Journal of Mathematics

Wavelets and Wavelet-Based Numerical Methods

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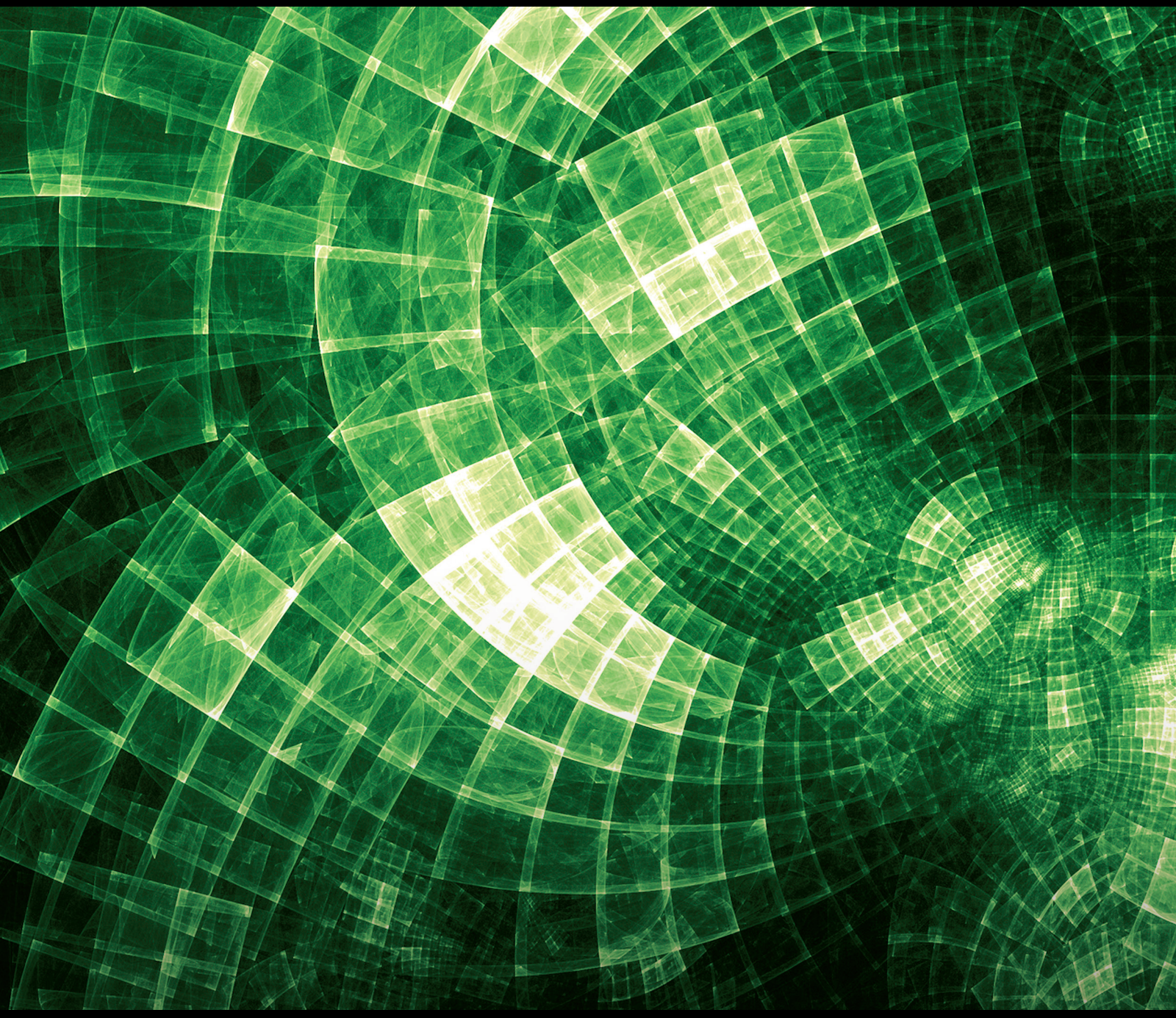

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Research Article

Some Properties of Fractional Boas Transforms of Wavelets

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In this paper, we introduce fractional Boas transforms and discuss some of their properties. We also introduce the notion of wavelets associated with fractional Boas transforms and give some results related to their vanishing moments. Finally, a comparative study of Hilbert transforms and fractional Boas transforms is done.

1. Introduction

The ordinary transformations have been replaced with the fractional ones, which play a significant role in information processing. This transition has occurred naturally due to its various applications in quantum mechanics and optics and purvey us a tool, to characterize a signal completely, in the form of the fractional order, which happens to be the new degree of freedom or an appended parameter for encoding. Among all fractional transforms, the fractional Fourier transform (FRFT), a generalization of the Fourier transform, has been widely studied. In the last three decades, the fractional Fourier transform (FRFT) has played a substantial role in signal processing, optical systems, and quantum physics [1–3]. Another important variation of FRFT is canonical fractional FT [4], which is very effective in optical information processing, since it is easily achievable using simple optical setups and it renders a mere rotation of the two important phase-space distributions: the Wigner distribution and the ambiguity function. The canonical fractional FT was first introduced in [5] more than 90 years ago, which was later improvised by various researchers for applications in quantum mechanics [3], optics [6], and signal processing [2]. Another fractional transform, the complex fractional FT, closely related to the canonical fractional FT has been introduced in [7]. The generalization of Legendre transformation to the fractional Legendre transformation was formulated on the lines of FRFT in [8]. Based on the approach of eigenfunction kernel decomposition similar to the one given in [9], some new fractional integral transforms, including the fractional Mellin transform, a fractional transform associated with the Jacobi polynomials, a Riemann-Liouville fractional derivative operator, and a fractional Riemann-Liouville integral, have been proposed in [10]. In the analogy with canonical fractional Fourier and Hankel transforms, the fractional Laplace and Barut-Girardello transforms have been introduced in [11]. The applications of these transforms in science and engineering are still subject of research.

In order to process one-sided signals, fractional cosine (CT) and sine transforms (ST) were employed. Their digital application along with that of fractional Hartley transforms (HaT) was discussed in [9, 12]. One may refer to [13] for image watermarking scheme classified on the basis of variant fractional transforms such as fractional discrete FT, HaT, CT, and ST.

Gabor [14] introduced the Hilbert transform (HT), an important tool in optics, by constructing an analytic signal from a one-dimensional signal. In 1950, its optical implementation was performed in two different approaches, when Kastler [15] employed it for image processing, primarily for edge enhancement, and Wolter [16] utilized it for spectroscopy. Further advancements in HT can be seen in [17]. The efficacy of HT was raised with the origination of fractional Hilbert transform (FRHT) by Lohmann et al. [18] in

1996, which proffered an additional degree of freedom in the form of a fractional order. Two ways of fractional HT were proposed, which resulted in increase of fractional order and provided improvements in image processing. The first method was based on a spatial filter with a fractional parameter and the other was based on the FRFT. For details, one may see [6, 19]. The FRHT for two-dimensional objects was presented in [20]. Later, Tseng and Pei [13] formulated an SSB modulation by considering the parameter of the fractional phase in FRHT as a secret key. Zayed [21] generalized the HT in a distinctive way and suppressed the negative frequency component of the signal in the FRF domain to obtain the analytic part of a signal. Using FRHT, Cusmariu [22] proposed three possible versions of fractional analytic signals. Tao [23] employed FRFT and FRHT and presented a secured SSB modulation system.

Paley and Wiener [24] studied a class of square integrable functions whose Fourier transforms vanish outside the intervals $[-\kappa, \kappa]$ in great details. This class denoted by B_κ^2 was later named by Paley-Weiner class of entire functions and a member of this class is a function band limited to $[-\kappa, \kappa]$. Contrary to this study, Boas was curious in examining the properties of square integrable functions whose Fourier transforms vanish on $[-\kappa, \kappa]$, that is, the class $B_{\mathbb{R}-[-\kappa,\kappa]}$. Boas noticed that these properties were not trivial and led to the introduction of Boas transforms (BT) in [25]. Later, BT was studied by Goldberg [26], Heywood [27], and Zaidi [28] who played a substantial role in outlining the properties and the results. For complete review of BT, one may read [29]. In a roundabout way, it was employed in the theory of filters in electrical engineering. Recall from [29] that any finite energy signal f on passing through a high pass filter whose system transfer function is given by $H(w) = \begin{cases} Ae^{it_o w}, & \text{if } |w| \geq 1; \\ 0, & \text{if otherwise} \end{cases}$ gives an output g such that $\widehat{g}(w) = H(w)\widehat{f}(w)$. Thus, $\widehat{g}$ vanishes on $(-1, 1)$. Using Boas' theorems, one can characterize the output of the high-pass filter in two ways: (i) A signal g is the output of a high-pass filter if and only if $\mathfrak{B}(\mathfrak{B}g) = -g$. (ii) If g is an output of high-pass filter, then $\mathfrak{B}g = \mathfrak{H}g$. Not much research has been done about it until 2019, when Khanna et al. introduced the notion of BT of wavelets (in a preprint form), which was later published in [30]. During the same year, Khanna and Kathuria [31] studied convolution of Boas transforms of wavelets. The motivation behind this study was the relationship between Boas and Fourier transforms of wavelets and the observation that wavelets $\psi(x)$ for which the Fourier transform $\widehat{\psi}(\eta)$ vanishes almost everywhere on $(-1, 1)$ can be characterized by the Boas transform of wavelets $\mathfrak{B}\psi(x)$. Since Boas transforms are closely related to Hilbert transforms, readers must be interested in reading Hilbert transforms of wavelets. For more details, see [30–40].

1.1. Plan of the Work. The paper is organized as follows: In Section 2, we introduce the notion of fractional Boas transforms (FRBT) and give some properties in the form of observations. Titchmarsh-type and Tricomi-type results are established and a relationship between FRHT and FRBT is given followed by an inversion formula of FRBT and fractional Boas transform product theorem. In Section 3, wavelets associated with FRBT are introduced and a relationship between two wavelets in terms of $(\mathfrak{B}_\theta - \mathfrak{H}_\theta)$ operator is given. We give a necessary and sufficient condition under which FRBT of a given wavelet is multiple of the first-order derivative of the given wavelet. We further give sufficient condition for the higher vanishing moments of FRBT of wavelets. Finally, we give a sufficient condition on two wavelets to obtain a two-dimensional wavelet and the number of vanishing moments of their convolution is given.

2. Fractional Boas Transforms

The Boas transform of a function $f \in L^2(\mathbb{R})$, denoted by $\mathfrak{B}f(x)$ in terms of principal value integral, is defined as

$$\begin{aligned} \mathfrak{B}f(x) &= \frac{1}{\pi} p.v. \int_0^\infty \frac{f(x+u) - f(x-u)}{u^2} \sin(u) \mathrm{d}u \\ &= \frac{1}{\pi} p.v. \int_{-\infty}^\infty \frac{f(x+u)}{u^2} \sin(u) \mathrm{d}u, \end{aligned} \tag{1}$$

for any x for which the integral exists.

The relationship between the Boas transform $\mathfrak{B}$ and the Hilbert transform $\mathfrak{H}$ of a function f is given by

$$(\mathfrak{B}f)(x) = (\mathfrak{H}f)(x) - \{\mathfrak{H}f * g\}(x), \tag{2}$$

where

$$g(x) = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{1 - \cos(x)}{x^2}\right). \tag{3}$$

The FRHT of $f \in L^2(\mathbb{R})$ is defined as $\mathfrak{H}_\theta f(x) = \cos(\theta) f(x) + \sin(\theta)\mathfrak{H}f(x), -\pi/2 \leq \theta \leq \pi/2$. It can be easily verified that the operator $\mathfrak{H}_\theta$ satisfies the properties of linearity, translation-invariance, dilation-invariance, orthogonality, unitary nature, and linear independence. The linear independence property endorses one to induce a novel base from a given set of linearly independent functions.

Now, we define an operator $\mathfrak{B}_\theta$ on $L^2(\mathbb{R})$ by

$$\begin{aligned} \mathfrak{B}_\theta &= \cos(\theta) I + \sin(\theta)\mathfrak{B}, \quad \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ &= \mathfrak{H}_\theta - \sin(\theta)(\mathfrak{H} * g), \end{aligned} \tag{4}$$

where $\mathfrak{B}_\theta$ is called a fractional Boas transform. For $\theta = \pi/2, \mathfrak{B}_\theta = \mathfrak{B}$.

Observations

(i) $\mathfrak{B}_\theta$ is translation-invariant; that is, if $e_\theta(x) = \{\mathfrak{B}_\theta f(t)\}(x)$, then for translation operator T_c, we have

$$T_c e_\theta(x) = e_\theta(x - c) = \mathfrak{B}_\theta\{f(t - c)\}(x) = \mathfrak{B}_\theta\{T_c f\}(x). \tag{5}$$

(ii) $\mathfrak{B}_\theta$ is dilation-invariant. Let D_α denote the dilation operator defined as $D_\alpha f(x) = f(\alpha x), \alpha \in \mathbb{R}^+$. Then $\mathfrak{B}_\theta D_\alpha f(x) = D_\alpha \mathfrak{B}_\theta f(x) = e_\theta(\alpha x)$. Thus, fractional Boas transform operator commutes with D_α.

(iii) The transformation $\mathfrak{B}_\theta$: $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is a nonsurjective bounded linear transform on $L^2(\mathbb{R})$. In fact, we have

$$\begin{aligned}\left\|\mathfrak{B}_\theta f\right\|_2 &\le \left\|\mathfrak{H}_\theta f\right\|_2 + \|\sin(\theta)(\mathfrak{H} f * g)\|_2 \\ &\le \left\|\mathfrak{H}_\theta f\right\|_2 + |\sin(\theta)| \|\mathfrak{H} f\|_2 \|g\|_1.\end{aligned} \tag{6}$$

Now, $\widehat{\mathfrak{H}_\theta f}(\eta) = e^{-i\theta(\operatorname{sgn}\eta)} \widehat{f}(\eta)$. Since $f \in L^2(\mathbb{R})$, it follows from Parseval's identity that $\|\mathfrak{H}_\theta f\|_2 = \|f\|_2$. Thus,

$$\left\|\mathfrak{B}_\theta f\right\|_2 \le \|f\|_2 (1 + \sqrt{2\pi}\, \sin(\theta)) < +\infty. \tag{7}$$

(iv) Let $h, f \in L^2(\mathbb{R})$. Then $\int_{\mathbb{R}} h(x) \mathfrak{B}_\theta f(x) \mathrm{d}x = \int_{\mathbb{R}} \mathfrak{B}_{-\theta} h(x) f(x) \mathrm{d}x$. In particular, if $h = f$ and $\theta \ne 0$, then $\int_{\mathbb{R}} f(x) \mathfrak{B}_\theta f(x) \mathrm{d}x = 0$. Further, $\int_{\mathbb{R}} (\mathfrak{B}_\theta f(x))^2 \mathrm{d}x = \int_{\mathbb{R}} (\mathfrak{B}_{-\theta} f(x))^2 \mathrm{d}x$.

Indeed, we have

$$\begin{aligned}\int_{\mathbb{R}} h(x) \mathfrak{B}_\theta f(x) \mathrm{d}x &= \cos(\theta) \int_{\mathbb{R}} h(x) f(x) \mathrm{d}x + \sin(\theta) \int_{\mathbb{R}} h(x) \mathfrak{B} f(x) \mathrm{d}x \\ &= \int_{\mathbb{R}} \mathfrak{B}_{-\theta} h(x) f(x) \mathrm{d}x.\end{aligned} \tag{8}$$

If $h = f$, then

$$\int_{\mathbb{R}} f(x) \left(\mathfrak{B}_\theta f(x) - \mathfrak{B}_{-\theta} f(x)\right) \mathrm{d}x = 0, \tag{9}$$

which gives $\int_{\mathbb{R}} f(x) \mathfrak{B} f(x) \mathrm{d}x = 0$. Further, if $h(x) = \mathfrak{B}_\theta f(x)$, then

$$\begin{aligned}\int_{\mathbb{R}} \left(\mathfrak{B}_\theta f(x)\right)^2 \mathrm{d}x &= \int_{\mathbb{R}} \mathfrak{B}_{-\theta} \{\mathfrak{B}_\theta f\}(x) f(x) \mathrm{d}x \\ &= \int_{\mathbb{R}} \cos^2(\theta) f^2(x) \mathrm{d}x - \sin^2(\theta) \int_{\mathbb{R}} \mathfrak{B}^2 f(x) f(x) \mathrm{d}x \\ &= \int_{\mathbb{R}} \left(\mathfrak{B}_{-\theta} f(x)\right)^2 \mathrm{d}x.\end{aligned} \tag{10}$$

(v) For $-\pi/2 \le \theta_1, \theta_2 \le \pi/2$, we have

$$\begin{aligned}\mathfrak{B}_{\theta_1}^2 f(x) = \mathfrak{B}_{2\theta_1} f(x) + \sin^2(\theta_1)(2(f * g)(x) \\ - ((f * g) * g)(x)).\end{aligned} \tag{11}$$

Indeed, we have

$$\begin{aligned}\mathfrak{B}_{\theta_1} \mathfrak{B}_{\theta_2} f(x) &= \mathfrak{B}_{\theta_1} \left\{\mathfrak{H}_{\theta_2} f(x) - \sin(\theta_2)(\mathfrak{H} f * g)(x)\right\} \\ &= \mathfrak{H}_{\theta_1} \mathfrak{H}_{\theta_2} f(x) - \sin(\theta_2) \mathfrak{H}_{\theta_1} \mathfrak{H}(f * g)(x) - \sin(\theta_1)\left(\mathfrak{H}\mathfrak{H}_{\theta_2} f * g\right)(x) + \sin(\theta_2)((f * g) * g)(x) \\ &= \mathfrak{H}_{\theta_1+\theta_2} f(x) - \sin(\theta_1 + \theta_2)(\mathfrak{H} f * g)(x) + 2\sin(\theta_1)\sin(\theta_2)(f * g)(x) \\ &\quad - \sin(\theta_1)\sin(\theta_2)((f * g) * g)(x)\left(\because \mathfrak{H}_{\theta_1} \mathfrak{H}_{\theta_2} f(x) = \mathfrak{H}_{\theta_1+\theta_2} f(x)\right) \\ &= \mathfrak{B}_{\theta_1+\theta_2} f(x) + 2\sin(\theta_1)\sin(\theta_2)(f * g)(x) - \sin(\theta_1)\sin(\theta_2)((f * g) * g)(x) \\ &= \mathfrak{B}_{\theta_1+\theta_2} f(x) + \sin(\theta_1)\sin(\theta_2)\left(\mathfrak{B}^2 f(x) + f(x)\right).\end{aligned} \tag{12}$$

Then, the iteration property of the fractional Boas transform (11) can be easily obtained by taking $\theta_2 = \theta_1$.

(vi) Let $f \in L^2(\mathbb{R})$ be a function such that $x^n f(x) \in L^2(\mathbb{R})$, for $n \in \mathbb{N}$. Then,

$$\mathfrak{B}_\theta\{x^n f(x)\} = x^n \mathfrak{H}_\theta f(x) - \sin(\theta)\left(\frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{\mathbb{R}} t^{n-1-k} f(t)\mathrm{d}t + \left(-\frac{1}{2\pi i}\right)^n \int_{-1}^{1} \widehat{f}^{(n)}(\eta) e^{2\pi i \eta x}(\mathrm{sgn}\eta - \eta)\mathrm{d}\eta\right). \tag{13}$$

Indeed, we have

$$\begin{aligned}
&\mathfrak{B}_\theta\{x^n f(x)\} \\
&= \cos(\theta) x^n f(x) + \sin(\theta)\left(x^n f(x) - \frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{\mathbb{R}} t^{n-1-k} f(t)\mathrm{d}t - \int_{\mathbb{R}} \mathfrak{H}\{t^n f(t)\} g(x-t)\mathrm{d}t\right) \\
&= x^n \mathfrak{H}_\theta f(x) - \sin(\theta)\left(\frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{\mathbb{R}} t^{n-1-k} f(t)\mathrm{d}t + \int_{\mathbb{R}} \mathrm{sgn}(\eta)\widehat{t^n f(t)}(\eta) E_x \widehat{g}(-\eta)\mathrm{d}\eta\right) \\
&= x^n \mathfrak{H}_\theta f(x) - \sin(\theta)\left(\frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{\mathbb{R}} t^{n-1-k} f(t)\mathrm{d}t + \int_{\mathbb{R}} \left(-\frac{1}{2\pi i}\right)^n \widehat{f}^{(n)}(\eta) e^{2\pi i \eta x}\mathrm{sgn}(\eta)(1-|\eta|)\mathrm{d}\eta\right) \\
&= x^n \mathfrak{H}_\theta f(x) - \sin(\theta)\left(\frac{1}{\pi}\sum_{k=0}^{n-1} x^k \int_{\mathbb{R}} t^{n-1-k} f(t)\mathrm{d}t + \left(-\frac{1}{2\pi i}\right)^n \int_{-1}^{1} \widehat{f}^{(n)}(\eta) e^{2\pi i \eta x}(\mathrm{sgn}(\eta) - \eta)\mathrm{d}\eta\right).
\end{aligned} \tag{14}$$

(vii) Let $\mathscr{R}$ denote the reflection operator, defined by $\mathscr{R} f(x) = f(-x), x \in \mathbb{R}$. Then, $\mathfrak{B}_\theta \mathscr{R} f(x) = \mathscr{R}\mathfrak{B}_\theta f(x) - 2\sin(\theta)\mathscr{R}\mathfrak{B} f(x)$.

We have

$$\begin{aligned}
\mathfrak{B}_\theta \mathscr{R} f(x) &= \mathfrak{H}_\theta \mathscr{R} f(x) - \sin(\theta)(\mathfrak{H}\mathscr{R} f * g)(x) \\
&= \cos(\theta)\mathscr{R} f(x) - \sin(\theta)\mathscr{R}\mathfrak{H} f(x) + \sin(\theta)(\mathfrak{H}\mathscr{R}(f * g))(x) \\
&= \cos(\theta)\mathscr{R} f(x) - \sin(\theta)\mathscr{R}\mathfrak{H} f(x) - \sin(\theta)(\mathscr{R}\mathfrak{H} f * g)(x) \\
&= \mathscr{R}\mathfrak{B}_\theta f(x) - 2\sin(\theta)\mathscr{R}\mathfrak{H} f(x).
\end{aligned} \tag{15}$$

(viii) It is easy to verify that if $h(x) = \mathfrak{B}_\theta f(x)$, then

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\mathfrak{B}_\theta f(x)\} = \mathfrak{B}_\theta\left\{\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right\}. \tag{16}$$

(ix) The fractional Boas transform of a convolution of two functions f and h can be expressed as a convolution of one of the functions with the fractional Boas transform of the other function; that is,

$$\mathfrak{B}_\theta(f * h)(x) = (\mathfrak{B}_\theta f * h)(x) = (f * \mathfrak{B}_\theta h)(x). \tag{17}$$

Next, we give a Titchmarsh-type result for the fractional Boas transform.

Proposition 1. *If $f, \vartheta \in L^2(\mathbb{R})$, then*

$$\int_{\mathbb{R}} \mathfrak{B}_\theta f(x)\mathfrak{B}_\theta \vartheta(x)\mathrm{d}x \le \int_{\mathbb{R}} f(x)\vartheta(x)\mathrm{d}x. \tag{18}$$

Proof. We compute

$$
\begin{aligned}
&\int_{\mathbb{R}} \mathfrak{B}_\theta f(x)\mathfrak{B}_\theta \vartheta(x)\mathrm{d}x \\
&= \int_{\mathbb{R}} (\cos(\theta) f(x) + \sin(\theta)\mathfrak{B} f(x))(\cos(\theta)\vartheta(x) + \sin(\theta)\mathfrak{B}\vartheta(x))\mathrm{d}x \\
&= \int_{\mathbb{R}} \Big(\cos^2(\theta) f(x)\vartheta(x) + \cos(\theta)\sin(\theta) f(x)\mathfrak{B}\vartheta(x) + \sin(\theta)\cos(\theta)\mathfrak{B} f(x)\vartheta(x) \\
&\quad + \sin^2(\theta)\mathfrak{B} f(x)\mathfrak{B}\vartheta(x)\mathrm{d}x \\
&= \cos^2(\theta)\int_{\mathbb{R}} f(x)\vartheta(x)\mathrm{d}x + \cos(\theta)\sin(\theta)\int_{\mathbb{R}} f(x)\mathfrak{B}\vartheta(x)\mathrm{d}x \\
&\quad - \sin(\theta)\cos(\theta)\int_{\mathbb{R}} f(x)\mathfrak{B}\vartheta(x)\mathrm{d}x + \sin^2(\theta)\int_{\mathbb{R}} \mathfrak{H} f(x)\mathfrak{H}\vartheta(x)\mathrm{d}x \\
&\quad - \sin^2(\theta)\int_{\mathbb{R}} \mathfrak{H} f(x)(\mathfrak{H}\vartheta * g)(x)\mathrm{d}x - \sin^2(\theta)\int_{\mathbb{R}} (\mathfrak{H} f * g)(x)\mathfrak{H}\vartheta(x)\mathrm{d}x \\
&\quad + \sin^2(\theta)\int_{\mathbb{R}} (\mathfrak{H} f * g)(x)(\mathfrak{H}\vartheta * g)(x)\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)\vartheta(x)\mathrm{d}x - 2\sin^2(\theta)\int_{\mathbb{R}} \widehat{f}(\eta)\widehat{\vartheta}(\eta)\widehat{g}(\eta)\mathrm{d}\eta + \sin^2(\theta)\int_{\mathbb{R}} \widehat{f}(\eta)\widehat{\vartheta}(\eta)(\widehat{g}(\eta))^2\mathrm{d}x \\
&= \int_{\mathbb{R}} f(x)\vartheta(x)\mathrm{d}x + \sin^2(\theta)\int_{-1}^{1} \widehat{f}(\eta)\widehat{\vartheta}(\eta)\Big(-2(1-|\eta|) + (1-|\eta|)^2\Big)\mathrm{d}\eta \\
&\le \int_{\mathbb{R}} f(x)\vartheta(x)\mathrm{d}x.
\end{aligned}
\tag{19}
$$

Next, we give a Tricomi-type result for the fractional Boas transform. □

Proposition 2. *Let f, ϑ be functions such that*

(i) $f, \vartheta, \widehat{f}, \widehat{\vartheta} \in L^1(\mathbb{R})$

(ii) $\widehat{f}(0) = 0$

(iii) $\widehat{f}(\eta)$ *vanishes for* $|\eta| > 1$ *and* $\widehat{\vartheta}(\eta)$ *vanishes for* $|\eta| \le 1$

Then, $\mathfrak{B}_\theta\{f(x)\mathfrak{B}_\theta\vartheta(x) - \vartheta(x)\mathfrak{B}_\theta f(x)\} = \sin(\theta)(\mathfrak{B}\vartheta(x)\mathfrak{B}_{-\theta} f(x) - \vartheta(x)\mathfrak{B}_{\pi/2-\theta} f(x)$.

Proof. We have

$$
\begin{aligned}
&\mathfrak{B}_\theta\{f(x)\mathfrak{B}_\theta\vartheta(x) - \vartheta(x)\mathfrak{B}_\theta f(x)\} \\
&= \mathfrak{B}_\theta\{f(x)(\cos(\theta)\vartheta(x) + \sin(\theta)\mathfrak{B}\vartheta(x)) - \vartheta(x)(\cos(\theta) f(x) + \sin(\theta)\mathfrak{B} f(x))\} \\
&= \mathfrak{B}_\theta\{\sin(\theta)(f(x)\mathfrak{B}\vartheta(x) - \vartheta(x)\mathfrak{B} f(x))\} \\
&= \sin(\theta)(\cos(\theta)(f(x)\mathfrak{B}\vartheta(x) - \vartheta(x)\mathfrak{B} f(x)) + \sin(\theta)\mathfrak{B}\{f(x)\mathfrak{B}\vartheta(x) - \vartheta(x)\mathfrak{B} f(x)\}) \\
&= \sin(\theta)\cos(\theta) f(x)\mathfrak{B}\vartheta(x) - \sin(\theta)\cos(\theta)\vartheta(x)\mathfrak{B} f(x) + \sin^2(\theta)\mathfrak{B}\{f(x)\mathfrak{B}\vartheta(x)\} - \sin^2(\theta)\mathfrak{B}\{\vartheta(x)\mathfrak{B} f(x)\}.
\end{aligned}
\tag{20}
$$

In view of Boas transform product theorem (Theorem 3.9, [31]), we have

$$
\begin{aligned}
&\mathfrak{B}_\theta\{f(x)\mathfrak{B}_\theta\vartheta(x) - \vartheta(x)\mathfrak{B}_\theta f(x)\} \\
&= \sin(\theta)\cos(\theta) f(x)\mathfrak{B}\vartheta(x) - \sin(\theta)\cos(\theta)\vartheta(x)\mathfrak{B} f(x) + \sin^2(\theta) f(x)\mathfrak{B}^2\vartheta(x) - \sin^2(\theta)\mathfrak{B}\vartheta(x)\mathfrak{B} f(x) \\
&= \sin(\theta)[\mathfrak{B}\vartheta(x)(\cos(\theta) f(x) - \sin(\theta)\mathfrak{B} f(x)) - \vartheta(x)(\cos(\theta)\mathfrak{B} f(x) + \sin(\theta) f(x))] \\
&= \sin(\theta)\big[\mathfrak{B}\vartheta(x)\mathfrak{B}_{-\theta} f(x) - \vartheta(x)\mathfrak{B}_{\pi/2-\theta} f(x)\big].
\end{aligned}
\tag{21}
$$

Next, we discuss fractional Boas transform of product of analytic functions (or signals). □

Proposition 3. *For analytic functions* $f_1(x) = g_1(x) + ih_1(x)$ *and* $f_2(x) = g_2(x) + ih_2(x)$, *we have*

$$\mathfrak{B}_\theta\{f_1(x)f_2(x)\} = \left(f_1 f_2 * \left(e^{-i\theta}\delta - i\sin(\theta)g\right)\right)(x). \tag{22}$$

Proof. We compute

$$\begin{aligned}\mathfrak{B}_\theta\{f_1(x)f_2(x)\} &= \cos(\theta)f_1(x)f_2(x) + \sin(\theta)\left(\mathfrak{H}\{f_1(x)f_2(x)\} - \mathfrak{H}\{f_1(x)f_2(x)\} * g(x)\right)\\ &= \cos(\theta)f_1(x)f_2(x) + \sin(\theta)\left(-if_1(x)f_2(x) - i\left(f_1 f_2 * g\right)(x)\right)\\ &= f_1(x)f_2(x)e^{-i\theta} - i\sin(\theta)\left(f_1 f_2 * g\right)(x)\\ &= \left(f_1 f_2 * \left(e^{-i\theta}\delta - i\sin(\theta)g\right)\right)(x).\end{aligned} \tag{23}$$

In particular, if $f_1 = f_2 = f$, then

$$\mathfrak{B}_\theta\left\{f^2(x)\right\} = \left(f^2 * \left(e^{-i\theta}\delta - i\sin(\theta)g\right)\right)(x). \tag{24}$$

The generalization to arbitrary powers is

$$\mathfrak{B}_\theta\{f^n(x)\} = \left(f^n * \left(e^{-i\theta}\delta - i\sin(\theta)g\right)\right)(x). \tag{25}$$

In the following result, we give a relationship between fractional Hilbert transform and fractional Boas transform. □

Proposition 4. *Let* $f, \widehat{f} \in L^1(\mathbb{R})$ *and let* $\widehat{f}(0) = 0$. *Then,*

$$\mathfrak{H}_\theta f(x) = \mathfrak{B}_\theta f(x) + \sum_{j=1}^{\infty}\left(\left(\mathfrak{B}_\theta f - \cos(\theta)f\right) * g_j\right)(x), \tag{26}$$

where $g_1 = g$ *and* $g_j = g_{j-1} * g$, *for* $j = 2, 3, 4, \ldots$.

Proof. We have

$$\begin{aligned}\widehat{\mathfrak{B}_\theta f}(\eta) &= \cos(\theta)\widehat{f}(\eta) + \sin(\theta)\widehat{\mathfrak{H}f}(\eta)(1 - \widehat{g}(\eta))\\ &= \left(\widehat{\mathfrak{B}_\theta f}(\eta) - \cos(\theta)\widehat{f}(\eta)\right)\sum_{j=0}^{\infty}(\widehat{g}(\eta))^j\\ &= \left(\mathfrak{B}_\theta f(x) - \cos(\theta)f(x)\right)^\wedge(\eta) + \sum_{j=1}^{\infty}\left(\left(\mathfrak{B}_\theta f - \cos(\theta)f\right) * g_j\right)^\wedge(\eta).\end{aligned} \tag{27}$$

Taking inverse Fourier transform, we have

$$\mathfrak{H}_\theta f(x) = \mathfrak{B}_\theta f(x) + \sum_{j=1}^{\infty}\left(\left(\mathfrak{B}_\theta f - \cos(\theta)f\right) * g_j\right)(x). \tag{28}$$

Note that, for any $M > N > 0$, we have

$$\begin{aligned}\left\|\sum_{j=N}^{M}\left(\left(\mathfrak{B}_\theta f - \cos(\theta)f\right) * g_j\right)\right\|_2 &= \left\|\sum_{j=N}^{M}\left(\widehat{\mathfrak{B}_\theta f} - \cos(\theta)\widehat{f}\right)(\widehat{g})^j\right\|_2\\ &\le \left\|\sum_{j=N}^{\infty}\left|\widehat{\mathfrak{B}_\theta f} - \cos(\theta)\widehat{f}\right|(\widehat{g})^j\right\|_2\\ &= \left\|(\widehat{g})^j\sum_{j=0}^{\infty}\left|\widehat{\mathfrak{B}_\theta f} - \cos(\theta)\widehat{f}\right|(\widehat{g})^j\right\|_2.\end{aligned} \tag{29}$$

Since $\sum_{j=0}^{\infty}|\widehat{\mathfrak{B}_\theta f} - \cos(\theta)\widehat{f}|(\widehat{g})^j \in L^2(\mathbb{R})$, it follows by Lebesgue convergence theorem that

$$\lim_{N\longrightarrow\infty}\left\|(\widehat{g})^N\sum_{j=0}^{\infty}\left|\widehat{\mathfrak{B}_\theta f}-tcosn(\theta)q\widehat{f}\right|(\widehat{g})^j\right\|_2=0. \tag{30}$$

Thus, the series in (26) converges in L^2 norm.

Next, we discuss the inversion of the fractional Boas transform. □

Proposition 5. *If $f\in L^2(\mathbb{R})$, then f can be retrieved from its fractional Boas transform by means of the formula*

$$f(x)=-\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\mathfrak{B}_\theta f-\mathfrak{B}_{-\theta}f\right)(x-z)\varphi(z)\mathrm{d}z, \tag{31}$$

where $\varphi(x)=\sum_{j=0}^{\infty}((-1)^j x^{2j-1}/(2j)!)+Si(x)$ with sine integral $Si(\cdot)$.

Proof. Let $\tau_\theta(x)=((\mathfrak{B}_\theta f-\mathfrak{B}_{-\theta}f)*\varphi)(x)$, where

$$\varphi(x)=\sum_{j=0}^{\infty}\frac{(-1)^j x^{2j-1}}{(2j)!}+\mathrm{Si}(x). \tag{32}$$

Then, we have

$$\begin{aligned}\widehat{\tau}_\theta(\eta)&=\left(\widehat{\mathfrak{B}_\theta f}(\eta)-\widehat{\mathfrak{B}_{-\theta}f}(\eta)\right)\widehat{\varphi}(\eta)\\&=2\sin(\theta)\widehat{\mathfrak{B}f}(\eta)\widehat{\varphi}(\eta).\end{aligned} \tag{33}$$

Now, observe that

$$\begin{aligned}\widehat{\varphi}(\eta)&=\int_{\mathbb{R}}e^{-i\eta x}\left(\sum_{j=0}^{\infty}\frac{(-1)^j x^{2j-1}}{(2j)!}+\mathrm{Si}(x)\right)\mathrm{d}x\\&=-\frac{\pi i}{2}\int_{\mathbb{R}}\frac{\sin((\eta+1)x)+\sin((\eta-1)x)}{x}\mathrm{d}x-\pi i\int_{\mathbb{R}}\frac{(\delta(x-\eta)-\delta(x+\eta))}{x}\mathrm{d}x\\&=\begin{cases}0, & \text{if } |\eta|<1\\ -\pi i\,\mathrm{sgn}(\eta), & \text{if } |\eta|>1\end{cases}+\begin{cases}-\pi i\,p.v.\,\dfrac{1}{\eta}, & \text{if } |\eta|<1,\\ 0, & \text{if } |\eta|>1,\end{cases}=\begin{cases}-\pi i\,p.v.\,\dfrac{1}{\eta}, & \text{if } |\eta|<1,\\ -\pi i\,\mathrm{sgn}(\eta), & \text{if } |\eta|>1.\end{cases}\end{aligned} \tag{34}$$

Therefore, $\widehat{\tau}_\theta(\eta)=-2\pi i\widehat{f}(\eta)$. Also, $(\widehat{\mathfrak{B}_\theta f}(\eta)-\widehat{\mathfrak{B}_{-\theta}f}(\eta)\widehat{\varphi}(\eta))=-2\pi i\widehat{f}(\eta)$.

