^{s}(\phi-s)^{p-1}\left(\log \frac{s}{r}\right)^{q-1} f(r, x(r)) \frac{d r}{r} d s
\end{align*}
$$

where $\phi \in\{t, b\}$. Also we use this one for a single RiemannLiouville and Hadamard fractional integrals of orders $p$ and $q$, respectively.

In this section, we will use fixed point theorems to prove the existence and uniqueness of solution for problems (1) and (2). To accomplish our purpose, we define the Banach space $\mathscr{C}=C([a, b], \mathbb{R})$, of all continuous functions on $[a, b]$ to $\mathbb{R}$ endowed with the norm $\|x\|=\sup \{|x(t)|, t \in[a, b]\}$. In addition, we define the operator $\mathscr{K}: \mathscr{C} \longrightarrow \mathscr{C}$ by

$$
\begin{align*}
\mathscr{K} x & (t) \\
= & \frac{\beta_{1}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(b) \\
& +\frac{\beta_{2}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{R L} I^{p}\left(f_{x}\right)(b)  \tag{26}\\
& +{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(t),
\end{align*}
$$

where $\Omega \neq 0$ is defined by (13) and $f_{x}(t)=f(t, x(t))$. Note that the separated boundary value problem (1) has solutions if and only if $x=\mathscr{K} x$ has fixed points.

For computational convenience we put

$$
\begin{align*}
& \Omega_{1} \\
& =\frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) \\
&  \tag{27}\\
& \quad+\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}(1)(b) \\
& \quad+{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) .
\end{align*}
$$

To prove the existence theorems of problem (2), we define the operator $\mathscr{A}: \mathscr{C} \longrightarrow \mathscr{C}$ by

$$
\begin{align*}
& \mathscr{A} x(t) \\
& =\begin{aligned}
& \beta_{1} \\
& \Omega^{*}\left(\alpha_{1} \frac{(t-a)^{p}}{\Gamma(p+1)}-\alpha_{2}\right){ }^{R L} I^{p}\left({ }^{H} I^{q}\left(f_{x}\right)\right)(b) \\
&+\frac{\beta_{2}}{\Omega^{*}}\left(\alpha_{1} \frac{(t-a)^{p}}{\Gamma(p+1)}-\alpha_{2}\right){ }^{H} I^{q}\left(f_{x}\right)(b) \\
&+{ }^{R L} I^{p}\left({ }^{H} I^{q}\left(f_{x}\right)\right)(t) .
\end{aligned}
\end{align*}
$$

Now, we prove the existence and uniqueness result for problem (1). For problem (2) the proof is similar and omitted.

Theorem 9. Suppose that
$\left(H_{1}\right)$ there exists a function $\psi(t)>0, t \in[a, b]$, such that

$$
\begin{align*}
& |f(t, x)-f(t, y)| \leq \psi(t)|x-y|  \tag{29}\\
& \quad \text { for all } t \in[a, b] \text { and } x, y \in \mathbb{R} .
\end{align*}
$$

If $\psi^{*} \Omega_{1}<1$, where $\psi^{*}=\sup \{\psi(t): t \in[a, b]\}$, then the separated boundary value problem (1) has a unique solution on $[a, b]$.

Proof. Firstly, we define a ball $B_{r}$ as $B_{r}=\{x \in \mathscr{C}:\|x\| \leq r\}$, where the constant $r$ satisfies

$$
\begin{equation*}
r \geq \frac{M \Omega_{1}}{1-\psi^{*} \Omega_{1}} \tag{30}
\end{equation*}
$$

where $M=\sup \{f(t, 0): t \in[a, b]\}$. Next, we will show that $\mathscr{K} B_{r} \subset B_{r}$. For any $x \in B_{r}$ and using the triangle inequality $\left|f_{x}\right| \leq\left|f_{x}-f_{0}\right|+\left|f_{0}\right|$, we have

$$
\begin{aligned}
& |\mathscr{K} x(t)| \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}\left(\left|f_{x}\right|\right)(b) \\
& \quad+{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(t) \\
& \quad \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}-f_{0}\right|+\left|f_{0}\right|\right)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{R L} I^{p}\left(\left|f_{x}-f_{0}\right|+\left|f_{0}\right|\right)(b) \\
& +{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}-f_{0}\right|+\left|f_{0}\right|\right)\right)(b) \\
& \quad \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\psi^{*} r+M\right)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}\left(\psi^{*} r+M\right) \\
& \quad \cdot(b)+{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\psi^{*} r+M\right)\right)(b)=\psi^{*} \Omega_{1} r+M \Omega_{1} \\
& \leq r
\end{aligned}
$$

which implies that $\mathscr{K} B_{r} \subset B_{r}$. Let $x, y \in B_{r}$, then

$$
\begin{align*}
& |\mathscr{K} x(t)-\mathscr{K} y(t)| \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}-f_{y}\right|\right)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}\left(\left|f_{x}-f_{y}\right|\right) \\
& \cdot(b)+{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}-f_{y}\right|\right)\right)(t) \\
& \quad \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \psi^{*}\|x-y\|  \tag{32}\\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \psi^{*}\|x-y\| \\
& .{ }^{R L} I^{p}(1)(b)+\psi^{*}\|x-y\|^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(t) \\
& =\psi^{*} \Omega_{1}\|x-y\|,
\end{align*}
$$

which yields that $\|\mathscr{K} x-\mathscr{K} y\| \leq \psi^{*} \Omega_{1}\|x-y\|$. Since $\psi^{*} \Omega_{1}<1$, we deduce that the operator $\mathscr{K}$ is a contraction. By Banach contraction mapping principle the operator $\mathscr{K}$ has a unique fixed point, which leads that problem (1) has a unique solution on $[a, b]$.

Theorem 10. Let $\left(H_{1}\right)$ in Theorem 9 holds. If $\psi^{*} \Omega_{1}^{*}<1$, where

$$
\begin{align*}
\Omega_{1}^{*}= & \frac{\left|\beta_{1}\right|}{\left|\Omega^{*}\right|}\left(\left|\alpha_{1}\right| \frac{(b-a)^{p}}{\Gamma(p+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}\left({ }^{H} I^{q}(1)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{\left|\Omega^{*}\right|}\left(\left|\alpha_{1}\right| \frac{(b-a)^{p}}{\Gamma(p+1)}+\left|\alpha_{2}\right|\right){ }^{H} I^{q}(1)(b)  \tag{33}\\
& +{ }^{R L} I^{p}\left({ }^{H} I^{q}(1)\right)(b)
\end{align*}
$$

then the separated boundary value problem (2) has a unique solution on $[a, b]$.

Our second existence result is based on Krasnoselskii's fixed point theorem.

Theorem 11 ((Krasnoselskii's fixed point theorem) [24]). Let $Q$ be a closed, bounded, convex, and nonempty subset of a Banach space X. Let A, B be operators such that
(a) $A x+B y \in Q$ where $x, y \in Q$;
(b) A is compact and continuous;
(c) $B$ is a contraction mapping.

Then there exists $z \in Q$ such that $z=A z+B z$.
Theorem 12. Let $f:[a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying $\left(H_{1}\right)$ in Theorem 9. In addition, assume that
$\left(H_{2}\right)|f(t, x)| \leq \varphi(t), \forall(t, x) \in[a, b] \times \mathbb{R}$ and $\varphi \in$ $C\left([a, b], \mathbb{R}^{+}\right)$.

If

$$
\begin{equation*}
\psi^{*}\left[{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b)\right]<1, \tag{34}
\end{equation*}
$$

then the separated boundary value problem (1) has at least one solution on $[a, b]$.

Proof. Let $B_{\rho}=\{x \in \mathscr{C}:\|x\| \leq \rho\}$, where a constant $\rho$ satisfying $\rho \geq \varphi^{*} \Omega_{1}$ and $\varphi^{*}=\sup \{\varphi(t): t \in[a, b]\}$. We decompose the operator $\mathscr{K}$ into two operators $\mathscr{K}_{1}$ and $\mathscr{K}_{2}$ on $B_{\rho}$ with

$$
\begin{align*}
& \mathscr{K}_{1} x(t) \\
& =\frac{\beta_{1}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(b) \\
& \quad+\frac{\beta_{2}}{\Omega}\left(\alpha_{1} \frac{(\log (t / a))^{q}}{\Gamma(q+1)}-\alpha_{2}\right){ }^{R L} I^{p}\left(f_{x}\right)(b),  \tag{35}\\
& \quad t \in[a, b], \\
& \mathscr{K}_{2} x(t)={ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(t), \quad t \in[a, b] .
\end{align*}
$$

Note that the ball $B_{\rho}$ is a closed, bounded, and convex subset of the Banach space $\mathscr{C}$.

Now, we will show that $\mathscr{K}_{1} x+\mathscr{K}_{2} y \in B_{\rho}$ for satisfying condition (a) of Theorem 11. Setting $x, y \in B_{\rho}$, then we have

$$
\begin{align*}
& \left|\mathscr{K}_{1} x(t)+\mathscr{K}_{2} y(t)\right| \\
& \quad \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& \quad \cdot{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(b) \\
& \quad+\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right)^{R L} I^{p}\left(\left|f_{x}\right|\right)(b) \\
& \quad+{ }^{H} I^{q R L} I^{p}\left(\left|f_{y}\right|\right)(t)  \tag{36}\\
& \quad \leq \varphi^{*}\left(\frac{\left|\alpha_{1} \beta_{1}\right|}{|\Omega|} \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\frac{\left|\beta_{1} \alpha_{2}\right|}{|\Omega|}\right){ }^{H} I^{q R L} I^{p}(1) \\
& \cdot(b)+\varphi^{*}\left(\frac{\left|\alpha_{1} \beta_{2}\right|}{|\Omega|} \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\frac{\left|\alpha_{2} \beta_{2}\right|}{|\Omega|}\right)^{R L} I^{p}(1) \\
& \cdot(b)+\varphi^{* H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b)=\varphi^{*} \Omega_{1} \leq \rho .
\end{align*}
$$

This means that $\mathscr{K}_{1} x+\mathscr{K}_{2} y \in B_{\rho}$. To prove that $\mathscr{K}_{2}$ is a contraction mapping, for $x, y \in B_{\rho}$, we have

$$
\begin{align*}
\left\|\mathscr{K}_{2} x-\mathscr{K}_{2} y\right\| & \leq{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}-f_{y}\right|\right)\right)(b) \\
& \leq \psi^{*}\left[{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b)\right]\|x-y\| \tag{37}
\end{align*}
$$

by condition $\left(H_{1}\right)$, which is a contraction, by (34). Therefore, the condition (c) of Theorem 11 is satisfied. Next we will show that the operator $\mathscr{K}_{1}$ is compact and continuous. By using the
continuity of the function $f$ on $[a, b] \times \mathbb{R}$, we can conclude that the operator $\mathscr{K}_{1}$ is continuous. For $x \in B_{\rho}$, it follows that

$$
\begin{equation*}
\left\|\mathscr{K}_{1} x\right\| \leq \varphi^{*} \Omega_{2} \tag{38}
\end{equation*}
$$

where
$\Omega_{2}$

$$
\begin{align*}
= & \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b)  \tag{39}\\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}(1)(b)
\end{align*}
$$

which implies that the set $\mathscr{K}_{1} B_{\rho}$ is uniformly bounded. Now we are going to prove that $\mathscr{K}_{1} B_{\rho}$ is equicontinuous. For $\tau_{1}$, $\tau_{2} \in[a, b]$ such that $\tau_{1}<\tau_{2}$ and for $x \in B_{\rho}$, we have

$$
\begin{align*}
& \left|\mathscr{K}_{1} x\left(\tau_{2}\right)-\mathscr{K}_{1} x\left(\tau_{1}\right)\right| \\
& \quad \leq \frac{\left|\alpha_{1} \beta_{1}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)^{q}\right)\right| \\
& \quad \cdot{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)(b) \\
& \quad+\frac{\left|\alpha_{1} \beta_{2}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)^{q}\right)\right|  \tag{40}\\
& \quad .{ }^{R L} I^{p}\left(f_{x}\right)(b) \\
& \quad \leq \varphi^{*} \Omega_{2}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)^{q}\right)\right|
\end{align*}
$$

which is independent of $x$ and also tends to zero as $\tau_{1} \longrightarrow$ $\tau_{2}$. Hence the set $\mathscr{K}_{1} B_{\rho}$ is equicontinuous. Therefore the set $\mathscr{K}_{1} B_{\rho}$ is relatively compact. By applying the Arzelá-Ascoli theorem, the operator $\mathscr{K}_{1}$ is compact on $B_{\rho}$. Therefore the operators $\mathscr{K}_{1}$ and $\mathscr{K}_{2}$ satisfy the assumptions of Theorem 11. By the conclusion of Theorem 11, we get that the separated boundary value problem (1) has at least one solution on $[a, b]$. This completes the proof.

Theorem 13. Assume that $\left(H_{1}\right)$ and $\left(H_{2}\right)$ are fulfilled. If $\psi^{*}\left[{ }^{R L} I^{p}\left({ }^{H} I^{q}(1)\right)(b)\right]<1$, then the separated boundary value problem (2) has at least one solution on $[a, b]$.

The above theorem can be proved by applying Krasnoselskii's fixed point theorem to the operator $\mathscr{A}$ defined in (28).

Remark 14. If the operators $\mathscr{K}_{1}$ and $\mathscr{K}_{2}$ are interchanged, then we have the existence results as follows:
(i) If $\psi^{*} \Omega_{2}<1$, then problem (1) has at least one solution on $[a, b]$.
(ii) If $\psi^{*} \Omega_{2}^{*}<1$, then problem (2) has at least one solution on $[a, b]$, where

$$
\begin{align*}
\Omega_{2}^{*}= & \frac{\left|\beta_{1}\right|}{\left|\Omega^{*}\right|}\left(\left|\alpha_{1}\right| \frac{(b-a)^{p}}{\Gamma(p+1)}+\left|\alpha_{2}\right|\right){ }^{R L} I^{p}\left({ }^{H} I^{q}(1)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{\left|\Omega^{*}\right|}\left(\left|\alpha_{1}\right| \frac{(b-a)^{p}}{\Gamma(p+1)}+\left|\beta_{2}\right|\right){ }^{H} I^{q}(1)(b) . \tag{41}
\end{align*}
$$

However, in application to existence theory, the computation of values ${ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b)$ and ${ }^{R L} I^{p}\left({ }^{H} I^{q}(1)\right)(b)$ is easier than $\Omega_{2}$ and $\Omega_{2}^{*}$, respectively.

The third existence result will be proved by applying Leray-Schauder nonlinear alternative.

Theorem 15 ((nonlinear alternative for single valued maps) [25]). Let E be a Banach space, C a closed, convex subset of $E, U$ an open subset of $C$, and $0 \in U$. Suppose that $\mathscr{D}: \bar{U} \longrightarrow C$ is a continuous; compact (that is, $\mathscr{D}(\bar{U})$ is a relatively compact subset of C) map. Then either
(i) $\mathscr{D}$ has a fixed point in $\bar{U}$ or
(ii) there is a $x \in \partial U$ (the boundary of $U$ in $C$ ) and $v \in$ $(0,1)$ with $x=\nu \mathscr{D}(x)$.

Let us state and prove the existence theorem.
Theorem 16. Suppose that
$\left(\mathrm{H}_{3}\right)$ there exist a continuous nondecreasing function $\xi$ : $[0, \infty) \longrightarrow(0, \infty)$ and a function $\eta \in C\left([a, b], \mathbb{R}^{+}\right)$such that

$$
\begin{equation*}
|f(t, x)| \leq \eta(t) \xi(|x|) \quad \text { for each }(t, x) \in[a, b] \times \mathbb{R} \tag{42}
\end{equation*}
$$

$\left(H_{4}\right)$ there exists a constant $K>0$ such that

$$
\begin{equation*}
\frac{K}{\|\eta\| \xi(K) \Omega_{1}}>1 \tag{43}
\end{equation*}
$$

Then the separated boundary value problem (1) has at least one solution on $[a, b]$.

Proof. Let the operator $\mathscr{K}$ be defined in (26). Let us prove that the operator $\mathscr{K}$ maps bounded sets (balls) into bounded sets in $\mathscr{C}$. For a constant $\lambda>0$, we define a bounded ball $B_{\lambda}=\{x \in \mathscr{C}:\|x\| \leq \lambda\}$. Then for $t \in[a, b]$, one has

$$
\begin{align*}
& |\mathscr{K} x(t)| \leq \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& \quad .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(b) \\
& +\frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right)^{R L} I^{p}\left(\left|f_{x}\right|\right)(b) \\
& \quad+{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(t) \leq\|\eta\| \xi(|x|)  \tag{44}\\
& \quad \cdot \frac{\left|\beta_{1}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right){ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) \\
& +\|\eta\| \xi(|x|) \frac{\left|\beta_{2}\right|}{|\Omega|}\left(\left|\alpha_{1}\right| \frac{(\log (b / a))^{q}}{\Gamma(q+1)}+\left|\alpha_{2}\right|\right) \\
& .{ }^{R L} I^{p}(1)(b)+\|\eta\| \xi(|x|)^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) \leq\|\eta\| \\
& \cdot \xi(\lambda) \Omega_{1},
\end{align*}
$$

which implies that

$$
\begin{equation*}
\|\mathscr{K} x\| \leq\|\eta\| \xi(\lambda) \Omega_{1} . \tag{45}
\end{equation*}
$$

After that we will show that the operator $\mathscr{K}$ maps bounded sets into equicontinuous sets of $\mathscr{C}$. Let $\tau_{1}, \tau_{2}$ be any two points in $[a, b]$ such that $\tau_{1}<\tau_{2}$. Then for $x \in B_{\lambda}$, we have

$$
\begin{align*}
& \left|(\mathscr{K} x)\left(\tau_{2}\right)-(\mathscr{K} x)\left(\tau_{1}\right)\right| \\
& \quad \leq \frac{\left|\alpha_{1} \beta_{1}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)\right)^{q}\right| \\
& \quad .{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(\left|f_{x}\right|\right)\right)(b) \\
& \quad+\frac{\left|\alpha_{1} \beta_{2}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)\right)^{q}\right| \\
& \quad .{ }^{R L} I^{p}\left(\left|f_{x}\right|\right)(b) \\
& \quad+\left|{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)\left(\tau_{2}\right)-{ }^{H} I^{q}\left({ }^{R L} I^{p}\left(f_{x}\right)\right)\left(\tau_{1}\right)\right|  \tag{46}\\
& \quad \leq \frac{\|\eta\| \xi(\lambda)\left|\alpha_{1} \beta_{1}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)\right)^{q}\right| \\
& .{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)(b) \\
& \quad+\frac{\|\eta\| \xi(\lambda)\left|\alpha_{1} \beta_{2}\right|}{|\Omega| \Gamma(q+1)}\left|\left(\log \left(\frac{\tau_{2}}{a}\right)\right)^{q}-\left(\log \left(\frac{\tau_{1}}{a}\right)\right)^{q}\right| \\
& .{ }^{R L} I^{p}(1)(b)+\|\eta\| \xi(\lambda) \\
& \quad .\left|{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)\left(\tau_{2}\right)-{ }^{H} I^{q}\left({ }^{R L} I^{p}(1)\right)\left(\tau_{1}\right)\right| .
\end{align*}
$$

As $\tau_{1} \longrightarrow \tau_{2}$, the right-hand side of the above inequality tends to zero independently of $x \in B_{\rho}$. Hence, by applying the Arzelá-Ascoli theorem, the operator $\mathscr{K}: \mathscr{C} \longrightarrow \mathscr{C}$ is completely continuous.

The result will be followed from the Leray-Schauder nonlinear alternative if we prove the boundedness of the set of the solutions to equation $x=\nu \mathscr{K} x$ for $v \in(0,1)$. Let $x$ be a solution of the operator equation $x=\mathscr{K} x$. Then, for $t \in[a, b]$, by directly computation, we have

$$
\begin{equation*}
|x(t)| \leq\|\eta\| \xi(\|x\|) \Omega_{1} \tag{47}
\end{equation*}
$$

which leads to

$$
\begin{equation*}
\frac{\|x\|}{\|\eta\| \xi(\|x\|) \Omega_{1}} \leq 1 \tag{48}
\end{equation*}
$$

From the assumption $\left(H_{4}\right)$, there exists a positive constant $K$ such that $\|x\| \neq K$. Let us set

$$
\begin{equation*}
U=\{x \in \mathscr{C}:\|x\|<K\} . \tag{49}
\end{equation*}
$$

It is easy to see that the operator $\mathscr{K}: \bar{U} \longrightarrow \mathscr{C}$ is continuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x=\nu \mathscr{K} x$ for some $v \in(0,1)$. Therefore, by the nonlinear alternative of Leray-Schauder type (Theorem 15), we deduce that the operator $\mathscr{K}$ has a fixed point $x \in \bar{U}$ which is a solution of problem (1). The proof is completed.

Theorem 17. Assume that the condition $\left(H_{3}\right)$ in Theorem 16 is satisfied. If a positive constant $K_{1}$ satisfying

$$
\begin{equation*}
\frac{K_{1}}{\|\eta\| \xi\left(K_{1}\right) \Omega_{1}^{*}}>1 \tag{50}
\end{equation*}
$$

then the separated boundary value problem (2) has at least one solution on $[a, b]$.

The next two special cases can be obtained by setting $\eta(t)=1, t \in[a, b]$ and $\xi(y)=E y+G, y \in[0, \infty)$ with two constants $E \geq 0, G>0$.

Corollary 18. Let $f:[a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function satisfying $|f(t, x)| \leq E|x|+G$, for all $x \in \mathbb{R}$. Then
(i) if $E \Omega_{1}<1$, then the separated boundary value problem (1) has at least one solution on $[a, b]$;
(ii) if $E \Omega_{1}^{*}<1$, then the separated boundary value problem (2) has at least one solution on $[a, b]$.

## 4. Examples

In this section, we present some examples to illustrate our results.

Example 1. Consider the following sequential CaputoHadamard fractional differential equations with separated boundary conditions

$$
\begin{aligned}
& { }^{{ }^{C} D^{1 / 2}\left({ }^{H} D^{1 / 3} x\right)(t)=f(t, x(t)),} \\
& \\
& \\
& t \in\left(\frac{1}{2}, \frac{5}{2}\right),
\end{aligned}
$$

$$
\begin{align*}
& \frac{1}{4} x\left(\frac{1}{2}\right)+\frac{3}{5}\left({ }^{H} D^{1 / 3} x\right)\left(\frac{1}{2}\right)=0  \tag{51}\\
& \frac{5}{8} x\left(\frac{5}{2}\right)+\frac{7}{9}\left({ }^{H} D^{1 / 3} x\right)\left(\frac{5}{2}\right)=0
\end{align*}
$$

Here $p=1 / 2, q=1 / 3, a=1 / 2, b=5 / 2, \alpha_{1}=1 / 4$, $\alpha_{2}=3 / 5, \beta_{1}=5 / 8$, and $\beta_{2}=7 / 9$. From given information, we find that $\Omega=-0.0244992447,{ }^{H} I^{1 / 3}\left({ }^{R L} I^{1 / 2}(1)\right)(5 / 2)=$ 1.622871815 , and ${ }^{R L} I^{1 / 2}(1)(5 / 2)=1.595769121$ which yield $\Omega_{1}=87.06444876$.
(i) Let $f:[1 / 2,5 / 2] \times \mathbb{R} \longrightarrow \mathbb{R}$ with

$$
\begin{equation*}
f(t, x)=\frac{\cos ^{2} \pi t}{2((t-1 / 2)+90)}\left(\frac{x^{2}+|x|}{|x|+1}\right)+\frac{1}{2} \tag{52}
\end{equation*}
$$

It follows that

$$
\begin{align*}
|f(t, x)-f(t, y)| & \leq \frac{\cos ^{2} \pi t}{((t-1 / 2)+90)}|x-y|  \tag{53}\\
& :=\psi(t)|x-y|
\end{align*}
$$

Then condition $\left(H_{1}\right)$ is satisfied with $\psi^{*}=1 / 90$. Thus $\psi^{*} \Omega_{1}=$ $0.9673827640<1$. Hence, by Theorem 9, problem (51) with (52) has a unique solution on $[1 / 2,5 / 2]$.
(ii) Given $f:[1 / 2,5 / 2] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
\begin{equation*}
f(t, x)=\frac{\cos ^{2} \pi t}{2((t-1 / 2)+2)}\left(\frac{x^{2}+|x|}{|x|+1}\right)+\frac{1}{2} \tag{54}
\end{equation*}
$$

Observe that the function $f$ defined in (54) satisfies $\left(H_{1}\right)$ with $\psi^{*}=1 / 2$. But the Theorem 9 can not be applied to this case because the value of $\psi^{*} \Omega_{1}=43.53222438>$ 1. However, by the benefit of Theorem 12, we have $\psi^{*}\left[{ }^{H} I^{1 / 3}\left({ }^{R L} I^{1 / 2}(1)\right)(5 / 2)\right]=0.8114359075<1$. By the conclusion of Theorem 12, problem (51) with (54) has at least one solution on [1/2,5/2].

Example 2. Consider the following sequential HadamardCaputo fractional differential equations with separated boundary conditions

$$
\begin{aligned}
& { }^{H} D^{3 / 4}\left({ }^{C} D^{2 / 5} x\right)(t)=g(t, x(t)) \\
& \\
& t \in\left(\frac{1}{8}, \frac{7}{8}\right),
\end{aligned}
$$

$$
\begin{align*}
\frac{3}{11} x\left(\frac{1}{8}\right)+\frac{\pi}{4}\left({ }^{C} D^{2 / 5} x\right)\left(\frac{1}{8}\right) & =0  \tag{55}\\
\frac{\sqrt{2}}{9} x\left(\frac{7}{8}\right)+\frac{2}{13}\left({ }^{C} D^{2 / 5} x\right)\left(\frac{7}{8}\right) & =0
\end{align*}
$$

Here $q=3 / 4, p=2 / 5, a=1 / 8, b=7 / 8, \alpha_{1}=3 / 11, \alpha_{2}=$ $\pi / 4, \beta_{1}=\sqrt{2} / 9$, and $\beta_{2}=2 / 13$. From above information, we can find that $\Omega^{*}=0.03840540910,{ }^{R L} I^{2 / 5}\left({ }^{H} I^{3 / 4}(1)\right)(7 / 8)=$ 1.526044488 , and ${ }^{H} I^{3 / 4}(1)(7 / 8)=1.792656288$ which can be computed the value of $\Omega_{1}^{*}=15.74791264$.
(i) The function $g:[1 / 8,7 / 8] \times \mathbb{R} \longrightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
g(t, x)=\frac{8 \sin ^{4} \pi t}{263+8 t}\left(\frac{x^{6}}{x^{4}+1}+1\right) \tag{56}
\end{equation*}
$$

Setting $\eta(t)=\left(8 \sin ^{4} \pi t /(263+8 t)\right)$ and $\xi(y)=y^{2}+1$, we see that the condition $\left(H_{3}\right)$ of Theorem 16 is satisfied with the above function $g(t, x)$. In addition, we can find that $\|\eta\|=1 / 33$. Then there exists a constant $K$ such that $K \in(0.7350333746,1.360482441)$ satisfying inequality (50). Therefore, applying Theorem 17, problem (55) with (56) has at least one solution on $[1 / 8,7 / 8]$.
(ii) Let $g:[1 / 8,7 / 8] \times \mathbb{R} \longrightarrow \mathbb{R}$ by

$$
\begin{equation*}
g(t, x)=\frac{e^{-(t-1 / 8)^{2}}}{16}\left(\frac{x^{8}}{|x|^{7}+1}\right)+\frac{3}{4\left(1+t^{2}\right)} \tag{57}
\end{equation*}
$$

It is easy to see that the function $g(t, x)$ defined in (57) can be expressed as $|g(t, x)| \leq(1 / 16)|x|+(3 / 4)$. Then $(1 / 16) \Omega_{1}^{*}=$ $0.9842445400<1$. Using (ii) of the Corollary 18, the problem (55) with (57) has at least one solution on [1/8, 7/8].

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# The Existence and Uniqueness of Solutions and Lyapunov-Type Inequality for CFR Fractional Differential Equations 

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In this paper, we research CFR fractional differential equations with the derivative of order $3<\alpha<4$. We prove existence and uniqueness theorems for CFR-type initial value problem. By Green's function and its corresponding maximum value, we obtain the Lyapunov-type inequality of corresponding equations. As for application, we study the eigenvalue problem in the sense of CFR.

## 1. Introduction

In the past decades, fractional calculus arose ([1-3]) and received extensive attention of many researchers. In recent years, it becomes more important because the subject of fractional calculus frequently appears in various fields such as science and engineering. Recently, some new fractional differential definitions have been created. With the rising of fractional computation, the research on the quantitative and qualitative properties of fractional differential equations has become a hot topic; see papers [4-15] and the references therein.

In [4-7], the authors presented new fractional derivatives with Mittsg-Leffler kernels and exponential-type kernels. The boundary value problem of fractional equations emerged as new branch in the field of differential equation due to its wide applications. In [8], Abdeljawad defined the higher order fractional derivative in the sense of Abdon and Baleanu and obtained a Lyapunov-type inequality for ABR fractional boundary value problem

$$
\begin{align*}
\left({ }_{a}^{A B R} D^{\alpha} y\right)(t)+q(t) y(t) & =0, \quad a<t<b, 2<\alpha<3  \tag{1}\\
y(a) & =y(\mathrm{~b})=0
\end{align*}
$$

and if $y$ is a nontrivial solution, then

$$
\begin{equation*}
\int_{a}^{b} T(s) d s>\frac{4}{b-a} \tag{2}
\end{equation*}
$$

where

$$
\begin{align*}
& T(s) \\
& \qquad=\left[\frac{3-\alpha}{B(\alpha-2)}|q(t)|+\frac{\alpha-2}{B(\alpha-2)}\left({ }_{a} I^{\alpha-2}|q(s)|\right)(t)\right] . \tag{3}
\end{align*}
$$

In [5], Abdeljawad defined the higher order fractional derivative in the sense of Caputo and Fabrizio and obtained a Lyapunov-type inequality for CFR fractional boundary value problem

$$
\begin{gather*}
\left({ }_{a}^{C F R} D^{\alpha} y\right)(t)+q(t) y(t)=0, \quad a<t<b, 2<\alpha<3  \tag{4}\\
y(a)=y(b)=0,
\end{gather*}
$$

and if $y$ is a nontrivial solution, then

$$
\begin{equation*}
\int_{a}^{b} R(s) d s>\frac{4}{b-a} \tag{5}
\end{equation*}
$$

where

$$
\begin{equation*}
R(s)=\left[\frac{3-\alpha}{B(\alpha-2)}|q(t)|+\frac{\alpha-2}{B(\alpha-2)} \int_{a}^{b}|q(s)| d s\right] . \tag{6}
\end{equation*}
$$

In this paper, we study the existence and uniqueness for initial problems

$$
\begin{align*}
\left({ }_{a}^{C F R} D^{\alpha} y\right)(t) & =f(t, y(t)), \quad a<t<b, 2<\alpha<3, \\
y(a) & =c_{1},  \tag{7}\\
y^{\prime}(a) & =c_{2},
\end{align*}
$$

and

$$
\begin{align*}
\left({ }_{a}^{C F R} D^{\alpha} y\right)(t) & =f(t, y(t)), \quad a<t<b, 3<\alpha<4, \\
y(a) & =c_{1}, \\
y^{\prime}(a) & =c_{2},  \tag{8}\\
y^{\prime \prime}(a) & =c_{3} .
\end{align*}
$$

We also consider the following boundary value problem:

$$
\begin{gather*}
\left({ }_{a}^{C F R} D^{\alpha} y\right)(t)+q(t) y(t)=0, \quad a<t<b, 3<\alpha<4, \\
y(a)=y^{\prime}(a)=y(b)=0, \tag{9}
\end{gather*}
$$

where is ${ }_{a}^{C F R} D^{\alpha}$ the Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio and $q:[a, b] \longrightarrow R$ is a continuous function.

Let us introduce the concepts of the Riemann-Liouville fractional integral and the Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio.

Definition 1 (see [1]). Let $\alpha>0$ and $f$ be a real function defined on $[a, b]$. The Riemann-Liouville fractional integral of order $\alpha$ is defined by

$$
\begin{align*}
\left({ }_{a} I^{\alpha} f\right)(t)=\frac{1}{\Gamma(\alpha)} \int_{a}^{t}(t-s)^{\alpha-1} f(s) d s &  \tag{10}\\
& \\
& \alpha>0, t \in[a, b]
\end{align*}
$$

where $\Gamma(\alpha)$ is defined by

$$
\begin{equation*}
\Gamma(\alpha)=\int_{0}^{\infty} t^{\alpha-1} e^{-t} d t, \quad \alpha>0 . \tag{11}
\end{equation*}
$$

This is fractionalizing of the $n$-iterated integral

$$
\begin{align*}
\left({ }_{a} I^{n} f\right)(t)=\frac{1}{(n-1)!} \int_{a}^{t}(t-s)^{n-1} f(s) d s, &  \tag{12}\\
& t \in[a, b] .
\end{align*}
$$

Definition 2 (see [5]). Let $f \in H^{1}[a, b], a<b, \alpha \in[0,1)$, then the Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio is defined by

$$
\begin{align*}
& \left({ }_{a}^{C F R} D^{\alpha} f\right)(t) \\
& \quad=\frac{B(\alpha)}{1-\alpha} \frac{d}{d t} \int_{a}^{t} f(x) e^{\left[-\alpha\left((t-x)^{\alpha} /(1-\alpha)\right)\right]} d x . \tag{13}
\end{align*}
$$

The associated fractional integral is

$$
\begin{equation*}
\left({ }_{a}^{C F} I^{\alpha} f\right)(t)=\frac{1-\alpha}{B(\alpha)} f(t)+\frac{\alpha}{B(\alpha)} \int_{a}^{t} f(s) d s \tag{14}
\end{equation*}
$$

where $B(\alpha)>0$ is a normalization function satisfying $B(0)=$ $B(1)=1$.

Definition 3 (see [5]). Let $n<\alpha \leq n+1$ and $f$ be such that $f^{(n)} \in H^{1}[a, b], a<b$. Set $\beta=\alpha-n, \beta \in(0,1)$, then the Riemann-Liouville fractional derivative in the sense of Caputo and Fabrizio has the following form:

$$
\begin{equation*}
\left({ }_{a}^{C F R} D^{\alpha} f\right)(t)=\left({ }_{a}^{C F R} D^{\beta} f^{(n)}\right)(t) . \tag{15}
\end{equation*}
$$

The associated fractional integral is

$$
\begin{equation*}
\left({ }_{a}^{C F} I^{\alpha} f\right)(t)=\left({ }_{a} I^{n}\left({ }_{a}^{C F} I^{\beta} f\right)\right)(t) . \tag{16}
\end{equation*}
$$

We also give a proposition which will be used in this article.

Proposition 4 (see [5]). For $u(t) \in[a, b]$ and $n \leq \alpha \leq n+1$, we have

$$
\begin{align*}
& \left({ }_{a}^{C F R} D^{\alpha} \cdot{ }_{a}^{C F} I^{\alpha} u\right)(t)=u(t), \\
& \left({ }_{a}^{C F} I^{\alpha} \cdot{ }_{a}^{C F R} D^{\alpha} u\right)(t)=u(t)-\sum_{k=0}^{n-1} \frac{u^{(k)}(a)}{k!}(t-a)^{k} . \tag{17}
\end{align*}
$$

## 2. Existence and Uniqueness Theorems

In this section, we establish existence and uniqueness theorems for CFR-type initial value problem and give corresponding proofs. We make some conditions as the mark $\left(A_{0}\right)$ :

$$
\begin{align*}
& \left(A_{0}\right)\left|f\left(t, y_{1}\right)-f\left(t, y_{2}\right)\right| \leq A\left|y_{1}-y_{2}\right| \\
& \quad A>0, f:[a, b] \times R \longrightarrow R, y:[a, b] \longrightarrow R . \tag{18}
\end{align*}
$$

Theorem 5. Consider the initial value problem (7). Suppose that $\left(A_{0}\right)$ holds; if

$$
\begin{equation*}
\frac{A}{B(\alpha-2)}\left[\frac{(3-\alpha)(b-a)^{2}}{2}+\frac{(\alpha-2)(b-a)^{3}}{6}\right]<1, \tag{19}
\end{equation*}
$$

then system (7) has a unique solution of the form

$$
\begin{equation*}
y(t)=c_{1}+c_{2}(t-a)+{ }_{a}^{C F} I^{\alpha} f(t, y(t)) . \tag{20}
\end{equation*}
$$

Proof. First, applying ${ }_{a}^{C F} I^{\alpha}$ to system (7) and using Proposition 4 with $\beta=\alpha-2$, then we have (20). On the other hand, if we apply ${ }_{a}^{C F R} D^{\alpha}$ to (20) and using Proposition 4, then we obtain (7). It is clear that $y(t)$ satisfies the system (7) if and only if it satisfies (20).

We endow the set $C[a, b]$ with the norm $\|x\|=$ $\max _{t \in[a, b]}|x(t)|$. We define the linear operator $T$ :

$$
\begin{equation*}
(T x)(t)=c_{1}+c_{2}(t-a)+{ }_{a}^{C F} I^{\alpha} f(t, x(t)) . \tag{21}
\end{equation*}
$$

Then, for arbitrary $x_{1}, x_{2} \in[a, b], \beta=\alpha-2$, we have

$$
\begin{align*}
&\left|\left(T x_{1}\right)(t)-\left(T x_{2}\right)(t)\right| \\
&=\left|{ }_{a}^{C F} I^{\alpha}\left[f\left(t, x_{1}(t)\right)-f\left(t, x_{2}(t)\right)\right]\right| \leq A_{a}^{C F} I^{\alpha} \mid x_{1} \\
&-x_{2} \mid=A_{a} I^{2}\left({ }_{a}^{C F} I^{\beta}\left|x_{1}-x_{2}\right|\right) \\
&=A_{a} I^{2}\left(\frac{1-\beta}{B(\beta)}\left|x_{1}-x_{2}\right|+\frac{\beta}{B(\beta)} \int_{a}^{t}\left|x_{1}-x_{2}\right| d s\right) \\
&=A\left(\frac{1-\beta}{B(\beta)} \int_{a}^{t}(t-s)\left|x_{1}-x_{2}\right| d s\right. \\
&\left.+\frac{\beta}{2 B(\beta)} \int_{a}^{t}(t-s)^{2}\left|x_{1}-x_{2}\right| d s\right) \\
&=A\left(\frac{3-\alpha}{B(\alpha-2)} \int_{a}^{t}(t-s)\left|x_{1}-x_{2}\right| d s\right.  \tag{22}\\
&\left.+\frac{\alpha-2}{2 B(\alpha-2)} \int_{a}^{t}(t-s)^{2}\left|x_{1}-x_{2}\right| d s\right) \\
& \leq A\left[\frac{3-\alpha}{B(\alpha-2)} \int_{a}^{t}(t-s) d s\right. \\
&\left.+\frac{\alpha-2}{2 B(\alpha-2)} \int_{a}^{t}(t-s)^{2} d s\right]\left\|x_{1}-x_{2}\right\| \\
& \leq \frac{A}{B(\alpha-2)}\left[\frac{(3-\alpha)(b-a)^{2}}{2}+\frac{(\alpha-2)(b-a)^{3}}{6}\right] \\
&+\left\|x_{1}-x_{2}\right\| \leq\left\|x_{1}-x_{2}\right\| .
\end{align*}
$$

Hence T is a contraction. Form the Banach contraction principle, there exists a unique $x$ such that $T x=x$. The proof is complete.

Theorem 6. Consider the initial value problem (8). Suppose that $\left(A_{0}\right)$ holds; if

$$
\begin{equation*}
\frac{A}{B(\alpha-3)}\left[\frac{(4-\alpha)(b-a)^{3}}{6}+\frac{(\alpha-3)(b-a)^{4}}{24}\right]<1 \tag{23}
\end{equation*}
$$

then system (8) has a unique solution of the form

$$
\begin{equation*}
y(t)=c_{1}+c_{2}(t-a)+c_{3}(t-a)^{2}+{ }_{a}^{C F} I^{\alpha} f(t, y(t)) \tag{24}
\end{equation*}
$$

Proof. Applying ${ }_{a}^{C F} I^{\alpha}$ to system (8) and using Proposition 4 with $\beta=\alpha-3$, then we have (24). On the other hand, if we apply ${ }_{a}^{C F R} D^{\alpha}$ to (24) and using Proposition 4, then we obtain (8). It is clear that $y(t)$ satisfies system (8) if and only if it satisfies (24).

We endow the set $C[a, b]$ with the norm $\|x\|=$ $\max _{t \in[a, b]}|x(t)|$. We define the linear operator T :

$$
\begin{align*}
(T x)(t)= & c_{1}+c_{2}(t-a)+c_{3}(t-a)^{2}  \tag{25}\\
& +{ }_{a}^{C F} I^{\alpha} f(t, y(t)) .
\end{align*}
$$

Then, for arbitrary $x_{1}, x_{2} \in[a, b], \beta=\alpha-3$, we have

$$
\begin{align*}
& \left|\left(T x_{1}\right)(t)-\left(T x_{2}\right)(t)\right| \\
& \quad=\left|{ }_{a}^{C F} I^{\alpha}\left[f\left(t, x_{1}(t)\right)-f\left(t, x_{2}(t)\right)\right]\right| \leq A_{a}^{C F} I^{\alpha} \mid x_{1} \\
& \quad-x_{2} \mid=A_{a} I^{3}\left({ }_{a}^{C F} I^{\beta}\left|x_{1}-x_{2}\right|\right) \\
& \quad=A_{a} I^{3}\left(\frac{1-\beta}{B(\beta)}\left|x_{1}-x_{2}\right|+\frac{\beta}{B(\beta)} \int_{a}^{t}\left|x_{1}-x_{2}\right| d s\right) \\
& \quad=A\left(\frac{1-\beta}{2 B(\beta)} \int_{a}^{t}(t-s)^{2}\left|x_{1}-x_{2}\right| d s\right. \\
& \left.\quad+\frac{\beta}{6 B(\beta)} \int_{a}^{t}(t-s)^{3}\left|x_{1}-x_{2}\right| d s\right) \\
& \quad=A\left(\frac{4-\alpha}{2 B(\alpha-3)} \int_{a}^{t}(t-s)^{2}\left|x_{1}-x_{2}\right| d s\right.  \tag{26}\\
& \left.\quad+\frac{\alpha-3}{6 B(\alpha-3)} \int_{a}^{t}(t-s)^{3}\left|x_{1}-x_{2}\right| d s\right) \\
& \quad \leq A\left[\frac{4-\alpha}{2 B(\alpha-3)} \int_{a}^{t}(t-s)^{2} d s\right. \\
& \left.\quad+\frac{\alpha-3}{6 B(\alpha-3)} \int_{a}^{t}(t-s)^{3} d s\right]\left\|x_{1}-x_{2}\right\| \\
& \quad \leq \frac{A}{B(\alpha-3)}\left[\frac{(4-\alpha)(b-a)^{3}}{6}+\frac{(\alpha-3)(b-a)^{4}}{24}\right] \\
& \quad\left\|x_{1}-x_{2}\right\| \leq\left\|x_{1}-x_{2}\right\|,
\end{align*}
$$

Hence T is a contraction. Form the Banach contraction principle, there exists a unique $x$ such that $T x=x$. The proof is complete.

## 3. Lyapunov Inequality for the CFR Boundary Value Problem

In this section, we establish some results for the CFR boundary value problem and give corresponding proofs.

Theorem 7. $y \in C[a, b]$ is a solution of the boundary value problem (9) if and only if $y$ satisfies the integral equation

$$
\begin{equation*}
y(t)=\int_{a}^{b} G(t, s) T(s, y(s)) d s \tag{27}
\end{equation*}
$$

where $G(t, s)$ is Green's function defined as

$$
\begin{align*}
& G(t, s) \\
& \quad= \begin{cases}\frac{(t-a)^{2}}{2(b-a)^{2}}(b-s)^{2}, & a \leq t \leq s \leq b \\
\frac{(t-a)^{2}}{2(b-a)^{2}}(b-s)^{2}-\frac{(t-s)^{2}}{2}, & a \leq s \leq t \leq b\end{cases} \tag{28}
\end{align*}
$$

and

$$
\begin{align*}
T(s, y(s))= & \left({ }_{a}^{C F} I^{\beta} q(\cdot) y(\cdot)\right)(s) \\
= & \frac{1-\beta}{B(\beta)} q(s) y(s)  \tag{29}\\
& +\frac{\beta}{B(\beta)}\left({ }_{a} I^{1} q(\cdot) y(\cdot)\right)(s),
\end{align*}
$$

$$
\beta=\alpha-3 .
$$

Proof. From (9), we have

$$
\begin{equation*}
{ }_{a}^{C F R} D^{\alpha} y(t)=-q(t) y(t) . \tag{30}
\end{equation*}
$$

Apply integral ${ }_{a}^{C F} I^{\alpha}$ on the (30); we have

$$
\begin{equation*}
{ }_{a}^{C F} I^{\alpha}\left({ }_{a}^{C F R} D^{\alpha} y(t)\right)={ }_{a}^{C F} I^{\alpha}(-q(t) y(t)) . \tag{31}
\end{equation*}
$$

Then, by Proposition 4 and Definitions 2 and 3, we obtain

$$
\begin{align*}
y & (t)+c_{1}+c_{2}(t-a)+c_{3}(t-a)^{2} \\
& =-{ }_{a} I^{3} \cdot{ }_{a}^{C F} I^{\beta} q(t) y(t) \\
& =-{ }_{a} I^{3}\left[\frac{1-\beta}{B(\beta)} q(t) y(t)+\frac{\beta}{B(\beta)}\left({ }_{a} I^{1} q(\cdot) y(\cdot)\right)(s)\right]  \tag{32}\\
& =-\frac{1}{2} \int_{a}^{b}(t-s)^{2} T(s, y(s)) d s
\end{align*}
$$

Then, we have

$$
\begin{align*}
y(t)= & c_{1}+c_{2}(t-a)+c_{3}(t-a)^{2} \\
& -\frac{1}{2} \int_{a}^{t}(t-s)^{2} T(s, y(s)) d s, \tag{33}
\end{align*}
$$

for some real constants $c_{i} \in R(i=1,2,3)$.
From $y(a)=0$, we get immediately that $c_{1}=0$. By the boundary condition $y^{\prime}(a)=0$, we can obtain that $c_{2}=0$. Hence,

$$
\begin{equation*}
y(t)=c_{3}(t-a)^{2}-\frac{1}{2} \int_{a}^{t}(t-s)^{2} T(s, y(s)) d s \tag{34}
\end{equation*}
$$

Using the boundary condition $y(b)=0$ yields

$$
\begin{equation*}
c_{3}=\frac{1}{2(b-a)^{2}} \int_{a}^{b}(b-s)^{2} T(s, y(s)) d s . \tag{35}
\end{equation*}
$$

Hence, equality (33) becomes

$$
\begin{align*}
y(t)= & \frac{(t-a)^{2}}{2(b-a)^{2}} \int_{a}^{b}(b-s)^{2} T(s, y(s)) d s  \tag{36}\\
& -\frac{1}{2} \int_{a}^{t}(t-s)^{2} T(s, y(s)) d s
\end{align*}
$$

By splitting the integral as follows:

$$
\begin{align*}
& \int_{a}^{b}(b-s)^{2} T(s, y(s)) d s \\
& \quad=\int_{a}^{t}(b-s)^{2} T(s, y(s)) d s  \tag{37}\\
& \quad+\int_{t}^{b}(b-s)^{2} T(s, y(s)) d s
\end{align*}
$$

We have that (27) holds. The proof is completed.
Corollary 8. For $s \in[a, b]$, the function $G$ in (28) satisfied the following properties:

$$
\begin{align*}
\max _{t \in[a, b]} G(t, s) & =G\left(t^{*}, s\right) \\
\max _{s \in[a, b]} G\left(t^{*}, s\right) & =\frac{(b-s)^{2}(s-a)}{2(2 b-s-a)}, \tag{38}
\end{align*}
$$

where

$$
\begin{equation*}
t^{*}=\frac{a((b-s) /(b-a))^{2}-s}{((b-s) /(b-a))^{2}-1} \tag{39}
\end{equation*}
$$

Proof. First, we define the function

$$
\begin{equation*}
g_{1}(t, s)=\frac{(t-a)^{2}}{2(b-a)^{2}}(b-s)^{2} \tag{40}
\end{equation*}
$$

and

$$
\begin{equation*}
g_{2}(t, s)=\frac{(t-a)^{2}}{2(b-a)^{2}}(b-s)^{2}-\frac{(t-s)^{2}}{2} . \tag{41}
\end{equation*}
$$

Differentiating $g_{1}(t, s)$ with respect to $t$, we get

$$
\begin{equation*}
\partial g_{1}(t, s)=\frac{(b-s)^{2}}{(b-a)^{2}}(t-a) \geq 0 \tag{42}
\end{equation*}
$$

Hence, $g_{1}(t, s)$ is an increasing function on $t$.
Then,

$$
\begin{align*}
g_{2} & (t, s)=\frac{(t-a)^{2}}{2(b-a)^{2}}(b-s)^{2}-\frac{(t-s)^{2}}{2} \\
& =\frac{1}{2}\left[\left(\frac{b-s}{b-a}\right)^{2}(t-a)^{2}-(t-s)^{2}\right] \\
& =\frac{1}{2}\left\{\left[\left(\frac{b-s}{b-a}\right)^{2}-1\right] t^{2}+\left[2 s-2 a\left(\frac{b-s}{b-a}\right)^{2}\right] t\right.  \tag{43}\\
& \left.+\left[\left(\frac{b-s}{b-a}\right)^{2} a^{2}-s^{2}\right]\right\}=\frac{((b-s) /(b-a))^{2}-1}{2} \\
& \cdot t^{2}+\left[s-a\left(\frac{(b-s)}{(b-a)}\right)^{2}\right] t \\
& +\frac{((b-s) /(b-a))^{2} a^{2}-s^{2}}{2} .
\end{align*}
$$

## Choose

$$
\begin{equation*}
t^{*}=-\frac{b}{2 a}=\frac{a((b-s) /(b-a))^{2}-s}{((b-s) /(b-a))^{2}-1} \tag{44}
\end{equation*}
$$

Then, we have

$$
\begin{equation*}
t^{*}-a=\frac{a-s}{((b-s) /(b-a))^{2}-1}>0 \tag{45}
\end{equation*}
$$

and

$$
\begin{aligned}
t^{*}-b & =\frac{(a-b)((b-s) /(b-a))^{2}-(s-b)}{((b-s) /(b-a))^{2}-1} \\
& <\frac{(s-b)((b-s) /(b-a))^{2}-(s-b)}{((b-s) /(b-a))^{2}-1}=s-b
\end{aligned}
$$

$$
<0
$$

It is clear that $t^{*} \in[a, b]$. Then,

$$
\begin{align*}
t^{*}-s & =\frac{a((b-s) /(b-a))^{2}-s}{((b-s) /(b-a))^{2}-1}-s \\
& =\frac{(a-s)((b-s) /(b-a))^{2}}{((b-s) /(b-a))^{2}-1}>0 . \tag{47}
\end{align*}
$$

Clearly, $t^{*}>s$. Hence, we get the maximum at $t^{*}$,

$$
\begin{aligned}
& \max _{t \in[a, b]} G(t, s)=G\left(t^{*}, s\right) \\
&= \frac{1}{2} \frac{(b-s)^{2}}{(b-a)^{2}}\left[\frac{a((b-s) /(b-a))^{2}-s}{((b-s) /(b-a))^{2}-1}-a\right]^{2} \\
&-\frac{1}{2}\left[\frac{a((b-s) /(b-a))^{2}-s}{((b-s) /(b-a))^{2}-1}-s\right]^{2} \\
&= \frac{1}{2} \frac{(b-s)^{2}}{(b-a)^{2}}\left[\frac{a-s}{((b-s) /(b-a))^{2}-1}\right]^{2} \\
&-\frac{1}{2}\left[\frac{(a-s)((b-s) /(b-a))^{2}}{((b-s) /(b-a))^{2}-1}\right]^{2} \\
&= \frac{1}{2} \frac{(b-s)^{2}}{(b-a)^{2}} \frac{(a-s)^{2}(b-a)^{4}}{\left[(b-s)^{2}-(b-a)^{2}\right]^{2}} \\
&-\frac{1}{2} \frac{(a-s)^{2}(b-s)^{4}}{\left[(b-s)^{2}-(b-a)^{2}\right]^{2}} \\
&= \frac{1}{2} \frac{(a-s)^{2}(b-s)^{2}}{\left[(b-s)^{2}-(b-a)^{2}\right]^{2}}\left[(b-a)^{2}-(b-s)^{2}\right] \\
&= \frac{(b-s)^{2}(s-a)}{2(2 b-s-a)}
\end{aligned}
$$

The proof is complete.

From (29), we have the following Corollary 9.
Corollary 9. For $y \in C[a, b], 3<\alpha<4$, and $\beta=\alpha-3$, for any $t \in[a, b]$, we have

$$
\begin{equation*}
|T(t, y(t))| \leq R(t)\|y\|, \tag{49}
\end{equation*}
$$

where

$$
\begin{equation*}
R(t)=\frac{4-\alpha}{B(\alpha-3)}|q(t)|+\frac{\alpha-3}{B(\alpha-3)} \int_{a}^{t}|q(s)| d s \tag{50}
\end{equation*}
$$

Proof. Form (29), we have

$$
\begin{align*}
\mid T & (s, y(s)) \mid \\
& =\left|\frac{1-\beta}{B(\beta)} q(s) y(s)+\frac{\beta}{B(\beta)}\left({ }_{a} I^{1} q(\cdot) y(\cdot)\right)(s)\right| \\
& \leq \frac{1-\beta}{B(\beta)}|q(s)||y(s)|+\frac{\beta}{B(\beta)} \int_{a}^{t}|q(s)| d s|y(s)|  \tag{51}\\
& \leq \frac{1-\beta}{B(\beta)}|q(s)|\|y(s)\|+\frac{\beta}{B(\beta)} \int_{a}^{t}|q(s)| d s\|y(s)\| \\
& =\left[\frac{1-\beta}{B(\beta)}|q(s)|+\frac{\beta}{B(\beta)} \int_{a}^{t}|q(s)| d s\right]\|y(s)\| .
\end{align*}
$$

By $\beta=\alpha-3$, we obtain (49). The proof is complete.
Theorem 10. If the boundary value problem (9) has a nontrivial continuous solution, then

$$
\begin{equation*}
\int_{a}^{b}(b-s)(s-a)|R(s)| d s \geq 4 \tag{52}
\end{equation*}
$$

Proof. Let $y \in Y=C[a, b]$ be a nontrivial solution of the boundary value problem (9) and

$$
\begin{equation*}
\|y\|=\sup _{t \in[a, b]}\{|y(t)|\} \tag{53}
\end{equation*}
$$

From Theorem 7, $y$ satisfies the integral equation

$$
\begin{equation*}
y(t)=\int_{a}^{b} G(t, s) T(s, y(s)) d s . \quad t \in[a, b] \tag{54}
\end{equation*}
$$

where $G(t, s)$ is defined in (28) and $T(s, y(s))$ is defined in (29).

By (48), we have

$$
\begin{align*}
|y(t)| & \leq \int_{a}^{b} \sup _{t \in[a, b]}|G(t, s)||T(s, y(s))| d s \\
& \leq \int_{a}^{b} \frac{(b-s)^{2}(s-a)}{2(2 b-s-a)}|T(s, y(s))| d s  \tag{55}\\
& \leq \int_{a}^{b} \frac{(b-s)^{2}(s-a)}{4(b-s)}|T(s, y(s))| d s \\
& =\frac{1}{4} \int_{a}^{b}(b-s)(s-a)|T(s, y(s))| d s .
\end{align*}
$$

Form Corollary 9, we obtain

$$
\begin{equation*}
\|y\| \leq \frac{1}{4}\left(\int_{a}^{b}(b-s)(s-a)|R(s)| d s\right)\|y\|, \tag{56}
\end{equation*}
$$

then we get the inequality in (52). This completes the proof.

Theorem 11. If the boundary value problem (9) has a nontrivial continuous solution, then

$$
\begin{equation*}
\int_{a}^{b}|R(s)| d s \geq \frac{16}{(a-b)^{2}}, \tag{57}
\end{equation*}
$$

where $R(t)$ is defined in (50).
Proof. Define the function

$$
\begin{equation*}
H(s)=(b-s)(s-a)=-s^{2}+(a+b) s-a b, \tag{58}
\end{equation*}
$$

$$
s \in[a, b] .
$$

Then, we have

$$
\begin{equation*}
H^{\prime}(s)=-2 s+(a+b) \tag{59}
\end{equation*}
$$

Observe that $H^{\prime}(s)=0$ if and only if

$$
\begin{equation*}
s=s^{*}=\frac{a+b}{2} \tag{60}
\end{equation*}
$$

It is easily to see that $s^{*} \in[a, b]$.
Hence, we get

$$
\begin{align*}
\max _{s \in[a, b]} H(s) & =H\left(s^{*}\right)=-\left(\frac{a+b}{2}\right)^{2}+\frac{(a+b)^{2}}{2}-a b  \tag{61}\\
& =\frac{(a-b)^{2}}{4}
\end{align*}
$$

Applying the result in (52), we have

$$
\begin{equation*}
\int_{a}^{b}|R(s)| d s \geq \frac{4}{H\left(s^{*}\right)}=\frac{16}{(a-b)^{2}} . \tag{62}
\end{equation*}
$$

The proof is complete.
Example 12. Consider the following fractional differential equation:

$$
\begin{gather*}
{ }_{a}^{C F R} D^{\alpha} y(t)+\lambda y(t)=0, \quad 0<t<1,3<\alpha<4, \\
y(0)=y^{\prime}(0)=y(1)=0 . \tag{63}
\end{gather*}
$$

If $\lambda$ is an eigenvalue to the boundary value problem (9), then

$$
\begin{equation*}
|\lambda| \geq 16\left[\frac{4-\alpha}{B(\alpha-3)}+\frac{\alpha-3}{2 B(\alpha-3)}\right]^{-1} . \tag{64}
\end{equation*}
$$

Proof. By using Corollary 9, we have

$$
\begin{align*}
R(t) & =\frac{4-\alpha}{B(\alpha-3)}|\lambda|+\frac{\alpha-3}{B(\alpha-3)}\left(\int_{0}^{1}|\lambda| d t\right)  \tag{65}\\
& =|\lambda|\left[\frac{4-\alpha}{B(\alpha-3)}+\frac{\alpha-3}{B(\alpha-3)} t\right] .
\end{align*}
$$

From Theorem 11, we have

$$
\begin{equation*}
\int_{0}^{1} R(s) d s=|\lambda|\left[\frac{4-\alpha}{B(\alpha-3)}+\frac{\alpha-3}{2 B(\alpha-3)}\right]>16 . \tag{66}
\end{equation*}
$$

Hence, we have

$$
\begin{equation*}
|\lambda| \geq 16\left[\frac{4-\alpha}{B(\alpha-3)}+\frac{\alpha-3}{2 B(\alpha-3)}\right]^{-1} . \tag{67}
\end{equation*}
$$

That concludes the proof.

## 4. Conclusions

In this paper, compared with existing results of fractional differential equations, we extend the order from $\alpha \in(2,3)$ to $\alpha \in(3,4)$. We prove existence and uniqueness theorems for initial value problem in the frame of CFR-type derivative of order $2<\alpha<3$ and $3<\alpha<4$ by using Banach Contraction Theorem. Then, we use our extension to obtain new Lyapunov-type inequality for CFR fractional boundary value problem with order $3<\alpha<4$ by Green's function and its corresponding maximum value. As for application, we give an example on the eigenvalue problem.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# Solutions for Integral Boundary Value Problems of Nonlinear Hadamard Fractional Differential Equations 

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In this paper using fixed point methods we establish some existence theorems of positive (nontrivial) solutions for integral boundary value problems of nonlinear Hadamard fractional differential equations.

## 1. Introduction

In this work we study the following integral boundary value problems of nonlinear Hadamard fractional differential equations

$$
\begin{align*}
D^{\beta}\left(\varphi_{p}\left(D^{\alpha} u(t)\right)\right) & =f(t, u(t)), \quad 1<t<e, \\
u(1) & =u^{\prime}(1)=u^{\prime}(e)=0, \\
D^{\alpha} u(1) & =0,  \tag{1}\\
\varphi_{p}\left(D^{\alpha} u(e)\right) & =\mu \int_{1}^{e} \varphi_{p}\left(D^{\alpha} u(t)\right) \frac{\mathrm{d} t}{t},
\end{align*}
$$

where $\alpha, \beta$, and $\mu$ are three positive real numbers with $\alpha \in$ $(2,3], \beta \in(1,2]$, and $\mu \in[0, \beta), \varphi_{p}(s)=|s|^{p-2} s$ is the $p$ Laplacian for $p>1, s \in \mathbb{R}$, and $f$ is a continuous function on $[1, e] \times \mathbb{R}$. Moreover, let $\varphi_{p}^{-1}=\varphi_{q}$ with $1 / p+1 / q=1$. In what follows, we offer some related definitions and lemmas for Hadamard fractional calculus.

Definition 1 (see [1, Page 111]). The $\alpha$ th Hadamard fractional order derivative of a function $y:[1,+\infty) \longrightarrow \mathbb{R}$ is defined by

$$
\begin{equation*}
D^{\alpha} y(t)=\frac{1}{\Gamma(n-\alpha)}\left(t \frac{\mathrm{~d}}{\mathrm{~d} t}\right)^{n} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{n-\alpha-1} y(s) \frac{\mathrm{d} s}{s}, \tag{2}
\end{equation*}
$$

where $\alpha>0, n=[\alpha]+1$, and $[\alpha]$ denotes the largest integer which is less than or equal to $\alpha$. Moreover, we here
also offer the $\alpha$ th Hadamard fractional order integral of $y$ : $[1,+\infty) \longrightarrow \mathbb{R}$ which is defined by

$$
\begin{equation*}
I^{\alpha} y(t)=\frac{1}{\Gamma(\alpha)} \int_{1}^{t}\left(\log \frac{t}{s}\right)^{\alpha-1} y(s) \frac{\mathrm{d} s}{s} \tag{3}
\end{equation*}
$$

where $\Gamma$ is the gamma function.
Lemma 2 (see [1, Theorem 2.3]). Let $\alpha>0, n=[\alpha]+1$. Then

$$
\begin{align*}
I^{\alpha} D^{\alpha} y(t)= & y(t)+c_{1}(\log t)^{\alpha-1}+c_{2}(\log t)^{\alpha-2}+\cdots \\
& +c_{n}(\log t)^{\alpha-n} \tag{4}
\end{align*}
$$

where $c_{i} \in \mathbb{R}, i=1,2, \ldots, n$.
In recent years, there have been some significant developments in the study of boundary value problems for nonlinear fractional differential equations; we refer to [2-11] and the references therein. For more related works, see also [12-49]. For example, by using monotone iterative methods, Wang et al. [3] investigated a class of boundary value problems of Hadamard fractional differential equations involving nonlocal multipoint discrete and Hadamard integral boundary conditions and established monotone iterative sequences, which can converge to the unique positive solution of their problems. Similar methods are also applied in [4, 5, 12-15].

For differential equations with the $p$-Laplacian, see, for example, $[6,7,15-20]$ and the references therein. In [6], Wang
considered the nonlinear Hadamard fractional differential equation with integral boundary condition and $p$-Laplacian operator

$$
\begin{align*}
D^{\beta} \varphi_{p}\left(D^{\alpha} u(t)\right) & =f(t, u(t)), \quad t \in(1, T), \\
u(T) & =\lambda I^{\sigma} u(\eta),  \tag{5}\\
D^{\alpha} u(1) & =0 \\
u(1) & =0
\end{align*}
$$

where $f$ grows ( $p-1$ )-sublinearly at $+\infty$, and by using the Schauder fixed point theorem, a solution existence result is obtained. In [7], Li and Lin used the Guo-Krasnosel'skii fixed point theorem to obtain the existence and uniqueness of positive solutions for (1) with $\mu=0$.

However, we note that these are seldom considered Hadamard fractional differential equations with the $p$ Laplacian in the literature; in this paper we are devoted to this direction. We first utilize the Guo-Krasnosel'skii fixed point theorem to obtain two positive solutions existence theorems when $f$ grows $(p-1)$-superlinearly and $(p-1)$-sublinearly with the $p$-Laplacian, and secondly by using the fixed point index, we obtain a nontrivial solution existence theorem without the $p$-Laplacian, but the nonlinearity can allow being
sign-changing and unbounded from below. This improves and generalizes some semipositone problems [21-31].

## 2. Preliminaries

In this section, we first calculate Green's functions associated with (1) and then transform the boundary value problem into its integral form. For this, we give the following lemma.

Lemma 3. Let $\alpha, \beta, \mu, \varphi_{p}$, and $D^{\alpha}, D^{\beta}$ be as in (1). Then (1) can take the integral form

$$
\begin{equation*}
u(t)=\int_{1}^{e} G(t, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}, \tag{6}
\end{equation*}
$$

for $t \in[1, e]$,
where

$$
\begin{align*}
& G(t, s)=\frac{1}{\Gamma(\alpha)} \\
& \quad \cdot \begin{cases}(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}-(\log t-\log s)^{\alpha-1}, & 1 \leq s \leq t \leq e, \\
(\log t)^{\alpha-1}(1-\log s)^{\alpha-2}, & 1 \leq t \leq s \leq e,\end{cases} \tag{7}
\end{align*}
$$

and

$$
\begin{align*}
& H(t, \tau)=H_{1}(t, \tau)+\frac{\mu}{(\beta-\mu) \Gamma(\beta)}(\log t)^{\beta-1} \log \tau(1-\log \tau)^{\beta-1}, \quad \text { for } t, \tau \in[1, e] \\
& H_{1}(t, \tau)=\frac{1}{\Gamma(\beta)} \begin{cases}(\log t)^{\beta-1}(1-\log \tau)^{\beta-1}-(\log t-\log \tau)^{\beta-1}, & 1 \leq \tau \leq t \leq e \\
(\log t)^{\beta-1}(1-\log \tau)^{\beta-1}, & 1 \leq t \leq \tau \leq e\end{cases} \tag{8}
\end{align*}
$$

Proof. Use $y(t)$ to replace $f(t, u)$ in (1). Let $D^{\beta}\left(\varphi_{p}\left(D^{\alpha} u(t)\right)\right)=$ $y(t)$. Then from Lemma 2 we have

$$
\begin{align*}
& \varphi_{p}\left(D^{\alpha} u(t)\right)=I^{\beta} y(t)+c_{1}(\log t)^{\beta-1}+c_{2}(\log t)^{\beta-2}  \tag{9}\\
& \text { for } c_{i}
\end{align*} \in \mathbb{R}, i=1,2 .
$$

Note that $D^{\alpha} u(1)=0$ implies $\varphi_{p}\left(D^{\alpha} u(1)\right)=0$, and then $c_{2}=$ 0 . Therefore, we obtain

$$
\begin{equation*}
\varphi_{p}\left(D^{\alpha} u(t)\right)=I^{\beta} y(t)+c_{1}(\log t)^{\beta-1} \tag{10}
\end{equation*}
$$

Next, we calculate $\varphi_{p}\left(D^{\alpha} u(e)\right)$ and $\mu \int_{1}^{e} \varphi_{p}\left(D^{\alpha} u(t)\right)(\mathrm{d} t / t)$ :

$$
\begin{align*}
\varphi_{p}\left(D^{\alpha} u(e)\right) & =I^{\beta} y(e)+c_{1} \\
& =c_{1}+\frac{1}{\Gamma(\beta)} \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}, \tag{11}
\end{align*}
$$

and

$$
\begin{aligned}
& \mu \int_{1}^{e} \varphi_{p}\left(D^{\alpha} u(t)\right) \frac{\mathrm{d} t}{t} \\
& \quad=\mu \int_{1}^{e} I^{\beta} y(t) \frac{\mathrm{d} t}{t}+\mu c_{1} \int_{1}^{e}(\log t)^{\beta-1} \frac{\mathrm{~d} t}{t}
\end{aligned}
$$

$$
\begin{equation*}
=\frac{\mu c_{1}}{\beta}+\frac{\mu}{\Gamma(\beta)} \int_{1}^{e} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} t}{t} . \tag{12}
\end{equation*}
$$

The condition $\varphi_{p}\left(D^{\alpha} u(e)\right)=\mu \int_{1}^{e} \varphi_{p}\left(D^{\alpha} u(t)\right)(\mathrm{d} t / t)$ enables us to obtain

$$
\begin{align*}
c_{1}= & \frac{\mu \beta}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} t}{t} \\
& -\frac{\beta}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} . \tag{13}
\end{align*}
$$

Substituting $c_{1}$ into (10) gives

$$
\begin{aligned}
& \varphi_{p}\left(D^{\alpha} u(t)\right)=\frac{1}{\Gamma(\beta)} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \\
& \quad-\frac{\beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \\
& \quad+\frac{\mu \beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} t}{t}
\end{aligned}
$$

$$
\begin{aligned}
& =\frac{1}{\Gamma(\beta)} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}-\frac{(\log t)^{\beta-1}}{\Gamma(\beta)} \\
& \cdot \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}+\frac{(\log t)^{\beta-1}}{\Gamma(\beta)} \\
& \cdot \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}-\frac{\beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}+\frac{\mu \beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \cdot \int_{1}^{e} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} t}{t}=-\frac{1}{\Gamma(\beta)} \\
& \cdot \int_{1}^{t}\left[(\log t)^{\beta-1}(1-\log \tau)^{\beta-1}-(\log t-\log \tau)^{\beta-1}\right] \\
& y(\tau) \frac{\mathrm{d} \tau}{\tau}-\frac{1}{\Gamma(\beta)} \int_{t}^{e}(\log t)^{\beta-1}(1-\log \tau)^{\beta-1} \\
& y(\tau) \frac{\mathrm{d} \tau}{\tau}+\frac{\mu \beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \cdot \int_{1}^{e} \int_{1}^{t}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} t}{t} \\
& -\frac{\mu(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \\
& =-\int_{1}^{e} H_{1}(t, \tau) y(\tau) \frac{\mathrm{d} \tau}{\tau}+\frac{\mu \beta(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \cdot \int_{1}^{e} \int_{\tau}^{e}(\log t-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} t}{t} \frac{\mathrm{~d} \tau}{\tau} \\
& -\frac{\mu(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau} \\
& =-\int_{1}^{e} H_{1}(t, \tau) y(\tau) \frac{\mathrm{d} \tau}{\tau}+\frac{\mu(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \cdot \int_{1}^{e}(1-\log \tau)^{\beta} y(\tau) \frac{\mathrm{d} \tau}{\tau}-\frac{\mu(\log t)^{\beta-1}}{(\beta-\mu) \Gamma(\beta)} \\
& \cdot \int_{1}^{e}(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}=-\int_{1}^{e} H_{1}(t, \tau) \\
& \cdot y(\tau) \frac{\mathrm{d} \tau}{\tau}-\int_{1}^{e} \frac{\mu}{(\beta-\mu) \Gamma(\beta)}(\log t)^{\beta-1} \\
& \cdot \log \tau(1-\log \tau)^{\beta-1} y(\tau) \frac{\mathrm{d} \tau}{\tau}=-\int_{1}^{e} H(t, \tau) \\
& y(\tau) \frac{\mathrm{d} \tau}{\tau} .
\end{aligned}
$$

Note that $-\varphi_{p}\left(D^{\alpha} u(t)\right)=\varphi_{p}\left(-D^{\alpha} u(t)\right)$, and hence we obtain

$$
\begin{equation*}
-D^{\alpha} u(t)=\varphi_{q}\left(\int_{1}^{e} H(t, \tau) y(\tau) \frac{\mathrm{d} \tau}{\tau}\right) \tag{15}
\end{equation*}
$$

for $\alpha \in(2,3], t \in[1, e]$.
Then, if we let $x(t)=\varphi_{q}\left(\int_{1}^{e} H(t, \tau) y(\tau)(\mathrm{d} \tau / \tau)\right), t \in[1, e]$, from Lemma 2 we obtain

$$
\begin{align*}
u(t)= & -I^{\alpha} x(t)+c_{1}(\log t)^{\alpha-1}+c_{2}(\log t)^{\alpha-2} \\
& +c_{3}(\log t)^{\alpha-3}, \quad \text { for } c_{i} \in \mathbb{R}, i=1,2,3 \tag{16}
\end{align*}
$$

The condition $u(1)=u^{\prime}(1)=0$ implies that $c_{2}=c_{3}=0$. Then we substitute $e$ into the first derivative of $u$, and we calculate $c_{1}$ as follows:

$$
\begin{equation*}
c_{1}=\frac{1}{\Gamma(\alpha)} \int_{1}^{e}(1-\log s)^{\alpha-2} x(s) \frac{\mathrm{d} s}{s} . \tag{17}
\end{equation*}
$$

As a result, from (16) we have

$$
\begin{align*}
u(t)= & -\frac{1}{\Gamma(\alpha)} \int_{1}^{t}(\log t-\log s)^{\alpha-1} x(s) \frac{\mathrm{d} s}{s} \\
& +\frac{1}{\Gamma(\alpha)} \int_{1}^{e}(\log t)^{\alpha-1}(1-\log s)^{\alpha-2} x(s) \frac{\mathrm{d} s}{s}  \tag{18}\\
= & \int_{1}^{e} G(t, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) y(\tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}
\end{align*}
$$

for $t \in[1, e]$.
This completes the proof.
Lemma 4. Green's functions G, H defined by (7) and (8) have the following properties:
(i) $G, H$ are continuous, nonnegative functions on $[1, e] \times$ [1,e],
(ii) $(\log t)^{\alpha-1}\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right] \leq \Gamma(\alpha) G(t, s) \leq$ $(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}$, for $t, s \in[1, e]$.

From [7, Lemma 7] and [8, Lemma 2.2] we easily obtain this lemma, so we omit its proof.

Let

$$
G_{1}(t, s)=\int_{1}^{e} G(t, \tau) H(\tau, s) \frac{\mathrm{d} \tau}{\tau}
$$

$$
\begin{equation*}
\phi(s)=\frac{1}{\Gamma(\alpha)} \tag{19}
\end{equation*}
$$

$$
\int_{1}^{e}\left[(1-\log t)^{\alpha-2}-(1-\log t)^{\alpha-1}\right] H(t, s) \frac{\mathrm{d} t}{t}
$$

for $t, s \in[1, e]$.
Then we obtain the following lemma.
Lemma 5. There exist $\kappa_{1}=\int_{1}^{e}(\log t)^{\alpha-1} \phi(t)(\mathrm{d} t / t), \kappa_{2}=$ $\int_{1}^{e} \phi(t)(\mathrm{d} t / t)$ such that

$$
\begin{equation*}
\kappa_{1} \phi(s) \leq \int_{1}^{e} G_{1}(t, s) \phi(t) \frac{\mathrm{d} t}{t} \leq \kappa_{2} \phi(s), \tag{20}
\end{equation*}
$$

Proof. We only prove the left inequality above. From Lemma 4(ii) we have

$$
\begin{align*}
& \int_{1}^{e} G_{1}(t, s) \phi(t) \frac{\mathrm{dt}}{t}=\int_{1}^{e} \int_{1}^{e} G(t, \tau) \\
& \quad \cdot H(\tau, s) \frac{\mathrm{d} \tau}{\tau} \phi(t) \frac{\mathrm{d} t}{t} \geq \frac{1}{\Gamma(\alpha)}  \tag{21}\\
& \cdot \int_{1}^{e} \int_{1}^{e}(\log t)^{\alpha-1}\left[(1-\log \tau)^{\alpha-2}-(1-\log \tau)^{\alpha-1}\right] \\
& \quad \cdot H(\tau, s) \frac{\mathrm{d} \tau}{\tau} \phi(t) \frac{\mathrm{d} t}{t}=\kappa_{1} \phi(s)
\end{align*}
$$

This completes the proof.
Let $\mathscr{E}=C[1, e]$ be the Banach space equipped with the norm $\|u\|=\max _{t \in[1, e]}|u(t)|$. Then we define two sets on $\mathscr{E}$ as follows:

$$
\begin{align*}
P & =\{u \in \mathscr{E}: u(t) \geq 0, \forall t \in[1, e]\}, \\
P_{0} & =\left\{u \in \mathscr{E}: u(t) \geq(\log t)^{\alpha-1}\|u\|, \forall t \in[1, e]\right\} \tag{22}
\end{align*}
$$

Consequently, $P, P_{0}$ are cones on $\mathscr{E}$. From Lemma 3 we can define an operator $A$ on $\mathscr{E}$ as follows:

$$
\begin{align*}
& (A u)(t) \\
& =\int_{1}^{e} G(t, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}  \tag{23}\\
& \\
& \quad \text { for } u \in \mathscr{E}, t \in[1, e] .
\end{align*}
$$

The continuity of $G, H, f$ implies that $A: \mathscr{E} \longrightarrow \mathscr{E}$ is a completely continuous operator and the existence of solutions for (1) if and only if the existence of fixed points for $A$.

Lemma 6 (see [50]). Let $\mathscr{E}$ be a Banach space and $\Omega$ a bounded open set in $\mathscr{E}$. Suppose that $A: \Omega \longrightarrow \mathscr{E}$ is a continuous compact operator. If there exists $u_{0} \in E \backslash\{0\}$ such that

$$
\begin{equation*}
u-A u \neq \mu u_{0}, \quad \forall u \in \partial \Omega, \mu \geq 0 \tag{24}
\end{equation*}
$$

then the topological degree $\operatorname{deg}(I-A, \Omega, 0)=0$.
Lemma 7 (see [50]). Let $\mathscr{E}$ be a Banach space and $\Omega$ a bounded open set in $\mathscr{E}$ with $0 \in \Omega$. Suppose that $A: \Omega \longrightarrow \mathscr{E}$ is a continuous compact operator. If

$$
\begin{equation*}
A u \neq \mu u, \quad \forall u \in \partial \Omega, \mu \geq 1 \tag{25}
\end{equation*}
$$

then the topological degree $\operatorname{deg}(I-A, \Omega, 0)=1$.
Lemma 8 (see [50]). Let $\mathscr{E}$ be a Banach space and $P \subset \mathscr{E}$ a cone in $\mathscr{E}$. Assume that $\Omega_{1}, \Omega_{2}$ are open subsets of $\mathscr{E}$ with $0 \in \Omega_{1} \subset \bar{\Omega}_{1} \subset \Omega_{2}$, and let $A: P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \longrightarrow P$ be $a$ completely continuous operator such that either
(G1) $\|A u\| \leq\|u\|, u \in \partial \Omega_{1} \cap P$, and $\|A u\| \geq\|u\|, u \in$ $\partial \Omega_{2} \cap P$,
or
(G2) $\|A u\| \geq\|u\|, u \in \partial \Omega_{1} \cap P$, and $\|A u\| \leq\|u\|, u \in$ $\partial \Omega_{2} \cap P$.

Then $A$ has a fixed point in $P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

## 3. Positive Solutions for (1)

Let $B_{\varrho}:=\{u \in \mathscr{E}:\|u\|<\varrho\}$ for $\varrho>0$. Now, we first list our assumptions on $f$ :
(H1) $f \in C\left([0,1] \times \mathbb{R}^{+}, \mathbb{R}^{+}\right)$,
(H2) there exist $\delta_{1} \in(1, e), t_{0} \in(1, e)$ such that $\liminf _{u \rightarrow+\infty}\left(f(t, u) / \varphi_{p}(u)\right) \geq \varphi_{p}\left(N_{1}\right), \liminf _{u \rightarrow 0^{+}}(f(t, u) /$ $\left.\varphi_{p}(u)\right) \geq \varphi_{p}\left(N_{2}\right)$, uniformly on $t \in\left[\delta_{1}, e\right]$, where $2 N_{1}^{-1}$, $N_{2}^{-1} \in\left(0,\left(\log \delta_{1}\right)^{\alpha-1} \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau)(\mathrm{d} \tau / \tau)\right)(\mathrm{d} s / s)\right)$,
(H3) there exists $\rho_{1}>0$ such that $f(t, u) \leq \varphi_{p}\left(N_{3} \rho_{1}\right)$, $\forall u \in\left[0, \rho_{1}\right], t \in[1, e]$, where $N_{3}^{-1}>\int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s\right.$, $\tau)(\mathrm{d} \tau / \tau))(\mathrm{d} s / s)$,
(H4) $\lim \sup _{u \rightarrow+\infty}\left(f(t, u) / \varphi_{p}(u)\right) \leq \varphi_{p}\left(M_{1}\right)$, $\lim \sup _{u \rightarrow 0^{+}}\left(f(t, u) / \varphi_{p}(u)\right) \leq \varphi_{p}\left(M_{2}\right)$, uniformly on $t \in[1, e]$, where $\left(2 M_{1}\right)^{-1}, M_{2}^{-1}>\int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau)(\mathrm{d} \tau /\right.$ $\tau)(\mathrm{d} s / s)$,
(H5) there exist $\rho_{2}>0, \delta_{1} \in(1, e), t_{0} \in(1, e)$ such that $f(t, u) \geq \varphi_{p}\left(M_{3} \rho_{2}\right), \forall u \in\left[\left(\log \delta_{1}\right)^{\alpha-1} \rho_{2}, \rho_{2}\right], t \in\left[\delta_{1}, e\right]$, where

$$
\begin{equation*}
M_{3}^{-1} \in\left(0, \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}\right) . \tag{26}
\end{equation*}
$$

Lemma 9. Suppose that (H1) holds. Then $A(P) \subset P_{0}$.
Proof. If $u \in P$, from Lemma 4 we have

$$
\begin{align*}
& (A u)(t) \leq \frac{1}{\Gamma(\alpha)} \int_{1}^{e}\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right] \\
& \cdot \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \tag{27}
\end{align*}
$$

$$
\forall t \in[1, e]
$$

On the other hand,

$$
\begin{align*}
& (A u)(t) \geq(\log t)^{\alpha-1} \cdot \frac{1}{\Gamma(\alpha)} \\
& \cdot \int_{1}^{e}\left[(1-\log s)^{\alpha-2}-(1-\log s)^{\alpha-1}\right]  \tag{28}\\
& \cdot \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \geq(\log t)^{\alpha-1}\|A u\|, \quad \forall t \in[1, e] .
\end{align*}
$$

This completes the proof.
Remark 10. Our aim is to find operator equation $u=A u$ has fixed points in $P$, and from Lemma 9, these fixed points must belong to the cone $P_{0}$. Therefore, our work space can be chosen $P_{0}$ rather than $P$.

In what follows, we discuss the existence of positive solutions for (1) in $P_{0}$.

Theorem 11. Suppose that (H1)-(H3) hold. Then (1) has at least two positive solutions.

Proof. From (H3), when $u \in \partial B_{\rho_{1}} \cap P_{0}$, we have
$(A u)(t)$

$$
\begin{aligned}
& \leq \max _{t \in[1, e]} \int_{1}^{e} G(t, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \leq \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \varphi_{p}\left(N_{3} \rho_{1}\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& =N_{3} \rho_{1} \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}<\rho_{1}
\end{aligned}
$$

$$
\forall t \in[1, e]
$$

Hence, we obtain

$$
\begin{equation*}
\|A u\|<\|u\|, \quad \text { for } u \in \partial B_{\rho_{1}} \cap P_{0} . \tag{30}
\end{equation*}
$$

On the other hand, by the second limit inequality in (H2), there exists $r_{1} \in\left(0, \rho_{1}\right)$ such that

$$
\begin{equation*}
f(t, u) \geq \varphi_{p}\left(N_{2} u\right), \quad \forall u \in\left[0, r_{1}\right], t \in\left[\delta_{1}, e\right] \tag{31}
\end{equation*}
$$

Note that if $u \in \partial B_{r_{1}} \cap P_{0}, t \in\left[\delta_{1}, e\right]$, from the definition of $P_{0}$ we have

$$
\begin{equation*}
u(t) \geq\left(\log \delta_{1}\right)^{\alpha-1}\|u\| \tag{32}
\end{equation*}
$$

This, together with (31), implies that

$$
\begin{aligned}
& \|A u\|=\max _{t \in[1, e]}(A u)(t) \geq(A u)\left(t_{0}\right)=\int_{1}^{e} G\left(t_{0}, s\right) \\
& \cdot \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \geq \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau)\right. \\
& \left.\cdot \varphi_{p}\left(N_{2}\left(\log \delta_{1}\right)^{\alpha-1}\|u\|\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& =N_{2}\left(\log \delta_{1}\right)^{\alpha-1}\|u\| \int_{1}^{e} G\left(t_{0}, s\right) \\
& \cdot \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}>\|u\|,
\end{aligned}
$$

for $u \in \partial B_{r_{1}} \cap P_{0}$.
By the first limit inequality in (H2), there exist $R_{1}>\rho_{1}$ and $C_{1}>0$ such that

$$
\begin{equation*}
f(t, u) \geq \varphi_{p}\left(N_{1} u\right)-C_{1}, \quad \forall u \in \mathbb{R}^{+}, t \in\left[\delta_{1}, e\right] \tag{34}
\end{equation*}
$$

Note that $R_{1}$ can be chosen large enough, and if $u \in \partial B_{R_{1}} \cap P_{0}$, together with (32), there exists $C_{2}>0$ such that

$$
\begin{equation*}
f(t, u) \geq \varphi_{p}\left(N_{1}\left(\log \delta_{1}\right)^{\alpha-1} R_{1}-C_{2}\right) \tag{35}
\end{equation*}
$$

Combining this and (33), we find

$$
\begin{align*}
& \|A u\| \geq \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau)\right. \\
& \left.\cdot \varphi_{p}\left(N_{1}\left(\log \delta_{1}\right)^{\alpha-1} R_{1}-C_{2}\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \quad=\left(N_{1}\left(\log \delta_{1}\right)^{\alpha-1} R_{1}-C_{2}\right) \int_{1}^{e} G\left(t_{0}, s\right)  \tag{36}\\
& \cdot \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \geq 2 R_{1}-C_{3}
\end{align*}
$$

where $C_{3}=C_{2} \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau)(\mathrm{d} \tau / \tau)\right)(\mathrm{d} s / s)$. Consequently, we have

$$
\begin{equation*}
\|A u\|>\|u\|, \quad \text { for } \partial B_{R_{1}} \cap P_{0}, \text { if }\|u\| \longrightarrow \infty \tag{37}
\end{equation*}
$$

In summary, from (30), (33), and (37) with $R_{1}>\rho_{1}>r_{1}$, Lemma 8 enables us to obtain that (1) has at least two positive solutions in $\left(\bar{B}_{R_{1}} \backslash B_{\rho_{1}}\right) \cap P_{0}$ and $\left(\bar{B}_{\rho_{1}} \backslash B_{r_{1}}\right) \cap P_{0}$. This completes the proof.

Theorem 12. Suppose that (H1), (H4)-(H5) hold. Then (1) has at least two positive solutions.

Proof. If $u \in \partial B_{\rho_{2}} \cap P_{0}$, we have $\|u\|=\rho_{2}$, and $u \in$ $\left[\left(\log \delta_{1}\right)^{\alpha-1} \rho_{2}, \rho_{2}\right]$, for $u \in P_{0}, t \in\left[\delta_{1}, e\right]$. Hence, from (H5) we obtain
$\|A u\|$

$$
\begin{align*}
& \geq \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \geq \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau) \varphi_{p}\left(M_{3} \rho_{2}\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}  \tag{38}\\
& \geq M_{3} \rho_{2} \int_{1}^{e} G\left(t_{0}, s\right) \varphi_{q}\left(\int_{\delta_{1}}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}>\rho_{2} .
\end{align*}
$$

This indicates that

$$
\begin{equation*}
\|A u\|>\|u\|, \quad \text { for } u \in \partial B_{\rho_{2}} \cap P_{0} \tag{39}
\end{equation*}
$$

On the other hand, by the second limit inequality in (H4), there exists $r_{2} \in\left(0, \rho_{2}\right)$ such that

$$
\begin{equation*}
f(t, u) \leq \varphi_{p}\left(M_{2} u\right), \quad \forall u \in\left[0, r_{2}\right], t \in[1, e] . \tag{40}
\end{equation*}
$$

This, if $u \in \partial B_{r_{2}} \cap P_{0}$, implies that
$\|A u\|$

$$
\begin{align*}
& \leq \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \varphi_{p}\left(M_{2} u(\tau)\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \leq \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \varphi_{p}\left(M_{2} r_{2}\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}  \tag{41}\\
& =M_{2} r_{2} \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}<r_{2} .
\end{align*}
$$

This gives

$$
\begin{equation*}
\|A u\|<\|u\|, \quad \text { for } u \in \partial B_{r_{2}} \cap P_{0} \tag{42}
\end{equation*}
$$

By the first limit inequality in (H4), there exist $R_{2}>\rho_{2}$ and $C_{4}>0$ such that

$$
\begin{equation*}
f(t, u) \leq \varphi_{p}\left(M_{1} u+C_{4}\right), \quad \forall u \in \mathbb{R}^{+}, t \in[1, e] . \tag{43}
\end{equation*}
$$

Consequently, if $u \in \partial B_{R_{2}} \cap P_{0}$ with $R_{2}$ large enough, we obtain

$$
\begin{align*}
& \|A u\| \leq \int_{1}^{e} G(e, s) \\
& \quad \cdot \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \varphi_{p}\left(M_{1} R_{2}+C_{4}\right) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s}  \tag{44}\\
& =\left(M_{1} R_{2}+C_{4}\right) \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau) \frac{\mathrm{d} \tau}{\tau}\right) \frac{\mathrm{d} s}{s} \\
& \quad \leq \frac{1}{2} R_{2}+C_{5}
\end{align*}
$$

where $C_{5}=C_{4} \int_{1}^{e} G(e, s) \varphi_{q}\left(\int_{1}^{e} H(s, \tau)(\mathrm{d} \tau / \tau)\right)(\mathrm{d} s / s)$. Hence, we have

$$
\begin{equation*}
\|A u\|<\|u\|, \quad \text { for } u \in \partial B_{R_{2}} \cap P_{0}, \text { if }\|u\| \longrightarrow \infty . \tag{45}
\end{equation*}
$$

In a word, from (39), (42), and (45) with $R_{2}>\rho_{2}>r_{2}$, Lemma 8 enables us to obtain that (1) has at least two positive
solutions in $\left(\bar{B}_{R_{2}} \backslash B_{\rho_{2}}\right) \cap P_{0}$ and $\left(\bar{B}_{\rho_{2}} \backslash B_{r_{2}}\right) \cap P_{0}$. This completes the proof.

Example 13. Let

$$
\begin{align*}
& f(t, u) \\
& \quad= \begin{cases}\rho_{1}^{p-1-\gamma_{1}} N_{3}^{p-1} u^{\gamma_{1}}, & u \in\left(\rho_{1},+\infty\right), t \in[1, e], \\
\rho_{1}^{p-1-\gamma_{2}} N_{3}^{p-1} u^{\gamma_{2}}, & u \in\left[0, \rho_{1}\right], t \in[1, e]\end{cases} \tag{46}
\end{align*}
$$

where $\gamma_{1} \in(p-1,+\infty), \gamma_{2} \in(0, p-1)$, and $N_{3}, \rho_{1}$ are defined by (H3). Then

$$
\begin{align*}
\liminf _{u \rightarrow+\infty} \frac{f(t, u)}{\varphi_{p}(u)} & =\liminf _{u \rightarrow+\infty} \frac{\rho_{1}^{p-1-\gamma_{1}} N_{3}^{p-1} u^{\gamma_{1}}}{u^{p-1}}=+\infty \\
& \geq \varphi_{p}\left(N_{1}\right),  \tag{47}\\
\liminf _{u \rightarrow 0^{+}} \frac{f(t, u)}{\varphi_{p}(u)} & =\liminf _{u \rightarrow 0^{+}} \frac{\rho_{1}^{p-1-\gamma_{2}} N_{3}^{p-1} u^{\gamma_{2}}}{u^{p-1}}=+\infty \\
& \geq \varphi_{p}\left(N_{2}\right) .
\end{align*}
$$

Moreover, for $u \in\left[0, \rho_{1}\right], t \in[1, e]$ we have

$$
\begin{equation*}
f(t, u) \leq \rho_{1}^{p-1-\gamma_{2}} N_{3}^{p-1} \rho_{1}^{\gamma_{2}}=\left(N_{3} \rho_{1}\right)^{p-1} . \tag{48}
\end{equation*}
$$

Therefore, (H1)-(H3) hold.
Example 14. Let

$$
f(t, u)= \begin{cases}\left(\log \delta_{1}\right)^{-(\alpha-1) \gamma_{3}} \rho_{2}^{p-1-\gamma_{3}} M_{3}^{p-1} u^{\gamma_{3}}, & u \in\left[\left(\log \delta_{1}\right)^{\alpha-1} \rho_{2},+\infty\right), t \in[1, e],  \tag{49}\\ \left(\log \delta_{1}\right)^{-(\alpha-1) \gamma_{4}} \rho_{2}^{p-1-\gamma_{4}} M_{3}^{p-1} u^{\gamma_{4}}, & u \in\left[0,\left(\log \delta_{1}\right)^{\alpha-1} \rho_{2}\right), t \in[1, e],\end{cases}
$$

where $\gamma_{3} \in(0, p-1), \gamma_{4} \in(p-1,+\infty)$, and $M_{3}, \rho_{2}$ are defined by (H5). Then

$$
\begin{align*}
& \limsup _{u \rightarrow+\infty} \frac{f(t, u)}{\varphi_{p}(u)} \\
& \quad=\limsup _{u \rightarrow+\infty} \frac{\left(\log \delta_{1}\right)^{-(\alpha-1) \gamma_{3}} \rho_{2}^{p-1-\gamma_{3}} M_{3}^{p-1} u^{\gamma_{3}}}{u^{p-1}}=0 \\
& \quad \leq \varphi_{p}\left(M_{1}\right) \\
& \limsup _{u \rightarrow 0^{+}} \frac{f(t, u)}{\varphi_{p}(u)}  \tag{50}\\
& \quad=\limsup _{u \rightarrow 0^{+}} \frac{\left(\log \delta_{1}\right)^{-(\alpha-1) \gamma_{4}} \rho_{2}^{p-1-\gamma_{4}} M_{3}^{p-1} u^{\gamma_{4}}}{u^{p-1}}=0 \\
& \quad \leq \varphi_{p}\left(M_{2}\right)
\end{align*}
$$

Moreover, for $u \in\left[\left(\log \delta_{1}\right)^{\alpha-1} \rho_{2}, \rho_{2}\right], t \in\left[\delta_{1}, e\right]$ we have

$$
f(t, u) \geq\left(\log \delta_{1}\right)^{-(\alpha-1) \gamma_{3}} \rho_{2}^{p-1-\gamma_{3}} M_{3}^{p-1} u^{\gamma_{3}}
$$

$$
\begin{equation*}
=\left(M_{3} \rho_{2}\right)^{p-1} \tag{51}
\end{equation*}
$$

Therefore, (H1), (H4)-(H5) hold.

## 4. Nontrivial Solutions for (1)

In this section we consider the boundary value problem (1) without the $p$-Laplacian, i.e., $p=2$. In this case, (1) can be transformed into its integral form as follows:

$$
\begin{align*}
u(t) & =\int_{1}^{e} G(t, s) \int_{1}^{e} H(s, \tau) f(\tau, u(\tau)) \frac{\mathrm{d} \tau}{\tau} \frac{\mathrm{~d} s}{s}  \tag{52}\\
& =\int_{1}^{e} G_{1}(t, s) f(s, u(s)) \frac{\mathrm{d} s}{s}, \quad \text { for } t \in[1, e] .
\end{align*}
$$

As said in Section 3, we define an operator, still denoted by $A$, as follows:

$$
\begin{equation*}
(A u)(t)=\int_{1}^{e} G_{1}(t, s) f(s, u(s)) \frac{\mathrm{d} s}{s} \tag{53}
\end{equation*}
$$

for $u \in \mathscr{E}, t \in[1, e]$.

In what follows, we aim to find the existence of fixed points of $A$. For this, we list our assumptions on $f$ :
(H6) $f \in C([1, e] \times \mathbb{R}, \mathbb{R})$,
(H7) There exist nonnegative functions $a(t), b(t) \in \mathscr{E}$ with $b \not \equiv 0$ and $K(u) \in C\left(\mathbb{R}, \mathbb{R}^{+}\right)$such that

$$
\begin{equation*}
f(t, u) \geq-a(t)-b(t) K(u), \quad \forall u \in \mathbb{R}, t \in[1, e] . \tag{54}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\lim _{|u| \rightarrow \infty} \frac{K(u)}{|u|}=0 \tag{55}
\end{equation*}
$$

(H8) $\liminf _{|u| \rightarrow \infty}(f(t, u) /|u|)>\kappa_{1}^{-1}$, uniformly in $t \in$ [1,e],
(H9) $\lim \inf _{|u| \rightarrow 0}(|f(t, u)| /|u|)<\kappa_{2}^{-1}$, uniformly in $t \in$ [1,e].

Theorem 15. Suppose that (H6)-(H9) hold. Then (1) has at least one nontrivial solution.

Proof. From (H9) there exist $\varepsilon_{3} \in\left(0, \kappa_{2}^{-1}\right)$ and $r_{3}>0$ such that

$$
\begin{equation*}
|f(t, u)| \leq\left(\kappa_{2}^{-1}-\varepsilon_{3}\right)|u|, \quad \forall t \in[1, e], \quad|u| \in\left[0, r_{3}\right) . \tag{56}
\end{equation*}
$$

For this $r_{3}$, we show that

$$
\begin{equation*}
A u \neq \mu u, \quad u \in \partial B_{r_{3}}, \mu \geq 1 \tag{57}
\end{equation*}
$$

If otherwise, there exist $u_{1} \in \partial B_{r_{3}}, \mu_{1} \geq 1$ such that

$$
\begin{equation*}
A u_{1}=\mu_{1} u_{1} \tag{58}
\end{equation*}
$$

and hence, we obtain

$$
\begin{align*}
\left|u_{1}(t)\right| & =\frac{1}{\mu_{1}}\left|\left(A u_{1}\right)(t)\right| \leq\left|\left(A u_{1}\right)(t)\right| \\
& \leq \int_{1}^{e} G_{1}(t, s)\left|f\left(s, u_{1}(s)\right)\right| \frac{\mathrm{d} s}{s}  \tag{59}\\
& \leq\left(\kappa_{2}^{-1}-\varepsilon_{3}\right) \int_{1}^{e} G_{1}(t, s)\left|u_{1}(s)\right| \frac{\mathrm{d} s}{s} .
\end{align*}
$$

$$
\begin{equation*}
R_{3} \geq \max \left\{\frac{\left(\kappa_{1}^{-1}+2\left(\varepsilon_{4}-\|b\| \epsilon\right)\right) \int_{1}^{e} W(s)\left(a(s)+\|b\| K^{*}+C_{6}\right)(\mathrm{d} s / s)}{\left(\varepsilon_{4}-\|b\| \epsilon\right) \Gamma(\alpha)-\|b\| \epsilon\left(\kappa_{1}^{-1}+2\left(\varepsilon_{4}-\|b\| \epsilon\right)\right) \int_{1}^{e} W(s)(\mathrm{d} s / s)}, \frac{\int_{1}^{e} W(s)\left(a(s)+\|b\| K^{*}+C_{6}\right)(\mathrm{d} s / s)}{\Gamma(\alpha)-\|b\| \epsilon \int_{1}^{e} W(s)(\mathrm{d} s / s)}\right\} \tag{66}
\end{equation*}
$$

where $W(s)=\int_{1}^{e}(1-\log \tau)^{\alpha-2} H(\tau, s)(\mathrm{d} \tau / \tau)$, for $s \in[1, e]$. Now we prove that

$$
\begin{equation*}
u-A u \neq \mu \phi, \quad \forall u \in \partial B_{R_{3}}, \mu \geq 0 \tag{67}
\end{equation*}
$$

where $\phi$ is defined by (19). Indeed, if (67) is not true, then there exists $u_{2} \in \partial B_{R_{3}}$ and $\mu_{0}>0$ such that

$$
\begin{equation*}
u_{2}-A u_{2}=\mu_{0} \phi \tag{68}
\end{equation*}
$$

Multiply both sides of the above inequality by $\phi(t)$ and integrate from 1 to $e$ and together with Lemma 5 we obtain

$$
\begin{align*}
& \int_{1}^{e}\left|u_{1}(t)\right| \phi(t) \frac{\mathrm{d} t}{t} \\
& \quad \leq\left(\kappa_{2}^{-1}-\varepsilon_{3}\right) \int_{1}^{e} \int_{1}^{e} G_{1}(t, s)\left|u_{1}(s)\right| \frac{\mathrm{d} s}{s} \phi(t) \frac{\mathrm{d} t}{t}  \tag{60}\\
& \quad \leq\left(\kappa_{2}^{-1}-\varepsilon_{3}\right) \kappa_{2} \int_{1}^{e}\left|u_{1}(t)\right| \phi(t) \frac{\mathrm{d} t}{t} .
\end{align*}
$$

This implies that $\int_{1}^{e}\left|u_{1}(t)\right| \phi(t)(\mathrm{d} t / t)=0$, and $u_{1} \equiv 0$ for the fact that $\phi(t) \not \equiv 0$, for $t \in[1, e]$, which contradicts $u_{1} \in \partial B_{r_{3}}$. Therefore, (57) is true, and from Lemma 7 we obtain

$$
\begin{equation*}
\operatorname{deg}\left(I-A, B_{r_{3}}, 0\right)=1 \tag{61}
\end{equation*}
$$

On the other hand, by (H8), there exist $\varepsilon_{4}>0$ and $X_{0}>0$ such that

$$
\begin{equation*}
f(t, u) \geq\left(\kappa_{1}^{-1}+\varepsilon_{4}\right)|u|, \quad \forall t \in[1, e], \quad|u|>X_{0} \tag{62}
\end{equation*}
$$

For every fixed $\epsilon$ with $\|b\| \epsilon \in\left(0, \varepsilon_{4}\right),\|b\|=\max _{t \in[1, e]}|b(t)|$, and from (H7), there exists $X_{1}>X_{0}$ such that

$$
\begin{equation*}
K(u) \leq \epsilon|u|, \quad \forall|u|>X_{1} . \tag{63}
\end{equation*}
$$

Combining the two inequalities above, (H7) enables us to find

$$
\begin{align*}
f(t, u) \geq & \geq\left(\kappa_{1}^{-1}+\varepsilon_{4}\right)|u|-a(t)-b(t) K(u) \\
\geq & \left(\kappa_{1}^{-1}+\varepsilon_{4}\right)|u|-a(t)-\epsilon b(t)|u|  \tag{64}\\
\geq & \left(\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon\right)|u|-a(t) \\
& \forall|u|>X_{1}, \quad t \in[1, e] .
\end{align*}
$$

If we take $C_{6}=\left(\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon\right) X_{1}+\max _{t \in[1, e],|u| \leq X_{1}}|f(t, u)|$, $K^{*}=\max _{|u| \leq X_{1}} K(u)$. Then we easily have

$$
\begin{align*}
& f(t, u) \geq\left(\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon\right)|u|-a( (t)-C_{6}  \tag{65}\\
& \forall u \\
& \forall \mathbb{R}, t \in[1, e] .
\end{align*}
$$

Note that $\epsilon$ can be chosen arbitrarily small, and we let

Let $\widetilde{u}(t)=\int_{1}^{e} G_{1}(t, s)\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right](\mathrm{d} s / s)$. Then
$\widetilde{u} \in P_{0}$ and $\tilde{u} \in P_{0}$ and

$$
\begin{aligned}
\tilde{u}(t) & =\int_{1}^{e} G_{1}(t, s)\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \\
\leq & \int_{1}^{e} \int_{1}^{e} \frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1}(1-\log \tau)^{\alpha-2} H(\tau, s) \frac{\mathrm{d} \tau}{\tau}
\end{aligned}
$$

$$
\begin{align*}
& \cdot\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \\
& \leq \frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1} \int_{1}^{e} \int_{1}^{e}(1-\log \tau)^{\alpha-2} H(\tau, s) \frac{\mathrm{d} \tau}{\tau} \\
& \cdot\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \\
& =\frac{1}{\Gamma(\alpha)}(\log t)^{\alpha-1} \int_{1}^{e} W(s) \\
& \cdot\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \tag{69}
\end{align*}
$$

Consequently, we have

$$
\begin{align*}
&\|\widetilde{u}\| \leq \frac{1}{\Gamma(\alpha)} \int_{1}^{e} W(s)\left[a(s)+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \\
& \leq \frac{1}{\Gamma(\alpha)} \int_{1}^{e} W(s)\left(a(s)+C_{6}\right) \frac{\mathrm{d} s}{s} \\
&+\frac{\|b\|}{\Gamma(\alpha)}\left(\int_{\left|u_{2}\right| \leq X_{1}} W(s) K\left(u_{2}(s)\right) \frac{\mathrm{d} s}{s}\right. \\
&\left.\quad+\int_{\left|u_{2}\right|>X_{1}} W(s) K\left(u_{2}(s)\right) \frac{\mathrm{d} s}{s}\right) \leq \frac{1}{\Gamma(\alpha)}  \tag{70}\\
& \quad \cdot \int_{1}^{e} W(s)\left(a(s)+\|b\| K^{*}+C_{6}\right) \frac{\mathrm{d} s}{s}+\frac{\|b\| \epsilon}{\Gamma(\alpha)} \\
& \quad \cdot \int_{\left|u_{2}\right|>X_{1}}^{e} W(s)\left|u_{2}(s)\right| \frac{\mathrm{d} s}{s} \leq \frac{1}{\Gamma(\alpha)} \\
& \quad \cdot \int_{1}^{e} W(s)\left(a(s)+\|b\| K^{*}+\|b\| \epsilon R_{3}+C_{6}\right) \frac{\mathrm{d} s}{s} .
\end{align*}
$$

Plus $\tilde{u}$ into (68) gives

$$
\begin{align*}
u_{2} & (t)+\widetilde{u}(t)=\left(A u_{2}\right)(t)+\tilde{u}(t)+\mu_{0} \phi(t) \\
& =\int_{1}^{e} G_{1}(t, s)\left[f\left(s, u_{2}(s)\right)+a(s)+b(s) K\left(u_{2}(s)\right)\right.  \tag{71}\\
& \left.+C_{6}\right] \frac{\mathrm{d} s}{s}+\mu_{0} \phi(t)
\end{align*}
$$

Note that $f\left(s, u_{2}(s)\right)+a(s)+b(s) K\left(u_{2}(s)\right)+C_{6} \in P, s \in[1, e]$ and $\phi \in P_{0}$. Lemma 9 enables us to know that $u_{2}+\tilde{u} \in P_{0}$. From (65) we have

$$
\begin{aligned}
& \left(A u_{2}\right)(t)+\widetilde{u}(t)=\int_{1}^{e} G_{1}(t, s)\left[f\left(s, u_{2}(s)\right)+a(s)\right. \\
& \left.\quad+b(s) K\left(u_{2}(s)\right)+C_{6}\right] \frac{\mathrm{d} s}{s} \geq \int_{1}^{e} G_{1}(t, s) \\
& \quad \cdot\left[f\left(s, u_{2}(s)\right)+a(s)+C_{6}\right] \frac{\mathrm{d} s}{s} \geq \int_{1}^{e} G_{1}(t, s) \\
& \cdot\left(\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon\right)\left|u_{2}(s)\right| \frac{\mathrm{d} s}{s} \geq \int_{1}^{e} G_{1}(t, s)\left(\kappa_{1}^{-1}\right. \\
& \left.\quad+\varepsilon_{4}-\|b\| \epsilon\right) u_{2}(s) \frac{\mathrm{d} s}{s} .
\end{aligned}
$$

On the other hand, we have

$$
\begin{align*}
\kappa_{1}^{-1} & \int_{1}^{e} G_{1}(t, s)\left[u_{2}(s)+\widetilde{u}(s)\right] \frac{\mathrm{d} s}{s} \\
& +\left(\varepsilon_{4}-\|b\| \epsilon\right) \int_{1}^{e} G_{1}(t, s) u_{2}(s) \frac{\mathrm{d} s}{s} \\
& -\kappa_{1}^{-1} \int_{1}^{e} G_{1}(t, s) \widetilde{u}(s) \frac{\mathrm{d} s}{s}  \tag{73}\\
\geq & \kappa_{1}^{-1} \int_{1}^{e} G_{1}(t, s)\left[u_{2}(s)+\widetilde{u}(s)\right] \frac{\mathrm{d} s}{s}
\end{align*}
$$

This inequality holds if

$$
\begin{align*}
&\left(\varepsilon_{4}-\|b\| \epsilon\right) \int_{1}^{e} G_{1}(t, s) u_{2}(s) \frac{\mathrm{d} s}{s} \\
&-\kappa_{1}^{-1} \int_{1}^{e} G_{1}(t, s) \widetilde{u}(s) \frac{\mathrm{d} s}{s} \geq 0 \tag{74}
\end{align*}
$$

Indeed, $u_{2}+\widetilde{u} \in P_{0}$ implies that $u_{2}(t)+\widetilde{u}(t) \geq(\log t)^{\alpha-1} \| u_{2}+$ $\widetilde{u} \| \geq(\log t)^{\alpha-1}\left(\left\|u_{2}\right\|-\|\tilde{u}\|\right)$, for $t \in[1, e]$. Consequently,

$$
\begin{align*}
& \left(\varepsilon_{4}-\|b\| \epsilon\right) \int_{1}^{e} G_{1}(t, s)\left[u_{2}(s)+\widetilde{u}(s)\right] \frac{\mathrm{d} s}{s} \\
& \quad-\left(\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon\right) \int_{1}^{e} G_{1}(t, s) \widetilde{u}(s) \frac{\mathrm{d} s}{s} \\
& \quad \geq\left(\varepsilon_{4}-\|b\| \epsilon\right)\left(R_{3}-\|\widetilde{u}\|\right) \int_{1}^{e} G_{1}(t, s)(\log s)^{\alpha-1} \frac{\mathrm{~d} s}{s} \\
& \quad-\frac{\kappa_{1}^{-1}+\varepsilon_{4}-\|b\| \epsilon}{\Gamma(\alpha)}  \tag{75}\\
& \quad \cdot \int_{1}^{e} W(s)\left(a(s)+\|b\| K^{*}+\|b\| \epsilon R_{3}+C_{6}\right) \frac{\mathrm{d} s}{s} \\
& \cdot \int_{1}^{e} G_{1}(t, s)(\log s)^{\alpha-1} \frac{\mathrm{~d} s}{s} \geq 0
\end{align*}
$$

As a result, we have

$$
\begin{align*}
\left(A u_{2}\right)(t)+\widetilde{u}(t) & \geq \kappa_{1}^{-1} \int_{1}^{e} G_{1}(t, s)\left[u_{2}(s)+\tilde{u}(s)\right] \frac{\mathrm{d} s}{s}  \tag{76}\\
& :=\kappa_{1}^{-1} T\left(u_{2}+\tilde{u}\right)(t), \quad \forall t \in[1, e]
\end{align*}
$$

where $(T u)(t)=\int_{1}^{e} G_{1}(t, s) u(s)(\mathrm{d} s / s)$, for $u \in \mathscr{E}, t \in[1, e]$. Using (68) we obtain

$$
\begin{align*}
u_{2}+\widetilde{u} & =A u_{2}+\tilde{u}+\mu_{0} \phi \geq \kappa_{1}^{-1} T\left(u_{2}+\tilde{u}\right)+\mu_{0} \phi  \tag{77}\\
& \geq \mu_{0} \phi .
\end{align*}
$$

Define

$$
\begin{equation*}
\mu^{*}=\sup \left\{\mu>0: u_{2}+\tilde{u} \geq \mu \phi\right\} \tag{78}
\end{equation*}
$$

Note that $\mu_{0} \in\left\{\mu>0: u_{2}+\widetilde{u} \geq \mu \phi\right\}$, and then $\mu^{*} \geq \mu_{0}$, $u_{2}+\tilde{u} \geq \mu^{*} \phi$. From Lemma 5 we have

$$
\begin{equation*}
\kappa_{1}^{-1} T\left(u_{2}+\widetilde{u}\right) \geq \mu^{*} \kappa_{1}^{-1} T \phi \geq \mu^{*} \phi \tag{79}
\end{equation*}
$$

and hence

$$
\begin{equation*}
u_{2}+\tilde{u} \geq \kappa_{1}^{-1} T\left(u_{2}+\tilde{u}\right)+\mu_{0} \phi \geq\left(\mu_{0}+\mu^{*}\right) \phi, \tag{80}
\end{equation*}
$$

which contradicts the definition of $\mu^{*}$. Therefore, (67) holds, and from Lemma 6 we obtain

$$
\begin{equation*}
\operatorname{deg}\left(I-A, B_{R_{3}}, 0\right)=0 . \tag{81}
\end{equation*}
$$

This, together with (61), implies that

$$
\begin{align*}
& \operatorname{deg}\left(I-A, B_{R_{3}} \backslash \bar{B}_{r_{3}}, 0\right)  \tag{82}\\
& \quad=\operatorname{deg}\left(I-A, B_{R_{3}}, 0\right)-\operatorname{deg}\left(I-A, B_{r_{3}}, 0\right)=-1 .
\end{align*}
$$

Therefore the operator $A$ has at least one fixed point in $B_{R_{3}} \backslash$ $\bar{B}_{r_{3}}$, and (1) has at least one nontrivial solution. This completes the proof.

Example 16. Let $f(t, u)=a|u|-b k(u), k(u)=\ln (|u|+1), u \in$ $\mathbb{R}, t \in[1, e]$, where $a \in\left(\kappa_{1}^{-1},+\infty\right)$ and $b \in\left(a, a+\kappa_{2}^{-1}\right)$. Then $\lim _{|u| \rightarrow+\infty}(k(u) /|u|)=0$, and $\lim _{|u| \rightarrow+\infty}((a|u|-b \ln (|u|+$ 1)) $/|u|)=a>\kappa_{1}^{-1}, \lim _{|u| \rightarrow 0}(|a| u|-b \ln (|u|+1)| /|u|)=\mid a-$ $b \mid<\kappa_{2}^{-1}$. Therefore, (H6)-(H9) hold.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no competing interests.

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# The Conjugate Gradient Viscosity Approximation 

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#### Abstract

In this paper, we study a kind of conjugate gradient viscosity approximation algorithm for finding a common solution of split generalized equilibrium problem and variational inequality problem. Under mild conditions, we prove that the sequence generated by the proposed iterative algorithm converges strongly to the common solution. The conclusion presented in this paper is the generalization, extension, and supplement of the previously known results in the corresponding references. Some numerical results are illustrated to show the feasibility and efficiency of the proposed algorithm.


## 1. Introduction

Let $H_{1}$ and $H_{2}$ be real Hilbert spaces with inner product $\langle\cdot, \cdot\rangle$ and norm $\|\cdot\|$. Let $C$ and $Q$ be nonempty closed convex subsets of $H_{1}$ and $H_{2}$, respectively. Let $\left\{x_{n}\right\}$ be a sequence in $H_{1}$; then $x_{n} \longrightarrow x$ (respectively, $x_{n} \rightharpoonup x$ ) will denote strong (respectively, weak) convergence of the sequence $\left\{x_{n}\right\}$. Assume $w_{\omega}\left(x_{k}\right)=\left\{x: \exists x_{k_{j}} \rightharpoonup x\right\}$ to stand for the weak $\omega$ limit set of $x_{k}$.

The split feasibility problem (SFP) originally introduced by Censor and Elfving [1] is to find

$$
\begin{equation*}
x \in C \text { such that } A x \in Q \text {, } \tag{1}
\end{equation*}
$$

where $A: H_{1} \longrightarrow H_{2}$ is a bounded linear operator. It serves as a model for many inverse problems where constraints are imposed on the solutions in the domain of a linear operator as well as in these operator's ranges. The applications of the split feasibility problem are very comprehensive such as CT in medicine, intelligence antennas, and the electronic warning systems in military, the development of fast image processing technology and HDTV, etc. Many authors generalize SFP to a lot of important problems, such as multiple-sets split feasibility problem, split equality fixed point problem, split variational inequality problem, split variational inclusion
problem, and split equilibrium problem, and the theories and algorithms are studied and details can be seen in [2-15] and references therein.

The fixed point problem (FPP) for the mapping $T$ is to find $x \in C$ such that

$$
\begin{equation*}
T x=x . \tag{2}
\end{equation*}
$$

We denote $\operatorname{Fix}(T):=\{x \in C: T x=x\}$ the set of solution of FPP.

Let $B: C \longrightarrow H_{1}$ be a nonlinear mapping. The variational inequality problem (VIP) is to find $x \in C$ such that

$$
\begin{equation*}
\langle B x, y-x\rangle \geq 0, \quad \forall y \in C \tag{3}
\end{equation*}
$$

The solution set of VIP is denoted by $\mathrm{VI}(C, B)$. It is well known that if $B$ is strongly monotone and Lipschitz continuous mapping on $C$, then VIP has a unique solution.

For finding a common problem of $\operatorname{Fix}(T) \cap \mathrm{VI}(C, B)$, Takahashi and Toyoda [16] introduced the following iterative scheme:

$$
\begin{gather*}
x_{0} \quad \text { chosen arbitrary } \\
x_{n+1}=\alpha_{n} x_{n}+\left(1-\alpha_{n}\right) T P_{C}\left(x_{n}-\lambda_{n} B x_{n}\right), \quad \forall n \geq 0 \tag{4}
\end{gather*}
$$

where $B$ is $\rho$-inverse-strongly monotone, $\left\{\alpha_{n}\right\}$ is a sequence in $(0,1)$, and $\left\{\lambda_{n}\right\}$ is a sequence in $(0,2 \rho)$. They showed that if $\operatorname{Fix}(T) \cap \operatorname{VI}(C, B) \neq \emptyset$, then the sequence $\left\{x_{n}\right\}$ generated by (4) converges weakly to $z_{0} \in \operatorname{Fix}(T) \cap \operatorname{VI}(C, B)$.

On the other hand, there are several numerical methods for solving variational inequalities and related optimization problems; see [5, 17-24] and the references therein.

In 2013, Kazmi and Rizvi [25] introduced the split generalized equilibrium problem (SGEP). Let $F_{1}, h_{1}: C \times$ $C \longrightarrow R$ and $F_{2}, h_{2}: Q \times Q \longrightarrow R$ be nonlinear bifunctions and $A: H_{1} \longrightarrow H_{2}$ be a bounded linear operator; then the split generalized equilibrium problem (SGEP) is to find $x^{*} \in C$ such that

$$
\begin{equation*}
F_{1}\left(x^{*}, x\right)+h_{1}\left(x^{*}, x\right) \geq 0, \quad \forall x \in C \tag{5}
\end{equation*}
$$

and such that

$$
\begin{align*}
y^{*}=A x^{*} \in Q \text { solves } F_{2}\left(y^{*}, y\right)+h_{2}\left(y^{*}, y\right) \geq 0 & , \\
& \forall y \in Q . \tag{6}
\end{align*}
$$

Denote the solution sets of generalized equilibrium problem (GEP) (5) and GEP (6) by $\operatorname{GEP}\left(F_{1}, h_{1}\right)$ and $\operatorname{GEP}\left(F_{2}, h_{2}\right)$, respectively. The solution set of SGEP is denoted by $\Gamma=$ $\left\{x^{*} \in \operatorname{GEP}\left(F_{1}, h_{1}\right): A x^{*} \in \operatorname{GEP}\left(F_{2}, h_{2}\right)\right\}$. They proposed the following iterative method for finding a common solution of split generalized equilibrium and fixed point problem.

Let $\left\{x_{n}\right\}$ and $\left\{u_{n}\right\}$ be the sequences generated by $x_{0} \in C$ and

$$
\begin{align*}
u_{n}= & T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\delta A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right), \\
x_{n+1}= & \alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n}  \tag{7}\\
& +\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) \frac{1}{s_{n}} \int_{0}^{s_{n}} T(s) u_{n} d s,
\end{align*}
$$

where $S=\{T(s): 0 \leq s<\infty\}$ is a nonexpansive semigroup on $C$ and $\operatorname{Fix}(S) \cap \Gamma \neq \emptyset, s_{n}$ is a positive real sequence which diverges to $+\infty,\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\} \subset(0,1),\left\{r_{n}\right\} \subset(0, \infty)$, and $\delta \in$ $(0,1 / L), L$ is the spectral radius of the operator $A^{*} A, A^{*}$ is the adjoint of $A$, and

$$
\begin{align*}
& T_{r}^{\left(F_{1}, h_{1}\right)}(x)=\left\{z \in C: F_{1}(z, y)+h_{1}(z, y)\right.  \tag{8}\\
& \left.\quad+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, \forall y \in C\right\} \\
& T_{s}^{\left(F_{2}, h_{2}\right)}(w)=\left\{d \in Q: F_{2}(d, e)+h_{2}(d, e)\right. \\
& \left.\quad+\frac{1}{s}\langle e-d, d-w\rangle \geq 0, \forall e \in Q\right\} . \tag{9}
\end{align*}
$$

Under suitable conditions, they proved a strong convergence theorem for the sequence generated by the proposed iterative scheme. But the calculation of integral is generally not easy. Therefore, it is necessary to reconsider the algorithm for solving this kind of problem.

Motivated by Kazmi and Rivi [25] as well as Che and Li [2], we introduce and study a kind of conjugate gradient viscosity approximation algorithm for finding a common solution of split generalized equilibrium problem and
variational inequality problem. Under mild conditions, we prove that the sequence generated by the proposed iterative algorithm converges strongly to the common solution of $\mathrm{VI}(C, B)$ and SGEP. The results presented in this paper are the generalization, extension, and supplement of the previously known results in the corresponding references. Numerical results show the feasibility and efficiency of the proposed algorithm.

## 2. Preliminaries

In this section, we introduce some concepts and results which are needed in sequel.

A mapping $T: H_{1} \longrightarrow H_{1}$ is called
(1) contraction, if there exists a constant $\alpha \in(0,1)$ such that

$$
\begin{equation*}
\|T x-T y\| \leq \alpha\|x-y\|, \quad \forall x, y \in H_{1} . \tag{10}
\end{equation*}
$$

If $\alpha=1$, then $T$ is called nonexpansive.
(2) monotone, if

$$
\begin{equation*}
\langle T x-T y, x-y\rangle \geq 0, \quad \forall x, y \in H_{1} \tag{11}
\end{equation*}
$$

(3) $\eta$-strongly monotone, if there exists a positive constant $\eta$ such that

$$
\begin{equation*}
\langle T x-T y, x-y\rangle \geq \eta\|x-y\|^{2}, \quad \forall x, y \in H_{1} . \tag{12}
\end{equation*}
$$

(4) $\alpha$-inverse strongly monotone ( $\alpha$-ism), if there exists a constant $\alpha>0$ such that

$$
\begin{equation*}
\langle T x-T y, x-y\rangle \geq \alpha\|T x-T y\|^{2}, \quad \forall x, y \in H_{1} . \tag{13}
\end{equation*}
$$

(5) firmly nonexpansive, if

$$
\begin{equation*}
\langle T x-T y, x-y\rangle \geq\|T x-T y\|^{2}, \quad \forall x, y \in H_{1} . \tag{14}
\end{equation*}
$$

A mapping $P_{C}$ is said to be metric projection of $H_{1}$ onto $C$ if, for every point $x \in H_{1}$, there exists a unique nearest point in $C$ denoted by $P_{C} x$ such that

$$
\begin{equation*}
\left\|x-P_{C} x\right\| \leq\|x-y\|, \quad \forall y \in C . \tag{15}
\end{equation*}
$$

It is well known that $P_{C}$ is a nonexpansive mapping and is characterized by the following properties:

$$
\begin{align*}
&\left\|P_{C} x-P_{C} y\right\|^{2} \leq\left\langle x-y, P_{C} x-P_{C} y\right\rangle  \tag{16}\\
& \forall x, y \in H_{1}, \\
&\left\langle x-P_{C} x, y-P_{C} x\right\rangle \leq 0, \quad \forall x \in H_{1}, y \in C,  \tag{17}\\
&\|x-y\|^{2} \geq\left\|x-P_{C} x\right\|^{2}+\left\|y-P_{C} x\right\|^{2},  \tag{18}\\
& \forall x \in H_{1}, y \in C
\end{align*}
$$

and

$$
\begin{align*}
& \left\|(x-y)-\left(P_{C} x-P_{C} y\right)\right\|^{2}  \tag{19}\\
& \quad \geq\|x-y\|^{2}-\left\|P_{C} x-P_{C} y\right\|^{2}, \quad \forall x, y \in H_{1} .
\end{align*}
$$

A linear bounded operator $B$ is strongly positive if there exists a constant $\bar{\gamma}>0$ with the property

$$
\begin{equation*}
\langle B x, x\rangle \geq \bar{\gamma}\|x\|^{2}, \quad \forall x \in H_{1} . \tag{20}
\end{equation*}
$$

A mapping $T: H_{1} \longrightarrow H_{1}$ is said to be averaged if and only if it can be written as the average of the identity mapping and a nonexpansive mapping; i.e.,

$$
\begin{equation*}
T=(1-\alpha) I+\alpha S \tag{21}
\end{equation*}
$$

where $\alpha \in(0,1), S: H_{1} \longrightarrow H_{1}$ is nonexpansive, and $I$ is the identity operator on $H_{1}$. More precisely, we say that $T$ is $\alpha$-averaged. We note that averaged mapping is nonexpansive. Furthermore, firmly nonexpansive mapping (in particular, projection on nonempty closed and convex subset) is averaged.

Let $B$ be a monotone mapping of $C$ into $H_{1}$. In the context of the variational inequality problem, the characterization of projection (17) implies the following relation:

$$
\begin{equation*}
u \in \mathrm{VI}(C, B) \Longleftrightarrow u=P_{C}(u-\lambda B u), \quad \lambda>0 . \tag{22}
\end{equation*}
$$

In the proof of our results, we need the following assumptions and lemmas.

Lemma 1 (see [26]). If $x, y, z \in H_{1}$, then
(i) $\|x+y\|^{2} \leq\|x\|^{2}+2\langle y, x+y\rangle$.
(ii) For any $\lambda \in[0,1]$,

$$
\begin{align*}
\|\lambda x+(1-\lambda) y\|^{2}= & \lambda\|x\|^{2}+(1-\lambda)\|y\|^{2}  \tag{23}\\
& -\lambda(1-\lambda)\|x-y\|^{2}
\end{align*}
$$

(iii) For $a, b, c \in[0,1]$ with $a+b+c=1$,

$$
\begin{align*}
\|a x+b y+c z\|^{2}= & a\|x\|^{2}+b\|y\|^{2}+c\|z\|^{2} \\
& -a b\|x-y\|^{2}-a c\|x-z\|^{2}  \tag{24}\\
& -b c\|y-z\|^{2} .
\end{align*}
$$

Assumption 2. Let $F: C \times C \longrightarrow R$ be a bifunction satisfying the following assumption:
(i) $F(x, x) \geq 0, \forall x \in C$.
(ii) $F$ is monotone; i.e., $F(x, y)+F(y, x) \leq 0, \forall x \in C$.
(iii) $F$ is upper hemicontinuous; i.e., for each $x, y, z \in C$,

$$
\begin{equation*}
\limsup _{t \rightarrow 0} F(t z+(1-t) x, y) \leq F(x, y) \tag{25}
\end{equation*}
$$

(iv) For each $x \in C$ fixed, the function $y \longmapsto F(x, y)$ is convex and lower semicontinuous.

Let $h: C \times C \longrightarrow R$ such that
(i) $h(x, x) \geq 0, \forall x \in C$
(ii) for each $y \in C$ fixed, the function $x \longrightarrow h(x, y)$ is upper semicontinuous
(iii) for each $x \in C$ fixed, the function $y \longrightarrow h(x, y)$ is convex and lower semicontinuous

And assume that, for fixed $r>0$ and $z \in C$, there exists a nonempty compact convex subset $K$ of $H_{1}$ and $x \in C \cap K$ such that

$$
\begin{align*}
F(y, x)+h(y, x)+\frac{1}{r}\langle y-x, x-z\rangle<0, &  \tag{26}\\
& \forall y \in C \backslash K .
\end{align*}
$$

Lemma 3 (see [25]). Assume that $F_{1}, h_{1}: C \times C \longrightarrow R$ satisfy Assumption 2. Let $r>0$ and $x \in H_{1}$. Then, there exists $z \in C$ such that

$$
\begin{align*}
F_{1}(z, y)+h_{1}(z, y)+\frac{1}{r}\langle y-z, z-x\rangle \geq 0, &  \tag{27}\\
& \forall y \in C .
\end{align*}
$$

Lemma 4 (see [1]). Assume that the bifunctions $F_{1}, h_{1}: C \times$ $C \longrightarrow R$ satisfy Assumption 2 and $h_{1}$ is monotone. For $r>0$ and for all $x \in H_{1}$, define a mapping $T_{r}^{\left(F_{1}, h_{1}\right)}: H_{1} \longrightarrow C$ as (8). Then, the following conclusions hold.
(1) $T_{r}^{\left(F_{1}, h_{1}\right)}$ is single-valued.
(2) $T_{r}^{\left(F_{1}, h_{1}\right)}$ is firmly nonexpansive.
(3) $\operatorname{Fix}\left(T_{r}^{\left(F_{1}, h_{1}\right)}\right)=\operatorname{GEP}\left(F_{1}, h_{1}\right)$.
(4) $\operatorname{GEP}\left(F_{1}, h_{1}\right)$ is compact and convex.

Furthermore, assume that $F_{2}, h_{2}: Q \times Q \longrightarrow R$ satisfy Assumption 2. For $s>0$ and, for all $w \in H_{2}, T_{s}^{\left(F_{2}, h_{2}\right)}: H_{2} \longrightarrow$ $Q$ is defined as (9). By Lemma 4, we easily observe that $T_{s}^{\left(F_{2}, h_{2}\right)}$ is single-valued and firmly nonexpansive, $\operatorname{GEP}\left(F_{2}, h_{2}, Q\right)$ is compact and convex, and $\operatorname{Fix}\left(T_{s}^{\left(F_{2}, h_{2}\right)}\right)=\operatorname{GEP}\left(F_{2}, h_{2}, Q\right)$, where $\operatorname{GEP}\left(F_{2}, h_{2}, Q\right)$ is the solution set of the following generalized equilibrium problem, which is to find $y^{*} \in Q$ such that $F_{2}\left(y^{*}, y\right)+h_{2}\left(y^{*}, y\right) \geq 0, \forall y \in Q$.

We observe that $\operatorname{GEP}\left(F_{2}, h_{2}\right) \subset \operatorname{GEP}\left(F_{2}, h_{2}, Q\right)$. Further, it is easy to prove that $\Gamma$ is a closed and convex set.

Lemma 5 (see [27, 28]). Assume that $T: H_{1} \longrightarrow H_{1}$ is nonexpansive operator. For all $(x, y) \in H_{1} \times H_{1}$, the following inequality is true

$$
\begin{align*}
& \langle(x-T(x))-(y-T(y)), T(y)-T(x)\rangle \\
& \quad \leq \frac{1}{2}\|(T(x)-x)-(T(y)-y)\|^{2} . \tag{28}
\end{align*}
$$

And for all $(x, y) \in H_{1} \times \operatorname{Fix}(T)$, one has

$$
\begin{equation*}
\langle x-T(x), y-T(x)\rangle \leq \frac{1}{2}\|T(x)-x\|^{2} . \tag{29}
\end{equation*}
$$

Lemma 6 (see [29]). Assume A is a strongly positive linear bounded operator on Hilbert space $H_{1}$ with coefficient $\bar{\gamma}>0$ and $0<\rho \leq\|A\|^{-1}$. Then $\|I-\rho A\| \leq 1-\rho \bar{\gamma}$.

Lemma 7 (see $[30,31]$ ). Assume that $\left\{a_{n}\right\}$ is a sequence of nonnegative real numbers such that

$$
\begin{equation*}
a_{n+1} \leq\left(1-\gamma_{n}\right) a_{n}+\gamma_{n} \delta_{n}+\beta_{n}, \quad n \geq 0, \tag{30}
\end{equation*}
$$

where $\left\{\gamma_{n}\right\}$ and $\left\{\beta_{n}\right\}$ are sequences in $(0,1)$ and $\left\{\delta_{n}\right\}$ is a sequence in $R$, such that
(i) $\sum_{n=0}^{\infty} \gamma_{n}=\infty$.
(ii) $\lim _{n \rightarrow \infty} \delta_{n} \leq 0$ or $\sum_{n=0}^{\infty}\left|\gamma_{n} \delta_{n}\right|<\infty$.
(iii) $\sum_{n=0}^{\infty} \beta_{n}<\infty$.

Then $\lim _{n \rightarrow \infty} a_{n}=0$.
According to [28], it is easy to prove the following Lemma.
Lemma 8. Let $F_{1}, h_{1}: C \times C \longrightarrow R$ be bifunctions satisfying Assumption 2 and, for $r>0$, the mapping $T_{r}^{\left(F_{1}, h_{1}\right)}$ is defined as (8). Let $x, y \in H_{1}$, and $r_{1}, r_{2}>0$. Then

$$
\begin{align*}
\left\|T_{r_{2}}^{\left(F_{1}, h_{1}\right)} y-T_{r_{1}}^{\left(F_{1}, h_{1}\right)} x\right\| \leq & \|y-x\| \\
& +\left|\frac{r_{2}-r_{1}}{r_{2}}\right|\left\|T_{r_{2}}^{\left(F_{1}, h_{1}\right)} y-y\right\| . \tag{31}
\end{align*}
$$

Lemma 9 (see [32,33]). Let $T: H \longrightarrow H$ be given. We have the following:
(i) $T$ is nonexpansive, iff the complement $I-T$ is $1 / 2$-ism
(ii) if $T$ is $v$-ism, then, for $\gamma>0, \gamma T$ is $v / \gamma$-ism
(iii) $T$ is averaged, iff the complement $I-T$ is $v$-ism for some $v>1 / 2$; indeed, for $\alpha \in(0,1), T$ is $\alpha$-averaged, iff $I-T$ is $1 / 2 \alpha$-ism

Lemma 10 (see [4, 32]). Let the operators S, $T, V: H \longrightarrow H$ be given.
(i) If $T=(1-\alpha) S+\alpha V$, where $S$ is averaged, $V$ is nonexpansive, and $\alpha \in(0,1)$, then $T$ is averaged.
(ii) $T$ is firmly nonexpansive, iff the complement $I-T$ is firmly nonexpansive.
(iii) If $T=(1-\alpha) S+\alpha V$, where $S$ is firmly nonexpansive, $V$ is nonexpansive, and $\alpha \in(0,1)$, then $T$ is averaged.
(iv) The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\left\{T_{i}\right\}_{i=1}^{N}$ is averaged, then so is the composite $T_{1}, \ldots, T_{N}$. In particular, if $T_{1}$ is $\alpha_{1}$ averaged and $T_{2}$ is $\alpha_{2}$-averaged, where $\alpha_{1}, \alpha_{2} \in(0,1)$, then the composite $T_{1} T_{2}$ is $\alpha$-averaged, where $\alpha=\alpha_{1}+\alpha_{2}-\alpha_{1} \alpha_{2}$.
(v) If the mappings $\left\{T_{i}\right\}_{i=1}^{N}$ are averaged and have a common fixed point, then

$$
\begin{equation*}
\bigcap_{i=1}^{N} \operatorname{Fix}\left(T_{i}\right)=\operatorname{Fix}\left(T_{1}, \ldots, T_{N}\right) . \tag{32}
\end{equation*}
$$

In the following, we give the relation between the projection operator and average mapping.

Lemma 11. Assume that the variational inequality problem (3) is solvable. If $B$ is $\beta$-ism from $C$ into $H_{1}$, then $P_{C}(I-\lambda B)$ is $(2 \beta+\lambda) / 4 \beta$-averaged.

Proof. Note that $B$ is $\beta$-ism, which implies that $\lambda B$ is $\beta / \lambda$-ism; i.e., $I-(I-\lambda B)$ is $\beta / \lambda$-ism. By Lemma 9(iii), we can see that $I-$ $\lambda B$ is $\lambda / 2 \beta$-averaged. Since the projection $P_{C}$ is $1 / 2$-averaged, it is easy to see from Lemma 10 that the composite $P_{C}(I-\lambda B)$ is $(2 \beta+\lambda) / 4 \beta$-averaged for $0<\lambda<2 \beta$ according to

$$
\begin{equation*}
\frac{1}{2}+\frac{\lambda}{2 \beta}-\frac{1}{2} \cdot \frac{\lambda}{2 \beta}=\frac{2 \beta+\lambda}{4 \beta} \tag{33}
\end{equation*}
$$

which completes the proof.

As a result we have that, for each $n, P_{C}\left(I-\lambda_{n} B\right)$ is $(2 \beta+$ $\left.\lambda_{n}\right) / 4 \beta$-averaged. Therefore, we can write

$$
\begin{align*}
P_{C}\left(I-\lambda_{n} B\right) & =\frac{2 \beta-\lambda_{n}}{4 \beta} I+\frac{2 \beta+\lambda_{n}}{4 \beta} T_{n}  \tag{34}\\
& =\left(1-b_{n}\right) I+b_{n} T_{n},
\end{align*}
$$

where $T_{n}$ is nonexpansive and $b_{n}=\left(2 \beta+\lambda_{n}\right) / 4 \beta \in[1 / 2,1]$.
Lemma 12 (see [28]). Let $\left\{x_{n}\right\}$ and $\left\{z_{n}\right\}$ be bounded sequence in a Banach space $X$ and let $\left\{\beta_{n}\right\}$ be a sequence in $[0,1]$ with $0<\liminf _{n \rightarrow \infty} \beta_{n} \leq \lim \sup _{n \rightarrow \infty} \beta_{n}<1$. Suppose $x_{n+1}=$ $\left(1-\beta_{n}\right) z_{n}+\beta_{n} x_{n}$ for all integers $n \geq 0$ and $\lim \sup _{n \rightarrow \infty}\left(\| z_{n+1}-\right.$ $\left.z_{n}\|-\| x_{n+1}-x_{n} \|\right) \leq 0$. Then $\lim _{n \rightarrow \infty}\left\|z_{n}-x_{n}\right\|=0$.

## 3. Main Results

In this section, we give the main results of this paper. First, we describe the algorithm for finding a common solution of split generalized equilibrium and variational inequality problems.

Throughout the rest of this paper, let $f$ be a contraction of $H_{1}$ into itself with coefficient $\eta \in(0,1), A$ be a bounded linear operator, $B$ be a $\beta$-inverse strongly monotone mapping from $C$ into $H_{1}$, and $D$ be a strongly positive linear bounded selfadjoint operator on $H_{1}$ with coefficient $\bar{\gamma}>0$ and $0<\gamma<\bar{\gamma} / \eta$.

Now, we give the description of the algorithm.
Algorithm 13. Let $x_{0} \in H_{1}$ be arbitrary and $\alpha>0$. Assume that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\} \subset(0,1),\left\{\lambda_{n}\right\} \subset(0,2 \beta),\left\{r_{n}\right\} \subset(0, \infty)$, and $\xi \in(0,1 / L)$, where $L$ is the spectral radius of the operator $A A^{*}$ and $A^{*}$ is the adjoint of $A$. Calculate sequences $\left\{u_{n}\right\}$, $\left\{y_{n}\right\}$, and $\left\{x_{n}\right\}$ by the following iteration formula.

$$
\begin{align*}
u_{n} & =T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right), \\
y_{n} & =u_{n}+\alpha d_{n+1},  \tag{35}\\
x_{n+1} & =\alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n}+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n},
\end{align*}
$$

where $d_{n+1}=(1 / \alpha)\left(T_{n} u_{n}-u_{n}\right)+\gamma_{n} d_{n}, d_{0}=(1 / \alpha)\left(T_{0} u_{0}-u_{0}\right)$ and $T_{n}$ is defined by (34).

As follows, we propose the convergence analysis of Algorithm 13.

Theorem 14. Let $H_{1}$ and $H_{2}$ be two real Hilbert spaces and $C \subset$ $H_{1}, Q \subset H_{2}$ be nonempty closed convex subsets. Let $F_{1}, h_{1}$ : $C \times C \longrightarrow R$ and $F_{2}, h_{2}: Q \times Q \longrightarrow R$ satisfy Assumption 2; $h_{1}, h_{2}$ are monotone and $F_{2}$ is upper semicontinuous in the first argument. Assume that $\Omega:=V I(C, B) \cap \Gamma \neq \emptyset$ and $\left\{x_{n}\right\},\left\{u_{n}\right\}$, and $\left\{y_{n}\right\}$ are generated by (35). Suppose that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\} \subset(0,1)$, $\left\{\gamma_{n}\right\} \subset(0,1 / 2),\left\{\lambda_{n}\right\} \subset(0,2 \beta),\left\{r_{n}\right\} \subset(0, \infty)$ satisfy the following conditions:
(C1) $\lim _{n \rightarrow \infty} \alpha_{n}=0, \sum_{n=0}^{\infty} \alpha_{n}=\infty$.
(C2) $0<\liminf _{n \rightarrow \infty} \beta_{n} \leq \lim \sup _{n \rightarrow \infty} \beta_{n}<1$.
(C3) $\gamma_{n}=o\left(\alpha_{n}\right)$.
(C4) $\lim \inf _{n \rightarrow \infty} \lambda_{n}>0, \lim _{n \rightarrow \infty}\left|\lambda_{n+1}-\lambda_{n}\right|=0$.
(C5) $\lim _{n \rightarrow \infty}\left(\left|r_{n+1}-r_{n}\right| / r_{n+1}\right)=0$.
(C6) $\left\{T_{n} u_{n}-u_{n}\right\}$ is bounded.

Then the sequence $\left\{x_{n}\right\}$ converges strongly to $q \in \Omega$, where $q=P_{\Omega}(I-D+\gamma f) q$, which is the unique solution of the variational inequality problem

$$
\begin{equation*}
\langle(D-\gamma f) q, x-q\rangle \geq 0, \quad \forall x \in \Omega \tag{36}
\end{equation*}
$$

or equivalently, $q$ is the unique solution to the minimization problem

$$
\begin{equation*}
\min _{x \in \Omega} \frac{1}{2}\langle D x, x\rangle-h(x), \tag{37}
\end{equation*}
$$

where $h$ is a potential function for $\gamma f$ such that $h^{\prime}(x)=\gamma f(x)$ for $x \in H_{1}$.

Proof. Some equalities and inequalities in the following can be obtained according to the proof of Theorem 1 in [25]. However, we give the detailed proof process in order to read handily.

From (C1) and (C2), without loss of generality we assume that $\alpha_{n} \leq\left(1-\beta_{n}\right)\|D\|^{-1}$ for all $n \in N$. By Lemma 6, we have $\|I-\rho D\| \leq 1-\rho \bar{\gamma}$ if $0<\rho \leq\|D\|^{-1}$. Now suppose that $\|I-D\| \leq 1-\bar{\gamma}$. Since $D$ is a strongly positive linear bounded self-adjoint operator on $H_{1}$, we obtain

$$
\begin{equation*}
\|D\|=\sup \left\{|\langle D x, x\rangle|: x \in H_{1},\|x\|=1\right\} . \tag{38}
\end{equation*}
$$

Notice that

$$
\begin{align*}
& \left\langle\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) x, x\right\rangle=1-\beta_{n}-\alpha_{n}\langle D x, x\rangle  \tag{39}\\
& \quad \geq 1-\beta_{n}-\alpha_{n}\|D\| \geq 0,
\end{align*}
$$

which shows that $\left(1-\beta_{n}\right) I-\alpha_{n} D$ is positive definite. Furthermore, we have

$$
\begin{align*}
& \left\|\left(1-\beta_{n}\right) I-\alpha_{n} D\right\| \\
& \quad=\sup \left\{\left|\left\langle\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) x, x\right\rangle\right|: x \in H_{1},\|x\|\right. \\
& \quad=1\}=\sup \left\{1-\beta_{n}-\alpha_{n}\langle D x, x\rangle: x \in H_{1},\|x\|=1\right\}  \tag{40}\\
& \quad \leq 1-\beta_{n}-\alpha_{n} \bar{\gamma} .
\end{align*}
$$

Since $f$ is a contraction mapping with constant $\eta \in(0,1)$, for all $x, y \in H_{1}$, we have

$$
\begin{align*}
& \left\|P_{\Omega}(I-D+\gamma f)(x)-P_{\Omega}(I-D+\gamma f)(y)\right\| \\
& \quad \leq\|(I-D+\gamma f)(x)-(I-D+\gamma f)(y)\| \\
& \quad \leq\|I-D\|\|x-y\|+\gamma\|f(x)-f(y)\|  \tag{41}\\
& \quad \leq(1-\bar{\gamma})\|x-y\|+\gamma \eta\|x-y\| \\
& \quad \leq(1-(\bar{\gamma}-\gamma \eta))\|x-y\|
\end{align*}
$$

which implies that $P_{\Omega}(I-D+\gamma f)$ is a contraction mapping from $H_{1}$ into itself. It follows from the Banach contraction principle that there exists an element $q \in \Omega$ such that $q=$ $P_{\Omega}(I-D+\gamma f) q$.

Step 1 (we show that $\left\{x_{n}\right\}$ is bounded). Let $x^{*} \in \Omega$; i.e., $x^{*} \in \Gamma$; we have $x^{*}=T_{r_{n}}^{\left(F_{1}, h_{1}\right)} x^{*}$ and $A x^{*}=T_{r_{n}}^{\left(F_{2}, h_{2}\right)} A x^{*}$.

In the following, we compute

$$
\begin{align*}
& \left\|u_{n}-x^{*}\right\|^{2}=\| T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& -x^{*}\left\|^{2}=\right\| T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& -T_{r_{n}}^{\left(F_{1}, h_{1}\right)} x^{*}\left\|^{2} \leq\right\| x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}  \tag{42}\\
& -x^{*}\left\|^{2}=\right\| x_{n}-x^{*}\left\|^{2}+\xi^{2}\right\| A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n} \|^{2} \\
& +2 \xi\left\langle x_{n}-x^{*}, A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle .
\end{align*}
$$

Thus, we have

$$
\begin{align*}
\| u_{n} & -x^{*} \|^{2} \\
\leq & \left\|x_{n}-x^{*}\right\|^{2} \\
& +\xi^{2}\left\langle\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}, A A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle  \tag{43}\\
& +2 \xi\left\langle x_{n}-x^{*}, A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle .
\end{align*}
$$

On the other hand, we have

$$
\begin{align*}
& \xi^{2}\left\langle\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}, A A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle \\
& \quad \leq L \xi^{2}\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2} \tag{44}
\end{align*}
$$

and

$$
\begin{align*}
2 \xi & \left\langle x_{n}-x^{*}, A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle \\
& =2 \xi\left\langle A\left(x_{n}-x^{*}\right),\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle \\
& =2 \xi\left\langle A\left(x_{n}-x^{*}\right)+\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right. \\
& \left.-\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n},\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle \\
& =2 \xi\left\{\left\langle T_{r_{n}}^{\left(F_{2}, h_{2}\right)} A x_{n}-A x^{*},\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle\right.  \tag{45}\\
& \left.-\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2}\right\} \\
& \leq 2 \xi\left\{\frac{1}{2}\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2}\right. \\
& \left.-\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2}\right\} \\
& \leq-\xi\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2},
\end{align*}
$$

where the first inequality is derived from (29).
From (43)-(45), we have

$$
\begin{align*}
\left\|u_{n}-x^{*}\right\|^{2} \leq & \left\|x_{n}-x^{*}\right\|^{2}  \tag{46}\\
& +\xi(L \xi-1)\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2} .
\end{align*}
$$

Noticing that $\xi \in(0,1 / L)$, we obtain

$$
\begin{equation*}
\left\|u_{n}-x^{*}\right\|^{2} \leq\left\|x_{n}-x^{*}\right\|^{2} \tag{47}
\end{equation*}
$$

As follows, we prove that $\left\{d_{n}\right\}$ is bounded. The proof is by induction. It is true trivially for $n=0$. Let $M_{1}=$ $\max \left\{\left\|d_{0}\right\|,(2 / \alpha) \sup _{n \in N}\left\|T_{n} u_{n}-u_{n}\right\|\right\}$. From (C6), it is shown that $M_{1}<\infty$. Assume that $\left\|d_{n}\right\| \leq M_{1}$ for some $n$; we prove that it holds for $n+1$. According to the triangle inequality, we obtain

$$
\begin{align*}
\left\|d_{n+1}\right\| & =\left\|\frac{1}{\alpha}\left(T_{n} u_{n}-u_{n}\right)+\gamma_{n} d_{n}\right\| \\
& \leq \frac{1}{\alpha}\left\|T_{n} u_{n}-u_{n}\right\|+\gamma_{n}\left\|d_{n}\right\| \leq \frac{1}{\alpha} \cdot \frac{\alpha}{2} M_{1}+\frac{M_{1}}{2}  \tag{48}\\
& =M_{1}
\end{align*}
$$

which implies that $\left\|d_{n}\right\| \leq M_{1}$ for all $n \in N$; i.e., $\left\{d_{n}\right\}$ is bounded.

It is easy to see that $x^{*} \in \operatorname{VI}(C, B)$ according to $x^{*} \in \Omega$. By (22), we have $P_{C}(I-\lambda B) x^{*}=x^{*}$, which, together with (34), implies that

$$
\begin{equation*}
b_{n} T_{n} x^{*}=P_{C}(I-\lambda B) x^{*}-\left(1-b_{n}\right) x^{*}=b_{n} x^{*} ; \tag{49}
\end{equation*}
$$

that is,

$$
\begin{equation*}
T_{n} x^{*}=x^{*} \tag{50}
\end{equation*}
$$

By the definition of $\left\{y_{n}\right\}$, (47), and $\left\{T_{n}\right\}$ being nonexpansive, we have

$$
\begin{align*}
\left\|y_{n}-x^{*}\right\| & =\left\|u_{n}+\alpha d_{n+1}-x^{*}\right\| \\
& =\left\|T_{n}\left(u_{n}\right)+\alpha \gamma_{n} d_{n}-T_{n} x^{*}\right\|  \tag{51}\\
& \leq\left\|u_{n}-x^{*}\right\|+\alpha \gamma_{n} M_{1} \\
& \leq\left\|x_{n}-x^{*}\right\|+\alpha \gamma_{n} M_{1} .
\end{align*}
$$

As a result, it follows from (51), Lemma 6, and the fact that $\alpha_{n} \longrightarrow 0$ and $\gamma_{n}=o\left(\alpha_{n}\right)$ that when $n$ is large enough,

$$
\begin{align*}
& \left\|x_{n+1}-x^{*}\right\|=\| \alpha_{n}\left(\gamma f\left(x_{n}\right)-D x^{*}\right)+\beta_{n}\left(x_{n}-x^{*}\right) \\
& \quad+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right)\left(y_{n}-x^{*}\right)\left\|\leq \alpha_{n}\right\| \gamma f\left(x_{n}\right) \\
& \quad-D x^{*}\left\|+\beta_{n}\right\| x_{n}-x^{*} \|+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \\
& \quad \cdot\left(\left\|x_{n}-x^{*}\right\|+\alpha \gamma_{n} M_{1}\right) \leq \alpha_{n} \gamma\left\|f\left(x_{n}\right)-f\left(x^{*}\right)\right\| \\
& \quad+\alpha_{n}\left\|\gamma f\left(x^{*}\right)-D x^{*}\right\|+\beta_{n}\left\|x_{n}-x^{*}\right\|+\left(1-\beta_{n}\right. \\
& \left.\quad-\alpha_{n} \bar{\gamma}\right)\left(\left\|x_{n}-x^{*}\right\|+\alpha \gamma_{n} M_{1}\right) \leq \alpha_{n} \gamma \eta\left\|x_{n}-x^{*}\right\|  \tag{52}\\
& \quad+\alpha_{n}\left\|\gamma f\left(x^{*}\right)-D x^{*}\right\|+\left(1-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-x^{*}\right\|+(1 \\
& \left.\quad-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \alpha \gamma_{n} M_{1} \leq\left(1-\alpha_{n}(\bar{\gamma}-\gamma \eta)\right)\left\|x_{n}-x^{*}\right\| \\
& \left.\quad+\alpha_{n}(\bar{\gamma}-\gamma \eta) \frac{\alpha M_{1}+\left\|\gamma f\left(x^{*}\right)-D x^{*}\right\|}{\bar{\gamma}-\gamma \eta}\right\} \\
& \quad \leq \max \left\{\left\|x_{n}-x^{*}\right\|, \frac{\alpha M_{1}+\left\|\gamma f\left(x^{*}\right)-D x^{*}\right\|}{\bar{\gamma}-\gamma \eta}\right\}
\end{align*}
$$

where the third inequality is true because $\beta_{n} \in(0,1), \alpha_{n} \longrightarrow$ 0 , and $\gamma_{n}=o\left(\alpha_{n}\right)$. As a result,

$$
\begin{equation*}
\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \alpha \gamma_{n} M_{1} \leq \alpha \alpha_{n} M_{1} \tag{53}
\end{equation*}
$$

when $n$ is large enough.
Hence, $\left\{x_{n}\right\}$ is bounded and so are $\left\{u_{n}\right\}$ and $\left\{y_{n}\right\}$.
Step 2 (we show that $\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=0$ ). Since $T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}$ and $T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}$ both are firmly nonexpansive, for $\xi \in(0,1 / L)$, the mapping $T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(I+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A\right)$ is nonexpansive; see [34, 35]. Noticing that $u_{n}=T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right)$ and $u_{n+1}=T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n+1}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n+1}\right)$, we have from Lemma 8 that

$$
\begin{align*}
& \left\|u_{n+1}-u_{n}\right\| \\
& \quad \leq \| T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n+1}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n+1}\right) \\
& \quad-T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \| \\
& \quad+\| T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& \quad-T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right)\|\leq\| x_{n+1} \\
& \quad-x_{n}\|+\| x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n} \\
& \quad-\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \|+\left|1-\frac{r_{n}}{r_{n+1}}\right|  \tag{54}\\
& \quad \cdot \| T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& \quad-\left(x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right)\|\leq\| x_{n+1}-x_{n} \| \\
& \quad+\xi\|A\| \| T_{r_{n+1}^{\left(F_{2}, h_{2}\right)} A x_{n}-T_{r_{n}}^{\left(F_{2}, h_{2}\right)} A x_{n}\left\|+\zeta_{n} \leq\right\| x_{n+1}} \\
& \quad-x_{n}\|+\xi\| A\left\|\left|1-\frac{r_{n}}{r_{n+1}}\right|\right\| T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)} A x_{n}-A x_{n} \|+\zeta_{n} \\
& \quad=\left\|x_{n+1}-x_{n}\right\|+\xi\|A\| \sigma_{n}+\zeta_{n},
\end{align*}
$$

where $\sigma_{n}:=\left|1-r_{n} / r_{n+1}\right|| | T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)} A x_{n}-A x_{n} \|$ and $\zeta_{n}:=\mid 1-$ $r_{n} / r_{n+1}\| \| T_{r_{n+1}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left({ }_{2}+h_{2}\right)}-I\right) A x_{n}\right)-\left(x_{n}+\xi A^{*}\left(T_{r_{n+1}}^{\left(F_{2}, h_{2}\right)}-\right.\right.$ I) $\left.A x_{n}\right) \|$.

Furthermore, one has

$$
\begin{align*}
\left\|y_{n+1}-y_{n}\right\|= & \left\|u_{n+1}+\alpha d_{n+2}-\left(u_{n}+\alpha d_{n+1}\right)\right\| \\
\leq & \left\|T_{n+1} u_{n+1}-T_{n} u_{n}\right\| \\
& +\alpha\left\|\gamma_{n+1} d_{n+1}-\gamma_{n} d_{n}\right\| \\
\leq & \left\|T_{n+1} u_{n+1}-T_{n+1} u_{n}\right\| \\
& +\left\|T_{n+1} u_{n}-T_{n} u_{n}\right\|  \tag{55}\\
& +\alpha\left\|\gamma_{n+1} d_{n+1}-\gamma_{n} d_{n}\right\| \\
\leq & \left\|u_{n+1}-u_{n}\right\|+\left\|T_{n+1} u_{n}-T_{n} u_{n}\right\| \\
& +\alpha M_{1}\left(\gamma_{n+1}+\gamma_{n}\right)
\end{align*}
$$

It follows from (34) that

$$
\begin{aligned}
& \left\|T_{n+1} u_{n}-T_{n} u_{n}\right\|=\| \frac{P_{C}\left(I-\lambda_{n+1} B\right)-\left(1-b_{n+1}\right) I}{b_{n+1}} u_{n} \\
& \\
& -\frac{P_{C}\left(I-\lambda_{n} B\right)-\left(1-b_{n}\right) I}{b_{n}} u_{n} \| \\
& \quad=\| \frac{4 \beta P_{C}\left(I-\lambda_{n+1} B\right)-\left(2 \beta-\lambda_{n+1}\right) I}{2 \beta+\lambda_{n+1}} u_{n} \\
& \quad-\frac{4 \beta P_{C}\left(I-\lambda_{n} B\right)-\left(2 \beta-\lambda_{n}\right) I}{2 \beta+\lambda_{n}} u_{n}\|=\| \frac{4 \beta P_{C}\left(I-\lambda_{n+1} B\right)}{2 \beta+\lambda_{n+1}} u_{n} \\
& \quad-\frac{4 \beta P_{C}\left(I-\lambda_{n} B\right)}{2 \beta+\lambda_{n}} u_{n}+\frac{\left(2 \beta-\lambda_{n}\right)}{2 \beta+\lambda_{n}} u_{n}-\frac{\left(2 \beta-\lambda_{n+1}\right)}{2 \beta+\lambda_{n+1}} u_{n} \| \\
& \quad \leq \frac{4 \beta\left[\left(2 \beta+\lambda_{n}\right) P_{C}\left(I-\lambda_{n+1} B\right)-\left(2 \beta+\lambda_{n+1}\right) P_{C}\left(I-\lambda_{n} B\right)\right]}{\left(2 \beta+\lambda_{n+1}\right)\left(2 \beta+\lambda_{n}\right)} \\
& \quad \cdot u_{n}\|+\| \frac{4 \beta\left(\lambda_{n+1}-\lambda_{n}\right)}{\left(2 \beta+\lambda_{n+1}\right)\left(2 \beta+\lambda_{n}\right)} u_{n} \| \\
& \quad \leq\left\|\frac{4 \beta\left(\lambda_{n}-\lambda_{n+1}\right) P_{C}\left(I-\lambda_{n+1} B\right)}{\left(2 \beta+\lambda_{n+1}\right)\left(2 \beta+\lambda_{n}\right)} u_{n}\right\| \\
& \quad+\left\|\frac{4 \beta\left(2 \beta+\lambda_{n+1}\right)\left[P_{C}\left(I-\lambda_{n+1} B\right)-P_{C}\left(I-\lambda_{n} B\right)\right]}{\left(2 \beta+\lambda_{n+1}\right)\left(2 \beta+\lambda_{n}\right)} u_{n}\right\| \\
& \quad+\left\|\frac{4 \beta\left(\lambda_{n+1}-\lambda_{n}\right)}{\left(2 \beta+\lambda_{n+1}\right)\left(2 \beta+\lambda_{n}\right)} u_{n}\right\| \leq \frac{1}{\beta}\left|\lambda_{n}-\lambda_{n+1}\right| \\
& \quad+\left(\left\|P_{C}\left(I-\lambda_{n+1} B\right) u_{n}\right\|+4 \beta\left\|B u_{n}\right\|\right. \\
& \left.\quad+\left\|u_{n}\right\|\right) \leq M_{2}\left|\lambda_{n+1}-\lambda_{n}\right|,
\end{aligned}
$$

where $M_{2}=\sup _{n \in N}\left\{(1 / \beta)\left(\left\|P_{C}\left(I-\lambda_{n+1} B\right) u_{n}\right\|+4 \beta\left\|B u_{n}\right\|+\right.\right.$ $\left.\left.\left\|u_{n}\right\|\right)\right\}$.

Hence from (54)-(56), we get

$$
\begin{align*}
\left\|y_{n+1}-y_{n}\right\| \leq & \left\|x_{n+1}-x_{n}\right\|+\xi\|A\| \sigma_{n}+\zeta_{n} \\
& +M_{2}\left|\lambda_{n+1}-\lambda_{n}\right|+\alpha M_{1}\left(\gamma_{n+1}+\gamma_{n}\right) . \tag{57}
\end{align*}
$$

Set $x_{n+1}=\left(1-\beta_{n}\right) z_{n}+\beta_{n} x_{n}$; it follows that

$$
\begin{align*}
z_{n} & =\frac{x_{n+1}-\beta_{n} x_{n}}{1-\beta_{n}} \\
& =\frac{\alpha_{n} \gamma f\left(x_{n}\right)+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}}{1-\beta_{n}} . \tag{58}
\end{align*}
$$

As a result,

$$
\begin{aligned}
& \left\|z_{n+1}-z_{n}\right\| \\
& \quad=\| \frac{\alpha_{n+1} \gamma f\left(x_{n+1}\right)+\left(\left(1-\beta_{n+1}\right) I-\alpha_{n+1} D\right) y_{n+1}}{1-\beta_{n+1}} \\
& \quad-\frac{\alpha_{n} \gamma f\left(x_{n}\right)+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}}{1-\beta_{n}} \|
\end{aligned}
$$

$$
\begin{align*}
& =\| \frac{\alpha_{n+1}\left(\gamma f\left(x_{n+1}\right)-D y_{n+1}\right)}{1-\beta_{n+1}} \\
& -\frac{\alpha_{n}\left(\gamma f\left(x_{n}\right)-D y_{n}\right)}{1-\beta_{n}}+y_{n+1}-y_{n} \| \\
& \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f\left(x_{n+1}\right)-D y_{n+1}\right\| \\
& +\frac{\alpha_{n}}{1-\beta_{n}}\left\|\gamma f\left(x_{n}\right)-D y_{n}\right\|+\left\|y_{n+1}-y_{n}\right\| \\
& \leq \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left\|\gamma f\left(x_{n+1}\right)-D y_{n+1}\right\| \\
& +\frac{\alpha_{n}}{1-\beta_{n}}\left\|\gamma f\left(x_{n}\right)-D y_{n}\right\|+\left\|x_{n+1}-x_{n}\right\|+\xi\|A\| \\
& \cdot \sigma_{n}+\zeta_{n}+M_{2}\left|\lambda_{n+1}-\lambda_{n}\right|+\alpha M_{1}\left(\gamma_{n+1}+\gamma_{n}\right) \tag{59}
\end{align*}
$$

Letting $n \longrightarrow \infty$, from (C1)-(C5), we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left(\left\|z_{n+1}-z_{n}\right\|-\left\|x_{n+1}-x_{n}\right\|\right) \leq 0 \tag{60}
\end{equation*}
$$

By Lemma 12, we obtain

$$
\begin{equation*}
\lim _{n \longrightarrow \infty}\left\|z_{n}-x_{n}\right\|=0 \tag{61}
\end{equation*}
$$

Further,

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=\lim _{n \longrightarrow \infty}\left(1-\beta_{n}\right)\left\|z_{n}-x_{n}\right\|=0 . \tag{62}
\end{equation*}
$$

Step 3 (we show that $\lim _{n \rightarrow \infty}\left\|u_{n}-x_{n}\right\|=0$ ). Since $x^{*} \in \Omega$, $x^{*}=T_{r_{n}}^{\left(F_{1}, h_{1}\right)} x^{*}$, and $T_{r_{n}}^{\left(F_{1}, h_{1}\right)}$ is firmly nonexpansive, we obtain

$$
\begin{align*}
& \left\|u_{n}-x^{*}\right\|^{2}=\| T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& -x^{*}\left\|^{2}=\right\| T_{r_{n}}^{\left(F_{1}, h_{1}\right)}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right) \\
& -T_{r_{n}}^{\left(F_{1}, h_{1}\right)} x^{*} \|^{2} \leq\left\langle u_{n}-x^{*}, x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right)\right. \\
& \left.\quad-A x_{n}-x^{*}\right\rangle=\frac{1}{2}\left\{\left\|u_{n}-x^{*}\right\|^{2}+\| x_{n}\right.  \tag{63}\\
& \quad+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}-x^{*}\left\|^{2}-\right\|\left(u_{n}-x^{*}\right) \\
& \left.-\left[x_{n}+\xi A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}-x^{*}\right] \|^{2}\right\}=\frac{1}{2}\left\{\| u_{n}\right. \\
& -x^{*}\left\|^{2}+\right\| x_{n}-x^{*}\left\|^{2}-\right\| u_{n}-x_{n} \|^{2}+2 \xi\left\langle u_{n}\right. \\
& \left.\left.-x^{*}, A^{*}\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\rangle\right\} .
\end{align*}
$$

Hence, we obtain

$$
\begin{align*}
&\left\|u_{n}-x^{*}\right\|^{2} \\
& \leq\left\|x_{n}-x^{*}\right\|^{2}-\left\|u_{n}-x_{n}\right\|^{2}  \tag{64}\\
&+2 \xi\left\|A\left(u_{n}-x^{*}\right)\right\|\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\| .
\end{align*}
$$

Furthermore,

$$
\begin{align*}
\left\|y_{n}-x^{*}\right\|^{2} & =\left\|u_{n}+\alpha d_{n+1}-x^{*}\right\|^{2} \\
& \leq\left\|u_{n}-x^{*}\right\|^{2}+2 \alpha \gamma_{n}\left\langle y_{n}-x^{*}, d_{n}\right\rangle  \tag{65}\\
& \leq\left\|u_{n}-x^{*}\right\|^{2}+M_{3} \gamma_{n}
\end{align*}
$$

where $M_{3}=\sup _{n \in N^{2}} 2 \alpha\left\langle y_{n}-x^{*}, d_{n}\right\rangle$.
By Lemma 1 (iii), (46), and (65), we have

$$
\begin{aligned}
&\left\|x_{n+1}-x^{*}\right\|^{2}=\| \alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n} \\
&+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}-x^{*} \|^{2} \\
& \quad=\| \alpha_{n}\left(\gamma f\left(x_{n}\right)-D x^{*}\right)+\beta_{n}\left(x_{n}-x^{*}\right) \\
&+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right)\left(y_{n}-x^{*}\right) \|^{2} \\
& \quad \leq\left(\alpha_{n}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|+\beta_{n}\left\|x_{n}-x^{*}\right\|\right. \\
&\left.\quad+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|\right)^{2} \\
& \quad=\left(\alpha_{n} \bar{\gamma}\left\|\frac{1}{\bar{\gamma}}\left(\gamma f\left(x_{n}\right)-D x^{*}\right)\right\|+\beta_{n}\left\|x_{n}-x^{*}\right\|\right. \\
&\left.\quad+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|\right)^{2} \leq \alpha_{n} \bar{\gamma} \frac{1}{\bar{\gamma}^{2}} \| \gamma f\left(x_{n}\right) \\
& \quad-D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*}\left\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\right\| y_{n} \\
& \quad-x^{*}\left\|^{2} \leq \frac{\alpha_{n}}{\bar{\gamma}}\right\| \gamma f\left(x_{n}\right)-D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*} \|^{2} \\
&+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|^{2} \leq \frac{\alpha_{n}}{\bar{\gamma}} \| \gamma f\left(x_{n}\right) \\
&-D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*} \|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \\
& \quad+\left(\left\|x_{n}-x^{*}\right\|^{2}+\xi(L \xi-1)\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2}\right. \\
&\left.+M_{3} \gamma_{n}\right)=\frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\beta_{n}\left\|x_{n}-x^{*}\right\|^{2} \\
&+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-x^{*}\right\|^{2}-\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \xi(1 \\
&-L \xi)\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} .
\end{aligned}
$$

As a result,

$$
\begin{aligned}
(1- & \left.\beta_{n}-\alpha_{n} \bar{\gamma}\right) \xi(1-L \xi)\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|^{2} \\
\leq & \left\|x_{n}-x^{*}\right\|^{2}-\left\|x_{n+1}-x^{*}\right\|^{2} \\
& +\frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} \\
\leq & \left(\left\|x_{n}-x^{*}\right\|+\left\|x_{n+1}-x^{*}\right\|\right)\left\|x_{n}-x_{n+1}\right\| \\
& +\frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n}
\end{aligned}
$$

According to $\alpha_{n} \longrightarrow 0, \gamma_{n}=o\left(\alpha_{n}\right),\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \xi(1-L \xi)>$ 0 , and $\lim _{n \rightarrow \infty}\left\|x_{n+1}-x_{n}\right\|=0$, we obtain

$$
\begin{equation*}
\lim _{n \longrightarrow \infty}\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|=0 \tag{68}
\end{equation*}
$$

From (65), (64), and Lemma 1 (iii), we obtain

$$
\begin{align*}
& \left\|x_{n+1}-x^{*}\right\|^{2}=\| \alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n} \\
& +\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}-x^{*} \|^{2} \\
& =\| \alpha_{n}\left(\gamma f\left(x_{n}\right)-D x^{*}\right)+\beta_{n}\left(x_{n}-x^{*}\right) \\
& +\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right)\left(y_{n}-x^{*}\right) \|^{2} \\
& \leq\left(\alpha_{n}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|+\beta_{n}\left\|x_{n}-x^{*}\right\|\right. \\
& \left.+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|\right)^{2} \\
& =\left(\alpha_{n} \bar{\gamma}\left\|\frac{1}{\bar{\gamma}}\left(\gamma f\left(x_{n}\right)-D x^{*}\right)\right\|+\beta_{n}\left\|x_{n}-x^{*}\right\|\right. \\
& \left.+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|\right)^{2} \leq \alpha_{n} \bar{\gamma} \frac{1}{\bar{\gamma}^{2}} \| \gamma f\left(x_{n}\right) \\
& -D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*}\left\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\right\| y_{n} \\
& -x^{*}\left\|^{2} \leq \frac{\alpha_{n}}{\bar{\gamma}}\right\| \gamma f\left(x_{n}\right)-D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*} \|^{2}  \tag{69}\\
& +\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|y_{n}-x^{*}\right\|^{2} \leq \frac{\alpha_{n}}{\bar{\gamma}} \| \gamma f\left(x_{n}\right) \\
& -D x^{*}\left\|^{2}+\beta_{n}\right\| x_{n}-x^{*} \|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \\
& \cdot\left(\left\|u_{n}-x^{*}\right\|^{2}+M_{3} \gamma_{n}\right) \leq \frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2} \\
& +\beta_{n}\left\|x_{n}-x^{*}\right\|^{2}+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left(\left\|x_{n}-x^{*}\right\|^{2}\right. \\
& -\left\|u_{n}-x_{n}\right\|^{2} \\
& \left.+2 \xi\left\|A\left(u_{n}-x^{*}\right)\right\|\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|+M_{3} \gamma_{n}\right) \\
& =\frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\beta_{n}\left\|x_{n}-x^{*}\right\|^{2}+\left(1-\beta_{n}\right. \\
& \left.-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-x^{*}\right\|^{2}-\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|u_{n}-x_{n}\right\|^{2} \\
& +2 \xi\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|A\left(u_{n}-x^{*}\right)\right\| \\
& \cdot\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} .
\end{align*}
$$

Therefore, one has

$$
\begin{aligned}
& \left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|u_{n}-x_{n}\right\|^{2} \leq \frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2} \\
& \quad+\left(1-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-x^{*}\right\|^{2}-\left\|x_{n+1}-x^{*}\right\|^{2} \\
& \quad+2 \xi\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|A\left(u_{n}-x^{*}\right)\right\|
\end{aligned}
$$

$$
\begin{align*}
& \cdot\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} \\
& \leq \frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\left\|x_{n}-x^{*}\right\|^{2} \\
& -\left\|x_{n+1}-x^{*}\right\|^{2}+2 \xi\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)\left\|A\left(u_{n}-x^{*}\right)\right\| \\
& \cdot\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\|+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} \\
& \leq \frac{\alpha_{n}}{\bar{\gamma}}\left\|\gamma f\left(x_{n}\right)-D x^{*}\right\|^{2}+\left\|x_{n}-x_{n+1}\right\| \\
& \cdot\left(\left\|x_{n}-x^{*}\right\|+\left\|x_{n+1}-x^{*}\right\|\right)+2 \xi\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \\
& \cdot\left\|A\left(u_{n}-x^{*}\right)\right\|\left\|\left(T_{r_{n}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n}\right\| \\
& +\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{3} \gamma_{n} . \tag{70}
\end{align*}
$$

According to $\alpha_{n} \longrightarrow 0, \gamma_{n}=o\left(\alpha_{n}\right),(62)$, and (68), we obtain

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n}-x_{n}\right\|=0 \tag{71}
\end{equation*}
$$

Step 4 (we show that $\lim _{n \rightarrow \infty}\left\|T_{n} u_{n}-u_{n}\right\|=0$ ). From (58), we have

$$
\begin{equation*}
z_{n}-y_{n}=\frac{\alpha_{n}}{1-\beta_{n}} \gamma f\left(x_{n}\right)-\frac{\alpha_{n}}{1-\beta_{n}} D y_{n} . \tag{72}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left\|z_{n}-y_{n}\right\| \leq \frac{\alpha_{n} \gamma}{1-\beta_{n}}\left|f\left(x_{n}\right)\right|+\frac{\alpha_{n}}{1-\beta_{n}}\left\|D y_{n}\right\| \tag{73}
\end{equation*}
$$

From ( Cl ) and ( C 2 ), one has

$$
\begin{equation*}
\lim _{n \longrightarrow \infty}\left\|z_{n}-y_{n}\right\|=0 \tag{74}
\end{equation*}
$$

By (74) and (61), we have

$$
\begin{equation*}
\lim _{n \longrightarrow \infty}\left\|x_{n}-y_{n}\right\|=0 \tag{75}
\end{equation*}
$$

Combining (71) and (75), one has

$$
\begin{equation*}
\lim _{n \rightarrow \infty}\left\|u_{n}-y_{n}\right\|=0 \tag{76}
\end{equation*}
$$

Noting that

$$
\begin{equation*}
y_{n}-u_{n}=T_{n} u_{n}-u_{n}+\alpha \gamma_{n} d_{n}, \tag{77}
\end{equation*}
$$

one has

$$
\begin{equation*}
T_{n} u_{n}-u_{n}=y_{n}-u_{n}-\alpha \gamma_{n} d_{n} \tag{78}
\end{equation*}
$$

It follows from (76) and $\alpha_{n} \longrightarrow 0, \gamma_{n}=o\left(\alpha_{n}\right)$ that

$$
\begin{equation*}
\lim _{n \longrightarrow \infty}\left\|T_{n} u_{n}-u_{n}\right\|=0 \tag{79}
\end{equation*}
$$

We claim that

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle(D-\gamma f) q, q-x_{n}\right\rangle \leq 0 \tag{80}
\end{equation*}
$$

where $q$ is the unique solution of the variational inequality $\langle(D-\gamma f) q, x-q\rangle \geq 0, \forall x \in \Omega$.

To show this inequality, we choose a subsequence $\left\{x_{n_{i}}\right\}$ of $\left\{x_{n}\right\}$ such that

$$
\begin{align*}
& \limsup _{n \rightarrow \infty}\left\langle(D-\gamma f) q, q-x_{n}\right\rangle  \tag{81}\\
& \quad=\lim _{i \rightarrow \infty}\left\langle(D-\gamma f) q, q-x_{n_{i}}\right\rangle
\end{align*}
$$

Since $\left\{x_{n_{i}}\right\}$ is bounded, there exists a subsequence of $\left\{x_{n_{i}}\right\}$ which converges weakly to $z \in C$. Without loss of generality, we can assume that $x_{n_{i}} \rightharpoonup z$.
Step 5 (we show that $z \in \Omega$ ). First, we show that $z \in \operatorname{VI}(C, B)$. Let $M: H_{1} \longrightarrow 2^{H_{1}}$ be a set-valued mapping defined by

$$
M v= \begin{cases}B v+N_{C} v, & v \in C  \tag{82}\\ \emptyset, & v \notin C\end{cases}
$$

where $N_{C} v:=\left\{z \in H_{1}:\langle v-u, z\rangle \geq 0, \forall u \in C\right\}$ is the normal cone to $C$ at $v \in C$. Then $M$ is maximal monotone and $0 \in$ $M v$ if and only if $v \in \operatorname{VI}(C, B)$ (see [36]). Let $(v, u) \in G(M)$. Therefore, we have

$$
\begin{equation*}
u \in M v=B v+N_{C} v \tag{83}
\end{equation*}
$$

and so

$$
\begin{equation*}
u-B v \in N_{C} v \tag{84}
\end{equation*}
$$

According to $u_{n} \in C$, we obtain

$$
\begin{equation*}
\left\langle v-u_{n}, u-B v\right\rangle \geq 0 \tag{85}
\end{equation*}
$$

On the other hand, according to

$$
\begin{equation*}
P_{C}\left(\left(I-\lambda_{n} B\right) u_{n}\right)=u_{n}+b_{n}\left(T_{n} u_{n}-u_{n}\right) \tag{86}
\end{equation*}
$$

where $b_{n}=\left(2 \beta+\lambda_{n}\right) / 4 \beta$, for $\forall n \in N$, and $v \in H_{1}$, we have

$$
\begin{align*}
& \left\langle v-u_{n}-b_{n}\left(T_{n} u_{n}-u_{n}\right), u_{n}+b_{n}\left(T_{n} u_{n}-u_{n}\right)\right. \\
& \left.\quad-\left(u_{n}-\lambda_{n} B u_{n}\right)\right\rangle \geq 0 \tag{87}
\end{align*}
$$

Therefore,

$$
\begin{align*}
& \left\langle v-u_{n}, b_{n}\left(T_{n} u_{n}-u_{n}\right)+\lambda_{n} B u_{n}\right\rangle-\left\|b_{n}\left(T_{n} u_{n}-u_{n}\right)\right\|^{2}  \tag{88}\\
& \quad-\left\langle b_{n}\left(T_{n} u_{n}-u_{n}\right), \lambda_{n} B u_{n}\right\rangle \geq 0 .
\end{align*}
$$

As a result,

$$
\begin{align*}
\langle v & \left.-u_{n}, \frac{b_{n}}{\lambda_{n}}\left(T_{n} u_{n}-u_{n}\right)+B u_{n}\right\rangle  \tag{89}\\
& -\left\langle b_{n}\left(T_{n} u_{n}-u_{n}\right), B u_{n}\right\rangle \geq 0
\end{align*}
$$

Furthermore, according to (85) and (89), for $\forall n \in N$, one has

$$
\begin{align*}
\left\langle v-u_{n}, u\right\rangle \geq & \left\langle v-u_{n}, B v\right\rangle \\
& -\left\langle v-u_{n}, \frac{b_{n}}{\lambda_{n}}\left(T_{n} u_{n}-u_{n}\right)+B u_{n}\right\rangle \\
& +\left\langle b_{n}\left(T_{n} u_{n}-u_{n}\right), B u_{n}\right\rangle \\
= & \left\langle v-u_{n}, B v-B u_{n}\right\rangle  \tag{90}\\
& -\left\langle v-u_{n}, \frac{b_{n}}{\lambda_{n}}\left(T_{n} u_{n}-u_{n}\right)\right\rangle \\
& +\left\langle b_{n}\left(T_{n} u_{n}-u_{n}\right), B u_{n}\right\rangle
\end{align*}
$$

Replacing $n$ by $n_{i}$, one has

$$
\begin{align*}
\left\langle v-u_{n_{i}}, u\right\rangle \geq & \left\langle v-u_{n_{i}}, B v-B u_{n_{i}}\right\rangle \\
& -\left\langle v-u_{n_{i}}, \frac{b_{n_{i}}}{\lambda_{n_{i}}}\left(T_{n_{i}} u_{n_{i}}-u_{n_{i}}\right)\right\rangle  \tag{91}\\
& +\left\langle b_{n_{i}}\left(T_{n_{i}} u_{n_{i}}-u_{n_{i}}\right), B u_{n_{i}}\right\rangle .
\end{align*}
$$

Since $\left\|u_{n}-x_{n}\right\| \longrightarrow 0$ and $x_{n_{i}} \rightharpoonup z$, we have $u_{n_{i}} \rightharpoonup z$. Noting that $B$ is $\beta$-ism, from (91) and (79), we have

$$
\begin{equation*}
\langle v-z, u\rangle \geq 0 . \tag{92}
\end{equation*}
$$

Since $M$ is maximal monotone, one has $z \in M^{-1} 0$. Hence $z \in V I(C, B)$.

Next, we prove $z \in \Gamma$.
According to Algorithm 13, we have

$$
\begin{equation*}
u_{n_{i}}=T_{r_{n_{i}}}^{\left(F_{1}, h_{1}\right)}\left(x_{n_{i}}+\xi A^{*}\left(T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n_{i}}\right) \tag{93}
\end{equation*}
$$

By (8), for any $w \in C$, one has

$$
\begin{aligned}
0 \leq & F_{1}\left(u_{n_{i}}, w\right)+h_{1}\left(u_{n_{i}}, w\right)+\frac{1}{r_{n_{i}}}\left\langle w-u_{n_{i}}, u_{n_{i}}\right. \\
& \left.-\left(x_{n_{i}}+\xi A^{*}\left(T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n_{i}}\right)\right\rangle=F_{1}\left(u_{n_{i}}, w\right) \\
& +h_{1}\left(u_{n_{i}}, w\right)+\frac{1}{r_{n_{i}}}\left\langle w-u_{n_{i}}, u_{n_{i}}-x_{n_{i}}\right\rangle-\frac{\xi}{r_{n_{i}}}\langle A w \\
& \left.-A u_{n_{i}},\left(T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n_{i}}\right\rangle \leq F_{1}\left(u_{n_{i}}, w\right) \\
& +h_{1}\left(u_{n_{i}}, w\right)+\frac{1}{r_{n_{i}}}\left\|w-u_{n_{i}}\right\|\left\|u_{n_{i}}-x_{n_{i}}\right\|+\frac{\xi}{r_{n_{i}}} \| A w \\
& -A u_{n_{i}}\| \|\left(T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n_{i}} \|
\end{aligned}
$$

According to the monotonicity of $F_{1}$, we have

$$
\begin{align*}
& F_{1}\left(w, u_{n_{i}}\right) \\
& \qquad \begin{array}{l}
h_{1}\left(u_{n_{i}}, w\right)+\frac{1}{r_{n_{i}}}\left\|w-u_{n_{i}}\right\|\left\|u_{n_{i}}-x_{n_{i}}\right\| \\
\\
\quad+\frac{\xi}{r_{n_{i}}}\left\|A w-A u_{n_{i}}\right\|\left\|\left(T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)}-I\right) A x_{n_{i}}\right\|
\end{array} . \tag{95}
\end{align*}
$$

From Assumption $2(i v)$ on $F$ and (ii) on $h$, (68), and (71), one has

$$
\begin{equation*}
F_{1}(w, z) \leq h_{1}(z, w) . \tag{96}
\end{equation*}
$$

It follows from the monotonicity of $h_{1}$ that

$$
\begin{equation*}
F_{1}(w, z)+h_{1}(w, z) \leq 0, \quad \forall w \in C \tag{97}
\end{equation*}
$$

For any $t \in(0,1]$ and $w \in C$, let $w_{t}=t w+(1-t) z$. Since $z \in C$ and $C$ is convex, we obtain that $w_{t} \in C$. Hence

$$
\begin{equation*}
F_{1}\left(w_{t}, z\right)+h_{1}\left(w_{t}, z\right) \leq 0 . \tag{98}
\end{equation*}
$$

From Assumption $2(i),(i v)$ on $F$ and (i), (iii) on $h$, we have

$$
\begin{align*}
0 \leq & F_{1}\left(w_{t}, w_{t}\right)+h_{1}\left(w_{t}, w_{t}\right) \\
\leq & t\left[F_{1}\left(w_{t}, w\right)+h_{1}\left(w_{t}, w\right)\right] \\
& +(1-t)\left[F_{1}\left(w_{t}, z\right)+h_{1}\left(w_{t}, z\right)\right]  \tag{99}\\
\leq & t\left[F_{1}\left(w_{t}, w\right)+h_{1}\left(w_{t}, w\right)\right],
\end{align*}
$$

which implies that

$$
\begin{equation*}
F_{1}\left(w_{t}, w\right)+h_{1}\left(w_{t}, w\right) \geq 0, \quad \forall w \in C \tag{100}
\end{equation*}
$$

Letting $t \longrightarrow 0$ and by Assumption 2 (iii) on $F$ and (ii) on $h$, we obtain

$$
\begin{equation*}
F_{1}(z, w)+h_{1}(z, w) \geq 0, \quad \forall w \in C ; \tag{101}
\end{equation*}
$$

that is, $z \in \operatorname{GEP}\left(F_{1}, h_{1}\right)$.
As follows, we prove $A z \in \operatorname{GEP}\left(F_{2}, h_{2}\right)$.
Since $A$ is a bounded linear operator, one has $A x_{n_{i}} \rightharpoonup A z$.
Now, set $\zeta_{n_{i}}=A x_{n_{i}}-T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)} A x_{n_{i}}$. It follows from (68) that $\lim _{i \rightarrow \infty} \zeta_{n_{i}}=0$. Since $A x_{n_{i}}-\zeta_{n_{i}}=T_{r_{n_{i}}}^{\left(F_{2}, h_{2}\right)} A x_{n_{i}}$, by (9) we have

$$
\begin{align*}
& F_{2}\left(A x_{n_{i}}-\zeta_{n_{i}}, \tilde{z}\right)+h_{2}\left(A x_{n_{i}}-\zeta_{n_{i}}, \tilde{z}\right) \\
& \quad \quad+\frac{1}{r_{n_{i}}}\left\langle\widetilde{z}-\left(A x_{n_{i}}-\zeta_{n_{i}}\right),\left(A x_{n_{i}}-\zeta_{n_{i}}\right)-A x_{n_{i}}\right\rangle  \tag{102}\\
& \quad \geq 0, \quad \forall \widetilde{z} \in Q .
\end{align*}
$$

Furthermore, one has

$$
\begin{align*}
& F_{2}\left(A x_{n_{i}}-\zeta_{n_{i}}, \widetilde{z}\right)+h_{2}\left(A x_{n_{i}}-\zeta_{n_{i}}, \tilde{z}\right) \\
& \quad+\frac{1}{r_{n_{i}}}\left\langle\widetilde{z}-A x_{n_{i}}+\zeta_{n_{i}},-\zeta_{n_{i}}\right\rangle \geq 0, \quad \forall \widetilde{z} \in Q \tag{103}
\end{align*}
$$

From the upper semicontinuity of $F_{2}(x, y)$ and $h_{2}(x, y)$ on $x$, we have

$$
\begin{equation*}
F_{2}(A z, \tilde{z})+h_{2}(A z, \tilde{z}) \geq 0, \quad \forall \tilde{z} \in Q \tag{104}
\end{equation*}
$$

which means that $A z \in \operatorname{GEP}\left(F_{2}, h_{2}\right)$. As a result, $z \in \Gamma$.
Therefore, $z \in \Omega$.
Since $q=P_{\Omega}(I-D+\gamma f) q$ is the unique solution of the variational inequality problem $\langle(D-\gamma f) q, x-q\rangle \geq 0, \forall x \in \Omega$, by (81) and $z \in \Omega$, we have

$$
\begin{equation*}
\limsup _{n \rightarrow \infty}\left\langle(D-\gamma f) q, q-x_{n}\right\rangle \leq 0 \tag{105}
\end{equation*}
$$

Step 6 (finally, we show that $\left\{x_{n}\right\}$ converges strongly to $q$ ). It is obvious that

$$
\begin{align*}
\left\|y_{n}-q\right\| & =\left\|T_{n} u_{n}-q+\alpha \gamma_{n} d_{n}\right\| \\
& \leq\left\|T_{n} u_{n}-T_{n} q\right\|+\alpha \gamma_{n}\left\|d_{n}\right\|  \tag{106}\\
& \leq\left\|u_{n}-q\right\|+\alpha M_{1} \gamma_{n}=\left\|u_{n}-q\right\|+M_{4} \gamma_{n} \\
& \leq\left\|x_{n}-q\right\|+M_{4} \gamma_{n}
\end{align*}
$$

where $M_{4}=\alpha M_{1}$. And the first inequality is true because $q \in$ $\Omega$ and $T_{n} q=q$ according to the same reasoning to equality (50). The last inequality is obtained by $q \in \Omega$ and the same reasoning to inequality (47). Thus, from Lemma $1(i)$, we have

$$
\begin{align*}
& \left\|x_{n+1}-q\right\|^{2}=\| \alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n} \\
& \quad+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}-q \|^{2} \\
& \quad=\| \alpha_{n}\left(\gamma f\left(x_{n}\right)-D q\right)+\beta_{n}\left(x_{n}-q\right) \\
& \quad+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right)\left(y_{n}-q\right)\left\|^{2} \leq\right\| \beta_{n}\left(x_{n}-q\right) \\
& \quad+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right)\left(y_{n}-q\right) \|^{2}+2 \alpha_{n}\left\langle\gamma f\left(x_{n}\right)\right. \\
& \left.\quad-D q, x_{n+1}-q\right\rangle \leq\left(\beta_{n}\left\|x_{n}-q\right\|\right. \\
& \left.\left.\quad+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) \| y_{n}-q\right) \|\right)^{2}+2 \alpha_{n} \gamma\left\langle f\left(x_{n}\right)\right.  \tag{107}\\
& \left.\quad-f(q), x_{n+1}-q\right\rangle+2 \alpha_{n}\left\langle\gamma f(q)-D q, x_{n+1}-q\right\rangle \\
& \quad \leq\left(\left(1-\alpha_{n} \bar{\gamma}\right)\left\|x_{n}-q\right\|+\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{4} \gamma_{n}\right)^{2} \\
& \quad+2 \alpha_{n} \gamma \eta\left\|x_{n}-q\right\|\left\|x_{n+1}-q\right\|+2 \alpha_{n}\langle\gamma f(q) \\
& \left.\quad-D q, x_{n+1}-q\right\rangle \leq\left(1-\alpha_{n} \bar{\gamma}\right)^{2}\left\|x_{n}-q\right\|^{2}+M_{5} \gamma_{n} \\
& \quad+\alpha_{n} \gamma \eta\left\|x_{n}-q\right\|^{2}+\alpha_{n} \gamma \eta\left\|x_{n+1}-q\right\|^{2} \\
& \quad+2 \alpha_{n}\left\langle\gamma f(q)-D q, x_{n+1}-q\right\rangle,
\end{align*}
$$

where $M_{5}=\sup _{n \in N}\left\{2\left(1-\alpha_{n} \bar{\gamma}\right)\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right) M_{4}\left\|x_{n}-q\right\|+\right.$ $\left.\left(1-\beta_{n}-\alpha_{n} \bar{\gamma}\right)^{2} M_{4}^{2} \gamma_{n}\right\}$.

As a result,

$$
\begin{aligned}
&\left(1-\alpha_{n} \gamma \eta\right)\left\|x_{n+1}-q\right\|^{2} \\
& \leq\left(1-2 \alpha_{n} \bar{\gamma}+\alpha_{n}^{2} \bar{\gamma}^{2}+\alpha_{n} \gamma \eta\right)\left\|x_{n}-q\right\|^{2}+M_{5} \gamma_{n} \\
&+2 \alpha_{n}\left\langle\gamma f(q)-D q, x_{n+1}-q\right\rangle
\end{aligned}
$$

which implies that

$$
\begin{align*}
& \left\|x_{n+1}-q\right\|^{2} \leq\left(1-\frac{2(\bar{\gamma}-\gamma \eta) \alpha_{n}}{1-\alpha_{n} \gamma \eta}\right)\left\|x_{n}-q\right\|^{2} \\
& \quad+\frac{2(\bar{\gamma}-\gamma \eta) \alpha_{n}}{1-\alpha_{n} \gamma \eta}\left(\frac{\alpha_{n} \bar{\gamma}^{2}\left\|x_{n}-q\right\|^{2}+M_{5}\left(\gamma_{n} / \alpha_{n}\right)}{2(\bar{\gamma}-\gamma \eta)}\right.  \tag{109}\\
& \left.\quad+\frac{1}{\bar{\gamma}-\gamma \eta}\left\langle\gamma f(q)-D q, x_{n+1}-q\right\rangle\right)=\left(1-t_{n}\right) \\
& \quad \cdot\left\|x_{n}-q\right\|^{2}+t_{n} \delta_{n}
\end{align*}
$$

where $t_{n}=2(\bar{\gamma}-\gamma \eta) \alpha_{n} /\left(1-\alpha_{n} \gamma \eta\right), \delta_{n}=\left(\alpha_{n} \bar{\gamma}^{2}\left\|x_{n}-q\right\|^{2}+\right.$ $\left.M_{5}\left(\gamma_{n} / \alpha_{n}\right)\right) / 2(\bar{\gamma}-\gamma \eta)+(1 /(\bar{\gamma}-\gamma \eta))\left\langle\gamma f(q)-D q, x_{n+1}-q\right\rangle$.

According to (80), $\left(C_{1}\right),\left(C_{3}\right)$, and $\eta \gamma<\bar{\gamma}$, we have $\sum_{n=0}^{\infty} t_{n}=\infty$ and $\lim \sup _{n \rightarrow \infty} \delta_{n} \leq 0$.

By Lemma 7, $x_{n} \longrightarrow q$, which completes the proof.

## 4. Consequently Results

In the above section, we discuss the iterative algorithm and prove the strong convergence theorem for finding a common solution of split generalized equilibrium and variational inequality problems. In this section, we give some corollaries, which can find a common solution of the special issues obtained from split generalized equilibrium and variational inequality problems.

If $h_{1}=h_{2}=0$, then SGEP (5)-(6) reduces to the following split equilibrium problem (SEP).

Let $F_{1}: C \times C \longrightarrow R$ and $F_{2}: Q \times Q \longrightarrow R$ be nonlinear bifunctions and $A: H_{1} \longrightarrow H_{2}$ be a bounded linear operator; then SEP is to find $x^{*} \in C$ such that

$$
\begin{equation*}
F_{1}\left(x^{*}, x\right) \geq 0, \quad \forall x \in C \tag{110}
\end{equation*}
$$

and such that

$$
\begin{equation*}
y^{*}=A x^{*} \in Q \text { solves } F_{2}\left(y^{*}, y\right) \geq 0, \quad \forall y \in Q \tag{111}
\end{equation*}
$$

The solution set of SEP (110)-(111) is denoted by $\Gamma_{1}$. And

$$
\begin{align*}
& T_{r}^{F_{1}}(x)=\left\{z \in C: F_{1}(z, y)+\frac{1}{r}\langle y-z, z-x\rangle\right. \\
& \quad \geq 0, \forall y \in C\},  \tag{112}\\
& T_{s}^{F_{2}}(w)=\left\{d \in Q: F_{2}(d, e)+\frac{1}{s}\langle e-d, d-w\rangle\right. \\
& \quad \geq 0, \forall e \in Q\} . \tag{113}
\end{align*}
$$

According to Theorem 14, we can obtain the following corollary.

Corollary 15. Let $H_{1}$ and $H_{2}$ be two real Hilbert spaces and $C \subset H_{1}, Q \subset H_{2}$ be nonempty closed convex subsets. Let $F_{1}$ : $C \times C \longrightarrow R$ and $F_{2}: Q \times Q \longrightarrow R$ satisfy Assumption 2 and $F_{2}$ is upper semicontinuous in the first argument. Assume that
$\Omega:=\mathrm{VI}(C, B) \cap \Gamma_{1} \neq \emptyset, x_{0} \in H_{1}$ and $\left\{u_{n}\right\},\left\{y_{n}\right\}$, and $\left\{x_{n}\right\}$ are generated by the following iterative scheme:

$$
\begin{align*}
u_{n} & =T_{r_{n}}^{F_{1}}\left(x_{n}+\xi A^{*}\left(T_{r_{n}}^{F_{2}}-I\right) A x_{n}\right), \\
y_{n} & =u_{n}+\alpha d_{n+1}  \tag{114}\\
x_{n+1} & =\alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n}+\left(\left(1-\beta_{n}\right) I-\alpha_{n} D\right) y_{n}
\end{align*}
$$

where $d_{n+1}=(1 / \alpha)\left(T_{n} u_{n}-u_{n}\right)+\gamma_{n} d_{n}, d_{0}=(1 / \alpha)\left(T_{0} u_{0}-u_{0}\right)$, $\alpha>0$, and $T_{n}$ is defined by (34). Suppose that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\} \subset$ $(0,1),\left\{\gamma_{n}\right\} \subset(0,1 / 2),\left\{\lambda_{n}\right\} \subset(0,2 \beta),\left\{r_{n}\right\} \subset(0, \infty)$ satisfy conditions (C1)-(C6) in Theorem 14. Then sequence $\left\{x_{n}\right\}$ converges strongly to $q \in \Omega$, where $q=P_{\Omega}(I-D+\gamma f) q$, which is the unique solution of the variational inequality problem

$$
\begin{equation*}
\langle(D-\gamma f) q, x-q\rangle \geq 0, \quad \forall x \in \Omega \tag{115}
\end{equation*}
$$

or equivalently, $q$ is the unique solution to the minimization problem

$$
\begin{equation*}
\min _{x \in Q} \frac{1}{2}\langle D x, x\rangle-h(x) \tag{116}
\end{equation*}
$$

where $h$ is a potential function for $\gamma f$ such that $h^{\prime}(x)=\gamma f(x)$ for $x \in H_{1}$.

Furthermore, if $F_{1}=F_{2}=F, H_{1}=H_{2}=H$, and $A=$ 0 , then SEP (110)-(111) reduce to the following equilibrium problem (EP).

Let $F: C \times C \longrightarrow R$ be nonlinear bifunction; then EP is to find $x^{*} \in C$ such that

$$
\begin{equation*}
F\left(x^{*}, x\right) \geq 0, \quad \forall x \in C . \tag{117}
\end{equation*}
$$

The solution set of EP (117) is denoted by $\Gamma_{2}$. And

$$
\begin{align*}
& T_{r}^{F}(x)=\left\{z \in C: F(z, y)+\frac{1}{r}\langle y-z, z-x\rangle\right. \\
& \quad \geq 0, \forall y \in C\} . \tag{118}
\end{align*}
$$

According to Corollary 15 , let $D=I$, and we can obtain the following corollary.

Corollary 16. Let $H$ be real Hilbert space and $C \subset H$ be nonempty closed convex subset. Let $F: C \times C \longrightarrow R$ satisfy Assumption 2. Assume that $\Omega:=\mathrm{VI}(C, B) \cap \Gamma_{2} \neq \emptyset, x_{0} \in H_{1}$, and $\left\{u_{n}\right\},\left\{y_{n}\right\}$, and $\left\{x_{n}\right\}$ are generated by the following iterative scheme:

$$
\begin{align*}
u_{n} & =T_{r_{n}}^{F}\left(x_{n}\right) \\
y_{n} & =u_{n}+\alpha d_{n+1}  \tag{119}\\
x_{n+1} & =\alpha_{n} \gamma f\left(x_{n}\right)+\beta_{n} x_{n}+\left(1-\beta_{n}-\alpha_{n}\right) y_{n}
\end{align*}
$$

where $d_{n+1}=(1 / \alpha)\left(T_{n} u_{n}-u_{n}\right)+\gamma_{n} d_{n}, d_{0}=(1 / \alpha)\left(T_{0} u_{0}-u_{0}\right)$, $\alpha>0$, and $T_{n}$ is defined by (34). Suppose that $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\} \subset$ $(0,1),\left\{\gamma_{n}\right\} \subset(0,1 / 2),\left\{\lambda_{n}\right\} \subset(0,2 \beta),\left\{r_{n}\right\} \subset(0, \infty)$ satisfy conditions (C1)-(C6) in Theorem 14. Then sequence $\left\{x_{n}\right\}$ converges strongly to $q \in \Omega$, where $q=P_{\Omega} \gamma f(q)$.

Table 1: The final value and the cpu time for different initial points.

| Init. | Fina. | Sec. |
| :--- | :---: | :---: |
| $x=(0.7060,0.0318)^{T}$ | $\bar{x}=(4.9924,3.9939)^{T}$ | 1.14 |
| $x=(5.4722,1.3862)^{T}$ | $\bar{x}=(4.9924,3.9939)^{T}$ | 1.35 |
| $x=(89.0903,95.9291)^{T}$ | $\bar{x}=(4.9924,3.9939)^{T}$ | 1.36 |

## 5. Numerical Examples

In this section, we show some insight into the behavior of Algorithm 13. The whole codes are written in Matlab 7.0. All the numerical results are carried out on a personal Lenovo Thinkpad computer with $\operatorname{Intel}(\mathrm{R})$ Core(TM) i7-6500U CPU 2.50 GHz and RAM 8.00 GB .

Example 1. In the variational inequality problem (3) as well as the split generalized equilibrium problem (5) and (6), let $H_{1}=$ $H_{2}=R^{2}, C=\left\{\left(x_{1}, x_{2}\right)^{T} \in R^{2} \mid 1 \leq x_{1} \leq 5,0.5 \leq x_{2} \leq 4\right\}$, $Q=\left\{\left(y_{1}, y_{2}\right)^{T} \in R^{2} \mid 2 \leq y_{1} \leq 10,3 \leq y_{2} \leq 24\right\}, F_{1}(x, y)=$ $h_{1}(x, y)=e^{T}(x-y), \forall x, y \in C, F_{2}(x, y)=h_{2}(x, y)=5 e^{T}(x-$ $y), \forall x, y \in Q$, and $A=\left(\begin{array}{cc}2 & 0 \\ 0 & 6\end{array}\right), B=\left(\begin{array}{cc}1 / 2 & 0 \\ 0 & 1 / 2\end{array}\right)$, where $e=(1,1)^{T}$.

It is easy to see that the optimal solution of Example 1 is $x^{*}=(5,4)^{T}$. Now, we use Algorithm 13 to compute the solution of the problem. Let $\alpha=1 / 2, \xi=1 / 7, D=\left(\begin{array}{cc}9 / 10 & 0 \\ 0 & 9 / 10\end{array}\right)$, $\gamma=1.2, r_{n}=1 / 10+1 / 2 n, \lambda_{n}=1 / n, \alpha_{n}=1 / \sqrt{n}, \beta_{n}=$ $1 / 10+1 / n^{2}, \gamma_{n}=1 / n^{2}$, and $f(x)=x / 2, \forall x \in H_{1}$. The stopping criterion is $\left\|x_{n+1}-x_{n}\right\| \leq \epsilon$. For $\epsilon=10^{-7}$ and different initial points which are presented randomly, such as

$$
\begin{align*}
& x=\operatorname{rand}(2,1), \\
& x=10 * \operatorname{rand}(2,1),  \tag{120}\\
& x=1000 * \operatorname{rand}(2,1),
\end{align*}
$$

separately. Table 1 shows the initial value, the final value, and the cpu time for the above three cases. We denote Init., Fina., and Sec. the initial value, the final value, and the cpu time in seconds, respectively.

From Table 1, we can see that the final value $\bar{x}$ is not influenced by the initial value.

To show the changing tendency of the final value $\bar{x}$ for different $\epsilon$, Table 2 gives the different $\epsilon$, the initial value, the final value, and the cpu time in seconds.

Now, we further express the status that the final value $\bar{x}$ tends to optimal solution $x^{*}$ through Figures 1 and 2. We carry out 100 experiments for different $\epsilon$ from $10^{-7}$ to $10^{-9}$. From Figures 1 and 2, we know that the final value $\bar{x}$ tends to the optimal solution $x^{*}$ when $\epsilon$ tends to 0 , which illustrates the efficiency of Algorithm 13.

## 6. Conclusion

In this paper, we study the split generalized equilibrium problem and variational inequality problem. For finding their common solution, we propose a kind of conjugate gradient viscosity approximation algorithm. Under mild conditions, we prove that the sequence generated by the

Table 2: The final value and the cpu time for different $\epsilon$.

| $\epsilon$ | Init. | Fina. | Sec. |
| :--- | :--- | :--- | :--- |
| $10^{-5}$ | $x=(8.0028,1.4189)^{T}$ | $\bar{x}=(4.9640,3.9712)^{T}$ | 0.08 |
| $10^{-6}$ | $x=(7.4313,3.9223)^{T}$ | $\bar{x}=(4.9835,3.9868)^{T}$ | 0.30 |
| $10^{-7}$ | $x=(5.4722,1.3862)^{T}$ | $\bar{x}=(4.9924,3.9939)^{T}$ | 1.35 |



Figure 1: The behaviors of first coordinate of $\bar{x}$ for different $\epsilon$.


Figure 2: The behaviors of the second coordinate of $\bar{x}$ for different $\epsilon$.
iterative algorithm converges strongly to the common solution. In comparison to [25], the authors introduce and study an effective iterative algorithm to approximate a common solution of a split generalized equilibrium problem and a fixed point problem. Under suitable conditions, they proved a strong convergence theorem for the sequence generated by the iterative scheme. Furthermore, we can study the iterative algorithm for finding a common solution of the split generalized equilibrium problem, variational inequality problem, and fixed point problem.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Some Properties for Solutions of Riemann-Liouville Fractional Differential Systems with a Delay 

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#### Abstract

In this paper, we study properties for solutions of Riemann-Liouville ( $\mathrm{R}-\mathrm{L}$ ) fractional differential systems with a delay. Some results on integral inequalities are first presented by Hölder inequality. Then we investigate properties on solutions for R-L fractional systems with a delay by using the obtained inequalities and obtain upper bound of solutions. Finally, an illustrative example is considered to support our new results.


## 1. Introduction

Fractional differential equations have been studied for several centuries. At first the researches were only on the pure theoretical aspect. In the last few years, more and more fractional differential equations have been applied to described some actual researches, such as mechanics, aerodynamics, chemistry, and the electrodynamics of complex mediums [18].

Integral inequalities play an important role in researches not only on properties of solutions for various differential and integral equations [9-12], but also on some fractional differential equations. Recently, some results are obtained on properties of solutions for a fractional differential equation with or without delays. For example, Ma [13] obtained upper bounds for solutions of a class of nonlinear fractional differential systems by a result of two dimensional linear integral inequalities. Ye [14] studied the Henry-Gronwall type retarded integral inequalities and then obtained a certain properties of fractional differential equations with delay. This paper studies some properties for solutions of R-L fractional differential systems with a delay. First, we obtain some results on the integral inequalities by Hölder inequality. Then, using the obtained inequalities, properties are investigated on solutions for R-L fractional systems with a delay, and upper bound of solutions is obtained. Moreover, an illustrative
example is studied to show that new results presented in this paper work very well.

## 2. Main Results

This section is devoted to studying properties of solutions for R-L fractional differential systems with a delay and presents the main result of this paper. First, we give some lemmas on integral inequalities.

Lemma 1 (let $I=\left[t_{0},+\infty\right)$ and $\mathbb{R}_{+}=[0,+\infty)$ ). Suppose $a_{i}(t), b_{i}(t)$, and $c_{i}(t) \in C\left(I, \mathbb{R}_{+}\right), \phi_{i}(t) \in C\left(\left[t_{0}-\tau, t_{0}\right], \mathbb{R}_{+}\right)$, $a_{i}\left(t_{0}\right)=\phi_{i}\left(t_{0}\right), i=1,2$, and $\tau>0$ is a contant. If $u_{i}(t) \in$ $C\left(\left[t_{0}-\tau,+\infty\right), \mathbb{R}_{+}\right)$and

$$
\begin{aligned}
u_{1}(t) \leq & a_{1}(t) \\
& +\int_{t_{0}}^{t}\left[b_{1}(s) u_{1}(s-\tau)+c_{1}(s) u_{2}(s-\tau)\right] d s, \\
& t \in\left[t_{0},+\infty\right), \\
u_{2}(t) \leq & a_{2}(t) \\
& +\int_{t_{0}}^{t}\left[b_{2}(s) u_{1}(s-\tau)+c_{2}(s) u_{2}(s-\tau)\right] d s, \\
& t \in\left[t_{0},+\infty\right),
\end{aligned}
$$

$$
\begin{array}{ll}
u_{1}(t) \leq \phi_{1}(t), & t \in\left[t_{0}-\tau, t_{0}\right)  \tag{2}\\
u_{2}(t) \leq \phi_{2}(t), & t \in\left[t_{0}-\tau, t_{0}\right)
\end{array}
$$

$$
\left[\begin{array}{l}
u_{1}(t) \\
u_{2}(t)
\end{array}\right] \leq A(t)+F(t), \quad t \in I,
$$

(1)
then ।
$F(t)$

$$
= \begin{cases}\exp \left\{\int_{t_{0}+\tau}^{t} H(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} H(s) \Phi(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi, & t \in\left[t_{0}+\tau,+\infty\right)  \tag{3}\\ \int_{t_{0}}^{t} H(s) \Phi(s-\tau) d s & t \in\left[t_{0}, t_{0}+\tau\right]\end{cases}
$$

$H(t)=\left[\begin{array}{ll}b_{1}(t) & c_{1}(t) \\ b_{2}(t) & c_{2}(t)\end{array}\right]$,
$\Phi(t)=\left[\begin{array}{l}\phi_{1}(t) \\ \phi_{2}(t)\end{array}\right]$.

Proof. Set $v_{1}(t)=\int_{t_{0}}^{t}\left[b_{1}(s) u_{1}(s-\tau)+c_{1}(s) u_{2}(s-\tau)\right] d s, v_{2}(t)=$ $\int_{t_{0}}^{t}\left[b_{2}(s) u_{1}(s-\tau)+c_{2}(s) u_{2}(s-\tau)\right] d s$. Then, $u_{i}(t) \leq a_{i}(t)+v_{i}(t)$, $v_{i}(t) \geq 0(i=1,2)$, are nondecreasing for $t \in\left[t_{0},+\infty\right)$. Hence, for $t \in\left[t_{0}+\tau,+\infty\right)$, we have

$$
\begin{align*}
v_{1}^{\prime}(t)= & b_{1}(t) u_{1}(t-\tau)+c_{1}(t) u_{2}(t-\tau) \\
\leq & b_{1}(t)\left[a_{1}(t-\tau)+v_{1}(t-\tau)\right] \\
& +c_{1}(t)\left[a_{2}(t-\tau)+v_{2}(t-\tau)\right] \\
= & {\left[b_{1}(t) a_{1}(t-\tau)+c_{1}(t) a_{2}(t-\tau)\right] }  \tag{5}\\
& +b_{1}(t) v_{1}(t-\tau)+c_{1}(t) v_{2}(t-\tau) \\
\leq & {\left[b_{1}(t) a_{1}(t-\tau)+c_{1}(t) a_{2}(t-\tau)\right] } \\
& +b_{1}(t) v_{1}(t)+c_{1}(t) v_{2}(t), \\
v_{2}^{\prime}(t)= & b_{2}(t) u_{1}(t-\tau)+c_{2}(t) u_{2}(t-\tau) \\
\leq & b_{2}(t)\left[a_{1}(t-\tau)+v_{1}(t-\tau)\right] \\
& +c_{2}(t)\left[a_{2}(t-\tau)+v_{2}(t-\tau)\right] \\
= & {\left[b_{2}(t) a_{1}(t-\tau)+c_{2}(t) a_{2}(t-\tau)\right] }  \tag{6}\\
& +b_{2}(t) v_{1}(t-\tau)+c_{2}(t) v_{2}(t-\tau) \\
\leq & {\left[b_{2}(t) a_{1}(t-\tau)+c_{2}(t) a_{2}(t-\tau)\right] } \\
& +b_{2}(t) v_{1}(t)+c_{2}(t) v_{2}(t) .
\end{align*}
$$

Set $W(t)=\left[\begin{array}{l}v_{1}(t) \\ v_{2}(t)\end{array}\right] .(5)$ and (6) can be rewritten as a matrix form

$$
\begin{equation*}
W^{\prime}(t) \leq H(t) A(t-\tau)+H(t) W(t) . \tag{7}
\end{equation*}
$$

Then we have

$$
\begin{align*}
& {\left[\exp \left\{-\int_{t_{0}+\tau}^{t} H(s) d s\right\} W(t)\right]^{\prime}}  \tag{8}\\
& \quad \leq \exp \left\{-\int_{t_{0}+\tau}^{t} H(s) d s\right\} H(t) A(t-\tau)
\end{align*}
$$

Hence,

$$
\begin{align*}
& \int_{t_{0}+\tau}^{t}\left[\exp \left\{-\int_{t_{0}+\tau}^{\xi} H(s) d s\right\} W(\xi)\right]^{\prime} d \xi  \tag{9}\\
& \quad \leq \int_{t_{0}+\tau}^{t}\left[\exp \left\{-\int_{t_{0}+\tau}^{\xi} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi
\end{align*}
$$

That is,

$$
\begin{align*}
& \exp \left\{-\int_{t_{0}+\tau}^{t} H(s) d s\right\} W(t)-W\left(t_{0}+\tau\right) \\
& \quad \leq \int_{t_{0}+\tau}^{t}\left[\exp \left\{-\int_{t_{0}+\tau}^{\xi} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi \tag{10}
\end{align*}
$$

Then
$W(t)$

$$
\begin{align*}
& \leq \exp \left\{\int_{t_{0}+\tau}^{t} H(s) d s\right\} W\left(t_{0}+\tau\right)  \tag{11}\\
& \quad+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi
\end{align*}
$$

When $t \in\left[t_{0}, t_{0}+\tau\right]$, from (1),

$$
W(t)=\left[\begin{array}{l}
v_{1}(t) \\
v_{2}(t)
\end{array}\right]
$$

$$
\leq\left[\begin{array}{l}
\int_{t_{0}}^{t}\left[b_{1}(s) \phi_{1}(s-\tau)+c_{1}(s) \phi_{2}(s-\tau)\right] d s  \tag{12}\\
\int_{t_{0}}^{t}\left[b_{2}(s) \phi_{1}(s-\tau)+c_{2}(s) \phi_{2}(s-\tau)\right] d s
\end{array}\right] \quad=\int_{t_{0}}^{t} H(s) \Phi(s-\tau) d s
$$

$F(t)$

$$
= \begin{cases}\exp \left\{\int_{t_{0}+\tau}^{t} H(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} H(s) \Phi(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi, & t \in\left[t_{0}+\tau,+\infty\right)  \tag{13}\\ \int_{t_{0}}^{t} H(s) \Phi(s-\tau) d s & t \in\left[t_{0}, t_{0}+\tau\right]\end{cases}
$$

Thus,

$$
\left[\begin{array}{l}
u_{1}(t)  \tag{14}\\
u_{2}(t)
\end{array}\right] \leq A(t)+W(t) \leq A(t)+F(t), \quad t \in I
$$

Therefore, (2) is satisfied and the proof is completed.

$$
\begin{align*}
& \quad \cdot\left[b_{2}(s) u_{1}(s-\tau)+c_{2}(s) u_{2}(s-\tau)\right] d s, \\
& t \in\left[t_{0},+\infty\right), \\
& u_{1}(t) \leq \phi_{1}(t), \quad t \in\left[t_{0}-\tau, t_{0}\right), \\
& u_{2}(t) \leq \phi_{2}(t), \quad t \in\left[t_{0}-\tau, t_{0}\right), \tag{15}
\end{align*}
$$

Lemma 2. Suppose $a_{i}(t), b_{i}(t)$, and $c_{i}(t) \in C\left(I, \mathbb{R}_{+}\right), \phi_{i}(t) \in$ $C\left(\left[t_{0}-\tau, t_{0}\right], \mathbb{R}_{+}\right), a_{i}\left(t_{0}\right)=\phi_{i}\left(t_{0}\right), i=1,2$, and $\tau>0$ and $0<\beta<1$ are constants. If $u_{i}(t) \in C\left(\left[t_{0}-\tau,+\infty\right), \mathbb{R}_{+}\right)$and

$$
\begin{aligned}
& u_{1}(t) \leq a_{1}(t)+\int_{t_{0}}^{t}(t-s)^{\beta-1} \\
& \quad \cdot\left[b_{1}(s) u_{1}(s-\tau)+c_{1}(s) u_{2}(s-\tau)\right] d s, \\
& \quad t \in\left[t_{0},+\infty\right), \\
& u_{2}(t) \leq a_{2}(t)+\int_{t_{0}}^{t}(t-s)^{\beta-1}
\end{aligned}
$$

where
then,
(i) when $1 / 2<\beta<1$, set

$$
\left[\begin{array}{l}
u_{1}(t)  \tag{16}\\
u_{2}(t)
\end{array}\right]=\left[\begin{array}{ll}
e^{t} w_{1}^{1 / 2} & (t) \\
e^{t} w_{2}^{1 / 2} & (t)
\end{array}\right]
$$

and we have

$$
W(t)=\left[\begin{array}{l}
w_{1}(t)  \tag{17}\\
w_{2}(t)
\end{array}\right] \leq A(t)+Q(t)
$$

$Q(t)$

$$
=\left\{\begin{array}{l}
\exp \left\{\int_{t_{0}+\tau}^{t} H(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} H(s) \Psi(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi  \tag{18}\\
\int_{t_{0}}^{t} H(s) \Psi(s-\tau) d s
\end{array}\right.
$$

$A(t)=\left[\begin{array}{l}\alpha_{1}(t) \\ \alpha_{2}(t)\end{array}\right]$,
$H(t)=\left[\begin{array}{ll}l_{1}(t) & m_{1}(t) \\ l_{2}(t) & m_{2}(t)\end{array}\right]$,
$\Psi(t)\left[\begin{array}{l}\psi_{1}(t) \\ \psi_{2}(t)\end{array}\right]=\left[\begin{array}{l}e^{-2 t} \phi_{1}^{2}(t) \\ e^{-2 t} \phi_{2}^{2}(t)\end{array}\right]$,
and $\alpha_{i}(t)=2 e^{-2 t} a_{i}^{2}(t), l_{i}(t)=\left(\Gamma(2 \beta-1) / 2^{2 \beta-3}\right) e^{-2 \tau} b_{i}^{2}(t), \quad$ and we have $m_{i}(t)=\left(\Gamma(2 \beta-1) / 2^{2 \beta-3}\right) e^{-2 \tau} c_{i}^{2}(t), i=1,2$;
(ii) when $0<\beta \leq 1 / 2$, let $p=1+\beta, q=1+1 / \beta$

$$
\left[\begin{array}{l}
u_{1}(t)  \tag{20}\\
u_{2}(t)
\end{array}\right]=\left[\begin{array}{l}
e^{t} \widetilde{w}_{1}^{1 / q}(t) \\
e^{t} \widetilde{w}_{2}^{1 / q}(t)
\end{array}\right]
$$

$\widetilde{\mathrm{Q}}(t)$

$$
\begin{align*}
&= \begin{cases}\exp \left\{\int_{t_{0}+\tau}^{t} \widetilde{H}(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} \widetilde{H}(s) \widetilde{\Psi}(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} \widetilde{H}(s) d s\right\} \widetilde{H}(\xi) \widetilde{A}(\xi-\tau)\right] d \xi, & t \in\left[t_{0}+\tau,+\infty\right), \\
\int_{t_{0}}^{t} \widetilde{H}(s) \widetilde{\Psi}(s-\tau) d s, & t \in\left[t_{0}, t_{0}+\tau\right]\end{cases}  \tag{22}\\
& \widetilde{A}(t)=\left[\begin{array}{l}
\widetilde{\alpha}_{1}(t) \\
\widetilde{\alpha}_{2}(t)
\end{array}\right], \\
& \widetilde{H}(t)=\left[\begin{array}{ll}
\widetilde{l}_{1}(t) & \widetilde{m}_{1}(t) \\
\widetilde{l}_{2}(t) & \widetilde{m}_{2}(t)
\end{array}\right],  \tag{23}\\
& \widetilde{\Psi}(t)=\left[\begin{array}{l}
\widetilde{\psi}_{1}(t) \\
\widetilde{\psi}_{2}(t)
\end{array}\right]=\left[\begin{array}{l}
e^{-q t} \phi_{1}^{q}(t) \\
e^{-q t} \phi_{2}^{q}(t)
\end{array}\right]
\end{align*}
$$

where
and $\widetilde{\alpha}_{i}(t)=2^{q-1}\left[e^{-t} a_{i}(t)\right]^{q}, \widetilde{l}_{i}(t)=\left(4^{1 / \beta} \Gamma^{1 / \beta}\left(\beta^{2}\right) / p^{\beta}\right) e^{-q \tau} b_{i}^{q}(t)$, $\widetilde{m}_{i}(t)=\left(4^{1 / \beta} \Gamma^{1 / \beta}\left(\beta^{2}\right) / p^{\beta}\right) e^{-q \tau} c_{i}^{q}(t), i=1,2$.

Proof. (i) Suppose $\beta>1 / 2$. By Hölder inequality and ( $a+$ $b)^{2} \leq 2\left(a^{2}+b^{2}\right)$, for $t \in\left[t_{0}+\tau,+\infty\right)$ and $i=1,2$, we have

$$
\begin{aligned}
& u_{i}(t) \leq a_{i}(t)+\int_{t_{0}}^{t}(t-s)^{\beta-1} e^{s} e^{-s}\left[b_{i}(s) u_{1}(s-\tau)\right. \\
& \left.\quad+c_{i}(s) u_{2}(s-\tau)\right] d s \leq a_{i}(t)+\left[\int_{t_{0}}^{t}(t\right. \\
& \left.\quad-s)^{2(\beta-1)} e^{2 s} d s\right]^{1 / 2}\left\{\int _ { t _ { 0 } } ^ { t } e ^ { - 2 s } \left[b_{i}(s) u_{1}(s-\tau)\right.\right. \\
& \left.\left.\quad+c_{i}(s) u_{2}(s-\tau)\right]^{2} d s\right\}^{1 / 2} \leq a_{i}(t) \\
& \quad+2^{1 / 2}\left[\int_{t_{0}}^{t}(t-s)^{2(\beta-1)} e^{2 s} d s\right]^{1 / 2} \\
& \quad .\left\{\int _ { t _ { 0 } } ^ { t } \left[b_{i}^{2}(s) e^{-2 s} u_{1}^{2}(s-\tau)\right.\right. \\
& \left.\left.\quad+c_{i}^{2}(s) e^{-2 s} u_{2}^{2}(s-\tau)\right] d s\right\}^{1 / 2}=a_{i}(t) \\
& \quad+2^{1 / 2}\left[-\int_{t-t_{0}}^{0} s^{2(\beta-1)} e^{2(t-s)} d s\right]^{1 / 2}
\end{aligned}
$$

$$
\widetilde{W}(t)=\left[\begin{array}{l}
\widetilde{w}_{1}(t)  \tag{21}\\
\widetilde{w}_{2}(t)
\end{array}\right] \leq \widetilde{A}(t)+\widetilde{Q}(t),
$$

$$
\begin{align*}
& +\left\{\int _ { t _ { 0 } } ^ { t } \left[b_{i}^{2}(s) e^{-2 s} u_{1}^{2}(s-\tau)\right.\right. \\
& \left.\left.+c_{i}^{2}(s) e^{-2 s} u_{2}^{2}(s-\tau)\right] d s\right\}^{1 / 2} \leq a_{i}(t) \\
& +\frac{e^{t}}{2^{\beta-1}}\left[\int_{0}^{+\infty}(2 s)^{(2 \beta-1)-1} e^{-2 s} d(2 s)\right]^{1 / 2} \\
& \cdot\left\{\int _ { t _ { 0 } } ^ { t } \left[b_{i}^{2}(s) e^{-2 s} u_{1}^{2}(s-\tau)\right.\right. \\
& \left.\left.+c_{i}^{2}(s) e^{-2 s} u_{2}^{2}(s-\tau)\right] d s\right\}^{1 / 2} \tag{24}
\end{align*}
$$

Thus,

$$
\begin{align*}
& u_{i}^{2}(t) \leq 2 a_{i}^{2}(t)+\frac{e^{2 t}}{2^{2 \beta-3}} \Gamma(2 \beta-1) \\
& \quad \cdot \int_{t_{0}}^{t}\left[b_{i}^{2}(s) e^{-2 s} u_{1}^{2}(s-\tau)\right.  \tag{25}\\
& \left.\quad+c_{i}^{2}(s) e^{-2 s} u_{2}^{2}(s-\tau)\right] d s, \quad(i=1,2)
\end{align*}
$$

Let $w_{i}(t)=\left[e^{-t} u_{i}(t)\right]^{2}, \alpha_{i}(t)=2 e^{-2 t} a_{i}^{2}(t), l_{i}(t)=(\Gamma(2 \beta-$ 1) $\left./ 2^{2 \beta-3}\right) e^{-2 \tau} b_{i}^{2}(t), m_{i}(t)=\left(\Gamma(2 \beta-1) / 2^{2 \beta-3}\right) e^{-2 \tau} c_{i}^{2}(t), i=$ 1,2 . Then, for $t \in\left[t_{0},+\infty\right)$, we have

$$
\begin{align*}
w_{i}(t) \leq & \alpha_{i}(t) \\
& +\int_{t_{0}}^{t}\left[l_{i}(t) w_{1}(s-\tau)+m_{i}(t) w_{2}(s-\tau)\right] d s \tag{26}
\end{align*}
$$

Set $\psi_{i}(t)=e^{-2 t} \phi_{i}^{2}(t)(i=1,2)$. For $t \in\left[t_{0}-\tau, t_{0}\right]$

$$
\begin{equation*}
w_{i}(t) \leq \psi_{i}(t) \quad(i=1,2) . \tag{27}
\end{equation*}
$$

By (26), (27), and Lemma 1, (17) is satisfied.
(ii) Suppose $0<\beta \leq 1 / 2$. Set $p=\beta+1$ and $q=1+1 / \beta$.

Then $1 / p+1 / q=1$. By Hölder inequality and $(a+b)^{l} \leq$ $2^{l-1}\left(a^{l}+b^{l}\right)(0<l<1)$, for $t \in\left[t_{0}+\tau,+\infty\right)$ and $i=1,2$, we have

$$
\begin{align*}
& u_{i}(t) \leq a_{i}(t)+\int_{t_{0}}^{t}(t-s)^{\beta-1} e^{s} e^{-s}\left[b_{i}(s) u_{1}(s-\tau)\right. \\
& \left.\quad+c_{i}(s) u_{2}(s-\tau)\right] d s \leq a_{i}(t)+\left[\int_{t_{0}}^{t}(t\right. \\
& \left.-s)^{p(\beta-1)} e^{p s} d s\right]^{1 / p}\left\{\int _ { t _ { 0 } } ^ { t } e ^ { - q s } \left[b_{i}(s) u_{1}(s-\tau)\right.\right. \\
& \left.\left.\quad+c_{i}(s) u_{2}(s-\tau)\right]^{q} d s\right\}^{1 / q} \leq a_{i}(t) \\
& \quad+2^{1 / p} e^{t}\left[\int_{0}^{\infty} s^{\beta^{2}-1} e^{-p s} d s\right]^{1 / p}\left\{\int _ { t _ { 0 } } ^ { t } \left[b_{i}^{q}(s)\right.\right.  \tag{28}\\
& \left.\left.\quad \cdot e^{-q s} u_{1}^{q}(s-\tau)+c_{i}^{q}(s) e^{-q s} u_{2}^{q}(s-\tau)\right] d s\right\}^{1 / q} \\
& \quad \leq a_{i}(t)+\frac{2^{1 / p} e^{t} \Gamma^{1 / p}\left(\beta^{2}\right)}{p^{\beta^{2} / p}}\left\{\int _ { t _ { 0 } } ^ { t } \left[b_{i}^{q}(s) e^{-q s} u_{1}^{q}(s-\tau)\right.\right. \\
& \left.\left.\quad+c_{i}^{q}(s) e^{-q s} u_{2}^{q}(s-\tau)\right] d s\right\}^{1 / q} .
\end{align*}
$$

Thus,

$$
\begin{align*}
& {\left[e^{-t} u_{i}(t)\right]^{q} \leq 2^{q-1}\left[e^{-t} a_{i}(t)\right]^{q}+\frac{2^{2 / \beta} \Gamma^{1 / \beta}\left(\beta^{2}\right)}{p^{\beta}}} \\
& \quad \cdot \int_{t_{0}}^{t}\left[b_{i}^{q}(s) e^{-q s} u_{1}^{q}(s-\tau)\right.  \tag{29}\\
& \left.\quad+c_{i}^{q}(s) e^{-q s} u_{2}^{q}(s-\tau)\right] d s, \quad i=1,2 .
\end{align*}
$$

Let $\widetilde{w}_{i}(t)=\left[e^{-t} u_{i}(t)\right]^{q}, \widetilde{\alpha}_{i}(t)=2^{q-1}\left[e^{-t} a_{i}(t)\right]^{q}, \widetilde{l}_{i}(t)=$ $\left(4^{1 / \beta} \Gamma^{1 / \beta}\left(\beta^{2}\right) / p^{\beta}\right) e^{-q \tau} b_{i}^{q}(t), \quad \widetilde{m}_{i}(t)=\left(4^{1 / \beta} \Gamma^{1 / \beta}\left(\beta^{2}\right) /\right.$ $\left.p^{\beta}\right) e^{-q \tau} b_{i}^{q}(t), i=1,2$. Then, we have

$$
\begin{align*}
& \widetilde{w}_{i}(t) \leq \\
& \widetilde{\alpha}_{i}(t)  \tag{30}\\
&+\int_{t_{0}}^{t}\left[\widetilde{l}_{i}(s) \widetilde{w}_{1}(s-\tau)+\widetilde{m}_{i}(s) \widetilde{w}_{2}(s-\tau)\right] d s \\
& t \in\left[t_{0},+\infty\right) .
\end{align*}
$$

Set $\widetilde{\psi_{i}}(t)=e^{-q t} \phi_{i}^{q}(t)(i=1,2)$. For $t \in\left[t_{0}-\tau, t_{0}\right]$,

$$
\begin{equation*}
\widetilde{w}_{i}(t) \leq \widetilde{\psi}_{i}(t) \quad(i=1,2) . \tag{31}
\end{equation*}
$$

By (30), (31), and Lemma 1, (21) is satisfied and the proof is completed.

Now, we introduce some definitions on RiemannLiouville fractional derivative and fractional primitive.

Definition 3 (see [2]). The fractional derivative of order $0<$ $\alpha<1$ of a function $x(t) \in C(\mathbb{R}, \mathbb{R})$ is given by

$$
\begin{equation*}
D_{\alpha}=\frac{1}{\Gamma(1-\alpha)} \frac{d}{d x} \int_{a}^{t}(s-t)^{-\alpha} x(s) d s \tag{32}
\end{equation*}
$$

Definition 4 (see [2]). The fractional derivative of order $0<$ $\alpha<1$ of a function $x(t) \in C(\mathbb{R}, \mathbb{R})$ is given by

$$
\begin{equation*}
I_{\alpha}=\frac{1}{\Gamma(\alpha)} \int_{a}^{t}(s-t)^{\alpha-1} x(s) d s \tag{33}
\end{equation*}
$$

Consider the following Riemann-Liouville fractional differential system with a delay:

$$
\begin{align*}
D^{\alpha} x(t) & =f(t, x(t-\tau), y(t-\tau)), \\
D^{\alpha} y(t) & =g(t, x(t-\tau), y(t-\tau)), \\
& t \in\left[t_{0},+\infty\right),  \tag{34}\\
D^{\alpha-1} x(t) & =\xi \\
D^{\alpha-1} y(t) & =\eta,  \tag{35}\\
& t \in\left[t_{0}-\tau, t_{0}\right]
\end{align*}
$$

where $0<\alpha<1, \tau>0, \xi, \eta$ are constants, and $f, g$ : $\left[t_{0},+\infty\right) \times \mathbb{R}^{2} \longrightarrow \mathbb{R}$ are continuous functions. Then, a result can be obtained on the solutions of system (34).

Theorem 5. Consider system (34). $f(t, x, y)$ and $g(t, x, y) \in$ $C\left(\left[t_{0},+\infty\right) \times \mathbb{R}^{2}, \mathbb{R}\right)$ and satisfy the following condition:

$$
\begin{align*}
& |f(t, x, y)| \leq b_{1}(t)|x|+c_{1}(t)|y|  \tag{36}\\
& |g(t, x, y)| \leq b_{2}(t)|x|+c_{2}(t)|y|
\end{align*}
$$

where $b_{i}(t)$ and $c_{i}(t) \in C\left(\left[t_{0},+\infty\right), \mathbb{R}_{+}\right)$. Then, solutions of system (34) satisfy that
(i) when $1 / 2<\alpha<1$,

$$
\begin{align*}
& {\left[\begin{array}{l}
|x(t)| \\
|y(t)|
\end{array}\right]=\left[\begin{array}{c}
e^{t} w_{1}^{1 / 2}(t) \\
e^{t} w_{2}^{1 / 2}(t)
\end{array}\right],}  \tag{37}\\
& {\left[\begin{array}{l}
w_{1}(t) \\
w_{2}(t)
\end{array}\right] \leq A(t)+R(t),} \tag{38}
\end{align*}
$$

where
$R(t)$

$$
= \begin{cases}\exp \left\{\int_{t_{0}+\tau}^{t} H(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} H(s) \Psi(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-\tau)\right] d \xi, & t \in\left[t_{0}+\tau,+\infty\right),  \tag{39}\\ \int_{t_{0}}^{t} H(s) \Psi(s-\tau) d s, & t \in\left[t_{0}, t_{0}+\tau\right],\end{cases}
$$

$\Psi(t)=\left[\begin{array}{l}e^{-2 t} a_{1}^{2}(t) \\ e^{-2 t} a_{2}^{2}(t)\end{array}\right]$,
$A(t)=\left[\begin{array}{l}\alpha_{1}(t) \\ \alpha_{2}(t)\end{array}\right]$,
$H(t)=\left[\begin{array}{ll}l_{1}(t) & m_{1}(t) \\ l_{2}(t) & m_{2}(t)\end{array}\right]$,
$a_{1}(t)=(\xi / \Gamma(\alpha)) t^{\alpha-1}, a_{2}(t)=(\eta / \Gamma(\alpha)) t^{\alpha-1}$ and $\alpha_{i}(t)=$ $2 e^{-2 t} a_{i}^{2}(t), l_{i}(t)=\left(\Gamma(2 \alpha-1) / 2^{2 \alpha-3}\right) e^{-2 \tau} b_{i}^{2}(t), m_{i}(t)=(\Gamma(2 \alpha-$ 1) $\left./ 2^{2 \alpha-3}\right) e^{-2 \tau} c_{i}^{2}(t), i=1,2$;
(ii) when $0<\alpha \leq 1 / 2$,

$$
\begin{align*}
& {\left[\begin{array}{l}
|x(t)| \\
|y(t)|
\end{array}\right]=\left[\begin{array}{l}
e^{t} \widetilde{w}_{1}^{1 / q}(t) \\
e^{t} \widetilde{w}_{2}^{1 / q}(t)
\end{array}\right],}  \tag{41}\\
& {\left[\begin{array}{l}
\widetilde{w}_{1}(t) \\
\widetilde{w}_{2}(t)
\end{array}\right] \leq \widetilde{A}(t)+\widetilde{R}(t),} \tag{42}
\end{align*}
$$

where
$\widetilde{R}(t)$

$$
= \begin{cases}\exp \left\{\int_{t_{0}+\tau}^{t} \widetilde{H}(s) d s\right\} \int_{t_{0}}^{t_{0}+\tau} \widetilde{H}(s) \widetilde{\Psi}(s-\tau) d s+\int_{t_{0}+\tau}^{t}\left[\exp \left\{\int_{\xi}^{t} \widetilde{H}(s) d s\right\} \widetilde{H}(\xi) \widetilde{A}(\xi-\tau)\right] d \xi, & t \in\left[t_{0}+\tau,+\infty\right),  \tag{43}\\ \int_{t_{0}}^{t} \widetilde{H}(s) \widetilde{\Psi}(s-\tau) d s, & t \in\left[t_{0}, t_{0}+\tau\right],\end{cases}
$$

$\widetilde{\Psi}(t)=\left[\begin{array}{l}\widetilde{\psi}_{1}(t) \\ \widetilde{\psi}_{2}(t)\end{array}\right]=\left[\begin{array}{l}e^{-q t} \phi_{1}^{q}(t) \\ e^{-q t} \phi_{2}^{q}(t)\end{array}\right]$
$p=1+\alpha, q=1+1 / \alpha$ and $\widetilde{\alpha}_{i}(t)=2^{q-1}\left[e^{-t} a_{i}(t)\right]^{q}$, $y(t)$
$\widetilde{l}_{i}(t) \quad=\quad\left(4^{1 / \alpha} \Gamma^{1 / \alpha}\left(\alpha^{2}\right) / p^{\alpha}\right) e^{-q \tau} b_{i}^{q}(t), \quad \widetilde{m}_{i}(t) \quad=$ $\left(4^{1 / \alpha} \Gamma^{1 / \alpha}\left(\alpha^{2}\right) / p^{\alpha}\right) e^{-q \tau} c_{i}^{q}(t), i=1,2$.

$$
=\frac{\eta}{\Gamma(\alpha)} t^{\alpha-1}
$$

$$
+\frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t}(t-s)^{\alpha-1} g(s, x(s-\tau), y(s-\tau)) d s
$$

$$
=\frac{\xi}{\Gamma(\alpha)} t^{\alpha-1}
$$

$$
t \in\left[t_{0},+\infty\right)
$$

$$
+\frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t}(t-s)^{\alpha-1} f(s, x(s-\tau), y(s-\tau)) d s, \quad x(t)=\frac{\xi}{\Gamma(\alpha)} t^{\alpha-1}
$$

$$
y(t)=\frac{\eta}{\Gamma(\alpha)} t^{\alpha-1}
$$

$$
\begin{equation*}
t \in\left[t_{0}-\tau, t_{0}\right] . \tag{45}
\end{equation*}
$$

By condition (36), we have

$$
\begin{align*}
& |x(t)| \leq \frac{|\xi|}{\Gamma(\alpha)} t^{\alpha-1}+\frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t}(t-s)^{\alpha-1} \\
& \quad \cdot\left[b_{1}(s)|x(s-\tau)|+c_{1}(s)|y(s-\tau)|\right] d s, \\
& |y(t)| \leq \frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}+\frac{1}{\Gamma(\alpha)} \int_{t_{0}}^{t}(t-s)^{\alpha-1} \\
& \quad \cdot\left[b_{2}(s)|x(s-\tau)|+c_{2}(s)|y(s-\tau)|\right] d s,  \tag{46}\\
& t \in\left[t_{0},+\infty\right), \\
& |x(t)|=\frac{|\xi|}{\Gamma(\alpha)} t^{\alpha-1}, \\
& |y(t)|=\frac{|\eta|}{\Gamma(\alpha)} t^{\alpha-1}, \\
& t \in\left[t_{0}-\tau, t_{0}\right] .
\end{align*}
$$

Set $a_{1}(t)=(|\xi| / \Gamma(\alpha)) t^{\alpha-1}$ and $a_{2}(t)=(|\eta| / \Gamma(\alpha)) t^{\alpha-1}$. Then, according to Lemma 2, the result is obtained and the proof is completed.

## 3. An Illustrative Example

In this section, we give an illustrative example to show effectiveness of results obtained in this paper.

Example 1. Consider the following fractional differential equation:

$$
\begin{align*}
D^{3 / 4} x(t) & =f(t, x(t-1), y(t-1)) \\
D^{3 / 4} y(t) & =g(t, x(t-1), y(t-1)), \quad t \in[1,+\infty)  \tag{47}\\
D^{-1 / 4} x(t) & =D^{-1 / 4} y(t)=\Gamma\left(\frac{3}{4}\right), \quad t \in[0,1]
\end{align*}
$$

where $f(t, x, y)=g(t, x, y)=(e / \sqrt{\Gamma(1 / 2)})(t-1)^{1 / 4}[x+y]$, $t \in[1,+\infty)$.

It is obvious that $|f(t, x, y)|=|g(t, x, y)| \leq(e /$ $\sqrt{\Gamma(1 / 2)})(t-1)^{1 / 4}[|x|+|y|], t \in[1,+\infty)$. From (47) and Theorem 5, we obtain $a_{1}(t)=a_{2}(t)=t^{-1 / 4}, \alpha_{1}(t)=\alpha_{2}(t)=$ $2 t^{-1 / 2} e^{-2 t}, l_{i}(t)=m_{i}(t)=2 \sqrt{t-1}, i=1,2$. Thus, $A(t)=$ $2 t^{-1 / 2} e^{-2 t}\left[\begin{array}{l}1 \\ 1\end{array}\right], H(t)=2 \sqrt{t-1}\left[\begin{array}{ll}1 & 1 \\ 1 & 1\end{array}\right], \Psi(t)=t^{-1 / 2} e^{-2 t}\left[\begin{array}{l}1 \\ 1\end{array}\right]$.

Set

$$
\left[\begin{array}{l}
|x(t)|  \tag{48}\\
|y(t)|
\end{array}\right]=\left[\begin{array}{l}
e^{t} w_{1}^{1 / 2}(t) \\
e^{t} w_{2}^{1 / 2}(t)
\end{array}\right]
$$

Therefore,

$$
\left[\begin{array}{l}
w_{1}(t)  \tag{49}\\
w_{2}(t)
\end{array}\right] \leq A(t)+R(t)
$$

where for $t \in[1,2]$,

$$
\begin{align*}
R(t) & =\int_{1}^{t} H(s) \Psi(s-1) d s=2\left[\begin{array}{l}
1 \\
1
\end{array}\right] \int_{1}^{t} e^{-2(s-1)} d s  \tag{50}\\
& =\left(1-e^{-2(t-1)}\right)\left[\begin{array}{l}
1 \\
1
\end{array}\right]
\end{align*}
$$

and for $t \in[2,+\infty)$,

$$
\begin{align*}
& R(t)=\exp \left\{\int_{2}^{t} H(s) d s\right\}\left[\int_{1}^{2} H(s) \Psi(s-1) d s\right. \\
& \left.\quad+\int_{2}^{t} \exp \left\{\int_{\xi}^{t} H(s) d s\right\} H(\xi) A(\xi-1) d \xi\right] \leq 2 \\
& \quad \cdot \exp \left\{\frac{4}{3}(t-1)^{3 / 2}\right\}\left[2\left(1-e^{-2}\right)\right.  \tag{51}\\
& \left.\quad+16 e^{(4 / 3)(t-1)^{3 / 2}} \int_{1}^{t} e^{-2 \xi} d \xi\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right] \leq 4\left[e^{(4 / 3)(t-1)^{3 / 2}}\right. \\
& \left.\quad+4 e^{(8 / 3)(t-1)^{3 / 2}-4}\right]\left[\begin{array}{l}
1 \\
1
\end{array}\right]
\end{align*}
$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# Existence Results for Impulsive Fractional $q$-Difference Equation with Antiperiodic Boundary Conditions 

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#### Abstract

In this paper, we investigate the impulsive fractional $q$-difference equation with antiperiodic conditions. The existence and uniqueness results of solutions are established via the theorem of nonlinear alternative of Leray-Schauder type and the Banach contraction mapping principle. Two examples are given to illustrate our results.


## 1. Introduction

In this paper, we are concerned with the existence and uniqueness of solutions for the following impulsive fractional $q$-difference equation with antiperiodic boundary conditions

$$
\begin{align*}
{ }^{c} D_{q}^{\alpha} u(t) & =f(t, u(t), T u(t), S u(t)), \\
& t \in J^{\prime}=J \backslash\left\{t_{1}, t_{2}, \ldots, t_{m}\right\}, \\
\left.\Delta u\right|_{t=t_{k}} & =I_{k}\left(u\left(t_{k}^{-}\right)\right), \\
\left.\Delta D_{q} u\right|_{t=t_{k}} & =I_{k}^{*}\left(u\left(t_{k}^{-}\right)\right),  \tag{1}\\
& k=1,2, \ldots, m, \\
u(0) & =-u(1), \\
{ }^{c} D_{q}^{\beta} u(0) & =-{ }^{c} D_{q}^{\beta} u(1),
\end{align*}
$$

where $q \in(0,1), 1<\alpha \leq 2,0<\beta<1, \alpha-\beta-1>0$, $J=[0,1], D_{q}$ is $q$-derivative, ${ }^{c} D_{q}^{\alpha}$, and ${ }^{c} D_{q}^{\beta}$ denote the Caputo $q$-derivative of orders $\alpha$ and $\beta$, respectively. $f \in C(J \times \mathbb{R} \times \mathbb{R} \times$ $\mathbb{R}, \mathbb{R}), I_{k}, I_{k}^{*} \in C(\mathbb{R}, \mathbb{R})(k=1,2, \ldots, m), \mathbb{R}$ is the set of all real numbers, and $0=t_{0}<t_{1}<\cdots<t_{m}<t_{m+1}=1 . T$ and $S$ are linear operators defined by

$$
\begin{align*}
& T u(t)=\int_{0}^{t} k(t, s) u(s) d_{q} s \\
& S u(t)=\int_{0}^{1} h(t, s) u(s) d_{q} s, \tag{2}
\end{align*}
$$

$$
t \in J
$$

where $k \in C(D, \mathbb{R}), h \in C(J \times J, \mathbb{R}), D=\{(t, s) \in J \times J: t \geq s\}$. $\left.\Delta u\right|_{t=t_{k}}=u\left(t_{k}^{+}\right)-u\left(t_{k}^{-}\right)$, where $u\left(t_{k}^{+}\right)$and $u\left(t_{k}^{-}\right)$represent the right and left limits of $u(t)$ at $t=t_{k},\left.\Delta D_{q} u\right|_{t=t_{k}}=D_{q} u\left(t_{k}^{+}\right)-$ $D_{q} u\left(t_{k}^{-}\right)$has a similar meaning.

Fractional $q$-difference calculus plays a very important role in modern applied mathematics due to their deep physical background and has been studied extensively [14]. Impulsive differential equations are important in both theory and applications. Considerable effort has been devoted to differential equations with or without impulse, for example, [5-21]. In recent years, impulsive fractional difference and differential equations with antiperiodic conditions have received much attention; see [22-27] and the references therein. Zhang and Wang [24] have applied cone contraction fixed point theorem to establish the existence of solutions to nonlinear fractional differential equation with impulses and antiperiodic boundary conditions

$$
\begin{aligned}
{ }^{c} D^{\alpha} u(t) & =f(t, u(t)), \\
& 1<\alpha \leq 2, t \in J \backslash\left\{t_{1}, t_{2}, \ldots, t_{m}\right\}, J=[0, T], \\
\left.\Delta u\right|_{t=t_{k}} & =I_{k}\left(u\left(t_{k}\right)\right),
\end{aligned}
$$

$$
\begin{align*}
\left.\Delta u^{\prime}\right|_{t=t_{k}} & =I_{k}^{*}\left(u\left(t_{k}\right)\right), \\
u(0) & =-u(T), \\
u^{\prime}(0) & =-u^{\prime}(T),
\end{align*}
$$

where ${ }^{c} D^{\alpha}$ is the Caputo fractional derivative, $f \in C(J \times$ $\mathbb{R}, \mathbb{R}), I_{k}, I_{k}^{*} \in C(\mathbb{R}, \mathbb{R})$. By using Banach fixed point theorem, Schaefer fixed point theorem, and nonlinear alternative of Leray-Schauder type theorem, some existence results of solutions for problem (3) are obtained in [25]. Ahmad et al. [28] studied existence of solutions for the following antiperiodic boundary value problem (BVP for short) of impulsive fractional $q$-difference equation

$$
\begin{align*}
& { }_{t_{k}}^{c} D_{q_{k}}^{\alpha_{k}} x(t)=f(t, x(t)), \quad t \in J_{k} \subseteq[0, T], t \neq t_{k}, \\
& \left.\Delta x\right|_{t=t_{k}}=x\left(t_{k}^{+}\right)-x\left(t_{k}\right)=\varphi_{k}\left({ }_{t_{k-1}} I_{q_{k-1}}^{\beta_{k-1} x} x\left(t_{k}\right)\right), \\
& \\
& k=1,2, \ldots, m,  \tag{4}\\
& { }_{t_{k}} D_{q_{k}} x\left(t_{k}^{+}\right)-{ }_{t_{k-1}} D_{q_{k-1}} x\left(t_{k}\right)=\varphi_{k}^{*}\left({ }_{t_{k-1}} I_{q_{k-1}}^{\gamma_{k-1}} x\left(t_{k}\right)\right), \\
& \\
& k=1,2, \ldots, m, \\
& x(0)=-x(T), \\
& { }_{0} D_{q 0} x(0)=-{ }_{t_{m}} D_{q_{m}} x(T),
\end{align*}
$$

where ${ }_{t_{k}}^{c} D_{q_{k}}^{\alpha_{k}}$ denotes the Caputo $q_{k}$-fractional derivative of order $\alpha_{k}$ on $J_{k}, 1<\alpha_{k} \leq 2,0<q_{k}<1, f \in C(J \times$ $\mathbb{R}, \mathbb{R}), \varphi_{k}, \varphi_{k}^{*} \in C(\mathbb{R}, \mathbb{R}), k=1,2, \ldots, m .{ }_{t_{k}} I_{q_{k}}^{\beta_{k}}$ and ${ }_{t_{k}} I_{q_{k}}^{\gamma_{k}}$ denote the Riemann-Liouville $q_{k}$-integral of orders $\beta_{k}$ and $\gamma_{k}$, respectively.

In this paper we are concerned with the existence and uniqueness of solutions for impulsive fractional $q$-difference equation antiperiodic BVP. By applying the theorem of nonlinear alternative of Leray-Schauder type and Banach contraction mapping principle, we show the existence and uniqueness of solutions for the BVP (1). Some ideas of this paper are from [29, 30].

## 2. Preliminaries and Lemmas

For $q \in(0,1)$, let

$$
\begin{align*}
{[a]_{q} } & =\frac{1-q^{a}}{1-q} \\
(a ; q)_{\infty} & =\prod_{i=0}^{\infty}\left(1-a q^{i}\right),  \tag{5}\\
(a ; q)_{\alpha} & =\frac{(a ; q)_{\infty}}{\left(a q^{\alpha} ; q\right)_{\infty}}
\end{align*}
$$

$$
(a, \alpha \in \mathbb{R}) .
$$

We define the $q$-analogue of the power function $(a-b)^{n}$ with $n \in \mathbb{N}_{0}$ is

$$
\begin{align*}
& (a-b)^{0}=1 \\
& (a-b)^{n}=\prod_{k=0}^{n-1}\left(a-b q^{k}\right), \tag{6}
\end{align*}
$$

$$
n \in \mathbb{N}, a, b \in \mathbb{R}
$$

and, for $\alpha \in \mathbb{R}$,

$$
\begin{equation*}
(a-b)^{(\alpha)}=a^{\alpha} \prod_{n=0}^{\infty} \frac{a-b q^{n}}{a-b q^{\alpha+n}} . \tag{7}
\end{equation*}
$$

The $q$-derivative of $f$ is defined by

$$
\begin{align*}
& \left(D_{q} f\right)(x)=\frac{f(x)-f(q x)}{(1-q) x}  \tag{8}\\
& \left(D_{q} f\right)(0)=\lim _{x \rightarrow 0}\left(D_{q} f\right)(x)
\end{align*}
$$

and $q$-derivative of higher order by

$$
\begin{align*}
& \left(D_{q}^{0} f\right)(x)=f(x), \\
& \left(D_{q}^{n} f\right)(x)=D_{q}\left(D_{q}^{n-1} f\right)(x), \quad n \in \mathbb{N} . \tag{9}
\end{align*}
$$

The $q$-integral of $f$ is defined by

$$
\begin{equation*}
\left(I_{q} f\right)=\int_{0}^{x} f(t) d_{q} t=x(1-q) \sum_{n=0}^{\infty} f\left(x q^{n}\right) q^{n} \tag{10}
\end{equation*}
$$

$$
x \in[0, b] .
$$

Lemma 1 (see [31]). (1) $I f|f|$ is $q$-integral on the interval $[0, x]$, then $\left|\int_{0}^{x} f(t) d_{q} t\right| \leq \int_{0}^{x}|f(t)| d_{q} t$.
(2) If $f$ and $g$ are $q$-integral on the interval $[0, x], f(t) \leq$ $g(t)$ for all $t \in[0, x]$, then $\int_{0}^{x} f(t) d_{q} t \leq \int_{0}^{x} g(t) d_{q} t$.

Definition 2 (see [2]). Let $\alpha \geq 0$ and $f$ be a function defined on $[0, b]$. The fractional $q$-integral of the Riemann-Liouville type is defined by $\left(I_{q}^{0} f\right)(x)=f(x)$ and

$$
\begin{align*}
\left(I_{q}^{\alpha} f\right)(x)=\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{x}(x-q t)^{(\alpha-1)} & f(t) d_{q} t  \tag{11}\\
& \\
& \alpha>0, x \in[0, b]
\end{align*}
$$

Definition 3 (see [3]). The fractional $q$-derivative of the Caputo type of order $\alpha \geq 0$ is defined by

$$
\begin{equation*}
\left({ }^{c} D_{q}^{\alpha} f\right)(x)=\left(I_{q}^{[\alpha]-\alpha} D_{q}^{[\alpha]} f\right)(x), \quad \alpha \geq 0 \tag{12}
\end{equation*}
$$

where $[\alpha]$ is the smallest integer greater than or equal to $\alpha$. If $f(x)=x^{\beta-1}, \beta>0$, then ${ }^{c} D_{q}^{\alpha} f(x)=\left(\Gamma_{q}(\beta) / \Gamma_{q}(\beta-\alpha)\right) x^{\beta-\alpha-1}$.

Lemma 4 (see $[2,3]$ ). Let $\alpha, \beta \geq 0$ and $f$ be a function defined on $[0, b]$. The following formulas hold:
(1) $\left(I_{q}^{\beta} I_{q}^{\alpha} f\right)(x)=\left(I_{q}^{\alpha+\beta} f\right)(x)$;
(2) $\left(D_{q}^{\alpha} I_{q}^{\alpha} f\right)(x)=f(x)$.

Lemma 5 (see [3]). Let $\alpha \in \mathbb{R}^{+} \backslash \mathbb{N}$ and $a<x$. Then

$$
\begin{equation*}
\left(I_{q}^{\alpha c} D_{q}^{\alpha} f\right)(x)=f(x)-\sum_{k=0}^{[\alpha]-1} \frac{\left(D_{q}^{k} f\right)(a)}{\Gamma_{q}(k+1)} x^{k}\left(\frac{a}{x}: q\right)_{k} \tag{13}
\end{equation*}
$$

If $\alpha \geq m \geq \beta$, then ${ }^{c} D_{q}^{\beta} I_{q}^{\alpha} f(x)=I_{q}^{m-\beta} I_{q}^{\alpha-m} f(x)=I_{q}^{\alpha-\beta} f(x)$.
Lemma 6 (see [3]). For $\beta \in \mathbb{R}^{+}, \lambda \in(-1,+\infty)$ and $0 \leq a<$ $t \leq b$,

$$
\begin{equation*}
I_{q}^{\beta}\left((t-a)^{(\lambda)}\right)=\frac{\Gamma_{q}(\lambda+1)}{\Gamma_{q}(\beta+\lambda+1)}(t-a)^{(\beta+\lambda)} \tag{14}
\end{equation*}
$$

In particular, when $\lambda=0$ and $a=0$, using $q$-integration by part,

$$
\begin{equation*}
\left(I_{q}^{\beta} 1\right)(t)=\frac{1}{\Gamma_{q(\beta)}} \int_{0}^{t}(t-q s)^{(\beta-1)} d_{q} s=\frac{1}{\Gamma_{q}(\beta+1)} t^{\beta} \tag{15}
\end{equation*}
$$

Lemma 7 (see [32] (nonlinear alternative of Leray-Schauder type)). Let $X$ be a Banach space, $U$ be a bounded open subset
of $X$ with $0 \in U$, and $P: \bar{U} \longrightarrow X$ be a completely continuous operator. Then, either there exists $x \in \partial U$ such that $x=\lambda P x$ for $\lambda \in(0,1)$ or there exists a fixed point $x^{*} \in \bar{U}$.

Let $P C(J, \mathbb{R})=\{u: u$ ia a map from $J$ into $\mathbb{R}$ such that $u(t)$ is continuous at $t \neq t_{k}$, left continuous at $t=t_{k}$ and its right limit at $t=t_{k}$ exists for $\left.k=1, \ldots, m\right\}$; then $P C(J, \mathbb{R})$ is a Banach space with the norm $\|u\|_{P C}=\sup \{|u(t)|: t \in J\}$.

Lemma 8. For $h \in P C(J, \mathbb{R})$, the solution of impulsive $B V P$,

$$
\begin{align*}
{ }^{c} D_{q}^{\alpha} u(t) & =h(t), \quad t \in J^{\prime}, \\
\left.\Delta u\right|_{t=t_{k}} & =I_{k}\left(u\left(t_{k}^{-}\right)\right), \\
\left.\Delta D_{q} u\right|_{t=t_{k}} & =I_{k}^{*}\left(u\left(t_{k}^{-}\right)\right),  \tag{16}\\
& k=1,2, \ldots, m, \\
u(0) & =-u(1), \\
{ }^{c} D_{q}^{\beta} u(0) & =-{ }^{c} D_{q}^{\beta} u(1),
\end{align*}
$$

is given by

$$
\begin{align*}
& u(t) \\
& = \begin{cases}\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s-\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s-\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right) \\
\quad+\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s+\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)-\sum_{i=1}^{m} t I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right), \quad t \in\left[0, t_{1}\right), \\
\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s-\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s & \\
\quad+\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s+\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) & \\
\quad-\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)+\sum_{i=k+1}^{m}\left(t_{i}-t\right) I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right), & t \in\left[t_{k}, t_{k+1}\right), k=1, \ldots, m-1, \\
\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s-\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s & \\
\quad+\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s+\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right), & t \in\left[t_{m}, 1\right] .\end{cases}
\end{align*}
$$

Proof. In view of Definitions 2 and 3 and Lemma 5, for $t \in$ $J_{k}=\left[t_{k}, t_{k+1}\right], k=0,1,2, \ldots, m$, we have

$$
\begin{align*}
u(t) & =I_{q}^{\alpha} h(t)+d_{k}+e_{k} t \\
& =\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s+d_{k}+e_{k} t \tag{18}
\end{align*}
$$

and

$$
\begin{equation*}
\left(D_{q} u\right)(t)=\frac{1}{\Gamma_{q}(\alpha-1)} \int_{0}^{t}(t-q s)^{(\alpha-2)} h(s) d_{q} s+e_{k} . \tag{19}
\end{equation*}
$$

It follows from Definition 3, Lemma 5, and (18) that

$$
\begin{align*}
{ }^{c} D_{q}^{\beta} u(t)= & \frac{1}{\Gamma_{q}(\alpha-\beta)} \int_{0}^{t}(t-q s)^{(\alpha-\beta-1)} h(s) d_{q} s  \tag{20}\\
& +e_{k} \frac{t^{1-\beta}}{\Gamma_{q}(2-\beta)}, \quad t \in J_{k} .
\end{align*}
$$

Applying ${ }^{c} D_{q}^{\beta} u(0)=-{ }^{c} D_{q}^{\beta} u(1)$ in (20), we obtain

$$
\begin{equation*}
e_{m}=-\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s . \tag{21}
\end{equation*}
$$

Note boundary conditions $\left.\Delta u\right|_{t=t_{k}}=I_{k}\left(u\left(t_{k}^{-}\right)\right)$and $\left.\Delta D_{q} u\right|_{t=t_{k}}=I_{k}^{*}\left(u\left(t_{k}^{-}\right)\right)$; we get

$$
\begin{align*}
d_{k}-d_{k-1}+\left(e_{k}-e_{k-1}\right) t_{k} & =I_{k}\left(u\left(t_{k}^{-}\right)\right) \\
e_{k}-e_{k-1} & =I_{k}^{*}\left(u\left(t_{k}^{-}\right)\right)  \tag{22}\\
& k=1,2, \ldots, m
\end{align*}
$$

Applying (21) and (22), we have

$$
\begin{align*}
e_{k}= & e_{m}-\sum_{i=k+1}^{m} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
= & -\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s  \tag{23}\\
& -\sum_{i=k+1}^{m} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)
\end{align*}
$$

Thanks to $u(0)=-u(1)$, it is derived that

$$
\begin{equation*}
d_{0}=-\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s-d_{m}-e_{m} \tag{24}
\end{equation*}
$$

and

$$
\begin{align*}
d_{m}+d_{0}= & -\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s-e_{m} \\
= & -\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s  \tag{25}\\
& +\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-\mathrm{q} s)^{(\alpha-\beta-1)} h(s) d_{q} s .
\end{align*}
$$

By (22), we get

$$
\begin{equation*}
d_{m}-d_{0}=\sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \tag{26}
\end{equation*}
$$

Combining (25) and (26), we have

$$
\begin{aligned}
d_{m} & =\frac{1}{2}\left[-\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s\right. \\
& +\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s \\
& \left.+\sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)\right]
\end{aligned}
$$

$$
\begin{align*}
d_{k} & =d_{m}-\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)+\sum_{i=k+1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& =-\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta)} \\
& \cdot \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s+\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right) \\
& -\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)-\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right) \\
& +\sum_{i=k+1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) . \tag{27}
\end{align*}
$$

Therefore, for $t \in J_{k}, k=0,1,2, \ldots, m-1$,
$u(t)$

$$
\begin{align*}
& =\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s \\
& -\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s \\
& +\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& -\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)+\sum_{i=k+1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& -\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} t \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s  \tag{28}\\
& -t \sum_{i=k+1}^{m} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& =\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s \\
& -\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s \\
& +\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& -\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)+\sum_{i=k+1}^{m}\left(t_{i}-t\right) I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right),
\end{align*}
$$

and, for $t \in J_{m}$,

$$
\begin{aligned}
& u(t) \\
& =\frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)} h(s) d_{q} s \\
& \\
& \quad-\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)} h(s) d_{q} s \\
& \quad+\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} h(s) d_{q} s \\
& \\
& \quad+\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) .
\end{aligned}
$$

## 3. Main Results

Define an operator $A: P C(J, \mathbb{R}) \longrightarrow P C(J, \mathbb{R})$ by

$$
\begin{aligned}
& A u(t)=\frac{1}{\Gamma_{q}(\alpha)} \\
& \quad \cdot \int_{0}^{t}(t-q s)^{(\alpha-1)} f(s, u(s), T u(s), S u(s)) d_{q} s \\
& \quad-\frac{1}{2 \Gamma_{q}(\alpha)} \\
& \quad \cdot \int_{0}^{1}(1-q s)^{(\alpha-1)} f(s, u(s), T u(s), S u(s)) d_{q} s \\
& \quad+\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\frac{1}{2}-t\right) \\
& \quad \cdot \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} f(s, u(s), T u(s), S u(s)) d_{q} s \\
& \quad+\frac{1}{2} \sum_{i=1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)-\frac{1}{2} \sum_{i=1}^{m} t_{i} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& \quad-\sum_{i=k+1}^{m} I_{i}\left(u\left(t_{i}^{-}\right)\right)+\sum_{i=k+1}^{m}\left(t_{i}-t\right) I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right) \\
& \quad
\end{aligned}
$$

## Theorem 9. Assume that

$\left(H_{1}\right)$ There exist nonnegative functions $L_{j}(t) \in C(J)(j=$ $1,2,3)$ such that

$$
\begin{aligned}
& \left|f\left(t, u_{1}, v_{1}, w_{1}\right)-f\left(t, u_{2}, v_{2}, w_{2}\right)\right| \\
& \quad \leq L_{1}(t)\left|u_{1}-u_{2}\right|+L_{2}(t)\left|v_{1}-v_{2}\right| \\
& \quad+L_{3}(t)\left|w_{1}-w_{2}\right|
\end{aligned}
$$

for $t \in J, u_{i}, v_{i}, \omega_{i} \in \mathbb{R}, i=1,2$.
$\left(\mathrm{H}_{2}\right)$ There exist positive numbers $N$ and $N^{*}$ such that

$$
\begin{align*}
&\left|I_{k}(u)-I_{k}(v)\right| \leq N|u-v| \\
&\left|I_{k}^{*}(u)-I_{k}^{*}(v)\right| \leq N^{*}|u-v| \tag{32}
\end{align*}
$$

$u, v \in \mathbb{R}, k=1,2, \ldots, m$.
$\left(H_{3}\right)$

$$
\begin{align*}
\chi= & \frac{3\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{q}(\alpha+1)} \\
& +\frac{\Gamma_{q}(2-\beta)\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{q}(\alpha-\beta+1)}+\frac{3}{2} m N  \tag{33}\\
& +\frac{5}{2} m N^{*}<1,
\end{align*}
$$

where $\overline{L_{i}}=\max \left\{L_{i}(t): t \in J\right\}, \quad i=1,2,3, k_{0}=\max \{|k(t, s)|$ : $(t, s) \in D\}, h_{0}=\max \left\{|h(t, s)|:(t, s) \in D_{0}\right\}$.

Then BVP (1) has a unique solution.
Proof. For $u, v \in P C(J, \mathbb{R})$ and $t \in J$, we have
$|(A u)(t)-(A v)(t)|$

$$
\begin{aligned}
& \left.\leq \int_{0}^{t} \frac{(t-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} \right\rvert\, f(s, u(s), T u(s), S u(s)) \\
& -f(s, v(s), T v(s), S v(s)) \left\lvert\, d_{q} s+\frac{1}{2}\right. \\
& \left.\cdot \int_{0}^{1} \frac{(1-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} \right\rvert\, f(s, u(s), T u(s), S u(s)) \\
& -f(s, v(s), T v(s), S v(s)) \mid d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)}
\end{aligned}
$$

$$
\cdot \mid f(s, u(s), T u(s), S u(s))
$$

$$
-f(s, v(s), T v(s), S v(s)) \left\lvert\, d_{q} s+\frac{1}{2}\right.
$$

$$
\left.\cdot \sum_{i=1}^{m}\left|I\left(u\left(t_{i}^{-}\right)\right)-I_{i}\left(v\left(t_{i}^{-}\right)\right)\right|_{i}+\frac{1}{2} \sum_{i=1}^{m} t_{i} \right\rvert\, I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)
$$

$$
-I_{i}^{*}\left(v\left(t_{i}^{-}\right)\right)\left|+\sum_{i=1}^{m}\right| I_{i}\left(u\left(t_{i}^{-}\right)\right)-I_{i}\left(v\left(t_{i}^{-}\right)\right) \mid
$$

$$
+\sum_{i=1}^{m} t_{i}\left|I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)-I_{i}^{*}\left(v\left(t_{i}^{-}\right)\right)\right|+t \sum_{i=1}^{m} \mid I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)
$$

$$
-I_{i}^{*}\left(v\left(t_{i}^{-}\right)\right) \left\lvert\, \leq \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t-q s)^{(\alpha-1)}\right.
$$

$$
\cdot\left(L_{1}(s)|u(s)-v(s)|\right.
$$

$$
\begin{aligned}
& +L_{2}(s)\left|\int_{0}^{s} k(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right| \\
& \left.+L_{3}(s)\left|\int_{0}^{1} h(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right|\right) d_{q} s \\
& +\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1-q s)^{(\alpha-1)}\left(L_{1}(s)|u(s)-v(s)|\right. \\
& +L_{2}(s)\left|\int_{0}^{s} k(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right| \\
& \left.+L_{3}(s)\left|\int_{0}^{1} h(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right|\right) d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} \\
& \cdot\left(L_{1}(s)|u(s)-v(s)|\right. \\
& +L_{2}(s)\left|\int_{0}^{s} k(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right| \\
& \left.+L_{3}(s)\left|\int_{0}^{1} h(s, \tau)(u(\tau)-v(\tau)) d_{q} \tau\right|\right) d_{q} s \\
& +\frac{3}{2} \sum_{i=1}^{m}\left|I_{i}\left(u\left(t_{i}^{+}\right)\right)-I_{i}\left(v\left(t_{i}^{-}\right)\right)\right|+\sum_{i=1}^{m}\left(\frac{3}{2} t_{i}+t\right) \\
& \cdot\left|I_{i}^{*}\left(u\left(t_{i}^{+}\right)\right)-I_{i}^{*}\left(v\left(t_{i}^{-}\right)\right)\right| \leq \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{t}(t \\
& -q s)^{(\alpha-1)}\left(\overline{L_{1}}\|u-v\|_{P C}+\overline{L_{2}} k_{0}\|u-v\|_{P C}\right. \\
& \left.+\overline{L_{3}} h_{0}\|u-v\|_{P C}\right) d_{q} s+\frac{1}{2 \Gamma_{q}(\alpha)} \int_{0}^{1}(1 \\
& -q s)^{(\alpha-1)}\left(\overline{L_{1}}\|u-v\|_{P C}+\overline{L_{2}} k_{0}\|u-v\|_{P C}\right. \\
& \left.+\overline{L_{3}} h_{0}\|u-v\|_{P C}\right) d_{q} s+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta)} \int_{0}^{1}(1 \\
& -q s)^{(\alpha-\beta-1)}\left(\overline{L_{1}}\|u-v\|_{P C}+\overline{L_{2}} k_{0}\|u-v\|_{P C}\right. \\
& \left.+\overline{L_{3}} h_{0}\|u-v\|_{P C}\right) d_{q} s+\frac{3}{2} m N\|u-v\|_{P C} \\
& +\frac{5}{2} m N^{*}\|u-v\|_{P C} \leq\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right) \\
& \cdot\left(\frac{1}{\Gamma_{q}(\alpha+1)}+\frac{1}{2 \Gamma_{q}(\alpha+1)}+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta+1)}\right) \| u \\
& -v\left\|_{P C}+\left(\frac{3}{2} m N+\frac{5}{2} m N^{*}\right)\right\| u-v \|_{P C} \\
& =\left\{\frac{3\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{q}(\alpha+1)}\right.
\end{aligned}
$$

$$
\begin{align*}
& +\frac{\Gamma_{q}(2-\beta)\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{q}(\alpha-\beta+1)}+\frac{3}{2} m N+\frac{5}{2} \\
& \left.\cdot m N^{*}\right\}\|u-v\|_{P C}=\chi\|u-v\|_{P C}, \tag{34}
\end{align*}
$$

and then $\|A u-A v\|_{P C} \leq \chi\|u-v\|_{P C}$, and hence $A$ is a contraction operator. It follows from Banach contraction mapping principle that BVP (1) has a unique solution.

Theorem 10. Assume the following:
$\left(H_{4}\right)$ There exist continuous and nondecreasing function $g$ : $[0,+\infty) \longrightarrow(0,+\infty)$ and $a(t) \in C[0,1]$ such that

$$
\begin{align*}
& |f(t, u, v, w)| \leq a(t) g(\max \{|u|,|v|,|w|\}), \\
& \quad t \in[0,1], u, v, w \in \mathbb{R} . \tag{35}
\end{align*}
$$

$\left(H_{5}\right)$ There exist continuous and nondecreasing functions $\varphi, \psi:[0,+\infty) \longrightarrow(0,+\infty)$ such that

$$
\begin{align*}
& \left|I_{k}(u)\right| \leq \varphi(|u|), \\
& \left|I_{k}^{*}(u)\right| \leq \psi(|u|), \tag{36}
\end{align*}
$$

$$
u \in \mathbb{R}, k=1, \ldots, m
$$

$\left(H_{6}\right)$ There exists constant $M>0$ such that

$$
\begin{align*}
M> & a^{\prime} g\left(\max \left\{M, k_{0} M, h_{0} M\right\}\right) \\
& \cdot\left(\frac{3}{2 \Gamma_{q}(\alpha+1)}+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta+1)}\right)+\frac{3 m}{2} \varphi(M)  \tag{37}\\
& +\frac{5}{2} m \psi(M),
\end{align*}
$$

where $a^{\prime}=\max \{a(t): t \in[0,1]\}$.
Then BVP (1) has at least one solution.

Proof. The continuity of $f, I_{k}, I_{k}^{*}$ implies that operator $A$ : $P C(J, \mathbb{R}) \longrightarrow P C(J, \mathbb{R})$ is continuous. Let $B \subset P C(J, \mathbb{R})$ be bounded; then there exist positive constants $P_{1}, P_{2}$, and $P_{3}$ such that $|f(t, u(t), T u(t), S u(t))| \leq P_{1},\left|I_{i}\left(u\left(t_{i}^{-}\right)\right)\right| \leq P_{2}$, and $\left|I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)\right| \leq P_{3}$ for all $t \in J, u \in B, i=1,2, \ldots, m$. Thus, we have

$$
\begin{aligned}
|(A u)(t)| \leq & \int_{0}^{t} \frac{(t-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} P_{1} d_{q} s \\
& +\frac{1}{2} \int_{0}^{1} \frac{(1-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} P_{1} d_{q} s
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{2} \Gamma_{q}(2-\beta) \int_{0}^{1} \frac{(1-q s)^{(\alpha-\beta-1)}}{\Gamma_{q}(\alpha-\beta)} P_{1} d_{q} s \\
& +\frac{m}{2} P_{2}+\frac{P_{3}}{2} \sum_{i=1}^{m} t_{i}+m P_{2}+P_{3} \sum_{i=1}^{m} t_{i}+m P_{3} \\
\leq & \frac{3 P_{1}}{2 \Gamma_{q}(\alpha+1)}+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta+1)} P_{1}+\frac{3 m}{2} P_{2} \\
& +\frac{5}{2} m P_{3} . \tag{38}
\end{align*}
$$

Consequently, operator $A$ is uniformly bounded on $B$.
On the other hand, for $t_{k} \leq \tau_{1}<\tau_{2} \leq t_{k+1}, u \in B$, we have

$$
\begin{aligned}
& \left|(A u)\left(\tau_{2}\right)-(A u)\left(\tau_{1}\right)\right|=\left\lvert\, \frac{1}{\Gamma_{q}(\alpha)} \int_{0}^{\tau_{2}}\left(\tau_{2}-q s\right)^{(\alpha-1)}\right. \\
& \cdot f(s, u(s), T u(s), S u(s)) d_{q} s-\frac{1}{\Gamma_{q}(\alpha)} \\
& \cdot \int_{0}^{\tau_{1}}\left(\tau_{1}-q s\right)^{(\alpha-1)} \\
& \text { - } f(s, u(s), T u(s), S u(s)) d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)}\left(\tau_{1}-\tau_{2}\right) \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} \\
& \text { - } f(s, u(s), T u(s), S u(s)) d_{q} s \\
& +\sum_{i=k+1}^{m} I_{i}^{*}\left(u\left(t_{i}^{-}\right)\right)\left(\tau_{1}-\tau_{2}\right) \left\lvert\, \leq \frac{1}{\Gamma_{q}(\alpha)}\right. \\
& \cdot \int_{0}^{\tau_{1}}\left|\left(\tau_{2}-q s\right)^{(\alpha-1)}-\left(\tau_{1}-q s\right)^{(\alpha-1)}\right| \\
& \cdot|f(s, u(s), T u(s), S u(s))| d_{q} s+\frac{1}{\Gamma_{q}(\alpha)} \\
& \cdot \int_{\tau_{1}}^{\tau_{2}}\left|\left(\tau_{2}-q s\right)^{(\alpha-1)} f(s, u(s), T u(s), S u(s))\right| d_{q} s \\
& +\frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta)} P_{1}\left|\tau_{2}-\tau_{1}\right| \int_{0}^{1}(1-q s)^{(\alpha-\beta-1)} d_{q} s \\
& +m P_{3}\left|\tau_{2}-\tau_{1}\right| \leq P_{1} \frac{\tau_{2}^{\alpha}-\tau_{1}^{\alpha}}{\Gamma_{q}(\alpha+1)}+P_{1} \\
& \cdot \frac{\Gamma_{q}(2-\beta)}{\Gamma_{q}(\alpha-\beta+1)}\left|\tau_{2}-\tau_{1}\right|+m P_{3}\left|\tau_{2}-\tau_{1}\right|,
\end{aligned}
$$

which tends to zero as $\tau_{2} \longrightarrow \tau_{1}$; then $A$ is equicontinuous on $J_{k}$. Hence by PC-type Arzela-Ascoli Theorem ([33]), operator $A: P C(J, \mathbb{R}) \longrightarrow P C(J, \mathbb{R})$ is completely continuous.

Let $u \in P C(J, \mathbb{R})$ be such that $u(t)=\lambda(A u)(t)$ for some $\lambda \in(0,1)$; then

$$
\begin{align*}
& |u(t)|=|\lambda(A u)(t)| \leq|(A u)(t)| \\
& \quad \leq \int_{0}^{t} \frac{(t-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} a^{\prime} g\left(\operatorname { m a x } \left\{\|u\|_{P C}, k_{0}\|u\|_{P C},\right.\right. \\
& \left.\left.h_{0}\|u\|_{P C}\right\}\right) d_{q} s+\frac{1}{2} \\
& \cdot \int_{0}^{1} \frac{(1-q s)^{(\alpha-1)}}{\Gamma_{q}(\alpha)} a^{\prime} g\left(\operatorname { m a x } \left\{\|u\|_{P C}, k_{0}\|u\|_{P C},\right.\right. \\
& \left.\left.h_{0}\|u\|_{P C}\right\}\right) d_{q} s+\frac{1}{2} \Gamma_{q}(2-\beta) \\
& \cdot \int_{0}^{1} \frac{(1-q s)^{(\alpha-\beta-1)}}{\Gamma_{q}(\alpha-\beta)} a^{\prime} g\left(\operatorname { m a x } \left\{\|u\|_{P C}, k_{0}\|u\|_{P C}\right.\right.  \tag{40}\\
& \left.\left.h_{0}\|u\|_{P C}\right\}\right) d_{q} s+\frac{1}{2} m \varphi\left(\|u\|_{P C}\right)+\frac{1}{2} \\
& \cdot \psi\left(\|u\|_{P C}\right) \sum_{i=1}^{m} t_{i}+m \varphi\left(\|u\|_{P C}\right)+\psi\left(\|u\|_{P C}\right) \sum_{i=1}^{m} t_{i} \\
& \quad+m \psi\left(\|u\|_{P C}\right) \leq a^{\prime} g\left(\operatorname { m a x } \left\{\|u\|_{P C}, k_{0}\|u\|_{P C},\right.\right. \\
& \left.\left.h_{0}\|u\|_{P C}\right\}\right)\left(\frac{3}{2 \Gamma_{q}(\alpha+1)}+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta+1)}\right)+\frac{3 m}{2} \\
& \cdot \varphi\left(\|u\|_{P C}\right)+\frac{5}{2} m \psi\left(\|u\|_{P C}\right)
\end{align*}
$$

and hence
$\|u\|_{P C} \leq a^{\prime} g\left(\max \left\{\|u\|_{P C}, k_{0}\|u\|_{P C}, h_{0}\|u\|_{P C}\right\}\right)$

$$
\begin{align*}
& \cdot\left(\frac{3}{2 \Gamma_{q}(\alpha+1)}+\frac{\Gamma_{q}(2-\beta)}{2 \Gamma_{q}(\alpha-\beta+1)}\right)+\frac{3 m}{2}  \tag{41}\\
& \cdot \varphi\left(\|u\|_{P C}\right)+\frac{5}{2} m \psi\left(\|u\|_{P C}\right) .
\end{align*}
$$

Let $U=\{u \in P C(J, \mathbb{R}):\|u\|<M\}$; then operator $A: \bar{U} \longrightarrow$ $P C(J, \mathbb{R})$ is completely continuous. By $\left(H_{6}\right)$, one has $u \neq \lambda A u$ for any $\lambda \in(0,1)$ and $u \in \partial U$. By Lemma 7, BVP (1) has at least one solution.

## 4. Examples

Example 1. Consider the BVP

$$
\begin{aligned}
{ }^{c} D_{1 / 2}^{3 / 2} u(t)= & \frac{u(t)+1}{100}+\frac{1}{50+t^{2}} \int_{0}^{t} \frac{u(s)}{e^{(t+s)}} d_{q} s \\
& +\frac{e^{-t}}{80} \int_{0}^{1} \frac{u(s)}{2+t+s} d_{q} s \\
& t \in[0,1] \backslash\left\{\frac{1}{2}\right\},
\end{aligned}
$$

$$
\begin{align*}
\left.\Delta u\right|_{t=1 / 2} & =\frac{|u(1 / 2)|}{10+|u(1 / 2)|}, \\
\left.\Delta D_{1 / 2} u\right|_{t=1 / 2} & =\frac{|u(1 / 2)|}{20+|u(1 / 2)|}, \\
u(0) & =-u(1), \\
{ }^{c} D_{1 / 2}^{1 / 4} u(0) & =-{ }^{c} D_{1 / 2}^{1 / 4} u(1) . \tag{42}
\end{align*}
$$

Let

$$
\begin{align*}
f(t, u, v, w) & =\frac{u+1}{100}+\frac{1}{50+t^{2}} v+\frac{e^{-t}}{80} w, \\
(T u)(t) & =\int_{0}^{t} e^{-(t+s)} u(s) d_{q} s,  \tag{43}\\
(S u)(t) & =\int_{0}^{1} \frac{u(s)}{2+t+s} d_{q} s .
\end{align*}
$$

By direct computation, $k_{0}=\max \left\{e^{-(t+s)}: 0 \leq s \leq t \leq 1\right\}=$ 1 , $h_{0}=\max \{1 /(2+t+s): 0 \leq s, t \leq 1\}=1 / 2$. For any $u_{1}, u_{2}, v_{1}, v_{2}, \omega_{1}, \omega_{2} \in \mathbb{R}$ and $t \in J$, we have

$$
\begin{align*}
& \left|f\left(t, u_{1}, v_{1}, w_{1}\right)-f\left(t, u_{2}, v_{2}, w_{2}\right)\right| \\
& \quad \leq \frac{1}{100}\left|u_{1}-u_{2}\right|+\frac{1}{50}\left|v_{1}-v_{2}\right|+\frac{1}{80}\left|w_{1}-w_{2}\right| \\
& \left|I_{k}(u)-I_{k}(v)\right| \leq \frac{1}{10}|u-v|  \tag{44}\\
& \left|I_{k}^{*}(u)-I_{k}^{*}(v)\right| \leq \frac{1}{20}|u-v| .
\end{align*}
$$

Let $L_{1}(t)=1 / 100, L_{2}(t)=1 / 50, L_{3}(t)=1 / 80, N=1 / 10$, and $N^{*}=1 / 20$; then

$$
\begin{align*}
\chi= & \frac{3\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{1 / 2}(5 / 2)} \\
& +\frac{\Gamma_{1 / 2}(7 / 4)\left(\overline{L_{1}}+\overline{L_{2}} k_{0}+\overline{L_{3}} h_{0}\right)}{2 \Gamma_{1 / 2}(9 / 4)}+\frac{3}{2} N+\frac{5}{2} N^{*}  \tag{45}\\
\approx & 0.307<1 .
\end{align*}
$$

Then, $\left(H_{1}\right)-\left(H_{3}\right)$ hold. It follows from Theorem 9 that BVP (42) has a unique solution.

Example 2. Consider the BVP

$$
\begin{aligned}
& { }^{c} D_{q}^{3 / 2} u(t)=\frac{t^{2}}{60}\left(\sin u(t)+\int_{0}^{t} \cos (u(s)) d_{q} s\right. \\
& \left.\quad+\int_{0}^{1} \frac{1}{(u(s))^{2}+t^{2}+1} d_{q} s\right), \quad t \in[0,1] \backslash\left\{\frac{1}{2}\right\}, \\
& \left.\Delta u\right|_{t=1 / 2}=\frac{|u(1 / 2)|}{20+|u(1 / 2)|},
\end{aligned}
$$

$$
\begin{align*}
& \left.\Delta D_{q} u\right|_{t=1 / 2}=\frac{|u(1 / 2)|}{30+|u(1 / 2)|}, \\
& u(0)=-u(1) \\
& { }^{c} D_{q}^{1 / 4} u(0)=-{ }^{c} D_{q}^{1 / 4} u(1) \tag{46}
\end{align*}
$$

Let

$$
\begin{align*}
f(t, u, v, w) & =\frac{t^{2}}{60}(\sin u+v+w) \\
(T u)(t) & =\int_{0}^{t} \cos (u(s)) d_{q} s  \tag{47}\\
(S u)(t) & =\int_{0}^{1} \frac{d_{q} s}{(u(s))^{2}+t^{2}+1}
\end{align*}
$$

then

$$
\begin{align*}
& |f(t, u(t), T u(t), S u(t))| \leq \frac{1}{60}(1+t+1)  \tag{48}\\
& \quad \leq 6(t+1)
\end{align*}
$$

Let $g(r)=6, a(t)=t+1$; then $a^{\prime}=\max \{t+1: t \in[0,1]\}=2$. Choose $\varphi(u)=1$ and $\psi(u)=1$; we have

$$
\begin{align*}
2 & \times 6 \times\left(\frac{3}{2 \Gamma_{1 / 2}(5 / 2)}+\frac{\Gamma_{1 / 2}(7 / 4)}{2 \Gamma_{1 / 2}(9 / 4)}\right)+\frac{3}{2}+\frac{5}{2}  \tag{49}\\
& \approx 19.87
\end{align*}
$$

Let $M=20$; then condition $\left(H_{6}\right)$ holds. Therefore, by Theorem 10, BVP (46) has at least one solution.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest.

## Authors' Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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# Existence of Generalized Nash Equilibrium in $n$-Person Noncooperative Games under Incomplete Preference 

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#### Abstract

To prove the existence of Nash equilibrium by traditional ways, a common condition that the preference of players must be complete has to be considered. This paper presents a new method to improve it. Based on the incomplete preference corresponding to equivalence class set being a partial order set, we translate the incomplete preference problems into the partial order problems. Using the famous Zorn lemma, we get the existence theorems of fixed point for noncontinuous operators in incomplete preference sets. These new fixed point theorems provide a new way to break through the limitation. Finally, the existence of generalized Nash equilibrium is strictly proved in the $n$-person noncooperative games under incomplete preference.


## 1. Introduction and Preliminaries

As a kind of strategy combinations, Nash equilibrium is closely bound up with many important mathematical problems, and many problems in economy and engineering technology can also be described as a Nash equilibrium problem. Recently, the existence of the Nash equilibrium of noncooperative games has been studied [1-4]. In [1], the existence of uncertainty for generalized Nash equilibrium is proved by introducing the uncertainty to study generalized games. Using maximization theorem, the author presented the existence of Nash equilibrium in generalized games, and in these results, the strategy set is noncompact and has infinite players [2]. In [3] the existence theorem of generalized Nash equilibrium in games is given where strategy space has abstract convex structure. Assuming strategy set is a $H$ space, the equilibrium existence theorems have been given in [4]. In the above studies, either the partial order relation on policy sets is required to satisfy, or every total order subset in policy sets must have an upper bound or certain convexity condition.

For a long time, the preference of rational decisionmakers on management and economics should satisfy the completeness. But in practice, they show the indecision on many major issues. Since the preference without
completeness is a kind of more general order structure, it can make preference relation and the partial order relation unified completely. The research of preference without completeness is started from the von Neumann and Morgenstern in [5]. Aumman and Bewley have made the classic study in [ 6,7$]$; Schmeidler has studied the existence of economic equilibrium with infinite number of institutions under incomplete preference in [8].

In noncooperative games, the policy set composed of player's selection strategies is a set which cannot meet the needs of completeness. If preference does not meet the completeness, Pareto optimality is meaningless and the traditional method of partial order will lose effectiveness without the antisymmetry axiom inevitably. Therefore, it is consistent with the realistic decision-making environment to study the existence of generalized Nash equilibrium of noncooperative game, but the study of this part has seldom been seen in the past literature research.

In this article, based on the equivalence class set which corresponds to the elements of incomplete preference set being a partial order set, the problem under incomplete preference is translated into the problem with partial order. This method overcomes the difficulties which are brought about by the elements in the set without the completeness.

Using the famous Zorn lemma, we get the existence theorems of fixed point for noncontinuous operators in incomplete preference sets. The fixed point theorems provide a new way for breaking through the limitations. The existence of generalized Nash equilibrium is strictly proven in the $n$-person noncooperative games under incomplete preference.

Here some concepts and theorems are given, which are related to incomplete preference.

Let $E$ be a nonempty set. An ordering relation $\leq$ on $E$ may satisfy the following axiom:
Reflexive: $x \leq x$, for any $x \in E$;
Symmetry: If $x \leq y$ and $y \leq x$, then $x=y$, for any $x, y \in E$;
Transitive: If $x \leq y$ and $y \leq z$, then $x \leq z$, for any $x, y, z \in E$;
Complete: If $x \leq y, y \leq x$, for any $x, y \in E$, there is at least one inequality to be established.

Definition 1 (see [9]). Let $E$ be a nonempty set. An order relation $\leq$ defined among certain elements of $E$ is said to be partial order if the order relation satisfies reflexive, transitivity, and antisymmetry axioms. Then ( $E \leq$ ) is called a poset.

Definition 2 (see [10]). Let $E$ be a nonempty set. An order relation defined among certain elements of $E$ is said to be incomplete preference order if the order relation satisfies reflexive and transitivity axioms, which is denoted by $\preceq$. If completeness axiom is still satisfied for incomplete preference order, the order relation is said to be preference order, which is still denoted by $\preceq$. Then $(E \preceq)$ is called an incomplete preference set.

Definition 3 (see [11]). Let $E$ be an incomplete preference set. For any $x, y \in E$, we say that $x, y$ are indifference, which is denoted by $x \sim y$, whenever both $x \leq y$ and $y \leq x$ hold.

Remark 4. $x \sim y \nRightarrow x=y$, but $x=y \Longrightarrow x \sim y$.
Remark 5. The indifference relation $\sim$ the equivalence relation.

Definition 6 (see [11]). Let $E$ be an incomplete preference set. If for any complete preference subset of $E$, there is denumerable set $\left\{x_{n}\right\} \subset M$ such that if $x \in M, x \neq \sup M$, there is $x_{n_{0}} \in\left\{x_{n}\right\}$ which satisfies $x \preceq x_{n_{0}}$, then $E$ is said to be pseudo separable in incomplete preference.