Taking the inverse Fourier transform, we obtain

$$f(x)=-\frac{1}{2\pi i}\int_{\mathbb{R}}\left(\mathfrak{B}_\theta f-\mathfrak{B}_{-\theta}f\right)(x-z)\varphi(z)\mathrm{d}z. \tag{35}$$

Towards the end, we give fractional Boas transform product theorem. □

Proposition 6. *Let f,ϑ be functions such that*

(i) $f,\vartheta,\widehat{f}_1,\widehat{f}_2\in L^1(\mathbb{R})$,

(ii) $\widehat{f}_1(\eta)$ vanishes for $|\eta|>1$ and $\widehat{f}_2(\eta)$ vanishes for $|\eta|\le 1$.

Then $\mathfrak{B}_\theta\{f_1(x)f_2(x)\}=f_1(x)\mathfrak{B}_\theta f_2(x)$.

Proof. In view of Theorem 3.1 in [31], we have

$$\begin{aligned}\mathfrak{B}_\theta\{f_1(x)f_2(x)\}&=\cos(\theta)f_1(x)f_2(x)\\&\quad+\sin(\theta)f_1(x)\mathfrak{B}f_2(x)\\&=f_1(x)\mathfrak{B}_\theta\{f_2\}(x).\end{aligned} \tag{36}$$

□

3. Fractional Boas Transforms of Wavelets

The wavelet theory operates with the general properties of the wavelets and the wavelet transform. A wavelet function is chosen according to the application; for example, for space-frequency analysis, a wavelet that is localized in terms of both spatial width and frequency bandwidth is preferred, whereas a smooth wavelet is more appropriate in dealing with smooth signals. In case of analysis of a signal with certain discontinuities, wavelets with good spatial localization to scrupulously track swift changes in the signal are required. For more details on wavelets, one may read [41–45].

Now, we give a sufficient condition under which fractional Boas transform of a wavelet is again a wavelet.

Theorem 1. *Let $\psi\in L^1(\mathbb{R})$ be a wavelet such that $\widehat{\psi}\in L^1(\mathbb{R})$ and $\widehat{\psi}(0)=0$. Then, $\mathfrak{B}_\theta\psi$ is again a wavelet.*

Proof. In view of Theorem 2.1 in [30], $\mathfrak{B}_\theta\psi\in L^2(\mathbb{R})$. To verify the admissibility condition, we have

$$\int_{\mathbb{R}}\frac{\left|\widehat{\mathfrak{B}_\theta\psi}(\eta)\right|^2}{|\eta|}\mathrm{d}\eta\le|\cos(\theta)|^2\int_{\mathbb{R}}\frac{|\widehat{\psi}(\eta)|^2}{|\eta|}\mathrm{d}\eta+|\sin(\theta)|^2\int_{\mathbb{R}}\frac{|\widehat{\mathfrak{B}\psi}(\eta)|^2}{|\eta|}\mathrm{d}\eta+|\sin(2\theta)|\int_{\mathbb{R}}\frac{|\widehat{\psi}(\eta)\widehat{\mathfrak{B}\psi}(\eta)|}{|\eta|}\mathrm{d}\eta. \tag{37}$$

Now, since

$$\begin{aligned}\int_{\mathbb{R}} \frac{|\widehat{\psi}(\eta)\widehat{\mathfrak{B}\psi}(\eta)|}{|\eta|}\mathrm{d}\eta &= \int_{|\eta|\le 1} \frac{|\widehat{\psi}(\eta)(-i\operatorname{sgn}(\eta)|\eta|)\widehat{\psi}(\eta)|}{|\eta|}\mathrm{d}\eta + \int_{|\eta|>1} \frac{|\widehat{\psi}(\eta)(-i\operatorname{sgn}(\eta))\widehat{\psi}(\eta)|}{|\eta|}\mathrm{d}\eta \\ &\le \int_{|\eta|\le 1} \frac{|\widehat{\psi}(\eta)|^2}{|\eta|}\mathrm{d}\eta + \int_{|\eta|>1} \frac{|\widehat{\psi}(\eta)|^2}{|\eta|}\mathrm{d}\eta \\ &= \int_{\mathbb{R}} \frac{|\widehat{\psi}(\eta)|^2}{|\eta|}\mathrm{d}\eta,\end{aligned} \tag{38}$$

we deduce that

$$\int_{\mathbb{R}} \frac{\left|\widehat{\mathfrak{B}_\theta\psi}(\eta)\right|^2}{|\eta|}\mathrm{d}\eta < +\infty. \tag{39}$$

□

Remark 1. The condition that $\psi, \widehat{\psi} \in L^1(\mathbb{R})$ such that $\widehat{\psi}(0) = 0$ is not necessary for $\mathfrak{B}_\theta\psi$ to be a wavelet. Indeed, let ψ be a Haar wavelet defined by

$$\psi(x) = \begin{cases} 1, & \text{if } 0 \le x < \frac{1}{2}, \\ -1, & \text{if } \frac{1}{2} \le x < 1, \\ 0, & \text{if otherwise.} \end{cases} \tag{40}$$

Since $\|\mathfrak{B}_\theta\psi\|_2 \le l\|\psi\|_2 < +\infty$, where $l = |\cos(\theta)| + |\sin(\theta)| + \sqrt{2\pi}$ and $\int_{\mathbb{R}} (|\widehat{\mathfrak{B}_\theta\psi}(\eta)|^2/|\eta|)\mathrm{d}\eta < +\infty$, we conclude that $\mathfrak{B}_\theta\psi$ is a wavelet. However, $\widehat{\psi} \notin L^1(\mathbb{R})$.

In the next result, we give a relationship between two wavelets in terms of $(\mathfrak{B}_\theta - \mathfrak{H}_\theta)$ operator.

Theorem 2. *Let ψ be a wavelet such that $\psi, \widehat{\psi} \in L^1(\mathbb{R})$ and let ρ be a function such that $\rho, \rho^{(1)}, \widehat{\rho}^{(1)} \in L^1(\mathbb{R})$. If $\widehat{\psi}(0) = 0$ and $\widehat{\psi}(\eta) = (\widehat{g}(\eta)t + n|\eta|)\widehat{\rho}(\eta)$, then ρ is a wavelet such that*

$$\{(\mathfrak{B}_\theta - \mathfrak{H}_\theta)\psi\}(x) = \{(\mathfrak{B}_\theta - \mathfrak{H}_\theta)\rho\}(x). \tag{41}$$

Proof. Note that $\{(\mathfrak{B}_\theta - \mathfrak{H}_\theta)\psi\}(x) = \sin(\theta)\{(\mathfrak{B} - \mathfrak{H})\psi\}(x)$. Thus, in view of Theorem 2.3 in [30], ρ is a wavelet such that $\{(\mathfrak{B}_\theta - \mathfrak{H}_\theta)\psi\}(x) = \{(\mathfrak{B}_\theta - \mathfrak{H}_\theta)\rho\}(x)$.

The following result gives a necessary and sufficient condition under which fractional Boas transform of a given wavelet is multiple of the first-order derivative of the given wavelet. □

Theorem 3. *Let $\psi(x)$ be a wavelet such that $\psi, \widehat{\psi} \in L^1(\mathbb{R})$ and $\widehat{\psi}(0) = 0$. Then $\mathfrak{B}_\theta\psi(x) = \cos(\theta)\psi^{(1)}(x)$ if and only if $\widehat{\psi}(\eta) = 0$ for every $|\eta| > 1$.*

Proof. Let $\mathfrak{B}_\theta\psi(x) = \cos(\theta)\psi^{(1)}(x)$. Then, taking Fourier transform on both sides, we have

$$\widehat{\psi}(\eta)(\cos(\theta)(1 + i\eta) - i\sin(\theta)\operatorname{sgn}(\eta)(1 - \widehat{g}(\eta))) = 0. \tag{42}$$

If $|\eta| > 1$, then $\widehat{\psi}(\eta)\mathcal{S}_\phi(\eta) = 0$, where

$$\mathcal{S}_\phi(\eta) = \begin{cases} e^{-i\phi} + i\eta\cos(\phi), & \text{if } \eta > 1, \\ e^{i\phi} + i\eta\cos(\phi), & \text{if } \eta < -1. \end{cases} \tag{43}$$

Since $\mathcal{S}_\phi(x) \ne 0$, we get $\widehat{\psi}(\eta) = 0$.

The proof of the converse part is straightforward.

A wavelet $\psi(x)$ is said to have n vanishing moments if $\int_{\mathbb{R}} x^q\psi(x)\mathrm{d}x = 0$, for $0 \le q \le n-1$. This property actually represents the regularity of the wavelet function and ability of wavelet transform to capture the localized information. If a wavelet with large number of vanishing moments is employed, then the corresponding wavelet series of a smooth function will converge very rapidly to the function. Thus, only few wavelet coefficients are required in order to obtain a good approximation. During image compression, it requires only to keep a few wavelet coefficients, where the image is smooth and, in contrary to this, more coefficients are needed at the edges. For more details, see [33–40].

Next, we define the notion of G-function of order n. □

Definition 1. Let f be a function such that $f, f^{(1)}, \widehat{f} \in L^1(\mathbb{R})$. Then, f is said to be a G-function of order n if $\int_{\mathbb{R}} x^q G(x)\mathrm{d}x = 0$, for $0 \le q \le n$, where $G(x) = \int_{-1}^{1} (1 - (1/|\eta|))e^{-2\pi i\eta x}\widehat{f^{(1)}}(-\eta)\mathrm{d}\eta$.

Recall from [44] that a function f is said to have fast decay with decay rate $l \in \mathbb{N}$, if there exists a constant C_l such that $|f(x)| \le (C_l/1 + |x|^l)$, for all $x \in \mathbb{R}$.

In the following result, we give a sufficient condition for the higher vanishing moments of fractional Boas transform of wavelets.

Theorem 4. *If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal system on $\mathbb{R}$, then*

$$\int_{\mathbb{R}} x^s\mathfrak{B}_\theta\psi(x)\mathrm{d}x = 0, \quad \text{for all } s = 0, 1, 2, \ldots, p; s + 1 < l, \tag{44}$$

where ψ such that $\psi, \widehat{\psi}, \psi^{(1)} \in L^1(\mathbb{R})$ is a G-function of order p, and $\psi \in \mathscr{C}^p(\mathbb{R})$ has fast decay with decay exponent $l \in \mathbb{N}$ such that $\psi^{(s)} \in L^\infty(\mathbb{R})$, $s = 1, 2, \ldots, p$.

Proof. We have

$$\begin{aligned}\int_{\mathbb{R}} x^s \mathfrak{B}_\theta \psi(x)\mathrm{d}x &= \int_{\mathbb{R}} x^s (\cos(\theta)\psi(x) + \sin(\theta)\mathfrak{B}\psi(x))\mathrm{d}x \\ &= \cos(\theta)\int_{\mathbb{R}} x^s \psi(x)\mathrm{d}x + \sin(\theta)\int_{\mathbb{R}} x^s \mathfrak{B}\psi(x)\mathrm{d}x \\ &= \cos(\theta)\int_{\mathbb{R}} x^s \psi(x)\mathrm{d}x + \sin(\theta)\int_{\mathbb{R}} x^s \mathfrak{H}\psi(x)\mathrm{d}x \\ &\quad - \frac{1}{2\pi}\int_{\mathbb{R}} x^s \int_{-1}^{1}\left(1-\frac{1}{|\eta|}\right)e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)\mathrm{d}\eta\mathrm{d}x\left(\because \psi, \psi^{(1)}, \widehat{\psi} \in L^1(\mathbb{R})\right) \\ &= \cos(\theta)\int_{\mathbb{R}} x^s \psi(x)\mathrm{d}x + \sin(\theta)\left(\int_{\mathbb{R}} x^s \mathfrak{H}\psi(x)\mathrm{d}x\right) - \frac{1}{2\pi}\int_{\mathbb{R}} x^s G(x)\mathrm{d}x \\ &= \cos(\theta)\int_{\mathbb{R}} x^s \psi(x)\mathrm{d}x + \sin(\theta)\int_{\mathbb{R}} x^s \mathfrak{H}\psi(x)\mathrm{d}x,\end{aligned} \tag{45}$$

where $G(x) = \int_{-1}^{1}(1-(1/|x|))e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)\mathrm{d}\eta$.

In view of Theorem 3.1 in [30], we have $\int_{\mathbb{R}} x^s \psi(x)dx = 0$, for all $s = 0, 1, \ldots, p$.

Also, since $\psi(x), x^p\psi(x) \in L^2(\mathbb{R})$ and $x^s\psi(x) \in L^2(\mathbb{R})$ for $s = 0, 1, \ldots, p$, using the moment formula for the Hilbert transform, we have

$$\int_{\mathbb{R}} x^s \mathfrak{B}_\theta \psi(x)\mathrm{d}x = \sin\theta\left(\int_{\mathbb{R}} \mathfrak{H}\{x^s\psi(x)\}\mathrm{d}x + \frac{1}{\pi}\sum_{j=0}^{s-1}\int_{\mathbb{R}} x^j \int_{\mathbb{R}} z^{s-1-j}\psi(z)\mathrm{d}z\mathrm{d}x\right). \tag{46}$$

Now $x^s\psi(x) \in L^2(\mathbb{R})$, for $s = 0, 1, \ldots, p$. Hence, it follows that $\int_{\mathbb{R}} x^s \mathfrak{B}_\theta \psi(x)\mathrm{d}x = 0$ for $s = 0, 1, \ldots, p$. □

Recall from [46] that a two-dimensional function $\Psi \in L^2(\mathbb{R}^2)$ is called an admissible wavelet if it satisfies the admissibility condition

$$C_\Psi = (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\widehat{\Psi}(\boldsymbol{\eta})|^2}{|\boldsymbol{\eta}|^2}\mathrm{d}\boldsymbol{\eta} < \infty, \tag{47}$$

where $|\eta|^2 = \eta_1^2 + \eta_2^2$.

In the following result, we give a sufficient condition on two wavelets ψ_1 and ψ_2 such that the product $\cos(\theta)\psi_i^{(1)}(u)\psi_i(v), i = 1, 2$ forms a two-dimensional wavelet. □

Theorem 5. *Let ψ_1, ψ_2 be wavelets such that*

(i) $\psi_i, \psi_i^{(1)}, \widehat{\psi}_i \in L^1(\mathbb{R})$

(ii) $\widehat{\psi}_i(0) = 0$ and $\widehat{\psi}_i(\eta) = 0, i = 1, 2$ for $|\eta| > 1$

Then, $\Psi_i(u, v) = \cos(\theta)\psi_i^{(1)}(u)\psi_i(v), i = 1, 2$ are admissible wavelets in $L^2(\mathbb{R}^2)$.

Proof. Note that

$$\begin{aligned}\Psi_i(u, v) &= \mathfrak{B}_\theta \psi_i(u)\psi_i(v), \quad i = 1, 2 \\ &= \cos(\theta)\psi_i(u)\psi_i(v) + \sin(\theta)\mathfrak{B}\psi_i(u)\psi_i(v).\end{aligned} \tag{48}$$

Since $\widehat{\psi}_i \in L^1(\mathbb{R})$, for $i \in 1, 2$, it follows that ψ_i is bounded. Also, $\psi_i \in L^1(\mathbb{R}), i = 1, 2$ is bounded and so it must be in $L^2(\mathbb{R})$. Thus, clearly $\Psi_i \in L^2(\mathbb{R}^2)$ for $i = 1, 2$.

Also, we have

$$\begin{aligned}C_{\Psi_i} &= (2\pi)^2 \int_{\mathbb{R}^2} \frac{|\widehat{\Psi}_i(\boldsymbol{\eta})|^2}{|\boldsymbol{\eta}|^2}\mathrm{d}\boldsymbol{\eta} \\ &\le (2\pi)^2|\cos(\theta)|^2 \int_{\mathbb{R}} \frac{|\widehat{\psi}_i(\eta_1)|^2}{|\eta_1|}\mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{|\widehat{\psi}_i(\eta_2)|^2}{|\eta_2|}\mathrm{d}\eta_2 \\ &\quad + (2\pi)^2|\sin(\theta)|^2 \int_{\mathbb{R}} \frac{|\widehat{\mathfrak{B}\psi_i}(\eta_1)|^2}{|\eta_1|}\mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{|\widehat{\psi}_i(\eta_2)|^2}{|\eta_2|}\mathrm{d}\eta_2 \\ &\quad + (2\pi)^2|\sin(2\theta)| \int_{\mathbb{R}} \frac{|\widehat{\psi}_i(\eta_1)|^2}{|\eta_1|}\mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{|\widehat{\psi}_i(\eta_2)|^2}{|\eta_2|}\mathrm{d}\eta_2 \\ &< \infty.\end{aligned} \tag{49}$$

Then, for $i = 1, 2$, Ψ_i is an admissible wavelet in $L^2(\mathbb{R}^2)$.

Next, we give a sufficient condition on two wavelets ψ_1 and ψ_2 such that the convolution of two-dimensional wavelets $\Psi_i, i = 1, 2$ again forms a wavelet with $(4p-2)$ vanishing moments. □

Theorem 6. *Let ψ_1, ψ_2 be wavelets as defined in Theorem 5 such that*

(i) $u^p \psi_i(u) \in L^2(\mathbb{R}), i = 1, 2,$ and $[((\psi_1 * \psi_2) * g) * (2\delta - g)](x) = 0$, where δ is Dirac delta function and g is given by (3)

(ii) $\psi_i, i = 1, 2$ are G-function of order p

(iii) $(\widehat{\psi_1 * \psi_2})(0) = 0$, and $\psi_i, i = 1, 2$ have p vanishing moments

Then, $\Psi_i(u, v) = \cos(\theta)\psi_i^{(1)}(u)\psi_i(v), i = 1, 2$ have $2p$ vanishing moments. Further, if $\Psi_3(u, v) = (\Psi_1 * \Psi_2)(u, v)$, then Ψ_3 is admissible wavelet with $(4p - 2)$ vanishing moments.

Proof. We have

$$\int_{\mathbb{R}^2} u^r v^s \Psi_i(u, v) \mathrm{d}u\mathrm{d}v = \cos(\theta) M_r M_{n-r} + \sin(\theta) \widetilde{M_r} M_{n-r} \, (r + s = n), \tag{50}$$

where $M_r = \int_{\mathbb{R}} z^r \psi_i(z)\mathrm{d}z$ and $\widetilde{M_r} = \int_{\mathbb{R}} z^r \mathfrak{B}\psi_i(z)\mathrm{d}z$.

Since $\psi_i, i = 1, 2$ have p vanishing moments, using the arguments given in Theorem 3.1 in [30], we conclude that $\mathfrak{B}\psi_i, i = 1, 2$ have $(p + 1)$ vanishing moments.

Assume that $n \le 2p - 1$. If $n - r \le p - 1$, then $M_{n-r} = 0$; otherwise, $r \le p$, which gives $M_r = \widetilde{M}_r = 0$. Thus, the number of vanishing moments of $\Psi_i, i = 1, 2$ is $2p$.

Also, we compute

$$\begin{aligned}
\Psi_3(u, v) &= \Psi_1(u, v) * \Psi_2(u, v) \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} \Psi_1(t_1, t_2)\Psi_2(u - t_1, v - t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} (\cos(\theta\psi_1(t_1)\psi_1(t_2) + \sin(\theta)\mathfrak{B}\psi_1(t_1)\psi_1(t_2)) \\
&\quad \cdot (\cos(\theta)\psi_2(u - t_1)\psi_2(v - t_2) + \sin(\theta)\mathfrak{B}\psi_2(u - t_1)\psi_2(v - t_2))\mathrm{d}t_1\mathrm{d}t_2 \\
&= \cos^2(\theta)\int_{\mathbb{R}}\int_{\mathbb{R}} \psi_1(t_1)\psi_2(u - t_1)\psi_1(t_2)\psi_2(v - t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + \cos(\theta)\sin(\theta)\int_{\mathbb{R}}\int_{\mathbb{R}} \psi_1(t_1)\mathfrak{B}\psi_2(u - t_1)\psi_1(t_2)\psi_2(v - t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + \cos(\theta)\sin(\theta)\int_{\mathbb{R}}\int_{\mathbb{R}} \mathfrak{B}\psi_1(t_1)\psi_2(u - t_1)\psi_1(t_2)\psi_2(v - t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&\quad + \sin^2(\theta)\int_{\mathbb{R}}\int_{\mathbb{R}} \mathfrak{B}\psi_1(t_1)\mathfrak{B}\psi_2(u - t_1)\psi_1(t_2)\psi_2(v - t_2)\mathrm{d}t_1\mathrm{d}t_2 \\
&= \cos^2(\theta)(\psi_1 * \psi_2)(u)(\psi_1 * \psi_2)(v) + \cos(\theta)\sin(\theta)(\psi_1 * \mathfrak{B}\psi_2)(u)(\psi_1 * \psi_2)(v) \\
&\quad + \cos(\theta)\sin(\theta)(\mathfrak{B}\psi_1 * \psi_2)(u)(\psi_1 * \psi_2)(v) + \sin^2(\theta)(\mathfrak{B}\psi_1 * \mathfrak{B}\psi_2)(u)(\psi_1 * \psi_2)(v) \\
&= \cos(2\theta)(\psi_1 * \psi_2)(u)(\psi_1 * \psi_2)(v) + \sin(2\theta)(\psi_1 * \mathfrak{B}\psi_2)(u)(\psi_1 * \psi_2)(v).
\end{aligned} \tag{51}$$

Further, note that

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\Psi_3(u, v)|^2 \mathrm{d}u\mathrm{d}v \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} |\cos(2\theta)(\psi_1 * \psi_2)(u)(\psi_1 * \psi_2)(v) + \sin(2\theta)(\psi_1 * \mathfrak{B}\psi_2)(u)(\psi_1 * \psi_2)(v)|^2 \mathrm{d}u\mathrm{d}v \\
&\le |\cos(2\theta)|^2 \int_{\mathbb{R}} |(\psi_1 * \psi_2)(u)|^2 \mathrm{d}u \int_{\mathbb{R}} |(\psi_1 * \psi_2)(v)|^2 \mathrm{d}v + |\sin(2\theta)|^2 \int_{\mathbb{R}} |(\psi_1 * \mathfrak{B}\psi_2)(u)|^2 \mathrm{d}u \\
&\quad \cdot \int_{\mathbb{R}} |(\psi_1 * \psi_2)(v)|^2 \mathrm{d}v + |\sin(4\theta)| \int_{\mathbb{R}} |(\psi_1 * \psi_2)(u)(\psi_1 * \mathfrak{B}\psi_2)(u)| \mathrm{d}u \int_{\mathbb{R}} |(\psi_1 * \psi_2)(v)|^2 \mathrm{d}v.
\end{aligned} \tag{52}$$

Now, since $\psi_1, \widehat{\psi}_1, \psi_2 \in L^1(\mathbb{R})$ and $(\widehat{\psi_1 * \psi_2})(0) = 0$, it follows that $\int_{\mathbb{R}^2} |\Psi_3(u, v)|^2 \mathrm{d}u\mathrm{d}v < \infty$.

Again, since $\Psi_i \in L^1(\mathbb{R}^2)$ for $i = 1, 2$, we have

$$\begin{aligned}
\widehat{\Psi}_3(\eta_1, \eta_2) &= \widehat{\Psi}_1(\eta_1, \eta_2)\widehat{\Psi}_2(\eta_1, \eta_2) \\
&= \left(\cos(\theta)\widehat{\psi}_1(\eta_1)\widehat{\psi}_1(\eta_2) + \sin(\theta)\widehat{\mathfrak{B}\psi}_1(\eta_1)\widehat{\psi}_1(\eta_2)\right)\left(\cos(\theta)\widehat{\psi}_2(\eta_1)\widehat{\psi}_2(\eta_2) + \sin(\theta)\widehat{\mathfrak{B}\psi}_2(\eta_1)\widehat{\psi}_2(\eta_2)\right) \\
&= \cos^2(\theta)\widehat{\psi}_1(\eta_1)\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_1)\widehat{\psi}_2(\eta_2) + \sin(2\theta)\widehat{\psi}_1(\eta_1)\widehat{\psi}_1(\eta_2)\widehat{\mathfrak{B}\psi}_2(\eta_1)\widehat{\psi}_2(\eta_2) \\
&\quad + \sin^2(\theta)\widehat{\mathfrak{B}\psi}_1(\eta_1)\widehat{\psi}_1(\eta_2)\widehat{\mathfrak{B}\psi}_2(\eta_1)\widehat{\psi}_2(\eta_2).
\end{aligned} \tag{53}$$

Therefore, we obtain

$$\begin{aligned}
C_{\Psi_3} &= (2\pi)^2 \int_{\mathbb{R}^2} \frac{\left|\widehat{\Psi}_3(\eta_1, \eta_2)\right|^2}{|\boldsymbol{\eta}|^2} \mathrm{d}\eta_1 \mathrm{d}\eta_2 \\
&\leq (2\pi)^2 |\cos(\theta)|^4 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_1)\widehat{\psi}_2(\eta_1)\right|^2}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2 \\
&\quad + (2\pi)^2 |\sin(2\theta)|^2 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_1)\widehat{\mathfrak{B}\psi}_2(\eta_1)\right|^2}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2 \\
&\quad + (2\pi)^2 |\sin(\theta)|^2 \int_{\mathbb{R}} \frac{\left|\widehat{\mathfrak{B}\psi}_1(\eta_1) t \widehat{\mathfrak{B}\psi}_2 n(\eta_1)\right|^2}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2 \\
&\quad + 2(2\pi)^2 \left|\cos^2(\theta)\sin(2\theta)\right| \overset{I_1}{\frown} \int_{\mathbb{R}} \frac{\left|\left(\widehat{\psi}_1(\eta_1)\right)^2 \widehat{\psi}_2(\eta_1)\widehat{\mathfrak{B}\psi}_2(\eta_1)\right|^2}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2 \\
&\quad + 2(2\pi)^2 \left|\sin(2\theta)\sin^2(\theta)\right| \overset{I_2}{\frown} \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_1)\widehat{\mathfrak{B}\psi}_1(\eta_1)\left(\widehat{\mathfrak{B}\psi}_2(\eta_1)\right)^2\right|}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2 \\
&\quad + 2(2\pi)^2 \left|\cos(\theta)\sin(\theta)\right|^2 \overset{I_3}{\frown} \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_1)\widehat{\mathfrak{B}\psi}_1(\eta_1)\widehat{\psi}_2(\eta_1)\widehat{\mathfrak{B}\psi}_2(\eta_1)\right|}{|\eta_1|} \mathrm{d}\eta_1 \int_{\mathbb{R}} \frac{\left|\widehat{\psi}_1(\eta_2)\widehat{\psi}_2(\eta_2)\right|^2}{|\eta_2|} \mathrm{d}\eta_2.
\end{aligned} \tag{54}$$

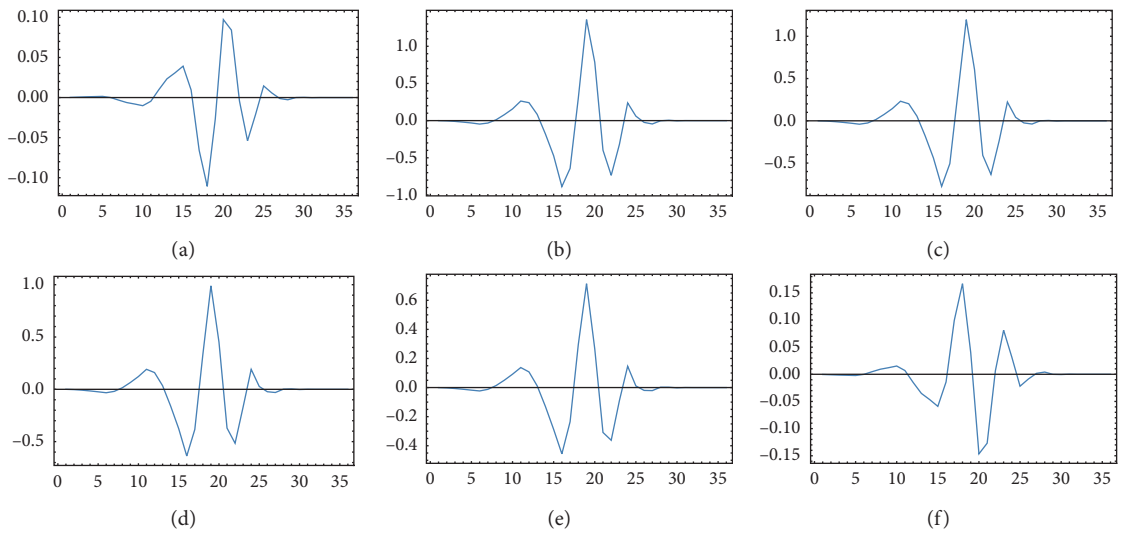

Figure 1: HT of Daubechies wavelet and FRBTs of Daubechies wavelet at $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2$. (a) HT of Daubechies wavelet. (b) FRBT of Daubechies wavelet with $\theta = 0$. (c) FRBT of Daubechies wavelet with $\theta = \pi/6$. (d) FRBT of Daubechies wavelet with $\theta = \pi/4$. (e) FRBT of Daubechies wavelet with $\theta = \pi/3$. (f) FRBT of Daubechies wavelet with $\theta = \pi/2$.

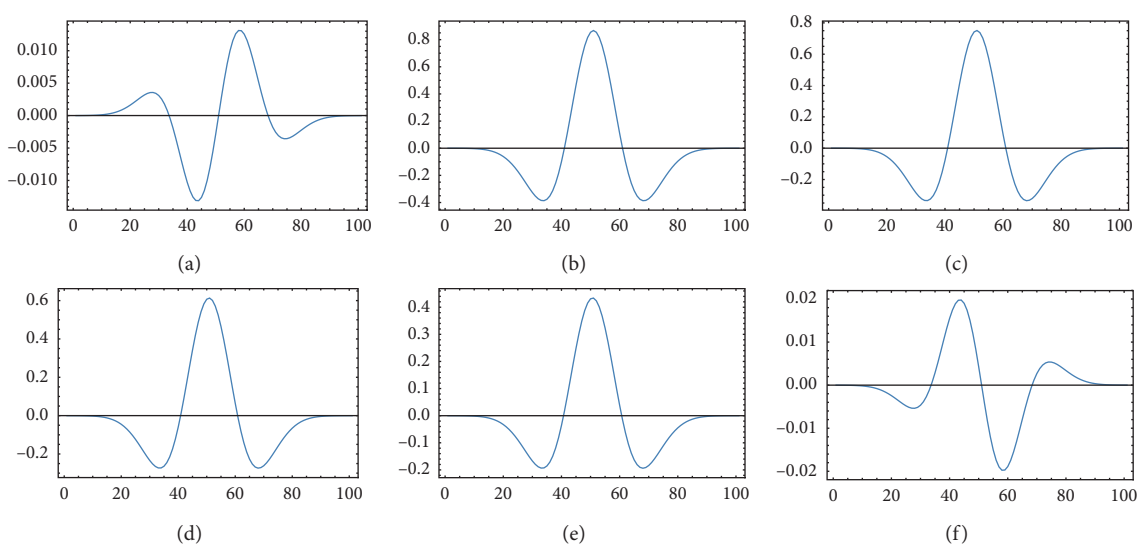

Figure 2: HT of Mexican Hat wavelet and FRBTs of Mexican Hat wavelet at $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2$. (a) HT of Mexican Hat wavelet. (b) FRBT of Mexican Hat wavelet with $\theta = 0$. (c) FRBT of Mexican Hat wavelet with $\theta = \pi/6$. (d) FRBT of Mexican Hat wavelet with $\theta = \pi/4$. (e) FRBT of Mexican Hat wavelet with $\theta = \pi/3$. (f) FRBT of Mexican Hat wavelet with $\theta = \pi/2$.

Since $\psi_2 \in L^1(\mathbb{R})$ and $\widehat{\psi}_2(0) = 0$, it follows that $\widehat{\psi}_2, \widehat{\mathfrak{B}\psi_2}$ are bounded. Thus, we have

$$I_1 = \int_{\mathbb{R}} \frac{\left|\{\widehat{\psi}_1(\eta_1)\}^2 \widehat{\psi}_2(\eta_1)\widehat{\mathfrak{B}\psi_2}(\eta_1)\right|^2}{|\eta_1|} \mathrm{d}\eta_1 \le K_1 K_2 \int_{\mathbb{R}} \frac{|\widehat{\psi}_1(\eta_1)|^2}{|\eta_1|^2} \mathrm{d}\eta_1, \tag{55}$$

where K_1 and K_2 are constants. A similar argument works for I_2 and I_3. Therefore, we have $C_{\Psi_3} < +\infty$. Now, observe that

$$\begin{aligned}\int_{\mathbb{R}^2} u^r v^s \Psi_3(u,v)\mathrm{d}u\mathrm{d}v &= \cos(2\theta)\int_{\mathbb{R}} u^r(\psi_1 * \psi_2)(u)\mathrm{d}u \int_{\mathbb{R}} v^s(\psi_1 * \psi_2)(v)\mathrm{d}v \\ &\quad + \sin(2\theta)\int_{\mathbb{R}} u^r(\psi_1 * \mathfrak{B}\psi_2)(u)\mathrm{d}u \int_{\mathbb{R}} v^s(\psi_1 * \psi_2)(v)\mathrm{d}v \\ &= \cos(2\theta)M_r^* M_s^* + \sin(2\theta)\widetilde{M}_r^* M_s^*,\end{aligned} \tag{56}$$

where $M_i^* = \int_{\mathbb{R}} z^i(\psi_1 * \psi_2)(z)\mathrm{d}z$ and $\widetilde{M}_i^* = \int_{\mathbb{R}} z^i(\psi_1 * \mathfrak{B}\psi_2)(z)\mathrm{d}z$.

Taking $r + s = t$, we obtain

$$\int_{\mathbb{R}^2} u^r v^s \Psi_3(u,v)\mathrm{d}u\mathrm{d}v = \cos(2\theta)M_r^* M_{t-r}^* + \sin(2\theta)\widetilde{M}_r^* M_{t-r}^*. \tag{57}$$

Thus, in view of Theorem 4.3 in [33], $(\psi_1 * \psi_2)$ and $(\psi_1 * \mathfrak{B}\psi_2)$ have, respectively, $2p-1$ and $2p$ vanishing moments.

If $r \le 2p-1$, then $M_r^* = 0$ and $\widetilde{M_r}^* = 0$. If not, then $t - r \le 2p - 2$. Thus, $M_{t-r}^* = 0$.

Therefore, M_r^*, $\widetilde{M_r}^*$, and M_{t-r}^* all vanish if $r + s \le 4p - 3$.

Hence, the number of vanishing moments of $\Psi_3(u,v)$ is $4p - 2$. □

4. Conclusion

In this paper, we define and study the notion of fractional Boas transforms (FRBT) with the aim of obtaining better comparative results. Various properties of FRBT are discussed, and several results are obtained. A comparative study is done to show that the FRBT of wavelets gives better results as compared to the usual wavelets of the classical Boas transform. We illustrate this study by considering the HT and the FRBTs of Daubechies wavelet and Mexican Hat wavelet, respectively, through Figures 1 and 2 and show that the FRBT of wavelet at $\theta = 0$ is the wavelet itself, and the FRBT of wavelet at $\theta = \pi/2$ is the BT of wavelet. It is easy to conclude that, after applying FRBT on a wavelet, the resulting wavelet is approximately equal to the original wavelet. Hence, it is better to employ FRBT of the wavelet instead of using BT of the wavelet, or HT of the wavelet.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors equally contributed to this paper and read and approved the final manuscript.

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Research Article

Fourier–Boas-Like Wavelets and Their Vanishing Moments

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In this paper, we propose Fourier–Boas-Like wavelets and obtain sufficient conditions for their higher vanishing moments. A sufficient condition is given to obtain moment formula for such wavelets. Some properties of Fourier–Boas-Like wavelets associated with Riesz projectors are also given. Finally, we formulate a variation diminishing wavelet associated with a Fourier–Boas-Like wavelet.

1. Introduction

Fourier analysis was initiated with the reconstruction and analysis of periodic functions with the help of Fourier series. It was later extended to Fourier transform in order to study the nonperiodic signals. The Fourier transform is an effective analytical tool to study the continuous and discrete-time signals, but there is a limitation in employing this transform as it does not exhibit the temporal information of the signal. Thus, to overcome this limitation, Gabor [1] introduced the concept of short-time Fourier transform, where an analysis window of fixed length slides over the time axis to give time-localized frequency information. The inherent limitation of short-time Fourier transform lies in the fact that the window width remains the same for all the frequencies, thereby making the localization extent to be constant for different frequencies. The resolution to this problem was given by Grossmann and Morlet [2] who introduced the notion of wavelet transforms in 1984. A function ψ with finite energy, E_ψ, i.e., $\psi \in L^2(\mathbb{R})$, is said to be a wavelet if it satisfies the admissibility condition given by $\mathfrak{C}_\psi = \int_{\mathbb{R}} ((|\widehat{\psi}(\eta)|^2)/|\eta|)\mathrm{d}\eta < +\infty$, where $\widehat{\psi}$ denotes the Fourier transform of a wavelet ψ. The most significant work that has driven the progress of wavelets to great heights was attributed to Mallat [3] and Meyer [4] who both were responsible in the development of the concept of multiresolution analysis (MRA), another way of constructing wavelets.

An MRA consists of a sequence of closed subspaces $\mathscr{V}_j$, $j \in \mathbb{Z}$ of $L^2(\mathbb{R})$ satisfying the following conditions: (i) $\mathscr{V}_j \subset \mathscr{V}_{j+1}$, for all $j \in \mathbb{Z}$; (ii) $f \in \mathscr{V}_j$ if and only if $f(2\cdot) \in \mathscr{V}_{j+1}$, for all $j \in \mathbb{Z}$; (iii) $\cap_{j\in\mathbb{Z}} \mathscr{V}_j = \{0\}$; (iv) $\overline{\cup_{j\in\mathbb{Z}} \mathscr{V}_j} = L^2(\mathbb{R})$; (v) if there exists $\phi \in \mathscr{V}_0$ so that $\{\phi(\cdot - k): k \in \mathbb{Z}\}$ constitutes an orthonormal basis for $\mathscr{V}_0$, then there exists an orthonormal wavelet basis $\psi_{j,k}$ such that $\mathrm{P}_j f = \mathrm{P}_{j-1} f + \sum_{k\in\mathbb{Z}} \langle f, \psi_{j,k}\rangle \psi_{j,k}$ holds, where P_j denotes the orthogonal projection operator onto $\mathscr{V}_j$. The function ϕ is called a scaling function of the given MRA. It solves the dilation equation $\phi(x) = 2\sum_{k\in\mathbb{Z}} a_k \phi(2x+k)$ with $|\widehat{\phi}(0)| = 1$ and ψ is a function associated to ϕ, which is defined by $\psi(x) = 2\sum_{k\in\mathbb{Z}} b_k \phi(2x+k)$. The functions ϕ and ψ are usually known as father and mother wavelets, respectively. It is well established in [5] that a pair of quadrature mirror filter coefficients, $(a_k)_{k\in\mathbb{Z}}, (b_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$, is associated to the MRA and the following relations $\widehat{\phi}(\eta) = m_0(\eta/2)\widehat{\phi}(\eta/2)$, $\widehat{\psi}(\eta) = m_1(\eta/2)\widehat{\phi}(\eta/2)$ are satisfied for $\eta \in \mathbb{R}$, where m_0 and m_1 are given by $m_0(\eta) = \sum_{k\in\mathbb{Z}} a_k e^{ik\eta}$, $m_1(\eta) = \sum_{k\in\mathbb{Z}} b_k e^{ik\eta} = e^{i\eta}\overline{m_0(\eta+\pi)}$. For more details on wavelets, one may refer to [6–14].

Boas [15] proposed an integral transform related to the Hilbert transform, which arose due to the study of the class of functions having Fourier transforms vanishing on a finite interval. This transform is known as Boas transform, which finds an application in the theory of filters in electrical

engineering. Goldberg [16], in 1960, analyzed this transform in detail and gave some significant results and properties. Subsequently, it was further examined by Heywood [17] in 1963 and Zaidi [18] in 1976. For various details pertaining to Boas transform, one may refer to [19].

Khanna et al. [9] introduced Boas transforms of wavelets and furnished various results related to their higher vanishing moments. Later, Khanna and Kathuria [8] studied convolution of these resulting wavelets to analyze Boas transform of convolution of signals. Recently, Khanna et al. [20] introduced fractional Boas transforms and the associated wavelets. These new wavelets appeared to be more prominent than the Boas transforms of wavelets due to an additional degree of freedom in terms of fractional order.

1.1. Framework. This work is streamlined as follows. In Section 2, Fourier–Boas-Like wavelets are introduced and studied. Some results related to higher vanishing moments of such wavelets are obtained. Also, the moment formula for such wavelets by enforcing sufficient condition on the wavelet is derived. Further, some properties of Fourier–Boas-Like wavelets associated with Riesz projectors are given. Finally, a variation diminishing wavelet associated with Fourier–Boas-Like wavelet is constructed.

2. Fourier–Boas-Like Wavelets

Let $f \in L^2(\mathbb{R})$. Then, the Boas transform of f in terms of principal value integral is defined as

$$\begin{aligned}\mathfrak{B}f(x) &= \frac{1}{\pi} p.v. \int_0^{\infty} \frac{f(x+\mathfrak{z}) - f(x-\mathfrak{z})}{\mathfrak{z}^2} \sin(\mathfrak{z}) d\mathfrak{z} \\ &= \frac{1}{\pi} p.v. \int_{-\infty}^{\infty} \frac{f(x+\mathfrak{z})}{\mathfrak{z}^2} \sin(\mathfrak{z}) d\mathfrak{z},\end{aligned} \tag{1}$$

for any x for which the integral exists.

The relationship between the Boas transform and the Hilbert transform of a function is given by

$$(\mathfrak{B}f)(x) = (\mathfrak{H}f)(x) - \{\mathfrak{H}f * \mathfrak{g}\}(x), \tag{2}$$

where

$$\mathfrak{g}(x) = \left(\frac{2}{\pi}\right)^{(1/2)} \left(\frac{1-\cos(x)}{\pi x^2}\right), \tag{3}$$

whereas the Fourier transform and the Hilbert transform share a relationship, specified by

$$\widehat{\mathfrak{H}f}(\eta) = \begin{cases} -i\widehat{f}(\eta), & \text{if } \eta > 0, \\ i\widehat{f}(\eta), & \text{if } \eta < 0, \\ 0, & \text{if } \eta = 0. \end{cases} \tag{4}$$

Taking Fourier transform on both the sides of (2), we have

$$\widehat{\mathfrak{B}f}(\eta) = \widehat{\mathfrak{H}f}(x) - \mathfrak{F}\{\mathfrak{H}f * \mathfrak{g}\}(\eta). \tag{5}$$

If $\mathfrak{H}f(x) \in L^1(\mathbb{R})$, then using (2), (4), and (5), we obtain $\widehat{\mathfrak{B}f}(\eta) = -i\operatorname{sgn}(\eta)\widehat{f}(\eta)(1-\widehat{\mathfrak{g}}(\eta))$, where

$$\widehat{\mathfrak{g}}(\eta) = \begin{cases} 0, & \text{if } |\eta| > 1, \\ 1-|\eta|, & \text{if } |\eta| \le 1. \end{cases} \tag{6}$$

Soares et al. [21] introduced Fourier-like wavelets, defined by $\mathfrak{F}_{\text{Like}}\{\psi\}(x) = (1/\sqrt{2})(\psi(x) - i\mathfrak{H}\psi(x))$, using the concept of the Fourier kernel $e^{i\eta}$. Further, a factor of $(1/\sqrt{2})$ was imposed on function $\psi(x) - i\mathfrak{H}\psi(x)$ in order to aver the same energy and admissibility coefficient of its generating wavelet. Later, Khanna et al. [12] defined an improved and natural version of such wavelets by employing Riesz projectors on wavelets. The main idea behind these wavelets was to perlustrate both even and odd symmetries of an asymmetric signal.

One may note that $\mathfrak{B}\{\sin(x)\} = \cos(x)$ and $\mathfrak{B}\{\cos(x)\} = -\sin(x)$, and thus the Fourier kernel can be written as $e^{ix} = \cos(x) - i\mathfrak{B}\{\cos(x)\}$. This observation persuaded us to analyze Boas transform further and we define Fourier–Boas-Like wavelets as $\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x) = \psi(x) - i\mathfrak{B}\psi(x)$. There are two main reasons for defining these new wavelets: (i) to characterize the wavelets whose Fourier transform vanishes a.e. on $]-1,1[$ and (ii) to reinforce the incompetency of wavelets to study both the symmetries of an asymmetric signal.

Next, we give a sufficient condition under which $\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi$ forms a wavelet.

Proposition 1. *Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\widehat{\psi} \in L^1(\mathbb{R})$ and $\widehat{\psi}(0) = 0$. Then, $\mathfrak{B}^{\mathfrak{F}}_{Like}\psi$ is again a wavelet.*

Proof. Note that

$$E_{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi} = \int_{\mathbb{R}} \left|\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\right|^2 dx \le \int_{\mathbb{R}} \left(|\psi(x)|^2 + |\mathfrak{B}\psi(x)|^2\right) dx < +\infty. \tag{7}$$

Also, we have

$$\begin{aligned}\mathfrak{C}_{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi} &= \int_{\mathbb{R}} \frac{\left|\mathfrak{F}\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\right\}(\eta)\right|^2}{|\eta|} d\eta \\ &= \int_{\mathbb{R}} \frac{|\widehat{\psi}(\eta) - \operatorname{sgn}(\eta)(1-\widehat{\mathfrak{g}}(\eta))\widehat{\psi}(\eta)|^2}{|\eta|} d\eta \\ &< +\infty.\end{aligned} \tag{8}$$

Clearly, $\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi$ has finite energy, and it also satisfies the admissibility condition. □

Vanishing moments bestow orthogonality relative to subspaces of polynomials and thus perform a significant task in signal processing. For many applications of wavelets such as reconstruction of a signal, compression of images and to examine the regularity of the analyzed signal, a large number of vanishing moments of a wavelet are needed. Theoretically, a large number of vanishing moments insinuate the competency of the scaling function in representing more complex signals scrupulously. A function $f(x)$ is said to

have s vanishing moments if $\int_{\mathbb{R}} x^u f(x)\mathrm{d}x = 0, 0 \le u \le s-1$, where the given integral is called the uth moment of $f(x)$. For more details, see [10].

Daubechies [22] constructed compactly supported orthonormal wavelets having smoothness of a fixed degree. She observed that if $\phi, \psi \in C^s(\mathbb{R})$, then the low-pass filter m_0 takes the form $m_0(\eta) = ((1+e^{-i\eta})/2)^{s+1}\mathscr{F}(\eta)$, where $\mathscr{F} \in C^s(\mathbb{R})$ is 2π-periodic function. This can be construed using the below stated result which also helps us in deducing that there does not exist any compactly supported orthonormal wavelet $\psi \in C^\infty(\mathbb{R})$.

Theorem 1. *(see [5]). Let $\psi \in C^s(\mathbb{R})$ $(s \in \mathbb{N}_0)$ be a function such that $|\psi(x)| \le \mathscr{M}(1+|x|)^{-s-1-\varrho}$ for some $\varrho > 0$, and that $\psi^{(u)} \in L^\infty(\mathbb{R})$ for $u = 1, 2, \ldots, s$. If $\{\psi_{j,k}\colon j, k \in \mathbb{Z}\}$ is an orthonormal system in $L^2(\mathbb{R})$, then $\int_{\mathbb{R}} x^u \psi(x)dx = 0$, for all $u = 0, 1, \ldots, s$.*

Recall from [13] that a function f is said to have fast decay with decay rate $p \in \mathbb{N}$, if there exists a constant C_p such that $|f(x)| \le (C_p/(1+|x|^p))\forall x \in \mathbb{R}$. The moment formula for the Hilbert transform of f is given by

$$\mathfrak{H}\{x^s f(x)\} = x^s \mathfrak{H} f(x) - \frac{1}{\pi}\sum_{m=0}^{s-1} x^m \int_{\mathbb{R}} q^{s-1-m} f(q)\mathrm{d}q, \quad s \ge 0. \tag{9}$$

Note that the above formula holds if $x^s f(x) \in L^q(\mathbb{R}), 1 < q < \infty$.

In the following result, we give the relationship between the higher vanishing moments of Fourier–Boas-Like wavelets and the fast decay of wavelet ψ.

Theorem 2. *Let ψ, $\psi^{(1)}$, $\widehat{\psi} \in L^1(\mathbb{R})$ such that $\psi \in C^s(\mathbb{R})$ is having a fast decay with decay exponent $p \in \mathbb{N}$ and $\psi^{(u)} \in L^\infty(\mathbb{R})$, $u = 1, 2, \ldots, s$. Also, let $x^s\psi(x) \in L^2(\mathbb{R})$, and $\int_{\mathbb{R}} x^u G(x)dx = 0$, for $u = 0, 1, \ldots, s$, where $G(x) = \int_{-1}^{1}(1-(1/|\eta|))e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)d\eta$. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ constitutes an orthonormal system in $L^2(\mathbb{R})$, then $\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{Like}\{\psi(x)\}dx = 0$, for all $u = 0, 1, \ldots, s$, where $u+1 < p$.*

Proof. We compute

$$\begin{aligned}
\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\{\psi(x)\}\mathrm{d}x &= \int_{\mathbb{R}} x^u(\psi(x) - i\mathfrak{B}\psi(x))\mathrm{d}x \\
&= \int_{\mathbb{R}} x^u(\psi(x) - i(\mathfrak{H}\psi(x) - (\mathfrak{H}\psi * \mathfrak{g})(x)))\mathrm{d}x \\
&= \int_{\mathbb{R}} x^u\left(\psi(x) - i\left(\mathfrak{H}\psi(x) - \int_{\mathbb{R}} T_{-x}\mathfrak{H}\psi(-t)\mathfrak{g}(t)\mathrm{d}t\right)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} x^u\left(\psi(x) - i\left(\mathfrak{H}\psi(x) - \int_{\mathbb{R}} \mathfrak{F}\{\mathfrak{H}T_{-x}\psi\}(-\eta)\widehat{\mathfrak{g}}(\eta)\mathrm{d}\eta\right)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} x^u\left(\psi(x) - i\left(\mathfrak{H}\psi(x) + \int_{-1}^{1}\mathrm{sgn}(-\eta)\mathfrak{F}\{T_{-x}\psi\}(-\eta)(1-|\eta|)\mathrm{d}\eta\right)\right)\mathrm{d}x.
\end{aligned} \tag{10}$$

Since ψ, $\psi^{(1)}$, $\widehat{\psi} \in L^1(\mathbb{R})$, we have

$$\begin{aligned}
\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi(x)\mathrm{d}x &= \int_{\mathbb{R}} x^u\left(\psi(x) - i\left(\mathfrak{H}\psi(x) - \frac{1}{2\pi}\int_{-1}^{1}\left(1-\frac{1}{|\eta|}\right)e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)\mathrm{d}\eta\right)\right)\mathrm{d}x \\
&= \int_{\mathbb{R}} x^u\left(\psi(x) - i\left(\mathfrak{H}\psi(x) - \frac{1}{2\pi}G(x)\right)\right)\mathrm{d}x.
\end{aligned} \tag{11}$$

Now, since $\psi, \widehat{\psi} \in L^1(\mathbb{R})$ and $x^s\psi(x) \in L^2(\mathbb{R})$, it follows that $x^u\psi(x) \in L^2(\mathbb{R})$, for $u = 0, 1, \ldots, s$. Using the moment formula for Hilbert transform, (11) can be written as

$$\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi(x)\mathrm{d}x = \int_{\mathbb{R}}\left(x^u\psi(x) - i\left(\mathfrak{H}\{x^u\psi(x)\} + \frac{1}{\pi}\sum_{j=0}^{u-1} x^j\int_{\mathbb{R}} q^{u-1-j}\psi(q)\mathrm{d}q\right)\right)\mathrm{d}x. \tag{12}$$

In view of Theorem 1 and since $x^u\psi(x) \in L^2(\mathbb{R})$, for $u = 0, 1, \ldots, s$, it follows that $\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\mathrm{d}x = 0$ for $u = 0, 1, 2, \ldots, s$. □

In [14], a sufficient condition was presented in order to obtain higher vanishing moments of wavelets. The result is stated as the following.

Theorem 3. *Let $\psi(x)$ be such that for some $s \in \mathbb{N}$, $x^s\psi(x), \eta^{s+1}\widehat{\psi}(\eta) \in L^1(\mathbb{R})$. If $\{\psi_{j,k}(x)\}_{j,k\in\mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then*

$$\int_{\mathbb{R}} x^u\psi(x)\mathrm{d}x = 0, \quad \text{for } 0 \le u \le s. \tag{13}$$

Next result generalizes Theorem 3 for Fourier–Boas-Like wavelets.

Theorem 4. *Let ψ, $\psi^{(1)}$, $\widehat{\psi} \in L^1(\mathbb{R})$ be such that for some $s \in \mathbb{N}$, $x^s\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, $\eta^{s+1}\widehat{\psi}(\eta) \in L^1(\mathbb{R})$, and $\int_{\mathbb{R}} x^u G(x)dx = 0$, for $0 \le u \le s$, where $G(x) = \int_{-1}^{1}(1-(1/|\eta|))e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)d\eta$. If $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthogonal system on $\mathbb{R}$, then $\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{Like}\psi(x)dx = 0$, for $0 \le u \le s$.*

Proof. We have

$$\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\mathrm{d}x = \int_{\mathbb{R}} x^u\psi(x)\mathrm{d}x - i\int_{\mathbb{R}} \mathfrak{H}\{x^u\psi(x)\}\mathrm{d}x - \frac{i}{\pi}\sum_{j=0}^{u-1}\int_{\mathbb{R}} x^j \int_{\mathbb{R}} q^{u-1-j}\psi(q)\mathrm{d}q\mathrm{d}x. \tag{14}$$

Since $x^u\psi(x) \in L^2(\mathbb{R})$ for $u = 0, 1, \ldots, s$, by Theorem 3, it follows that $\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\mathrm{d}x = 0$, for $u = 0, 1, 2, \ldots, s$. □

In [6], regularity of orthonormal wavelet bases was studied and a relationship between the regularity of wavelet and the multiplicity of the zero at $\eta = 0$ of $\widehat{\psi}$ was observed. This observation can be seen and deduced using the following result which is given in more generalized form.

Theorem 5. *Let $f, \widetilde{f}$ be two functions (not identically constant) such that*

(i) $\langle f_{j,k}, \widetilde{f}_{j_1,k_1}\rangle = \delta_{jj_1}\delta_{kk_1}$, where $f_{j,k}(x) = 2^{j/2} f(2^j x - k)$, $\widetilde{f}_{j,k}(x) = 2^{j/2}\widetilde{f}(2^j x - k)$.

(ii) $|\widetilde{f}(x)| \le M(1+|x|)^{-\gamma}$, with $\gamma > s+1$.

(iii) $f \in C^s$, with $f^{(u)}$ bounded for $u \le s$.

Then, $\int_{\mathbb{R}} x^u \widetilde{f}(x)dx = 0$ for $u = 0, 1, \ldots, s$.

The next result generalizes Theorem 5 and depicts the relationship among the regularity of orthonormal wavelets ψ and vanishing moments of Fourier–Boas-Like wavelets.

Theorem 6. *Let ψ, $\psi^{(1)}$, $\widehat{\psi} \in L^1(\mathbb{R})$ and let the system $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$ form an orthonormal set in $L^2(\mathbb{R})$ such that $|\psi(x)| \le M(1+|x|)^{-s-1-\varrho}$, where $\varrho > 0$ and $\psi \in C^s(\mathbb{R})$ such that $\psi^{(u)}$ is bounded for $u \le s$. Also, if $x^s\psi(x) \in L^2(\mathbb{R})$ and $\int_{\mathbb{R}} x^u G(x)dx = 0$, for $0 \le u \le s$, where $G(x) = \int_{-1}^{1}(1-(1/|\eta|))e^{-2\pi i\eta x}\widehat{\psi^{(1)}}(-\eta)d\eta$, then $\int_{\mathbb{R}} x^u \mathfrak{B}_{Like}\mathfrak{F}\{\psi(x)\}dx = 0$ for $u = 0, 1, 2, \ldots, s$.*

Proof. Note that

$$\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\{\psi(x)\}\mathrm{d}x = \int_{\mathbb{R}} x^u\psi(x)\mathrm{d}x - i\int_{\mathbb{R}} \mathfrak{H}\{x^u\psi(x)\}\mathrm{d}x - \frac{i}{\pi}\sum_{j=0}^{u-1}\int_{\mathbb{R}} x^j \int_{\mathbb{R}} q^{u-1-j}\psi(q)\mathrm{d}q\mathrm{d}x. \tag{15}$$

Therefore, in view of Theorem 5 and the fact that $x^u\psi(x) \in L^2(\mathbb{R})$, for $u = 0, 1, \ldots, s$, it follows that $\int_{\mathbb{R}} x^u \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\mathrm{d}x = 0$, for $u = 0, 1, 2, \ldots, s$. □

Recall from [12] that wavelets associated with projection operators $\mathbb{P}_+$ and $\mathbb{P}_-$ involving the Hilbert transform are defined as $\mathbb{P}_+ = (1/2)(\mathfrak{I} + i\mathfrak{H})$ and $\mathbb{P}_- = (1/2)(\mathfrak{I} - i\mathfrak{H})$, where $\mathfrak{I}$ and $\mathfrak{H}$ represent the identity operator and Hilbert transform operator, respectively. These projection operators are also known as Riesz projectors.

In the following result, the moment formula for Fourier–Boas-Like wavelets is given.

Theorem 7. *Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\widehat{\psi} \in L^1(\mathbb{R})$ and $x^s\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, for some $s \in \mathbb{N}$. Then,*

$$\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\{x^s\psi(x)\} = 2x^s\mathbb{P}_-\psi(x) + \frac{i}{\pi}\sum_{p=0}^{s-1} x^p \int_{\mathbb{R}} q^{s-1-p}\psi(q)\mathrm{d}q + \left(\frac{-1}{2\pi i}\right)^s \int_{-1}^{1} \widehat{\psi}^{(s)}(\eta)e^{-2\pi i\eta x}(\operatorname{sgn}(\eta) - \eta)\mathrm{d}\eta, \tag{16}$$

where $\mathbb{P}_-\psi$ is a wavelet associated with Riesz projectors.

Proof. We compute

$$\begin{aligned}\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\{x^s\psi(x)\} &= x^s\psi(x) - i\mathfrak{B}\{x^s\psi(x)\} = 2x^s\mathbb{P}_-\psi(x) + \frac{i}{\pi}\sum_{p=0}^{s-1}x^p\int_{\mathbb{R}}q^{s-1-p}\psi(q)\mathrm{d}q \\ &+ i\int_{\mathbb{R}}\mathfrak{F}\{\mathfrak{H}\{t^s\psi(t)\}\}(\eta)\mathfrak{F}\{T_x\mathfrak{g}(t)\}(\eta)\mathrm{d}\eta = 2x^s\mathbb{P}_-\psi(x) + \frac{i}{\pi}\sum_{p=0}^{s-1}x^p\int_{\mathbb{R}}q^{s-1-p}\psi(q)\mathrm{d}q \\ &+ i\int_{\mathbb{R}} - i\operatorname{sgn}(\eta)\{\mathfrak{F}\{t^s\psi(t)\}\}(\eta)E_{-x}\widehat{\mathfrak{g}}(\eta)\mathrm{d}\eta = 2x^s\mathbb{P}_-\psi(x) + \frac{i}{\pi}\sum_{p=0}^{s-1}x^p\int_{\mathbb{R}}q^{s-1-p}\psi(q)\mathrm{d}q \\ &+ \left(\frac{-1}{2\pi i}\right)^s\int_{-1}^{1}\widehat{\psi}^{(s)}(\eta)e^{-2\pi i\eta x}(\operatorname{sgn}(\eta)-\eta)\mathrm{d}\eta.\end{aligned} \tag{17}$$

A continual filter is defined by $\varphi(f(x)) = \mathbb{P}_-f(x)$, where $f \in L^2(\mathbb{R})$ and φ: $L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$. Note that φ is said to be a convolution filter if $\varphi(x) = w * x$ for any x, where $w \in (L^1 \cap L^2)(\mathbb{R})$ is a weight function and $\widehat{w}(\eta)$ is known as the transfer function. For more details, one may see [12].

In the next result, we give a sufficient condition under which $\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right\}(x)$, i.e., Fourier–Boas-Like wavelets associated with Riesz projectors form a convolution filter and Fourier transform of $\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right\}(x)$ vanishes for all positive frequencies.

Theorem 8. *Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\widehat{\psi} \in L^1(\mathbb{R})$ and $\widehat{\psi}(0) = 0$. Then, $\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{Like}\psi\right\}(x)$ is a convolution filter with transfer function*

$$\begin{cases} 2, & \text{if } \eta < -1, \\ 1-\eta, & \text{if } -1 \le \eta < 0, \\ \dfrac{1}{2}, & \text{if } \eta = 0, \\ 0, & \text{if } \eta > 0, \end{cases} \tag{18}$$

and Fourier transform of $\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{Like}\psi\right\}(x)$ vanishes for all positive frequencies.

Proof. Note that

$$\begin{aligned}\varphi\left(\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\right) &= \mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right\}(x) \\ &= \frac{1}{2}\left(\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x) - i\left(\frac{1}{\pi}p.v.\frac{1}{x} * \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi(x)\right)\right) \\ &= \left(\mathbb{P}_-\delta * \mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right)(x) \\ &= (\mathbb{P}_-\delta * \psi)(x) - i(\mathbb{P}_-\delta * (\mathfrak{H}\psi - \mathfrak{H}\psi * \mathfrak{g}))(x) \\ &= (\mathbb{P}_-\delta * \psi)(x) - i\mathfrak{B}(\mathbb{P}_-\delta * \psi)(x).\end{aligned} \tag{19}$$

Thus, $\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right\}(x)$ is a convolution filter. Clearly, we calculate

$$\mathfrak{F}\left\{\mathfrak{B}^{\mathfrak{F}}_{Like}\{\mathbb{P}_-\delta\}\right\}(\eta) = \mathfrak{F}\{\mathbb{P}_-\delta\}(\eta)[1 - \operatorname{sgn}(\eta)(1 - \widehat{\mathfrak{g}}(\eta))]$$

$$= \begin{cases} 2, & \text{if } \eta < -1, \\ 1-\eta, & \text{if } -1 \le \eta < 0, \\ \dfrac{1}{2}, & \text{if } \eta = 0, \\ 0, & \text{if } \eta > 0. \end{cases} \tag{20}$$

Also, we evaluate

$$\mathfrak{F}\left\{\mathbb{P}_-\left\{\mathfrak{B}^{\mathfrak{F}}_{\text{Like}}\psi\right\}\right\}(\eta) = \begin{cases} 2\widehat{\psi}(\eta), & \text{if } \eta < -1, \\ (1-\eta)\widehat{\psi}(\eta), & \text{if } -1 \le \eta < 0, \\ \dfrac{\widehat{\psi}(\eta)}{2}, & \text{if } \eta = 0, \\ 0, & \text{if } \eta > 0. \end{cases} \tag{21}$$

□

Let $\alpha(x) \in L^1(\mathbb{R})$ and let $h(x) = (\alpha * \psi)(x) = \int_{\mathbb{R}}\alpha(x-t)\psi(t)\mathrm{d}t$, where $\psi(t)$ is continuous and bounded. The kernel $\alpha(x)$ is variation diminishing if $\eta[\alpha * \psi] \le \eta[\psi]$, where $\eta[\psi]$ denotes number of changes of sign of ψ on $\mathbb{R}$. It was found that $\alpha(x)$ is variation diminishing if and only if

$$\widehat{\alpha}(\eta) = \int_{\mathbb{R}}\alpha(x)e^{-i\eta x}\mathrm{d}x = \left(e^{l\eta^2+im\eta}\prod_{n\in\mathbb{N}}\left(1-\frac{i\eta}{d_n}\right)e^{i\eta/d_n}\right)^{-1}, \tag{22}$$

where $l, m \neq 0$ and d_n $(n \in \mathbb{N})$ are real numbers with $\sum_{n\in\mathbb{N}}d_n^{-2} < \infty$.

This gives

$$\alpha(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\eta x}\left(e^{l\eta^2+im\eta}\prod_{n\in\mathbb{N}}\left(1-\frac{i\eta}{d_n}\right)e^{i\eta/d_n}\right)^{-1}\mathrm{d}\eta, \tag{23}$$

$$\alpha^{(s)}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} (i\eta)^s e^{i\eta x}\left(e^{l\eta^2+im\eta}\prod_{n\in\mathbb{N}}\left(1-\frac{i\eta}{d_n}\right)e^{i\eta/d_n}\right)^{-1}\mathrm{d}\eta. \tag{24}$$

Let ψ be a wavelet. Then, for a variation diminishing kernel given by (23) such that $\sum_{n\in\mathbb{N}} d_n^{-2} < \infty$, $\psi * \alpha$ is known as wavelet, a wavelet with specific changes of sign. For further details, one may read [23, 24].

In the given result, we show that $\mathfrak{B}^{\mathfrak{F}}_{Like}\psi * \alpha$ is also a variation diminishing wavelet.

Theorem 9. *Let $\psi \in L^1(\mathbb{R})$ be a wavelet such that $\widehat{\psi} \in L^1(\mathbb{R})$ and $\widehat{\psi}(0) = 0$. Let $\alpha \in L^1(\mathbb{R})$ be a variation diminishing kernel. Then, $\mathfrak{B}^{\mathfrak{F}}_{Like}\psi * \alpha$ is a variation diminishing wavelet.*

Proof. Since $\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi \in L^2(\mathbb{R})$ and $\alpha \in L^1(\mathbb{R})$, it follows that $\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi * \alpha \in L^2(\mathbb{R})$.

Also note that

$$\begin{aligned}\left\|\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi\right\|_1 &= \|\psi - i\mathfrak{B}\psi\|_1\\ &= \|\psi - i\mathfrak{H}\psi + i\mathfrak{H}\psi * \mathfrak{g}\|_1\\ &\le \|\psi\|_1 + \|\mathfrak{H}\psi\|_1 + \|\mathfrak{H}\psi * \mathfrak{g}\|_1.\end{aligned} \tag{25}$$

Since $\widehat{\psi} \in L^1(\mathbb{R})$ and $\widehat{\psi}(0) = 0$, it follows that $\mathfrak{H}\psi \in L^1(\mathbb{R})$. Thus,

$$\left\|\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi\right\|_1 \le \|\psi\|_1 + \|\mathfrak{H}\psi\|_1 + \|\mathfrak{H}\psi\|_1\|\mathfrak{g}\|_1 < +\infty. \tag{26}$$

Hence,

$$\begin{aligned}\mathfrak{C}_{\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi * \alpha} &= \int_{\mathbb{R}} \frac{\left|\mathfrak{F}\left\{\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi * \alpha\right\}(\eta)\right|^2}{|\eta|}\mathrm{d}\eta,\\ &= \int_{\mathbb{R}} \frac{\left|\mathfrak{F}\left\{\mathfrak{B}^{\mathfrak{F}}_{\mathrm{Like}}\psi\right\}(\eta)\right|^2|\widehat{\alpha}(\eta)|^2}{|\eta|}\mathrm{d}\eta\\ &\le \begin{cases} 4\,\mathfrak{D}\mathfrak{C}_{\psi}, & \text{if } \eta < -1,\\ \mathfrak{D}\displaystyle\int_{\mathbb{R}} \frac{|\psi(\eta)(1-\eta)|^2}{|\eta|}\mathrm{d}\eta, & \text{if } |\eta| \le 1, \eta \ne 0,\\ \mathfrak{D}\mathfrak{C}_{\psi}, & \text{if } \eta = 0,\\ 0, & \text{if } \eta > 1,\end{cases}\\ &\le \begin{cases} 4\mathfrak{A}\mathfrak{C}_{\psi}, & \text{if } \eta < -1,\\ 4\mathfrak{A}\mathfrak{C}_{\psi}, & \text{if } |\eta| \le 1, \eta \ne 0,\\ \mathfrak{A}\mathfrak{C}_{\psi}, & \text{if } \eta = 0,\\ 0, & \text{if } \eta > 1,\end{cases}\\ &< +\infty,\end{aligned} \tag{27}$$

where $\mathfrak{D} = \sup_{\eta} |e^{i\eta^2+im\eta}\prod_{n\in\mathbb{N}}(1-(i\eta/d_n))e^{(i\eta/d_n)}|^{-2}$ and $\mathfrak{A} = \sup_{\eta}(1/(\prod_{n\in\mathbb{N}}(1-(\eta^2/d_n^2))))$. □

3. Conclusions

New wavelet functions, called Fourier–Boas-Like wavelets, have been established. These wavelets have been found to be better than the earlier proposed wavelets in [12, 21] derived from the Riesz projectors and Fourier kernels, respectively. Various results related to higher vanishing moments of Fourier–Boas-Like wavelets have been given, and it is observed that regularity and fast decay are significant attributes for the vanishing moments of Fourier–Boas-Like wavelets. It has also been investigated that under some conditions, Fourier–Boas-Like wavelets associated with Riesz projectors form a convolution filter. Further, using Schoenberg's theory of variation diminishing integral operators of convolution type, variation diminishing wavelet associated with Fourier–Boas-Like wavelet is constructed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this paper and read and approved the final manuscript.

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Research Article

Woven Frames in Quaternionic Hilbert Spaces

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In this paper, we introduce and study woven frames in quaternionic Hilbert spaces. We also give some properties of woven frames and give some conditions on family of frames under which it is woven in quaternionic Hilbert spaces. Also, a characterization of weaving frames in terms of a surjective-bounded right linear operator is given.

1. Introduction and Preliminaries

In 1952, Duffin and Schaeffer [1] defined frames as follows:

"if $\mathscr{H}$ is a Hilbert space, then $\{u_n\}_{n\in\mathbb{N}} \subset \mathscr{H}$ is said to be a *frame* for $\mathscr{H}$ if there exist finite constants c_l and c_u with $0 < c_l \le c_u$ such that

$$c_l\|u\|^2 \le \sum_{n\in\mathbb{N}} \left|\langle u, u_n\rangle\right|^2 \le c_u\|u\|^2, \quad \text{for all } u \in \mathscr{H}." \tag{1}$$

The positive constants c_l and c_u are called lower and upper frame bounds for the frame $\{u_n\}_{n\in\mathbb{N}}$, respectively. Inequality (1) is called the frame inequality for the frame $\{u_n\}_{n\in\mathbb{N}}$. A frame $\{u_n\}_{n\in\mathbb{N}}$ in $\mathscr{H}$ is said to be

(i) *tight*: if it is possible to choose $c_l = c_u$

(ii) *Parseval*: if it is a tight frame with $c_l = c_u = 1$

Frames have many properties of bases, but lack a very important one, namely, uniqueness. In fact, a frame is an overcomplete sequence of vectors. Members of a frame need not be linearly independent, but still serve as building blocks and are able to recover every vector of the space completely. This property of frames turns out to be very useful particularly in image and signal processing and in communications in general, since this redundancy yields robustness, i.e., less sensitivity to truncation and transmission errors. Frames can be approached in different ways. Two most important ways are to consider frame theory as a branch of functional analysis, which belongs to the branch of pure mathematics, and to consider a class of frames, which is best suited in applications, which lies in the branch of applied mathematics.

Dependencies among the coefficients of the overcomplete representations guarantee a better stability in presence of noise, quantization, and erasures, as well as greater freedom of design comparing to the bases. The redundant counterpart of a basis is called a frame. The role played by redundancy varies with specific applications. One important role is its robustness. That is, by spreading our information over a wider range of vectors, we are more capable to sustain losses and still have accurate reconstruction.

Frame work is an important tool in the study of signal and image processing [2], filter bank theory [3], wireless communications [4], and sigma-delta quantization [5], refer [4, 6–12] for more literature, applications, and various generalizations of frames in Hilbert spaces.

1.1. Notations. Throughout this paper, we will denote $\mathfrak{Q}$ to be the noncommutative field of quaternions, $\mathbb{N}$ be the set of natural numbers, and $\mathbb{H}^R(\mathfrak{Q})$ be a separable right quaternionic Hilbert space. By the term "right linear operator," we mean a "right $\mathfrak{Q}$-linear operator," and $\mathfrak{B}(\mathbb{H}^R(\mathfrak{Q}))$ denotes the set of all bounded (right $\mathfrak{Q}$-linear) operators of $\mathbb{H}^R(\mathfrak{Q})$ and $\mathbb{N}_m$ denotes the set of first m natural numbers.