Let $E$ be an incomplete preference set, and $\Omega$ is a subset in $E$. The order relation $\leq$ in quotient set $\Omega / \sim$ is elicited by the incomplete preference relation $\leq$ in $E$. Let $[x]=\{y \in \Omega \mid$ $x \sim y\}$, and $[x]$ is an equivalence class set in $\Omega$.

Definition 7 (see [11]). For any $[x],[y] \in \Omega / \sim$, if there are $u \in[x], v \in[y]$ such that $u \leq v$, we write $[x] \leq[y]$.

Lemma 8 (see [11]). Let E be an incomplete preference set, and $\Omega$ is a subset in $E$. The order relation $\leq$ in quotient set $\Omega / \sim$ which is elicited by the incomplete preference relation $\leq$ in $E$ is a partial order. Then the quotient set $\Omega / \sim$ is a poset.

Lemma 9 (see [11]). If $\Omega$ is incomplete preference pseudo separable, then $\Omega / \sim$ is incomplete preference pseudo separable.

Lemma 10 (see [12] (Zorn Lemma)). Let $E$ be a nonempty partial ordered set. If every total ordered subset in $E$ has an upper bound in $E$, then there is a maximal element in $E$.

## 2. Existence Theorems for Fixed Point on Incomplete Preference Sets

Partial order method is discussed and applied greatly in mathematics, and the conclusion on the partial order is becoming a very complete system [13-23]. But few scholars study the fixed point and extreme value theorems on incomplete preference set.

Definition 11. Let $\left(E, \preceq^{E}\right),\left(U, \preceq^{U}\right)$ be incomplete preference sets, and let $T: E \longrightarrow 2^{U}$ be an order-increasing set-valued mapping. $T$ is said to be order-increasing upward, if $x \preceq^{E} y$ in $E$, for any $u \in T(x)$; there is $v \in T(y)$ such that $u \leq{ }^{U} v$; $T$ is said to be order-increasing downward, if $x \leq^{E} y$ in $E$, for any $v \in T(y)$; there is $u \in T(x)$ such that $u \preceq^{U} v$. If $T$ is both order-increasing upward and order-increasing downward, $T$ is said to be order-increasing.

Definition 12. Let $(E, \preceq)$ be incomplete preference set, and let $T: E \longrightarrow 2^{E} \backslash \Phi$ be an order-increasing set-valued mapping. An element $x \in E$ is called a generalized fixed point of $T$, if there are $x^{*} \in E, u \in T x^{*}$ such that $x^{*} \sim u$.

Let $(E, \preceq)$ be incomplete preference set, and let $T: E \longrightarrow$ $2^{u}$ be an order-increasing set-valued mapping. The following notation will be used in Theorem 13:

$$
\begin{equation*}
S T(x)=\{x \in E \mid x \leq u, y \in T(x)\} . \tag{1}
\end{equation*}
$$

Theorem 13. Let $(E, \preceq)$ be an incomplete preference pseudo separable set, and let $T: E \longrightarrow 2^{E}$ be an order-increasing setvalued mapping. If T satisfies the following conditions:
$\left(A_{1}\right)$ Every increasing sequence in $S T(x)$ has an upper bound in $S T(x)$
$\left(A_{2}\right)$ There is a $x_{0} \in E$ with $x_{0} \preceq u$, for some $u \in T x_{0}$
then $T$ has a generalized fixed point; that is, there are $x^{*} \in$ $E, u \in T x^{*}$ such that $x^{*} \sim u$.

Proof. Let $\Omega=\{x \in E \mid x \leq u, y \in T(x)\}$. From the condition $\left(A_{2}\right)$, it implies that $\Omega$ is a nonempty set in $E$. Take an arbitrary total ordered subset $M \subset \Omega$. Since $M$ is also an arbitrary total ordered subset of incomplete preference pseudo separable set $(E, \preceq)$, there is denumerable set $\left\{x_{n}\right\} \subset$ $M$ such that if $x \in M, x \neq \sup M$, there is $x_{n_{0}} \in\left\{x_{n}\right\}$ which satisfy $x \leq x_{n_{0}}$.

Let

$$
\begin{align*}
& z_{1}=x_{1} \\
& z_{n}=\max \left\{x_{n}, z_{n-1}\right\}, \quad n=2,3 \cdots .  \tag{2}\\
& z_{n} \subset M \subset \Omega(x) .
\end{align*}
$$

Since $M$ is an arbitrary total ordered subset, $\left\{z_{n}\right\}$ is well defined. So

$$
\begin{equation*}
z_{1} \preceq z_{2} \preceq \cdots z_{n} \preceq \cdots \tag{3}
\end{equation*}
$$

For any $x \in M, x \neq \sup M$, by the condition $\left(A_{1}\right)$, there is a point $z^{*} \in \Omega$ such that $z_{n} \leq z^{*}$, and since $(E, \preceq)$ is an incomplete preference pseudo separable set, there is a $x_{n_{0}} \in\left\{x_{n}\right\}$ such that $x \leq x_{n_{0}}$. By the definition of $\left\{z_{n}\right\}$, we get

$$
\begin{equation*}
x \leq x_{n_{0}} \leq z_{n} \leq z^{*} . \tag{4}
\end{equation*}
$$

That is, $z^{*} \in \Omega$ is an upper bound of total ordered subset $M$.
Let $[x]=\{y \in \Omega \mid x \sim y\}$; then $[x]$ is equivalence class set in $\Omega$. Assuming that $\Omega / \sim=\{[x], x \in \Omega\}$ is a quotient set corresponding to the equivalence relation $\sim$, then applying Lemmas 8 and 9 , we get that the order relation $\leq$ in quotient set $\Omega / \sim$ which is elicited by the incomplete preference relation $\leq$ in $\Omega$ is a partial order.

Take an arbitrary total ordered subset $N \subset \Omega / \sim$, next, to show that the set $N$ has an upper bound in $\Omega / \sim$.

Let

$$
\begin{equation*}
W=\bigcup[x] \subset \Omega \tag{5}
\end{equation*}
$$

It is easy to know that $W$ is total ordered subset in $\Omega$. In fact, for any $x, y \in W$, there are $[x],[y] \in N$. Since $N$ is total ordered subset in $\Omega / \sim$, we can get that either $[x] \leq[y]$ or $[y] \leq[x]$ is valid. According to the relation between partial order and incomplete preference, we have that either $x \leq y$ or $y \leq x$ is valid. So $W$ is total ordered subset in $\Omega$.

For any $[z] \in N$, by the definition of $W$, we get $z \in W$. Since every total ordered subset in $\Omega$ has an upper bound, there is $x_{0} \in W$ such that $z \leq x_{0}$. So $[z] \leq\left[x_{0}\right]$; that is, every total ordered subset in $\Omega / \sim$ has an upper bound. Then applying Zorn lemma, we have that there is a maximal element $\left[x^{*}\right]$ in $\Omega / \sim$. By the definition of $\Omega / \sim$, we have that $x^{*}$ is the maximal element in $\Omega$.

Since $x^{*}$ is the maximal element in $\Omega$, there is $u \in T\left(x^{*}\right)$ such that $x^{*} \preceq u$. Supposing that $u \npreceq x^{*}$, the monotonicity of $T$ together with $u \in T\left(x^{*}\right), x^{*} \preceq u$ implies that there is $v \in$ $T(u)$ such that $u \leq v$. It means $u \in \Omega$, and it is contradictory with which $x^{*}$ is the maximal element in $\Omega$. So $u \leq x^{*}$ is proved. Hence there are $x^{*} \in E, u \in T x^{*}$ such that $x^{*} \sim u$; that is, $T$ has a generalized fixed point.

Corollary 14. Let $(E, \preceq)$ be an incomplete preference pseudo separable set, and let $T: E \longrightarrow 2^{E} \backslash \Phi$ be an order-increasing set-valued mapping. If $T$ satisfies the following conditions:
$\left(A_{1}\right)$ Every increasing sequence in $S T(x)$ has an upper bound in ST(x)
$\left(A_{2}\right)$ There is $x_{0} \in E$ with $x_{0} \leq u$, for some $u \in T x_{0}$
$\left(A_{3}\right)$ If $x \sim y, x, y \in E$, then $T x=T y$
then $T$ has a fixed point; that is, there are $x^{*} \in E, u \in T x^{*}$ such that $x^{*} \in T x^{*}$.

## 3. Existence of Nash Equilibrium Points in Generalized Games under Incomplete Preferences

The incomplete preference we present in the paper is more general order relation than the preference in the field of economic management and coincident with the reality of economic phenomenon. It can be applied to the existence
of generalized Nash equilibrium of noncooperative game theory.

Definition 15. Let $n$ be a positive integer greater than 1 . An $n$-person noncooperative game consists of the following elements:
(1) The set of $n$ players is denoted by $I=\{i=1,2,3, \ldots, n\}$;
(2) For any $i \in I$, let $S_{i}$ be the strategy set of player $i$ and $\left(S_{i}, \preceq_{i}\right)$ be an incomplete preference pseudo separable set; denote $S=S_{1} \times S_{2} \times \cdots \times S_{n}$;
(3) Let $P_{i}: S \longrightarrow U, i=1,2,3, \ldots, n$ be the payoff function for player $i$; denote $P=\left\{P_{1}, P_{2}, \ldots, P_{n}\right\}$.

The game is denoted by $\Gamma=(N, S, P, U)$.
Every player in the $n$-person noncooperative game independently chooses his own strategy $x_{i} \in S_{i}, i=1,2,3, \ldots, n$, to maximize his payoff function $P_{i}\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in U$. For any $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S$, denote

$$
\begin{align*}
& x_{-i}=\left(x_{1}, x_{2}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{n}\right) \in S  \tag{6}\\
& S_{-i}=S_{1} \times S_{2} \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_{n} .
\end{align*}
$$

Then $x_{-i} \in S_{-i}$, and $x$ can be written as $x=\left(x_{i}, x_{-i}\right)$.
Definition 16. Let $\Gamma=(N, S, P, U)$ be an $n$-person noncooperative game. The strategy $\widehat{x}=\left(\widehat{x}_{1}, \widehat{x}_{2}, \ldots, \widehat{x}_{n}\right) \in S$ is said to be a generalized Nash equilibrium in the noncooperative game $\Gamma=(N, S, P, U)$ under the incomplete preference, if there is strategy $\widehat{x}=\left(\widehat{x}_{1}, \widehat{x}_{2}, \ldots, \widehat{x}_{n}\right) \in S$, for every $i=1,2,3, \ldots, n$; the following order inequality holds

$$
\begin{equation*}
P_{i}\left(x_{i}, \widehat{x}_{-i}\right) \not ¥^{U} P_{i}\left(\widehat{x}_{i}, \widehat{x}_{-i}\right), \quad \forall x_{i} \in S_{i} . \tag{7}
\end{equation*}
$$

Lemma 17. Let $\left(S_{i}, \leq^{s_{i}}\right)$ be an incomplete preference pseudo separable set. $S=S_{1} \times S_{2} \times \cdots \times S_{n}$ is a coordinate ordering set composed of $S_{1}, S_{2}, \ldots, S_{n}$, for any $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right), y=$ $\left(y_{1}, y_{2}, \ldots, y_{n}\right) \in S$, the order relation $\leq^{s}$ in $S$ induced by the partial order $\leq^{S_{i}}$, denoted as the following:

$$
\begin{equation*}
x \leq^{s} y \Longleftrightarrow x_{i} \leq^{s_{i}} y_{i}, \quad \forall i=1,2,3, \ldots, n . \tag{8}
\end{equation*}
$$

Then $\left(S, \preceq^{s}\right)$ is an incomplete preference pseudo separable set.
Proof. First we show that ( $S, \preceq^{s}$ ) is an incomplete preference set. Since $\left(S_{i}, \preceq^{s_{i}}\right)$ is incomplete preference set, for any $x=$ $\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S$, we can get $x_{i} \simeq^{s_{i}} x_{i}, \forall i=1,2,3, \ldots, n$. So $x \preceq^{s} x$. Hence the order relation $\preceq^{s}$ satisfies reflexive axiom.

For any $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right), y=\left(y_{1}, y_{2}, \ldots, y_{n}\right)$, and $z=\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in S$, which satisfy $x \preceq^{s} y, y \preceq^{s} z \in S$. By Definition 15, we have $x_{i} \leq^{S_{i}} y_{i}, y_{i} \leq^{S_{i}} z_{i}$. Since ( $S_{i}, \leq^{s_{i}}$ ) is incomplete preference set, we have $x_{i} \leq^{s_{i}} z_{i}$. So $x \unlhd^{s} z$; that is, $S=S_{1} \times S_{2} \times \cdots \times S_{n}$ is incomplete preference set.

Next we prove that the incomplete preference set ( $S, \preceq^{s}$ ) is pseudo separable, for any $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S$, since ( $S_{i}, \leq^{s_{i}}$ ) is pseudo separable,

Let $M_{i}$ be an arbitrary total ordered subset of set $S_{i}$; then there is denumerable set $\left\{x_{i}^{n}\right\} \subset M_{i}$ such that if $x_{i} \in M_{i}, x_{i} \neq$ $\sup M_{i}$, there is $x_{i_{n_{0}}}^{n} \in\left\{x_{i}^{n}\right\}$ which satisfies $x_{i}^{n} \leq x_{i_{n_{0}}}^{n}$.

Define the following

$$
\begin{equation*}
M=M_{1} \times M_{2} \times \cdots \times M_{n} . \tag{9}
\end{equation*}
$$

Let $x^{n}=\left\{x_{1}^{n}, x_{2}^{n}, \ldots, x_{n}^{n}\right\}, x^{n}{ }_{n_{0}}=\left\{x_{1 n_{0}}^{n}, x_{2 n_{0}}^{n}, \ldots, x_{n_{0}}^{n}\right\}$; then by the definition of $M$, we have $x^{n} \in M$. This together with $x^{n}{ }_{n_{0}} \in\left\{x^{n}\right\}$ implies that $x^{n} \leq^{s} x^{n}{ }_{n_{0}}$. Hence the incomplete preference set $\left(S, \preceq^{s}\right)$ is pseudo separable.

Theorem 18. Let $\Gamma=(N, S, P, U)$ be an $n$-person noncooperative game. Suppose that, for any $x \in S$, the payoff function $P_{i}, i=1,2,3, \ldots, n$, satisfies the following conditions:
$\left(G_{1}\right)$ Every total ordered subset in $P_{i}\left(S_{i}, x_{-i}\right)$ has an upper bound in $P_{i}\left(S_{i}, x_{-i}\right)$;
$\left(G_{2}\right)$ Every increasing sequence in the inverse image $\left\{z_{i} \in\right.$ $S_{i}: P_{i}\left(S_{i}, x_{-i}\right)$ is a maximal element of $\left.P_{i}\left(S_{i}, x_{-i}\right)\right\}$ has an upper bound;
$\left(G_{3}\right)$ For any $x, y \in S, x \preceq^{S} y$, if there is $z_{i} \in S_{i}$ with $P_{i}\left(z_{i}, x_{-i}\right)$ to be a maximal element of $P_{i}\left(S_{i}, x_{-i}\right)$, then there is $\omega_{i} \in S_{i}$ with $z_{i} \leq^{s_{i}} \omega_{i}$ such that $P_{i}\left(\omega_{i}, y_{-i}\right)$ is a maximal element of $P_{i}\left(S_{i}, y_{-i}\right)$;
$\left(G_{4}\right)$ If there are $p, q \in S$ such that $p \leq^{s} q$ and $P_{i}\left(q_{i}, p_{-i}\right)$ is a maximal element of $P_{i}\left(S_{i}, p_{-i}\right)$;
$\left(G_{5}\right)$ If $p \sim^{s} q, p, q \in S$ such that $P_{i}\left(p_{i}, x_{-i}\right)=P_{i}\left(q_{i}, x_{-i}\right)$.
Then there is a generalized Nash equilibrium in the $n$-person noncooperative game $\Gamma=(N, S, P, U)$.

Proof. Since $\left(S_{i}, \preceq^{s_{i}}\right)$ is an incomplete preference pseudo separable set, for every $i=1,2,3, \cdots, n$, then, from Lemma 17 , $\left(S, \unlhd^{s}\right)$ is also an incomplete preference pseudo separable set equipped with the product order $\preceq^{s}$.

For every fixed $i=1,2,3, \ldots, n$, define a set-valued mapping $T_{i}: S \longrightarrow 2^{S_{i}} \backslash \Phi$ as the following:

$$
\begin{align*}
& T_{i}(x)=\left\{z_{i} \in S_{i}:\right.  \tag{10}\\
& \left.\quad P_{i}\left(S_{i}, x_{-i}\right) \text { is a maximal element of } P_{i}\left(S_{i}, x_{-i}\right)\right\}
\end{align*}
$$

for all $x=\left(x_{1}, x_{2}, \ldots, x_{n}\right) \in S$.
From assumption $G_{1}$ of this theorem, for every fixed element $x \in S$, every total ordered subset in $P_{i}\left(S_{i}, x_{-i}\right)$ has an upper bound in $\left(U, \preceq^{U}\right)$. Then applying Zorn Lemma, the set $\left.P_{i}\left(S_{i}, x_{-i}\right)\right\}$ has a maximal element. Therefore, $T_{i}(x)$ is a nonempty subset of $S_{i}$. Then we define

$$
\begin{equation*}
T(x)=T_{1}(x) \times T_{2}(x) \times \cdots \times T_{n}(x), \quad x \in S \tag{11}
\end{equation*}
$$

For any arbitrary $x \in S$, with respect to the set $T(x)$, we write

$$
\begin{equation*}
S T(x)=\left\{x \in S: x \leq^{s} z, z \in T(x)\right\} . \tag{12}
\end{equation*}
$$

For every $i=1,2,3, \ldots, n$, we have

$$
\begin{align*}
& S T_{i}(x)=\left\{x_{i} \in S_{i}: x_{i} \leq^{s_{i}} z_{i}, z_{i} \in T_{i}(x)\right\}=\left\{x_{i} \in S_{i}: x_{i}\right. \\
& \quad \leq^{s_{i}} z_{i}, z_{i} \in S_{i},  \tag{13}\\
& \left.\quad P_{i}\left(z_{i}, x_{-i}\right) \text { is a maximal element of } P_{i}\left(S_{i}, x_{-i}\right)\right\} .
\end{align*}
$$

Now we prove that the operator $T$ satisfies the conditions in Theorem 18. Firstly, we will show that the operator $T$ is order-increasing. For any given $x \preceq^{S} y$ in $S$, and for any $z=$ $\left(z_{1}, z_{2}, \ldots, z_{n}\right) \in T(x)$, for every $i=1,2, \ldots, n$, we have
$z_{i} \in T_{i}(x)$; that is, $P_{i}\left(z_{i}, x_{-i}\right)$ is a maximal element of $P_{i}\left(S_{i}, x_{-i}\right)$. Then from hypothesis $G_{3}$ of this theorem, there is $\omega_{i} \in S_{i}$ with $z_{i} \leq^{s_{i}} \omega_{i}$ such that $P_{i}\left(\omega_{i}, y_{-i}\right)$ is a maximal element of $P_{i}\left(S_{i}, y_{-i}\right)$; that is, $\omega_{i} \in T_{i}(y)$. Let $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{n}\right)$. We obtain that $z \preceq^{S} \omega$ and $\omega \in D(y)$. Hence the operator $T$ is order-increasing.

From assumption $G_{2}$ of this theorem, every increasing sequence in $T_{i}(x)$ has an upper bound. Then we can similarly show that every increasing sequence in $S T_{i}(x)$ has an upper bound. In fact, take an arbitrary total ordered subset $M \subset$ $S T(x)$. Since $M$ is also a total ordered subset of incomplete preference pseudo separable set $(E, \succeq)$, there is denumerable set $\left\{x_{n}\right\} \subset M$ such that for any $x \in M, x \neq \sup M$, there are $x_{n_{0}} \in\left\{x_{n}\right\}, u(x) \in T(x)$, which satisfy $x \leq^{S} x_{n_{0}}, x \leq^{S} u(x)$.

From the above discussion, we have that there is $e(x) \in$ $T\left(x_{n_{0}}\right)$ such that $u(x) \preceq^{S} e(x)$. Thus we obtain a mapping $e$ : $S T(x) \longrightarrow T(x)$ satisfying the following order inequality: $x \preceq u(x) \preceq^{S} e(x), u(x) \in T(x)$ with $e(x) \in T(x)$. For any an increasing sequence $\left\{x_{n}\right\}$ in $\operatorname{ST}(x)$, we have that there is $z^{*} \in \Omega$ such that $x_{n} \preceq^{S} z^{*}$. By the definition of $\left\{z_{n}\right\}$, we get

$$
\begin{equation*}
x \leq^{S} x_{n_{0}} . \tag{14}
\end{equation*}
$$

So we have that $\left\{x_{n}\right\}$ is an increasing sequence; then $\left\{e\left(x_{n}\right)\right\}$ is an increasing sequence in $T(x)$. since every increasing sequence in $T(x)$ has an upper bound in $T(x)$, we have that there is $z^{*} \in T(x)$ such that $e\left(x_{n}\right) \preceq^{S} z^{*}$. Since $x_{n} \unlhd^{S} e\left(x_{n}\right)$, we have $x_{n} \preceq^{S} z^{*}$. Hence every increasing sequence in $\operatorname{ST}(x)$ has an upper bound in $S T(x)$.

Then applying Theorem 13, it implies that the operator $T$ has a generalized point; that is, there are $x^{*} \in E, u \in T x^{*}$ such that $x^{*} \sim^{s} u$. Since the operator $T$ is order-increasing, there are $u \in T\left(x^{*}\right)$ and $v \in T(u)$ such that $u \leq^{s} v$.

From assumption $G_{5}$ of this theorem, if $x^{*} \sim^{s} u$, we can get

$$
\begin{equation*}
P_{i}\left(x_{i}^{*}, x_{-i}\right)=P_{i}\left(u_{i}, x_{-i}\right) . \tag{15}
\end{equation*}
$$

Since $u \in T\left(x^{*}\right)$, we have that $P_{i}\left(u_{i}, x_{-i}^{*}\right)$ is a maximal element of $P_{i}\left(S_{i}, x_{-i}^{*}\right)$. This together with $P_{i}\left(x_{i}^{*}, x_{-i}\right)=$ $P_{i}\left(u_{i}, x_{-i}\right)$ implies that $P_{i}\left(x_{i}^{*}, x_{-i}^{*}\right)$ is a maximal element of $P_{i}\left(S_{i}, x_{-i}^{*}\right)$. It implies that for any $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right) \in S$, $i=1,2,3, \ldots, n$, the following inequality is established

$$
\begin{equation*}
P_{i}\left(x_{i}, x_{-i}^{*}\right) \not \not ㇒^{U} P_{i}\left(x_{i}^{*}, x_{-i}\right), \quad \forall x_{i} \in S_{i} . \tag{16}
\end{equation*}
$$

This shows that $x^{*}=\left(x_{1}^{*}, x_{2}^{*}, \ldots, x_{n}^{*}\right) \in S$ is a generalized Nash equilibrium in the $n$-person noncooperative game $\Gamma=$ $(N, S, P, U)$.

## 4. Conclusion

Incomplete preference is more general order relation than complete preference in the field of economic management, because restriction on order relation is eased. So it is more consistent with the reality of economic management phenomenons. The generalized game model under the incomplete preference can play an important application in economic management problems. Generalized game plays an important role to prove existence of general equilibrium.

But many economic problems ultimately come down to nonlinear problems which are denoted by the utility function without preference in an order infinite dimensional space. The traditional general game model cannot deal with the problems such as the utility function without preference, incomplete preference, order infinite dimension space, or nonlinear problem. Now since there are no ready-made methods to deal with the problems, new research methods must be sought. The generalized game model under the incomplete preference, which is proposed in this paper, can deal with the problems. But in this paper, the research is limited to the existence of the equilibrium. The stability of equilibrium and new game model which are more close to the reality are our next step research direction.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

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# Positive Solutions for Higher Order Nonlocal Fractional Differential Equation with Integral Boundary Conditions 

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#### Abstract

In this paper, by using the spectral analysis of the relevant linear operator and Gelfand's formula, some properties of the first eigenvalue of a fractional differential equation were obtained; combining fixed point index theorem, sufficient conditions for the existence of positive solutions are established. An example is given to demonstrate the application of our main results.


## 1. Introduction

Fractional differential equations describe many phenomena in various fields of engineering and scientific disciplines such as physics, biophysics, chemistry, control theory, signal and image processing, and aerodynamics (see [1, 2]). In the last decade, boundary value problems of fractional calculus have received a great attention, based on various analytic techniques, and a variety of results concerning the existence of solutions can be found in the literature [3-19]. For example, Zhang [17] studied the following singular fractional boundary value problem (FBVP):

$$
\begin{align*}
D_{0+}^{\alpha} y(t)+q(t) f\left(t, y(t), y^{\prime}(t), \ldots, y^{(n-2)}(t)\right) & =0, \\
0 & <t<1, \tag{1}
\end{align*}
$$

$$
y(0)=y^{\prime}(0)=\cdots=y^{(n-2)}(0)=y^{(n-2)}(1)=0
$$

where $\alpha \in(n-1, n], n \geq 2, D_{0+}^{\alpha}$ is the standard RiemannLiouville derivative, the nonlinear term $f=g+h$, and $g$ and $h$ have different monotone properties. By using a mixed monotone method, a unique positive solution is obtained.

Integral boundary conditions arise in thermal conduction problems [20], semiconductor problems [21], and hydrodynamic problems [22]. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations with integral boundary value problems have received a great deal of attention. To identify a few, see [23-27] and the references therein for integer order integral boundary
value problems and [28-33] for fractional order integral boundary value problems. In [28], authors using monotone iterative technique investigated the existence and uniqueness of the positive solutions of higher-order nonlocal fractional differential equations of the type

$$
\begin{array}{r}
D_{0+}^{\alpha} y(t)+f(t, y(t))=0, \quad 0<t<1, n-1<\alpha \leq n \\
y(0)=y^{\prime}(0)=\cdots=y^{(n-2)}(0)=0  \tag{2}\\
y(1)=\lambda[y]
\end{array}
$$

where $f \in C\left((0,1) \times \mathbb{R}^{+}, \mathbb{R}^{+}\right)$. By means of Schauders fixed point theorem, FBVP (2) are also studied in [29]. In [30], the authors investigated problems (2) with $f(t, y(t))$ replaced by $q(t) f(t, y(t))$, and the existence and multiplicity of positive solutions are obtained by means of the fixed point index theory in cones.

Inspired by the work of the above papers, the aim of this paper is to establish the existence of positive solutions for the following nonlinear fractional differential equation involving Stieltjes integrals conditions:

$$
-D_{0+}^{\alpha} y(t)=f\left(t, y(t), y^{\prime}(t), \ldots, y^{(n-2)}(t)\right)
$$

$$
0<t<1,
$$

$$
\begin{align*}
& y(0)=y^{\prime}(0)=\cdots=y^{(n-2)}(0)=0  \tag{3}\\
& \qquad y^{(n-2)}(1)=\lambda\left[y^{(n-2)}\right]
\end{align*}
$$

where $D_{0+}^{\alpha}$ is the Riemann-Liouville fractional derivative, $n-1<\alpha \leq n, n \geq 2, \lambda[z]=\int_{0}^{1} z(t) d A(t)$ is a linear functional on $C[0,1]$ given by a Stieltjes integral with $A$ representing a suitable function of bounded variation, and $d A$ can be a signed measure. $f:(0,1) \times\left(\mathbb{R}_{0}^{+}\right)^{n-1} \longrightarrow \mathbb{R}^{+}$is a continuous function, the nonlinearity $f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)$ may be singular at $t=0,1$, and $z_{1}=z_{2}=\cdots=z_{n-1}=0$. Here $\mathbb{R}^{+}=[0,+\infty)$ and $\mathbb{R}_{0}^{+}=(0,+\infty)$.

Our work presented in this paper has the following features. First of all, we discuss the boundary value problem with the Stieltjes integral boundary conditions $\lambda[z]=\int_{0}^{1} z(t) d A(t)$ appearing in the boundary conditions of FBVP (2) as more general which was not considered in the literature [15], covering two-point, multipoint, and nonlocal problems as special cases. The second new feature is that the nonlinearity $f$ is allowed to depend on higher order derivatives of unknown function $y(t)$ up to $n-2$ order which was not considered in the literature [28-30]. Thirdly, we allow that the nonlinearity $f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)$ may be singular at $t=0,1$ and $z_{1}=z_{2}=$ $\cdots=z_{n-1}=0$. Therefore, our study improves and extends the previous results to some degree in the relevant literature [17, 28-30].

The rest of this paper is organized as follows. In Section 2, we present some lemmas that are used to prove our main results. In Section 3, the existence of positive solutions is established under some sufficient conditions. In Section 4, an example is given to demonstrate the application of our theoretical results.

## 2. Basic Definitions and Preliminaries

In this paper, our work is based on the theory of fractional calculus, and, for details on definitions of Riemann-Liouville fractional calculus, we refer the reader to $[1,2]$.

Let $E=C[0,1]$, and then $E$ is a Banach space with the norm $\|y\|=\max _{0 \leq t \leq 1}|y(t)|$, for any $y \in E$. Let $P=\{y \in E$ : $y(t) \geq 0$ for $t \in[0,1]\}$ be a cone in $E$ and construct a subcone of $P$ as follows:

$$
\begin{equation*}
K=\left\{y \in P: y(t) \geq t^{\alpha-n+1}(1-t)\|y\| \text { for } t \in[0,1]\right\} \tag{4}
\end{equation*}
$$

For any $r>0$, let $K_{r}=\{y \in K:\|y\|<r\}, \partial K_{r}=\{y \in K$ : $\|y\|=r\}$, and $\bar{K}_{r}=\{y \in K:\|y\| \leq r\}$.

Set

$$
\begin{align*}
& G(t, s)=\frac{1}{\Gamma(\alpha-n+2)} \\
& \quad \cdot \begin{cases}{[t(1-s)]^{\alpha-n+1}-(t-s)^{\alpha-n+1},} & 0 \leq s \leq t \leq 1, \\
{[t(1-s)]^{\alpha-n+1},} & 0 \leq t \leq s \leq 1 .\end{cases} \tag{5}
\end{align*}
$$

Lemma 1 (see [17]). Let $q \in C_{r}[0,1]\left(C_{r}[0,1]=\{q \in\right.$ $\left.\left.C[0,1], t^{r} q \in C[0,1], 0 \leq r<1\right\}\right)$. Then the boundary value problem

$$
\begin{aligned}
& -D_{0+}^{\alpha-n+2} u(t)=q(t), \\
& \quad 0<t<1, n-1<\alpha \leq n, n \geq 2, \\
& u(0)=0, \quad u(1)=0
\end{aligned}
$$

has a unique solution

$$
\begin{equation*}
u(t)=\int_{0}^{1} G(t, s) q(s) d s \tag{7}
\end{equation*}
$$

According to the theory of fractional calculus, the unique solution of the problem

$$
\begin{align*}
& -D_{0+}^{\alpha-n+2} u(t)=q(t) \\
& \quad 0<t<1, n-1<\alpha \leq n, n \geq 2,  \tag{8}\\
& u(0)=0, \quad u(1)=\lambda[u]
\end{align*}
$$

is $\gamma(t)=t^{\alpha-n+1}$ with $\lambda[u]$ replaced by 1 and $q(t) \equiv 0$. As in [27], Green's function for boundary value problem (8) is given by

$$
\begin{equation*}
H(t, s)=\frac{\gamma(t)}{1-\lambda[\gamma]} \mathscr{G}(s)+G(t, s) \tag{9}
\end{equation*}
$$

where $\mathscr{G}(s):=\int_{0}^{1} G(t, s) d A(t)$.
Lemma 2. Suppose that $\mathscr{G}(s) \geq 0$ for $s \in[0,1]$ and $0 \leq \lambda[\gamma]<$ 1 , and then the functions $G(t, s)$ and $H(t, s)$ have the following properties:
(1) $G(t, s)$ and $H(t, s)$ are nonnegative and continuous for $(t, s) \in[0,1] \times[0,1]$.
(2) $G(t, s)$ satisfies

$$
\begin{align*}
& \frac{t^{\alpha-n+1}(1-t) s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \leq G(t, s)  \tag{10}\\
& \quad \leq \frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)}, \quad \text { for } t, s \in[0,1]
\end{align*}
$$

(3) $H(t, s)$ satisfies

$$
\begin{equation*}
t^{\alpha-n+1}(1-t) \Phi(s) \leq H(t, s) \leq \Phi(s), \tag{11}
\end{equation*}
$$

$$
\text { for } t, s \in[0,1]
$$

where

$$
\begin{equation*}
\Phi(s)=\frac{\mathscr{G}(s)}{1-\lambda[\gamma]}+\frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \tag{12}
\end{equation*}
$$

Throughout this paper, we adopt the following assumptions:
$\left(\mathbf{H}_{0}\right) A$ is a function of bounded variation, $\mathscr{G}(s) \geq 0$ for $s \in[0,1]$ and $0 \leq \lambda[\gamma]<1$.
$\left(\mathbf{H}_{1}\right) f:(0,1) \times\left(\mathbb{R}_{0}^{+}\right)^{n-1} \longrightarrow \mathbb{R}^{+}$is a continuous function and, for any $0<r<R<+\infty$,

$$
\begin{align*}
& \lim _{m \rightarrow+\infty} \sup _{\substack{z_{1} \in K_{R /(n-2)!} \\
z_{n-2}-K_{R} \\
z_{n-1} \in \overline{K_{R}} \backslash K_{r}}} \int_{H(m)} \Phi(s) \\
& \quad \cdot f\left(s, z_{1}(s), z_{2}(s), \ldots, z_{n-1}(s)\right) d s=0, \tag{13}
\end{align*}
$$

where $H(m)=[0,1 / m] \cup[(m-1) / m, 1]$.

In order to overcome the difficulty due to the dependance of $f$ on derivatives, we consider the following modified problem:

$$
\begin{aligned}
& -D_{0+}^{\alpha-n+2} x(t) \\
& \quad=f\left(t, I_{0+}^{n-2} x(t), I_{0+}^{n-3} x(t), \ldots, I_{0+}^{1} x(t), x(t)\right), \\
& \qquad 0<t<1, \\
& x(0)=0, \quad x(1)=\lambda[x]
\end{aligned}
$$

where $n-1<\alpha \leq n, n \geq 2$.
Lemma 3 (see [15]). The nonlocal FBVP (3) has a positive solution $y(t)=I_{0+}^{n-2} x(t)$ if and only if the nonlinear fractional integrodifferential equation (14) has a positive solution $x(t)$.

Define a nonlinear operator $T: K \backslash\{0\} \longrightarrow P$ and a linear operator $L: E \longrightarrow E$ as follows:

$$
\begin{align*}
& (T x)(t)=\int_{0}^{1} H(t, s) f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots,\right.  \tag{15}\\
& \left.\quad I_{0+}^{1} x(s), x(s)\right) d s, \quad t \in[0,1] \\
& (L x)(t)=\int_{0}^{1} H(t, s) x(s) d s, \quad t \in[0,1] \tag{16}
\end{align*}
$$

Observe that problem (14) has solutions if the operator equation $x=T x$ has fixed points.

Lemma 4 (Krein-Rutmann [34]). Let $L: E \longrightarrow E$ be a continuous linear operator, $P$ be a total cone, and $L(P) \subset P$. If there exist $\psi \in E \backslash(-P)$ and a positive constant $c$ such that $c L(\psi) \geq \psi$, then the spectral radius $r(L) \neq 0$ and $L$ has a positive eigenfunction corresponding to its first eigenvalue $\lambda=(r(L))^{-1}$.

Lemma 5 (Gelfand's formula [34]). For a bounded linear operator $L$ and the operator norm $\|\cdot\|$, the spectral radius of $L$ satisfies

$$
\begin{equation*}
r(L)=\lim _{m \rightarrow+\infty}\left\|L^{m}\right\|^{1 / m} \tag{17}
\end{equation*}
$$

Lemma 6. Assume that $\left(H_{0}\right)$ holds. Then $L: K \longrightarrow K$ defined by (16) is a completely continuous linear operator, and the spectral radius $r(L) \neq 0$; moreover $L$ has a positive eigenfunction $\varphi^{*}$ corresponding to its first eigenvalue $\lambda_{1}=$ $(r(L))^{-1}$.

Proof. For any $x \in K$, by Lemma 2, we can obtain

$$
\begin{equation*}
\|L x\|=\max _{t \in[0,1]} \int_{0}^{1} H(t, s) x(s) d s \leq \int_{0}^{1} \Phi(s) x(s) d s \tag{18}
\end{equation*}
$$

On the other hand, from Lemma 2, we also have

$$
\begin{align*}
(L x)(t) & =\int_{0}^{1} H(t, s) x(s) d s \\
& \geq t^{\alpha-n+1}(1-t) \int_{0}^{1} \Phi(s) x(s) d s \tag{19}
\end{align*}
$$

Then (18) and (19) yield that

$$
\begin{equation*}
(L x)(t) \geq t^{\alpha-n+1}(1-t)\|L x\| \tag{20}
\end{equation*}
$$

Consequently, $L: K \longrightarrow K$. And, from the uniform continuity of $H(t, s),(t, s) \in[0,1] \times[0,1]$, we have that $L: K \longrightarrow K$ is a completely continuous linear operator.

Next we will show that $L$ has the first eigenvalue $\lambda_{1}>0$ by using Krein-Rutmann's theorem. In fact, by Lemma 2, there exists $t_{0} \in(0,1)$ such that $H\left(t_{0}, t_{0}\right)>0$. Thus there exists $[a, b] \subset(0,1)$ such that $t_{0} \in(a, b)$ and $H(t, s)>0$ for all $t, s \in[a, b]$. Choose $x \in K$ such that $x\left(t_{0}\right)>0$ and $x(t)=0$ for all $t \notin[a, b]$. Then, for any $t \in[a, b]$, we have

$$
\begin{equation*}
(L x)(t)=\int_{0}^{1} H(t, s) x(s) d s \geq \int_{a}^{b} H(t, s) x(s) d s \tag{21}
\end{equation*}
$$

$$
>0
$$

So there exists $c>0$ such that $c(L x)(t) \geq x(t)$ for $t \in[0,1]$. From Lemma 4, we know that the spectral radius $r(L) \neq 0$ and $L$ has a positive eigenfunction $\varphi^{*}$ corresponding to its first eigenvalue $\lambda_{1}=(r(L))^{-1}$; i.e., $\lambda_{1} L \varphi^{*}=\varphi^{*}$. The proof is completed.

Lemma 7. Assume that $\left(H_{0}\right),\left(H_{1}\right)$ hold; then $T: \bar{K}_{R} \backslash K_{r} \longrightarrow$ $K$ is completely continuous.

Proof. First, we prove $T\left(\bar{K}_{R} \backslash K_{r}\right) \subset K$. In fact, for any $x \in$ $\bar{K}_{R} \backslash K_{r}$, Lemma 2 implies that

$$
\begin{gather*}
(T x)(t)=\int_{0}^{1} H(t, s) f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots,\right. \\
\left.I_{0+}^{1} x(s), x(s)\right) d s \leq \int_{0}^{1} \Phi(s) f\left(s, I_{0+}^{n-2} x(s)\right.  \tag{22}\\
\left.I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s, \quad t \in[0,1]
\end{gather*}
$$

Hence,

$$
\begin{align*}
& \|T x\| \leq \int_{0}^{1} \Phi(s) f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s)\right.  \tag{23}\\
& x(s)) d s
\end{align*}
$$

On the other hand, from Lemma 2, we also have

$$
\begin{align*}
& (T x)(t)=\int_{0}^{1} H(t, s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \geq t^{\alpha-n+1}(1-t) \int_{0}^{1} \Phi(s)  \tag{24}\\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \geq t^{\alpha-n+1}(1-t)\|T x\|
\end{align*}
$$

Therefore, $T\left(\bar{K}_{R} \backslash K_{r}\right) \subset K$.

Next, for any $r>0$, we assert that

$$
\begin{align*}
& \sup _{x \in \partial K_{r}} \int_{0}^{1} \Phi(s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s  \tag{25}\\
& \quad<+\infty
\end{align*}
$$

which implies that $T: K \backslash\{0\} \longrightarrow P$ is well defined. In fact, it follows from $\left(H_{1}\right)$ that there exists a natural number $m$ such that

$$
\begin{aligned}
& \sup _{x \in \partial K_{r}} \int_{H(m)} \Phi(s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s
\end{aligned}
$$

$$
<1
$$

For any $x \in \partial K_{r}$, as $x \in K$, we have

$$
\begin{equation*}
t^{\alpha-n+1}(1-t)\|x\| \leq x(t) \leq\|x\|=r, \quad t \in[0,1] \tag{27}
\end{equation*}
$$

and

$$
\begin{align*}
& \frac{r t^{\alpha-n+i+1}(1-t) \Gamma(\alpha-n+2)}{\Gamma(\alpha-n+i+2)} \\
& =\frac{\|x\| t^{\alpha-n+i+1}(1-t) \mathbf{B}(i, \alpha-n+2)}{\Gamma(i)} \\
& \quad \leq \frac{\|x\|}{\Gamma(i)} \int_{0}^{t}(t-s)^{i-1} s^{\alpha-n+1}(1-t) d s \leq I_{0+}^{i} x(t)  \tag{28}\\
& =\frac{1}{\Gamma(i)} \int_{0}^{t}(t-s)^{i-1} x(s) d s \leq \frac{1}{i!}\|x\|=\frac{r}{i!} \\
& \quad t \in[0,1], i=1,2, \ldots, n-2
\end{align*}
$$

where $\mathbf{B}(\cdot, \cdot)$ is the beta function. So, for any $1 / m \leq t \leq 1-$ $1 / m,(27)$ and (28) yield that

$$
\begin{aligned}
\frac{r}{m^{\alpha-n+2}} \leq x(t) & \leq r \\
\frac{\Gamma(\alpha-n+2) r}{m^{\alpha-n+i+2} \Gamma(\alpha-n+i+2)} \leq I_{0+}^{i} x(t) & \leq \frac{r}{i!}, \\
& i=1,2, \ldots, n-2 .
\end{aligned}
$$

From (26) and (29), we have

$$
\begin{align*}
& \sup _{x \in \partial K_{r}} \int_{0}^{1} \Phi(s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \leq \sup _{x \in \partial K_{r}} \int_{H(m)} \Phi(s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s  \tag{30}\\
& \quad+\sup _{x \in \partial K_{r}} \int_{1 / m}^{1-1 / m} \Phi(s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad<1+M_{1} \int_{0}^{1} \Phi(s) d s<+\infty,
\end{align*}
$$

where

$$
\begin{align*}
M_{1} & =\max \left\{f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right):\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)\right. \\
& \in\left[\frac{1}{m}, 1-\frac{1}{m}\right] \times\left[\frac{\Gamma(\alpha-n+2) r}{m^{\alpha} \Gamma(\alpha)}, \frac{r}{(n-2)!}\right]  \tag{31}\\
& \times\left[\frac{\Gamma(\alpha-n+2) r}{m^{\alpha-1} \Gamma(\alpha-1)}, \frac{r}{(n-3)!}\right] \times \cdots \\
& \left.\times\left[\frac{\Gamma(\alpha-n+2) r}{m^{\alpha-n+3} \Gamma(\alpha-n+3)}, r\right] \times\left[\frac{r}{m^{\alpha-n+2}}, r\right]\right\} .
\end{align*}
$$

Thus (25) is true which implies that $T$ is uniformly bounded on any bounded set.

Now we will show that $T: \bar{K}_{R} \backslash K_{r} \longrightarrow K$ is continuous. Let $x_{j}, x_{0} \in \bar{K}_{R} \backslash K_{r}$ and $\left\|x_{j}-x_{0}\right\| \longrightarrow 0(j \longrightarrow \infty)$. For any $\varepsilon>0$, by $\left(H_{1}\right)$, there exists a natural number $k>0$ such that

$$
\begin{equation*}
\sup _{x \in \bar{K}_{R} \backslash K_{r}} \int_{H(k)} \Phi(s) f\left(s, I_{0+}^{n-2} x(s), \ldots, x(s)\right) d s<\frac{\varepsilon}{4} \tag{32}
\end{equation*}
$$

Since $f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)$ is uniformly continuous on $[1 / k, 1-1 / k] \times\left[\Gamma(\alpha-n+2) r / k^{\alpha} \Gamma(\alpha), r /(n-2)!\right] \times \cdots \times$ $\left[r / k^{\alpha-n+2}, r\right]$, we have that

$$
\begin{align*}
& \lim _{j \rightarrow+\infty} \mid f\left(t, I_{0+}^{n-2} x_{j}(t), \ldots, x_{j}(t)\right)  \tag{33}\\
& \quad-f\left(t, I_{0+}^{n-2} x_{0}(t), \ldots, x_{0}(t)\right) \mid=0
\end{align*}
$$

holds uniformly on $t \in[1 / k, 1-1 / k]$. It follows from the Lebesgue control convergence theorem that

$$
\begin{align*}
& \int_{1 / k}^{1-1 / k} \Phi(s) \mid f\left(s, I_{0+}^{n-2} x_{j}(s), \ldots, x_{j}(s)\right) \\
& \quad-f\left(s, I_{0+}^{n-2} x_{0}(s), \ldots, x_{0}(s)\right) \mid d s \longrightarrow 0 \tag{34}
\end{align*}
$$

as $j \longrightarrow \infty$.

Thus, for the above $\varepsilon>0$, there exists a natural number $J$ such that, for $j>J$, we have

$$
\begin{align*}
& \int_{1 / k}^{1-1 / k} \Phi(s) \mid f\left(s, I_{0+}^{n-2} x_{j}(s), \ldots, x_{j}(s)\right)  \tag{35}\\
& \quad-f\left(s, I_{0+}^{n-2} x_{0}(s), \ldots, x_{0}(s)\right) \left\lvert\, d s<\frac{\varepsilon}{2}\right.
\end{align*}
$$

It follows from (32) and (35) that when $j>J$,

$$
\begin{align*}
& \left\|T x_{j}-T x_{0}\right\| \leq \int_{H(k)} \Phi(s) \mid f\left(s, I_{0+}^{n-2} x_{j}(s), \ldots, x_{j}(s)\right) \\
& \quad-f\left(s, I_{0+}^{n-2} x_{0}(s), \ldots, x_{0}(s)\right) \mid d s \\
& \quad+\int_{1 / k}^{1-1 / k} \Phi(s) \mid f\left(s, I_{0+}^{n-2} x_{j}(s), \ldots, x_{j}(s)\right)  \tag{36}\\
& \quad-f\left(s, I_{0+}^{n-2} x_{0}(s), \ldots, x_{0}(s)\right) \mid d s \leq 2 \\
& \quad \sup _{x \in \bar{K}_{R} \backslash K_{r}} \int_{H(k)} \Phi(s) f\left(s, I_{0+}^{n-2} x(s), \ldots, x(s)\right) d s \\
& \quad+\frac{\varepsilon}{2}<2 \times \frac{\varepsilon}{4}+\frac{\varepsilon}{2}=\varepsilon .
\end{align*}
$$

Hence, $T: \bar{K}_{R} \backslash K_{r} \longrightarrow K$ is continuous.
For any bounded set $B \subset \bar{K}_{R} \backslash K_{r}$, we show that $T(B)$ is equicontinuous. In fact, by $\left(H_{1}\right)$, for any $\varepsilon>0$, there exists a natural number $l$ such that

$$
\begin{equation*}
\sup _{x \in \bar{K}_{R} \backslash K_{r}} \int_{H(l)} \Phi(s) f\left(s, I_{0+}^{n-2} x(s), \ldots, x(s)\right) d s<\frac{\varepsilon}{4} \tag{37}
\end{equation*}
$$

Let

$$
\begin{align*}
M_{2} & =\max \left\{f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right):\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)\right. \\
& \in\left[\frac{1}{l}, 1-\frac{1}{l}\right] \times\left[\frac{\Gamma(\alpha-n+2) r}{l^{\alpha} \Gamma(\alpha)}, \frac{r}{(n-2)!}\right]  \tag{38}\\
& \times\left[\frac{\Gamma(\alpha-n+2) r}{l^{\alpha-1} \Gamma(\alpha-1)}, \frac{r}{(n-3)!}\right] \times \cdots \\
& \left.\times\left[\frac{\Gamma(\alpha-n+2) r}{l^{\alpha-n+3} \Gamma(\alpha-n+3)}, r\right] \times\left[\frac{r}{l^{\alpha-n+2}}, r\right]\right\} .
\end{align*}
$$

Since $H(t, s)$ is uniformly continuous on $[0,1] \times[0,1]$, for the above $\varepsilon>0$ and a fixed $s \in[1 / l, 1-1 / l]$, there exists $\delta>0$ such that

$$
\begin{equation*}
\left|H(t, s)-H\left(t^{\prime}, s\right)\right|<\frac{\varepsilon}{2\left(M_{2}+1\right)} \tag{39}
\end{equation*}
$$

for $\left|t-t^{\prime}\right|<\delta, t, t^{\prime} \in[0,1]$. Hence, for $\left|t-t^{\prime}\right|<\delta, t, t^{\prime} \in[0,1]$, and $x \in B$, we have

$$
\begin{align*}
& \left|T x(t)-T x\left(t^{\prime}\right)\right| \leq \int_{H(k)}\left|H(t, s)-H\left(t^{\prime}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad+\int_{1 / k}^{1-1 / k}\left|H(t, s)-H\left(t^{\prime}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \leq 2 \\
& \quad \cdot \sup _{x \in \bar{K}_{R} \backslash K_{r}} \int_{e(k)} \Phi(s)  \tag{40}\\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad+\sup _{x \in \bar{K}_{R} \backslash K_{r}} \int_{1 / k}^{1-1 / k}\left|H(t, s)-H\left(t^{\prime}, s\right)\right| \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s<2 \times \frac{\varepsilon}{4} \\
& \quad+\frac{\varepsilon}{2}=\varepsilon
\end{align*}
$$

which show that $T(B)$ is equicontinuous. According to the Ascoli-Arzela theorem, $T: \bar{K}_{R} \backslash K_{r} \longrightarrow K$ is completely continuous. The proof is completed.

Lemma 8 (see [35]). Let $K$ be a cone in Banach space E. Suppose that $T: \bar{K}_{r} \longrightarrow K$ is a completely continuous operator.
(i) If there exists $y_{0} \in K \backslash\{\theta\}$ such that $y-T y \neq \mu y_{0}$ for any $y \in \partial K_{r}$ and $\mu \geq 0$, then $i\left(T, K_{r}, K\right)=0$.
(ii) If $T y \neq \mu y$ for any $y \in \partial K_{r}$ and $\mu \geq 1$, then $i\left(T, K_{r}, K\right)=1$.

## 3. Main Results

Theorem 9. Suppose that the conditions $\left(H_{0}\right)$ and $\left(H_{1}\right)$ are satisfied, and

$$
\begin{equation*}
\liminf _{\substack{z_{1} \rightarrow 0 \\ z_{n-1} \rightarrow 0}} \frac{f\left(t, z_{1}, \ldots, z_{n-1}\right)}{z_{1}+\cdots+z_{n-1}}>\lambda_{1} \tag{41}
\end{equation*}
$$

uniformly on $t \in[0,1]$,
$\underset{\substack{z_{1}+\cdots+z_{n-1} \xrightarrow{z_{n-1}}++\infty}}{\lim \sup ^{2}} \frac{f\left(t, z_{1}, \ldots, z_{n-1}\right)}{z_{n-1}}<\lambda_{1}$,
uniformly on $t \in[0,1]$,
where $\lambda_{1}$ is the first eigenvalue of $L$ defined by (16). Then the FBVP (3) has at least one positive solution.

Proof. By (41), there exists $r>0$ such that

$$
\begin{align*}
& f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right) \geq \lambda_{1}\left(z_{1}+z_{2}+\cdots+z_{n-1}\right) \\
& \qquad\left|z_{1}\right| \leq \frac{r}{(n-2)!}, \ldots,\left|z_{n-1}\right| \leq r, t \in[0,1] . \tag{43}
\end{align*}
$$

Let $\varphi^{*}$ be the positive eigenfunction of $L$ corresponding to $\lambda_{1}$, and thus $\varphi^{*}=\lambda_{1} L \varphi^{*}$. For any $x \in \partial K_{r}$, by virtue of (27) and (28), that is

$$
\begin{align*}
\left|I_{0+}^{n-2} x(s)\right| \leq \frac{r}{(n-2)!}, \ldots,\left|I_{0+}^{1} x(s)\right| \leq r &  \tag{44}\\
& |x(s)| \leq r, t \in[0,1],
\end{align*}
$$

from (43) and (44), we have

$$
\begin{aligned}
& (T x)(t)=\int_{0}^{1} H(t, s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \geq \lambda_{1} \int_{0}^{1} H(t, s)\left(I_{0+}^{n-2} x(s)+I_{0+}^{n-3} x(s)+\cdots\right. \\
& \left.\quad+I_{0+}^{1} x(s)+x(s)\right) d s \geq \lambda_{1}(L x)(t)
\end{aligned}
$$

$$
t \in[0,1] .
$$

We may suppose that $T$ has no fixed point on $\partial K_{r}$ (otherwise, the proof is finished). Now we show that

$$
\begin{equation*}
x-T x \neq \mu \varphi^{*}, \quad x \in \partial K_{r}, \quad \mu \geq 0 \tag{46}
\end{equation*}
$$

If, otherwise, there exist $x_{0} \in \partial K_{r}$ and $\mu_{0} \geq 0$ such that $x_{0}-T x_{0}=\mu_{0} \varphi^{*}$, obviously, $\mu_{0}>0$ and $x_{0}=T x_{0}+\mu_{0} \varphi^{*} \geq$ $\mu_{0} \varphi^{*}$. Let $\bar{\mu}=\sup \left\{\mu \mid x_{0} \geq \mu \varphi^{*}\right\}$, and then $\bar{\mu} \geq \mu_{0}>0$ and $x_{0} \geq \bar{\mu} \varphi^{*}$. Since $L(K) \subset K$, we have $\lambda_{1} L x_{0} \geq \lambda_{1} \bar{\mu} L \varphi^{*}=\bar{\mu} \varphi^{*}$. Therefore, by (45), we have

$$
\begin{align*}
x_{0} & =T x_{0}+\mu_{0} \varphi^{*} \geq \lambda_{1} L x_{0}+\mu_{0} \varphi^{*} \geq \bar{\mu} \varphi^{*}+\mu_{0} \varphi^{*}  \tag{47}\\
& =\left(\bar{\mu}+\mu_{0}\right) \varphi^{*},
\end{align*}
$$

which contradicts the definition of $\bar{\mu}$. Hence (46) is true and it follows from Lemma 8 that

$$
\begin{equation*}
i\left(T, K_{r}, K\right)=0 \tag{48}
\end{equation*}
$$

Now we choose a constant $0<\rho<1$ and define a linear operator $\widetilde{L} x=\rho \lambda_{1} L x$, and then $\widetilde{L}: E \longrightarrow E$ is a bounded linear operator and $\widetilde{L}(K) \subset K$. Moreover $\widetilde{L} \varphi^{*}=\rho \lambda_{1} L \varphi^{*}=$ $\rho \varphi^{*}$, so the spectral radius of $\widetilde{L}$ is $r(\widetilde{L})=\rho$ and $\widetilde{L}$ also has the first eigenvalue $(r(\widetilde{L}))^{-1}=\rho^{-1}>1$. By Gelfand's formula, we know

$$
\begin{equation*}
\rho=\lim _{m \rightarrow+\infty}\left\|\widetilde{L}^{m}\right\|^{1 / m} \tag{49}
\end{equation*}
$$

Let $\varepsilon_{0}=(1 / 2)(1-\rho)$; by (49), there exists a sufficiently large natural number $M$ such that $m \geq M$ implies that $\left\|\widetilde{L}^{m}\right\| \leq$ $\left(\rho+\varepsilon_{0}\right)^{m}$. For any $x \in E$, define

$$
\begin{equation*}
\|x\|^{*}=\sum_{i=1}^{M}\left(\rho+\varepsilon_{0}\right)^{M-i}\left\|\widetilde{L}^{i-1} x\right\| \tag{50}
\end{equation*}
$$

where $\widetilde{L}^{0}=I$ is the identity operator. Clearly, $\|\cdot\|^{*}$ is also the norm of $E$.

On the other hand, it follows from (42) that there exists $R_{0}>r$ such that

$$
\begin{align*}
& f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right) \leq \rho \lambda_{1} z_{n-1} \\
& \qquad\left|z_{1}+\cdots+z_{n-1}\right| \geq R_{0},\left|z_{n-1}\right| \geq R_{0} . \tag{51}
\end{align*}
$$

Let

$$
\begin{align*}
& M_{0}=\sup _{x \in \partial K_{R_{0}}} \int_{0}^{1} \Phi(s)  \tag{52}\\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s,
\end{align*}
$$

and then, by (25), we know that $M_{0}<+\infty$. For the sake of convenience, let $M_{0}^{*}=\left\|M_{0}\right\|^{*}$ and choose

$$
\begin{equation*}
R>\max \left\{R_{0}, \frac{2 M_{0}^{*}}{\varepsilon_{0}\left(\rho+\varepsilon_{0}\right)^{M-1}}\right\} \tag{53}
\end{equation*}
$$

In the following we will prove that

$$
\begin{equation*}
T x \neq \mu x, \quad x \in \partial K_{R}, \quad \mu \geq 1 \tag{54}
\end{equation*}
$$

If, otherwise, there exist $x_{1} \in \partial K_{R}$ and $\mu_{1} \geq 1$ such that $T x_{1}=\mu_{1} x_{1}$. It follows from $R>R_{0}$ and $x_{1} \in C[0,1]$ that there exists $0<t_{1} \leq 1$ such that $x_{1}\left(t_{1}\right)=R_{0}$. Thus let $\widetilde{x}(t)=$ $\min \left\{x_{1}(t), R_{0}\right\}$, and then we have $\|\tilde{x}\|=R_{0}$ and $\tilde{x} \in \partial K_{R_{0}}$. Now set

$$
\begin{equation*}
D\left(x_{1}\right)=\left\{t \in[0,1]: x_{1}(t)>R_{0}\right\}, \tag{55}
\end{equation*}
$$

and then, for any $t \in D\left(x_{1}\right)$, we have $\left|x_{1}(t)\right|>R_{0}$ and $\left|I_{0+}^{n-2} x_{1}(t)+I_{0+}^{n-3} x_{1}(t)+\cdots+I_{0+}^{1} x_{1}(t)+x_{1}(t)\right|>R_{0}$. It follows from (51) and Lemma 2 that

$$
\begin{align*}
& \left(T x_{1}\right)(t)=\int_{0}^{1} H(t, s) f\left(s, I_{0+}^{n-2} x_{1}(s), I_{0+}^{n-3} x_{1}(s), \ldots,\right. \\
& \left.I_{0+}^{1} x_{1}(s), x_{1}(s)\right) d s \leq \int_{D\left(x_{1}\right)} H(t, s) f(s, \\
& \left.I_{0+}^{n-2} x_{1}(s), I_{0+}^{n-3} x_{1}(s), \ldots, I_{0+}^{1} x_{1}(s), x_{1}(s)\right) d s \\
& \quad+\int_{[0,1] \backslash D\left(x_{1}\right)} H(t, s) f\left(s, I_{0+}^{n-2} x_{1}(s), I_{0+}^{n-3} x_{1}(s), \ldots,\right.  \tag{56}\\
& \left.I_{0+}^{1} x_{1}(s), x_{1}(s)\right) d s \leq \rho \lambda_{1} \int_{0}^{1} H(t, s) x_{1}(s) d s \\
& \quad+\int_{0}^{1} \Phi(s) f\left(s, I_{0+}^{n-2} x_{1}(s), I_{0+}^{n-3} x_{1}(s), \ldots, I_{0+}^{1} x_{1}(s),\right. \\
& \left.x_{1}(s)\right) d s \leq\left(\widetilde{L} x_{1}\right)(t)+M_{0}, \quad t \in[0,1] .
\end{align*}
$$

Noticing that $\widetilde{L}(K) \subset K$ is a bounded linear operator, and from (56), we have

$$
\begin{align*}
& 0 \leq\left(\tilde{L}^{j}\left(T x_{1}\right)\right)(t) \leq\left(\tilde{L}^{j}\left(\widetilde{L} x_{1}+M_{0}\right)\right)(t)  \tag{57}\\
& \quad j=0,1,2, \ldots, M-1 .
\end{align*}
$$

Then (57) yields that

$$
\begin{align*}
\left\|\widetilde{L}^{j}\left(T x_{1}\right)\right\| \leq\left\|\tilde{L}^{j}\left(\widetilde{L} x_{1}+M_{0}\right)\right\| &  \tag{58}\\
& j=0,1,2, \ldots, M-1
\end{align*}
$$

which leads to

$$
\begin{align*}
\left\|T x_{1}\right\|^{*} & =\sum_{i=1}^{M}\left(\rho+\varepsilon_{0}\right)^{M-i}\left\|\widetilde{L}^{i-1}\left(T x_{1}\right)\right\| \\
& \leq \sum_{i=1}^{M}\left(\rho+\varepsilon_{0}\right)^{M-i}\left\|\widetilde{L}^{i-1}\left(\widetilde{L} x_{1}+M_{0}\right)\right\|  \tag{59}\\
& =\left\|\widetilde{L} x_{1}+M_{0}\right\|^{*}
\end{align*}
$$

Since $x_{1} \in \partial K_{R}$ and $\left\|x_{1}\right\|=R$, from (50), we have

$$
\begin{equation*}
\left\|x_{1}\right\|^{*}>\left(\rho+\varepsilon_{0}\right)^{M-1}\left\|x_{1}\right\|=\left(\rho+\varepsilon_{0}\right)^{M-1} R>\frac{2}{\varepsilon_{0}} M_{0}^{*} \tag{60}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
M_{0}^{*}<\frac{\varepsilon_{0}}{2}\left\|x_{1}\right\|^{*} \tag{61}
\end{equation*}
$$

By (50), (59), and (61), we have

$$
\begin{align*}
\mu_{1}\left\|x_{1}\right\|^{*}= & \left\|T x_{1}\right\|^{*} \leq\left\|\widetilde{L} x_{1}\right\|+M_{0}^{*} \\
= & \sum_{i=1}^{M}\left(\rho+\varepsilon_{0}\right)^{M-i}\left\|\widetilde{L}^{i-1}\left(L x_{1}\right)\right\|+M_{0}^{*} \\
= & \left(\rho+\varepsilon_{0}\right) \sum_{i=1}^{M-1}\left(\rho+\varepsilon_{0}\right)^{M-i-1}\left\|\widetilde{L}^{i} x_{1}\right\| \\
& +\left\|\widetilde{L}^{M} x_{1}\right\|+M_{0}^{*} \\
\leq & \left(\rho+\varepsilon_{0}\right) \sum_{i=1}^{M-1}\left(\rho+\varepsilon_{0}\right)^{M-i-1}\left\|\widetilde{L}^{i} x_{1}\right\|  \tag{62}\\
& +\left(\rho+\varepsilon_{0}\right)^{M}\left\|x_{1}\right\|+M_{0}^{*} \\
= & \left(\rho+\varepsilon_{0}\right) \sum_{i=1}^{M}\left(\rho+\varepsilon_{0}\right)^{M-i}\left\|\widetilde{L}^{i-1} x_{1}\right\|+M_{0}^{*} \\
= & \left(\rho+\varepsilon_{0}\right)\left\|x_{1}\right\|^{*}+M_{0}^{*} \\
\leq & \left(\rho+\varepsilon_{0}\right)\left\|x_{1}\right\|^{*}+\frac{\varepsilon_{0}}{2}\left\|x_{1}\right\|^{*}=\frac{\rho+3}{4}\left\|x_{1}\right\|^{*} .
\end{align*}
$$

Notice that $\mu_{1} \geq 1$, we have $(\rho+3) / 4 \geq 1$, and then $\rho \geq 1$, which is a contradiction with $0<\rho<1$. Thus (54) is indeed true, and, by Lemma 8, we have

$$
\begin{equation*}
i\left(T, K_{R}, K\right)=1 \tag{63}
\end{equation*}
$$

It follows from (48) and (63) that

$$
\begin{equation*}
i\left(T, K_{R} \backslash \bar{K}_{r}, K\right)=i\left(T, K_{R}, K\right)-i\left(T, K_{r}, K\right)=1 \tag{64}
\end{equation*}
$$

Then $T$ has at least one fixed point on $K_{R} \backslash \bar{K}_{r}$. Consequently, problem (14) has at least one positive solution, which implies that the FBVP (3) has at least one positive solution.

Now we consider another case of problem (14). For this, we define a linear operator $L_{\tau}$ for any sufficiently small $0<$ $\tau<1$ as follows:

$$
\begin{equation*}
\left(L_{\tau} x\right)(t)=\int_{\tau}^{1-\tau} H(t, s) x(s) d s, \quad t \in[0,1] \tag{65}
\end{equation*}
$$

From Lemma 6, we know $L_{\tau}$ is also a completely continuous linear operator, and the spectral radius $r\left(L_{\tau}\right) \neq 0$, and moreover $L_{\tau}$ has a positive eigenfunction $\varphi_{\tau}$ corresponding to its first eigenvalue $\lambda_{\tau}=\left(r\left(L_{\tau}\right)\right)^{-1}$.

Lemma 10. Suppose that $\left(H_{0}\right)$ holds, and there exists an eigenvalue $\widetilde{\lambda}_{1}$ of $L$ such that

$$
\begin{equation*}
\lim _{\tau \rightarrow 0^{+}} \lambda_{\tau}=\tilde{\lambda}_{1} \tag{66}
\end{equation*}
$$

Proof. Take $\tau_{1} \geq \tau_{2} \geq \cdots \geq \tau_{k} \geq \cdots$ and $\tau_{k} \longrightarrow 0(k \longrightarrow$ $+\infty)$. So, for any $j>k$ and $\varphi \in E$, we have

$$
\begin{equation*}
\left(L_{\tau_{k}} \varphi\right)(t) \leq\left(L_{\tau_{j}} \varphi\right)(t) \leq(L \varphi)(t), \quad t \in[0,1] \tag{67}
\end{equation*}
$$

and

$$
\begin{align*}
& \left(L_{\tau_{k}}^{i} \varphi\right)(t) \leq\left(L_{\tau_{j}}^{i} \varphi\right)(t) \leq\left(L^{i} \varphi\right)(t)  \tag{68}\\
& \\
& \quad t \in[0,1], i=2,3, \ldots,
\end{align*}
$$

where $L_{\tau_{k}}^{i}=L\left(L_{\tau_{k}}^{i-1}\right), i=2,3, \cdots$. Consequently, $\left\|L_{\tau_{k}}^{i}\right\| \leq$ $\left\|L_{\tau_{j}}^{i}\right\| \leq\left\|L^{i}\right\|, i=1,2, \cdots$. From Gelfand's formula, we get $\lambda_{1} \leq \lambda_{\tau_{j}} \leq \lambda_{\tau_{k}}$, where $\lambda_{1}$ is the first eigenvalue of $L$. Since $\left\{\lambda_{\tau_{k}}\right\}$ is monotonous with lower boundedness $\lambda_{1}$, let

$$
\begin{equation*}
\lim _{k \rightarrow+\infty} \lambda_{\tau_{k}}=\tilde{\lambda}_{1} \tag{69}
\end{equation*}
$$

Now we shall show that $\tilde{\lambda}_{1}$ is an eigenvalue of $L$. Suppose $\varphi_{\tau_{k}}$ is a positive eigenfunction of $L_{\tau_{k}}$ corresponding to $\lambda_{\tau_{k}}$ with $\left\|\varphi_{\tau_{k}}\right\|=1, k=1,2, \ldots$; i.e.,

$$
\begin{array}{r}
\varphi_{\tau_{k}}(t)=\lambda_{\tau_{k}} \int_{\tau_{k}}^{1-\tau_{k}} H(t, s) \varphi_{\tau_{k}}(s) d s=\lambda_{\tau_{k}} L_{\tau_{k}} \varphi_{\tau_{k}}(t),  \tag{70}\\
t \in[0,1] .
\end{array}
$$

Notice that

$$
\begin{array}{r}
\left\|\lambda_{\tau_{k}} \varphi_{\tau_{k}}\right\|=\max _{0 \leq t \leq 1} \int_{\tau_{k}}^{1-\tau_{k}} H(t, s) \varphi_{\tau_{k}}(s) d s \leq \int_{0}^{1} \Phi(s) d s  \tag{71}\\
\\
k=1,2, \ldots
\end{array}
$$

and thus $\left\{\lambda_{\tau_{k}} \varphi_{\tau_{k}}\right\} \subset E$ is uniformly bounded.
On the other hand, for any $k$ and $t_{1}, t_{2} \in[0,1]$, we have

$$
\begin{align*}
& \left|L_{\tau_{k}} \varphi_{\tau_{k}}\left(t_{1}\right)-L_{\tau_{k}} \varphi_{\tau_{k}}\left(t_{2}\right)\right| \\
& \quad \leq \int_{\tau_{k}}^{1-\tau_{k}}\left|H\left(t_{1}, s\right)-H\left(t_{2}, s\right)\right| \varphi_{\tau_{k}}(s) d s . \tag{72}
\end{align*}
$$

It follows from (72) that $\left\{L_{\tau_{k}} \varphi_{\tau_{k}}\right\} \subset E$ is equicontinuous since $H(t, s)$ is uniformly continuous on $[0,1] \times[0,1]$. Without loss of generality, we may suppose that, by Arzela-Ascoli theorem and $\lim _{k \rightarrow+\infty} \lambda_{\tau_{k}}=\tilde{\lambda}_{1}$, we get that $\varphi_{\tau_{k}} \longrightarrow \varphi_{0}$ as $k \longrightarrow+\infty$. This leads to $\left\|\varphi_{0}\right\|=1$ and then, by (70), we have

$$
\begin{equation*}
\varphi_{0}(t)=\tilde{\lambda}_{1} \int_{0}^{1} H(t, s) \varphi_{0}(s) d s, \quad t \in[0,1] ; \tag{73}
\end{equation*}
$$

that is, $\varphi_{0}=\tilde{\lambda}_{1} L \varphi_{0}$. The proof is completed.
Theorem 11. Suppose that conditions $\left(H_{0}\right)$ and $\left(H_{1}\right)$ are satisfied, and

$$
\begin{align*}
& \liminf _{\substack{z_{1} \cdots 0 \\
z_{n-1} \rightarrow 0}} \frac{f\left(t, z_{1}, \ldots, z_{n-1}\right)}{z_{n-1}}<\lambda_{1},  \tag{74}\\
& \quad \text { uniformly on } t \in[0,1], \\
& \lim _{z_{1}+\cdots+z_{n-1} \rightarrow+\infty} \frac{f\left(t, z_{1}, \ldots, z_{n-1}\right)}{z_{1}+\cdots+z_{n-1}}>\tilde{\lambda}_{1},  \tag{75}\\
& \text { uniformly on } t \in[0,1],
\end{align*}
$$

where $\lambda_{1}, \widetilde{\lambda}_{1}$ are the eigenvalues of $L$ and $\lambda_{1}$ is the first eigenvalue of $L$. Then the FBVP (3) has at least one positive solution.

Proof. It follows from (74) that there exists $r>0$ such that

$$
\begin{align*}
& f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right) \leq \lambda_{1} z_{n-1} \\
& \qquad\left|z_{1}\right| \leq \frac{r}{(n-2)!}, \ldots,\left|z_{n-1}\right| \leq r, t \in[0,1] . \tag{76}
\end{align*}
$$

Thus for any $x \in \partial K_{r}$, noticing

$$
\begin{align*}
\left|I_{0+}^{n-2} x(s)\right| \leq \frac{r}{(n-2)!}, \ldots,\left|I_{0+}^{1} x(s)\right| \leq r &  \tag{77}\\
& |x(s)| \leq r, t \in[0,1]
\end{align*}
$$

by (76), we have

$$
\begin{align*}
& (T x)(t)=\int_{0}^{1} H(t, s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \leq \lambda_{1} \int_{0}^{1} H(t, s) x(s) d s=\lambda_{1}(L x)(t) \tag{78}
\end{align*}
$$

$$
t \in[0,1] .
$$

In fact, we may suppose that $T$ has no fixed point on $\partial K_{r}$ (otherwise, the proof is finished). Now we show that

$$
\begin{equation*}
T x \neq \mu x, \quad x \in \partial K_{r}, \quad \mu \geq 1 \tag{79}
\end{equation*}
$$

Otherwise, there exist $x_{0} \in \partial K_{r}$ and $\mu_{0} \geq 1$ such that $T x_{0}=\mu_{0} x_{0}$. We know $\mu_{0}>1$ and from (78) we have

$$
\begin{equation*}
\mu_{0} x_{0}=T x_{0} \leq \lambda_{1} L x_{0} \tag{80}
\end{equation*}
$$

By using induction for (80), we have

$$
\begin{equation*}
\mu_{0}^{m} x_{0} \leq \lambda_{1}^{m} L^{m} x_{0}, \quad m=1,2, \cdots \tag{81}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left\|L^{m}\right\| \geq \frac{\left\|L^{m} x_{0}\right\|}{\left\|x_{0}\right\|} \geq \frac{\mu_{0}^{m}\left\|x_{0}\right\|}{\lambda_{1}^{m}\left\|x_{0}\right\|}=\frac{\mu_{0}^{m}}{\lambda_{1}^{m}} \tag{82}
\end{equation*}
$$

By Gelfand's formula, we know

$$
\begin{equation*}
r(L)=\lim _{m \rightarrow+\infty}\left\|L^{m}\right\|^{1 / m} \geq \frac{\mu_{0}}{\lambda_{1}}>\frac{1}{\lambda_{1}} \tag{83}
\end{equation*}
$$

which is a contradiction with $r(L)=\lambda_{1}^{-1}$. Hence (79) is true and, by Lemma 8, we have

$$
\begin{equation*}
i\left(T, K_{r}, K\right)=1 \tag{84}
\end{equation*}
$$

For a fixed sufficiently small $0<\tau<1$, let

$$
\begin{align*}
\chi_{\tau}= & \frac{\tau^{\alpha} \Gamma(\alpha-n+2)}{\Gamma(\alpha)}+\frac{\tau^{\alpha-1} \Gamma(\alpha-n+2)}{\Gamma(\alpha-1)}+\cdots \\
& +\frac{\tau^{\alpha-n+3} \Gamma(\alpha-n+2)}{\Gamma(\alpha-n+3)}+\tau^{\alpha-n+2} . \tag{85}
\end{align*}
$$

So it follows from (74) and $\lim _{\tau \rightarrow 0^{+}} \lambda_{\tau}=\widetilde{\lambda}_{1}$ that there exist a sufficiently small $\tau>0$ and $R>r$ such that

$$
\begin{align*}
f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right) \geq \lambda_{\tau}\left(z_{1}+z_{2}+\cdots+z_{n-1}\right),  \tag{86}\\
z_{1}+z_{2}+\cdots+z_{n-1} \geq \chi_{\tau} R, t \in[0,1]
\end{align*}
$$

where $\lambda_{\tau}$ is the first eigenvalue of $L_{\tau}$.
Let $\varphi_{\tau}$ be the positive eigenfunction of $L_{\tau}$ corresponding to $\lambda_{\tau}$, and then $\varphi_{\tau}=\lambda_{\tau} L_{\tau} \varphi_{\tau}$. For any $x \in \partial K_{R}, s \in[\tau, 1-\tau]$, we have

$$
\begin{align*}
& I_{0+}^{n-2} x(s)+I_{0+}^{n-3} x(s)+\cdots, I_{0+}^{1} x(s)+x(s) \\
& \quad=\frac{1}{\Gamma(n-2)} \int_{0}^{s}(s-\tau)^{n-3} x(\tau) d \tau+\frac{1}{\Gamma(n-3)} \\
& \quad \cdot \int_{0}^{s}(s-\tau)^{n-4} x(\tau) d \tau+\cdots+\frac{1}{\Gamma(1)} \\
& \quad \cdot \int_{0}^{s}(s-\tau)^{1-1} x(\tau) d \tau+x(s) \geq \frac{1}{\Gamma(n-2)} \\
& \quad \cdot \int_{0}^{s}(s-\tau)^{n-3} \tau^{\alpha-n+1}(1-\tau) d \tau\|x\| \\
& \quad+\frac{1}{\Gamma(n-3)} \int_{0}^{s}(s-\tau)^{n-4} \tau^{\alpha-n+1}(1-\tau) d \tau\|x\|+\cdots  \tag{87}\\
& \quad+\frac{1}{\Gamma(1)} \int_{0}^{s}(s-\tau)^{1-1} \tau^{\alpha-n+1}(1-\tau) d \tau\|x\| \\
& \quad+s^{\alpha-n+1}(1-s)\|x\| \geq\left(\frac{\tau^{\alpha} \Gamma(\alpha-n+2)}{\Gamma(\alpha)}\right. \\
& \quad+\frac{\tau^{\alpha-1} \Gamma(\alpha-n+2)}{\Gamma(\alpha-1)}+\cdots+\frac{\tau^{\alpha-n+3} \Gamma(\alpha-n+2)}{\Gamma(\alpha-n+3)} \\
& \left.\quad+\tau^{\alpha-n+2}\right) R=\chi_{\tau} R .
\end{align*}
$$

By (86) and (87), we have

$$
\begin{align*}
& (T x)(t)=\int_{0}^{1} H(t, s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s \\
& \quad \geq \int_{\tau}^{1-\tau} H(t, s) \\
& \quad \cdot f\left(s, I_{0+}^{n-2} x(s), I_{0+}^{n-3} x(s), \ldots, I_{0+}^{1} x(s), x(s)\right) d s  \tag{88}\\
& \quad \geq \lambda_{\tau} \int_{\tau}^{1-\tau} H(t, s)\left(I_{0+}^{n-2} x(s)+I_{0+}^{n-3} x(s)\right. \\
& \left.\quad+\cdots, I_{0+}^{1} x(s)+x(s)\right) d s \geq \lambda_{\tau} \int_{\tau}^{1-\tau} H(t, s) \\
& \quad \cdot x(s) d s=\lambda_{\tau}\left(L_{\tau} x\right)(t), \quad t \in[0,1] .
\end{align*}
$$

Proceeding as for the proof of Theorem 9, we get

$$
\begin{equation*}
x-T x \neq \mu \varphi_{\tau}, \quad x \in \partial K_{R}, \quad \mu \geq 0 \tag{89}
\end{equation*}
$$

By Lemma 8, we have

$$
\begin{equation*}
i\left(T, K_{R}, K\right)=0 \tag{90}
\end{equation*}
$$

It follows from (84) and (90) that

$$
\begin{equation*}
i\left(T, K_{R} \backslash \bar{K}_{r}, K\right)=i\left(T, K_{R}, K\right)-i\left(T, K_{r}, K\right)=-1 \tag{91}
\end{equation*}
$$

Then $T$ has at least one fixed point on $K_{R} \backslash \bar{K}_{r}$. Consequently, problem (14) has at least one positive solution, which implies that the FBVP (3) has at least one positive solution.

Remark 12. In this work, the nonlinearity $f\left(t, z_{1}, z_{2}, \ldots, z_{n-1}\right)$ may be singular at $t=0,1$ and $z_{1}=z_{2}=\cdots=z_{n-1}=0$. To the best of our knowledge, very little work has been done for the case where $f$ can be singular at $z_{1}=z_{2}=\cdots=z_{n-1}=0$.

## 4. An Example

Example 1. Consider the following problem:

$$
\begin{align*}
& \begin{array}{l}
D_{0+}^{7 / 2} y(t)+\frac{\left(y+y^{\prime}+y^{\prime \prime}\right)^{-1 / 4}+\ln y^{\prime \prime}}{\sqrt{1-t}}=0, \\
\\
\quad 0<t<1, \\
y(0)=y^{\prime}(0)=y^{\prime \prime}(0)=0, \quad y^{\prime \prime}(1)=\lambda\left[y^{\prime \prime}\right],
\end{array}, l
\end{align*}
$$

where $\alpha=7 / 2, f\left(t, z_{1}, z_{2}, z_{3}\right)=\left(\left(z_{1}+z_{2}+z_{3}\right)^{-1 / 4}+\right.$ $\left.\ln z_{3}\right) / \sqrt{1-t}$, and $\left(t, z_{1}, z_{2}, z_{3}\right) \in(0,1) \times\left(\mathbb{R}_{0}^{+}\right)^{3}$. Obviously, $f$ is singular at $t=1$ and $z_{1}=z_{2}=z_{3}=0$. Clearly,

$$
\begin{aligned}
& G(t, s) \\
& = \begin{cases}G_{1}(t, s)=\frac{[t(1-s)]^{1 / 2}}{\Gamma(3 / 2)}, & 0 \leq t \leq s \leq 1, \\
G_{2}(t, s)=\frac{[t(1-s)]^{1 / 2}-(t-s)^{1 / 2}}{\Gamma(3 / 2)}, & 0 \leq s \leq t \leq 1 .\end{cases}
\end{aligned}
$$

In the following we discuss that condition $\left(H_{0}\right)$ holds when $\lambda[\cdot]$ take different cases.
(1) Let $\lambda\left[y^{\prime \prime}\right]=0$. In this case, we have

$$
\begin{align*}
& \lambda[\gamma]=0 \\
& \mathscr{G}(s)=0  \tag{94}\\
& \Phi(s)=\frac{s(1-s)^{1 / 2}}{\Gamma(1 / 2)} \leq \frac{13(1-s)^{1 / 2}}{\Gamma(1 / 2)} .
\end{align*}
$$

(2) Now, let $\lambda\left[y^{\prime \prime}\right]=\int_{0}^{1} y^{\prime \prime}(t)(4-5 t) d t$. Note that the function $h(t)=4-5 t$ changes the sign on the interval $[0,1]$. In this case, we have

$$
\begin{align*}
\lambda[\gamma] & =\int_{0}^{1} t^{1 / 2} d A(t)=\int_{0}^{1} t^{1 / 2}(4-5 t) d t=\frac{2}{3}<1 \\
\mathscr{G}(s) & =\int_{0}^{1} G(t, s)(4-5 t) d t \\
& =\frac{1}{\Gamma(5 / 2)}(1-s)^{1 / 2} s(3-2 s) \geq 0, \quad s \in[0,1] \\
\Phi(s) & =\frac{\mathscr{G}(s)}{1-\lambda[\gamma]}+\frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)}  \tag{95}\\
& =\frac{3 s(1-s)^{1 / 2}(3-2 s)}{\Gamma(5 / 2)}+\frac{s(1-s)^{1 / 2}}{\Gamma(1 / 2)} \\
& \leq \frac{13(1-s)^{1 / 2}}{\Gamma(1 / 2)}
\end{align*}
$$

(3) Let $\lambda\left[y^{\prime \prime}\right]=(1 / 2) y^{\prime \prime}(1 / 2)$. In this case, we have

$$
\begin{align*}
\lambda[\gamma] & =\int_{0}^{1} t^{1 / 2} d A(t)=\frac{\sqrt{2}}{4} \approx 0.353553<1, \\
0 & \leq \mathscr{G}(s)=\frac{1}{2} G\left(\frac{1}{2}, s\right) \\
& = \begin{cases}\frac{[(1 / 2)(1-s)]^{1 / 2}}{\Gamma(3 / 2)}, & \frac{1}{2} \leq s \leq 1, \\
\frac{[(1 / 2)(1-s)]^{1 / 2}-(1 / 2-s)^{1 / 2}}{\Gamma(3 / 2)}, & 0 \leq s \leq \frac{1}{2}\end{cases}  \tag{96}\\
& \leq \frac{(\sqrt{2} / 4)(1-s)^{1 / 2}}{\Gamma(3 / 2)}, \quad s \in[0,1]
\end{align*}
$$

$\Phi(s)=\frac{\mathscr{G}(s)}{1-\lambda[\gamma]}+\frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)}$

$$
\leq \frac{\sqrt{2}(1-s)^{1 / 2}}{(4-\sqrt{2}) \Gamma(3 / 2)}+\frac{s(1-s)^{1 / 2}}{\Gamma(1 / 2)} \leq \frac{13(1-s)^{1 / 2}}{\Gamma(1 / 2)}
$$

(4) Let $\lambda\left[y^{\prime \prime}\right]=2 y^{\prime \prime}(1 / 2)-y^{\prime \prime}(3 / 4)$. Note that the coefficients $b_{1}=2, b_{2}=-1$; i.e., not all of the
coefficients must be positive, some coefficients of $b_{i}$ can be negative. In this case, we have

$$
\begin{align*}
& \lambda[\gamma]=\int_{0}^{1} t^{1 / 2} d A(t)=\sqrt{2}-\frac{\sqrt{3}}{2} \approx 0.548188<1, \\
& \mathscr{G}(s)=\left\{\begin{array}{lc}
2 G_{2}\left(\frac{1}{2}, s\right)-G_{2}\left(\frac{3}{4}, s\right), \quad 0 \leq s<\frac{1}{2}, \\
2 G_{1}\left(\frac{1}{2}, s\right)-G_{2}\left(\frac{3}{4}, s\right), \quad \frac{1}{2} \leq s \leq \frac{3}{4}, \\
2 G_{1}\left(\frac{1}{2}, s\right)-G_{1}\left(\frac{3}{4}, s\right), \quad \frac{3}{4}<s \leq 1 .
\end{array}\right. \tag{97}
\end{align*}
$$

Then $0 \leq \mathscr{G}(s)<1, s \in[0,1]$, and

$$
\begin{align*}
\Phi(s) & =\frac{\mathscr{G}(s)}{1-\lambda[\gamma]}+\frac{s(1-s)^{\alpha-n+1}}{\Gamma(\alpha-n+1)} \\
& \leq \frac{2 \sqrt{2}(1-s)^{1 / 2}}{(2-2 \sqrt{2}+\sqrt{3}) \Gamma(3 / 2)}+\frac{s(1-s)^{1 / 2}}{\Gamma(1 / 2)}  \tag{98}\\
& \leq \frac{13(1-s)^{1 / 2}}{\Gamma(1 / 2)}
\end{align*}
$$

Seen from above, condition $\left(H_{0}\right)$ holds in different cases $1-4$. Now we define a cone

$$
\begin{align*}
K & =\left\{y \in C[0,1]: y(t) \geq t^{1 / 2}(1-t)\|y\| \text { for } t\right.  \tag{99}\\
& \in[0,1]\} .
\end{align*}
$$

For any $0<r<R<+\infty$ and $z_{3} \in \bar{K}_{R} \backslash K_{r}$, we have

$$
\begin{aligned}
& 0 \leq r t^{5 / 2}(1-t) \leq t^{1 / 2}(1-t) r=t^{1 / 2}(1-t)\left\|z_{3}\right\| \\
& \leq z_{3}(t) \leq R, \quad t \in[0,1] \\
& 0 \leq \frac{2}{3} r t^{5 / 2}(1-t) \leq \frac{2}{3} r t^{3 / 2}(1-t) \\
&=\frac{r t^{3 / 2}(1-t) \Gamma(3 / 2)}{\Gamma(5 / 2)} \leq z_{2}(t)=I_{0+}^{1} z_{3}(t) \leq R \\
& \quad t \in[0,1]
\end{aligned}
$$

$$
\begin{align*}
0 & \leq \frac{4}{15} r t^{5 / 2}(1-t)=\frac{r t^{5 / 2}(1-t) \Gamma(3 / 2)}{\Gamma(7 / 2)} \leq z_{1}(t) \\
& =I_{0+}^{2} z_{3}(t) \leq \frac{R}{2}, \quad t \in[0,1] \tag{100}
\end{align*}
$$

Noticing that $\ln z_{3}$ is decreasing on $(0,1)$ and is increasing on $[1,+\infty)$, so we have

$$
\begin{align*}
& \left|\ln z_{3}(t)\right| \leq\left|\ln r t^{5 / 2}(1-t)\right|+|\ln R| \\
& \quad \leq|\ln r|+|\ln R|+\left|\ln t^{5 / 2}(1-t)\right| \\
& \left(z_{1}(t)+z_{2}(t)+z_{3}(t)\right)^{-1 / 4}  \tag{101}\\
& \quad \leq\left(\frac{29}{15} r\right)^{-1 / 4} t^{-5 / 8}(1-t)^{-1 / 4} .
\end{align*}
$$

Consequently

$$
\begin{align*}
\int_{0}^{1} & {\left[\left|\ln t^{1 / 2}(1-t)\right|+\left(\frac{29}{15} r\right)^{-1 / 4} t^{-5 / 8}(1-t)^{-1 / 4}\right] d t } \\
\leq & \frac{1}{2} \int_{0}^{1}|\ln t| d t+\int_{0}^{1}|\ln (1-t)| d t  \tag{102}\\
& +\left(\frac{29}{15} r\right)^{-1 / 4} \int_{0}^{1} t^{-5 / 8}(1-t)^{-1 / 4} d t \\
& \leq \frac{3}{2}+\left(\frac{29}{15} r\right)^{-1 / 4} \mathbf{B}\left(\frac{3}{8}, \frac{3}{4}\right)<+\infty
\end{align*}
$$

The absolute continuity of integral leads to

$$
\begin{align*}
& \lim _{m \rightarrow+\infty} \int_{H(m)}\left[\left|\ln t^{1 / 2}(1-t)\right|\right. \\
& \left.\quad+\left(\frac{29}{15} r\right)^{-1 / 4} t^{-5 / 8}(1-t)^{-1 / 4}\right] d t=0 . \tag{103}
\end{align*}
$$

Hence
$\lim _{m \rightarrow+\infty} \sup _{\substack{z_{1} \in K_{R / 2} \\ z_{2} \in K_{R} \\ z_{3} \in \bar{K}_{R} \backslash K_{r}}} \int_{H(m)} \Phi(s) f\left(s, z_{1}(s), z_{2}(s), z_{3}(s)\right) d s \leq \lim _{m \rightarrow+\infty} \sup _{\substack{z_{1} \in K_{R / 2} \\ z_{2} \in K_{R} \\ z_{3} \in \bar{K}_{R} \backslash K_{r}}} \int_{H(m)} \frac{13(1-s)^{1 / 2}}{\Gamma(1 / 2)}$
$\times \frac{\left(z_{1}(s)+z_{2}(s)+z_{3}(s)\right)^{-1 / 4}+\left|\ln z_{3}(s)\right|}{\sqrt{1-s}} d s$
$\leq \lim _{m \rightarrow+\infty} \sup _{\substack{z_{1} \in K_{R / 2} \\ z_{2} \in K_{R} \\ z_{3} \in \bar{K}_{R} \backslash K_{r}}} \int_{H(m)} \frac{13}{\Gamma(1 / 2)}\left[|\ln R|+|\ln r|+\left|\ln s^{1 / 2}(1-s)\right|+\left(\frac{29}{15} r\right)^{-1 / 4} s^{-5 / 8}(1-s)^{-1 / 4}\right] d s=\frac{26}{\Gamma(1 / 2)}[|\ln R|$
$+|\ln r|] \lim _{m \rightarrow+\infty} \frac{1}{m}+\frac{13}{\Gamma(1 / 2)} \lim _{m \rightarrow+\infty} \int_{H(m)}\left[\left|\ln s^{1 / 2}(1-s)\right|+\left(\frac{29}{15} r\right)^{-1 / 4} s^{-5 / 8}(1-s)^{-1 / 4}\right] d s=0$,

On the other hand, it is obvious that

$$
\begin{align*}
& \liminf _{\substack{z_{1} \longrightarrow 0 \\
z_{2} \longrightarrow 0 \\
z_{3} \longrightarrow 0}} \frac{f\left(t, z_{1}, z_{2}, z_{3}\right)}{z_{1}+z_{2}+z_{3}} \\
& \quad=\liminf _{\substack{z_{1} \longrightarrow 0 \\
z_{2} \longrightarrow 0 \\
z_{3} \longrightarrow 0}} \frac{\left(z_{1}+z_{2}+z_{3}\right)^{-1 / 2}+\ln z_{3}}{\sqrt{t(1-t)}\left(z_{1}+z_{2}+z_{3}\right)}=+\infty  \tag{105}\\
& \limsup _{\substack{z_{1}+z_{2}+z_{3} \longrightarrow+\infty \\
z_{3} \longrightarrow+\infty}} \frac{f\left(t, z_{1}, z_{2}, z_{3}\right)}{z_{3}} \\
& \quad=\limsup _{\substack{z_{1}+z_{2}+z_{3} \longrightarrow+\infty \\
z_{3} \longrightarrow+\infty}} \frac{\left(z_{1}+z_{2}+z_{3}\right)^{-1 / 2}+\ln z_{3}}{z_{3} \sqrt{t(1-t)}=0,}
\end{align*}
$$

uniformly on $t \in[0,1]$, which implies that

$$
\begin{aligned}
& \limsup _{\substack{z_{1}+z_{2}+z_{3} \longrightarrow+\infty \\
z_{3} \longrightarrow+\infty}} \frac{f\left(t, z_{1}, z_{2}, z_{3}\right)}{z_{3}}<\lambda_{1} \\
& \quad<\liminf _{\substack{z_{1} \longrightarrow 0 \\
z_{2} \longrightarrow 0 \\
z_{3} \longrightarrow 0}} \frac{f\left(t, z_{1}, z_{2}, z_{3}\right)}{z_{1}+z_{2}+z_{3}} .
\end{aligned}
$$

Therefore, all conditions of Theorem 9 are satisfied. Thus Theorem 9 ensures that FBVP (92) has at least one positive solution.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Hopf Bifurcation of a Delayed Ecoepidemic Model with Ratio-Dependent Transmission Rate 

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#### Abstract

A delayed ecoepidemic model with ratio-dependent transmission rate has been proposed in this paper. Effects of the time delay due to the gestation of the predator are the main focus of our work. Sufficient conditions for local stability and existence of a Hopf bifurcation of the model are derived by regarding the time delay as the bifurcation parameter. Furthermore, properties of the Hopf bifurcation are investigated by using the normal form theory and the center manifold theorem. Finally, numerical simulations are carried out in order to validate our obtained theoretical results.


## 1. Introduction

In recent years, many dynamical models characterizing the propagation of infectious disease [1-3], spread of computer viruses [4-6], and dynamics of some other systems [7-10] are studied by scholars. Ecoepidemiological research deals with the study of the spread of diseases among interacting populations, where the epidemic and demographic aspects are merged within one model. And they have been investigated by many scholars at home and abroad since the pioneer work of Kermack and McKendrick [11], and the interests in investigating the dynamics of ecoepidemic models will be increasing steadily due to its importance from both the mathematical and the ecological points of view.

Many scholars studied different predator-prey models with disease infection in the prey. Chakraborty et al. [12] studied a ratio-dependent ecoepidemic model with prey harvesting and they assumed that both the susceptible and infected prey are subjected to combined harvesting. Upadhyay and Roy [13] proposed an ecoepidemic model with simple law of mass action and modified Holling type II functional response based on the model in [14]. They analyzed stability (linear and nonlinear) of the model. Zhang et al. [15] proposed a three species ecoepidemic model perturbed by white noise
and they studied stochastic stability and longtime behavior of the model. Zhou et al. [16] studied local and global stability of a modified Leslie-Gower predator-prey model with prey infection. Some delayed ecoepidemic models with disease infection in the prey have been proposed, and the effect of the delay on the models has been investigated [17-19]. Similarly, some scholars proposed and investigated the ecoepidemic models with disease in predators. Sarwardi et al. [20] and Shaikh et al. [21] studied a Leslie-Gower Holling type II pre-dator-prey model with disease in predator and Leslie-Gower Holling type III predator-prey model with disease in predator, respectively. Some other ecoepidemic models with disease in predators one can refer to include [22-29].

Clearly, most of the epidemic models above are formulated based on the bilinear transmission rate, which is based on the law of mass action. As stated in [30], transmission rate plays an important role in the modelling of epidemic dynamics and the infection probability per contact is likely influenced by the number of infective individuals. Thus, it can be concluded that nonlinear transmission rate seems more reasonable than the bilinear one. To study the effect of a nonlinear incidence rate on the dynamics of an ecoepidemic model, Maji et al. [31] proposed the following ecoepidemic model based the work of Morozov [32]:

$$
\begin{align*}
& \frac{d S(t)}{d t}=S(t)\left[r\left(1-\frac{S(t)+I(t)}{K}\right)\right. \\
& \left.\quad-\left(\lambda_{0}+\frac{a P(t)}{1+b P(t)}\right) \frac{I(t)}{S(t)+I(t)}\right] \\
& \frac{d I(t)}{d t}=\left(\lambda_{0}+\frac{a P(t)}{1+b P(t)}\right) \frac{S(t) I(t)}{S(t)+I(t)}-d I(t)  \tag{1}\\
& \quad-\frac{\alpha_{1} I(t) P(t)}{1+\beta I(t)} \\
& \frac{d P(t)}{d t}=\frac{\alpha_{2} I(t) P(t)}{1+\beta I(t)}-\delta P(t)
\end{align*}
$$

where $S(t)>0, I(t) \geq 0$, and $P(t)>0$ present the densities of the healthy prey, the infected prey, and the predator population, respectively. More parameters are listed in Table 1. They studied stability and persistence of system (1).

As we know, delay differential equations exhibit much more complicated dynamics than ordinary differential equations, and delays can make a dynamical system lose its stability and can induce various oscillations and periodic solutions [17, 23, 26, 33-38]. It is interesting to study the effect of time delay on system (1). To this end, and considering the time required for the gestation of the predator, we incorporate time delay due to the gestation of the predator into system (1) and get the following delayed ecoepidemic system:

$$
\begin{align*}
& \frac{d S(t)}{d t}=S(t)\left[r\left(1-\frac{S(t)+I(t)}{K}\right)\right. \\
& \left.\quad-\left(\lambda_{0}+\frac{a P(t)}{1+b P(t)}\right) \frac{I(t)}{S(t)+I(t)}\right] \\
& \frac{d I(t)}{d t}=\left(\lambda_{0}+\frac{a P(t)}{1+b P(t)}\right) \frac{S(t) I(t)}{S(t)+I(t)}-d I(t)  \tag{2}\\
& \quad-\frac{\alpha_{1} I(t) P(t)}{1+\beta I(t)} \\
& \frac{d P(t)}{d t}=\frac{\alpha_{2} I(t-\tau) P(t-\tau)}{1+\beta I(t-\tau)}-\delta P(t)
\end{align*}
$$

subjected to the initial condition:

$$
\begin{align*}
& S(\theta)=\phi_{1}(\theta)>0, \\
& I(\theta)=\phi_{2}(\theta)>0,  \tag{3}\\
& P(\theta)=\phi_{3}(\theta)>0, \quad \theta \in[-\tau, 0)
\end{align*}
$$

where $\tau$ is the time delay due to the gestation of the predator.
This paper is organized as follows. Section 2 deals with local stability and existence of the Hopf bifurcation. In Section 3, direction and stability of the Hopf bifurcation are obtained by using center manifold and normal form theory. In Section 4, some numerical simulations are presented in order to verify the analytical findings. Conclusions and discussions are presented in Section 5.

## 2. Local Stability of the Positive Equilibrium

By direct computation, we can conclude that if $\alpha_{2}>\delta \beta$, then system (2) has positive equilibrium $E_{*}\left(S_{*}, I_{*}, P_{*}\right)$, where

$$
\begin{align*}
& I_{*}=\frac{\delta}{\alpha_{2}-\delta \beta} \\
& P_{*}=\frac{C_{2} S_{*}^{2}+C_{1} S_{*}+C_{0}}{D_{2} S_{*}^{2}+D_{1} S_{*}+D_{0}} \tag{4}
\end{align*}
$$

where $S_{*}$ is the positive root of (5)

$$
\begin{equation*}
K_{5} S^{5}+K_{4} S^{4}+K_{3} S^{3}+K_{2} S^{2}+K_{1} S+K_{0}=0 \tag{5}
\end{equation*}
$$

with

$$
\begin{align*}
K_{0}= & -\left(A_{2} C_{0}^{2}+A_{1} C_{0} D_{0}+A_{0} D_{0}^{2}\right), \\
K_{1}= & B_{2} C_{0}^{2}+B_{1} C_{0} D_{0}+B_{0} D_{0}^{2}-2 A_{2} C_{0} C_{1}-A_{1} C_{1} D_{0} \\
& -A_{1} C_{0} D_{1}-2 A_{0} D_{0} D_{1}, \\
K_{2}= & 2 B_{2} C_{0} C_{1}+B_{1} C_{1} D_{0}+B_{1} C_{0} D_{1}+2 B_{0} D_{0} D_{1} \\
& -A_{2} C_{1}^{2}-2 A_{2} C_{0} C_{2}-A_{1} C_{2} D_{0}-A_{1} C_{1} D_{1} \\
& -A_{1} C_{0} D_{2}-A_{0} D_{1}^{2}-2 A_{0} D_{0} D_{2},  \tag{6}\\
K_{3}= & B_{2} C_{1}^{2}+2 B_{2} C_{0} C_{2}+B_{1} C_{2} D_{0}+B_{1} C_{1} D_{1} \\
& +B_{1} C_{0} D_{2}+B_{0} D_{1}^{2}+2 B_{0} D_{0} D_{2}-2 A_{2} C_{1} C_{2} \\
& -A_{1} C_{2} D_{1}-A_{1} C_{1} D_{2}-2 A_{0} D_{1} D_{2}, \\
K_{4}= & 2 B_{2} C_{1} C_{2}+B_{1} C_{2} D_{1}+B_{1} C_{1} D_{2}+2 B_{0} D_{1} D_{2} \\
& -A_{2} C_{2}^{2}-A_{1} C_{2} D_{2}-A_{0} D_{2}^{2}, \\
K_{5}= & B_{2} C_{2}^{2}+B_{1} C_{2} D_{2},
\end{align*}
$$

and

$$
\begin{align*}
& A_{0}=d I_{*}\left(1+\beta I_{*}\right), \\
& A_{1}=\alpha_{1} I_{*}+b d I_{*}\left(1+\beta I_{*}\right), \\
& A_{2}=b \alpha_{1} I_{*} \\
& B_{0}=\left(\lambda_{0}-d I_{*}\right)\left(1+\beta I_{*}\right), \\
& B_{1}=\left[a+b\left(\lambda_{0}-d I_{*}\right)\right]\left(1+\beta I_{*}\right)-\alpha_{1}, \\
& B_{2}=-b \alpha_{1} \\
& C_{0}=r\left(K-I_{*}\right) I_{*}-k \lambda_{0} I_{*},  \tag{7}\\
& C_{1}=R\left(K-2 I_{*}\right), \\
& C_{2}=-r, \\
& D_{0}=K I_{*}\left(b \lambda_{0}+a\right), \\
& D_{1}=b r I_{*}, \\
& D_{2}=b r .
\end{align*}
$$

Table 1: Parameters and their meanings in this paper.

| Parameter | Description |
| :--- | :--- |
| $K$ | The carrying capacity of the environment |
| $r$ | The maximal per capita growth rate of the healthy prey |
| $\lambda_{0}$ | The transmission rate in the absence of predator |
| $a$ | The predator density mediated additional disease transmission rate |
| $b$ | The inhibitory effect |
| $d$ | The death rate of the infected prey population |
| $\alpha_{1}$ | The per capita predator consumption rate |
| $\alpha_{2}$ | The conversion efficiency of the predator |
| $\beta$ | The encounter rate between the predator and the infected prey |

The Jacobian matrix of system (2) at $E_{*}\left(S_{*}, I_{*}, P_{*}\right)$ is

$$
J\left(E_{*}\right)=\left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13}  \tag{8}\\
a_{21} & a_{22} & a_{23} \\
0 & b_{32} e^{-\lambda \tau} & a_{33}+b_{33} e^{-\lambda \tau}
\end{array}\right)
$$

where

$$
\begin{align*}
& a_{11}=\frac{S_{*} I_{*}}{\left(S_{*}+I_{*}\right)^{2}}\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right)-\frac{r S_{*}}{K}, \\
& a_{12}=-\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*}^{2}}{\left(S_{*}+I_{*}\right)^{2}}-\frac{r S_{*}}{K}, \\
& a_{13}=-\frac{a S_{*} I_{*}}{\left(S_{*}+I_{*}\right)\left(1+b P_{*}\right)^{2}}, \\
& a_{21}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}^{2}}{\left(S_{*}+I_{*}\right)^{2}}, \\
& a_{22}=\frac{\alpha_{1} \beta I_{*} P_{*}}{\left(1+\beta I_{*}\right)^{2}}-\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*} I_{*}}{\left(S_{*}+I_{*}\right)^{2}},  \tag{9}\\
& a_{23}=\frac{a S_{*} I_{*}}{\left(S_{*}+I_{*}\right)\left(1+b P_{*}\right)^{2}}-\frac{\alpha_{1} I_{*}}{1+\beta I_{*}}, \\
& a_{33}=-\delta, \\
& b_{32}=\frac{\alpha_{2} P_{*}}{\left(1+\beta I_{*}\right)^{2}}, \\
& b_{33}=\frac{\alpha_{2} I_{*}}{1+\beta I_{*}}
\end{align*}
$$

Thus, the characteristic equation of $J\left(E_{*}\right)$ about the positive equilibrium $E_{*}$ is given by

$$
\begin{align*}
\lambda^{3}+ & A_{02} \lambda^{2}+A_{01} \lambda+A_{00} \\
& +\left(B_{02} \lambda^{2}+B_{01} \lambda+B_{00}\right) e^{-\lambda \tau}=0 \tag{10}
\end{align*}
$$

with

$$
\begin{aligned}
& A_{00}=a_{33}\left(a_{12} a_{21}-a_{11} a_{22}\right) \\
& A_{01}=a_{11} a_{22}+a_{11} a_{33}+a_{22} a_{33}-a_{12} a_{21}
\end{aligned}
$$

$$
\begin{align*}
& A_{02}=-\left(a_{11}+a_{22}+a_{33}\right) \\
& B_{00}=b_{32}\left(a_{11} a_{23}-a_{13} a_{21}\right)+b_{33}\left(a_{12} a_{21}-a_{11} a_{22}\right) \\
& B_{01}=b_{33}\left(a_{11}+a_{22}\right)-a_{23} b_{32} \\
& B_{02}=-b_{33} \tag{11}
\end{align*}
$$

When $\tau=0$, (10) becomes

$$
\begin{equation*}
\lambda^{3}+p_{2} \lambda^{2}+p_{1} \lambda+p_{0}=0 \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
& p_{0}=A_{00}+B_{00} \\
& p_{1}=A_{01}+B_{01}  \tag{13}\\
& p_{2}=A_{02}+B_{02}
\end{align*}
$$

Based on the Routh-Hurwitz criterion and the discussion in [31], it follows that the positive equilibrium $E_{*}$ is locally asymptotically stable if the following condition holds: $\left(H_{1}\right)$ : $p_{0}>0, p_{1}>0$ and $p_{1} p_{2}>p_{0}$.

For $\tau>0$, let $\lambda=i \omega(\omega>0)$ be the root of (10); then

$$
\begin{align*}
& B_{01} \sin \tau \omega+\left(B_{00}-B_{02} \omega^{2}\right) \cos \tau \omega=A_{02} \omega^{2}-A_{00} \\
& B_{01} \cos \tau \omega-\left(B_{00}-B_{02} \omega^{2}\right) \sin \tau \omega=\omega^{3}-A_{01} \omega \tag{14}
\end{align*}
$$

Thus,

$$
\begin{equation*}
\omega^{6}+l_{2} \omega^{4}+l_{1} \omega^{2}+l_{0} \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
& l_{0}=A_{00}^{2}-B_{00}^{2} \\
& l_{1}=A_{01}^{2}-B_{01}^{2}-2 A_{00} A_{02}+2 B_{00} B_{02}  \tag{16}\\
& l_{2}=A_{02}^{2}-B_{02}^{2}-2 A_{01}
\end{align*}
$$

Suppose that
$\left(H_{2}\right)(15)$ has at least one positive root $\omega_{0}$.
For $\omega_{0}$, from (14)

$$
\begin{equation*}
\tau_{0}=\frac{1}{\omega_{0}} \times \arccos \left\{\frac{\left(B_{01}-A_{02} B_{02}\right) \omega_{0}^{4}+\left(A_{00} B_{02}+A_{02} B_{22}-A_{01} B_{01}\right) \omega_{0}^{2}-A_{00} B_{00}}{B_{01}^{2} \omega_{0}^{2}+\left(B_{00}-B_{02} \omega_{0}^{2}\right)^{2}}\right\} . \tag{17}
\end{equation*}
$$

Differentiating both sides of (10) with respect to $\tau$ yields

$$
\begin{align*}
{\left[\frac{d \lambda}{d \tau}\right]^{-1}=} & -\frac{3 \lambda^{2}+2 A_{02} \lambda+A_{01}}{\lambda\left(\lambda^{3}+A_{02} \lambda^{2}+A_{01} \lambda+A_{00}\right)}  \tag{18}\\
& +\frac{2 B_{02} \lambda+B_{01}}{\lambda\left(B_{02} \lambda^{2}+B_{01} \lambda+B_{00}\right)}-\frac{\tau}{\lambda}
\end{align*}
$$

Further, we have

$$
\begin{equation*}
\operatorname{Re}\left[\frac{d \lambda}{d \tau}\right]_{\tau=\tau_{0}}^{-1}=\frac{f^{\prime}\left(v_{* *}\right)}{\left(\beta_{1} \omega_{0}-\beta_{3} \omega_{0}^{3}\right)^{2}+\left(\beta_{0}-\beta_{2} \omega_{0}^{2}\right)^{2}} \tag{19}
\end{equation*}
$$

where $f(v)=v^{3}+l_{2} v^{2}+l_{1} v+l_{0}$ and $v=\omega^{2}, v_{* *}=\omega_{0}^{2}$.
Obviously, if the condition $\left(H_{3}\right) f^{\prime}\left(\omega_{0}^{2}\right) \neq 0$ holds, then $\operatorname{Re}[d \lambda / d \tau]_{\tau=\tau_{0}}^{-1} \neq 0$. Therefore, based on the Hopf bifurcation theorem in [39], we can obtain the following results.

Theorem 1. Suppose that the conditions $\left(H_{1}\right)-\left(H_{3}\right)$ hold for system (2). The positive equilibrium $E_{*}\left(S_{*}, I_{*}, P_{*}\right)$ is locally asymptotically stable when $\tau \in\left[0, \tau_{0}\right)$ and a Hopf bifurcation occurs at the positive equilibrium $E_{*}\left(S_{*}, I_{*}, P_{*}\right)$ when $\tau=\tau_{0}$.

## 3. Property of the Hopf Bifurcation

Let $\tau=\tau_{0}+\mu, \mu \in R$; then $\mu=0$ is the Hopf bifurcation value of system (2). Rescaling the time delay $t \longrightarrow(t / \tau)$, then system (2) can be transformed into a functional differential equation in $C=C\left([-1,0], R^{3}\right)$ as

$$
\begin{equation*}
\dot{u}(t)=L_{\mu} u_{t}+F\left(\mu, u_{t}\right) \tag{20}
\end{equation*}
$$

where

$$
\begin{equation*}
L_{\mu} \phi=\left(\tau_{0}+\mu\right)\left(M_{1} \phi(0)+M_{2} \phi(-1)\right) \tag{21}
\end{equation*}
$$

and

$$
\begin{equation*}
F(\mu, \phi)=\left(\tau_{0}+\mu\right)\left(F_{1}, F_{2}, F_{3}\right)^{T} \tag{22}
\end{equation*}
$$

with

$$
\begin{align*}
& M_{1}=\left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & 0 & a_{33}
\end{array}\right),  \tag{23}\\
& M_{2}=\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & b_{32} & b_{33}
\end{array}\right),
\end{align*}
$$

and

$$
\begin{align*}
F_{1}= & g_{1} \phi_{1}^{2}(0)+g_{2} \phi_{1}(0) \phi_{2}(0)+g_{3} \phi_{1}(0) \phi_{3}(0) \\
& +g_{4} \phi_{2}(0) \phi_{3}(0)+g_{5} \phi_{2}^{2}(0)+g_{6} \phi_{3}^{2}(0) \\
& +g_{7} \phi_{1}^{3}(0)+g_{8} \phi_{2}^{3}(0)+g_{9} \phi_{3}^{3}(0) \\
& +g_{10} \phi_{1}(0) \phi_{2}^{2}(0)+\cdots, \\
F_{2}= & h_{1} \phi_{1}^{2}(0)+h_{2} \phi_{1}(0) \phi_{2}(0)+h_{3} \phi_{1}(0) \phi_{3}(0) \\
& +h_{4} \phi_{2}(0) \phi_{3}(0)+h_{5} \phi_{2}^{2}(0)+h_{6} \phi_{3}^{2}(0)  \tag{24}\\
& +h_{7} \phi_{1}^{3}(0)+h_{8} \phi_{2}^{3}(0)+h_{9} \phi_{3}^{3}(0) \\
& +h_{10} \phi_{1}(0) \phi_{2}^{2}(0)+\cdots, \\
F_{3}= & k_{1} \phi_{2}^{2}(-1)+k_{2} \phi_{2}(-1) \phi_{3}(-1)+k_{3} \phi_{2}^{3}(-1) \\
& +k_{4} \phi_{2}^{2}(-1) \phi_{3}(-1)+\cdots,
\end{align*}
$$

with
$g_{1}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}\left(I_{*}-S_{*}\right)}{2\left(S_{*}+I_{*}\right)^{3}}-\frac{r}{2 K}$,
$g_{2}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}\left(S_{*}-I_{*}\right)}{2\left(S_{*}+I_{*}\right)^{3}}$,
$g_{3}=\frac{a S_{*} I_{*}}{\left(S_{*}+I_{*}\right)^{2}\left(1+b P_{*}\right)^{2}}$,
$g_{4}=\frac{a S_{*}^{2}}{\left(S_{*}+I_{*}\right)^{2}\left(1+b P_{*}\right)^{2}}$,
$g_{5}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*}^{2}}{\left(S_{*}+I_{*}\right)^{3}}$,
$g_{6}=\frac{a b S_{*} I_{*}}{\left(S_{*}+I_{*}\right)\left(1+b P_{*}\right)^{3}}$,
$g_{7}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}\left(2 S_{*}-I_{*}\right)}{6\left(S_{*}+I_{*}\right)^{4}}$,
$g_{8}=-\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*}^{2}}{6\left(S_{*}+I_{*}\right)^{4}}$,
$g_{9}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*}\left(2 I_{*}-S_{*}\right)}{6\left(S_{*}+I_{*}\right)^{4}}$,
$g_{10}=\frac{2 I_{*}\left(S_{*}-I_{*}\right)}{2\left(S_{*}+I_{*}\right)^{4}}$,

$$
\begin{align*}
& h_{1}=-\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}^{2}}{\left(S_{*}+I_{*}\right)^{3}}, \\
& h_{2}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{2 S_{*} I_{*}}{\left(S_{*}+I_{*}\right)^{3}}, \\
& h_{3}=\frac{a I_{*}^{2}}{2\left(1+b P_{*}\right)^{2}\left(S_{*}+I_{*}\right)^{2}} \text {, } \\
& h_{4}=\frac{\alpha_{1} \beta I_{*}}{\left(1+\beta I_{*}\right)^{2}}-\frac{a S_{*} I_{*}}{\left(1+b P_{*}\right)^{2}\left(S_{*}+I_{*}\right)^{2}}, \\
& h_{5}=\frac{\alpha_{1} \beta P_{*}(1-\beta)}{2\left(1+\beta I_{*}\right)^{3}} \\
& +\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{S_{*}\left(I_{*}-S_{*}\right)}{2\left(1+\beta I_{*}\right)^{3}}, \\
& h_{6}=-\frac{a b S_{*} I_{*}}{\left(S_{*}+I_{*}\right)\left(1+b P_{*}\right)^{3}}, \\
& h_{7}=\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{I_{*}^{2}}{\left(S_{*}+I_{*}\right)^{4}}, \\
& h_{8}=\frac{2 \alpha_{1} \beta^{3} I_{*} P_{*}}{\left(1+\beta I_{*}\right)^{4}}+\frac{2 S_{*}\left(S_{*}-I_{*}\right)}{\left(S_{*}+I_{*}\right)^{4}}, \\
& h_{9}=\frac{a b^{2} S_{*} I_{*}}{\left(S_{*}+I_{*}\right)\left(1+b P_{*}\right)^{4}}, \\
& h_{10}=-\left(\lambda_{0}+\frac{a P_{*}}{1+b P_{*}}\right) \frac{3 S_{*} I_{*}}{\left(S_{*}+I_{*}\right)^{4}}, \\
& k_{1}=-\frac{\alpha_{2} \beta P_{*}}{\left(1+\beta I_{*}\right)^{3}}, \\
& k_{2}=\frac{\alpha_{2}}{\left(1+\beta I_{*}\right)^{2}}, \\
& k_{3}=\frac{\alpha_{2} \beta^{2} P_{*}}{\left(1+\beta I_{*}\right)^{4}}, \\
& k_{4}=-\frac{\alpha_{2} \beta}{\left(1+\beta I_{*}\right)^{3}} \text {. } \tag{25}
\end{align*}
$$

Thus, there exists a $3 \times 3$ matrix function $\eta(\theta, \mu), \theta \in[-1,0]$, such that

$$
\begin{equation*}
L_{\mu} \phi=\int_{-1}^{0} d \eta(\theta, \mu) \phi(\theta), \quad \phi \in C \tag{26}
\end{equation*}
$$

In view of (21), we choose

$$
\begin{equation*}
\eta(\theta, \mu)=\left(\tau_{0}+\mu\right)\left(M_{1} \delta(\theta)+M_{2} \delta(\theta+1)\right) \tag{27}
\end{equation*}
$$

where $\delta$ is the Dirac delta function.

For $\phi \in C\left([-1,0], R^{3}\right)$, define

$$
A(\mu) \phi= \begin{cases}\frac{d \phi(\theta)}{d \theta}, & -1 \leq \theta<0  \tag{28}\\ \int_{-1}^{0} d \eta(\theta, \mu) \phi(\theta), & \theta=0\end{cases}
$$

and

$$
R(\mu) \phi= \begin{cases}0, & -1 \leq \theta<0  \tag{29}\\ F(\mu, \phi), & \theta=0\end{cases}
$$

Then system (20) is equivalent to

$$
\begin{equation*}
\dot{u}(t)=A(\mu) u_{t}+R(\mu) u_{t} . \tag{30}
\end{equation*}
$$

where $u_{t}(\theta)=u(t+\theta)$ for $\theta \in[-1,0]$.
For $\varphi \in C^{1}\left([0,1],\left(R^{3}\right)^{*}\right)$, define

$$
A^{*}(\varphi)= \begin{cases}-\frac{d \varphi(s)}{d s}, & 0<s \leq 1  \tag{31}\\ \int_{-1}^{0} d \eta^{T}(s, 0) \varphi(-s), & s=0\end{cases}
$$

and a bilinear inner product

$$
\begin{align*}
\langle\varphi(s), \phi(\theta)\rangle= & \bar{\varphi}(0) \phi(0) \\
& -\int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta) d \eta(\theta) \phi(\xi) d \xi \tag{32}
\end{align*}
$$

where $\eta(\theta)=\eta(\theta, 0)$. Then $A(0)$ and $A^{*}$ are adjoint operators.
Next, we suppose that $\rho(\theta)=\left(1, \rho_{2}, \rho_{3}\right)^{T} e^{i \omega_{0} \tau_{0} \theta}$ is the eigenvector of $A(0)$ belonging to $+i \omega_{0} \tau_{0}$ and $\rho^{*}(s)=$ $D\left(1, \rho_{2}^{*}, \rho_{3}^{*}\right) e^{i \omega_{0} \tau_{0} s}$ is the eigenvector of $A^{*}(0)$ belonging to $-i \omega_{0} \tau_{0}$. According to the definition of $A(0)$ and $A^{*}$, we can obtain

$$
\begin{align*}
& \rho_{2}=\frac{a_{21}+a_{23} \rho_{3}}{i \omega_{0}-a_{22}} \\
& \rho_{3}=\frac{\left(i \omega_{0}-a_{11}\right)\left(i \omega_{0}-a_{22}\right)-a_{12} a_{21}}{a_{13}\left(i \omega_{0}-a_{22}\right)-a_{12} a_{23}}  \tag{33}\\
& \rho_{2}^{*}=-\frac{i \omega_{0}+a_{22}}{a_{21}} \\
& \rho_{3}^{*}=\frac{\left(i \omega_{0}+a_{11}\right)\left(i \omega_{0}+a_{22}\right)-a_{12} a_{21}}{b_{32} e^{i \tau_{0} \omega_{0}}}
\end{align*}
$$

From (32), we can get

$$
\begin{align*}
\bar{D} & =\left[1+\rho_{2} \bar{\rho}_{2}^{*}+\rho_{3} \bar{\rho}_{3}^{*}\right. \\
& \left.+\tau_{0} e^{-i \tau_{0} \omega_{0}}\left(b_{32} \rho_{2} \bar{\rho}_{2}^{*}+b_{33} \rho_{3} \bar{\rho}_{3}^{*}\right)\right]^{-1} \tag{34}
\end{align*}
$$

such that $\left\langle\rho^{*}, \rho\right\rangle=1$.

Following the method in [39] and using similar computation process in [40], we can get the following coefficients:

$$
\begin{aligned}
g_{20} & =2 \tau_{0} \bar{D}\left[g_{1}+g_{2} \rho_{2}+g_{3} \rho_{3}+g_{4} \rho_{2} \rho_{3}+g_{5} \rho_{2}^{2}\right. \\
& +g_{6} \rho_{3}^{2}+\bar{\rho}_{2}^{*}\left(h_{1}+h_{2} \rho_{2}+h_{3} \rho_{3}+h_{4} \rho_{2} \rho_{3}+h_{5} \rho_{2}^{2}\right. \\
& \left.\left.+h_{6} \rho_{3}^{2}\right)+\bar{\rho}_{3}^{*}\left(k_{1} \rho_{2}^{2} e^{-2 i \tau_{0} \omega_{0}}+k_{2} \rho_{2} \rho_{3} e^{-2 i \tau_{0} \omega_{0}}\right)\right], \\
g_{11} & =\tau_{0} \bar{D}\left[2 g_{1}+g_{2}\left(\rho_{2}+\bar{\rho}_{2}\right)+g_{3}\left(\rho_{3}+\bar{\rho}_{3}\right)\right. \\
& +g_{4}\left(\rho_{2} \bar{\rho}_{3}+\bar{\rho}_{2}\right)+2 g_{5} \rho_{2} \bar{\rho}_{2}+2 g_{6} \rho_{3} \bar{\rho}_{3}+\rho_{2}^{*}\left(2 h_{1}\right. \\
& +h_{2}\left(\rho_{2} \mid \bar{\rho}_{2}\right)+h_{3}\left(\rho_{3}+\bar{\rho}_{3}\right)+h_{4}\left(\rho_{2} \bar{\rho}_{3}+\bar{\rho}_{2}\right) \\
& \left.+2 h_{5} \rho_{2} \bar{\rho}_{2}+2 h_{6} \rho_{3} \bar{\rho}_{3}\right)+\bar{\rho}_{3}^{*}\left(2 k_{1} \rho_{2} \bar{\rho}_{2}+k_{2}\left(\rho_{2} \bar{\rho}_{3}\right.\right. \\
& \left.\left.\left.+\bar{\rho}_{2} \rho_{3}\right)\right)\right],
\end{aligned}
$$

$$
g_{02}=2 \tau_{0} \bar{D}\left[g_{1}+g_{2} \bar{\rho}_{2}+g_{3} \bar{\rho}_{3}+g_{4} \bar{\rho}_{2} \bar{\rho}_{3}+g_{5} \bar{\rho}_{2}^{2}\right.
$$

$$
+g_{6} \bar{\rho}_{3}^{2}+\bar{\rho}_{2}^{*}\left(h_{1}+h_{2} \bar{\rho}_{2}+h_{3} \bar{\rho}_{3}+h_{4} \bar{\rho}_{2} \bar{\rho}_{3}+h_{5} \bar{\rho}_{2}^{2}\right.
$$

$$
\left.\left.+h_{6} \bar{\rho}_{3}^{2}\right)+\bar{\rho}_{3}^{*}\left(k_{1} \bar{\rho}_{2}^{2} e^{2 i \tau_{0} \omega_{0}}+k_{2} \bar{\rho}_{2} \bar{\rho}_{3} e^{2 \tau_{0} \omega_{0}}\right)\right],
$$

$$
g_{21}=2 \tau_{0} \bar{D}\left[g_{1}\left(2 W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right)+g_{2}\left(W_{11}^{(1)}(0)\right.\right.
$$

$$
\left.\cdot \rho_{2}+\frac{1}{2} W_{20}^{(1)}(0) \bar{\rho}_{2}+W_{11}^{(2)}(0)+\frac{1}{2} W_{20}^{(2)}(0)\right)
$$

$$
+g_{3}\left(W_{11}^{(1)}(0) \rho_{3}+\frac{1}{2} W_{20}^{(1)}(0) \bar{\rho}_{3}+W_{11}^{(2)}(0)+\frac{1}{2}\right.
$$

$$
\left.\cdot W_{20}^{(2)}(0)\right)+g_{4}\left(W_{11}^{(2)}(0) \rho_{3}+\frac{1}{2} W_{20}^{(2)}(0) \bar{\rho}_{3}\right.
$$

$$
\left.+W_{11}^{(3)}(0) \rho_{2}+\frac{1}{2} W_{20}^{(3)}(0) \bar{\rho}_{2}\right)+g_{5}\left(2 W_{11}^{(2)}(0) \rho_{2}\right.
$$

$$
\left.+W_{20}^{(2)}(0) \bar{\rho}_{2}\right)+g_{6}\left(2 W_{11}^{(3)}(0) \rho_{3}+W_{20}^{(3)}(0) \bar{\rho}_{3}\right)
$$

$$
+3 g_{7}+3 g_{8} \rho_{2}^{2} \bar{\rho}_{2}+3 g_{9} \rho_{3}^{2} \bar{\rho}_{3}+g_{10}\left(\bar{\rho}_{2}+2 \rho_{2} \bar{\rho}_{2}\right)
$$

$$
+\bar{\rho}_{2}^{*}\left(h_{1}\left(2 W_{11}^{(1)}(0)+W_{20}^{(1)}(0)\right)+h_{2}\left(W_{11}^{(1)}(0) \rho_{2}\right.\right.
$$

$$
\left.+\frac{1}{2} W_{20}^{(1)}(0) \bar{\rho}_{2}+W_{11}^{(2)}(0)+\frac{1}{2} W_{20}^{(2)}(0)\right)
$$

$$
+h_{3}\left(W_{11}^{(1)}(0) \rho_{3}+\frac{1}{2} W_{20}^{(1)}(0) \bar{\rho}_{3}+W_{11}^{(2)}(0)\right.
$$

$$
\left.+\frac{1}{2} W_{20}^{(2)}(0)\right)+h_{4}\left(W_{11}^{(2)}(0) \rho_{3}+\frac{1}{2} W_{20}^{(2)}(0) \bar{\rho}_{3}\right.
$$

$$
\left.+W_{11}^{(3)}(0) \rho_{2}+\frac{1}{2} W_{20}^{(3)}(0) \bar{\rho}_{2}\right)+h_{5}\left(2 W_{11}^{(2)}(0) \rho_{2}\right.
$$

$$
\left.+W_{20}^{(2)}(0) \bar{\rho}_{2}\right)+g_{6}\left(2 W_{11}^{(3)}(0) \rho_{3}+W_{20}^{(3)}(0) \bar{\rho}_{3}\right)
$$

$$
\left.+3 h_{7}+3 h_{8} \rho_{2}^{2} \bar{\rho}_{2}+3 h_{9} \rho_{3}^{2} \bar{\rho}_{3}+h_{10}\left(\bar{\rho}_{2}+2 \rho_{2} \bar{\rho}_{2}\right)\right)
$$

$$
+\bar{\rho}_{3}^{*}\left(k_{1}\left(2 W_{11}^{(2)}(-1) \rho_{2} e^{-i \tau_{0} \omega_{0}}\right)\right.
$$

$$
\begin{align*}
& +k_{2}\left(W_{11}^{(2)}(-1) \rho_{3} e^{-i \tau_{0} \omega_{0}}+\frac{1}{2} W_{20}^{(2)}(-1) \bar{\rho}_{3} e^{i \tau_{0} \omega_{0}}\right. \\
& \left.+W_{11}^{(3)}(-1) \rho_{2} e^{-i \tau_{0} \omega_{0}}+\frac{1}{2} W_{20}^{(3)}(-1) \bar{\rho}_{2} e^{i \tau_{0} \omega_{0}}\right) \\
& +3 k_{3} \rho_{2} e^{-2 i \tau_{0} \omega_{0}}+k_{4}\left(\rho_{2} e^{-i \tau_{0} \omega_{0}} \bar{\rho}_{3}\right. \\
& \left.\left.+2 \rho_{2} \bar{\rho}_{2} \rho_{3} e^{-i \tau_{0} \omega_{0}}\right)\right] \tag{35}
\end{align*}
$$

with

$$
\begin{align*}
W_{20}(\theta)= & \frac{i g_{20} \rho(0)}{\tau_{0} \omega_{0}} e^{i \tau_{0} \omega_{0} \theta}+\frac{i \bar{g}_{02} \bar{\rho}(0)}{3 \tau_{0} \omega_{0}} e^{-i \tau_{0} \omega_{0} \theta} \\
& +E_{1} e^{2 i \tau_{0} \omega_{0} \theta}, \tag{36}
\end{align*}
$$

$$
W_{11}(\theta)=-\frac{i g_{11} \rho(0)}{\tau_{0} \omega_{0}} e^{i \tau_{0} \omega_{0} \theta}+\frac{i \bar{g}_{11} \bar{\rho}(0)}{\tau_{0} \omega_{0}} e^{-i \tau_{0} \omega_{0} \theta}+E_{2}
$$

where $E_{1}$ and $E_{2}$ can be determined by the following two equations:

$$
\begin{align*}
& \left(\begin{array}{ccc}
2 i \omega_{0}-a_{11} & -a_{12} & -a_{13} \\
-a_{21} & 2 i \omega_{0}-a_{22} & -a_{23} \\
0 & -b_{32} e^{-2 i \tau_{0} \omega_{0}} & 2 i \omega_{0}-a_{33}-b_{33} e^{-2 i \tau_{0} \omega_{0}}
\end{array}\right) E_{1} \\
& =2\left(\begin{array}{l}
E_{11} \\
E_{12} \\
E_{13}
\end{array}\right)  \tag{37}\\
& \left(\begin{array}{ccc}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
0 & b_{32} & a_{33}+b_{33}
\end{array}\right) E_{2}=-\left(\begin{array}{l}
E_{21} \\
E_{22} \\
E_{23}
\end{array}\right)
\end{align*}
$$

and

$$
\begin{align*}
E_{11}= & g_{1}+g_{2} \rho_{2}+g_{3} \rho_{3}+g_{4} \rho_{2} \rho_{3}+g_{5} \rho_{2}^{2}+g_{6} \rho_{3}^{2} \\
E_{12}= & h_{1}+h_{2} \rho_{2}+h_{3} \rho_{3}+h_{4} \rho_{2} \rho_{3}+h_{5} \rho_{2}^{2}+h_{6} \rho_{3}^{2} \\
E_{13}= & k_{1} \rho_{2}^{2} e^{-2 i \tau_{0} \omega_{0}}+k_{2} \rho_{2} \rho_{3} e^{-2 i \tau_{0} \omega_{0}}, \\
E_{21}= & 2 g_{1}+g_{2}\left(\rho_{2}+\bar{\rho}_{2}\right)+g_{3}\left(\rho_{3}+\bar{\rho}_{3}\right)  \tag{38}\\
& +g_{4}\left(\rho_{2} \bar{\rho}_{3}+\bar{\rho}_{2}\right)+2 g_{5} \rho_{2} \bar{\rho}_{2}+2 g_{6} \rho_{3} \bar{\rho}_{3} \\
E_{22}= & 2 h_{1}+h_{2}\left(\rho_{2} \mid \bar{\rho}_{2}\right)+h_{3}\left(\rho_{3}+\bar{\rho}_{3}\right) \\
& +h_{4}\left(\rho_{2} \bar{\rho}_{3}+\bar{\rho}_{2}\right)+2 h_{5} \rho_{2} \bar{\rho}_{2}+2 h_{6} \rho_{3} \bar{\rho}_{3} \\
E_{23}= & 2 k_{1} \rho_{2} \bar{\rho}_{2}+k_{2}\left(\rho_{2} \bar{\rho}_{3}+\bar{\rho}_{2} \rho_{3}\right) .
\end{align*}
$$

Then, we can get the following coefficients which determine the properties of the Hopf bifurcation:

$$
\begin{aligned}
C_{1}(0) & =\frac{i}{2 \tau_{0} \omega_{0}}\left(g_{11} g_{20}-2\left|g_{11}\right|^{2}-\frac{\left|g_{02}\right|^{2}}{3}\right)+\frac{g_{21}}{2}, \\
\mu_{2} & =-\frac{\operatorname{Re}\left\{C_{1}(0)\right\}}{\operatorname{Re}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}},
\end{aligned}
$$

$$
\begin{align*}
& \beta_{2}=2 \operatorname{Re}\left\{C_{1}(0)\right\}, \\
& T_{2}=-\frac{\operatorname{Im}\left\{C_{1}(0)\right\}+\mu_{2} \operatorname{Im}\left\{\lambda^{\prime}\left(\tau_{0}\right)\right\}}{\tau_{0} \omega_{0}} . \tag{39}
\end{align*}
$$

In conclusion, we have the following results.
Theorem 2. For system (2), If $\mu_{2}>0\left(\mu_{2}<0\right)$, then the Hopf bifurcation is supercritical (subcritical). If $\beta_{2}<0\left(\beta_{2}>0\right)$, then the bifurcating periodic solutions are stable (unstable). If $T_{2}>$ $0\left(T_{2}<00\right)$, then the bifurcating periodic solutions increase (decrease).

## 4. Numerical Simulation

We choose the same parameters of system (2) as those in [21]: $r=3, K=5, \lambda_{0}=1.5, a=1, b=1, d=0.5, \alpha_{1}=1$, $\alpha_{2}=1, \beta=1$, and $\delta=0.5$, while setting $\tau$ as the bifurcation parameter. Then, we get the specific case of system (2) as follows:

$$
\begin{align*}
& \frac{d S(t)}{d t}=S(t)\left[3\left(1-\frac{S(t)+I(t)}{5}\right)\right. \\
& \left.\quad-\left(1.5+\frac{P(t)}{1+P(t)}\right) \frac{I(t)}{S(t)+I(t)}\right] \\
& \frac{d I(t)}{d t}=\left(1.5+\frac{P(t)}{1+P(t)}\right) \frac{S(t) I(t)}{S(t)+I(t)}-0.5 I(t)  \tag{40}\\
& \quad-\frac{I(t) P(t)}{1+I(t)}, \\
& \frac{d P(t)}{d t}=\frac{I(t-\tau) P(t-\tau)}{1+I(t-\tau)}-0.5 P(t)
\end{align*}
$$

from which we can obtain the unique positive equilibrium $E_{*}(3.107,1,2.328)$. Numerically for $\tau=0$ we have drawn the figure of Lyapunov exponents (Figure 1). Since all the LEs are negative, the system is stable for $\tau=0$. Further, we can obtain $\omega_{0}=0.0042$ and the critical value $\tau_{0}=0.3408$ at which a Hopf bifurcation occurs. As is shown in Figure 2, $E_{*}$ is locally asymptotically stable when $\tau=0.265<\tau_{0}$. In this case, the three species in system (40) can coexist in an ideal stable state. However, $E_{*}$ loses its stability and a family of periodic solutions bifurcate from $E_{*}$ when $\tau=0.405>\tau_{0}$, which can be illustrated by Figure 3.

On the other hand, by some complex calculations, we can obtain $\lambda^{\prime}\left(\tau_{0}\right)=0.002582+0.102144 i$ and $C-1(0)=$ $-0.005236+0.000094 i$. And further we have $\mu_{2}=2.0279>0$, $\beta_{2}=-0.0105<0$ and $T_{2}=-144.7797<0$. Thus, based on the Theorem 2, we can conclude that the Hopf bifurcation is supercritical and the bifurcating periodic solutions are stable and decrease. Since the bifurcating periodic solutions are stable, the three species in system (40) can coexist in an oscillatory mode under some given conditions. This is valuable from the viewpoint of biology.


Figure 1: Lyapunov exponents for $\tau=0$, depicting a stable system.

## 5. Conclusions

In the present paper, we propose a delayed ecoepidemic model by incorporating the time delay due to the gestation of the predator in the model studied in [31]. Compared with the work in [31], we mainly consider the effect of the time delay on the stability of system (2). The model investigated in our paper is more general since the time required for the gestation of the predator and the results we obtained are suitable complements to the literature [31]. By regarding the time delay due to the gestation of the predator as the bifurcation parameter, sufficient conditions for the local stability of the model and the critical value $\tau_{0}$ at which a Hopf bifurcation occurs are derived. It is found that when the value of the time delay is suitablely small, system (2) is locally asymptotically stable. In this case, the densities of the healthy prey, the infected prey, and the predator population will tend to stabilization. Namely, the densities of the three species will be in ideal stable state and the disease spreading among the prey can be controlled. Once the value of the time delay passes through the critical value $\tau_{0}$, system (2) loses stability and a family of periodic solutions bifurcate from the positive equilibrium $E_{*}$, which shows that the delay due to the gestation of the predator plays a very complicated role in destabilizing the stability of system (2). In this case, the densities of the three species may coexist in an oscillatory and the disease spreading among the prey will be out of control. In addition, the explicit formulae determining stability and direction of the Hopf bifurcation are derived by using the normal form theory and then center manifold theorem for the further investigation.

It should be pointed out that predator-prey models involving delays and also spatial diffusion are increasingly applied to the study of a variety of situations. Based on this consideration, we will investigate the dynamics of the ecoepidemic model with diffusion based on the delayed model in our present paper in the near future.

## Data Availability

All the data can be accessed in our manuscript in the Numerical Simulation.


Figure 2: $E_{*}$ is locally asymptotically stable when $\tau=0.265<\tau_{0}$.


Figure 3: $E_{*}$ loses its stability when $\tau=0.405>\tau_{0}$.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Uniqueness of Successive Positive Solution for Nonlocal Singular Higher-Order Fractional Differential Equations Involving Arbitrary Derivatives 

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#### Abstract

In this article, by means of fixed point theorem on mixed monotone operator, we establish the uniqueness of positive solution for some nonlocal singular higher-order fractional differential equations involving arbitrary derivatives. We also give iterative schemes for approximating this unique positive solution.


## 1. Introduction

We are interested in investigating the existence and iterative schemes of the unique positive solution for the following fractional differential equation (FDE):

$$
\begin{align*}
& D_{0+}^{\alpha} u(t)+f\left(t, u(t), D_{0+}^{\delta} u(t)\right)=0, \quad 0<t<1 \\
& D_{0+}^{\delta} u(0)=D_{0+}^{\delta+1} u(0)=\cdots=D_{0+}^{\delta+n-2} u(0)=0  \tag{1}\\
& D_{0+}^{9} u(1)=\lambda \int_{0}^{\eta} h(t) D_{0+}^{\iota} u(t) \mathrm{d} t
\end{align*}
$$

where $D_{0+}^{\alpha}$ is the standard Riemann-Liouville derivative, $n-$ $1<\alpha \leq n, n \geq 3, \vartheta-\delta \geq 1, \alpha-\vartheta-1>0,0 \leq \delta<1,0<\eta \leq 1$, $\lambda$ is a positive parameter with $0 \leq \lambda \Gamma(\alpha-\vartheta) \int_{0}^{\eta} h(t) t^{\alpha-\iota-1} \mathrm{~d} t<$ $\Gamma(\alpha-\imath), f \in C\left(J \times R_{+}^{\prime} \times R_{+}^{\prime}, R_{+}\right), J=(0,1), R_{+}^{\prime}=(0,+\infty), R_{+}=$ $[0,+\infty), 0<\lambda \Gamma(\alpha-\vartheta) \int_{0}^{\eta} h(t) t^{\alpha-\iota-1} \mathrm{~d} t<\Gamma(\alpha-\imath), h \in L^{1}[0,1]$ is nonnegative, $f\left(t, x_{1}, x_{2}\right)$ permits singularities at $x_{i}=0(i=$ $1,2)$, and $t=0,1$.

In recent years, fractional calculus and fractional models play more and more significant role in describing a wide spectrum of nonlinear phenomena in natural sciences,
engineering, economics, biology, and signal and image processing; see books and monographs [1-3] and references [4-32] to name a few. More and more attention has been paid to nonlocal problem of fractional differential equation because of its wide applications to applied mathematics and physics such as chemical engineering, underground water flow, heat conduction, thermoelasticity, and plasma physics. Under different conjugate type integral conditions such as no parameters, only one or two parameters involved in boundary conditions, [8-16, 33, 34] investigate the existence, uniqueness, and multiplicity of positive solutions for FDEs when $f$ is either continuous or singular. Very recently, in [16], we give two uniqueness results of solution for the following FDE:

$$
\begin{align*}
D_{0+}^{\alpha} u(t)+f(t, u(t)) & =0, \quad 0 \leq t \leq 1, \\
u(0) & =u^{\prime}(0)=\cdots=u^{(n-2)}(0)=0,  \tag{2}\\
D_{0+}^{\beta} u(1) & =\lambda \int_{0}^{\eta} h(t) D_{0+}^{\gamma} u(t) \mathrm{d} t,
\end{align*}
$$

where $f \in C(I \times R, R), I=[0,1], R=(-\infty,+\infty)$, and $h \in L^{1}[0,1]$ is nonnegative. The whole discussion is based
on the Banach contraction map principle and the theory of $u_{0}$-positive linear operator.

Motivated by the above papers, in this article we aim to obtain the uniqueness result of solution for BVP (1) by means of theory on mixed monotone operator. This article admits some new features. First, compared to [6-16] the problem considered in this paper performs more general form since another parameter $\delta$ is contained in boundary conditions. Second, the nonlinearity $f$ is not only singular on space variable $u$ but also relative with $\delta$-order derivative of an unknown variable $u$. Finally, the method used in this paper is different from that in [16].

## 2. Preliminaries and Several Lemmas

Definition 1 (see [3]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $x:(0, \infty) \longrightarrow R$ is given by

$$
\begin{equation*}
I_{0+}^{\alpha} x(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} x(s) \mathrm{d} s \tag{3}
\end{equation*}
$$

provided the right-hand side is pointwise defined on $(0, \infty)$.
Definition 2 (see [3]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $x$ : $(0, \infty) \longrightarrow R$ is given by

$$
\begin{equation*}
D_{0+}^{\alpha} x(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{\mathrm{d}}{\mathrm{~d} t}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{\alpha-n+1}} \mathrm{~d} s \tag{4}
\end{equation*}
$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the number $\alpha$, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 3 (see [23]). (1) If $x \in L(0,1), v>\sigma>0$, then

$$
\begin{align*}
I_{0^{+}}^{v} I_{0^{+}}^{\sigma} x(t) & =I_{0^{+}}^{v+\sigma} x(t), \\
D_{0^{+}}^{\sigma} I_{0^{+}}^{v} x(t) & =I_{0^{+}}^{v-\sigma} x(t),  \tag{5}\\
D_{0^{+}}^{\sigma} I_{0^{+}}^{\sigma} x(t) & =x(t) .
\end{align*}
$$

(2) If $v>0, \sigma>0$, then

$$
\begin{equation*}
D_{0^{+}}^{\nu} t^{\sigma-1}=\frac{\Gamma(\sigma)}{\Gamma(\sigma-\nu)} t^{\sigma-\nu-1} \tag{6}
\end{equation*}
$$

Let $u(t)=I_{0^{+}}^{\delta} x(t), x(t) \in C[0,1]$. According to the definition of Riemann-Liouville derivative and Lemma 3, we have

$$
\begin{aligned}
D_{0^{+}}^{\alpha} u(t) & =\frac{\mathrm{d}^{n}}{\mathrm{~d} t^{n}} I_{0^{+}}^{n-\alpha} u(t)=\frac{\mathrm{d}^{n}}{\mathrm{~d} t^{n}} I_{0^{+}}^{n-\alpha} I_{0^{+}}^{\delta} x(t) \\
& =\frac{\mathrm{d}^{n}}{\mathrm{~d} t^{n}} I_{0^{+}}^{n-\alpha+\delta} x(t)=D_{0^{+}}^{\alpha-\delta} x(t) \\
D_{0^{+}}^{\delta} u(t) & =D_{0^{+}}^{\delta} I_{0^{+}}^{\delta} x(t)=x(t) \\
D_{0^{+}}^{\delta+1} u(t) & =x^{\prime}(t), \cdots, D_{0^{+}}^{\delta+n-2} u(t)=x^{(n-2)}(t) ;
\end{aligned}
$$

$$
\begin{align*}
& D_{0^{+}}^{\beta+\delta} u(t)=D_{0^{+}}^{\beta} x(t) \\
& D_{0^{+}}^{\gamma+\delta} u(t)=D_{0^{+}}^{\gamma} x(t) \tag{7}
\end{align*}
$$

Let $\beta=\mathcal{\vartheta}-\delta, \gamma=\imath-\delta$. Then, by (7), BVP (1) can reduce to the following modified fractional boundary value problems (MFBVP):

$$
\begin{align*}
& D_{0+}^{\alpha-\delta} x(t)+f\left(t, I_{0^{+}}^{\delta} x(t), x(t)\right)=0, \quad 0<t<1, \\
& x(0)=x^{\prime}(0)=\cdots=x^{(n-2)}(0)=0,  \tag{8}\\
& D_{0+}^{\beta} x(1)=\lambda \int_{0}^{\eta} h(t) D_{0+}^{\gamma} x(t) \mathrm{d} t .
\end{align*}
$$

In a similar way, we can transform (8) into the form (1). Thus, MFBVP (8) is equivalent to BVP (1).