1.2. Quaternionic Hilbert Spaces. The noncommutative field of quaternions $\mathfrak{Q}$ is a four-dimensional real algebra with unity. In $\mathfrak{Q}$, 0 denotes the null element and 1 denotes the identity with respect to multiplication. It also includes three so-called imaginary units, denoted by i, j, and k, i.e.,

$$\mathfrak{Q} = \{r_0 + r_1 i + r_2 j + r_3 k\colon r_0, r_1, r_2, r_3 \in \mathbb{R}\}, \tag{2}$$

where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, and $ki = -ik = j$. For each $q = r_0 + r_1 i + r_2 j + r_3 k \in \mathfrak{Q}$, define conjugate of q denoted by $\overline{q}$ as $\overline{q} = r_0 - r_1 i - r_2 j - r_3 k \in \mathfrak{Q}$. If $q = r_0 + r_1 i + r_2 j + r_3 k$ is a quaternion, then r_0 is called the real part of q and $r_1 i + r_2 j + r_3 k$ is called the imaginary part of q. The modulus of a quaternion $q = r_0 + r_1 i + r_2 j + r_3 k$ is defined as

$$|q| = (\overline{q}q)^{1/2} = (q\overline{q})^{1/2} = \sqrt{r_0^2 + r_1^2 + r_2^2 + r_3^2}. \tag{3}$$

Definition 1. A right quaternionic pre-Hilbert space or right quaternionic inner product space $\mathbb{V}_R(\mathfrak{Q})$ is a right quaternionic vector space together with the binary mapping $\langle . | . \rangle\colon \mathbb{V}_R(\mathfrak{Q}) \times \mathbb{V}_R(\mathfrak{Q}) \longrightarrow \mathfrak{Q}$ (called the Hermitian quaternionic inner product) which satisfies the following properties:

(a) $\overline{\langle v_1 | v_2 \rangle} = \langle v_2 | v_1 \rangle$ for all $v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$

(b) $\langle v | v \rangle > 0$ if $v \neq 0$

(c) $\langle v | v_1 + v_2 \rangle = \langle v | v_1 \rangle + \langle v | v_2 \rangle$ for all $v, v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$

(d) $\langle v | uq \rangle = \langle v | u \rangle q$ for all $v, u \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$

A right quaternionic inner product space $\mathbb{V}_R(\mathfrak{Q})$ also has the following properties:

(i) $\langle vq | u \rangle = \overline{q} \langle v | u \rangle$ for all $v, u \in \mathbb{V}_R(\mathfrak{Q})$ and $q \in \mathfrak{Q}$

(ii) $v_1 p + v_2 q \in \mathbb{V}_R(\mathfrak{Q})$, for all $v_1, v_2 \in \mathbb{V}_R(\mathfrak{Q})$ and $p, q \in \mathfrak{Q}$

Let $\mathbb{V}_R(\mathfrak{Q})$ be right quaternionic inner product space with the Hermitian inner product $\langle . | . \rangle$. Define the quaternionic norm $\|.\|\colon \mathbb{V}_R(\mathfrak{Q}) \longrightarrow \mathbb{R}^+$ on $\mathbb{V}_R(\mathfrak{Q})$ by

$$\|u\| = \sqrt{\langle u | u \rangle},\ u \in \mathbb{V}_R(\mathfrak{Q}). \tag{4}$$

Definition 2. The right quaternionic pre-Hilbert space is called a right quaternionic Hilbert space if it is complete with respect to norm (4) and is denoted by $\mathbb{H}^R(\mathfrak{Q})$.

Theorem 1 (the Cauchy–Schwarz inequality, see [13]). *If $\mathbb{H}^R(\mathfrak{Q})$ is a right quaternionic Hilbert space, then*

$$|\langle u | v \rangle|^2 \leq \langle u | u \rangle \langle v | v \rangle, \quad \text{for all } u, v \in \mathbb{H}^R(\mathfrak{Q}). \tag{5}$$

For more literature related to quaternionic Hilbert spaces, one may refer to [13, 14].

1.3. Frames in Quaternionic Hilbert Spaces. Khokulan et al. [15] introduced and studied frames for finite-dimensional quaternionic Hilbert spaces, which were further studied in [16]. Recently, Sharma and Goel [17] introduced and studied frames for separable quaternionic Hilbert spaces. Frames in separable right quaternionic Hilbert spaces $\mathbb{H}^R(\mathfrak{Q})$ are defined as follows.

Definition 3 (see [17]). Let $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space and $\{u_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathbb{H}^R(\mathfrak{Q})$. Then, $\{u_i\}_{i\in\mathbb{N}}$ is said to be a frame for $\mathbb{H}^R(\mathfrak{Q})$ if there exist two finite real constants with $0 < r_1 \leq r_2$ such that

$$r_1 \|u\|^2 \leq \sum_{i\in\mathbb{N}} |\langle u_i | u \rangle|^2 \leq r_2 \|u\|^2, \quad \text{fo all } u \in \mathbb{H}^R(\mathfrak{Q}). \tag{6}$$

The positive constants r_1 and r_2 are called lower frame and upper frame bounds for the frame $\{u_i\}_{i\in\mathbb{N}}$, respectively. Inequality (6) is called frame inequality for the frame $\{u_i\}_{i\in\mathbb{N}}$. A sequence $\{u_i\}_{i\in\mathbb{N}}$ is called a Bessel sequence for the right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ with bound r_2 if $\{u_i\}_{i\in\mathbb{N}}$ satisfies the right-hand side of inequality (6). A sequence $\{u_i\}_{i\in\mathbb{N}}$ is a tight frame for right quaternionic Hilbert space $\mathbb{H}^R(\mathfrak{Q})$ if there exist positive r_1 and r_2 satisfying inequality (6) with $r_1 = r_2$, Parseval frame if it is tight with $r_1 = r_2 = 1$, and exact if it ceases to be a frame in case any one of its element is removed.

If $\{u_i\}_{i\in\mathbb{N}}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$, then the right linear operator $T\colon \ell_2(\mathfrak{Q}) \longrightarrow \mathbb{H}^R(\mathfrak{Q})$ defined by

$$T(\{q_i\}_{i\in\mathbb{N}}) = \sum_{i\in\mathbb{N}} u_i q_i,\ \{q_i\} \in \ell_2(\mathfrak{Q}), \tag{7}$$

is called the *(right) synthesis operator* and the adjoint operator T^* is called the *(right) analysis operator* is given by

$$T^*(u) = \{\langle u_i | u \rangle\}_{i\in\mathbb{N}}, \quad u \in \mathbb{H}^R(\mathfrak{Q}). \tag{8}$$

Also, the (right) frame operator $S\colon \mathbb{H}^R(\mathfrak{Q}) \longrightarrow \mathbb{H}^R(\mathfrak{Q})$ for the frame $\{u_i\}_{i\in\mathbb{N}}$ is a right linear operator given by

$$\begin{aligned} S(u) &= TT^*(u) = T(\{\langle u_i | u \rangle\}_{i\in\mathbb{N}}) \\ &= \sum_{i\in\mathbb{N}} u_i \langle u_i | u \rangle,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{9}$$

1.4. Woven Frames in Hilbert Spaces. Woven frame for separable Hilbert spaces were introduced and studied by Bemrose et al. [18]. They gave the following definition:

Definition 4 (see [18]). A family of frames $\mathfrak{F} = \{\{u_{ij}\}_{i\in\mathbb{N}}\colon j \in \mathbb{N}_m\}$ is said to be woven frame for a Hilbert space $\mathscr{H}$ if there are universal constants c_l and c_u such that, for every partition $P = \{\sigma_j\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, the family $\mathfrak{F}_P = \{u_{ij}\}_{i\in\sigma_j, j\in\mathbb{N}_m}$ is a frame for $\mathscr{H}$ with lower and upper frame bounds c_l and c_u, respectively.

Distributed signal processing is the main motivation behind the concept of woven frames. Furthermore, woven frames have inherent applications in wireless sensor networks which are based upon distributed processing under family of frames. For more literature on woven frame, one may refer to [19–21].

1.5. Outline of the Paper. In this paper, we introduce and study woven frames in quaternionic Hilbert spaces. Example is given in support of its existence. We also discuss some properties of woven Bessel family and woven frames. Some conditions under which a family of frames is woven in quaternionic Hilbert spaces are given. Furthermore, a characterization of weaving frames in terms of a surjective-bounded right linear operator is given.

2. Woven Frames in Quaternionic Hilbert Spaces

We begin this section with the following definition of woven frames in quaternionic Hilbert spaces.

Definition 5. Let $\mathbb{N}_m$ be the set of first m natural numbers, $\mathbb{H}^R(\mathfrak{Q})$ be a right quaternionic Hilbert space, and $\mathfrak{F} = \left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ be a family of frames for $\mathbb{H}^R(\mathfrak{Q})$. Then, $\mathfrak{F}$ is said to be woven if there are universal positive real numbers r_1 and r_2 so that, for every partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, the family $\mathfrak{F}_P = \left\{u_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds r_1 and r_2, respectively. Each family $\mathfrak{F}_P$ is called a weaving. If every weaving is a Bessel sequence, then $\mathfrak{F}$ is called a woven Bessel sequence for $\mathbb{H}^R(\mathfrak{Q})$.

Next, we give the following lemma:

Lemma 1. *Let $\{v_i\}_{i\in\mathbb{N}}$ be a frame for $\mathbb{H}^R(\mathfrak{Q})$ with frame bounds r_1 and r_2 and $\{q_i\}_{i\in\mathbb{N}} \subset \mathfrak{Q}$ be a sequence such that $r_3 < \inf_{i\in\mathbb{N}} |q_i| < sup_{i\in\mathbb{N}} |q_i| < r_4$. Define a family $\mathfrak{F} = \left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}} = \{jv_iq_i\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$. Then, $\mathfrak{F}$ is a woven frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, we have

$$\begin{aligned} r_1r_3^2\|u\|^2 &\le \sum_{i\in\mathbb{N}} \left|\langle v_iq_i|u\rangle\right|^2 \\ &\le \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle jv_iq_i|u\rangle\right|^2 \\ &\le \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle u_{ij}|u\rangle\right|^2 \\ &\le m^2 \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle v_iq_i|u\rangle\right|^2 \\ &\le m^2r_2r_4^2\|u\|^2,\ u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{10}$$

Thus, $\mathfrak{F}$ is a woven frame for $\mathbb{H}^R(\mathfrak{Q})$ with lower and upper frame bounds $r_1r_3^2$ and $m^2r_2r_4^2$, respectively.

Next, we generalize Lemma 1 in the form of the following theorem. □

Theorem 2. *Let $\mathfrak{F} = \left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ be a woven frame for $\mathbb{H}^R(\mathfrak{Q})$ with universal frame bounds r_1 and r_2 and $Q = \left\{\left\{q_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ be a family of sequences in $\mathfrak{Q}$ such that $0 < r_3 \le \inf|q_{ij}|^2 \le sup|q_{ij}|^2 \le r_4$, $j \in \mathbb{N}_m$. Then, $\mathfrak{F}Q = \left\{\left\{u_{ij}q_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ is a woven frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. For any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, we have

$$\begin{aligned} r_1r_3\|u\|^2 &\le r_3 \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle u_{ij}|u\rangle\right|^2 \\ &\le \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} |\langle u_{ij}q_{ij}|u\rangle|^2 \\ &\le r_4 \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle u_{ij}|u\rangle\right|^2 \le r_2r_4\|u\|^2, \quad u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{11}$$

Thus, $\mathfrak{F}Q$ is a woven frame for $\mathbb{H}^R(\mathfrak{Q})$ with universal lower and upper frame bounds r_1r_3 and r_2r_4, respectively.

Next, we prove that any finite family of Bessel sequences in a quaternionc Hilbert space is a family of woven Bessel family. □

Theorem 3. *Let $\mathfrak{F} = \left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ be a family of Bessel sequences for $\mathbb{H}^R(\mathfrak{Q})$ with Bessel bounds r_j, $j \in \mathbb{N}_m$. Then, $\mathfrak{F}$ is woven Bessel sequence with universal Bessel bound $\sum_{j\in\mathbb{N}_m} r_j$.*

Proof. For any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, we have

$$\begin{aligned} \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|\langle u_{ij}|u\rangle\right|^2 &\le \sum_{j\in\mathbb{N}_m}\sum_{i\in\mathbb{N}} \left|\langle u_{ij}|u\rangle\right|^2 \\ &\le \left(\sum_{j\in\mathbb{N}_m} r_j\right)\|u\|^2, \quad u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{12}$$

Thus, $\mathfrak{F}$ is woven Bessel sequence with universal Bessel bound $\sum_{j\in\mathbb{N}_m} r_j$.

Let $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ be any partition of $\mathbb{N}$, and define the space

$$\bigoplus_{j\in\mathbb{N}_m} \ell_2(\mathfrak{Q})_{\sigma_j} = \left\{\left\{q_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m} \subset \mathfrak{Q}: \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \left|q_{ij}\right|^2 < \infty\right\}. \tag{13}$$

Then, $\oplus_{j\in\mathbb{N}_m} \ell_2(\mathfrak{Q})_{\sigma_j}$ is a right quaternionic Hilbert space with the quaternionic inner product:

$$\left\langle \left\{p_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m} | \left\{q_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m} \right\rangle = \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} \overline{p_{ij}q_{ij}}. \tag{14}$$

Let $\mathfrak{F} = \left\{U_j = \left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_m\right\}$ be a woven frame for $\mathbb{H}^R(\mathfrak{Q})$, for any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, then $\mathfrak{F}_P = \left\{u_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}$ is a frame for $\mathbb{H}^R(\mathfrak{Q})$ and the operator $T_{\mathfrak{F}_P}: \oplus_{j\in\mathbb{N}_m} \ell_2(\mathfrak{Q})_{\sigma_j} \longrightarrow \mathbb{H}^R(\mathfrak{Q})$ defined by

$$T_{\mathfrak{F}_P}\left(\left\{q_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}\right) = \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} u_{ij}q_{ij} \in \mathbb{H}^R(\mathfrak{Q}), \tag{15}$$

is the synthesis operator of frame $\mathfrak{F}_P$. The adjoint operator $T^*_{\mathfrak{F}_P}: \mathbb{H}^R(\mathfrak{Q}) \longrightarrow \oplus_{j\in\mathbb{N}_m} \ell_2(\mathfrak{Q})_{\sigma_j}$ of $T_{\mathfrak{F}_P}$ is given by

$$T^*_{\mathfrak{F}_P}(u) = \left\{\langle u_{ij}|u\rangle\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}, \quad u \in \mathbb{H}^R(\mathfrak{Q}). \tag{16}$$

In view of the above discussion, if we consider D_{σ_j} to be a diagonal matrix such that

$$d_{ii} = \begin{cases} 1, & \text{if } i \in \sigma_j, \\ 0, & \text{otherwise}, \end{cases} \tag{17}$$

then one can observe that

$$\begin{aligned} T_{\mathfrak{F}_P}\left(\left\{q_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}\right) &= \sum_{j\in\mathbb{N}_m} T_{U_j} D_{\sigma_j}\left(\left\{q_{ij}\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}\right), \\ T^*_{\mathfrak{F}_P}(u) &= \sum_{j\in\mathbb{N}_m} D_{\sigma_j} T^*_{U_j}(u), \end{aligned} \tag{18}$$

where T_{U_j} is the synthesis operator of the frame U_j.

The frame operator $S_{\mathfrak{F}_P}: \mathbb{H}^R(\mathfrak{Q}) \longrightarrow \mathbb{H}^R(\mathfrak{Q})$ of the frame $\mathfrak{F}_P$ is given by

$$\begin{aligned} S_{\mathfrak{F}_P}(u) &= T_{\mathfrak{F}_P} T^*_{\mathfrak{F}_P}(u) \\ &= T_{\mathfrak{F}_P}\left(\left\{\langle u_{ij}|u\rangle\right\}_{i\in\sigma_j, j\in\mathbb{N}_m}\right) \\ &= \sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j} u_{ij}\langle u_{ij}|u\rangle, \quad u \in \mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{19}$$

One may easily verify that, for any partition P, the frame operator $S_{\mathfrak{F}_P}$ is positive, self-adjoint, and invertible.

In the next result, we give a characterization of weaving tight frame. □

Theorem 4. *Let $\mathfrak{F} = \left\{U_j = \left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_2\right\}$ be a woven frame for $\mathbb{H}^R(\mathfrak{Q})$, where T_{U_j} is the synthesis operator of the frame U_j, $j \in \mathbb{N}_2$. For any partition $P = \{\sigma_1, \sigma_2 = \mathbb{N} \sim \sigma_1\}$, the weaving $\left\{\left\{u_{i1}\right\}_{i\in\sigma_1} \cup \left\{u_{i2}\right\}_{i\in\sigma_2}\right\}$ is a tight frame for $\mathbb{H}^R(\mathfrak{Q})$ with the frame bound r if and only if $\sum_{j\in\mathbb{N}_2} T_{U_j} D\sigma_j T^*_{U_j} = r\mathcal{I}_{\mathbb{H}^R(\mathfrak{Q})}$, where T_{U_j} is the synthesis operator of the frame U_j and D_{σ_j}, a diagonal matrix such that $d_{ii} = \begin{cases} 1, \text{ if } i \in \sigma_j, \\ 0, \text{ otherwise.} \end{cases}$*

Proof. For any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_2}$ of $\mathbb{N}$, the synthesis operator of the tight frame $U = \left\{\left\{u_{i1}\right\}_{i\in\sigma_1} \cup \left\{u_{i2}\right\}_{i\in\sigma_2}\right\}$ is $\sum_{j\in\mathbb{N}_2} T_{U_j} D_j$. Therefore, the frame operator of S_U is

$$\begin{aligned} r\mathcal{I}_{\mathbb{H}^R(\mathfrak{Q})} &= S_U \\ &= \left(\sum_{j\in\mathbb{N}_2} T_{U_j} D_j\right)\left(\sum_{j\in\mathbb{N}_2} T_{U_j} D_j\right)^* \\ &= \sum_{j\in\mathbb{N}_2} T_{U_j} D\sigma_j T^*_{U_j}. \end{aligned} \tag{20}$$

In the following, we construct a woven frame with the help of a given woven frame. □

Theorem 5. *Let $\mathfrak{F} = \left\{U_j = \left\{u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_2\right\}$ be a woven frame for $\mathbb{H}^R(\mathfrak{Q})$ with universal frame bounds r_1 and r_2 and S_{U_j} be the frame operators for the frames U_j, $j \in \mathbb{N}_2$. If, for $j \in \mathbb{N}_2$, $\|S^{-1}_{U_j}\|\|S_{U_1} - S_{U_2}\| < r_1/r_2$, then $\mathfrak{G} = \left\{S^{-1}_{U_j} \quad U_j = \left\{S^{-1}_{U_j} u_{ij}\right\}_{i\in\mathbb{N}}: j \in \mathbb{N}_2\right\}$ is also a woven frame for $\mathbb{H}^R(\mathfrak{Q})$.*

Proof. Let $\|S^{-1}_{U_2}\|\|S_{U_1} - S_{U_2}\| < r_1/r_2$. For any partition $P = \left\{\sigma_j\right\}_{j\in\mathbb{N}_2}$ of $\mathbb{N}$ and $u \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned} \left(\sum_{j\in\mathbb{N}_2}\sum_{i\in\sigma_j}\left|\langle S^{-1}_{U_j} u_{ij}|u\rangle\right|^2\right)^{1/2} &= \left(\sum_{j\in\mathbb{N}_2}\sum_{i\in\sigma_j}\left|\langle u_{ij}|S^{-1}_{U_j} u\rangle\right|^2\right)^{1/2} \\ &= \left(\sum_{i\in\sigma_1}\left|\langle u_{i1}|S^{-1}_{U_1} u\rangle\right|^2 + \sum_{i\in\sigma_2}\left|\langle u_{i2}|S^{-1}_{U_1} + \left(S^{-1}_{U_2} - S^{-1}_{U_1}\right)u\rangle\right|^2\right)^{1/2} \\ &\geq \left(\sum_{i\in\sigma_1}\left|\langle u_{i1}|S^{-1}_{U_1} u\rangle\right|^2 + \sum_{i\in\sigma_2}\left|\langle u_{i2}|S^{-1}_{U_1} u\rangle\right|^2\right)^{1/2} - \left(\sum_{i\in\sigma_2}\left|\langle u_{i2}|\left(S^{-1}_{U_2} - S^{-1}_{U_1}\right)u\rangle\right|^2\right)^{1/2} \\ &\geq \sqrt{r_1}\left\|S^{-1}_{U_1} u\right\| - \left(\sum_{i\in\mathbb{N}}\left|\langle u_{i2}|\left(S^{-1}_{U_2} - S^{-1}_{U_1}\right)u\rangle\right|^2\right)^{1/2} \\ &\geq \sqrt{r_1}\left\|S^{-1}_{U_1} u\right\| - \sqrt{r_2}\left\|S^{-1}_{U_2} - S^{-1}_{U_1}\right\|\|u\| \\ &\geq \left(\frac{\sqrt{r_1}}{\left\|S^{-1}_{U_1}\right\|} - \sqrt{r_2}\left\|S^{-1}_{U_2} - S^{-1}_{U_1}\right\|\right)\|u\|. \end{aligned} \tag{21}$$

Also,

$$\sum_{j\in\mathbb{N}_2}\sum_{i\in\sigma_j}\left|\langle S_{U_j}^{-1}u_{ij}|u\rangle\right|^2 \le \sum_{j\in\mathbb{N}_2}\sum_{i\in\mathbb{N}}\left|\langle S_{U_j}^{-1}u_{ij}|u\rangle\right|^2 \le r_2\left(\sum_{j\in\mathbb{N}_2}\left\|S_{U_j}\right\|^2\right)\|u\|^2. \tag{22}$$

Next, we give a characterization for the existence of the woven frame in terms of bounded surjective right linear operator. □

Theorem 6. *Let $\mathfrak{F}=\left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}$ be a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$ with universal frame bounds r_1 and r_2 and T be a bounded right linear operator on $\mathbb{H}^R(\mathfrak{Q})$. Then, $T\mathfrak{F}=\left\{\left\{Tu_{ij\,i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}\right\}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$ if and only if T is surjective.*

Proof. As $T\mathfrak{F}=\left\{\left\{Tu_{ij\,i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}\right\}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$, therefore, for the partition $P=\left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$,

$$\sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j}Tu_{ij}\langle Tu_{ij}|u\rangle = \mathrm{TS}_PT^*(u),\quad u\in\mathbb{H}^R(\mathfrak{Q}), \tag{23}$$

where S_P be the frame operator for the frame $\left\{u_{ij}\right\}_{i\in\sigma_j,j\in\mathbb{N}_m}$. Therefore, TS_PT^* is the frame operator of $\left\{Tu_{ij}\right\}_{i\in\sigma_j,j\in\mathbb{N}_m}$, so TS_PT^* is invertible and T is surjective.

Conversely, let T is surjective. Then, $\operatorname{ran}T=\mathbb{H}^R(\mathfrak{Q})$ is closed. Therefore, by Theorem 2.19 in [13], $\ker T\oplus\operatorname{ran}T^*=\mathbb{H}^R(\mathfrak{Q})=\ker T^*\oplus\operatorname{ran}T$. Now, we shall show that TT^* is bijective.

If $TT^*u=0$, $u\in\mathbb{H}^R(\mathfrak{Q})$, then $T^*u\in\ker T\cap\operatorname{ran}T^*=\{0\}$. Therefore, $T^*u=0$. Now, let $u\in\ker T^*=(\operatorname{ran}T)^\perp=(\mathbb{H}^R(\mathfrak{Q}))^\perp=\{0\}$. This give $u=0$. Thus, TT^* is injective.

As TT^* is a positive invertible operator on $\mathbb{H}^R(\mathfrak{Q})$, therefore,

$$0\le\left(TT^*\right)^{-1}\le\left\|\left(TT^*\right)^{-1}\right\|\mathcal{I}_{\mathbb{H}}^R(\mathfrak{Q}). \tag{24}$$

So, we have

$$TT^*-\left\|\left(TT^*\right)^{-1}\right\|^{-1}\mathcal{I}_{\mathbb{H}}^R(\mathfrak{Q})\ge 0. \tag{25}$$

This gives

$$\left\|T^*u\right\|^2\ge\left\|\left(TT^*\right)^{-1}\right\|^{-1}\|u\|^2,\quad u\in\mathbb{H}^R(\mathfrak{Q}). \tag{26}$$

Thus, for any partition $P=\left\{\sigma_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, we have

$$\begin{aligned} r_1\left\|\left(TT^*\right)^{-1}\right\|^{-1}\|u\|^2 &\le r_1\left\|T^*u\right\|^2\\ &\le\sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j}\left|\langle u_{ij}|T^*u\rangle\right|^2\\ &=\sum_{j\in\mathbb{N}_m}\sum_{i\in\sigma_j}\left|\langle Tu_{ij}|u\rangle\right|^2\\ &\le r_2\|T\|^2\|u\|^2,\quad u\in\mathbb{H}^R(\mathfrak{Q}). \end{aligned} \tag{27}$$

Hence, $T\mathfrak{F}=\left\{\left\{Tu_{ij\,i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}\right\}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$ with universal lower and upper frame bounds $r_1\|(TT^*)^{-1}\|^{-1}$ and $r_2\|T\|^2$, respectively. □

Corollary 1. *Let $\mathfrak{F}=\left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}$ be a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$. Then, one of the frame in the family $\mathfrak{F}$ is considered to be Parseval by considering the family $\sqrt{S_\lambda}\mathfrak{F}$, where S_λ is the frame operator of the frame $\{u_{i\lambda}\}_{i\in\mathbb{N}}$, for some $\lambda\in\mathbb{N}_m$.*

Next, we give a condition under which a subfamily from a woven frame is removed so that remaining family is a woven frame.

Theorem 7. *Let $\mathfrak{F}=\left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}$ be a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$ with universal lower and upper frame bounds r_1 and r_2, respectively. If $\mathbb{J}\subset\mathbb{N}$ and for some $\lambda\in\mathbb{N}_m$, there exists positive real constant $r_3<r_1$ such that*

$$\sum_{i\in\mathbb{J}}\left|\langle u_{i\lambda}|u\rangle\right|^2\le r_3\|u\|^2,\quad u\in\mathbb{H}^R(\mathfrak{Q}). \tag{28}$$

Then, $\mathfrak{F}_{\mathbb{N},\mathbb{J}}=\left\{\left\{u_{ij}\right\}_{i\in\mathbb{N},\mathbb{J}}: j\in\mathbb{N}_m\right\}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$.

Proof. For any partition $P=\left\{\eta_j\right\}_{j\in\mathbb{N}_m}$ of $\mathbb{N}$, $\mathbb{J}$, we have

$$\begin{aligned}\sum_{j\in\mathbb{N}_m}\sum_{i\in\eta_j}\left|\langle u_{ij}|u\rangle\right|^2 &\le\sum_{j\in\mathbb{N}_m}\sum_{i\in\eta_j\cup\mathbb{J}}\left|\langle u_{ij}|u\rangle\right|^2\\ &\le r_2\|u\|^2,\quad u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned} \tag{29}$$

Also,

$$\begin{aligned}\sum_{j\in\mathbb{N}_m}\sum_{i\in\eta_j}\left|\langle u_{ij}|u\rangle\right|^2 &= \left(\sum_{i\in\eta_\lambda\cup\mathbb{J}}\left|\langle u_{i\lambda}|u\rangle\right|^2-\sum_{i\in\mathbb{J}}\left|\langle u_{i\lambda}|u\rangle\right|^2\right)+\sum_{\substack{j\in\mathbb{N}_m\\ j\ne\lambda}}\sum_{i\in\eta_j}\left|\langle u_{ij}|u\rangle\right|^2\\ &\ge(r_1-r_3)\|u\|^2,\quad u\in\mathbb{H}^R(\mathfrak{Q}).\end{aligned} \tag{30}$$

Hence, $\mathfrak{F}_{\mathbb{N},\mathbb{J}}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$.

In the next result, we give a positive right linear operator with the help of family of frames, whose existence ensures the weaving of frame family. □

Theorem 8. *Let $\mathfrak{F}=\left\{\left\{u_{ij}\right\}_{i\in\mathbb{N}}: j\in\mathbb{N}_m\right\}$ be a family of frames for $\mathbb{H}^R(\mathfrak{Q})$ with frame bounds r_{1j} and r_{2j}, $j\in\mathbb{N}_m$. For any $\mathbb{J}\subset\mathbb{N}$ and fix $\lambda\in\mathbb{N}_m$, let*

$$T_j(u) = \sum_{i\in\mathbb{J}} u_{ij}\langle u_{ij}|u\rangle - \sum_{i\in\mathbb{J}} u_{i\lambda}\langle u_{i\lambda}|u\rangle, \quad j \in \mathbb{N}_m, j \neq \lambda. \tag{31}$$

If T_j is a positive right linear operator, then $\mathfrak{F}$ is a woven family.

Proof. Let $P = \{\eta_j\}_{j\in\mathbb{N}_m}$ be a partition of $\mathbb{N}$. Then, for any $u \in \mathbb{H}^R(\mathfrak{Q})$, we have

$$\begin{aligned}
r_{1\lambda}\|u\|^2 &\leq \sum_{i\in\mathbb{N}} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&= \sum_{i\in\eta_1} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_j} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_m} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&= \sum_{i\in\eta_1} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \langle u| \sum_{i\in\eta_j} u_{i\lambda}\langle u_{i\lambda}|u\rangle\rangle + \cdots + \sum_{i\in\eta_m} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&\leq \sum_{i\in\eta_1} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \langle u| \sum_{i\in\eta_j} u_{ij}\langle u_{ij}|u\rangle - T_j(u)\rangle + \cdots + \sum_{i\in\eta_m} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&\leq \sum_{i\in\eta_1} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \langle u| \sum_{i\in\eta_j} u_{ij}\langle u_{ij}|u\rangle\rangle + \cdots + \sum_{i\in\eta_m} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&= \sum_{i\in\eta_1} \left|\langle u_{i\lambda}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_j} \left|\langle u_{ij}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_m} \left|\langle u_{i\lambda}|u\rangle\right|^2 \\
&\leq \sum_{i\in\eta_1} \left|\langle u_{ij}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_j} \left|\langle u_{ij}|u\rangle\right|^2 + \cdots + \sum_{i\in\eta_m} \left|\langle u_{ij}|u\rangle\right|^2 \\
&\leq \left(\sum_{j\in\mathbb{N}_m} r_{1j}\right)\|u\|^2.
\end{aligned} \tag{32}$$

Hence, $\mathfrak{F}$ is a family of woven frames for $\mathbb{H}^R(\mathfrak{Q})$ with universal lower and upper frame bounds $r_{1\lambda}$ and $(\sum_{j\in\mathbb{N}_m} r_{1j})$, respectively. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

On Discrete Time Wilson Systems

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In this paper, we define the discrete time Wilson frame (DTW frame) for $l^2(\mathbb{Z})$ and discuss some properties of discrete time Wilson frames. Also, we give an interplay between DTW frames and discrete time Gabor frames. Furthermore, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are given. Moreover, the frame operator for the DTW frame is obtained. Finally, we discuss dual pair of frames for discrete time Wilson systems and give a sufficient condition for their existence.

1. Introduction

The idea of frame as a redundant peer of a basis was originated in 1952 by Duffin and Schaeffer [1]. It came to limelight only with the historic paper of Daubechies et al. [2]. A sequence of vectors $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathscr{H}$ is termed as a *frame* (or *Hilbert frame*) for a separable Hilbert space $\mathscr{H}$ if there exist constants $\mathscr{A}_l, \mathscr{A}_u > 0$ such that

$$\mathscr{A}_l \|u\|^2 \le \sum_{j\in\mathbb{N}} \left|\langle u, u_j\rangle\right|^2 \le \mathscr{A}_u \|u\|^2, \quad \text{for all } u \in \mathscr{H}. \tag{1}$$

The positive numbers $\mathscr{A}_l$ and $\mathscr{A}_u$ are termed as *lower* and *upper frame bounds* of the frame, respectively. The bounds may not be unique. If $\mathscr{A}_l = \mathscr{A}_u$, then $\{u_j\}_{j\in\mathbb{N}}$ is called an $\mathscr{A}_l$*-tight frame*, and if $\mathscr{A}_l = \mathscr{A}_u = 1$, then $\{u_j\}_{j\in\mathbb{N}}$ is said to be a *Parseval frame*. The inequality in (1) is recognized as the *frame inequality* of the frame $\{u_j\}_{j\in\mathbb{N}}$.

A sequence of vectors $\{u_j\}_{j\in\mathbb{N}} \subseteq \mathscr{H}$ is called a *Riesz basis* if $\{u_j\}_{j\in\mathbb{N}}$ is complete and there are positive constants $\mathscr{A}_l$ and $\mathscr{A}_u$ such that

$$\mathscr{A}_l \sum_{j\in\mathbb{N}} \left|\alpha_j\right|^2 \le \left\|\sum_{j\in\mathbb{N}} \alpha_j u_j\right\|^2 \le \mathscr{A}_u \sum_{j\in\mathbb{N}} \left|\alpha_j\right|^2, \quad \text{for all } \{\alpha_j\}_{j\in\mathbb{N}} \in \ell^2(\mathbb{N}). \tag{2}$$

Gabor frame for $L^2(\mathbb{R})$ (which is a Riesz basis) has bad localization properties in either time or frequency. Thus, a system to replace Gabor systems which do not have bad localization properties in time and frequency was required. Wilson [3, 4] suggested a system of functions which are localized around the positive and negative frequency of the same order. The idea of Wilson was used by Daubechies et al. [5] to construct orthonormal "Wilson bases" which consist of functions given by

$$\psi_j^k(x) = \begin{cases} \varepsilon_k \cos(2k\pi x)\, w\left(x - \dfrac{j}{2}\right), & \text{if } j \text{ is even,} \\ 2\sin(2(k+1)\pi x)\, w\left(x - \dfrac{j+1}{2}\right), & \text{if } j \text{ is odd,} \end{cases}$$
$$\varepsilon_k = \begin{cases} \sqrt{2}, & \text{if } k = 0, \\ 2, & \text{if } k \in \mathbb{N}, \end{cases} \tag{3}$$

with a smooth well-localized window function w. For such bases, the disadvantage described in the Balian–Low theorem is completely removed. Independent of the work of Daubechies et al. [5], orthonormal local trigonometric bases consisting of the functions $w_j \cos((k+(1/2))\pi(-j))$, $j \in \mathbb{Z}$, $k \in \mathbb{N}_0$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, were introduced by Malvar [6], where window functions are assumed to be compactly supported, and only two immediately neighbouring windows are allowed to have overlapping support. Some generalizations of Malvar bases were studied in [7, 8]. To obtain more freedom for the choice of window functions, biorthogonal bases were investigated in [9]. A drawback of Malvar's construction is the restriction on the support of the window functions. Therefore, it was preferred to consider Wilson bases of Daubechies et al. [5].

Feichtinger et al. [10] proved that Wilson bases of exponential decay are not unconditional bases for all modulation spaces on $\mathbb{R}$ including the classical Bessel potential spaces and the Schwartz spaces. Also, Wilson bases are not unconditional bases for the ordinary L^p spaces for $p \neq 2$, as shown in [10]. Approximation properties of Wilson bases are studied by Bittner [11], and Wilson bases for general time-frequency lattices are studied by Kutyniok and Strohmer [12]. Generalizations of Wilson bases to nonrectangular lattices are discussed by Sullivan et al. [3], with motivation from wireless communication and cosines modulated filter banks. Wojdyllo [13] studied modified Wilson bases and discussed Wilson system for triple redundancy in [14]. Discrete time Wilson frames with general lattices are studied by Lian et al. [15]. Motivated by the fact that one has different trigonometric functions for odd and even indices, Bittner [11, 16] considered Wilson bases introduced by Daubechies et.al [5] with nonsymmetrical window functions for odd and even indices. This generalized system of Bittner was later studied extensively by Kaushik and Panwar [17–19] and Jarrah and Panwar [20].

In this article, we consider the system defined by Bittner [16] to define the discrete time Wilson frame (DTWF) and give examples for its existence. Some observations related to properties of discrete time Wilson frames are given. Also, a relationship between DTW frames and the discrete time Gabor frames is discussed. Furthermore, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are obtained and the frame operator for the DTW frame is constructed. Finally, dual pair of frames for discrete time Wilson systems is defined and a sufficient condition for its existence is given.

The discrete time Wilson (DTW) system associated with $g_0, g_{-1} \in l^2(\mathbb{Z})$ is defined as

$$\psi_{\frac{m}{M},kL} = \begin{cases} \left(E_{(m/M)}T_{(kL/2)} + E_{(-m/M)}T_{(kL/2)}\right)g_0, & \text{if } k \in 2\mathbb{Z},\ k \neq 0, \\ \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}, & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{(-m/M)}\right)g_0, & \text{if } k = 0, \end{cases} \tag{4}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, M-1$.

The DTW system given by (4) can be rewritten for any $n \in \mathbb{Z}$ as

$$\psi_{(m/M),kL}(n) = \begin{cases} \sqrt{2}\cos\left(\frac{2\pi mn}{M}\right)g_0(n), & \text{if } k = 0, \\ 2\cos\left(\frac{2\pi mn}{M}\right)g_0\left(n - \frac{kL}{2}\right), & \text{if } k \in 2\mathbb{Z},\ k \neq 0, \\ 2\sin\left(\frac{2\pi(m+1)n}{M}\right)g_{-1}\left(n - \frac{(k+1)L}{2}\right), & \text{if } k \in 2\mathbb{Z}+1. \end{cases} \tag{5}$$

Remark 1. For $g_0 = g_{-1} = g$, the DTW system has the form

$$\psi_{(m/M),kL}g = \begin{cases} \left(E_{(m/M)}T_{(kL/2)} + E_{(-m/M)}T_{(kL/2)}\right)g, & \text{if } k \in 2\mathbb{Z}, k \neq 0, \\ \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)} - E_{-(m+1/M)}T_{((k+1)L/2)}\right)g, & \text{if } k \in 2\mathbb{Z}+1, \\ \frac{1}{\sqrt{2}}\left(E_{(m/M)} + E_{(-m/M)}\right)g, & \text{if } k = 0, \end{cases} \tag{6}$$

where $k \in \mathbb{Z}$, $L, M \in \mathbb{N}$ and $m = 0, 1, 2, \ldots, M-1$.

2. Outline of the Paper

In this article, we define discrete time Wilson frames (DTW frames) and discuss various properties of DTW frames (see Observations (I) to (VIII)). An interplay between DTW frames and discrete time Gabor frames has been given in Theorem 1. Also, a necessary and a sufficient condition for the DTW frame in terms of Zak transform are given in Theorem 3 and 4, respectively. The construction of the frame operator for the DTW frame is discussed in Theorem 5. Finally, we discuss dual pair of frames for discrete time Wilson systems and give a sufficient condition for its existence. Various examples are given to illustrate the discussion.

3. Discrete Time Wilson Frames

In this section, we define the discrete time Wilson frame based on the Wilson system considered by Bittner [11, 16], explore their existence through examples, and investigate various properties including its relationship with discrete time Gabor systems. We begin with the following definition.

Definition 1. The discrete time Wilson system:

$$\left\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z}), k \in \mathbb{Z}, L, M \in \mathbb{N}, m = 0, 1, 2, \ldots, M-1\right\}, \tag{7}$$

where $\psi_{(m/M),kL}$ is as defined in (4) and is called a discrete time Wilson frame (DTWF) if there exist constants $0 < \mathscr{A}_l \leq \mathscr{A}_u < \infty$ satisfying

$$\mathscr{A}_l\|f\|^2 \leq \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 \leq \mathscr{A}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{8}$$

The constants $\mathscr{A}_l$ and $\mathscr{A}_u$ are called lower and upper frame bounds for the DTWF $\left\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\right\}$. The supremum of all lower frame bounds and the infimum of all upper frame bounds are called optimal lower and optimal upper frames bounds, respectively.

In case the system $\left\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z}), k \in \mathbb{Z}, L, M \in \mathbb{N}, m = 0, 1, 2, \ldots, M-1\right\}$ satisfy only the right-hand side of inequality (8), then the system is called a discrete time Wilson Bessel sequence for $l^2(\mathbb{Z})$.

In order to show the existence of discrete time Wilson Bessel sequences which are not DTWF for $l^2(\mathbb{Z})$, we give the following examples.

Example 1

(i) Let $\{g(n)\}_{n\in\mathbb{Z}} = e_n$, $n \in \mathbb{N}$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}|f(n+kL)|^2 \\ &= M\sum_{k\in\mathbb{Z}}|f(n+kL)|^2 \\ &\leq M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}).\end{aligned} \tag{9}$$

Therefore, we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}g\rangle\right|^2 \leq 4M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{10}$$

Hence, $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4M$.

However, it is a DTW frame if and only if $L = 1$.

(ii) Let $g(n) = \begin{cases}(1/n), n=1,2,3,\ldots,A, \text{where} A<L<M, L\geq 2, \\ 0, \quad \text{otherwise.}\end{cases}$

Then, we have

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \leq M\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{11}$$

Hence, $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4M$. Furthermore, it is not a frame as it does not satisfy the lower frame condition for $\{f(n)\}_{n\in\mathbb{Z}} = e_L \in l^2(\mathbb{Z})$. Moreover, note that

(1) If $A = L$, then $\left\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $A = 2M$ and $B = 4M$.

(2) If $A = L = M$, then $\{\psi_{(m/M),kL}: g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $A = M$ and $B = 4M$.

Next, we give examples of Wilson systems which are discrete time Wilson frames for $l^2(\mathbb{Z})$.

Example 2

(i) Let $g(n) = \begin{cases} (1/\sqrt{M}), & n = 0, 1, 2, \ldots, L-1, L < M, \\ 0, & \text{otherwise.} \end{cases}$

Then, using the fact that $\sum_{m=0}^{M-1} e^{2\pi i (m/M)(q-p)} = \begin{cases} M, & \text{if} \quad q - p \in M\mathbb{Z}, \\ 0, & \text{if} \quad q - p \notin M\mathbb{Z}, \end{cases}$ we have

$$\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\langle f, E_{(m/M)} T_{kL} g\rangle\right|^2 = \frac{1}{M} \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \sum_{p=0}^{L-1} |f(p+kL)|^2 = \|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}),$$

$$\sum_{m=0}^{M-1} \left|\left\langle f, \cos\left(\frac{2\pi m \cdot}{M}\right) g(\cdot)\right\rangle\right|^2 \le \frac{3}{2}\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{12}$$

Therefore, $\{\psi_{(m/M),kL}: g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a discrete time Wilson frame for $l^2(\mathbb{Z})$.

(ii) Let $g(n) = \begin{cases} (1/2^M), & n \in [0, L] \cap \mathbb{Z}, L < M, \\ 0, & \text{otherwise.} \end{cases}$

Note that $\sum_{m=0}^{M-1} e^{2\pi i (m/M)(q-p)} = \begin{cases} M, & \text{if} \quad q - p \in M\mathbb{Z}, \\ 0, & \text{if} \quad q - p \notin M\mathbb{Z}. \end{cases}$

Therefore,

$$\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\langle f, E_{(m/M)} T_{kL} g\rangle\right|^2 = \frac{M}{2^{2M}} \left(\|f\|^2 + \sum_{k\in\mathbb{Z}} |f((k+1)L)|^2 \right),$$
$$2\sum_{m=0}^{M-1} \left|\left\langle f, \cos\left(\frac{2\pi m \cdot}{M}\right) g(\cdot)\right\rangle\right|^2 \le \frac{3M}{2^{2M}}\|f\|^2, \quad f \in l^2(\mathbb{Z}). \tag{13}$$

Hence, $\{\psi_{(m/M),kL}: g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$.

In view of the above discussion, we have the following observations in relation to DTW frames.

(I)Let $f, g \in l^2(\mathbb{Z})$ and let T_{kL} be the translation operator on $l^2(\mathbb{Z})$, where $k \in \mathbb{Z}$ and $L \in \mathbb{N}$. Then,

$$\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \operatorname{Im}\left(\left\langle f, \cos\left(\frac{2\pi m (.)}{M}\right) T_{kL} g(.)\right\rangle \cdot \left\langle \overline{f}, \sin\left(\frac{2\pi m (.)}{M}\right) \overline{T_{kL} g(.)}\right\rangle \right) = 0. \tag{14}$$

Indeed, it follows from the fact that

$$\sum_{m=0}^{M-1} e^{2\pi i (m/M)(q-p)} = \begin{cases} M, & \text{if} \quad q - p \in M\mathbb{Z}, \\ 0, & \text{if} \quad q - p \notin M\mathbb{Z}. \end{cases} \tag{15}$$

(II) Let $\{\psi_{(m/M),kL}: g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTW system for $l^2(\mathbb{Z})$. Then,

$$\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\left\langle f, \cos\left(\frac{2\pi m (\cdot)}{M}\right) T_{kL} g_0(\cdot)\right\rangle\right|^2 + 4\sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \left|\left\langle f, \sin\left(\frac{2\pi m (\cdot)}{M}\right) T_{kL} g_{-1}(\cdot)\right\rangle\right|^2 - 2\sum_{m=0}^{M-1} \left|\left\langle f, \cos\left(\frac{2\pi m (\cdot)}{M}\right) g_0(\cdot)\right\rangle\right|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{16}$$

Indeed, one can compute that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 &= 4\sum_{\substack{k\in 2\mathbb{Z}\\ k\neq 0}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)T_{(kL/2)}g_0(\cdot)\right\rangle\right|^2\\ &\quad+4\sum_{\substack{k\in 2\mathbb{Z}+1\\ k\neq 0}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi(m+1).}{M}\right)T_{((k+1)L/2)}g_{-1}(\cdot)\right\rangle\right|^2+2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2\\ &=4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2-4\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2\\ &\quad+4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m.}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2+2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m.}{M}\right)g_0(\cdot)\right\rangle\right|^2.\end{aligned}\tag{17}$$

In view of Observations (I) and (II), we obtain (III).

(III) Let $\left\{\psi_{(m/M),kL}\colon g_0,g_{-1}\in l^2(\mathbb{Z})\right\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTW system for $l^2(\mathbb{Z})$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0(\cdot)\rangle\right|^2+4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}(\cdot)\rangle\right|^2\\ &\quad-4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m.}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2-2\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m}{M}\right)g_0(\cdot)\right\rangle\right|^2\\ &\quad-4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m\cdot}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2,\quad f\in l^2(\mathbb{Z}).\end{aligned}\tag{18}$$

Using Observations (II) and (III), we have (IV).

(IV) For all $f\in l^2(\mathbb{Z})$,

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0(\cdot)\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}(\cdot)\rangle\right|^2\\ &=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_0(\cdot)\right\rangle\right|^2\\ &\quad+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\cos\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2+\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\left\langle f,\sin\left(\frac{2\pi m(\cdot)}{M}\right)T_{kL}g_{-1}(\cdot)\right\rangle\right|^2.\end{aligned}\tag{19}$$

(V) If $g_0 = g_{-1} = g$ for the DTW system $\{\psi_{(m/M),kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$, then for all $f \in l^2(\mathbb{Z})$, and we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g(.)\rangle\right|^2 - 2\sum_{m=0}^{M-1}\left|\langle f,\cos\left(\frac{2\pi m}{M}\right)g(.)\rangle\right|^2. \tag{20}$$

(VI) Let $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTG Bessel sequences with Bessel bounds B_1 and B_2, respectively. Then, the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence with Bessel bound $4(B_1+B_2)$.

Indeed, using observation (III) and the hypothesis, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}\rangle\right|^2 &\le 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0(.)\rangle\right|^2 + 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}(.)\rangle\right|^2\\ &\le 4(B_1+B_2)\|f\|^2, \quad \text{for all } f\in l^2(\mathbb{Z}).\end{aligned} \tag{21}$$

Remark 2. The converse of observation (VI) may not be true even if additionally we assume that the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a frame for $l^2(\mathbb{Z})$.

Example 3. Let $L=2, M=4$, $L=2, M=4$, and $\{g_{-1}(n)\}_{n\in\mathbb{Z}} = e_1$. Then,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_0\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}|f(2k)|^2, \quad \text{for all } f\in l^2(\mathbb{Z}),\\ \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,E_{(m/M)}T_{kL}g_{-1}\rangle\right|^2 &= 4\sum_{k\in\mathbb{Z}}|f(2k+1)|^2, \quad \text{for all } f\in l^2(\mathbb{Z}).\end{aligned} \tag{22}$$

Thus, the systems $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are not DTG frames for $l^2(\mathbb{Z})$. Now, using observation (III), we obtain

$$8\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f,\psi_{(m/M),kL}g\rangle\right|^2 \le 16\|f\|^2, \quad \text{for all } f\in l^2(\mathbb{Z}). \tag{23}$$

Hence, $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ with frame bounds 8 and 16.

(VII) Let $g_0, g_{-1}\in l^2(\mathbb{Z})$ be such that

$$\begin{aligned}B_1 &= M\sup_{n\in[0,L]\cap\mathbb{N}}\sum_{p\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}}g_0(n-kL)\overline{g_0(n-kL-pM)}\right| < \infty,\\ B_2 &= M\sup_{n\in[0,L]\cap\mathbb{N}}\sum_{p\in\mathbb{Z}}\left|\sum_{k\in\mathbb{Z}}g_{-1}(n-kL)\overline{g_{-1}(n-kL-pM)}\right| < \infty.\end{aligned} \tag{24}$$

Then, the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$ with Bessel bound $4(B_1+B_2)$.

(VIII) If $g_0, g_{-1} \in l^2(\mathbb{Z})$ are functions having bounded support, then the DTW system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$.

Indeed, one may perceive that, since the functions g_0 and g_{-1} have bounded support, B_1 and B_2 as defined in observation (VII) are finite, and hence the DTW system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence for $l^2(\mathbb{Z})$.

Now, we prove a result related to DTW systems for the particular case when $g_0 = g_{-1} = g$.

Lemma 1. *For $f, g \in l^2(\mathbb{Z})$, we have*

$$2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 \le 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2. \tag{25}$$

Proof. Using observation (V), we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = 4\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g(\cdot)\rangle\right|^2 - 2\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m.}{M}\right)g(\cdot)\rangle\right|^2. \tag{26}$$

Hence, we compute

$$\begin{aligned} 2\sum_{m=0}^{M-1}\left|\langle f, \cos\left(\frac{2\pi m.}{M}\right)g(.)\rangle\right|^2 &= 2\sum_{m=0}^{M-1}\left|\langle f, \frac{1}{2}\left(E_{(m/M)} + E_{(-m/M)}\right)g(.)\rangle\right|^2 \\ &= \frac{1}{2}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}g\rangle + \langle f, E_{(-m/M)}g\rangle\right|^2 \\ &\le 2\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}g\rangle\right|^2 \le 2\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2. \end{aligned} \tag{27}$$

In the following result, we give an interplay between the DTW frame and DTG frame for $l^2(\mathbb{Z})$. □

Theorem 1. *The Wilson system $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ if and only if $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$.*

Proof. Let $\mathscr{A}_l$ and $\mathscr{A}_u$ be the positive constants such that

$$\mathscr{A}_l\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{\frac{m}{M},kL}\rangle\right|^2 \le \mathscr{A}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{28}$$

Then, using Lemma 1, it is easy to conclude that $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$ with frame bounds $(\mathscr{A}_l/4)$ and $(\mathscr{A}_u/2)$.

Conversely, let $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTG frame for $l^2(\mathbb{Z})$. Then, there exist positive constants $\mathscr{B}_l$ and $\mathscr{B}_u$ such that

$$\mathscr{B}_l\|f\|^2 \le \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 \le \mathscr{B}_u\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{29}$$

Again, by utilizing Lemma 1, we deduce that $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame with frame bounds $2\mathscr{B}_l$ and $4\mathscr{B}_u$.

Now, we define discrete time tight Wilson frame for $l^2(\mathbb{Z})$ and investigate their relationship with discrete time Gabor frame for $l^2(\mathbb{Z})$. □

Definition 2. The discrete time Wilson system $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ given by (4) is called a discrete time tight Wilson frame (DTTWF) if there exists a constant $\mathscr{C}_\infty \ge 0$ such that

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, \psi_{(m/M),kL}\rangle\right|^2 = \mathscr{C}_1\|f\|^2, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{30}$$

If $\mathscr{C}_\infty = 1$, then the frame $\{\psi_{(m/M),kL}\colon g_0, g_{-1} \in l^2(\mathbb{Z})\}$ is called Discrete Time Parseval frame.

Next, we state two results whose proofs can be worked out using Lemma 1.

Proposition 1. *Let $\{\psi_{(m/M),kL}\colon g \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a DTTW frame for $l^2(\mathbb{Z})$ with frame bound $\mathscr{C}_\infty$. Then, $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame with frame bounds $(\mathscr{C}_\infty/4)$ and $(\mathscr{C}_\infty/2)$.*

Proposition 2. *Let* $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *be a DTTG frame for* $l^2(\mathbb{Z})$ *with frame bound* $\mathscr{C}_\infty$. *Then,* $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $2\mathscr{C}_\infty$ *and* $4\mathscr{C}_\infty$.

4. Discrete Zak Transform and Discrete Time Wilson Frames

Various properties of the Zak transform (continuous version) were studied by Janssen [21,22] and the discrete version is discussed by Heil [23] who gave the following definition of discrete Zak transform.

Definition 3 (see [23]). The discrete Zak transform of a sequence $f\in l^2(\mathbb{Z})$ is given by

$$Zf(n,x)=\sum_{j\in\mathbb{Z}}f(n+ja)e^{2\pi ijx},\quad \forall(n,x)\in\mathbb{Z}\times\widehat{\mathbb{R}},\tag{31}$$

where $a\in\mathbb{Z}^+$ is a fixed parameter and $\widehat{\mathbb{R}}$ is the dual group of $\mathbb{R}$.

Next, we state a result related to Zak transform proved by Heil [23].

Theorem 2 (see [23]). *Given a fixed* $g\in L^2(\mathbb{R})$ *and* $L\in\mathbb{Z}^+$. *If* $L=M$, *then the system* $\{E_{(m/M)}T_{kL}g\}$ *is a frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_1$ *and* $\mathscr{D}_2\Leftrightarrow 0<M^{-1}\mathscr{D}_1\le|Zg|^2\le M^{-1}\mathscr{D}_2<\infty$ *a.e.*

Now, we give a necessary condition for DTW frame in terms of the discrete Zak transform.

Theorem 3. *Let* $L=M$. *If* $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_l$ *and* $\mathscr{D}_u$, *then*

$$0<\frac{L^{-1}\mathscr{D}_l}{4}\le|Zg|^2\le\frac{L^{-1}\mathscr{D}_u}{2}<\infty\text{ a.e.}\tag{32}$$

Proof. Since $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW frame for $l^2(\mathbb{Z})$ with frame bounds $\mathscr{D}_l$ and $\mathscr{D}_u$, using Theorem 1, the system $\{E_{(m/M)}T_{kL}g\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTG frame for $l^2(\mathbb{Z})$ with frame bounds $(\mathscr{D}_l/4)$ and $(\mathscr{D}_u/2)$.

Hence, the result follows using Theorem 2.

Towards, the converse of Theorem 3, we have the following result. □

Theorem 4. *Let* $L=M$. *If there exists* $\mathscr{D}_l>0$ *and* $\mathscr{D}_u>0$ *such that the following inequality holds*

$$0<\frac{L^{-1}\mathscr{D}_l}{2}\le|Zg|^2\le\frac{L^{-1}\mathscr{D}_u}{4}<\infty.\text{ a.e.},\tag{33}$$

then $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is a DTW frame for* $l^2(\mathbb{Z})$ *with frame bounds* $\mathscr{D}_l$ *and* $\mathscr{D}_u$.

Proof. It can be worked out using Theorem 1 and Theorem 2. □

Remark 3. For $L>M$, the system $\{\psi_{(m/M),kL}\colon g\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is not a frame for $l^2(\mathbb{Z})$.

5. Discrete Time Wilson Frame Operator

The frame operator for a frame is constructed by the composition of two important operators, namely, the analysis operator and the synthesis operator. The frame operator is positive, bounded, invertible, and self-adjoint. It ensures the existence of a canonical dual frame of a given frame, i.e., if $\{f_n\}$ is a frame and S is the frame operator, then $\{S^{-1}f_n\}$ is a frame called the canonical dual of the frame $\{f_n\}$. It is known that the canonical tight frame leads to a perfect reconstruction when used for both analysis and synthesis. Keeping this in mind, we make an attempt to construct the frame operator for the discrete time Wilson frame. We begin with the following definition.

Definition 4. Let $\{\psi_{(m/M),kL}\colon g_0,g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a discrete time Bessel sequence for $l^2(\mathbb{Z})$. Then, DTWF operator $S\colon l^2(\mathbb{Z})\longrightarrow l^2(\mathbb{Z})$ is defined as

$$Sf=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,\psi_{(m/M),kL}\rangle\psi_{(m/M),kL},\quad\forall f\in l^2(\mathbb{Z}).\tag{34}$$

In the following result, we construct the frame operator for the discrete time Wilson frame with the help of the frame operators of the two associated discrete time Gabor Bessel sequences.

Theorem 5. *For* $g_0,g_{-1}\in l^2(\mathbb{Z})$, *let* $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *and* $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *be DTG Bessel sequences with frame operators* S_1 *and* S_2, *respectively. Then, the frame operator* S *for DTW system* $\{\psi_{(m/M),kL}\colon g_0,g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ *is given by* $S=2(S_1+S_2+P_1-P_2+R)$, *where*

$$\begin{aligned}
S_1f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(m/M)}T_{kL}g_0,\\
S_2f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_{-1}\rangle E_{(m/M)}T_{kL}g_{-1},\\
P_1f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(-m/M)}T_{kL}g_0,\\
P_2f&=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_{-1}\rangle E_{(-m/M)}T_{kL}g_{-1},\\
Rf&=\sum_{m=0}^{M-1}\langle f,E_{(m/M)}g_0\rangle\cos\left(2\pi\frac{m}{M}(\cdot)\right)g_0(\cdot).
\end{aligned}\tag{35}$$

Proof. By hypothesis, we have $S_1f=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}g_0\rangle E_{(m/M)}T_{kL}g_0$ and $S_2f=\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f,E_{(m/M)}T_{kL}$

$g_{-1}\rangle E_{(m/M)}T_{kL}g_{-1}$. Since $\{E_{(m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are DTG Bessel sequences, we obtain

$$\sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{(m/M)}T_{kL}g\rangle\right|^2 = \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\left|\langle f, E_{-(m/M)}T_{kL}g\rangle\right|^2. \tag{36}$$

Also, using observation (VI), we deduce that the systems $\{E_{(-m/M)}T_{kL}g_0\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{E_{(-m/M)}T_{kL}g_{-1}\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ are DTG Bessel sequences and the system $\{\psi_{(m/M),kL}\colon g_0, g_{-1}\in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence with their frame operators denoted by K_1, K_2, and S, respectively. Then, for all $f\in l^2(\mathbb{Z})$, we obtain

$$\begin{aligned}
K_1 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_0\rangle E_{-(m/M)}T_{kL}g_0,\\
K_2 f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{kL}g_{-1}\rangle E_{-(m/M)}T_{kL}g_{-1},\\
Sf &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1}\langle f, \psi_{(m/M),kL}\rangle \psi_{(m/M),kL}\\
&= \sum_{m=0}^{M-1}\langle f, \frac{1}{\sqrt{2}}\left(E_{(m/M)}+E_{-(m/M)}\right)g_0\rangle \frac{1}{\sqrt{2}}\left(E_{(m/M)}+E_{-(m/M)}\right)g_0\\
&\quad+\sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, \left(E_{(m/M)}T_{(kL/2)}+E_{-(m/M)}T_{(kL/2)}\right)g_0\rangle\left(E_{(m/M)}T_{(kL/2)}+E_{-(m/M)}T_{(kL/2)}\right)g_0\\
&\quad+\sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)}-E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}\rangle \frac{1}{i}\left(E_{(m+1/M)}T_{((k+1)L/2)}-E_{-(m+1/M)}T_{((k+1)L/2)}\right)g_{-1}\\
&= \frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{(m/M)}g_0+\frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{(m/M)}g_0+\frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}g_0\rangle E_{-(m/M)}g_0\\
&\quad+\frac{1}{2}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}g_0\rangle E_{-(m/M)}g_0+\sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{(kL/2)}g_0\rangle E_{(m/M)}T_{(kL/2)}g_0\\
&\quad+\sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{(kL/2)}g_0\rangle E_{(m/M)}T_{(kL/2)}g_0+\sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{(m/M)}T_{(kL/2)}g_0\rangle E_{-(m/M)}T_{(kL/2)}g_0\\
&\quad+\sum_{k\in 2\mathbb{Z},k\neq 0}\sum_{m=0}^{M-1}\langle f, E_{-(m/M)}T_{(kL/2)}g_0\rangle E_{-(m/M)}T_{(kL/2)}g_0+\sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad-\sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad-\sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\\
&\quad+\sum_{k\in 2\mathbb{Z}+1}\sum_{m=0}^{M-1}\langle f, E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}\rangle E_{-(m+1/M)}T_{((k+1)L/2)}g_{-1}mn,\\
&= S_1 f+S_2 f+K_1 f+K_2 f+P_1 f+T_1 f-P_2 f-T_2 f-R_1 f-R_2 f-R_3 f-R_4 f,
\end{aligned} \tag{37}$$

where

$$
\begin{aligned}
P_1 f &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{(m/M)} T_{kL} g_0 \rangle E_{-(m/M)} T_{kL} g_0, \\
T_1 f &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{-(m/M)} T_{kL} g_0 \rangle E_{(m/M)} T_{kL} g_0, \\
P_2 f &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{(m/M)} T_{kL} g_{-1} \rangle E_{-(m/M)} T_{kL} g_{-1}, \\
T_2 f &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{-(m/M)} T_{kL} g_{-1} \rangle E_{(m/M)} T_{kL} g_{-1}, \\
R_1 f &= \frac{1}{2} \sum_{m=0}^{M-1} \langle f, E_{(m/M)} g_0 \rangle E_{(m/M)} g_0, \\
R_2 f &= \frac{1}{2} \sum_{m=0}^{M-1} \langle f, E_{-(m/M)} g_0 \rangle E_{(m/M)} g_0, \\
R_3 f &= \frac{1}{2} \sum_{m=0}^{M-1} \langle f, E_{(m/M)} g_0 \rangle E_{-(m/M)} g_0, \\
R_4 f &= \frac{1}{2} \sum_{m=0}^{M-1} \langle f, E_{-(m/M)} g_0 \rangle E_{-(m/M)} g_0.
\end{aligned}
\tag{38}
$$

Now, for all $h \in l^2(\mathbb{Z})$, we compute

$$
\begin{aligned}
\langle T_1 f, h \rangle &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{(-m/M)} T_{kL} g_0 \rangle \langle E_{(m/M)} T_{kL} g_0, h \rangle \\
&= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} f(p) \overline{g_0(p-kL)} g_0(q-kL) \overline{h(q)} \exp^{2\pi i (m/M)(p+q)} \\
&= M \sum_{k\in\mathbb{Z}} \sum_{p,q\in\mathbb{Z}, p+q\in M\mathbb{Z}} f(p) \overline{g_0(p-kL)} g_0(q-kL) \overline{h(q)}, \\
\langle P_1 f, h \rangle &= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \langle f, E_{(m/M)} T_{kL} g_0 \rangle \langle E_{(-m/M)} T_{kL} g_0, h \rangle \\
&= \sum_{k\in\mathbb{Z}} \sum_{m=0}^{M-1} \sum_{p\in\mathbb{Z}} \sum_{q\in\mathbb{Z}} f(p) \overline{g_0(p-kL)} g_0(q-kL) \overline{h(q)} \exp^{-2\pi i (m/M)(p+q)} \\
&= M \sum_{k\in\mathbb{Z}} \sum_{p,q\in\mathbb{Z}, p+q\in M\mathbb{Z}} f(p) \overline{g_0(p-kL)} g_0(q-kL) \overline{h(q)}.
\end{aligned}
\tag{39}
$$

Thus, $T_1 f = P_1 f$, for all $f \in l^2(\mathbb{Z})$. Similarly, it can be proved that $T_2 f = P_2 f$, $S_1 f = K_1 f$, $S_2 f = K_2 f$, $R_1 f = R_4 f$, and $R_2 f = R_3 f$, for all $f \in l^2(\mathbb{Z})$.

Hence, we conclude that $S = 2(S_1 + S_2 + P_1 - P_2 + R)$. □

6. Dual Pair of Frames for Discrete Time Wilson Systems

In this section, we study dual pair of frames and obtain a sufficient condition for the existence of a dual pair of discrete time Wilson systems. First, we state the definition of a dual pair of frames discussed by Christensen [24, 25].

Definition 5 (see [25]). Let H be a Hilbert space and let $\{f_i\}_{i\in I}$, $\{g_i\}_{i\in I}$, $\{p_j\}_{j\in J}$, and $\{q_i\}_{j\in J}$ be Bessel sequences. Then, $F = \{f_i\}_{i\in I} \cup \{p_j\}_{j\in J}$ and $G = \{g_i\}_{i\in I} \cup \{q_j\}_{j\in J}$ are dual pair of frames if

$$
f = \sum_{i\in I} \langle f, f_i \rangle g_i + \sum_{j\in J} \langle f, p_j \rangle q_j, \quad \text{for all } f \in H. \tag{40}
$$

In the following result, we give a sufficient condition for the existence of a dual pair of discrete time Wilson systems.

Theorem 6. *Let $L \le M$ and let $\{\psi_{(m/M),kL}: g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}: w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTW Bessel sequences for $l^2(\mathbb{Z})$. Then, there exist DTW Bessel sequences $\{P_{(m/M),kL}: p_0, p_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{Q_{(m/M),kL}: q_0, q_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ such that $P = \{\psi_{(m/M),kL}\} \cup \{P_{(m/M),kL}\}$ and $Q = \{\xi_{(m/M),kL}\} \cup \{Q_{(m/M),kL}\}$ are dual pair of frames for $l^2(\mathbb{Z})$.*

Proof. Let T and U be the preframe operators for the DTW Bessel sequences $\{\psi_{(m/M),kL}: g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}: w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$, respectively. Then, $T: l^2(\mathbb{Z}\times\mathbb{Z}_M) \longrightarrow l^2(\mathbb{Z})$ and $U: l^2(\mathbb{Z}\times\mathbb{Z}_M) \longrightarrow l^2(\mathbb{Z})$ are given by

$$\begin{aligned} T(\{c_{m,k}\}) &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} c_{m,k}\psi_{(m/M),kL}, \\ U(\{c_{m,k}\}) &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} c_{m,k}\xi_{(m/M),kL}, \end{aligned} \tag{41}$$

where $\mathbb{Z}_M = \{0, 1, 2, \ldots, M-1\}$. Then,

$$UT^* f = \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} \langle f, \psi_{(m/M),kL}\rangle \xi_{(m/M),kL}, \quad \text{for all } f \in l^2(\mathbb{Z}). \tag{42}$$

Also, the operator $\Phi = I - UT^*$ is bounded on $l^2(\mathbb{Z})$. Furthermore, $\Phi^* = I - TU^*$. Let $R = \{R_{(m/M),kL}: r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $S = \{S_{(m/M),kL}: s_0, s_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be a pair of DTW dual frames for $l^2(\mathbb{Z})$. Then, using Proposition 2.1 of [25], we compute

$$\begin{aligned} f &= \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} \langle f, \psi_{(m/M),kL}\rangle \xi_{(m/M),kL} \\ &\quad + \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} \langle f, \Phi^* R_{(m/M),kL}\rangle S_{(m/M),kL}. \end{aligned} \tag{43}$$

Also, using Lemma 6.3.2 of [24], we deduce that $R = \{\psi_{(m/M),kL}\} \cup \{\Phi^* R_{(m/M),kL}\}$ and $S = \{\xi_{(m/M),kL}\} \cup \{S_{(m/M),kL}\}$ form a dual pair of frames for $l^2(\mathbb{Z})$ if $\{\Phi^* R_{(m/M),kL}\}$ is a DTW Bessel sequence. Now, observe that $\{\Phi^* R_{(m/M),kL}\}$ is a DTW system given by $\{\Phi^* R_{(m/M),kL}\} = \{R^*_{(m/M),kL}\} = \{R_{(m/M),kL}: \Phi^* r_0, \Phi^* r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$. Since $R = \{R_{(m/M),kL}: r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ is a DTW Bessel sequence and Φ is a bounded operator, $\{\Phi^* R_{(m/M),kL}\}$ is a DTW Bessel sequence.

Finally, we prove a result related to compact support of functions generating DTW Bessel sequences. □

Theorem 7. *Let $L \le M$, and let $\{\psi_{(m/M),kL}: g_0, g_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $\{\xi_{(m/M),kL}: w_0, w_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be two DTW Bessel sequences for $l^2(\mathbb{Z})$. If the functions g_0, g_{-1}, w_0, and w_{-1} are compactly supported, then the functions p_0, p_{-1}, q_0, and q_{-1} are also compactly supported.*

Proof. Suppose that $R = \{R_{(m/M),kL}: r_0, r_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ and $S = \{S_{(m/M),kL}: s_0, s_{-1} \in l^2(\mathbb{Z})\}_{m=0,k\in\mathbb{Z},L,M\in\mathbb{N}}^{M-1}$ be such that r_0, r_{-1}, s_0, and s_{-1} be compactly supported. Then, in view of the proof of Theorem 6, one can conclude that the functions p_0, p_{-1}, q_0, and q_{-1} are compactly supported if $\Phi^* r_0$ and $\Phi^* r_{-1}$ are compactly supported.

By assumption, $g_0, g_{-1}, w_0, w_{-1}, r_0, r_{-1}, s_0$, and s_{-1} are all compactly supported. Therefore, there exists an $N \in \mathbb{N}$ such that

$$\begin{aligned} g_0(n) &= g_{-1}(n) = w_0(n) = w_{-1}(n) = s_0(n) \\ &= s_{-1}(n) = r_0(n) = r_{-1}(n) = 0, \quad \text{for all } n \notin [-N, N]. \end{aligned} \tag{44}$$

Since

$$\Phi^* r_0 = (I - TU^*) r_0 = r_0 - \sum_{k\in\mathbb{Z}}\sum_{m=0}^{M-1} \langle r_0, \xi_{(m/M),kL}\rangle \psi_{(m/M),kL}, \tag{45}$$

$\Phi^* r_0$ is compactly supported. Similarly, $\Phi^* r_{-1}$ is compactly supported. □

7. Conclusion

Gabor frame for $L^2(\mathbb{R})$ (which is a Riesz basis) has bad localization properties in either time or frequency. Wilson [3, 4] suggested a system of functions which are localized around the positive and negative frequency of the same order. Based on the Wilson systems, Wilson frames for $L^2(\mathbb{R})$ were introduced and studied in [17–20]. In this article, discrete time Wilson frames (DTWF) are defined and their relationship with discrete time Gabor frames is investigated. Also, frame operator for the DTWF has been constructed. Finally, keeping duality in mind, dual pair of frames for the discrete time Wilson systems have been studied.

Data Availability

No data were used to support the findings of the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

An Intertwining of Curvelet and Linear Canonical Transforms

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In this article, we introduce a novel curvelet transform by combining the merits of the well-known curvelet and linear canonical transforms. The motivation towards the endeavour spurts from the fundamental question of whether it is possible to increase the flexibility of the curvelet transform to optimize the concentration of the curvelet spectrum. By invoking the fundamental relationship between the Fourier and linear canonical transforms, we formulate a novel family of curvelets, which is comparatively flexible and enjoys certain extra degrees of freedom. The preliminary analysis encompasses the study of fundamental properties including the formulation of reconstruction formula and Rayleigh's energy theorem. Subsequently, we develop the Heisenberg-type uncertainty principle for the novel curvelet transform. Nevertheless, to extend the scope of the present study, we introduce the semidiscrete and discrete analogues of the novel curvelet transform. Finally, we present an example demonstrating the construction of novel curvelet waveforms in a lucid manner.

1. Introduction

The wavelet transform is a multiscale integral transform, which serves as one of the corner stones of nonstationary signal processing. It can be used in time-frequency analysis, wherein the scale and frequency are inverse to each other. The wavelet transform decomposes a signal into components determined by the translations and dilations of a single function known as the mother wavelet. By applying these local decomposition filters, the wavelet transform has proved to be of substantial importance in capturing the local characteristics of nonstationary signals and has paved its way to a number of fields including signal and image processing, sampling theory, geophysics, astrophysics, and quantum mechanics [1–4]. However, the efficiency of the wavelet transform fades away in the realm of higher-dimensional signal processing due to the fact that the wavelet transform employs isotropic scalings in dimensions $n \geq 2$. Such isotropic scalings are incompetent to capture the edges and corners in higher-dimensional signals appearing due to the spatial occlusion between different objects; for instance, in medical imaging curves separate bones and different kinds of soft tissue. Therefore, the key problem in multidimensional signal analysis is to extract and characterize the relevant and directional information regarding the occurrence of curves and boundaries in signals. As a result, some off-shoots of the wavelet transform, such as the Stockwell transform [5, 6], ridgelet transform [7], curvelet transform [8, 9], contourlet transform [10], and the shearlet transform [11], have been introduced to address these shortcomings of the wavelet transform.