Lemma 4. Let $u(t)=I_{0^{+}}^{\delta} x(t), x(t) \in C[0,1]$. Then $B V P$ (1) can transform to (8). In addition, if $x \in C[0,1]$ is a positive solution for (8), then $I_{0^{+}}^{\delta} x$ is a positive solution for BVP (1).

Proof. Substituting $u(t)=I_{0^{+}}^{\delta} x(t)$ into (1), we know from Lemma 3 and (7) that

$$
\begin{align*}
D_{0^{+}}^{\alpha} u(t) & =D_{0^{+}}^{\alpha-\delta} x(t) \\
D_{0^{+}}^{\delta} u(t) & =x(t) \\
D_{0^{+}}^{\delta+1} u(t) & =x^{\prime}(t), \cdots, D_{0^{+}}^{\delta+n-2} u(t)=x^{(n-2)}(t)  \tag{9}\\
D_{0^{+}}^{9} u(t) & =D_{0^{+}}^{\beta+\delta} u(t)=D_{0^{+}}^{\beta} x(t) \\
D_{0^{+}}^{\iota} u(t) & =D_{0^{+}}^{\gamma+\delta} u(t)=D_{0^{+}}^{\gamma} x(t)
\end{align*}
$$

Considering this together with the boundary value conditions, we have

$$
\begin{align*}
x(0) & =D_{0^{+}}^{\delta} u(0)=0, \\
x^{(i)}(t) & =D_{0^{+}}^{\delta+i} u(0)=0, \\
D_{0+}^{\beta} x(1) & =D_{0+}^{9} u(1)=\lambda \int_{0}^{\eta} h(t) D_{0+}^{\iota} u(t) \mathrm{d} t \\
& =\lambda \int_{0}^{\eta} h(t) D_{0+}^{\gamma} x(t) \mathrm{d} t, \tag{10}
\end{align*}
$$

and

$$
\begin{equation*}
D_{0+}^{\alpha-\delta} x(t)=-f\left(t, I_{0^{+}}^{\delta} x(t), x(t)\right) \tag{11}
\end{equation*}
$$

Thus, (1) is converted to (8).
Additionally, suppose that $x \in C[0,1]$ is a positive of (8). Let $u(t)=I_{0^{+}}^{\delta} x(t)$. Then, by Lemma 3, one gets

$$
\begin{align*}
D_{0^{+}}^{\alpha} u(t) & =D_{0^{+}}^{\alpha} I_{0^{+}}^{\delta} x(t)=D_{0^{+}}^{\alpha-\delta} x(t) \\
& =-f\left(t, I_{0^{+}}^{\delta} x(t), x(t)\right)  \tag{12}\\
& =-f\left(t, u(t), D_{0^{+}}^{\delta} u(t)\right) .
\end{align*}
$$

The boundary condition $x(0)=x^{\prime}(0)=\cdots=x^{(n-2)}(0)=0$, $D_{0+}^{\beta} x(1)=\lambda \int_{0}^{\eta} h(t) D_{0+}^{\gamma} x(t) \mathrm{d} t$ together with (9) implies that

$$
\begin{align*}
& D_{0+}^{\delta} u(0)=D_{0+}^{\delta+1} u(0)=\cdots=D_{0+}^{\delta+n-2} u(0)=0 \\
& D_{0+}^{9} u(1)=\lambda \int_{0}^{\eta} h(t) D_{0+}^{\iota} u(t) \mathrm{d} t \tag{13}
\end{align*}
$$

That is to say, $I_{0^{+}}^{\delta} x(t)$ is a positive solution of BVP (1).
Remark 5. Direct computation implies that

$$
\begin{align*}
I_{0^{+}}^{\delta} t^{\alpha-\delta-1} & =\frac{1}{\Gamma(\delta)} \int_{0}^{t}(t-s)^{\delta-1} s^{\alpha-\delta-1} \mathrm{~d} s \\
& =\frac{B(\delta, \alpha-\delta)}{\Gamma(\delta)} t^{\alpha-1}=\frac{\Gamma(\alpha-\delta)}{\Gamma(\alpha)} t^{\alpha-1} \tag{14}
\end{align*}
$$

The following two lemmas are isomorphic forms of those in [16].

Lemma 6 (see [16]). Assume that $\lambda \Gamma(\alpha-\delta-$ $\beta) \int_{0}^{\eta} h(t) t^{\alpha-\delta-\gamma-1} \mathrm{~d} t \neq \Gamma(\alpha-\delta-\gamma)$. Then for any $y \in L(0,1)$, the unique solution of the boundary value problems

$$
\begin{align*}
D_{0+}^{\alpha-\delta} x(t)+y(t) & =0, \quad 0<t<1 \\
x(0) & =x^{\prime}(0)=\cdots=x^{(n-2)}(0)=0,  \tag{15}\\
D_{0+}^{\beta} x(1) & =\lambda \int_{0}^{\eta} h(t) D_{0+}^{\gamma} x(t) \mathrm{d} t
\end{align*}
$$

solves

$$
\begin{equation*}
x(t)=\int_{0}^{1} G(t, s) y(s) \mathrm{d} s, \quad t \in[0,1] \tag{16}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, s)=G_{1}(t, s)+G_{2}(t, s), \\
& G_{1}(t, s) \\
& \quad= \begin{cases}\frac{t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1}-(t-s)^{\alpha-\delta-1}}{\Gamma(\alpha-\delta)}, & 0 \leq s \leq t \leq 1, \\
\frac{t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1}}{\Gamma(\alpha-\delta)}, & 0 \leq t \leq s \leq 1,\end{cases} \\
& G_{2}(t, s)=\frac{\lambda \Gamma(\alpha-\delta-\beta) t^{\alpha-\delta-1}}{\Gamma(\alpha-\delta-\gamma)-\lambda \Gamma(\alpha-\delta-\beta) \int_{0}^{\eta} h(t) t^{\alpha-\delta-\gamma-1} \mathrm{~d} t} \\
& H(t, s) \\
& \quad= \begin{cases}\eta \\
\quad h(t) H(t, s) \mathrm{d} t, & 0 \leq t \leq s \leq 1, \\
\frac{t^{\alpha-\delta-\gamma-1}(1-s)^{\alpha-\delta-\beta-1}-(t-s)^{\alpha-\delta-\gamma-1}}{\Gamma(\alpha-\delta)}, & 0 \leq t \leq 1, \\
\frac{t^{\alpha-\delta-\gamma-1}(1-s)^{\alpha-\delta-\beta-1}}{\Gamma(\alpha-\delta)}, & 0 \leq t\end{cases}
\end{align*}
$$

Here, $G(t, s)$ is called the Green function of $B V P(15)$. Obviously, $G(t, s)$ is continuous on $[0,1] \times[0,1]$.

Clearly, $x$ is a positive solution of BVP (8) if and only if $x \in C[0,1]$ is a solution of the following nonlinear integral equation:

$$
\begin{equation*}
x(t)=\int_{0}^{1} G(t, s) f\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right) \mathrm{d} s . \tag{21}
\end{equation*}
$$

Lemma 7 (see [16]). The functions $G_{1}(t, s)$ and $G(t, s)$ given by (18) and (17), respectively, admit the following properties:
$\left(a_{1}\right) G_{1}(t, s) \geq(1 / \Gamma(\alpha-\delta)) t^{\alpha-\delta-1} s(1-s)^{\alpha-\delta-\beta-1}, \forall t, s \in$ [0, 1];
$\left(a_{2}\right) G_{1}(t, s) \leq(1 / \Gamma(\alpha-\delta))(\alpha-\delta-1) s(1-s)^{\alpha-\delta-\beta-1}, \forall t, s \in$ [0, 1];
$\left(a_{3}\right) G(t, s) \leq J(s), J(s)=(1 / \Gamma(\alpha-\delta))(\alpha-\delta-1) s(1-$ $s)^{\alpha-\delta-\beta-1}+\lambda \Gamma(\alpha-\delta-\beta) /(\Gamma(\alpha-\delta-\gamma)-\lambda \Gamma(\alpha-\delta-$ ر) $\left.\int_{0}^{\eta} h(t) t^{\alpha-\delta-\gamma-1} \mathrm{~d} t\right) \cdot \int_{0}^{\eta} h(t) H(t, s) \mathrm{d} t, \quad \forall t, s \in[0,1]$;
$\left(a_{4}\right)(1 /(\alpha-\delta-1)) t^{\alpha-\delta-1} J(s) \leq G(t, s) \leq(1 / \Gamma(\alpha-$ $\delta)) \Lambda t^{\alpha-\delta-1}(1-s)^{\alpha-\delta-\beta-1}$, here $\Lambda=(\alpha-\delta-1)+(\lambda \Gamma(\alpha-$ $\left.\delta-\beta) /\left(\Gamma(\alpha-\delta-\gamma)-\lambda \Gamma(\alpha-\delta-\beta) \int_{0}^{\eta} h(t) t^{\alpha-\delta-\gamma-1} \mathrm{~d} t\right)\right)$. $\int_{0}^{\eta} h(t) t^{\alpha-\delta-\gamma-1} \mathrm{~d} t, \forall t, s \in[0,1]$.

Let $P$ be a normal cone in a Banach space $E$ and nonzero element $e \in P$ with $\|e\| \leq 1$. A subset of cone $P$ is given as follows:

$$
\begin{align*}
Q_{e} & =\{x \in P \mid \text { there exist constants } m, M  \tag{22}\\
& >0 \text { such that } m e \leq x \leq M e\} .
\end{align*}
$$

Definition 8 (see [35]). Assume that $A: Q_{e} \times Q_{e} \longrightarrow Q_{e}$. A is said to be mixed monotone if $A(x, y)$ is nondecreasing in $x$ and nonincreasing in $y$, i.e., if $x_{1} \leq x_{2}\left(x_{1}, x_{2} \in Q_{e}\right)$ implies $A\left(x_{1}, y\right) \leq A\left(x_{2}, y\right)$ for any $y \in Q_{e}$, and $y_{1} \leq y_{2}\left(y_{1}, y_{2} \in\right.$ $Q_{e}$ ) implies $A\left(x, y_{1}\right) \leq A\left(x, y_{2}\right)$ for any $x \in Q_{e}$. The element $x^{*} \in Q_{e}$ is said to be a fixed point of $A$ if $A\left(x^{*}, x^{*}\right)=x^{*}$.

Lemma 9 (see [17]). Suppose that $A: Q_{e} \times Q_{e} \longrightarrow Q_{e}$ is a mixed monotone operator and there exists a constant $\sigma, 0 \leq$ $\sigma<1$, such that

$$
\begin{equation*}
A\left(t x, \frac{1}{t} y\right) \geq t^{\sigma} A(x, y), \quad \forall x, y \in Q_{e}, 0<t<1 \tag{23}
\end{equation*}
$$

Then $A$ has a unique fixed point $x^{*} \in Q_{e}$. Moreover, for any $\left(x_{0}, y_{0}\right) \in Q_{e} \times Q_{e}$,

$$
\begin{align*}
& x_{k}=A\left(x_{k-1}, y_{k-1}\right) \\
& y_{k}=A\left(y_{k-1}, x_{k-1}\right) \tag{24}
\end{align*}
$$

$$
k=1,2, \cdots
$$

satisfy

$$
\begin{align*}
& x_{k} \longrightarrow x^{*}  \tag{25}\\
& y_{k} \longrightarrow x^{*}
\end{align*}
$$

where

$$
\begin{align*}
& \left\|x_{k}-x^{*}\right\|=o\left(1-r^{\sigma^{k}}\right)  \tag{26}\\
& \left\|y_{k}-x^{*}\right\|=o\left(1-r^{\sigma^{k}}\right)
\end{align*}
$$

where, $r$ is a constant, $0<r<1$, and dependent on $x_{0}, y_{0}$.

## 3. Main Result

Throughout this paper, we adopt the following assumptions:
$\left(\mathrm{H}_{1}\right) f\left(t, x_{1}, x_{2}\right)=\phi\left(t, x_{1}, x_{2}\right)+\psi\left(t, x_{1}, x_{2}\right)$, where, $\phi, \psi$ : $J \times R_{+}^{\prime} \times R_{+}^{\prime} \longrightarrow R_{+}$is continuous, $\phi\left(t, x_{1}, x_{2}\right)$ is nondecreasing on $x_{i}$, and $\psi\left(t, x_{1}, x_{2}\right)$ is nonincreasing on $x_{i}(i=1,2)$.
$\left(\mathrm{H}_{2}\right)$ There exists $\sigma \in J$ such that for $x_{i}>0(i=$ 1,2), $t, c \in J$

$$
\begin{align*}
\phi\left(t, c x_{1}, c x_{2}\right) & \geq c^{\sigma} \phi\left(t, x_{1}, x_{2}\right) \\
\psi\left(t, c^{-1} x_{1}, c^{-1} x_{2}\right) & \geq c^{\sigma} \psi\left(t, x_{1}, x_{2}\right) \tag{27}
\end{align*}
$$

$\left(\mathrm{H}_{3}\right)$
$0<\int_{0}^{1}(1-s)^{\alpha-9-1} s^{-\sigma(\alpha-1)} \psi(s, 1,1) \mathrm{d} s<+\infty ;$
$0<\int_{0}^{1}(1-s)^{\alpha-9-1} \phi(s, 1,1) \mathrm{d} s<+\infty$.
Remark 10. According to $\left(\mathrm{H}_{2}\right)$, for any $t \in J, c \geq 1, x_{i}>0$ ( $i=$ 1,2 ), one has

$$
\begin{align*}
\phi\left(t, c x_{1}, c x_{2}\right) & \leq c^{\sigma} \phi\left(t, x_{1}, x_{2}\right)  \tag{33}\\
\psi\left(t, c^{-1} x_{1}, c^{-1} x_{2}\right) & \leq c^{\sigma} \psi\left(t, x_{1}, x_{2}\right) . \tag{29}
\end{align*}
$$

Theorem 11. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold. Then, the BVP (1) has a unique solution $u^{*}$, and there exists a constant $D>1$ such that

$$
\begin{equation*}
\frac{\Gamma(\alpha-\delta)}{D \Gamma(\alpha)} t^{\alpha-1} \leq u^{*}(t) \leq \frac{D \Gamma(\alpha-\delta)}{\Gamma(\alpha)} t^{\alpha-1} \tag{30}
\end{equation*}
$$

$$
\forall t \in[0,1]
$$

Moreover, for any $u_{0}$, we construct a successive sequence

$$
\begin{align*}
& u_{k+1}(t)=I_{0^{+}}^{\delta}\left\{\int _ { 0 } ^ { 1 } G ( t , s ) \left[\phi\left(s, u_{k}(s), D_{0^{+}}^{\delta} u_{k}(s)\right)\right.\right. \\
& \left.\left.\quad+\psi\left(s, u_{k}(s), D_{0^{+}}^{\delta} u_{k}(s)\right)\right] \mathrm{d} s\right\}, \quad k=1,2, \cdots \tag{31}
\end{align*}
$$

and we have $\left\|u_{k}-u^{*}\right\| \longrightarrow 0$ as $k \longrightarrow \infty$; the convergence rate is

$$
\begin{equation*}
\left\|u_{k}-u^{*}\right\|=\left\|I_{0^{+}}^{\delta} x_{k}-I_{0^{+}}^{\delta} x^{*}\right\|=o\left(1-r^{\sigma^{k}}\right) \tag{28}
\end{equation*}
$$

where $r$ is a constant with $0<r<1$ and is dependent on $u_{0}$.
Proof. Let $e(t)=t^{\alpha-\delta-1}$, and we define

$$
Q_{e}=\left\{x \in C[0,1] \left\lvert\, \frac{1}{D} e(t) \leq x(t) \leq D e(t)\right.\right\},
$$

where

$$
\begin{align*}
D & >\max \left\{\left[\frac{1}{\Gamma(\alpha-\delta)} \int_{0}^{1} s(1-s)^{\alpha-9-1}\left(\rho^{\sigma} s^{\sigma(\alpha-1)} \phi(s, 1,1)+2^{-\sigma} b^{-\sigma} \psi(s, 1,1)\right) \mathrm{d} s\right]^{1 /(\sigma-1)}, 1,2 \rho\right. \\
& {\left.\left[\frac{\Lambda}{\Gamma(\alpha-\delta)} \int_{0}^{1}(1-s)^{\alpha-9-1}\left(2^{\sigma} b^{\sigma} \phi(s, 1,1)+\rho^{-\sigma} s^{-\sigma(\alpha-1)} \psi(s, 1,1)\right) \mathrm{d} s\right]^{1 /(1-\sigma)}\right\} } \tag{34}
\end{align*}
$$

where $b=\max \{\Gamma(\alpha-\delta) / \Gamma(\alpha), 1\}, \rho=\min \{1, \Gamma(\alpha-\delta) / \Gamma(\alpha)\}$. We consider the existence of positive solution for (8). For any $x, y \in Q_{e}$, define an operator $T$ as follows:

$$
\begin{align*}
& T(x, y)(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right.  \tag{35}\\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s .
\end{align*}
$$

It is clear that $x$ is a positive solution of BVP (8) if and only if $x \in C[0,1]$ is a fixed point of the operator $T$.

First, we are in position to show that $T: Q_{e} \times Q_{e} \longrightarrow Q_{e}$ is well defined. By $\left(\mathrm{H}_{2}\right)$ and Remark 5, for any $x, y \in Q_{e}$ we have

$$
\begin{aligned}
& \phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right) \\
& \quad \leq \phi\left(s, \frac{D \Gamma(\alpha-\delta)}{\Gamma(\alpha)} s^{\alpha-1}, D s^{\alpha-\delta-1}\right)
\end{aligned}
$$

$$
\begin{align*}
& \leq \phi(s, D b+1, D b+1) \leq(D b+1)^{\sigma} \phi(s, 1,1) \\
& \leq 2^{\sigma} b^{\sigma} D^{\sigma} \phi(s, 1,1), \quad s \in(0,1) \tag{36}
\end{align*}
$$

and

$$
\begin{align*}
\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right) & \leq \psi\left(s, \frac{\Gamma(\alpha-\delta)}{D \Gamma(\alpha)} s^{\alpha-1}, \frac{1}{D} s^{\alpha-\delta-1}\right) \\
& \leq \psi\left(s, \frac{\rho}{D} s^{\alpha-1}, \frac{\rho}{D} s^{\alpha-1}\right) \\
& \leq\left(\frac{\rho}{D} s^{\alpha-1}\right)^{-\sigma} \psi(s, 1,1)  \tag{37}\\
& =\rho^{-\sigma} D^{\sigma} s^{-\sigma(\alpha-1)} \psi(s, 1,1)
\end{align*}
$$

$$
s \in(0,1) .
$$

Considering the fact that $(\rho / D) s^{\alpha-1}<1$, we have from (34), ( $\mathrm{H}_{2}$ ), and Remark 5 that

$$
\begin{aligned}
& \phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right) \geq \phi\left(s, \frac{\Gamma(\alpha-\delta)}{D \Gamma(\alpha)} s^{\alpha-1}, \frac{1}{D} s^{\alpha-\delta-1}\right) \\
& \geq \phi\left(s, \frac{\rho}{D} s^{\alpha-1}, \frac{\rho}{D} s^{\alpha-\delta-1}\right) \\
& \geq \phi\left(s, \frac{\rho}{D} s^{\alpha-1}, \frac{\rho}{D} s^{\alpha-1}\right) \\
& \geq\left(\frac{\rho}{D} s^{\alpha-1}\right)^{\sigma} \phi(s, 1,1) \\
& \geq \rho^{\sigma} D^{-\sigma} s^{\sigma(\alpha-1)} \phi(s, 1,1) \\
& s \in(0,1)
\end{aligned}
$$

and

$$
\begin{aligned}
& \psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right) \\
& \quad \geq \psi\left(s, \frac{D \Gamma(\alpha-\delta)}{\Gamma(\alpha)} s^{\alpha-1}, D s^{\alpha-\delta-1}\right) \\
& \geq \psi\left(s, D b s^{\alpha-1}, D b s^{\alpha-\delta-1}\right) \geq \psi(s, D b+1, D b+1) \\
& \geq(D b+1)^{-\sigma} \psi(s, 1,1) \geq 2^{-\sigma} b^{-\sigma} D^{-\sigma} \psi(s, 1,1) \\
& \quad s \in(0,1)
\end{aligned}
$$

Thus, it follows from (36), (37), Lemma 7, and $\left(\mathrm{H}_{3}\right)$ that

$$
\begin{aligned}
& \int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s \leq \frac{1}{\Gamma(\alpha-\delta)} \\
& \cdot \Lambda t^{\alpha-\delta-1} \int_{0}^{1}(1-s)^{\alpha-\delta-\beta-1}\left[2^{\sigma} b^{\sigma} D^{\sigma} \phi(s, 1,1)\right. \\
& \left.\quad+\rho^{-\sigma} D^{\sigma} s^{-\sigma(\alpha-1)} \psi(s, 1,1)\right] \mathrm{d} s=\frac{1}{\Gamma(\alpha-\delta)} \\
& \quad \cdot \Lambda t^{\alpha-\delta-1} \int_{0}^{1}(1-s)^{\alpha-9-1}\left[2^{\sigma} b^{\sigma} D^{\sigma} \phi(s, 1,1)\right. \\
& \left.\quad+\rho^{-\sigma} D^{\sigma} s^{-\sigma(\alpha-1)} \psi(s, 1,1)\right] \mathrm{d} s<+\infty
\end{aligned}
$$

$\forall t \in[0,1]$.

This means that $T: Q_{e} \times Q_{e} \longrightarrow P$ is well defined.
On the other hand, we can easily see from (34) and (40) that

$$
\begin{equation*}
T(x, y)(t) \leq D t^{\alpha-\delta-1}=D e(t), \quad \forall t \in[0,1] . \tag{41}
\end{equation*}
$$

At the same time, by (38), (39), and Lemma 7, we know that

$$
\begin{align*}
& T(x, y)(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s \geq \frac{1}{\Gamma(\alpha-\delta)} \\
& \quad \cdot t^{\alpha-\delta-1} \int_{0}^{1} s(1-s)^{\alpha-\delta-\beta-1}\left[\rho^{\sigma} D^{-\sigma} s^{\sigma(\alpha-1)} \phi(s, 1,1)\right. \\
& \left.\quad+2^{-\sigma} b^{-\sigma} D^{-\sigma} \psi(s, 1,1)\right] \mathrm{d} s=\frac{1}{\Gamma(\alpha-\delta)}  \tag{42}\\
& \quad \cdot t^{\alpha-\delta-1} \int_{0}^{1} s(1-s)^{\alpha-9-1}\left[\rho^{\sigma} D^{-\sigma} s^{\sigma(\alpha-1)} \phi(s, 1,1)\right. \\
& \left.\quad+2^{-\sigma} b^{-\sigma} D^{-\sigma} \psi(s, 1,1)\right] \mathrm{d} s \geq \frac{1}{D} t^{\alpha-\delta-1}=\frac{1}{D} \\
& \quad \cdot e(t), \quad \forall t \in[0,1] .
\end{align*}
$$

It follows from (40)-(42) that $T: Q_{e} \times Q_{e} \longrightarrow Q_{e}$ is well defined.

Next, we shall prove that $T: Q_{e} \times Q_{e} \longrightarrow Q_{e}$ is a mixed monotone operator. To this end, let $x_{1}, x_{2} \in Q_{e}$ with $x_{1} \leq$ $x_{2}$. For any $y \in Q_{e}$, it follows from $\left(\mathrm{H}_{1}\right)$ together with the monotonicity of the operator $I_{0^{+}}^{\delta}$ that

$$
\begin{align*}
& T\left(x_{1}, y\right)(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x_{1}(s), x_{1}(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s \leq \int_{0}^{1} G(t, s)  \tag{43}\\
& \quad \cdot\left[\phi\left(s, I_{0^{+}}^{\delta} x_{2}(s), x_{2}(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s=T\left(x_{2}, y\right)(t),
\end{align*}
$$

which implies that $T(x, y)$ is nondecreasing in $x$ for any $y \in$ $Q_{e}$. In a similar manner, for any $x, y_{1}, y_{2} \in Q_{e}$ with $y_{1} \leq y_{2}$, we have

$$
\begin{align*}
& T\left(x, y_{1}\right)(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y_{1}(s), y_{1}(s)\right)\right] \mathrm{d} s \geq \int_{0}^{1} G(t, s)  \tag{44}\\
& \quad \cdot\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right. \\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} y_{2}(s), y_{2}(s)\right)\right] \mathrm{d} s=T\left(x, y_{2}\right)(t)
\end{align*}
$$

This is to say, $T(x, y)$ is nonincreasing in $y$ for any $x \in Q_{e}$. Thus, $T: Q_{e} \times Q_{e} \longrightarrow Q_{e}$ is a mixed monotone operator.

Finally, by $\left(\mathrm{H}_{2}\right)$, one has

$$
\begin{aligned}
& T\left(c x, c^{-1} y\right)(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, c I_{0^{+}}^{\delta} x(s), c x(s)\right)\right. \\
& \left.\quad+\psi\left(s, c^{-1} I_{0^{+}}^{\delta} y(s), c^{-1} y(s)\right)\right] \mathrm{d} s \geq \int_{0}^{1} G(t, s)
\end{aligned}
$$

$$
\begin{aligned}
& \cdot c^{\sigma}\left[\phi\left(s, I_{0^{+}}^{\delta} x(s), x(s)\right)\right. \\
& \left.+\psi\left(s, I_{0^{+}}^{\delta} y(s), y(s)\right)\right] \mathrm{d} s=c^{\sigma} T(x, t)(t)
\end{aligned}
$$

$$
\begin{equation*}
\forall t \in[0,1], x, y \in Q_{e} \tag{45}
\end{equation*}
$$

which means that (23) in Lemma 9 is satisfied. Hence, Lemma 9 guarantees that there exists a unique positive solution $x^{*}$ for BVP (8). Let $u^{*}(t)=I_{0^{+}}^{\delta} x^{*}(t)$; then $u^{*}$ is the unique positive solution for BVP (1).

In addition, for any $u_{0}(t)=I_{0^{+}}^{\delta} x_{0}(t) \in Q_{e}$, by Lemma 9 , constructing a successive sequence

$$
\begin{align*}
& x_{k+1}(t)=\int_{0}^{1} \mathrm{G}(t, s)\left[\phi\left(s, I_{0^{+}}^{\delta} x_{k}(s), x_{k}(s)\right)\right.  \tag{46}\\
& \left.\quad+\psi\left(s, I_{0^{+}}^{\delta} x_{k}(s), x_{k}(s)\right)\right] \mathrm{d} s, \quad k=1,2, \cdots
\end{align*}
$$

and $u_{k+1}(t)=I_{0^{+}}^{\delta} x_{k+1}(t)$, then

$$
\begin{align*}
u_{k+1}(t)= & I_{0^{+}}^{\delta}\left\{\int _ { 0 } ^ { 1 } G ( t , s ) \left[\phi\left(s, u_{k}(s), D_{0^{+}}^{\delta} u_{k}(s)\right)\right.\right. \\
\left.\left.+\psi\left(s, u_{k}(s), D_{0^{+}}^{\delta} u_{k}(s)\right)\right] \mathrm{d} s\right\}, &  \tag{47}\\
& k=1,2, \cdots
\end{align*}
$$

and we have $\left\|u_{k}-u^{*}\right\|=\left\|I_{0^{+}}^{\delta} x_{k}-I_{0^{+}}^{\delta} x^{*}\right\| \longrightarrow 0$ as $k \longrightarrow \infty$; the convergence rate is

$$
\begin{equation*}
\left\|u_{k}-u^{*}\right\|=\left\|I_{0^{+}}^{\delta} x_{k}-I_{0^{+}}^{\delta} x^{*}\right\|=o\left(1-r^{\sigma^{k}}\right) \tag{48}
\end{equation*}
$$

where $r$ is a constant with $0<r<1$ and is dependent on $u_{0}$. Moreover, by Remark 5, we get

$$
\begin{equation*}
\frac{\Gamma(\alpha-\delta)}{D \Gamma(\alpha)} t^{\alpha-1} \leq u^{*}(t)=I_{0^{+}}^{\delta} x^{*}(t) \leq \frac{D \Gamma(\alpha-\delta)}{\Gamma(\alpha)} t^{\alpha-1} \tag{49}
\end{equation*}
$$

$\forall t \in[0,1]$.

## 4. An Example

Example 1. Consider the following fractional differential equation integral boundary value problems:

$$
\begin{aligned}
& D_{0+}^{9 / 2} u(t)+f\left(t, u(t), D_{0^{+}}^{1 / 4} u(t)\right)=0, \quad 0<t<1, \\
& D_{0^{+}}^{1 / 4} u(0)=D_{0^{+}}^{5 / 4} u(0)=D_{0^{+}}^{9 / 4} u(0)=D_{0^{+}}^{13 / 4} u(0)=0, \\
& D_{0+}^{7 / 4} u(1)=\frac{2}{3} \int_{0}^{4 / 5} t^{-3 / 4} D_{0+}^{3 / 2} u(t) \mathrm{d} t,
\end{aligned}
$$

where $\alpha=9 / 2, n=5, \delta=1 / 4, \vartheta=7 / 4, \iota=3 / 2, \lambda=$ $2 / 3, h(t)=t^{-3 / 4}$,

$$
\begin{align*}
f\left(t, u(t), D_{0^{+}}^{1 / 4} u(t)\right)= & \left(t^{-2 / 3}+\cos t\right) u^{1 / 8}(t) \\
& +t^{-1 / 4} u^{-1 / 9}(t) \\
& +6 t^{4 / 5}\left(D_{0^{+}}^{1 / 4} u(t)\right)^{1 / 7}  \tag{51}\\
& +(3-2 t)\left(D_{0^{+}}^{1 / 4} u(t)\right)^{-1 / 7}
\end{align*}
$$

By simple computation, we have $\lambda \Gamma(\alpha-\vartheta) \int_{0}^{\eta} h(t) t^{\alpha-\iota-1} \mathrm{~d} t=$ $2 / 3 \times \Gamma(11 / 4) \times \int_{0}^{4 / 5} t^{-3 / 4} \cdot t^{2} \mathrm{~d} t=2 / 3 \times 1.6084 \times 0.2690=$ $0.2884<2=\Gamma(\alpha-\iota)$. It is easy to know that $\left(\mathrm{H}_{1}\right)$ holds for

$$
\begin{align*}
& \phi\left(t, x_{1}, x_{2}\right)=\left(t^{-2 / 3}+\cos t\right) x_{1}^{1 / 8}+6 t^{4 / 5} x_{2}^{1 / 7}  \tag{52}\\
& \psi\left(t, x_{1}, x_{2}\right)=t^{-1 / 4} x_{1}^{-1 / 9}+(3-2 t) x_{2}^{-1 / 7}
\end{align*}
$$

At the same time, for any $\left(t, x_{1}, x_{2}\right) \in J \times R_{+}^{\prime} \times R_{+}^{\prime}$ and $c \in J$, one has

$$
\begin{align*}
\phi\left(t, c x_{1}, c x_{2}\right)= & c^{1 / 8}\left(t^{-2 / 3}+\cos t\right) x_{1}^{1 / 8} \\
& +c^{1 / 7} 6 t^{4 / 5} x_{2}^{1 / 7} \\
\geq & c^{1 / 7} \phi\left(t, x_{1}, x_{2}\right) \\
\psi\left(t, c^{-1} x_{1}, c^{-1} x_{2}\right)= & c^{1 / 9} t^{-1 / 4} x_{1}^{-1 / 9}  \tag{53}\\
& +c^{1 / 7}(3-2 t) x_{2}^{-1 / 7} \\
\geq & c^{1 / 7} \psi\left(t, x_{1}, x_{2}\right) .
\end{align*}
$$

Thus, $\left(\mathrm{H}_{2}\right)$ is valid for $\sigma=1 / 7$. Notice that $\phi(s, 1,1)=s^{-2 / 3}+$ $\cos s+6 s^{4 / 5}, \psi(s, 1,1)=s^{-1 / 4}+(3-2 s)$, and one gets

$$
\begin{align*}
& \int_{0}^{1}(1-s)^{\alpha-9-1} s^{-\sigma(\alpha-1)} \psi(s, 1,1) \mathrm{d} s \\
& \quad=\int_{0}^{1}(1-s)^{7 / 4} s^{-1 / 2}\left(s^{-1 / 4}+3-2 s\right) \mathrm{d} s \approx 5.9264  \tag{54}\\
& \quad<+\infty
\end{align*}
$$

and

$$
\begin{align*}
& \int_{0}^{1}(1-s)^{\alpha-9-1} \phi(s, 1,1) \mathrm{d} s \\
& \quad=\int_{0}^{1}(1-s)^{7 / 4}\left(s^{-2 / 3}+\cos s+6 s^{4 / 5}\right) \mathrm{d} s  \tag{55}\\
& \quad<\int_{0}^{1}(1-s)^{7 / 4}\left(s^{-2 / 3}+1+6 s^{4 / 5}\right) \mathrm{d} s \approx 3.0765 \\
& \quad<+\infty
\end{align*}
$$

which implies that $\left(\mathrm{H}_{3}\right)$ is also satisfied. Thus, by Theorem 11 we know that BVP (50) has a unique positive solution.

In addition, for any initial $u_{0}=I_{0^{+}}^{1 / 4} x_{0} \in Q_{e}$, we construct a successive sequence

$$
\begin{align*}
& x_{k+1}(t)=\int_{0}^{1} G(t, s)\left[\phi\left(s, I_{0^{+}}^{1 / 4} x_{k}(s), x_{k}(s)\right)\right.  \tag{56}\\
& \left.\quad+\psi\left(s, I_{0^{+}}^{1 / 4} x_{k}(s), x_{k}(s)\right)\right] \mathrm{d} s, \quad k=1,2, \cdots
\end{align*}
$$

and $u_{k+1}(t)=I_{0^{+}}^{1 / 4} x_{k+1}(t)$; then

$$
\begin{align*}
& u_{k+1}(t)=I_{0^{+}}^{1 / 4}\left\{\int _ { 0 } ^ { 1 } G ( t , s ) \left[\phi\left(s, u_{k}(s), D_{0^{+}}^{1 / 4} u_{k}(s)\right)\right.\right. \\
& \left.\left.+\psi\left(s, u_{k}(s), D_{0^{+}}^{1 / 4} u_{k}(s)\right)\right] \mathrm{d} s\right\},  \tag{57}\\
&
\end{align*}
$$

and we have $\left\|u_{k}-u^{*}\right\|=\left\|I_{0^{+}}^{1 / 4} x_{k}-I_{0^{+}}^{1 / 4} x^{*}\right\| \longrightarrow 0$ as $k \longrightarrow \infty$; the convergence rate is

$$
\begin{equation*}
\left\|u_{k}-u^{*}\right\|=\left\|I_{0^{+}}^{\delta} x_{k}-I_{0^{+}}^{1 / 4} x^{*}\right\|=o\left(1-r^{(1 / 7)^{k}}\right) \tag{58}
\end{equation*}
$$

where $r$ is a constant with $0<r<1$ and is dependent on $u_{0}$.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All the authors read and approved the final manuscript.

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# Research Article 

# Existence and Nonexistence of Positive Solutions for Mixed Fractional Boundary Value Problem with Parameter and $p$-Laplacian Operator 

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#### Abstract

This paper mainly studies a class of mixed fractional boundary value problem with parameter and $p$-Laplacian operator. Based on the Guo-Krasnosel'skii fixed point theorem, results on the existence and nonexistence of positive solutions for the fractional boundary value problem are established. An example is then presented to illustrate the effectiveness of the results.


## 1. Introduction

In this paper, we consider the existence and nonexistence of positive solution for the following mixed fractional boundary value problem (BVP):

$$
\begin{align*}
& D_{0^{+}}^{\alpha}\left(\varphi_{p}\left({ }^{c} D_{0^{+}}^{\beta} u(t)\right)\right)+\lambda f(t, u(t))=0, \quad 0<t<1, \\
& u(1)=u^{\prime}(0)=u^{\prime \prime}(0)=\cdots=u^{(n-1)}(0)=0, \\
& { }^{c} D_{0^{+}}^{\beta} u(0)=0,  \tag{1}\\
& { }^{c} D_{0^{+}}^{\beta} u(1)=a^{c} D_{0^{+}}^{\beta} u(\eta),
\end{align*}
$$

where $\lambda>0$ is a parameter, $1<\alpha \leq 2, n-1<\beta \leq n, n \geq 2$, $D_{0^{+}}^{\alpha}$ is the Riemann-Liouville derivative operator, ${ }^{c} D_{0^{+}}^{\beta}$ is the Caputo fractional derivative operator, $\varphi_{p}$ is the $p$-Laplacian operator defined by $\varphi_{p}(s)=|s|^{p-2} s,\left(\varphi_{p}\right)^{-1}=\varphi_{q}, 1 / p+1 / q=$ $1, p>1, \eta \in(0,1)$, and $a>0$ and satisfies $1-a^{p-1} \eta^{\alpha-1}>0$, and $f:[0,1] \times[0,+\infty) \longrightarrow[0,+\infty)$ is a continuous function.

The theory of fractional differential equation has gained interesting by many researchers due to its deep real world background and, in recent years, more and more papers concern the boundary value problems for fractional-order differential equations; see [1-27]. In [28], Wang, Xiang and

Liu use Krasnoselskii's fixed point theorem and LeggettWilliams theorem to obtain the existence results of BVP, which is given in the following:

$$
\begin{align*}
D_{0^{+}}^{\gamma}\left(\varphi_{p}\left(D_{0^{+}}^{\alpha} u(t)\right)\right) & =f(t, u(t)), \quad 0<t<1, \\
D_{0^{+}}^{\alpha} u(0) & =u(0)=0, \\
u^{\prime}(1) & =a u(\xi), \tag{2}
\end{align*}
$$

$$
D_{0^{+}}^{\alpha} u(1)=b D_{0^{+}}^{\alpha} u(\eta)
$$

where $1<\alpha, \gamma \leq 2,0 \leq a, b \leq 1,0<\xi, \eta<1, D_{0^{+}}^{\alpha}$ and $D_{0^{+}}^{\gamma}$ are Riemann-Liouville derivative operators, $\varphi_{p}$ is the $p-$ Laplacian operator defined by $\varphi_{p}(s)=|s|^{p-2} s$, and $f:[0,1] \times$ $[0,+\infty) \longrightarrow[0,+\infty)$ is a continuous function.

Lu et al. in [29] investigated the BVP

$$
\begin{align*}
D_{0^{+}}^{\alpha}\left(\varphi_{p}\left(D_{0^{+}}^{\beta} u(t)\right)\right) & =f(t, u(t)), \quad 0<t<1, \\
u(0) & =u^{\prime}(0)=u^{\prime}(1)=0,  \tag{3}\\
D_{0^{+}}^{\beta} u(0) & =D_{0^{+}}^{\beta} u(1)=0,
\end{align*}
$$

where $1<\alpha \leq 2,2<\beta \leq 3, D_{0^{+}}^{\alpha}$ and $D_{0^{+}}^{\beta}$ are Riemann-Liouville derivative operators, $\varphi_{p}$ is the $p$-Laplacian operator defined by $\varphi_{p}(s)=|s|^{p-2} s$, and $f:[0,1] \times$
$[0,+\infty) \longrightarrow[0,+\infty)$ is a continuous function. By the use of Guo-Krasnosel'skii fixed point theorem, the existence and multiplicity results of BVP (3) are obtained.

In this paper, we study the existence and nonexistence of positive solutions for the mixed fractional boundary value problem as BVP (1), which leads to lots of difference and new features. On one hand, compared to the papers mentioned above, which only involve one derivative, our study involves both the Riemann-Liouville fractional derivative and the Caputo fractional derivative, which making the studied problems difficult. On the other hand, under different combinations of superlinearity and sublinearity of the function $f$, results on the existence and nonexistence of positive solutions are received and the impact of the parameter on the existence and nonexistence of positive solutions is also obtained.

## 2. Preliminaries and Lemmas

Definition 1 (see [30, 31]). The Caputo fractional-order derivative of orders $\alpha>0$ and $n-1<\alpha<n, n \in \mathbb{N}$, is defined as

$$
\begin{equation*}
{ }^{c} D_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-s)^{n-\alpha-1} u^{(n)}(s) d s \tag{4}
\end{equation*}
$$

where $u \in C^{n}(J, \mathbb{R}), \mathbb{R}=(-\infty,+\infty), \mathbb{N}$ denotes the natural number set, $n=[\alpha]+1$, and $[\alpha]$ denotes the integer part of $\alpha$.

Definition 2 (see [30, 31]). Let $\alpha>0$ and let $u$ be piecewise continuous on $(0,+\infty)$ and integrable on any finite subinterval of $J$. Then for $t>0$, we call

$$
\begin{equation*}
I_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s \tag{5}
\end{equation*}
$$

the Riemann-Liouville fractional integral of $u$ of order $\alpha$.
Lemma 3 (see $[30,31]$ ). Let $n-1<\alpha \leq n, u \in C^{n}[0,1]$. Then

$$
\begin{equation*}
I_{0^{+}}^{\alpha}\left({ }^{c} D_{0^{+}}^{\alpha} u\right)(t)=u(t)+c_{0}+c_{1} t+c_{2} t^{2}+\cdots+c_{n-1} t^{n-1} \tag{6}
\end{equation*}
$$

where $c_{i} \in \mathbb{R}(i=1,2, \ldots, n-1)$ and $n$ is the smallest integer greater than or equal to $\alpha$.

Let $\varphi_{p}\left({ }^{c} D_{0^{+}}^{\beta} u(t)\right)=v(t)$; then $v(0)=0, v(1)=a^{p-1} v(\eta)$, and we now consider the following BVP:

$$
\begin{align*}
D_{0^{+}}^{\alpha} v(t)+y(t) & =0, \quad 0<t<1 \\
v(0) & =0  \tag{7}\\
v(1) & =a^{p-1} v(\eta)
\end{align*}
$$

Lemma 4 (see [32]). If $y \in C[0,1]$, then $B V P(7)$ has a unique solution

$$
\begin{equation*}
v(t)=\int_{0}^{1} H(t, s) y(s) d s \tag{8}
\end{equation*}
$$

where

$$
\begin{align*}
& H(t, s)=h(t, s)+\frac{a^{p-1} t^{\alpha-1}}{1-a^{p-1} \eta^{\alpha-1}},  \tag{9}\\
& h(t, s)= \begin{cases}\frac{(t(1-s))^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\
\frac{(t(1-s))^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1\end{cases} \tag{10}
\end{align*}
$$

For any given $y \in C[0,1]$, consider the following BVP:

$$
\begin{align*}
& D_{0^{+}}^{\alpha}\left(\varphi_{p}\left({ }^{c} D_{0^{+}}^{\beta} u(t)\right)\right)+y(t)=0, \quad 0<t<1, \\
& u(1)=u^{\prime}(0)=u^{\prime \prime}(0)=\cdots=u^{(n-1)}(0)=0, \\
& { }^{c} D_{0^{+}}^{\beta} u(0)=0,  \tag{11}\\
& { }^{c} D_{0^{+}}^{\beta} u(1)=a^{c} D_{0^{+}}^{\beta} u(\eta) .
\end{align*}
$$

By analysis, we know that (11) can be decomposed into the BVP (7) and the BVP

$$
\begin{equation*}
{ }^{c} D_{0^{+}}^{\beta} u(t)+\varphi_{q}\left(\int_{0}^{1} H(t, s) y(s) d s\right)=0, \quad 0<t<1, \tag{12}
\end{equation*}
$$

$$
u(1)=u^{\prime}(0)=u^{\prime \prime}(0)=\cdots=u^{(n-1)}(0)=0
$$

Lemma 5. If $y \in C[0,1]$, then $B V P$

$$
\begin{align*}
{ }^{c} D_{0^{+}}^{\beta} u(t)+y(t) & =0, \quad 0<t<1, \\
u(1) & =u^{\prime}(0)=u^{\prime \prime}(0)=\cdots=u^{(n-1)}(0)  \tag{13}\\
& =0
\end{align*}
$$

has a unique solution

$$
\begin{equation*}
u(t)=\int_{0}^{1} G(t, s) y(s) d s \tag{14}
\end{equation*}
$$

where

$$
G(t, s)= \begin{cases}\frac{(1-s)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq t \leq 1  \tag{15}\\ \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq t \leq s \leq 1\end{cases}
$$

Proof. By Lemma 3, BVP (13) is equivalent to the following integral equation:

$$
\begin{align*}
u(t)= & -\int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s) d s+c_{1}+c_{2} t+c_{3} t^{2} \cdots  \tag{16}\\
& +c_{n} t^{n-1}
\end{align*}
$$

Conditions $u^{\prime}(0)=u^{\prime \prime}(0)=\cdots=u^{(n-1)}(0)=0$ imply that $c_{2}=c_{3}=\cdots=c_{n}=0$. That is,

$$
\begin{equation*}
u(t)=-\int_{0}^{t} \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} y(s) d s+c_{1} \tag{17}
\end{equation*}
$$

By $u(1)=0$, we get

$$
\begin{equation*}
c_{1}=-\int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma\left(\alpha_{1}\right)} y(s) d s \tag{18}
\end{equation*}
$$

Combining (16) and (18), we can obtain (14). The proof is completed.

Lemma 6. The Green functions $H(t, s)$ and $G(t, s)$ defined separately by (9) and (15) have the following properties:
(1) $H(t, s)$ and $G(t, s):[0,1] \times[0,1] \longrightarrow[0,+\infty)$ are continuous.
(2) $(1-s)^{\beta-1}\left(1-t^{\beta-1}\right) / \Gamma(\beta) \leq G(t, s) \leq(1-s)^{\beta-1} / \Gamma(\beta)$.

Proof. Obviously, (1) holds, in the following, and we proof (3). From the definition of $G(t, s)$, for $0 \leq t \leq s \leq 1$, we know that (3) holds.

For $0 \leq s \leq t \leq 1$, we have $t-t s \geq t-s$ and then

$$
\begin{align*}
& (1-s)^{\beta-1}-(t-s)^{\beta-1} \geq(1-s)^{\beta-1}-(t-t s)^{\beta-1} \\
& \quad \geq(1-s)^{\beta-1}-t^{\beta-1}(1-s)^{\beta-1}  \tag{19}\\
& \quad=(1-s)^{\beta-1}\left(1-t^{\beta-1}\right)
\end{align*}
$$

so we know that $(1-s)^{\beta-1}\left(1-t^{\beta-1}\right) / \Gamma(\beta) \leq G(t, s)$. It is also defined by $G(t, s)$, and we obtain that $G(t, s) \leq(1-s)^{\beta-1} / \Gamma(\beta)$. Thus, we get that (3) holds. The proof is completed.

Let $X=C[0,1]$; then $X$ is a Banach space with the norm $\|u\|=\sup _{t \in[0,1]}|u(t)|$. For any $\gamma_{1}, \gamma_{2} \in(0,1), \gamma_{1}<\gamma_{2}$, denote

$$
\begin{equation*}
K=\left\{u \in X: \min _{t \in\left[\gamma_{1}, \gamma_{2}\right]} u(t) \geq \omega\|u\|\right\}, \tag{20}
\end{equation*}
$$

where $\omega=\min _{t \in\left[\gamma_{1}, \gamma_{2}\right]}\left(1-t^{\beta-1}\right)$ and then $K$ is a positive cone in $X$. Define an integral operator $T: K \longrightarrow X$ by

$$
\begin{align*}
& T u(t)=\varphi_{q}(\lambda) \\
& \qquad \begin{array}{l}
\cdot \int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s \\
\\
t \in[0,1] .
\end{array} \tag{21}
\end{align*}
$$

We know that $u$ is a positive solutions of BVP (1) if and only if $u$ is a fixed point of $T$ in $K$.

Lemma 7. $T: K \longrightarrow K$ is a completely continuous operator.
Proof. By the routine discussion, $T: K \longrightarrow X$ is well defined and we only prove $T(K) \subseteq K$. For any $u \in K, t \in[0,1]$, by Lemma 6, we have

$$
\begin{aligned}
& T u(t)=\varphi_{q}(\lambda) \\
& \quad \cdot \int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s \\
& \quad \leq \frac{\varphi_{q}(\lambda)}{\Gamma(\beta)} \\
& \quad \cdot \int_{0}^{1}(1-s)^{\beta-1} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s
\end{aligned}
$$

and then

$$
\begin{align*}
& \|T u\| \leq \frac{\varphi_{q}(\lambda)}{\Gamma(\beta)}  \tag{23}\\
& \quad \cdot \int_{0}^{1}(1-s)^{\beta-1} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s
\end{align*}
$$

On the other hand, for $t \in\left[\gamma_{1}, \gamma_{2}\right]$, by Lemma 6 , we get

$$
\begin{align*}
& T u(t) \geq \frac{\varphi_{q}(\lambda)}{\Gamma(\beta)} \int_{0}^{1}(1-s)^{\beta-1}\left(1-t^{\beta-1}\right) \\
& \cdot \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s \\
& =\frac{\left(1-t^{\beta-1}\right) \varphi_{q}(\lambda)}{\Gamma(\beta)} \int_{0}^{1}(1-s)^{\beta-1}  \tag{24}\\
& \cdot \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s \geq \frac{\omega \varphi_{q}(\lambda)}{\Gamma(\beta)} \\
& \cdot \int_{0}^{1}(1-s)^{\beta-1} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s
\end{align*}
$$

By (23) and (24), we can prove that $T u(t) \geq \omega\|T u\|$. Therefore, we have $T(K) \subseteq K$.

According to the Ascoli-Arzela theorem and the continuity of $f$, we get that $T: K \longrightarrow K$ is completely continuous. The proof is completed.

Lemma 8 (see [33]). Let $P$ be a positive cone in a Banach space; $E, \Omega_{1}$, and $\Omega_{\underline{2}}$ are bounded open sets in $E, \theta \in \Omega_{1}$, and $\bar{\Omega}_{1} \subset \Omega_{2}, A: P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \longrightarrow P$ is a completely continuous operator. If the conditions
$\|A x\| \leq\|x\|, \forall x \in P \cap \partial \Omega_{1},\|A x\| \geq\|x\|$, and $\forall x \in$ $P \cap \partial \Omega_{2}$,
or
$\|A x\| \geq\|x\|, \forall x \in P \cap \partial \Omega_{1},\|A x\| \leq\|x\|$, and $\forall x \in$ $P \cap \partial \Omega_{2}$,
are satisfied, then $A$ has at least one fixed point in $P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

## 3. Main Results

Denote

$$
\begin{aligned}
& f_{0}=\liminf _{x \rightarrow 0^{+}} \inf _{t \in\left[\gamma_{1}, \gamma_{2}\right]} \frac{f(t, x)}{\varphi_{p}(x)}, \\
& f^{0}=\limsup _{x \rightarrow 0^{+}} \sup _{t \in[0,1]} \frac{f(t, x)}{\varphi_{p}(x)}, \\
& f_{\infty}=\liminf _{x \rightarrow+\infty} \inf _{t \in\left[\gamma_{1}, \gamma_{2}\right]} \frac{f(t, x)}{\varphi_{p}(x)}, \\
& f^{\infty}=\limsup _{x \rightarrow+\infty} \sup _{t \in[0,1]} \frac{f(t, x)}{\varphi_{p}(x)},
\end{aligned}
$$

$$
\begin{align*}
& L_{1}=\int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) d \tau\right) d s \\
& L_{2}=\omega^{2} \int_{\gamma_{1}}^{\gamma_{2}} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{\gamma_{1}}^{\gamma_{2}} H(s, \tau) d \tau\right) d s \tag{25}
\end{align*}
$$

### 3.1. Existence of $B V P$ (1)

Theorem 9. Assume $f_{\infty} \varphi_{p}\left(L_{1}^{-1}\right)>f^{0} \varphi_{p}\left(L_{2}^{-1}\right)$; then $B V P$ (1) has at least one positive solution for

$$
\begin{equation*}
\lambda \in\left(\frac{\varphi_{p}\left(L_{2}^{-1}\right)}{f_{\infty}}, \frac{\varphi_{p}\left(L_{1}^{-1}\right)}{f^{0}}\right) \tag{26}
\end{equation*}
$$

where we impose $1 / f_{\infty}=0$, if $f_{\infty}=+\infty$, and $1 / f^{0}=+\infty$, if $f^{0}=0$.

Proof. For any $\lambda$ satisfying (26), there exists $\varepsilon_{0}>0$ such that

$$
\begin{equation*}
\frac{\varphi_{p}\left(L_{2}^{-1}\right)}{f_{\infty}-\varepsilon_{0}} \leq \lambda \leq \frac{\varphi_{p}\left(L_{1}^{-1}\right)}{f^{0}+\varepsilon_{0}} . \tag{27}
\end{equation*}
$$

By the definition of $f^{0}$, there exists $R_{1}>0$ such that

$$
\begin{equation*}
f(t, x) \leq\left(f^{0}+\varepsilon_{0}\right) \varphi_{p}(x), \quad 0 \leq x \leq R_{1}, t \in[0,1] . \tag{28}
\end{equation*}
$$

Let $K_{R_{1}}=\left\{u \in K:\|u\|<R_{1}\right\}$. For any $u \in \partial K_{R_{1}}, t \in[0,1]$, by the definition of $\|\cdot\|$, we know that

$$
\begin{equation*}
u(t) \leq|u(t)| \leq\|u\| \leq R_{1}, \quad t \in[0,1] . \tag{29}
\end{equation*}
$$

Thus, for any $u \in \partial K_{R_{1}}$, by (28) and (29), we have

$$
\begin{equation*}
f(t, u(t)) \leq\left(f^{0}+\varepsilon_{0}\right) \varphi_{p}(u(t)), \quad t \in[0,1] . \tag{30}
\end{equation*}
$$

Hence, for any $u \in \partial K_{R_{1}}$, by Lemmas 6 and (30), we conclude that

$$
\begin{align*}
& T u(t)=\varphi_{q}(\lambda) \int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau)\right. \\
& \cdot f(\tau, u(\tau)) d \tau) d s \leq \varphi_{q}(\lambda) \\
& \cdot \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau)\left(f^{0}+\varepsilon_{0}\right)\right.  \tag{31}\\
& \left.\cdot \varphi_{p}(u(\tau)) d \tau\right) d s \leq \varphi_{q}\left(\lambda\left(f^{0}+\varepsilon_{0}\right)\right)\|u\| \\
& \cdot \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) d \tau\right) d s \leq\|u\| .
\end{align*}
$$

Then, we have

$$
\begin{equation*}
\|T u\| \leq\|u\|, \quad u \in \partial K_{R_{1}} \tag{32}
\end{equation*}
$$

On the other hand, by the definition of $f_{\infty}$, there exists $R^{\prime}>0$ such that

$$
\begin{equation*}
f(t, x) \geq\left(f_{\infty}-\varepsilon_{0}\right) \varphi_{p}(x), \quad x \geq R^{\prime}, t \in\left[\gamma_{1}, \gamma_{2}\right] \tag{33}
\end{equation*}
$$

Choose $R_{2}=\max \left\{R^{\prime} / \omega, 2 R_{1}\right\}$. Let $K_{R_{2}}=\left\{u \in K:\|u\|<R_{2}\right\}$. For any $u \in \partial K_{R_{2}}$, by the definition of $\|\cdot\|$, we have

$$
\begin{equation*}
u(t) \geq \omega\|u\| \geq \omega R_{2} \geq R^{\prime}, \quad t \in\left[\gamma_{1}, \gamma_{2}\right] . \tag{34}
\end{equation*}
$$

For any $u \in \partial K_{R_{2}}$, by (33) and (34), we have

$$
\begin{align*}
f(t, u(t)) & \geq\left(f_{\infty}-\varepsilon_{0}\right) \varphi_{p}(u(t)) \\
& \geq\left(f_{\infty}-\varepsilon_{0}\right) \varphi_{p}\left(\omega R_{2}\right), \quad t \in\left[\gamma_{1}, \gamma_{2}\right] . \tag{35}
\end{align*}
$$

Then, for any $u \in \partial K_{R_{2}}$, by Lemma 6 and (35), we have

$$
\begin{aligned}
& T u(t) \geq \varphi_{q}(\lambda) \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)}\left(1-t^{\beta-1}\right) \\
& \cdot \varphi_{q}\left(\int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) d \tau\right) d s \geq \varphi_{q}(\lambda) \\
& \cdot \min _{t \in\left[\gamma_{1}, \gamma_{2}\right]}\left(1-t^{\beta-1}\right) \int_{\gamma_{1}}^{\gamma_{2}} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{\gamma_{1}}^{\gamma_{2}} H(s, \tau)\right. \\
& \cdot f(\tau, u(\tau)) d \tau) d s \geq \varphi_{q}(\lambda) \\
& \cdot \omega \int_{\gamma_{1}}^{\gamma_{2}} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{\gamma_{1}}^{\gamma_{2}} H(s, \tau)\left(f_{\infty}-\varepsilon_{0}\right)\right. \\
& \left.\cdot \varphi_{p}(u(\tau)) d \tau\right) d s \geq \varphi_{q}\left(\lambda\left(f_{\infty}-\varepsilon_{0}\right)\right)\|u\| \\
& \cdot \omega^{2} \int_{\gamma_{1}}^{\gamma_{2}} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{\gamma_{1}}^{\gamma_{2}} H(s, \tau) d \tau\right) d s \\
& \geq\|u\| .
\end{aligned}
$$

It follows from the above discussion, (32), (36), and Lemmas 7 and 8 , for any $\lambda \in\left(\varphi_{p}\left(L_{2}^{-1}\right) / f_{\infty}, \varphi_{p}\left(L_{1}^{-1}\right) / f^{0}\right)$, that $T$ has a fixed point $u \in \bar{K}_{R_{2}} \backslash K_{R_{1}}$, so BVP (1) has at least one positive solution $u$; moreover $u$ satisfies $R_{1} \leq\|u\| \leq R_{2}$. The proof is completed.

By the similar proof as Theorem 9, the following Theorem 10 holds.

Theorem 10. Assume that $f_{0} \varphi_{p}\left(L_{1}^{-1}\right)>f^{\infty} \varphi_{p}\left(L_{2}^{-1}\right)$; then $B V P$ (1) has at least one positive solution for $\lambda \in\left(\varphi_{p}\left(L_{2}^{-1}\right)\right.$ / $\left.f_{0}, \varphi_{p}\left(L_{1}^{-1}\right) / f^{\infty}\right)$, where we impose $1 / f_{0}=0$, if $f_{0}=+\infty$, and $1 / f^{\infty}=+\infty$, if $f^{\infty}=0$.

### 3.2. Nonexistence of BVP (1)

Theorem 11. Assume that $f^{\infty}<+\infty$ and $f^{0}<+\infty$; then there exists $\lambda_{0}>0$, such that for, $\lambda \in\left(0, \lambda_{0}\right)$, BVP (1) has no positive solution.

Proof. By $f^{\infty}<+\infty$ and $f^{0}<+\infty$, there exist positive constants $M_{1}$ and $M_{2}$ and $r_{1}$ and $r_{2}\left(r_{1}<r_{2}\right)$, such that

$$
\begin{array}{ll}
f(t, x) \leq M_{1} \varphi_{p}(x), & 0 \leq x \leq r_{1}, t \in[0,1],  \tag{37}\\
f(t, x) \leq M_{2} \varphi_{p}(x), & x \geq r_{2}, t \in[0,1] .
\end{array}
$$

Setting

$$
\begin{equation*}
M_{0}=\max \left\{M_{1}, M_{2}, \max _{\substack{t \in[0,1] \\ r_{1} \leq x \leq r_{2}}} \frac{f(t, x)}{\varphi_{p}(x)}\right\}, \tag{38}
\end{equation*}
$$

we get

$$
\begin{equation*}
f(t, x) \leq M_{0} \varphi_{p}(x), \quad x \geq 0, t \in[0,1] . \tag{39}
\end{equation*}
$$

Assume that $u$ is a positive solution of BVP (1); we will show that it leads to a contradiction. Define $\lambda_{0}=\left(M_{0}\right)^{-1} \varphi_{p}\left(L_{1}^{-1}\right)$, and since $\lambda \in\left(0, \lambda_{0}\right)$, we conclude that

$$
\begin{align*}
& u(t)=T u(t)=\varphi_{q}(\lambda) \int_{0}^{1} G(t, s) \varphi_{q}\left(\int_{0}^{1} H(s, \tau)\right. \\
& \cdot f(\tau, u(\tau)) d \tau) d s \leq \varphi_{q}(\lambda) \\
& \cdot \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau)\right. \\
& \left.\cdot M_{0} \varphi_{p}(u(\tau)) d \tau\right) d s \leq \varphi_{q}\left(\lambda M_{0}\right)\|u\|  \tag{40}\\
& \cdot \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) d \tau\right) d s \\
& \quad<\varphi_{q} \\
& \cdot \int_{0}^{1} \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} \varphi_{q}\left(\int_{0}^{1} H(s, \tau) d \tau\right) d s<\|u\| .
\end{align*}
$$

Then, we have $\|u\|<\|u\|$, which is a contradiction. Therefore, BVP (1) has no positive solution for $\lambda \in\left(\varphi_{p}\left(L_{2}^{-1}\right) /\right.$ $\left.f_{0}, \varphi_{p}\left(L_{1}^{-1}\right) / f^{\infty}\right)$. The proof is completed.

By the similar proof as Theorem 11, the following Theorem 12 holds.

Theorem 12. Assume that $f_{\infty}>0, f_{0}>0, f(t, x)>0$ for $t \in\left[\gamma_{1}, \gamma_{2}\right]$, and $x>0$; then there exists $\lambda_{*}>0$, such that, for $\lambda \in\left(\lambda_{*},+\infty\right)$, BVP (1) has no positive solution.

## 4. Examples

Consider the BVP

$$
\begin{aligned}
& D_{0^{+}}^{3 / 2}\left(\varphi_{3 / 2}\left({ }^{c} D_{0^{+}}^{3 / 2} u(t)\right)\right)+\lambda f(t, u(t))=0, \\
& \qquad 0<t<1, \\
& u(1)=u^{\prime}(0)=0,
\end{aligned}
$$

$$
\begin{align*}
& { }^{c} D_{0^{+}}^{3 / 2} u(0)=0, \\
& { }^{c} D_{0^{+}}^{3 / 2} u(1)=a^{c} D_{0^{+}}^{3 / 2} u\left(\frac{1}{2}\right), \tag{41}
\end{align*}
$$

where $\lambda>0$ is a parameter, $\alpha=\beta=3 / 2$, and $\eta=1 / 2$. Let $f(t, x)=x^{3} /(1+t)$; choose $[1 / 4,1 / 3] \subset(0,1)$, andthen $f_{\infty}=+\infty, f^{0}=0$. By Theorem 9, BVP (41) has at least one positive solution for $\lambda \in(0,+\infty)$.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that they have no conflicts of interest.

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# Research Article 

# Existence Results for Generalized Bagley-Torvik Type Fractional Differential Inclusions with Nonlocal Initial Conditions 

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#### Abstract

In this article, we prove the existence of solutions for the generalized Bagley-Torvik type fractional order differential inclusions with nonlocal conditions. It allows applying the noncompactness measure of Hausdorff, fractional calculus theory, and the nonlinear alternative for Kakutani maps fixed point theorem to obtain the existence results under the assumptions that the nonlocal item is compact continuous and Lipschitz continuous and multifunction is compact and Lipschitz, respectively. Our results extend the existence theorems for the classical Bagley-Torvik inclusion and some related models.


## 1. Introduction

In this article, we will consider the following generalized Bagley-Torvik type fractional differential inclusions:

$$
\begin{align*}
{ }^{c} D^{v_{1}} z(t)-\chi^{c} D^{v_{2}} z(t) & \in G(t, z(t)), \quad t \in(0,1]=B, \\
z(0) & =h(z) \tag{1}
\end{align*}
$$

where ${ }^{c} D^{\nu_{1}}$ and ${ }^{c} D^{\nu_{2}}$ are Caputo fractional derivatives with $0<\nu_{1} \leq 1$ and $0<\nu_{2}<\nu_{1}, \chi \in \mathbb{R}$ is a constant, and $G$ is a multifunction.

By introducing nonlocal conditions into the initial-value problems, Byszewski and Lakshmikantham [1] provided a more accurate model for the nonlocal initial valued problem since more information was incorporated in the experiment. As a result, the negative impact of single initial value can be significantly reduced. Concerning the initial-value problems, the most recent developments can be referred to in $[2,3]$. On the other hand, the fractional calculus and fractional differential equations have many real applications in biology, physics, and natural sciences and a number of results on this topic have emerged in the last decade [4-6]. In [7], EI-Sayed and Ibrahim initiated the research on fractional multivalued differential inclusions. After that, many authors were devoted to study the existence of solutions for fractional
differential inclusions [8-11]. Very recently, Wang et al. [12, 13] studied the controllability and topological structure of the solution set for fractional impulsive differential inclusions.

For the problem of fractional differential inclusions, multiterm fractional differential equations are a hot research direction owning to their wide use in practice and technique sciences, for example, physics, mechanics, and chemistry. An important result on multiterm fractional calculus is formulated by Bagley and Torvik in [14]. Here the authors deduced and tested a relation $A x^{\prime \prime}(t)+B^{c} D^{3 / 2} x(t)+C x(t)=$ $g(t)$, where $x(t)$ is a function describing the motion of thin plates in Newtonian fluids, $A=m$, the mass of thin rigid plate, $B=2 s \sqrt{v \rho}$, where $s$ is area of the plate immersed in Newtonian fluid, $v$ is viscosity, $\rho$ is the fluid density, $C=$ $k$, the stiffness of the spring, and $g(t)$ is an external force. Later, the above equation was called Bagley-Torvik equation [15]. Based on this model, the nonlinear multiterm fractional differential equations were rediscovered and popularized by Kaufmann and Yao in [16]. As far as the author knows, there are few papers on the existence of the generalized BagleyTorvik type fractional differential inclusion (1) except for Ibrahim, Dong, and Fan [17]. They studied the following equation:

$$
\begin{align*}
{ }^{c} D^{\theta} v(s)-a^{c} D^{\delta} v(s)+g(s, v(s)) & =0 ; \quad 0<s<1 \\
v(0) & =v_{0} ;  \tag{2}\\
v(1) & =v_{1}
\end{align*}
$$

where ${ }^{c} D^{\theta}$ and ${ }^{c} D^{\delta}$ are the Caputo fractional derivatives, $1<\theta \leq 2,1 \leq \delta<$ $\theta$.

In this article, we shall be concerned with the existence of the generalized Bagley-Torvik type fractional differential inclusions (1) by employing the noncompactness measure of Hausdorff, fractional calculus, and the nonlinear alternative for Kakutani maps fixed point theorem, when $h$ is compact continuous and Lipschitz continuous and $G$ is compact and Lipschitz, respectively. Our theory improves the results in [ $6,8-10,16,17]$ and extends (generalizes) the corresponding results of Bagley-Torvik equation.

The structure of this article is as follows: some preliminary knowledge is introduced in Section 2; some existence criteria are derived from (1) in Section 3; in the end, we use an example to illustrate an application of the main result.

## 2. Preliminaries

Let $Y$ be a metric space and $W$ a normed space; $K(W)=$ $\{\Delta \subseteq W$ : nonempty $\} ; K_{b}(W)=\{\Delta \subseteq K(W)$ : bounded $\} ;$ $K_{b f}(W)=\{\Delta \subseteq K(W)$ : closed and bounded $\} ; K_{p c}(W)=$ $\{\Delta \subseteq K(W)$ : compact and convex $\} ; K_{f c}(W)=\{\Delta \subseteq K(W)$ : closed and convex\}.

Let $E$ be a Banach space. $L^{p}(\bar{B}, E):=\{u: \bar{B} \longrightarrow$ $E$ is measurable and $\left.\int_{0}^{1}\|u(t)\|^{p} d t<+\infty\right\}$. For $u \in L^{p}(\bar{B}, E)$, $1<p<\infty$, the norm is defined as $\|u\|_{p}=\left(\int_{0}^{1}\|u(t)\|^{p} d t\right)^{1 / p}$, and $p^{\prime}$ is the conjugate exponent; that is, $1 / p+1 / p^{\prime}=1$. Denote the Hausdorff measure of noncompactness (MNC) $\gamma: K_{b}(E) \longrightarrow \mathbb{R}^{+}$as

$$
\begin{equation*}
\gamma(D)=\inf \{\rho \tag{3}
\end{equation*}
$$

$>0$ : there exist finite point $x_{1}, x_{2}, \ldots, x_{n}$

$$
\left.\in E \text { with } D \subset \bigcup_{i=1}^{n} B\left(x_{i}, \rho\right)\right\}
$$

Proposition 1 (see [18]). The MNC $\gamma$ enjoy the following properties:
(1) monotone, if $\Delta_{1}, \Delta_{2} \in K_{b}(E), \Delta_{1} \subseteq \Delta_{2} \Longrightarrow \gamma\left(\Delta_{1}\right) \leq$ $\gamma\left(\Delta_{2}\right)$
(2) nonsingular, if $\gamma(\{b\} \cup \Delta)=\gamma(\Delta)$, for all $b \in E$ and $\Delta \subseteq K_{b}(E)$
(3) regular, if $\gamma(\Delta)=0 \Longleftrightarrow \Delta$ is relative compact
(4) algebraically semiadditive, if $\gamma\left(\Delta_{1}+\Delta_{2}\right) \leq \gamma\left(\Delta_{1}\right)+$ $\gamma\left(\Delta_{2}\right)$, for $\Delta_{1}, \Delta_{2} \in K_{b}(E)$
(5) $\gamma(\omega \Delta)=|\omega| \gamma(\Delta)$

We now denote the sequential MNC $\gamma_{0}$ as follows:

$$
\begin{equation*}
\gamma_{0}(\Delta)=\sup \left\{\gamma\left(\left\{x_{m}: m \geq 1\right\}\right):\left\{x_{m}\right\}_{m=1}^{\infty} \subseteq \Delta\right\} \tag{4}
\end{equation*}
$$

One can easily have that

$$
\begin{equation*}
\gamma_{0}(\Delta) \leq \gamma(\Delta) \leq 2 \gamma_{0}(\Delta) \tag{5}
\end{equation*}
$$

Moreover, $\gamma_{0}(\Delta)=\gamma(\Delta)$, if $E$ is separable.
Proposition 2 (see [18]). For every bounded set $\Delta \subseteq C(\bar{B}, E)$, $s \in \bar{B}$, the inequality

$$
\begin{equation*}
\gamma(\Delta(s)) \leq \gamma_{c}(\Delta) \tag{6}
\end{equation*}
$$

holds, where $\Delta(s)=\{\delta(s): \delta \in \Delta\} \subset E$ and $\gamma_{c}$ is the $M N C$ defined in $C(\bar{B}, E)$. Moreover, if $\Delta$ is equicontinuous, then $\gamma(\Delta(s))$ is continuous. Moreover,

$$
\begin{equation*}
\gamma_{c}(\Delta)=\sup \{\gamma(\Delta(s)): s \in \bar{B}\} \tag{7}
\end{equation*}
$$

Lemma 3 (see [19]). Suppose $\left\{g_{m}\right\}_{m=1}^{\infty} \subset L^{1}(\bar{B}, E)$, $a \in$ $L^{1}\left(\bar{B}, \mathbb{R}^{+}\right)$; the inequality $\left|g_{m}(t)\right| \leq a(t)$ holds; for every $t \in \bar{B}$, $m \geq 1$. Then $\gamma\left(\left\{g_{m}(t)\right\}_{m=1}^{\infty}\right) \in L^{1}\left(\bar{B}, \mathbb{R}^{+}\right)$, and

$$
\begin{equation*}
\gamma\left(\left\{\int_{0}^{t} g_{m}(s) d s: m \geq 1\right\}\right) \leq 2 \int_{0}^{t} \gamma\left(\left\{g_{m}(s)\right\}_{m=1}^{\infty}\right) d s \tag{8}
\end{equation*}
$$

In what follows, we will recall some necessary definitions and lemmas of the fractional order differential and integra theory, which can be found in the literature [15].

Definition 4. Suppose $g \in L^{1}(\bar{B}, E), \iota>0$. If $\int_{0}^{s}((s-$ $\left.\eta)^{t-1} / \Gamma(t)\right) g(\eta) d \eta<\infty$, then

$$
\begin{equation*}
I_{0^{+}}^{\iota} g(s)=\int_{0}^{s} \frac{(s-\eta)^{\iota-1}}{\Gamma(\iota)} g(\eta) d \eta \tag{9}
\end{equation*}
$$

is called $\iota$ order Riemann-Liouville fractional integral of $g$.
Definition 5. Suppose $g \in L^{1}(\bar{B}, E), \iota>0$. we define

$$
\begin{equation*}
{ }^{c} D_{0^{+}}^{\iota} g(s)=\frac{1}{\Gamma(n-\iota)} \int_{0}^{s} \frac{g^{(n)}(\eta)}{(s-\eta)^{\iota-n+1}} d \eta \tag{10}
\end{equation*}
$$

as the $\iota$ order Caputo fractional derivative of $g$, where $n=$ $[\iota]+1$.

Lemma 6. Suppose $\iota>0$ and $g \in L^{1}(\bar{B}, E)$. Consider the following differential equation:

$$
\begin{equation*}
{ }^{c} D_{0^{+}}^{\iota} g(t)=0 . \tag{11}
\end{equation*}
$$

Then there exists some constants $d_{k} \in E, k=0,1,2, \ldots n-1$ such that

$$
\begin{equation*}
g(t)=d_{0}+d_{1} t+d_{2} t^{2}+\cdots+d_{n-1} t^{n-1} \tag{12}
\end{equation*}
$$

where $n=[\iota]+1$.

Lemma 7. Suppose $g \in L^{1}(\bar{B}, E)$, and $g^{(n)} \in L^{1}(\bar{B}, E)$; then there exist $d_{k} \in E, k=0,1,2, \ldots n-1$ satisfying

$$
\begin{equation*}
I_{0^{+}}^{c} D_{0^{+}}^{t} g(t)=g(t)+d_{0}+d_{1} t+d_{2} t^{2}+\cdots+d_{n-1} t^{n-1} \tag{13}
\end{equation*}
$$

For simplicity, we denote ${ }^{c} D_{0^{+}}^{\iota}$ and $I_{0^{+}}^{l}$ by ${ }^{c} D^{\iota}$ and $I^{l}$, respectively. And let $\left((p-1) /\left(v_{1} p-1\right)\right)^{(p-1) / p}=C, v_{1}-v_{2}=C_{1}$.

Lemma 8. Let $0<\nu_{1} \leq 1,0<\nu_{2}<\nu_{1}$. Assume that $g \in$ $L^{1}(\bar{B}, E)$. Then the solution of the equation,

$$
\begin{align*}
{ }^{c} D^{v_{1}} z(s)-\chi^{c} D^{v_{2}} z(s) & =g(s), \quad \text { a.e. } s \in B, \\
z(0) & =z_{0}, \tag{14}
\end{align*}
$$

has the following form:

$$
\begin{align*}
z(s)= & \left(1-\frac{\chi s^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) z_{0} \\
& +\chi \int_{0}^{s} \frac{(s-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} z(\eta) d \eta  \tag{15}\\
& +\int_{0}^{s} \frac{(s-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta .
\end{align*}
$$

Proof. In light of $0<\nu_{1} \leq 1$, from Lemma 7, there exists $d_{0}$ satisfying

$$
\begin{equation*}
I^{\nu_{1} c} D^{\nu_{1}} z(s)=z(s)+d_{0}, \quad s \in \bar{B} . \tag{16}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
I^{\nu_{1} c} D^{\nu_{1}} z(s)=\chi I^{\nu_{1} c} D^{\nu_{2}} z(s)+I^{\nu_{1}} g(s) . \tag{17}
\end{equation*}
$$

By Lemma 6, it follows that

$$
\begin{align*}
I^{\nu_{1} c} D^{\nu_{2}} z(s) & =I^{C_{1}}\left(I^{\nu_{2} c} D^{\nu_{2}} z(s)\right)=I^{C_{1}}\left(z(s)+d_{0}\right) \\
& =I^{C_{1}} z(s)+\frac{d_{0} s^{C_{1}}}{\Gamma\left(C_{1}+1\right)} . \tag{18}
\end{align*}
$$

So we have

$$
\begin{equation*}
z(s)+d_{0}=\chi I^{C_{1}} z(s)+\chi \frac{d_{0} s^{C_{1}}}{\Gamma\left(C_{1}+1\right)}+I^{\nu_{1}} g(s) \tag{19}
\end{equation*}
$$

Since $z(0)=z_{0}$, we obtain $d_{0}=-z_{0}$. Substituting the value of $d_{0}$ into (19), the proof is complete.

Given $z_{0} \in E, g \in L^{1}(\bar{B}, E)$. Define the operator $S$ by the formula

$$
\begin{align*}
\operatorname{Sg}(s)= & \left(1-\frac{\chi s^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) z_{0} \\
& +\chi \int_{0}^{s} \frac{(s-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} z(\eta) d \eta  \tag{20}\\
& +\int_{0}^{s} \frac{(s-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta, \quad \forall s \in \bar{B} ;
\end{align*}
$$

that is, $S g$ is the solution of the above system (14).

For any $w$ belonging to $C(\bar{B}, E)$, set

$$
\begin{align*}
& \text { Sel }(w) \\
& \qquad=\left\{g \in L^{1}(\bar{B}, E): g(s) \in G(s, w(s)) \text { a.e. } s \in \bar{B}\right\} . \tag{21}
\end{align*}
$$

For given $g^{w} \in \operatorname{Sel}(w)$, we define $S_{h(w)} g^{w}$ as a set of solutions to the generalized Bagley-Torvik type fractional differential system

$$
\begin{gather*}
{ }^{c} D^{v_{1}} z(s)-\chi^{c} D^{v_{2}} z(s)=g^{w}(s), \quad \text { a.e } s \in B,  \tag{22}\\
z(0)=h(w) .
\end{gather*}
$$

Before ending this section, we define the solution of the generalized Bagley-Torvik type fractional differential inclusions (1).

Definition 9. If $z(\cdot) \in C(\bar{B}, E)$ and

$$
\begin{align*}
z(t)= & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) g(z) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} z(\eta) d \eta  \tag{23}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta, \quad t \in \bar{B},
\end{align*}
$$

where $g \in \operatorname{Sel}(z)$, then $z$ is a solution of the generalized Bagley-Torvik type fractional differential inclusions (1).

## 3. Main Results

Here, we shall derive some existence criteria for the generalized Bagley-Torvik type differential system (1), when $h$ is compact continuous and Lipschitz continuous and $G$ is compact and Lipschitz, respectively. Our basic tools are noncompactness measure of Hausdorff, fractional calculus, and the nonlinear alternative for Kakutani maps fixed point theorem. Before proceeding, we assume that $p>1 / \nu_{1}$ and $h$ and $G$ satisfy the following conditions:
(h1) $h: C(\bar{B}, E) \longrightarrow E$ is compact and continuous mapping, and there exist positive constants $c_{1}, c_{2}$ satisfying $\|h(z)\| \leq c_{1}\|z\|+c_{2}\left(H G_{1}\right) \forall z \in E, G(\cdot, E): \bar{B} \longrightarrow K_{f_{c}}(E)$ admits a measurable selection
$\left(H G_{2}\right) G(t, \cdot): E \longrightarrow K_{f c}(E)$ is upper semicontinuous, for almost every $t \in \bar{B}\left(H G_{3}\right)$ For every $z \in E$ and $t \in \bar{B}$, $\|G(t, z)\|:=\sup \{\|\zeta\|: \zeta \in G(t, z)\} \leq l(t) \Sigma(\|z\|)$, where $l \in L^{p}\left(\bar{B}, \mathbb{R}^{+}\right)$, and $\Sigma: \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$is an increasing and continuous function
$\left(H G_{4}\right)$ For almost every $t \in \bar{B}$, there exists $b \in L^{p}\left(\bar{B}, \mathbb{R}^{+}\right)$; the inequality $\gamma(G(t, \Delta)) \leq b(t) \gamma(\Delta)$ holds, where $\Delta \subset E$ is a bounded set.

In the sequel, we introduce some important lemmas which are crucial to derive existence results.

Lemma 10 (see [20]). Under assumptions $\left(H G_{1}\right)-\left(H G_{4}\right)$, if there exists sequence $\left\{w_{m}\right\}_{m=1}^{\infty} \subset C(\bar{B}, E),\left\{g_{m}\right\}_{m=1}^{\infty} \subset L^{1}(\bar{B}, E)$,
and $g_{m} \in \operatorname{Sel}\left(w_{m}\right), m \geq 1$, such that $w_{m} \longrightarrow w$ and $g_{m} \rightharpoonup g$, then $g \in \operatorname{Sel}(w)$.

Lemma 11. Suppose that (h1) is satisfied; then, for any bounded set $\Delta \subset C(\bar{B}, E)$, we obtain

$$
\begin{align*}
& \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \quad \leq\left[\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right] \gamma_{c}(\Delta) . \tag{24}
\end{align*}
$$

Proof. According to (5), for any $\epsilon>0$, there exist $\left\{w_{m}\right\}_{m=1}^{+\infty} \subset$ $\Delta$ and $\left\{g_{m}\right\}_{m=1}^{+\infty} \subset L^{1}(\bar{B}, E)$ satisfying $g_{m} \in \operatorname{Sel}\left(w_{m}\right)$ for all $m \geq 1$ and

$$
\begin{align*}
& \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \quad \leq 2 \gamma\left(\left\{S_{h\left(w_{m}\right)} g_{m}(t): m \geq 1\right\}\right)+\epsilon \tag{25}
\end{align*}
$$

Since

$$
\begin{align*}
S_{h\left(w_{m}\right)} g_{m}(t)= & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h\left(w_{m}\right) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w_{m}(\eta) d \eta  \tag{26}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g_{m}(\eta) d \eta
\end{align*}
$$

now by (h1) and Proposition 1, it follows that

$$
\begin{align*}
\gamma & \left(\left\{S_{h\left(w_{m}\right)} g_{m}(t): m \geq 1\right\}\right) \\
& =\gamma\left(\left\{\left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h\left(w_{m}\right)\right.\right. \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w_{m}(\eta) d \eta \\
& \left.\left.\left.+\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g_{m}(\eta) d \eta\right): m \geq 1\right\}\right)  \tag{27}\\
& =\gamma\left(\left\{\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w_{m}(\eta) d \eta\right.\right. \\
& \left.\left.\left.+\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g_{m}(\eta) d \eta\right): m \geq 1\right\}\right)
\end{align*}
$$

By $\left(H G_{4}\right)$, we obtain

$$
\begin{aligned}
\gamma\left(\left\{g_{m}(s)\right\}_{m=1}^{+\infty}\right) & \leq \gamma\left(G\left(s,\left\{w_{m}(s)\right\}_{m=1}^{+\infty}\right)\right. \\
& \leq b(s) \gamma\left(\left\{w_{m}(s)\right\}_{m=1}^{+\infty}\right) \\
& \leq b(s) \gamma(\Delta(s)) .
\end{aligned}
$$

Applying Proposition 1 and Lemma 3, one can easily achieve

$$
\begin{align*}
& \gamma\left(\left\{S_{h\left(w_{m}\right)} g_{m}(t): m \geq 1\right\}\right) \\
& \quad \leq \frac{2|\chi|}{\Gamma\left(C_{1}+1\right)} \gamma_{c}(\Delta) \\
& \quad+\frac{2}{\Gamma\left(v_{1}\right)} \int_{0}^{t}(t-\eta)^{v_{1}-1} b(\eta) \gamma(\Delta(\eta)) d \eta  \tag{29}\\
& \quad \leq\left[\frac{2|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{2 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right] \gamma_{c}(\Delta) .
\end{align*}
$$

It follows from (25) that

$$
\begin{align*}
& \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \quad \leq\left[\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right] \gamma_{c}(\Delta)+\epsilon \tag{30}
\end{align*}
$$

Since $\epsilon>0$ is arbitrary, we have

$$
\begin{align*}
& \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \quad \leq\left[\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right] \gamma_{c}(\Delta) \tag{31}
\end{align*}
$$

A nonlinear alternative for Kakutani maps, which is significant to develop our main results, is introduced as follows.

Theorem 12 (see [21]). Let $\Delta \subset E$ be a closed convex set and $W \subseteq \Delta$ be an open set with $0 \in W$. Suppose $T: \bar{W} \longrightarrow K_{f c}(\Delta)$ is compact, upper semicontinuous. If there exist no $v \in \partial W$ and $0<\mu<1$ satisfying $v \in \mu T(v)$, then $T$ has at least one fixed point.

Next, we shall prove the existence result when $h$ is compact continuous.

Theorem 13. Suppose $(h 1)$ and $\left(H G_{1}\right)-\left(H G_{4}\right)$ are satisfied. If there exists $L>0$ such that

$$
\begin{align*}
& \frac{\Gamma\left(v_{1}\right)\left[\Gamma\left(C_{1}+1\right)-\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{1}-|\chi|\right] L}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{2}+\Gamma\left(C_{1}+1\right) \Sigma(L)\|l\|_{p} C}  \tag{32}\\
& \quad>1
\end{align*}
$$

and

$$
\begin{equation*}
\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}-1<0 \tag{33}
\end{equation*}
$$

then the generalized Bagley-Torvik type differential inclusion (1) has at least one solution in $C(\bar{B}, E)$.

Proof. From ([22]), according to conditions $\left(H G_{1}\right)-\left(H G_{3}\right)$, $\operatorname{Sel}(w)$ is not empty, for every $w \in C(\bar{B}, E)$. Using (23), we define the following multivalued operator:

$$
\begin{equation*}
T: C(\bar{B}, E) \longrightarrow K(C(\bar{B}, E) \tag{34}
\end{equation*}
$$

as

$$
\begin{align*}
& T(w)=\{z \in C(\bar{B}, E): z(t) \\
& \quad=\left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h(w)  \tag{35}\\
& \quad+\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w(\eta) d \eta \\
& \left.\quad+\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta, g \in \operatorname{Sel}(w)\right\} .
\end{align*}
$$

Clearly, if $w$ is a fixed point of $T$, then $w$ is a solution of (1). Let $r>0 ; B_{r}$ is a bounded ball, defined as $B_{r}=\{w \in C(B, E)$ : $\|w\| \leq r\}$. Suppose $\Delta \subset B_{r}$ is a bounded set and belongs to $\overline{c o}(\{0\} \cup T(\Delta))$. In order to utilize the nonlinear alternative for Kakutani maps, we first need to show that $\gamma(\Delta)=0$. Let $s_{1}, s_{2} \in \bar{B}, 0<s_{1}<s_{2}, z \in T(\Delta)$; there exist $w \in \Delta$ and $h \in \operatorname{Sel}(w)$ such that

$$
\begin{align*}
z(t)= & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h(w) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w(\eta) d \eta  \tag{36}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta, \quad t \in \bar{B} .
\end{align*}
$$

By Hölder inequality and $\left(\mathrm{HG}_{3}\right)$, we have

$$
\begin{aligned}
& \left|\int_{0}^{s_{2}}\left(s_{2}-\eta\right)^{v_{1}-1} g(\eta) d \eta-\int_{0}^{s_{1}}\left(s_{1}-\eta\right)^{v_{1}-1} g(\eta) d \eta\right| \\
& \quad \leq \int_{0}^{s_{1}}\left|\left(s_{2}-\eta\right)^{v_{1}-1}-\left(s_{1}-\eta\right)^{v_{1}-1}\right||g(\eta)| d \eta \\
& \quad+\int_{s_{1}}^{s_{2}}\left|\left(s_{2}-\eta\right)^{v_{1}-1}\right||g(\eta)| d \eta \leq\left|v_{1}-1\right| \\
& \quad \cdot \int_{0}^{s_{1}}\left|s_{1}-s\right|^{v_{1}-2}\left(s_{2}-s_{1}\right)|g(\eta)| d \eta+\|l\|_{p} \Phi(r) \\
& \cdot\left(s_{2}-s_{1}\right)^{v_{1}-1 / p} C \leq\left|v_{1}-1\right| \\
& \cdot\left(\int_{0}^{s_{1}}\left|s_{1}-\eta\right|^{\left(v_{1}-1\right) p^{\prime}} d \eta\right)^{\left(2-v_{1}\right) /\left(1-v_{1}\right) p^{\prime}} \\
& \quad \cdot\left(\int_{0}^{s_{1}}\left(s_{2}-s_{1}\right)^{\left(v_{1}-1\right) p^{\prime}} d \eta\right)^{1 /\left(v_{1}-1\right) p^{\prime}} \\
& \cdot\left(\int_{0}^{s_{1}}|l(\eta)|^{p} d \eta\right)^{1 / p} \Sigma(r)+\|l\|_{p} \Sigma(r)\left(s_{2}\right. \\
& \left.\quad-s_{1}\right)^{v_{1}-1 / p} C \leq\left|v_{1}-1\right| s_{1}\left|s_{2}-s_{1}\right|
\end{aligned}
$$

$$
\begin{align*}
& \cdot\left(\int_{0}^{s_{1}}\left|s_{1}-\eta\right|^{\left(v_{1}-1\right) p^{\prime}} d \eta\right)^{\left(2-v_{1}\right) /\left(1-v_{1}\right) p^{\prime}}\|l\|_{p} \Sigma(r) \\
& +\|l\|_{p} \Sigma(r)\left(s_{2}-s_{1}\right)^{v_{1}-1 / p} C . \tag{37}
\end{align*}
$$

Taking into account the above inequality, one has

$$
\begin{align*}
& \left|z\left(s_{2}\right)-z\left(s_{1}\right)\right| \leq \frac{|\chi|\left|\left(s_{2}\right)^{C_{1}}-\left(s_{1}\right)^{C_{1}}\right|}{\Gamma\left(C_{1}+1\right)}|h(w)| \\
& \left.\quad+\frac{|\chi|}{\Gamma\left(C_{1}\right)} \right\rvert\, \int_{0}^{s_{2}}\left(s_{2}-\eta\right)^{C_{1}-1} w(\eta) d \eta \\
& \quad-\int_{0}^{s_{1}}\left(s_{1}-\eta\right)^{C_{1}-1} w(\eta) d \eta \mid \\
& \left.\quad+\frac{1}{\Gamma\left(v_{1}\right)} \right\rvert\, \int_{0}^{s_{2}}\left(s_{2}-\eta\right)^{v_{1}-1} g(\eta) d \eta \\
& \quad-\int_{0}^{s_{1}}\left(s_{1}-\eta\right)^{v_{1}-1} g(\eta) d \eta \mid \\
& \quad \leq \frac{|\chi|\left|\left(s_{2}\right)^{C_{1}}-\left(s_{1}\right)^{C_{1}}\right|}{\Gamma\left(C_{1}+1\right)}\left(c_{1} r+c_{2}\right)  \tag{38}\\
& \quad+\frac{|\chi| r}{\Gamma\left(C_{1}+1\right)}\left[\left|\left(s_{2}\right)^{C_{1}}-\left(s_{1}\right)^{C_{1}}\right|+2\left(s_{2}-s_{1}\right)^{C_{1}}\right] \\
& \quad+\frac{1}{\Gamma\left(v_{1}\right)}\left[\left|v_{1}-1\right| s_{1}\left|s_{2}-s_{1}\right|\right. \\
& \quad+\left(\int_{0}^{s_{1}}\left|s_{1}-\eta\right|^{(v-1) p^{\prime}} d \eta\right)^{\left(2-v_{1}\right) /\left(1-v_{1}\right) p^{\prime}}\|l\|_{p} \Sigma(r) \\
& \left.\quad+\|l\|_{p} \Sigma(r)\left(s_{2}-s_{1}\right)^{v_{1}-1 / p} C\right]
\end{align*}
$$

Applying the above inequality, $\left|z\left(s_{2}\right)-z\left(s_{1}\right)\right| \longrightarrow 0$, as $s_{1} \longrightarrow$ $s_{2}$. Hence, $T(\Delta)$ and $\Delta \subseteq \overline{c o}(\{0\} \cup T(\Delta))$ are equicontinuous.

Also because, for every $t \in \bar{B}$,

$$
\begin{equation*}
T \Delta(t) \subset\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\} \tag{39}
\end{equation*}
$$

It follows from Lemma 11 that

$$
\begin{align*}
\gamma(T \Delta(t)) & \leq \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \leq\left[\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(\nu_{1}\right)}\right] \gamma(\Delta) . \tag{40}
\end{align*}
$$

Thus, one can obtain

$$
\begin{align*}
\gamma(\overline{c o}(\{0\} \cup T(\Delta))) & =\gamma(T(\Delta)) \\
& \leq\left[\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right] \gamma(\Delta) . \tag{41}
\end{align*}
$$

Since $\Delta \subseteq \overline{c o}(\{0\} \cup T(\Delta))$ and by (33), we know that $\gamma(\Delta)=0$.

Clearly, the multioperator $T$ has convex values. We still need to prove the multioperator $T$ is closed on $\Delta$. Suppose $\left\{w_{m}\right\}_{m=1}^{\infty} \subset \Delta$ with $w_{m} \longrightarrow w$ in $C(\bar{B}, E)$ and $z_{m} \in T\left(w_{m}\right)$ with $z_{m} \longrightarrow z$ in $C(\bar{B}, E)$. Moreover, assume $\left\{g_{m}\right\}_{m=1}^{+\infty} \subset L^{1}(\bar{B}, E)$ is a sequence satisfying $g_{m} \in \operatorname{Sel}\left(w_{m}\right)$ for any $m \geq 1$ and

$$
\begin{align*}
z_{m}(t)= & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h\left(w_{m}\right) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w_{m}(\eta) d \eta  \tag{42}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g_{m}(\eta) d \eta
\end{align*}
$$

Then the set $\left\{g_{m}\right\}_{m=1}^{\infty}$ is integrally bounded. From $\left(H G_{3}\right)$, one can achieve that

$$
\begin{align*}
&\left.\left\|g_{m}(\eta)\right\| \leq \| G\left(\eta, w_{m}\right)(\eta)\right) \| \leq l(\eta) \Sigma\left(\left\|w_{m}\right\|\right) \\
& \text { a.e. } \eta \in \bar{B}, m=1,2, \ldots \tag{43}
\end{align*}
$$

Since $w_{m} \longrightarrow w$ in $C(\bar{B}, E)$, therefore $\left\{w_{m}: m \geq 1\right\}$ is bounded and belongs to $C(\bar{B}, E)$. Invoking $\left(H G_{4}\right)$, we infer that

$$
\begin{align*}
\gamma\left(\left\{g_{m}(\eta)\right\}_{m=1}^{\infty}\right) & \leq \gamma\left(G\left(\eta,\left\{v_{m}(\eta)\right\}_{m=1}^{\infty}\right)\right) \\
& \leq b(\eta) \gamma\left(\left\{w_{m}(\eta)\right\}_{m=1}^{\infty}\right)=0 . \tag{44}
\end{align*}
$$

So the sequence $\left\{g_{m}\right\}_{m=1}^{\infty}$ is semicompact. Applying the proposition (see [22]), $\left\{g_{m}\right\}_{m=1}^{\infty}$ is weakly compact. Therefore, there is $g \in L^{1}(\bar{B}, E)$ satisfying $g_{m} \rightharpoonup g$. Moreover, one has

$$
\begin{equation*}
\int_{0}^{t} \frac{(t-\eta)^{\nu_{1}-1}}{\Gamma\left(v_{1}\right)} g_{m}(\eta) d \eta \longrightarrow \int_{0}^{t} \frac{(t-\eta)^{\nu_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta \tag{45}
\end{equation*}
$$

which, together with (42) and $g$ being continuous, implies that

$$
\begin{align*}
z_{m}(t) \longrightarrow & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h(w) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w(\eta) d \eta  \tag{46}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta .
\end{align*}
$$

Then, one can obtain

$$
\begin{align*}
z(t)= & \left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) g(w) \\
& +\chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w(\eta) d \eta  \tag{47}\\
& +\int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta
\end{align*}
$$

Thus $g \in \operatorname{Sel}(w)$; i.e., $z \in T(w)$. Equivalently, $\operatorname{graph}(T)$ is closed, and $T$ is closed on $\Delta$. Based on the above discussion, it can be concluded that $T$ is u.s.c. on $\Delta$ (see [23]).

In the following, we are to find an open set $W$, which satisfies the conditions of Theorem 12. Let $\mu \in(0,1), w \in$ $\mu T(w)$. Then there exist $w \in \partial W$ and $g \in \operatorname{Sel}(w)$ such that

$$
\begin{align*}
w(t)= & \mu\left(1-\frac{\chi t^{C_{1}}}{\Gamma\left(C_{1}+1\right)}\right) h(w) \\
& +\mu \chi \int_{0}^{t} \frac{(t-\eta)^{C_{1}-1}}{\Gamma\left(C_{1}\right)} w(\eta) d \eta  \tag{48}\\
& +\mu \int_{0}^{t} \frac{(t-\eta)^{v_{1}-1}}{\Gamma\left(v_{1}\right)} g(\eta) d \eta
\end{align*}
$$

From $(h 1)$ and $\left(H G_{3}\right)$, one can obtain

$$
\begin{align*}
\|w(t)\| \leq & \left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right)|h(w)| \\
& +\frac{|\chi|}{\Gamma\left(C_{1}\right)} \int_{0}^{t}(t-\eta)^{C_{1}-1}|w(\eta)| d \eta \\
& +\frac{1}{\Gamma\left(v_{1}\right)} \int_{0}^{t}(t-\eta)^{v_{1}-1}|g(\eta)| d \eta \\
\leq & \left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right)\left(c_{1}\|w\|+c_{2}\right) \\
& +\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\|w\|  \tag{49}\\
& +\frac{\Sigma(\|w\|)}{\Gamma\left(v_{1}\right)} \int_{0}^{t}(t-\eta)^{v_{1}-1} l(\eta) d \eta \\
\leq & \left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right)\left(c_{1}\|w\|+c_{2}\right) \\
& +\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\|w\|+\frac{\sum(\|w\|)\|l\|_{p}}{\Gamma\left(v_{1}\right)} C .
\end{align*}
$$

Consequently,

$$
\begin{equation*}
\frac{\Gamma\left(v_{1}\right)\left[\Gamma\left(C_{1}+1\right)-\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{1}-|\chi|\right]\|w\|}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{2}+\Gamma\left(C_{1}+1\right) \Sigma(\|w\|)\|l\|_{p} C} \tag{50}
\end{equation*}
$$

$$
\leq 1
$$

It follows from (32) that there exists $L$ which satisfies $\|w\| \neq L$. Denote

$$
\begin{equation*}
W=\{w \in C(\bar{B}, E):\|w\|<L\} . \tag{51}
\end{equation*}
$$

By the definition of $W$, for every $\mu \in(0,1)$, there exists no $w \in \partial W$ satisfying $w \in \mu T(w)$. Since $T: \bar{W} \longrightarrow K(C(\bar{B}, E))$ is compact and u.s.c., according to Theorem $12, T$ has at least one fixed point in $\bar{W}$.

Next, we assume the nonlocal item $h$ is Lipschitz continuous; that is,
(h2) for every $x_{1}, x_{2} \in C(\bar{B}, E)$, there is $k$ satisfying $\left|h\left(x_{1}\right)-h\left(x_{2}\right)\right| \leq k\left|x_{1}-x_{2}\right|$.

Now, we give the existence criterion for the generalized Bagley-Torvik type fractional differential inclusions (1) when the nonlocal item satisfies (h2).

Theorem 14. Suppose conditions (h2) and $\left(H G_{1}\right)-\left(H G_{4}\right)$ are satisfied. If there exists $L>0$, such that

$$
\begin{aligned}
& \frac{\Gamma\left(v_{1}\right)\left[\Gamma\left(C_{1}+1\right)-\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) k-|\chi|\right] L}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right)|h(0)|+\Gamma\left(C_{1}+1\right) \Sigma(L)\|l\|_{p} C} \\
& \quad>1,
\end{aligned}
$$

and

$$
\begin{equation*}
\left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k+\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}-1<0 \tag{53}
\end{equation*}
$$

then the generalized Bagley-Torvik type differential inclusion (1) has a solution in $C(\bar{B}, E)$.

Proof. According to Theorem 13, the multioperator $T$ is upper semicontinuous. Suppose $\Delta \subseteq B_{r}$ is bounded, and belongs to $\overline{c o}(\{0\} \cup T(\Delta))$. We need to prove $\gamma(\Delta)=0$.

From (h2) and Lemma 11, one can infer that

$$
\begin{aligned}
& \gamma(T \Delta(t)) \leq \gamma\left(\left\{S_{h(w)} g^{w}(t): w \in \Delta, g^{w} \in \operatorname{Sel}(w)\right\}\right) \\
& \quad \leq\left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k \gamma(\Delta)+\frac{2|\chi|}{\Gamma\left(C_{1}+1\right)} \gamma(\Delta) \\
& \quad+\frac{4}{\Gamma\left(v_{1}\right)} \int_{0}^{t}(t-\eta)^{v_{1}-1} b(s) \gamma(\Delta(\eta)) d \eta \\
& \quad \leq\left[\left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k+\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right]
\end{aligned}
$$

$$
\cdot \gamma(\Delta)
$$

Thus, one has

$$
\begin{align*}
& \gamma(\Delta) \leq \gamma(\overline{c o}(\{0\} \cup T(\Delta)))=\gamma(T(\Delta)) \\
& \quad \leq\left[\left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k+\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}\right]  \tag{55}\\
& \quad \cdot \gamma(\Delta) .
\end{align*}
$$

Together with (53), one can conclude that $\gamma(\Delta)=0$.
From a proof similar to Theorem 13, we can prove that the multioperator $T$ has at least one fixed point, which is a solution of the generalized Bagley-Torvik type fractional differential inclusions (1).
Remark 15. Assume $\Omega(x)=k_{1} x^{p}+k_{2}, \forall x>0$ for some $k_{1}, k_{2}>0$ and $0<p<1$; then conditions (32) and (52) are automatically satisfied.

Subsequently, we will investigate the case where $G$ is Lipschitz-type about the Hausdorff metric. Before proceeding, let us introduce the following definition.

Suppose $(E, d)$ is metric spaces, corresponding to $(E,\|\cdot\|)$. Let $A_{1}, A_{2} \in K_{b f}(E), a_{1} \in A_{1}$. Denote

$$
\begin{align*}
& D\left(a_{1}, A_{2}\right)=\inf \left\{d\left(a_{1}, a_{2}\right): a_{2} \in A_{2}\right\} \\
& \rho\left(A_{1}, A_{2}\right)=\sup \left\{D\left(a_{1}, A_{2}\right): a_{1} \in A_{1}\right\} . \tag{56}
\end{align*}
$$

A function $H: K_{b f}(E) \times K_{b f}(E) \longrightarrow \mathbb{R}^{+}$is called the Hausdorff metric on $E$, if

$$
\begin{equation*}
H\left(A_{1}, A_{2}\right)=\max \left\{\rho\left(A_{1}, A_{2}\right), \rho\left(A_{2}, A_{1}\right)\right\} \tag{57}
\end{equation*}
$$

Now, we make the following assumption:
$\left(H G_{5}\right)$ there exists $\omega \in L^{p}\left(\bar{B}, \mathbb{R}^{+}\right)$, satisfying

$$
\begin{align*}
H\left(G\left(s, z_{1}(s)\right), G\left(s, z_{2}(z)\right)\right) & \leq \omega(s)\left\|z_{1}-z_{2}\right\| \\
\forall s & \in \bar{B}, z_{1}, z_{2} \in C(\bar{B}, E), \tag{58}
\end{align*}
$$

and for almost every $s \in \bar{B},\|G(s, z(s))\| \leq \omega(s)(1+\|z\|)$, $z \in C(\bar{B}, E)$.

Noting that if $\left(H G_{5}\right)$ is satisfied, $\left(H G_{3}\right)$ and $\left(H G_{4}\right)$ are automatically satisfied, in this case, the following theorems automatically hold.

Theorem 16. Let $h$ and $G$ satisfy conditions (h1), $\left(H G_{1}\right)$, $\left(H G_{2}\right)$, and $\left(H G_{5}\right)$. If there exists $L>0$, such that

$$
\begin{equation*}
\frac{\left[\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)-\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{1}-|\chi|\right)-\left(\Gamma\left(C_{1}+1\right)\|\omega\|_{p} C\right] L\right.}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) c_{2}+\Gamma\left(C_{1}+1\right)\|\omega\|_{p} C}>1, \tag{59}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|\omega\|_{p}}{\Gamma\left(\nu_{1}\right)}-1<0 \tag{60}
\end{equation*}
$$

then the fractional differential inclusion (1) has at least one solution.

Theorem 17. Assume $h$ and $G$ satisfy conditions ( $h 2$ ), $\left(H G_{1}\right)$, $\left(H G_{2}\right)$, and $\left(H G_{5}\right)$. If there exists $L>0$, such that

$$
\begin{equation*}
\frac{\left[\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)-(\Gamma(v+1)+|\chi|) k-|\chi|\right)-\left(\Gamma\left(C_{1}+1\right)\|\omega\|_{p} C\right] L\right.}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right)|g(0)|+\Gamma\left(C_{1}+1\right)\|\omega\|_{p} C}>1, \tag{61}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k+\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|\omega\|_{p}}{\Gamma\left(v_{1}\right)}<1 \tag{62}
\end{equation*}
$$

then the fractional differential inclusion (1) has at least one solution.

## 4. Example

Consider the following nonlocal problem of fractional differential inclusion.

Example 18. Discuss the following differential inclusion:

$$
\begin{gather*}
{ }^{c} D z(s)-\left(\frac{\sqrt{\pi}}{16}\right)^{c} D^{1 / 2} z(s) \in G(s, z(s)), \quad s \in B \\
z(0)=\sum_{j=1}^{m} \beta_{j} z\left(s_{j}\right), \tag{63}
\end{gather*}
$$

where $s_{j}(j=1,2, \ldots, m)$ are certain constants, with $0<s_{1}<$ $s_{2}<\cdots<s_{m}<1, v_{1}=1, v_{2}=1 / 2, C_{1}=1 / 2, \sum_{j=1}^{m} \beta_{j}=1 / 90$, $\chi=\sqrt{\pi} / 16$. Let $p=2$ and $G(s, z)=\left[0,(1 / 3) s^{4} z+1 / 6\right]$.

Clearly, for all $s \in \bar{B},\|G(s, z)\| \leq(1 / 3)\|z\|+1 / 6=$ $l(s) \Sigma(\|z\|)$, and then $\|l\|_{2}=1, \Sigma(\|z\|)=(1 / 3)\|z\|+1 / 6$.

Further, we suppose $h(z)=\sum_{j=1}^{m} \beta_{j} z\left(s_{j}\right)$. Obviously, $h(0)=0$. Moreover,

$$
\begin{align*}
& H\left(G\left(s, z_{1}\right), G\left(s, z_{2}\right)\right) \leq \frac{1}{3} s^{4}\left|z_{1}-z_{2}\right| \\
& \forall \quad \forall z_{1}, z_{2} \in C(\bar{B}, \mathbb{R}), s \in \bar{B}, \\
& \left|h\left(z_{1}\right)-h\left(z_{2}\right)\right| \leq \sum_{j=1}^{m} \beta_{j}\left|z_{1}\left(s_{j}\right)-z_{2}\left(s_{j}\right)\right|  \tag{64}\\
& \leq \sum_{j=1}^{m} \beta_{j} \max _{s_{j} \in \bar{B}}\left\{\left|z_{1}\left(s_{j}\right)-z_{2}\left(s_{j}\right)\right|\right\}, \\
& \quad z_{1}, z_{2} \in C(\bar{B}, \mathbb{R}) .
\end{align*}
$$

Take $b(s)=(1 / 3) s^{4}, k=\sum_{j=1}^{m} \beta_{j}=1 / 90$. Then, one can easily obtain that $\|b\|_{2}=1 / 9$.

It is easy to check that (h2) and $\left(H G_{1}\right)-\left(H G_{4}\right)$ hold. Further, we can find $L=3$, such that

$$
\begin{align*}
& \frac{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)-\left(\Gamma\left(C_{1}+1\right)+|\chi|\right) k-|\chi|\right) L}{\Gamma\left(v_{1}\right)\left(\Gamma\left(C_{1}+1\right)+|\chi|\right)|h(0)|+\Gamma\left(C_{1}+1\right) \Sigma(L)\|l\|_{p} C} \\
& \quad=\frac{3 \times(\sqrt{\pi} / 2-\sqrt{\pi} / 160-\sqrt{\pi} / 16)}{(\sqrt{\pi} / 2)((1 / 3) \times 3+1 / 6)}>1, \tag{65}
\end{align*}
$$

and

$$
\begin{align*}
& \left(1+\frac{|\chi|}{\Gamma\left(C_{1}+1\right)}\right) k+\frac{4|\chi|}{\Gamma\left(C_{1}+1\right)}+\frac{4 C\|b\|_{p}}{\Gamma\left(v_{1}\right)}  \tag{66}\\
& \quad=\frac{1}{80}+\frac{1}{2}+\frac{4}{9}<1 .
\end{align*}
$$

Therefore, according to Theorem 14, there exists at least one solution in $C(\bar{B}, \mathbb{R})$ for the problem (63).

## 5. Conclusion

This paper has studied the generalized Bagley-Torvik type fractional order differential inclusions with nonlocal conditions. By employing the noncompactness measure of Hausdorff and the nonlinear alternative for Kakutani maps fixed point theorem, the existence results have been derived when the nonlocal item is compact and Lipschitz continuous and multifunction is compact and Lipschitz. An example has been used to illustrate applications of the main result. Future research directions include the extension of the present results to other relevant cases, for example, controllability and topological structure of the solution set $[12,13]$.

## Data Availability

No date were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# On the Effective Reducibility of a Class of Quasi-Periodic Linear Hamiltonian Systems Close to Constant Coefficients 

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In this paper, we consider the effective reducibility of the quasi-periodic linear Hamiltonian system $\dot{x}=(A+\varepsilon Q(t, \varepsilon)) x, \varepsilon \in\left(0, \varepsilon_{0}\right)$, where $A$ is a constant matrix with possible multiple eigenvalues and $Q(t, \varepsilon)$ is analytic quasi-periodic with respect to $t$. Under nonresonant conditions, it is proved that this system can be reduced to $\dot{y}=\left(A^{*}(\varepsilon)+\varepsilon R^{*}(t, \varepsilon)\right) y, \varepsilon \in\left(0, \varepsilon^{*}\right)$, where $R^{*}$ is exponentially small in $\varepsilon$, and the change of variables that perform such a reduction is also quasi-periodic with the same basic frequencies as $Q$.

## 1. Introduction

The question about the reducibility of quasi-periodic systems plays an important role in the theory of ordinary differential equations. In general, in order to understand the qualitative behavior of a system, we need to obtain the information about the existence and stability of solutions. During the last two decades, the study of the existence of solutions for differential equations has attracted the attention of many researchers; see [ $1-10$ ] and the references therein. Some classical tools have been used to study the existence of solutions for differential equations in the literature, including the method of upper and lower solutions, degree theory, some fixed point theorems in cones for completely continuous operators, Schauder's fixed point theorem, and a nonlinear Leray-Schauder alternative principle.

Compared with the existence of solutions, the study on the dynamical stability behaviors of such equations is more difficult, and the results are fewer in the literature. Here we refer the reader to [11-16].

Before stating our problem, we give some definitions and notations. A function $f$ is said to be a quasi-periodic function with a vector of basic frequencies $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{r}\right)$ if $f(t)=F\left(\theta_{1}, \theta_{2}, \ldots, \theta_{r}\right)$, where $F$ is $2 \pi$ periodic in all its arguments and $\theta_{j}=\omega_{j} t$ for $j=1,2, \ldots, r$. Moreover, if
$F(\theta)\left(\theta=\left(\theta_{1}, \theta_{2}, \ldots, \theta_{r}\right)\right)$ is analytic on $D_{\rho}=\left\{\theta \in \mathbb{C}^{r}:\left|\operatorname{Im} \theta_{j}\right|\right.$ $\leq \rho, j=1,2, \ldots, r\}$, we say that $f(t)$ is analytic quasi-periodic on $D_{\rho}$.

It is well known that an analytic quasi-periodic function $f(t)$ can be expanded as Fourier series

$$
\begin{equation*}
f(t)=\sum_{k \in \mathbb{Z}^{r}} f_{k} e^{\langle k, \omega\rangle \sqrt{-1} t} \tag{1}
\end{equation*}
$$

with Fourier coefficients defined by

$$
\begin{equation*}
f_{k}=\frac{1}{(2 \pi)^{r}} \int_{\mathbb{\pi} r} F\left(\theta_{1}, \theta_{2}, \ldots, \theta_{r}\right) e^{-\langle k, \theta\rangle \sqrt{-1}} d \theta \tag{2}
\end{equation*}
$$

We denote by $\|f\|_{\rho}$ the norm

$$
\begin{equation*}
\|f\|_{\rho}=\sum_{k \in \mathbb{Z}^{r}}\left|f_{k}\right| e^{|k| \rho} \tag{3}
\end{equation*}
$$

An $n \times n$ matrix $Q(t)=\left(q_{i j}\right)_{1 \leq i, j \leq n}$ is said to be analytic quasiperiodic on $D_{\rho}$ with frequencies $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{r}\right)$, if all $q_{i j}(i, j=1,2, \ldots, n)$ are analytic quasi-periodic on $D_{\rho}$ with frequencies $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{r}\right)$. Define the norm of $Q$ by

$$
\begin{equation*}
\|Q\|_{\rho}=\max _{1 \leq i \leq n} \sum_{j=1}^{n}\left\|q_{i j}\right\|_{\rho} \tag{4}
\end{equation*}
$$

It is easy to see that $\left\|Q_{1} Q_{2}\right\|_{\rho} \leq\left\|Q_{1}\right\|_{\rho}\left\|Q_{2}\right\|_{\rho}$. If $Q$ is a constant matrix, write $\|Q\|=\|Q\|_{\rho}$ for simplicity. Denote the average of $Q(t)$ by $\bar{Q}=\left(\bar{q}_{i j}\right)_{1 \leq i, j \leq n}$, where

$$
\begin{equation*}
\bar{q}_{i j}=\lim _{T \rightarrow \infty} \frac{1}{2 T} \int_{-T}^{T} q_{i j}(t) d t \tag{5}
\end{equation*}
$$

for the existence of the limit, see [17].
Let $A(t)$ be an $n \times n$ quasi-periodic matrix; the differential equations $\dot{x}=A(t) x, x \in \mathbb{R}^{n}$, are called reducible if there exists a nonsingular quasi-periodic change of variables $x=\phi(t) y$, such that $\phi(t)$ and $\phi^{-1}(t)$ are quasi-periodic and bounded, which changes $\dot{x}=A(t) x$ to $\dot{y}=B y$, where $B$ is a constant matrix. The well-known Floquet theorem states that any periodic differential equations $\dot{x}=A(t) x$ can be reduced to constant coefficient differential equations $\dot{y}=B y$ by means of a periodic change of variables with the same period as $A(t)$. But this is not true for the quasi-periodic coefficient system; see [18]. Johnson and Sell [19] proved that $\dot{x}=A(t) x$ is reducible if the quasi-periodic coefficient matrix $A(t)$ satisfies "full spectrum" condition.

Recently, many authors [20-23] considered the reducibility of the following system which is close to constant coefficients matrix:

$$
\begin{equation*}
\dot{x}=(A+\varepsilon Q(t)) x . \tag{6}
\end{equation*}
$$

This problem was first considered by Jorba and Simó in [20]. Suppose that $A$ is a constant matrix with different eigenvalues; they proved that if the eigenvalues of $A$ and the frequencies of $Q$ satisfy some nonresonant conditions, then for sufficiently small $\varepsilon_{0}>0$, there exists a nonempty Cantor set $E \subset\left(0, \varepsilon_{0}\right)$, such that, for any $\varepsilon \in E$, system (6) is reducible. Moreover, the relative measure of the set $\left(0, \varepsilon_{0}\right) \backslash E$ in $\left(0, \varepsilon_{0}\right)$ is exponentially small in $\varepsilon_{0}$. In [23], Xu obtained the similar result for the multiple eigenvalues case.

In [21], Jorba and Simó extended the conclusion of the linear system to the nonlinear system

$$
\begin{equation*}
\dot{x}=(A+\varepsilon Q(t, \varepsilon)) x+\varepsilon g(t)+h(x, t), \quad x \in \mathbb{R}^{n} . \tag{7}
\end{equation*}
$$

Suppose that $A$ has $n$ different nonzero eigenvalues; they proved that, under some nonresonant conditions and nondegeneracy conditions, there exists a nonempty Cantor set $E \subset\left(0, \varepsilon_{0}\right)$, such that, for all $\varepsilon \in E$, system (7) is reducible. Later, in [24], Wang and Xu considered the nonlinear quasiperiodic system

$$
\begin{equation*}
\dot{x}=A x+f(x, t), \quad x \in \mathbb{R}^{2}, \tag{8}
\end{equation*}
$$

and they proved without any nondegeneracy condition that one of two results holds: (1) system (8) is reducible to $\dot{y}=B y+$ $O(y)$ for all $\varepsilon \in\left(0, \varepsilon_{0}\right)$; (2) there exists a nonempty Cantor set $E \subset\left(0, \varepsilon_{0}\right)$, such that system (8) is reducible to $\dot{y}=B y+O\left(y^{2}\right)$ for all $\varepsilon \in E$.

These papers above all deal with a total reduction to constant coefficients. In [25], instead of a total reduction to constant coefficients, Jorba, Ramirez-ros, and Villanueva considered the effective reducibility of the following quasiperiodic system:

$$
\begin{equation*}
\dot{x}=(A+\varepsilon Q(t, \varepsilon)) x, \quad|\varepsilon| \leq \varepsilon_{0} \tag{9}
\end{equation*}
$$

where $A$ is a constant matrix with different eigenvalues. They proved that, under nonresonant conditions, by a quasiperiodic transformation, system (9) is reducible to a quasiperiodic system

$$
\begin{equation*}
\dot{y}=\left(A^{*}(\varepsilon)+\varepsilon R^{*}(t, \varepsilon)\right) y, \quad|\varepsilon| \leq \varepsilon_{*} \leq \varepsilon_{0}, \tag{10}
\end{equation*}
$$

where $R^{*}$ is exponentially small in $\varepsilon$. In [26], Li and Xu obtained the similar result for Hamiltonian systems.

In this paper, we consider the case that $A$ has multiple eigenvalues. Under some nonresonant conditions, we can obtain the effective reducibility for system (9) similar to [25, 26].

Now we are in a position to state the main result.
Theorem 1. Consider the following linear Hamiltonian system:

$$
\begin{equation*}
\dot{x}=(A+\varepsilon Q(t, \varepsilon)) x, \quad x \in \mathbb{R}^{n} \tag{11}
\end{equation*}
$$

where $A$ is a constant matrix with eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$, $Q(t, \varepsilon)$ is an analytic quasi-periodic function on $D_{\rho}$ with the frequencies $\omega=\left(\omega_{1}, \ldots, \omega_{r}\right)$, and $\varepsilon \in\left(0, \varepsilon_{0}\right)$ is a small parameter.

If $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{n}\right)$ and $\omega=\left(\omega_{1}, \omega_{2}, \ldots, \omega_{r}\right)$ satisfy the nonresonant conditions,

$$
\begin{equation*}
\left|(k, \omega) \sqrt{-1}-\lambda_{i}+\lambda_{j}\right| \geq \frac{\alpha}{|k|^{\tau}} \tag{12}
\end{equation*}
$$

for all $k \in \mathbb{Z}^{r} \backslash\{0\}, 0 \leq i, j \leq n$, where $\alpha>0$ is a small constant and $\tau>r-1$. In addition, we assume that $A+\varepsilon \bar{Q}$ has $n$ different eigenvalues $\mu_{1}, \ldots, \mu_{n}$, and $\delta:=\min \left\{\varepsilon^{-1}\left|\mu_{i}-\mu_{j}\right|: 0 \leq i, j \leq\right.$ $n, i \neq j\}$ is a positive constant independent of $\varepsilon$.

Then there exists some $\varepsilon^{*}>0$ such that, for any $\varepsilon \in\left(0, \varepsilon^{*}\right)$, there is an analytic quasi-periodic symplectic transformation $x=\psi(t, \varepsilon) y$ on $D_{\rho}$, where $\psi(t, \varepsilon)$ has same frequencies as $Q(t, \varepsilon)$, which changes system (11) into the following linear system:

$$
\begin{equation*}
\dot{y}=\left(A^{*}(\varepsilon)+\varepsilon R^{*}(t, \varepsilon)\right) y \tag{13}
\end{equation*}
$$

where $A^{*}$ is a constant matrix with

$$
\begin{equation*}
\left\|A^{*}-A\right\| \leq \frac{e(\beta+1) q}{e-1} \varepsilon \tag{14}
\end{equation*}
$$

$R^{*}(t, \varepsilon)$ is an analytic quasi-periodic function on $D_{\rho}$ with the frequencies $\omega$, and

$$
\begin{align*}
\left\|R^{*}(t, \varepsilon)\right\|_{\rho-s} \leq \frac{e^{2} \beta^{2} q}{e-1} \exp \left(-\left(\frac{d}{\varepsilon^{1 / 2}}\right)^{1 / \tau} s\right) &  \tag{15}\\
& s \in(0, \rho]
\end{align*}
$$

Furthermore, a general explicit computation of $\varepsilon^{*}$ and $d$ is possible:

$$
\begin{equation*}
\varepsilon^{*}=\min \left\{\varepsilon_{0},\left(\frac{\delta(e-1)}{(3 n-1) e \beta q}\right)^{2}, \frac{1}{\beta^{2}}\right\}, \quad d=\frac{\alpha}{12 e \beta q} \tag{16}
\end{equation*}
$$

where $\beta$ is the condition number of a matrix $S$ such that $S^{-1}(A+\varepsilon \bar{Q}) S$ is diagonal, that is, $\beta=C(S)=\left\|S^{-1}\right\|\|S\|$, and the constant $q$ is the bound of $Q(t, \varepsilon)$ on $D_{\rho}$, that is, $\|Q(t, \varepsilon)\|_{\rho} \leq q$.

Remark 2. In general, $Q$ depends on $\varepsilon$, so does the average $\bar{Q}$. Below for simplicity, we do not indicate this dependence explicitly.

Remark 3. In Hamiltonian system (11), $n$ is an even number. In fact, a Hamiltonian system is $2 m$-dimensional; moreover, the eigenvalues $\lambda_{1}, \ldots, \lambda_{2 m}$ of a $2 m \times 2 m$ Hamiltonian matrix may be ordered so that $\lambda_{k+m}=-\lambda_{k} \quad(k=1, \ldots, m)$.

Now we give some remarks on this result. Firstly, here we deal with the Hamiltonian system and have to find the symplectic transformation, which is different from that in [20, 23, 25]. Secondly, compared with [26], we can allow the matrix $A$ to have multiple eigenvalues. Of course, if the eigenvalues of $A$ are different, the nondegeneracy condition holds naturally, then our result is just the same as in [26].

## 2. Some Lemmas

We need some lemmas which are provided in this section for the proof of Theorem 1.

Lemma 4. Let $Q(t)=\sum_{k \in \mathbb{Z}^{r}} Q_{k} e^{\langle k, \omega\rangle \sqrt{-1} t}$ be analytic quasiperiodic on $D_{\rho}$ with frequencies $\omega$. Let $\widetilde{Q}(t)=Q(t)-\bar{Q}$,

$$
\begin{equation*}
Q^{\geq M}(t)=\sum_{k \in \mathbb{Z}^{r},|k| \geq M} Q_{k} e^{\langle k, \omega\rangle \sqrt{-1} t}, \tag{17}
\end{equation*}
$$

and $\widetilde{Q}^{M}=\widetilde{Q}-Q^{\geq M}$, where $M>0$. Then we have the following results:
(1) $\|\bar{R}\|,\|\widetilde{Q}\|_{\rho},\left\|\widetilde{Q}^{M}\right\|_{\rho} \leq\|Q\|_{\rho}$.
(2) $\left\|Q^{\geq M}\right\|_{\rho-s} \leq\|Q\|_{\rho} e^{-M s}, \forall s \in(0, \rho]$.

This lemma can be seen in [25].
The next lemma will be used to show the convergence.
Lemma 5. Let $\left(q_{m}\right)_{m},\left(a_{m}\right)_{m}$, and $\left(r_{m}\right)_{m}$ be sequences defined by

$$
\begin{align*}
& q_{m+1}=q_{m}^{2} \\
& a_{m+1}=a_{m}+q_{m+1}  \tag{18}\\
& r_{m+1}=\frac{2+q_{m}}{2-q_{m}} r_{m}+q_{m+1}
\end{align*}
$$

with initial values $q_{1}=a_{1}=r_{1}=e^{-1}$. Then $\left(q_{m}\right)_{m}$ is decreasing to zero and $\left(a_{m}\right)_{m},\left(r_{m}\right)_{m}$ are increasing and convergent to some values $a_{\infty}$ and $r_{\infty}$, respectively, with $a_{\infty}<1 /(e-1), r_{\infty}<$ $e /(e-1)$.

The proof of this lemma can be found in [25].
Lemma 6. Let $D$ be an $n \times n$ diagonal matrix with different eigenvalues $\lambda_{1}, \ldots, \lambda_{n}$, and $\delta=\min _{i \neq j}\left(\left|\lambda_{i}-\lambda_{j}\right|\right)$. Then if $A$ verifies $\|A-D\| \leq b \leq \delta /(3 n-1)$, the following conditions hold:
(1) A has $n$ different eigenvalues $\mu_{1}, \ldots, \mu_{n}$ and $\left|\lambda_{j}-\mu_{j}\right| \leq$ $b, j=1, \ldots, n$.
(2) There exists a regular matrix $S$ such that $S^{-1} A S=D^{*}=$ $\operatorname{diag}\left(\mu_{1}, \ldots, \mu_{n}\right)$ satisfying $C(S) \leq 2$.

This lemma can be seen in [20].

## 3. Proof of Theorem 1

By the assumptions of Theorem $1, A+\varepsilon \bar{Q}$ has $n$ different eigenvalues $\mu_{1}, \cdots, \mu_{n}$, then there exists a symplectic matrix $S$ such that

$$
\begin{equation*}
S^{-1}(A+\varepsilon \bar{Q}) S=D=\operatorname{diag}\left(\mu_{1}, \ldots, \mu_{n}\right) \tag{19}
\end{equation*}
$$

Under the change of variables $x=S x_{1}$, system (11) is changed into

$$
\begin{equation*}
\dot{x}_{1}=\left(D+\varepsilon S^{-1}(Q(t)-\bar{Q}) S\right) x_{1}=(D+\varepsilon \widetilde{Q}(t)) x_{1} \tag{20}
\end{equation*}
$$

where $\widetilde{Q}(t)=S^{-1}(Q(t)-\bar{Q}) S$; it is easy to see that $\overline{\widetilde{Q}}=0$.
Now we can consider the iteration step.
In the $m$-th step, we consider the system

$$
\begin{equation*}
\dot{x}_{m}=\left(A_{m}(\varepsilon)+\varepsilon Q_{m}(t)+\varepsilon R_{m}(t)\right) x_{m}, \quad m \geq 1 \tag{21}
\end{equation*}
$$

where $A_{1}=D, Q_{1}=\widetilde{Q}^{M}, R_{1}=\widetilde{Q}^{\geq M}$ are Hamiltonian. Suppose $A_{m}, Q_{m}$, and $R_{m}$ are Hamiltonian. Assume

$$
\begin{gather*}
\left\|A_{m}-D\right\| \leq q^{*} a_{m} \varepsilon^{3 / 2} \\
\left\|Q_{m}\right\|_{\rho} \leq q^{*} q_{m}  \tag{22}\\
\left\|R_{m}\right\|_{\rho-s} \leq q^{*} r_{m} e^{-M(\varepsilon) s}
\end{gather*}
$$

where $q^{*}=\beta e q, s \in(0, \rho], a_{m}, q_{m}$, and $r_{m}$ are defined in Lemma 5.

Let the change of variables be $x_{m}=e^{\varepsilon P_{m}} x_{m+1}$; under this symplectic transformation, system (21) is changed to

$$
\begin{align*}
& \dot{x}_{m+1}=\left(e^{-\varepsilon P_{m}}\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right) e^{\varepsilon P_{m}}\right. \\
& \left.\quad+e^{-\varepsilon P_{m}}\left(\varepsilon \dot{P}_{m} e^{\varepsilon P_{m}}-\frac{d}{d t}\left(e^{\varepsilon P_{m}}\right)\right)+\varepsilon e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}\right) \\
& \quad \cdot x_{m+1}=\left(\left(I-\varepsilon P_{m}+\widetilde{B}_{m}\right)\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right)\right.  \tag{23}\\
& \quad \cdot\left(I+\varepsilon P_{m}+B_{m}\right)+e^{-\varepsilon P_{m}}\left(\varepsilon \dot{P}_{m} e^{\varepsilon P_{m}}-\frac{d}{d t}\left(e^{\varepsilon P_{m}}\right)\right) \\
& \left.\quad+\varepsilon e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}\right) x_{m+1}=\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right. \\
& \left.\quad+\varepsilon A_{m} P_{m}-\varepsilon P_{m} A_{m}+\varepsilon Q_{m}^{*}+\varepsilon e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}\right) x_{m+1},
\end{align*}
$$

where

$$
\begin{aligned}
\varepsilon Q_{m}^{*}= & -\varepsilon^{2} P_{m}\left(Q_{m}-\dot{P}_{m}\right)+\varepsilon^{2}\left(Q_{m}-\dot{P}_{m}\right) P_{m} \\
& -\varepsilon^{2} P_{m}\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right) P_{m} \\
& -\varepsilon P_{m}\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right) B_{m} \\
& +\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right) B_{m}
\end{aligned}
$$

$$
\begin{align*}
& +\widetilde{B}_{m}\left(A_{m}+\varepsilon Q_{m}-\varepsilon \dot{P}_{m}\right) e^{\varepsilon P_{m}} \\
& +e^{-\varepsilon P_{m}}\left(\varepsilon \dot{P}_{m} e^{\varepsilon P_{m}}-\frac{d}{d t} e^{\varepsilon P_{m}}\right), \\
e^{\varepsilon P_{m}}= & I+\varepsilon P_{m}+B_{m}, \\
e^{-\varepsilon P_{m}}= & I-\varepsilon P_{m}+\widetilde{B}_{m}, \tag{24}
\end{align*}
$$

and

$$
\begin{align*}
& B_{m}=\frac{\left(\varepsilon P_{m}\right)^{2}}{2!}+\frac{\left(\varepsilon P_{m}\right)^{3}}{3!}+\cdots \\
& \widetilde{B}_{m}=\frac{\left(\varepsilon P_{m}\right)^{2}}{2!}-\frac{\left(\varepsilon P_{m}\right)^{3}}{3!}+\cdots \tag{25}
\end{align*}
$$

We would like to have

$$
\begin{equation*}
Q_{m}+A_{m} P_{m}-\dot{P}_{m}-P_{m} A_{m}=0 \tag{26}
\end{equation*}
$$

and this is equivalent to

$$
\begin{equation*}
\dot{P}_{m}=A_{m} P_{m}-P_{m} A_{m}+Q_{m} . \tag{27}
\end{equation*}
$$

Now we want to solve (27) to obtain an analytic quasiperiodic Hamiltonian solution $P_{m}(t)$ on $D_{\rho}$ with the frequencies $\omega$.

From (22), it follows that

$$
\begin{equation*}
\left\|A_{m}-D\right\| \leq q^{*} a_{m} \varepsilon^{3 / 2} \leq \frac{\delta \varepsilon}{3 n-1}, \quad \varepsilon \in\left(0, \varepsilon^{*}\right) . \tag{28}
\end{equation*}
$$

Thus by Lemma 6, $A_{m}$ has $n$ different eigenvalues $\lambda_{1}^{m}, \ldots, \lambda_{n}^{m}$ and

$$
\begin{equation*}
\left|\lambda_{i}^{m}-\mu_{i}\right| \leq q^{*} a_{m} \varepsilon^{3 / 2} . \tag{29}
\end{equation*}
$$

Since $A_{m}$ is Hamiltonian, from the discussion in Section 15 of [17], it follows that there exists a symplectic matrix $S_{m}$ such that

$$
\begin{equation*}
S_{m}^{-1} A_{m} S_{m}=D_{m}=\operatorname{diag}\left(\lambda_{1}^{m}, \ldots, \lambda_{n}^{m}\right) ; \tag{30}
\end{equation*}
$$

moreover, $C\left(S_{m}\right) \leq 2$, where we let $D_{1}=D, \lambda_{i}^{1}=\mu_{i}(i=$ $1,2, \ldots, n)$.

If

$$
\begin{equation*}
\left|\langle k, \omega\rangle \sqrt{-1}-\lambda_{i}^{m}+\lambda_{j}^{m}\right| \geq L \varepsilon^{1 / 2} \tag{31}
\end{equation*}
$$

for $0<|k|<M, 1 \leq i, j \leq n$, where $L>0, M>0$ are constants.

Making the change of variable $P_{m}=S_{m} X_{m} S_{m}^{-1}$ and defining $Y_{m}=S_{m}^{-1} Q_{m} S_{m}$, (27) becomes

$$
\begin{equation*}
\dot{X}_{m}=D_{m} X_{m}-X_{m} D_{m}+Y_{m}, \quad \bar{Y}_{m}=0 . \tag{32}
\end{equation*}
$$

Expand $X_{m}$ and $Y_{m}$ into Fourier series

$$
\begin{align*}
& X_{m}(t)=\sum_{k \in \mathbb{Z}^{r}, 0<|k|<M} x_{m}^{k} e^{\langle k, \omega\rangle \sqrt{-1} t}, \\
& Y_{m}(t)=\sum_{k \in \mathbb{Z}^{r}, 0<|k|<M} y_{m}^{k} e^{\langle k, \omega\rangle \sqrt{-1} t}, \tag{33}
\end{align*}
$$

where $x_{m}^{k}=\left(x_{i j, m}^{k}\right)_{1 \leq i, j \leq n}$ and $y_{m}^{k}=\left(y_{i j, m}^{k}\right)_{1 \leq i, j \leq n}$.

Thus the coefficients must be

$$
\begin{equation*}
x_{i j, m}^{k}=\frac{y_{i j, m}^{k}}{\langle k, \omega\rangle \sqrt{-1}-\lambda_{i}^{m}+\lambda_{j}^{m}} . \tag{34}
\end{equation*}
$$

By (31), we have

$$
\begin{align*}
\left\|X_{m}\right\|_{\rho} & \leq\left(L \varepsilon^{1 / 2}\right)^{-1}\left\|Y_{m}\right\|_{\rho} \leq\left(L \varepsilon^{1 / 2}\right)^{-1} C\left(S_{m}\right)\left\|Q_{m}\right\|_{\rho}  \tag{35}\\
& \leq 2\left(L \varepsilon^{1 / 2}\right)^{-1}\left\|Q_{m}\right\|_{\rho}
\end{align*}
$$

which implies

$$
\begin{equation*}
\left\|P_{m}\right\|_{\rho} \leq C\left(S_{m}\right)\left\|X_{m}\right\|_{\rho} \leq 4\left(L \varepsilon^{1 / 2}\right)^{-1}\left\|Q_{m}\right\|_{\rho} \tag{36}
\end{equation*}
$$

Now we prove that $P_{m}$ is Hamiltonian. To this end, we only need to prove that $X_{m}$ is Hamiltonian. Since $D_{m}$ and $Y_{m}$ are Hamiltonian, then $D_{m}=J D_{m J}$ and $Y_{m}=J Y_{m J}$, where $D_{m J}$ and $Y_{m J}$ are symmetric. Let $X_{m J}=J^{-1} X_{m}$, if $X_{m J}$ is symmetric, then $X_{m}$ is Hamiltonian. Below we prove that $X_{m J}$ is symmetric. Substituting $X_{m}=J X_{m J}$ into (32) yields that

$$
\begin{equation*}
\dot{X}_{m J}=D_{m J} J X_{m J}-X_{m J} J D_{m J}+Y_{m J}, \tag{37}
\end{equation*}
$$

and transposing (37), we get

$$
\begin{equation*}
\dot{X}_{m J}^{T}=D_{m J} J X_{m J}^{T}-X_{m J}^{T} J D_{m J}+Y_{m J} . \tag{38}
\end{equation*}
$$

It is easy to see that $J X_{m J}$ and $J X_{m J}^{T}$ are solutions of (32); moreover, $\overline{J X_{m J}}=\overline{J X_{m J}^{T}}=0$. Since the solution of (32) with $\bar{X}_{m}=0$ is unique, we have that $J X_{m J}=J X_{m J}^{T}$, which implies that $X_{m}$ is Hamiltonian. Since $S_{m}$ is symplectic, it is easy to see that $P_{m}=S_{m} X_{m} S_{m}^{-1}$ is Hamiltonian.

Thus, under the symplectic transformation $x_{m}=$ $e^{\varepsilon P_{m}} x_{m+1}$, system (21) is changed into the system

$$
\begin{equation*}
\dot{x}_{m+1}=\left(A_{m}+\varepsilon Q_{m}^{*}+\varepsilon e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}\right) x_{m+1}, \tag{39}
\end{equation*}
$$

where

$$
\begin{align*}
\varepsilon Q_{m}^{*}= & -\varepsilon^{2} P_{m}\left(-A_{m} P_{m}+P_{m} A_{m}\right) \\
& +\varepsilon^{2}\left(-A_{m} P_{m}+P_{m} A_{m}\right) P_{m} \\
& -\varepsilon^{2} P_{m}\left(A_{m}+\varepsilon P_{m} A_{m}-\varepsilon A_{m} P_{m}\right) P_{m} \\
& -\varepsilon P_{m}\left(A_{m}+\varepsilon P_{m} A_{m}-\varepsilon A_{m} P_{m}\right) B_{m}  \tag{40}\\
& +\left(A_{m}+\varepsilon P_{m} A_{m}-\varepsilon A_{m} P_{m}\right) B_{m} \\
& +\widetilde{B}_{m}\left(A_{m}+\varepsilon P_{m} A_{m}-\varepsilon A_{m} P_{m}\right) e^{\varepsilon P_{m}} \\
& +e^{-\varepsilon P_{m}}\left(\varepsilon \dot{P}_{m} e^{\varepsilon P_{m}}-\frac{d}{d t} e^{\varepsilon P_{m}}\right) .
\end{align*}
$$

System (39) can be written in the following system:

$$
\begin{equation*}
\dot{x}_{m+1}=\left(A_{m+1}+\varepsilon Q_{m+1}+\varepsilon e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}\right) x_{m+1}, \tag{41}
\end{equation*}
$$

where $A_{m+1}=A_{m}+\varepsilon \bar{Q}_{m}^{*}, \quad Q_{m+1}=\left(Q_{m}^{*}-\bar{Q}_{m}^{*}\right)^{M}$, and $R_{m+1}=$ $\left(Q_{m}^{*}-\bar{Q}_{m}^{*}\right)^{\geq M}+e^{-\varepsilon P_{m}} R_{m} e^{\varepsilon P_{m}}$ are Hamiltonian and analytic quasi-periodic on $D_{\rho}$ with the frequencies $\omega$.

Now we prove the convergence of the iteration as $m \longrightarrow$ $\infty$.

We first prove (22) holds by mathematical induction. By Lemma 4, it is easy to verify that

$$
\begin{gather*}
\left\|A_{1}-D\right\|=0 \leq q^{*} a_{1} \varepsilon^{3 / 2}, \\
\left\|Q_{1}\right\|_{\rho} \leq q^{*} q_{1}  \tag{42}\\
\left\|R_{1}\right\|_{\rho} \leq q^{*} r_{1} e^{-M(\varepsilon) s},
\end{gather*}
$$

where $a_{1}=q_{1}=r_{1}=e^{-1}, \beta=C(S), q^{*}=\beta e q, s \in(0, \rho]$, $\varepsilon \in\left(0, \varepsilon^{*}\right)$.

Assume that (22) holds at the $m$-th step. By (22) and (36), we have

$$
\begin{equation*}
\left\|P_{m}\right\|_{\rho} \leq 4\left(L \varepsilon^{1 / 2}\right)^{-1}\left\|Q_{m}\right\|_{\rho} \leq 4\left(L \varepsilon^{1 / 2}\right)^{-1} q^{*} q_{m} \tag{43}
\end{equation*}
$$

Hence

$$
\begin{equation*}
\left\|\varepsilon^{1 / 2} P_{m}\right\|_{\rho} \leq \frac{q_{m}}{2}<\frac{1}{2}, \quad \varepsilon \in\left(0, \varepsilon^{*}\right) \tag{44}
\end{equation*}
$$

where $L=8 q^{*}$ is a constant. It is easy to see that

$$
\begin{equation*}
\left\|\varepsilon P_{m}\right\|_{\rho} \leq\left\|\varepsilon^{1 / 2} P_{m}\right\|_{\rho} \leq \frac{q_{m}}{2}<\frac{1}{2}, \quad \varepsilon \in\left(0, \varepsilon^{*}\right) . \tag{45}
\end{equation*}
$$

Thus

$$
\begin{equation*}
\left\|e^{ \pm \varepsilon P_{m}}\right\|_{\rho}<2 \tag{46}
\end{equation*}
$$

From (22), (44), (46), Lemmas 4 and 5, it follows that

$$
\begin{equation*}
\left\|Q_{m}^{*}\right\|_{\rho} \leq c \varepsilon^{1 / 2} q^{*} q_{m}^{2} \varepsilon^{1 / 2} \leq q^{*} q_{m}^{2} \varepsilon^{1 / 2}=q^{*} q_{m+1} \varepsilon^{1 / 2} \tag{47}
\end{equation*}
$$

where $c$ is a positive constant,

$$
\begin{align*}
\left\|A_{m+1}-D\right\| & \leq\left\|A_{m}-D\right\|+\left\|\varepsilon Q_{m}^{*}\right\|_{\rho} \\
& \leq q^{*}\left(a_{m}+q_{m+1}\right) \varepsilon^{3 / 2}=q^{*} a_{m+1} \varepsilon^{3 / 2} \\
\left\|Q_{m+1}\right\|_{\rho} & \leq\left\|Q_{m}^{*}\right\|_{\rho} \leq q^{*} q_{m+1} \varepsilon^{1 / 2} \leq q^{*} q_{m+1} \\
\left\|R_{m+1}\right\|_{\rho-s} & \leq\left\|\left(Q_{m}^{*}\right)^{\geq M}\right\|_{\rho-s}+\frac{1+\left\|\varepsilon P_{m}\right\|}{1-\left\|\varepsilon P_{m}\right\|}\left\|R_{m}\right\|_{\rho-s}  \tag{48}\\
& \leq\left(q_{m+1}+\frac{2+q_{m}}{2-q_{m}} r_{m}\right) q^{*} e^{-M(\varepsilon) s} \\
& =q^{*} r_{m+1} e^{-M(\varepsilon) s}, \quad s \in(0, \rho]
\end{align*}
$$

By the mathematical induction, then (22) holds.

Below we prove (31) holds. If $k \in \mathbb{Z}^{r}$ and $0<|k|<M(\varepsilon)=$ $\left(d / \varepsilon^{1 / 2}\right)^{1 / \tau}$, from the nonresonant conditions of Theorem 1 and (29), it follows that

$$
\begin{align*}
& \left|\langle k, \omega\rangle \sqrt{-1}-\lambda_{i}^{m}+\lambda_{j}^{m}\right| \\
& \quad \geq\left|\langle k, \omega\rangle \sqrt{-1}-\lambda_{i}+\lambda_{j}\right|-\left|\lambda_{i}^{m}-\mu_{i}\right|-\left|\lambda_{j}^{m}-\mu_{j}\right| \\
& \quad-\left|\mu_{i}-\lambda_{i}\right|-\left|\mu_{j}-\lambda_{j}\right| \\
& \geq \frac{\alpha}{|k|^{\tau}}-2 q^{*} a_{m} \varepsilon^{3 / 2}-2 q \beta \varepsilon  \tag{49}\\
& \geq \frac{\alpha}{|k|^{\tau}}-2 q^{*} a_{\infty} \varepsilon^{3 / 2}-2 q \beta \varepsilon \\
& \geq \\
& \geq \frac{\alpha}{|k|^{\tau}}-2 q^{*} a_{\infty} \varepsilon^{1 / 2}-2 q \varepsilon^{1 / 2}>\left(\frac{\alpha}{d}-4 q^{*}\right) \varepsilon^{1 / 2} \\
& \quad=L \varepsilon^{1 / 2}
\end{align*}
$$

where $d=\alpha / 12 q^{*}$ and $L=8 q^{*}$. So for any $m \geq 1$, (31) holds.
Consequently, the iterative process can be carried out. The composition of all of the changes $e^{\varepsilon P_{m}}$ is convergent because $\left\|e^{\varepsilon P_{m}}\right\|_{\rho} \leq 1+q_{m}$. That is, there exists an analytic quasiperiodic function $\varphi(t, \varepsilon)$ on $D_{\rho}$ with the frequencies $\omega$, such that the composition of all of the changes $e^{\varepsilon P_{m}}$ converges to $\varphi(t, \varepsilon)$ as $m \longrightarrow \infty$.