The curvelet transform aims to deal with certain interesting phenomena occurring along curved edges in higher-dimensional signals. Unlike the wavelet transform, the curvelet transform provides time-frequency localization with a reasonable directionality and anisotropy by using angled polar wedges or angled trapezoid windows in frequency domain. The intrinsic multiscale and anisotropic nature of curvelet waveforms leads to optimally sparse representations of objects which display curve-punctuated smoothness, that is, smoothness except for discontinuity along a general curve with bounded curvature. Another remarkable property of curvelets is that they elegantly model the geometry of wave propagation; curvelets may be viewed as coherent waveforms with enough frequency localization to behave like waves but, at the same time, with sufficient

spatial localization to behave like particles [12]. For more about curvelets and their applications, we refer to the monographs in [12–19]. Keeping in view the merits of the curvelet transform, in the present study, we aim to answer the fundamental question of whether it is possible to increase the flexibility of the curvelet transform to optimize the concentration of the curvelet spectrum. The answer to this question is affirmative and lies in intertwining the curvelet transform with the well-known linear canonical transform, an integral transform known for its flexibility and higher degrees of freedom in modelling physical phenomenon [20].

The highlights of the article are given as follows:

(i) We introduce the notion of novel curvelet transform by combining the merits of the curvelet and linear canonical transforms

(ii) We study the fundamental properties of the proposed transform including the reconstruction and Rayleigh's energy formulae

(iii) We formulate a Heisenberg-type uncertainty principle associated with the novel curvelet transform

(iv) To extend the scope of the study, we introduce both the semidiscrete and discrete analogues of the novel curvelet transform

(v) Finally, we present an example regarding the construction of novel curvelets

The rest of the article is structured as follows: In Section 2, we recapitulate the linear canonical transform and the ordinary curvelet transform. In Section 3, we present the formal aspects of the study, which are continued to Section 4, and Section 5 is devoted to illustrating the construction of novel curvelets. Finally, in Section 6, we extract a conclusion and provide an impetus to the future research work in the realm of novel curvelet transform.

2. Linear Canonical and Curvelet Transforms

In this section, we shall present a gentle overview of the linear canonical and curvelet transforms, which facilitates the formulation of the proposed novel curvelet transform.

2.1. Two-Dimensional Linear Canonical Transform. The origin of the theory of linear canonical transforms dates back to early 1970s with the independent seminal works of Collins [21] in paraxial optics and Moshinsky and Quesne [22] in quantum mechanics to study the conservation of information and uncertainty under linear maps of phase space. It was only in 1990s that both these independent works began to be referred to jointly in the open literature. The linear canonical transform (LCT) encompasses several well-known signal processing transforms as special cases including the Fourier transform, the fractional Fourier transform, the Fresnel transform, and even simple multiplication by quadratic phase factors [20]. As of now, the theory of linear canonical transforms has expanded into an independent and broad field of research with numerous applications to optics, mathematical physics, and signal and image processing. For more about LCT and its applications, the reader is referred to the monographs in [20–27].

Below, we shall present the formal definition of the two-dimensional LCT [25]. For notational convenience, we shall write a 2×2 matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ as $M = (A, B\colon C, D)$.

Definition 1. For any $f \in L^2(\mathbb{R}^2)$, the two-dimensional LCT with respect to a real, unimodular matrix $M = (A, B\colon C, D)$ is denoted by $\mathscr{L}_M[f]$ and is defined as

$$\mathscr{L}_M[f](\xi) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} f(\mathbf{t})\mathscr{K}_M(\mathbf{t},\xi)\mathrm{d}\mathbf{t}, & B \neq 0, \\ \displaystyle\sqrt{D}\,\exp\left\{\frac{iCD|\xi|^2}{2}\right\} f(D\xi), & B = 0, \end{cases} \tag{1}$$

where $\mathscr{K}_M(\mathbf{t},\xi)$, with $\mathbf{t} = (t_1, t_2)^T$ and $\xi = (\xi_1, \xi_2)^T$, denotes the kernel of the two-dimensional LCT and is given by

$$\mathscr{K}_M(\mathbf{t},\xi) = \frac{1}{2\pi B}\exp\left\{\frac{i\left(A|\mathbf{t}|^2 - 2\mathbf{t}^T\xi + D|\xi|^2\right)}{2B}\right\}, \quad B \neq 0. \tag{2}$$

It is pertinent to mention that, for the case $B = 0$, the two-dimensional LCT (1) corresponds to a chirp multiplication operation. Moreover, the case $B < 0$ is also of no particular interest to us. As such, in the rest of the article, we shall focus our attention on the case $B > 0$. We also note that the phase-space transform (1) is lossless if and only if the matrix M is unimodular; that is, $AD - BC = 1$. The inversion formula corresponding to the two-dimensional LCT (1) is given by

$$f(\mathbf{t}) = \mathscr{L}_M^{-1}\left(\mathscr{L}_M[f](\xi)\right)(\mathbf{t}) = \int_{\mathbb{R}^2} \mathscr{L}_M[f](\xi)\overline{\mathscr{K}_M(\mathbf{t},\xi)}\mathrm{d}\xi. \tag{3}$$

Also, Parseval's formula associated with (1) reads

$$\langle f, g\rangle_2 = \langle \mathscr{L}_M[f], \mathscr{L}_M[g]\rangle_2, \quad \forall f, g \in L^2\left(\mathbb{R}^2\right). \tag{4}$$

In the remaining part of this subsection, we shall present an analogue of the two-dimensional LCT using the polar coordinates. We emphasize that the polar LCT plays a key role in the development of the novel curvelet transform. For $\xi_1 = r\cos\omega, \xi_2 = r\sin\omega$ and $t_1 = \rho\cos\eta$, $t_2 = \rho\sin\eta$, where $r, \rho \geq 0$ and $\omega, \eta \in [0, 2\pi)$, the polar LCT is given by

$$\mathscr{L}_M[f](r,\omega) = \frac{1}{2\pi B}\int_0^{2\pi}\int_0^{\infty} f(\rho,\eta)\exp\left\{\frac{i\left(A\rho^2 + Dr^2 - 2\rho r\cos(\eta-\omega)\right)}{2B}\right\}\rho\, d\rho\, d\eta. \tag{5}$$

Also, the inversion formula corresponding to (5) is given by

$$f(\rho,\eta) = \frac{1}{2\pi B}\int_0^{2\pi}\int_0^{\infty} \mathscr{L}_M[f](r,\omega)\exp\left\{-\frac{i\left(A\rho^2 + Dr^2 - 2\rho r\cos(\eta-\omega)\right)}{2B}\right\} r\mathrm{d}r\mathrm{d}\omega. \tag{6}$$

Remarks 1. The aforementioned definitions (1) and (5) embody several well-known integral transforms, some of which are listed below:

(i) As a special case when $M = (0, 1: -1, 0)$, the LCT definitions (1) and (5) reduce to their respective counterparts of the Fourier transform

(ii) Plugging the matrix $M = (\cos\alpha, \sin\alpha: -\sin\alpha, \cos\alpha)$, $\alpha \neq n\pi, n \in \mathbb{Z}$ (1) and (5), yields the respective counterparts of the fractional Fourier transform

(iii) For the matrix $M = (1, B: 0, 1)$, $B \neq 0$, the LCT definitions (1) and (5) boil down to their analogues for the Fresnel transform

2.2. Ordinary Curvelet Transform. In this subsection, we shall recapitulate the mathematical frameworks of the classical curvelet transform, which serve as preliminaries for the development of the novel curvelet transform.

Consider the frequency plane $\mathbb{R}^2$ and let (r,ω), $r \geq 0, \omega \in [0, 2\pi)$, denote the polar coordinates of an arbitrary point $\xi \in \mathbb{R}^2$. We choose a pair of window functions $W: (0,\infty) \longrightarrow (0,\infty)$, called "radial window," and $V: (-\infty,\infty) \longrightarrow (0,\infty)$, called "angular window," satisfying the following admissibility conditions:

$$\int_0^{\infty} |W(r)|^2 \frac{\mathrm{d}r}{r} = 1, \operatorname{supp}(W) \subseteq \left(\frac{1}{2}, 2\right), \tag{7}$$

$$(2\pi)^2 \int_{-1}^{1} |V(\omega)|^2 \mathrm{d}\omega = 1, \ \operatorname{supp}(V) \subseteq [-1, 1]. \tag{8}$$

The window functions (7) and (8) are used to construct a family of complex-valued waveforms adopted to scale $a > 0$ location $\mathbf{b} \in \mathbb{R}^2$ and orientation $\theta \in [0, 2\pi)$ or $(-\pi, \pi)$ according to convenience. For a fixed scale $a \in (0, a_0)$ where $a_0 < \pi^2$ the basic curvelet $\Psi_a: \mathbb{R}^2 \longrightarrow \mathbb{C}$ is defined via the polar Fourier transform as

$$\mathscr{F}[\Psi_a](r,\omega) = a^{3/4} W(ar) V\left(\frac{\omega}{\sqrt{a}}\right), \tag{9}$$

where $\mathscr{F}$ denotes the well-known Fourier transform defined by

$$\mathscr{F}[f](\xi) = \frac{1}{2\pi}\int_{\mathbb{R}^2} f(\mathbf{t}) e^{-it^T\xi}\mathrm{d}\mathbf{t}, \tag{10}$$

which can be expressed via the polar coordinates as

$$\mathscr{F}[f](r,\omega) = \frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty} f(\rho,\eta)\exp\{-i\rho r\cos(\eta-\omega)\}\rho\mathrm{d}\rho\mathrm{d}\eta. \tag{11}$$

Consequently, the family of analyzing waveforms $\Psi_{a,\mathbf{b},\theta}(\mathbf{t})$ called curvelets is generated by translation and rotation of the basic element $\Psi_a(\mathbf{t})$; that is,

$$\Psi_{a,\mathbf{b},\theta}(\mathbf{t}) = \Psi_a\left(R_\theta(\mathbf{t}-\mathbf{b})\right), \quad \mathbf{t} \in \mathbb{R}^2, \tag{12}$$

where $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ denotes the 2×2 rotation matrix affecting the planar rotation by θ radians. From (9), we note that the support of the basic element Ψ_a in the frequency domain is a polar wedge governed by the respective supports of the radial and angular windows. The scaling in the radial and angular windows is parabolic in nature with ω being the "thin" variable. The coarsest scale a_0 is fixed once for all and must obey $a_0 < \pi^2$. These elements become increasingly needle-like at fine scales. Formally, we have the following definition of the ordinary curvelet transform [8, 9].

Definition 2. Given a function $f \in L^2(\mathbb{R}^2)$, the ordinary curvelet transform is defined as

$$[\Gamma_\Psi f](a,\mathbf{b},\theta) = \int_{\mathbb{R}^2} f(\mathbf{t})\overline{\Psi_{a,b,\theta}}(\mathbf{t})\mathrm{d}\mathbf{t}, \tag{13}$$

where $a < a_0, \mathbf{b} \in \mathbb{R}^2$, $\theta \in [0, 2\pi)$, and $\Psi_{a,\mathbf{b},\theta}(\mathbf{t})$ is given by (12).

3. Novel Curvelet Transform

In this section, our aim is to introduce the notion of the novel curvelet transform and formulate the associated reconstruction formula and Rayleigh's energy theorem. Subsequently, we shall also study the support and oscillation properties of the proposed novel curvelet transform.

For a fixed scale $a \in (0, a_0)$ where $a_0 < \pi^2$, consider a basic waveform $\Psi_a: \mathbb{R}^2 \longrightarrow \mathbb{C}$ defined via the polar LCT (5) as

$$\mathscr{L}_M[\Psi_a](r,\omega) = a^{3/4} W(ar) V\left(\frac{\omega}{\sqrt{a}}\right), \tag{14}$$

where the radial and angular windows $W(r)$ and $V(\omega)$ satisfy the slightly modified set of admissibility conditions given by

$$B^2\int_0^\infty |W(r)|^2\frac{\mathrm{d}r}{r}=1,\ \mathrm{supp}(W)\subseteq\left(\frac{1}{2},2\right), \tag{15}$$

$$(2\pi)^2\int_{-1}^{1}|V(\omega)|^2\mathrm{d}\omega=1,\ \mathrm{supp}(V)\subseteq[-1,1]. \tag{16}$$

Applying the inverse LCT (6) on both sides of the expression (14), we have

$$\begin{aligned}\Psi_a(\rho,\eta)&=\frac{a^{3/4}}{2\pi B}\int_0^{2\pi}\int_0^\infty W(ar)V\left(\frac{\omega}{\sqrt{a}}\right)\exp\left\{-\frac{i\left(A\rho^2+Dr^2-2\rho r\cos(\eta-\omega)\right)}{2B}\right\}r\mathrm{d}r\mathrm{d}\omega,\\&=\frac{a^{3/4}}{2\pi B}\exp\left\{-\frac{iA\rho^2}{2B}\right\}\int_0^{2\pi}\int_0^\infty\exp\left\{-\frac{iDr^2}{2B}\right\}W(ar)V\left(\frac{\omega}{\sqrt{a}}\right)\times\exp\left\{\frac{i\rho r\cos(\eta-\omega)}{B}\right\}r\mathrm{d}r\mathrm{d}\omega.\end{aligned} \tag{17}$$

and upon simplifying (17), we obtain a novel basic waveform $\Psi_a^M(\mathbf{t})$ via the following expression:

$$\mathcal{F}\left[\Psi_a^M\right](r,\omega)=a^{3/4}B\exp\left\{-\frac{iDBr^2}{2}\right\}W(aBr)V\left(\frac{\omega}{\sqrt{a}}\right), \tag{18}$$

where $\Psi_a^M(\mathbf{t})=\exp\left\{iA|\mathbf{t}|^2/2B\right\}\Psi_a(\mathbf{t})$.

Hence, the family of novel curvelets $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ (or linear canonical curvelets) is obtained by translating the basic waveform $\Psi_a^M(\mathbf{t})$ by $\mathbf{b}\in\mathbb{R}^2$ and then inducing a rotation of $\theta\in[0,2\pi)$ radians; that is,

$$\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})=\Psi_a^M\left(R_\theta(\mathbf{t}-\mathbf{b})\right),\quad \mathbf{t}\in\mathbb{R}^2. \tag{19}$$

Having formulated a new family of curvelets $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ by invoking the two-dimensional linear canonical transform (1), we are ready to introduce the formal definition of the novel curvelet transform.

Definition 3. Given a real, unimodular matrix $M=(A,B:C,D)$ with $B>0$, for any square-integrable function f on $\mathbb{R}^2$, the novel curvelet transform is defined as

$$\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)=\langle f,\Psi_{a,\mathbf{b},\theta}^M\rangle_2=\int_{\mathbb{R}^2}f(\mathbf{t})\overline{\Psi_{a,\mathbf{b},\theta}^M}(\mathbf{t})\mathrm{d}t, \tag{20}$$

where $a<a_0,\mathbf{b}\in\mathbb{R}^2,\theta\in[0,2\pi)$, and $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ is given by (19).

Definition 3 embodies many new integral transforms that are yet to be reported in the open literature. Below we point out some important deductions.

(i) Choosing the matrix $M=(\cos\alpha,\sin\alpha:-\sin\alpha,\cos\alpha)$, $\alpha\neq n\pi, n\in\mathbb{Z}$, Definition 3 yields a new curvelet transform combining the merits of the ordinary curvelet transform and the well-known fractional Fourier transform

(ii) For $M=(1,B:0,1)$, $B\neq 0$, Definition 3 intertwines the advantages of the ordinary curvelet and the well-known Fresnel transforms into a new curvelet transform

(iii) Nevertheless, when $M=(0,1:-1,0)$, Definition 3 boils down to the ordinary curvelet transform (13)

Next, we shall present a proposition that interlinks the Fourier transform of the novel curvelet transform $[\Gamma_\Psi^M f](a,\mathbf{b},\theta)$ as a function of the translation variable $\mathbf{b}$, with the respective Fourier transforms of the given function f and the basic waveform Ψ_a^M.

Proposition 1. *Given any $f\in L^2(\mathbb{R}^2)$, the novel curvelet transform $[\Gamma_\Psi^M f](a,\mathbf{b},\theta)$ defined in (20) can be expressed as*

$$\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right)(\xi)=2\pi\mathcal{F}[f](\xi)\mathcal{F}\left[\Psi_a^M\right](R_\theta\xi). \tag{21}$$

Proof. To accomplish the motive, we shall firstly compute the Fourier transform of the novel curvelet family $\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})$ defined in (19). We proceed as

$$\begin{aligned}\mathcal{F}\left[\Psi_{a,\mathbf{b},\theta}^M\right](\xi)&=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Psi_{a,\mathbf{b},\theta}^M(\mathbf{t})e^{-it^T\xi}\mathrm{d}\mathbf{t}\\&=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Psi_a^M\left(R_\theta(\mathbf{t}-\mathbf{b})\right)e^{-it^T\xi}\mathrm{d}\mathbf{t}\\&=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Psi_a^M(\mathbf{z})e^{-i\left(b+R_{-\theta}z\right)^T\xi}\mathrm{d}\mathbf{z}\\&=\frac{e^{-ib^T\xi}}{2\pi}\int_{\mathbb{R}^2}\Psi_a^M(z)e^{-iz^T\left(R_\theta\xi\right)}\mathrm{d}\mathbf{z}\\&=e^{-ib^T\xi}\mathcal{F}\left[\Psi_a^M\right](R_\theta\xi).\end{aligned} \tag{22}$$

Let (σ,μ), (ρ,η), and (r,ω) denote the polar coordinates of the variables $\mathbf{b}$, $\mathbf{t}$, and ξ, respectively. Then, we can rewrite (22) as follows:

$$\begin{aligned}\mathscr{F}\left[\Psi_{a,b,\theta}^{M}\right](r,\omega) &= e^{-ir\sigma\cos(\mu-\omega)}\mathscr{F}\left[\Psi_{a}^{M}\right](r,\omega-\theta)\\ &= e^{-ir\sigma\cos(\mu-\omega)}a^{3/4}B\exp\left\{-\frac{i\,DB\,r^{2}}{2}\right\}\\ &\quad W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right).\end{aligned} \tag{23}$$

Finally, using Definition 3 and invoking the well-known Parseval's formula in polar coordinates, we have

$$\begin{aligned}\left[\Gamma_{\Psi}^{M}f\right](a,\mathbf{b},\theta) &= \langle f,\Psi_{a,b,\theta}^{M}\rangle_{2}\\ &= a^{3/4}B\int_{0}^{2\pi}\int_{0}^{\infty}e^{ir\sigma\cos(\mu-\omega)}\mathscr{F}[f](r,\omega)\exp\left\{\frac{i\,DB\,r^{2}}{2}\right\}W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right)r\mathrm{d}r\mathrm{d}\omega,\end{aligned} \tag{24}$$

Next, translating the expression (24) into cartesian coordinates yields the following:

$$\begin{aligned}\left[\Gamma_{\Psi}^{M}f\right](a,\mathbf{b},\theta) &= \int_{\mathbb{R}^{2}}\mathscr{F}[f](\xi)\overline{\mathscr{F}\left[\Psi_{a}^{M}\right]}(R_{\theta}\xi)e^{ib^{T}\xi}\mathrm{d}\xi\\ &= 2\pi\mathscr{F}^{-1}\left(\mathscr{F}[f](\xi)\overline{\mathscr{F}\left[\Psi_{a}^{M}\right]}(R_{\theta}\xi)\right)(\mathbf{b}),\end{aligned} \tag{25}$$

Applying the Fourier transform on both sides of (25), we obtain the desired result

$$\mathscr{F}\left(\left[\Gamma_{\Psi}^{M}f\right](a,\mathbf{b},\theta)\right)(\xi) = 2\pi\mathscr{F}[f](\xi)\mathscr{F}\left[\Psi_{a}^{M}\right](R_{\theta}\xi). \tag{26}$$

This completes the proof of Proposition 1.

Next, we shall analyze the support and oscillatory behaviour of the novel curvelet transform by invoking Proposition 1. We shall demonstrate that the proposed transform enjoys a certain degree of freedom as the radial window is comparatively more flexible with the degree of flexibility governed by the matrix parameter B. As such, the proposed transform is capable of optimizing the concentration of the curvelet spectrum.

Let $(\sigma,\mu),\sigma\geq 0,\mu\in[0,2\pi)$ be the polar coordinates of the translation variable $\mathbf{b}$. Then, as a consequence of Proposition 1, we can express the novel curvelet transform (20) as

$$\begin{aligned}\left[\Gamma_{\Psi}^{M}f\right](a,\mathbf{b},\theta) &= \left[\Gamma_{\Psi}^{M}f\right](a,(\sigma,\mu),\theta)\\ &= a^{3/4}B\int_{0}^{2\pi}\int_{0}^{\infty}e^{ir\sigma\cos(\mu-\omega)}\mathscr{F}[f](r,\omega)\exp\left\{\frac{i\,DB\,r^{2}}{2}\right\}W(aBr)V\left(\frac{\omega-\theta}{\sqrt{a}}\right)r\mathrm{d}r\mathrm{d}\omega.\end{aligned} \tag{27}$$

From (23), we observe that the support of the analyzing elements $\Psi_{a,\mathbf{b},\theta}^{M}$ in the frequency domain is completely determined by the support of the radial window $W(aBr)$ and the angular window $V(\omega-\theta/\sqrt{a})$. Moreover, we observe that

$$\begin{aligned}\operatorname{supp}(W(aBr)) &\subseteq \left(\frac{1}{2aB},\frac{2}{aB}\right)\text{and}\operatorname{supp}\left(V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right)\\ &\subseteq[-\sqrt{a}+\theta,\sqrt{a}+\theta].\end{aligned} \tag{28}$$

Hence, we conclude that the support of the analyzing elements $\Psi_{a,\mathbf{b},\theta}^{M}$ in the frequency domain depends upon the choice of the matrix parameter B and is completely independent of the translation parameter $\mathbf{b}$. Therefore, an appropriate matrix parameter B can be chosen to optimize the concentration of novel curvelet spectrum.

On the other hand, since the curvelet functions $\Psi_{a,\mathbf{b},\theta}^{M}$ have compact support in the frequency domain, the well-known Heisenberg's uncertainty principle implies that the novel curvelet functions cannot have compact support in the time domain. We note that, for large $|\mathbf{t}|$, the decay of the novel curvelet functions $\Psi_{a,\mathbf{b},\theta}^{M}(\mathbf{t})$ depends upon the smoothness of the corresponding Fourier transform; the smoother $\mathscr{F}[\Psi_{a,\mathbf{b},\theta}^{M}](\xi)$ is, the faster the decay is. Moreover, by definition, $\mathscr{F}[\Psi_{a}^{M}](\xi)$ is supported away from the vertical axis $\xi_{1}=0$ but near the horizontal axis $\xi_{2}=0$. Hence, for smaller values of $a<a_{0}$, the basic waveform $\Psi_{a}^{M}(\mathbf{t})$ is less oscillatory in t_{2} direction and more oscillatory in t_{1} direction.

Below, we shall present the formal reconstruction formula associated with the novel curvelet transform. We note that the said reconstruction formula is valid for high-frequency signals. The analogue for low-frequency signals will be dealt with afterwards. To facilitate the narrative, we need the following definition. □

Definition 4. Given any two functions $f,g\in L^{2}(\mathbb{R}^{2})$, the convolution operation is denoted by $\circledast$ and is defined as

$$(f \circledast g)(\mathbf{z}) = \int_{\mathbb{R}^2} f(\mathbf{t}) g(\mathbf{z}-\mathbf{t}) \mathrm{d}t, \tag{29}$$

Moreover, the convolution theorem corresponding to (29) reads

$$\mathcal{F}[f \circledast g](\xi) = 2\pi \mathcal{F}[f](\xi) \mathcal{F}[g](\xi). \tag{30}$$

Theorem 1 (Reconstruction Formula). *For any $f \in L^2(\mathbb{R}^2)$ satisfying $\mathcal{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, a_0 < \pi^2$, the reconstruction formula for the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in (20) is given by*

$$f(\mathbf{t}) = \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} [\Gamma_\Psi^M f](a, b, \theta) \Psi_{a,b,\theta}^M(t) \frac{\mathrm{d}a \mathrm{d}b \mathrm{d}\theta}{a^3}, \tag{31}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (15) and (16).

Proof. We note that the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in (20) can be expressed via the convolution $\circledast$ as follows:

$$\begin{aligned}\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) &= \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_a^M}(R_\theta(\mathbf{t}-\mathbf{b})) \mathrm{d}\mathbf{t} \\ &= \int_{\mathbb{R}^2} f(\mathbf{t}) \overline{\Psi_{a,0,\theta}^M}(-(\mathbf{b}-\mathbf{t})) \mathrm{d}\mathbf{t} \\ &= \left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b}), \widetilde{\Psi}^M(\mathbf{t}) = \overline{\Psi^M}(-\mathbf{t}).\end{aligned} \tag{32}$$

Next, we define a function

$$F_{a,\theta}^M(\mathbf{t}) = \int_{\mathbb{R}^2} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,b,\theta}^M(\mathbf{t}) \mathrm{d}\mathbf{b}, \tag{33}$$

Invoking (32), we can express (33) as follows:

$$F_{a,\theta}^M(\mathbf{t}) = \left(\left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)(\mathbf{b}) \circledast \Psi_{a,0,\theta}^M(\mathbf{b})\right)(\mathbf{t}). \tag{34}$$

Applying the convolution theorem (30), we can compute the Fourier transform of the function $F_{a,\theta}(\mathbf{t})$ as

$$\begin{aligned}\mathcal{F}\left[F_{a,\theta}^M\right](\xi) &= 2\pi \mathcal{F}\left[\left(f \circledast \widetilde{\Psi}_{a,0,\theta}^M\right)\right](\xi) \mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi) \\ &= (2\pi)^2 \mathcal{F}[f](\xi) \left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2.\end{aligned} \tag{35}$$

Consequently, we have

$$\int_0^{a_0} \int_0^{2\pi} \mathcal{F}\left[F_{a,\theta}^M\right](\xi) \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^3} = (2\pi)^2 \mathcal{F}[f](\xi) \int_0^{a_0} \int_0^{2\pi} \left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](\xi)\right|^2 \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^3}. \tag{36}$$

Next, we shall evaluate the integral on the right-hand side of (36). To do so, we shall use the polar coordinates of ξ and invoke the admissibility conditions (15) and (16). For $r \geq 2/a_0 B, a_0 < \pi^2$, we have

$$\begin{aligned}&(2\pi)^2 \int_0^{a_0} \int_0^{2\pi} \left|\mathcal{F}\left[\Psi_{a,0,\theta}^M\right](r, \omega)\right|^2 \frac{\mathrm{d}\theta \mathrm{d}a}{a^3} \\ &= (2\pi B)^2 \int_0^{a_0} \int_0^{2\pi} |W(aBr)|^2 \left|V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right|^2 \frac{\mathrm{d}\theta \, \mathrm{d}a}{a^{3/2}} \\ &= B^2 \int_0^{a_0} |W(aBr)|^2 \left\{(2\pi)^2 \int_0^{2\pi} \left|V\left(\frac{\omega-\theta}{\sqrt{a}}\right)\right|^2 \mathrm{d}\theta\right\} \frac{\mathrm{d}a}{a^{3/2}} \\ &= B^2 \int_0^{a_0} |W(aBr)|^2 \frac{\mathrm{d}a}{a} \\ &= B^2 \int_0^{a_0 Br} |W(r)'|^2 \frac{\mathrm{d}r'}{r'} = 1.\end{aligned} \tag{37}$$

Implementing (37) in (36), we obtain

$$\mathcal{F}[f](\xi) = \int_0^{a_0} \int_0^{2\pi} \mathcal{F}\left[F_{a,\theta}^M\right](\xi) \frac{\mathrm{d}\theta \mathrm{d}a}{a^3}. \tag{38}$$

That is,

$$f(\mathbf{t}) = \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,b,\theta}^M(\mathbf{t}) \frac{\mathrm{d}a \mathrm{d}\mathbf{b} \mathrm{d}\theta}{a^3}. \tag{39}$$

This completes the proof of Theorem 1. □

Theorem 2 (Rayleigh's Energy Formula). *For any $f \in L^2(\mathbb{R}^2)$ satisfying $\mathcal{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, \quad a_0 < \pi^2$, we have*

$$\left\|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right\|_2^2 = \|f\|_2^2. \tag{40}$$

That is, the total energy of the signal is preserved from the natural domain $L^2(\mathbb{R}^2)$ to transformed domain $L^2((0, a_0) \times \mathbb{R}^2 \times [0, 2\pi))$, where $a_0 < \pi^2$.

Proof. Invoking the well-known Parseval's formula and using (30), we have

$$\begin{aligned}
\left\|\left[\Gamma_{\Psi}^{M} f\right](a,\mathbf{b},\theta)\right\|_2^2 &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\left[\Gamma_{\Psi}^{M} f\right](a,\mathbf{b},\theta)\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\
&= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\left(f\circledast\widetilde{\Psi}_{a,0,\theta}^{M}\right)(\mathbf{b})\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\
&= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left|\mathcal{F}\left[\left(f\circledast\widetilde{\Psi}_{a,0,\theta}^{M}\right)\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\
&= (2\pi)^2\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\mathcal{F}[f](\xi)|^2\left|\mathcal{F}\left[\Psi_{a,0,\theta}^{M}\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\
&= \int_{\mathbb{R}^2}|\mathcal{F}[f](\xi)|^2\left\{(2\pi)^2\int_0^{a_0}\int_0^{2\pi}\left|\mathcal{F}\left[\Psi_{a,0,\theta}^{M}\right](\xi)\right|^2\frac{\mathrm{d}\theta\mathrm{d}a}{a^3}\right\}\mathrm{d}\xi\\
&= \int_{\mathbb{R}^2}|\mathcal{F}[f](\xi)|^2\mathrm{d}\xi = \|f\|_2^2,
\end{aligned} \tag{41}$$

which evidently completes the proof. □

Remark 2. From (40), we infer that the novel curvelet transform defined in (20) is an isometry from the space of signals $L^2(\mathbb{R}^2)$ to the space of transforms $L^2((0,a_0)\times\mathbb{R}^2\times[0,2\pi))$, where $a_0<\pi^2$.

We note that the reconstruction formula (31) is concerned for those signals $f\in L^2(\mathbb{R}^2)$ satisfying $\mathcal{F}[f](\xi)=0, \forall|\xi|<2/a_0B, a_0<\pi^2$. In order to have a complete reconstruction formula, we need to take care of the other frequency components as well. To facilitate the narrative, we consider an arbitrary square integrable function f on $\mathbb{R}^2$ and define

$$\begin{aligned}
(T_1 f)(\mathbf{t}) &= \int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}\left[\Gamma_{\Psi}^{M} f\right](a,\mathbf{b},\theta)\Psi_{a,\mathbf{b},\theta}^{M}(\mathbf{t})\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\\
&= \int_0^{2\pi}\int_0^{a_0}\left(\left(f\circledast\widetilde{\Psi}_{a,0,\theta}^{M}\right)(\mathbf{b})\circledast\Psi_{a,0,\theta}^{M}(\mathbf{b})\right)(\mathbf{t})\frac{\mathrm{d}a\mathrm{d}\theta}{a^3},
\end{aligned} \tag{42}$$

$$(T_0 f)(\mathbf{t}) = f(\mathbf{t}) - (T_1 f)(\mathbf{t}). \tag{43}$$

Here, we note that

$$\begin{aligned}
\mathcal{F}[(T_1 f)](\xi) &= (2\pi)^2\int_0^{2\pi}\int_0^{a_0}\mathcal{F}[f](\xi)\left|\mathcal{F}\left[\Psi_{a,0,\theta}^{M}\right](\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\theta}{a^3}\\
&= B^2\mathcal{F}[f](\xi)\int_0^{a_0B|\xi|}|W(a)|^2\frac{\mathrm{d}a}{a}\\
&= (2\pi)^2\mathcal{F}[f](\xi)\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2,
\end{aligned} \tag{44}$$

where $(\mathcal{F}[\Omega^M](\xi))^2 = B^2/(2\pi)^2\int_0^{a_0B|\xi|}|W(a)|^2\mathrm{d}a/a$.

Furthermore, using additivity of the Fourier transform, we observe that

$$\begin{aligned}
\mathcal{F}[(T_0 f)](\xi) &= \mathcal{F}[f](\xi) - \mathcal{F}[(T_1 f)](\xi)\\
&= \mathcal{F}[f](\xi)\left(1-(2\pi)^2\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2\right)\\
&= (2\pi)^2\mathcal{F}[f](\xi)\left(\frac{1}{(2\pi)^2}-\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2\right)\\
&= (2\pi)^2\mathcal{F}[f](\xi)\left(\mathcal{F}\left[\Phi^M\right](\xi)\right)^2,
\end{aligned} \tag{45}$$

where $(\mathcal{F}[\Phi^M](\xi))^2 = 1/(2\pi)^2 - (\mathcal{F}[\Omega^M](\xi))^2$.

Also, thanks to the convolution theorem (30), we infer from (44) and (45) that

$$\begin{aligned}
(T_0 f)(\mathbf{t}) &= \left(f\circledast\Phi^M\circledast\Phi^M\right)(\mathbf{t}),\\
(T_1 f)(\mathbf{t}) &= \left(f\circledast\Omega^M\circledast\Omega^M\right)(\mathbf{t}).
\end{aligned} \tag{46}$$

Moreover, we note that

$$(2\pi)^2\left[\left(\mathcal{F}\left[\Omega^M\right](\xi)\right)^2+\left(\mathcal{F}\left[\Phi^M\right](\xi)\right)^2\right] = 1. \tag{47}$$

Also,

$$\begin{aligned}
\mathcal{F}\left[\Phi^M\right](\xi) &= 0, \quad |\xi|>\frac{2}{a_0B},\\
\mathcal{F}\left[\Phi^M\right](\xi) &= \frac{1}{2\pi}, \quad |\xi|<\frac{1}{2a_0B}.
\end{aligned} \tag{48}$$

Finally, we define the father wavelet $\Phi_{\mathbf{b}}^M(\mathbf{t}) = \Phi^M(\mathbf{t}-\mathbf{b})$, so that

$$(T_0 f)(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) \mathrm{d}\mathbf{b}. \tag{49}$$

Consequently, (43) implies that

$$f(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) \mathrm{d}\mathbf{b} + \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta) \Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) \frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}. \tag{50}$$

Therefore, we conclude that the complete reconstruction formula for the novel curvelet transform (20) is composed of both curvelet waveforms and isotropic father wavelets. The above discussion can be summarized into the following theorem:

Theorem 3 (Complete Reconstruction Formula). *For any $f \in L^2(\mathbb{R}^2)$, the reproducing formula for the novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ defined in [20] is given by*

$$f(\mathbf{t}) = \int_{\mathbb{R}^2} \langle f, \Phi_{\mathbf{b}}^M \rangle_2 \Phi_{\mathbf{b}}^M(\mathbf{t}) \mathrm{d}\mathbf{b} + \int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} [\Gamma_\Psi^M f](a, \mathbf{b}, \theta) \Psi_{a,\mathbf{b},\theta}^M(\mathbf{t}) \frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}, \tag{51}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (15) and (16).

The classical Heisenberg's uncertainty principle in harmonic analysis gives information about the spread of a signal and its Fourier transform by asserting that a signal cannot be sharply localized in both the time and frequency domains [29]. That is, if we limit the behaviour of one, we lose control over the other. The essence of the uncertainty principle is that it provides a lower bound for optimal resolution of a signal in both the time and frequency domains. This classical uncertainty inequality has been extended in different settings and, as of now, many analogues have appeared in the literature [28–31]. In analogy to the uncertainty principles governing the simultaneous localization of a function f and its Fourier transform, a different class of uncertainty principles comparing the localization of f with the localization of its Gabor or wavelet transform were studied by Wilczok [28]. Motivated by this fact, we shall also obtain an uncertainty inequality comparing the localization of the Fourier transform of a function f with the corresponding novel curvelet transform $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$, regarded as a function of the translation variable $\mathbf{b}$.

Theorem 4 (Heisenberg-Type Uncertainty Principle). *If $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ is the novel curvelet transform of any nontrivial function $f \in L^2(\mathbb{R}^2)$, satisfying $\mathcal{F}[f](\xi) = 0, \forall |\xi| < 2/a_0 B, a_0 < \pi^2$, the following uncertainty inequality holds:*

$$\left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 \frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}|\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 |\mathcal{F}[f](\xi)|^2 \mathrm{d}\xi\right\}^{1/2} \geq \frac{1}{2} \|f\|_2^2. \tag{52}$$

Proof. The classical Heisenberg-Pauli-Weyl inequality is given by [29]

$$\left\{\int_{\mathbb{R}^2} |\mathbf{t}|^2 |f(\mathbf{t})|^2 \mathrm{d}t\right\}^{1/2} \left\{\int_{\mathbb{R}^2} \xi^2 |\mathcal{F}[f](\xi)|^2 \mathrm{d}\xi\right\}^{1/2} \geq \frac{1}{2} \left\{\int_{\mathbb{R}^2} |f(\mathbf{t})|^2 \mathrm{d}\mathbf{t}\right\}. \tag{53}$$

Identifying $[\Gamma_\Psi^M f](a, \mathbf{b}, \theta)$ as a function of the translation variable $\mathbf{b}$ and invoking (53), we have

$$\left\{\int_{\mathbb{R}^2} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 \mathrm{d}\mathbf{b}\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 \mathrm{d}\xi\right\}^{1/2} \geq \frac{1}{2} \left\{\int_{\mathbb{R}^2} \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 \mathrm{d}\mathbf{b}\right\}. \tag{54}$$

Integrating (54) with respect to the measure $\mathrm{d}a\mathrm{d}\theta/a^3$, we obtain

$$\int_0^{2\pi} \int_0^{a_0} \left\{\int_{\mathbb{R}^2} |\mathbf{b}|^2 \left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)|^2 \mathrm{d}\mathbf{b}\right\}^{1/2} \cdot \left\{\int_{\mathbb{R}^2} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 \mathrm{d}\xi\right\}^{1/2} \frac{\mathrm{d}a\mathrm{d}\theta}{a^3} \geq \frac{1}{2} \left\{\int_0^{2\pi} \int_0^{a_0} \int_{\mathbb{R}^2} \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 \frac{\mathrm{d}\mathbf{b}\mathrm{d}a\mathrm{d}\theta}{a^3}\right\}. \tag{55}$$

As a consequence of the Cauchy-Schwartz's inequality, Fubini's theorem, and (40), the above inequality can be expressed as

$$\left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\mathbf{b}|^2 \left|\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right|^2 \frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\right\}^{1/2} \times \left\{\int_0^{2\pi} \int_{\mathbb{R}^2} \int_0^{a_0} |\xi|^2 \left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a, \mathbf{b}, \theta)\right)(\xi)\right|^2 \frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\right\}^{1/2} \geq \frac{1}{2} \|f\|_2^2. \tag{56}$$

Invoking (21) and noting that $f \in L^2(\mathbb{R}^2)$ satisfies $\mathcal{F}[f](\xi) = 0, \forall r < 2/a_0 B, a_0 < \pi^2$, we have

$$\begin{aligned}
&\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\xi|^2\left|\mathcal{F}\left(\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right)(\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\
&=(2\pi)^2\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\xi|^2|\mathcal{F}[f](\xi)|^2\left|\mathcal{F}\left[\Psi_a^M\right](R_\theta\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\xi\mathrm{d}\theta}{a^3}\\
&=(2\pi)^2\int_{\mathbb{R}^2}|\xi|^2|\mathcal{F}[f](\xi)|^2\left\{\int_0^{2\pi}\int_0^{a_0}\left|\mathcal{F}\left[\Psi_a^M\right](R_\theta\xi)\right|^2\frac{\mathrm{d}a\mathrm{d}\theta}{a^3}\right\}\mathrm{d}\xi\\
&=\int_{\mathbb{R}^2}|\xi|^2|\mathcal{F}[f](\xi)|^2\left\{B^2\int_0^{a_0}|W(aBr)|^2\frac{\mathrm{d}a}{a}\right\}\mathrm{d}\xi\\
&=\int_{\mathbb{R}^2}|\xi|^2|\mathcal{F}[f](\xi)|^2\left\{B^2\int_0^{a_0Br}W(r')|^2\frac{\mathrm{d}r'}{r'}\right\}\mathrm{d}\xi\\
&=\int_{\mathbb{R}^2}|\xi|^2|\mathcal{F}[f](\xi)|^2\mathrm{d}\xi.
\end{aligned}\tag{57}$$

Plugging (57) in (56), we obtain the desired Heisenberg-type uncertainty inequality as

$$\begin{aligned}
&\left\{\int_0^{2\pi}\int_{\mathbb{R}^2}\int_0^{a_0}|\mathbf{b}|^2\left|\left[\Gamma_\Psi^M f\right](a,\mathbf{b},\theta)\right|^2\frac{\mathrm{d}a\mathrm{d}\mathbf{b}\mathrm{d}\theta}{a^3}\right\}^{1/2}\\
&\cdot\left\{\int_{\mathbb{R}^2}|\xi|^2|\mathcal{F}[f](\xi)|^2\mathrm{d}\xi\right\}^{1/2}\geq\frac{1}{2}\|f\|_2^2.
\end{aligned}\tag{58}$$

This completes the proof of Theorem 4. □

4. Novel Semidiscrete and Discrete Curvelet Transforms

In this section, our main aim is to study both the semidiscrete and discrete analogues of the proposed novel curvelet transform defined in [20]. In the beginning of the section, we formulate the definition of the novel semidiscrete curvelet transform, wherein the spatial variable $\mathbf{b}$ is continuous, whereas the scalings and orientations vary over a discrete grid. In the sequel, we obtain a reconstruction formula associated with the novel semidiscrete curvelet transform. Towards the culmination, we introduce the notion of the novel discrete curvelet transform by extending the aforementioned discretization to the spatial variable $\mathbf{b}$.

4.1. Novel Semidiscrete Curvelet Transform. To formulate the semidiscrete analogue of the proposed transform (20), we shall discretize the scaling parameter a and the rotation parameter θ in the following manner:

(i) For $\lambda>1$, we choose the jth scale as $a_j=\lambda^{-j}$, $j\geq 0$, and $j\in\mathbb{Z}$.

(ii) For a fixed $L_0\in\mathbb{Z}$, we sample the rotation parameter θ into L_0 equispaced pieces as

$$\theta_\ell=\frac{2\pi\ell}{L_0},\quad \text{where } \ell\in\mathbb{Z}_{L_0}=\{0,1,2,\ldots,L_0-1\}.\tag{59}$$

To prevent the expansion of the angular part as the radial parameter moves away from origin, it is desirable to make the spacing between the consecutive angles scale-dependent. As such, we choose $L_0=\lambda^{\lfloor j/2\rfloor}$, where $\lfloor j/2\rfloor$ denotes the integer part of $\lfloor j/2\rfloor$. Consequently, the scale-dependent angular discretization is given below:

$$\theta_{\ell_j}=\frac{2\pi\ell}{\lambda^{\lfloor j/2\rfloor}},\quad \text{where } \ell\in\mathbb{Z}_{\lambda^{\lfloor j/2\rfloor}}=\left\{0,1,2,\ldots,\lambda^{\lfloor j/2\rfloor}-1\right\}.\tag{60}$$

Now, for a given unimodular matrix $M=(A,B\colon C,D)$, with $B>0$, the radial and angular windows W and V are chosen to satisfy the discrete admissibility conditions:

$$B^2\sum_{j=-\infty}^{\infty}\left|W\left(\lambda^j r\right)\right|^2=1,\quad \lambda>1, r>0,\tag{61}$$

$$(2\pi)^2\sum_{\ell=-\infty}^{\infty}|V(y-\ell)|^2=1,\quad y\in\mathbb{R}.\tag{62}$$

Having discretized the scale and angular parameters, we define a semidiscrete family of linear canonical curvelets as

$$\Psi_{j,b,\ell}^M(\mathbf{t})=\Psi_j^M\left(R_{\theta_{\ell_j}}(\mathbf{t}-\mathbf{b})\right),\quad t\in\mathbb{R}^2,\tag{63}$$

where the novel basic waveform $\Psi_j^M(\mathbf{t})$ is defined in the polar coordinate setting as

$$\begin{aligned}
\mathcal{F}\left[\Psi_j^M\right](r,\omega)&:=\lambda^{-3j/4}B\exp\left\{-\frac{iDBr^2}{2}\right\}W\left(\frac{Br}{\lambda^j}\right)V\left(\frac{\omega}{\theta_{1_j}}\right)\\
&=\lambda^{-3j/4}B\exp\left\{-\frac{iDBr^2}{2}\right\}W\left(\frac{Br}{\lambda^j}\right)V\left(\frac{\lambda^{\lfloor j/2\rfloor}\omega}{2\pi}\right),
\end{aligned}\tag{64}$$

with $\Psi_j^M(\mathbf{t})=\exp\{iA|\mathbf{t}|^2/2B\}\Psi_j(\mathbf{t})$.

With the semidiscrete family of novel curvelets $\Psi_{j,\mathbf{b},\ell}^M(\mathbf{t})$ at hand, we have the following definition.

Definition 5. Given a real, unimodular matrix $M=(A,B\colon C,D)$, with $B>0$, the novel semidiscrete curvelet transform corresponding to any $f\in L^2(\mathbb{R}^2)$ is defined as

$$\left[\Gamma_\Psi^M f\right](j,\mathbf{b},\ell)=\langle f,\Psi_{j,\mathbf{b},\ell}^M\rangle_2=\int_{\mathbb{R}^2}f(\mathbf{t})\overline{\Psi_{j,\mathbf{b},\ell}^M}(\mathbf{t})\mathrm{d}\mathbf{t},\tag{65}$$

where the novel semidiscrete family $\Psi_{j,\mathbf{b},\ell}^M(\mathbf{t})$ is given by (63).

We now intend to establish a reconstruction formula associated with the novel semidiscrete curvelet transform defined in (65).

Theorem 5 (Reconstruction Formula). *For any $f\in L^2(\mathbb{R}^2)$ satisfying $\mathcal{F}[f](\xi)=0,\forall|\xi|<2/a_0B, a_0<\pi^2$, we have*

$$f(\mathbf{t})=\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{j/2-1}}\int_{\mathbb{R}^2}\left[\Gamma_\Psi^M f\right](j,b,\ell)\Psi_{j,b,\ell}^M(\mathbf{t})\frac{\mathrm{d}\mathbf{b}}{\lambda^{-3j/2}},\tag{66}$$

where the radial and angular windows W and V satisfy their respective admissibility conditions (61) and (62).

Proof. For $\lambda > 1$, we define the function

$$F_{j,\ell}^{M}(\mathbf{t}) = \int_{\mathbb{R}^2} \left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\Psi_{j,\mathbf{b},\ell}^{M}(\mathbf{t})\frac{\mathrm{d}\mathbf{b}}{\lambda^{-3j/2}}. \tag{67}$$

Then, we observe that

$$\begin{aligned}\mathcal{F}\left[F_{j,\ell}^{M}\right](\xi) &= (2\pi)^2\mathcal{F}[f](\xi)\left|\mathcal{F}\left[\Psi_{j,0,\ell}^{M}\right](\xi)\right|^2\\ &= (2\pi B)^2\mathcal{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &\quad\cdot\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2,\end{aligned} \tag{68}$$

Noting that $\mathcal{F}[f](\xi) = 0, \forall|\xi| < 2/a_0B, a_0 < \pi^2$, and invoking the admissibility condition (61), we have

$$\begin{aligned}B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2 &= B^2\sum_{j=-\infty}^{-1}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2 + B^2\sum_{j=0}^{\infty}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &= B^2\sum_{j=-\infty}^{\infty}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &= B^2\sum_{j=-\infty}^{\infty}\left|W\left(\lambda^j Br\right)\right|^2 = 1.\end{aligned} \tag{69}$$

Invoking the admissibility condition (62) yields

$$\begin{aligned}&(2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2,\\ &= (2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\omega}{2\pi}-\ell\right)\right|^2,\\ &= (2\pi)^2\sum_{\ell=-\left(\lambda^{j/2}/2\right)}^{\left(\lambda^{\lfloor j/2\rfloor}/2\right)-1}|V(y-\ell)|^2 = 1,\end{aligned} \tag{70}$$

where y is proportional to the distance from ω to the nearest θ_{ℓ_j}. Thus, we have

$$\begin{aligned}&\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\mathcal{F}\left[F_{j,\ell}^{M}\right](\xi),\\ &= B^2\mathcal{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\left((2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\right),\\ &= B^2\mathcal{F}[f](\xi)\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2.\end{aligned} \tag{71}$$

Hence,

$$\begin{aligned}\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\mathcal{F}\left[F_{j,\ell}^{M}\right](\xi) &= \mathcal{F}[f](\xi)\left(B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\right)\\ &= \mathcal{F}[f](\xi).\end{aligned} \tag{72}$$

From (72), we obtain the desired reconstruction formula as

$$f(\mathbf{t}) = \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}\left[\Gamma_{\Psi}^{M} f\right](j,b,\ell)\Psi_{j,b,\ell}^{M}(t)\frac{\mathrm{d}b}{\lambda^{-3j/2}}. \tag{73}$$

This completes the proof of Theorem 5. □

Corollary 1. *Invoking (69) and (70), we observe that*

$$\begin{aligned}&\sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\lambda^{3j/2}\left\|\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right\|_2^2,\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\lambda^{3j/2}\left\|\mathcal{F}\left(\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right)(\xi)\right\|_2^2,\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}\left|\mathcal{F}\left(\left[\Gamma_{\Psi}^{M} f\right](j,\mathbf{b},\ell)\right)(\xi)\right|^2\frac{\mathrm{d}\xi}{\lambda^{-3j/2}},\\ &= \sum_{j\geq 0}\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}\lambda^{\lfloor j/2\rfloor-1}}\int_{\mathbb{R}^2}(2\pi B)^2\lambda^{-3j/2}\left|\mathcal{F}[f](\xi)\right|^2\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\\ &\quad\cdot\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\frac{\mathrm{d}\xi}{\lambda^{-3j/2}},\\ &\quad\cdot(j/2) = \int_{\mathbb{R}^2}\left|\left|\mathcal{F}[f](\xi)\right|^2\left(B^2\sum_{j\geq 0}\left|W\left(\frac{Br}{\lambda^j}\right)\right|^2\right)\right.\\ &\quad\cdot\left((2\pi)^2\sum_{\ell=0}^{\lambda^{\lfloor j/2\rfloor-1}}\left|V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right)\right|^2\right)\mathrm{d}\xi,\\ &= \int_{\mathbb{R}^2}\left|\mathcal{F}[f](\xi)\right|^2\mathrm{d}\xi = \|f\|_2^2.\end{aligned} \tag{74}$$

4.2. Novel Discrete Curvelet Transform. In this subsection, we shall present a complete discrete analogue of the proposed novel curvelet transform defined in (20). Having formulated the semidiscrete analogue, we need to discretize the spatial variable $\mathbf{b}$ by taking both the previous discretizations of the scale and angular parameters into consideration. For $\mathbf{m} = (m_1, m_2) \in \mathbb{Z}^2$ and $\beta_1, \beta_2 > 0$, we sample the spatial variable $\mathbf{b}$ as

$$\mathbf{b}_{\mathbf{m}}^{j\ell} := R_{-\theta_{\ell_j}}\mathscr{K}(\mathbf{m},\beta_1,\beta_2,j) = R_{-\theta_{\ell_j}}\left(\frac{\beta_1 m_1}{\lambda^j},\frac{\beta_2 m_2}{\lambda^{j/2}}\right)^T. \quad (75)$$

Consequently, the novel discrete family of curvelets takes the following form:

$$\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t}) = \Psi_j^M\left(R_{\theta_{\ell_j}}\mathbf{t} - \mathscr{K}(\mathbf{m},\beta_1,\beta_2,j)\right), \quad t \in \mathbb{R}^2, \quad (76)$$

where the basic waveform $\Psi_j^M(\mathbf{t})$ is given by (64). Moreover, an easy computation yields that

$$\begin{aligned}\mathscr{F}\left[\Psi_{j,\mathbf{m},\ell}^M\right](\xi) &= \exp\left\{-i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T\xi\right\}\mathscr{F}\left[\Psi_j^M\right]\left(R_{\theta_{\ell_j}}\xi\right),\\ &= \lambda^{-3j/4}B \quad \exp\left\{-i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T\xi\right\}\exp\left\{-\frac{i\ DB\ r^2}{2}\right\}\\ &\quad\cdot W\left(\frac{Br}{\lambda^j}\right)V\left(\frac{\lambda^{\lfloor j/2\rfloor}\left(\omega-\theta_{\ell_j}\right)}{2\pi}\right).\end{aligned} \quad (77)$$

The formal definition of the novel discrete curvelet transform is given below.

Definition 6. Given a real, unimodular matrix $M = (A, B\colon C, D)$, with $B > 0$, the novel discrete curvelet transform corresponding to any $f \in L^2(\mathbb{R}^2)$ is defined as

$$\left[\Gamma_\Psi^M f\right](j,\mathbf{m},\ell) = \langle f, \Psi_{j,\mathbf{m},\ell}^M\rangle_2 = \int_{\mathbb{R}^2} f(\mathbf{t})\overline{\Psi_{j,\mathbf{m},\ell}^M}(\mathbf{t})\mathrm{d}\mathbf{t}, \quad (78)$$

where the novel discrete family of curvelets $\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t})$ is given by (76).

By implementing Parseval's formula for the Fourier transform and taking the benefit of (77), we can express the above definition as

$$\begin{aligned}\left[\Gamma_\Psi^M f\right](j,\mathbf{m},\ell) &= \langle f, \Psi_{j,\mathbf{m},\ell}^M\rangle_2 = \langle \mathscr{F}[f], \mathscr{F}\left[\Psi_{j,\mathbf{m},\ell}^M\right]\rangle_2\\ &= \int_{\mathbb{R}^2}\exp\left\{i\left(\mathbf{b}_{\mathbf{m}}^{j\ell}\right)^T\xi\right\}\mathscr{F}[f](\xi)\overline{\mathscr{F}\left[\Psi_j^M\right]}\left(R_{\theta_{\ell_j}}\xi\right)\mathrm{d}\xi.\end{aligned} \quad (79)$$

In analogy to the continuous case, we need to take care of the low-frequency signals. We introduce another radial window $W_0(r)$ satisfying

$$\left|W_0(Br)\right|^2 + \sum_{j\geq 0} W\left(\frac{Br}{\lambda^j}\right) = \frac{1}{(2\pi)^2}. \quad (80)$$

And, for $\mathbf{m} \in \mathbb{Z}^2$, the father wavelet $\Phi_{\mathbf{m}}^M$ is defined by

$$\Phi_{\mathbf{m}}^M(\mathbf{t}) = \Phi^M(\mathbf{t}-\mathbf{m}), \text{where } \mathscr{F}\left[\Phi^M\right](\xi) = W_0(B|\xi|). \quad (81)$$

These father wavelets are nondirectional in nature. Therefore, the complete family of novel discrete curvelets $\mathscr{F}_{\Phi,\Psi}$ takes the following form:

$$\mathscr{F}_{\Phi,\Psi} := \left\{\Phi_{\mathbf{m}}^M(t)\colon \mathbf{m} \in \mathbb{Z}^2\right\}\bigcup\left\{\Psi_{j,\mathbf{m},\ell}^M(\mathbf{t})\colon j \geq 0, \mathbf{m} \in \mathbb{Z}^2, \ell \in \mathbb{Z}_{\lambda^{j/2}}\right\}. \quad (82)$$

5. Construction of Novel Curvelets: An Example

In this section, we shall present a lucid construction of the radial and angular window functions W and V satisfying the prescribed admissibility conditions. As is evident from (18) and (64), the construction of basic curvelet waveforms is governed by the admissible radial and angular window functions W and V; therefore, the upcoming example also guides the construction of novel basic curvelet waveforms. Consequently, the family of novel curvelets can be obtained by appropriately translating and rotating the basic waveform. It is pertinent to mention that our approach is motivated by [19].

Example 1. Given a 2×2 real, unimodular matrix $M = (A, B\colon C, D)$, with $B > 0$, we consider the following window functions:

$$W(r) = \begin{cases}\frac{1}{B}\cos\left[\frac{\pi}{2}(\nu(5-6r))\right], & 2/3 \leq r \leq 5/6,\\ \frac{1}{B}, & 5/6 \leq r \leq 4/3,\\ \frac{1}{B}\cos\left[\frac{\pi}{2}(\nu(3r-4))\right], & 4/3 \leq r \leq 5/3,\\ 0, & \text{elsewhere},\end{cases}$$

$$V(\omega) = \begin{cases}\frac{1}{2\pi}\cos\left[\frac{\pi}{2}(\nu(-3\omega-1))\right], & -2/3 \leq \omega \leq -1/3,\\ \frac{1}{2\pi}, & -1/3 \leq \omega \leq 1/3,\\ \frac{1}{2\pi}\cos\left[\frac{\pi}{2}(\nu(3\omega-1))\right], & -1/3 \leq \omega \leq 2/3,\\ 0, & \text{elsewhere},\end{cases} \quad (83)$$

where ν is a smooth function, such that

$$\nu(y) = \begin{cases}0, & y \leq 0,\\ 1, & y \geq 1,\end{cases} \quad (84)$$
$$\nu(y) + \nu(1-y) = 1, \quad y \in \mathbb{R}.$$

Certain choices of the function ν include $\nu(y) = y$ or even smoother polynomials like $\nu(y) = 3y^2 - 2y^3$ and $\nu(y) = y^5 - 5y^4 + 5y^3$. We note that the smoothness of the window functions W and V is governed by the function ν. The smoother ν is, the smoother the window functions are and consequently the faster the decay of curvelets is. As an

example, one of the sufficiently smooth functions is given below:

$$\nu(y) = \begin{cases} -0, & y \le 0, \\ \dfrac{\alpha(y-1)}{\alpha(y-1)+\alpha(y)}, & 0 < y < 1, \text{ where,} \quad \alpha(y) = \exp\left\{-\dfrac{1}{(1+y)^2} - \dfrac{1}{(1-y)^2}\right\}. \\ -1, & y \ge 1. \end{cases} \tag{85}$$

Next, we show that the aforementioned window functions obey the admissibility conditions (61) and (62). By definition, we have $\operatorname{supp} V \subseteq [-2/3, 2/3]$. Firstly, we shall compute the sum $\sum_{\ell=-\infty}^{\infty} |V(\omega-\ell)|^2$, where $\omega \in \mathbb{R}$. For a fixed $\omega \in \mathbb{R}$, the aforementioned sum contains only two nonvanishing terms, and for $t \in [1/3, 2/3]$ we have

$$\begin{aligned} (2\pi)^2 \sum_{\ell=-\infty}^{\infty} |V(\omega-\ell)|^2 &= (2\pi)^2\left(|V(\omega)|^2 + |V(\omega-1)|^2\right), \\ &= \cos^2\left[\frac{\pi}{2}(\nu(3\omega-1))\right] + \cos^2\left[\frac{\pi}{2}(\nu(-3\omega+2))\right] \\ &= \cos^2\left[\frac{\pi}{2}(\nu(x))\right] + \cos^2\left[\frac{\pi}{2}(\nu(1-x))\right] \\ &= \cos^2\left[\frac{\pi}{2}(\nu(x))\right] + \cos^2\left[\frac{\pi}{2}(1-\nu(x))\right] \\ &= \cos^2\left[\frac{\pi}{2}(\nu(x))\right] + \sin^2\left[\frac{\pi}{2}(\nu(x))\right] = 1. \end{aligned} \tag{86}$$

In order to show that the admissibility condition (61) holds for the window function W we choose the scale $\lambda = 2$. Since $\operatorname{supp} W \subset [1/2, 2]$, it follows that $\operatorname{Supp} W(2^j r) \subseteq [2^{-j-1}, 2^{1-j}]$. Consequently, the sum on the left-hand side of (61) has only two nonvanishing terms corresponding to $r \in [1/2, 1]$, namely, $|W(r)|^2$ and $|W(2r)|^2$. Thus, for $r \in [1/2, 1]$, we have

$$\begin{aligned} B^2 \sum_{j=-\infty}^{\infty} \left|W\left(2^j r\right)\right|^2 &= B^2\left(|W(r)|^2 + |W(2r)|^2\right), \\ &= \begin{cases} 1, & 1/2 \le r \le 2/3, \\ \cos^2\left[\dfrac{\pi}{2}(\nu(6r-4))\right] + \cos^2\left[\dfrac{\pi}{2}(\nu(5-6r))\right], & 2/3 \le r \le 5/6, \\ 1, & 5/6 \le r \le 1, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \tag{87}$$

Moreover, we observe that

$$\begin{aligned}
&\cos^2\left[\frac{\pi}{2}(\nu(6r-4))\right]+\cos^2\left[\frac{\pi}{2}(\nu(5-6r))\right]\\
&=\cos^2\left[\frac{\pi}{2}(\nu(z))\right]+\cos^2\left[\frac{\pi}{2}(\nu(1-z))\right]\\
&=\cos^2\left[\frac{\pi}{2}(\nu(z))\right]+\cos^2\left[\frac{\pi}{2}(1-\nu(z))\right]\\
&=\cos^2\left[\frac{\pi}{2}(\nu(z))\right]+\sin^2\left[\frac{\pi}{2}(\nu(z))\right]=1,
\end{aligned} \tag{88}$$

Plugging equation (88) in equation (87), we obtain

$$B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^j r\right)\right|^2=1. \tag{89}$$

Finally, if we choose $\ln 2\, W'(r)=W(r)$, then we shall demonstrate that the window functions W' and V satisfy the admissibility conditions (15) and (16). We proceed with

$$\begin{aligned}
1&=(2\pi)^2\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2=(2\pi)^2\int_0^1\sum_{\ell=-\infty}^{\infty}|V(\omega-\ell)|^2\mathrm{d}\omega\\
&=(2\pi)^2\int_{-\infty}^{\infty}|V(\omega)|^2\mathrm{d}\omega.
\end{aligned} \tag{90}$$

Finally, for $r\in(0,\infty)$, we take $r=2^x, x\in(-\infty,\infty)$ so that we have

$$\begin{aligned}
1&=B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^j r\right)\right|^2=B^2\sum_{j=-\infty}^{\infty}\left|W\left(2^{j+x}\right)\right|^2\\
&=B^2\int_0^1\sum_{j=-\infty}^{\infty}\left|W\left(2^{j+x}\right)\right|^2\mathrm{d}x\\
&=B^2\sum_{j=-\infty}^{\infty}\frac{1}{\ln 2}\int_{2^j}^{2^{j+1}}|W(y)|^2\frac{\mathrm{d}y}{y}\\
&=B^2\sum_{j=-\infty}^{\infty}\int_{2^j}^{2^{j+1}}\left|W'(y)\right|^2\frac{\mathrm{d}y}{y}=B^2\int_0^{\infty}\left|W'(y)\right|^2\frac{\mathrm{d}y}{y}.
\end{aligned} \tag{91}$$

6. Conclusion and Future Work

In the present study, we intertwined the advantages of the curvelet and linear canonical transforms and introduced the notion of the novel curvelet transform. The prime advantage of this intertwining lies in the fact that the novel curvelet transform enjoys certain degrees of freedom and the new radial window achieves higher flexibility, which in turn can be employed in optimizing the concentration of the curvelet spectrum. As such, the proposed transform serves as a significant addition to the contemporary tools of signal and image processing. Nevertheless, the present study, in itself, appeals several ramifications and developments thereon. An immediate concern is to study the frame theory associated with the novel discrete curvelet transform.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

On the Multiwavelets Galerkin Solution of the Volterra–Fredholm Integral Equations by an Efficient Algorithm

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We develop the multiwavelet Galerkin method to solve the Volterra–Fredholm integral equations. To this end, we represent the Volterra and Fredholm operators in multiwavelet bases. Then, we reduce the problem to a linear or nonlinear system of algebraic equations. The interesting results arise in the linear type where thresholding is employed to decrease the nonzero entries of the coefficient matrix, and thus, this leads to reduction in computational efforts. The convergence analysis is investigated, and numerical experiments guarantee it. To show the applicability of the method, we compare it with other methods and it can be shown that our results are better than others.

1. Introduction

A mathematical model of the spatiotemporal development of an epidemic yields the following Volterra–Fredholm integral equation (VFIE):

$$u(x) = f(x) + \int_0^x k_1(x, s, u(s))\mathrm{d}s + \int_0^1 k_2(x, s, u(s))\mathrm{d}s, \quad x \in \Omega := [0, 1], \tag{1}$$

where the given functions $f\colon \Omega \longrightarrow \mathbb{R}$ and $k_1\colon S \times \mathbb{R} \longrightarrow \mathbb{R}$ with $S := \{(x, s)\colon x, s \in \Omega\}$ are assumed to be continuous functions. Furthermore, we consider $k_2 := p(x, s)h(u(s))$ to be integrable, where $h(x)$ is a nonlinear function and $p(x, s)\colon S \longrightarrow \mathbb{R}$ is a continuous function. Besides, the given functions are selected so that equation (1) has a unique solution.

The parabolic boundary value problems lead to these types of integral equations and widely arise from various physical and biological models. The VFIE also appears in the literature in mixed form as

$$u(x) = f(x) + \int_0^x \int_0^1 k(t, s)u(s)\mathrm{d}s\,\mathrm{d}t, \tag{2}$$

where f and k are analytic functions. Many authors studied the mixed form of VFIE numerically. Among these, we can mention collocation method [1], projection method [2], spline collocation method [3], wavelet collocation method [4], Adomian decomposition method [5], and so on [6–8]. Among these studies, we focus on a paper that uses the multiwavelet Galerkin method to solve linear mixed VFIE as mentioned in [9]. In [9], the wavelet transform matrix and the operational matrix of integration are utilized to reduce the problem of linear mixed VFIE to a sparse linear system of algebraic equations. By searching among the sources, we can find a small number of papers in the field of numerical and analytical solutions to problem (1). In [10], the Lagrange collocation method is employed to solve this problem. Wang and Wang [11] applied the Taylor collocation method to find the numerical solution of the equation. Also, the convergence analysis is investigated for the proposed method. The Taylor collocation method was applied by Karamete and Sezer [12] to solve this equation as well as the high-order linear Fredholm–Volterra integrodifferential equations [13]. The Bell polynomials have been employed for solving this equation [14].

The motivation of our work is to develop the multiwavelet Galerkin method to solve (1). We split the problem into two configurations, linear and nonlinear. After using

the multiwavelet Galerkin method, both types reduce to the system of the linear and nonlinear algebraic equations, respectively. The interesting results arise in the linear type where thresholding is employed to decrease the nonzero entries of the coefficient matrix. This gives the sparse system. This property is very useful to reduce the computational cost. We use Alpert's multiwavelet bases constructed in [15] following [16], and these bases have been used to solved PDEs, ODEs, and integral equations [9, 17–19]. These bases allow us to have high vanishing moments, compact support, and properties such as orthogonality and interpolation [15]. These characteristics of Alpert's multiwavelets lead to a sparse representation of differential and integral operators [16, 20].

This paper is organized as follows: In Section 2, we briefly introduce Alpert's multiwavelet bases. In Section 3, the Multiwavelets Galerkin method is used to solve this proble, and the conditions for convergence of the proposed method are discussed. Section 5 contains some numerical results to confirm the validity and efficiency of the proposed method, and Section 6 contains a few concluding remarks.

2. Alpert's Multiwavelet Bases

Let $J \in \mathbb{Z}^+ \cup \{0\}$. We consider the uniform finite discretizations $\Omega := [0,1] = \cup_{b\in\mathscr{B}} X_{J,b}$ where the subintervals $X_{J,b} := [x_b, x_{b+1}]$ are determined by the point $x_b := (b/(2^J))$ with $\mathscr{B} := \{0,\ldots,2^J-1\}$. For $k \in \mathscr{R} := \{0,1,\ldots,r-1\}$, we introduce the subspace V_J^r as a space of piecewise polynomial bases of degree less than multiplicity parameter r that is spanned by

$$V_J^r := \operatorname{span}\left\{\phi_{j,b}^k := \mathscr{D}_{2^j}\mathscr{T}_b\phi^k, b \in \mathscr{B}_j, k \in \mathscr{R}\right\} \subset L^2(\Omega), \quad r \geq 0, \tag{3}$$

where $\mathscr{D}$ and $\mathscr{T}$ are the dilation and translation operators, respectively, and $\left\{\phi^k\right\}_{k\in\mathscr{R}}$ are the primal interpolating scaling bases introduced by Alpert et al. [16]. Given nodes $\{\tau_k\}_{k\in\mathscr{R}}$, which are the roots of Legendre polynomial of degree r, the interpolating scaling bases are defined as

$$\phi^k(t) = \begin{cases} \sqrt{\dfrac{2}{\omega_k}} L_k(2t-1), & t \in [0,1], \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

where $\{L_k(t)\}_{k\in\mathscr{R}}$ are the Lagrange interpolating polynomials at the point $\{\tau_k\}_{k\in\mathscr{R}}$ and $\{\omega_k\}_{k\in\mathscr{R}}$ are the Gauss–Legendre quadrature weights [16, 18]. These bases form an orthonormal bases on Ω with respect to the L^2-inner product. Due to the definition of the space V_J^r, the spaces $\left\{V_j^r\right\}_{j\in\mathbb{Z}^+\cup\{0\}}$ have dimension $N := 2^J r$ and obviously are nested:

$$V_0^r \subset V_1^r \subset \cdots \subset V_J^r \subset \cdots \subset L^2(\Omega). \tag{5}$$

Hence, we consider the complement subspace W_J^r of V_J^r in V_{J+1}^r such that

$$V_{J+1}^r = V_J^r \oplus W_J^r, W_J^r \perp V_J^r, \tag{6}$$

where $\oplus$ denotes orthogonal sums. According to (6), the space V_J may be inductively decomposed to

$$V_J^r = V_0^r \oplus \left(\oplus_{j=0}^{J-1} W_j^r\right). \tag{7}$$

The complement subspace W_J^r has dimension $2^J r$ and is spanned by multiwavelet bases $\{\psi^k\}_{k\in\mathscr{R}}$, as

$$W_J^r = \operatorname{span}\left\{\psi_{J,b}^k \cong \mathrm{W}_{2^J}\mathrm{W}_b\psi^k: \quad b \in \mathscr{B}_J, k \in \mathscr{R}\right\}. \tag{8}$$

Because Alpert's multiwavelets are completely introduced in [16], we avoid this and refer the readers to [15, 16, 19].

Every function $p \in L^2(\Omega)$ can be represented in the form

$$p \approx \mathrm{V}_J^r(p) = \sum_{b\in\mathscr{B}_J}\sum_{k\in\mathscr{R}} p_{J,b}^k \phi_{J,b}^k, \tag{9}$$

where $\mathscr{P}_J^r$ is the orthogonal projection that maps $L^2(\Omega)$ onto the subspace V_J^r. To find the coefficients $p_{J,b}^k$ that are determined by $\langle p, \phi_{J,b}^k\rangle = \int_{X_{J,b}} f(x)\phi_{J,b}^k(x)\mathrm{d}x$, we shall compute the integrals. We apply the r-point Gauss–Legendre quadrature by a suitable choice of the weights ω_k and nodes τ_k for $k \in \mathscr{R}$ to avoid these integrals [16, 19], via

$$p_{J,b}^k \approx 2^{-(J/2)}\sqrt{\frac{\omega_k}{2}}\, p\left(2^{-J}\left(\frac{\tau_k+1}{2}+b\right)\right), \quad k \in \mathscr{R}, b \in \mathscr{B}_J. \tag{10}$$

Convergence analysis of the projection $\mathrm{P}_J^r(p)$ is investigated for the r-times continuously on differentiable function $p \in \mathbb{C}^r(\Omega)$:

$$\left\|\mathrm{P}_J^r(p) - p\right\| \leq 2^{-\mathrm{J}r}\frac{2}{4^r r!}\sup_{x\in[0,1]}\left|p^{(r)}(x)\right|. \tag{11}$$

For the full proof of this approximation and further details, we refer the readers to [15]. Thus, we can conclude that $\mathrm{P}_J^r(p)$ converges to p with rate of convergence $O(2^{-\mathrm{J}r})$.