From (22) and Lemma 5, it follows that

$$
\begin{equation*}
\lim _{m \rightarrow \infty} Q_{m}=0 . \tag{50}
\end{equation*}
$$

By (22) and (41), we have

$$
\begin{equation*}
\left\|A_{m+1}-A_{m}\right\| \leq \varepsilon\left\|\bar{Q}_{m}^{*}\right\| \leq \varepsilon\left\|Q_{m}^{*}\right\|_{\rho} \leq q^{*} q_{m+1} \varepsilon^{3 / 2} \tag{51}
\end{equation*}
$$

Hence, according to Lemma 5, $A_{m}$ and $R_{m}$ are convergent as $m \longrightarrow \infty$. Let

$$
\begin{align*}
& \lim _{m \rightarrow \infty} A_{m}=A_{\infty},  \tag{52}\\
& \lim _{m \rightarrow \infty} R_{m}=R_{\infty} .
\end{align*}
$$

Then the final equation is

$$
\begin{equation*}
\dot{x}_{\infty}=\left(A_{\infty}(\varepsilon)+\varepsilon R_{\infty}(t, \varepsilon)\right) x_{\infty}, \quad \varepsilon \in\left(0, \varepsilon^{*}\right) . \tag{53}
\end{equation*}
$$

By (22) and Lemma 5, we have

$$
\begin{equation*}
\left\|A_{\infty}(\varepsilon)-D\right\| \leq q^{*} a_{\infty} \varepsilon^{3 / 2} \leq \frac{e \beta q}{e-1} \varepsilon^{3 / 2} \tag{54}
\end{equation*}
$$

and

$$
\begin{align*}
\left\|R_{\infty}(t, \varepsilon)\right\|_{\rho-s} & \leq q^{*} r_{\infty} e^{-M(\varepsilon) s} \\
& \leq \frac{e^{2} \beta q}{e-1} \exp \left(-\left(\frac{d}{\varepsilon^{1 / 2}}\right)^{1 / \tau} s\right), \tag{55}
\end{align*}
$$

$$
s \in(0, \rho]
$$

where $d=\alpha / 12 \beta$ eq.

Under the change of variables $x_{\infty}=S^{-1} y$, system (53) is changed into (13) with

$$
\begin{align*}
& A^{*}=S A_{\infty} S^{-1} \\
& R^{*}=S R_{\infty} S^{-1} \tag{56}
\end{align*}
$$

Moreover,

$$
\begin{align*}
&\left\|A^{*}-A\right\|=\left\|A^{*}-(A+\varepsilon \bar{Q})+\varepsilon \bar{Q}\right\| \\
& \leq\left\|S A_{\infty} S^{-1}-S D S^{-1}\right\|+\|\varepsilon Q\|_{\rho} \\
& \leq \beta \frac{e \beta q}{e-1} \varepsilon^{3 / 2}+\varepsilon q \leq \frac{e(\beta+1) q}{e-1} \varepsilon, \\
& \quad \varepsilon \in\left(0, \varepsilon^{*}\right) .  \tag{57}\\
&\left\|R^{*}\right\|_{\rho-s} \leq \frac{e^{2} \beta^{2} q}{e-1} \exp \left(-\left(\frac{d}{\varepsilon^{1 / 2}}\right)^{1 / \tau} s\right), \\
& s \in(0, \rho], \quad \varepsilon \in\left(0, \varepsilon^{*}\right),
\end{align*}
$$

where $d=\alpha / 12 e \beta q$.
Thus, under the symplectic transformation $x=\varphi(t$, $\varepsilon) S^{-1} y=\psi(t, \varepsilon) y$, Hamiltonian system (11) is changed into Hamiltonian system (13). Therefore, Theorem 1 is proved completely.

## Data Availability

There is no additional data in the manuscript, because the main result is theoretical proof.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# A Note on the Fractional Generalized Higher Order KdV Equation 

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We obtain exact solutions to the fractional generalized higher order Korteweg-de Vries (KdV) equation using the complex method. It has showed that the applied method is very useful and is practically well suited for the nonlinear differential equations, those arising in mathematical physics.

## 1. Introduction

Nonlinear fractional differential equations (NFDEs) are universally applied in signal processing, electrical networks, acoustics, fluid dynamics, biology, chemistry, physics, etc. For example, the singular behaviours [1-9] and impulsive phenomena [10-19] often exhibit some blow-up properties [20-25] which occur in a lot of complex physical processes. NFDEs have been attracted extensive attention and have been widely investigated [26-38]. Exact solutions of NFDEs play an important role in the study of mathematical physics phenomena. Therefore, seeking exact solutions of NFDEs is an interesting and significant subject.

The fractional generalized higher order KdV equation is a useful model. Applying the generalized $\exp (-\Phi(\zeta))-$ expansion method, Lu et al. [39] obtained exact solutions of this equation. In this article, we would like to utilize the complex method [40-43] to seek exact solutions to the fractional generalized higher order KdV equation.

## 2. Preliminaries

Let $f: R \longrightarrow R, x \longrightarrow f(x)$ be a continuous function and $\nu>0$ denote a constant discrete span. Define the operator $F W(\nu)$ as follows:
then the fractional difference of $f(x)$ of order $\mu$ can be expressed as

$$
\begin{align*}
\Delta^{\mu} f(x) & =(F W-1)^{\mu} f(x) \\
& =\sum_{n=0}^{\infty}(-1)^{n}\binom{\mu}{n} f[x+(\mu-n) \nu], \tag{2}
\end{align*}
$$

where $0<\mu \leq 1$, and its fractional derivative of order $\mu$ can be expressed as

$$
\begin{equation*}
f^{(\mu)}(x)=\lim _{\nu \rightarrow 0}\left(\frac{\sum_{n=0}^{\infty}(-1)^{n}\binom{\mu}{n} f[x+(\mu-n) \nu]}{\nu^{\mu}}\right) \tag{3}
\end{equation*}
$$

The above is expressed as

$$
\begin{array}{r}
\frac{1}{\Gamma(-\mu)} \int_{0}^{x}(x-z)^{-\mu-1} f(z) d z, \quad \mu<0 \\
\frac{1}{\Gamma(1-\mu)} \frac{d}{d_{x}} \int_{0}^{x}(x-z)^{-\mu}[f(z)-f(0)] d z \tag{4}
\end{array}
$$

$$
0<\mu<1,
$$

$$
\left(f^{(\mu-n)}(x)\right)^{(n)}, \quad n \leq \mu \leq n+1, n \geq 1
$$

Further, Jumarie's modified Riemann-Liouville derivative $[44,45]$ is given by

$$
\begin{equation*}
D_{x}^{\mu} x^{\gamma}=\frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\mu)} x^{(\gamma-\mu)}, \quad \gamma>0 \tag{5}
\end{equation*}
$$

then its related NFDE is given by

$$
\begin{align*}
& f\left(u, u_{t}, u_{x}, D_{t}^{\mu} u, D_{x}^{\mu} u, D_{t}^{\gamma} u, D_{x}^{\gamma} u, \ldots\right)=0,  \tag{6}\\
& \quad 0<\mu \leq 1 .
\end{align*}
$$

Let $m \in \mathbb{N}:=\{1,2,3, \ldots\}, r_{j} \in\{0,1,2, \ldots\}, j=0,1, \ldots, m$, $r=\left(r_{0}, r_{1}, \ldots, r_{m}\right)$, and

$$
\begin{equation*}
K_{r}[w](z):=\prod_{j=0}^{m}\left[w^{(j)}(z)\right]^{r_{j}} \tag{7}
\end{equation*}
$$

then the degree of $K_{r}[w]$ is denoted by $d(r):=\sum_{j=0}^{m} r_{j}$. We define the differential polynomial as

$$
\begin{equation*}
F\left(w, w^{\prime}, \ldots, w^{(m)}\right):=\sum_{r \in H} a_{r} K_{r}[w], \tag{8}
\end{equation*}
$$

in which $H$ is a finite index set and $a_{r}$ are constants. The degree of $F\left(w, w^{\prime}, \ldots, w^{(m)}\right)$ can be denoted by $\operatorname{deg} F\left(w, w^{\prime}, \ldots, w^{(m)}\right):=\max _{r \in H}\{d(r)\}$.

The ordinary differential equation (ODE) is given by

$$
\begin{equation*}
F\left(w, w^{\prime}, \ldots, w^{(m)}\right)=c w^{n}+d, \tag{9}
\end{equation*}
$$

where $c \neq 0, d$ are constants, $n \in \mathbb{N}$.
Suppose that the meromorphic solutions $w$ of (9) have at least one pole. Let $p, q \in \mathbb{N}$ and insert the Laurent series

$$
\begin{equation*}
w(z)=\sum_{\tau=-q}^{\infty} \beta_{\tau} z^{\tau}, \quad \beta_{-q} \neq 0, q>0 \tag{10}
\end{equation*}
$$

into (9); if it is determined $p$ different Laurent singular parts:

$$
\begin{equation*}
\sum_{\tau=-q}^{-1} \beta_{\tau} z^{\tau} \tag{11}
\end{equation*}
$$

then (9) is said to satisfy the weak $\langle p, q\rangle$ condition.
Give two complex numbers $v_{1}, v_{2}$ such that $\operatorname{Im}\left(v_{1} / \nu_{2}\right)>$ 0 , let $L$ be the discrete subset $L\left[2 \nu_{1}, 2 \nu_{2}\right]:=\left\{\nu \mid \nu=2 c_{1} \nu_{1}+\right.$ $\left.2 c_{2} v_{2}, c_{1}, c_{2} \in \mathbb{Z}\right\}$, and L is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. Let the discriminant $\Delta=\Delta\left(b_{1}, b_{2}\right):=b_{1}^{3}-27 b_{2}^{2}$ and

$$
\begin{equation*}
h_{n}:=h_{n}(L):=\sum_{\nu \in L \backslash\{0\}} \frac{1}{v^{n}} . \tag{12}
\end{equation*}
$$

A meromorphic function $\wp(z):=\wp\left(z, g_{2}, g_{3}\right)$ with double periods $2 v_{1}, 2 v_{2}$, which satisfies the equation

$$
\begin{equation*}
\left(\wp^{\prime}(z)\right)^{2}=4 \wp^{3}(z)-g_{2} \wp(z)-g_{3}, \tag{13}
\end{equation*}
$$

in which $\Delta\left(g_{2}, g_{3}\right) \neq 0, g_{2}=60 h_{4}$, and $g_{3}=140 h_{6}$, is called the Weierstrass elliptic function.

If a meromorphic function $\varsigma$ is a rational function of $z$, an elliptic function, or a rational function of $e^{\alpha z}, \alpha \in \mathbb{C}$, then we say that $\varsigma$ belongs to the class $W$.

In 2009, Eremenko et al. [46] considered the following $m$-order Briot-Bouquet equation (BBEq):

$$
\begin{equation*}
F\left(w, w^{(m)}\right)=\sum_{j=0}^{n} F_{j}(w)\left(w^{(m)}\right)^{j}=0, \tag{14}
\end{equation*}
$$

where $m \in \mathbb{N}$ and $F_{j}(w)$ are constant coefficient polynomials. For the $m$-order BBEq, we have the lemma as follows.

Lemma 1 (see [41, 47, 48]). Let $m, s, n, p \in \mathbb{N}$, and $\operatorname{deg} F\left(w, w^{(m)}\right)<n$, and the $m$-order BBEq

$$
\begin{equation*}
F\left(w, w^{(m)}\right)=c w^{n}+d \tag{15}
\end{equation*}
$$

satisfies weak $\langle p, q\rangle$ condition, then the meromorphic solutions $w(z)$ belong to the class $W$. Suppose that for some values of parameters such solutions $w$ exist, then other meromorphic solutions should form one-parametric family $\left(z-z_{0}\right), z_{0} \in \mathbb{C}$. Then each elliptic solution with a pole at $z=0$ can be expressed as

$$
\begin{align*}
w(z)= & \sum_{i=1}^{s-1} \sum_{j=2}^{q} \frac{(-1)^{j} \beta_{-i j}}{(j-1)!} \frac{d^{j-2}}{d z^{j-2}} \\
& \cdot\left(\frac{1}{4}\left[\frac{\wp^{\prime}(z)+C_{i}}{\wp(z)-D_{i}}\right]^{2}-\wp(z)\right)+\sum_{i=1}^{s-1} \frac{\beta_{-i 1}}{2}  \tag{16}\\
& \quad \frac{\wp^{\prime}(z)+C_{i}}{\wp(z)-D_{i}}+\sum_{j=2}^{q} \frac{(-1)^{j} \beta_{-s j}}{(j-1)!} \frac{d^{j-2}}{d z^{j-2}} \wp(z) \\
& +\beta_{0}
\end{align*}
$$

in which $\beta_{-i j}$ are determined by (10), $\sum_{i=1}^{s} \beta_{-i 1}=0$, and $C_{i}^{2}=$ $4 D_{i}^{3}-g_{2} D_{i}-g_{3}$.

Each rational function solution $R(z)$ can be expressed as

$$
\begin{equation*}
R(z)=\sum_{i=1}^{s} \sum_{j=1}^{q} \frac{\beta_{i j}}{\left(z-z_{i}\right)^{j}}+\beta_{0} \tag{17}
\end{equation*}
$$

and it has $s(\leq p)$ distinct poles of multiplicity $q$.
Each simply periodic solution $R(\eta)$ is a rational function of $\eta=e^{\alpha z}(\alpha \in \mathbb{C})$ and can be expressed as

$$
\begin{equation*}
R(\eta)=\sum_{i=1}^{s} \sum_{j=1}^{q} \frac{\beta_{i j}}{\left(\eta-\eta_{i}\right)^{j}}+\beta_{0}, \tag{18}
\end{equation*}
$$

and it has $s(\leq p)$ distinct poles of multiplicity $q$.
Lemma 2 (see [48, 49]). Weierstrass elliptic functions $\wp(z)$ have the following addition formula:

$$
\begin{align*}
\wp\left(z-z_{0}\right)= & -\wp\left(z_{0}\right)-\wp(z) \\
& +\frac{1}{4}\left[\frac{\wp^{\prime}(z)+\wp^{\prime}\left(z_{0}\right)}{\wp(z)-\wp\left(z_{0}\right)}\right]^{2} . \tag{19}
\end{align*}
$$

When $g_{2}=g_{3}=0$, it can be degenerated to rational functions according to

$$
\begin{equation*}
\wp(z, 0,0)=\frac{1}{z^{2}} \tag{20}
\end{equation*}
$$

When $\Delta\left(g_{2}, g_{3}\right)=0$, it can also be degenerated to simple periodic functions according to

$$
\begin{equation*}
\wp\left(z, 3 d^{2},-d^{3}\right)=2 d-\frac{3 d}{2} \operatorname{coth}^{2} \sqrt{\frac{3 d}{2}} z . \tag{21}
\end{equation*}
$$

## 3. Main Results

The fractional generalized higher order KdV equation [39] is given as

$$
\begin{equation*}
D_{t}^{\mu}+u u_{x}-u u_{x x x}+u_{x x x x x}=0 \tag{22}
\end{equation*}
$$

Substituting traveling wave transform

$$
\begin{equation*}
u(x, t)=w(z), \quad z=k\left(x+\frac{\lambda t^{\mu}}{\Gamma(1+\mu)}\right) \tag{23}
\end{equation*}
$$

into (22), we get

$$
\begin{equation*}
k \lambda w^{\prime}+k w w^{\prime}-k^{3} w w^{\prime \prime \prime}+k^{5} w^{(5)}=0 \tag{24}
\end{equation*}
$$

Integrating (24) yields

$$
\begin{equation*}
\lambda w+\frac{w^{2}}{2}+k^{2} w w^{\prime \prime}-\frac{k^{2}}{2}\left(w^{\prime}\right)^{2}+k^{4} w^{(4)}+\delta=0 \tag{25}
\end{equation*}
$$

where $k$ and $\lambda$ are constants and $\delta$ is the integral constant.
Theorem 3. If $k \neq 0$, then the meromorphic solutions $w(z)$ of (25) have the following forms.
(I) The rational function solutions

$$
\begin{equation*}
w_{r}(z)=\frac{-30 k^{2}}{\left(z-z_{0}\right)^{2}}+\frac{5}{2}, \tag{26}
\end{equation*}
$$

where $\lambda=-5 / 2, \delta=25 / 8$, and $z_{0} \in \mathbb{C}$.
(II) The simply periodic solutions

$$
\begin{equation*}
w_{s}(z)=-\frac{15 k^{2} \alpha^{2}}{2} \operatorname{coth}^{2} \frac{\alpha\left(z-z_{0}\right)}{2}+\frac{10 k^{2} \alpha^{2}+5}{2} \tag{27}
\end{equation*}
$$

where $\lambda=\left(3 k^{4} \alpha^{4}-5\right) / 2, \delta=\left(30 k^{6} \alpha^{6}-55 k^{4} \alpha^{4}+25\right) / 8$, and $z_{0} \in \mathbb{C}$.
(III) The elliptic function solutions

$$
\begin{align*}
w_{d}(z)= & -30 k^{2}\left(-\wp(z)+\frac{1}{4}\left(\frac{\wp^{\prime}(z)+D}{\wp(z)-C}\right)^{2}\right)  \tag{28}\\
& +30 k^{2} C+\frac{5}{2}
\end{align*}
$$

where $C^{2}=4 D^{3}-g_{2} D-g_{3}, g_{2}=(2 \lambda+5) / 36 k^{4}$, and $g_{3}=$ $-(100+55 \lambda+12 \delta) / 9720 k^{6}$.

Proof. Substituting (10) into (25) we have $p=1, q=2, \beta_{-2}=$ $-30 k^{2}, \beta_{-1}=0, \beta_{0}=5 / 2, \beta_{1}=0, \beta_{2}=-(2 \lambda+5) / 24 k^{2}$, and $\beta_{3}$ is an arbitrary constant.

Therefore, (25) satisfies the weak $\langle 1,2\rangle$ condition. In the following, we will show the meromorphic solutions of (25).

By (17), we infer that the indeterminant rational solutions of (25) are

$$
\begin{equation*}
R_{1}(z)=\frac{\beta_{11}}{z^{2}}+\frac{\beta_{12}}{z}+\beta_{10} \tag{29}
\end{equation*}
$$

with pole at $z=0$.

Inserting $R_{1}(z)$ into (25), we have

$$
\begin{align*}
& \frac{1}{2} \beta_{10}^{2}+\lambda \beta_{10}+\delta+\frac{\lambda \beta_{11}+\beta_{11} \beta_{10}}{z} \\
& \quad+\frac{2 \beta_{12} \beta_{10}+\beta_{11}^{2}+2 \lambda \beta_{12}}{2 z^{2}} \\
& \quad+\frac{2 k^{2} \beta_{10} \beta_{11}+\beta_{12} \beta_{11}}{z^{3}}  \tag{30}\\
& \quad+\frac{3 k^{2} \beta_{11}^{2}+12 k^{2} \beta_{10} \beta_{12}+\beta_{12}^{2}}{2 z^{4}} \\
& \quad+\frac{24 k^{4} \beta_{11}+6 k^{2} \beta_{11} \beta_{12}}{z^{5}}+\frac{120 k^{4} \beta_{12}+4 k^{2} \beta_{12}^{2}}{z^{6}} \\
& =0
\end{align*}
$$

then we get $\beta_{12}=-30 k^{2}, \beta_{11}=0$, and $\beta_{10}=5 / 2$.
So we can determine that

$$
\begin{equation*}
R_{1}(z)=\frac{-30 k^{2}}{z^{2}}+\frac{5}{2} \tag{31}
\end{equation*}
$$

where $\lambda=-5 / 2$ and $\delta=25 / 8$.
Therefore the rational solutions of (25) are

$$
\begin{equation*}
w_{r}(z)=\frac{-30 k^{2}}{\left(z-z_{0}\right)^{2}}+\frac{5}{2}, \tag{32}
\end{equation*}
$$

where $\lambda=-5 / 2, \delta=25 / 8$, and $z_{0} \in \mathbb{C}$.
Let $\eta=e^{\alpha z}$. To derive simply periodic solutions, we substitute $w=R(\eta)$ into (25) to yield

$$
\begin{gather*}
k^{4} \alpha^{4}\left(R^{(4)} \eta^{4}+6 R^{\prime \prime \prime} \eta^{3}+7 R^{\prime \prime} \eta^{2}+R^{\prime} \eta\right)-\frac{k^{2}}{2}\left(\alpha R^{\prime} \eta\right)^{2} \\
+k^{2} \alpha^{2} R\left(\eta R^{\prime}+\eta^{2} R^{\prime \prime}\right)+\frac{R^{2}}{2}+\lambda R+\delta=0 \tag{33}
\end{gather*}
$$

Substituting

$$
\begin{equation*}
R_{2}(z)=\frac{\beta_{21}}{(\eta-1)^{2}}+\frac{\beta_{22}}{(\eta-1)}+\beta_{20} \tag{34}
\end{equation*}
$$

into (33), we obtain that

$$
\begin{equation*}
R_{2}(z)=-\frac{30 k^{2} \alpha^{2}}{(\eta-1)^{2}}-\frac{30 k^{2} \alpha^{2}}{(\eta-1)}-\frac{5 k^{2} \alpha^{2}}{2}+\frac{5}{2} \tag{35}
\end{equation*}
$$

where $\lambda=\left(3 k^{4} \alpha^{4}-5\right) / 2$ and $\delta=\left(30 k^{6} \alpha^{6}-55 k^{4} \alpha^{4}+25\right) / 8$.
Substituting $\eta=e^{\alpha z}$ into (35), we achieve simply periodic solutions to (25) with pole at $z=0$

$$
\begin{align*}
w_{s 0}(z) & =-\frac{30 k^{2} \alpha^{2}}{\left(e^{\alpha z}-1\right)^{2}}-\frac{30 k^{2} \alpha^{2}}{\left(e^{\alpha z}-1\right)}-\frac{5 k^{2} \alpha^{2}}{2}+\frac{5}{2} \\
& =-30 k^{2} \alpha^{2} \frac{e^{\alpha z}}{\left(e^{\alpha z}-1\right)^{2}}-\frac{5 k^{2} \alpha^{2}}{2}+\frac{5}{2}  \tag{36}\\
& =-\frac{15 k^{2} \alpha^{2}}{2} \operatorname{coth}^{2} \frac{\alpha z}{2}+\frac{10 k^{2} \alpha^{2}+5}{2},
\end{align*}
$$

where $\lambda=\left(3 k^{4} \alpha^{4}-5\right) / 2$ and $\delta=\left(30 k^{6} \alpha^{6}-55 k^{4} \alpha^{4}+25\right) / 8$.

Thurs, simply periodic solutions of (25) are

$$
\begin{equation*}
w_{s}(z)=-\frac{15 k^{2} \alpha^{2}}{2} \operatorname{coth}^{2} \frac{\alpha\left(z-z_{0}\right)}{2}+\frac{10 k^{2} \alpha^{2}+5}{2} \tag{37}
\end{equation*}
$$

where $\lambda=\left(3 k^{4} \alpha^{4}-5\right) / 2, \delta=\left(30 k^{6} \alpha^{6}-55 k^{4} \alpha^{4}+25\right) / 8$, and $z_{0} \in \mathbb{C}$.

From (16) of Lemma 1, the elliptic solutions of (25) are expressed as

$$
\begin{equation*}
w_{d 0}(z)=\beta_{-2} \wp(z)+\beta_{30} \tag{38}
\end{equation*}
$$

with pole at $z=0$.
Putting $w_{d 0}(z)$ into (25), we get

$$
\begin{equation*}
w_{d 0}(z)=-30 k^{2} \wp(z)+\frac{5}{2} \tag{39}
\end{equation*}
$$

where $g_{2}=(2 \lambda+5) / 36 k^{4}$ and $g_{3}=-(100+55 \lambda+12 \delta) / 9720 k^{6}$.
So, the elliptic solutions of (25) are

$$
\begin{equation*}
w_{d}(z)=-30 k^{2} \wp\left(z-z_{0}\right)+\frac{5}{2} \tag{40}
\end{equation*}
$$

where $z_{0} \in \mathbb{C}$.
We can apply the addition formula to rewrite it as

$$
\begin{align*}
w_{d}(z)= & -30 k^{2}\left(-\wp(z)+\frac{1}{4}\left(\frac{\wp^{\prime}(z)+D}{\wp(z)-C}\right)^{2}\right)  \tag{41}\\
& +30 k^{2} C+\frac{5}{2}
\end{align*}
$$

where $C^{2}=4 D^{3}-g_{2} D-g_{3}, g_{2}=(2 \lambda+5) / 36 k^{4}$, and $g_{3}=$ $-(100+55 \lambda+12 \delta) / 9720 k^{6}$.

Substitute traveling wave transform into the meromorphic solutions $w(z)$ of (25) to get traveling wave exact solutions to the fractional generalized higher order KdV equation. So we obtain Theorem 4 as follows.

Theorem 4. If $k \neq 0$, then traveling wave solutions of (25) have the following forms.
(I) The rational function solutions

$$
\begin{equation*}
w_{r}(x, t)=w_{r}\left(k x-\frac{5 k t^{\mu}}{2 \Gamma(1+\mu)}\right) \tag{42}
\end{equation*}
$$

where $\lambda=-5 / 2, \delta=25 / 8$, and $z_{0} \in \mathbb{C}$.
(II) The simply periodic solutions

$$
\begin{equation*}
w_{s}(x, t)=w_{s}\left(k x+\frac{\left(3 k^{5} \alpha^{4}-5 k\right) t^{\mu}}{2 \Gamma(1+\mu)}\right) \tag{43}
\end{equation*}
$$

where $\lambda=\left(3 k^{4} \alpha^{4}-5\right) / 2, \delta=\left(30 k^{6} \alpha^{6}-55 k^{4} \alpha^{4}+25\right) / 8$, and $z_{0} \in \mathbb{C}$.
(III) The elliptic function solutions

$$
\begin{equation*}
w_{d}(x, t)=w_{d}\left(k x+\frac{k \lambda t^{\mu}}{\Gamma(1+\mu)}\right) \tag{44}
\end{equation*}
$$

where $C^{2}=4 D^{3}-g_{2} D-g_{3}, g_{2}=(2 \lambda+5) / 36 k^{4}$, and $g_{3}=$ $-(100+55 \lambda+12 \delta) / 9720 k^{6}$.

## 4. Conclusions

In this note, we have used the complex method to construct exact solutions to the mentioned NFDE. Although we do not show that the meromorphic solutions of the fractional generalized higher order KdV equation belong to the class $W$, we can still obtain the meromorphic solutions to this NFDE and then get its traveling wave exact solutions. The results demonstrate that the applied method is direct and efficient method, which allows us to do tedious and complicated algebraic calculation. We can utilize these ideas to other NFDEs.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# The Eigenvalue Problem for Caputo Type Fractional Differential Equation with Riemann-Stieltjes Integral Boundary Conditions 

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In this paper, we investigate the eigenvalue problem for Caputo fractional differential equation with Riemann-Stieltjes integral boundary conditions ${ }^{c} D_{0+}^{\theta} p(y)+\mu f(t, p(y))=0, y \in[0,1], p(0)=p^{\prime \prime}(0)=0, p(1)=\int_{0}^{1} p(y) d A(y)$, where ${ }^{c} D_{0+}^{\theta}$ is Caputo fractional derivative, $\theta \in(2,3]$, and $f:[0,1] \times[0,+\infty) \longrightarrow[0,+\infty)$ is continuous. By using the Guo-Krasnoselskii's fixed point theorem on cone and the properties of the Green's function, some new results on the existence and nonexistence of positive solutions for the fractional differential equation are obtained.

## 1. Introduction

The experience of the last few years has fully borne out the fact that the integer order calculus is not as widely used as fractional order calculus in some fields such as chemistry, control theory, and signal processing. On the remarkable survey of Agarwal, Benchohra, and Hamani [1] it is pointed out that fractional differential equations constitute a fundamental tool in the modeling of some phenomena (see also [2-4]). The use of fractional order is more accurate for the description of phenomena, so the study of fractional differential equations becomes the mainstream with the help of techniques of nonlinear analysis. We refer the reader to [536] for recent results. For example, in [9], the author studied the following fractional differential equation:

$$
\begin{equation*}
{ }^{c} D^{\alpha} x(y)+f(y, x(y))=0, \quad 0<y<1, \tag{1}
\end{equation*}
$$

with boundary conditions

$$
\begin{align*}
& x^{\prime}(0)=x^{\prime \prime}(0)=0 \\
& x(1)=\mu \int_{0}^{1} x(s) d s \tag{2}
\end{align*}
$$

where $\alpha \in(2,3], \mu \in[0,1)$, and ${ }^{c} D^{\alpha}$ is the Caputo derivative. They solved the above problem by means of classical fixed point theorems.

In [5], the boundary value problem for the following nonlinear fractional differential equation was discussed:

$$
\begin{equation*}
D_{0+}^{\alpha} x(y)+a(y) f\left(y, x(y), x^{\prime \prime}(y)\right)=0 \tag{3}
\end{equation*}
$$

$$
\begin{equation*}
0<y<1, \tag{3}
\end{equation*}
$$

with boundary conditions

$$
\begin{equation*}
x(0)=x^{\prime}(0)=x^{\prime \prime}(0)=x^{\prime \prime}(1)=0, \tag{4}
\end{equation*}
$$

where $D_{0+}^{\alpha}$ is the Riemann-Liouville differentiation, $\alpha \in$ $(3,4]$. By using a fixed point theorem, a new result of the existence of three positive solutions is obtained.

In [15], the authors investigated the following class of BVP:

$$
\begin{equation*}
{ }^{c} D_{0+}^{q} x(y)=f\left(y, x(y),{ }^{c} D_{0+}^{\sigma} x(y)\right), \quad 0<y<1, \tag{5}
\end{equation*}
$$

with boundary conditions

$$
\begin{align*}
x(0) & =x^{\prime \prime}(0)=0, \\
x^{\prime}(1) & =\alpha x^{\prime \prime}(1), \tag{6}
\end{align*}
$$

where $q \in(2,3), \sigma \in(1,2), f:[0,1] \times R \times R \longrightarrow R$ is a given function, and ${ }^{c} D_{0_{+}}^{q}$ denotes the Caputo differentiation. The author investigated this problem by using Banach contraction principle, Leray-Schauder nonlinear alternative, properties of the Green's function, and Guo-Krasnoselskii fixed point theorem on cone. Similar problems can be referred to in [25].

In this paper, we investigate the eigenvalue problem for Caputo fractional boundary value problem with RiemannStieltjes integral boundary conditions

$$
\begin{align*}
{ }^{c} D_{0+}^{\theta} p(y)+\mu f(y, p(y)) & =0, \quad y \in[0,1], \\
p(0) & =p^{\prime \prime}(0)=0,  \tag{7}\\
p(1) & =\int_{0}^{1} p(y) d A(y),
\end{align*}
$$

where $\theta \in(2,3), f:[0,1] \times[0, \infty) \longrightarrow(0, \infty)$ is continuous, $\mu>0$ is a parameter, ${ }^{c} D_{0+}^{\theta}$ is the Caputo fractional derivative, and $A$ is a bounded variation function with positive measures with

$$
\begin{equation*}
B=\int_{0}^{1} y d A(y)<1 \tag{8}
\end{equation*}
$$

Our proof is based on the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem on cone.

## 2. Preliminaries

In order to solve problem (7), we provide the properties related to problem (7).

Definition 1 (see [3]). The Caputo's fractional derivative of order $\theta>0$ for a function $x \in C^{n}[0,+\infty)$ is defined as

$$
\begin{align*}
&{ }^{c} D_{0+}^{\theta} x(y)=\frac{1}{\Gamma(n-\theta)} \int_{0}^{y}(y-s)^{n-\theta-1} x^{(n)}(s) d s  \tag{9}\\
& n-1<\theta<n
\end{align*}
$$

where $n$ is the smallest integer greater than or equal to $\theta$.

Lemma 2 (see [3]). Let $\theta>0$. If we assume $x \in C(0,1) \cap$ $L(0,1)$, then the fractional differential equation

$$
\begin{equation*}
{ }^{c} D_{0+}^{\theta} x(y)=0 \tag{10}
\end{equation*}
$$

has the general solution $x(y)=C_{0}+C_{1} y+\cdots+C_{n-1} y^{n-1}$, $C_{k} \in R, k=0,1 \ldots, n-1$, where $n$ is the smallest integer greater than or equal to $\theta$.

Lemma 3 (see [3]). Given that $x \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\theta$ that belongs to $C(0,1) \cap L(0,1)$. Then

$$
\begin{align*}
& I_{0+}^{\theta}{ }^{c} D_{0+}^{\theta} x(y)=x(y)+C_{0}+C_{1} y+\cdots+C_{n-1} y^{n-1}  \tag{11}\\
& \text { for } C_{k} \in R, k=0,1 \ldots, n-1
\end{align*}
$$

where $n$ is the smallest integer greater than or equal to $\theta$.
Firstly, we consider the following linear Caputo fractional differential equation:

$$
\begin{align*}
{ }^{c} D_{0+}^{\theta} p(y)+\sigma(y) & =0, \quad y \in[0,1] \\
p(0) & =p^{\prime \prime}(0)=0,  \tag{12}\\
p(1) & =\int_{0}^{1} p(y) d A(y) .
\end{align*}
$$

Lemma 4. Let $\theta \in(2,3]$ and assume that $\sigma \in C[0,1]$. Then $p$ is the solution of the above boundary value problem (12), if and only if $p$ satisfies the following integral equation:

$$
\begin{equation*}
p(y)=\int_{0}^{1} G(y, s) \sigma(s) d s \tag{13}
\end{equation*}
$$

where

$$
G(y, s)=\frac{1}{\Gamma(\theta)} \begin{cases}\frac{y}{1-B}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right]-(y-s)^{\theta-1}, & 0 \leq s \leq y \leq 1  \tag{14}\\ \frac{y}{1-B}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right], & 0 \leq y \leq s \leq 1\end{cases}
$$

and

$$
\begin{equation*}
B=\int_{0}^{1} y d A(y)<1 \tag{15}
\end{equation*}
$$

Proof. Applying the fractional integral of order $\theta$ to both sides of (12) for $y \in[0,1]$, we get the following formula:

$$
\begin{equation*}
I_{0+}^{\theta}{ }^{c} D_{0+}^{\theta} p(y)+I_{0+}^{\theta} \sigma(y)=0 \tag{16}
\end{equation*}
$$

According to $p(0)=p^{\prime \prime}(0)=0$ and Lemma 3, we obtain

$$
\begin{equation*}
p(y)=c y-\frac{1}{\Gamma(\theta)} \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s \tag{17}
\end{equation*}
$$

where $c \in R$. Since $p(1)=\int_{0}^{1} p(y) d A(y)$, we deduce that

$$
\begin{align*}
c- & \frac{1}{\Gamma(\theta)} \int_{0}^{1}(1-s)^{\theta-1} \sigma(s) d s \\
& =\int_{0}^{1}\left(c y-\frac{1}{\Gamma(\theta)} \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s\right) d A(y) . \tag{18}
\end{align*}
$$

Therefore,

$$
\begin{align*}
c & =\frac{1}{\Gamma(\theta)(1-B)}\left[\int_{0}^{1}(1-s)^{\theta-1} \sigma(s) d s\right.  \tag{19}\\
& \left.-\int_{0}^{1} \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s d A(y)\right] .
\end{align*}
$$

Substituting the above equality into (17), one has

$$
\begin{aligned}
& p(y)=\frac{y}{\Gamma(\theta)(1-B)}\left[\int_{0}^{1}(1-s)^{\theta-1} \sigma(s) d s\right. \\
& \left.\quad-\int_{0}^{1} \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s d A(y)\right]-\frac{1}{\Gamma(\theta)}
\end{aligned}
$$

$$
G(y, s)=\frac{1}{\Gamma(\theta)} \begin{cases}\frac{y}{1-B}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right]-(y-s)^{\theta-1}, & 0 \leq s \leq y \leq 1  \tag{21}\\ \frac{y}{1-B}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right], & 0 \leq y \leq s \leq 1\end{cases}
$$

The proof is completed.
Lemma 5. The Green's function $G(y, s)$ has the following properties:
(i) $\Gamma(\theta) G(y, s) \leq(1 /(1-B))(1-s)^{\theta-1}$, for $y, s \in[0,1]$;
(ii) $\Gamma(\theta) G(y, s) \geq N(1-s)^{\theta-1}$, for $y \in[1 / 4,3 / 4]$ and $s \in$ $[0,1]$, where

$$
\begin{gather*}
N=\min \left\{\frac{1-\int_{0}^{1} y^{\theta-1} d A(y)}{4(1-B)},\right.  \tag{22}\\
\left.\min _{y \in[1 / 4,3 / 4]} y(1-y)^{\theta-2}\right\}
\end{gather*}
$$

Proof. (i) Obviously, the inequality $\Gamma(\theta) G(y, s) \leq(1 /(1-$ $B)(1-s)^{\theta-1}$ holds from the representation of $G(y, s)$.
(ii) In view of $B=\int_{0}^{1} y d A(y)<1$ and $\theta \in(2,3)$, we have $1-\int_{s}^{1} y^{\theta-1} d A(y)>0$.

For $y \leq s$, we have

$$
\begin{align*}
& \frac{y}{(1-B)}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right] \\
& \geq \frac{y}{(1-B)}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-y s)^{\theta-1} d A(y)\right] \\
& \geq \frac{y}{(1-B)}(1-s)^{\theta-1}\left[1-\int_{s}^{1} y^{\theta-1} d A(y)\right]  \tag{23}\\
& \geq \frac{y\left(1-\int_{0}^{1} y^{\theta-1} d A(y)\right)}{(1-B)}(1-s)^{\theta-1}
\end{align*}
$$

$$
\begin{align*}
& \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s \\
& =\frac{y}{\Gamma(\theta)(1-B)}\left[\int_{0}^{1}(1-s)^{\theta-1} \sigma(s) d s\right. \\
& \left.-\int_{0}^{1} \int_{s}^{1}(y-s)^{\theta-1} d A(y) \sigma(s) d s\right]-\frac{1}{\Gamma(\theta)} \\
& \cdot \int_{0}^{y}(y-s)^{\theta-1} \sigma(s) d s \tag{20}
\end{align*}
$$

and

$$
\begin{aligned}
& \frac{y}{(1-B)}\left[(1-s)^{\theta-1}-\int_{s}^{1}(y-s)^{\theta-1} d A(y)\right]-(y \\
& -s)^{\theta-1} \geq \frac{y}{(1-B)}\left[(1-s)^{\theta-1}\right. \\
& \left.-\int_{s}^{1}(y-y s)^{\theta-1} d A(y)\right]-(y-y s)^{\theta-1} \\
& \quad=\frac{y(1-B+B)}{(1-B)}(1-s)^{\theta-1}-\frac{y}{(1-B)} \\
& \cdot \int_{s}^{1}(y-y s)^{\theta-1} d A(y)-y^{\theta-1}(1-s)^{\theta-1} \\
& =y(1-s)^{\theta-1}+\frac{y \int_{0}^{1} y d A(y)}{(1-B)}(1-s)^{\theta-1} \\
& -\frac{y}{(1-B)} \int_{s}^{1}(y-y s)^{\theta-1} d A(y)-y^{\theta-1}(1-s)^{\theta-1} \\
& =(1-s)^{\theta-1}\left(y-y^{\theta-1}\right) \\
& +\frac{y}{(1-B)}\left((1-s)^{\theta-1} \int_{0}^{1} y d A(y)\right. \\
& \left.-\int_{s}^{1}(y-y s)^{\theta-1} d A(y)\right)=(1-s)^{\theta-1}\left(y-y^{\theta-1}\right) \\
& +\frac{y}{(1-B)}(1-s)^{\theta-1}\left(\int_{0}^{1} y d A(y)\right. \\
& \quad y
\end{aligned}
$$

$$
\begin{align*}
& \left.-\int_{0}^{1} y^{\theta-1} d A(y)\right)=(1-s)^{\theta-1}\left(y-y^{\theta-1}\right) \\
& +\frac{y}{(1-B)}(1-s)^{\theta-1}\left(\int_{0}^{1}\left(y-y^{\theta-1}\right) d A(y)\right) \geq(1 \\
& -s)^{\theta-1}\left(y-y^{\theta-1}\right)=(1-s)^{\theta-1} y\left(1-y^{\theta-2}\right) \tag{24}
\end{align*}
$$

Thus, the above two inequalities yield the inequality in (ii). The proof is completed.

Let $X=C[0,1],\|p\|=\max _{y \in[0,1]}|p(y)|$; then $(X,\|\cdot\|)$ is a Banach space. We define the cone $P \subset X$ by

$$
\begin{equation*}
P=\left\{p \in X: p(y) \geq N(1-B)\|p\|, y \in\left[\frac{1}{4}, \frac{3}{4}\right]\right\} \tag{25}
\end{equation*}
$$

Let $A_{\mu}: X \longrightarrow X$ be the operator defined as

$$
\begin{equation*}
\left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \tag{26}
\end{equation*}
$$

Thus, the fixed point of the above integral equation is equivalent to the solution of the BVP (7).

Lemma 6. $A_{\mu}(P) \subset P$ and $A_{\mu}: P \longrightarrow P$ is a completely continuous operator, where $A_{\mu}$ is defined in (26).

Proof. By Lemma 5, for $\forall p \in P$, we have

$$
\begin{align*}
& \left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \\
& \quad \geq \frac{\mu N}{\Gamma(\theta)} \int_{0}^{1}(1-s)^{\theta-1} f(s, p(s)) d s  \tag{27}\\
& \quad \geq \mu N(1-B) \max _{t \in[0,1]} \int_{0}^{1} G(t, s) f(s, p(s)) d s \\
& \quad=N(1-B)\left\|A_{\mu} p\right\|, \quad y \in\left[\frac{1}{4}, \frac{3}{4}\right]
\end{align*}
$$

Hence we have $A_{\mu}(P) \subset P$. Let $\Omega \subset P$ be bounded. Then there exists a constant $M>0$ such that $\|p\| \leq M$ for $\forall p \in \Omega$. Let $L_{1}=\max _{y \in[0,1], p \in[0, M]}(\mu f(y, p)+1)$. Then

$$
\begin{aligned}
&\left(A_{\mu} p\right)(y) \leq \frac{L_{1}}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1} d s \\
& \\
& y \in[0,1]
\end{aligned}
$$

Thus, $A_{\mu}(\Omega)$ is bounded. Put $p \in \Omega$ and $y_{1}, y_{2} \in[0,1]$. We deduce that

$$
\begin{aligned}
& \left|\left(A_{\mu} p\right)\left(y_{1}\right)-\left(A_{\mu} p\right)\left(y_{2}\right)\right|=\mu \mid \int_{0}^{1} G\left(y_{1}, s\right) \\
& \quad \cdot f(s, p(s)) d s-\int_{0}^{1} G\left(y_{2}, s\right) f(s, p(s)) d s \mid
\end{aligned}
$$

$$
\begin{aligned}
& =\mu \left\lvert\, \int_{0}^{y_{1}}\left[\frac{y_{1}}{1-B}\left((1-s)^{\theta-1}-\int_{s}^{1}\left(y_{1}-s\right)^{\theta-1} d A(y)\right)\right.\right. \\
& \left.-\left(y_{1}-s\right)^{\theta-1}\right] f(s, p(s)) d s+\int_{y_{1}}^{y_{2}} \frac{y_{1}}{1-B}\left[(1-s)^{\theta-1}\right. \\
& \left.-\int_{s}^{1}\left(y_{1}-s\right)^{\theta-1} d A(y)\right] f(s, p(s)) d s \\
& +\int_{y_{2}}^{1} \frac{y_{1}}{1-B}\left[(1-s)^{\theta-1}-\int_{s}^{1}\left(y_{1}-s\right)^{\theta-1} d A(y)\right] \\
& \text { - } f(s, p(s)) d s \\
& -\int_{0}^{y_{1}}\left[\frac{y_{2}}{1-B}\left((1-s)^{\theta-1}-\int_{s}^{1}\left(y_{2}-s\right)^{\theta-1} d A(y)\right)\right. \\
& \left.-\left(y_{2}-s\right)^{\theta-1}\right] f(s, p(s)) d s \\
& -\int_{y_{1}}^{y_{2}}\left[\frac{y_{2}}{1-B}\left((1-s)^{\theta-1}-\int_{s}^{1}\left(y_{2}-s\right)^{\theta-1} d A(y)\right)\right. \\
& \left.-\left(y_{2}-s\right)^{\theta-1}\right] f(s, p(s)) d s-\int_{y_{2}}^{1} \frac{y_{2}}{1-B}\left[(1-s)^{\theta-1}\right. \\
& \left.-\int_{s}^{1}\left(y_{2}-s\right)^{\theta-1} d A(y)\right] f(s, p(s)) d s \mid \\
& =\mu \left\lvert\, \int_{0}^{y_{1}}\left[(1-s)^{\theta-1} \frac{y_{1}-y_{2}}{1-B}\right.\right. \\
& -\int_{s}^{1}\left(\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}-\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right) d A(y) \\
& \left.-\left(y_{1}-s\right)^{\theta-1}+\left(y_{2}-s\right)^{\theta-1}\right] f(s, p(s)) d s \\
& +\int_{y_{1}}^{y_{2}}\left[(1-s)^{\theta-1} \frac{y_{1}-y_{2}}{1-B}\right. \\
& -\int_{s}^{1}\left(\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}-\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right) d A(y) \\
& \left.+\left(y_{2}-s\right)^{\theta-1}\right] f(s, p(s)) d s \\
& +\int_{y_{2}}^{1}\left[(1-s)^{\theta-1} \frac{y_{1}-y_{2}}{1-B}\right. \\
& \left.-\int_{s}^{1}\left(\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}-\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right) d A(y)\right] \\
& \cdot f(s, p(s)) d s \left\lvert\, \geq \frac{\left(y_{1}-y_{2}\right) L_{1}}{1-B} \int_{0}^{y_{1}}(1-s)^{\theta-1} d s\right. \\
& +L_{1} \int_{s}^{1} \int_{0}^{y_{1}}\left[\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right. \\
& \left.-\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}\right] d s d A(y)
\end{aligned}
$$

$$
\begin{align*}
& +L_{1} \int_{0}^{y_{1}}\left[\left(y_{2}-s\right)^{\theta-1}-\left(y_{1}-s\right)^{\theta-1}\right] d s \\
& +\frac{\left(y_{1}-y_{2}\right) L_{1}}{1-B} \int_{y_{1}}^{y_{2}}(1-s)^{\theta-1} d s \\
& +L_{1} \int_{s}^{1} \int_{y_{1}}^{y_{2}}\left[\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right. \\
& \left.-\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}\right] d s d A(y) \\
& +L_{1} \int_{y_{1}}^{y_{2}}\left(y_{2}-s\right)^{\theta-1} d s+\frac{\left(y_{1}-y_{2}\right) L_{1}}{1-B} \int_{y_{2}}^{1}(1 \\
& -s)^{\theta-1} d s+L_{1} \int_{s}^{1} \int_{y_{2}}^{1}\left[\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right. \\
& \left.-\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}\right] d s d A(y) \\
& =\frac{\left(y_{1}-y_{2}\right) L_{1}}{1-B} \int_{0}^{1}(1-s)^{\theta-1} d s \\
& +L_{1} \int_{s}^{1} \int_{0}^{1}\left[\frac{y_{2}}{1-B}\left(y_{2}-s\right)^{\theta-1}\right. \\
& \left.-\frac{y_{1}}{1-B}\left(y_{1}-s\right)^{\theta-1}\right] d s d A(y) \\
& +L_{1}\left(\frac{y_{2}^{\theta}}{\theta}-\frac{y_{1}^{\theta}}{\theta}\right)^{\theta} \\
& +L_{1} \int_{y_{1}}^{y_{2}}\left(y_{2}-s\right)^{\theta-1} d s+L_{1} \int_{0}^{1}\left[\frac{y_{1}\left(y_{1}-1\right)^{\theta}}{(1-B) \theta}-\frac{y_{2}\left(y_{2}-1\right)^{\theta}}{(1-B) \theta}\right] d A(y) \\
& \left.-s)^{\theta-1}\right] d s=\frac{\left(y_{1}-y_{2}-s\right)^{\theta-1}-\left(y_{1}\right.}{(1-B) \theta} L_{1} \\
& \left.+B+\frac{y_{1}^{\theta+1}}{(1-B) \theta}\right] d A(y) \\
& +1 \tag{29}
\end{align*}
$$

Since $y, y(y-1)^{\theta}, y^{\theta+1}, y^{\theta}$ are uniformly continuous on $[0,1]$, $A_{\mu}(\Omega)$ is equicontinuous, by using Arzela-Ascoli's theorem, we can prove that $A_{\mu}: P \longrightarrow P$ is completely continuous. The proof is completed.

The following Guo-Krasnoselskii's fixed point theorem is used to prove the existence of positive solution of (7).

Theorem 7 (see [37]). Let P be a cone of real Banach space $X$ and let $\Omega_{1}$ and $\Omega_{2}$ be two bounded open sets in $X$ such that $0 \in \Omega_{1} \in \bar{\Omega}_{1} \in \Omega_{2}$. Let operator $A: P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right) \longrightarrow P$ be completely continuous operator. If one of the following two conditions holds:
(1) $\|A p\| \leq\|p\|$ for all $p \in P \cap \partial \Omega_{1},\|A p\| \geq\|p\|$ for all $p \in P \cap \partial \Omega_{2}$,
(2) $\|A p\| \geq\|p\|$ for all $p \in P \cap \partial \Omega_{1},\|A p\| \leq\|p\|$ for all $p \in P \cap \partial \Omega_{2}$,
then $A$ has at least one fixed point in $P \cap\left(\bar{\Omega}_{2} \backslash \Omega_{1}\right)$.

## 3. Existence of Positive Solutions

In this section, we investigate the existence of positive solutions for integral boundary value problems of fractional differential equation (7).

For convenience, we denote them by

$$
\begin{align*}
h_{0} & =\lim _{u \rightarrow 0^{+}} \sup _{y \in[0,1]} \frac{f(y, u)}{u}, \\
h_{\infty} & =\lim _{u \rightarrow+\infty} \sup _{y \in[0,1]} \frac{f(y, u)}{u}, \\
h_{0}^{*} & =\lim _{u \rightarrow 0^{+}} \inf _{y \in[1 / 4,3 / 4]} \frac{f(y, u)}{u},  \tag{30}\\
h_{\infty}^{*} & =\lim _{u \rightarrow+\infty} \inf _{y \in[1 / 4,3 / 4]} \frac{f(y, u)}{u}, \\
C & =\frac{1}{\Gamma(\theta+1)(1-B)}, \\
E & =\frac{N^{2}(1-B)}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} d s .
\end{align*}
$$

Theorem 8. Suppose that $C h_{0}<E h_{\infty}^{*}$ holds; then for $\mu \in$ $\left(1 / h_{\infty}^{*} E, 1 / C h_{0}\right)$, problem (7) has a positive solution. Here we impose $h_{0}^{-1}=+\infty$ if $h_{0}=0$ and $\left[h_{\infty}^{*}\right]^{-1}=0$ if $h_{\infty}^{*}=+\infty$.

Proof. Let $\mu \in\left(1 / h_{\infty}^{*} E, 1 / C h_{0}\right)$ and $\epsilon>0$ satisfy

$$
\begin{equation*}
\frac{1}{\left(h_{\infty}^{*}-\epsilon\right) E} \leq \mu \leq \frac{1}{C\left(h_{0}+\varepsilon\right)} . \tag{31}
\end{equation*}
$$

According to the definition of $h_{0}$, we know that there exists a constant $E_{1}>0$ such that

$$
\begin{equation*}
f(y, u) \leq\left(h_{0}+\varepsilon\right) u, \quad \text { for } u \in\left[0, E_{1}\right], y \in[0,1] \tag{32}
\end{equation*}
$$

Put $\Omega_{1}=\left\{p \in P:\|p\|<E_{1}\right\}$. Let $p \in P \cap \partial \Omega_{1}$. We have $\|p\|=E_{1}$ and

$$
\begin{align*}
& \left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \\
& \quad \leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1}\left(h_{0}+\varepsilon\right) p(s) d s  \tag{33}\\
& \quad \leq \frac{\mu\left(h_{0}+\varepsilon\right)}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1} d s \cdot\|p\| \\
& \quad=\left(h_{0}+\varepsilon\right) \mu C \cdot\|p\| \leq\|p\|, \quad y \in[0,1]
\end{align*}
$$

Therefore, $\left\|A_{\mu} p\right\| \leq\|p\|$ for $p \in P \cap \partial \Omega_{1}$.
By the definition of $h_{\infty}^{*}$, we know that there exists $E_{2}>0$ such that

$$
\begin{equation*}
f(y, u) \geq\left(h_{\infty}^{*}-\epsilon\right) u, \quad \text { for } u \geq E_{2} \text { and } y \in\left[\frac{1}{4}, \frac{3}{4}\right] . \tag{34}
\end{equation*}
$$

Let $E_{3}=\max \left\{2 E_{1}, E_{2} / N(1-B)\right\}, \Omega_{2}=\{p \in P:\|p\|<$ $\left.E_{3}\right\}$. Then for $p \in P \cap \partial \Omega_{2}$, by (25) we have

$$
\begin{equation*}
\min _{y \in[1 / 4,3 / 4]} p(y) \geq N(1-B)\|p\| \geq E_{2}, \tag{35}
\end{equation*}
$$

and thus

$$
\begin{align*}
& \left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \\
& \quad \geq \mu \int_{1 / 4}^{3 / 4} G(y, s) f(s, p(s)) d s \\
& \quad \geq \frac{\mu N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} f(s, p(s)) d s  \tag{36}\\
& \quad \geq \frac{\mu N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1}\left(h_{\infty}^{*}-\epsilon\right) p(s) d s \\
& \quad \geq \frac{\mu N^{2}\left(h_{\infty}^{*}-\epsilon\right)(1-B)}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1}\|p\| d s \\
& \quad=\mu E\left(h_{\infty}^{*}-\epsilon\right) \cdot\|p\| \geq\|p\| .
\end{align*}
$$

Therefore, $\left\|A_{\mu} p\right\| \geq\|p\|$ for $p \in P \cap \partial \Omega_{2}$.
By Theorem 7, if $\mu \in\left(1 / h_{\infty}^{*} E, 1 / C h_{0}\right)$, we assert that $A_{\mu}$ has a fixed point in $P \cap\left(\overline{\Omega_{2}} \backslash \Omega_{1}\right)$ and therefore problem (7) has at least one positive solution. The proof is completed.

Theorem 9. Assume that $h_{\infty} C<E h_{0}^{*}$ holds. Then for $\mu \in\left(1 / E h_{0}^{*}, 1 / h_{\infty} C\right)$, the problem (7) has at least a positive solution. Here we impose $\left[h_{0}^{*}\right]^{-1}=0$ if $h_{0}^{*}=+\infty$ and $\left[h_{\infty}\right]^{-1}=$ $+\infty$ if $h_{\infty}=0$.

Proof. Let $\mu \in\left(1 / E h_{0}^{*}, 1 / h_{\infty} C\right)$ and $\epsilon>0$ such that

$$
\begin{equation*}
\frac{1}{\left(h_{0}^{*}-\epsilon\right) E} \leq \mu \leq \frac{1}{C\left(h_{\infty}+\varepsilon\right)} . \tag{37}
\end{equation*}
$$

According to the definition of $h_{0}^{*}$, there exists a constant $E_{4}>$ 0 such that

$$
\begin{align*}
& f(y, u) \geq\left(h_{0}^{*}-\epsilon\right) u, \\
& \qquad \text { for } u \in\left(0, E_{4}\right] \text { and } y \in\left[\frac{1}{4}, \frac{3}{4}\right] . \tag{38}
\end{align*}
$$

Put $\Omega_{3}=\left\{p \in P:\|p\|<E_{4}\right\}$. Let $p \in P \cap \partial \Omega_{3}$; we have

$$
\begin{aligned}
& \left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \\
& \quad \geq \mu \int_{1 / 4}^{3 / 4} G(y, s) f(s, p(s)) d s \\
& \quad \geq \frac{\mu N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} f(s, p(s)) d s \\
& \quad \geq \frac{\mu N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1}\left(h_{0}^{*}-\epsilon\right) p(s) d s
\end{aligned}
$$

$$
\begin{align*}
& \geq \frac{\mu N^{2}\left(h_{0}^{*}-\epsilon\right)(1-B)}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1}\|p\| d s \\
& =\mu E\left(h_{0}^{*}-\epsilon\right) \cdot\|p\| \geq\|p\| . \tag{39}
\end{align*}
$$

Therefore, $\left\|A_{\mu} p\right\| \geq\|p\|$ for $p \in P \cap \partial \Omega_{3}$.
It follows from the definition of $h_{\infty}$ that there exists a constant $E_{5}>0$ such that

$$
\begin{equation*}
f(y, u) \leq\left(h_{\infty}+\frac{\varepsilon}{2}\right) u, \quad \text { for } u \geq E_{5} \text { and } y \in[0,1] . \tag{40}
\end{equation*}
$$

This together with the continuity of $f$ implies that

$$
\begin{equation*}
f(y, u) \leq\left(h_{\infty}+\frac{\varepsilon}{2}\right) u+M \tag{41}
\end{equation*}
$$

for $u \in R$ and $y \in[0,1]$
holds for some $M>0$.
Let $E_{6}=\max \left\{2 E_{4}, E_{5}, 2 M / \varepsilon\right\}, \Omega_{4}=\left\{p \in P:\|p\|<E_{6}\right\}$. For $\forall p \in P \cap \partial \Omega_{4}$, we conclude that

$$
\begin{align*}
& \left(A_{\mu} p\right)(y)=\mu \int_{0}^{1} G(y, s) f(s, p(s)) d s \\
& \quad \leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1}\left(h_{\infty}+\frac{\varepsilon}{2}\right)\|p\| d s \\
& \quad+\frac{M \mu}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1} d s  \tag{42}\\
& \quad=\mu C\left(\left(h_{\infty}+\frac{\varepsilon}{2}\right)\|p\|+M\right) \leq \mu C\left(h_{\infty}+\varepsilon\right)\|p\| \\
& \quad \leq\|p\|
\end{align*}
$$

Therefore, $\left\|A_{\mu} p\right\| \leq\|p\|$ for $\forall p \in P \cap \partial \Omega_{4}$.
By Theorem 7, if $\mu \in\left(1 / E h_{0}^{*}, 1 / h_{\infty} C\right)$, we conclude that $A_{\mu}$ has a fixed point in $P \cap\left(\overline{\Omega_{4}} \backslash \Omega_{3}\right)$, and so problem (7) has one positive solution. The proof is completed.

## 4. Nonexistence of Positive Solutions

In this section, we present some sufficient conditions for nonexistence of positive solution to integral boundary value problems of fractional differential equation (7).

Theorem 10. If $h_{0}<+\infty$ and $h_{\infty}<+\infty$, then there exists a $\mu^{*}>0$ such that problem (7) has no positive solution for $\mu \in\left(0, \mu^{*}\right)$.

Proof. Since $h_{0}<+\infty, h_{\infty}<+\infty$, we have $f(y, u) \leq n_{1} u$ for $u \in\left[0, r_{1}\right]$, and $f(y, u) \leq n_{2} u$ for $u \in\left[r_{2},+\infty\right)$, where $n_{1}, n_{2}, r_{1}, r_{2}$ are positive numbers with $r_{1}<r_{2}$. Let $n=$ $\max \left\{n_{1}, n_{2}, \max _{r_{1} \leq u \leq r_{2}}(f(y, u) / u)\right\}$; then we have $f(y, u) \leq$ $n u$ for $u \in[0,+\infty)$. Suppose $p_{0}(y)$ is a positive solution of problem (7); then we are going to prove that this leads to a
contradiction for $0<\mu<\mu^{*}:=1 / \mathrm{C}$. Since $\left(A_{\mu} p_{0}\right)(y)=$ $p_{0}(y)$, for $y \in[0,1]$, then

$$
\begin{align*}
\left\|p_{0}\right\| & =\left\|A_{\mu} p_{0}\right\| \\
& \leq \frac{\mu}{\Gamma(\theta)(1-B)} \int_{0}^{1}(1-s)^{\theta-1} f\left(y, p_{0}(y)\right) d s  \tag{43}\\
& \leq \mu C n\left\|p_{0}\right\|<\left\|p_{0}\right\|
\end{align*}
$$

which is a contradiction. Therefore this completes the proof.

Theorem 11. If $h_{0}^{*}>0, h_{\infty}^{*}>0, f(t, u)>0$ for $t \in[1 / 4,3 / 4]$ and $u>0$, then there exists a $\mu^{*}>0$ such that problem (7) has no positive solution for all $\mu>\mu^{*}$.

Proof. Since $h_{0}^{*}>+\infty, h_{\infty}^{*}>+\infty$, we have $f(y, u) \geq m_{1} u$ for $u \in\left[0, r_{1}\right]$, and $f(y, u) \geq m_{2} u$ for $u \in\left[r_{2},+\infty\right)$, where $m_{1}, m_{2}, r_{1}, r_{2}$ are positive numbers and $r_{1}<r_{2}$. Let $m=\min \left\{m_{1}, m_{2}, \min _{r_{1} \leq u \leq r_{2}}(f(y, u) / u)\right\}>0$; then we have $f(y, u) \geq m u$ for $u \in[0,+\infty)$. Suppose $p_{1}(y)$ is a positive solution of problem (7); then we are going to prove that this leads to a contradiction for $\mu>\mu^{*}:=1 / \operatorname{EmN}(1-B)$. Since $A_{\mu} p_{1}=p_{1}$, then

$$
\begin{align*}
\left\|p_{1}\right\| & =\left\|A_{\mu} p_{1}\right\| \geq \frac{\mu N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} f(s, p(s)) d s \\
& \geq \frac{\mu m N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} p(s) d s  \tag{44}\\
& \geq \frac{\mu m N}{\Gamma(\theta)} \int_{1 / 4}^{3 / 4}(1-s)^{\theta-1} N(1-B)\left\|p_{1}\right\| d s \\
& \geq \mu E m N(1-B)\left\|p_{1}\right\|>\left\|p_{1}\right\|
\end{align*}
$$

which is a contradiction. Therefore this completes the proof.

## 5. Example

Example 1. We consider the following fractional equation:

$$
\begin{align*}
& { }^{c} D_{0+}^{5 / 2} p(y)+\mu\left(y p^{3}(y)-p(y)+e^{p(y)}-1\right)=0, \\
& y \in[0,1], \\
& p(0)=p^{\prime \prime}(0)=0,  \tag{45}\\
& p(1)=\frac{1}{2} \int_{0}^{1} p(y) d y,
\end{align*}
$$

where $\theta=5 / 2, f(y, p)=y p^{3}-p+e^{p}-1, A(y)=(1 / 2) y$, $B=1 / 4<1$. We obtain $C=8 / 15 \Gamma(5 / 2), N=3 / 64, E=$ $27(9 \sqrt{3}-1) / 1310720 \Gamma(5 / 2)$.

It is easy to see that, for all $p>0$,

$$
\begin{equation*}
\sup _{y \in[0,1]} \frac{f(y, p)}{p}=p^{2}-1+\frac{e^{p}-1}{p} \tag{46}
\end{equation*}
$$

and

$$
\begin{equation*}
\inf _{y \in[1 / 4,3 / 4]} \frac{f(y, p)}{p}=\frac{1}{4} p^{2}-1+\frac{e^{p}-1}{p} . \tag{47}
\end{equation*}
$$

Then $h_{0}=0, h_{\infty}^{*}=\infty$; from Theorem 8, problem (45) has a positive solution.

Example 2. We consider the following fractional equation:

$$
\begin{align*}
& { }^{c} D_{0+}^{5 / 2} p(y)+\mu(\sqrt[3]{p(y)}+y \ln (p(y)+3))=0 \\
& y(0)=[0,1] \\
& p(1)=\frac{1}{2} \int_{0}^{1} p(y) d y  \tag{48}\\
& p
\end{align*}
$$

where $\theta=5 / 2, f(y, p)=\sqrt[3]{p}+y \ln (p+3), A(y)=(1 / 2) y$, and $B=1 / 4<1$. We obtain $C=8 / 15 \Gamma(5 / 2), N=3 / 64$, and $E=27(9 \sqrt{3}-1) / 1310720 \Gamma(5 / 2)$.

One can easily see that, for all $p>0$,

$$
\begin{equation*}
\sup _{y \in[0,1]} \frac{f(y, p)}{p}=\frac{\sqrt[3]{p}+\ln (p+3)}{p} \tag{49}
\end{equation*}
$$

and

$$
\begin{equation*}
\inf _{y \in[1 / 4,3 / 4]} \frac{f(y, p)}{p}=\frac{\sqrt[3]{p}+(1 / 4) \ln (p+3)}{p} \tag{50}
\end{equation*}
$$

Then $h_{\infty}=0, h_{0}^{*}=\infty$; from Theorem 9, problem (48) has a positive solution.

Example 3. We consider the following fractional equation:

$$
\begin{align*}
{ }^{c} D_{0+}^{5 / 2} p(y)+\mu y p(y) & =0, \quad y \in[0,1], \\
p(0) & =p^{\prime \prime}(0)=0,  \tag{51}\\
p(1) & =\frac{1}{2} \int_{0}^{1} p(y) d y,
\end{align*}
$$

where $\theta=5 / 2, f(y, p)=y p, A(y)=(1 / 2) y$, and $B=1 / 4<$ 1. We obtain $C=8 / 15 \Gamma(5 / 2), N=3 / 64$, and $E=27(9 \sqrt{3}-$ 1)/ $1310720 \Gamma(5 / 2)$.

By direct calculation, we obtain that

$$
\begin{equation*}
h_{0}=\lim _{p \rightarrow 0^{+}} \sup _{y \in[0,1]} \frac{f(y, p)}{p}=1<+\infty \tag{52}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{\infty}=\lim _{p \rightarrow+\infty} \sup _{y \in[0,1]} \frac{f(y, p)}{p}=1<+\infty . \tag{53}
\end{equation*}
$$

Take $n=1$ and $\mu^{*}=45 \sqrt{\pi} / 32>0$. By Theorem 10, problem (51) has no positive solution for $0<\mu<\mu^{*}$.

Example 4. We consider the following fractional equation:

$$
\begin{align*}
{ }^{c} D_{0+}^{5 / 2} p(y)+\mu y p(y) & =0, \quad y \in[0,1], \\
p(0) & =p^{\prime \prime}(0)=0,  \tag{54}\\
p(1) & =\frac{1}{2} \int_{0}^{1} p(y) d y,
\end{align*}
$$

where $\theta=5 / 2, f(y, p)=y p, A(y)=(1 / 2) y$, and $B=1 / 4<$ 1. We obtain $C=8 / 15 \Gamma(5 / 2), N=3 / 64$, and $E=27(9 \sqrt{3}-$ 1)/ $1310720 \Gamma(5 / 2)$.

Obviously, we can infer that

$$
\begin{equation*}
h_{0}^{*}=\lim _{p \rightarrow 0^{+}} \inf _{y \in[1 / 4,3 / 4]} \frac{f(y, p)}{p}=\frac{1}{4}>0 \tag{55}
\end{equation*}
$$

and

$$
\begin{equation*}
h_{\infty}^{*}=\lim _{p \rightarrow+\infty} \inf _{y \in[1 / 4,3 / 4]} \frac{f(y, p)}{p}=\frac{1}{4}>0 . \tag{56}
\end{equation*}
$$

Take $m=1 / 4$ and $\mu^{*}=3.4 \sqrt{\pi} / 81(9 \sqrt{3}-1) \times 10^{8}$. By Theorem 11, problem (54) has no positive solution for $\mu>\mu^{*}$.

## Data Availability

The dataset supporting the conclusions of this article is included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# Synchronization of Different Uncertain Fractional-Order Chaotic Systems with External Disturbances via T-S Fuzzy Model 

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#### Abstract

This paper presents an adaptive fuzzy synchronization control strategy for a class of different uncertain fractional-order chaotic/hyperchaotic systems with unknown external disturbances via T-S fuzzy systems, where the parallel distributed compensation technology is provided to design adaptive controller with fractional adaptation laws. T-S fuzzy models are employed to approximate the unknown nonlinear systems and tracking error signals are used to update the parametric estimates. The asymptotic stability of the closed-loop system and the boundedness of the states and parameters are guaranteed by fractional Lyapunov theory. This approach is also valid for synchronization of fractional-order chaotic systems with the same system structure. One constructive example is given to verify the feasibility and superiority of the proposed method.


## 1. Introduction

Fractional calculus is a mathematical topic being more than 300 years old, which can be traced back to the birth of integer-order calculus. The fundamentals results of fractional calculus were concluded in [1]. At present, researchers found that fractional differential equations not only improve the veracity in modeling physical systems but also generate a lot of applications in physics, electrical engineering, robotics, control systems, and chemical mixing [2-11]. In addition, the chaotic behavior has been discovered in many fractionalorder systems, for instance, the fractional-order Chen's system, the fractional-order Chua's system, and the fractionalorder Liu system. In view of chaotic potential value in control systems and secure communication [12], chaos synchronization was studied by more and more researches [13, 14].

The conventional nonlinear systems control approaches suffer from discontented performance resulting from structure and parametric uncertainties, external disturbances. Usually, it is very hard to provide accurate mathematical models [15-25]. To control these uncertain systems, adaptive fuzzy/neural-network control was proposed [26, 27]. This method is effective and superior for handling parametric
and structure uncertainties, external disturbances in integerorder nonlinear systems [28, 29], where tracking error is developed to update adjusted parameters and fuzzy logic systems or neural networks are introduced to model unknown physical systems as well as to approximate unknown nonlinear functions. There are two types of fuzzy logic systems: Mamdani type and T-S type. T-S fuzzy logic system is first proposed by Takagi and Sugeno [30]. Subsequently, many works found that T-S fuzzy systems can uniformly approximate any continuous functions on a compact set with random accuracy based on the Weierstrass approximation theorem [31]. Moreover, it was also shown that the approximation ability of T-S fuzzy systems was better than the Mamdani fuzzy systems [32]. Therefore, many studies focused on the chaos synchronization of fractional-order chaotic systems via T-S fuzzy models. For example, synchronization of fractional-order modified chaotic system via new linear control, backstepping control, and T-S fuzzy approaches was investigated in [33]. Impulsive control for fractional-order chaotic system was presented in [34]. Other results about the synchronization of a fractional-order chaotic system via TS fuzzy approaches can be found in [35, 36]. However, only chaos synchronization of fractional-order nonlinear systems
with same structure based on T-S fuzzy systems is considered in above previous works.

This work investigates the chaos synchronization of fractional-order chaotic systems with different structures based on T-S fuzzy systems, where external disturbances in slaves system are considered. T-S fuzzy systems with random rule consequents are introduced to model controlled systems, whereas T-S fuzzy systems that have the same rule consequents with Mamdani fuzzy systems are used to approximate unknown nonlinear functions. The asymptotic stability of closed-loop system is proofed based on fractional Lyapunov stability theory. Compared to previous literature, the main contributions of this paper are as follows:
(1) This paper first considers the chaos synchronization of the master system and slave system with different structure based on T-S fuzzy systems, and the external disturbances are assumed to be unknown. The required knowledge of the disturbances is weaker than above previous works, for example, in [34-36]. In these works, the external disturbances are assumed to be bounded with known upper bounds. However, in our control method, we do not need to know the exact value of the upper bounds of external disturbances.
(2) T-S fuzzy logic systems are used to model the controlled system and the final outputs of system can be obtained. By combining the adaptive fuzzy control method and parallel distributed compensation technique, an adaptive controller with fractional-order laws is designed. The proposed method is superior to some works based on linear matrix inequality (LMI) and modified LMI [37].

## 2. Fundamentals of Fractional Calculus and Fuzzy Logic Systems

2.1. Fractional Calculus. There are two frequently used definitions for fractional integration and differentiation: RiemannLiouville (denote R-L) and Caputo definitions. In this paper, we will consider Caputo's definition, whose initial conditions are as the same form of the integer-order one [38-40]. The fractional integral is designed as [1]

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{-\mu} u(t)=\frac{1}{\Gamma(\mu)} \int_{0}^{t} \frac{u(\xi)}{(t-\xi)^{1-\mu}} d \xi \tag{1}
\end{equation*}
$$

where $\mu>0, n-1 \leq \mu<n$, and $\Gamma(\cdot)$ is Euler's Gamma function, which is defined as $\Gamma(s)=\int_{0}^{\infty} t^{s-1} e^{-t} d t$. The fractional derivative operator is given as

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} u(t)=\frac{1}{\Gamma(n-\mu)} \int_{0}^{t} \frac{u^{(n)}(\xi)}{(t-\xi)^{\mu+1-n}} d \xi \tag{2}
\end{equation*}
$$

Some useful properties of fractional calculus that will be used in the controller design are listed as follows.

Property 1 (see [1, 41-44]). Caputo's fractional derivative and integral are linear operations with $\lambda_{1}, \lambda_{2}, \mu \in R$

$$
\begin{align*}
& { }_{0}^{C} D_{t}^{\mu}\left(\lambda_{1} u_{1}(t)+\lambda_{2} u_{2}(t)\right) \\
& \quad=\lambda_{1}{ }_{0}^{C} D_{t}^{\mu} u_{1}(t)+\lambda_{2}{ }_{0}^{C} D_{t}^{\mu} u_{2}(t) . \tag{3}
\end{align*}
$$

Property 2. Let $x(t) \in C^{n}[0, T]$. Then we have

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{-\mu}{ }_{0}^{C} D_{t}^{\mu} u(t)=u(t)-\sum_{k=0}^{n-1} \frac{u^{(k)}(0)}{k!} t^{k} . \tag{4}
\end{equation*}
$$

Property 3 (see [1, 45-47]). The Laplace transform of (2) is

$$
\begin{equation*}
\mathscr{L}\left[{ }_{0}^{C} D_{t}^{\mu} f(t)\right]=s^{\mu} F(s)-\sum_{k=0}^{n-1} s^{\mu-k-1} f^{(k)}(0) \tag{5}
\end{equation*}
$$

with $F(s)=\mathscr{L}[f(t)]$.
Definition 4. The two-parameter Mittag-Leffler function was defined by [1]

$$
\begin{equation*}
E_{\alpha, \beta}(z)=\sum_{k=0}^{+\infty} \frac{z^{k}}{\Gamma(k \alpha+\beta)}, \tag{6}
\end{equation*}
$$

with $\alpha, \beta>0$ and $z \in C$. The Laplace transform of the MittagLeffler function is given as

$$
\begin{equation*}
\mathscr{L}\left[t^{\beta-1} E_{\alpha, \beta}\left(-a t^{\alpha}\right)\right]=\frac{s^{\alpha-\beta}}{s^{\alpha}+a} . \tag{7}
\end{equation*}
$$

In the subsequent paper, we only consider the case that $\mu \in(0,1)$.
2.2. Takagi-Sugeno Fuzzy Logic Systems. Unlike the Mamdani fuzzy logic systems, the ith rule of a Multi-Input and Multioutput general fractional-order Takagi-Sugeno (T-S) fuzzy systems can be expressed as follows ( $i=1,2, \cdots, N$ ):
$R^{i}$ : If $x_{1}(t)$ is $F_{1}^{i}$ and $\cdots$ and $x_{n}(t)$ is $F_{n}^{i}$, then ${ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)=f_{i}(t, \mathbf{x}(t))$,
with $F_{j}^{i} \in R, j=1,2, \cdots, n$ are fuzzy sets, $\mathbf{x}(t)=$ $\left(x_{1}(t), x_{2}(t), \cdots, x_{n}(t)\right)^{T} \in R^{n}$ is a state vector, and $f_{i}(t, \mathbf{x}(t))$ is a random function. In this paper, singleton fuzzification, center average defuzzification, and product inference are adopted and a general fractional-order T-S fuzzy system can be rewritten in the form

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)=\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t)) f_{i}(t, \mathbf{x}(t)) \tag{8}
\end{equation*}
$$

where $\mu_{i}(\mathbf{x}(t))=\prod_{j=1}^{n} F_{j}^{i}\left(x_{j}(t)\right) / \sum_{i=1}^{N}\left(\prod_{j=1}^{n} F_{j}^{i}\left(x_{j}(t)\right)\right)$ satisfying $\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t))=1$ and $\mu_{i}(\mathbf{x}(t)) \geq 0$.

Depending on the above statements, a main difference of Mamdani fuzzy logic systems and T-S fuzzy systems is that the rule consequents are functions for T-S fuzzy system whereas the rule consequents are fuzzy sets for Mamdani fuzzy logic systems. Moreover, the T-S fuzzy logic systems are also universal approximators [31].

## 3. Adaptive Fuzzy Synchronization Control

3.1. Problem Statement. Consider the following fractionalorder chaotic system as the master system via T-S type fuzzy systems. The $i$ th rule can be expressed as ( $i=1,2, \cdots, N$ )

$$
\begin{aligned}
& R^{i}: \text { If } x_{1}(t) \text { is } F_{1}^{i} \text { and } \cdots \text { and } x_{n}(t) \text { is } F_{n}^{i} \text {, then } \\
& { }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)=A_{i} \mathbf{x}(t)+\mathbf{b}_{1},
\end{aligned}
$$

where $A_{i}$ is a constant matrix, $\mathbf{x}(t) \in D_{1} \subseteq R^{n}$ is the state $\operatorname{vector}\left(D_{1}\right.$ is a compact set), $\mathbf{b}_{1}$ is a constant vector, and $F_{j}^{i}, j=$ $1,2, \cdots, n$, are fuzzy sets. Hence, the final output of master system can be rewritten as

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)=\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t))\left[A_{i} \mathbf{x}(t)+\mathbf{b}_{1}\right] \tag{9}
\end{equation*}
$$

with $\mu_{i}(\mathbf{x}(t))=\prod_{j=1}^{n} F_{j}^{i}\left(x_{j}(t)\right) / \sum_{i=1}^{N}\left[\prod_{j=1}^{n} F_{j}^{i}\left(x_{j}(t)\right)\right]$ satisfy$\operatorname{ing} \mu_{i}(\mathbf{x}(t)) \geq 0$ and $\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t))=1$.

Consider the following fractional-order chaotic system with external disturbances in the equation as the slave system based on T-S fuzzy models. The $i$ th rule can be written in the following form $(i=1, \cdots, N)$ :

$$
\begin{aligned}
& R^{i} \text { : If } y_{1}(t) \text { is } \widehat{F}_{1}^{i} \text { and } y_{2}(t) \text { is } \widehat{F}_{2}^{i} \text { and } \cdots \text { and } y_{n}(t) \text { is } \widehat{F}_{n}^{i} \text {, } \\
& \text { then }{ }_{0}^{C} D_{t}^{\mu} \mathbf{y}(t)=B_{i} \mathbf{y}(t)+\mathbf{b}_{2}+\mathbf{u}(t)+\mathbf{d}_{i}(t, \mathbf{y}),
\end{aligned}
$$

where $B_{i}$ is a constant matrix, $\mathbf{y}(t) \in D_{2} \subseteq R^{n}$ is the state vector ( $D_{2}$ is a compact set), $\mathbf{b}_{2}$ is a constant vector, $\widehat{F}_{j}^{i}, j=$ $1,2, \cdots, n$, are fuzzy sets, $u(t) \in R$ is control input, and $\mathbf{d}_{i}(t, \mathbf{y}) \in R^{n}$ are unknown external disturbances. Hence, the final output of slave system can be obtained as

$$
\begin{align*}
& { }_{0}^{C} D_{t}^{\mu} \mathbf{y}(t) \\
& \quad=\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))\left[B_{i} \mathbf{y}(t)+\mathbf{b}_{2}+\mathbf{u}(t)+\mathbf{d}_{i}(t, \mathbf{y})\right], \tag{10}
\end{align*}
$$

with $\mu_{i}(\mathbf{y}(t))=\prod_{j=1}^{n} \widehat{F}_{j}^{i}\left(y_{j}(t)\right) / \sum_{i=1}^{N}\left[\prod_{j=1}^{n} \widehat{F}_{j}^{i}\left(y_{j}(t)\right)\right]$ satisfying $\mu_{i}(\mathbf{y}(t)) \geq 0$ and $\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))=1$.

The control objective of this work is to design a proper adaptive controller $u(t)$ to synchronize the above chaotic systems (9) and (10) with the tracking error signal

$$
\begin{equation*}
\mathbf{e}(t)=\mathbf{y}(t)-\mathbf{x}(t) \tag{11}
\end{equation*}
$$

asymptotically converging to zero with random accuracy, that is, $\lim _{t \rightarrow+\infty}\|\mathbf{e}(t)\|=0$. The norm adopts Euclid norm in this paper. In addition, all states and parameters in the closedloop system are bounded. The following assumptions are necessary.

Assumption 5. The structure of master system (9) and slave system (10) is different. The parameters and the structure of the master system are complete unknown or partial unknown, but the parameters and structure of the slave system are known.

Assumption 6. The unknown disturbances $\mathbf{d}_{i}(t, \mathbf{y})=$ $\left(d_{i}^{1}(t, \mathbf{y}), \cdots, d_{i}^{n}(t, \mathbf{y})\right)^{T}, \quad(i=1,2, \cdots, N)$ satisfying $\left|d_{i}^{j}(t, \mathbf{y})\right| \leq \rho_{i}^{j}(\mathbf{y})$ with $\rho_{i}^{j}(\mathbf{y})$ being a continuous function, where $\rho_{i}^{j}(\mathbf{y})$ is the estimated value of the observed value for $\left|d_{i}^{j}(t, \mathbf{y})\right|$, for all $\mathbf{y} \in D_{2}$.

Remark 7. It is worth pointing out that Assumptions 5 and 6 are rational. Due to the boundedness of chaos systems, we assume that $D_{1}$ and $D_{2}$ are compact sets. Since $\mathbf{d}_{i}(t, y)$ are unknown external disturbances and may be not continuous, they are assumed to be unknown measurable nonlinear functions. The slave systems and the controller lie on the receiving terminal; hence, the parameters and the structure of the master system may be complete unknown or partial unknown, but the parameters and structure of the slave system are known.
3.2. Control Design. The synchronization error dynamic equation can be obtained from (11) as

$$
\begin{align*}
{ }_{0}^{C} D_{t}^{\mu} \mathbf{e}(t)= & { }_{0}^{C} D_{t}^{\mu} \mathbf{y}(t)-{ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t) \\
= & \sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t)) B_{i} \mathbf{y}(t)-\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t)) A_{i} \mathbf{x}(t)  \tag{12}\\
& +\mathbf{m}+\mathbf{u}(t)+\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t)) \mathbf{d}_{i}(t, \mathbf{y}),
\end{align*}
$$

with $\mathbf{m}=\mathbf{b}_{2}-\mathbf{b}_{1}$ being a constant vector.
Based on T-S fuzzy logic system universal approximation theorem, T-S fuzzy systems $\hat{\rho}_{i}^{j}\left(\mathbf{y}, \theta_{i}^{j}(t)\right)$ that have the same rule consequents with the Mamdani type fuzzy logic systems are used to approximate to $\rho_{i}^{j}(\mathbf{y})$ in the Assumption 6, $j \in 1,2, \cdots, n$ and $i \in 1,2, \cdots, N$, where $\theta_{i}^{j}(t)$ are adjusted parameters in fuzzy systems. Denote $\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)=$ $\left(\hat{\rho}_{i}^{1}\left(\mathbf{y}, \theta_{i}^{1}(t)\right) . \cdots, \hat{\rho}_{i}^{n}\left(\mathbf{y}, \theta_{i}^{n}(t)\right)\right)^{T}$. Using [48-50], we obtain the ideal parameter as

$$
\begin{equation*}
\theta_{i}^{* j}=\arg \min _{\theta_{i}^{j}(t) \in R} \sup _{t \geq 0}\left|\hat{\rho}_{i}^{j}\left(\mathbf{y}, \theta_{i}^{j}(t)\right)-\rho_{i}^{j}(\mathbf{y})\right| . \tag{13}
\end{equation*}
$$

Then we obtain the optimal parameter vector as $\boldsymbol{\theta}_{i}^{*}=$ $\left(\theta_{i}^{* 1}, \cdots, \theta_{i}^{* n}\right)^{T}$. Hence, $\hat{\rho}_{i}^{j}\left(\mathbf{y}, \theta_{i}^{* j}\right)$ is the ideal approximator of $\rho_{i}^{j}(\mathbf{y})$; that is, $\hat{\boldsymbol{\rho}}_{i}\left(y, \boldsymbol{\theta}_{i}^{*}\right)$ is the ideal approximator of $\boldsymbol{\rho}_{i}(\mathbf{y})$. The minimum approximation errors and the ideal parameter errors of the fuzzy systems are defined as $(j=1,2, \cdots, n)$

$$
\begin{align*}
\varepsilon_{i}^{j}(\mathbf{y}) & =\rho_{i}^{j}(\mathbf{y})-\hat{\rho}_{i}^{j}\left(\mathbf{y}, \theta_{i}^{* j}\right),  \tag{14}\\
\widehat{\theta}_{i}^{j}(t) & =\theta_{i}^{j}(t)-\theta_{i}^{* j} \tag{15}
\end{align*}
$$

According to [29, 51, 52], the approximation errors $\varepsilon_{i}^{j}(\mathbf{y}), j=$ $1,2, \cdots, n)$ are assumed to be bounded, that is, $\left|\varepsilon_{i}^{j}(\mathbf{y})\right| \leq \varepsilon_{i}^{* j}$ with the $\varepsilon_{i}^{* j}$ being constants and $\widehat{\varepsilon}_{i}^{* j}(\mathbf{y})$ being the estimate value of $\varepsilon_{i}^{* j}$. Thus, from the above analysis, we can obtain the equations $\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)=\boldsymbol{\theta}_{i}^{T}(t) \varphi_{i}(\mathbf{y})$ and $\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}^{*}\right)=\boldsymbol{\theta}_{i}^{* T} \varphi_{i}(\mathbf{y})$,
where $\varphi_{i}(\mathbf{y})$ is fuzzy base functions. Denoting $\widetilde{\boldsymbol{\theta}}_{i}(t)=\boldsymbol{\theta}_{i}(t)-\boldsymbol{\theta}_{i}^{*}$ and $\boldsymbol{\varepsilon}_{i}(\mathbf{y})=\left(\varepsilon_{i}^{1}(\mathbf{y}), \cdots, \varepsilon_{i}^{n}(\mathbf{y})\right)^{T}$, one has

$$
\begin{align*}
\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)-\boldsymbol{\rho}_{i}(\mathbf{y})= & \widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)-\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}^{*}\right) \\
& +\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}^{*}\right)-\boldsymbol{\rho}_{i}(\mathbf{y}) \\
= & \widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)-\widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}^{*}\right)  \tag{16}\\
& -\boldsymbol{\varepsilon}_{i}(\mathbf{y}) \\
= & \widetilde{\boldsymbol{\theta}}_{i}^{T}(t) \varphi_{i}(\mathbf{y})-\boldsymbol{\varepsilon}_{i}(\mathbf{y}) .
\end{align*}
$$

Remark 8. As shown in [53], if the rule consequences of T-S fuzzy systems have the same form with the rule consequences of Mamdani type logic systems, then T-S type is equivalent to Mamdani type fuzzy system.

Based on above discussion, the controller is designed with the fuzzy system $\hat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)$ as well as the estimate value $\widehat{\varepsilon}_{i}^{* j}(\mathbf{y})$ as

$$
\begin{align*}
& \mathbf{u}(t)=\mathbf{u}_{d}(t)+\widetilde{\mathbf{u}}(t)+\mathbf{u}_{1}(t)=-\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))[K \mathbf{e}(t) \\
& \left.\quad+B_{i} \mathbf{y}(t)+H_{i} \operatorname{sign}(\mathbf{e}(t))+T \widehat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)+\mathbf{m}\right]  \tag{17}\\
& \quad+\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t)) A_{i} \mathbf{x}(t)
\end{align*}
$$

where the $i$ th rule of $\mathbf{u}_{d}(t)$ and $\widetilde{\mathbf{u}}(t)$ can be written as follows, respectively, $(i=1,2, \cdots, N)$ :

$$
R^{i}: \text { If } y_{1}(t) \text { is } \widehat{F}_{1}^{i} \text { and } y_{2}(t) \text { is } \widehat{F}_{2}^{i} \text { and } \cdots \text { and } y_{n}(t) \text { is } \widehat{F}_{n}^{i},
$$ then $\mathbf{u}_{d}(t)=-K \mathbf{e}(t)-B_{i} \mathbf{y}(t)-\mathbf{m}$, with $K$ being an adjusted control gain matrix.

$R^{i}$ : If $x_{1}(t)$ is $F_{1}^{i}$ and $\cdots$ and $x_{n}(t)$ is $F_{n}^{i}$, then $\widetilde{\mathbf{u}}(t)=$ $A_{i} \mathbf{x}(t)$.

Let us denote $\mathbf{u}_{1}(t)=-\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))\left[T \hat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)+\right.$ $\left.H_{i} \operatorname{sign}(\mathbf{e}(t))\right]$, where $T=\operatorname{diag}\left[\operatorname{sign}\left(\mathbf{e}_{1}(t)\right), \cdots, \operatorname{sign}\left(\mathbf{e}_{n}(t)\right)\right]$, $H_{i}=\operatorname{diag}\left[\widehat{\varepsilon}_{i}^{* 1}(\mathbf{y}), \cdots, \widehat{\varepsilon}_{i}^{* n}(\mathbf{y})\right]$.

In order to update parametric estimates, the fractional adaptation laws are designed as

$$
\begin{align*}
& { }_{0}^{C} D_{t}^{\mu} \widetilde{\theta}_{i}^{j}(t)=\alpha_{i} \mu_{i}(\mathbf{y}(t))\left|e_{j}(t)\right| \varphi_{i}^{j}(\mathbf{y}),  \tag{18}\\
& { }_{0}^{C} D_{t}^{\mu} \widetilde{\varepsilon}_{i}^{* j}(\mathbf{y})=\beta_{i} \mu_{i}(\mathbf{y}(\mathrm{t}))\left|e_{j}(t)\right|, \tag{19}
\end{align*}
$$

with $\alpha_{i}, \beta_{i}>0$ being adaptation rates which are constant parameters. Taking the control law (17) into (12) and letting $K=\operatorname{diag}\left[k_{1}, \cdots, k_{n}\right]\left(k_{j}>0\right)$, we have

$$
\begin{align*}
&{ }_{0}^{C} D_{t}^{\mu} \mathbf{e}(t)=-K \mathbf{e}(t)+\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))  \tag{20}\\
& \cdot {\left[\mathbf{d}_{i}(t, \mathbf{y})-T \hat{\boldsymbol{\rho}}_{i}\left(\mathbf{y}, \boldsymbol{\theta}_{i}(t)\right)-H_{i} \operatorname{sign}(\mathbf{e}(t))\right] . }
\end{align*}
$$

Multiplying both sides of (20) by $\mathbf{e}^{T}(t)$ and letting $\widetilde{\varepsilon}_{i}^{* j}(\mathbf{y})=$ $\widehat{\varepsilon}_{i}^{* j}(\mathbf{y})-\varepsilon_{i}^{* j}$, one gets

$$
\begin{align*}
& \mathbf{e}^{T}(t){ }_{0}^{C} D_{t}^{\mu} \mathbf{e}(t) \leq-\mathbf{e}^{T}(t) K \mathbf{e}(t)-\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t)) \\
& \quad \cdot \sum_{j=1}^{n}\left|e_{j}(t)\right|\left[\widehat{\varepsilon}_{i}^{* j}(\mathbf{y})+\widehat{\rho}_{i}^{j}\left(\mathbf{y}, \theta_{i}^{j}(t)\right)-\rho_{i}^{j}(\mathbf{y})\right] \\
& \quad \leq-\mathbf{e}^{T}(t) K \mathbf{e}(t)+\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t))  \tag{21}\\
& \quad \cdot \sum_{j=1}^{n}\left|e_{j}(t)\right|\left[\varepsilon_{i}^{* j}-\widehat{\varepsilon}_{i}^{* j}(\mathbf{y})-\widetilde{\theta}_{i}^{j}(t) \varphi_{i}^{j}(\mathbf{y})\right]=-\mathbf{e}^{T}(t) \\
& \quad \cdot K \mathbf{e}(t)-\sum_{i=1}^{N} \mu_{i}(\mathbf{y}(t)) \\
& \quad \cdot \sum_{j=1}^{n}\left|e_{j}(t)\right|\left[\widetilde{\varepsilon}_{i}^{* j}(\mathbf{y})+\widetilde{\theta}_{i}^{j}(t) \varphi_{i}^{j}(\mathbf{y})\right] .
\end{align*}
$$

3.3. Stability Analysis. Here, fractional Lyapunov's theory is used to analyze the stability in closed-loop system. The following Lemmas are proposed to simplify the stability analysis.

Lemma 9 (see [54]). If $\mu \in(0,1)$ and $\mathbf{x}(t) \in C^{1}$, then ${ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)^{T} \mathbf{x}(t) \leq 2 \mathbf{x}(t)^{T}{ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)$.

Lemma 10. If ${ }_{0}^{C} D_{t}^{\mu} x(t) \leq 0$, then one gets $\mathbf{x}(t) \leq x(0)$; if ${ }_{0}^{C} D_{t}^{\mu} x(t) \geq 0$, then one gets $x(t) \geq x(0)$, with $\mu \in(0,1)$ and $t \in[0,+\infty)$.

Proof. We only consider the front part. If ${ }_{0}^{C} D_{t}^{\mu} x(t) \leq 0$, let

$$
\begin{equation*}
h(t)=-{ }_{0}^{C} D_{t}^{\mu} x(t) . \tag{22}
\end{equation*}
$$

Both sides of (22) take Laplace transform and one obtains

$$
\begin{equation*}
\mathscr{L}\left[{ }_{0}^{C} D_{t}^{\mu} x(t)\right]+H(s)=0 . \tag{23}
\end{equation*}
$$

Using Property 3 , one obtains the following:

$$
\begin{equation*}
s^{\mu} X(s)-s^{\mu-1} x(0)+H(s)=0 \tag{24}
\end{equation*}
$$

Further, one gets

$$
\begin{equation*}
X(s)=\frac{x(0)}{s}-\frac{H(s)}{s^{\mu}} \tag{25}
\end{equation*}
$$

with $X(s)=\mathscr{L}[x(t)], H(s)=\mathscr{L}[h(t)]$. Both sides of (25) make Laplace inverse transform and using the fractional integral definition, one obtains

$$
\begin{align*}
x(t) & =x(0)-{ }_{0}^{C} D_{t}^{-\mu} h(t) \\
& =x(0)-\frac{1}{\Gamma(\mu)} \int_{0}^{t}(t-\xi)^{\mu-1} h(\xi) d \xi . \tag{26}
\end{align*}
$$

From the above equation, we get $x(t) \leq x(0), t \in[0,+\infty)$.

Lemma 11. Let $V(t)=(1 / 2) \mathbf{x}(t)^{T} \mathbf{x}(t)+(1 / 2) \mathbf{y}(t)^{T} \mathbf{y}(t)$ with $\mathbf{x}(t), \mathbf{y}(t) \in R^{n}$ be continuous and derivable functions. If there exists a constant $h>0$ such that

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} V(t) \leq-h \mathbf{x}(t)^{T} \mathbf{x}(t), \tag{27}
\end{equation*}
$$

then $\|\mathbf{x}(t)\|$ and $\|\mathbf{y}(t)\|$ are bounded and $\lim _{t \rightarrow+\infty}\|\mathbf{x}(t)\|=0$.
Proof. According to Lemma 10 and ${ }_{0}^{C} D_{t}^{\mu} V(t) \leq$ $-h \mathbf{x}(t)^{T} \mathbf{x}(t) \leq 0$, we obtain $V(t) \leq V(0)$. Further, we get the following:

$$
\begin{align*}
\|\mathbf{x}(t)\| & \leq \sqrt{2 V(0)}, \\
\|\mathbf{y}(t)\| & \leq \sqrt{2 V(0)} \tag{28}
\end{align*}
$$

This means that $\|\mathbf{x}(t)\|$ and $\|\mathbf{y}(t)\|$ are bounded.
We will proof that $\mathbf{x}(t)$ tends to $\mathbf{0}$ asymptotically below. Both sides of (27) commute with $\mu$-order integral; based on Property 2, one gets

$$
\begin{equation*}
V(t)-V(0) \leq-h_{0}^{C} D_{t}^{-\mu} \mathbf{x}(t)^{T} \mathbf{x}(t) \tag{29}
\end{equation*}
$$

Further, one obtains

$$
\begin{equation*}
\mathbf{x}(t)^{T} \mathbf{x}(t) \leq 2 V(0)-2 h_{0}^{C} D_{t}^{-\mu} \mathbf{x}(t)^{T} \mathbf{x}(t) \tag{30}
\end{equation*}
$$

Hence, the following can be obtained from (30) with a nonnegative function $z(t)$ as

$$
\begin{equation*}
\mathbf{x}(t)^{T} \mathbf{x}(t)+z(t)=2 V(0)-2 h_{0}^{C} D_{t}^{-\mu} \mathbf{x}(t)^{T} \mathbf{x}(t) \tag{31}
\end{equation*}
$$

Applying the Laplace transform to formula (31) and according to the Definition 4, we have

$$
\begin{align*}
\mathscr{L}\left[\mathbf{x}(t)^{T} \mathbf{x}(t)\right]= & 2 V(0) \frac{s^{\mu-1}}{s^{\mu}+2 h}-Z(s) \frac{s^{\mu}}{s^{\mu}+2 h} \\
= & \mathscr{L}\left[2 V(0) E_{\mu, 1}\left(-2 h t^{\mu}\right)\right]  \tag{32}\\
& -\mathscr{L}[z(t)] \mathscr{L}\left[t^{-1} E_{\mu, 0}\left(-2 h t^{\mu}\right)\right]
\end{align*}
$$

Hence, using the Laplace inverse transform to (32), we have

$$
\begin{align*}
\mathbf{x}(t)^{T} \mathbf{x}(t)= & 2 V(0) E_{\mu, 1}\left(-2 h t^{\mu}\right)-z(t) \\
& *\left[t^{-1}\right] E_{\mu, 0}\left(-2 h t^{\mu}\right), \tag{33}
\end{align*}
$$

with $*$ being the convolution operator. Since $t^{-1}$ and $E_{\mu, 0}\left(-2 h t^{\mu}\right)$ are nonnegative functions, then $\mathbf{x}(t)^{T} \mathbf{x}(t) \leq$ $2 V(0) E_{\mu, 1}\left(-2 h t^{\mu}\right)$. According to the results in [55], one obtains that $\mathbf{x}(t)$ is M-L stability and $\mathbf{x}(t)$ tends to 0 asymptotically (namely, $\lim _{t \rightarrow \infty}\|\mathbf{x}(t)\|=0$ ).

From above discussion, the boundedness of all signals in closed-loop system and the convergence of tracking error based on adaptive fuzzy control scheme via T-S fuzzy logic systems is presented in the following theorem.

Theorem 12. For the master system (9) and slave system (10) under the known initial conditions, if Assumptions 5 and 6 are satisfied and the adaptive controller is given as (17) with the fractional adaptation laws (18) and (19), then all signals in the closed-loop system are bounded and the tracking error signal tends to zero asymptotically.

Proof. Define the following Lyapunov function:

$$
\begin{align*}
V(t)= & \frac{1}{2} \mathbf{e}^{T}(t) \mathbf{e}(t)+\sum_{i=1}^{N} \frac{1}{2 \alpha_{i}} \tilde{\boldsymbol{\theta}}_{i}^{T}(t) \widetilde{\boldsymbol{\theta}}_{i}(t) \\
& +\sum_{i=1}^{N} \frac{1}{2 \beta_{i}} \widetilde{\boldsymbol{\varepsilon}}_{i}^{* T}(\mathbf{y}) \widetilde{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y}) \tag{34}
\end{align*}
$$

with $\widetilde{\boldsymbol{\theta}}_{i}(t)=\boldsymbol{\theta}_{i}(t)-\boldsymbol{\theta}_{i}^{*}$ and $\widetilde{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y})=\widehat{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y})-\boldsymbol{\varepsilon}_{i}^{*}$. Hence, using the Lemma 9, the $\mu$-order derivative of $V(t)$ with respect to time $t$ is obtained as

$$
\begin{align*}
{ }_{0}^{C} D_{t}^{\mu} V(t) \leq & \mathbf{e}^{T}(t){ }_{0}^{C} D_{t}^{\mu} \mathbf{e}(t)+\sum_{i=1}^{N} \frac{1}{\alpha_{i}} \widetilde{\boldsymbol{\theta}}_{i}^{T}(t){ }_{0}^{C} D_{t}^{\mu} \widetilde{\boldsymbol{\theta}}_{i}(t)  \tag{35}\\
& +\sum_{i=1}^{N} \frac{1}{\beta} \widetilde{\boldsymbol{\varepsilon}}_{i}^{* T}(\mathbf{y}){ }_{0}^{C} D_{t}^{\mu} \widetilde{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y})
\end{align*}
$$

Substituting (21) into (35), one gets

$$
\begin{align*}
& { }_{0}^{C} D_{t}^{\mu} V(t) \leq-\mathbf{e}^{T}(t) K \mathbf{e}(t) \\
& \quad+\sum_{i=1}^{N} \sum_{j=1}^{n}\left[\frac{1}{\alpha_{i}} \widetilde{\theta}_{i}^{j}(t){ }_{0}^{C} D_{t}^{\mu} \widetilde{\theta}_{i}^{j}(t)\right. \\
& \quad-\mu_{i}(\mathbf{y}(t))\left|e_{j}(t)\right| \widetilde{\theta}_{i}^{j}(t) \varphi_{i}^{j}(\mathbf{y})  \tag{36}\\
& \quad+\frac{1}{\beta_{i}} \widetilde{\varepsilon}_{i}^{* j}(\mathbf{y}){ }_{0}^{C} D_{t}^{\mu} \widetilde{\varepsilon}_{i}^{* j}(\mathbf{y}) \\
& \left.\quad-\mu_{i}(\mathbf{y}(t))\left|e_{j}(t)\right| \widetilde{\varepsilon}_{i}^{* j}(\mathbf{y})\right]
\end{align*}
$$

Taking (18) and (19) into (36), one gets the following inequality:

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} V \leq-\mathbf{e}^{T}(t) K \mathbf{e}(t) \leq-\lambda_{\min } \mathbf{e}^{T}(t) \mathbf{e}(t), \tag{37}
\end{equation*}
$$

where $\lambda_{\text {min }}$ is the least eigenvalue of the positive definite matrix $K$. According to Lemma 11 and above discussion, we know that the tracking error signal $\mathbf{e}(t)$ tends to 0 asymptotically (that is, $\lim _{t \rightarrow \infty}\|\mathbf{e}(t)\|=0$ ) and $\widetilde{\boldsymbol{\theta}}_{i}(t)$ and $\widetilde{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y})$ are bounded. Further, it means that $\boldsymbol{\theta}_{i}(t)$ and $\widehat{\boldsymbol{\varepsilon}}_{i}^{*}(\mathbf{y})$ are bounded. Because of the boundedness of $\mathbf{e}(t)$ and $\mathbf{x}(t)$, we know that $\mathbf{y}(t)$ is bounded. Based on the control design, $\mathbf{u}(t)$ is bounded. Therefore, we know that all signals in the closedloop system are bounded.

## 4. Simulation Example

In this section, in order to further illustrate the effectiveness of the proposed control method designed in previous
sections, one example about the synchronization for two different uncertain fractional-order chaotic system is given. The master system of a fractional-order chaotic system via T$S$ fuzzy model is given as

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\mu} \mathbf{x}(t)=\sum_{i=1}^{N} \mu_{i}(\mathbf{x}(t))\left[A_{i} \mathbf{x}(t)+\mathbf{b}_{1}\right] \tag{38}
\end{equation*}
$$

the ith rule of master system is given by

$$
\begin{aligned}
& R^{1}: \text { If } x_{1} \text { is } F_{1}^{1} \text { and } x_{2} \text { is } F_{2}^{1} \text { and } x_{3} \text { is } F_{3}^{1} \text {, then } \\
& { }_{0}^{C} D_{t}^{0.8} \mathbf{x}(t)=A_{1} \mathbf{x}(t)+\mathbf{b}_{1} \text {, } \\
& R^{2}: \text { If } x_{1} \text { is } F_{1}^{2} \text { and } x_{2} \text { is } F_{2}^{2} \text { and } x_{3} \text { is } F_{3}^{2} \text {, then } \\
& { }_{0}^{C} D_{t}^{0.8} \mathbf{x}(t)=A_{2} \mathbf{x}(t)+\mathbf{b}_{1} .
\end{aligned}
$$

The upper system is formulated to the alike form in (9) with

$$
\begin{align*}
A_{1} & =\left[\begin{array}{ccc}
-10 & 10 & 0 \\
40 & 0 & -62.6 \\
25.04 & 0 & -8
\end{array}\right], \\
A_{2} & =\left[\begin{array}{ccc}
-10 & 10 & 0 \\
40 & 0 & 60.7 \\
-24.28 & 0 & -8
\end{array}\right],  \tag{39}\\
\mathbf{b}_{1}(t) & =(0,0,0)^{T}
\end{align*}
$$

Figure 1 depicts the simulation results of the master system with the parameters $N=2, \mu=0.8$ with time step $h=$ 0.005 . Figure 1 shows $x_{1}(t), x_{2}(t)$ and $x_{3}(t) \in[-6.07,6.26]$, $[-10,10.6]$, and $[0.2,9.05]$, respectively, that is, $\mathbf{x}(t) \in D_{1}=$ $[-6.07,6.26] \times[-10,10.6] \times[0.2,9.05]$. Obviously, Chaos was found in system (38) with $\mu=0.8$.

Two fuzzy sets are defined for the state $x_{1}$ over the interval $[-6.07,6.26]$ with the membership functions as

$$
\begin{align*}
& F_{1}^{1}\left(x_{1}(t)\right)=\frac{1}{2}\left(1-\frac{0.095-x_{1}(t)}{6.165}\right),  \tag{40}\\
& F_{1}^{2}\left(x_{1}(t)\right)=\frac{1}{2}\left(1+\frac{0.095-x_{1}(t)}{6.165}\right) .
\end{align*}
$$

Two fuzzy sets are defined for the state $x_{2}$ over the interval $[-10,10.6]$ with the membership functions as

$$
\begin{align*}
& F_{2}^{1}\left(x_{2}(t)\right)=\frac{1}{2}\left(1-\frac{0.3-x_{2}(t)}{10.3}\right),  \tag{41}\\
& F_{2}^{2}\left(x_{2}(t)\right)=\frac{1}{2}\left(1-\frac{0.3-x_{2}(t)}{10.3}\right) .
\end{align*}
$$

Two fuzzy sets are defined for the state $x_{3}$ over the interval [ $0.2,9.05$ ] with the membership functions as

$$
\begin{align*}
& F_{3}^{1}\left(x_{3}(t)\right)=\frac{1}{2}\left(1-\frac{4.625-x_{3}(t)}{4.425}\right),  \tag{42}\\
& F_{3}^{2}\left(x_{3}(t)\right)=\frac{1}{2}\left(1-\frac{4.625-x_{3}(t)}{4.425}\right) .
\end{align*}
$$

The slave system of a fractional-order chaotic system with unknown disturbances via T-S fuzzy model is given as

$$
\begin{align*}
&{ }_{0}^{C} D_{t}^{0.8} \mathbf{y}(t) \\
&=\sum_{i=1}^{2} \mu_{i}(\mathbf{y}(t))\left[B_{i} \mathbf{y}(t)+\mathbf{b}_{2}+\mathbf{u}(t)+\mathbf{d}_{i}(t, \mathbf{y})\right] . \tag{43}
\end{align*}
$$

The ith rule of slave system is given by
$R^{1}$ : If $y_{1}$ is $\widehat{F}_{1}^{1}$ and $y_{2}$ is $\widehat{F}_{2}^{1}$ and $y_{3}$ is $\widehat{F}_{3}^{1}$, then ${ }_{0}^{C} D_{t}^{0.8} \mathbf{y}(t)=B_{1} \mathbf{y}(t)+\mathbf{b}_{2}+\mathbf{u}(t)+\mathbf{d}_{1}(t, \mathbf{y})$,
$R^{2}$ : If $y_{1}$ is $\widehat{F}_{1}^{2}$ and $y_{2}$ is $\hat{F}_{2}^{2}$ and $y_{3}$ is $\hat{F}_{3}^{2}$, then ${ }_{0}^{C} D_{t}^{0.8} \mathbf{y}(t)=B_{2} \mathbf{y}(t)+\mathbf{b}_{2}+\mathbf{u}(t)+\mathbf{d}_{2}(t, \mathbf{y})$.
The upper system is formulated to the alike form in (10) with

$$
\begin{align*}
& B_{1}=\left[\begin{array}{ccc}
-30 & 30 & 0 \\
0 & 22.2 & -21.52 \\
0 & 21.52 & -2.94
\end{array}\right], \\
& B_{2}=\left[\begin{array}{ccc}
-30 & 30 & 0 \\
0 & 22.2 & 29.21 \\
0 & -29.21 & -2.94
\end{array}\right],  \tag{44}\\
& \mathbf{b}_{2}(t)=(0,0,0)^{T} .
\end{align*}
$$

Figure 2 with $\mathbf{u}(t)=\mathbf{0}$ and without the external disturbance is depicted the simulation results of the slave system with the parameters below: $N=2, \mu=0.8$, for time step $h=0.005$. Moreover, Chaos was found in system (43) with $\mu=0.8$. Figure 2 shows $y_{1}(t), y_{2}(t)$ and $y_{3}(t) \in$ [ $-29.21,21.52],[-35.6,26.5]$ and $[0,53.6]$, respectively, that is, $\mathbf{y}(t) \in D_{2}=[-29.21,21.52] \times[-35.6,26.5] \times[0,53.6]$.

Two fuzzy sets are defined for the state $y_{1}$ over the interval [-29.21,21.52] with the membership functions as follows:

$$
\begin{align*}
& \widehat{F}_{1}^{1}\left(y_{1}(t)\right)=\frac{1}{2}\left(1-\frac{-3.845-y_{1}(t)}{25.365}\right), \\
& \widehat{F}_{1}^{2}\left(y_{1}(t)\right)=\frac{1}{2}\left(1+\frac{-3.845-y_{1}(t)}{25.365}\right) . \tag{45}
\end{align*}
$$

Two fuzzy sets are defined for the state $y_{2}$ over the interval [-35.6, 26.5] with the membership functions as follows:

$$
\begin{align*}
& \widehat{F}_{2}^{1}\left(y_{2}(t)\right)=\frac{1}{2}\left(1-\frac{-4.55-y_{2}(t)}{31.05}\right),  \tag{46}\\
& \widehat{F}_{2}^{2}\left(y_{2}(t)\right)=\frac{1}{2}\left(1-\frac{-4.55-y_{2}(t)}{31.05}\right) .
\end{align*}
$$

Two fuzzy sets are defined for the state $y_{3}$ over the interval [ $0,53.6$ ] with the membership functions as follows:

$$
\begin{align*}
& \widehat{F}_{3}^{1}\left(y_{3}(t)\right)=\frac{1}{2}\left(1-\frac{26.8-y_{3}(t)}{26.8}\right),  \tag{47}\\
& \widehat{F}_{3}^{2}\left(y_{3}(t)\right)=\frac{1}{2}\left(1-\frac{26.8-y_{3}(t)}{26.8}\right) .
\end{align*}
$$



Figure 1: Master system.

In the simulation, the initial conditions of master system and slave system are selected as $\mathbf{x}(0)=(1,1,-2)^{T}$ and $\mathbf{y}(0)=$ $(-2,-3,3)^{T}$. The parameters relating the synchronization problem are set to $K=E$ and $\rho_{1}(y)=\rho_{2}(y)=$ $\left(1.5 y_{1}, 1.5 y_{2}, 1.5 y_{3}\right)^{T}$. Let $\mathbf{e}(t)=\left(e_{1}(t), e_{2}(t), e_{3}(t)\right)^{T}$.

The controller is designed as

$$
\begin{aligned}
\mathbf{u}(t)= & \left(u_{1}(t), u_{2}(t), u_{3}(t)\right)^{T} \\
= & -\widehat{F}_{1}^{1}\left(y_{1}(t)\right) B_{1} \mathbf{y}(t)-T \hat{\boldsymbol{\rho}}(\mathbf{y}, \boldsymbol{\theta}(t))-\widehat{F}_{1}^{2} B_{2} \mathbf{y}(t) \\
& -H \operatorname{sign}(\mathbf{e}(t))-\mathbf{e}(t)+F_{1}^{1} A_{1} \mathbf{x}(t) \\
& +F_{1}^{2} A_{2} \mathbf{x}(t)
\end{aligned}
$$

Let $\hat{\boldsymbol{\rho}}(\mathbf{y}, \boldsymbol{\theta}(t))=\left(\hat{\rho}^{1}, \widehat{\rho}^{2}, \widehat{\rho}^{3}\right)^{T}, H=\left(\widehat{\varepsilon}^{* 1}(\mathbf{y}), \widehat{\varepsilon}^{* 2}(\mathbf{y})\right.$, and $\left.\widehat{\mathcal{\varepsilon}}^{* 3}(\mathbf{y})\right)^{T}$; then

$$
\begin{align*}
u_{1}(t)= & -e_{1}(t)-10 x_{1}+10 x_{2}+30 y_{1}-30 y_{2}-T \hat{\rho}^{1} \\
& -\widehat{\varepsilon}^{* 1}(\mathbf{y}), \\
u_{2}(t)= & -e_{2}(t)+40 x_{1}-10 x_{1} x_{3}-22.2 y_{2}+y_{1} y_{3} \\
& -T \widehat{\rho}^{2}-\widehat{\varepsilon}^{* 2}(\mathbf{y}),  \tag{49}\\
u_{3}(t)= & -e_{3}(t)-8 x_{3}+4 x_{1}^{2}-y_{1} y_{2}+2.94 y_{3}-T \widehat{\rho}^{3} \\
& -\widehat{\varepsilon}^{* 3}(\mathbf{y}) .
\end{align*}
$$



Figure 2: Slave system.

The fractional adaptation laws of $\widetilde{\theta}^{j}(\mathbf{y})$ and $\widetilde{\varepsilon}^{* j}(\mathbf{y}),(j=$ $1,2,3$ ), with $\alpha=700, \beta=0.05$ are designed to be

$$
\begin{align*}
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\theta}^{1}(t) & =\alpha\left|e_{1}(t)\right| \varphi^{1}(\mathbf{y}), \\
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\varepsilon}^{* 1}(\mathbf{y}) & =\beta\left|e_{1}(t)\right|, \\
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\theta}^{2}(t) & =\alpha\left|e_{2}(t)\right| \varphi^{2}(\mathbf{y}), \\
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\varepsilon}^{* 2}(\mathbf{y}) & =\beta\left|e_{2}(t)\right|,  \tag{50}\\
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\theta}^{3}(t) & =\alpha\left|e_{3}(t)\right| \varphi^{3}(\mathbf{y}), \\
{ }_{0}^{C} D_{t}^{\mu} \widetilde{\varepsilon}^{* 3}(\mathbf{y}) & =\beta\left|e_{3}(t)\right| .
\end{align*}
$$

The simulation results of the proposed adaptive control approach are shown in Figure 3, where subgraph (a) denotes the tracking error trajectory and subgraph (b) denotes the
control trajectory. Define the initial conditions of the approximation errors as $\widehat{\varepsilon}^{* 1}(0)=0, \widehat{\varepsilon}^{* 2}(0)=0, \widehat{\varepsilon}^{* 3}(0)=0$. In reducing the computation of the numerical simulation, $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are replaced by $\mathbf{e}(t)$. Four fuzzy sets are defined for the tracking errors $e_{1}(t), e_{2}(t), e_{3}(t)$ over the interval $[-3,3]$ with the Gaussian membership functions, where the first parameters are 1.1 and the second parameters are $-3,-1,1,3$, respectively. Comparing the conventional control method with the proposed method, we can see that the proposed approach can synchronize two chaotic plants to desired high accuracy and improve the performance as shown in Figure 3.

## 5. Conclusions

In this paper, synchronization of different fractional-order chaotic or hyperchaotic systems with unknown disturbances


Figure 3: (a) Synchronization error and (b) controller.
and parametric uncertainties is addressed with adaptive fuzzy control algorithm based on T-S fuzzy models. The distinctive features of the proposed control approach are that T-S fuzzy logic systems are introduced to approximate the unknown disturbances and to model the unknown controlled systems; both adaptive fuzzy controller and fractional adaptation laws are developed based on combined fractional Lyapunov stability theory and parallel distributed compensation technique. It is shown that the proposed control method can guarantee that all the signals in the closed-loop system remain bounded and the synchronization error converges towards an arbitrary small neighbourhood of the origin asymptotically. A simulation example is used for verifying the effectiveness of the proposed control strategy. Further works would focus on chaos synchronization control of different uncertain fractional-order chaotic systems with time delay and input saturation.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors do not have a direct financial relation with any commercial identity mentioned in their paper that might lead to conflicts of interest for any of the authors.

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## Research Article

# Positive Solutions for a Fractional Boundary Value Problem with a Perturbation Term 

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We obtain some new upper and lower estimates for the Green's function associated with a fractional boundary value problem with a perturbation term. Criteria for the existence of positive solutions of the problem are then obtained based on it.

## 1. Introduction

In this paper, we are investigating the existence of positive solution for fractional differential equation with a perturbation term

$$
\begin{equation*}
-D^{\alpha} x(t)+a(t) x(t)=f(t, x(t)), \quad t \in(0,1) \tag{1}
\end{equation*}
$$

with the boundary condition (BC)

$$
\begin{equation*}
x(0)=x^{\prime}(0)=x^{\prime}(1)=0 \tag{2}
\end{equation*}
$$

where $2<\alpha<3, a \in C[0,1]$, and $f \in C([0,1] \times[0,+\infty), \mathbb{R})$. Here, $D^{\alpha} x$ is the standard Riemann-Liouville derivative of order $\alpha>0$ of a continuous function $x:(0, \infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{align*}
D^{\alpha} x(t)= & \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{\alpha-n+1}} d s,  \tag{3}\\
& n-1 \leq \alpha<n
\end{align*}
$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Spurred by the extensively applicability of fractional derivatives in a variety of mathematical models in science and engineering [1-3], the subject of fractional differential equations with boundary value problems, which emerged as a new branch of differential equations, have attracted a great deal of attention for decades. As a small sampling of recent development, we refer the reader to [4-14]. When one
seeks the existence of solution of boundary value problems for fractional differential equations, the usual method is converted to a Fredholm integral equation and find the fixed points by using various techniques of nonlinear analysis such as Banach contraction map principle [13, 15], linear operator theory [16, 17], Leggett-Williams fixed point theorem [12, 18], Schauder fixed point theorem and Leray-Schauder nonlinear alternative theory [19], and Krasnosel'skii fixed point theorem [20]. It should be noted that the Green's functions play a vital role in the construction of an appropriate Fredholm integral equation. However, as a result of the unusual feature of the fractional calculus, the investigation on the Green's functions for fractional boundary value problems is still in the initial stage. Recently, based on the spectral theory, the authors in [21] give an associated Green's function for BVP (1) (2) as series of functions. This idea was also used in [22-24].

In the next section, we will study some new sharper upper and lower estimates for the Green's function of BVP (1) (2) than the ones given in [21]. In Section 3, we employ the new estimate to obtain the existence of a positive solution of BVP (1) (2). The idea of this paper may trace to [21-27].

## 2. Some New Upper and Lower Estimates for the Green's Function

Firstly, we present the Green's function for BVP (1) (2) which is given in [21]. Let $G_{0}:[0,1] \times[0,1] \longrightarrow \mathbb{R}$ be defined by

$$
G_{0}(t, s)
$$

$$
= \begin{cases}\frac{t^{\alpha-1}(1-s)^{\alpha-2}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1  \tag{4}\\ \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1\end{cases}
$$

It is well known that the function $G_{0}(t, s)$ is Green's function for BVP (1) (2) with $a(t) \equiv 0$. In the following lemma we present some properties of Green's function $G_{0}(t, s)$, see [28] for details. Listed properties will be used later for estimating the upper bound and the lower bound on Green's function $G(t, s)$ of BVP (1) (2).

Lemma 1 (see [28]). The function $G_{0}(t, s)$ defined by (4) satisfies the following conditions:
(i) $0 \leq G_{0}(t, s) \leq t^{\alpha-1}(1-s)^{\alpha-2} / \Gamma(\alpha), 0 \leq t, s \leq 1$,
(ii) $t^{\alpha-1} G_{0}(1, s) \leq G_{0}(t, s) \leq G_{0}(1, s)=s(1-s)^{\alpha-2} /$

$$
\Gamma(\alpha), \text { for } 0 \leq t, s \leq 1
$$

Let $M=\int_{0}^{1}\left(|a(s)|(1-s)^{\alpha-2} / \Gamma(\alpha)\right) s^{\alpha-1} d s, M_{1}=\int_{0}^{1}(|a(s)|$ $\left.(1-s)^{\alpha-2} / \Gamma(\alpha)\right) d s, A=\max _{t \in[0,1]}|a(t)|$. For $n=1,2, \ldots$, define

$$
\begin{align*}
G_{n}(t, s)= & \int_{0}^{1} a(\tau) G_{0}(t, \tau) G_{n-1}(\tau, s) d \tau \\
= & \int_{0}^{1} \cdots \int_{0}^{1} a\left(r_{1}\right) G_{0}\left(t, r_{1}\right)  \tag{5}\\
& \cdot a\left(r_{2}\right) G_{0}\left(r_{1}, r_{2}\right) \cdots a\left(r_{n}\right) G_{0}\left(r_{n-1}, r_{n}\right) \\
& \cdot G_{0}\left(r_{n}, s\right) d r_{1} \cdots d r_{n}
\end{align*}
$$

and

$$
\begin{equation*}
G(t, s)=\sum_{n=0}^{+\infty}(-1)^{n} G_{n}(t, s) \tag{6}
\end{equation*}
$$

It follows from Theorem 2.1 in [21] that the function $G(t, s)$ defined by (6) as a series of functions that converge uniformly is the Green's function for BVP (1) (2) if $A<(\alpha-$ 1) $\Gamma(\alpha+1)$ holds. Furthermore, the function $G(t, s)$ satisfies the following property:

$$
\begin{align*}
&(1-\delta) G_{0}(t, s) \leq G(t, s) \leq(1+\delta) G_{0}(t, s)  \tag{7}\\
& t, s \in[0,1]
\end{align*}
$$

provided $A<(\alpha-1) \Gamma(\alpha+1)(\alpha+1)^{-1}$, where $\delta=\alpha A /((\alpha-$ 1) $\Gamma(\alpha+1)-A)<1$.

The uniform convergence of (6) follows from the fact that $\|T\|<1$, where the operator $T$ is defined by the following form:

$$
\begin{equation*}
(T x)(t)=\int_{0}^{1} G_{0}(t, s) a(s) x(s) d s, \quad x \in C[0,1] \tag{8}
\end{equation*}
$$

Indeed, the uniform convergence of (6) can be obtained by $r(T)<1$ (see [21, 29]), where $r(T)$ is the spectral radius of $T$.

Lemma 2. If $M<1$ holds, then $r(T)<1$.
Proof. By Lemma 1, for any $x \in C[0,1]$, we have

$$
\begin{align*}
|(T x)(t)| & \leq \int_{0}^{1} G_{0}(t, s)|a(s) x(s)| d s \\
& \leq \frac{t^{\alpha-1} A\|x\|}{\Gamma(\alpha)} \int_{0}^{1}(1-s)^{\alpha-2} d s  \tag{9}\\
& =\frac{A\|x\|}{(\alpha-1) \Gamma(\alpha)} t^{\alpha-1}
\end{align*}
$$

Hence, we conclude that

$$
\begin{align*}
& \left|\left(T^{2} x\right)(t)\right| \leq \int_{0}^{1} G_{0}(t, s)|a(s)(T x)(s)| d s \\
& \quad \leq \frac{A\|x\|}{(\alpha-1) \Gamma(\alpha)} \int_{0}^{1} \frac{t^{\alpha-1}|a(s)|(1-s)^{\alpha-2}}{\Gamma(\alpha)} s^{\alpha-1} d s  \tag{10}\\
& \quad=\frac{A M\|x\|}{(\alpha-1) \Gamma(\alpha)} t^{\alpha-1}
\end{align*}
$$

By induction, one has

$$
\begin{equation*}
\left|\left(T^{n} x\right)(t)\right| \leq \frac{A M^{n-1}\|x\|}{(\alpha-1) \Gamma(\alpha)} t^{\alpha-1} \tag{11}
\end{equation*}
$$

which implies that

$$
\begin{equation*}
\left\|T^{n}\right\| \leq \frac{A M^{n-1}}{(\alpha-1) \Gamma(\alpha)} \tag{12}
\end{equation*}
$$

Note that $M<1$. Then by the Gelfand formula, we get

$$
\begin{equation*}
r(T)=\lim _{n \rightarrow+\infty} \sqrt[n]{\left\|T^{n}\right\|} \leq M<1 \tag{13}
\end{equation*}
$$

Lemma 3. If $M+M_{1}<1$ holds, then for any $t, s \in[0,1]$,

$$
\begin{align*}
\frac{1-M-M_{1}}{1-M} G_{0}(t, s) & \leq G(t, s) \\
& \leq \frac{1-M+M_{1}}{1-M} G_{0}(t, s) \tag{14}
\end{align*}
$$

Proof. By Lemma 1, for $n \geq 1$ and $t, s \in[0,1]$, we have

$$
\begin{aligned}
& \left|G_{n}(t, s)\right| \leq \int_{0}^{1} \cdots \int_{0}^{1}\left|a\left(r_{1}\right)\right| G_{0}\left(t, r_{1}\right) \cdot\left|a\left(r_{2}\right)\right| \\
& \cdot G_{0}\left(r_{1}, r_{2}\right) \cdots\left|a\left(r_{n}\right)\right| G_{0}\left(r_{n-1}, r_{n}\right) \\
& \cdot G_{0}\left(r_{n}, s\right) d r_{1} \cdots d r_{n} \\
& \quad \leq \int_{0}^{1} \cdots \int_{0}^{1}\left|a\left(r_{1}\right)\right| \frac{t^{\alpha-1}\left(1-r_{1}\right)^{\alpha-2}}{\Gamma(\alpha)} \cdot\left|a\left(r_{2}\right)\right| \\
& \quad \cdot \frac{r_{1}^{\alpha-1}\left(1-r_{2}\right)^{\alpha-2}}{\Gamma(\alpha)} \cdots \frac{\left|a\left(r_{n}\right)\right| r_{n-1}^{\alpha-1}\left(1-r_{n}\right)^{\alpha-2}}{\Gamma(\alpha)}
\end{aligned}
$$

$$
\begin{align*}
& \cdot \frac{s(1-s)^{\alpha-2}}{\Gamma(\alpha)} d r_{1} \cdots d r_{n}=\frac{t^{\alpha-1} s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \\
& \cdot \int_{0}^{1} \frac{\left|a\left(r_{1}\right)\right| r_{1}^{\alpha-1}\left(1-r_{1}\right)^{\alpha-2}}{\Gamma(\alpha)} d r_{1} \\
& \cdot \int_{0}^{1} \frac{\left|a\left(r_{2}\right)\right| r_{2}^{\alpha-1}\left(1-r_{2}\right)^{\alpha-2}}{\Gamma(\alpha)} d r_{2} \\
& \cdots \int_{0}^{1} \frac{\left|a\left(r_{n-1}\right)\right| r_{n-1}^{\alpha-1}\left(1-r_{n-1}\right)^{\alpha-2}}{\Gamma(\alpha)} d r_{n-1} \\
& \cdot \int_{0}^{1} \frac{\left|a\left(r_{n}\right)\right|\left(1-r_{n}\right)^{\alpha-2}}{\Gamma(\alpha)} d r_{n}=M_{1} M^{n-1} \\
& \cdot \frac{t^{\alpha-1} s(1-s)^{\alpha-2}}{\Gamma(\alpha)} \leq M_{1} M^{n-1} G_{0}(t, s), \\
& \quad t, s \in[0,1] . \tag{15}
\end{align*}
$$

Then introducing the above inequality into (6) can lead to

$$
\begin{align*}
\left|\sum_{n=1}^{+\infty}(-1)^{n} G_{n}(t, s)\right| & \leq \sum_{n=1}^{+\infty}\left|G_{n}(t, s)\right| \\
& \leq \sum_{n=1}^{+\infty} M_{1} M^{n-1} G_{0}(t, s)  \tag{16}\\
& =\frac{M_{1}}{1-M} G_{0}(t, s), \quad t, s \in[0,1]
\end{align*}
$$

It is easy to verify that if $M+M_{1}<1$, then $M_{1} /(1-M)<1$. Therefore, (14) follows from (6) and (16).

Similar to the proof of Lemmas 2 and 3, we can obtain the following results.

Lemma 4. If $A<\Gamma(2 \alpha-1) / \Gamma(\alpha-1)$ holds, then $r(T)<1$.
Lemma 5. If $A /(\alpha-1) \Gamma(\alpha)+A \Gamma(\alpha-1) / \Gamma(2 \alpha-1)<1$ holds, then for any $t, s \in[0,1]$,

$$
\begin{equation*}
(1-\gamma) G_{0}(t, s) \leq G(t, s) \leq(1+\gamma) G_{0}(t, s) \tag{17}
\end{equation*}
$$

where $\gamma=(A /(\alpha-1) \Gamma(\alpha))(1-A \Gamma(\alpha-1) / \Gamma(2 \alpha-1))^{-1}$.
By the properties of definite integral and $\alpha \in(2,3)$, we assert that

$$
\begin{equation*}
\int_{0}^{1} \frac{\tau^{\alpha-1}(1-\tau)^{\alpha-2}}{\Gamma(\alpha)} d \tau<\int_{0}^{1} \frac{\tau(1-\tau)^{\alpha-2}}{\Gamma(\alpha)} d \tau \tag{18}
\end{equation*}
$$

that is

$$
\begin{equation*}
\frac{\Gamma(\alpha-1)}{\Gamma(2 \alpha-1)}<\frac{1}{(\alpha-1) \Gamma(\alpha+1)} \tag{19}
\end{equation*}
$$

Thus, we obtain that

$$
\begin{align*}
\gamma & =\frac{A /(\alpha-1) \Gamma(\alpha)}{1-A \Gamma(\alpha-1) / \Gamma(2 \alpha-1)}  \tag{20}\\
& <\frac{\alpha A /(\alpha-1) \Gamma(\alpha+1)}{1-A /(\alpha-1) \Gamma(\alpha+1)}=\delta .
\end{align*}
$$

This means that Lemmas $2-5$ is more general and complements many known results.

Combining Lemmas 1 and 3, we obtain the following result.

Theorem 6. If $M+M_{1}<1$ holds. Then for any $t, s \in[0,1]$,

$$
\begin{align*}
& \frac{1-M-M_{1}}{1-M} t^{\alpha-1} G_{0}(1, s) \leq G(t, s)  \tag{21}\\
& \quad \leq \frac{1-M+M_{1}}{1-M} G_{0}(1, s) .
\end{align*}
$$

Theorem 7. If $x(t)$ satisfies the boundary conditions (2),

$$
\begin{equation*}
-D^{\alpha} x(t)+a(t) x(t) \geq 0 \tag{22}
\end{equation*}
$$

If $M+M_{1}<1$ holds, then

$$
\begin{equation*}
x(t) \geq \frac{1-M-M_{1}}{1-M+M_{1}} t^{\alpha-1}\|x\|, \quad t \in[0,1] \tag{23}
\end{equation*}
$$

## 3. Existence Theorems

In this section, we shall employ Theorem 6 to investigate the existence results for BVP (1) (2). Let $C[0,1]$ be the Banach space endowed with the maximum norm $\|x\|=$ $\max _{t \in[0,1]}|x(t)|$.

Theorem 8. Assume that there exist $c_{2}>c_{1}>0$ such that

$$
\begin{equation*}
\inf _{x \in \Omega} \int_{0}^{1} G_{0}(1, s) f(s, x(s)) d s \geq \frac{c_{1}(1-M)}{1-M-M_{1}} \tag{24}
\end{equation*}
$$

and

$$
\begin{equation*}
\sup _{x \in \Omega} \int_{0}^{1} G_{0}(1, s) f(s, x(s)) d s \leq \frac{c_{2}(1-M)}{1-M+M_{1}}, \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\Omega=\left\{x \in C[0,1]: c_{1} t^{\alpha-1} \leq x(t) \leq c_{2}, t \in[0,1]\right\} . \tag{26}
\end{equation*}
$$

Then BVP (1) (2) has at least one positive solution in $\Omega$.
Proof. Define an operator $T$ by

$$
\begin{equation*}
(S x)(t)=\int_{0}^{1} G(t, s) f(s, x(s)) d s, \quad x \in C[0,1] \tag{27}
\end{equation*}
$$

where $G(t, s)$ is given by (6). Obviously, $x(t)$ is a solution of BVP (1)(2) if and only if $x \in C[0,1]$ is a fixed point of $S$. Moreover, we can show that $S: C[0,1] \longrightarrow C[0,1]$ is completely continuous.

For any given $x \in \Omega$, by (24) and (25), we conclude that

$$
(S x)(t)
$$

$$
\begin{align*}
& \geq \frac{1-M-M_{1}}{1-M} t^{\alpha-1} \inf _{x \in S} \int_{0}^{1} G_{0}(1, s) f(s, x(s)) d s  \tag{28}\\
& \geq c_{1} t^{\alpha-1}
\end{align*}
$$

and

$$
\begin{align*}
(S x)(t) & \leq \frac{1-M+M_{1}}{1-M} \sup _{x \in S} \int_{0}^{1} G_{0}(1, s) f(s, x(s)) d s  \tag{29}\\
& \leq c_{2}
\end{align*}
$$

Therefore, $S(\Omega) \subset \Omega$. By Schauder's fixed point theorem, $S$ has a fixed point $x$ in $\Omega$ which implies that BVP (1) (2) has at least one positive solution in $\Omega$.

The following corollaries are direct results of Theorem 8.
Corollary 9. Assume that there exist $c_{2}>c_{1}>0$ such that for any $t \in[0,1], f(t, \cdot)$ is nondecreasing on $\left[0, c_{2}\right]$,

$$
\begin{equation*}
\int_{0}^{1} G_{0}(1, s) f\left(s, c_{1} s^{\alpha-1}\right) d s \geq \frac{c_{1}(1-M)}{1-M-M_{1}} \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{1} G_{0}(1, s) f\left(s, c_{2}\right) d s \leq \frac{c_{2}(1-M)}{1-M+M_{1}} \tag{31}
\end{equation*}
$$

Then BVP (1) (2) has at least one positive solution in $\Omega$.
Corollary 10. Assume that there exist $c_{2}>c_{1}>0$ such that for any $t \in[0,1], f(t, \cdot)$ is nonincreasing on $\left[0, c_{2}\right]$,

$$
\begin{equation*}
\int_{0}^{1} G_{0}(1, s) f\left(s, c_{1} s^{\alpha-1}\right) d s \leq \frac{c_{2}(1-M)}{1-M+M_{1}} \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
\int_{0}^{1} G_{0}(1, s) f\left(s, c_{2}\right) d s \geq \frac{c_{1}(1-M)}{1-M-M_{1}} \tag{33}
\end{equation*}
$$

Then $B V P$ (1) (2) has at least one positive solution in $\Omega$.
Example 11. Consider the BVP

$$
\begin{align*}
-D^{5 / 2} x(t)+t(1-t) x(t) & =\sqrt{x(t)}, \quad t \in(0,1)  \tag{34}\\
x(0) & =x^{\prime}(0)=x^{\prime}(1)=0
\end{align*}
$$

After simple computation, we have $M=\Gamma(\alpha+1) / \Gamma(2 \alpha+1)=$ $\sqrt{\pi} / 64, M_{1}=1 / \Gamma(\alpha+2)=16 \sqrt{\pi} / 105 \pi$.

Let $f(t, x)=\sqrt{x}$. It is easy to see that (30) and (31) hold when $c_{1}$ is small enough and $c_{2}$ is large enough. Then, by Corollary 9, BVP (34) has at least one solution.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Research Article 

# The Exact Iterative Solution of Fractional Differential Equation with Nonlocal Boundary Value Conditions 

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#### Abstract

We deal with a singular nonlocal fractional differential equation with Riemann-Stieltjes integral conditions. The exact iterative solution is established under the iterative technique. The iterative sequences have been proved to converge uniformly to the exact solution, and estimation of the approximation error and the convergence rate have been derived. An example is also given to demonstrate the results.


## 1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines; see [1-5]. Much attention has been paid to study fractional differential equations both with initial and boundary conditions; see, for example, $[6,7]$. In $[8,9]$, they focused on sign-changing solution for some fractional differential equations. In [10], they get the existence of solutions for impulsive fractional differential equations. In [11-13], they get the existence and multiplicity of nontrivial solutions for a class of fractional differential equations. The mainly techniques authors need are fixed point theory, variational method, and global bifurcation techniques.

Also, ordinary differential equations and partial differential equations involving nonlocal boundary conditions have been studied extensively in recent years, see [14-22], including integral boundary conditions and multipoint boundary conditions.

In [23], authors obtained results on the uniqueness of positive solution for problem

$$
\begin{aligned}
\mathrm{D}^{p} x(t)+p(t) f(t, x(t))+q(t) & =0, \quad t \in(0,1), \\
x(0) & =x^{\prime}(0)=0, \\
x(1) & =0,
\end{aligned}
$$

where $2<p \leq 3$ is a real number. Under the assumption that

$$
\begin{equation*}
|f(t, u)-f(t, v)| \leq k \lambda_{1}|u-v|, \tag{2}
\end{equation*}
$$

where $k \in[0,1)$, and $\lambda$ is the first eigenvalue of the corresponding linear operator.

Motivated by the above works, we study the following nonlocal boundary value problems:

$$
\begin{align*}
\mathrm{D}^{q} x(t)+f(t, x(t)) & =0, \quad t \in(0,1), \\
x(1) & =x^{\prime}(1)=0,  \tag{3}\\
x(0) & =\int_{0}^{1} x(t) d \Lambda(t),
\end{align*}
$$

where $D^{q} x$ denotes the left-handed Riemann-Liouville derivative of order q and $2<q \leq 3$ is a real number. $\lambda[x]=\int_{0}^{1} x(t) d \Lambda(t)$ denotes a Stieltjes integral with a suitable function $\Lambda$ of bounded variation. Different from [23] and other works, we only use the iterative methods to obtain the existence and uniqueness of positive solution. Moreover, the estimation of the approximation error and the convergence rate have also been derived.

For clarity in presentation, we also list below some assumptions to be used later in the paper.
$\left(H_{1}\right): f:(0,1) \times(0,+\infty) \longrightarrow[0,+\infty)$ is continuous, and for $(t, u) \in(0,1) \times(0,+\infty), f$ is increasing with respect to $u$ and there exists a constant $k \in(0,1)$ such that, for $\forall \sigma \in(0,1]$,

$$
\begin{equation*}
f(t, \sigma u) \geq \sigma^{k} f(t, u) \tag{4}
\end{equation*}
$$

It is easy to see that if $\sigma \in(1,+\infty)$, then $f(t, \sigma u) \leq \sigma^{k} f(t, u)$.

$$
\begin{aligned}
& \left(H_{2}\right): 0 \leq A=\int_{0}^{1}(1-t)^{q-1} d \Lambda(t)<1, q \in(2,3] \\
& \left(H_{3}\right): \int_{0}^{1} d \Lambda(t) \geq 0, \int_{0}^{1} d \Lambda(t) \neq 1
\end{aligned}
$$

## 2. Preliminaries

For the convenience of the reader, we present here some necessary definitions from fractional calculus theory. These definitions and properties can be found in the recent monograph [23].

Definition 1. The Riemann-Liouville fractional integral of order $q>0$ of a function $x:(0, \infty) \longrightarrow R$ is given by

$$
\begin{equation*}
I^{q} x(t)=\frac{1}{\Gamma(q)} \int_{0}^{t}(t-s)^{q-1} x(s) d s \tag{5}
\end{equation*}
$$

provided that the right-hand side is pointwise defined on ( $0, \infty$ ).

Definition 2. The Riemann-Liouville fractional derivative of order $q>0$ of a continuous function $x:(0, \infty) \longrightarrow R$ is given by

$$
\begin{equation*}
D^{q} x(t)=\frac{1}{\Gamma(n-q)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{x(s)}{(t-s)^{q-n+1}} d s \tag{6}
\end{equation*}
$$

where $n 1 \leq q<n, n=[q]+1, q>0$, provided that the right-hand side is pointwise defined on $(0, \infty)$. In particular,

$$
\begin{equation*}
D^{n} x(t)=x^{(n)}(t), \quad n=1,2,3, \ldots \tag{7}
\end{equation*}
$$

Lemma 3 (see [13]). Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold. Let $y \in$ $L^{1}((0,1),[0,+\infty))$. Then boundary value problem

$$
\begin{align*}
\mathrm{D}^{q} u(t)+y(t) & =0, \quad t \in(0,1), \\
u(1) & =u^{\prime}(1)=0,  \tag{8}\\
u(0) & =\int_{0}^{1} u(t) d \Lambda(t)
\end{align*}
$$

has the unique solution given by the following formula:

$$
\begin{equation*}
u(t)=\int_{0}^{1} G_{q}(t, s) y(s) d s \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
& G(t, s) \\
& \quad= \begin{cases}(1-t)^{q-1} s^{q-1}, & 0 \leq s \leq t \leq 1 ; \\
(1-t)^{q-1} s^{q-1}-(s-t)^{q-1}, & 0 \leq t \leq s \leq 1,\end{cases}  \tag{10}\\
& G_{q}(t, s)=G(t, s)+\frac{(1-t)^{q-1}}{1-A} \int_{0}^{1} G(t, s) d \Lambda(t) .
\end{align*}
$$

One can prove that $G(t, s), G_{q}(t, s)$ have the following properties.

Lemma 4. Note that $G_{q}(t, s)$ is the Green function of problem (8).

Lemma 5 (see [12]). For $t, s \in[0,1]$, one has

$$
\begin{align*}
& \frac{1}{\Gamma(q)} m(t) k(s) \leq G(t, s) \leq \frac{1}{\Gamma(q-1)} k(s), \\
& \frac{1}{\Gamma(q)} m(t) k(s) \leq G(t, s) \leq \frac{1}{\Gamma(q-1)} m(t), \tag{11}
\end{align*}
$$

where

$$
\begin{align*}
& k(s)=s^{q-1}(1-s)  \tag{12}\\
& m(t)=t(1-t)^{q-1}
\end{align*}
$$

## Lemma 6

$$
\begin{equation*}
g(s)(1-t)^{q-1} \leq G_{q}(t, s) \leq M(1-t)^{q-1} \tag{13}
\end{equation*}
$$

where $M>0$ is a constant and $g(s) \in L^{1}(0,1)$ is nonnegative for any $s \in(0,1)$.

Proof. We have the estimation

$$
\begin{align*}
& G_{q}(t, s) \leq \frac{1}{\Gamma(q-1)}(1-t)^{q-1}+\frac{(1-t)^{q-1}}{1-A} \\
& \quad \cdot \int_{0}^{1} \frac{1}{\Gamma(q-1)}(1-t)^{q-1} d \Lambda(t) \\
& \quad=\frac{1}{\Gamma(q-1)}\left[1+\frac{\int_{0}^{1}(1-t)^{q-1} d \Lambda(t)}{1-A}\right](1-t)^{q-1}  \tag{14}\\
& \quad=\left[\frac{1}{\Gamma(q-1)} \frac{1}{1-A}\right](1-t)^{q-1}=M(1-t)^{q-1} \\
& G_{q} \\
& (t, s) \geq \frac{1}{\Gamma(q)} \frac{(1-t)^{q-1}}{1-A} \\
& \quad \cdot \int_{0}^{1} t(1-t)^{q-1} s^{q-1}(1-s) d \Lambda(t)  \tag{15}\\
& \quad=\frac{1}{\Gamma(q)}\left[\frac{(1-t)^{q-1}}{1-A} \int_{0}^{1} t(1-t)^{q-1} d \Lambda(t)\right] \\
& \quad \cdot s^{q-1}(1-s) \\
& \quad=\left[\frac{1}{\Gamma(q)} \frac{\int_{0}^{1} t(1-t)^{q-1} d \Lambda(t)}{1-A} s^{q-1}(1-s)\right] \\
& \quad \cdot(1-t)^{q-1}=g(s)(1-t)^{q-1}
\end{align*}
$$

where $M=(1 / \Gamma(q-1))(1 /(1-A))$ and $g(s)=(1 / \Gamma(q))\left(\int_{0}^{1} t(1\right.$ $\left.-t)^{q-1} d \Lambda(t) /(1-A)\right) s^{q-1}(1-s)$. Thus, (13) holds.

## 3. The Main Results

Throughout this paper, we will work in the space $E=C[0,1]$, which is a Banach space if it is endowed with the norm $\|u\|=$ $\max _{t \in[0,1]}|u(t)|$ for any $u \in E$.

Define the set $P$ in $E$ as follows:
$P=\left\{u \in E \mid\right.$ there exists positive constants $0<l_{u}<1<$ $L_{u}$ such that $\left.l_{u}(1-t)^{q-1} \leq u(t) \leq L_{u}(1-t)^{q-1}, t \in[0,1]\right\}$.

And define the operator $T: E \longrightarrow E$.

$$
\begin{equation*}
T u(t)=\int_{0}^{1} G_{q}(t, s) f(s, u(s)) d s \tag{16}
\end{equation*}
$$

Evidently $(1-t)^{q-1} \in P$. Therefore, $P$ is not empty.
Theorem 7. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold. And

$$
\begin{equation*}
0<\int_{0}^{1} f\left(t,(1-t)^{q-1}\right) d t<\infty \tag{17}
\end{equation*}
$$

Then BVP (3) has at least one positive solution $u(t)$, and there exist constants $0<l_{u}<1<L_{u}$ satisfying

$$
\begin{equation*}
l_{u}(1-t)^{q-1} \leq u(t) \leq L_{u}(1-t)^{q-1}, \quad t \in[0,1] \tag{18}
\end{equation*}
$$

Proof. It is clear that $u$ is a solution of (3) if and only if $u$ is a fixed point of $T$.

Claim 1. The operator $T: P \longrightarrow P$ is nondecreasing.
In fact, for $u \in E$, it is obvious that $u \in E, T u(1)=$ $T u^{\prime}(1)=0, T u(0)=\int_{0}^{1} T u(t) d \Lambda(t)$, and $T u(t)>0$ for $t \in(0,1)$. For any $u \in P$, we have that, for $t \in[0,1]$,

$$
\begin{aligned}
T u(t) & =\int_{0}^{1} G_{q}(t, s) f(s, u(s)) d s \\
& \leq M(1-t)^{q-1} f\left(s, L_{u}(1-s)^{q-1}\right) d s \\
& \leq \int_{0}^{1} M(1-t)^{q-1} L_{u}^{k} f\left(s,(1-s)^{q-1}\right) \\
& \leq L_{T u}(1-t)^{q-1}
\end{aligned}
$$

and

$$
\begin{align*}
T u(t) & =\int_{0}^{1} G_{q}(t, s) f(s, u(s)) d s . \\
& \geq g(s)(1-t)^{q-1} f\left(s, l_{u}(1-s)^{q-1}\right) d s  \tag{20}\\
& \geq(1-t)^{q-1} l_{u}^{k} \int_{0}^{1} g(s) f\left(s,(1-s)^{q-1}\right) \\
& =l_{T u}(1-t)^{q-1},
\end{align*}
$$

where $L_{T u}$ and $l_{T u}$ are positive constants satisfying

$$
\begin{align*}
L_{T u} & >\max \left\{1, \int_{0}^{1} M L_{u}^{k} f\left(s,(1-s)^{q-1}\right)\right\}  \tag{21}\\
& 0<l_{T u}<\min \left\{1, l_{u}^{k} \int_{0}^{1} g(s) f\left(s,(1-s)^{q-1}\right)\right\} .
\end{align*}
$$

Thus, it follows that there are constants $0<l_{T u}<1<L_{T u}$ such that, for $t \in[0,1]$,

$$
\begin{equation*}
l_{T u}(1-t)^{q-1} \leq T u(t) \leq L_{T u}(1-t)^{q-1} . \tag{22}
\end{equation*}
$$

Therefore, for any $u(t) \in P, T u(t) \in P$, i.e., $T$ is the operator $P \longrightarrow P$. From (16), it is easy to see that $T$ is nondecreasing for $u$. Hence, Claim 1 holds.

Claim 2. We take $e(t)=(1-t)^{q-1}$. Let $\delta$ and $\gamma$ be fixed numbers satisfying

$$
\begin{align*}
& 0<\delta \leq l_{T e}^{1 /(1-k)} \\
& \gamma \geq L_{T e}^{1 /(1-k)} \tag{23}
\end{align*}
$$

and assume that

$$
\begin{align*}
& u_{0}=\delta e(t),  \tag{24}\\
& v_{0}=\gamma e(t), \\
& u_{n}=T u_{n-1}, \\
& v_{n}=T v_{n-1}, \tag{25}
\end{align*}
$$

$$
n=1,2,3, \ldots
$$

Then

$$
\begin{equation*}
u_{0} \leq u_{1} \leq \ldots \leq u_{n} \leq \ldots v_{n} \leq \ldots \leq v_{1} \leq v_{0} \tag{26}
\end{equation*}
$$

and there exists $u^{*} \in P$ such that

$$
\begin{align*}
& u_{n}(t) \longrightarrow u^{*}(t), \\
& v_{n}(t) \longrightarrow u^{*}(t) \tag{27}
\end{align*}
$$

uniformly on $[0,1]$.
In fact, $0<l_{T e}<1<L_{T e}$ since $T e \in P$. Therefore, $0<\delta<$ $1<\gamma$. From (24), we have $u_{0}, v_{0} \in P$ and $u_{0} \leq v_{0}$.

On the other hand,

$$
\begin{align*}
u_{1} & =T u_{0}(t)=\int_{0}^{1} G_{q}(t, s) f(s, \delta e(s)) d s \\
& \geq \delta^{k} \int_{0}^{1} G_{q}(t, s) f(s, e(s)) d s=\delta^{k} T e \geq \delta^{k} l_{T e} e(t) \\
& \geq \delta^{k} \delta^{1-k} e(t)=u_{0} \\
v_{1} & =T v_{0}(t)=\int_{0}^{1} G_{q}(t, s) f(s, \gamma e(s)) d s  \tag{28}\\
& \leq \gamma^{k} \int_{0}^{1} G_{q}(t, s) f(s, e(s)) d s=\gamma^{k} T e \leq \gamma^{k} l_{T e} e(t) \\
& \leq \gamma^{k} \gamma^{1-k} e(t)=v_{0}
\end{align*}
$$

and since $u_{0} \leq v_{0}$ and $T$ is nondecreasing, by induction, (26) holds.

Let $c_{0}=\delta / \gamma$, and then $0<c_{0}<1$. It follows from (4) that

$$
\begin{equation*}
T\left(c_{0} u\right) \geq c_{0}^{k} T u \tag{29}
\end{equation*}
$$

And for any natural number $n$,

$$
\begin{align*}
u_{n} & =T u_{n-1}=T^{n} u_{0}=T^{n}(\delta e(t))=T^{n}\left(c_{0} \gamma e(t)\right) \\
& \geq c_{0}^{k^{n}} T^{n}(\gamma e(t))=c_{0}^{k^{n}} v_{n} . \tag{30}
\end{align*}
$$

Thus, for any natural number $n$ and $p^{*}$, we have

$$
\begin{align*}
0 & \leq u_{n+p^{*}}-u_{n} \leq v_{n}-u_{n} \leq\left(1-c_{0}^{k^{n}}\right) v_{n}  \tag{31}\\
& \leq\left(1-c_{0}^{k^{n}}\right) \gamma e(t),
\end{align*}
$$

which implies that there exists $u^{*} \in P$ such that (27) holds and Claim 2 holds.

Letting $n \longrightarrow \infty$ in $u_{n}=T u_{n-1}$ and noting the fact that $T$ is continuous, we obtain $u^{*}(t)=T u^{*}(t)$, which is a positive solution of BVP (3). The proof of Theorem 7 is now complete.

Theorem 8. Assume that $\left(H_{1}\right)-\left(H_{3}\right)$ hold. Then
(i) BVP (3) has unique positive solution $u^{*}(t)$, and there exist constants $l$, $L$ with $0<l<1<L$ such that

$$
\begin{equation*}
l(1-t)^{q-1} \leq u^{*}(t) \leq L(1-t)^{q-1}, \quad t \in[0,1] \tag{32}
\end{equation*}
$$

(ii) For any initial value $x_{0} \in P$, there exists a sequence $x_{n}(t)$ that uniformly converges to the unique positive solution $u^{*}(t)$, and one has the error estimation

$$
\begin{equation*}
\max \left\{x_{n}(t)-u^{*}(t)\right\}=\circ\left(1-\lambda^{k^{n}}\right) \tag{33}
\end{equation*}
$$

where $\lambda$ is a constant with $0<\lambda<1$ and determined by $x_{0}$.
Proof. Let $u_{0}, v_{0}, u_{n}, v_{n}$ be defined in (24) and (25).
(i) It follows from Theorem 7 that BVP (3) has a positive solution $u^{*}(t) \in P$, which implies that there exist constants $l$ and $L$ with $0<1<L<1$ such that $u^{*}(t)$ satisfies (18). Let $v^{*}(t)$ be another positive solution of BVP (3); then from Theorem 7 we have that there exist constants $c_{1}$ and $c_{2}$ with $0<c_{1}<1<c_{2}$ such that

$$
\begin{equation*}
c_{1}(1-t)^{q-1} \leq v^{*}(t) \leq c_{2}(1-t)^{q-1}, \quad t \in[0,1] . \tag{34}
\end{equation*}
$$

Let $\delta$ defined in (23) be small enough such that $\delta<c_{1}$ and $\gamma$ defined in (23) be large enough such that $\gamma>c_{2}$. Then

$$
\begin{equation*}
u_{0}(t) \leq v^{*}(t) \leq v_{0}(t), \quad t \in[0,1] . \tag{35}
\end{equation*}
$$

Note that $T v^{*}=v^{*}$ and $T$ is nondecreasing; we have

$$
\begin{equation*}
u_{n}(t) \leq v^{*}(t) \leq v_{n}(t), \quad t \in[0,1] \tag{36}
\end{equation*}
$$

Letting $n \longrightarrow \infty$ in (36), we obtain that $v^{*}=u^{*}$. Hence, the positive solution of BVP (3) is unique.
(ii) From (i), we know that the positive solution $u^{*}$ to BVP (3) is unique. For any $x_{0} \in P$, there exist constants $l_{0}$ and $L_{0}$ with $0<l_{0}<1<L_{0}$ such that

$$
\begin{equation*}
l_{0}(1-t)^{q-1} \leq x_{0}(t) \leq L_{0}(1-t)^{q-1}, \quad t \in[0,1] . \tag{37}
\end{equation*}
$$

Similar to (i), we can let $\delta$ and $\gamma$ defined by (23) satisfy $\delta<l_{0}$ and $\gamma>L_{0}$. Then

$$
\begin{equation*}
u_{0}(t) \leq x_{0}(t) \leq v_{0}(t), \quad t \in[0,1] . \tag{38}
\end{equation*}
$$

Let $x_{n}=T x_{n-1}, n=1,2, \ldots$ Note that $T$ is nondecreasing; we have

$$
\begin{equation*}
u_{n}(t) \leq x_{n}(t) \leq v_{n}(t), \quad t \in[0,1] . \tag{39}
\end{equation*}
$$

Letting $n \longrightarrow \infty$ in (39), it follows that $x_{n}$ uniformly converges to the unique positive solution $u^{*}$ for BVP (3), where

$$
\begin{equation*}
x_{n}=\int_{0}^{1} G_{q}(t, s) f\left(s, x_{n-1}(s)\right) d s, \quad n=1,2, \ldots \tag{40}
\end{equation*}
$$

At the same time, (33) follows from (31). Thus, the proof of the theorem is complete.

## 4. An Example

$$
\begin{align*}
& \mathrm{D}^{5 / 2} x(t)+\sin (t)\left(x^{1 / 2}(t)+x^{1 / 3}(t)\right)=0, \\
& x(1)=x^{\prime}(1)=0, \\
& x(0)=\int_{0}^{1} x(t) d \Lambda(t), \tag{41}
\end{align*}
$$

where

$$
\Lambda(t)= \begin{cases}0, & t \in\left[0, \frac{1}{2}\right)  \tag{42}\\ 2, & t \in\left[\frac{1}{2}, \frac{3}{4}\right) \\ 1, & t \in\left[\frac{3}{4}, 1\right]\end{cases}
$$

Analysis 1. Let

$$
\begin{align*}
q & =\frac{5}{2}  \tag{43}\\
f(t, x) & =\sin (t)\left(x^{1 / 2}(t)+x^{1 / 3}(t)\right)
\end{align*}
$$

and then for any $\sigma \in(0,1)$, we take $k=1 / 2$ and have

$$
\begin{equation*}
f(t, \sigma u) \geq \sigma^{k} f(t, u) \tag{44}
\end{equation*}
$$

Then $\left(H_{1}\right)$ holds.
In addition, we have

$$
\begin{align*}
& 0<A=\int_{0}^{1}(1-t)^{q-1} d \Lambda(t)=\frac{\sqrt{2}}{2}-\frac{1}{8}<1 .  \tag{45}\\
& 0<\int_{0}^{1} d \Lambda(t)=\frac{3}{4}<1 .
\end{align*}
$$

Then $\left(H_{2}\right)$ and $\left(H_{3}\right)$ hold.

And

$$
\begin{equation*}
0<\int_{0}^{1} f\left(t,(1-t)^{5 / 2-1}\right) d t<\infty \tag{46}
\end{equation*}
$$

Hence all conditions of Theorem 7 are satisfied, and consequently we have the following corollary.

Corollary 9. Problem (41) has unique positive solution $x^{*}(t)$. For any initial value $x_{0} \in P$, the successive iterative sequence $x_{n}(t)$ generated by

$$
\begin{array}{r}
x_{n}(t)=\int_{0}^{1} G_{q}(t, s) \sin (s)\left(x_{n-1}^{1 / 2}(s)+x_{n-1}^{1 / 3}(s)\right) d s  \tag{47}\\
n=1,2, \ldots,
\end{array}
$$

uniformly converges to the unique positive solution $x^{*}(t)$ on $[0,1]$. One has the error estimation

$$
\begin{equation*}
\max \left\{x_{n}(t)-x^{*}(t)\right\}=\circ\left(1-\lambda^{(1 / 2)^{n}}\right) \tag{48}
\end{equation*}
$$

where $\lambda$ is a constant with $0<\lambda<1$ and determined by the initial value $x_{0}$. And there are constants $l, L$ with $0<l<1<L$ such that

$$
\begin{equation*}
l(1-t)^{3 / 2} \leq x^{*}(t) \leq L(1-t)^{3 / 2}, \quad t \in[0,1] \tag{49}
\end{equation*}
$$

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# Fixed-Point Theorems for Systems of Operator Equations and Their Applications to the Fractional Differential Equations 

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#### Abstract

We study the existence and uniqueness of positive solution for a class of nonlinear binary operator equations systems by means of the cone theory and monotone iterative technique, under more general conditions. Also, we give the iterative sequence of the solution and the error estimation of the system. Moreover, we use this new result to study the existence and uniqueness of the solutions for fractional differential equations systems involving integral boundary value conditions in ordered Banach spaces as an application. The results obtained in this paper are more general than many previous results and complement them.


## 1. Introduction

In this paper, using the cone theory and monotone iterative technique, we consider the following nonlinear nonmonotone binary equations systems in real Banach spaces,

$$
\begin{align*}
& x=A(x, x), \\
& x=B(x, x), \tag{1}
\end{align*}
$$

where $A, B: D \times D \longrightarrow E, D$ is a subset of real Banach spaces. There have appeared a series of research results concerning the nonlinear operator equation $x=A(x, x)$, $x=A x[1-7]$, the sum of several classes of mixed-monotone operator equations, and the nonlinear equations systems (1) [8], in recent years. The techniques they used are cone and semiorder [9, 10], the Granas fixed-point index theory [11], the equivalent classes (which are called components) of a real Banach space [12], the Ishikawa iteration process [13, 14], etc.

In [15], Zhang investigated the existence and uniqueness of solutions for a class of nonlinear operator equations $x=$ $A x$ in ordered Banach space, by using the cone theory and Banach contraction mapping principle. The assumption they used is

$$
\begin{align*}
&-B^{n_{0}}(x-y) \leq A x-A y \leq B^{n_{0}}(x-y) \\
& \forall x, y \in E, x \geq y \tag{2}
\end{align*}
$$

where $P$ is a generating normal cone, $A: E \longrightarrow E$ is a nonlinear operator, $B: E \longrightarrow E$ is a positive linear bounded operator with $r(B)<1$ (where $r(B)$ is the spectral radius of $B)$, and $n_{0}$ is a positive integer.

In [16], Zhang investigated the existence and uniqueness theorems of fixed points to a class of mixed-monotone operators with convexity and concavity, in which suppose that there exist $v>\theta, c>1 / 2$ such that $\theta<A(v, \theta) \leq v$ and

$$
\begin{equation*}
A(\theta, v) \geq c A(v, \theta) \tag{3}
\end{equation*}
$$

But in $[15,16]$, they did not consider the nonmonotone binary operator equations systems $x=A(x, x), x=B(x, x)$, where $A, B: D \times D \longrightarrow E$ are two nonlinear operators and $D$ is the order interval in $E$. In this article, the existence and uniqueness of positive solution for nonlinear nonmonotone binary operator equations systems are established under more general conditions, even not supposing the generating of cone $P$. Compared with [16], we do not need the assumption of convexity-concavity. Moreover, the operators we consider are $T^{m_{0}}$-increasing in $x$ and the different upper and lower bounds in formula (6) are more general; the results here cannot be obtained by using Banach contraction mapping principle. Also, different from the results in [15], the
iterative sequence of the solution and the error estimation of the system are obtained.

As an application, we study the existence and uniqueness of positive solutions and iterative approximation of the unique solution for the following fractional differential equations involving integral boundary value problems:

$$
\begin{align*}
&-D_{0+}^{\alpha} u(t)= f_{1}\left(t, u(t), v(t), D_{0+}^{\alpha_{k}} u(s), D_{0+}^{\delta_{k}} v(s)\right), \\
& t \in I, \\
&-D_{0+}^{\alpha} v(t)= f_{2}\left(t, v(t), u(t), D_{0+}^{\alpha_{k}} v(s), D_{0+}^{\delta_{k}} u(s)\right), \\
& t \in I, n-1<\alpha \leq n, \\
& u(0)= D_{0+}^{\gamma_{1}} u(0)=D_{0+}^{\gamma_{2}} u(0)=\cdots=D_{0+}^{\gamma_{n-2}} u(0) \\
&= 0, \\
& v(0)= D_{0+}^{\gamma_{1}} v(0)=D_{0+}^{\gamma_{2}} v(0)=\cdots=D_{0+}^{\gamma_{n-2}} v(0)  \tag{4}\\
&= 0, \\
& D_{0+}^{\beta_{1}} u(1)= \int_{0}^{\eta} h(s) D_{0+}^{\beta_{2}} u(s) d A s \\
&+\int_{0}^{1} a(s) D_{0+}^{\beta_{3}} u(s) d A(s), \\
& D_{0+}^{\beta_{1}} v(1)= \int_{0}^{\eta} h(s) D_{0+}^{\beta_{2}} v(s) d A s \\
&+\int_{0}^{1} a(s) D_{0+}^{\beta_{3}} v(s) d A(s),
\end{align*}
$$

where $n \geq 2, D_{0+}^{\alpha} u, D_{0+}^{\alpha_{k}} u, D_{0+}^{\delta_{k}} u(k=1,2, \ldots, n-2), D_{0+}^{\gamma_{k}} u$, $(k=1,2, \ldots, n-2), D_{0+}^{\beta_{i}} u(i=1,2,3)$ is the standard Riemann-Liouville derivatives and $n-1<\alpha \leq n, k-1<$ $\alpha_{k}, \delta_{k}, \gamma_{k} \leq k(k=1,2, \ldots, n-2), n-k-1<\alpha-\alpha_{k} \leq n-k$, $n-k-1<\alpha-\delta_{k} \leq n-k, n-k-1<\alpha-\gamma_{k} \leq n-k$, $(k=1,2, \ldots, n-2), \beta_{1} \geq \beta_{2}, \beta_{1} \geq \beta_{3}, \alpha-\beta_{1}>1, \beta-\beta_{1}>1$, $\beta_{i}-\alpha_{n-2} \geq 0, \beta_{i}-\delta_{n-2} \geq 0(i=1,2,3), f: I \times E^{4} \longrightarrow E$ is continuous and $a, h \in C\left(I, \mathbb{R}^{+}\right), A$ is a function of bounded variation, $\int_{0}^{\eta} D_{0+}^{\beta_{2}} v(s) d A(s)$, and $\int_{0}^{1} D_{0+}^{\beta_{3}} v(s) d A(s)$ denote the Riemann-Stieltjes integral with respect to $A$.

Recently, fractional differential equations, arising in the mathematical modeling of systems and processes, have drawn more and more attention of the research community due to their numerous applications in various fields of science such as engineering, chemistry, physics, and mechanics. Boundary value problems of fractional differential equations have been investigated for many authors [17-22]. Now, there are many papers dealing with the problem for different kinds of boundary value conditions such as multipoint boundary condition [23, 24], integral boundary condition [25-31], and many other boundary conditions [32]. From the application, we can see that the fixed-point theorems in this paper have extensive applied background. The results presented here are more general and complement many previous known results.

## 2. Preliminaries

Now we present briefly some definitions, lemmas, and basic results that are to be used in the article; for convenience of the reader, we refer the reader to $[3,4,10,33,34]$ for more details.

Suppose that $(E,\|\cdot\|)$ is a real Banach space, and $\theta$ is the zero element of $E$. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies (1) $x \in P, \lambda \geq 0 \Longrightarrow \lambda x \in P$; (2) $x \in P,-x \in P \Longrightarrow x=\theta$. The real Banach space $E$ can be partially ordered by a cone $P \subset E$, i.e., $x \leq y$ if and only if $y-x \in P$. If $x \leq y$ and $x \neq y$, then we denote $x<y$ or $y>x$. Let $C[I, E]=\{x(t): I \longrightarrow E \mid x(t)$ is continuous $\}$. Then $C[I, E]$ is a Banach space with the norm $\|x\|_{c}=\max _{t \in I}\|x(t)\|$, for $x \in C[I, E]$.

The cone $P$ is called normal if there exists a constant $N>$ 0 such that for all $x, y \in E, \theta \leq x \leq y$ implies $\|x\| \leq N\|y\|$, and the smallest $N$ is called the normality constant of $P$. If $x_{1}, x_{2} \in E, x_{1} \leq x_{2}$, the set $\left[x_{1}, x_{2}\right]=\left\{x \in E \mid x_{1} \leq x \leq x_{2}\right\}$ is called the order interval between $x_{1}$ and $x_{2}$.

Definition $1([3,4])$. Let $D$ be a subset of a real ordered Banach space $E ; A: D \times D \longrightarrow E$ is said to be a mixed-monotone operator if $A(x, y)$ is increasing in $x$ and decreasing in $y$; i.e., $x_{i}, y_{i} \in D(i=1,2), x_{1} \leq x_{2}, y_{1} \geq y_{2}$ imply $A\left(x_{1}, y_{1}\right) \leq$ $A\left(x_{2}, y_{2}\right)$. The element $x \in D$ is called a fixed point of $A$ if $A(x, x)=x$.

## 3. Main Results

Theorem 2. Let $E$ be a real Banach space, $P$ be a normal cone in $E$, and $D=\left[u_{0}, v_{0}\right]=\left\{x \in E \mid u_{0} \leq x \leq v_{0}\right\}$ be the order interval in $E$. Assume that $A, B: D \times D \longrightarrow E$ are two nonlinear operators, $T, L: E \longrightarrow E$ are two positive linear bounded operators and satisfy the following conditions:
$\left(J_{0}\right) u_{0} \leq A\left(u_{0}, v_{0}\right), B\left(v_{0}, u_{0}\right) \leq v_{0}$.
$\left(J_{1}\right)$ For all $x \in D, A(x, y)$ and $B(x, y)$ are decreasing in $y$, i.e., for any $x \in D, y_{1}, y_{2} \in D, y_{1} \leq y_{2}$ implies $A\left(x, y_{1}\right) \geq A\left(x, y_{2}\right), B\left(x, y_{1}\right) \geq B\left(x, y_{2}\right)$; and there exist two positive numbers $M_{i}>0(i=1,2)$ such that for all $y \in D$, $x_{1}, x_{2} \in D, x_{1} \leq x_{2}$,

$$
\begin{align*}
& A\left(x_{2}, y\right)-A\left(x_{1}, y\right) \geq-T^{m_{0}}\left(x_{2}-x_{1}\right),  \tag{5}\\
& B\left(x_{2}, y\right)-B\left(x_{1}, y\right) \geq-T^{m_{0}}\left(x_{2}-x_{1}\right) ;
\end{align*}
$$

$\left(J_{2}\right) I+T^{m_{0}}$ is reversible ( $I$ is the identity operator) and $\left(I+T^{m_{0}}\right) x \geq \theta \Longrightarrow x \in P$.
$\left(J_{3}\right) T L=L T$ and there exist two positive integers $m_{0}, n_{0}$ such that $r\left[\left(I+T^{m_{0}}\right)^{-1}\right]\left[r\left(L^{n_{0}}\right)+r\left(T^{m_{0}}\right)\right]<1$ (where $r(\cdot)$ is the spectral radius of linear bounded operator) and

$$
\begin{align*}
&-T^{m_{0}}(y-x) \leq B(y, x)-A(x, y) \leq L^{n_{0}}(y-x)  \tag{6}\\
& \forall x, y \in D, x \leq y .
\end{align*}
$$

Then the nonlinear operator equations system (1) has a unique solution $\left(x^{*}, x^{*}\right)$ in $D \times D$. And for any initial values $x_{0}, y_{0} \in D$, $x_{0} \leq y_{0}$, by constructing successively the sequences as follows:

$$
\begin{align*}
& x_{n}=\left(I+T^{m_{0}}\right)^{-1}\left[A\left(x_{n-1}, y_{n-1}\right)+T^{m_{0}} x_{n-1}\right] \\
& y_{n}=\left(I+T^{m_{0}}\right)^{-1}\left[B\left(y_{n-1}, x_{n-1}\right)+T^{m_{0}} y_{n-1}\right],  \tag{7}\\
& n=1,2, \ldots,
\end{align*}
$$

we have $x_{n} \longrightarrow x^{*}, y_{n} \longrightarrow x^{*}$ in $E$, as $n \longrightarrow \infty$. Moreover, for any $r\left[\left(I+T^{m_{0}}\right)^{-1}\right]\left[r\left(L^{n_{0}}\right)+r\left(T^{m_{0}}\right)\right]<\delta<1$, there exists $n_{1}$ such that

$$
\begin{align*}
& \left\|x_{n}-x^{*}\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|, \\
& \left\|y_{n}-x^{*}\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|, \tag{8}
\end{align*}
$$

$$
n \geq n_{1}
$$

Proof. Since $I+T^{m_{0}}$ is reversible, let

$$
\begin{align*}
& F(x, y)=\left(I+T^{m_{0}}\right)^{-1}\left[A(x, y)+T^{m_{0}} x\right], \\
& G(y, x)=\left(I+T^{m_{0}}\right)^{-1}\left[B(y, x)+T^{m_{0}} y\right], \tag{9}
\end{align*}
$$

$$
x, y \in D
$$

Then (7) can be written as

$$
\begin{align*}
& x_{n}=F\left(x_{n-1}, y_{n-1}\right), \\
& y_{n}=G\left(y_{n-1}, x_{n-1}\right), \tag{10}
\end{align*}
$$

$$
n=1,2, \ldots .
$$

By $\left(J_{1}\right),\left(J_{2}\right)$, it is easy to prove that $F$ and $G$ satisfy the following conditions:
$\left(A_{1}\right) \mathrm{By}\left(I+T^{m_{0}}\right) x \geq \theta \Longrightarrow x \in P$, we know $F, G: D \times$ $D \longrightarrow E$ are two mixed-monotone operators.
$\left(A_{2}\right)$ For all $x, y \in D$,

$$
\begin{align*}
G( & y, x)-F(x, y) \\
= & \left(I+T^{m_{0}}\right)^{-1}\left[B(y, x)+T^{m_{0}} y\right] \\
& -\left(I+T^{m_{0}}\right)^{-1}\left[A(x, y)+T^{m_{0}} x\right]  \tag{11}\\
= & \left(I+T^{m_{0}}\right)^{-1}\left[(B(y, x)-A(x, y))+T^{m_{0}}(y-x)\right] .
\end{align*}
$$

Combining with $\left(J_{3}\right)$, it is easy to prove that

$$
\theta \leq G(y, x)-F(x, y) \leq H(y-x)
$$

$$
\begin{equation*}
\forall x, y \in D, x \leq y \tag{12}
\end{equation*}
$$

where $H \triangleq\left(I+T^{m_{0}}\right)^{-1}\left(L^{n_{0}}+T^{m_{0}}\right)$ and $I$ is the identity operator.
$\left(A_{3}\right)$ Also, by $\left(J_{0}\right)$, we have

$$
\begin{align*}
F\left(u_{0}, v_{0}\right) & =\left(I+T^{m_{0}}\right)^{-1}\left[A\left(u_{0}, v_{0}\right)+T^{m_{0}} u_{0}\right] \\
& \geq\left(I+T^{m_{0}}\right)^{-1}\left[u_{0}+T^{m_{0}} u_{0}\right]=u_{0}  \tag{13}\\
G\left(v_{0}, u_{0}\right) & =\left(I+T^{m_{0}}\right)^{-1}\left[B\left(v_{0}, u_{0}\right)+T^{m_{0}} v_{0}\right]  \tag{14}\\
& \leq\left(I+T^{m_{0}}\right)^{-1}\left[v_{0}+T^{m_{0}} v_{0}\right]=v_{0} .
\end{align*}
$$

Thus, combining with (12), we have

$$
\begin{equation*}
u_{0} \leq F\left(u_{0}, v_{0}\right) \leq G\left(v_{0}, u_{0}\right) \leq v_{0} . \tag{15}
\end{equation*}
$$

Let $u_{n}=F\left(u_{n-1}, v_{n-1}\right), v_{n}=G\left(v_{n-1}, u_{n-1}\right) \quad(n=1,2, \ldots)$. Thus, by (15), we know

$$
\begin{equation*}
u_{0} \leq u_{1} \leq v_{1} \leq v_{0} \tag{16}
\end{equation*}
$$

Therefore, by $\left(A_{1}\right)$ and $\left(A_{2}\right)$, using mathematical induction, we can easily prove that

$$
\begin{equation*}
u_{0} \leq u_{1} \leq \cdots \leq u_{n} \leq \cdots \leq v_{n} \leq \cdots v_{1} \leq v_{0} \tag{17}
\end{equation*}
$$

Firstly, we prove that

$$
\begin{equation*}
\theta \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right), \quad n=1,2, \ldots \tag{18}
\end{equation*}
$$

By $\left(A_{2}\right)$, we can easily prove that

$$
\begin{equation*}
\theta \leq v_{1}-u_{1} \leq G\left(v_{0}, u_{0}\right)-F\left(u_{0}, v_{0}\right) \leq H\left(v_{0}-u_{0}\right), \tag{19}
\end{equation*}
$$

i.e., (18) holds for $n=1$. Suppose that (18) holds for $n=k$, i.e.,

$$
\begin{equation*}
\theta \leq v_{k}-u_{k} \leq H^{k}\left(v_{0}-u_{0}\right) \tag{20}
\end{equation*}
$$

Then, for $n=k+1$, by $\left(A_{1}\right)$ and $\left(A_{2}\right)$ we know

$$
\begin{align*}
u_{k+1} & =F\left(u_{k}, v_{k}\right) \leq G\left(v_{k}, u_{k}\right)=v_{k+1} \\
\theta & \leq v_{k+1}-u_{k+1}=G\left(v_{k}, u_{k}\right)-F\left(u_{k}, v_{k}\right)  \tag{21}\\
& \leq H\left(v_{k}-u_{k}\right) \leq H^{k+1}\left(v_{0}-u_{0}\right) .
\end{align*}
$$

By (19)-(21), using mathematical induction, we know (18) holds.

Next we prove that $\left\{x_{n}\right\}$ is a Cauchy sequence. From $T L=$ $L T$, we have $\left(I+T^{m_{0}}\right)^{-1}\left(L^{n_{0}}+T^{m_{0}}\right)=\left(L^{n_{0}}+T^{m_{0}}\right)\left(I+T^{m_{0}}\right)^{-1}$. Then

$$
\begin{align*}
r & {\left[\left(I+T^{m_{0}}\right)^{-1}\left(L^{n_{0}}+T^{m_{0}}\right)\right] } \\
& \leq r\left[\left(I+T^{m_{0}}\right)^{-1}\right] r\left(L^{n_{0}}+T^{m_{0}}\right)  \tag{22}\\
& \leq r\left[\left(I+T^{m_{0}}\right)^{-1}\right]\left[r\left(L^{n_{0}}\right)+r\left(T^{m_{0}}\right)\right]<1
\end{align*}
$$

Consequently, there exists $\delta>0$ such that

$$
\begin{align*}
\lim _{n \rightarrow \infty}\left\|H^{n}\right\|^{1 / n} & =r(H)=r\left[\left(I+T^{m_{0}}\right)^{-1}\left(L^{n_{0}}+T^{m_{0}}\right)\right] \\
& \leq r\left[\left(I+T^{m_{0}}\right)^{-1}\right]\left[r\left(L^{n_{0}}\right)+r\left(T^{m_{0}}\right)\right]  \tag{23}\\
& <\delta<1
\end{align*}
$$

Thus, there exists $n_{1}$ such that

$$
\begin{equation*}
\left\|H^{n}\right\|<\delta^{n}, \quad n \geq n_{1} \tag{24}
\end{equation*}
$$

Then by (17), we have

$$
\begin{equation*}
\theta \leq u_{n} \leq u_{n+p} \leq v_{n+p} \leq v_{n}, \quad n, p=1,2, \ldots \tag{25}
\end{equation*}
$$

Consequently, by (18) and (25), we have

$$
\begin{align*}
& \theta \leq u_{n+p}-u_{n} \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right), \\
& \theta \leq v_{n}-v_{n+p} \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right) \tag{26}
\end{align*}
$$

$$
n, p=1,2, \ldots .
$$

Therefore, by (24), (26) and the normality of cone $P$, we have

$$
\begin{align*}
\left\|u_{n+p}-u_{n}\right\| \leq N\left\|H^{n}\left(v_{0}-u_{0}\right)\right\| & \leq N \delta^{n}\left\|v_{0}-u_{0}\right\| \\
\left\|v_{n}-v_{n+p}\right\| \leq N\left\|H^{n}\left(v_{0}-u_{0}\right)\right\| & \leq N \delta^{n}\left\|v_{0}-u_{0}\right\|  \tag{27}\\
& n \geq n_{1}, p=1,2, \ldots
\end{align*}
$$

where $N$ is the normality constant of $P$. Consequently, $\left\{u_{n}\right\}$ and $\left\{v_{n}\right\}$ are Cauchy sequences. Since $E$ is complete, thus there exist $u^{*}, v^{*} \in D$, such that

$$
\begin{align*}
& \lim _{n \rightarrow \infty} u_{n}=u^{*},  \tag{28}\\
& \lim _{n \rightarrow \infty} v_{n}=v^{*} .
\end{align*}
$$

And by (17), we know

$$
\begin{equation*}
u_{n} \leq u^{*} \leq v^{*} \leq v_{n}, \quad n=0,1,2, \ldots \tag{29}
\end{equation*}
$$

Thus, $u^{*}, v^{*} \in D$. By (18) and (29), we have

$$
\theta \leq v^{*}-u^{*} \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right), ~(~ n=0,1,2, \ldots .
$$

Thus, by (26), (30) and the normality of cone $P$, we have

$$
\begin{array}{r}
\left\|v^{*}-u^{*}\right\| \leq N\left\|H^{n}\left(v_{0}-u_{0}\right)\right\| \leq N \delta^{n}\left\|v_{0}-u_{0}\right\|  \tag{31}\\
n \\
n \longrightarrow \infty,
\end{array}
$$

Consequently, $u^{*}=v^{*}$. Let $x^{*}:=u^{*}=v^{*} ;$ by $\left(A_{1}\right)$ and $\left(A_{2}\right)$, we have

$$
\begin{align*}
u_{n+1} & =F\left(u_{n}, v_{n}\right) \leq F\left(x^{*}, x^{*}\right) \leq G\left(x^{*}, x^{*}\right)  \tag{32}\\
& \leq G\left(v_{n}, u_{n}\right)=v_{n+1}, \quad n=1,2, \ldots .
\end{align*}
$$

Let $n \longrightarrow \infty$, by (32) and the normality of $P$ we have $F\left(x^{*}, x^{*}\right)=G\left(x^{*}, x^{*}\right)=x^{*}$. Therefore, by the definitions of $F$ and $G$, we have $x^{*}=A\left(x^{*}, x^{*}\right), x^{*}=B\left(x^{*}, x^{*}\right)$; i.e., ( $x^{*}, x^{*}$ ) is the solution of operator equation (1).

Finally, we prove that $\left(x^{*}, x^{*}\right)$ is the unique solution of operator equations systems (1) in $D \times D$. In fact, suppose $(\bar{x}, \bar{x})$ is another solution of equations systems (1) in $D \times D$, then by $\left(A_{1}\right)$, using mathematical induction, we can easily see that $u_{n} \leq \bar{x} \leq v_{n}(n=1,2, \ldots)$. Thus, by (28) and the normality of $P$, we have $\bar{x}=x^{*}$. Therefore, the operator equations systems (1) have a unique solution $\left(x^{*}, x^{*}\right)$ in $D \times D$.

Now for any initial points $x_{0}, y_{0} \in D, x_{0} \leq y_{0}$, we construct successively the sequences

$$
\begin{align*}
& x_{n}=F\left(x_{n-1}, y_{n-1}\right), \\
& y_{n}=G\left(y_{n-1}, x_{n-1}\right), \tag{33}
\end{align*}
$$

Since $x_{0}, y_{0} \in D$, i.e.,

$$
\begin{equation*}
u_{0} \leq x_{0} \leq y_{0} \leq v_{0} \tag{34}
\end{equation*}
$$

Suppose

$$
\begin{equation*}
u_{n-1} \leq x_{n-1} \leq y_{n-1} \leq v_{n-1} . \tag{35}
\end{equation*}
$$

Then

$$
\begin{align*}
u_{n} & =F\left(u_{n-1}, v_{n-1}\right) \leq x_{n}=F\left(x_{n-1}, y_{n-1}\right) \leq y_{n} \\
& =G\left(y_{n-1}, x_{n-1}\right) \leq v_{n}=G\left(v_{n-1}, u_{n-1}\right) . \tag{36}
\end{align*}
$$

Thus, by mathematical induction, we have

$$
\begin{equation*}
u_{n} \leq x_{n} \leq y_{n} \leq v_{n}, \quad n=0,1,2, \ldots \tag{37}
\end{equation*}
$$

By (18) and (37), we have

$$
\begin{align*}
& \theta \leq x_{n}-u_{n} \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right),  \tag{38}\\
& \theta \leq x^{*}-u_{n} \leq v_{n}-u_{n} \leq H^{n}\left(v_{0}-u_{0}\right) . \tag{39}
\end{align*}
$$

Thus,

$$
\begin{align*}
&\left\|x_{n}-x^{*}\right\| \leq\left\|x_{n}-u_{n}\right\|+\left\|x^{*}-u_{n}\right\| \\
& \leq 2 N\left\|H^{n}\left(v_{0}-u_{0}\right)\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|  \tag{40}\\
& n \geq n_{1} .
\end{align*}
$$

In the same way, we can prove that

$$
\begin{equation*}
\left\|y_{n}-x^{*}\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|, \quad n \geq n_{1} . \tag{41}
\end{equation*}
$$

Consequently, by (40) and (41) we know that (8) holds.
Taking $m_{0}=n_{0}=1$ in Theorem 2, then we get the following corollary.

Corollary 3. Let $E$ be a real Banach space, $P$ be a normal cone in $E$, and $D=\left[u_{0}, v_{0}\right]=\left\{x \in E \mid u_{0} \leq x \leq v_{0}\right\}$ be the order interval in $E$. Assume that $A, B: D \times D \longrightarrow E$ are two nonlinear operators, $T, L: E \longrightarrow E$ are two positive linear bounded operators and satisfy the following conditions:
$\left(Q_{0}\right) u_{0} \leq A\left(u_{0}, v_{0}\right), B\left(v_{0}, u_{0}\right) \leq v_{0}$.
$\left(Q_{1}\right)$ For all $x \in D, A(x, y)$ and $B(x, y)$ are decreasing in $y$, i.e., for any $x \in D, y_{1}, y_{2} \in D, y_{1} \leq y_{2}$ implies $A\left(x, y_{1}\right) \geq A\left(x, y_{2}\right), B\left(x, y_{1}\right) \geq B\left(x, y_{2}\right)$; and there exist two positive numbers $M_{i}>0(i=1,2)$ such that for all $y \in D$, $x_{1}, x_{2} \in D, x_{1} \leq x_{2}$,

$$
\begin{align*}
& A\left(x_{2}, y\right)-A\left(x_{1}, y\right) \geq-T\left(x_{2}-x_{1}\right) \\
& B\left(x_{2}, y\right)-B\left(x_{1}, y\right) \geq-T\left(x_{2}-x_{1}\right) \tag{42}
\end{align*}
$$

$\left(Q_{2}\right) T L=L T, r\left[(I+T)^{-1}\right][r(L)+r(T)]<1($ where $r(\cdot)$ is the spectral radius of linear bounded operator) and

$$
\begin{align*}
-T(y-x) \leq B(y, x)-A(x, y) \leq L(y-x), & \\
& \forall x, y \in D, x \leq y . \tag{43}
\end{align*}
$$

Then the nonlinear operator equations system (1) has a unique solution $\left(x^{*}, x^{*}\right)$ in $D \times D$. And for any initial values $x_{0}, y_{0} \in D$, $x_{0} \leq y_{0}$, by constructing successively the sequences as follows:

$$
\begin{align*}
& x_{n}=(I+T)^{-1}\left[A\left(x_{n-1}, y_{n-1}\right)+T x_{n-1}\right], \\
& y_{n}=(I+T)^{-1}\left[B\left(y_{n-1}, x_{n-1}\right)+T y_{n-1}\right], \tag{44}
\end{align*}
$$

$$
n=1,2, \ldots,
$$

we have $x_{n} \longrightarrow x^{*}, y_{n} \longrightarrow x^{*}$ in $E$, as $n \longrightarrow \infty$. Moreover, for any $r\left[(I+T)^{-1}\right][r(L)+r(T)]<\delta<1$, there exists $n_{1}$ such that

$$
\begin{array}{ll}
\left\|x_{n}-x^{*}\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|, & n \geq n_{1}  \tag{45}\\
\left\|y_{n}-x^{*}\right\| \leq 2 N \delta^{n}\left\|v_{0}-u_{0}\right\|, & n \geq n_{1}
\end{array}
$$

## 4. An Application for Fractional Differential Equations Involving Integral Boundary Value Problems

Now we present briefly some definitions, lemmas, and basic results that are to be used in the article for convenience of the reader. We refer the reader to [35-39] for more details.

Definition 4 ([35, 36, 38, 39]). The Riemann-Liouville fractional integral of order $\alpha>0$ of a function $u:(0,+\infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
I_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(\alpha)} \int_{0}^{t}(t-s)^{\alpha-1} u(s) d s \tag{46}
\end{equation*}
$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 5 ([35, 36, 38, 39]). The Riemann-Liouville fractional derivative of order $\alpha>0$ of a continuous function $u$ : $(0,+\infty) \longrightarrow \mathbb{R}$ is given by

$$
\begin{equation*}
D_{0^{+}}^{\alpha} u(t)=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{d t}\right)^{n} \int_{0}^{t} \frac{u(s)}{(t-s)^{\alpha-n+1}} d s \tag{47}
\end{equation*}
$$

where $n=[\alpha]+1,[\alpha]$ denotes the integer part of the number $\alpha$, provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Lemma 6 ([35, 36, 38, 39]). (1) If $u \in L^{1}(0,1)$ and $\alpha>\beta>0$, then

$$
\begin{align*}
I_{0^{+}}^{\alpha} I_{0^{+}}^{\beta} u(t) & =I_{0^{+}}^{\alpha+\beta} u(t), \\
D_{0^{+}}^{\beta} I_{0^{+}}^{\alpha} u(t) & =I_{0^{+}}^{\alpha-\beta} u(t),  \tag{48}\\
D_{0^{+}}^{\beta} I_{0^{+}}^{\beta} u(t) & =u(t)
\end{align*}
$$

(2) If $u \in L^{1}(0,1)$ and $\alpha>0$, then $D_{0^{+}}^{\alpha} u(t)=0$ has unique solution

$$
\begin{equation*}
u(t)=c_{1} t^{\alpha-1}+c_{2} t^{\alpha-2}+\cdots+c_{n} t^{\alpha-n} \tag{49}
\end{equation*}
$$

where $c_{i} \in \mathbb{R}(i=0,1,2, \ldots, n), n=[\alpha]+1$.
Lemma 7 ([35, 36, 38, 39]). Let $\alpha>0$ and let $f(x)$ be integrable; then

$$
\begin{equation*}
I_{0^{+}}^{\alpha} D_{0^{+}}^{\alpha} f(u)=f(u)+c_{1} u^{\alpha-1}+c_{2} u^{\alpha-2}+\cdots+c_{n} u^{\alpha-n} \tag{50}
\end{equation*}
$$

where $c_{i} \in \mathbb{R}(i=1,2, \ldots, n)$ and $n$ is the smallest integer greater than or equal to $\alpha$.

Lemma 8 ([37]). Let $f \in L^{1}(0,1) \cap C(0,1), n-1<\alpha \leq n$. Assume that the following condition is satisfied:

$$
\begin{equation*}
\text { (H) } \quad \frac{\Gamma\left(\alpha-\beta_{1}\right)}{\Gamma\left(\alpha-\beta_{2}\right)} \int_{0}^{\eta} h(t) t^{\alpha-\beta_{2}} d A(t)+\frac{\Gamma\left(\alpha-\beta_{1}\right)}{\Gamma\left(\alpha-\beta_{3}\right)} \int_{0}^{1} a(t) t^{\alpha-\beta_{3}} d A(t)<1 \tag{51}
\end{equation*}
$$

then

$$
\begin{align*}
D_{0^{+}}^{\alpha-\alpha_{n-2}} u(t)+f(t)= & 0, \quad n-1<\alpha \leq n, 0<t<1 \\
D_{0^{+}}^{\gamma_{n-2}-\alpha_{n-2}} u(0)= & 0 \\
D_{0^{+}}^{\beta_{1}-\alpha_{n-2}} u(1)= & \int_{0}^{\eta} h(s) D_{0^{+}}^{\beta_{2}-\alpha_{n-2}} u(s) d A(s)  \tag{52}\\
& +\int_{0}^{1} h(s) D_{0^{+}}^{\beta_{3}-\alpha_{n-2}} u(s) d A(s)
\end{align*}
$$

has a unique positive solution

$$
\begin{equation*}
u(t)=\int_{0}^{1} G(t, s) f(s) d s, \quad t \in[0,1] \tag{53}
\end{equation*}
$$

where

$$
\begin{align*}
G(t, s)= & G_{1}(t, s) \\
& +\frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma\left(\alpha-\beta_{2}\right)} \Delta^{-1} \int_{0}^{\eta} h(t) G_{2}(t, s) d A(t)  \tag{54}\\
& +\frac{t^{\alpha-\alpha_{n-2}-1}}{\Gamma\left(\alpha-\beta_{3}\right)} \Delta^{-1} \int_{0}^{1} a(t) G_{3}(t, s) d A(t),
\end{align*}
$$

in which

$$
\begin{aligned}
& \delta_{1}:=\int_{0}^{\eta} h(t) t^{\alpha-\beta_{2}-1} d A(t), \\
& \delta_{2}:=\int_{0}^{\eta} a(t) t^{\alpha-\beta_{3}-1} d A(t),
\end{aligned}
$$

$$
\begin{align*}
& \Delta=\frac{1}{\Gamma\left(\alpha-\beta_{1}\right)}-\frac{1}{\Gamma\left(\alpha-\beta_{2}\right)} \delta_{1}-\frac{1}{\Gamma\left(\alpha-\beta_{3}\right)} \delta_{2}, \\
& G_{1}(t, s)=\frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)} \\
& \quad \cdot \begin{cases}t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_{1}-1}-(t-s)^{\alpha-\alpha_{n-2}-1}, & 0 \leq s \leq t \leq 1, \\
t^{\alpha-\alpha_{n-2}-1}(1-s)^{\alpha-\beta_{1}-1}, & 0 \leq t \leq s \leq 1,\end{cases} \\
& G_{2}(t, s)=\frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)} \\
& \quad \cdot \begin{cases}t^{\alpha-\beta_{2}-1}(1-s)^{\alpha-\beta_{1}-1}-(t-s)^{\alpha-\beta_{2}-1}, & 0 \leq s \leq t \leq 1, \\
t^{\alpha-\beta_{2}-1}(1-s)^{\alpha-\beta_{1}-1}, & 0 \leq t \leq s \leq 1,\end{cases} \\
& G_{3}(t, s)=\frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)} \\
& \quad \begin{cases}t^{\alpha-\beta_{3}-1}(1-s)^{\alpha-\beta_{1}-1}-(t-s)^{\alpha-\beta_{3}-1}, & 0 \leq s \leq t \leq 1, \\
t^{\alpha-\beta_{2}-1}(1-s)^{\alpha-\beta_{3}-1}, & 0 \leq t \leq s \leq 1 .\end{cases} \tag{55}
\end{align*}
$$

Lemma 9 ([37]). If the condition (H) in Lemma 8 is satisfied, the Green function $G(t, s)$ has the following properties:
(1) $G(t, s)>0$, for all $t, s \in(0,1)$.
(2) For any $t, s \in[0,1]$, we have

$$
\begin{equation*}
\theta \leq t^{\alpha-\alpha_{n-2}-1} l_{1}(s) \leq G(t, s) \leq L_{1} t^{\alpha-\alpha_{n-2}-1} \leq L_{1}, \tag{56}
\end{equation*}
$$

where

$$
\begin{align*}
l_{1}(s)= & \frac{\Delta^{-1}}{\Gamma\left(\alpha-\beta_{2}\right)} \int_{0}^{\eta} h(t) G_{2}(t, s) d A(t) \\
& +\frac{\Delta^{-1}}{\Gamma\left(\alpha-\beta_{3}\right)} \int_{0}^{1} a(t) G_{3}(t, s) d A(t) \\
L_{1}= & \frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)}  \tag{57}\\
& +\frac{\Delta^{-1}}{\Gamma\left(\alpha-\beta_{2}\right)} \frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)} \int_{0}^{\eta} h(t) d A(t) \\
& +\frac{\Delta^{-1}}{\Gamma\left(\alpha-\beta_{3}\right)} \frac{1}{\Gamma\left(\alpha-\alpha_{n-2}\right)} \int_{0}^{1} a(t) d A(t)
\end{align*}
$$

In the following, we need the following assumptions: $\left(H_{0}\right) \max _{t \in I} \int_{0}^{1}|\bar{G}(t, \tau)| d \tau<1$, where

$$
\begin{align*}
& \bar{G}(t, \tau) \\
&=\left(r_{1}+r_{2}\right) G(t, \tau) \\
&+\frac{r_{3}}{\Gamma\left(\alpha_{n-2}-\alpha_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\alpha_{n-2}-\alpha_{k}-1} d s\right)  \tag{58}\\
&+\frac{r_{4}}{\Gamma\left(\delta_{n-2}-\delta_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\delta_{n-2}-\delta_{k}-1} d s\right) ;
\end{align*}
$$

$\left(H_{1}\right) f_{i} \in C\left[I \times E^{4}, E\right]$ and there exists $v_{0} \in C[I, E]$ such that

$$
\begin{align*}
& \int_{0}^{1} G(t, s) f_{2}\left(s, v_{0}(s), \theta, I_{0+}^{\alpha_{n-2}-\alpha_{k}} v_{0}(s), \theta\right) d s  \tag{59}\\
& \quad \leq v_{0}(t), \quad \forall t \in I
\end{align*}
$$

$\left(H_{2}\right)$ For all $x_{i}, y_{i} \in E(i=1,2,3,4), y_{1} \geq x_{1}, y_{2} \leq x_{2}, y_{3} \geq$ $x_{3}, y_{4} \leq x_{4}$,

$$
\begin{align*}
f_{i}\left(t, y_{1}, y_{2}, y_{3}, y_{4}\right)-f_{i}\left(t, x_{1}, x_{2}, x_{3}, x_{4}\right) & \geq 0,  \tag{60}\\
\forall t & \in I, i=1,2
\end{align*}
$$

$\left(H_{3}\right)$ There exist four constants $r_{i}>0(i=1,2,3,4)$ such that for any $t \in I, x_{i}, y_{i} \in E(i=1,2,3,4)$ with $x_{1} \leq y_{1}, x_{2} \geq y_{2}$, $x_{3} \leq y_{3}, x_{4} \geq y_{4}$,

$$
\begin{align*}
0 \leq & f_{2}\left(t, y_{1}, y_{2}, y_{3}, y_{4}\right)-f_{1}\left(t, x_{1}, x_{2}, x_{3}, x_{4}\right) \\
\leq & r_{1}\left(y_{1}-x_{1}\right)+r_{2}\left(x_{2}-y_{2}\right)+r_{3}\left(y_{3}-x_{3}\right)  \tag{61}\\
& +r_{4}\left(x_{4}-y_{4}\right) .
\end{align*}
$$

Theorem 10. Let $E$ be a real Banach space and $P$ be a normal cone in E. Assume that the conditions $(H)$ and $\left(\mathrm{H}_{0}\right)-\left(\mathrm{H}_{3}\right)$ are satisfied. Then the system of nonlinear differential equations (50) has unique positive symmetry solution $\left(w^{*}, w^{*}\right) \in D \times D$, where $D=\left[\theta, v_{0}\right] \subset C[I, E]$. Moreover, for any initial functions $x_{0}, y_{0} \in D$, there exist monotone iteration sequences $\left\{x_{n}\right\},\left\{y_{n}\right\}$, such that $x_{n} \longrightarrow w^{*}, y_{n} \longrightarrow w^{*}$ in $C[I, E]$, as $n \longrightarrow \infty$, where

$$
\begin{align*}
& x_{n}(t)=\int_{0}^{1} G(t, s) f_{1}\left(s, x_{n-1}(s), y_{n-1}(s),\right. \\
& \left.I_{0+}^{\alpha_{n-2}-\alpha_{k}} x_{n-1}(s), I_{0+}^{\delta_{n-2}-\delta_{k}} y_{n-1}(s)\right) d s, \\
& y_{n}(t)=\int_{0}^{1} G(t, s) f_{2}\left(s, y_{n-1}(s), x_{n-1}(s),\right.  \tag{62}\\
& \left.I_{0+}^{\alpha_{n-2}-\alpha_{k}} y_{n-1}(s), I_{0+}^{\delta_{n-2}-\delta_{k}} x_{n-1}(s)\right) d s, \\
& \quad t \in I, n=1,2,3, \ldots .
\end{align*}
$$

Proof. It is well known that $(u, v) \in C[I, E] \times C[I, E]$ is a solution of the system (50) if and only if $(u, v) \in C[I, E] \times$ $C[I, E]$ is a solution of the system of nonlinear integral equations

$$
\begin{align*}
& u(t)=\int_{0}^{1} G(t, s) f_{1}\left(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} u(s),\right. \\
& \left.I_{0+}^{\delta_{n-2}-\delta_{k}} v(s)\right) d s \\
& v(t)=\int_{0}^{1} G(t, s) f_{2}\left(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} v(s),\right.  \tag{63}\\
& \left.\quad I_{0+}^{\delta_{n-2}-\delta_{k}} u(s)\right) d s
\end{align*}
$$

Consider the operators $A, B: D \times D \longrightarrow C[I, E]$ as follows: $A(u, v)(t)=\int_{0}^{1} G(t, s) f_{1}\left(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} u(s)\right.$,
$\cdot f_{1}\left(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} u(s), I_{0+}^{\delta_{n-2}-\delta_{k}} v(s)\right) d s$ $\geq 0$.

$$
\begin{align*}
& B(v, u)(t)-A(u, v)(t)=\int_{0}^{1} G(t, s)  \tag{69}\\
& \quad \cdot f_{2}\left(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} v(s), I_{0+}^{\delta_{n-2}-\delta_{k}} u(s)\right) d s \\
& \quad-\int_{0}^{1} G(t, s)
\end{align*}
$$

$$
\cdot f_{1}\left(s, u(s), v(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} u(s), I_{0+}^{\delta_{n-2}-\delta_{k}} v(s)\right) d s
$$

$$
\leq \int_{0}^{1} G(t, s)\left[r_{1}(v(s)-u(s))+r_{2}(v(s)-u(s))\right.
$$

$$
+r_{3}\left(I_{0+}^{\alpha_{n-2}-\alpha_{k}} v(s)-I_{0+}^{\alpha_{n-2}-\alpha_{k}} u(s)\right)
$$

$$
\begin{equation*}
\left.+r_{4}\left(I_{0+}^{\delta_{n-2}-\delta_{k}} v(s)-I_{0+}^{\delta_{n-2}-\delta_{k}} u(s)\right)\right] d s \tag{65}
\end{equation*}
$$

$$
=\int_{0}^{1}\left(r_{1}+r_{2}\right) G(t, s)(v(s)-u(s)) d s+\int_{0}^{1} r_{3} G(t, s)
$$

$$
\cdot I_{0+}^{\alpha_{n-2}-\alpha_{k}}(v(s)-u(s)) d s+\int_{0}^{1} r_{4} G(t, s)
$$

$$
\cdot I_{0+}^{\delta_{n-2}-\delta_{k}}(v(s)-u(s)) d s
$$

$$
\begin{equation*}
=\int_{0}^{1}\left(r_{1}+r_{2}\right) G(t, s)(v(s)-u(s)) d s+\int_{0}^{1} r_{3} G(t, s) \tag{66}
\end{equation*}
$$

$$
\begin{equation*}
\cdot\left(\frac{1}{\Gamma\left(\alpha_{n-2}-\alpha_{k}\right)} \int_{0}^{s}(s-\tau)^{\alpha_{n-2}-\alpha_{k}-1}\right. \tag{70}
\end{equation*}
$$

Consequently, $A, B: D \times D \longrightarrow C[I, E]$ are mixed-monotone. By $\left(H_{1}\right)$, for all $t \in I$, we obtain

$$
\begin{align*}
& A\left(\theta, v_{0}\right)(t) \\
& \quad=\int_{0}^{1} G(t, s) f_{1}\left(s, 0, v_{0}(s), 0, I_{0+}^{\delta_{n-2}-\delta_{k}} v_{0}(s)\right) d s  \tag{67}\\
& \quad \geq 0 \\
& B\left(v_{0}, \theta\right)(t) \\
& \quad=\int_{0}^{1} G(t, s) f_{2}\left(s, v_{0}(s), 0, I_{0+}^{\alpha_{n-2}-\alpha_{k}} v_{0}(s), 0\right) d s  \tag{68}\\
& \quad \leq v_{0}(t) .
\end{align*}
$$

By $\left(H_{3}\right)$, for any $t \in I, u, v \in D$ with $u \leq v$,

$$
\cdot(v(\tau)-u(\tau)) d \tau) d s+\int_{0}^{1} r_{4} G(t, s)
$$

$$
\cdot\left(\frac{1}{\Gamma\left(\delta_{n-2}-\delta_{k}\right)} \int_{0}^{s}(s-\tau)^{\delta_{n-2}-\delta_{k}-1}\right.
$$

$$
\cdot(v(\tau)-u(\tau)) d \tau) d s
$$

$$
=\int_{0}^{1}\left(r_{1}+r_{2}\right) G(t, s)(v(s)-u(s)) d s
$$

$$
+\int_{0}^{1} \frac{r_{3}}{\Gamma\left(\alpha_{n-2}-\alpha_{k}\right)}\left(\int_{\tau}^{1} G(t, s)\right.
$$

$$
\left.\cdot(s-\tau)^{\alpha_{n-2}-\alpha_{k}-1} d s\right)(v(\tau)-u(\tau)) d \tau
$$

$$
\begin{aligned}
& B(v, u)(t)-A(u, v)(t)=\int_{0}^{1} G(t, s) \\
& \quad \cdot f_{2}\left(s, v(s), u(s), I_{0+}^{\alpha_{n-2}-\alpha_{k}} v(s), I_{0+}^{\delta_{n-2}-\delta_{k}} u(s)\right) d s \\
& \quad-\int_{0}^{1} G(t, s)
\end{aligned}
$$

$$
+\int_{0}^{1} \frac{r_{4}}{\Gamma\left(\delta_{n-2}-\delta_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\delta_{n-2}-\delta_{k}-1} d s\right)
$$

$$
\cdot(v(\tau)-u(\tau)) d \tau
$$

$$
=L(v-u)(t)
$$

where

$$
\begin{align*}
& L u(t)=\int_{0}^{1}\left[\left(r_{1}+r_{2}\right) G(t, \tau)\right. \\
& \quad+\frac{r_{3}}{\Gamma\left(\alpha_{n-2}-\alpha_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\alpha_{n-2}-\alpha_{k}-1} d s\right)  \tag{71}\\
& \left.\quad+\frac{r_{4}}{\Gamma\left(\delta_{n-2}-\delta_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\delta_{n-2}-\delta_{k}-1} d s\right)\right] \\
& \quad \cdot u(\tau) d \tau, \quad \forall t \in I .
\end{align*}
$$

Set

$$
\begin{align*}
& \bar{G}(t, \tau) \\
&=\left(r_{1}+r_{2}\right) G(t, \tau) \\
&+\frac{r_{3}}{\Gamma\left(\alpha_{n-2}-\alpha_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\alpha_{n-2}-\alpha_{k}-1} d s\right)  \tag{72}\\
&+\frac{r_{4}}{\Gamma\left(\delta_{n-2}-\delta_{k}\right)}\left(\int_{\tau}^{1} G(t, s)(s-\tau)^{\delta_{n-2}-\delta_{k}-1} d s\right), \\
& \forall t \in I ;
\end{align*}
$$

then $L u(t)=\int_{0}^{1} \bar{G}(t, \tau) u(\tau) d \tau$. Consequently, for any $t \in I$, for any $u, v \in D$ with $u \leq v$,

$$
\begin{equation*}
0 \leq B(v, u)(t)-A(u, v)(t) \leq L(v-u)(t) . \tag{73}
\end{equation*}
$$

In the following, we prove $r(L)<1$. In fact, by $\left(H_{0}\right)$, since $\max _{t \in I} \int_{0}^{1}|\bar{G}(t, s)| d s<1$, there exists a constant $m_{1}: 0<$ $m_{1}<1$ such that $\int_{0}^{1}|G(t, s)| d s \leq m_{1}<1$, for any $t \in I$. Thus, for all $t \in I, u \in D$,

$$
\begin{aligned}
\|(L u)(t)\| & =\left\|\int_{0}^{1} \bar{G}(t, s) u(s) d s\right\| \\
& \leq \int_{0}^{1}\|\bar{G}(t, s) u(s)\| d s \\
& \leq \int_{0}^{1}|\bar{G}(t, s)| d s\|u\|_{c} \\
& \leq m_{1}\|u\|_{c}, \\
\left\|\left(L^{2} u\right)(t)\right\| & =\left\|\int_{0}^{1} \bar{G}(t, s)(L u)(s) d s\right\| \\
& \leq \int_{0}^{1}\|\bar{G}(t, s)(L u)(s)\| d s \\
& \leq \int_{0}^{1}|\bar{G}(t, s)|\|(L u)(s)\| d s \\
& \leq\left(\int_{0}^{1}|\bar{G}(t, s)| d s\right) m_{1}\|u\|_{c} \\
& \leq m_{1}^{2}\|u\|_{c} .
\end{aligned}
$$

By mathematical induction, we can easily prove that for all natural number $n$,

$$
\begin{equation*}
\left\|\left(L^{n} u\right)(t)\right\| \leq m_{1}^{n}\|u\|_{c}, \quad t \in I \tag{76}
\end{equation*}
$$

therefore,

$$
\begin{equation*}
\left\|L^{n} u\right\|_{c}=\max _{t \in I}\left\|\left(L^{n} u\right)(t)\right\| \leq m_{1}^{n}\|u\|_{c}, \tag{77}
\end{equation*}
$$

consequently,

$$
\begin{equation*}
\left\|L^{n}\right\| \leq m_{1}^{n}, \tag{78}
\end{equation*}
$$

thus, $r(L)=\lim _{n \rightarrow \infty}\left\|L^{n}\right\|^{1 / n} \leq m_{1}<1$.
Thus, all conditions of Corollary 3 are satisfied; therefore, the conclusions of Theorem 10 hold. Consequently, the proof of Theorem 10 is completed.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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# Generalization of Hermite-Hadamard Type Inequalities via Conformable Fractional Integrals 

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#### Abstract

We establish a Hermite-Hadamard type identity and several new Hermite-Hadamard type inequalities for conformable fractional integrals and present their applications to special bivariate means.


## 1. Introduction

In the field of nonlinear programming and optimization theory, no one can ignore the role of convex sets and convex functions. For the class of convex functions, many inequalities have been introduced such as Jensen's, HermiteHadmard, and Slater's inequalities. Among those inequalities, the most famous and important inequality is the HermiteHadamard's inequality [1] which can be stated as follows.

Let $I \subseteq \mathbb{R}$ be an interval and $h: I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function defined on $I$. Then the double inequality

$$
\begin{equation*}
h\left(\frac{a_{1}+a_{2}}{2}\right) \leq \frac{1}{a_{2}-a_{1}} \int_{a_{1}}^{a_{2}} h(x) d x \leq \frac{h\left(a_{1}\right)+h\left(a_{2}\right)}{2} \tag{1}
\end{equation*}
$$

holds for all $a_{1}, a_{2} \in I$ with $a_{1}<a_{2}$. Both inequalities in (1) hold in the reverse direction if the function $h$ is concave on $I$.

In the last 60 years, many efforts have gone on generalizations, extensions, variants, and applications for the Hermite-Hadamard's inequality (see [2-13]). Anderson [14] and Sarikaya et al. [15] provide the important variants for the Hermite-Hadamard's inequality.

Recently, the author in [16] gave a new definition for the (conformable) fractional derivative as follows.

Let $0<\alpha \leq 1$ and $h:[0, \infty) \longrightarrow \mathbb{R}$ be a realvalued function. Then the $\alpha$-order (conformable) fractional derivative of $h$ at $s>0$ is defined by

$$
\begin{equation*}
D_{\alpha}(h)(s)=\lim _{\epsilon \rightarrow 0} \frac{h\left(s+\epsilon s^{1-\alpha}\right)-h(s)}{\epsilon} . \tag{2}
\end{equation*}
$$

$h$ is said to be $\alpha$-differentiable if the $\alpha$-order (conformable) fractional derivative of $h$ exists, and the $\alpha$-order (conformable) fractional derivative of $h$ at 0 is defined as $h^{\alpha}(0)=$ $\lim _{s \rightarrow 0^{+}} h^{\alpha}(s)$.

Now we discuss some theorems for the (conformable) fractional derivative.

Theorem 1. Let $\alpha \in(0,1]$ and $h_{1}, h_{2}$ be $\alpha$-differentiable at $s>$ 0 . Then one has the following:
(i) $\left(d_{\alpha} / d_{\alpha} s\right)\left(s^{n}\right)=n s^{n-\alpha}$ for all $n \in \mathbb{R}$.
(ii) $\left(d_{\alpha} / d_{\alpha} s\right)(c)=0$ for all constants $c \in \mathbb{R}$.
(iii) $\left(d_{\alpha} / d_{\alpha} s\right)\left(a_{1} h_{1}(s)+a_{2} h_{2}(s)\right)=a_{1}\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s)\right)+$ $a_{2}\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)$ for all constants $a_{1}, a_{2} \in \mathbb{R}$.
(iv) $\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s) h_{2}(s)\right)=h_{1}(s)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)+h_{2}(s)\left(d_{\alpha} /\right.$ $\left.d_{\alpha} s\right)\left(h_{1}(s)\right)$.
(v) $\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s) / h_{2}(s)\right)=\left(h_{2}(s)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}(s)\right)-h_{1}(s)\left(d_{\alpha} /\right.\right.$ $\left.\left.d_{\alpha} s\right)\left(h_{2}(s)\right)\right) /\left(h_{2}(s)\right)^{2}$.
(vi) $\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{1}\left(h_{2}(s)\right)\right)=h_{1}^{\prime}\left(h_{2}(s)\right)\left(d_{\alpha} / d_{\alpha} s\right)\left(h_{2}(s)\right)$ if $h_{1}$ is differentiable at $h_{2}(s)$.

In addition,

$$
\begin{equation*}
\frac{d_{\alpha}}{d_{\alpha} s}\left(h_{1}(s)\right)=s^{1-\alpha} \frac{d}{d s}\left(h_{1}(s)\right) \tag{3}
\end{equation*}
$$

if $h_{1}$ is differentiable.
Definition 2 (conformable fractional integral). Let $\alpha \in(0,1]$ and $0 \leq a_{1}<a_{2}$. Then the function $h_{1}:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ is said to be $\alpha$-fractional integrable on $\left[a_{1}, a_{2}\right.$ ] if the integral

$$
\begin{equation*}
\int_{a_{1}}^{a_{2}} h_{1}(x) d_{\alpha} x:=\int_{a_{1}}^{a_{2}} h_{1}(x) x^{\alpha-1} d x \tag{4}
\end{equation*}
$$

exists and is finite. All $\alpha$-fractional integrable functions on $\left[a_{1}, a_{2}\right]$ are indicated by $L_{\alpha}\left(\left[a_{1}, a_{2}\right]\right)$.

Remark 3.

$$
\begin{equation*}
I_{\alpha}^{a_{1}}\left(h_{1}\right)(s)=I_{1}^{a_{1}}\left(s^{\alpha-1} h_{1}\right)=\int_{a_{1}}^{s} \frac{h_{1}(x)}{x^{1-\alpha}} d x \tag{5}
\end{equation*}
$$

where the integral is the usual Riemann improper integral and $\alpha \in(0,1]$.

Recently, the conformable integrals and derivatives have attracted the attention of many researchers, and many remarkable properties and inequalities for the conformable integrals and derivatives can be found in the literature [1724]. Anderson [14] found the conformable integral version of the Hermite-Hadamard inequality as follows.

Theorem 4 (see [14]). If $\alpha \in(0,1]$ and $h_{1}:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ is an $\alpha$-fractional differentiable function such that $D_{\alpha}(h)$ is increasing, then we have the following inequality:

$$
\begin{equation*}
\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(x) d_{\alpha} x \leq \frac{h\left(a_{1}\right)+h\left(a_{2}\right)}{2} . \tag{6}
\end{equation*}
$$

Moreover if the function $h$ is decreasing on $\left[a_{1}, a_{2}\right]$, then we have

$$
\begin{equation*}
h\left(\frac{a_{1}+a_{2}}{2}\right) \leq \frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(x) d_{\alpha} x \tag{7}
\end{equation*}
$$

If $\alpha=1$, then inequalities (6) and (7) reduce to the classical Hermite-Hadamard's inequalities.

The main purpose of the article is to present the conformable fractional integrals version of the HermiteHadamard's inequality. We first establish an identity for the conformable fractional integrals (Lemma 5) and discuss their special cases. Then applying Jensen's inequality, power mean inequality, Hölder inequality, the convexity of the functions $x^{\alpha-1}$ and $-x^{\alpha}(x>0, \alpha \in(0,1])$, and the identity given by Lemma 5, we obtain inequalities for conformable fractional integrals version of the Hermite-Hadamard's inequality. At last, using particular classes of convex functions we find several new inequalities for some special bivariate means. For some related results, see [25, 26].

## 2. Main Results

The main results of our work can be calculated with the help of the following lemma associated with inequality (8).

Lemma 5. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}, \alpha \in(0,1]$, and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$-fractional differentiable function on $\left(a_{1}, a_{2}\right)$. Then the identity

$$
\begin{align*}
& \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \\
& \quad=\frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\int _ { 0 } ^ { 1 } \left(\left((1-t) a_{1}+t \eta\right)^{2 \alpha-1}\right.\right. \\
& \left.-\eta^{\alpha}\left((1-t) a_{1}+t \eta\right)^{\alpha-1}\right) \times D_{\alpha}(h)\left((1-t) a_{1}+t \eta\right)  \tag{8}\\
& \left.\cdot t^{1-\alpha} d_{\alpha} t\right]+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\int _ { 0 } ^ { 1 } \left(\left((1-t) a_{2}+t \eta\right)^{2 \alpha-1}\right.\right. \\
& \left.-\eta^{\alpha}\left((1-t) a_{2}+t \eta\right)^{\alpha-1}\right) \times D_{\alpha}(h)\left((1-t) a_{2}+t \eta\right) \\
& \left.\cdot t^{1-\alpha} d_{\alpha} t\right]
\end{align*}
$$

holds for any $\eta \in\left[a_{1}, a_{2}\right]$ if $D_{\alpha}(h) \in L_{\alpha}\left(\left[a_{1}, a_{2}\right]\right)$.
Proof. It follows from Theorem 1, Definition 2, and integrating by parts that

$$
\begin{align*}
& \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{1}+t \eta\right)^{2 \alpha-1}\right. \\
& \left.\quad-\eta^{\alpha}\left((1-t) a_{1}+t \eta\right)^{\alpha-1}\right) D_{\alpha}(h)\left((1-t) a_{1}+t \eta\right) d t \\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{2 \alpha-1}\right. \\
& \left.\quad-\eta^{\alpha}\left((1-t) a_{2}+t \eta\right)^{\alpha-1}\right) D_{\alpha}(h)\left((1-t) a_{2}+t \eta\right) d t  \tag{9}\\
& \quad=\frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{1}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) h^{\prime}\left((1-t) a_{1}\right. \\
& \quad+t \eta) d t+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}\right. \\
& \left.\quad-\eta^{\alpha}\right) h^{\prime}\left((1-t) a_{2}+t \eta\right) d t \\
& =\frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(\left((1-t) a_{1}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)\right. \\
& \left.\quad \cdot \frac{h\left((1-t) a_{1}+t \eta\right)}{\eta-a_{1}}\right|_{0} ^{1}-\int_{0}^{1} \alpha\left((1-t) a_{1}+t \eta\right)^{\alpha-1} \\
& \left.\quad \cdot\left(\eta-a_{1}\right) \frac{h\left((1-t) a_{1}+t \eta\right)}{\eta-a_{1}} d t\right]
\end{align*}
$$

$$
\begin{align*}
& +\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)\right. \\
& \left.\cdot \frac{h\left((1-t) a_{2}+t \eta\right)}{\eta-a_{2}}\right|_{0} ^{1}-\int_{0}^{1} \alpha\left((1-t) a_{2}+t \eta\right)^{\alpha-1} \\
& \left.\cdot\left(\eta-a_{2}\right) \frac{h\left((1-t) a_{2}+t \eta\right)}{\eta-a_{2}} d t\right] \\
& =\frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{\eta^{\alpha}-a_{1}^{\alpha}}{\eta-a_{1}} h\left(a_{1}\right)-\frac{\alpha}{\eta-a_{1}} \int_{a_{1}}^{\eta} h(s) d_{\alpha} s\right] \\
& +\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{a_{2}^{\alpha}-\eta^{\alpha}}{a_{2}-\eta} h\left(a_{2}\right)-\frac{\alpha}{a_{2}-\eta} \int_{\eta}^{a_{2}} h(s) d_{\alpha} s\right] \\
& =\frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s, \tag{10}
\end{align*}
$$

where we have used the changes of variable $s=(1-t) a_{1}+t \eta$ and $s=(1-t) a_{2}+t \eta$ to get the desired result.

Remark 6. Let $\alpha=1$. Then identity (8) becomes

$$
\begin{gather*}
\frac{\left(a_{2}-\eta\right) h\left(a_{2}\right)+\left(\eta-a_{1}\right) h\left(a_{1}\right)}{a_{2}-a_{1}}-\frac{1}{a_{2}-a_{1}} \int_{a_{1}}^{a_{2}} h(s) d s \\
=\frac{\left(\eta-a_{1}\right)^{2}}{a_{2}-a_{1}} \int_{0}^{1}(t-1) h^{\prime}\left((1-t) a_{1}+t \eta\right) d t  \tag{11}\\
\quad+\frac{\left(a_{2}-\eta\right)^{2}}{a_{2}-a_{1}} \int_{0}^{1}(1-t) h^{\prime}\left((1-t) a_{2}+t \eta\right) d t,
\end{gather*}
$$

which was proved by Kavurmaci et al. in [2].
Theorem 7. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}, \alpha \in(0,1]$, and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$-differentiable function on $\left(a_{1}, a_{2}\right)$. Then the inequality

$$
\begin{aligned}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \quad-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{1}{2} \eta^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|\right.\right. \\
& \quad-\frac{1}{3} a_{1}^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|-\frac{1}{6} \eta^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|+\frac{1}{2} \eta^{\alpha}\left|h^{\prime}(\eta)\right| \\
& \left.\quad-\frac{1}{6} a_{1}^{\alpha}\left|h^{\prime}(\eta)\right|-\frac{1}{3} \eta^{\alpha}\left|h^{\prime}(\eta)\right|\right] \\
& \quad+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{1}{4} a_{2}^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} \eta^{\alpha-1} a_{2}\left|h^{\prime}\left(a_{2}\right)\right|\right. \\
& \quad+\frac{1}{12} \eta a_{2}^{\alpha-1}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} \eta^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|
\end{aligned}
$$

$$
\begin{align*}
& -\frac{1}{2} \eta^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} a_{2}^{\alpha}\left|h^{\prime}(\eta)\right|+\frac{1}{12} \eta^{\alpha-1} a_{2}\left|h^{\prime}(\eta)\right| \\
& \left.+\frac{1}{12} \eta a_{2}^{\alpha-1}\left|h^{\prime}(\eta)\right|+\frac{1}{4} \eta^{\alpha}\left|h^{\prime}(\eta)\right|-\frac{1}{2} \eta^{\alpha}\left|h^{\prime}(\eta)\right|\right] \tag{12}
\end{align*}
$$

holds for any $\eta \in\left[a_{1}, a_{2}\right]$ if $D_{\alpha}(h) \in L_{\alpha}\left(\left[a_{1}, a_{2}\right]\right)$ and $\left|h^{\prime}\right|$ is convex on $\left[a_{1}, a_{2}\right]$.

Proof. Let $x>0, \varphi_{1}(x)=x^{\alpha-1}$, and $\varphi_{2}(x)=-x^{\alpha}$. Then we clearly see that both the functions $\varphi_{1}(x)$ and $\varphi_{2}(x)$ are convex. From Lemma 5 and the convexity of $\varphi_{1}, \varphi_{2}$, and $\left|h^{\prime}\right|$, we have

$$
\begin{aligned}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t=\frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha+1-1}-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha-1}\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)
\end{aligned}
$$

$$
\begin{align*}
& \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}  \tag{15}\\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) \\
& \cdot\left[(1-t)\left|h^{\prime}\left(a_{1}\right)\right|+t\left|h^{\prime}(\eta)\right|\right] d t+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left[(1-t)\left|h^{\prime}\left(a_{2}\right)\right|+t\left|h^{\prime}(\eta)\right|\right] d t .
\end{align*}
$$

$$
\begin{aligned}
& \leq \frac{\left(\eta-a_{1}\right)^{2}}{a_{2}-a_{1}}\left[\frac{\left|h^{\prime}(\eta)\right|+2\left|h^{\prime}\left(a_{1}\right)\right|}{6}\right] \\
& +\frac{\left(a_{2}-\eta\right)^{2}}{a_{2}-a_{1}}\left[\frac{\left|h^{\prime}(\eta)\right|+2\left|h^{\prime}\left(a_{2}\right)\right|}{6}\right],
\end{aligned}
$$

which was proved by Kavurmaci et al. in [2].
Theorem 9. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{2}>a_{1}, \alpha \in(0,1], p, q>1$ such that $1 / p+1 / q=1$ and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$ differentiable function on $\left(a_{1}, a_{2}\right)$. Then the inequality

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(A_{1}(\alpha, p)\right)^{1 / p}\right.\right. \\
& \left.\cdot\left(\frac{\left|h^{\prime}\left(a_{1}\right)\right|^{q}+\left|h^{\prime}(\eta)\right|^{q}}{2}\right)^{1 / q}\right]  \tag{16}\\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(B_{1}(\alpha, p)\right)^{1 / p}\right. \\
& \left.\cdot\left(\frac{\left|h^{\prime}\left(a_{2}\right)\right|^{q}+\left|h^{\prime}(\eta)\right|^{q}}{2}\right)^{1 / q}\right]
\end{align*}
$$

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \quad-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{1}{2} \eta^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|\right.\right. \\
& \quad-\frac{1}{3} a_{1}^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|-\frac{1}{6} \eta^{\alpha}\left|h^{\prime}\left(a_{1}\right)\right|+\frac{1}{2} \eta^{\alpha}\left|h^{\prime}(\eta)\right| \\
& \left.\quad-\frac{1}{6} a_{1}^{\alpha}\left|h^{\prime}(\eta)\right|-\frac{1}{3} \eta^{\alpha}\left|h^{\prime}(\eta)\right|\right]  \tag{14}\\
& \quad+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\frac{1}{4} a_{2}^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} \eta^{\alpha-1} a_{2}\left|h^{\prime}\left(a_{2}\right)\right|\right. \\
& \quad+\frac{1}{12} \eta a_{2}^{\alpha-1}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} \eta^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|  \tag{17}\\
& \quad-\frac{1}{2} \eta^{\alpha}\left|h^{\prime}\left(a_{2}\right)\right|+\frac{1}{12} a_{2}^{\alpha}\left|h^{\prime}(\eta)\right|+\frac{1}{12} \eta^{\alpha-1} a_{2}\left|h^{\prime}(\eta)\right| \\
& \left.\quad+\frac{1}{12} \eta a_{2}^{\alpha-1}\left|h^{\prime}(\eta)\right|+\frac{1}{4} \eta^{\alpha}\left|h^{\prime}(\eta)\right|-\frac{1}{2} \eta^{\alpha}\left|h^{\prime}(\eta)\right|\right] .
\end{align*}
$$

$$
\begin{aligned}
& A_{1}(\alpha, p)=\int_{0}^{1}\left(\eta^{\alpha}-\left(\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\right)^{p} d t \\
& B_{1}(\alpha, p) \\
& \quad=\int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)\right. \\
& \left.\quad-\eta^{\alpha}\right)^{p} d t
\end{aligned}
$$

Proof. It follows from inequality (13) that

$$
\left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right.
$$

$$
\begin{aligned}
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) \\
& \left.\cdot \mid h^{\prime}\left((1-t) a_{1}+t \eta\right)\right) d t+\frac{a_{2}-\eta}{\eta^{\alpha}-a_{1}^{\alpha}}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t
\end{aligned}
$$

Making use of Hölder's inequality, one has

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t \\
& \quad \leq\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)^{p} d t\right)^{1 / p}  \tag{20}\\
& \cdot\left(\int_{0}^{1}\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right|^{q} d t\right)^{1 / q}  \tag{19}\\
& \quad \leq\left(A_{1}(\alpha, p)\right)^{1 / p} \\
& \cdot\left(\int_{0}^{1}\left((1-t)\left|h^{\prime}\left(a_{1}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right) d t\right)^{1 / q} \\
& \quad=\left(A_{1}(\alpha, p)\right)^{1 / p}\left(\frac{\left|h^{\prime}\left(a_{1}\right)\right|^{q}+\left|h^{\prime}(\eta)\right|^{q}}{2}\right)^{1 / q}
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t  \tag{18}\\
& \quad \leq\left(\int _ { 0 } ^ { 1 } \left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)\right.\right. \\
& \left.\left.-\eta^{\alpha}\right)^{p} d t\right)^{1 / p} \\
& \quad \times\left(\int_{0}^{1}\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right|^{q} d t\right)^{1 / q} \leq\left(B_{1}(\alpha,\right. \\
& p))^{1 / p}\left(\int_{0}^{1}\left((1-t)\left|h^{\prime}\left(a_{2}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right) d t\right)^{1 / q} \\
& \leq\left(B_{1}(\alpha, p)\right)^{1 / p}\left(\frac{\left|h^{\prime}\left(a_{2}\right)\right|^{q}+\left|h^{\prime}(\eta)\right|^{q}}{2}\right)^{1 / q} .
\end{align*}
$$

$$
\begin{align*}
& \left|\frac{\left(\eta-a_{1}\right) h\left(a_{1}\right)+\left(a_{2}-\eta\right) h\left(a_{2}\right)}{a_{2}-a_{1}}-\frac{1}{a_{2}-a_{1}} \int_{a_{1}}^{a_{2}} h(x) d x\right| \\
& \quad \leq\left(\frac{1}{p+1}\right)^{1 / \eta}\left(\frac{1}{2}\right)^{1 / q}\left[\frac{\left(\eta-a_{1}\right)^{2}\left[\left|h^{\prime}(\eta)\right|^{q}+\left|h^{\prime}\left(a_{1}\right)\right|^{q}\right]^{1 / q}+\left(a_{2}-\eta\right)^{2}\left[\left|h^{\prime}(\eta)\right|^{q}+\left|h^{\prime}\left(a_{2}\right)\right|^{q}\right]^{1 / q}}{a_{2}-a_{1}}\right] \tag{21}
\end{align*}
$$

which was proved by Kavurmaci et al. in [2].
Theorem 11. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}, \alpha \in(0,1], q>$ 1 and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$-differentiable function on $\left(a_{1}, a_{2}\right)$. Then the inequality

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(A_{1}(\alpha)\right)^{1-1 / q}\right.\right. \\
& \left.\cdot\left\{A_{2}(\alpha)\left|h^{\prime}\left(a_{1}\right)\right|^{q}+A_{3}(\alpha)\left|h^{\prime}(\eta)\right|^{q}\right\}^{1 / q}\right]  \tag{22}\\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(B_{1}(\alpha)\right)^{1-1 / q}\right. \\
& \left.\cdot\left\{B_{2}(\alpha)\left|h^{\prime}\left(a_{2}\right)\right|+B_{3}(\alpha)\left|h^{\prime}(\eta)\right|\right\}^{1 / q}\right]
\end{align*}
$$

holds for any $\eta \in\left[a_{1}, a_{2}\right]$ if $D_{\alpha}(h) \in L_{\alpha}\left(\left[a_{1}, a_{2}\right]\right)$ and $\left|h^{\prime}\right|^{q}$ is convex on $\left[a_{1}, a_{2}\right]$, where

$$
\begin{align*}
& A_{1}(\alpha)=\frac{\eta^{\alpha}-a_{1}^{\alpha}}{2} \\
& B_{1}(\alpha)=\frac{2 a_{2}^{\alpha}+\eta^{\alpha-1} a_{2}+\eta a_{2}^{\alpha-1}-4 \eta^{\alpha}}{6} \\
& A_{2}(\alpha)=\frac{\eta^{\alpha}-a_{1}^{\alpha}}{3}  \tag{23}\\
& A_{3}(\alpha)=\frac{\eta^{\alpha}-a_{1}^{\alpha}}{6} \\
& B_{2}(\alpha)=\frac{3 a_{2}^{\alpha}+\eta^{\alpha-1} a_{2}+\eta a_{2}^{\alpha-1}-5 \eta^{\alpha}}{12} \\
& B_{3}(\alpha)=\frac{a_{2}^{\alpha}+\eta^{\alpha-1} a_{2}+\eta a_{2}^{\alpha-1}-3 \eta^{\alpha}}{12}
\end{align*}
$$

Proof. It follows from inequality (13) that

$$
\begin{aligned}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right.
\end{aligned}
$$

$$
\begin{align*}
& \cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{a_{2}-\eta}{\eta^{\alpha}-a_{1}^{\alpha}} \\
& \cdot \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \tag{24}
\end{align*}
$$

Making use of the power mean inequality, we get

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t \\
& \quad \leq\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) d t\right)^{1-1 / q}  \tag{25}\\
& \quad \times\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\right. \\
& \left.\quad \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right|^{q} d t\right)^{1 / q} .
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \quad \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \\
& \quad \leq\left(\int _ { 0 } ^ { 1 } \left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)\right.\right. \\
& \left.\left.\quad-\eta^{\alpha}\right) d t\right)^{1-1 / q} \times\left(\int _ { 0 } ^ { 1 } \left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\right.\right.  \tag{26}\\
& \cdot \\
& \left.\quad\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \left.\cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right|^{q} d t\right)^{1 / q}
\end{align*}
$$

From the convexity of $\left|h^{\prime}\right|^{q}$, we have

$$
\begin{aligned}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right|^{q} d t \\
& \quad \leq \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)
\end{aligned}
$$

$\cdot\left[(1-t)\left|h^{\prime}\left(a_{1}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right] d t=\left|h^{\prime}\left(a_{1}\right)\right|^{q}$
$\cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)(1-t) d t+\left|h^{\prime}(\eta)\right|^{q}$
$\cdot \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) t d t=\left|f^{\prime}\left(a_{1}\right)\right|^{q}\left(\frac{1}{2}\right.$
$\left.\cdot \eta^{\alpha}-\frac{1}{3} a_{1}^{\alpha}-\frac{1}{6} \eta^{\alpha}\right)+\left|h^{\prime}(\eta)\right|^{q}\left(\frac{1}{2} \eta^{\alpha}-\frac{1}{6} a_{1}^{\alpha}-\frac{1}{3}\right.$
$\left.\cdot \eta^{\alpha}\right)$
and

$$
\begin{aligned}
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \quad \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right|^{q} d t \\
& \quad \leq \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)\right. \\
& \left.\quad-\eta^{\alpha}\right)\left[(1-t)\left|h^{\prime}\left(a_{2}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right] d t \\
& \quad=\left|h^{\prime}\left(a_{2}\right)\right|^{q}\left(\frac{1}{4} a_{2}^{\alpha}+\frac{1}{12} \eta^{\alpha-1} a_{2}+\frac{1}{12} \eta a_{2}^{\alpha-1}+\frac{1}{12} \eta^{\alpha}\right. \\
& \left.\quad-\frac{1}{2} \eta^{\alpha}\right)+\left|h^{\prime}(\eta)\right|^{q}\left(\frac{1}{12} a_{2}^{\alpha}+\frac{1}{12} \eta^{\alpha-1} a_{2}+\frac{1}{12} \eta a_{2}^{\alpha-1}\right. \\
& \left.\quad+\frac{1}{4} \eta^{\alpha}-\frac{1}{2} \eta^{\alpha}\right),
\end{aligned}
$$

where we have also used the facts that

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) d t=A_{1}(\alpha) \\
& \quad=\eta^{\alpha}-\frac{1}{2} a_{1}^{\alpha}-\frac{1}{2} \eta^{\alpha} \\
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) d t  \tag{29}\\
& \quad=B_{1}(\alpha)=\frac{1}{3} a_{2}^{\alpha}+\frac{1}{6} \eta^{\alpha-1} a_{2}+\frac{1}{6} \eta a_{2}^{\alpha-1}+\frac{1}{3} \eta^{\alpha}-\eta^{\alpha} .
\end{align*}
$$

Hence, we have the result in (22).
Remark 12. If $\alpha=1$, then inequality (22) becomes

$$
\begin{align*}
& \left|\frac{\left(\eta-a_{1}\right) h\left(a_{1}\right)+\left(a_{2}-\eta\right) h\left(a_{2}\right)}{a_{2}-a_{1}}-\frac{1}{a_{2}-a_{1}} \int_{a_{1}}^{a_{2}} h(x) d x\right| \\
& \quad \leq \frac{1}{2}\left(\frac{1}{3}\right)^{1 / q}\left[\frac{\left(\eta-a_{1}\right)^{2}\left[\left|h^{\prime}(\eta)\right|^{q}+2\left|h^{\prime}\left(a_{1}\right)\right|^{q}\right]^{1 / q}+\left(a_{2}-\eta\right)^{2}\left[\left|h^{\prime}(\eta)\right|^{q}+2\left|h^{\prime}\left(a_{2}\right)\right|^{q}\right]^{1 / q}}{a_{2}-a_{1}}\right] \tag{30}
\end{align*}
$$

which can be found in [2].

Theorem 13. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}, \alpha \in(0,1], q>$ 1 and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$-differentiable function on $\left(a_{1}, a_{2}\right)$. Then the inequality

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(A_{1}(\alpha)\right)^{1-1 / q}\right.\right. \\
& \left.\cdot\left\{A_{2}(\alpha)\left|h^{\prime}\left(a_{1}\right)\right|^{q}+A_{3}(\alpha)\left|h^{\prime}(\eta)\right|^{q}\right\}^{1 / q}\right]  \tag{31}\\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(B_{1}(\alpha)\right)^{1-1 / q}\right. \\
& \left.\cdot\left\{B_{2}(\alpha)\left|h^{\prime}\left(a_{2}\right)\right|^{q}+B_{3}(\alpha)\left|h^{\prime}(\eta)\right|^{q}\right\}^{1 / q}\right]
\end{align*}
$$

is valid for any $\eta \in\left[a_{1}, a_{2}\right]$ if $D_{\alpha}(h) \in L_{\alpha}\left(\left[a_{1}, a_{2}\right]\right)$ and $\left|h^{\prime}\right|^{q}$ is convex on $\left[a_{1}, a_{2}\right]$, where

$$
\left.\begin{array}{rl}
A_{1}(\alpha)= & \eta^{\alpha}-\left[\frac{\eta^{\alpha+1}-a_{1}^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\right] \\
B_{1}(\alpha)= & {\left[\frac{\eta^{\alpha+1}-a_{2}^{\alpha+1}}{(\alpha+1)\left(a_{2}-\eta\right)}\right]-\eta^{\alpha},} \\
A_{2}(\alpha)= & \frac{1}{2} \eta^{\alpha} \\
& +\frac{a_{1}^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\left[\frac{(\alpha+2)\left(\eta-a_{1}\right)-a_{1}}{(\alpha+2)\left(\eta-a_{1}\right)}\right] \\
& -\frac{\eta^{\alpha+2}}{(\alpha+1)\left(\eta-a_{1}\right)^{2}(\alpha+2)}, \\
B_{2}(\alpha)= & -\frac{a_{2}^{\alpha+1}}{(\alpha+1)\left(a_{2}-\eta\right)}\left[\frac{(\alpha+2)\left(a_{2}-\eta\right)+a_{2}}{(\alpha+2)\left(a_{2}-\eta\right)}\right] \\
& +\frac{\eta^{\alpha+2}}{(\alpha+1)\left(a_{2}-\eta\right)^{2}(\alpha+2)}-\frac{1}{2} \eta^{\alpha}, \\
A_{3}(\alpha)= & \frac{1}{2} \eta^{\alpha} \\
& -\frac{\eta^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\left[\frac{(\alpha+2)\left(\eta-a_{1}\right)-\eta}{(\alpha+2)\left(\eta-a_{1}\right)}\right] \\
B_{3}(\alpha)= & \frac{-}{(\alpha+1)\left(a_{2}-\eta\right)} \frac{\eta^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)^{2}(\alpha+2)}, \\
(\alpha+1)\left(a_{2}-\eta\right)^{2}(\alpha+2) & \frac{(\alpha+2)\left(a_{2}-\eta\right)-\eta}{2} \eta^{\alpha} \\
(\alpha+2)\left(a_{2}-\eta\right)
\end{array}\right]
$$

Proof. From Theorem 1, Definition 2, and Lemma 5, we get

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \quad \cdot \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s|=| \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \quad \cdot \int_{0}^{1}\left(\left((1-t) a_{1}+t \eta\right)^{2 \alpha-1}\right. \\
& \left.\quad-\eta^{\alpha}\left((1-t) a_{1}+t \eta\right)^{\alpha-1}\right) D_{\alpha}(h)\left((1-t) a_{1}\right. \\
& \quad+t \eta) d t+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{2 \alpha-1}\right.  \tag{33}\\
& \left.\quad-\eta^{\alpha}\left((1-t) a_{2}+t \eta\right)^{\alpha-1}\right) D_{\alpha}(h)\left((1-t) a_{2}\right. \\
& \quad+t \eta) d t \left\lvert\, \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}\right.\right.\right. \\
& \left.\quad+t \eta)^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}} \\
& \quad \cdot \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) \mid h^{\prime}\left((1-t) a_{2}\right. \\
& \quad+t \eta) \mid d t .
\end{align*}
$$

Making use of power mean inequality, we get

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta^{\alpha}\right)\right| d t \\
& \quad \leq\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) d t\right)^{1-1 / q} \\
& \quad \times\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right)\right.  \tag{34}\\
& \left.\quad \cdot\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right|^{q} d t\right)^{1 / q}
\end{align*}
$$

Similarly, we have

$$
\begin{align*}
& \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \\
& \quad \leq\left(\int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) d t\right)^{1-1 / q}  \tag{35}\\
& \quad \times\left(\int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)\right. \\
& \left.\quad \cdot\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right|^{q} d t\right)^{1 / q}
\end{align*}
$$

It follows from the convexity of $\left|h^{\prime}\right|^{q}$ that

$$
\begin{aligned}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right|^{q} d t \\
& \quad \leq \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) \\
& \quad \cdot\left[(1-t)\left|h^{\prime}\left(a_{1}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right] d t \\
& =\left|h^{\prime}\left(a_{1}\right)\right|^{q} \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right)(1-t) d t \\
& \quad+\left|h^{\prime}(\eta)\right|^{q} \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) t d t \\
& =\left|h^{\prime}(a)_{1}\right|^{q}\left(\frac{1}{2} \eta^{\alpha}\right. \\
& \quad+\frac{a_{1}^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\left[\frac{(\alpha+2)\left(\eta-a_{1}\right)-a_{1}}{(\alpha+2)\left(\eta-a_{1}\right)}\right] \\
& \left.\quad-\frac{\eta^{\alpha+2}}{(\alpha+1)\left(\eta-a_{1}\right)^{2}(\alpha+2)}\right)+\left|h^{\prime}(\eta)\right|^{q}\left(\frac{1}{2} \eta^{\alpha}\right. \\
& \\
& \quad-\frac{\eta^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\left[\frac{(\alpha+2)\left(\eta-a_{1}\right)-\eta}{(\alpha+2)\left(\eta-a_{1}\right)}\right] \\
& \\
& \left.-\frac{a_{1}^{\alpha+2}}{(\alpha+1)\left(\eta-a_{1}\right)^{2}(\alpha+2)}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right|^{q} d t \\
& \quad \leq \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) \\
& \cdot\left[(1-t)\left|h^{\prime}\left(a_{2}\right)\right|^{q}+t\left|h^{\prime}(\eta)\right|^{q}\right] d t \\
& =\left|h^{\prime}\left(a_{2}\right)\right|^{q} \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right)(1-t) d t \\
& \quad+\left|h^{\prime}(\eta)\right|^{q} \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) t d t \\
& \quad=\left|h^{\prime}\left(a_{2}\right)\right|^{q} \\
& \cdot\left(-\frac{a_{2}^{\alpha+1}}{(\alpha+1)\left(a_{2}-\eta\right)}\left[\frac{(\alpha+2)\left(a_{2}-\eta\right)+a_{2}}{(\alpha+2)\left(a_{2}-\eta\right)}\right]\right. \\
& \left.\quad+\frac{\eta^{\alpha+2}}{(\alpha+1)\left(a_{2}-\eta\right)^{2}(\alpha+2)}-\frac{1}{2} \eta^{\alpha}\right)+\left|h^{\prime}(\eta)\right|^{q} \\
& \\
& \cdot\left(\frac{\eta^{\alpha+1}}{(\alpha+1)\left(a_{2}-\eta\right)}\left[\frac{(\alpha+2)\left(a_{2}-\eta\right)-\eta}{(\alpha+2)\left(a_{2}-\eta\right)}\right]\right. \\
& \\
& \left.\quad-\frac{a_{2}^{\alpha+2}}{(\alpha+1)\left(a_{2}-\eta\right)^{2}(\alpha+2)}-\frac{1}{2} \eta^{\alpha}\right)
\end{aligned}
$$

where we have used the identities

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}+t \eta\right)^{\alpha}\right) d t \\
& \quad=\eta^{\alpha}-\left[\frac{\eta^{\alpha+1}-a_{1}^{\alpha+1}}{(\alpha+1)\left(\eta-a_{1}\right)}\right]  \tag{38}\\
& \int_{0}^{1}\left(\left((1-t) a_{2}+t \eta\right)^{\alpha}-\eta^{\alpha}\right) d t \\
& \quad=\left[\frac{\eta^{\alpha+1}-a_{2}^{\alpha+1}}{(\alpha+1)\left(a_{2}-\eta\right)}\right]-\eta^{\alpha} \tag{36}
\end{align*}
$$

Hence, we have the result in (31).
Theorem 14. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}, \alpha \in(0,1]$ and $h:\left[a_{1}, a_{2}\right] \longrightarrow \mathbb{R}$ be an $\alpha$-differentiable function on $\left(a_{1}, a_{2}\right)$. Then the inequality

$$
\begin{align*}
& \left\lvert\, \frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\right. \\
& \left.\quad-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s \right\rvert\,  \tag{39}\\
& \quad \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[A_{1}(\alpha)\left|h^{\prime}\left(\frac{C_{1}(\alpha)}{A_{1}(\alpha)}\right)\right|\right] \\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[B_{1}(\alpha)\left|h^{\prime}\left(\frac{C_{2}(\alpha)}{B_{1}(\alpha)}\right)\right|\right]
\end{align*}
$$

$$
\begin{align*}
& A_{1}(\alpha)=\frac{\eta^{\alpha}-a_{1}^{\alpha}}{2} \\
& B_{1}(\alpha)=\frac{2 a_{2}^{\alpha}+\eta^{\alpha-1} a_{2}+\eta a_{2}^{\alpha-1}-4 \eta^{\alpha}}{6},  \tag{37}\\
& C_{1}(\alpha)=\frac{a_{1} \eta^{\alpha}-2 a_{1}^{\alpha+1}+\eta^{\alpha+1}}{6},  \tag{40}\\
& C_{2}(\alpha) \\
& =\frac{3 a_{2}^{\alpha+1}+\eta^{\alpha-1} a_{2}^{2}+2 \eta a_{2}^{\alpha}-4 a_{2} \eta^{\alpha}+\eta^{\alpha+1} a_{2}^{\alpha-1}-3 \eta^{\alpha+1}}{12} .
\end{align*}
$$

Proof. We clearly see that $\left|h^{\prime}\right|$ is concave because $\left|h^{\prime}\right|^{q}$ is concave for some $q>1$ (see [27]). From Theorem 1, Definition 2, Lemma 5, the concavity of $\left|h^{\prime}\right|$, and Jensen's integral inequality, we have

$$
\begin{align*}
& \left|\frac{\left(a_{2}^{\alpha}-\eta^{\alpha}\right) h\left(a_{2}\right)+\left(\eta^{\alpha}-a_{1}^{\alpha}\right) h\left(a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}-\frac{\alpha}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{a_{1}}^{a_{2}} h(s) d_{\alpha} s\right| \\
& \leq \frac{\eta-a_{1}}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t \\
& \quad+\frac{a_{2}-\eta}{a_{2}^{\alpha}-a_{1}^{\alpha}} \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t, \\
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left|h^{\prime}\left((1-t) a_{1}+t \eta\right)\right| d t \\
& \leq\left(\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) d t\right) \times\left|h^{\prime}\left(\frac{\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left((1-t) a_{1}+t \eta\right) d t}{\int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) d t}\right)\right|  \tag{41}\\
& \quad=A_{1}(\alpha) h^{\prime}\left(\frac{C_{1}(\alpha)}{A_{1}(\alpha)}\right), \\
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right)\left|h^{\prime}\left((1-t) a_{2}+t \eta\right)\right| d t \\
& \leq\left(\int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) d t\right) \\
& \quad \times\left|h^{\prime}\left(\frac{\int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right)\left((1-t) a_{2}+t \eta\right) d t}{\int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) d t}\right)\right|=B_{1}(\alpha) h^{\prime}\left(\frac{C_{2}(\alpha)}{B_{1}(\alpha)}\right),
\end{align*}
$$

where we have used the identities

$$
\begin{align*}
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right) d t=A_{1}(\alpha)  \tag{42}\\
& \quad=\eta^{\alpha}-\frac{1}{2} a_{1}^{\alpha}-\frac{1}{2} \eta^{\alpha} \\
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) d t  \tag{43}\\
& \quad=B_{1}(\alpha)=\frac{1}{3} a_{2}^{\alpha}+\frac{1}{6} \eta^{\alpha-1} a_{2}+\frac{1}{6} \eta a_{2}^{\alpha-1}+\frac{1}{3} \eta^{\alpha}-\eta^{\alpha} \\
& \int_{0}^{1}\left(\eta^{\alpha}-\left((1-t) a_{1}^{\alpha}+t \eta^{\alpha}\right)\right)\left((1-t) a_{1}+t \eta\right) d t \\
& \quad=C_{1}(\alpha)
\end{align*}
$$

$$
\begin{aligned}
= & \frac{1}{2} a_{1} \eta^{\alpha}-\frac{1}{3} a_{1}^{\alpha+1}-\frac{1}{6} a_{1} \eta^{\alpha}+\frac{1}{2} \eta^{\alpha+1}-\frac{1}{6} \eta a_{1}^{\alpha} \\
& -\frac{1}{3} \eta^{\alpha+1}
\end{aligned}
$$

and

$$
\begin{aligned}
& \int_{0}^{1}\left(\left((1-t) a_{2}^{\alpha-1}+t \eta^{\alpha-1}\right)\left((1-t) a_{2}+t \eta\right)-\eta^{\alpha}\right) \\
& \cdot\left((1-t) a_{2}+t \eta\right) d t=C_{2}(\alpha)=\frac{1}{4} a_{2}^{\alpha+1}+\frac{1}{12} \\
& \cdot \eta^{\alpha-1} a_{2}^{2}+\frac{1}{6} \eta a_{2}^{\alpha}-\frac{1}{3} a_{2} \eta^{\alpha}+\frac{1}{12} \eta^{\alpha+1} a_{2}^{\alpha-1}-\frac{1}{4} \eta^{\alpha+1}
\end{aligned}
$$

Remark 15. If $\alpha=1$, then inequality (39) becomes

$$
\begin{align*}
& \left|\frac{\left(\eta-a_{1}\right) h\left(a_{1}\right)+\left(a_{2}-\eta\right) h\left(a_{2}\right)}{a_{2}-a_{1}}-\frac{1}{a_{2}-a_{1}} \int_{a_{1}}^{a_{2}} h(x) d x\right| \\
& \quad \leq \frac{1}{2}\left[\frac{\left(\eta-a_{1}\right)^{2}\left|h^{\prime}\left(\left(\eta+2 a_{1}\right) / 3\right)\right|+\left(a_{2}-\eta\right)^{2}\left|h^{\prime}\left(\left(\eta+2 a_{2}\right) / 3\right)\right|}{a_{2}-a_{1}}\right] \tag{44}
\end{align*}
$$

## 3. Applications to Special Means

A bivariate function $M:(0, \infty) \times(0, \infty) \longmapsto(0, \infty)$ is said to be a mean if $\min \{x, y\} \leq M(x, y) \leq \max \{x, y\}$ for all $x, y \in(0, \infty)$. Recently, the mean value theory has been the subject of intensive research, and many remarkable inequalities and properties for various bivariate means can be found in the literature [28-33].

In this section, we use the results obtained in Section 2 to give some applications to the weighted arithmetic mean

$$
\begin{equation*}
A\left(a_{1}, a_{2} ; w_{1}, w_{2}\right)=\frac{w_{1} a_{1}+w_{2} a_{2}}{w_{1}+w_{2}} \quad\left(a_{1}, a_{2}>0\right) \tag{45}
\end{equation*}
$$

and $(\alpha, r)$-th generalized logarithmic mean

$$
\begin{align*}
& L_{(\alpha, r)}\left(a_{1}, a_{2}\right)=\left[\frac{\alpha\left(a_{2}^{\alpha+r}-a_{1}^{\alpha+r}\right)}{(\alpha+r)\left(a_{2}^{\alpha}-a_{1}^{\alpha}\right)}\right]^{1 / r}  \tag{46}\\
& \quad\left(a_{1}, a_{2}>0, a_{1} \neq a_{2}, r \in \mathbb{R}, r \neq 0, \alpha,-\alpha \in(0,1]\right) .
\end{align*}
$$

Proposition 16. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}$ and $r>1$. Then the inequality

$$
\begin{align*}
\mid A & \left(a_{1}^{r}, a_{2}^{r} ; \eta^{\alpha}-a_{1}^{\alpha}, a_{2}^{\alpha}-\eta^{\alpha}\right)-L_{(\alpha, r)}^{r}\left(a_{1}, a_{2}\right) \mid \\
& \leq \frac{r\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left\{\frac{1}{2} \eta^{\alpha}\left|a_{1}\right|^{r-1}-\frac{1}{3} a_{1}^{\alpha}\left|a_{1}\right|^{r-1}\right. \\
& -\frac{1}{6} \eta^{\alpha}\left|a_{1}\right|^{r-1}+\frac{1}{2} \eta^{\alpha}|\eta|^{r-1}-\frac{1}{6} a_{1}^{\alpha}|\eta|^{r-1} \\
& \left.-\frac{1}{3} \eta^{\alpha}|\eta|^{r-1}\right\}+\frac{r\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left\{\frac{1}{4} a_{2}^{\alpha}\left|a_{2}\right|^{r-1}\right.  \tag{47}\\
& +\frac{1}{12} \eta^{\alpha-1} a_{2}\left|a_{2}\right|^{r-1}+\frac{1}{12} \eta a_{2}^{\alpha-1}\left|a_{2}\right|^{r-1} \\
& +\frac{1}{12} \eta^{\alpha}\left|a_{2}\right|^{r-1}-\frac{1}{2} \eta^{\alpha}\left|a_{2}\right|^{r-1}+\frac{1}{12} a_{2}^{\alpha}|\eta|^{r-1} \\
& +\frac{1}{12} \eta^{\alpha-1} a_{2}|\eta|^{r-1}+\frac{1}{12} \eta a_{2}^{\alpha-1}|\eta|^{r-1}+\frac{1}{4} \eta^{\alpha}|\eta|^{r-1} \\
& \left.-\frac{1}{2} \eta^{\alpha}|\eta|^{r-1}\right\}
\end{align*}
$$

holds for any $\alpha \in(0,1]$ and $\eta \in\left[a_{1}, a_{2}\right]$.
Proof. Let $h(x)=x^{r}$. Then the result follows easily from Theorem 7 and the convexity of $h(x)$ on the interval $\left[a_{1}, a_{2}\right]$.

Proposition 17. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}$ and $r>1$. Then the inequality

$$
\begin{aligned}
& \left|A\left(a_{1}^{-1}, a_{2}^{-1} ; \eta^{\alpha}-a_{1}^{\alpha}, a_{2}^{\alpha}-\eta^{\alpha}\right)-L_{(\alpha,-1)}^{r}\left(a_{1}, a_{2}\right)\right| \\
& \quad \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left\{\frac{1}{2} \eta^{\alpha}\left|a_{1}\right|^{-2}-\frac{1}{3} a_{1}^{\alpha}\left|a_{1}\right|^{-2}-\frac{1}{6} \eta^{\alpha}\left|a_{1}\right|^{-2}\right.
\end{aligned}
$$

$$
\begin{align*}
& \left.+\frac{1}{2} \eta^{\alpha}|\eta|^{-2}-\frac{1}{6} a_{1}^{\alpha}|\eta|^{-2}-\frac{1}{3} \eta^{\alpha}|\eta|^{-2}\right\} \\
& +\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left\{\frac{1}{4} a_{2}^{\alpha}\left|a_{2}\right|^{-2}+\frac{1}{12} \eta^{\alpha-1} a_{2}\left|a_{2}\right|^{-2}\right. \\
& +\frac{1}{12} \eta a_{2}^{\alpha-1}\left|a_{2}\right|^{-2}+\frac{1}{12} \eta^{\alpha}\left|a_{2}\right|^{-2}-\frac{1}{2} \eta^{\alpha}\left|a_{2}\right|^{-2} \\
& +\frac{1}{12} a_{2}^{\alpha}|\eta|^{-2}+\frac{1}{12} \eta^{\alpha-1} a_{2}|\eta|^{-2}+\frac{1}{12} \eta a_{2}^{\alpha-1}|\eta|^{-2} \\
& \left.+\frac{1}{4} \eta^{\alpha}|\eta|^{-2}-\frac{1}{2} \eta^{\alpha}|\eta|^{-2}\right\} \tag{48}
\end{align*}
$$

holds for any $\alpha \in(0,1]$ and $\eta \in\left[a_{1}, a_{2}\right]$.
Proof. Let $h(x)=1 / x$. Then Proposition 17 follows from Theorem 7 and the convexity of the function $h(x)$ on the interval $\left[a_{1}, a_{2}\right]$.

Proposition 18. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}$ and $r>1$. Then the inequality

$$
\begin{align*}
& \left|A\left(a_{1}^{r}, a_{2}^{r} ; \eta^{\alpha}-a_{1}^{\alpha}, a_{2}^{\alpha}-\eta^{\alpha}\right)-L_{(\alpha, r)}^{r}\left(a_{1}, a_{2}\right)\right| \\
& \quad \leq \frac{r\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(A_{1}(\alpha)\right)^{1-1 / q}\right. \\
&  \tag{49}\\
& \left.\cdot\left\{A_{2}(\alpha)\left|a_{1}\right|^{(r-1) q}+A_{3}(\alpha)|\eta|^{(r-1) q}\right\}^{1 / q}\right] \\
& \\
& \quad+\frac{r\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(B_{1}(\alpha)\right)^{1-1 / q}\right. \\
& \\
& \left.\cdot\left\{B_{2}(\alpha)\left|a_{2}\right|^{(r-1) q}+B_{3}(\alpha)|\eta|^{(r-1) q}\right\}^{1 / q}\right]
\end{align*}
$$

holds for all $\alpha \in(0,1]$ and $\eta \in\left[a_{1}, a_{2}\right]$, where $A_{i}(\alpha)$ and $B_{i}(\alpha)(i=1,2,3)$ are defined as in Theorem 11.

Proof. Let $h(x)=x^{r}$. Then Proposition 18 follows from Theorem 11 and the convexity of $h(x)$ on $\left[a_{1}, a_{2}\right]$ immediately.

Proposition 19. Let $a_{1}, a_{2} \in \mathbb{R}^{+}$with $a_{1}<a_{2}$ and $r>1$. Then the inequality

$$
\begin{align*}
& \left|A\left(a_{1}^{-1}, a_{2}^{-1} ; \eta^{\alpha}-a_{1}^{\alpha}, a_{2}^{\alpha}-\eta^{\alpha}\right)-L_{(\alpha,-1)}^{r}\left(a_{1}, a_{2}\right)\right| \\
& \quad \leq \frac{\left(\eta-a_{1}\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(A_{1}(\alpha)\right)^{1-1 / q}\right. \\
& \left.\cdot\left\{A_{2}(\alpha)\left|a_{1}\right|^{-2 q}+A_{3}(\alpha)|\eta|^{-2 q}\right\}^{1 / q}\right]  \tag{50}\\
& \quad+\frac{\left(a_{2}-\eta\right)}{a_{2}^{\alpha}-a_{1}^{\alpha}}\left[\left(B_{1}(\alpha)\right)^{1-1 / q}\right. \\
& \left.\quad \cdot\left\{B_{2}(\alpha)\left|a_{2}\right|^{-2 q}+B_{3}(\alpha)|\eta|^{-2 q}\right\}^{1 / q}\right]
\end{align*}
$$

holds for all $\alpha \in(0,1]$ and $\eta \in\left[a_{1}, a_{2}\right]$, where $A_{i}(\alpha)$ and $B_{i}(\alpha)(i=1,2,3)$ are defined as in Theorem 11.

Proof. Let $h(x)=1 / x$. Then the result follows easily from Theorem 11 and the convexity of $h(x)$ on the interval [ $a_{1}$, $a_{2}$ ].

## 4. Conclusion

In the article, we establish an identity and several new inequalities of Hermite-Hadamard type for conformable fractional integrals by use of the convexity theory and Jesen's inequality, Hölder inequality, and power mean inequality and present their applications to special bivariate means. The given Hermite-Hadamard type inequalities for conformable fractional integrals are the generalizations of the corresponding results established by Kavurmaci, Avci, and Özdemir in [2], and the idea may stimulate further research in the theory of Hermite-Hadamard's inequalities, conformable fractional integrals, and generalized convex functions.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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# Fixed Point Theorems for Generalized $\alpha_{s}-\psi$-Contractions with Applications 

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#### Abstract

We study the sufficient conditions for the existence of a unique common fixed point of generalized $\alpha_{s}-\psi$-Geraghty contractions in an $\alpha_{s}$-complete partial $b$-metric space. We give an example in support of our findings. Our work generalizes many existing results in the literature. As an application of our findings we demonstrate the existence of the solution of the system of elliptic boundary value problems.


## 1. Introduction and Preliminaries

The Banach contraction mapping principle is very important in modern mathematics. For decades, several authors have studied existence of fixed points by contraction mappings, such as fuzzy mappings and others, and also get some important results, for details we can see [1-5, 5-12]. Now the theory of fixed point has been applied in many applied mathematics [13, 14] besides integral equations and differential equations [15]. For decades, people have done a lot of research on this issue and got a lot of important results [16-20].

As is well known, the existence of the solution of boundary value problems is an important of differential equations. In this paper we study the sufficient conditions for the existence of a unique common fixed point of generalized $\alpha_{s}$ -$\psi$-Geraghty contractions in an $\alpha_{s}$-complete partial $b$-metric space. As an application of our findings we demonstrate the existence of the solution of the system of elliptic boundary value problems.

We first give some conceptions of this paper. In 1973, Geraghty studied a generalization of Banach contraction principle. In 2013, Cho introduced the notion of $\alpha$-Geraghty contractive type mappings and established some unique fixed point theorems for such mappings in complete metric spaces.

Popescu defined the concept of triangular $\alpha$-orbital admissible mappings and proved the unique fixed point theorems for the mentioned mappings which are generalized $\alpha$-Geraghty contraction type mappings. On the other hand, Karapinar proved the existence of a unique fixed point theorem for a triangular $\alpha$-admissible mapping which is a generalized $\alpha-\psi$ Geraghty contraction type mapping. Shukla [21] introduced the concept of partial $b$-metric space and established some fixed point theorems. We have Figure 1 where arrows stand for inclusions. The inverse inclusions do not hold.

In this paper, we introduce the notion of generalized $\alpha_{s}$ -$\psi$-Geraghty contraction type mappings and develop some new common fixed point theorems for such mappings in an $\alpha_{s}$-complete partial $b$-metric space. An example and an application are given to support the theory.

We denote the set of natural numbers, rational numbers, $(-\infty,+\infty),(0,+\infty)$, and $[0,+\infty)$ by $\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{R}^{+}$, and $\mathbb{R}_{0}^{+}$, respectively.

First we recall some definitions and properties of a partial $b$-metric space.

Shukla generalized the notion of $b$-metric, as follows.
Definition 1 (see [21]). Let $X$ be a nonempty set and $s \geq 1$ be a real number. A mapping $p_{b}: X \times X \longrightarrow \mathbb{R}_{0}^{+}$is said

$$
\begin{array}{rll}
\text { metric space } & \longrightarrow & \text { b-metric space } \\
\downarrow & & \downarrow \\
\text { partial metric space } & \longrightarrow & \text { partial b-metric space }
\end{array}
$$

Figure 1
to be a partial $b$-metric if it satisfies following axioms, for all $x, y, z \in X$,
$\left(p_{b} 1\right) x=y$ if and only if $p_{b}(x, y)=p_{b}(x, x)=p_{b}(y, y)$
$\left(p_{b} 2\right) p_{b}(x, x) \leq p_{b}(x, y)$
$\left(p_{b} 3\right) p_{b}(x, y)=p_{b}(y, x)$;
$\left(p_{b} 4\right) \quad p_{b}(x, y) \leq s\left[p_{b}(x, z)+p_{b}(z, y)\right]-p_{b}(z, z)$.
The triplet $\left(X, p_{b}, s\right)$ is called a partial $b$-metric space.
Remark 1. The self-distance $p_{b}(x, x)$, referred to the size or weight of $x$, is a feature used to describe the amount of information contained in $x$.

Remark 2. Obviously, every partial metric space is a partial $b$ metric space with coefficient $s=1$ and every $b$-metric space is a partial $b$-metric space with zero self-distance. However, the converse of this fact needs not to hold.

Example 3. Let $X=\mathbb{R}^{+}$and $k>1$; the mapping $p_{b}: \mathrm{X} \times$ $X \longrightarrow \mathbb{R}^{+}$defined by

$$
\begin{equation*}
p_{b}(x, y)=\left\{(x \vee y)^{k}+|x-y|^{k}\right\} \quad \text { for all } x, y \in X \tag{1}
\end{equation*}
$$

is a partial $b$-metric on $X$ with $s=2^{k}$. For $x=y, p_{b}(x, x)=$ $x^{k} \neq 0$, so $p_{b}$ is not a $b$-metric on $X$.

Let $x, y, z \in X$ such that $x>z>y$. Then following inequality always holds:

$$
\begin{equation*}
(x-y)^{k}>(x-z)^{k}+(z-y)^{k} \tag{2}
\end{equation*}
$$

Since $p_{b}(x, y)=x^{k}+(x-y)^{k}$ and $p_{b}(x, z)+p_{b}(z, y)-p_{b}(z, z)=$ $x^{k}+(x-z)^{k}+(z-y)^{k}$,

$$
\begin{equation*}
p_{b}(x, y)>p_{b}(x, z)+p_{b}(z, y)-p_{b}(z, z) . \tag{3}
\end{equation*}
$$

This shows that $p_{b}$ is not a partial metric on $X$.
Example 4 (see [21]). Let $X$ be a nonempty set and $p$ be a partial metric defined on $X$. The mapping $p_{b}: X \times X \longrightarrow \mathbb{R}^{+}$ defined by

$$
\begin{equation*}
p_{b}(x, y)=[p(x, y)]^{q} \quad \text { for all } x, y \in X \text { and } q>1 \tag{4}
\end{equation*}
$$

defines a partial $b$-metric.
Definition 5. Let $\left(X, p_{b}, s\right)$ be a partial $b$-metric space. The mapping $d_{p_{b}}: X \times X \longrightarrow \mathbb{R}_{0}^{+}$defined by

$$
d_{p_{b}}(x, y)=2 p_{b}(x, y)-p_{b}(x, x)-p_{b}(y, y)
$$

$$
\begin{equation*}
\text { for all } x, y \in X \tag{5}
\end{equation*}
$$

defines a metric on $X$, called induced metric.

In partial $b$-metric space $\left(X, p_{b}, s\right)$, we immediately have a natural definition for the open balls:

$$
\begin{equation*}
B_{p_{b}}(x ; \epsilon)=\left\{y \in X \mid p_{b}(x, y)<p_{b}(x, x)+\epsilon\right\} \tag{6}
\end{equation*}
$$

for all $x \in X$.
Remark 6. The open balls in a partial $b$-metric space ( $X, p_{b}, s$ ) may not be open set.

The following example justifies Remark 6.
Example 7. Let $X=\{a, b, c\}$ and define $p_{b}$ as follows: $p_{b}(a, a)=p_{b}(c, c)=1, p_{b}(b, b)=1 / 2, p_{b}(a, b)=p_{b}(b, a)=3$, $p_{b}(a, c)=p(c, a)=3 / 2, p_{b}(b, c)=p_{b}(c, b)=1$. Then $p$ is a partial $b$-metric, $c \in B_{p_{b}}(a ; 1)$ but, for any $r>0, B_{p_{b}}(c ; r)$ does not lie in $B_{p_{b}}(a ; 1)$. This implies that $B_{p_{b}}(a ; 1)$ is not an open set in ( $X, p_{b}, s$ ).

Definition 8. Let $\left(X, p_{b}, s\right)$ be a partial $b$-metric space.
(1) A sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ in $\left(X, p_{b}, s\right)$ is called a Cauchy sequence if $\lim _{n, m \rightarrow \infty} p_{b}\left(x_{n}, x_{m}\right)$ exists and is finite.
(2) A partial $b$-metric space $\left(X, p_{b}, s\right)$ is said to be complete if every Cauchy sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ in $X$ converges with respect to topology induced by its convergence, to a point $v \in X$ such that

$$
\begin{equation*}
p_{b}(x, x)=\lim _{n, m \longrightarrow \infty} p_{b}\left(x_{n}, x_{m}\right) . \tag{7}
\end{equation*}
$$

Lemma 9 (see [21]). Let $\left(X, p_{b}, s\right)$ be a partial b-metric space. Then
(1) every Cauchy sequence in $\left(X, d_{p_{b}}\right)$ is also a Cauchy sequence in $\left(X, p_{b}, s\right)$ and vice versa;
(2) a partial b-metric $\left(X, p_{b}, s\right)$ is complete if and only if the metric space $\left(X, d_{p_{b}}\right)$ is complete;
(3) a sequence $\left\{x_{n}\right\}_{n \in \mathbb{N}}$ in $X$ converges to a point $v \in X$ if and only if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} p_{b}\left(v, x_{n}\right)=p_{b}(v, v)=\lim _{n \rightarrow \infty} p_{b}\left(x_{n}, x_{m}\right) \tag{8}
\end{equation*}
$$

Remark 10. We know that in a metric space limit of a convergent sequence is always unique but in a partial $b$ metric space the limit of a convergent sequence may not be unique. Indeed, if $X=\mathbb{R}^{+}$, let $\sigma>0$ be any constant. Define $p_{b}: X \times X \longrightarrow \mathbb{R}^{+}$by $p_{b}(x, y)=x \vee y+\sigma$ for all $x, y \in X$, then $\left(X, p_{b}, s\right)$ is a partial $b$-metric space with arbitrary coefficient $s \geq 1$. Define the sequence $\left\{x_{n}\right\}$ in $X$ by $x_{n}=\rho$ for all $n \in \mathbb{N}$. One can note that if $y \geq \rho$ then $p_{b}\left(x_{n}, y\right)=y+\sigma=p_{b}(y, y)$; thus $\lim _{n \rightarrow \infty} p_{b}\left(x_{n}, y\right)=p_{b}(y, y)$ for all $y \geq \rho$. Hence, the limit of a convergent sequence is not unique.

Definition 11 (see [22]). Let $T: X \longrightarrow X$ and $\alpha: X \times X \longrightarrow \mathbb{R}_{0}^{+}$ be two mappings. We say that $T$ is $\alpha$-admissible if $\alpha(x, y) \geq 1$ implies $\alpha(T x, T y) \geq 1$.

Definition 12 (see [22]). Let $T: X \longrightarrow X$ and $\alpha: X \times$ $X \longrightarrow \mathbb{R}_{0}^{+}$be two mappings. Then $T$ is said to be triangular $\alpha$-admissible if $T$ satisfies the following conditions:
(T1) $T$ is $\alpha$-admissible.
(T2) $\alpha(x, u) \geq 1$ and $\alpha(u, y) \geq 1$ imply $\alpha(x, y) \geq 1$.
Definition 13 (see [23]). Let $T: X \longrightarrow X$ and $\alpha: X \times X \longrightarrow$ $\mathbb{R}_{0}^{+}$be two mappings. Then $T$ is said to be $\alpha$-orbital admissible if
(T3) $\alpha(x, T x) \geq 1$ implies $\alpha\left(T x, T^{2} x\right) \geq 1$.
Definition 14 (see [23]). Let $T: X \longrightarrow X$ and $\alpha: X \times X \longrightarrow$ $\mathbb{R}_{0}^{+}$be two mappings. Then $T$ is said to be triangular $\alpha$-orbital admissible if $T$ is $\alpha$-orbital admissible and

$$
\text { (T4) } \alpha(x, y) \geq 1 \text { and } \alpha(y, T y) \geq 1 \text { imply } \alpha(x, T y) \geq 1 .
$$

Let $\Omega$ denote the class of all mappings $\beta: \mathbb{R}_{0}^{+} \longrightarrow[0,1 / s)$ such that, for any bounded sequence $\left\{t_{n}\right\}$ of positive reals, $\beta\left(t_{n}\right) \longrightarrow 1 / s$ implies $t_{n} \longrightarrow 0$.

We let $\Psi$ denote the class of the functions $\psi:[0, \infty) \longrightarrow$ $\mathbb{R}_{0}^{+}$satisfying the following conditions:
(1) $\psi$ is nondecreasing.
(2) $\psi$ is continuous.
(3) $\psi(t)=0$ if and only if $t=0$.

## 2. Main Results

Throughout this paper we let $X=\left(X, p_{b}, s\right)$ be a partial $b$ metric space, $\alpha_{s}: X \times X \longrightarrow \mathbb{R}_{0}^{+}$be a mapping, and

$$
\begin{align*}
& M(x, y)=\max \left\{p_{b}(x, y), p_{b}(x, S x), p_{b}(y, T y),\right. \\
& \left.\frac{p_{b}(x, T y)+p_{b}(y, S x)}{2 s}\right\} . \tag{9}
\end{align*}
$$

Definition 15. The space $\left(X, p_{b}, s\right)$ is said to be $\alpha_{s}$-complete if every Cauchy sequence $\left\{x_{n}\right\}$ in $X$ satisfying $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}$ for all $n \in \mathbb{N}$ converges in $X$.

Remark 16. If $X$ is a complete partial $b$-metric space, then $X$ is also an $\alpha_{s}$-complete partial $b$-metric space but the converse is not true. The following example explains this fact.

Example 17. Let $X=(0, \infty)$ and the partial $b$-metric $p_{b}: X \times$ $X \longrightarrow[0, \infty)$ be defined by $p_{b}(x, y)=(x \vee y)^{2}$, for all $x, y \in$ $X$. Define $\alpha_{2}: X \times X \longrightarrow[0, \infty)$ by

$$
\alpha_{2}(x, y)= \begin{cases}4 e^{|x-y|} & \text { if } x, y \in[1,3]  \tag{10}\\ 0 & \text { otherwise }\end{cases}
$$

It is easy to see that $\left(X, p_{b}, 2\right)$ is not a complete partial $b$ metric space, but $\left(X, p_{b}, 2\right)$ is an $\alpha_{2}$-complete partial $b$-metric space. Indeed, if $\left\{x_{n}\right\}$ is a Cauchy sequence in $X$ such that $\alpha_{2}\left(x_{n}, x_{n+1}\right) \geq 4$, for all $n \in \mathbb{N}$, then $x_{n} \in[1,3]$, for all $n \in \mathbb{N}$. Since $[1,3]$ is a closed subset of $\mathbb{R}$, we see that $\left([1,3], p_{b}, 2\right)$ is a complete partial b-metric space and then there exists $x \in$ $[1,3]$ such that $x_{n} \longrightarrow x$ as $n \longrightarrow \infty$.

Definition 18. Let ( $X, \preceq$ ) be a partially ordered set. Two mappings $S, T: X \longrightarrow X$ are said to be weakly increasing mappings, if $S(x) \preceq T S(x)$ and $T(y) \preceq S T(y)$ hold for all $x, y \in X$.

Example 19. Let $X=\mathbb{R}^{+}$. Define $S, T: X \longrightarrow X$ by

$$
\begin{align*}
& S(x)= \begin{cases}x^{1 / 2} & \text { if } x \in[0,1] \\
x^{2} & \text { if } x \in(1, \infty)\end{cases}  \tag{11}\\
& \text { and } T(x)= \begin{cases}x & \text { if } x \in[0,1] \\
2 x & \text { if } x \in(1, \infty)\end{cases}
\end{align*}
$$

Then, $S, T$ are weakly increasing mappings.
Definition 20. The self-mappings $S, T: X \longrightarrow X$ are said to be $\alpha_{s}$-orbital admissible if the following condition holds.

$$
\alpha_{s}(x, S x) \geq s^{2} \text { and } \alpha_{s}(x, T x) \geq s^{2} \text { imply } \alpha_{s}(S x, T S x) \geq s^{2}
$$ and $\alpha_{s}(T x, S T x) \geq s^{2}$.

We note that Definitions 13 and 18 are particular cases of Definition 20 (set $S=T$ and define $\alpha_{s}(x, y) \geq s^{2}$ whenever $x \leq y$ or $y \leq x$, respectively, in Definition 20).

Definition 21. Let $S, T: X \longrightarrow X$ be two mappings. The pair ( $S, T$ ) is said to be triangular $\alpha_{s}$-orbital admissible, if
(i) the self-mappings $S, T$ are $\alpha_{s}$-orbital admissible,
(ii) $\alpha_{s}(x, y) \geq s^{2}, \alpha_{s}(y, S y) \geq s^{2}$ and $\alpha_{s}(y, T y) \geq s^{2}$ imply $\alpha_{s}(x, S y) \geq s^{2}$ and $\alpha_{s}(x, T y) \geq s^{2}$.

Example 22. Let $M=\mathbb{R}_{0}^{+}$and $p_{b}\left(r_{1}, r_{2}\right)=\left(r_{1} \vee r_{2}\right)^{2}$ for all $r_{1}, r_{2} \in M$ be a partial $b$-metric with $s=2$ :

$$
\begin{align*}
& S(r)= \begin{cases}r & \text { if } r \in[0,1) ; \\
1 & \text { if } r \in[1, \infty),\end{cases}  \tag{12}\\
& T(r)= \begin{cases}r^{1 / 3} & \text { if } r \in[0,1) ; \\
1 & \text { if } r \in[1, \infty) .\end{cases}
\end{align*}
$$

Define $\alpha_{s}: M \times M \longrightarrow \mathbb{R}_{0}^{+}$by

$$
\alpha_{s}\left(r_{1}, r_{2}\right)= \begin{cases}4+r_{2}-r_{1} & \text { if } r_{1}, r_{2} \in[0,1)  \tag{13}\\ 0 & \text { if } r_{1}, r_{2} \in[1, \infty)\end{cases}
$$

Then it is easy to show that the mappings $S, T$ satisfy conditions (i) and (ii) in Definition 21.

Lemma 23. Let $S, T: X \longrightarrow X$ be two mappings such that the pair $(S, T)$ is triangular $\alpha_{s}$-orbital admissible. Assume that there exists $x_{0} \in X$ such that $\alpha_{s}\left(x_{0}, S x_{0}\right) \geq s^{2}$. Define a sequence $\left\{x_{n}\right\}$ in $X$ by $x_{2 i+1}=S\left(x_{2 i}\right)$ and $x_{2 i+2}=T\left(x_{2 i+1}\right)$, where $i=0,1,2, \ldots$. Then for $n, m \in \mathbb{N} \cup\{0\}$ with $m>n$, we have $\alpha_{s}\left(x_{n}, x_{m}\right) \geq s^{2}$.

Proof. Since $\alpha_{s}\left(x_{0}, S x_{0}\right)=\alpha_{s}\left(x_{0}, x_{1}\right) \geq s^{2}$ and $S, T$ are $\alpha_{s}$ orbital admissible self-mappings,

$$
\begin{align*}
\alpha_{s}\left(x_{0}, S x_{0}\right) & \geq s^{2} \text { implies } \\
\alpha_{s}\left(S x_{0}, T S x_{0}\right) & =\alpha_{s}\left(x_{1}, T x_{1}\right)=\alpha_{s}\left(x_{1}, x_{2}\right) \geq s^{2} \\
\alpha_{s}\left(x_{1}, T x_{1}\right) & \geq s^{2} \text { implies }  \tag{14}\\
\alpha_{s}\left(T x_{1}, S T x_{1}\right) & =\alpha_{s}\left(x_{2}, S x_{2}\right)=\alpha_{s}\left(x_{2}, x_{3}\right) \geq s^{2} \\
\alpha_{s}\left(x_{2}, S x_{2}\right) & \geq s^{2} \text { implies } \\
\alpha_{s}\left(S x_{2}, T S x_{2}\right) & =\alpha_{s}\left(x_{3}, T x_{3}\right)=\alpha_{s}\left(x_{3}, x_{4}\right) \geq s^{2} . \tag{15}
\end{align*}
$$

Applying the above argument repeatedly, we obtain $\alpha_{s}\left(x_{n}\right.$, $\left.x_{m}\right) \geq s^{2}$ for all $n, m \in \mathbb{N} \cup\{0\}$ with $m=n+1$. Since $S, T$ are triangular $\alpha_{s}$-orbital admissible mappings, $\alpha_{s}\left(x_{n}, x_{m}\right) \geq$ $s^{2}$ for all $n, m \in \mathbb{N} \cup\{0\}$ with $m>n$.

Definition 24. We say the self-mapping $S: X \longrightarrow X$ is an $\alpha_{s}-p_{b}$-continuous mapping if whenever $\left\{x_{n}\right\}$ is a sequence in $X$ with $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}$ for all $n \in \mathbb{N}$ and $x \in X$ such that $\lim _{n \rightarrow \infty} p_{b}\left(x_{n}, x\right)=0$, then $\lim _{n \rightarrow \infty} p_{b}\left(S\left(x_{n}\right), S(x)\right)=0$.

Now, we introduce the concept of generalized $\alpha_{s}-\psi$ Geraghty contractions as follows.

Definition 25. The self-mappings $S, T$ defined on $\left(X, p_{b}, s\right)$ are called generalized $\alpha_{s}-\psi$-Geraghty contractions with respect to $p_{b}$, if there exist $\beta \in \Omega, \psi \in \Psi$, and $L \geq 0$ such that

$$
\begin{align*}
& \psi\left(\alpha_{s}(x, y) p_{b}(S x, T y)\right) \\
& \quad \leq  \tag{16}\\
& \quad \beta(\psi(M(x, y))) \cdot \psi(M(x, y)) \\
& \quad+L \psi\left(\min \left\{p_{b}(x, S x), p_{b}(y, S x)\right\}\right)
\end{align*}
$$

for $x, y \in X$ satisfying $\alpha_{s}(x, y) \geq s^{2}$.
The main result of this section is given by the following:
Theorem 26. Let $\left(X, p_{b}, s\right)$ be an $\alpha_{s}$-complete partial b-metric space. Let $S, T: X \longrightarrow X$ be generalized $\alpha_{s}-\psi$-Geraghty contractions satisfying the following conditions:
(i) there exists $x_{0} \in X$ such that $\alpha_{s}\left(x_{0}, S x_{0}\right) \geq s^{2}$;
(ii) the mappings $S, T$ are triangular $\alpha_{s}$-orbital admissible; (iii)
(a) the mappings $S$, $T$ are $\alpha_{s}-p_{b}$-continuous
(b) $\left\{x_{n}\right\}$ is a sequence in $X$ such that $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq$ $s^{2}$ for all $n \in \mathbb{N}$ and $x_{n} \longrightarrow x^{*} \in X$ as $n \longrightarrow$ $\infty$, then there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\left\{x_{n}\right\}$ such that $\alpha_{s}\left(x_{n(k)}, x^{*}\right) \geq s^{2}$ for all $k \in \mathbb{N}$.

Then $S$ and $T$ have a common fixed point in $X$. In addition, if $y^{*}$ is also a common fixed point of the pair $(S, T)$ such that $\alpha_{s}\left(x^{*}, y^{*}\right) \geq s^{2}$, then $x^{*}=y^{*}$.
Proof. Firstly we prove that the self-mappings $S, T$ have at most one common fixed point. Suppose that $v$ and $\omega$ are two different common fixed points of $S$ and $T$. Then $S(v)=v \neq$ $\omega=T(\omega)$. It follows that $p_{b}(S(v), T(\omega))=p_{b}(v, \omega)>0$, $p_{b}(v, v)=0$ and $p_{b}(\omega, \omega)=0$. Since $\alpha_{s}(v, \omega) \geq s^{2}$, contractive condition (16) implies

$$
\begin{align*}
\psi\left(p_{b}(v, \omega)\right)= & \psi\left(p_{b}(S(v), T(\omega))\right) \\
\leq & \psi\left(\alpha_{s}(v, \omega) p_{b}(S(v), T(\omega))\right) \\
\leq & \beta(\psi(M(v, \omega))) \cdot \psi(M(v, \omega))  \tag{17}\\
& +L \psi\left(\min \left\{p_{b}(v, S v), p_{b}(\omega, S v)\right\}\right) \\
< & \psi(M(v, \omega))=\psi\left(p_{b}(v, \omega)\right) .
\end{align*}
$$

which is a contradiction. Hence, the pair $(S, T)$ has at most one common fixed point.
(a). By assumption (i) and Lemma 23, we have

$$
\begin{equation*}
\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}, \quad \text { for all } n \in \mathbb{N} . \tag{18}
\end{equation*}
$$

For $i \in \mathbb{N}$, we have

$$
\begin{align*}
0< & \psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right) \\
\leq & \psi\left(\alpha_{s}\left(x_{2 i}, x_{2 i+1}\right) p_{b}\left(S x_{2 i}, T x_{2 i+1}\right)\right)  \tag{19}\\
\leq & \beta\left(\psi\left(M\left(x_{2 i}, x_{2 i+1}\right)\right)\right) \cdot \psi\left(M\left(x_{2 i}, x_{2 i+1}\right)\right) \\
& +L \psi\left(\min \left\{p_{b}\left(x_{2 i}, S x_{2 i}\right), p_{b}\left(x_{2 i+1}, S x_{2 i}\right)\right\}\right)
\end{align*}
$$

where

$$
\begin{align*}
M\left(x_{2 i}, x_{2 i+1}\right) & =\max \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i}, S x_{2 i}\right), p_{b}\left(x_{2 i+1}, T x_{2 i+1}\right), \frac{p_{b}\left(x_{2 i}, T x_{2 i+1}\right)+p_{b}\left(x_{2 i+1}, S x_{2 i}\right)}{2 s}\right\} \\
& =\max \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i+1}, x_{2 i+2}\right), \frac{p_{b}\left(x_{2 i}, x_{2 i+2}\right)+p_{b}\left(x_{2 i+1}, x_{2 i+1}\right)}{2 s}\right\}  \tag{20}\\
& \leq \max \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i+1}, x_{2 i+2}\right), \frac{p_{b}\left(x_{2 i}, x_{2 i+1}\right)+p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)}{2 s}\right\} \\
& =\max \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right\} .
\end{align*}
$$

If $\max \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right\}=p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)$, then by (29) we have

$$
\begin{align*}
& \psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right) \\
& \quad \leq \beta\left(\psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right)\right) \cdot \psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right)  \tag{21}\\
& \quad<\psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right)
\end{align*}
$$

which is a contradiction. Thus we conclude that

$$
\begin{align*}
\max & \left\{p_{b}\left(x_{2 i}, x_{2 i+1}\right), p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right\} \\
& =p_{b}\left(x_{2 i}, x_{2 i+1}\right) . \tag{22}
\end{align*}
$$

By (29), we get that $\psi\left(p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)\right)<\psi\left(p_{b}\left(x_{2 i}, x_{2 i+1}\right)\right)$. Since $\psi$ is nondecreasing, we have

$$
\begin{equation*}
p_{b}\left(x_{2 i+1}, x_{2 i+2}\right)<p_{b}\left(x_{2 i}, x_{2 i+1}\right) . \tag{23}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
p_{b}\left(x_{n+1}, x_{n+2}\right)<p_{b}\left(x_{n}, x_{n+1}\right), \quad \text { for all } n \in \mathbb{N} . \tag{24}
\end{equation*}
$$

Hence, we deduce that the sequence $\left\{p_{b}\left(x_{n}, x_{n+1}\right)\right\}$ is nonincreasing. Therefore, there exists $r \geq 0$ such that $\lim _{n \rightarrow \infty} p_{b}\left(x_{n}, x_{n+1}\right)=r$. Now, we shall prove that $r=0$. Suppose that $r>0$. By (16), we have

$$
\begin{align*}
& \psi\left(p_{b}\left(x_{n+1}, x_{n+2}\right)\right) \\
& \quad \leq \psi\left(\alpha_{s}\left(x_{n}, x_{n+1}\right) p_{b}\left(S x_{n}, T x_{n+1}\right)\right) \\
& \quad \leq \beta\left(\psi\left(M\left(x_{n}, x_{n+1}\right)\right)\right) \cdot \psi\left(M\left(x_{n}, x_{n+1}\right)\right)  \tag{25}\\
& \quad+L \psi\left(\min \left\{p_{b}\left(x_{n}, S x_{n}\right), p_{b}\left(x_{n+1}, S x_{n}\right)\right\}\right),
\end{align*}
$$

which implies

$$
\begin{align*}
& \psi\left(p_{b}\left(x_{n+1}, x_{n+2}\right)\right) \\
& \quad \leq \beta\left(\psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right)\right) \cdot \psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right) . \tag{26}
\end{align*}
$$

Hence

$$
\begin{equation*}
\frac{\psi\left(p_{b}\left(x_{n+1}, x_{n+2}\right)\right)}{\psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right)} \leq \beta\left(\psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right)\right)<1 . \tag{27}
\end{equation*}
$$

This implies that $\lim _{n \rightarrow \infty} \beta\left(\psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right)\right)=1$. Since $\beta \in$ $\Omega$, we have

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} \psi\left(p_{b}\left(x_{n}, x_{n+1}\right)\right)=0, \tag{28}
\end{equation*}
$$

which yields

$$
\begin{equation*}
r=\lim _{n \longrightarrow \infty} p_{b}\left(x_{n}, x_{n+1}\right)=0, \tag{29}
\end{equation*}
$$

a contradiction. Now, we claim that $\left\{x_{n}\right\}$ is a Cauchy sequence in $\left(X, p_{b}, s\right)$. Suppose, on thw contrary, that $\left\{x_{n}\right\}$ is not a Cauchy sequence; that is, $\lim _{n, m \rightarrow \infty} p_{b}\left(x_{n}, x_{m}\right) \neq 0$. Then there exists $\epsilon>0$ for which we can find two subsequences $\left\{x_{m_{k}}\right\}$ and $\left\{x_{n_{k}}\right\}$ of $\left\{x_{n}\right\}$ such that $n_{k}$ is the smallest index for which $n_{k}>m_{k}>k$,

$$
\begin{equation*}
p_{b}\left(x_{m_{k}}, x_{n_{k}}\right) \geq \epsilon . \tag{30}
\end{equation*}
$$

This means that

$$
\begin{equation*}
p_{b}\left(x_{m_{k}}, x_{n_{k-1}}\right)<\epsilon . \tag{31}
\end{equation*}
$$

By the triangle inequality, we have

$$
\begin{align*}
\epsilon & \leq p_{b}\left(x_{m_{k}}, x_{n_{k}}\right) \\
\leq & s\left(p_{b}\left(x_{m_{k}}, x_{n_{k-1}}\right)+p_{b}\left(x_{n_{k-1}}, x_{n_{k}}\right)\right) \\
& -p_{b}\left(x_{n_{k-1}}, x_{n_{k-1}}\right)  \tag{32}\\
\leq & s\left(p_{b}\left(x_{m_{k}}, x_{n_{k-1}}\right)+p_{b}\left(x_{n_{k-1}}, x_{n_{k}}\right)\right) .
\end{align*}
$$

Thus,

$$
\begin{equation*}
\frac{\epsilon}{s} \leq p_{b}\left(x_{m_{k}}, x_{n_{k}}\right)<\epsilon+p_{b}\left(x_{n_{k-1}}, x_{n_{k}}\right) \tag{33}
\end{equation*}
$$

for all $k \in \mathbb{N}$. In the view of (33) and (29), we have

$$
\begin{equation*}
\epsilon \leq \lim _{k \rightarrow \infty} p_{b}\left(x_{m_{k}}, x_{n_{k}}\right)<s \epsilon . \tag{34}
\end{equation*}
$$

Again by triangle inequality, we have

$$
\begin{align*}
p_{b}\left(x_{m_{k}}, x_{n_{k}}\right) \leq & s\left(p_{b}\left(x_{m_{k}}, x_{m_{k+1}}\right)+p_{b}\left(x_{m_{k+1}}, x_{n_{k}}\right)\right) \\
& -p_{b}\left(x_{m_{k+1}}, x_{m_{k+1}}\right) \\
\leq & s\left(p_{b}\left(x_{m_{k}}, x_{m_{k+1}}\right)+p_{b}\left(x_{m_{k+1}}, x_{n_{k}}\right)\right) \\
\leq & s p_{b}\left(x_{m_{k}}, x_{m_{k+1}}\right)+s^{2} p_{b}\left(x_{m_{k+1}}, x_{n_{k+1}}\right)  \tag{35}\\
& +s^{2} p_{b}\left(x_{n_{k+1}}, x_{n_{k}}\right)-p_{b}\left(x_{n_{k+1}}, x_{n_{k+1}}\right) \\
\leq & s p_{b}\left(x_{m_{k}}, x_{m_{k+1}}\right)+s^{2} p_{b}\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \\
& +s^{2} p_{b}\left(x_{n_{k+1}}, x_{n_{k}}\right)
\end{align*}
$$

and

$$
\begin{align*}
& p_{b}\left(x_{m_{k+1}}, x_{n_{k+1}}\right) \\
& \leq s\left(p_{b}\left(x_{m_{k+1}}, x_{m_{k}}\right)+p_{b}\left(x_{m_{k}}, x_{n_{k+1}}\right)\right) \\
&-p_{b}\left(x_{m_{k}}, x_{m_{k}}\right) \\
& \leq s p_{b}\left(x_{m_{k+1}}, x_{m_{k}}\right)+s p_{b}\left(x_{m_{k}}, x_{n_{k+1}}\right) \\
& \leq s p_{b}\left(x_{m_{k+1}}, x_{m_{k}}\right)+s^{2} p_{b}\left(x_{m_{k}}, x_{n_{k}}\right)  \tag{36}\\
&+s^{2} p_{b}\left(x_{n_{k}}, x_{n_{k+1}}\right)-p_{b}\left(x_{n_{k}}, x_{n_{k}}\right) \\
& \leq s p_{b}\left(x_{m_{k+1}}, x_{m_{k}}\right)+s^{2} p_{b}\left(x_{m_{k}}, x_{n_{k}}\right) \\
&+s^{2} p_{b}\left(x_{n_{k}}, x_{n_{k+1}}\right) .
\end{align*}
$$

By (29) and (34), we deduce that

$$
\begin{equation*}
\frac{\epsilon}{s^{2}} \leq \lim _{k \rightarrow+\infty} p\left(x_{m_{k+1}}, x_{n_{k+1}}\right)<s^{3} \epsilon . \tag{37}
\end{equation*}
$$

Also by application of triangle inequality, it follows that

$$
\begin{equation*}
\frac{\epsilon}{s} \leq \lim _{k \rightarrow \infty} p_{b}\left(x_{n_{k+1}}, x_{m_{k}}\right) \leq s^{2} \epsilon . \tag{38}
\end{equation*}
$$

By Lemma 23, since $\alpha_{s}\left(x_{n_{k+1}}, x_{m_{k}}\right) \geq s^{2}$, we have

$$
\begin{align*}
\frac{\varepsilon}{s} & =\max \left\{\frac{\varepsilon}{s}, \frac{s \varepsilon}{4}\right\} \leq \lim _{k \rightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right) \\
& \leq \max \left\{s^{2} \varepsilon, \frac{s^{2} \varepsilon}{4}\right\}=s^{2} \varepsilon . \tag{39}
\end{align*}
$$

Similarly, we can show that

$$
\begin{align*}
\frac{\varepsilon}{s} & =\max \left\{\frac{\varepsilon}{s}, \frac{s \varepsilon}{4}\right\} \leq \lim _{k \rightarrow \infty} \inf M\left(x_{n(k)+1}, x_{m(k)}\right) \\
& \leq \max \left\{s^{2} \varepsilon, \frac{s^{2} \varepsilon}{4}\right\}=s^{2} \varepsilon . \tag{40}
\end{align*}
$$

Thus, concluding above arguments we have

$$
\begin{align*}
& \psi\left(s^{2} \varepsilon\right) \leq \psi\left(\alpha_{s}\left(x_{n(k)+1}, x_{m(k)}\right)\right. \\
& \left.\cdot \lim _{k \rightarrow \infty} \sup p_{b}\left(x_{n(k)+2}, x_{m(k)+1}\right)\right) \\
& \quad \leq \beta\left(\psi\left(\lim _{k \rightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right)\right)\right)  \tag{41}\\
& \cdot \psi\left(\lim _{k \rightarrow \infty} \sup M\left(x_{n(k)+1}, x_{m(k)}\right)\right)+0 \\
& \quad \leq \beta\left(\psi\left(s^{2} \varepsilon\right)\right) \psi\left(s^{2} \varepsilon\right)<\psi\left(s^{2} \varepsilon\right)
\end{align*}
$$

which is a contradiction. Thus $\left\{x_{n}\right\}$ is a Cauchy sequence in ( $X, p_{b}, s$ ). Since $\left(X, p_{b}, s\right)$ is an $\alpha_{s}$-complete partial $b$-metric space, by Lemma $9(2),\left(X, d_{p_{b}}\right)$ is an $\alpha_{s}$-complete $b$-metric space. Therefore, the sequence $\left\{x_{n}\right\}$ converges to some $x^{*} \in$ $\left(X, d_{p_{b}}\right)$. By Lemma 9(3), there exists $x^{*} \in X$ such that $\lim _{n \rightarrow \infty} d_{p_{b}}\left(x_{n}, x^{*}\right)=0$ if and only if

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} p_{b}\left(x^{*}, x_{n}\right)=p_{b}\left(x^{*}, x^{*}\right)=\lim _{n, m \longrightarrow \infty} p_{b}\left(x_{n}, x_{m}\right) \tag{42}
\end{equation*}
$$

Since $\left.d_{p_{b}}(x, y)=2 p_{b}(x, y)-p_{( } x, x\right)-p_{b}(y, y)$, thus, considering (29) and axiom ( $p_{b} 2$ ) with $\lim _{n \rightarrow \infty} d_{p_{b}}\left(x_{n}, x^{*}\right)=0$, we conclude that

$$
\begin{equation*}
\lim _{n \longrightarrow \infty} p_{b}\left(x_{n}, x_{m}\right)=0 \tag{43}
\end{equation*}
$$

Combining (42) and (43), we have

$$
\begin{align*}
\lim _{n \rightarrow \infty} p_{b}\left(x^{*}, x_{n}\right) & =p_{b}\left(x^{*}, x^{*}\right)=\lim _{n, m \longrightarrow \infty} p_{b}\left(x_{n}, x_{m}\right)  \tag{44}\\
& =0
\end{align*}
$$

Now $\lim _{n \rightarrow \infty} p_{b}\left(x^{*}, x_{n}\right)=0$ implies that $\lim _{i \rightarrow \infty} p_{b}\left(x_{2 i+1}\right.$, $\left.x^{*}\right)=0$ and $\lim _{i \rightarrow \infty} p_{b}\left(x_{2 i+2}, x^{*}\right)=0$. As $S$ and $T$ are $\alpha_{s}-p_{b}{ }^{-}$ continuous mappings, we $\lim _{i \rightarrow \infty} p_{b}\left(S x_{2 i+1}, S x^{*}\right)=0$. Thus

$$
\begin{align*}
p_{b}\left(x^{*}, S x^{*}\right) & =\lim _{i \rightarrow \infty} p_{b}\left(x_{2 i+2}, S x^{*}\right) \\
& =\lim _{i \rightarrow \infty} p_{b}\left(S x_{2 i+1}, S x^{*}\right)=0 \tag{45}
\end{align*}
$$

and so $x^{*}=S x^{*}$, and, similarly, $x^{*}=T x^{*}$. Therefore $S$ and $T$ have a common fixed point $x^{*} \in X$.
(b). From (a) we know that the sequence $\left\{x_{n}\right\}$ in $X$ defined by $x_{2 i+1}=S x_{2 i}$ and $x_{2 i+2}=T x_{2 i+1}$, where $i=0,1,2, \ldots$ with $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}$, for all $n \in \mathbb{N}$ converges to $x^{*} \in X$. There exists a subsequence $\left\{x_{n(k)}\right\}$ of $\left\{x_{n}\right\}$ such that $\alpha_{s}\left(x_{n(k)}, x^{*}\right) \geq s^{2}$ for all $k$. Therefore,

$$
\begin{align*}
& \psi\left(p_{b}\left(x_{2 n(k)+1}, T x^{*}\right)\right)=\psi\left(p_{b}\left(S x_{2 n(k)}, T x^{*}\right)\right) \\
& \leq \psi\left(\alpha_{s}\left(x_{2 n(k)}, x^{*}\right) p_{b}\left(S x_{2 n(k)}, T x^{*}\right)\right) \\
& \leq \beta\left(\psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)\right) \cdot \psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)  \tag{46}\\
& \quad+L \psi\left(\min \left\{p_{b}\left(x_{2 n(k)}, S x_{2 n(k)}\right), p_{b}\left(x^{*}, S x_{2 n(k)}\right)\right\}\right),
\end{align*}
$$

which implies

$$
\begin{align*}
& \psi\left(p_{b}\left(x_{2 n(k)+1}, T x^{*}\right)\right) \\
& \leq  \tag{47}\\
& \quad \beta\left(\psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)\right) \cdot \psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right) \\
& \quad+L \psi\left(\min \left\{p_{b}\left(x_{2 n(k)}, S x_{2 n(k)}\right), p_{b}\left(x^{*}, S x_{2 n(k)}\right)\right\}\right)
\end{align*}
$$

where

$$
\begin{align*}
& M\left(x_{2 n(k)}, x^{*}\right)=\max \left\{p_{b}\left(x_{2 n(k)}, x^{*}\right),\right. \\
& \quad p_{b}\left(x_{2 n(k)}, S x_{2 n(k)}\right), p_{b}\left(x^{*}, T x^{*}\right) \\
& \left.\quad \frac{p_{b}\left(x_{2 n(k)}, S x^{*}\right)+p_{b}\left(x^{*}, T x_{2 n(k)}\right)}{2 s}\right\}  \tag{48}\\
& \quad \leq \max \left\{p_{b}\left(x_{2 n(k)}, x^{*}\right), p_{b}\left(x_{2 n(k)}, x_{2 n(k)+1}\right),\right. \\
& \left.\quad p_{b}\left(x^{*}, T x^{*}\right), \frac{p_{b}\left(x_{2 n(k)}, T x^{*}\right)+p_{b}\left(x^{*}, S x_{2 n(k)}\right)}{2 s}\right\} .
\end{align*}
$$

Since

$$
\begin{align*}
& \lim _{k \rightarrow \infty} \sup \frac{p_{b}\left(x_{2 n(k)}, T x^{*}\right)+p_{b}\left(x^{*}, S x_{2 n(k)}\right)}{2 s} \\
& \quad \leq \frac{p_{b}\left(x^{*}, T x^{*}\right)+p_{b}\left(x^{*}, x^{*}\right)}{2 s}, \tag{49}
\end{align*}
$$

then by letting $k \longrightarrow \infty$ we have $\lim _{k \rightarrow \infty} M\left(x_{2 n(k)}, x^{*}\right)=$ $p_{b}\left(x^{*}, T x^{*}\right)$. Suppose that $p_{b}\left(x^{*}, T x^{*}\right)>0$. By (47), we have

$$
\begin{equation*}
\frac{\psi\left(p_{b}\left(x_{2 n(k)+1}, T x^{*}\right)\right)}{\psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)} \leq \beta\left(\psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)\right)<1 . \tag{50}
\end{equation*}
$$

Letting $k \longrightarrow \infty$ in above inequality, we obtain that

$$
\begin{equation*}
\lim _{k \rightarrow \infty} \beta\left(\psi\left(M\left(x_{2 n(k)}, x^{*}\right)\right)\right)=1 . \tag{51}
\end{equation*}
$$

So $\lim _{k \rightarrow \infty} M\left(x_{2 n(k)}, x^{*}\right)=0$. Hence $p_{b}\left(x^{*}, T x^{*}\right)=0$, and due to $\left(p_{b} 1\right)$ and $\left(p_{b} 2\right)$ we obtain so $x^{*}=T x^{*}$. Similarly we can show that $x^{*}=S x^{*}$. Thus $S$ and $T$ have a common fixed point $x^{*} \in X$.

Remark 27. We note that Theorem 26 is more general than the results established in [24-26].

Example 28. Let $X=[0,1]$. Define a function $p_{b}: X \times X \longrightarrow$ $[0,+\infty)$ by $p_{b}(x, y)=(x \vee y)^{2}+(x-y)^{2}$. Clearly, $\left(X, p_{b}, s\right)$ is a complete partial $b$-metric space with the constant $s=4$. Let $\beta$ be a function on $[0, \infty]$ defined by $\beta(t)=1 /(4+t)$ for all $t \geq 0$. Then $\beta \in \Omega$. Also, $\psi$ be a function on $[0,+\infty)$ defined by $\psi(t)=t / 2$. Then $\psi \in \Psi$. Define the mappings $S, T: X \longrightarrow X$ by

$$
T(x)= \begin{cases}\frac{2}{245} x, & \text { if } x \in\left[0, \frac{1}{2}\right]  \tag{52}\\ 1, & \text { if } x \in\left(\frac{1}{2}, 1\right]\end{cases}
$$

and $S(x)=0$
$\forall x \in X$.
Also, we define the function $\alpha_{s}: X \times X \longrightarrow[0, \infty)$ by

$$
\alpha_{s}(x, y)= \begin{cases}s^{2}, & \text { if } 0 \leq x, y \leq \frac{1}{2}  \tag{53}\\ 0, & \text { otherwise }\end{cases}
$$

If $\left\{x_{n}\right\}$ is a Cauchy sequence such that $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}$ for all $n \in \mathbb{N}$, then $\left\{x_{n}\right\} \subseteq[0,1 / 2]$. Since $\left([0,1 / 2], p_{b}\right)$ is a complete partial $b$-metric space, then the sequence $\left\{x_{n}\right\}$ converges in $[0,1 / 2] \subseteq X$. Thus $\left(X, p_{b}\right)$ is an $\alpha_{s}$-complete partial $b$-metric space. Let $\alpha_{s}(x, S x) \geq s^{2}$ and $\alpha_{s}(x, T x) \geq s^{2}$, and thus $x \in[0,1 / 2]$ and $S x, T x \in[0,1 / 2]$ and so $\alpha_{s}(S x, T S x) \geq s^{2}$ and $\alpha_{s}(T x, S T x) \geq s^{2}$. Thus, $(S, T)$ is $\alpha_{s}$-orbital admissible. Let $x, y \in X$ be such that $\alpha_{s}(x, y) \geq^{2}, \alpha_{s}(y, S y) \geq 1$ and $\alpha_{s}(y, T y) \geq s^{2}$. Then we have $x, y, S y, T y \in[0,1 / 2]$, which implies that $\alpha_{s}(x, S y) \geq s^{2}$ and $\alpha_{s}(x, T y) \geq s^{2}$. Therefore $(S, T)$ is triangular $\alpha_{s}$-orbital admissible. Let $\left\{x_{n}\right\}$ be a Cauchy sequence such that $x_{n} \longrightarrow x$ as $n \longrightarrow \infty$ and $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq$ $s^{2}$ for all $n \in \mathbb{N}$. Then $\left\{x_{n}\right\} \subseteq[0,1 / 2]$ for all $n \in \mathbb{N}$. So $\lim _{n \rightarrow \infty} T x_{n}=\lim _{n \rightarrow \infty}(2 / 245) x_{n}=(2 / 245) x=T x$. Hence $T$ is $\alpha_{s}$-continuous. Similarly, we can show that $S$ is $\alpha_{s}$-continuous. Let $x_{0}=1 / 4$. Then

$$
\begin{equation*}
\alpha_{s}\left(\frac{1}{4}, S\left(\frac{1}{4}\right)\right)=\alpha_{s}\left(\frac{1}{4}, 0\right) \geq s^{2} . \tag{54}
\end{equation*}
$$

Let $x, y \in X$ such that $\alpha_{s}(x, y) \geq s^{2}$. Then $x, y \in[0,1 / 2]$ and hence

$$
\begin{align*}
\frac{16}{245} \leq & \psi\left(\alpha_{s}(x, y) p_{b}(S x, T y)\right) \\
\leq & \beta(\psi(M(x, y))) \cdot \psi(M(x, y))  \tag{55}\\
& +L \psi\left(\min \left\{p_{b}(x, S x), p_{b}(y, S x)\right\}\right) \leq \frac{1}{9}
\end{align*}
$$

with $L \geq 0$. Thus all conditions of Theorem 26 are satisfied. Hence $S$ and $T$ have a common fixed point $(x=0)$.

## 3. Consequences

Corollary 29. Let $\left(X, p_{b}, s\right)$ be an $\alpha_{s}$-complete partialb-metric space. Assume that
(i) there exist $\beta \in \Omega$ and $L \geq 0$ such that, for all $x, y \in$ $X$ with $\alpha_{s}(x, y) \geq s^{2}$ the self-mappings $S, T$ satisfy the following inequality:

$$
\begin{align*}
& \alpha_{s}(x, y) p_{b}(S x, T y) \\
& \leq  \tag{56}\\
& \quad \beta((M(x, y))) \cdot(M(x, y)) \\
& \quad+L\left(\min \left\{p_{b}(x, S x), p_{b}(y, S x)\right\}\right)
\end{align*}
$$

(ii) $S, T$ are triangular $\alpha_{s}$-orbital admissible mappings;
(iii) there exists $x_{0} \in X$ such that $\alpha_{s}\left(x_{0}, S x_{0}\right) \geq s^{2}$
(iv)
(a) $S$ and $T$ are $\alpha_{s}$-continuous mappings;
(b) $\left\{x_{n}\right\}$ is a sequence in $X$ with $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq s^{2}$ for all $n \in \mathbb{N}$ such that $x_{n} \longrightarrow x^{*} \in X$ as $n \longrightarrow$ $\infty$, then there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\left\{x_{n}\right\}$ such that $\alpha_{s}\left(x_{n(k)}, x^{*}\right) \geq s^{2}$ for all $k \in \mathbb{N}$.

Then $S$ and $T$ have a common fixed point $x^{*} \in X$. In addition, if $y^{*}$ is also a common fixed point of the pair $(S, T)$ such that $\alpha_{s}\left(x^{*}, y^{*}\right) \geq s^{2}$, then $x^{*}=y^{*}$.

Proof. Define $\psi: \mathbb{R}_{0}^{+} \longrightarrow \mathbb{R}_{0}^{+}$by $\psi(t)=t$ for all $t \in \mathbb{R}_{0}^{+}$.
Corollary 30. Let $Y=(Y, d, s)$ be an $\alpha_{s}$-complete $b$ metric space. Let $S, T: Y \longrightarrow Y$ be a generalized $\alpha_{s}-\psi$ Geraghty contractions with respect to $d$ satisfying the following conditions:
(i) There exists $y_{0} \in Y$ such that $\alpha_{s}\left(y_{0}, S y_{0}\right) \geq s^{2}$.
(ii) The mappings $S, T$ are triangular $\alpha_{s}$-orbital admissible.
(iii)
(a) The mappings $S, T$ are $\alpha_{s}$ - $d$-continuous.
(b) $\left\{y_{n}\right\}$ is a sequence in $Y$ such that $\alpha_{s}\left(y_{n}, y_{n+1}\right) \geq s^{2}$ for all $n \in \mathbb{N}$ and $y_{n} \longrightarrow y^{*} \in Y$ as $n \longrightarrow \infty$, then there exists a subsequence $\left\{y_{n(k)}\right\}$ of $\left\{y_{n}\right\}$ such that $\alpha_{s}\left(y_{n(k)}, y^{*}\right) \geq s^{2}$ for all $k \in \mathbb{N}$.

Then $S$ and $T$ have a common fixed point in $Y$. In addition, if $y^{*}$ is also a common fixed point of the pair $(S, T)$ such that $\alpha_{s}\left(x^{*}, y^{*}\right) \geq s^{2}$, then $x^{*}=y^{*}$.

Proof. Set $p_{b}(x, x)=0$ for all $x \in X$ in Theorem 26.
Corollary 31. Let $(X, \leq)$ be a partially ordered set and ( $X, \preceq$ $\left., p_{b}, s\right)$ be an ordered complete partial b-metric space. Assume that the weakly increasing mappings $S, T: X \longrightarrow X$ satisfy the following conditions:
(i) there exist $\beta \in \Omega \psi \in \Psi$ and $L \geq 0$ such that

$$
\begin{align*}
& \psi\left(s^{2} p_{\mathrm{b}}(S x, T y)\right) \\
& \quad \leq  \tag{57}\\
& \quad \beta(\psi(M(x, y))) \cdot \psi(M(x, y)) \\
& \quad+L \psi\left(\min \left\{p_{b}(x, S x), p_{b}(y, S x)\right\}\right)
\end{align*}
$$

for all comparable $x, y \in X$ (i.e. $x \leq y$ or $y \leq x$ );
(ii) there exists $x_{0} \in X$ such that $x_{0} \leq S x_{0}$
(iii)
(a) either $S$ or $T$ is continuous;
(b) $\left\{x_{n}\right\}$ is a nondecreasing sequence such that $x_{n} \longrightarrow$ $x^{*} \in X$ as $n \longrightarrow \infty$, then there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\left\{x_{n}\right\}$ such that $x_{n(k)} \leq x^{*}$ for all $k \in \mathbb{N}$.

Then $S$ and $T$ have a common fixed point $x^{*} \in X$. In addition, if $y^{*}$ is also a common fixed point of the pair $(S, T)$ such that $x^{*} \preceq y^{*}$, then $x^{*}=y^{*}$.

Proof. Define the relation $\leq$ on $X$ by

$$
\alpha_{s}(x, y)= \begin{cases}s^{2}, & \text { if } x \leq y \text { or } y \leq x  \tag{58}\\ 0, & \text { otherwise }\end{cases}
$$

Proof follows from the proof of Theorem 26.
Definition 32. The self-mappings $T$ defined on $\left(X, p_{b}, s\right)$ is called a generalized $\alpha_{s}-\psi$-Geraghty contraction if there exist $\beta \in \Omega, \psi \in \Psi$, and $L \geq 0$ such that

$$
\begin{align*}
\psi\left(\alpha_{s}\right. & \left.(x, y) p_{b}(T x, T y)\right) \\
\leq & \beta(\psi(C(x, y))) \cdot \psi(C(x, y))  \tag{59}\\
& +L \psi\left(\min \left\{p_{b}(x, T x), p_{b}(y, T x)\right\}\right)
\end{align*}
$$

for $x, y \in X$ satisfying $\alpha_{s}(x, y) \geq s^{2}$, where

$$
\begin{align*}
& C(x, y)=\max \left\{p_{b}(x, y), p_{b}(x, T x), p_{b}(y, T y)\right. \\
& \left.\quad \frac{p_{b}(x, T y)+p_{b}(y, T x)}{2 s}\right\} . \tag{60}
\end{align*}
$$

Corollary 33. Let $\left(X, p_{b}, s\right)$ be an $\alpha_{s}$-complete partialb-metric space. Let $T: X \longrightarrow X$ be a generalized $\alpha_{s}-\psi$-Geraghty contraction satisfying the following conditions:
(i) there exists $x_{0} \in X$ such that $\alpha_{s}\left(x_{0}, T x_{0}\right) \geq s^{2}$;
(ii) the mapping $T$ is triangular $\alpha_{s}$-orbital admissible; (iii)
(a) the mapping $T$ is $\alpha_{s}-p_{b}$-continuous;
(b) $\left\{x_{n}\right\}$ is a sequence in $X$ such that $\alpha_{s}\left(x_{n}, x_{n+1}\right) \geq$ $s^{2}$ for all $n \in \mathbb{N}$ and $x_{n} \longrightarrow x^{*} \in X$ as $n \longrightarrow$ $\infty$, then there exists a subsequence $\left\{x_{n(k)}\right\}$ of $\left\{x_{n}\right\}$ such that $\alpha_{s}\left(x_{n(k)}, x^{*}\right) \geq s^{2}$ for all $k \in \mathbb{N}$.

Then $T$ has a fixed point in $X$. In addition, if $y^{*}$ is also a common fixed point of the pair $(S, T)$ such that $\alpha_{s}\left(x^{*}, y^{*}\right) \geq s^{2}$, then $x^{*}=y^{*}$.

Proof. Set $S=T$ in Theorem 26.

We extend Definition 25 for all $x, y \in X$ as follows
Definition 34. The self-mappings $S, T$ defined on $\left(X, p_{b}, s\right)$ are called generalized $\psi$-Geraghty contractions, if there exist $\beta \in$ $\Omega, \psi \in \Psi$, and $L \geq 0$ such that

$$
\begin{align*}
& \psi\left(p_{b}(S x, T y)\right) \\
& \quad \leq  \tag{61}\\
& \quad \beta(\psi(M(x, y))) \cdot \psi(M(x, y)) \\
& \quad+L \psi\left(\min \left\{p_{b}(x, S x), p_{b}(y, S x)\right\}\right)
\end{align*}
$$

for $x, y \in X$.
Theorem 35. Let $S, T: X \longrightarrow X$ be two $p_{b}$-continuous generalized $\psi$-Geraghty contractions defined on a complete partial b-metric space $\left(X, p_{b}, s\right)$; then $S$ and $T$ have a common fixed point.

Proof. The arguments follow as the same lines in proof of Theorem 26.

## 4. Application

In this section, we present an application on existence of a solution of a pair of elliptic boundary value problems. Let $C(I)$ be the space of all continuous function defined on $I=$ $[0,1]$. Consider the following pair of differential equations:

$$
\begin{align*}
-\frac{d^{2} x}{d t^{2}} & =f(t, x(t)), \quad t \in[0,1] \\
x(0) & =x(1)=0 \\
\text { and }-\frac{d^{2} y}{d t^{2}} & =K(t, y(t)), \quad t \in[0,1]  \tag{62}\\
y(0) & =y(1)=0
\end{align*}
$$

where $f, K: I \times C(I) \longrightarrow \mathbb{R}$ are continuous functions. The Green function associated with (62) is defined by

$$
G(t, s)= \begin{cases}t(1-\tau), & 0 \leq t \leq \tau \leq 1  \tag{63}\\ \tau(1-t), & 0 \leq \tau \leq t \leq 1\end{cases}
$$

Define the function $p_{b}: C(I) \times C(I) \longrightarrow[0, \infty)$ by

$$
\begin{aligned}
& p_{b}(x, y)=\sup _{t \in I}|x(t)-y(t)|^{2}+k \\
& \text { for all } x, y \in C(I) \text { and } k>0 .
\end{aligned}
$$

It is known that $\left(C(I), p_{b}\right)$ is a complete partial $b$-metric space with constant $s=4$. Now, define the operators $S, T: C(I) \longrightarrow$ $C(I)$ defined by

$$
\begin{align*}
S x(t) & =\int_{0}^{1} G(t, \tau) f(\tau, x(\tau)) d \tau \\
\text { and } T x(t) & =\int_{0}^{1} G(t, \tau) K(\tau, y(\tau)) d \tau \tag{65}
\end{align*}
$$

for all $t \in I$. Remark that (62) has a solution if and only if operators $S$ and $T$ have a common fixed point.

Theorem 36. Assume that there exist continuous functions $f, K: I \times C(I) \longrightarrow \mathbb{R}$ such that, for all $x, y \in C(I)$ and $\rho \in \mathbb{R}$, we have

$$
\begin{array}{r}
|f(t, x)-K(t, y)|^{2} \leq 64 \ln \left(\frac{M(x(t), y(t))+1}{\rho}\right)  \tag{66}\\
\text { for all } t \in I
\end{array}
$$

where $M(x(t), y(t))$ is defined by (9) such that $M(x(t), y(t))>$ $\rho>0$

Proof. It is well known that $x^{*} \in C^{2}(I)$ is a solution of (62) if and only if $x^{*} \in C(I)$ is a common solution of the integral equations given by (65). Define the mappings $S, T: C(I) \longrightarrow$ $C(I)$ by (65). Hence the solution of (62) is equivalent to find a common fixed point $x^{*} \in C(I)$ of $T$ and $S$. Let $x, y \in C(I)$. By (i), we get

$$
\begin{align*}
& \mid S x(t)-\left.T y(t)\right|^{2} \\
&=\left[\left|\int_{0}^{1} G(t, \tau)[f(\tau, x(\tau))-K(\tau, y(\tau))] d \tau\right|\right]^{2} \\
& \leq\left[\int_{0}^{1} G(t, \tau)|f(\tau, x(\tau))-K(\tau, y(\tau))| d \tau\right]^{2} \\
& \leq\left[8 \int_{0}^{1} G(t, \tau) \sqrt{\ln \left(\frac{M(x(\tau), y(\tau))+1}{\rho}\right)} d \tau\right]^{2}  \tag{67}\\
& \leq\left[8 \int_{0}^{1} G(t, \tau) \sqrt{\ln \left(\frac{M(x(\tau), y(\tau))+1}{\rho}\right)} d \tau\right]^{2} \\
& \quad=8^{2} \ln \left(\frac{M(x(\tau), y(\tau))+1}{\rho}\right) \\
& \cdot\left(\sup _{t \in I}\left[\int_{0}^{1} G(t, \tau) d \tau\right]^{2}\right) .
\end{align*}
$$

Since $\int_{0}^{1} G(t, \tau) d \tau=-t^{2} / 2+t / 2$ for all $t \in I$, then we have $\left(\sup _{t \in I}\left[\int_{0}^{1} G(t, \tau) d \tau\right]^{2}\right)=1 / 8^{2}$, which implies that

$$
\begin{equation*}
|S x(t)-T y(t)|^{2} \leq \ln \left(\frac{M(x(\tau), y(\tau))+1}{\rho}\right) . \tag{68}
\end{equation*}
$$

It can easily be proved that $\mathscr{M}(x, y)=\sup _{\tau \in I} \mathscr{M}(x(\tau), y(\tau))$. Thus,

$$
\begin{align*}
& \ln \left(p_{b}(S x, T y)+1\right) \leq \ln (\ln (M(x, y)+1)+1) \\
& \quad=\frac{\ln (\ln (M(x, y)+1)+1)}{\ln (M(x, y)+1)} \ln (M(x, y)+1) \tag{69}
\end{align*}
$$

Define the functions $\psi:[0, \infty) \longrightarrow[0, \infty)$ and $\beta:[0, \infty)$ $\longrightarrow[0,1 / s)$ by

$$
\begin{align*}
\psi(x) & =\ln (x+1)  \tag{70}\\
\text { and } \beta(x) & = \begin{cases}\frac{\psi(x)}{x}, & \text { if } x \geq 10 \\
0, & \text { otherwise. }\end{cases} \tag{71}
\end{align*}
$$

Note that $\psi:[0, \infty) \longrightarrow[0, \infty)$ is continuous, nondecreasing, positive in $(0, \infty), \psi(0)=0$, and $\psi(x)<x$.

Hence $\beta \in \Omega$ and

$$
\begin{equation*}
\psi\left(p_{b}(S x, T y)\right) \leq \beta(\psi(M(x, y))) \cdot \psi(M(x, y)) \tag{72}
\end{equation*}
$$

for all $x, y \in C(I)$. Therefore all assumptions of Theorem 35 are satisfied with $L=0$. Hence $S$ and $T$ have a common fixed point $x^{*} \in C(I)$; that is, $S x^{*}=T x^{*}=x^{*}$ which is a solution of (62).

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# A New Sufficient Condition for Checking the Robust Stabilization of Uncertain Descriptor Fractional-Order Systems 

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#### Abstract

We consider the robust asymptotical stabilization of uncertain a class of descriptor fractional-order systems. In the state matrix, we require that the parameter uncertainties are time-invariant and norm-bounded. We derive a sufficient condition for the system with the fractional-order $\alpha$ satisfying $1 \leq \alpha<2$ in terms of linear matrix inequalities (LMIs). The condition of the proposed stability criterion for fractional-order system is easy to be verified. An illustrative example is given to show that our result is effective.


## 1. Introduction

Descriptor systems arise naturally in many applications such as aerospace engineering, social economic systems, and network analysis. Sometimes we also call descriptor systems singular systems. Descriptor system theory is an important part in control systems theory. Since 1970s, descriptor systems have been wildly studied, for example, descriptor linear systems [1], descriptor nonlinear systems [2-4], and discrete descriptor systems [5-7]. In particular, Dai has systematically introduced the theoretical basis of descriptor systems in [8], which is the first monograph on this subject. A detailed discussion of descriptor systems and their applications can be found in $[9,10]$.

It is well known that fractional-order systems have been studied extensively in the last 20 years, since the fractional calculus has been found many applications in viscoelastic systems [11-14], robotics [15-18], finance system [19-21], and many others [22-26]. Studying on fractional-order calculus has become an active research field. To the best of our knowledge, although stability analysis is a basic problem in control theory, very few works existed for the stability analysis for descriptor fractional-order systems.

Many problems related to stability of descriptor fractional-order control systems are still challenging and unsolved. For the nominal stabilization case, N'Doye et al. [27] study the stabilization of one descriptor fractional-order system with the fractional-order $\alpha, 1<\alpha<2$, in terms of LMIs. N'Doye et al. [28] derive some sufficient conditions for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order $\alpha$ satisfying $0<\alpha<2$. Furthermore, Ma et al. [29] study the robust stability and stabilization of fractional-order linear systems with positive real uncertainty. Note that, in Example 1, by applying Theorem 2 [27], it is harder to determine whether the uncertain descriptor fractionalorder system (6) is asymptotically stable. Therefore, it is valuable to seek sufficient conditions, for checking the robust asymptotical stabilization of uncertain descriptor fractional-order systems.

In this paper, we study the stabilization of a class of descriptor fractional-order systems with the fractional-order $\alpha, 1 \leq \alpha<2$, in terms of LMIs. We derive a new sufficient condition for checking the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order $\alpha$ satisfying $1 \leq \alpha<2$, in terms of

LMIs. It should be mentioned that, compared with some prior works, our main contributions consist in the following: (1) we assume that the matrix of uncertain parameters in the uncertain descriptor fractional-order system is diagonal. Thus, compared with the results in [28], our conclusion, Theorem 8, is more feasible and effective and has wider applications; (2) compared with some stability criteria of fractional-order nonlinear systems, for example, in [9, 22], our method is easier to be used.

Notations: throughout this paper, $\mathbb{R}^{m \times n}$ stands for the set of $m$ by $n$ matrices with real entries, $M^{T}$ stands for the transpose of $M, \operatorname{Sym}\{X\}$ denotes the expression $X^{T}+X$, $I_{n}$ denotes the identity matrix of order $n, \operatorname{diag}\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ denotes the diagonal matrix, and $\bullet$ will be used in some matrix expressions to indicate a symmetric structure; i.e., if given matrices $H_{1}=H_{1}^{T} \in \mathbb{R}^{m \times m}$ and $H_{2}=H_{2}^{T} \in \mathbb{R}^{n \times n}$, then

$$
\left(\begin{array}{cc}
H_{1} & \bullet  \tag{1}\\
L & H_{2}
\end{array}\right)=\left(\begin{array}{cc}
H_{1} & L^{T} \\
L & H_{2}
\end{array}\right) .
$$

## 2. Preliminary Results

Consider the following class of linear fractional-order systems:

$$
\begin{align*}
{ }_{0}^{C} D_{t}^{\alpha} x(t) & =A x(t),  \tag{2}\\
x(0) & =x_{0},
\end{align*}
$$

where $0<\alpha<2$ is the fractional-order, $x(t) \in \mathbb{R}^{n}$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is a constant matrix, and ${ }_{0}^{C} D_{t}^{\alpha}$ represent the fractional-order derivative, which can be expressed as

$$
\begin{equation*}
{ }_{0}^{C} D_{t}^{\alpha} f(t)=\frac{1}{\Gamma(n-\alpha)} \int_{0}^{t}(t-\tau)^{n-\alpha-1} f^{(n)}(\tau) d \tau \tag{3}
\end{equation*}
$$

where $\Gamma(\cdot)$ is the Euler Gamma function. For convenience, we use $D^{\alpha}$ to replace ${ }_{0}^{C} D_{t}^{\alpha}$ in the rest of this paper. It is well known that system (2) is stable if [30-32]

$$
\begin{equation*}
|\arg (\operatorname{spec}(A))|>\alpha \frac{\pi}{2} \tag{4}
\end{equation*}
$$

where $0<\alpha<2$ and $\operatorname{spec}(A)$ is the spectrum of all eigenvalues of $A$.

The next lemma, given by Chilali et al. [33], contains the necessary and sufficient conditions of (4) in terms of LMI, when the fractional-order $\alpha$ belongs to $1 \leq \alpha<2$.

Lemma 1 (see [33]). Let $A \in \mathbb{R}^{n \times n}$ be a real matrix and $1 \leq$ $\alpha<2$. Then $|\arg (\operatorname{spec}(A))|>(\pi / 2) \alpha$ if and only if there exists $P>0$ such that

$$
\left(\begin{array}{lc}
\left(A P+P A^{T}\right) \sin \theta & \bullet  \tag{5}\\
\left(P A^{T}-A P\right) \cos \theta & \left(A P+P A^{T}\right) \sin \theta
\end{array}\right)<0 .
$$

Consider the following uncertain descriptor fractionalorder systems:

$$
\begin{align*}
E D^{\alpha} x(t) & =\left(A+\Delta_{A}\right) x(t)+B u(t) \\
x(0) & =x_{0} \tag{6}
\end{align*}
$$

where $1 \leq \alpha<2, x(t) \in \mathbb{R}^{n}$ is the semistate vector, $u(t) \in \mathbb{R}^{m}$ is the control input, $E \in \mathbb{R}^{n \times n}$ is singular, $A \in \mathbb{R}^{n \times n}$ and $B \in$ $\mathbb{R}^{n \times m}$ are constant matrices, and the time-invariant matrix $\Delta_{A}$ corresponds to a norm-bounded parameter uncertainty, which is the following form:

$$
\begin{equation*}
\Delta_{A}=M_{A} \Delta N_{A} \tag{7}
\end{equation*}
$$

where $M_{A}$ and $N_{A}$ are real constant matrices of appropriate sizes, and the uncertain matrix $\Delta=\left(\gamma_{i j}\right)_{p \times q}$ satisfies

$$
\begin{equation*}
\Delta \Delta^{T} \leq I_{p} \tag{8}
\end{equation*}
$$

Remark 2. Condition $\Delta \Delta^{T} \leq I_{p}$ is rational because a lot of system uncertainties satisfy this inequality. Besides, this condition can also be used in many literatures, for example, in [9, 34-39].

It is well known that the following system

$$
\begin{align*}
E D^{\alpha} x(t) & =A x(t)+B u(t)  \tag{9}\\
x(0) & =x_{0}
\end{align*}
$$

is normalizable if and only if

$$
\operatorname{rank}\left[\begin{array}{ll}
E & B \tag{10}
\end{array}\right]=n
$$

Further we have that the uncertain descriptor fractionalorder systems (6) is normalizable if and only if the nominal descriptor fractional-order system (9) is normalizable.

Lemma 3 (see [28], Theorem 1). System (6) is normalizable if and only if there exist a nonsingular matrix $P$ and a matrix $Y$ such that the following LMI

$$
\begin{equation*}
E P+B Y+P^{T} E^{T}+Y^{T} B^{T}<0 \tag{11}
\end{equation*}
$$

is satisfied. In this case, the gain matrix $L$ is given by

$$
\begin{equation*}
L=Y P^{-1} . \tag{12}
\end{equation*}
$$

Assume that (6) is normalizable; by applying LMI (11), we obtain $L \in \mathbb{R}^{m \times n}$ such that $\operatorname{rank}(E+B L)=n$. Consider the feedback control for (6) in the following form:

$$
\begin{equation*}
u(t)=-L D^{\alpha} x(t)+K x(t) \tag{13}
\end{equation*}
$$

where $K \in \mathbb{R}^{m \times n}$ is one gain matrix such that the obtained normalized system is asymptotically stable. Then we have the closed-loop system:

$$
\begin{equation*}
(E+B L) D^{\alpha} x(t)=\left(A+\Delta_{A}+B K\right) x(t), \tag{14}
\end{equation*}
$$

that is,

$$
\begin{equation*}
D^{\alpha} x(t)=\left(A_{1}+B_{1} K+E_{1} \Delta_{A}\right) x(t) \tag{15}
\end{equation*}
$$

where

$$
\begin{align*}
& E_{1}=(E+B L)^{-1}, \\
& A_{1}=E_{1} A,  \tag{16}\\
& B_{1}=E_{1} B .
\end{align*}
$$

To facilitate the description of our main results, we need the following results.

In [28], N'Doye et al. derive a sufficient condition for the robust asymptotical stabilization of uncertain descriptor fractional-order systems with the fractional-order $\alpha$ satisfying $1 \leq \alpha<2$ in terms of LMIs.

Lemma 4 (see [28], Theorem 2). Assume that (6) is normalizable; then there exists gain matrix $K$ such that the uncertain descriptor fractional-order system (6) with fractional-order $1 \leq$ $\alpha<2$ controlled by the control (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m \times n}, P_{0}=P_{0}^{T}>0 \in \mathbb{R}^{n \times n}$ and a real scalar $\delta>0$, such that

$$
\left[\begin{array}{cccc}
\Omega_{11} & \bullet & \bullet & \bullet  \tag{17}\\
\Omega_{21} & \Omega_{22} & \bullet & \bullet \\
N_{A} P_{0} & 0 & -\delta I & \bullet \\
0 & N_{A} P_{0} & 0 & -\delta I
\end{array}\right]<0
$$

where

$$
\begin{align*}
\Omega_{11}= & \Omega_{22} \\
= & \left(P_{0} A_{1}^{T}+A_{1} P_{0}+B_{1} X+X^{T} B_{1}^{T}\right) \sin \theta \\
& +\delta E_{1} M_{A}\left(E_{1} M_{A}\right)^{T}  \tag{18}\\
\Omega_{21}= & \left(P_{0} A_{1}^{T}-A_{1} P_{0}+X^{T} B_{1}^{T}-B_{1} X\right) \cos \theta,
\end{align*}
$$

with $\theta=\pi-\alpha(\pi / 2)$ and matrices $P$ and $Y$ are given by LMI (11).

Moreover, the gain matrix $K$ is given by

$$
\begin{equation*}
K=X P_{0}^{-1} \tag{19}
\end{equation*}
$$

Lemma 5 (see [40]). For any matrices $X$ and $Y$ with appropriate sizes, we have

$$
\begin{equation*}
X^{T} Y+Y^{T} X \leq \epsilon X^{T} X+\epsilon^{-1} Y^{T} Y \tag{20}
\end{equation*}
$$

for any $\epsilon>0$.
Lemma 6 (see [41]). Let $X, Y$, and $Z$ be real matrices of appropriate sizes. Then, for any $x \in \mathbb{R}^{n}$,

$$
\begin{gather*}
\max \left\{\left(x^{T} X F Y x\right)^{2}: F^{T} F \leq I\right\}  \tag{21}\\
=\left(x^{T} X X^{T} x\right)\left(x^{T} Y^{T} Y x\right)
\end{gather*}
$$

## 3. Main Result

In this section, we present a new sufficient condition to design the gain matrix $K$. In the following theorem, $\Delta_{M}$ and $\Delta_{N}$ are given nonsingular matrices, such that

$$
\begin{equation*}
\Delta_{M}^{-1} \Delta \Delta_{N}^{-1}\left(\Delta_{M}^{-1} \Delta \Delta_{N}^{-1}\right)^{T} \leq I_{p} \tag{22}
\end{equation*}
$$

From now on, we denote $\widehat{\Delta}=\Delta_{M}^{-1} \Delta \Delta_{N}^{-1}, \widehat{\mathbf{M}}=E_{1} M_{A} \Delta_{M}$, and $\widehat{\mathbf{N}}=\Delta_{N} N_{A} P$. It is obvious that $\widehat{\Delta} \widehat{\Delta}^{T} \leq I_{p}$. Thus, for any
$\epsilon_{1}>0$ and $\epsilon_{2}>0$, by using Lemmas 5 and 6 and $\widehat{\Delta} \widehat{\Delta}^{T} \leq I_{p}$, we have

$$
\begin{align*}
& {\left[\begin{array}{cc}
\left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta & 0 \\
0 & \left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta
\end{array}\right] } \\
&= {\left[\begin{array}{ccc}
\widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta) & 0 \\
0 & \widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta)
\end{array}\right]\left[\begin{array}{ll}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right] } \\
&+\left[\begin{array}{ll}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right]^{T}\left[\begin{array}{cc}
\widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta) & 0 \\
0 & \widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta)
\end{array}\right]^{T}  \tag{23}\\
&\left.+\epsilon_{1}\left[\begin{array}{cc}
\widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta) & 0 \\
0 & \widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta)
\end{array}\right]\left[\begin{array}{cc}
\widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta) & 0 \\
0 & \widehat{\mathbf{M}}(\widehat{\Delta} \sin \theta)
\end{array}\right]^{\widehat{\mathbf{N}}} \begin{array}{l}
0 \\
0
\end{array}\right]^{T}\left[\begin{array}{ll}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right] \\
& \leq \epsilon_{1}\left[\begin{array}{ll}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]^{T}+\frac{1}{\epsilon_{1}}\left[\begin{array}{ll}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right]^{T}\left[\begin{array}{ll}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right]
\end{align*}
$$

and

$$
\begin{align*}
& {\left[\begin{array}{cc}
0 & \left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}-\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \cos \theta \\
\left(\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}-\widehat{\mathbf{M}} \widehat{\mathbf{N}}\right) \cos \theta & 0
\end{array}\right]} \\
& \leq \epsilon_{2}\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]^{T}+\frac{1}{\epsilon_{2}}\left[\begin{array}{cc}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right]^{T}\left[\begin{array}{cc}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right] \tag{24}
\end{align*}
$$

that is,

$$
\begin{align*}
& {\left[\begin{array}{l}
\left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta \\
\left(\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}-\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}-\widehat{\mathbf{N}}^{T} \widehat{\mathbf{N}}\right) \cos \theta\left(\widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \cos \theta \\
\quad \leq\left(\epsilon_{1}+\epsilon_{2}\right)\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \left.\widehat{\mathbf{M}}+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta
\end{array}\right] \\
\quad+\left(\frac{1}{\epsilon_{1}}+\frac{1}{\epsilon_{2}}\right)\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]^{T} \\
0
\end{array} \widehat{\widehat{\mathbf{N}}}\right]^{T}\left[\begin{array}{cc}
\widehat{\mathbf{N}} & 0 \\
0 & \widehat{\mathbf{N}}
\end{array}\right]}
\end{align*}
$$

Remark 7. Note that, when $\delta=2$, we have $\epsilon_{1}+\epsilon_{2} \leq 2$ and

$$
\begin{equation*}
\frac{1}{\epsilon_{1}}+\frac{1}{\epsilon_{2}} \geq 2>\frac{1}{\delta}=\frac{1}{2} \tag{26}
\end{equation*}
$$

That is, for any real scalar $\delta>0$, and two matrices $X \in \mathbb{R}^{m \times n_{1}}$ and $Y \in \mathbb{R}^{n_{2} \times m}$, we cannot obtain real scalars $\epsilon_{1}>0$ and $\epsilon_{2}>0$ such that

$$
\begin{equation*}
\left(\epsilon_{1}+\epsilon_{2}\right) X X^{T}+\left(\frac{1}{\epsilon_{1}}+\frac{1}{\epsilon_{2}}\right) Y^{T} Y \leq \delta X X^{T}+\frac{1}{\delta} Y^{T} Y \tag{27}
\end{equation*}
$$

where

$$
\begin{align*}
& X=\left[\begin{array}{cc}
E_{1} M_{A} \Delta_{M} & 0 \\
0 & E_{1} M_{A} \Delta_{M}
\end{array}\right],  \tag{28}\\
& Y=\left[\begin{array}{cc}
\Delta_{N} N_{A} P & 0 \\
0 & \Delta_{N} N_{A} P
\end{array}\right] .
\end{align*}
$$

Theorem 8. Assume that (6) is normalizable; then there exists a gain matrix $K$ such that the uncertain descriptor fractionalorder system (6) with fractional-order $1 \leq \alpha<2$ controlled by the controller (13) is asymptotically stable, if there exist matrices $X \in \mathbb{R}^{m \times n}, P=P^{T}>0 \in \mathbb{R}^{n \times n}$ and two real scalars $\epsilon_{1}>0$ and $\epsilon_{2}>0$, such that

$$
\left[\begin{array}{cccc}
\widehat{\Omega_{11}} & \bullet & \bullet & \bullet  \tag{29}\\
\widehat{\Omega_{21}} & \widehat{\Omega_{22}} & \bullet & \bullet \\
\Delta_{N} N_{A} P & 0 & -\epsilon_{1} I & \bullet \\
0 & \Delta_{N} N_{A} P & 0 & -\epsilon_{1} I
\end{array}\right]<0
$$

where

$$
\begin{align*}
\widehat{\Omega_{11}}= & \widehat{\Omega_{22}} \\
= & \left(P A_{1}^{T}+A_{1} P+B_{1} X+X^{T} B_{1}^{T}\right) \sin \theta \\
& +\left(\epsilon_{1}+\epsilon_{2}\right) E_{1} M_{A} \Delta_{M}\left(E_{1} M_{A} \Delta_{M}\right)^{T},  \tag{30}\\
\widehat{\Omega_{21}}= & \left(P A_{1}^{T}-A_{1} P+X^{T} B_{1}^{T}-B_{1} X\right) \cos \theta
\end{align*}
$$

$$
\left.\left.\left.\begin{array}{l}
{\left[\left(\left(A_{1} P+B_{1} X\right)+\left(P A_{1}^{T}+X^{T} B_{1}^{T}\right)\right) \sin \theta\left(\left(A_{1} P+B_{1} X\right)-\left(P A_{1}^{T}+X^{T} B_{1}^{T}\right)\right) \cos \theta\right.}  \tag{33}\\
\left(\left(P A_{1}^{T}+X^{T} B_{1}^{T}\right)-\left(A_{1} P+B_{1} X\right)\right) \cos \theta\left(\left(A_{1} P+B_{1} X\right)+\left(P A_{1}^{T}+X^{T} B_{1}^{T}\right)\right) \sin \theta
\end{array}\right]+\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]\left[\begin{array}{cc}
\widehat{\mathbf{M}} & 0 \\
0 & \widehat{\mathbf{M}}
\end{array}\right]^{T}\right)
$$

Write $K=X P^{-1}$. It follows from applying (25) that

$$
\begin{align*}
& {\left[\begin{array}{cc}
\left(\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right) P+P\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right)^{T}\right) \sin \theta & \left(\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right) P-P\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right)^{T}\right) \cos \theta \\
\left(-\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right) P+P\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right)^{T}\right) \cos \theta & \left(\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right) P+P\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right)^{T}\right) \sin \theta
\end{array}\right]} \\
& \quad=\left[\begin{array}{lc}
\left(\left(A_{1} P+B_{1} K P\right)+\left(P A_{1}^{T}+P K^{T} B_{1}^{T}\right)\right) \sin \theta & \left(\left(A_{1} P+B_{1} K P\right)-\left(P A_{1}^{T}+P K^{T} B_{1}^{T}\right)\right) \cos \theta \\
\left(\left(P A_{1}^{T}+P K^{T} B_{1}^{T}\right)-\left(A_{1} P+B_{1} K P\right)\right) \cos \theta & \left(\left(A_{1} P+B_{1} K P\right)+\left(P A_{1}^{T}+P K^{T} B_{1}^{T}\right)\right) \sin \theta
\end{array}\right]  \tag{34}\\
& +\left[\begin{array}{ccc}
\left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta & 0 & 0
\end{array}\right]+\left[\begin{array}{cc}
\left(\widehat{\mathbf{M}} \widehat{\Delta} \widehat{\mathbf{N}}-\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \cos \theta \\
0 & \left(\widehat{\mathbf{M}} \widehat{\Delta} \Delta_{N} N_{A} P+\widehat{\mathbf{N}}^{T} \widehat{\Delta}^{T} \widehat{\mathbf{M}}^{T}\right) \sin \theta
\end{array}\right]+0 .
\end{align*}
$$

By using the above inequality (34) and Lemma 1, we obtain

$$
\begin{equation*}
\left|\arg \left(\operatorname{spec}\left(A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A}\right)\right)\right|>\frac{\pi}{2} \alpha . \tag{35}
\end{equation*}
$$

Therefore, system (6) is asymptotically stable. This ends the proof.

Remark 9. Write

$$
\begin{align*}
T= & {\left[\begin{array}{cccc}
\widehat{\Omega_{11}} & \bullet & \bullet & \bullet \\
\widehat{\Omega_{21}} & \widehat{\Omega_{22}} & \bullet & \bullet \\
\Delta_{N} N_{A} P & 0 & -\epsilon_{1} I & \bullet \\
0 & \Delta_{N} N_{A} P & 0 & -\epsilon_{1} I
\end{array}\right] }  \tag{36}\\
& -\left[\begin{array}{cccc}
\Omega_{11} & \bullet & \bullet & \bullet \\
\Omega_{21} & \Omega_{22} & \bullet & \bullet \\
N_{A} P_{0} & 0 & -\delta I & \bullet \\
0 & N_{A} P_{0} & 0 & -\delta I
\end{array}\right]
\end{align*}
$$

Note that if we choose $\Delta_{M}=I_{p}$ and $\Delta_{N}=I_{q}$ in LMI (29),

$$
T=\left[\begin{array}{cccc}
\left(\epsilon_{1}+\epsilon_{2}-\delta\right) E_{1} M_{A}\left(E_{1} M_{A}\right)^{T} & \bullet & \bullet & \bullet  \tag{37}\\
0 & \left(\epsilon_{1}+\epsilon_{2}-\delta\right) E_{1} M_{A}\left(E_{1} M_{A}\right)^{T} & \bullet & \bullet \\
0 & 0 & -\left(\epsilon_{1}-\delta\right) I & \bullet \\
0 & 0 & 0 & -\left(\epsilon_{1}-\delta\right) I
\end{array}\right]
$$

It is easy to see the following:
(1) For given $\delta$, when $\epsilon_{1}-\delta>0$, it is always true that $\epsilon_{1}+\epsilon_{2}-\delta>0$; that is, there do not exist $\epsilon_{1}$ and $\epsilon_{2}$ such that $T<0$. Therefore, Theorem 8 is not a special case of Lemma 4 [28, Theorem 2], when $\Delta_{M}=I_{p}$ and $\Delta_{N}=I_{q}$.
(2) For given $\epsilon_{1}$ and $\epsilon_{2}$, when $\epsilon_{1}-\delta<0$, there exists $\epsilon_{2}$ such that $T$ is positive definite; that is, there exists $\delta$ such that $T>0$. Since conditions in Lemma 4 and Theorem 8 are both sufficient, we cannot derive Lemma 4 by applying Theorem 8 ; that is, Theorem 8 is not a generalization of Lemma 4 [28, Theorem 2].

## 4. A Numerical Example

In this section, we assume that the matrix of uncertain parameters $\Delta$ in the uncertain descriptor fractional-order system (6) is diagonal. We provide a numerical example to illustrate that Theorem 8 is feasible and effective with wider applications.

Example 1. Consider the uncertain descriptor fractionalorder system described in (6) with parameters as follows:

$$
\begin{align*}
E & =\left[\begin{array}{lll}
1 & 0 & 0.5 \\
2 & 1 & 1 \\
0 & 0 & 0
\end{array}\right], \\
A & =\left[\begin{array}{ccc}
2.4 & 0.2 & 1.2 \\
4 & 1.5 & 2 \\
0 & 0 & 0
\end{array}\right], \\
B & =\left[\begin{array}{ll}
4 & 1 \\
1 & 1 \\
1 & 2
\end{array}\right]  \tag{38}\\
M_{A} & =\left[\begin{array}{ccc}
0.1 & 0 & 0 \\
0.1 & 0.3 & 4.8 \\
0 & 0.2 & 0
\end{array}\right], \\
N_{A} & =\left[\begin{array}{ccc}
0 & 0 & 0 \\
0.2 & 0 & \frac{1}{30} \\
0 & 0.1 & 0
\end{array}\right],
\end{align*}
$$

where $\alpha=1.23$.

It is easy to check that $\operatorname{rank}(E)=2<3$; that is, $E$ is singular. By applying the LMI (11), we obtain

$$
\begin{align*}
& P=10^{8} \times\left[\begin{array}{ccc}
0.2613 & -1.5587 & 1.0635 \\
-1.5587 & -1.2010 & -0.7272 \\
1.0635 & -0.7272 & 0.4664
\end{array}\right]  \tag{39}\\
& Y=10^{8} \times\left[\begin{array}{ccc}
-1.4992 & 0.7485 & 0.3345 \\
0.6535 & -0.2534 & -2.4425
\end{array}\right]
\end{align*}
$$

and the gain matrix $L$

$$
L=Y P^{-1}=\left[\begin{array}{ccc}
1.0224 & -0.5004 & -2.3944  \tag{40}\\
-2.8016 & 1.6203 & 3.6775
\end{array}\right]
$$

It follows from (16) that

$$
\begin{align*}
& E_{1}=\left[\begin{array}{ccc}
-0.1025 & 0.3106 & -0.2545 \\
0.2781 & 0.3221 & 0.1544 \\
-0.2483 & 0.1088 & -0.1188
\end{array}\right], \\
& A_{1}=\left[\begin{array}{ccc}
0.9963 & 0.4454 & 0.4981 \\
1.9559 & 0.5388 & 0.9780 \\
-0.1605 & 0.1136 & -0.0802
\end{array}\right],  \tag{41}\\
& B_{1}=\left[\begin{array}{cc}
-0.3541 & -0.3010 \\
1.5889 & 0.9091 \\
-1.0030 & -0.3770
\end{array}\right]
\end{align*}
$$

Firstly, we compute $P_{0}, X, \delta$, and $K$ by using Lemma 4 [28, Theorem 2]. A feasible solution of LMI (11) is as follows:

$$
\begin{align*}
P_{0} & =10^{-7} \times\left[\begin{array}{ccc}
0.0560 & -0.0060 & -0.1056 \\
-0.0060 & 0.0489 & -0.0407 \\
-0.1056 & -0.0407 & 0.2552
\end{array}\right], \\
X & =\left[\begin{array}{ccc}
-0.0022 & -0.0010 & 0.0056 \\
0.0043 & 0.0001 & -0.0087
\end{array}\right],  \tag{42}\\
\delta & =10^{-15} \times 1.0293 \\
K & =X P_{0}^{-1}=10^{8} \times\left[\begin{array}{ccc}
-2.0350 & -1.0959 & -1.0150 \\
1.7432 & 0.9354 & 0.8674
\end{array}\right]
\end{align*}
$$

We choose

$$
\Delta=\left[\begin{array}{ccc}
\cos (0.8) & 0 & 0  \tag{43}\\
0 & e^{-0.8} & 0 \\
0 & 0 & \sin (0.1)
\end{array}\right]
$$

It follows from (15) that

$$
\begin{align*}
S & =A_{1}+B_{1} K+E_{1} M_{A} \Delta N_{A} \\
& =10^{8} \times\left[\begin{array}{ccc}
0.1959 & 0.1065 & 0.0983 \\
-1.6487 & -0.8909 & -0.8242 \\
1.3839 & 0.7465 & 0.6910
\end{array}\right] \tag{44}
\end{align*}
$$

and the arguments of all eigenvalues of $S$ are

$$
\begin{align*}
& 3.1416 \\
& 3.1416 \tag{45}
\end{align*}
$$

## 0.

Based on those results, it is debatable whether or not system (6) is stable.

In the second way, we compute $P_{0}, X, \epsilon_{1}, \epsilon_{2}$, and $K$ by using Theorem 8; we choose

$$
\begin{align*}
& \Delta_{M}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{12}
\end{array}\right],  \tag{46}\\
& \Delta_{N}=\left[\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 12
\end{array}\right] .
\end{align*}
$$

It is easy to check that

$$
\begin{aligned}
\Delta_{M}^{-1} \Delta \Delta_{N}^{-1} & =\left[\begin{array}{ccc}
\cos (0.8) & 0 & 0 \\
0 & e^{-0.8} & 0 \\
0 & 0 & \sin (0.1)
\end{array}\right] \\
M_{A} \Delta_{M} & =\left[\begin{array}{ccc}
0.1 & 0 & 0 \\
0.1 & 0.3 & 0.4 \\
0 & 0.2 & 0
\end{array}\right] \\
\Delta_{N} N_{A} & =\left[\begin{array}{ccc}
0 & 0 & 0 \\
0.2 & 0 & 0.4 \\
0 & 0.1 & 0
\end{array}\right]
\end{aligned}
$$

It follows that a feasible solution of LMI (11) is

$$
\begin{align*}
& P_{0}=\left[\begin{array}{ccc}
3074 & -5431 & -1331 \\
-5431 & 11221 & 885 \\
-1331 & 885 & 1912
\end{array}\right], \\
& X=\left[\begin{array}{ccc}
-6379 & 7189 & 4104 \\
12249 & -13814 & -6269
\end{array}\right],  \tag{48}\\
& \epsilon_{1}=1055.1 \\
& \epsilon_{2}=173.0328
\end{align*}
$$

asymptotically stabilizing state-feedback gain is

$$
K=X P_{0}^{-1}=\left[\begin{array}{ccc}
-324.0313 & -143.8217 & -156.8345  \tag{49}\\
840.6966 & 373.4044 & 409.0759
\end{array}\right]
$$

$$
\widehat{S}=A_{1}+B_{1} K+E_{1} \Delta_{A}
$$

$$
=\left[\begin{array}{ccc}
-137.3101 & -61.0208 & -67.0911  \tag{50}\\
251.3914 & 111.4837 & 123.6975 \\
7.9010 & 3.5938 & 3.0047
\end{array}\right]
$$

and the arguments of all eigenvalues of $\widehat{S}$ are
3.1416,
3.1416,
3.1416 .

Therefore, system (6) is stable.

## 5. Conclusion

In this paper, the robust asymptotical stability of uncertain descriptor fractional-order systems (6) with the fractionalorder $\alpha$ belonging to $1 \leq \alpha<2$ has been studied. We derive a new sufficient condition for checking the robust asymptotical stabilization of (6) in terms of LMIs. Out results can be seen as a generalization of [28, Theorem 2]. By adding appropriate parameters into LMIs, our result has wider applications. One special numerical example has shown that our results are feasible and easy to be used.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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## Research Article

# Positive Solutions for a System of Semipositone Fractional Difference Boundary Value Problems 

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Using the fixed point index, we establish two existence theorems for positive solutions to a system of semipositone fractional difference boundary value problems. We adopt nonnegative concave functions and nonnegative matrices to characterize the coupling behavior of our nonlinear terms.

## 1. Introduction

In this paper we study the existence of positive solutions for the system of fractional difference boundary value problems involving semipositone nonlinearities:

$$
\begin{align*}
-\Delta_{v-3}^{v} x(t)= & f(t+v-1, x(t+v-1), \\
& y(t+v-1)), \\
-\Delta_{v-3}^{v} y(t)= & g(t+v+2]_{\mathbb{N}_{0}}, \\
x(v-3)= & \left.t \in[0, b+2]_{\mathbb{N}_{0}}^{\alpha} x(t)\right]\left.\right|_{t=v-\alpha-2} \\
= & {\left.\left[\Delta_{\gamma-3}^{\beta} x(t)\right]\right|_{t=v+b+2-\beta}=0, }  \tag{1}\\
y(\nu-3)= & {\left.\left[\Delta_{\gamma-3}^{\alpha} y(t)\right]\right|_{t=v-\alpha-2} } \\
= & {\left.\left[\Delta_{\nu-3}^{\beta} y(t)\right]\right|_{t=v+b+2-\beta}=0, }
\end{align*}
$$

where $2<\nu \leq 3,1<\beta<2, \nu-\beta>1,0<\alpha<1, b>$ $3(b \in \mathbb{N})$, and $\Delta_{\nu-3}^{v}$ is a discrete fractional operator. For the nonlinear terms $f, g$, we assume the following.
(H0) $f, g:[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+} \longrightarrow \mathbb{R}$ are two continuous functions; moreover, there exists a positive constant $M>0$ such that

$$
\begin{align*}
& f, g(t, x, y) \geq-M \\
& \quad \text { for all }(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+} . \tag{2}
\end{align*}
$$

Note that, in this paper, we use $[a, b]_{\mathbb{N}_{a}}$ to stand for $\{a, a+$ $1, a+2, \ldots, b\}$ with $b-a \in \mathbb{N}_{1}$, where $\mathbb{N}_{a}:=\{a, a+1, a+2, \cdots\}$.

Fractional calculus has been applied in physics, chemistry, aerodynamics, biophysics, and blood flow phenomena. For example, $\mathrm{CD} 4^{+} \mathrm{T}$ cells' infections can be depicted by a fractional order model

$$
\begin{align*}
D^{\alpha_{1}}(T) & =s-K V T-d T+b I, \\
D^{\alpha_{2}}(I) & =K V T-(b+\delta) I,  \tag{3}\\
D^{\alpha_{3}}(V) & =N \delta I-c V,
\end{align*}
$$

where $D^{\alpha_{i}}(i=1,2,3)$ are fractional derivatives (see [1,2]); we also refer the reader to [1-45] and the references therein. In [3], the authors considered the existence of positive solutions for the semipositone discrete fractional system

$$
\begin{aligned}
& -\Delta^{\nu_{1}} y_{1}(t) \\
& =\lambda_{1} f_{1}\left(t+v_{1}-1, y_{1}\left(t+v_{1}-1\right), y_{2}\left(t+v_{2}-1\right)\right), \\
& \\
& t \in[1, b+1]_{\mathbb{N}}
\end{aligned}
$$

$$
\begin{align*}
& -\Delta^{v_{2}} y_{2}(t) \\
& \quad=\lambda_{2} f_{2}\left(t+v_{2}-1, y_{1}\left(t+v_{1}-1\right), y_{2}\left(t+v_{2}-1\right)\right) \\
& \quad t \in[1, b+1]_{\mathbb{N}}, \\
& y_{1}\left(v_{1}-2\right)=y_{1}\left(v_{1}+b+1\right)=0, \\
& y_{2}\left(v_{2}-2\right)=y_{2}\left(v_{2}+b+1\right)=0, \tag{6}
\end{align*}
$$

where $\nu_{1}, \nu_{2} \in(1,2]$. Using the Guo-Krasnosel'skiĭ fixed point theorem, the authors showed that the problem has positive solutions for sufficiently small values of $\lambda_{1}, \lambda_{2}>$ 0 . The growth conditions on $f_{i}(i=1,2)$ are superlinear; i.e.,

$$
\begin{align*}
\lim _{y_{1}+y_{2} \longrightarrow+\infty} \frac{f_{i}(t, x, y)}{y_{1}+y_{2}} & =+\infty \\
\lim _{y_{1}+y_{2} \rightarrow 0^{+}} \frac{f_{i}(t, x, y)}{y_{1}+y_{2}} & =0 \tag{5}
\end{align*}
$$

uniformly for $t \in\left[v_{i}, v_{i}+b\right]_{\mathbb{N}_{v_{i}-2}}$. Using conditions of (5) type the existence of solutions for various fractional boundary value problems was considered in [1, 4-6, 9, 11-13].

In this paper, we use the fixed point index to obtain two existence theorems for positive solutions to (1) with semipositone nonlinearities. We adopt some appropriate nonnegative concave functions and nonnegative matrices to characterize the coupling behavior of our nonlinear terms. Moreover, the growth conditions on $+\infty$ of our nonlinearities $f, g$ are an improvement of (5); see conditions (H1) and (H3) in Section 3.

## 2. Preliminaries

We first recall some background materials from discrete fractional calculus; for more details we refer the reader to [10].

Definition 1. We define $t^{\underline{\nu}}:=\Gamma(t+1) / \Gamma(t+1-v)$ for any $t, v \in \mathbb{R}$ for which the right-hand side is well-defined. We use the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}}=0$.

Definition 2. For $v>0$, the $v$-th fractional sum of a function $f$ is

$$
\begin{equation*}
\Delta_{a}^{-v} f(t)=\frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-v}(t-s-1)^{\nu-1} f(s), \quad \text { for } t \in \mathbb{N}_{a+v} \tag{4}
\end{equation*}
$$

We also define the $v$-th fractional difference for $v>0$ by

$$
\begin{equation*}
\Delta_{a}^{v} f(t)=\Delta^{N} \Delta_{a}^{v-N} f(t), \quad \text { for } t \in \mathbb{N}_{a+N-v} \tag{7}
\end{equation*}
$$

where $N \in \mathbb{N}$ with $0 \leq N-1<\nu \leq N$.
Let $h:[\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \longrightarrow \mathbb{R}$ be a continuous function. Then we consider the fractional difference boundary value problems

$$
\begin{align*}
-\Delta_{v-3}^{v} y(t) & =h(t+v-1), \quad t \in[0, b+2]_{\mathbb{N}_{0}} \\
y(v-3) & =\left.\left[\Delta_{v-3}^{\alpha} y(t)\right]\right|_{t=v-\alpha-2}  \tag{8}\\
& =\left.\left[\Delta_{v-3}^{\beta} y(t)\right]\right|_{t=v+b+2-\beta}=0
\end{align*}
$$

where $v, \alpha, \beta, b$ are as in (1). The following two lemmas are in [9], so we omit their proofs.

Lemma 3 (see [9], Lemma 4). Problem (8) has a unique solution

$$
\begin{equation*}
y(t)=\sum_{s=0}^{b+2} G(t, s) h(s+v-1) \tag{9}
\end{equation*}
$$

$$
t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}},
$$

where

$$
G(t, s)=\frac{1}{\Gamma(\nu)} \begin{cases}\frac{t^{\nu-1}(\nu+b-\beta-s+1)^{\nu-\beta-1}}{(\nu+b-\beta+2)^{\nu-\beta-1}}-(t-s-1)^{\frac{\nu-1}{}}, & 0 \leq s<t-v+1 \leq b+2  \tag{10}\\ \frac{t^{\nu-1}(\nu+b-\beta-s+1)^{\nu-\beta-1}}{(\nu+b-\beta+2)^{\nu-\beta-1}}, & 0 \leq t-v+1 \leq s \leq b+2\end{cases}
$$

Lemma 4 (see [9], Lemma 5). Green's function (10) has the following properties.
(i) $G(t, s)>0,(t, s) \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}} \times[0, b+2]_{\mathbb{N}_{0}}$,
(ii) $q^{*}(t) G(b+\nu+1, s) \leq G(t, s) \leq G(b+v+1, s),(t, s) \in$ $[\nu-1, b+v+1]_{\mathbb{N}_{\nu-1}} \times[0, b+2]_{\mathbb{N}_{0}}$, where $^{*}(t)=t^{\nu-1} /(b+\nu+1)^{\nu-1}$.

Let $\varphi(s+\nu-1)=G(b+\nu+1, s)$ for $s \in[0, b+2]_{\mathbb{N}_{0}}$. Then $\varphi(t)=G(b+v+1, t-v+1)$ for $t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}$. From Lemma ??, the following inequalities are satisfied:

$$
\begin{align*}
& \sum_{t=v-1}^{b+v+1} q^{*}(t) \varphi(t) \cdot \varphi(s+v-1) \leq \sum_{t=v-1}^{b+v+1} G(t, s) \varphi(t)  \tag{11}\\
& \quad \leq \sum_{t=v-1}^{b+v+1} \varphi(t) \cdot \varphi(s+v-1), \quad \text { for } s \in[0, b+2]_{\mathbb{N}_{0}}
\end{align*}
$$

For convenience, we let

$$
\begin{align*}
& \kappa_{1}=\sum_{t=\nu-1}^{b+\nu+1} q^{*}(t) \varphi(t) \text { and } \\
& \kappa_{2}=\sum_{t=\gamma-1}^{b+\nu+1} \varphi(t) \tag{12}
\end{align*}
$$

Let $E$ be the collection of all maps from $[v-3, b+v+1]_{\mathbb{N}_{v-3}}$ to $\mathbb{R}$ equipped with the max norm, $\|\cdot\|$. Then $E$ is a Banach space. Define a set $P \subset E$ by $P=\{y \in E: y(t) \geq 0, t \in$ $\left.[v-1, b+v+1]_{\mathbb{N}_{\nu-1}}\right\}$. Then $P$ is a cone in $E$. Note that $E \times E$ is a Banach space with the norm $\|(x, y)\|:=\max \{\|x\|,\|y\|\}$, and $P \times P$ is a cone in $E \times E$.

From Lemma 3, for all $t \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}}$, we have that (1) is equivalent to

$$
\begin{align*}
& x(t)=\sum_{s=0}^{b+2} G(t, s) \\
& \cdot f(s+v-1, x(s+v-1), y(s+v-1)) \\
& y(t)=\sum_{s=0}^{b+2} G(t, s)  \tag{13}\\
& \quad \cdot g(s+v-1, x(s+v-1), y(s+v-1))
\end{align*}
$$

where $G$ is defined in (10).
Lemma 5 (see [46]). Let $E$ be a real Banach space and $P a$ cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set and that $A: \bar{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists $\omega_{0} \in P \backslash\{0\}$ such that

$$
\begin{equation*}
\omega-A \omega \neq \lambda \omega_{0}, \quad \forall \lambda \geq 0, \omega \in \partial \Omega \cap P \tag{14}
\end{equation*}
$$

then $i(A, \Omega \cap P, P)=0$, where $i$ denotes the fixed point index on $P$.

Lemma 6 (see [46]). Let E be a real Banach space and P a cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A: \bar{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If

$$
\begin{equation*}
\omega-\lambda A \omega \neq 0, \quad \forall \lambda \in[0,1], \omega \in \partial \Omega \cap P \tag{15}
\end{equation*}
$$

then $i(A, \Omega \cap P, P)=1$.

## 3. Main Results

Let $\omega$ be a solution of

$$
\begin{align*}
-\Delta_{\nu-3}^{v} y(t) & =1, \quad t \in[0, b+2]_{\mathbb{N}_{0}}, \\
y(\nu-3) & =\left.\left[\Delta_{\nu-3}^{\alpha} y(t)\right]\right|_{t=v-\alpha-2}  \tag{16}\\
& =\left.\left[\Delta_{\nu-3}^{\beta} y(t)\right]\right|_{t=v+b+2-\beta}=0
\end{align*}
$$

where $v, \alpha, \beta, b$ are as in (1). Define $z=M \omega$, and then, from Lemmas 3 and ??, we have

$$
\begin{align*}
z(t) & =M \omega(t)=M \sum_{s=0}^{b+2} G(t, s) \leq M \sum_{s=0}^{b+2} \varphi(s+\nu-1) \\
& =M \sum_{s=v-1}^{b+v+1} \varphi(s)=M \kappa_{2} \tag{17}
\end{align*}
$$

We note that (1) has a positive solution $(x, y) \in(P \times P) \backslash\{\mathbf{0}\}$ if and only if $(\tilde{x}, \tilde{y})=(x+z, y+z)$ is a solution of the fractional difference boundary value problems

$$
\begin{align*}
& -\Delta_{\nu-3}^{v} x(t)=\tilde{f}(t+v-1, x(t+v-1) \\
& -z(t+v-1), y(t+v-1)-z(t+v-1)), \\
& t \in[0, b+2]_{\mathbb{N}_{0}}, \\
& -\Delta_{v-3}^{v} y(t)=\widetilde{g}(t+v-1, x(t+v-1) \\
& -z(t+v-1), y(t+v-1)-z(t+v-1)), \\
& t \in[0, b+2]_{\mathbb{N}_{0}},  \tag{18}\\
& x(\nu-3)=\left.\left[\Delta_{\nu-3}^{\alpha} x(t)\right]\right|_{t=\gamma-\alpha-2} \\
& =\left.\left[\Delta_{\nu-3}^{\beta} x(t)\right]\right|_{t=v+b+2-\beta}=0, \\
& y(\nu-3)=\left.\left[\Delta_{\nu-3}^{\alpha} y(t)\right]\right|_{t=\nu-\alpha-2} \\
& =\left.\left[\Delta_{\gamma-3}^{\beta} y(t)\right]\right|_{t=\gamma+b+2-\beta}=0,
\end{align*}
$$

and $(\widetilde{x}, \tilde{y})(t) \geq(z, z)(t)$ for $t \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}}$, where $\nu, \alpha, \beta, b$ are as in (1) and

$$
\begin{align*}
& \tilde{f}(t, x, y) \\
& = \begin{cases}f(t, x, y)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, x, y \geq 0, \\
f(t, x, 0)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, \\
f(t, 0, y)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}}, \\
f<0, y>0, \\
f(t, 0,0)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, x, y<0,\end{cases} \tag{19}
\end{align*}
$$

and

$$
\begin{align*}
& \tilde{g}(t, x, y) \\
& = \begin{cases}g(t, x, y)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, x, y \geq 0, \\
g(t, x, 0)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, \\
g>0, y<0, \\
g(t, 0, y)+M, & t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}, x<0, y>0, \\
g(t, 0,0)+M, & t \in[v-1, b+v+1]_{N_{v-1}}, x, y<0 .\end{cases} \tag{20}
\end{align*}
$$

Note that for $\left(x_{1}, y_{1}\right)(t) \geq\left(x_{2}, y_{2}\right)(t)$, we mean $x_{1}(t) \geq x_{2}(t)$, $y_{1}(t) \geq y_{2}(t)$ for all $t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}$.

For $(x, y) \in P \times P$, and $t \in[\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}$, we define the operators

$$
\begin{align*}
& B_{1}(x, y)(t)=\sum_{s=0}^{b+2} G(t, s) \tilde{f}(s+v-1, x(s+v-1) \\
& \quad-z(s+v-1), y(s+v-1)-z(s+v-1))  \tag{21}\\
& B_{2}(x, y)(t)=\sum_{s=0}^{b+2} G(t, s) \tilde{g}(s+v-1, x(s+v-1) \\
& \quad-z(s+v-1), y(s+v-1)-z(s+v-1))
\end{align*}
$$

and

$$
\begin{equation*}
B(x, y)(t)=\left(B_{1}, B_{2}\right)(x, y)(t) \tag{22}
\end{equation*}
$$

Then (H0) and using the Arzelà-Ascoli theorem in a standard way establish that $B: P \times P \longrightarrow P \times P$ is a completely continuous operator. It is clear that $(x, y) \in(P \times P) \backslash\{\mathbf{0}\}$ is a positive solution for (18) if and only if $(x, y) \in(P \times P) \backslash\{\mathbf{0}\}$ is a fixed point of $B$.

Let $P_{0}=\left\{y \in P: y(t) \geq q^{*}(t)\|y\|, \forall t \in[\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}\right\}$. Then from Lemma ?? we have

$$
\begin{equation*}
B_{i}(P \times P) \subset P_{0}, \quad i=1,2 \tag{23}
\end{equation*}
$$

If we seek a fixed point $(\widetilde{x}, \tilde{y})$ of $B$ then $\tilde{x}, \tilde{y} \in P_{0}$ and

$$
\begin{align*}
& w(t)-z(t) \geq q^{*}(t)\|w\|-M \kappa_{2} \geq q_{0}\|w\|-M \kappa_{2}  \tag{24}\\
& \text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}},
\end{align*}
$$

where $w=\tilde{x}, \tilde{y}$, and $q_{0}=\min _{t \in[\gamma-1, b+\nu+1]_{N_{\nu-1}}} q^{*}(t)>0$, so as a result if $\|\widetilde{x}\|,\|\tilde{y}\| \geq q_{0}^{-1} M \kappa_{2}$ then $(\widetilde{x}, \tilde{y})(t) \geq(z, z)(t)$ for $t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}$ (i.e., $(\widetilde{x}-z, \widetilde{y}-z)(t)$ is a positive solution for (1)).

For convenience, we use $c_{1}, c_{2}, \ldots$ to stand for different positive constants. Let $B_{\varrho}:=\{x \in E:\|x\|<\varrho\}$ for $\varrho>0$. Now, we list our assumptions on $\tilde{f}, \widetilde{g}$ (the first two are needed for Theorem 10 and the last two are needed for Theorem 11):
(H1) There exist $p, q \in C\left(\mathbb{R}^{+}, \mathbb{R}^{+}\right)$such that
(i) $p$ is concave and strictly increasing on $\mathbb{R}^{+}$,
(ii) there exists $c_{1}>0$ such that

$$
\begin{align*}
\widetilde{f}(t, x, y) & \geq x+p(y)-c_{1} \\
\widetilde{g}(t, x, y) & \geq q(x)-c_{1}  \tag{25}\\
\forall & (t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+}
\end{align*}
$$

(iii) there is a $\gamma_{1}>0$ such that $\kappa_{1}\left(1+\gamma_{1} \kappa_{1}\right)>1$ and

$$
\begin{align*}
& p\left(\kappa_{2} q(x(t+v-1)-z(t+v-1))\right) \\
& \quad \geq \kappa_{2} \gamma_{1}(x(t+v-1)-z(t+v-1))-c_{1} \tag{26}
\end{align*}
$$

for $x \in \mathbb{R}^{+}$and $t \in[0, b+2]_{\mathbb{N}_{0}}$.
(H2) For any $(t, x, y) \in[\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}} \times\left[0, q_{0}^{-1} M \kappa_{2}\right] \times$ $\left[0, q_{0}^{-1} M \kappa_{2}\right]$, assume

$$
\begin{equation*}
\tilde{f}, \tilde{g}(t, x, y)<q_{0}^{-1} M \tag{27}
\end{equation*}
$$

(H3) There exist $e_{i} \geq 0(i=1,2,3,4)$ with $e_{1}^{2}+e_{2}^{2} \neq 0, e_{3}^{2}+$ $e_{4}^{2} \neq 0$ such that
(i) $\kappa=\left(1-e_{1} \kappa_{2}\right)\left(1-e_{4} \kappa_{2}\right)-e_{2} e_{3} \kappa_{2}^{2}>0, e_{1}, e_{4}<\kappa_{2}^{-1}$,
(ii) there exist $c_{2}>0$ such that

$$
\begin{align*}
&\binom{\tilde{f}(t, x, y)}{\tilde{g}(t, x, y)} \leq\binom{ e_{1} x+e_{2} y+c_{2}}{e_{3} x+e_{4} y+c_{2}}  \tag{28}\\
& \forall(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+}
\end{align*}
$$

(H4) For any $(t, x, y) \in[\nu-1, b+\nu+1]_{\mathbb{N}_{v-1}} \times\left[0, q_{0}^{-1} M \kappa_{2}\right] \times$ [ $0, q_{0}^{-1} M \kappa_{2}$ ], assume

$$
\begin{equation*}
\tilde{f}, \tilde{g}(t, x, y)>q_{0}^{-2} M \tag{29}
\end{equation*}
$$

Example 7. Let $p(y)=y^{19 / 20}$ and $q(x)=x^{2}$ for $x, y \in \mathbb{R}^{+}$. Then, for any $\omega>0$, we have

$$
\begin{equation*}
\liminf _{x \rightarrow+\infty} \frac{p(\omega q(x))}{x}=\liminf _{x \rightarrow+\infty} \frac{\omega^{19 / 20} x^{19 / 10}}{x}=+\infty \tag{30}
\end{equation*}
$$

Let $f(t, x, y)=\left(1 / 2 \kappa_{2}\right) x+\left(1 / 2 \kappa_{2} e^{\beta_{1}+\cos (t x)}\right) y-M$ and $g(t, x, y)=\left(1 / e^{\beta_{2}+\sin (t x)}\right)\left(q_{0}^{-1} M\right)^{1-\beta_{3}} \kappa_{2}^{-\beta_{3}} x^{\beta_{3}}-M$, where $\beta_{1}, \beta_{2}>1, \beta_{3}>2$, for $(t, x, y) \in[\nu-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+}$. Then, for any $(t, x, y) \in[\nu-1, b+\nu+1]_{\mathbb{N}_{v-1}} \times\left[0, q_{0}^{-1} M \kappa_{2}\right] \times$ [ $0, q_{0}^{-1} M \kappa_{2}$ ], we have

$$
\begin{align*}
f(t, x, y)+M & =\frac{1}{2 \kappa_{2}} x+\frac{1}{2 \kappa_{2} e^{\beta_{1}+\cos (t x)}} y \\
& <\frac{1}{2 \kappa_{2}} q_{0}^{-1} M \kappa_{2}+\frac{1}{2 \kappa_{2}} q_{0}^{-1} M \kappa_{2}  \tag{31}\\
& =q_{0}^{-1} M
\end{align*}
$$

and

$$
\begin{align*}
g(t, x, y)+M & =\frac{1}{e^{\beta_{2}+\sin (t x)}}\left(q_{0}^{-1} M\right)^{1-\beta_{3}} \kappa_{2}^{-\beta_{3}} x^{\beta_{3}} \\
& <\left(q_{0}^{-1} M\right)^{1-\beta_{3}} \kappa_{2}^{-\beta_{3}}\left(q_{0}^{-1} M \kappa_{2}\right)^{\beta_{3}}  \tag{32}\\
& =q_{0}^{-1} M .
\end{align*}
$$

Also,
$\liminf _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{f(t, x, y)+M}{x+p(y)}$

$$
\begin{equation*}
=\liminf _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{\left(1 / 2 \kappa_{2}\right) x+\left(1 / 2 \kappa_{2} e^{\beta_{1}+\cos (t x)}\right) y}{x+y^{19 / 20}} \tag{33}
\end{equation*}
$$

$$
=+\infty, \quad \text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}},
$$

and

$$
\begin{aligned}
& \liminf _{x \rightarrow+\infty} \frac{g(t, x, y)+M}{q(x)} \\
& \quad=\liminf _{x \longrightarrow+\infty} \frac{\left(1 / e^{\beta_{2}+\sin (t x)}\right)\left(q_{0}^{-1} M\right)^{1-\beta_{3}} \kappa_{2}^{-\beta_{3}} x^{\beta_{3}}}{x^{2}} \\
& \quad=+\infty,
\end{aligned}
$$

uniformly on $(t, y) \in[\nu-1, b+v+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^{+}$.
Thus, (H1)-(H2) are satisfied.
Example 8. Let $f(t, x, y)=\left(q_{0}^{-2} M e^{q_{0}^{-1} M \kappa_{2}}+e^{|\sin (t x y)|}\right) e^{-(x+y) / 2}-$ $M$ and $g(t, x, y)=\left(q_{0}^{-2} M e^{q_{0}^{-1} M \kappa_{2}}+e^{|\cos (t x y)|}\right) e^{-(x+y) / 2}-M$, for $(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+}$. Then, for any $(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times\left[0, q_{0}^{-1} M \kappa_{2}\right] \times\left[0, q_{0}^{-1} M \kappa_{2}\right]$, we have

$$
\begin{align*}
& f(t, x, y)+M, g(t, x, y)+M \\
& \quad>q_{0}^{-2} M e^{q_{0}^{-1} M \kappa_{2}} e^{-q_{0}^{-1} M \kappa_{2}}=q_{0}^{-2} M \tag{35}
\end{align*}
$$

Also,

$$
\begin{aligned}
& \limsup _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{f(t, x, y)+M}{e_{1} x+e_{2} y} \\
& =\limsup _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{\left(q_{0}^{-2} M e^{q_{0}^{-1} M \kappa_{2}}+e^{|\sin (t x y)|}\right) e^{-(x+y) / 2}}{e_{1} x+e_{2} y} \\
& =0,
\end{aligned}
$$

and

$$
\begin{align*}
& \limsup _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{g(t, x, y)+M}{e_{3} x+e_{4} y} \\
& =\lim _{x \rightarrow+\infty, y \rightarrow+\infty} \frac{\left(q_{0}^{-2} M e^{q_{0}^{-1} M \kappa_{2}}+e^{|\cos (t x y)|}\right) e^{-(x+y) / 2}}{e_{3} x+e_{4} y}  \tag{37}\\
& =0,
\end{align*}
$$

for $t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}$. Thus, (H3)-(H4) hold.
Remark 9. (i) In (H1), the growth condition for nonlinear term $f$ depends on two variables $x, y$; however, in [7], this corresponding condition only involves one variable.
(ii) When nonlinear terms $f, g$ grow sublinearly at $+\infty$, nonnegative matrices are used to depict the coupling behavior of our nonlinearities. This is different from condition (H4) in [7].

Theorem 10. Suppose that (H0)-(H2) hold. Then (1) has at least one positive solution.

Proof. We first claim that there exists a sufficiently large positive number $R>q_{0}^{-1} M \kappa_{2}$ such that

$$
\begin{align*}
& (x, y) \neq B(x, y)+\lambda\left(x_{0}, y_{0}\right), \\
& \quad \forall(x, y) \in \partial\left(B_{R} \times B_{R}\right) \cap(P \times P), \quad \lambda \geq 0, \tag{38}
\end{align*}
$$

where $x_{0}, y_{0} \in P_{0}$ are two given functions. Suppose not. Then there exist $(x, y) \in \partial\left(B_{R} \times B_{R}\right) \cap(P \times P)$ and $\lambda \geq 0$ such that $(x, y)=B(x, y)+\lambda\left(x_{0}, y_{0}\right)$, and so

$$
\begin{align*}
& x(t)=B_{1}(x, y)(t)+\lambda x_{0}(t), \\
& y(t)=B_{2}(x, y)(t)+\lambda y_{0}(t), \tag{39}
\end{align*}
$$

$$
\text { for } t \in[\nu-1, b+\nu+1]_{\mathbb{N}_{v-1}} .
$$

This implies $x(t) \geq B_{1}(x, y)(t)$, and $y(t) \geq B_{2}(x, y)(t)$ for $t \in[\nu-1, b+v+1]_{\mathbb{N}_{v-1}}$. From (H1) we have

$$
x(t) \geq B_{1}(x, y)(t) \geq \sum_{s=0}^{b+2} G(t, s)[x(s+v-1)
$$

$$
-z(s+v-1)+p(y(s+v-1)-z(s+v-1))
$$

$$
\left.-c_{1}\right] \geq \sum_{s=0}^{b+2} G(t, s) x(s+v-1)+\sum_{s=0}^{b+2} G(t, s)
$$

$$
\begin{equation*}
p(y(s+v-1)-z(s+v-1))-c_{3} \geq \sum_{s=0}^{b+2} G(t, s) \tag{40}
\end{equation*}
$$

$$
x(s+v-1)+\sum_{s=0}^{b+2} G(t, s)[p(y(s+v-1))
$$

$$
-p(z(s+v-1))]-c_{3} \geq \sum_{s=0}^{b+2} G(t, s) x(s+v-1)
$$

$$
+\sum_{s=0}^{b+2} G(t, s) p(y(s+v-1))-c_{4}
$$

$$
\text { for } t \in[\nu-1, b+v+1]_{\mathbb{N}_{\nu-1}},
$$

and

$$
\begin{align*}
& y(t) \geq B_{2}(x, y)(t) \\
& \quad \geq \sum_{s=0}^{b+2} G(t, s)\left[q(x(s+v-1)-z(s+v-1))-c_{1}\right] \\
& \geq \sum_{s=0}^{b+2} G(t, s) q(x(s+v-1)-z(s+v-1))-c_{3},  \tag{41}\\
& \quad \text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} .
\end{align*}
$$

As a result, for $t \in[0, b+2]_{\mathbb{N}_{0}}$, we have

$$
\begin{aligned}
& p(y(t+v-1))+p\left(c_{3}\right) \geq p\left(y(t+v-1)+c_{3}\right) \\
& \quad \geq p\left[\sum_{s=0}^{b+2} G(t+v-1, s)\right. \\
& \quad \cdot q(x(s+v-1)-z(s+v-1))] \\
& \quad=p\left[\sum_{s=0}^{b+2} G(t+v-1, s)\right. \\
& \left.\quad \kappa_{2}^{-1} \kappa_{2} q(x(s+v-1)-z(s+v-1))\right] \geq \sum_{s=0}^{b+2} G(t \\
& \quad+\nu-1, s) \kappa_{2}^{-1} p\left(\kappa_{2} q(x(s+\nu-1)-z(s+v-1))\right) \\
& \quad \geq \sum_{s=0}^{b+2} G(t+v-1, s) \\
& \quad \kappa_{2}^{-1}\left[\kappa_{2} \gamma_{1}(x(s+v-1)-z(s+v-1))-c_{1}\right] \\
& \geq \gamma_{1} \sum_{s=0}^{b+2} G(t+v-1, s)(x(s+v-1) \\
& -1)-c_{6} .
\end{aligned}
$$

Thus

$$
p(y(t+v-1)) \geq \gamma_{1} \sum_{s=0}^{b+2} G(t+v-1, s) x(s+v-1)
$$

$$
-c_{7}
$$

and, therefore,

$$
\begin{aligned}
x(t) & \geq \sum_{s=0}^{b+2} G(t, s) x(s+v-1) \\
& +\sum_{s=0}^{b+2} G(t, s) p(y(s+v-1))-c_{4} \\
& \geq \sum_{s=0}^{b+2} G(t, s) x(s+v-1)+\sum_{s=0}^{b+2} G(t, s) \\
& \cdot\left[\gamma_{1} \sum_{\tau=0}^{b+2} G(s+v-1, \tau) x(\tau+v-1)-c_{7}\right]-c_{4}
\end{aligned}
$$

$$
\begin{align*}
& \geq \sum_{s=0}^{b+2} G(t, s) x(s+v-1)+\gamma_{1} \sum_{s=0}^{b+2} G(t, s) \\
& \cdot \sum_{\tau=0}^{b+2} G(s+v-1, \tau) x(\tau+v-1)-c_{8} \\
& \quad \text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \tag{44}
\end{align*}
$$

Multiply both sides of the above inequality by $\varphi(t)$ and sum from $v-1$ to $b+v+1$ and together with (11) we obtain

$$
\begin{aligned}
& \sum_{t=\gamma-1}^{b+v+1} x(t) \varphi(t)=\sum_{t=0}^{b+2} x(t+v-1) \varphi(t+v-1) \\
& \quad \geq \sum_{t=0}^{b+2} \varphi(t+v-1)\left[\sum_{s=0}^{b+2} G(t+v-1, s) x(s+v-1)\right. \\
& \quad+\gamma_{1} \sum_{s=0}^{b+2} G(t+v-1, s) \\
& \left.\quad \cdot \sum_{\tau=0}^{b+2} G(s+v-1, \tau) x(\tau+v-1)-c_{8}\right] \geq\left(\kappa_{1}\right. \\
& \left.\quad+\gamma_{1} \kappa_{1}^{2}\right) \sum_{t=0}^{b+2} x(t+v-1) \varphi(t+v-1)-c_{9}=\left(\kappa_{1}\right.
\end{aligned}
$$

$$
\left.+\gamma_{1} \kappa_{1}^{2}\right) \sum_{t=\nu-1}^{b+\nu+1} x(t) \varphi(t)-c_{9} .
$$

From (23), (39), and $x_{0} \in P_{0}$ we have $x \in P_{0}$. This implies

$$
\begin{align*}
\kappa_{1}\|x\| & =\|x\| \sum_{t=\nu-1}^{b+\nu+1} q^{*}(t) \varphi(t) \leq \sum_{t=\nu-1}^{b+\nu+1} x(t) \varphi(t)  \tag{46}\\
& \leq \frac{c_{9}}{\kappa_{1}\left(1+\gamma_{1} \kappa_{1}\right)-1},
\end{align*}
$$

and

$$
\begin{equation*}
\|x\| \leq \frac{\kappa_{1}^{-1} c_{9}}{\kappa_{1}\left(1+\gamma_{1} \kappa_{1}\right)-1} . \tag{47}
\end{equation*}
$$

Note that, from (23), (39), and $y_{0} \in P_{0}$, we find $y \in P_{0}$. Moreover, we may assume $y(t) \not \equiv 0$, for $t \in[\nu-1, b+\nu+1]_{\mathbb{N}_{\gamma-1}}$. Then $\|y\|>0$ and $p(\|y\|)>0$. Thus, from the concavity of $p$, we have

$$
\begin{align*}
& \kappa_{1}\|y\|=\|y\| \sum_{t=v-1}^{b+v+1} q^{*}(t) \varphi(t) \leq \sum_{t=v-1}^{b+v+1} y(t) \varphi(t) \\
& =\sum_{t=0}^{b+2} y(t+v-1) \varphi(t+v-1) \\
& =\frac{\|y\|}{p(\|y\|)} \sum_{t=0}^{b+2} \frac{y(t+v-1)}{\|y\|} p(\|y\|) \varphi(t+v-1)  \tag{48}\\
& \quad \leq \frac{\|y\|}{p(\|y\|)} \sum_{t=0}^{b+2} p(y(t+v-1)) \varphi(t+v-1)
\end{align*}
$$

This implies that

$$
\begin{equation*}
p(\|y\|) \leq \kappa_{1}^{-1} \sum_{t=0}^{b+2} p(y(t+v-1)) \varphi(t+v-1) \tag{49}
\end{equation*}
$$

From (40) and Lemma ?? we obtain

$$
\begin{align*}
x(t)+c_{4} & \geq \sum_{s=0}^{b+2} G(t, s) p(y(s+v-1)) \\
& \geq \sum_{s=0}^{b+2} q^{*}(t) \varphi(s+v-1) p(y(s+v-1))  \tag{50}\\
& \geq q_{0} \sum_{t=0}^{b+2} p(y(t+v-1)) \varphi(t+v-1)
\end{align*}
$$

Combining the above two inequalities, we get

$$
\begin{align*}
p(\|y\|) & \leq\left(\kappa_{1} q_{0}\right)^{-1}\left(x(t)+c_{4}\right) \\
& \leq\left(\kappa_{1} q_{0}\right)^{-1}\left[\frac{\kappa_{1}^{-1} c_{9}}{\kappa_{1}\left(1+\gamma_{1} \kappa_{1}\right)-1}+c_{4}\right] . \tag{51}
\end{align*}
$$

From (H1), $\lim _{z \rightarrow+\infty} p(z)=+\infty$, and thus there exists $\mathscr{M}_{1}>$ 0 such that $\|y\| \leq \mathscr{M}_{1}$.

Hence, we have $\|x\| \leq \kappa_{1}^{-1} c_{9} /\left(\kappa_{1}\left(1+\gamma_{1} \kappa_{1}\right)-1\right)$ and $\|y\| \leq$ $\mathscr{M}_{1}$. As a result, choosing $R>\max \left\{q_{0}^{-1} M \kappa_{2}, \kappa_{1}^{-1} c_{9} /\left(\kappa_{1}(1+\right.\right.$ $\left.\left.\left.\gamma_{1} \kappa_{1}\right)-1\right), \mathscr{M}_{1}\right\}$ we have a contradiction (recall in general $\partial(A \times$ $B)=(\partial A \times \bar{B}) \cup(\bar{A} \times \partial B))$. Thus (38) is true. Consequently Lemma 5 (with $R$ chosen above) implies

$$
\begin{equation*}
i\left(B,\left(B_{R} \times B_{R}\right) \cap(P \times P), P \times P\right)=0 \tag{52}
\end{equation*}
$$

Now we show that

$$
\begin{align*}
& (x, y) \neq \lambda B(x, y), \\
& \forall(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P), \lambda \in[0,1] . \tag{53}
\end{align*}
$$

Suppose not. Then there exist $(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap$ $(P \times P), \lambda_{0} \in[0,1]$ such that $(x, y)=\lambda_{0} B(x, y)$. This implies that

$$
\begin{align*}
& x(t) \leq B_{1}(x, y)(t), \\
& y(t) \leq B_{2}(x, y)(t), \tag{54}
\end{align*}
$$

Hence, $\|x\| \leq\left\|B_{1}(x, y)\right\|$ and $\|y\| \leq\left\|B_{2}(x, y)\right\|$. However, from (H2) we have

$$
\begin{align*}
& B_{1}(x, y)(t)=\sum_{s=0}^{b+2} G(t, s) \widetilde{f}(s+v-1, x(s+v-1) \\
& \quad-z(s+v-1), y(s+v-1)-z(s+v-1))  \tag{55}\\
& \quad<\sum_{s=0}^{b+2} \varphi(s+v-1) q_{0}^{-1} M=q_{0}^{-1} M \kappa_{2}
\end{align*}
$$

for all $t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}}$. This implies $\left\|B_{1}(x, y)\right\|<$ $q_{0}^{-1} M \kappa_{2}$. Similarly, $\left\|B_{2}(x, y)\right\|<q_{0}^{-1} M \kappa_{2}$. Thus, note that $(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P)$, and we have

$$
\begin{align*}
\|(x, y)\| & =\max \{\|x\|,\|y\|\} \\
& \leq \max \left\{\left\|B_{1}(x, y)\right\|,\left\|B_{2}(x, y)\right\|\right\}<q_{0}^{-1} M \kappa_{2}  \tag{56}\\
& =\|(x, y)\|
\end{align*}
$$

Clearly, this is a contradiction. Thus (53) is true. It follows from Lemma 6 that

$$
\begin{equation*}
i\left(B,\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P), P \times P\right)=1 \tag{57}
\end{equation*}
$$

From (52) and (57) we have

$$
\begin{align*}
& i\left(B,\left(\left(B_{R} \times B_{R}\right) \backslash\left(\bar{B}_{q_{0}^{-1} M \kappa_{2}} \times \bar{B}_{q_{0}^{-1} M \kappa_{2}}\right)\right) \cap(P \times P), P\right. \\
& \quad \times P)=0-1=-1 . \tag{58}
\end{align*}
$$

Therefore the operator $B$ has at least one fixed point $(x, y)$ in $\left(\left(B_{R} \times B_{R}\right) \backslash\left(\bar{B}_{q_{0}^{-1} M \kappa_{2}} \times \bar{B}_{q_{0}^{-1} M \kappa_{2}}\right)\right) \cap(P \times P)$ with $\|x\|,\|y\| \geq$ $q_{0}^{-1} M \kappa_{2}$, and then $(x-z, y-z)(t)$ is a positive solution for (1). This completes the proof.

Theorem 11. Suppose that (H0), (H3), and (H4) hold. Then (1) has at least one positive solution.

Proof. We show there exists a positive constant $R>q_{0}^{-1} M \kappa_{2}$ such that

$$
\begin{align*}
(x, y) & \neq \lambda B(x, y), \\
& \forall(x, y) \in \partial\left(B_{R} \times B_{R}\right) \cap(P \times P), \lambda \in[0,1] . \tag{59}
\end{align*}
$$

Suppose not. Then there exist $(x, y) \in \partial\left(B_{R} \times B_{R}\right) \cap(P \times$ $P), \lambda_{0} \in[0,1]$ such that $(x, y)=\lambda_{0} B(x, y)$. This implies that

$$
\begin{align*}
& x(t) \leq B_{1}(x, y)(t), \\
& y(t) \leq B_{2}(x, y)(t), \tag{60}
\end{align*}
$$

$$
\text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}} .
$$

From (H3) we have

$$
\begin{aligned}
& x(t) \leq \sum_{s=0}^{b+2} G(t, s)\left[e_{1}(x(s+v-1)-z(s+v-1))\right. \\
& \left.\quad+e_{2}(y(s+v-1)-z(s+v-1))+c_{2}\right] \\
& \quad \leq \sum_{s=0}^{b+2} G(t, s)\left[e_{1} x(s+v-1)+e_{2} y(s+v-1)\right] \\
& \quad+c_{10}
\end{aligned}
$$

$$
\forall(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+}
$$

Similarly, we have

$$
\begin{align*}
y(t) \leq & \sum_{s=0}^{b+2} G(t, s)\left[e_{3} x(s+v-1)+e_{4} y(s+v-1)\right] \\
& +c_{10},  \tag{62}\\
& \forall(t, x, y) \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}} \times \mathbb{R}^{+} \times \mathbb{R}^{+} .
\end{align*}
$$

Consequently, for all $t \in[0, b+2]_{\mathbb{N}_{0}}$, multiply both sides of the above two inequalities by $\varphi(t)$ and sum from $v-1$ to $b+v+1$ and together with (11) we obtain

$$
\begin{align*}
& \sum_{t=v-1}^{b+v+1} x(t) \varphi(t)=\sum_{t=0}^{b+2} x(t+v-1) \varphi(t+v-1) \\
& \quad \leq \sum_{t=0}^{b+2} \varphi(t+v-1) \\
& \quad\left[\sum_{s=0}^{b+2} G(t, s)\left[e_{1} x(s+v-1)+e_{2} y(s+v-1)\right]\right.  \tag{63}\\
& \left.\quad+c_{10}\right] \leq e_{1} \kappa_{2} \sum_{t=v-1}^{b+v+1} x(t) \varphi(t) \\
& \quad+e_{2} \kappa_{2} \sum_{t=v-1}^{b+v+1} y(t) \varphi(t)+c_{10} \sum_{t=v-1}^{b+v+1} \varphi(t)
\end{align*}
$$

Similarly, we have

$$
\begin{aligned}
& \sum_{t=\nu-1}^{b+\nu+1} y(t) \varphi(t)=\sum_{t=0}^{b+2} y(t+\nu-1) \varphi(t+v-1) \\
& \quad \leq \sum_{t=0}^{b+2} \varphi(t+\nu-1) \\
& \cdot\left[\sum_{s=0}^{b+2} G(t, s)\left[e_{3} x(s+v-1)+e_{4} y(s+v-1)\right]\right. \\
& \left.\quad+c_{10}\right] \leq e_{3} \kappa_{2} \sum_{t=\nu-1}^{b+v+1} x(t) \varphi(t)
\end{aligned}
$$

$$
+e_{4} \kappa_{2} \sum_{t=\nu-1}^{b+\nu+1} y(t) \varphi(t)+c_{10} \sum_{t=\nu-1}^{b+\nu+1} \varphi(t)
$$

Consequently, we obtain

$$
\begin{align*}
& \left(\begin{array}{cc}
1-e_{1} \kappa_{2} & -e_{2} \kappa_{2} \\
-e_{3} \kappa_{2} & 1-e_{4} \kappa_{2}
\end{array}\right)\binom{\sum_{t=\nu-1}^{t+\nu+1} x(t) \varphi(t)}{\sum_{t=\nu-1}^{b+\nu+1} y(t) \varphi(t)}  \tag{65}\\
& \quad \leq\binom{\kappa_{2} c_{10}}{\kappa_{2} c_{10}}
\end{align*}
$$

From (H3)(i) we have

$$
\begin{align*}
& \left(\begin{array}{l}
\sum_{t=\gamma-1}^{t=\gamma-1} \\
\sum_{t=\gamma-1}^{b+v+1}
\end{array} x(t) \varphi(t) \varphi(t)\right. \\
& \quad \leq \frac{1}{\kappa}\left(\begin{array}{cc}
1-e_{4} \kappa_{2} & e_{2} \kappa_{2} \\
e_{3} \kappa_{2} & 1-e_{1} \kappa_{2}
\end{array}\right)\binom{\kappa_{2} c_{10}}{\kappa_{2} c_{10}}  \tag{66}\\
& =\binom{\frac{\kappa_{2} c_{10}\left(1+e_{2} \kappa_{2}-e_{4} \kappa_{2}\right)}{\kappa}}{\frac{\kappa_{2} c_{10}\left(1+e_{3} \kappa_{2}-e_{1} \kappa_{2}\right)}{\kappa}}
\end{align*}
$$

Note that $x, y \in P_{0}$ from the fact that $B_{i}(P \times P) \subset P_{0}(i=1,2)$. This implies

$$
\begin{align*}
\|x\| \sum_{t=\nu-1}^{b+\gamma+1} q^{*}(t) \varphi(t) & \leq \sum_{t=\nu-1}^{b+\nu+1} x(t) \varphi(t) \\
& \leq \frac{\kappa_{2} c_{10}\left(1+e_{2} \kappa_{2}-e_{4} \kappa_{2}\right)}{\kappa}, \\
\|y\| \sum_{t=\nu-1}^{b+\nu+1} q^{*}(t) \varphi(t) & \leq \sum_{t=\nu-1}^{b+\nu+1} y(t) \varphi(t)  \tag{67}\\
& \leq \frac{\kappa_{2} c_{10}\left(1+e_{3} \kappa_{2}-e_{1} \kappa_{2}\right)}{\kappa} .
\end{align*}
$$

Hence

$$
\begin{align*}
& \|x\| \leq \frac{\kappa_{2} c_{10}\left(1+e_{2} \kappa_{2}-e_{4} \kappa_{2}\right)}{\kappa_{1} \kappa}, \\
& \|y\| \leq \frac{\kappa_{2} c_{10}\left(1+e_{3} \kappa_{2}-e_{1} \kappa_{2}\right)}{\kappa_{1} \kappa} \tag{68}
\end{align*}
$$

Thus if we choose $R>\max \left\{q_{0}^{-1} M \kappa_{2}, \kappa_{2} c_{10}\left(1+e_{2} \kappa_{2}-e_{4} \kappa_{2}\right) / \kappa_{1} \kappa\right.$, and $\left.\kappa_{2} c_{10}\left(1+e_{3} \kappa_{2}-e_{1} \kappa_{2}\right) / \kappa_{1} \kappa\right\}$ we have a contradiction. Thus (59) is true. Lemma 6 (with $R$ chosen above) implies

$$
\begin{equation*}
i\left(B,\left(B_{R} \times B_{R}\right) \cap(P \times P), P \times P\right)=1 \tag{69}
\end{equation*}
$$

We next prove that

$$
\begin{align*}
& (x, y) \neq B(x, y)+\lambda\left(x_{0}, y_{0}\right) \\
& \quad \forall(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P), \lambda \geq 0 \tag{70}
\end{align*}
$$

where $x_{0}, y_{0} \in P$ are two fixed functions. Indeed, if not, there exist $(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P), \lambda_{0} \geq 0$ such that $(x, y)=B(x, y)+\lambda_{0}\left(x_{0}, y_{0}\right)$. This implies that

$$
\begin{align*}
& x(t) \geq B_{1}(x, y)(t), \\
& y(t) \geq B_{2}(x, y)(t),  \tag{71}\\
& \quad \text { for } t \in[v-1, b+v+1]_{\mathbb{N}_{v-1}} .
\end{align*}
$$

Hence, $\|x\| \geq\left\|B_{1}(x, y)\right\|$ and $\|y\| \geq\left\|B_{2}(x, y)\right\|$. However, from (H4) we have

$$
\begin{align*}
& B_{1}(x, y)(t)=\sum_{s=0}^{b+2} G(t, s) \tilde{f}(s+v-1, x(s+v-1) \\
& \quad-z(s+v-1), y(s+v-1)-z(s+v-1))  \tag{72}\\
& \quad>\sum_{s=0}^{b+2} q^{*}(t) \varphi(s+v-1) q_{0}^{-2} M \geq q_{0}^{-1} M \kappa_{2}
\end{align*}
$$

for all $t \in[v-1, b+v+1]_{\mathbb{N}_{\nu-1}}$. This implies $\left\|B_{1}(x, y)\right\|>$ $q_{0}^{-1} M \kappa_{2}$. Similarly, $\left\|B_{2}(x, y)\right\|>q_{0}^{-1} M \kappa_{2}$. Thus, note that $(x, y) \in \partial\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P)$, and we have

$$
\begin{align*}
\|(x, y)\| & =\max \{\|x\|,\|y\|\} \\
& \geq \max \left\{\left\|B_{1}(x, y)\right\|,\left\|B_{2}(x, y)\right\|\right\}>q_{0}^{-1} M \kappa_{2}  \tag{73}\\
& =\|(x, y)\| .
\end{align*}
$$

This is a contradiction. So (70) is true. It follows from Lemma 5 that

$$
\begin{equation*}
i\left(B,\left(B_{q_{0}^{-1} M \kappa_{2}} \times B_{q_{0}^{-1} M \kappa_{2}}\right) \cap(P \times P), P \times P\right)=0 . \tag{74}
\end{equation*}
$$

From (69) and (74) we have

$$
\begin{align*}
& i\left(B,\left(\left(B_{R} \times B_{R}\right) \backslash\left(\bar{B}_{q_{0}^{-1} M \kappa_{2}} \times \bar{B}_{q_{0}^{-1} M \kappa_{2}}\right)\right) \cap(P \times P), P\right. \\
& \quad \times P)=1-0=1 . \tag{75}
\end{align*}
$$

Therefore the operator $B$ has at least one fixed point $(x, y)$ in $\left(\left(B_{R} \times B_{R}\right) \backslash\left(\bar{B}_{q_{0}^{-1} M \kappa_{2}} \times \bar{B}_{q_{0}^{-1} M \kappa_{2}}\right)\right) \cap(P \times P)$ with $\|x\|,\|y\| \geq$ $q_{0}^{-1} M \kappa_{2}$, and then $(x-z, y-z)(t)$ is a positive solution for (1). This completes the proof.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## Research Article

# Non-Nehari Manifold Method for Fractional p-Laplacian Equation with a Sign-Changing Nonlinearity 

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We consider the following fractional p-Laplacian equation: $(-\Delta)_{p}^{\alpha} u+V(x)|u|^{p-2} u=f(x, u)-\Gamma(x)|u|^{q-2} u, x \in \mathbb{R}^{N}$, where $N \geq 2$, $p_{\alpha}^{*}>q>p \geq 2, \alpha \in(0,1),(-\Delta)_{p}^{\alpha}$ is the fractional $p$-Laplacian, and $\Gamma \in L^{\infty}\left(\mathbb{R}^{N}\right)$ and $\Gamma(x) \geq 0$ for a.e. $x \in \mathbb{R}^{N}$. $f$ has the subcritical growth but higher than $\Gamma(x)|u|^{q-2} u$; however, the nonlinearity $f(x, u)-\Gamma(x)|u|^{q-2} u$ may change sign. If $V$ is coercive, we investigate the existence of ground state solutions for p -Laplacian equation.

## 1. Introduction

Consider the following nonlinear Schrödinger equation with fractional $p$-Laplacian:

$$
\begin{align*}
(-\Delta)_{p}^{\alpha} u+V(x)|u|^{p-2} u=f(x, u)-\Gamma(x)|u|^{q-2} u & , \\
& x \in \mathbb{R}^{N}, \tag{1}
\end{align*}
$$

where $N \geq 2, p_{\alpha}^{*}>q>p \geq 2, \alpha \in(0,1)$, and $(-\Delta)_{p}^{\alpha}$ is the fractional $p$-Laplacian. $V(x), \Gamma(x)$, and $f(x, u): \mathbb{R}^{N} \times \mathbb{R} \longrightarrow$ $\mathbb{R}$ satisfy the following assumptions:
$(V) V \in C\left(\mathbb{R}^{N}, \mathbb{R}\right), V_{0}:=\inf _{x \in \mathbb{R}^{N}} V(x)>0$, there exists a constant $d_{0}>0$ such that
$\lim _{|y| \rightarrow \infty} \operatorname{meas}\left\{x \in \mathbb{R}^{N}:|x-y| \leq d_{0}, V(x) \leq M\right\}=0$,

$$
\begin{equation*}
\forall M>0, \tag{2}
\end{equation*}
$$

where meas ( $\cdot$ ) denotes the Lebesgue measure in $\mathbb{R}^{N}$;
( Г ) $\Gamma \in L^{\infty}\left(\mathbb{R}^{N}\right), \Gamma(x) \geq 0$ for a.e. $x \in \mathbb{R}^{N}$;
$\left(f_{1}\right) f(x, t): \mathbb{R}^{N} \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable, continuous in $t \in \mathbb{R}$ for a.e. $x \in \mathbb{R}^{N}$ and there are $C>0$ and $2 \leq p<q<r<p_{\alpha}^{*}$ such that

$$
\begin{equation*}
|f(x, t)| \leq C\left(1+|t|^{r-1}\right) \quad \text { for all } t \in \mathbb{R}, x \in \mathbb{R}^{N} \tag{3}
\end{equation*}
$$

where $p_{\alpha}^{*}=N p /(N-p \alpha)$;
$\left(f_{2}\right) f(x, t)=o\left(|t|^{p-1}\right)$ as $|t| \longrightarrow 0$ uniformly in $x \in \mathbb{R}^{N}$;
$\left(f_{3}\right) F(x, t) /|t|^{q} \longrightarrow \infty$ uniformly in $x$ as $|t| \longrightarrow \infty$, where $F(x, t)=\int_{0}^{t} f(x, \tau) d \tau ;$
$\left(f_{4}\right) t \longmapsto f(x, t) /|t|^{q-1}$ is nondecreasing on $(-\infty, 0) \cup$ $(0, \infty)$.

When $p=2$, (1) arises in the study of the nonlinear Fractional Schrödinger equation

$$
\begin{equation*}
(-\Delta)^{\alpha} u+V(x) u=f(x, u)-\Gamma(x)|u|^{q-2} u \tag{4}
\end{equation*}
$$

Problems with this type have occurred in many different fields, such as continuum mechanics, phase transition phenomena, population dynamics, and game theory, as they are
the typical outcome of stochastically stabilization of Lévy processes; see [1-4].

When $\alpha=1$ and $\Gamma=0$, (4) reduces to be the nonlinear Schrödinger equation

$$
\begin{equation*}
-\Delta u+V(x) u=f(x, u), \quad x \in \mathbb{R}^{N} \tag{5}
\end{equation*}
$$

Using the Nehari-type monotonicity condition, Szulkin and Weth [5] obtained the existence of ground state solutions for (5). But in this paper, the Nehari manifold is usually not smooth and the Nehari-type monotonicity condition for the nonlinearity is not satisfied; then the Nehari manifold method is invalid. In this paper, we are aimed to obtain ground state solutions for (1) by the so-called non-Nehari manifold method, which is established by Tang [6, 7]. Unlike the Nehari manifold method, the main idea of our approach lies on finding a minimizing sequence for the energy functional outside the Nehari manifold by using the diagonal method.

Now, we are ready to state the main result of this paper.
Theorem 1. Suppose that $(V),(\Gamma)$ and $\left(f_{1}\right)-\left(f_{4}\right)$ hold. Then (1) has a nontrivial ground state solution.

## 2. Preliminaries

In the paper, we will denote $o_{n}(1)$ by the infinitesimal as $n \longrightarrow$ $+\infty$. For the sake of simplicity, the norm of the space $L^{p}\left(\mathbb{R}^{N}\right)$ will be denoted by $\|\cdot\|_{p}$, and integrals over the whole $\mathbb{R}^{N}$ will be written $\int$.

We define the Gagliardo seminorm by

$$
\begin{equation*}
[u]_{\alpha, p}=\left(\int \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+\alpha p}} d x d y\right)^{1 / p} \tag{6}
\end{equation*}
$$

where $u: \mathbb{R}^{N} \longrightarrow \mathbb{R}$ is a measurable function. Then fractional Sobolev space $W^{\alpha, p}\left(\mathbb{R}^{N}\right)$ is given by

$$
\begin{align*}
& W^{\alpha, p}\left(\mathbb{R}^{N}\right) \\
& \quad=\left\{u \in L^{p}\left(\mathbb{R}^{N}\right): u \text { is measurable and }[u]_{\alpha, p}<\infty\right\} \tag{7}
\end{align*}
$$

endowed with the norm

$$
\begin{equation*}
\|u\|_{\alpha, p}=\left([u]_{\alpha, p}^{p}+\|u\|_{p}^{p}\right)^{1 / p} \tag{8}
\end{equation*}
$$

For the basic properties of fractional Sobolev spaces, we refer the interested reader to [8]. By condition $(V)$, we define the fractional Sobolev space with potential $V(x)$ by

$$
\begin{equation*}
E:=\left\{u \in W^{\alpha, p}: \int V(x)|u|^{p} d x<\infty\right\} \tag{9}
\end{equation*}
$$

endowed with the norm

$$
\begin{equation*}
\|u\|:=\left([u]_{\alpha, p}^{p}+\int V(x)|u|^{p} d x\right)^{1 / p} \tag{10}
\end{equation*}
$$

The energy functional $J: E \longrightarrow \mathbb{R}$ defined by

$$
\begin{align*}
J(u)= & \frac{1}{p} \int \frac{|u(x)-u(y)|^{p}}{|x-y|^{N+\alpha p}} d x d y \\
& +\frac{1}{p} \int V(x)|u(x)|^{p} d x  \tag{11}\\
& -\int\left(F(x, u)-\frac{1}{q} \Gamma(x)|u|^{q}\right) d x .
\end{align*}
$$

Under our hypotheses, $J$ is well defined on $E$. It is well known that $J \in C^{1}(E, \mathbb{R})$, and its derivative is given by

$$
\begin{align*}
& \left\langle J^{\prime}(u), v\right\rangle \\
& =\int_{\mathbb{R}^{2 N}} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+\alpha p}} d x d y  \tag{12}\\
& \quad+\int_{\mathbb{R}^{N}} V(x)|u|^{p-2} u v d x \\
& \quad-\int\left(f(x, u) v-\Gamma(x)|u|^{q-2} u v\right) d x,
\end{align*}
$$

for $u, v \in E$. It is standard to verify that the weak solutions of (1) correspond to the critical points of $J$. Now, we review the main embedding result for the space $E$.

Lemma 2 ([9, Lemma 1]). Under assumption $(V)$, the embed$\operatorname{ding} E \hookrightarrow L^{r}\left(\mathbb{R}^{N}\right)$ is compact for any $r \in\left[p, p_{\alpha}^{*}\right)$.

In the following lemma, we will show that $J$ has Mountain Pass geometric structure.

Lemma 3. Suppose that $(V),(\Gamma)$, and $\left(f_{1}\right)-\left(f_{4}\right)$ hold.
(i) There is $\delta_{0}>0$ such that $\rho_{0}:=\inf _{\|u\|=\delta_{0}} J(u)>J(0)=0$.
(ii) For any $u \neq 0$, there exists $t>0$ such that $J(t u)<0$.

Proof. (i) $\operatorname{By}\left(f_{1}\right)$ and ( $f_{2}$ ), we have

$$
\begin{align*}
& |f(x, u)| \leq \varepsilon|u|^{p-1}+C_{\varepsilon}|u|^{r-1} \\
& |F(x, u)| \leq \frac{\varepsilon}{p}|u|^{p}+\frac{C_{\varepsilon}}{r}|u|^{r} . \tag{13}
\end{align*}
$$

By $E \hookrightarrow L^{s}\left(\mathbb{R}^{N}\right)$ for $s \in\left[p, p_{\alpha}^{*}\right)$ and (13), we have

$$
\begin{align*}
J(u) & =\frac{1}{p}\|u\|^{p}-\int F(x, u) d x+\frac{1}{q} \int \Gamma(x)|u|^{q} d x  \tag{14}\\
& \geq \frac{1}{p}\|u\|^{p}-\varepsilon C_{1}\|u\|^{p}-C_{\varepsilon} C_{2}\|u\|^{r} .
\end{align*}
$$

By the arbitrariness of $\varepsilon$ and $p<r$, we get the conclusion.
(ii) Fix $u \neq 0$; by $\left(f_{3}\right)$, we have

$$
\begin{align*}
\frac{J(t u)}{t^{q}}= & \frac{1}{p t^{q-p}}\|u\|^{p} \\
& -\int\left(\frac{F(x, t u)}{(t u)^{q}} u^{q}-\frac{1}{q} \Gamma(x)|u|^{q}\right) d x \tag{15}
\end{align*}
$$

$$
\longrightarrow-\infty
$$

as $t \longrightarrow+\infty$. Thus, there exists $t>0$ such that $J(t u)<0$.

Now, we define the Nehari manifold by

$$
\begin{equation*}
\mathcal{N}:=\left\{u \in E \backslash\{0\}:\left\langle J^{\prime}(u), u\right\rangle=0\right\} . \tag{16}
\end{equation*}
$$

It is easy to prove that $\mathcal{N}$ is not empty. And we have the following lemma.

Lemma 4. Suppose that $(V),(\Gamma)$ and $\left(f_{1}\right)-\left(f_{4}\right)$ hold. Let $\theta \in$ $[0,1) \cup(1, \infty)$ and $u \in \mathcal{N}$; then $J(\theta u)<J(u)$.

Proof. By $\left(f_{4}\right)$, we have

$$
\begin{align*}
f(x, s) \leq \frac{f(x, t)}{|t|^{q-1}}|s|^{q-1}, \quad s<t  \tag{17}\\
f(s) \geq \frac{f(x, t)}{|t|^{q-1}}|s|^{q-1}, \quad s>t
\end{align*}
$$

Then

$$
\begin{array}{r}
F(x, t)-F(x, \theta t)=\int_{\theta t}^{t} f(x, s) d s \leq \frac{1-\theta^{q}}{q} f(x, t) t  \tag{18}\\
\forall \theta \geq 0, t \in \mathbb{R} .
\end{array}
$$

Let $h(\theta)=p \theta^{q}-q \theta^{p}$; then $h^{\prime}(\theta)=p q\left(\theta^{q-1}-\theta^{p-1}\right)$. By a simple calculation, we have $h^{\prime}(1)=0$ and $h(\theta)>h(1)=p-q$ for all $\theta \in[0,1) \cup(1, \infty)$. Thus,

$$
\begin{equation*}
\frac{\theta^{p}-1}{p}-\frac{\theta^{q}-1}{q}=\frac{p-q-h(\theta)}{p q}<0 . \tag{19}
\end{equation*}
$$

Let $u \in \mathcal{N}$, it follows from (18) and (19) that
$J(\theta u)$

$$
\begin{aligned}
= & J(u)+\left(J(\theta u)-J(u)-\frac{\theta^{q}-1}{q}\left\langle J^{\prime}(u), u\right\rangle\right) \\
= & J(u)+\left[\frac{\theta^{p}-1}{p}-\frac{\theta^{q}-1}{q}\right]\|u\|^{p} \\
& +\int\left[\frac{\theta^{q}-1}{q} f(x, u) u-F(x, \theta u)+F(x, u)\right] d x
\end{aligned}
$$

$$
<J(u)
$$

Lemma 5. Suppose that $(V),(\Gamma)$ and $\left(f_{1}\right)-\left(f_{4}\right)$ hold; let $m:=$ $\inf _{\mathcal{N}} J$; then there exist $\left\{u_{n}\right\} \in E, c_{*} \in\left(\rho_{0}, m\right]$ satisfying

$$
\begin{align*}
J\left(u_{n}\right) & \longrightarrow c_{*}, \\
\left(1+\left\|u_{n}\right\|\right)\left\|J^{\prime}\left(u_{n}\right)\right\| & \longrightarrow 0 \tag{21}
\end{align*}
$$

Proof. By (i) of Lemma 3, there exist $\delta_{0}>0$ and $\rho_{0}>0$ such that

$$
\begin{equation*}
u \in E, \quad\|u\|=\delta_{0} \Longrightarrow J(u) \geq \rho_{0} \tag{22}
\end{equation*}
$$

Choose $v_{k} \in \mathcal{N}$ such that

$$
\begin{equation*}
m \leq J\left(v_{k}\right)<m+\frac{1}{k}, \quad k \in \mathbb{N} . \tag{23}
\end{equation*}
$$

Since $J\left(t v_{k}\right)<0$ for large $t>0$, Mountain Pass Lemma implies that there exists $\left\{u_{k, n}\right\}_{n \in \mathbb{N}} \subset E$ satisfying

$$
\begin{align*}
J\left(u_{k, n}\right) & \longrightarrow c_{k} \\
\left\|J^{\prime}\left(u_{k, n}\right)\right\|\left(1+\left\|u_{k, n}\right\|\right) & \longrightarrow 0 \tag{24}
\end{align*}
$$

$$
k \in \mathbb{N}
$$

where $c_{k} \in\left[\rho_{0}, \sup _{t \geq 0} J\left(t v_{k}\right)\right]$. By Lemma 4 , we have $J\left(v_{k}\right)=$ $\sup _{t \geq 0} J\left(t v_{k}\right)$. Hence, by (23) and (24), we have

$$
\begin{align*}
J\left(u_{k, n}\right) & \longrightarrow c_{k}<m+\frac{1}{k}, \\
\left\|J^{\prime}\left(u_{k, n}\right)\right\|\left(1+\left\|u_{k, n}\right\|\right) & \longrightarrow 0 \tag{25}
\end{align*}
$$

$$
k \in \mathbb{N}
$$

Now, we can choose a sequence $\left\{n_{k}\right\} \subset \mathbb{N}$ such that

$$
\begin{align*}
J\left(u_{k, n_{k}}\right) & <m+\frac{1}{k} \\
\left\|J^{\prime}\left(u_{k, n_{k}}\right)\right\|\left(1+\left\|u_{k, n_{k}}\right\|\right) & <\frac{1}{k} \tag{26}
\end{align*}
$$

$$
k \in \mathbb{N}
$$

Let $u_{k}=u_{k, n_{k}}, k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$
\begin{align*}
J\left(u_{n}\right) & \longrightarrow c_{*} \in\left[\rho_{0}, m\right] \\
\left\|J^{\prime}\left(u_{n}\right)\right\|\left(1+\left\|u_{n}\right\|\right) & \longrightarrow 0 \tag{27}
\end{align*}
$$

## 3. Proof of Theorem 1

Proof of Theorem 1 . In view of Lemma 5, we find a Cerami sequence $\left\{u_{n}\right\}$ satisfying (21). By (18), we have

$$
\begin{equation*}
\frac{1}{q} f\left(x, u_{n}\right) u_{n}-F\left(x, u_{n}\right) \geq 0 \tag{28}
\end{equation*}
$$

$$
\text { for all } x \in \mathbb{R}^{N}, u_{n} \in E \text {. }
$$

Combining (21) and (28), for $n$ big enough, we have

$$
\begin{align*}
c_{*}+1 \geq & J\left(u_{n}\right)-\frac{1}{q}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle \\
= & \left(\frac{1}{p}-\frac{1}{q}\right)\left\|u_{n}\right\|^{p}  \tag{29}\\
& +\int\left[\frac{1}{q} f\left(x, u_{n}\right) u_{n}-F\left(x, u_{n}\right)\right] d x \\
\geq & \left(\frac{1}{p}-\frac{1}{q}\right)\left\|u_{n}\right\|^{p} .
\end{align*}
$$

It follows that $\left\|u_{n}\right\|$ is bounded. Passing to a subsequence, we have $u_{n} \rightharpoonup u_{0}$ in $E$. By Lemma 2, we have $u_{n} \longrightarrow u_{0}$ in $L^{r}\left(\mathbb{R}^{N}\right)$ for $r \in\left[p, p_{\alpha}^{*}\right)$. Then, by (13) and the Hölder inequality, we have

$$
\begin{align*}
& \left|\int\left(f\left(x, u_{n}\right)-f\left(x, u_{0}\right)\right)\left(u_{n}-u_{0}\right) d x\right| \\
& \quad \leq \varepsilon \int\left(\left|u_{n}\right|^{p-1}+\left|u_{0}\right|^{p-1}\right)\left|u_{n}-u_{0}\right| d x \\
& \quad+C_{\varepsilon} \int\left(\left|u_{n}\right|^{r-1}+\left|u_{0}\right|^{r-1}\right)\left|u_{n}-u_{0}\right| d x  \tag{30}\\
& \quad \leq \varepsilon\left(\left\|u_{n}\right\|_{p}^{p-1}+\left\|u_{0}\right\|_{p}^{p-1}\right)\left\|u_{n}-u_{0}\right\|_{p} \\
& \quad+C_{\varepsilon}\left(\left\|u_{n}\right\|_{r}^{r-1}+\left\|u_{0}\right\|_{r}^{r-1}\right)\left\|u_{n}-u_{0}\right\|_{r} \longrightarrow 0
\end{align*}
$$

and

$$
\begin{align*}
& \left|\int \Gamma(x)\left(\left|u_{n}\right|^{q-2} u_{n}-\left|u_{0}\right|^{q-2} u_{0}\right)\left(u_{n}-u_{0}\right) d x\right| \\
& \quad \leq\|\Gamma\|_{\infty} \int\left(\left|u_{n}\right|^{q-1}+\left|u_{0}\right|^{q-1}\right)\left|u_{n}-u_{0}\right| d x  \tag{31}\\
& \quad \leq\left(\left\|u_{n}\right\|_{q}^{q-1}+\left\|u_{0}\right\|_{q}^{q-1}\right)\left\|u_{n}-u_{0}\right\|_{q} \longrightarrow 0 .
\end{align*}
$$

It follows from (30), (31) and Simon inequality $\left(\left(|a|^{p-2} a-\right.\right.$ $\left.\left.|b|^{p-2} b\right)(a-b) \geq\left(1 / 2^{p-2}\right)|a-b|^{p}\right)$ that

$$
\begin{align*}
& \left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}\left(u_{0}\right), u_{n}-u_{0}\right\rangle \\
& \quad=\int \frac{1}{|x-y|^{N+\alpha p}}\left[\left|u_{n}(x)-u_{n}(y)\right|^{p-2}\right. \\
& \cdot \\
& \left.\left.\quad \cdot\left(u_{n}(x)-u_{n}(y)\right)-\mid u_{0}(x)-u_{0}(y)\right)\right]\left[u_{n}(x)-\left.u_{n}(y)\right|^{p-2}\right. \\
& \left.\quad+u_{0}(y)\right] d x d y+\int V(x)\left[\left|u_{n}\right|^{p-2} u_{n}(x)\right. \\
& \left.\quad-\left|u_{0}\right|^{p-2} u\right]\left(u_{n}-u_{0}\right) d x-\int\left(f\left(x, u_{n}\right)\right.  \tag{32}\\
& \left.\quad-f\left(x, u_{0}\right)\right)\left(u_{n}-u_{0}\right) d x+\int \Gamma(x)\left(\left|u_{n}\right|^{q-2} u_{n}\right. \\
& \left.\quad-\left|u_{0}\right|^{q-2} u_{0}\right)\left(u_{n}-u_{0}\right) d x \geq \frac{1}{2^{p-2}} \\
& \quad \cdot \int \frac{\left|\left(u_{n}(x)-u_{n}(y)\right)-\left(u_{0}(x)-u_{0}(y)\right)\right|^{p}}{|x-y|^{N+\alpha p}} d x d y \\
& \quad+\frac{1}{2^{p-2}} \int V(x)\left|u_{n}-u_{0}\right|^{p} d x+o_{n}(1)=\frac{1}{2^{p-2}} \| u_{n} \\
& \quad-u_{0} \|^{p}+o_{n}(1) .
\end{align*}
$$

On the other hand, by $\left\langle J^{\prime}\left(u_{n}\right), u_{n}-u_{0}\right\rangle \longrightarrow 0$ and $u_{n} \rightharpoonup u_{0}$, we have

$$
\begin{equation*}
\left\langle J^{\prime}\left(u_{n}\right)-J^{\prime}\left(u_{0}\right), u_{n}-u_{0}\right\rangle \longrightarrow 0 \tag{33}
\end{equation*}
$$

Combining (32) and (33), we have $u_{n} \longrightarrow u_{0}$ in $E$. Then, by $J \in C^{1}(E, \mathbb{R})$, we have $J^{\prime}\left(u_{0}\right)=0$. By (28), Lemma 5 , and Fatou's lemma, we have

$$
\begin{align*}
m \geq & c_{*}=\lim _{n \rightarrow \infty}\left[J\left(u_{n}\right)-\frac{1}{q}\left\langle J^{\prime}\left(u_{n}\right), u_{n}\right\rangle\right] \\
= & \left(\frac{1}{p}-\frac{1}{q}\right) \lim _{n \rightarrow \infty}\left\|u_{n}\right\|^{p} \\
& +\lim _{n \rightarrow \infty} \int\left[\frac{1}{q} f\left(x, u_{n}\right) u_{n}-F\left(x, u_{n}\right)\right] d x  \tag{34}\\
\geq & \left(\frac{1}{p}-\frac{1}{q}\right)\left\|u_{0}\right\|^{p} \\
& +\int\left[\frac{1}{q} f\left(x, u_{0}\right) u_{0}-F\left(x, u_{0}\right)\right] d x \\
= & J\left(u_{0}\right)-\frac{1}{q}\left\langle J^{\prime}\left(u_{0}\right), u_{0}\right\rangle=J\left(u_{0}\right) .
\end{align*}
$$

This shows that $J\left(u_{0}\right) \leq m$ and so $J\left(u_{0}\right)=m=\inf _{\mathcal{N}} J>$ 0.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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# Research Article 

# $C^{*}$-Algebra-Valued G-Metric Spaces and Related Fixed-Point Theorems 

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#### Abstract

We introduce the notion of the $C^{*}$-algebra-valued $G$-metric space. The existence and uniqueness of some fixed-point theorems for self-mappings with contractive or expansive conditions on complete $C^{*}$-algebra-valued $G$-metric spaces are proved. As an application, we prove the existence and uniqueness of the solution of a type of differential equations.


## 1. Introduction

As is known to all, the proverbial fixed-point theorem of Banach has been widely used in many branches of mathematics and physics. There are a large number of generalizations for such a theorem. In general, the theorem has been extended in two directions. On the one hand, the usual contractive condition is replaced with weakly contractive conditions [1-7]. On the other hand, the action spaces are replaced with different types of metric spaces. Particularly, in 2006, Mustafa and Sims [8] introduced the concept of generalized metric spaces ( $G$-metric spaces). Since then, many scholars studied fixed-point theory in $G$-metric spaces and many meaningful results are obtained.

Let us recall the basic definitions and conclusions on $G$ metric spaces. Details can be seen in [8-20].

Definition 1 (see [8]). Let $X$ be a nonempty set. Suppose that $G: X \times X \times X \longrightarrow \mathbb{R}^{+}$is a mapping satisfying
(1) $G(x, y, z)=0$ if $x=y=z$;
(2) $0<G(x, x, y)$ for all $x, y \in X$ with $x \neq y$;
(3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $z \neq y$;
(4) $G(x, y, z)=G(x, z, y)=G(y, z, x)=\cdots$ (symmetry in all three variables);
(5) $G(x, y, z) \leq G(x, a, a)+G(a, y, z)$ for all $x, y, z, a \in$ $X$ (rectangle inequality).

Then $G$ is called a generalized metric, or a $G$-metric on $X$. The pair $(X, G)$ is called a $G$-metric space.

Definition 2 (see [8]). Let $(X, G)$ be a $G$-metric space, $\left\{x_{n}\right\} \subseteq$ $X$. If there exists $x \in X$, such that $\lim _{n, m \rightarrow \infty} G\left(x, x_{n}, x_{m}\right)=0$, we say that the sequence $\left\{x_{n}\right\}$ is called $G$-convergent to $x$. If for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that, for all $m, n, l>N$,

$$
\begin{equation*}
G\left(x_{n}, x_{m}, x_{l}\right)<\varepsilon, \tag{1}
\end{equation*}
$$

$\left\{x_{n}\right\}$ is called a $G$-Cauchy sequence.
Notice that, in a $G$-metric space $(X, G)$, the following statements are equivalent:
(1) $\left\{x_{n}\right\}$ is $G$-convergent to $x$;
(2) $G\left(x_{n}, x_{n}, x\right) \longrightarrow 0$ as $n \longrightarrow \infty$;
(3) $G\left(x_{n}, x, x\right) \longrightarrow 0$ as $n \longrightarrow \infty$.

As we have known that $C^{*}$-algebra, which was first proposed for its use in quantum mechanics to model algebras of physical observable, is an important research field of modern mathematics [21-29]. In 1947, I. Segal [28] introduced the term " $C^{*}$-algebra" to describe a "uniformly
closed, self-adjoint algebra of bounded operators on a Hilbert space".

Throughout this paper, $\mathscr{A}$ will denote a unital $C^{*}$-algebra with a unit $I$; namely, $\mathscr{A}$ is a unital Banach algebra with an involution $*$ such that $\left\|A^{*} A\right\|=\|A\|^{2}(A \in \mathscr{A})$. Let $\mathscr{H}$ be a Hilbert space and $B(\mathscr{H})$ the set of all bounded linear operators on $\mathscr{H}$, then $B(\mathscr{H})$ is a $C^{*}$-algebra with the operator norm. Let $\mathscr{A}_{\text {sa }}$ be the set of all self-adjoint elements in $\mathscr{A}$, and define the spectrum of $A \in \mathscr{A}$ to be the set $\sigma(A)=\{\lambda \in$ $\mathbb{C} \mid \lambda I-A$ is not invertible $\}$. An element $A \in \mathscr{A}$ is positive (denoted by $A \geq \theta$ ) if $A \in \mathscr{A}_{s a}$ and $\sigma(A) \subseteq \mathbb{R}_{+}$, set $\mathscr{A}_{+}=$ $\{A \in \mathscr{A} \mid A \geq \theta\}$, then $\mathscr{A}_{+}=\left\{A^{*} A \mid A \in \mathscr{A}\right\}$ [27]. Using the positive element, one can define a partial ordering " $\leq$ " on $\mathscr{A}_{s a}$ as follows: $A \leq B$ if and only if $B-A \geq \theta$. It is clear that if $A, B \in \mathscr{A}_{s a}$ and $C \in \mathscr{A}$, then $A \leq B \Longrightarrow C^{*} A C \leq C^{*} B C$, and that if $A, B \in \mathscr{A}_{+}$are invertible, then $A \leq B \Longrightarrow \theta \leq$ $B^{-1} \leq A^{-1}$.

Using the partial ordering " $\leq$ " on $\mathscr{A}_{s a}$, Ma introduced the notion of $C^{*}$-algebra-valued metric spaces and gave the fixedpoint theory for contractive or expansion mapping on such a space $[30,31]$. Let us recall the definition first.

Definition 3 (see [30]). Let $X$ be a nonempty set. If $d: X \times$ $X \longrightarrow \mathscr{A}_{+}$is a mapping satisfying
(1) $d(x, y)=\theta \Longleftrightarrow x=y$;
(2) $d(x, y)=d(y, x)$ for all $x, y \in X$;
(3) $d(x, y) \leq d(x, z)+d(y, z)$ for all $x, y, z \in X$,
then $d$ is called a $C^{*}$-algebra-valued metric on $X .(X, \mathscr{A}, d)$ is called a $C^{*}$-algebra-valued metric space.

Definition 4 (see [30]). Let $(X, \mathscr{A}, d)$ be a $C^{*}$-algebra-valued metric space, $\left\{x_{n}\right\} \subset X, x \in X$. If for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that for all $n>N,\left\|d\left(x_{n}, x\right)\right\|<\varepsilon$, then we say this $\left\{x_{n}\right\}$ converges to $x$. We denote it by $\lim _{n \rightarrow \infty} x_{n}=x$.

If for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that, for all $m, n>$ $N,\left\|d\left(x_{m}, x_{n}\right)\right\|<\varepsilon$, then we say $\left\{x_{n}\right\}$ is a Cauchy sequence with respect to $\mathscr{A}$. We say that $(X, \mathscr{A}, d)$ is a complete $C^{*}$ -algebra-valued metric space if every Cauchy sequence with respect to $\mathscr{A}$ is convergent.

Definition 5 (see [30]). Let $(X, \mathscr{A}, d)$ be a $C^{*}$-algebra-valued metric space. We call a mapping $T: X \longrightarrow X$ is a contractive mapping on $(X, \mathscr{A}, d)$ if there exists $A \in \mathscr{A}$ with $\|A\|<1$ such that $d(T x, T y) \leq A^{*} d(x, y) A, \forall x, y \in X$.

Theorem 6 (see [30]). If $(X, \mathscr{A}, d)$ is a complete $C^{*}$-algebravalued metric space and $T$ is a contractive mapping, there exists a unique fixed point in $X$.

In this paper, we will define $C^{*}$-algebra-valued $G$-metric spaces and prove some fixed point theorems on such spaces. We also provide an application of the theory for a type of differential equations.

## 2. Main Results

In this section, we first give the definition of a $C^{*}$-algebravalued $G$-metric spaces.

Definition 7. Let $X$ be a nonempty set and $S_{3}$ be the permutation group on $\{1,2,3\}$. If $G: X \times X \times X \longrightarrow \mathscr{A}_{+}$is a mapping satisfying
(1) $G\left(x_{1}, x_{2}, x_{3}\right)=\theta \Longleftrightarrow x_{1}=x_{2}=x_{3}$;
(2) $G\left(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right)=G\left(x_{1}, x_{2}, x_{3}\right)$ for all $x_{1}, x_{2}, x_{3} \in$ $X, \sigma \in S_{3}$;
(3) $G\left(x_{1}, x_{1}, x_{2}\right) \leq G\left(x_{1}, x_{2}, x_{3}\right)$ for all $x_{1}, x_{2}, x_{3} \in X$ with $x_{2} \neq x_{3}$;
(4) $G\left(x_{1}, x_{2}, x_{3}\right) \leq G\left(x_{1}, a, a\right)+G\left(a, x_{2}, x_{3}\right)$ for all $x_{1}, x_{2}$, $x_{3}, a \in X$,
$G$ is called a $C^{*}$-algebra-valued $G$-metric on $X$, and $(X, \mathscr{A}, G)$ is called a $C^{*}$-algebra-valued $G$-metric space.

Example 8. Let $X=\{x, y\}$. Notice that $M_{2}(\mathbb{C})$ of $2 \times 2$ matrices with entries in $\mathbb{C}$ is identified with $B\left(\mathbb{C}^{2}\right)$. This is a $C^{*}$-algebra. Let

$$
\begin{align*}
& G(x, x, x)=G(y, y, y)=\theta \\
& G(x, x, y)=I  \tag{2}\\
& G(x, y, y)=2 I
\end{align*}
$$

and extend $G$ to all of $X \times X \times X$ by symmetry in the variables. Then $G$ is a $C^{*}$-algebra-valued $G$-metric on $X$, and ( $X$, $\left.M_{2}(\mathbb{C}), G\right)$ is a $C^{*}$-algebra-valued $G$-metric space.

Definition 9. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$-metric space, $\left\{x_{n}\right\} \subseteq X, x \in X$. If for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that for all $m, n>N,\left\|G\left(x_{m}, x_{n}, x\right)\right\|<\varepsilon$, then we say $\left\{x_{n}\right\}$ is $G$-convergent to $x$, and denote by $x_{n} \xrightarrow{G} x,(n \longrightarrow \infty)$.

Proposition 10. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$ metric space, $\left\{x_{n}\right\} \subseteq X, x \in X$. The following statements are equivalent:
(1) $x_{n} \xrightarrow{G} x(n \longrightarrow \infty)$.
(2) $G\left(x_{n}, x_{n}, x\right) \longrightarrow \theta$, as $n \longrightarrow \infty$.
(3) $G\left(x_{n}, x, x\right) \longrightarrow \theta$, as $n \longrightarrow \infty$.

Proof. (1) $\Longrightarrow$ (2) If $x_{n} \xrightarrow{G} x$, that is, for all $\varepsilon>0, \exists N \in \mathbb{N}$, such that, for all, $m, n>N,\left\|G\left(x_{m}, x_{n}, x\right)\right\|<\varepsilon$, especially, $\left\|G\left(x_{n}, x_{n}, x\right)\right\|<\varepsilon$. Hence $G\left(x_{n}, x_{n}, x\right) \longrightarrow \theta$, as $n \longrightarrow \infty$.
(2) $\Longrightarrow$ (3) If $\forall \varepsilon>0, \exists N \in \mathbb{N}$, such that, for all $n>$ $N,\left\|G\left(x_{n}, x_{n}, x\right)\right\|<\varepsilon / 2$, then when $n>N$,

$$
\begin{align*}
\left\|G\left(x_{n}, x, x\right)\right\| & =\left\|G\left(x, x_{n}, x\right)\right\| \\
& \leq\left\|G\left(x, x_{n}, x_{n}\right)\right\|+\left\|G\left(x_{n}, x_{n}, x\right)\right\|<\varepsilon, \tag{3}
\end{align*}
$$

that is, $G\left(x_{n}, x, x\right) \longrightarrow \theta$, as $n \longrightarrow \infty$.
(3) $\Longrightarrow$ (1) If $G\left(x_{n}, x, x\right) \longrightarrow \theta$, as $n \longrightarrow \infty$, then, for any $\varepsilon>0, \exists N_{1} \in \mathbb{N}$, such that for all $n>N_{1},\left\|G\left(x_{n}, x, x\right)\right\|<\varepsilon / 2$; $\exists N_{2} \in \mathbb{N}$, such that, for all $m>N_{2},\left\|G\left(x_{m}, x, x\right)\right\|<\varepsilon / 2$. Let $N=N_{1}+N_{2}$, for $m, n>N$,

$$
\begin{equation*}
\left\|G\left(x_{m}, x_{n}, x\right)\right\| \leq\left\|G\left(x_{m}, x, x\right)\right\|+\left\|G\left(x, x_{n}, x\right)\right\|<\varepsilon \tag{4}
\end{equation*}
$$

that is, $x_{n} \xrightarrow{G} x(n \longrightarrow \infty)$.

Definition 11. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$-metric space, $\left\{x_{n}\right\} \subseteq X$. If for any $\varepsilon>0$, there is an $N \in \mathbb{N}$ such that for all $m, n, l>N,\left\|G\left(x_{m}, x_{n}, x_{l}\right)\right\|<\varepsilon$, then the sequence $\left\{x_{n}\right\}$ is called a $G$-Cauchy sequence. If any $G$-Cauchy sequence in $(X, \mathscr{A}, G)$ is $G$-convergent, then $(X, \mathscr{A}, G)$ is called a complete $C^{*}$-algebra-valued $G$-metric space.

Proposition 12. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$ metric space, $\left\{x_{n}\right\} \subseteq X$. Then $\left\{x_{n}\right\}$ is a $G$-Cauchy sequence if and only if for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that, for all $m, n>N,\left\|G\left(x_{m}, x_{n}, x_{n}\right)\right\|<\varepsilon$.

Proof. It suffices to show the necessity. For any $\varepsilon>0, \exists N_{1} \in$ $\mathbb{N}$, such that for all $m, n>N_{1},\left\|G\left(x_{m}, x_{n}, x_{n}\right)\right\|<\varepsilon / 2 ; \exists N_{2} \in$ $\mathbb{N}$, such that for all $n, l>N_{2},\left\|G\left(x_{l}, x_{n}, x_{n}\right)\right\|<\varepsilon / 2$. So for the above $\varepsilon$, let $N=N_{1}+N_{2}$, when $m, n, l>N$,

$$
\begin{align*}
\left\|G\left(x_{m}, x_{n}, x_{l}\right)\right\| & \leq\left\|G\left(x_{m}, x_{n}, x_{n}\right)\right\|+\left\|G\left(x_{n}, x_{n}, x_{l}\right)\right\|  \tag{5}\\
& <\varepsilon
\end{align*}
$$

that is, $\left\{x_{n}\right\}$ is a $G$-Cauchy sequence.
Example 13. Set $X=\mathbb{C}, \mathscr{A}=M_{3}(\mathbb{C})$. Let $\alpha, \beta>0$; set

$$
\begin{align*}
& G(x, y, z) \\
& \quad=\left(\begin{array}{ccc}
g(x, y, z) & 0 & 0 \\
0 & \alpha g(x, y, z) & 0 \\
0 & 0 & \beta g(x, y, z)
\end{array}\right) \tag{6}
\end{align*}
$$

where $g(x, y, z)=|x-y|+|y-z|+|z-x|$, then $G$ is a $C^{*}$ -algebra-valued $G$-metric on $X$, and $(X, \mathscr{A}, G)$ is a complete $C^{*}$-algebra-valued $G$-metric space.

It is easy to see that $(X, \mathscr{A}, G)$ is a $C^{*}$-algebra-valued $G$ metric space. We only need to prove the completeness. Let $\left\{x_{n}\right\} \subseteq(X, \mathscr{A}, G)$ be a $G$-Cauchy sequence. Then for any $\varepsilon>0$, there is an $N \in \mathbb{N}$, such that for all $m, n>N$,

$$
\begin{align*}
& \left\|G\left(x_{m}, x_{n}, x_{n}\right)\right\|=\max \left\{g\left(x_{m}, x_{n}, x_{n}\right), \alpha g\left(x_{m}, x_{n}, x_{n}\right),\right.  \tag{7}\\
& \left.\quad \beta g\left(x_{m}, x_{n}, x_{n}\right)\right\}<\varepsilon .
\end{align*}
$$

So $g\left(x_{m}, x_{n}, x_{n}\right)=2\left|x_{m}-x_{n}\right|<\varepsilon$. Since $X$ is complete, there exists $x \in X$, such that $x_{n} \longrightarrow x$. Hence there is $N_{0} \in \mathbb{N}$ such that, for any $n>N_{0},\left|x_{n}-x\right|<\varepsilon / 2$. It follows that

$$
\begin{align*}
& \left\|G\left(x_{n}, x, x\right)\right\| \\
& \quad=\max \left\{2\left|x-x_{n}\right|, 2 \alpha\left|x-x_{n}\right|, 2 \beta\left|x-x_{n}\right|\right\}  \tag{8}\\
& \quad=2 \max \{1, \alpha, \beta\}\left|x_{n}-x\right|<\max \{1, \alpha, \beta\} \varepsilon .
\end{align*}
$$

Therefore, $x_{n} \xrightarrow{G} x$, and $(X, \mathscr{A}, G)$ is complete.
Example 14. Suppose $\Omega$ is a compact Hausdorff space and $\mu$ is a positive regular Borel measure on $\Omega$. Let $X=L^{\infty}(\Omega, \mu)$, the set of all essentially bounded complex-valued measurable functions on $\Omega$, then $X$ is a Banach space with the essential supremum norm $\|f\|_{\infty}$. Set $\mathscr{H}=L^{2}(\Omega, \mu)=\{f: \Omega \longrightarrow \mathbb{C} \mid$
$\left.\int_{\Omega}|f(\omega)|^{2} d \mu(\omega)<\infty\right\} ; \mathscr{H}$ is a Hilbert space with the innerproduct

$$
\begin{equation*}
\langle f, g\rangle=\int_{\Omega} f(\omega) \overline{g(\omega)} d \mu(\omega) \tag{9}
\end{equation*}
$$

For $f \in L^{\infty}(\Omega, \mu)$, define

$$
\begin{align*}
M_{f}: \mathscr{H} & \longrightarrow \mathscr{H}  \tag{10}\\
\varphi & \longmapsto f \varphi .
\end{align*}
$$

Then $M_{f}$ is bounded and moreover $\left\|M_{f}\right\|=\|f\|_{\infty}$. Let $G$ : $X \times X \times X \longrightarrow B(\mathscr{H})$ by

$$
\begin{equation*}
G(f, g, h)=M_{|f-g|+|g-h|+|h-f|} . \tag{11}
\end{equation*}
$$

Then $G$ is a $C^{*}$-algebra-valued $G$-metric and $(X, B(\mathscr{H}), G)$ is a complete $C^{*}$-algebra-valued $G$-metric space. We omit its proof and leave it to readers.

Next, we define the contractive mapping on $C^{*}$-algebravalued $G$-metric space and prove the fixed point theorem for contractive mappings.

Definition 15. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$-metric space and $T: X \longrightarrow X$ is a mapping. If there exists $A \in \mathscr{A}$ with $\|A\|<1$ such that

$$
\begin{equation*}
G(T x, T y, T z) \leq A^{*} G(x, y, z) A, \quad \forall x, y, z \in X \tag{12}
\end{equation*}
$$

then $T$ is called a contractive mapping on $(X, \mathscr{A}, G)$.
Theorem 16. Let $(X, \mathscr{A}, G)$ be a complete $C^{*}$-algebra-valued $G$-metric space. If $T: X \longrightarrow X$ is a contractive mapping on $(X$, $\mathscr{A}, G)$, then there is a unique fixed point of $T$ on $X$.

Proof. Let $d(x, y)=G(x, x, y)+G(x, y, y)$.
We first show that $d$ is a $C^{*}$-algebra-valued metric on $X$. It suffices to show that

$$
\begin{equation*}
d(x, y) \leq d(x, z)+d(y, z), \quad \forall x, y, z \in X \tag{13}
\end{equation*}
$$

That is,

$$
\begin{align*}
& G(x, x, y)+G(x, y, y) \\
& \quad \leq G(x, x, z)+G(x, z, z)+G(y, y, z)  \tag{14}\\
& \quad+G(y, z, z) .
\end{align*}
$$

Since

$$
\begin{align*}
G(x, & x, y)+G(x, y, y)=G(y, x, x)+G(x, y, y) \\
\leq & G(y, z, z)+G(z, x, x)+G(x, z, z) \\
& +G(z, y, y)  \tag{15}\\
= & G(x, x, z)+G(x, z, z)+G(y, z, z) \\
& +G(y, y, z)
\end{align*}
$$

we have $d(x, y) \leq d(x, z)+d(y, z), \forall x, y, z \in X$.

Next, we show $(X, \mathscr{A}, d)$ is a complete $C^{*}$-algebra-valued metric space. Let $\left\{x_{n}\right\} \subseteq X$ be a Cauchy sequence with respect to $\mathscr{A}$. Then for any $\varepsilon>0$, there is an $N \in \mathbb{N}$ such that for all $m, n>N,\left\|d\left(x_{n}, x_{m}\right)\right\|<\varepsilon$, that is,

$$
\begin{equation*}
\left\|G\left(x_{n}, x_{m}, x_{m}\right)+G\left(x_{n}, x_{n}, x_{m}\right)\right\|<\varepsilon \tag{16}
\end{equation*}
$$

Since $\theta \leq G\left(x_{n}, x_{m}, x_{m}\right) \leq G\left(x_{n}, x_{m}, x_{m}\right)+G\left(x_{n}, x_{n}, x_{m}\right)$,

$$
\begin{equation*}
\left\|G\left(x_{n}, x_{m}, x_{m}\right)\right\| \leq\left\|G\left(x_{n}, x_{m}, x_{m}\right)+G\left(x_{n}, x_{n}, x_{m}\right)\right\| \tag{17}
\end{equation*}
$$

$$
<\varepsilon .
$$

So $\left\{x_{n}\right\} \subseteq X$ is a $G$-Cauchy sequence. By the completeness of $(X, \mathscr{A}, G)$, there exists an $x \in X$, such that $x_{n} \xrightarrow{G} x(n \longrightarrow \infty)$. That is, for any $\varepsilon>0$, there is an $N_{1} \in \mathbb{N}$ such that, for all $n>N_{1},\left\|G\left(x_{n}, x, x\right)\right\|<\varepsilon / 2$; there is an $N_{2} \in \mathbb{N}$ such that for all $m>N_{2},\left\|G\left(x_{m}, x_{m}, x\right)\right\|<\varepsilon / 2$. Let $N=N_{1}+N_{2}$, then for all $n>N$, we have

$$
\begin{align*}
& \left\|G\left(x_{n}, x, x\right)\right\|<\frac{\varepsilon}{2}, \\
& \left\|G\left(x_{n}, x_{n}, x\right)\right\|<\frac{\varepsilon}{2} . \tag{18}
\end{align*}
$$

Therefore,

$$
\begin{align*}
\left\|d\left(x_{n}, x\right)\right\| & =\left\|G\left(x_{n}, x, x\right)+G\left(x_{n}, x_{n}, x\right)\right\|  \tag{19}\\
& \leq\left\|G\left(x_{n}, x, x\right)\right\|+\left\|G\left(x_{n}, x_{n}, x\right)\right\|<\varepsilon
\end{align*}
$$

Hence $\lim _{n \rightarrow \infty} x_{n}=x$, and $(X, \mathscr{A}, d)$ is complete.
Moreover, $T$ is a contractive mapping on ( $X, \mathscr{A}, d$ ). In fact,

$$
\begin{align*}
d(T x, T y) & =G(T x, T x, T y)+G(T x, T y, T y) \\
& \leq A^{*} G(x, x, y) A+A^{*} G(x, y, y) A \\
& =A^{*}(G(x, x, y)+G(x, y, y)) A  \tag{20}\\
& =A^{*} d(x, y) A .
\end{align*}
$$

It follows from Theorem 6 that there is an $x \in X$ such that $T x=x$.

Finally, we show the uniqueness of this fixed point. Let $y$ is another fixed point of $T$. If $x \neq y$, then

$$
\begin{equation*}
\theta \leqq G(x, y, y)=G(T x, T y, T y) \leq A^{*} G(x, y, y) A . \tag{21}
\end{equation*}
$$

Since $\|G(x, y, y)\| \neq 0$,

$$
\begin{align*}
\|G(x, y, y)\| & \leq\left\|A^{*} G(x, y, y) A\right\| \\
& \leq\|A\|^{2}\|G(x, y, y)\|<\|G(x, y, y)\| . \tag{22}
\end{align*}
$$

This is a contradiction. So $x=y$.
Remark 17. (1) In the theorem above, the completeness of $X$ is essential. For example, let $X=(0,1) \subseteq \mathbb{R}$ and $G: X \times X \times$ $X \longrightarrow B(\mathscr{H})$ satisfy $G(x, y, z)=(|x-y|+|y-z|+|z-x|) I$,
then $(X, B(\mathscr{H}), G)$ is a $C^{*}$-algebra-valued $G$-metric space, but $(X, B(\mathscr{H}), G)$ is not complete. Considering the mapping

$$
\begin{align*}
T: X & \longrightarrow X \\
x & \longrightarrow \frac{1}{2} x \tag{23}
\end{align*}
$$

$T$ is a contractive mapping, but $T$ has no fixed point.
(2) In the definition of contractive mapping, the element $A \in \mathscr{A}$ does not depend on the choice of $x, y, z$. If $A$ depends on $x, y, z$, then $T$ may not have a fixed point.

Indeed, let $X=\mathbb{R}$ and $G: X \times X \times X \longrightarrow B(\mathscr{H})$ be defined by $G(x, y, z)=(|x-y|+|y-z|+|z-x|) I$. Then $(X, B(\mathscr{H}), G)$ is a complete $C^{*}$-algebra-valued $G$-metric space. Let $f(x)=\pi /$ $2+x-\arctan x$. If $x<y<z$, then

$$
\begin{align*}
G & (f(x), f(y), f(z)) \\
\quad & (|x-y-\arctan x+\arctan y| \\
& +|y-z-\arctan y+\arctan z| \\
& +|z-x-\arctan z+\arctan x|) I \\
& =\left(\left|x-y+\frac{y-x}{1+\xi^{2}}\right|+\left|y-z+\frac{z-y}{1+\eta^{2}}\right|\right.  \tag{24}\\
& \left.+\left|z-x+\frac{x-z}{1+\zeta^{2}}\right|\right) I=\left(\frac{\xi^{2}}{1+\xi^{2}}|x-y|\right. \\
& \left.+\frac{\eta^{2}}{1+\eta^{2}}|y-z|+\frac{\zeta^{2}}{1+\zeta^{2}}|z-x|\right) I \leq \alpha(|x-y| \\
& +|y-z|+|z-x|) I=\sqrt{\alpha} I G(x, y, z) \sqrt{\alpha} I,
\end{align*}
$$

where $\xi \in(x, y), \eta \in(y, z), \zeta \in(x, z)$, and $\alpha=\max \left\{\xi^{2} /(1+\right.$ $\left.\left.\xi^{2}\right), \eta^{2} /\left(1+\eta^{2}\right), \zeta^{2} /\left(1+\zeta^{2}\right)\right\}<1$ depends on $x, y, z$, but $f$ has no fixed point.
(3) When $\|A\|=1, T$ may not have a unique fixed point.

Consider $X=l^{\infty}(\mathbb{N})=\left\{\left(x_{1}, x_{2}, \cdots\right) \mid x_{n} \in \mathbb{C}, n \in \mathbb{N}\right.$ and $\left.\sup _{n}\left|x_{n}\right|<+\infty\right\}$, for $x=\left(x_{1}, x_{2}, \cdots\right) \in l^{\infty}(\mathbb{N})$, and let $\|x\|_{\infty}=$ $\sup _{n}\left|x_{n}\right|$. Define $G: X \times X \times X \longrightarrow M_{2}(\mathbb{C})$ by

$$
\begin{equation*}
G(x, y, z)=\left(\|x-y\|_{\infty}+\|y-z\|_{\infty}+\|z-x\|_{\infty}\right) I . \tag{25}
\end{equation*}
$$

Then $\left(X, M_{2}(\mathbb{C}), G\right)$ is a complete $C^{*}$-algebra-valued $G$ metric space. Let

$$
\begin{equation*}
T\left(x_{1}, x_{2}, x_{3}, \cdots\right)=\left(1+x_{2}, \frac{1}{2}+x_{3}, \frac{1}{2^{2}}+x_{4}, \cdots\right) \tag{26}
\end{equation*}
$$

Then

$$
\begin{equation*}
G(T x, T y, T z) \leq G(x, y, z) \tag{27}
\end{equation*}
$$

But for each $\alpha \in \mathbb{C}, x=\left(\alpha, \alpha-1, \alpha-3 / 2, \cdots, \alpha-\sum_{i=0}^{n-1}\left(1 / 2^{i}\right)\right.$, $\cdots$ ) is a fixed point, which means the fixed point is not unique.

What follows is the definition of the expansion mapping on $C^{*}$-algebra-valued $G$-metric space and the fixed-point theorem for expansion mappings.

Definition 18. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$-metric space. If $T: X \longrightarrow X$ satisfies the condition

$$
\begin{equation*}
G(T x, T y, T z) \geq A^{*} G(x, y, z) A \tag{28}
\end{equation*}
$$

where $A \in \mathscr{A}$ is invertible and $\left\|A^{-1}\right\|<1$, we call $T$ an expansion mapping on $(X, \mathscr{A}, G)$.

Theorem 19. Let $(X, \mathscr{A}, G)$ be a complete $C^{*}$-algebra-valued $G$-metric space and $T: X \longrightarrow X$ a expansion mapping on $(X$, $\mathscr{A}, G)$. If $T$ is surjective, then there is a unique fixed point for T.

Proof. First we show $T$ is injective. Indeed, if $T x=T y$, then

$$
\begin{equation*}
\theta=G(T x, T y, T y) \geq A^{*} G(x, y, y) A \tag{29}
\end{equation*}
$$

Since $A^{*} G(x, y, y) A \in \mathscr{A}_{+}, A^{*} G(x, y, y) A=\theta$, and since $A$ is invertible, $G(x, y, y)=\theta$. Therefore $x=y$.

Next we show that $T$ has a unique fixed point in $X$. In fact, $T$ is bijective and so $T$ is invertible. Since $\forall x, y, z \in X$,

$$
\begin{equation*}
G(T x, T y, T z) \geq A^{*} G(x, y, z) A \tag{30}
\end{equation*}
$$

Replace $x, y, z$ by $T^{-1} x, T^{-1} y, T^{-1} z$, respectively, we get

$$
\begin{equation*}
G(x, y, z) \geq A^{*} G\left(T^{-1} x, T^{-1} y, T^{-1} z\right) A \tag{31}
\end{equation*}
$$

Hence

$$
\begin{equation*}
G\left(T^{-1} x, T^{-1} y, T^{-1} z\right) \leq\left(A^{*}\right)^{-1} G(x, y, z) A^{-1} \tag{32}
\end{equation*}
$$

Thus

$$
\begin{equation*}
G\left(T^{-1} x, T^{-1} y, T^{-1} z\right) \leq\left(A^{-1}\right)^{*} G(x, y, z) A^{-1} \tag{33}
\end{equation*}
$$

By Theorem 16, there is a unique $x \in X$, such that $T^{-1} x=x$, and therefore there is a unique $x \in X$, such that $T x=x$.

The following lemma is necessary for another fixed-point theorem, for detail, see [27].

Lemma 20. Set $\mathscr{A}^{\prime}=\{A \in \mathscr{A}: A B=B A, \forall B \in \mathscr{A}\}$.
(1) If $A, B \in \mathscr{A}_{+}$and $A B=B A$, then $A B \in \mathscr{A}_{+}$.
(2) If $A \in \mathscr{A}^{\prime}, B, C \in \mathscr{A}$ with $B \geq C \geq \theta$ and $I-A \in \mathscr{A}^{\prime}$ is invertible, then $(I-A)^{-1} B \geq(I-A)^{-1} C$.

Theorem 21. Let $(X, \mathscr{A}, G)$ be a complete $C^{*}$-algebra-valued $G$-metric space, $T: X \longrightarrow X$. If there exists an $A \in\left(\mathscr{A}^{\prime}\right)_{+},\|A\|$ $<1 / 6$ such that for any $x, y \in X$,

$$
\begin{align*}
& G(T x, T y, T z) \\
& \quad \leq A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z)) \tag{34}
\end{align*}
$$

or

$$
\begin{align*}
& G(T x, T y, T z) \\
& \leq A(G(T x, T y, z)+G(x, T y, T z)+G(T x, y, T z)) \tag{35}
\end{align*}
$$

then $T$ has a unique fixed point in $X$.

Proof. Without loss of generality, we can assume $A \neq \theta$.
(1) Suppose that $T$ satisfies

$$
\begin{align*}
& G(T x, T y, T z) \\
& \begin{aligned}
& \leq A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z)) \\
& \forall x, y, z \in X
\end{aligned} \tag{36}
\end{align*}
$$

Since $A \in\left(\mathscr{A}^{\prime}\right)_{+}$,

$$
\begin{equation*}
A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z)) \geq \theta \tag{37}
\end{equation*}
$$

For $x_{0} \in X$, set $x_{n+1}=T x_{n}=T^{n+1} x_{0}, n=1,2, \cdots$, and $B=G\left(x_{1}, x_{0}, x_{0}\right)$. Then

$$
\begin{align*}
G & \left(x_{n+1}, x_{n}, x_{n}\right)=G\left(T x_{n}, T x_{n-1}, T x_{n-1}\right) \\
& \leq A\left(G\left(T x_{n}, x_{n-1}, x_{n-1}\right)+G\left(x_{n}, T x_{n-1}, x_{n-1}\right)\right. \\
& \left.+G\left(x_{n}, x_{n-1}, T x_{n-1}\right)\right)=A G\left(x_{n+1}, x_{n-1}, x_{n-1}\right) \\
& +2 A G\left(x_{n}, x_{n}, x_{n-1}\right) \leq A\left(G\left(x_{n+1}, x_{n}, x_{n}\right)\right.  \tag{38}\\
& \left.+G\left(x_{n}, x_{n-1}, x_{n-1}\right)\right)+2 A\left(G\left(x_{n}, x_{n-1}, x_{n-1}\right)\right. \\
& \left.+G\left(x_{n-1}, x_{n}, x_{n-1}\right)\right)=A G\left(x_{n+1}, x_{n}, x_{n}\right) \\
& +5 A G\left(x_{n}, x_{n-1}, x_{n-1}\right)
\end{align*}
$$

That is,

$$
\begin{equation*}
(I-A) G\left(x_{n+1}, x_{n}, x_{n}\right) \leq 5 A G\left(x_{n}, x_{n-1}, x_{n-1}\right) \tag{39}
\end{equation*}
$$

Since $A \in\left(\mathscr{A}^{\prime}\right)_{+}$with $\|A\|<1 / 6$, we have $(I-A)^{-1} \in\left(\mathscr{A}^{\prime}\right)_{+}$ and furthermore $A(I-A)^{-1} \in\left(\mathscr{A}^{\prime}\right)_{+}$with $\left\|5 A(I-A)^{-1}\right\|<1$. Therefore,

$$
\begin{equation*}
G\left(x_{n+1}, x_{n}, x_{n}\right) \leq 5 A(I-A)^{-1} G\left(x_{n}, x_{n-1}, x_{n-1}\right) \tag{40}
\end{equation*}
$$

Let $t=5 A(I-A)^{-1}$, for $n+1>m$,

$$
\begin{align*}
G( & \left.x_{n+1}, x_{m}, x_{m}\right) \\
& \leq G\left(x_{n+1}, x_{n}, x_{n}\right)+G\left(x_{n}, x_{n-1}, x_{n-1}\right) \cdots \\
& +G\left(x_{m+1}, x_{m}, x_{m}\right) \\
\leq & \left(t^{n}+t^{n-1}+\cdots+t^{m}\right) G\left(x_{1}, x_{0}, x_{0}\right)=\sum_{k=m}^{n} t^{k} B \\
= & \sum_{k=m}^{n}\left|t^{k / 2} B^{1 / 2}\right|^{2} \leq\left\|\sum_{k=m}^{n}\left|t^{k / 2} B^{1 / 2}\right|^{2}\right\| I  \tag{41}\\
\leq & \sum_{k=m}^{n}\left\|t^{k / 2}\right\|^{2}\left\|B^{1 / 2}\right\|^{2} I \leq\left\|B^{1 / 2}\right\|^{2} \sum_{k=m}^{n}\|t\|^{k} I \\
\leq & \left\|B^{1 / 2}\right\|^{2} \frac{\|t\|^{m}}{1-\|t\|} I \longrightarrow \theta \quad(m \longrightarrow \infty) .
\end{align*}
$$

This implies that $\left\{x_{n}\right\}$ is a $G$-Cauchy sequence in $X$. Since $(X, \mathscr{A}, G)$ is complete, there exists an $x \in X$ such that $x_{n} \xrightarrow{G}$ $x(n \longrightarrow \infty)$, i.e., $T x_{n-1} \xrightarrow{G} x(n \longrightarrow \infty)$. Since

$$
\begin{align*}
G & (T x, x, x) \leq G\left(T x, T x_{n}, T x_{n}\right)+G\left(T x_{n}, x, x\right) \\
\quad & \leq A\left(G\left(T x, x_{n}, x_{n}\right)+G\left(x, T x_{n}, x_{n}\right)\right. \\
& \left.+G\left(x, x_{n}, T x_{n}\right)\right)+G\left(x_{n+1}, x, x\right) \leq A G(T x, x, x)  \tag{42}\\
& +A G\left(x, x_{n}, x_{n}\right)+2 A G\left(x, x_{n}, x_{n+1}\right) \\
& +G\left(x_{n+1}, x, x\right)
\end{align*}
$$

That is

$$
\begin{align*}
(I-A) G(T x, x, x) \leq & A G\left(x, x_{n}, x_{n}\right) \\
& +2 A G\left(x, x_{n}, x_{n+1}\right)  \tag{43}\\
& +G\left(x_{n+1}, x, x\right) .
\end{align*}
$$

Then

$$
\begin{align*}
& \|G(T x, x, x)\| \leq\left\|A(I-A)^{-1}\right\| \\
& \quad \cdot\left(\left\|G\left(x, x_{n}, x_{n}\right)\right\|+2\left\|G\left(x, x_{n}, x_{n+1}\right)\right\|\right)  \tag{44}\\
& \quad+\left\|(I-A)^{-1}\right\|\left\|G\left(x_{n+1}, x, x\right)\right\| \longrightarrow 0 \quad(n \longrightarrow \infty)
\end{align*}
$$

and hence $T x=x$ and $x$ is the fixed point of $T$ on $X$.
Next, we show the uniqueness of $x$. If there exists another fixed point $y \in X$, then

$$
\begin{align*}
\theta & \leq G(x, x, y)=G(T x, T x, T y) \\
& \leq A(G(T x, x, y)+G(x, T x, y)+G(x, x, T y))  \tag{45}\\
& =A G(x, x, y)+2 A G(x, x, y)
\end{align*}
$$

i.e.,

$$
\begin{equation*}
G(x, x, y) \leq 2 A(I-A)^{-1} G(x, x, y) \tag{46}
\end{equation*}
$$

Since $\left\|5 A(I-A)^{-1}\right\|<1,\left\|2 A(I-A)^{-1}\right\|<1$. Hence $G(x, x$, $y)=\theta$, and $x=y$.
(2) The case when

$$
\begin{align*}
& G(T x, T y, T z) \\
& \begin{array}{r}
\leq A(G(T x, T y, z)+G(x, T y, T z)+G(T x, y, T z)) \\
\forall x, y, z \in X
\end{array} \tag{47}
\end{align*}
$$

can be proved similarly and we omitted it.
Definition 22. Let $(X, \mathscr{A}, G)$ be a $C^{*}$-algebra-valued $G$-metric space. We say $G$ is symmetric if $G(x, x, y)=G(x, y, y)$ for all $x, y \in X$.

It is easy to show that, in Example 8, $G$ is not symmetric and, in Example 13, $G$ is symmetric.

Theorem 23. Let $(X, \mathscr{A}, G)$ be a complete $C^{*}$-algebra-valued $G$-metric space and $G$ symmetric. If $T: X \longrightarrow X$ is a mapping satisfying that for $x, y \in X$

$$
\begin{align*}
& G(T x, T y, T z)  \tag{48}\\
& \quad \leq A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z))
\end{align*}
$$

or

$$
\begin{align*}
& G(T x, T y, T z) \\
& \leq A(G(T x, T y, z)+G(x, T y, T z)+G(T x, y, T z)), \tag{49}
\end{align*}
$$

where $A \in\left(\mathscr{A}^{\prime}\right)_{+}$and $\|A\|<1 / 4$, then $T$ has a unique fixed point in $X$.

Proof. Without loss of generality, one can assume $A \neq \theta$.
We only consider the case when $T$ satisfies

$$
\begin{align*}
& G(T x, T y, T z) \\
& \quad \leq A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z)) . \tag{50}
\end{align*}
$$

Since $A \in\left(\mathscr{A}^{\prime}\right)_{+}, A(G(T x, y, z)+G(x, T y, z)+G(x, y, T z)) \geq$ $\theta$.

For $x_{0} \in X$, set $x_{n+1}=T x_{n}=T^{n+1} x_{0}, n=1,2, \cdots$, and $B=G\left(x_{1}, x_{0}, x_{0}\right)$. If $G$ is symmetric,

$$
\begin{align*}
& G\left(x_{n+1}, x_{n}, x_{n}\right)=G\left(T x_{n}, T x_{n-1}, T x_{n-1}\right) \\
& \quad \leq A\left(G\left(T x_{n}, x_{n-1}, x_{n-1}\right)+G\left(x_{n}, T x_{n-1}, x_{n-1}\right)\right. \\
& \left.\quad+G\left(x_{n}, x_{n-1}, T x_{n-1}\right)\right)=A G\left(x_{n+1}, x_{n-1}, x_{n-1}\right)  \tag{51}\\
& \quad+2 A G\left(x_{n}, x_{n}, x_{n-1}\right) \leq A\left(G\left(x_{n+1}, x_{n}, x_{n}\right)\right. \\
& \left.\quad+G\left(x_{n}, x_{n-1}, x_{n-1}\right)\right)+2 A G\left(x_{n}, x_{n-1}, x_{n-1}\right) \\
& \quad=A G\left(x_{n+1}, x_{n}, x_{n}\right)+3 A G\left(x_{n}, x_{n-1}, x_{n-1}\right) .
\end{align*}
$$

That is,

$$
\begin{equation*}
(I-A) G\left(x_{n+1}, x_{n}, x_{n}\right) \leq 3 A G\left(x_{n}, x_{n-1}, x_{n-1}\right) \tag{52}
\end{equation*}
$$

Since $A \in\left(\mathscr{A}^{\prime}\right)_{+}$with $\|A\|<1 / 4$, then $I-A$ is invertible and $(I-A)^{-1} \in\left(\mathscr{A}^{\prime}\right)_{+}$and $\left\|3 A(I-A)^{-1}\right\|<1$. Therefore

$$
\begin{equation*}
G\left(x_{n+1}, x_{n}, x_{n}\right) \leq 3 A(I-A)^{-1} G\left(x_{n}, x_{n-1}, x_{n-1}\right) . \tag{53}
\end{equation*}
$$

Just like the proof of Theorem 21, we can prove that, for $n+1>$ $m$,

$$
\begin{equation*}
G\left(x_{n+1}, x_{m}, x_{m}\right) \longrightarrow \theta \quad(m \longrightarrow \infty) \tag{54}
\end{equation*}
$$

This implies that $\left\{x_{n}\right\}$ is a $G$-Cauchy sequence in $X$; thus there exists an $x \in X$ such that $x_{n} \xrightarrow{G} x(n \longrightarrow \infty)$ i.e., $T x_{n-1} \xrightarrow{G}$ $x(n \longrightarrow \infty)$. Similarly, we can show that $T x=x, x$ is the fixed point of $T$ on $X$.

The uniqueness of the fixed point can be proved similarly.

## 3. Applications

For fixed-point theorems, there are a number of applications in differential equations and integral equations.

Consider the second-order differential equation:

$$
\begin{align*}
& \frac{d^{2} u}{d t^{2}}=K(t, u(t))  \tag{*}\\
& u(0)=u(1)=0
\end{align*}
$$

where $u:[0,1] \longrightarrow[0, \infty)$ and $K:[0,1] \times \mathbb{R}^{+} \longrightarrow \mathbb{R}^{+}$are continuous. Then the equation has a unique solution if and only if $u(t)=\int_{0}^{1} g(t, s) K(s, u(s)) d s, t \in[0,1]$ has a unique solution, where

$$
g(t, s)= \begin{cases}t(1-s), & 0 \leq s<t \leq 1  \tag{55}\\ s(1-t), & 0 \leq t<s \leq 1\end{cases}
$$

Let $X=C\left([0,1], \mathbb{R}^{+}\right) \subseteq L^{\infty}\left([0,1], \mathbb{R}^{+}\right), \mathscr{A}=B\left(L^{2}[0,1]\right)$, $G(u, v, w)=M_{|u-v|+|v-w|+|w-u|}$, then $(X, \mathscr{A}, G)$ is a complete $C^{*}$-algebra-valued $G$-metric space. Let $T: X \longrightarrow X$ be a mapping defined by

$$
\begin{equation*}
T u(t)=\int_{0}^{1} g(t, s) K(s, u(s)) d s, \tag{56}
\end{equation*}
$$

then ( $*$ ) has a unique solution if and only if $T$ has a unique fixed point.

Theorem 24. If $|K(s, x)-K(s, y)| \leq|x-y|, \forall s \in[0,1]$, $x, y \in \mathbb{R}^{+}$, equation (*) has a unique solution.

Proof.

$$
\begin{aligned}
& \|G(T u, T v, T w)\|=\left\|M_{|T u-T v|+|T v-T w|+|T w-T u|}\right\|=\| T u \\
& \quad-T v\left\|_{\infty}+\right\| T v-T w\left\|_{\infty}+\right\| T w-T u \|_{\infty} \\
& \quad=\max _{t \in[0,1]}\left|\int_{0}^{1} g(t, s)(K(s, u(s))-K(s, v(s))) d s\right| \\
& \quad+\max _{t \in[0,1]}\left|\int_{0}^{1} g(t, s)(K(s, v(s))-K(s, w(s))) d s\right| \\
& \quad+\max _{t \in[0,1]}\left|\int_{0}^{1} g(t, s)(K(s, w(s))-K(s, u(s))) d s\right| \\
& \quad \leq \max _{t \in[0,1]} \int_{0}^{1} g(t, s)|K(s, u(s))-K(s, v(s))| d s \\
& \quad+\max _{t \in[0,1]} \int_{0}^{1} g(t, s)|K(s, v(s))-K(s, w(s))| d s \\
& \quad+\max _{t \in[0,1]} \int_{0}^{1} g(t, s)|K(s, w(s))-K(s, u(s))| d s \\
& \quad \leq \max _{t \in[0,1]} \int_{0}^{1} g(t, s)|u(s)-v(s)| d s+\max _{t \in[0,1]} \int_{0}^{1} g(t, s) \\
& \quad \cdot|v(s)-w(s)| d s+\max _{t \in[0,1]} \int_{0}^{1} g(t, s) \mid w(s)
\end{aligned}
$$

$$
\begin{align*}
& -u(s) \mid d s=\max _{t \in[0,1]} \int_{0}^{1} g(t, s)(|u(s)-v(s)| \\
& +|v(s)-w(s)|+|w(s)-u(s)|) d s \leq \| G(u, \\
& v, w) \| \max _{t \in[0,1]} \int_{0}^{1} g(t, s) d s \tag{57}
\end{align*}
$$

For a fixed $t \in[0,1]$,

$$
\begin{align*}
\int_{0}^{1} g(t, s) d s & =\int_{0}^{t} t(1-s) d s+\int_{t}^{1} s(1-t) d s  \tag{58}\\
& =\frac{1}{2}+\frac{t^{2}}{2}-\frac{t}{2} \leq \frac{1}{2}
\end{align*}
$$

Therefore,

$$
\begin{equation*}
\|G(T u, T v, T w)\| \leq \frac{1}{2}\|G(u, v, w)\| \tag{59}
\end{equation*}
$$

and $T$ has a unique fixed point.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# The Tensor Padé-Type Approximant with Application in Computing Tensor Exponential Function 

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#### Abstract

Tensor exponential function is an important function that is widely used. In this paper, tensor Padé-type approximant (TPTA) is defined by introducing a generalized linear functional for the first time. The expression of TPTA is provided with the generating function form. Moreover, by means of formal orthogonal polynomials, we propose an efficient algorithm for computing TPTA. As an application, the TPTA for computing the tensor exponential function is presented. Numerical examples are given to demonstrate the efficiency of the proposed algorithm.


## 1. Introduction

Tensor exponential function is an important function that is widely used, owing to its key role in the solution of tensor differential equations [1-4]. For instance, Markovian master equation can be written as tensor differential equations $(\partial / \partial t) \mathbf{P}(t)=\mathbf{A} \cdot \mathbf{P}(t)$, where the probabilities tensor $\mathbf{P}(t) \in$ $\mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{d}}$ [5]. Consider the initial value problem defined by the tensor ordinary differential equation $[6,7]$

$$
\begin{align*}
& \dot{\mathscr{Y}}(t)=\mathscr{A} \mathscr{Y}(t), \\
& \mathscr{Y}\left(t_{0}\right)=\mathscr{Y}_{0}, \tag{1}
\end{align*}
$$

where the superimposed dot denotes differentiation with respect to $t$ and $\mathscr{A}$ and $\mathscr{Y}_{0}$ are given constant tensors. The solution to system (1) is $\mathscr{y}(t)=\exp \left[\left(t-t_{0}\right) \mathscr{A}\right] \mathscr{Y}_{0}$, and $\exp (\cdot)$ is the tensor exponential function.

To solve (1), we need to calculate an exponential function about tensor $\mathscr{A}$. Recently, tensor computation, especially eigenvalues of tensors, has attracted attention of many scholars; some important results have been found in the current literature [8-15]. For instance, computation of $\exp (\mathscr{A} t)$ is required in applications such as finite strain hyperelasticbased multiplicative plasticity models [7, 16-19]. Explicitly,
for a generic tensor $\mathscr{A}$, the tensor exponential can be expressed by means of its series representation [7] (see p.749).

$$
\begin{equation*}
\exp (\mathscr{A})=\sum_{n=0}^{\infty} \frac{1}{n!} \mathscr{A}^{n} \tag{2}
\end{equation*}
$$

and the preceding series is absolutely convergent for any argument $\mathscr{A}$ and, as its scalar counterpart, can be used to evaluate the tensor exponential function to any prescribed degree of accuracy [16]. The computation of (2) is carried out by simply truncating the infinite series with $n_{\max }$ terms:

$$
\begin{equation*}
\exp (\mathscr{A})=\sum_{n=0}^{n_{\max }} \frac{1}{n!} \mathscr{A}^{n} \tag{3}
\end{equation*}
$$

with $n_{\max }$ being such that $\left(1 / n_{\max }\right)\left\|\mathscr{A}^{n_{\max }}\right\| \leq \epsilon_{\text {tol }}$.
However, the accuracy and effectiveness of the preceding algorithm is limited by round-off and choice of termination criterion [16]. Padé approximant has become by far the most widely used one in calculation of exponential function or formal power series due to the following reasons: first, the series may converge too slowly to be of any use and the approximation can accelerate its convergence; second, only few coefficients of the series may be known and a good approximation to the series is needed to obtain properties of
the function that it represents [20]. For instance, matrix Padétype approximant (MPTA) [21] can be used to simplify the high degree multivariable system by approximating transfer function matrix $G(s)$ that can be expanded into a power series with matrix coefficients, i.e., $G(s)=\sum_{i=0}^{\infty} c_{i} s^{i}$, where $c_{i} \in \mathbb{C}^{s \times t}$. The key to construct TPTA is to maintain the same order of tensor $\mathscr{A}$ with different powers. For this issue, we introduce t-product [22, 23] of two tensors to solve it. In addition, in order to give the definition of TPTA, we introduce a generalized linear functional in the tensor space for the first time.

This paper is organized as follows. In Section 2, we provide some preliminaries. First, we introduce the t-product of two tensors; then, we show the definitions of tensor exponential function and the Frobenius norm of a tensor. In Section 3.1, we define the tensor Padé-type approximant by using generalized linear functional; the expression of TPTA is of the form of tensor numerator and scalar denominator; and then we introduce the definition of orthogonal polynomial with respect to generalized linear functional and sketch an algorithm to compute the TPTA. Numerical examples are given and analyzed in Section 4. Finally, we finish the paper with concluding remarks in Section 5.

## 2. Preliminaries

There arise mainly a problem for approximating tensor exponential function. That is how to expand $e^{\mathscr{A} t}$ into the power series for order- $p(p \geq 3)$ tensors. For a Symmetric and second-order tensor $A$, higher powers of $A$ can be computed by the Cayley-Hamilton theorem [24], but it fails for the order- $p(p \geq 3)$ tensors. Therefore, we shall utilize the tproduct to obtain higher powers of order- $p(p \geq 3)$ tensors in this section. Firstly, we introduce some notations and basic definitions which will be used in the sequel. Throughout this paper tensors are denoted by calligraphic letters (e.g., $\mathscr{A}, \mathscr{B}$ ), while capital letters represent matrices, and lowercase letters refer to scalars.

An order- $p$ tensor, $\mathscr{A}$, can be written as

$$
\begin{equation*}
\mathscr{A}=\left(a_{i_{1} i_{2} \cdots i_{p}}\right) \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}} . \tag{4}
\end{equation*}
$$

Thus, a matrix is considered a second-order tensor, and a vector is a first-order tensor [22], for $i=1, \ldots, n_{p}$, denoted by $\mathscr{A}_{i} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p-1}}$, the tensor whose order is $p-1$ and is created by holding the $p$ th index of $\mathscr{A}$ fixed at $i$. For example, consider a third-order tensor, $\mathscr{A}=\left(a_{i_{1} i_{2} i_{3}}\right) \in \mathbb{R}^{3 \times 3 \times 3}$. Fixing the 3 rd index of $\mathscr{A}$, we can get three $3 \times 3$ matrices, namely, 2-order tensor, which are $A_{1}, A_{2}$, and $A_{3}$ and with elements

$$
\begin{align*}
& A_{1}: a_{111} \\
& a_{121}  \tag{5}\\
& a_{131}
\end{align*} a_{211} a_{221} a_{231} a_{311} a_{321} a_{331},
$$

respectively.
Now, we will define the t-product of two tensors.

Definition 1 (see [22, 23]). Let $\mathscr{A} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$. Then the block circulant pattern tensor of $\mathscr{A}$ is denoted by

$$
\operatorname{circ}(\mathscr{A})=\left(\begin{array}{ccccc}
\mathscr{A}_{1} & \mathscr{A}_{n_{p}} & \mathscr{A}_{n_{p}-1} & \cdots & \mathscr{A}_{2}  \tag{6}\\
\mathscr{A}_{2} & \mathscr{A}_{1} & \mathscr{A}_{n_{p}} & \cdots & \mathscr{A}_{3} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\mathscr{A}_{n_{p}} & \mathscr{A}_{n_{p}-1} & \cdots & \mathscr{A}_{2} & \mathscr{A}_{1}
\end{array}\right)
$$

where $\mathscr{A}_{i} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p-1}}, i=1,2, \ldots, n_{p}$.
Define $\operatorname{unfold}(\cdot)$ to an $n_{1} \times n_{2} \times \cdots \times n_{p}$ tensor by an $n_{1} n_{p} \times$ $n_{2} \times \cdots \times n_{p-1}$ block tensor in the following way:

$$
\operatorname{unfold}(\mathscr{A})=\left(\begin{array}{c}
\mathscr{A}_{1}  \tag{7}\\
\mathscr{A}_{2} \\
\vdots \\
\mathscr{A}_{n_{p}}
\end{array}\right)
$$

If $\mathscr{A}$ is order-3 tensor, then $\operatorname{unfold}(\mathscr{A})$ is a block vector. Similarly, define fold $(\cdot)$ as the inverse operation, which takes an $n_{1} n_{p} \times n_{2} \times \cdots \times n_{p-1}$ block tensor and returns an $n_{1} \times n_{2} \times$ $\cdots \times n_{p}$ tensor; then

$$
\begin{equation*}
\text { fold }(\operatorname{unfold}(\mathscr{A}))=\mathscr{A} \tag{8}
\end{equation*}
$$

Definition 2 (see [23]). Let $\mathscr{A}$ be $\mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$ and $\mathscr{B}$ be $\mathbb{R}^{n_{2} \times l \times n_{3} \cdots \times n_{p}}$. Then the t-product $\mathscr{A}^{*} \mathscr{B}$ is the $n_{1} \times l \times n_{3} \times \cdots \times n_{p}$ tensor defined recursively as

$$
\begin{equation*}
\mathscr{A}^{*} \mathscr{B}=\operatorname{fold}\left(\operatorname{circ}(\mathscr{A})^{\star} \operatorname{unfold}(\mathscr{B})\right) . \tag{9}
\end{equation*}
$$

Remark 3. If $\mathscr{A}$ and $\mathscr{B}$ are order-2 tensors, then the product
"*" can be replaced by standard matrix multiplication.
Remark 4. The $k$ times power of $\mathscr{A}$ is defined as $\mathscr{A}^{k}=$ $\mathscr{A}^{*} \mathscr{A}^{*} \ldots{ }^{*} \mathscr{A}$ ( $k$ times); "*" denotes the t-product.

Example 5. Letting

$$
\mathscr{A}=\left(\begin{array}{ccc|ccc}
1 & 0 & -1 & 3 & 1 & 0  \tag{10}\\
2 & 1 & 0 & 2 & 0 & 0 \\
0 & -1 & 1 & 1 & -1 & 2
\end{array}\right) \in \mathbb{R}^{3 \times 3 \times 2}
$$

then, from Definition 2, we have

$$
\begin{align*}
\mathscr{A}^{4} & =\mathscr{A}^{*} \mathscr{A}^{*} \mathscr{A}^{*} \mathscr{A} \\
& =\left(\begin{array}{ccc|ccc}
271 & 113 & -98 & 237 & 104 & -119 \\
242 & 97 & -96 & 242 & 96 & -96 \\
-73 & -46 & 56 & -94 & -52 & 43
\end{array}\right) \tag{11}
\end{align*}
$$

Remark 6. One of the characteristic features of t -product is that it ensures that the order of multiplication result of two tensors does not change, whereas other tensor multiplications do not have the feature; that is why we chose the t-product as the multiplication of tensors.

The tensor exponential function is a tensor function on tensors analogous to the ordinary exponential function, which can be defined as follows.

Definition 7. Let $\mathscr{A}$ be an $n_{1} \times n_{2} \times \cdots \times n_{p}$ real or complex tensor. The tensor exponential function of $t$, denoted by $e^{\mathscr{A} t}$ or $\exp (\mathscr{A} t)$, is the $n_{1} \times n_{2} \times \cdots \times n_{p}$ tensor given by the power series

$$
\begin{equation*}
e^{\mathscr{A} t}=\sum_{k=0}^{\infty} \frac{1}{k!}(\mathscr{A} t)^{k} \tag{12}
\end{equation*}
$$

where $\mathscr{A}^{0}$ is defined to be the identity tensor $\mathscr{F}$ (see Definition 8) with the same orders as $\mathscr{A}$.

Definition 8 (see [23]). The $n \times n \times l_{1} \times \cdots \times l_{p-2}$ order- $p$ identity tensor $(p>3) \mathscr{J}$ is the tensor such that $\mathscr{J}_{1}$ is the $n \times n \times l_{1} \times \cdots \times l_{p-3}$ order- $(p-1)$ identity tensor, and $\mathscr{J}_{j}$ is the order- $(p-1)$ zero tensor, for $j=2,3, \ldots, l_{p-2}$.

By Definition 8, we can define tensor inverse, transpose, and orthogonality. However, we do not discuss these works here, as it is beyond the scope of the present work. For the details of these definitions of tensor, we refer to reader to [22, 23,25 ] and the references therein.

Let $\mathscr{A} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$; then the norm of a tensor is the square root of the sum of the squares of all its entries [25]; i.e.,

$$
\begin{equation*}
\|\mathscr{A}\|=\sqrt{\sum_{i_{1}=1 i_{2}=1}^{n_{1}} \sum_{i_{p}=1}^{n_{2}} \cdots \sum_{i_{1} i_{2} \cdots i_{p}}^{n_{p}} a^{2}} \tag{13}
\end{equation*}
$$

This is analogous to the matrix Frobenius norm. The inner product of two same-sized tensors $\mathscr{A}, \mathscr{B} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$ is the sum of the products of their elements [25]; i.e.,

$$
\begin{equation*}
(\mathscr{A}, \mathscr{B})=\sum_{i_{1}=1 i_{2}=1}^{n_{1}} \sum_{i_{p}=1}^{n_{2}} \cdots \sum_{i_{1} i_{2} \ldots i_{p}} b_{i_{1} i_{2} \cdots i_{p}} . \tag{14}
\end{equation*}
$$

It follows immediately that $(\mathscr{A}, \mathscr{A})=\|\mathscr{A}\|^{2}$.

## 3. Tensor Padé-Type Approximant

Let $f(x)$ be a given power series with tensor coefficients; i.e.,

$$
\begin{align*}
& f(x)=\mathscr{A}_{0}+\mathscr{A}_{1} x+\mathscr{A}_{2} x^{2}+\cdots+\mathscr{A}_{n} x^{n}+\cdots,  \tag{15}\\
& \mathscr{A}_{i} \in \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}, x \in \mathbb{R} .
\end{align*}
$$

Let $\mathbf{P}$ denote the set of scalar polynomials in one real variable whose coefficients belong to the real field $\mathbb{R}$ and $\mathbf{P}_{k}$ denote the set of elements of $\mathbf{P}$ of degree less than or equal to $k$.

Let $\phi: \mathbf{P} \rightarrow \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$ be a linear functional on $\mathbf{P}$. Let it act on $t$ by

$$
\begin{equation*}
\phi\left(t^{i}\right)=\mathscr{A}_{i}, \quad i=0,1,2, \ldots \tag{16}
\end{equation*}
$$

Then, by the linearity of $\phi$, we have

$$
\begin{align*}
\phi\left((1-t x)^{-1}\right)= & \phi\left(1+t x+(t x)^{2}+\cdots+(t x)^{n}+\cdots\right) \\
= & \mathscr{A}_{0}+\mathscr{A}_{1} x+\mathscr{A}_{2} x^{2}+\cdots+\mathscr{A}_{n} x^{n}  \tag{17}\\
& +\cdots=f(x) .
\end{align*}
$$

3.1. Definition of Tensor Padé-Type Approximant. Let $v_{n}(x)=$ $b_{0}+b_{1} x+\cdots+b_{n} x^{n}\left(b_{n} \neq 0\right)$ be a scalar polynomial of $\mathbf{P}_{n}$ of exact degree $n$. In this case, $v_{n}(x)$ is said to be quasi-monic. Define the tensor polynomial $\mathscr{W}_{n-1}(x)$ associated with $v_{n}(x)$ with tensor-valued coefficients, by

$$
\begin{equation*}
\mathscr{W}_{n-1}(x)=\phi\left(\frac{v_{n}(x)-v_{n}(t)}{x-t}\right) . \tag{18}
\end{equation*}
$$

It is easily seen that $\mathscr{W}_{n-1}(x)$ is a tensor polynomial of exact degree $n-1$ in $x$. Set

$$
\begin{gather*}
\widetilde{v}_{n}(x)=x^{n} v_{n}\left(x^{-1}\right),  \tag{19}\\
\widetilde{\mathscr{W}}_{n-1}(x)=x^{n-1} \mathscr{W}_{n-1}\left(x^{-1}\right) . \tag{20}
\end{gather*}
$$

Then, the polynomials $\widetilde{v}_{n}(x)$ and $\widetilde{\mathscr{V}}_{n-1}(x)$ are obtained from $v_{n}(x)$ and $\mathscr{W}_{n-1}(x)$, respectively, by reversing the numbering of the coefficients. By the procedure given above, the following conclusion is obtained.

Theorem 9. Let $\widetilde{v}_{n}(0) \neq 0$; then $\widetilde{\mathscr{W}}_{n-1}(x) / \widetilde{v}_{n}(x)-f(x)=$ $O\left(x^{n}\right)$.

Proof. Expanding $\left(v_{n}(x)-v_{n}(t)\right) /(x-t)$ in (18) and applying $\phi$ yields that

$$
\begin{align*}
\widetilde{\mathscr{W}}_{n-1}(x) & =\sum_{l=0}^{n-1}\left(\sum_{i=0}^{n-l-1} b_{l+i+1} \mathscr{A}_{i}\right) x^{l} \\
& =\sum_{l=0}^{n-1}\left(\sum_{i=0}^{l} b_{n-l+i} \mathscr{A}_{i}\right) x^{l} . \tag{21}
\end{align*}
$$

Computing $\widetilde{v}_{n}(x) f(x)$, we get

$$
\begin{align*}
\widetilde{v}_{n}(x) f(x) & =\left(\sum_{j=0}^{n} b_{n-j} x^{j}\right)\left(\sum_{i=0}^{\infty} \mathscr{A}_{i} x^{i}\right)  \tag{22}\\
& =\sum_{l=0}^{\infty}\left(\sum_{i=0}^{l} b_{n-l+i} \mathscr{A}_{i}\right) x^{l} .
\end{align*}
$$

Thus,

$$
\begin{align*}
& \widetilde{v}_{n}(x) f(x)-\widetilde{\mathscr{W}}_{n-1}(x)=\sum_{l=n}^{\infty}\left(\sum_{i=0}^{l} b_{n-l+i} \mathscr{A}_{i}\right) x^{l}  \tag{23}\\
& \quad=O\left(x^{n}\right) .
\end{align*}
$$

Definition 10. $R_{n-1, n}(x)=\widetilde{\mathscr{W}}_{n-1}(x) / \widetilde{v}_{n}(x)$ is called a tensor Padé-type approximant with order $n$ for the given power series (15) and is denoted by $(n-1 / n)_{f}(x)$.

Remark 11. The polynomial $v_{n}(x)$, called the generating polynomial of $(n-1 / n)_{f}$ with respect to power series $f(x)$, can be arbitrarily chosen.

Remark 12. The tensor Padé-type approximant $(n-1 / n)_{f}$ possesses the degree constraint, which is caused by its construction process. The constraint implies that the method does not construct tensor Padé-type approximant of type $(m / n)$ when $m$ is different from $n-1$.

To fill this gap, we define a new tensor Padé-type approximant by introducing a generalized linear functional.

Let $\phi^{(q)}: \mathbf{P} \rightarrow \mathbb{R}^{n_{1} \times n_{2} \times \cdots \times n_{p}}$ be a generalized linear functional on $\mathbf{P}$. Let it act on $t$ by

$$
\begin{equation*}
\phi^{(q)}\left(t^{i}\right)=\mathscr{A}_{q+i}, \quad i=0,1,2, \ldots . \tag{24}
\end{equation*}
$$

Similarly to what was done for $\mathscr{W}_{n-1}(x)$, we consider the polynomial $\mathscr{W}_{q}(x)$ associated with $v_{n}(x)$, and defined by

$$
\begin{equation*}
\mathscr{W}_{q}(x)=\phi^{(q)}\left(\frac{v_{n}(x)-v_{n}(t)}{x-t}\right), \quad q=m-n+1 . \tag{25}
\end{equation*}
$$

Set

$$
\begin{equation*}
\widetilde{\mathscr{W}}_{q}(x)=x^{n-1} \mathscr{W}_{q}\left(x^{-1}\right), \tag{26}
\end{equation*}
$$

and define

$$
\begin{equation*}
\mathscr{P}_{m n}(x)=\widetilde{v}_{n}(x) \sum_{i=0}^{m-n} \mathscr{A}_{i} x^{i}+x^{m-n+1} \widetilde{\mathscr{W}}_{q}(x), \quad m \geq n \tag{27}
\end{equation*}
$$

Then we have the following conclusion.
Theorem 13. Let $\widetilde{v}_{n}(0) \neq 0$; then $\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)-f(x)=$ $O\left(x^{m+1}\right)$.

Proof. Let $f_{m-n+1}$ be the formal power series

$$
\begin{equation*}
f_{m-n+1}(x)=\sum_{j=0}^{\infty} \mathscr{A}_{m-n+1+j} x^{j} \tag{28}
\end{equation*}
$$

Then

$$
\begin{equation*}
x^{m-n+1} f_{m-n+1}(x)=f(x)-\sum_{i=0}^{m-n} \mathscr{A}_{i} x^{i} . \tag{29}
\end{equation*}
$$

Expanding (25) and using (26) we obtain

$$
\begin{equation*}
\widetilde{\mathscr{W}}_{q}(x)=\sum_{q=0}^{n-1}\left(\sum_{i=0}^{q} b_{n-q+i} \mathscr{A}_{m-n+1+i}\right) x^{q} . \tag{30}
\end{equation*}
$$

Computing the product $\widetilde{v}_{n}(x) f_{m-n+1}(x)$, we find that

$$
\begin{equation*}
\widetilde{v}_{n}(x) f_{m-n+1}(x)=\sum_{q=0}^{\infty}\left(\sum_{i=0}^{q} b_{n-q+i} \mathscr{A}_{m-n+1+i}\right) x^{q} . \tag{31}
\end{equation*}
$$

By Theorem 9, one has

$$
\begin{equation*}
\frac{\widetilde{\mathscr{W}}_{q}(x)}{\widetilde{v}_{n}(x)}=\left(n-\frac{1}{n}\right)_{f_{m-n+1}}(x) \tag{32}
\end{equation*}
$$

Then, for $m \geq n$ we deduce from (27) and (29) that

$$
\begin{aligned}
& \widetilde{v}_{n}(x) f(x)-\mathscr{P}_{m n}(x) \\
&= \widetilde{v}_{n}(x)\left(\sum_{i=0}^{m-n} \mathscr{A}_{i} x^{i}+x^{m-n+1} f_{m-n+1}(x)\right) \\
&-\widetilde{v}(x) \sum_{i=0}^{m-n} \mathscr{A}_{i} x_{i}+x^{m-n+1} \widetilde{\mathscr{W}}_{q}(x) \\
&= x^{m-n+1}\left(\widetilde{v}_{n}(z) f_{m-n+1}-\widetilde{\mathscr{W}}_{q}(x)\right) \\
&= x^{m-n+1}\left\{\sum_{q=n}^{\infty}\left(\sum_{i=0}^{q} b_{n-q+i} \mathscr{A}_{m-n+1+i}\right) x^{q}\right\} \\
&=O\left(x^{m+1}\right) .
\end{aligned}
$$

Now, we can achieve $(m / n)_{f}(x)$ by the above procedure, and it will be denoted by $\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)$.

Definition 14. $R_{m, n}(x)=\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)$ is called TPTA with order $m+1$ and is denoted by $(m / n)_{f}(x)$.

Algorithm 15 (compute $R_{m, n}(x)=\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)$ with $v_{n}(x)$ being arbitrarily chosen).
(1) Set $q=m-n+1$ and chose a quasi-monic polynomial $v_{n}(x)$.
(2) Use (19) to compute $\widetilde{v}_{n}(x)$.
(3) Compute $\mathscr{W}_{q}(x)$ and $\widetilde{\mathscr{W}}_{q}(x)$ by (25) and (26), respectively.
(4) Substitute $\widetilde{v}_{n}(x)$ and $\widetilde{\mathscr{W}}_{q}(x)$ into (27) to compute $\mathscr{P}_{m n}(x)$.
(5) Set $R_{m, n}(x)=\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)$.

Example 16. Let

$$
\begin{align*}
f(x)= & \left(\begin{array}{cc|cc|cc}
1 & 0 & 0 & 1 & 1 & -1 \\
0 & 2 & -1 & 2 & 2 & 1
\end{array}\right) \\
& +\left(\begin{array}{cc|cc|cc}
1 & 0 & 2 & 1 & 1 & -2 \\
1 & 2 & -1 & 2 & 3 & 1
\end{array}\right) x  \tag{34}\\
& +\left(\begin{array}{cc|cc|cc}
1 & 2 & 1 & 1 & 0 & -1 \\
0 & 2 & -1 & 3 & 2 & 0
\end{array}\right) x^{2}+\cdots \\
= & \mathscr{A}_{0}+\mathscr{A}_{1} x+\mathscr{A}_{2} x^{2}+\cdots .
\end{align*}
$$

Now we apply Algorithm 15 to compute TPTA of type (2/2) for this example.
(1) Chose $v_{2}(x)=x^{2}-2 x+4, q=m-n+1=1$.
(2) Use (19) to compute $\widetilde{v}_{2}(x)$ :

$$
\begin{equation*}
\widetilde{v}_{2}(x)=1-2 x+4 x^{2} \tag{35}
\end{equation*}
$$

(3) By using (25) and (26) we get

$$
\begin{aligned}
& \mathscr{W}_{1}(x) \\
& =\left(\begin{array}{cc|cc|cc}
x-1 & 2 & 2 x+1 & x-1 & x-2 & 1-2 x \\
x & 2 x-2 & -x+1 & 2 x-1 & 3 x-2 & x-2
\end{array}\right)
\end{aligned}
$$

and

$$
\begin{align*}
& \widetilde{\mathscr{W}}_{1}(x) \\
& \quad=\left(\begin{array}{cc}
1-x & 2 x \\
0 & 2-2 x
\end{array}\left|\begin{array}{cc}
x & 1-x \\
x-1 & 2-x
\end{array}\right| \begin{array}{cc}
1-2 x & x-1 \\
2-2 x & 1-2 x
\end{array}\right) \tag{37}
\end{align*}
$$

(4) Substituting $\widetilde{v}_{2}(x)$ and $\widetilde{\mathscr{V}}_{1}(x)$ into (27), we obtain

$$
\begin{align*}
\mathscr{P}_{22}(x) & =\widetilde{v}_{2}(x) \mathscr{A}_{0}+x \widetilde{\mathscr{W}}_{1}(x) \\
& =\left(\begin{array}{cc|cc|cc}
3 x^{2}-x+1 & 2 x^{2} & -3 x^{2}+2 x & 3 x^{2}-x+1 & 2 x^{2}-x+1 & -x^{2}-1 \\
-2 x^{2}+x & 6 x^{2}-2 x+2 & -3 x^{2}+x-1 & 7 x^{2}-2 x+2 & 4 x^{2}-x+2 & 2 x^{2}-x+1
\end{array}\right) . \tag{38}
\end{align*}
$$

(5) Set $R_{22}(x)=\mathscr{P}_{22}(x) / \widetilde{v}_{2}(x)$. It is easy to verify that

$$
\begin{equation*}
\frac{\mathscr{P}_{22}(x)}{\widetilde{v}_{2}(x)}-f(x)=O\left(x^{3}\right) \tag{39}
\end{equation*}
$$

3.2. Algorithm for Computing TPTA. Generally, the precision of TPTA is limited, since the denominator polynomials of TPTA are arbitrarily prescribed. In this subsection, in order to improve the precision of approximation, we propose an algorithm for computing the denominator polynomials and illustrate the efficiency of this algorithm in next section.

First, we give the following conclusion.
Theorem 17 (error formula).

$$
\begin{equation*}
f(x)-\left(n-\frac{1}{n}\right)_{f}(x)=\frac{x^{n}}{\widetilde{v}_{n}(x)} \phi\left(\frac{v_{n}(t)}{1-t x}\right) \tag{40}
\end{equation*}
$$

Proof. Note that $\phi$ is a linear functional on $\mathbf{P}$, only acting on $t$. From (18) and (20) we deduced that

$$
\begin{align*}
\widetilde{\mathscr{W}}_{n-1}(x) & =x^{n-1} \mathscr{W}_{n-1}\left(x^{-1}\right) \\
& =x^{n-1} \phi\left(\frac{x v_{n}\left(x^{-1}\right)-x v_{n}(t)}{1-t x}\right) \\
& =\phi\left(\frac{x^{n} v_{n}\left(x^{-1}\right)-x^{n} v_{n}(t)}{1-t x}\right)  \tag{41}\\
& =\phi\left(\frac{x^{n} v_{n}\left(x^{-1}\right)}{1-t x}\right)-x^{n} \phi\left(\frac{v_{n}(t)}{1-t x}\right) \\
& =\widetilde{v}_{n}(x) f(x)-x^{n} \phi\left(\frac{\left.v_{n}(t)\right)}{1-t x}\right)
\end{align*}
$$

and then this error formula holds.

In terms of the error formula, it holds that

$$
\begin{align*}
& f(x)-\left(n-\frac{1}{n}\right)_{f}(x)=\frac{x^{n}}{\widetilde{v}_{n}(x)} \phi\left(\frac{v_{n}(t)}{1-t x}\right)=\frac{x^{n}}{\widetilde{v}_{n}(x)} \\
& \quad \cdot \phi\left(v_{n}(t)+v_{n}(t) t x+v_{n}(t) t^{2} x^{2}+\cdots\right) \\
& \quad=\frac{x^{n}}{\widetilde{v}_{n}(x)}\left(\phi\left(v_{n}(t)\right)+\phi\left(v_{n}(t) t\right) x\right.  \tag{42}\\
& \left.\quad+\phi\left(v_{n}(t) t^{2}\right) x^{2}+\cdots\right)
\end{align*}
$$

If we impose that $v_{n}(t)$ satisfies the condition $\phi\left(v_{n}(t)\right)=$ 0 , then the first term of (42) disappears, and the order of approximation becomes $n+1$. If, in addition, we also impose the condition $\phi\left(t v_{n}(t)\right)=0$, the second term in the expansion of the error also disappears, and the order of approximation becomes $n+2$, and so on. We indicate that $v_{n}(x)$ depends on $n+1$ arbitrary constants; however, on the other side, a rational function is defined apart from a multiplying factor in its numerator and its denominator. It implies that $(n-1 / n)_{f}(x)$ depends on $n$ arbitrary constants. So let us take $v_{n}(t)$ such that

$$
\begin{equation*}
\phi\left(v_{n}(t) t^{k}\right)=0, \quad k=0,1,2, \ldots, n-1 \tag{43}
\end{equation*}
$$

Definition 18. $v_{n}(t)$ in (43) is called an orthogonal polynomial with respect to the linear functional $\phi$ and $(n-1 / n)_{f}(x)$ in (42) is also called a TPTA for the given power series (15) when (43) is satisfied.

From (43) we obtain

$$
\begin{equation*}
\phi\left(v_{n}(t) t^{k}\right)=\sum_{i=0}^{n} b_{i} \mathscr{A}_{i+k}=0, \quad k=0,1,2, \ldots, n-1 \tag{44}
\end{equation*}
$$

Let $b_{n}=1$ in (44); then it follows that

$$
\begin{equation*}
\sum_{i=0}^{n-1} b_{i} \mathscr{A}_{i+k}=-\mathscr{A}_{k+n}, \quad k=0,1,2, \ldots, n-1 \tag{45}
\end{equation*}
$$

Forming the scalar product of both sides of (45) with $\mathscr{A}_{0}, \mathscr{A}_{1}, \ldots, \mathscr{A}_{n-1}$, respectively, we get

$$
\begin{equation*}
\sum_{i=0}^{n-1} b_{i}\left(\mathscr{A}_{i+k}, \mathscr{A}_{k}\right)=-\left(\mathscr{A}_{k+n}, \mathscr{A}_{k}\right) \tag{46}
\end{equation*}
$$

$$
k=0,1,2, \ldots, n-1
$$

Denote

$$
\begin{align*}
& H_{n}\left(\mathscr{A}_{0}\right) \\
& =\left(\begin{array}{cccc}
\left(\mathscr{A}_{0}, \mathscr{A}_{0}\right) & \left(\mathscr{A}_{1}, \mathscr{A}_{0}\right) & \cdots & \left(\mathscr{A}_{n-1}, \mathscr{A}_{0}\right) \\
\left(\mathscr{A}_{1}, \mathscr{A}_{1}\right) & \left(\mathscr{A}_{2}, \mathscr{A}_{1}\right) & \cdots & \left(\mathscr{A}_{n}, \mathscr{A}_{1}\right) \\
\vdots & \vdots & \ddots & \vdots \\
\left(\mathscr{A}_{n-1}, \mathscr{A}_{n-1}\right) & \left(\mathscr{A}_{n}, \mathscr{A}_{n-1}\right) & \cdots & \left(\mathscr{A}_{2 n-2}, \mathscr{A}_{n-1}\right)
\end{array}\right),  \tag{47}\\
& \vec{h}_{n}=\left(\begin{array}{c}
-\left(\mathscr{A}_{n}, \mathscr{A}_{0}\right) \\
-\left(\mathscr{A}_{n+1}, \mathscr{A}_{1}\right) \\
\vdots \\
-\left(\mathscr{A}_{2 n-1}, \mathscr{A}_{n-1}\right)
\end{array}\right),
\end{align*}
$$

and call $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right)$ the Hankel determinant of $f(x)$ with respect to the coefficients $\mathscr{A}_{0}, \mathscr{A}_{1}, \ldots, \mathscr{A}_{n-1}$.

Then (46) is converted into

$$
\begin{equation*}
H_{n}\left(\mathscr{A}_{0}\right) x=\vec{h}_{n}, \quad x=\left(b_{0}, b_{1}, \ldots, b_{n-1}\right)^{T} . \tag{48}
\end{equation*}
$$

In the case of TPTA, $v_{n}(x)$ is not arbitrarily chosen any more but is determined by the preceding system. The choice of $v_{n}(x)$ can help to improve the accuracy of approximation, but unfortunately we have not been able to guarantee that the solution of system (48) comes into existence, so far. We only give the following basic theorem about system (48) on the basis of linear algebra.

Theorem 19. The solution of (48) exists if and only if $\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right)\right)=\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right): \vec{h}_{n}\right)$. Moreover, the solution is unique if $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right) \neq 0$.

Proof. The proof of the assertion follows from the simple fact that, for a system of linear equations, described by $A x=b$, where $A$ is matrix and $x, b$ are vectors, the solution of system (48) comes into existence for $x$ if and only if $\operatorname{rank}(A)=$ $\operatorname{rank}(A: b)$; i.e., the right-hand vector must be in the vector space spanned by the columns of the coefficient matrix $A$. Moreover, if $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right) \neq 0$, according to Cramer's rule, the solution is unique.

Theorem 20 (existence). Let $f(x)$ be given power series (15); then $(n-1 / n)_{f}$ and $(m / n)_{f}$ exist and are unique if and only if $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right) \neq 0$.

Proof. " $\Leftarrow$ " By Theorem 19, if $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right) \neq 0$, it means that nonhomogeneous equation (48) exists as a unique solution $b_{0}, b_{1}, \ldots, b_{n-1}$ for $v_{n}(x)$. From (19), it also means that $\widetilde{v}_{n}(x)$ and $\widetilde{\mathscr{W}}_{n-1}(x)$ exist. Hence, by the construction of $(n-1 / n)_{f}$, existence holds.
$" \Rightarrow "$ Let $(n-1 / n)_{f}=\widetilde{\mathscr{W}}_{n-1}(x) / \widetilde{v}_{n-1}(x)$ exist and be unique; then it implies that

$$
\begin{equation*}
\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right)\right)=\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right): \vec{h}_{n}\right) \tag{49}
\end{equation*}
$$

and if $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right)=0$, then $\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right): \vec{h}_{n}\right)<n$; the fact that equation (46) has $n-\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right): \vec{h}_{n}\right)$ solutions, namely, that we can construct $n-\operatorname{rank}\left(H_{n}\left(\mathscr{A}_{0}\right)\right.$ :
$\vec{h}_{n}$ ) generating polynomials, which is contradictory to the uniqueness of $(n-1 / n)_{f}$, holds.

The proof of existence and uniqueness of $(m / n)_{f}$ is similar to the preceding process.

Theorem 21. Let $\operatorname{det}\left(H_{n}\left(\mathscr{A}_{0}\right)\right) \neq 0$; then $(n-1 / n)_{f}(x)=$ $\widetilde{\mathscr{W}}_{n-1}(x) / \widetilde{v}_{n}(x)$, where the generating polynomial $v_{n}(x)$ is given by

$$
v_{n}(x)=\operatorname{det}\left(\begin{array}{cc}
H_{n}\left(\mathscr{A}_{0}\right) & \vec{\gamma}  \tag{50}\\
\vec{\delta}^{T} & x^{n}
\end{array}\right)
$$

where $\vec{\gamma}=\left(\left(\mathscr{A}_{n}, \mathscr{A}_{0}\right),\left(\mathscr{A}_{n+1}, \mathscr{A}_{1}\right), \ldots,\left(\mathscr{A}_{2 n-1}, \mathscr{A}_{n-1}\right)\right)^{T}, \vec{\delta}^{T}=$ $\left(1, x, \ldots, x^{n-1}\right)$, and $\widetilde{v}_{n}(x)$ and $\widetilde{\mathscr{V}}_{n-1}(x)$ are given by (19) and (20), respectively.

Now, we can derive an algorithm to calculate $(m / n)_{f}(x)$ using (26), (27), and (50).

Algorithm $22\left(\right.$ compute $\left.(m / n)_{f}(x)=\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)\right)$.
(1) Use (14) to calculate $H_{n}\left(\mathscr{A}_{0}\right)$ and $\vec{\gamma}$.
(2) Use (50) and (19) to compute $v_{n}(x)$ and $\widetilde{v}_{n}(x)$, respectively.
(3) Set $q=m-n+1$ and compute $\mathscr{W}_{q}(x)$ and $\widetilde{\mathscr{W}}_{q}(x)$ by

$$
\begin{align*}
& \mathscr{W}_{q}(x)=\phi^{(m-n+1)}\left(\frac{v_{n}(x)-v_{n}(t)}{x-t}\right),  \tag{51}\\
& \widetilde{\mathscr{W}}_{q}(x)=x^{n-1} \mathscr{W}_{q}\left(x^{-1}\right)
\end{align*}
$$

(4) Compute the numerator of TPTA by

$$
\begin{equation*}
\mathscr{P}_{m n}(x)=\widetilde{v}_{n}(x) \sum_{i=0}^{m-n} \mathscr{A}_{i} x^{i}+x^{m-n+1} \widetilde{\mathscr{W}}_{q}(x) . \tag{52}
\end{equation*}
$$

(5) Obtain $(m / n)_{f}(x)=\mathscr{P}_{m n}(x) / \widetilde{v}_{n}(x)$.

## 4. Application for Computing the Tensor Exponential Function

The method of truncated infinite series has abroad applications in finite single crystal plasticity for computing tensor exponential function [16]. However, the accuracy and effectiveness of such algorithm are limited by round-off and choice of termination criterion. In this section, we will utilize the
method of TPTA to compute tensor exponential function. We start by briefly reviewing some basic equations that model the behaviour of single crystals in the finite strain range [16].

Consider a single crystal model

$$
\begin{equation*}
F=F^{e} F^{p} \tag{53}
\end{equation*}
$$

where $F^{e}$ and $F^{p}$ denote elastic part and plastic part, respectively.

For a single crystal with a total number $n_{\text {syst }}$ of slip systems, the evolution of the inelastic deformation gradient, $F^{p}$, is defined by means of the following rate form:

$$
\begin{equation*}
\dot{F}^{p} F^{p-1}=\sum_{\alpha=1}^{n_{\text {syst }}} \dot{\gamma}^{\alpha} s_{0}^{\alpha} \otimes m_{0}^{\alpha}, \tag{54}
\end{equation*}
$$

where $\dot{\gamma}^{\alpha}$ denotes the contribution of slip system $\alpha$ to the total inelastic rate of deformation. The vectors $s_{0}^{\alpha}$ and $m_{0}^{\alpha}$ denote, respectively, the slip direction and normal direction of slip system $\alpha$.

The above tensor differential equation can be discretized in an implicit fashion with use of the tensor exponential function. The implicit exponential approximation to the inelastic flow equation results in the following discrete form:

$$
\begin{equation*}
F_{n+1}^{p}=\exp \left(\sum_{\alpha=1}^{n_{s y s t}} \Delta \gamma^{\alpha} s_{0}^{\alpha} \otimes m_{0}^{\alpha}\right) F_{n}^{p} \tag{55}
\end{equation*}
$$

The above formula is analogous to the exact solution of initial value problem (1) and it is necessary to calculate

$$
\begin{equation*}
\exp \left(\sum_{\alpha=1}^{n_{\text {syst }}} \Delta \gamma^{\alpha} s_{0}^{\alpha} \otimes m_{0}^{\alpha}\right) \tag{56}
\end{equation*}
$$

In [7], the author used Algorithm 23 to calculate (56).
Algorithm 23 (truncated infinite series method [7] (p.749)).
(1) Given tensor $\mathscr{X}$, initialise $n=0$ and $\exp (\mathscr{X}):=\mathscr{F}$.
(2) Increment counter $n:=n+1$.
(3) Compute $n$ ! and $X^{n}$.
(4) Add new term to the series $\exp (\mathscr{X}):=\exp (\mathscr{X})+$ $(1 / n!) X^{n}$.
(5) Check convergence, if

$$
\begin{equation*}
\frac{\left\|X^{n}\right\|}{n!}<\epsilon_{\text {tol }}, \quad \text { then exit, else goto } \tag{57}
\end{equation*}
$$

Example 24. Consider a tensor exponential function $\exp (\mathscr{A} x)$; the entries of $\mathscr{A}$ are $a_{121}=1, a_{221}=-2, a_{122}=2$, $a_{222}=-1$, and zero elsewhere.

To find a tensor Padé-type approximation of type (3/3) for the tensor exponential function, first we should expand $\exp (\mathscr{A} x)$ into power series by means of Definition 7. We can obtain

$$
\begin{align*}
\exp (\mathscr{A} x)= & \left(\begin{array}{ll|ll}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{array}\right)+\left(\begin{array}{cc|cc}
0 & 1 & 0 & 2 \\
0 & -2 & 0 & -1
\end{array}\right) x \\
& +\left(\begin{array}{cc|cc}
0 & -2 & 0 & -\frac{5}{2} \\
0 & \frac{5}{2} & 0 & 2
\end{array}\right) x^{2} \\
& +\left(\begin{array}{cc|cc}
0 & \frac{13}{6} & 0 & \frac{14}{6} \\
0 & -\frac{14}{6} & 0 & -\frac{13}{6}
\end{array}\right) x^{3}  \tag{58}\\
& +\left(\begin{array}{cc|cc}
0 & -\frac{40}{24} & 0 & -\frac{41}{24} \\
0 & \frac{41}{24} & 0 & \frac{40}{24}
\end{array}\right) x^{4}+\cdots \\
= & \mathscr{A}_{0}+\mathscr{A}_{1} x+\mathscr{A}_{2} x^{2}+\mathscr{A}_{3} x^{3}+\mathscr{A}_{4} x^{4} \\
& +\cdots .
\end{align*}
$$

By Algorithm 22, the following can be done.
(1) Use (14) to compute $H_{3}\left(\mathscr{A}_{0}\right)$ and $\vec{\gamma}$ :

$$
\begin{align*}
H_{3}\left(\mathscr{A}_{0}\right) & =\left(\begin{array}{ccc}
2 & -2 & \frac{5}{2} \\
10 & -14 & \frac{41}{3} \\
\frac{41}{2} & -\frac{61}{3} & \frac{365}{24}
\end{array}\right),  \tag{59}\\
\vec{\gamma} & =\left(\begin{array}{c}
-\frac{7}{3} \\
-\frac{61}{6} \\
-\frac{547}{60}
\end{array}\right)
\end{align*}
$$

(2) Use (50) to calculate $v_{3}(x)$ :

$$
\begin{equation*}
v_{3}(x)=\frac{15041}{1080}+\frac{3947}{60} x+\frac{1189}{10} x^{2}+\frac{1493}{18} x^{3} \tag{60}
\end{equation*}
$$

and compute $\widetilde{\gamma_{3}}(x)$ by (19), so we get

$$
\begin{equation*}
\widetilde{v}_{3}(x)=\frac{15041}{1080} x^{3}+\frac{3947}{60} x^{2}+\frac{1189}{10} x+\frac{1493}{18} . \tag{61}
\end{equation*}
$$

(3) Set $q=m-n+1=1$ and compute $\mathscr{W}_{1}(x), \widetilde{\mathscr{W}}_{1}(x)$ :

$$
\mathscr{W}_{1}(x)=\left(\begin{array}{cc|cc}
0 & \frac{1493}{18} x^{2}-\frac{4229}{90} x+\frac{1039}{135} & 0 & \frac{1493}{9} x^{2}+\frac{5479}{180} x+\frac{15041}{540}  \tag{62}\\
0 & -\frac{1493}{9} x^{2}-\frac{5479}{180} x-\frac{15041}{540} & 0 & -\frac{1493}{18} x^{2}+\frac{4229}{90} x-\frac{1039}{135}
\end{array}\right)
$$

Table 1: Numerical results of Example 24 at different points by using Algorithm 22.

| $x$ |  | $(1,2,1)$ | $(2,2,1)$ | $(1,2,2)$ | (2, 2, 2) | Res |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 0.1 | $e^{d x}$ | 0.08200959 | 0.82282781 | 0.17717281 | -0.08200959 | 8.34e-12 |
|  | $(3 / 3)_{e^{8 / x}}(x)$ | 0.08200778 | 0.82282688 | 0.17717311 | -0.08200778 |  |
| 0.2 | $e^{\mathscr{A} x}$ | 0.13495955 | 0.68377119 | 0.31622880 | -0.13495955 | 1.62e-9 |
|  | $(3 / 3)_{e^{8 x}(x)}$ | 0.13493452 | 0.68375764 | 0.31624235 | -0.13493452 |  |
| 0.3 | $e^{8 / x}$ | 0.16712428 | 0.57369394 | 0.42630605 | -0.16712428 | 3.14e-8 |
|  | $(3 / 3)_{e^{8 x}(x)}$ | 0.16701602 | 0.57363058 | 0.42636941 | -0.16701602 |  |
| 0.4 | $e^{8 / x}$ | 0.18456291 | 0.48575712 | 0.51424287 | -0.18456291 | 2.38e-7 |
|  | $(3 / 3)_{e^{8 x}(x)}$ | 0.18427224 | 0.48557038 | 0.51442961 | -0.18427224 |  |

Table 2: The exact value of $\exp (\mathscr{A})$.

|  | $(1,2,1)$ | $(2,2,1)$ | $(1,2,2)$ | $(2,2,2)$ |
| :--- | :---: | :---: | :---: | :---: |
| $\exp (\mathscr{A})$ | 0.15904705 | 0.20883238 | 0.79116761 | -0.15904705 |

and

$$
\widetilde{\mathscr{V}}_{1}(x)=\left(\begin{array}{cc|cc}
0 & \frac{1493}{18}-\frac{4229}{90} x+\frac{1039}{135} x^{2} & 0 & \frac{1493}{9}+\frac{5479}{180} x+\frac{15041}{540} x^{2}  \tag{63}\\
0 & -\frac{1493}{9}-\frac{5479}{180} x-\frac{15041}{540} x^{2} & 0 & -\frac{1493}{18}+\frac{4229}{90} x-\frac{1039}{135} x^{2}
\end{array}\right)
$$

(4) Substitute $\widetilde{v}_{3}(x)$ and $\widetilde{\mathscr{V}}_{1}(x)$ into (27) to compute $\mathscr{P}_{33}(x)$ :

$$
\begin{align*}
& \mathscr{P}_{33}(x) \\
& =\left(\left.\begin{array}{cc}
\frac{15041}{1080} x^{3}+\frac{3947}{60} x^{2}+\frac{1189}{10} x+\frac{1493}{18} & \frac{1039}{135} x^{3}-\frac{4229}{90} x^{2}+\frac{1493}{18} x \\
0 & -\frac{15041}{1080} x^{3}+\frac{3181}{90} x^{2}-\frac{4229}{90} x+\frac{1493}{18}
\end{array} \right\rvert\, \begin{array}{cc}
\frac{15041}{540} x^{3}+\frac{5479}{180} x^{2}+\frac{1493}{9} x \\
0 & -\frac{1039}{135} x^{3}+\frac{4229}{90} x^{2}-\frac{1493}{18} x
\end{array}\right) \tag{64}
\end{align*}
$$

(5) Obtain $(m / n)_{f}(x)=\mathscr{P}_{33}(x) / \widetilde{v}_{3}(x)$.

In Table 1 we compare the number of exact figures given by the method of TPTA of type (3/3) with corresponding exact value of $\exp (\mathscr{A} x)$ referring to the entries of $(1,2,1)$, $(2,2,1),(1,2,2)$, and $(2,2,2)$. We also compute the norm of absolute residual tensor (denoted by Res). Here,

$$
\begin{equation*}
\operatorname{Res}\left(x_{i}\right)=\left\|\exp \left(\mathscr{A} x_{i}\right)-\left(\frac{3}{3}\right)_{e^{a d x}}\left(x_{i}\right)\right\|^{2} \tag{65}
\end{equation*}
$$

where the operation $\|\cdot\|$ is defined by (13).
From Table 1, it is observed that the estimates from TPTA can reach the desired accuracy.

Example 25. Let $f(x)$ be given by Example 24.

By Algorithm 22 for preceding example again, we calculate $(m / m)_{f}(1), m=1,2,3,4,5$. The exact value and approximant value associated with the entries of $(1,2,1)$, $(2,2,1),(1,2,2)$, and $(2,2,2)$ are listed in Tables 2 and 3, respectively, where $x=1$.

From Table 3, we can see that $(3 / 3)_{e^{a x}}$ has the best approximation for this example. We also compute $\exp (\mathscr{A})$ by using Algorithm 23, and the corresponding numerical results are listed in Table 4. By comparison of Table 3 with Table 4, we find that it requires at most 6 coefficients (since $m=3$ ) of power series expansion of $\exp (\mathscr{A} x)$ to achieve an error of $10^{-5}$ in Algorithm 22, while requiring 11 coefficients in Algorithm 23. It is straightforward to understand that Algorithm 23 is more expensive than Algorithm 22 especially for higher order tensor exponential function. In practical applications, only few coefficients of the series may be known,

Table 3: Numerical approximations of $\exp (\mathscr{A})$ using Algorithm 22 for Example 25.

| $m$ | $(1,2,1)$ | $(2,2,1)$ | $(1,2,2)$ | $(2,2,2)$ | Res |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 0.50000000 | 0.00000000 | 1.00000000 | -0.50000000 | $3.19 \mathrm{e}-1$ |
| 2 | -2.12500000 | 0.75000000 | 0.25000000 | 2.12500000 | $1.10 \mathrm{e}+1$ |
| 3 | 0.15503865 | 0.20377270 | 0.79622729 | -0.15503865 | $8.33 \mathrm{e}-5$ |
| 4 | 0.17454584 | 0.22682585 | 0.77317481 | -0.17454584 | $1.12 \mathrm{e}-3$ |
| 5 | 0.17625313 | 0.19112365 | 0.80887636 | -0.17625313 | $1.21 \mathrm{e}-3$ |

Table 4: Numerical results of Example 25 by using Algorithm 23.

| $n_{\max }$ | $(1,2,1)$ | $(2,2,1)$ | $(1,2,2)$ | $(2,2,2)$ | Res |
| :--- | :---: | :---: | :---: | :---: | :---: |
| 1 | 1 | -1 | 2 | 2 | 4.33 |
| 2 | -1 | 1.5 | -0.5 | 1 | 6.02 |
| 3 | 1.16666666 | -0.83333333 | 1.83333333 | -1.16666666 | 4.20 |
| 4 | -0.50000000 | 0.87500000 | 0.12500000 | 0.50000000 | 1.75 |
| 5 | 0.50833333 | -0.14166666 | 1.14166666 | -0.50833333 | $4.89 \mathrm{e}-1$ |
| 6 | 0.00277777 | 0.36527777 | 0.63472222 | -0.00277777 | $9.77 \mathrm{e}-2$ |
| 7 | 0.21964285 | 0.14821428 | 0.85178571 | -0.21964285 | $1.47 \mathrm{e}-2$ |
| 8 | 0.13829365 | 0.22958829 | 0.77041170 | -0.13829365 | $1.72 \mathrm{e}-3$ |
| 9 | 0.16541280 | 0.20246638 | 0.79753361 | -0.16541280 | $1.62 \mathrm{e}-4$ |
| 10 | 0.15735119 | 0.21060267 | 0.78939732 | -0.15735119 | $1.22 \mathrm{e}-5$ |
| 11 | 0.15957013 | 0.20838371 | 0.79161628 | -0.15957013 | $8.77 \mathrm{e}-7$ |

so, we may get the desired results by means of TPTA. Thus the effectiveness of the proposed Algorithm 22 is verified.

## 5. Conclusion

In this paper, we presented tensor Padé-type approximant method for computing tensor exponential function; the expression of TPTA is of the form of tensor numerator and scalar denominator. In order to have a tensor Padétype approximant with the higher possible precision of approximation, we proposed an algorithm for computing denominator polynomials of TPTA, and its effectiveness has been investigated in one example of tensor exponential function. The key to the TPTA to be applied to the tensor exponential function is that it can be expanded into power series with the same order tensors coefficients by means of $t$ product. Of course, there are several ways to multiply tensors [26-30], but the order of the resulting tensor may be changed. For example, if $\mathscr{A}$ is $n_{1} \times n_{2} \times n_{3}$ and $\mathscr{B}$ is $n_{1} \times m_{2} \times m_{3}$, then the contracted product [26] of $\mathscr{A}$ and $\mathscr{B}$ is $n_{2} \times n_{3} \times m_{2} \times m_{3}$. So, the choice of the multiplication of two tensors is an open question for expanding tensor exponential function, and the corresponding tensor Padé approximant theoretic is a subject of further research.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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# A New Approach to the Existence of Quasiperiodic Solutions for Second-Order Asymmetric p-Laplacian Differential Equations 

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For $p \geq 2$ and $\phi_{p}(s):=|s|^{p-2} s$, we propose a new estimate approach to study the existence of Aubry-Mather sets and quasiperiodic solutions for the second-order asymmetric $p$-Laplacian differential equations $\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}+\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)=\psi(t, x)$, where $\lambda$ and $\mu$ are two positive constants satisfying $\lambda^{-1 / p}+\mu^{-1 / p}=2 / \omega$ with $\omega \in \mathbb{R}^{+}, \psi(t, x) \in C^{0,1}\left(\mathbf{S}^{p} \times \mathbb{R}\right)$ is a continuous function, $2 \pi_{p^{-}}$ periodic in the first argument and continuously differentiable in the second one, $x^{ \pm}=\max \{ \pm x, 0\}, \pi_{p}=2 \pi(p-1)^{1 / p} / p \sin (\pi / p)$, and $\mathbf{S}^{p}=\mathbb{R} / 2 \pi_{p} \mathbb{Z}$. Using the Aubry-Mather theorem given by Pei, we obtain the existence of Aubry-Mather sets and quasiperiodic solutions under some reasonable conditions. Particularly, the advantage of our approach is that it not only gives a simpler estimation procedure, but also weakens the smoothness assumption on the function $\psi(t, x)$ in the existing literature.

## 1. Introduction

In recent years, all kinds of nonlinear dynamic behavior, such as the existence of positive solutions [1-16] and signchanging solutions [17, 18], the existence and uniqueness of solutions [19-25], the existence and multiplicity results [26-30], and the existence of unbounded solutions[31, 32], have been widely investigated for some nonlinear ordinary differential equations and partial differential equations due to the application in many fields such as physics, mechanics, and the engineering technique fields. In the present paper, we deal with the existence of Aubry-Mather sets and quasiperiodic solutions for the second-order differential equations with a $p$-Laplacian and an asymmetric nonlinear term

$$
\begin{equation*}
\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}+\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)=\psi(t, x) \tag{1}
\end{equation*}
$$

where $\phi_{p}(s):=|s|^{p-2} s, p \geq 2, x^{ \pm}=\max \{ \pm x, 0\}$, and $\lambda$ and $\mu$ are positive constants satisfying

$$
\begin{equation*}
\frac{1}{\lambda^{1 / p}}+\frac{1}{\mu^{1 / p}}=\frac{2}{\omega}, \quad \omega \in \mathbb{R}^{+} \tag{2}
\end{equation*}
$$

where $\psi(t, x) \in C^{0,1}\left(\mathbf{S}^{p} \times \mathbb{R}\right)$ is a continuous function, $2 \pi_{p^{-}}$ periodic in the first argument and continuously differentiable in the second one, where $\mathbf{S}^{p}=\mathbb{R} / 2 \pi_{p} \mathbb{Z}$. Since the pioneering works of Aubry [33] and Mather [34], the existence of AubryMather sets and quasiperiodic solutions for a variety of differential equations, such as Hamiltonian systems [35-41], and reversible systems [42-44] had been widely investigated due to the application in many fields such as one-dimensional crystal model of solid state physics, differential geometry, and dynamical systems (see [45, 46]).

If $p=2$, then $\pi_{p}=\pi$ and (1) reduces to the following piecewise linear equation:

$$
\begin{equation*}
x^{\prime \prime}+\lambda x^{+}-\mu x^{-}=\psi(t, x) \tag{3}
\end{equation*}
$$

and (2) becomes

$$
\begin{equation*}
\frac{1}{\sqrt{\lambda}}+\frac{1}{\sqrt{\mu}}=\frac{2}{\omega}, \quad \omega \in \mathbb{R}^{+} \tag{4}
\end{equation*}
$$

The first result is due to Capietto and Liu [38], who proved that the existence of Aubry-Mather sets and quasiperiodic solutions of (3) for some $\omega \in \mathbb{Q}^{+}$in (4), provided that
$\psi(t, x)=e(t)-\phi(x), e(t) \in C^{2}(\mathbb{R} / 2 \pi \mathbb{Z})$, and the perturbation term $\phi(x) \in C^{2}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$ satisfies some growth conditions. Recently, this result was extended to a much weaker smoothness nonlinearity $\psi(t, x)$. In [41], by using the Aubry-Mather theorem generalized by Pei [37], the present author [41] studied the existence of Aubry-Mather sets and quasiperiodic solutions of (3), under the condition that $\omega \in$ $\mathbb{R}^{+}$in (4) and $\psi(t, x) \in C^{0,1}\left(\mathbf{S}^{p} \times \mathbb{R}\right)$ can be allowed to be either a bounded function or an unbounded function, which differs from above existing results.

In [39], (3) has been generalized to the following $p$ -Laplacian-like nonlinear differential equation:

$$
\begin{align*}
& (p-1)^{-1}\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}+\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)+g(x)  \tag{5}\\
& \quad=h(t)
\end{align*}
$$

where $p>1, \lambda$ and $\mu$ are positive constants satisfying (2) with $\omega=n \in \mathbb{N}, g(x) \in C^{1}(\mathbb{R}) \cap L^{\infty}(\mathbb{R})$, and $h(t) \in C^{2}\left(\mathbf{S}^{p}\right)$ is a $2 \pi_{p^{-}}$ periodic function. They considered the existence of AubryMather sets and quasiperiodic solutions of (5) when $g(x)$ satisfies some further approximate properties at infinity. We notice that in [39], to overcome the barriers of weak smoothness, they made use of exchange of the role of time and angle variables skills and showed the existence of Aubry-Mather sets and quasiperiodic solutions by employing a version of Aubry-Mather theorem obtained by Pei [37]. Moreover, the results in [39] need the smoothness requirement of the perturbation function at least to $C^{2}$ smooth in $t$.

Now a natural question to ask is whether the smoothness of the function $h(t)$ in (5) is further reduced; we can also obtain the same results as [13]. In this paper, we will deal with this interesting problem and answer this question in the form of Theorem 1 with more general case (1) than that of (5). Because of the presence of weak smoothness nonlinearity, the methods of seeking the existence of Aubry-Mather sets and quasiperiodic solutions for problems as $[38,39]$ do not seem to be applicable to (1). This phenomenon provokes some mathematical difficulties, which make the study of (1) particularly interesting. Our approach here is mainly based on the direct proof of the Poincare map of the transformed system satisfying monotone twist property and is developed from the present author (see the recent papers [41, 44]) but is more subtle than the ones in [38-40]. More efforts have to be made to estimate the monotone twist property for the Poincaré map of the transformed system, but the procedure is a little simpler than those in [38-40]. One important advantage of our approach is that it does not require any high smoothness assumptions on function $\psi(t, x)$. Our results improve and generalize some results of the previous studies $[39,41]$ to some extent.

The main result of this paper is the following theorem.
Theorem 1. Suppose that (2) holds. Moreover, $\psi(t, x) \in$ $C^{0,1}\left(\mathbf{S}^{p} \times \mathbb{R}\right)$ satisfies the following conditions:
$\left(A_{1}\right)$ The limit is

$$
\begin{equation*}
\lim _{|x| \rightarrow+\infty} \psi_{x}(t, x)=0, \quad \text { uniformly in } t \in\left[0,2 \pi_{p}\right] . \tag{6}
\end{equation*}
$$

$\left(A_{2}\right)$ There exist constants $d \geq 0, \beta>0$, such that

$$
\begin{align*}
& \operatorname{sgn}(x)\left[(p-1) \psi(t, x)-x \psi_{x}(t, x)\right]>\beta \\
& \qquad \text { for } \forall|x| \geq d \text { and } \forall t \in\left[0,2 \pi_{p}\right] . \tag{7}
\end{align*}
$$

Then there exists $\varepsilon_{0}>0$, such that, for any $\alpha \in\left(2 \omega \pi_{p}\right.$, $\left.2 \omega \pi_{p}+\varepsilon_{0}\right)$, (1) possesses an Aubry-Mather type solution $z_{\alpha}(t)=$ $\left(x_{\alpha}(t), x_{\alpha}^{\prime}(t)\right)$ with rotation number $\alpha$; that is,
(i) if $\alpha=k / m$ is rational and $(k, m)=1$, the solutions $z_{\alpha}^{i}(t)=z_{\alpha}\left(t+2 \pi_{p} i\right), 0 \leq i \leq m-1$, are mutually unlinked periodic solutions of period m;
(ii) if $\alpha$ is irrational, the solution $z_{\alpha}(t)$ is either a usual quasiperiodic solution or a generalized one.

Remark 2. A solution is called generalized quasiperiodic one if the closed set

$$
\begin{equation*}
M_{\alpha} \equiv \overline{\left\{z_{\alpha}\left(2 \pi_{p} i\right), i \in \mathbf{Z}\right\}} \tag{8}
\end{equation*}
$$

is Denjoy's minimal set (see its definition in [47]).
Remark 3. Using the rule of L'Hospital to condition $\left(A_{1}\right)$, it can easily be seen that

$$
\lim _{|x| \rightarrow+\infty} \frac{\psi(t, x)}{x}=0, \quad \text { uniformly in } t \in\left[0,2 \pi_{p}\right] . \quad\left(\psi_{0}\right)
$$

Remark 4. We noticed that the perturbations $g(x)$ and $h(t)$ in [39] need to be bounded. But from $\left(A_{1}\right)$ and $\left(A_{2}\right)$ of this paper, it is easy to verify that the perturbation $\psi(t, x)$ can be either a bounded function or an unbounded function. For example, we can set $\psi(t, x)$ to be a bounded function $\arctan x \cdot\left(1+\sin ^{2}\left(\left(\pi / \pi_{p}\right) t\right)\right)$ or an unbounded function $x^{1 / 3} \cdot\left(1+\cos ^{2}\left(\left(\pi / \pi_{p}\right) t\right)\right)$ for $t \in\left[0,2 \pi_{p}\right]$ when $d=1$ and $\beta=\pi / 4-1 / 2$ in Theorem 1. Moreover, positive constant $\omega=n \in \mathbb{N}$ satisfying (1.2) in [39] has been extended to the case $\omega \in \mathbb{R}^{+}$in this paper. Thus, our situation is more general than the results obtained in [39] for $p \geq 2$.

Remark 5. If $p=2$, let us point out that the results in Theorem 1 have covered the conclusions obtained by Wang [41]. Besides, the estimation process in this paper is much more meticulous than that in [41] since the $p$-Laplacian $\left(\phi_{p}\left(x^{\prime}\right)\right)^{\prime}$ of a function $x(t)$, with $p>2$, is no longer linear. Therefore, the results obtained in this paper are natural generalizations and refinements of the results obtained in [41].

The main idea of our proof is acquired from [39, 41]. The proof of Theorem 1 is based on an Aubry-Mather theorem due to Pei [37]. The rest of this manuscript is as follows. In Section 2, we introduce some action-angle variables transformation to transform system (1) into an equivalent integral Hamiltonian system and then present some growth properties on the corresponding action and angle variables functions. In Section 3, we provide some crucial estimates by some lemmas which say that the Poincare mapping of the new system is monotone twist around the infinity. At last, Section 4 gives the proof of Theorem 1 by using Pei's AubryMather theorem [37].

## 2. Preliminaries

2.1. The Action and Angle Variables. Let $S_{p}(t)=\sin _{p} t$ be the solution of

$$
\begin{equation*}
\left(\phi_{p}\left(u^{\prime}\right)\right)^{\prime}+\phi_{p}(u)=0 \tag{9}
\end{equation*}
$$

satisfying the initial condition $u(0)=0, u^{\prime}(0)=1$. Then it follows from [48] that $S_{p}(t)=\sin _{p} t$ is a $2 \pi_{p}$-periodic $C^{2}$ odd function with $\sin _{p}\left(\pi_{p}-t\right)=\sin _{p} t$, for $t \in\left[0, \pi_{p} / 2\right]$, and $\sin _{p}\left(2 \pi_{p}-t\right)=-\sin _{p} t$ for $t \in\left[\pi_{p}, 2 \pi_{p}\right]$. Moreover, for $t \in$ $\left(0, \pi_{p} / 2\right), S_{p}(t)>0, S_{p}^{\prime}(t)>0$ and $S_{p}:\left[0, \pi_{p} / 2\right] \rightarrow[0,(p-$ $1)^{1 / p}$ ] can be implicitly given by

$$
\begin{equation*}
\int_{0}^{\sin _{p} t} \frac{d s}{\left(1-s^{p} /(p-1)\right)^{1 / p}}=t \tag{10}
\end{equation*}
$$

Introducing a new variable $y=-\phi_{p}\left(x^{\prime}\right)$, then (9) is equivalent to the planar system

$$
\begin{align*}
& x^{\prime}=-\phi_{q}(y) \\
& y^{\prime}=\phi_{p}(x) \tag{11}
\end{align*}
$$

where $q$ is the conjugate exponent of $p: p^{-1}+q^{-1}=1$. Letting $(x, y)=\left(C_{p}(t), S_{p}(t)\right)$ be the unique solution of (11) satisfying $\left(C_{p}(0), S_{p}(0)\right)=(1,0)$, then the functions $C_{p}(t)$ and $S_{p}(t)$ are much similar to cosine and sine. It follows from [49] that $C_{p}(t) \in C^{2}$ and $S_{p}(t) \in C^{1}$ are $2 \pi_{p}$-periodic, and for $\forall n \in \mathbb{Z}$, $C_{p}(t)=0$ iff $t=\pi_{p} / 2+n \pi_{p}$, and $S_{p}(t)=0$ iff $t=n \pi_{p}$. Moreover, $C_{p}^{\prime}(t)=-\phi_{q}\left(S_{p}(t)\right)$ and $S_{p}^{\prime}(t)=\phi_{p}\left(C_{p}(t)\right)$, and $(1 / p)\left|C_{p}(t)\right|^{1 / p}+(1 / q)\left|S_{p}(t)\right|^{1 / q} \equiv 1 / p$.

Now we consider (1). Set $y=-\phi_{p}\left(x^{\prime}\right)$ in (1); then (1) can be rewritten as a planar system

$$
\begin{align*}
& x^{\prime}=-\phi_{q}(y)  \tag{12}\\
& y^{\prime}=\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)-\psi(t, x)
\end{align*}
$$

where $q=p /(p-1)$ is the conjugate exponent of $p$.
Lemma 6. For $p \geq 2$ and for any $\left(x_{0}, y_{0}\right) \in \mathbb{R}^{2}, t_{0} \in \mathbb{R}$, the solution

$$
\begin{equation*}
z(t)=\left(x\left(t, t_{0}, x_{0}, y_{0}\right), y\left(t, t_{0}, x_{0}, y_{0}\right)\right) \tag{13}
\end{equation*}
$$

of (12) satisfying the initial condition $z\left(t_{0}\right)=\left(x_{0}, y_{0}\right)$ is unique and exists on the whole $t$-axis.

Proof. The proof of uniqueness can be established similarly to the proof of Proposition 2 in [50]; the global existence result can be acquired similarly to Lemma 3.1 in [51].

Let $(C(t), S(t))$ be the solution of the following homogeneous system:

$$
\begin{aligned}
x^{\prime} & =-\phi_{q}(y) \\
y^{\prime} & =\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right), \\
x(0) & =1 \\
y(0) & =0
\end{aligned}
$$

Then, by using (2) and direct computation, one obtains the following.

Lemma 7. (i) Both $C(t) \in C^{2}$ and $S(t) \in C^{1}$ are $2 \pi_{p} / \omega$ periodic functions, and $C(t)$ can be given by

$$
\begin{align*}
& C(t) \\
& =\left\{\begin{array}{lr}
C_{p}\left(\lambda^{1 / p} t\right), & 0 \leq|t| \leq \frac{\pi_{p}}{2 \lambda^{1 / p}} \\
C_{p}\left(\mu^{1 / p}\left(t-\frac{\pi_{p}}{2 \lambda^{1 / p}}+\frac{\pi_{p}}{2 \mu^{1 / p}}\right)\right), & \frac{\pi_{p}}{2 \lambda^{1 / p}} \leq t \leq \frac{\pi_{p}}{\omega}
\end{array}\right. \tag{15}
\end{align*}
$$

(ii) $C^{\prime}(t)=-\phi_{q}(S(t))$ and $S^{\prime}(t)=\lambda \phi_{p}\left(C(t)^{+}\right)-$ $\mu \phi_{p}\left(C(t)^{-}\right)$.
(iii) $(1 / q)|S(t)|^{q}+(1 / p)\left(\lambda\left|C(t)^{+}\right|^{p}+\mu\left|C(t)^{-}\right|^{p}\right)=\lambda / p$.

Now we introduce an action-angle variables transformation by the mapping $\Psi: \mathbb{S}^{p} \times(0,+\infty) \rightarrow \mathbb{R}^{2} \backslash\{(0,0)\}$, where $(x, y)=\Psi(\theta, I)$ defined by the formula

$$
\begin{align*}
& x=(\gamma I)^{1 / p} C\left(\frac{\theta}{\omega}\right),  \tag{16}\\
& y=(\gamma I)^{1 / q} S\left(\frac{\theta}{\omega}\right),
\end{align*}
$$

where $\gamma=\lambda^{-1} \omega p$ is a constant. This transformation is said to be a generalized symplectic transformation because its Jacobian is equal to 1 .
2.2. Some Properties on Action and Angle Variables Functions. Under the transformation $\Psi$ and using Lemma 7 (iii), (12) is changed into

$$
\begin{align*}
\dot{\theta} & =\Phi_{1}(t, \theta, I),  \tag{17}\\
\dot{I} & =\Phi_{2}(t, \theta, I),
\end{align*}
$$

where $\Phi_{1}(t, \theta, I)=\omega+x(\theta, I) \psi(t, x(\theta, I)) / p I, \Phi_{2}(t, \theta, I)=$ $\phi_{q}(y(\theta, I)) \psi(t, x(\theta, I)) / \omega$.

We notice that the relation between (17) and (12) is that if $\theta(t)=\theta\left(t ; \theta_{0}, I_{0}\right), I(t)=I\left(t ; \theta_{0}, I_{0}\right)$ are the solutions of (17) with the initial value condition $\theta(0)=\theta_{0}, I(0)=I_{0}$, then

$$
\begin{align*}
x(t) & =x\left(t ; \theta_{0}, I_{0}\right)=x\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right) \\
& =\left(\gamma I\left(t ; \theta_{0}, I_{0}\right)\right)^{1 / p} C\left(\frac{\theta\left(t ; \theta_{0}, I_{0}\right)}{\omega}\right) \tag{18}
\end{align*}
$$

and

$$
\begin{align*}
y(t) & =y\left(t ; \theta_{0}, I_{0}\right)=y\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right) \\
& =\left(\gamma I\left(t ; \theta_{0}, I_{0}\right)\right)^{1 / q} S\left(\frac{\theta\left(t ; \theta_{0}, I_{0}\right)}{\omega}\right) \tag{19}
\end{align*}
$$

are the solutions of (12) with initial data $x(0)=$ $x\left(0 ; \theta_{0}, I_{0}\right), y(0)=y\left(0 ; \theta_{0}, I_{0}\right)$. By Lemma 6, (17) has a unique solution for $I_{0}>0$ and $\theta_{0} \in \mathbb{R}$. Moreover, this
solution has continuous derivatives with respect to initial data $\theta_{0}$ and $I_{0}$.

For notional convenience, hereinafter, we write $x, y, \theta, I$ instead of $x\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right), y\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right)$, $\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)$, respectively.

Firstly, by some simple calculations, we have the following.

Lemma 8. (i) $\partial x / \partial I=x / p I, \quad \partial y / \partial I=y / q I, \partial x / \partial \theta=$ $-(1 / \omega) \phi_{q}(y), \partial y / \partial \theta=\left(\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)\right) / \omega$.
(ii) $|C(t)| \leq \max \left\{1,(\lambda / \mu)^{1 / p}\right\}:=C_{\infty},|S(t)| \leq(q \lambda /$ $p)^{1 / q}:=S_{\infty}$.

Now we are concerned with the growth estimates with regard to $I\left(t ; \theta_{0}, I_{0}\right)$ and $\theta\left(t ; \theta_{0}, I_{0}\right)$.

Lemma 9. The limit

$$
\begin{equation*}
\lim _{I_{0} \rightarrow+\infty} I\left(t ; \theta_{0}, I_{0}\right)=+\infty \tag{20}
\end{equation*}
$$

holds uniformly on $t \in\left[0,2 \pi_{p}\right]$.
Proof. In view of $\left(\psi_{0}\right)$ and (16), there exist constants $D>0$, $K>0$, such that

$$
\begin{array}{r}
\left|I^{\prime}(t)\right|=\left|\frac{\phi_{q}(y(\theta, I)) \psi(t, x(\theta, I))}{\omega}\right| \leq D I(t)+K \\
\forall I \neq 0 .
\end{array}
$$

Then, by the Gronwall inequality, one has

$$
\begin{gather*}
e^{-2 \pi D} I_{0}-\frac{K}{D}\left(1-e^{-2 \pi D}\right) \leq I(t) \\
\quad \leq e^{2 \pi D} I_{0}+\frac{K}{D}\left(e^{2 \pi D}-1\right) \tag{22}
\end{gather*}
$$

for all $t \in\left[0,2 \pi_{p}\right]$.
So, by (22), $I\left(t ; \theta_{0}, I_{0}\right) \rightarrow+\infty$ as $I_{0} \rightarrow+\infty$ uniformly for $t \in\left[0,2 \pi_{p}\right]$.

According to (22), it is easy to see the following.
Corollary 10. $\forall \theta_{0} \in \mathbb{R}$ and $\forall t \in\left[0,2 \pi_{p}\right]$, there exist constants $\rho_{2}>\rho_{1}>0$ and $\bar{I}>0$, such that

$$
\begin{equation*}
\rho_{1} I_{0} \leq I\left(t ; \theta_{0}, I_{0}\right) \leq \rho_{2} I_{0} \tag{23}
\end{equation*}
$$

when $I_{0} \geq \bar{I}$.
Lemma 11. $\forall \theta_{0} \in \mathbb{R}$ and $\forall t \in\left[0,2 \pi_{p}\right]$, there exists constant $\bar{I}>0$, such that

$$
\begin{equation*}
\frac{\omega}{2} \leq \theta^{\prime}\left(t ; \theta_{0}, I_{0}\right) \leq 2 \omega \tag{24}
\end{equation*}
$$

if $I_{0} \geq \bar{I}$.
Proof. Since $\left(\psi_{0}\right)$ holds, then, for every $\varepsilon>0$, there exists $M=M(\varepsilon)>0$, such that

$$
\begin{equation*}
|\psi(t, x)| \leq \varepsilon|x| \tag{25}
\end{equation*}
$$

if $|x| \geq M$ and $\forall t \in\left[0,2 \pi_{p}\right]$. Hence,

$$
\begin{equation*}
\frac{d \theta}{d t}=\omega+\frac{x \psi(t, x)}{p I} \geq \omega-\frac{\varepsilon x^{2}}{p I} \tag{26}
\end{equation*}
$$

Thus, by using action-angle variables transformation (16) and Lemma 9, there exists $\overline{I_{1}}>0$ such that $d \theta / d t \geq \omega / 2$ if $I_{0} \geq \overline{I_{1}}$.

For if $|x| \leq M$, we assume that $|\psi(t, x)| \leq \psi_{\infty}$, where $\psi_{\infty}=\max \left\{|\psi(t, x)|: t \in\left[0,2 \pi_{p}\right],|x| \leq M\right\}$, and then

$$
\begin{equation*}
\frac{d \theta}{d t}=\omega+\frac{x \psi(t, x)}{p I} \geq \omega-\frac{\psi_{\infty}|x|}{p I} . \tag{27}
\end{equation*}
$$

So, by (16), Lemma 8 (ii), and Lemma 9, there exists a constant $\overline{I_{2}}>0$, such that $d \theta / d t \geq \omega / 2$ if $I_{0} \geq \overline{I_{2}}$.

If we choose $\bar{I}=\max \left\{\overline{I_{1}}, \overline{I_{2}}\right\}$, then $I_{0} \geq \bar{I}$ implies $d \theta / d t \geq$ $\omega / 2$.

Exploiting the same arguments, one can show that the inequality on the right side of (i) holds.

## 3. Twist Property and Proof of Theorem 1

Let the Poincaré mapping $P$ of equation (17) be

$$
\begin{equation*}
P:\left(\theta_{0}, I_{0}\right) \longmapsto\left(\theta\left(2 \pi_{p}, \theta_{0}, I_{0}\right), I\left(2 \pi_{p}, \theta_{0}, I_{0}\right)\right) \tag{28}
\end{equation*}
$$

In order to apply the Aubry-Mather theorem developed by Pei [37], we only need to show that the Poincaré mapping $P$ is a monotone twist map around the infinity; that is, it is enough to show $\partial \theta\left(2 \pi ; \theta_{0}, I_{0}\right) / \partial I_{0}<0$ if $I_{0} \gg 1$. In the following we are going to give its detailed proofs by some lemmas.

Similarly, for notional convenience, hereinafter, we also write $x, y, \theta, I$ instead of $x\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right)$, $y\left(\theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)\right), \theta\left(t ; \theta_{0}, I_{0}\right), I\left(t ; \theta_{0}, I_{0}\right)$, respectively.

Lemma 12. The following convergences hold uniformly on $t \in$ $\left[0,2 \pi_{p}\right]$ :
(i) $x \psi(t, x) / I \rightarrow 0 ; x^{2} \psi_{x}(t, x) / I \rightarrow 0$, as $I_{0} \rightarrow+\infty$.
(ii) $\phi_{q}(y) \psi(t, x) / I \rightarrow 0 ; \phi_{q}(y) x \psi_{x}(t, x) / I \rightarrow 0$, as $I_{0} \rightarrow$ $+\infty$.
(iii) $x \psi^{2}(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x) / I^{2} \rightarrow 0 ; x \psi(t, x) \psi_{x}(t, x) \phi_{q}^{2}(y) /$ $I^{2} \rightarrow 0 ; x^{2} \psi_{x}(t, x) \psi(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x) / I^{2} \rightarrow 0 ;$ $x^{2} \psi_{x}^{2}(t, x) \phi_{q}^{2}(y) / I^{2} \rightarrow 0$, as $I_{0} \rightarrow+\infty$.

Proof. If $\left(A_{1}\right)$ and $\left(\psi_{0}\right)$ hold, then to each $\varepsilon>0$ there corresponds a positive number $M=M(\varepsilon)>0$, such that

$$
\begin{equation*}
\left|\psi_{x}(t, x)\right| \leq \frac{\varepsilon}{2 \gamma^{2 / p}} \tag{29}
\end{equation*}
$$

and

$$
\begin{equation*}
|\psi(t, x)| \leq \frac{\varepsilon}{2 \gamma^{2 / p}}|x| \tag{30}
\end{equation*}
$$

when $|x| \geq M$ and $t \in\left[0,2 \pi_{p}\right]$, where $\gamma=\lambda^{-1} \omega p$ is a constant given in (16).

Denote $V_{1}(\varepsilon)=\max \left\{|\psi(t, x)|: t \in\left[0,2 \pi_{p}\right],|x| \leq M\right\}$, $V_{2}(\varepsilon)=\max \left\{\left|\psi_{x}(t, x)\right|: t \in\left[0,2 \pi_{p}\right],|x| \leq M\right\}$.
(i) By action-angle variables transformation (16) and Lemma 8 (ii) and $p \geq 2$, we have

$$
\begin{align*}
\left|\frac{x \psi(t, x)}{I}\right| & \leq \frac{M V_{1}(\varepsilon)}{I}+\frac{\varepsilon x^{2}}{2 \gamma^{2 / p} I} \\
& \leq \frac{M V_{1}(\varepsilon)}{I}+\frac{\varepsilon}{2 I^{1-2 / p}} \\
\left|\frac{x^{2} \psi_{x}(t, x)}{I}\right| & \leq \frac{M^{2} V_{2}(\varepsilon)}{I}+\frac{\varepsilon x^{2}}{2 \gamma^{2 / p} I}  \tag{31}\\
& \leq \frac{M^{2} V_{2}(\varepsilon)}{I}+\frac{\varepsilon}{2 I^{1-2 / p}}
\end{align*}
$$

Then, given $\bar{I}>0$, choosing $I_{0}$ so that $I_{0} \geq \bar{I}$, by using Corollary 10, provided

$$
\begin{equation*}
I(t) \geq \max \left\{\frac{2 M V_{1}(\varepsilon)}{\varepsilon} ; \frac{2 M^{2} V_{2}(\varepsilon)}{\varepsilon} ; 1\right\} \tag{32}
\end{equation*}
$$

we have

$$
\begin{align*}
\left|\frac{x \psi(t, x)}{I}\right| & \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \leq \varepsilon \\
\left|\frac{x^{2} \psi_{x}(t, x)}{I}\right| & \leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \leq \varepsilon . \tag{33}
\end{align*}
$$

Since $\varepsilon>0$ is arbitrary, the proof of (i) is complete.
For (ii), observe that $p \geq 2$, and, combining (16) and Lemma 8 (ii), one has

$$
\begin{align*}
\left|\frac{\phi_{q}(y) \psi(t, x)}{I}\right| \leq & \frac{\varepsilon|y|^{q-1}|x|}{2 \gamma^{2 / p} I}+\frac{V_{1}(\varepsilon)|y|^{q-1}}{I} \\
\leq & \frac{\varepsilon}{2} C_{\infty} S_{\infty}^{q-1} I^{2 / p-1} \\
& +V_{1}(\varepsilon) \gamma^{1 / p} S_{\infty}^{q-1} I^{-1 / q} ; \\
\left|\frac{\phi_{q}(y) x \psi_{x}(t, x)}{I}\right| \leq & \frac{\varepsilon|y|^{q-1}|x|}{2 \gamma^{2 / p} I}+\frac{M|y|^{q-1} V_{2}(\varepsilon)}{I}  \tag{34}\\
\leq & \frac{\varepsilon}{2} C_{\infty} S_{\infty}^{q-1} I^{2 / p-1} \\
& +M \gamma^{1 / p} V_{2}(\varepsilon) S_{\infty}^{q-1} I^{-1 / q} .
\end{align*}
$$

Then, given $\bar{I}>0$, choosing $I_{0}$ so that $I_{0} \geq \bar{I}$, by using Corollary 10, provided

$$
\begin{align*}
& I^{1 / q}(t) \\
& \quad \geq \max \left\{\frac{2 V_{1}(\varepsilon) S_{\infty}^{q-1} \gamma^{1 / p}}{\varepsilon} ; \frac{2 M V_{2}(\varepsilon) S_{\infty}^{q-1} \gamma^{1 / p}}{\varepsilon} ; 1\right\}, \tag{35}
\end{align*}
$$

we have

$$
\begin{array}{r}
\left|\frac{\phi_{q}(y) \psi(t, x)}{I}\right|<\frac{\sigma \varepsilon}{2}+\frac{\varepsilon}{2} \\
\left|\frac{\phi_{q}(y) x \psi_{x}(t, x)}{I}\right|<\frac{\sigma \varepsilon}{2}+\frac{\varepsilon}{2} \tag{36}
\end{array}
$$

where $\sigma=C_{\infty} S_{\infty}^{q-1}$. Since $\varepsilon>0$ is arbitrary, (ii) is proved.
(iii) Set $\eta_{1}=q C_{\infty}^{p+2} S_{\infty}^{q-2}, \eta_{2}=q M^{p+2} S_{\infty}^{q-2} \gamma^{1-2 / q}, \nu_{1}=$ $C_{\infty}^{2} S_{\infty}^{2(q-1)}, v_{2}=M V_{2}(\varepsilon) \gamma^{2 / p} S_{\infty}^{2(q-1)}$. In view of (16), Lemma 7 (ii), and $p \geq 2$, we can get

$$
\begin{aligned}
& \left|\frac{x \psi^{2}(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x)}{I^{2}}\right| \\
& \quad=\left|\frac{(q-1) x \psi^{2}(t, x) \phi_{p}(x) y^{q-2}}{I^{2}}\right| \\
& \quad \leq \frac{q|x|^{p+2}|y|^{q-2} \varepsilon^{2}}{4 \gamma^{4 / p} I^{2}}+\frac{q M^{p+2}|y|^{q-2}}{I^{2}} \\
& \quad \leq \frac{\varepsilon^{2}}{4} q I^{4 / p-2} C_{\infty}^{p+2} S_{\infty}^{q-2}+\frac{q M^{p+2} S_{\infty}^{q-2} \gamma^{1-2 / q}}{I^{1+2 / q}} \\
& \quad \leq \frac{\varepsilon^{2} \eta_{1}}{2} I^{4 / p-2}+\frac{\eta_{2}}{2 I^{1+2 / q}} ;
\end{aligned}
$$

$$
\left|\frac{x \psi(t, x) \psi_{x}(t, x) \phi_{q}^{2}(y)}{I^{2}}\right|
$$

$$
\leq \frac{|y|^{2(q-1)}|x|^{2} \varepsilon^{2}}{4 \gamma^{4 / p} I^{2}}+\frac{M V_{1}(\varepsilon) V_{2}(\varepsilon)|y|^{2(q-1)}}{I^{2}}
$$

$$
\leq \frac{\varepsilon^{2}}{4} I^{4 / p-2} C_{\infty}^{2} S_{\infty}^{2(q-1)}+\frac{M V_{1}(\varepsilon) V_{2}(\varepsilon) \gamma^{2 / p} S_{\infty}^{2(q-1)}}{I^{2 / q}}
$$

$$
\leq \frac{\varepsilon^{2} \nu_{1} I^{4 / p-2}}{2}+\frac{V_{1}(\varepsilon) v_{2}}{2 I^{2 / q}}
$$

$$
\left|\frac{x^{2} \psi_{x}(t, x) \psi(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x)}{I^{2}}\right|
$$

$$
=\left|\frac{(q-1) x^{2} \psi(t, x) \psi_{x}(t, x) \phi_{p}(x)|y|^{q-2}}{I^{2}}\right|
$$

$$
\leq \frac{q|x|^{p+2}|y|^{q-2} \varepsilon^{2}}{4 \gamma^{4 / p} I^{2}}+\frac{q M^{p+1} V_{1}(\varepsilon) V_{2}(\varepsilon)|y|^{q-2}}{I^{2}}
$$

$$
\leq \frac{\varepsilon^{2}}{4} q I^{4 / p-2} C_{\infty}^{p+2} S_{\infty}^{q-2}
$$

$$
+\frac{q M^{p+2} V_{1}(\varepsilon) V_{2}(\varepsilon) S_{\infty}^{q-2} \gamma^{1-2 / q}}{I^{1+2 / q}}
$$

$$
\begin{align*}
& \leq \frac{\varepsilon^{2} \eta_{1}}{2} I^{4 / p-2}+\frac{V_{1}(\varepsilon) V_{2}(\varepsilon) \eta_{2}}{2 I^{1+2 / q}} ; \\
\mid & \left.\frac{x^{2} \psi_{x}^{2}(t, x) \phi_{q}^{2}(y)}{I^{2}} \right\rvert\, \\
& \leq \frac{|x|^{2}|y|^{2(q-1)} \varepsilon^{2}}{4 \gamma^{4 / p} I^{2}}+\frac{M^{2} V_{2}(\varepsilon)|y|^{2(q-1)}}{I^{2}} \\
& \leq \frac{\varepsilon^{2}}{4} I^{4 / p-2} C_{\infty}^{2} S_{\infty}^{2(q-1)}+\frac{M^{2} V_{2}(\varepsilon) \gamma^{2 / p} S_{\infty}^{2(q-1)}}{I^{2 / q}} \\
& \leq \frac{\varepsilon^{2} v_{1} I^{4 / p-2}}{2}+\frac{M v_{2}}{2 I^{2 / q}} . \tag{37}
\end{align*}
$$

Then, given $\bar{I}>0$, choosing $I_{0}$ so that $I_{0} \geq \bar{I}$, by using Corollary 10, provided

$$
\begin{align*}
& I^{1 / q}(t) \geq \max \left\{\left(\frac{\eta_{2}}{\varepsilon}\right)^{1 /(q+2)} ;\left(\frac{V_{1}(\varepsilon) \nu_{2}}{\varepsilon}\right)^{1 / 2} ;\right. \\
& \left.\quad\left(\frac{V_{1}(\varepsilon) V_{2}(\varepsilon) \eta_{2}}{\varepsilon}\right)^{1 /(q+2)} ;\left(\frac{M v_{2}}{\varepsilon}\right)^{1 / 2} ; 1\right\}, \tag{38}
\end{align*}
$$

we have

$$
\begin{align*}
\left|\frac{x \psi^{2}(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x)}{I^{2}}\right| & \leq \frac{\eta_{1} \varepsilon^{2}}{2}+\frac{\varepsilon}{2} \\
\left|\frac{x \psi(t, x) \psi_{x}(t, x) \phi_{q}^{2}(y)}{I^{2}}\right| & \leq \frac{v_{1} \varepsilon^{2}}{2}+\frac{\varepsilon}{2} ; \\
\left|\frac{x^{2} \psi_{x}(t, x) \psi(t, x) \phi_{q}^{\prime}(y) \phi_{p}(x)}{I^{2}}\right| & \leq \frac{\eta_{1} \varepsilon^{2}}{2}+\frac{\varepsilon}{2} ;  \tag{39}\\
\left|\frac{x^{2} \psi_{x}^{2}(t, x) \phi_{q}^{2}(y)}{I^{2}}\right| & \leq \frac{v_{1} \varepsilon^{2}}{2}+\frac{\varepsilon}{2}
\end{align*}
$$

Since $\varepsilon>0$ is arbitrary, the proof of (iii) is finished.
$\forall t \in\left[0,2 \pi_{p}\right]$, set

$$
\begin{align*}
b_{1}(t)= & \frac{\partial \Phi_{1}}{\partial I}=\frac{-x\left[(p-1) \psi(t, x)-x \psi_{x}(t, x)\right]}{p^{2} I^{2}} \\
b_{2}(t)= & \frac{\partial \Phi_{1}}{\partial \theta}=\frac{-\phi_{q}(y)}{\omega p I}\left[\psi(t, x)+x \psi_{x}(t, x)\right] \\
b_{3}(t)= & \frac{\partial \Phi_{2}}{\partial \theta}  \tag{40}\\
= & \frac{\left[\lambda \phi_{p}\left(x^{+}\right)-\mu \phi_{p}\left(x^{-}\right)\right] \phi_{q}^{\prime}(y) \psi(t, x)}{\omega^{2}} \\
& -\frac{\phi_{q}^{2}(y) \psi_{x}(t, x)}{\omega^{2}}
\end{align*}
$$

As a result of Lemma 9, Corollary 10, and Lemma 12, we have the following.

Lemma 13. $\forall t, s \in\left[0,2 \pi_{p}\right]$, the following conclusions hold:
(i) $b_{1}(t)=o\left(1 / I_{0}\right)$, as $I_{0} \rightarrow+\infty$.
(ii) $b_{2}(t)=o(1)$, as $I_{0} \rightarrow+\infty$.
(iii) $b_{1}(t) \cdot b_{3}(s)=o(1)$, as $I_{0} \rightarrow+\infty$.

Let us consider the variational equation of (17) with respect to the initial value $I_{0}$. One can verify that

$$
\begin{align*}
& \dot{\theta}_{I_{0}}=b_{1}(t) \frac{\partial I}{\partial I_{0}}+b_{2}(t) \frac{\partial \theta}{\partial I_{0}} \\
& \dot{I}_{I_{0}}=-b_{2}(t) \frac{\partial I}{\partial I_{0}}+b_{3}(t) \frac{\partial \theta}{\partial I_{0}} . \tag{41}
\end{align*}
$$

Lemma 14. For all $t \in\left(0,2 \pi_{p}\right], I_{0} \rightarrow+\infty$, one has
(i) $\theta_{I_{0}}\left(t ; \theta_{0}, I_{0}\right) \rightarrow 0$;
(ii) $I_{I_{0}}\left(t ; \theta_{0}, I_{0}\right)=1+o(1)$;
(iii) $\theta_{\theta_{0}}\left(t ; \theta_{0}, I_{0}\right)=1+o(1)$.

Proof. From variational equations (41) and Lemma 13, one has

$$
\begin{align*}
\theta_{I_{0}}(t) & =e^{\int_{0}^{t} b_{2}(s) d s} \cdot \int_{0}^{t} e^{-\int_{0}^{s} b_{2}(t) d t} b_{1}(s) \cdot I_{I_{0}}(s) d s \\
= & (1+o(1)) \int_{0}^{t} b_{1}(s) \cdot I_{I_{0}}(s) d s, \\
I_{I_{0}}(t) & =e^{-\int_{0}^{t} b_{2}(s) d s} \cdot\left(1+\int_{0}^{t} e^{\int_{0}^{s} b_{2}(t) d t} b_{3}(s) \cdot \theta_{I_{0}}(s) d s\right)  \tag{42}\\
= & 1+o(1) \\
& +(1+o(1)) \int_{0}^{t} b_{3}(s) \cdot\left(\int_{0}^{s} b_{1}(t) \cdot I_{I_{0}}(t) d t\right) d s \\
= & 1+o(1)+o(1) \int_{0}^{t} \int_{0}^{s} I_{I_{0}}(t) d t d s,
\end{align*}
$$

and here we have used $\theta_{I_{0}}(0)=0$ and $I_{I_{0}}(0)=1$.
Hence, for all $t \in\left(0,2 \pi_{p}\right], I_{0} \rightarrow+\infty$, we have $I_{I_{0}}(t)=$ $1+o(1)$ and $\theta_{I_{0}}(t)=(1+o(1)) \int_{0}^{t} b_{1}(s) d s \rightarrow 0$. Thus, (i) and (ii) are proved.

To prove (iii), we consider the variational equation of (17) about $\theta_{0}$; one can get

$$
\begin{align*}
& \dot{\theta}_{\theta_{0}}=b_{1}(t) \frac{\partial I}{\partial \theta_{0}}+b_{2}(t) \frac{\partial \theta}{\partial \theta_{0}},  \tag{43}\\
& \dot{I}_{\theta_{0}}=-b_{2}(t) \frac{\partial I}{\partial \theta_{0}}+b_{3}(t) \frac{\partial \theta}{\partial \theta_{0}} .
\end{align*}
$$

By using a similar argument in (ii), we can also show that $\theta_{\theta_{0}}\left(t ; \theta_{0}, I_{0}\right)=1+o(1), \forall t \in\left(0,2 \pi_{p}\right]$, as $I_{0} \rightarrow+\infty$. This completes the proof of Lemma 14.

Next, we will develop an estimate of upper bound and lower bound for $b_{1}(t)$.

Lemma 15. Let $d \geq 0$ satisfy $\left(A_{2}\right)$.
(i) If $\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \leq d$ for all $t \in\left[0,2 \pi_{p}\right]$, then there exists a constant $E_{d}>0$, such that $\left|b_{1}(t)\right| \leq E_{d} / I^{2}(t)$.
(ii) If $\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \geq d$ for all $t \in\left[0,2 \pi_{p}\right]$, then there exists a constant $F_{d}>0$, such that $\left|b_{1}(t)\right| \geq F_{d} / I^{2}(t)$. Moreover, if $\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \geq d$, then $b_{1}(t)<0$.

Proof. (i) If $\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \leq d$, take $M_{d}=\max _{|x| \leq d, t \in\left[0,2 \pi_{p}\right]} \mid(p-$ 1) $\psi(x, t)-x \psi_{x}(x, t) \mid$. Then

$$
\begin{align*}
\left|b_{1}(t)\right| & =\left|\frac{-x\left[(p-1) \psi(t, x)-x \psi_{x}(t, x)\right]}{p^{2} I^{2}}\right|  \tag{44}\\
& \leq \frac{|x| M_{d}}{p^{2} I^{2}(t)} \leq \frac{d M_{d}}{p^{2} I^{2}(t)}
\end{align*}
$$

Writing $E_{d}=d M_{d} / p^{2}$, we have $\left|b_{1}(t)\right| \leq E_{d} / I^{2}(t)$.
(ii) If $\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \geq d$, with condition $\left(A_{2}\right)$, it is easy to know that $(p-1) x \psi(t, x)-x^{2} \psi_{x}(t, x)>\beta x$ when $x \geq d$ and $(p-1) x \psi(t, x)-x^{2} \psi_{x}(t, x)>-\beta x$ when $x \leq-d$. Hence, $b_{1}(t)<0$ and

$$
\begin{align*}
\left|b_{1}(t)\right| & =\left|\frac{-x\left[(p-1) \psi(t, x)-x \psi_{x}(t, x)\right]}{p^{2} I^{2}}\right|  \tag{45}\\
& \geq \frac{\beta|x|}{p^{2} I^{2}(t)} \geq \frac{d \beta}{p^{2} I^{2}(t)}
\end{align*}
$$

Therefore, setting $F_{d}=d \beta / p^{2}$, we obtain $\left|b_{1}(t)\right| \geq F_{d} /$ $I^{2}(t)$. The proof is complete.

Let $b_{1}(t)=b_{1}^{+}(t)-b_{1}^{-}(t)$ with $b_{1}^{ \pm}(t)=\max \left\{ \pm b_{1}(t), 0\right\}$. To estimate that the integral of $b_{1}^{+}(t)$ on $[0,2 \pi]$ is smaller than the integral of $b_{1}^{-}(t)$ on $[0,2 \pi]$, we need the following lemma.

Lemma 16. Let $d \geq 0$ be as in Theorem 1. Define $\Delta t=\{t \in$ $\left[0,2 \pi_{p}\right]\left|\left|x\left(t ; \theta_{0}, I_{0}\right)\right| \leq d\right\}$. Then there exist $\overline{I_{0}}>0, T>0$, such that

$$
\begin{equation*}
|\Delta t| \leq \frac{T}{I_{0}^{1 / q}} \tag{46}
\end{equation*}
$$

for all $I_{0} \geq \bar{I}_{0}$.
Proof. According to Lemma 11, we see that $\Delta t \rightarrow 0$ if and only if $\Delta \theta \rightarrow 0$.

By the action-angle variables transformation (16), it is not difficult to verify that there exists $\tau>0$ such that $|\tan \Delta \theta| \leq$ $\tau d / I^{1 / q}(t)$ when $\Delta \theta \rightarrow 0$. Therefore, by using Corollary 10 , we know that there exist $\overline{I_{0}}>0, T>0$, such that

$$
\begin{equation*}
|\Delta t| \leq \frac{T}{I_{0}^{1 / q}} \tag{47}
\end{equation*}
$$

for all $I_{0} \geq \overline{I_{0}}$. Thus, we prove Lemma 16 .
The next lemma gives the estimates of $\partial \theta\left(2 \pi_{p} ; \theta_{0}, I_{0}\right) / \partial I_{0}$ for $I_{0} \gg 1$.

Lemma 17. For $I_{0} \gg 1$, one gets $\theta_{I_{0}}\left(2 \pi_{p}\right)<0$.
Proof. The following results immediately from Corollary 10, Lemma 15, and Lemma 16:

$$
\begin{align*}
\theta_{I_{0}} & \left(2 \pi_{p}\right)=(1+o(1)) \int_{0}^{2 \pi_{p}} b_{1}(s) d s \\
& =(1+o(1))\left(\int_{b_{1}(t) \leq 0} b_{1}(s) d s+\int_{b_{1}(t) \geq 0} b_{1}(s) d s\right) \\
& \leq-(1+o(1))\left(\frac{F_{d}}{\rho_{2}^{2} I_{0}^{2}}\left(2 \pi_{p}-|\Delta t|\right)-\frac{E_{d}}{\rho_{1}^{2} I_{0}^{2}}|\Delta t|\right)  \tag{48}\\
& \leq(1+o(1))\left(\frac{\left(\rho_{1}^{2} F_{d}+\rho_{2}^{2} E_{d}\right) T}{\rho_{1}^{2} \rho_{2}^{2} I_{0}^{2+1 / q}}-\frac{2 \pi_{p} F_{d}}{\rho_{2}^{2} I_{0}^{2}}\right)
\end{align*}
$$

So, if $I_{0}^{1 / q}>\left(\rho_{1}^{2} F_{d}+\rho_{2}^{2} E_{d}\right) T / 2 \pi_{p} L_{d} \rho_{1}^{2}$, we have $\theta_{I_{0}}\left(2 \pi_{p}\right)<$
0.

## 4. Proof of Theorem 1

Now we start to give the proof of Theorem 1.
Proof of Theorem 1. Based on Lemma 17 and the AubryMather theorem [37], we can see that the Poincaré map $P$ of system (17) is a monotone twist map when $I_{0} \gg 1$. At last, using similar arguments as in [37], we may broaden the Poincaré map $P$ to a new map $\widehat{P}$ which is a whole monotone twist homeomorphism on the cylinder $S^{1} \times \mathbb{R}$ and agree with $P$ on $\mathbf{S}^{1} \times\left[I_{0},+\infty\right)$ with a fixed constant $I_{0} \gg 1$. Hence, the existence of Aubry-Mather sets $M_{\sigma}$ of $\widehat{P}$ is ensured by the Aubry-Mather theorem due to Pei [37]. Moreover, for some small $\varepsilon_{0}>0$, all those Aubry-Mather sets with rotation number $\alpha \in\left(2 \omega \pi_{p}, 2 \omega \pi_{p}+\varepsilon_{0}\right)$ lie in the domain $\mathbf{S}^{1} \times\left[I_{0},+\infty\right)$. Therefore, they happen to be the Aubry-Mather sets of the Poincaré map of $P$. From the above discussions, we have showed the existence of Aubry-Mather sets; this implies that (1) has an Aubry-Mather type solution $u_{\alpha}(t)=$ $\left(x_{\alpha}(t), x_{\alpha}^{\prime}(t)\right)$ with rotation number $\alpha$. This completes the proof of Theorem 1 .

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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