Assume that the vector function $\Phi_J^r := [\Phi_{r,J,0},\ldots,\Phi_{r,J,2^J-1}]^T$ with $\Phi_{r,J,b} := [\phi_{J,b}^0,\ldots,\phi_{J,b}^{r-1}]$ includes the scaling functions and is called multiscaling function. Approximation (9) may be rewritten using the vector P that includes entries $P_{\mathrm{br}+k+1} := p_{J,b}^k$ as follows:

$$\Phi_J^r(p) = P^T\Phi_J^r, \tag{12}$$

where P is an N-dimensional vector. The building blocks of these bases construction can be applied to approximate a higher-dimensional function. To this end, one can introduce the two-dimensional subspace $V_J^{r,2} := V_J^r \times V_J^r \subset L^2(\Omega)^2$ that is spanned by

$$\left\{\phi_{J,b}^k\phi_{J,b'}^{k'}: b, \quad b' \in \mathscr{B}_J, k, k' \in \mathscr{R}\right\}. \tag{13}$$

Thus, by this assumption, to derive an approximation of the function $p \in L^2(\Omega)^2$ by the projection operator $\mathscr{P}_J^r$, we have

$$p(x,y) \approx \Phi_J^r(p)(x,y) = \Phi_J^{rT}(x)P\Phi_J^r(y), \tag{14}$$

where the components of the square matrix P of order N are obtained by

$$\mathscr{P}_{rb+(k+1),rb'+(k'+1)} \approx 2^{-J}\sqrt{\frac{\omega_k}{2}}\sqrt{\frac{\omega_{k'}}{2}}p\left(2^{-J}\left(\widehat{\tau}_k+b\right),2^{-J}\left(\widehat{\tau}_{k'}+b'\right)\right), \tag{15}$$

where $\widehat{\tau}_k = (\tau_k + 1)/2$. Consider the $2r$-th partial derivatives of p: $\Omega^2 \longrightarrow \mathbb{R}$ to be continuous. Utilizing this assumption, the error of this approximation can be bounded as follows:

$$\left\|\mathscr{P}_J^r p - p\right\| \le \mathscr{M}_{\max}\frac{2^{1-rJ}}{4^r r!}\left(2+\frac{2^{1-Jr}}{4^r r!}\right), \tag{16}$$

where $\mathscr{M}_{\max}$ is a constant.

Given orthogonal projection operator $\mathscr{Q}_j^r = \mathscr{P}_{j+1}^r - \mathscr{P}_j^r$ that maps $L^2(\Omega)$ onto W_j^r, the multiscale projection operator $\mathscr{M}_J^r$ can be represented as

$$\mathscr{M}_J^r = \mathscr{Q}_0^r + \sum_{j=0}^{J-1}\mathscr{Q}_j^r, \tag{17}$$

and consequently, any function $p \in L^2(\Omega)$ can be approximated as a linear combination of multiwavelets and single-scale interpolating scaling functions as

$$p \approx \mathscr{M}_J^r(p) = \sum_{k=0}^{r-1} p_{0,0}^k\phi_{0,0}^k + \sum_{j=0}^{J-1}\sum_{b\in\mathscr{B}_j}\sum_{k\in\mathscr{R}}\widetilde{p}_{j,b}^k\psi_{j,b}^k, \tag{18}$$

where

$$\begin{aligned} p_{0,0}^k &\cong \langle p,\phi_{0,0}^k\rangle,\\ \widetilde{p}_{j,b}^k &\cong \langle p,\psi_{j,b}^k\rangle. \end{aligned} \tag{19}$$

Note that we can compute the coefficients $p_{0,0}^k$ by using (10). But in many cases, multiwavelet coefficients from zero up to higher-level $J-1$ must be evaluated numerically. To avoid this, we use multiwavelet transform matrix T_J, introduced in [18, 19]. This matrix connects multiwavelet bases and multiscaling functions, via

$$\Psi_J^r = T_J\Phi_J^r, \tag{20}$$

where $\Psi_J^r := [\Phi_{r,0,b},\Psi_{r,0,b},\Psi_{r,1,b},\ldots,\Psi_{r,J-1,b}]^T$ is a vector with the same dimension Φ_J^r (here $\Psi_{r,j,b} := [\psi_{j,b}^0,\ldots,\psi_{j,b}^{r-1}]$). This representation helps to rewrite equation (18) as to form

$$p \approx \mathscr{M}_J^r(p) = \widetilde{P}_J^T\Psi_J^r, \tag{21}$$

where we have the N-dimensional vector $\widetilde{P}_J$ whose entries are $p_{0,0}^k$ and $\widetilde{p}_{j,b}^k$ and is given by employing the multiwavelet transform matrix T_J as $\widetilde{P}_J = T_J P_J$.

The multiwavelet coefficients (details) become small when the underlying function is smooth (locally) with increasing refinement levels. If the multiwavelet bases have N_ψ^r vanishing moments [19, 21], then details decay at the rate of $2^{-JN_\psi^r}$ [17]. Because vanishing moments of Alpert's multiwavelets are equal to r, consequently, one can obtain $\widetilde{p}_{J,b}^k \approx O(2^{-Jr})$. This allows us to truncate the full wavelet transforms while preserving most of the necessary data. Thus, we can set to zero all details that satisfy a certain constraint ε using thresholding operator $\mathscr{C}_\varepsilon$:

$$\mathscr{C}_\varepsilon\left(\widetilde{P}_J\right) = \overline{P}_J, \tag{22}$$

and the elements of $\overline{P}_J$ are determined by

$$\overline{p}_{j,b}^k \cong \begin{cases}\widetilde{p}_{j,b}^k, & (j,b,k)\in D_\varepsilon,\\ 0, & \text{else},\end{cases} \quad b\in\mathscr{B}_j, j=0,\ldots,J-1, k\in\mathscr{R}, \tag{23}$$

where $D_\varepsilon := \left\{(j,b,k): |\widetilde{p}_{j,b}^k| > \varepsilon\right\}$. Now, we can bound the approximation error after thresholding [17] via

$$\left\|\mathscr{P}_J^r p - \mathscr{P}_{J,D_\varepsilon}^r p\right\|_{L^2(\Omega)} \le C_{\text{thr}}\varepsilon, \tag{24}$$

where $\mathscr{P}_{J,D_\varepsilon}^r(p)$ is the projection operator after thresholding with the threshold ε and $C_{\text{thr}} > 0$ is a constant independent of J and ε.

3. Multiwavelet Galerkin Method

In order to obtain multiwavelet Galerkin solution of (1), assume that solution can be approximated as an expansion of the Alpert's multiwavelets, i.e.,

$$u(x) \approx \mathscr{P}_J^r(u)(x) = U^T\Psi_J^r(x), \tag{25}$$

where the N dimension vector U of unknowns must be specified. This solution is selected such that it satisfies (1) approximately. Also, it is obtained from the solution of the minimization problem

$$\left\|u - \mathscr{P}_J^r(u)\right\| = \min_{z\in L^2(\Omega)}\|u-z\|. \tag{26}$$

Since $L^2(\Omega)$ is an inner product space with finite dimension, it can be shown that this minimization problem has a unique solution [22].

Let us rewrite (1) in the operator form

$$(I-\mathscr{F})u = g, \tag{27}$$

where $g(x) := f(x) + \mathrm{V}(u)(x)$ with $\mathscr{V}(u)(x) := \int_0^x k_1(x,s,u(s))\mathrm{d}s$. Furthermore, the operator $\mathscr{F}(u) := \int_0^1 p(x,s)h(u(s))\mathrm{d}s$ is assumed to be compact on $L^2(\Omega)$ to $L^2(\Omega)$ and k_1 is a given continuous function. Due to these assumptions, $\mathrm{V}(u)(x)$ is a continuous function and, consequently, g is also continuous function. Due to (11), the function $g(x)$ can be approximated at a rate of at least 2^{-Jr}:

$$\begin{aligned} g(x) &\approx \mathscr{P}_J^r(g)(x) = F^T\Phi_J^r(x) + K_1^T\Phi_J^r(x)\\ &= F^T T_J^{-1}\Psi_J^r(x) + K_1^T T_J^{-1}\Psi_J^r(x)\\ &= \widetilde{F}^T\Psi_J^r(x) + \begin{cases}\mathscr{U}^T\mathrm{A}_1\Psi_J^r(x), & \text{linear},\\ \mathscr{C}_1\Psi_J^r(x), & \text{nonlinear},\end{cases}\\ &= \widetilde{G}^T\Psi_J^r(x) = \mathscr{M}_J^r(x)(x), \end{aligned} \tag{28}$$

where $\mathscr{A}_1$ and $\mathscr{C}_1$ are $N\times N$ matrices and the rest are N-dimensional vectors. Now, we introduce the residual in the approximation of (1):

$$r_J(x) = u_J(x) - \mathcal{M}_J^r(g)(x) - \mathcal{M}_J^r\left(\mathcal{F}_J^r\right)(x), \tag{29}$$

where $u_J := \mathcal{M}_J^r(u)$ and

$$\begin{aligned}\mathcal{M}_J^r\mathcal{F}_J^r\left(u_J\right) &= \mathcal{M}_J^r\left(\int_0^1 \mathcal{M}_J^r\left(k_2\left(x,s,u_J(s)\right)\right)\mathrm{d}s\right)\\ &= \mathcal{M}_J^r\left(\int_0^1 \mathcal{M}_J^r\left(p(x,s)h\left(u_J(s)\right)\right)\mathrm{d}s\right)\\ &= \int_0^1 H^T\Psi_J^r(s)\Psi_J^{rT}(s)P\Psi_J^r(x)\mathrm{d}s = H^TP\Psi_J^r(x)\\ &= \begin{cases} U^T\Psi_2\Psi_J^r(x), & \text{linear},\\ \Psi_2\Psi_J^r(x), & \text{nonlinear}.\end{cases}\end{aligned} \tag{30}$$

Symbolically,

$$r_J = \mathcal{M}_J^r\left(\left(I - \mathcal{F}_J^r\right)u_J - g\right). \tag{31}$$

To find u_J, it requires that the approximate solution u_J satisfies

$$\langle r_J, \left[\Psi_J^r\right]_i\rangle = 0, \quad i = 1,\ldots,N. \tag{32}$$

This is multiwavelet Galerkin's method and yields a linear or nonlinear system that must be solved to obtain the approximate solution. Note that $\mathcal{M}_J^r(z) = 0$ if and only if $\langle z, [\Psi_J^r]_i\rangle = 0$. Thus, we can rewrite (31) as

$$\mathcal{M}_J^r\left(r_J\right) = 0, \tag{33}$$

or equivalently,

$$\mathcal{M}_J^r\left(I - \mathcal{F}_J^r\right)u_J = \mathcal{M}_J^r(g). \tag{34}$$

Note that $\mathcal{M}_J^r(u_J) = u_J$ whenever $u_J \in V_J^r$. Due to this, equation (34) can be rewritten as

$$\left(I - \mathcal{M}_J^r\mathcal{F}_J^r\right)\left(u_J\right) = \mathcal{M}_J^r(g). \tag{35}$$

According to (35), we obtain the system of linear or nonlinear equations. Due to the higher vanishing moments r and increasing refinement level J, for the linear type of this equation, we can discard coefficients by hard thresholding introduced in the previous section. We can reduce the computational efforts using proper methods for this type of system such as the GMRES method. The GMRES method is introduced by Saad and Shultz [23] for solving sparse and large linear systems. The GMRES generates an approximate solution whose residual norm is minimum by using a Krylov subspace. In this paper, we use restarted GMRES Algorithm 2 [24]. To use this method, we must first define Arnoldi's algorithm. Arnoldi' s procedure is an algorithm for building an orthogonal basis of the Krylov subspace κ_m. The N-th Krylov subspace is defined as follows:

$$\kappa_m(\Lambda, w) = \operatorname{span}\left\{w_1, \Lambda w_1, \ldots, \Lambda^{m-1}w_1\right\}. \tag{36}$$

Here, we assume that system (35) of the linear type to be of form $\Lambda U = D$ and W_m is a $N \times m$ matrix with column vectors $w_1, \ldots, w_m$. Also, $\overline{H}_m$ is a $(m+1 \times m)$ Hessenberg matrix whose nonzero entries $h_{i,j}$ are defined by Algorithm 1.

(1) Compute $r_0 = D - \Lambda U_0$, $\beta = \|r_0\|_2$ *and* $w_1 = r_0/\beta$
(2) Generate the Arnoldi basis and the matrix $\overline{H}_m$ using the Arnoldi algorithm starting with w_1
(3) Compute y_m the minimizer of $\|\beta e_1 - \widetilde{H}_m y\|_2$ and $U_m = U_0 + W_m y_m$
(4) If satisfied then stop, else set $U_0 = U_m$ and go to 1

To investigate the convergence analysis, one can prove that $\|\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\| = O(2^{-\mathrm{Jr}})$. Thus, $\|\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\| \longrightarrow 0$ when $J \longrightarrow \infty$ because $\mathcal{F}$ is a compact operator. Now, we can raise the convergence theorem.

Theorem 1. *Let $\mathcal{F}$ be a compact operator and $I - \mathcal{F}$ be injective. Assume that the sequence $\mathcal{F}_J^r$: $L^2(\Omega) \longrightarrow L^2(\Omega)$ is collectively compact and pointwise convergent to $\mathcal{F}$.*

Then, $(I - \mathcal{F}_J^r)^{-1}$ exists and is uniformly bounded. Also, the solution of (27) and (31) satisfy the error estimate

$$\|u - u_J\| \le \frac{\left\|\left(I - \mathcal{F}_J^r\right)^{-1}\right\|}{1 - \varepsilon_{J_0}\left\|\left(I - \mathcal{F}_J^r\right)^{-1}\right\|}\left(\|u - \mathcal{M}_J^r u\| + \left\|\left(\mathcal{M}_J^r\mathcal{F} - \mathcal{M}_J^r\mathcal{F}_J^r\right)u\right\|\right). \tag{37}$$

Proof. Because the sequence $\mathcal{F}_J^r$ converges pointwise to $\mathcal{F}$ in $L^2(\Omega)$ and is collectively compact, we conclude that

$$\left\|\left(\mathcal{F}_J^r - \mathcal{F}\right)\mathcal{F}_J^r\right\| \longrightarrow 0, \quad \text{as } J \longrightarrow \infty. \tag{38}$$

For all sufficiently large J, we have

$$\left\|(I - \mathcal{F})^{-1}\left(\mathcal{F}_J^r - \mathcal{F}\right)\mathcal{F}_J^r\right\| < 1, \tag{39}$$

and as a consequence of this, $I - \mathcal{F}_J^r$ is reversible. Note that the inverse operator $(I - \mathcal{F})$ exists due to the Riesz theorem. The investigation is based on the approximation of $I - \mathcal{F}_J^r$ by $I - \mathcal{M}_J^r\mathcal{F}_J^r$,

$$\begin{aligned} I - \mathcal{M}_J^r\mathcal{F}_J^r &= \left(I - \mathcal{F}_J^r\right) + \left(\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\right)\\ &= \left(I - \mathcal{F}_J^r\right)\left(I + \left(I - \mathcal{F}_J^r\right)^{-1}\left(\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\right)\right).\end{aligned} \tag{40}$$

To prove the existence of $(I - \mathcal{M}_J^r\mathcal{F}_J^r)^{-1}$, assume that

$$\varepsilon_{J_0} \equiv \sup_{J \ge J_0}\left\|\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\right\| < \frac{1}{\left\|\left(I - \mathcal{F}_J^r\right)^{-1}\right\|}. \tag{41}$$

Thus, $(I + (I - \mathcal{F}_J^r)^{-1}(\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r))^{-1}$ exists and is uniformly bounded due to geometric series theorem, i.e.,

$$\left\|\left(I + \left(I - \mathcal{F}_J^r\right)^{-1}\left(\mathcal{F}_J^r - \mathcal{M}_J^r\mathcal{F}_J^r\right)\right)^{-1}\right\| \le \frac{1}{1 - \varepsilon_{J_0}\left\|\left(I - \mathcal{F}_J^r\right)^{-1}\right\|}. \tag{42}$$

Using (42), $(I - \mathcal{M}_J^r\mathcal{F}_J^r)$ exists. Taking norm and using (42), we obtain

$$\left\|\left(I-\mathcal{M}_J^r\mathcal{F}_J^r\right)^{-1}\right\| \le \frac{\left\|\left(I-\mathcal{F}_J^r\right)^{-1}\right\|}{1-\varepsilon_{J_0}\left\|\left(I-\mathcal{F}_J^r\right)^{-1}\right\|}. \tag{43}$$

Applying the operator $\mathcal{M}_J^r$ to both sides of (27) and then rearranging, we can obtain

$$\left(I-\mathcal{M}_J^r\mathcal{F}_J^r\right)u = \mathcal{M}_J^r g + \left(u-\mathcal{M}_J^r u\right) + \left(\mathcal{M}_J^r\mathcal{F} - \mathcal{M}_J^r\mathcal{F}_J^r\right)u. \tag{44}$$

Subtracting (35) from (44), we obtain

$$\left(I-\mathcal{M}_J^r\mathcal{F}_J^r\right)\left(u-u_J\right) = \left(u-\mathcal{M}_J^r u\right) + \left(\mathcal{M}_J^r\mathcal{F} - \mathcal{M}_J^r\mathcal{F}_J^r\right)u. \tag{45}$$

Taking norm and employing (43),

$$\|u-u_J\| \le \frac{\left\|\left(I-\mathcal{F}_J^r\right)^{-1}\right\|}{1-\varepsilon_{J_0}\left\|\left(I-\mathcal{F}_J^r\right)^{-1}\right\|}\left(\|u-\mathcal{M}_J^r u\| + \left\|\left(\mathcal{M}_J^r\mathcal{F} - \mathcal{M}_J^r\mathcal{F}_J^r\right)u\right\|\right). \tag{46}$$

This is equivalent to (37). It is straightforward to show that $\|(\mathcal{M}_J^r\mathcal{F} - \mathcal{M}_J^r\mathcal{F}_J^r)u\| \longrightarrow 0$ as $J \longrightarrow \infty$. Assume that $\{u_J\}$ be a sequence of continuous functions so that $u_J \longrightarrow u$ as $J \longrightarrow \infty$. Since the orthonormal projection $\mathcal{M}_J^r$ satisfies $\|\mathcal{M}_J^r\| = 1$, we can obtain

$$\begin{aligned}\left\|u-\mathcal{M}_J^r u\right\| &\le \|u-u_J\| + \left\|u_J-\mathcal{M}_J^r u_J^r\right\| + \left\|\mathcal{M}_J^r\left(u-u_J\right)\right\| \\ &\le 2\left\|u-u_J^r\right\| + \left\|u_J^r-\mathcal{M}_J^r u_J^r\right\|.\end{aligned} \tag{47}$$

Thus, for each real number $\varepsilon > 0$, there exists a number J_0 such that, for every number $J_0 \le J$, one can write $\|u-u_J\| \le (\varepsilon/4)$. This then implies that

$$\left\|u-\mathcal{M}_J^r u\right\| \le \frac{\varepsilon}{2} + \left\|u_J^r-\mathcal{M}_J^r u_J^r\right\|. \tag{48}$$

This implies that $\|u-\mathcal{M}_J^r u\| \le \varepsilon$, for sufficiently large value of J and because ε is arbitrary, consequently, $\mathcal{M}_J^r u \longrightarrow u$ as $J \longrightarrow \infty$. □

4. Numerical Experiments

To verify the accuracy and efficiency of the proposed method, we consider a series of numerical examples. In the linear type of equation (1), we aim to generate a sparse matrix to reduce the computational costs. We illustrate the rate of sparsity S_ε which is defined by [21,25]

$$S_\varepsilon = \frac{N_0 - N_\varepsilon}{N_0} \times 100\%, \tag{49}$$

where ε is the threshold (small positive number) and N_ε and N_0 are the number of elements remaining after thresholding and the total number of elements, respectively.

Example 1. Consider the linear VFIE given in [11] as

$$u(x) = e^{-x} - e^x(x-1) + \int_0^x e^{x+s}u(s)\mathrm{d}s - \int_0^1 e^{x+s}u(s)\mathrm{d}s. \tag{50}$$

The exact solution is e^{-x} [11].

The effects of the multiplicity parameter r, the refinement level J, and thresholding with different thresholds are reported in Table 1 and Figure 1. The results confirm the theoretical claims and demonstrate the effectiveness of the method. Note that the L^2 error decreases as parameters r and J increase due to the rate of convergence $O(2^{-\mathrm{J}r})$. We compare the error of the proposed method, the Lagrange collocation method [10], Taylor collocation method [11], and Taylor polynomial method [26] in Table 2. Due to Table 2, our method is flexible than other methods, and without changing the multiplicity parameter r, we can improve the results. Figure 2 illustrate the effect of thresholding with different threshold parameters ε on the coefficient matrix. It can be seen that the number of matrix elements decreases when the threshold parameter increases.

Example 2. Let us consider the following VFIE:

$$\begin{aligned}u(x) &= \frac{e^{2x}-1}{2}\cos(x) + e^x + \frac{1-e^2}{2}\sin(x) \\ &\quad + \int_0^x e^s\cos(x)u(s)\mathrm{d}s - \int_0^1 e^s\sin(x)u(s)\mathrm{d}s.\end{aligned} \tag{51}$$

In Table 3 and Figure 3, we show the effect of the parameters r, J, and ε on sparsity and L_2-error. It is obvious that increasing the parameters r and J reduces the error. A comparison of the proposed method with other methods such as Taylor method [26] and the Lagrange collocation [10] is reported in Table 4. In Figure 4, we illustrate the effect of thresholding on the coefficient matrix by taking different threshold ε when $r = 8$ and $J = 2$.

Example 3. Let us consider nonlinear VFIE (1) with

$$\begin{aligned}u(x) &= \sin(x) + \left(1-e^{\sin(1)}\right)x^3 - \frac{x^2-\sin^2(x)}{4} \\ &\quad - \int_0^x (x-s)u^2(s)\mathrm{d}s - \int_0^1 x^3\cos(s)e^{u(s)}\mathrm{d}s.\end{aligned} \tag{52}$$

The exact solution of this equation is $u(x) = \sin(x)$.

In Figure 5, we illustrate the effect of parameter r and J on the L^2-error and the error of approximation is plotted in Figure 6 taking $r = 7$ and $J = 2$.

Example 4. Consider the nonlinear VFIE as

$$u(x) = f(x) + \int_0^x \cos(x+s)u^2(s)\mathrm{d}s + \int_0^1 \sin(s-x)u^2(s)\mathrm{d}s, \tag{53}$$

with

$$\begin{aligned}f(x) &= \frac{(-2\cos(2x)-\sin(2x))e^{2x}}{5} + \frac{2e^2\sin(-1+x)}{5} \\ &\quad + \frac{e^2\cos(-1+x)}{5} + \frac{\cos(x)}{5} + e^x - \frac{\sin(x)}{5}.\end{aligned} \tag{54}$$

(1) Choose a vector w_1, such that $\|w_1\|_2 = 1$
(2) For $j = 1, 2, \ldots, m$ do
(3) Compute $h_{i,j} = (\Lambda w_j, w_i)$ for $i = 1, 2, \ldots, j$
(4) $w_j = (\Lambda w_j, w_i)$ for $i = 1, 2, \ldots, j$
(5) $h_{j+1,j} = \|w_j\|_2$
(6) If $h_{2j+1,j} = 0$ then stop
(7) $w_{j+1} = (w_j / h_{j+1,j})$
(8) End do

Algorithm 1: Arnoldi"s algorithm.

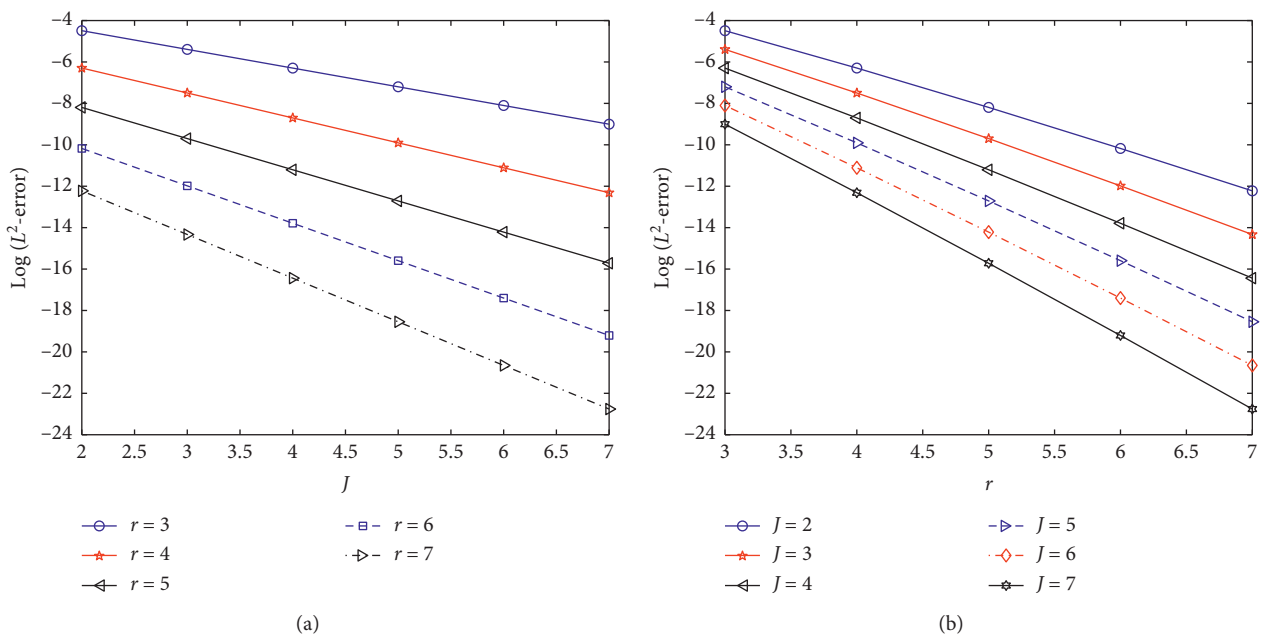

Figure 1: Effects of the refinement level J and multiplicity parameter r for Example 1.

(1) Compute $r_0 = D - \Lambda U_0$, $\beta = \|r_0\|_2$ *and* $w_1 = r_0/\beta$
(2) Generate the Arnoldi basis and the matrix $\overline{H}_m$ using the Arnoldi algorithm starting with w_1
(3) Compute y_m the minimizer of $\|\beta e_1 - \tilde{H}_m y\|_2$ and $U_m = U_0 + W_m y_m$
(4) If satisfied then stop, else set $U_0 = U_m$ and go to 1

Algorithm 2: Restarted (GMRES).

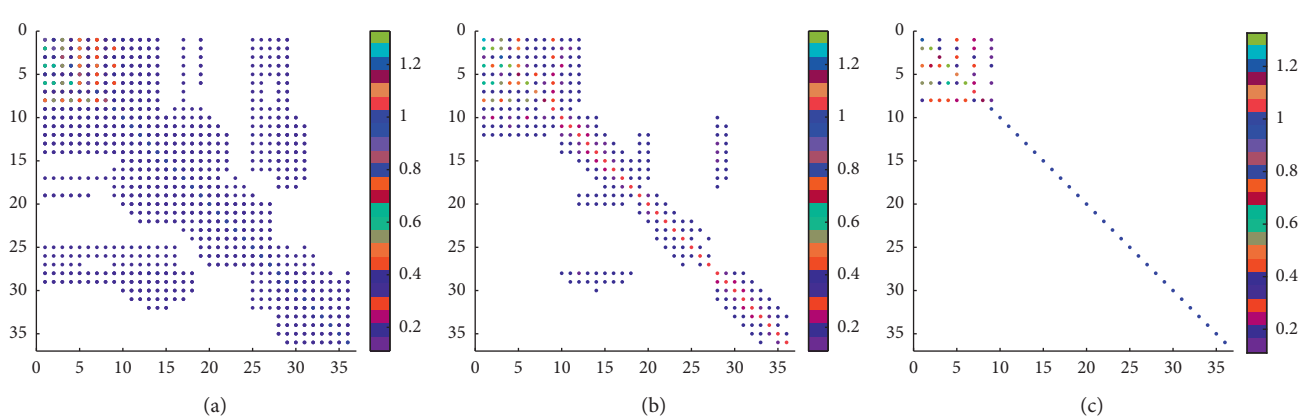

Figure 2: The effect of thresholding with threshold $\varepsilon = 10^{-5}$ (left), $\varepsilon = 10^{-3}$ (middle), and $\varepsilon = 10^{-1}$ (right) on the coefficient matrix for Example 1.

Table 1: Effects of parameters r, J, and ε on sparsity and L_2-error for Example 1.

r	J	Without thresholding		$\varepsilon = 10^{-5}$		$\varepsilon = 10^{-3}$		$\varepsilon = 10^{-1}$	
		S_ε	L_2-error	S_ε	L_2-error	S_ε	L_2-error	S_ε	L_2-error
5	2	0	$6.39e-9$	24.8	$1.20e-6$	54	$2.74e-4$	91	$2.96e-2$
	3	0	$2.00e-10$	59.8	$4.87e-6$	78.8	$2.78e-4$	96.5	$2.96e-2$
7	2	0	$5.97e-13$	43	$1.72e-6$	65.9	$1.38e-4$	97.5	$7.98e-3$
	3	0	$4.71e-15$	72.4	$2.65e-6$	85.4	$1.38e-4$	93.8	$7.98e-3$

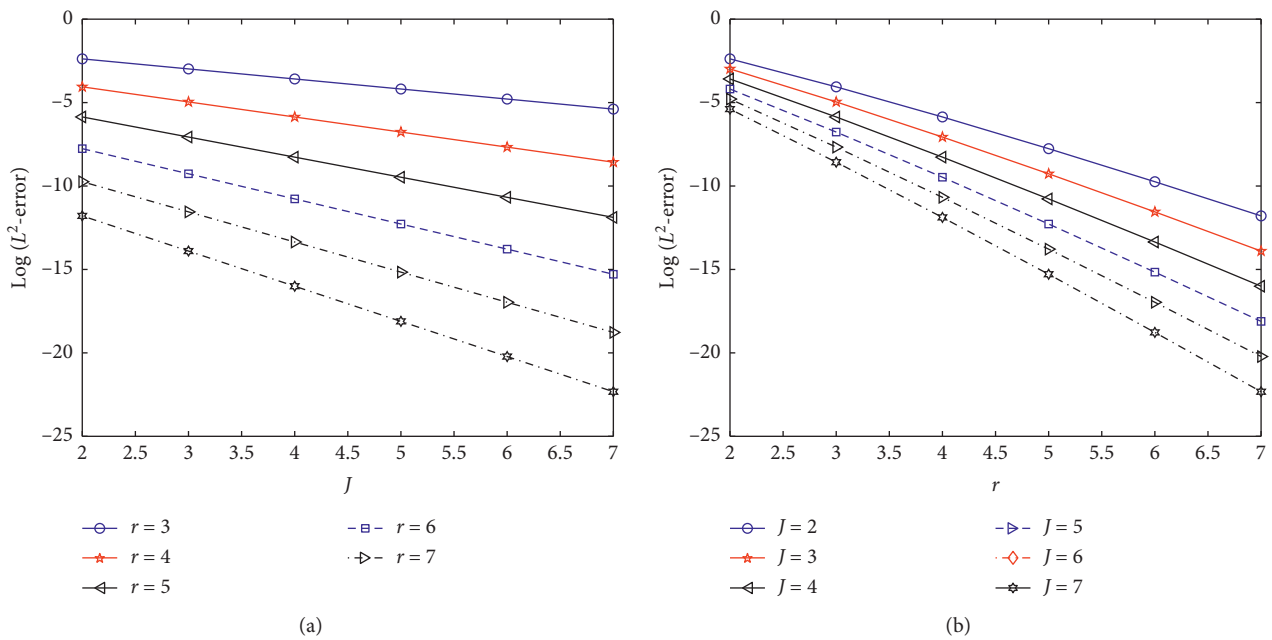

Figure 3: Effects of the refinement level J and multiplicity parameter r for Example 2.

Table 2: Comparison of the error for Example 1.

r	Presented method	Taylor collocation	Taylor polynomial method	Lagrange collocation
2	$1.55e-3$	$7.87e-2$	$3.41e-2$	$7.87e-2$
5	$6.39e-9$	$6.23e-5$	$3.68e-4$	$6.23e-5$
8	$4.63e-15$	$1.89e-8$	$1.24e-5$	$1.77e-7$
9	$4.01e-16$	$2.35e-8$	$3.46e-7$	$7.21e-6$

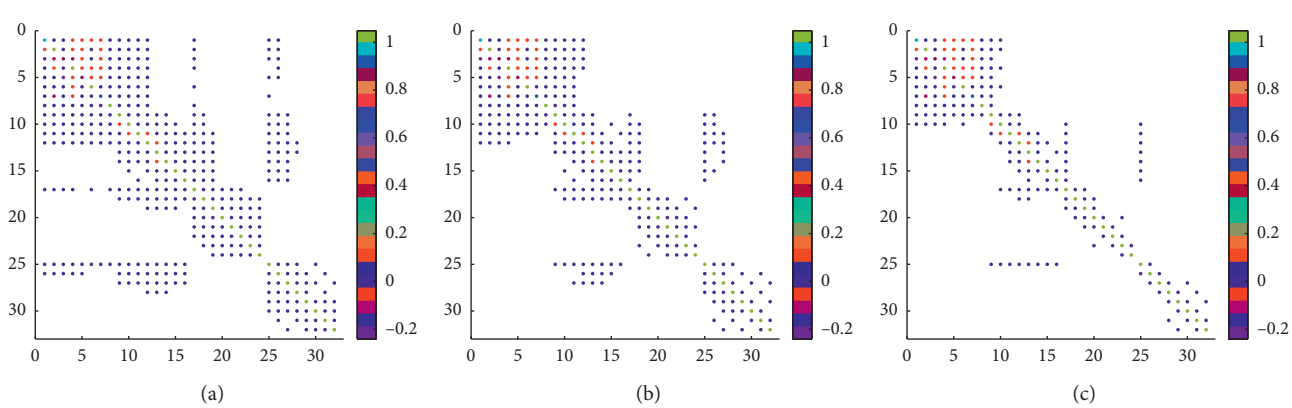

Figure 4: The effect of thresholding with threshold $\varepsilon = 10^{-5}$ (left), $\varepsilon = 10^{-4}$ (middle), and $\varepsilon = 10^{-3}$ (right) on the coefficient matrix for Example 2.

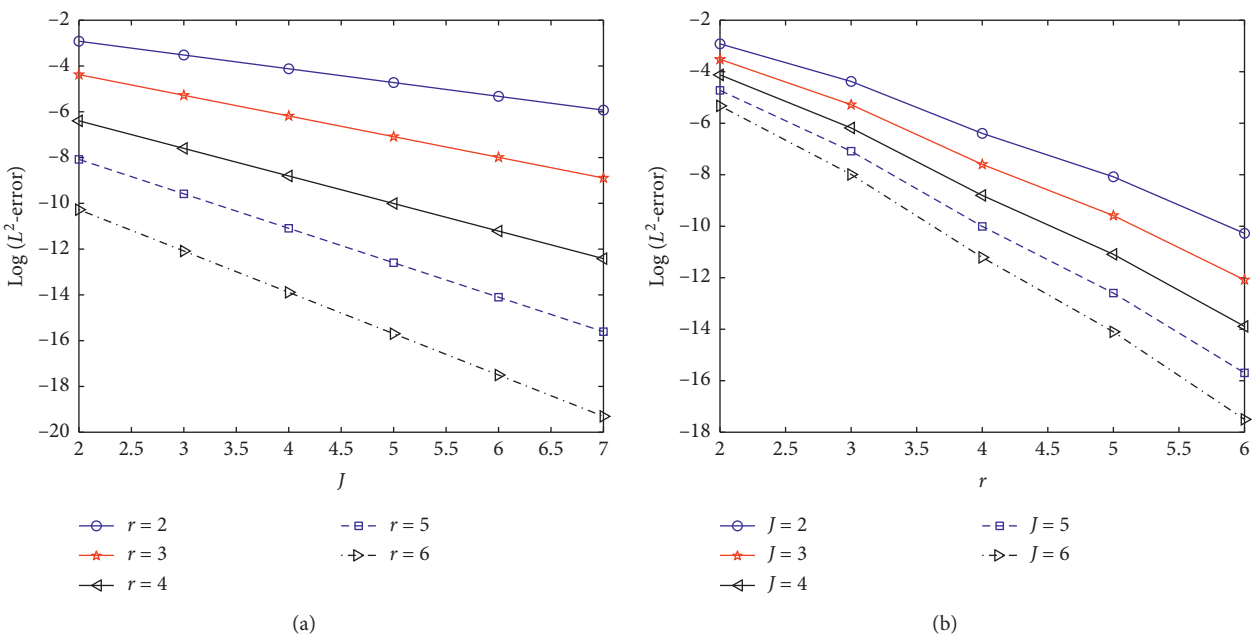

Figure 5: Effects of the refinement level J and multiplicity parameter r for Example 3.

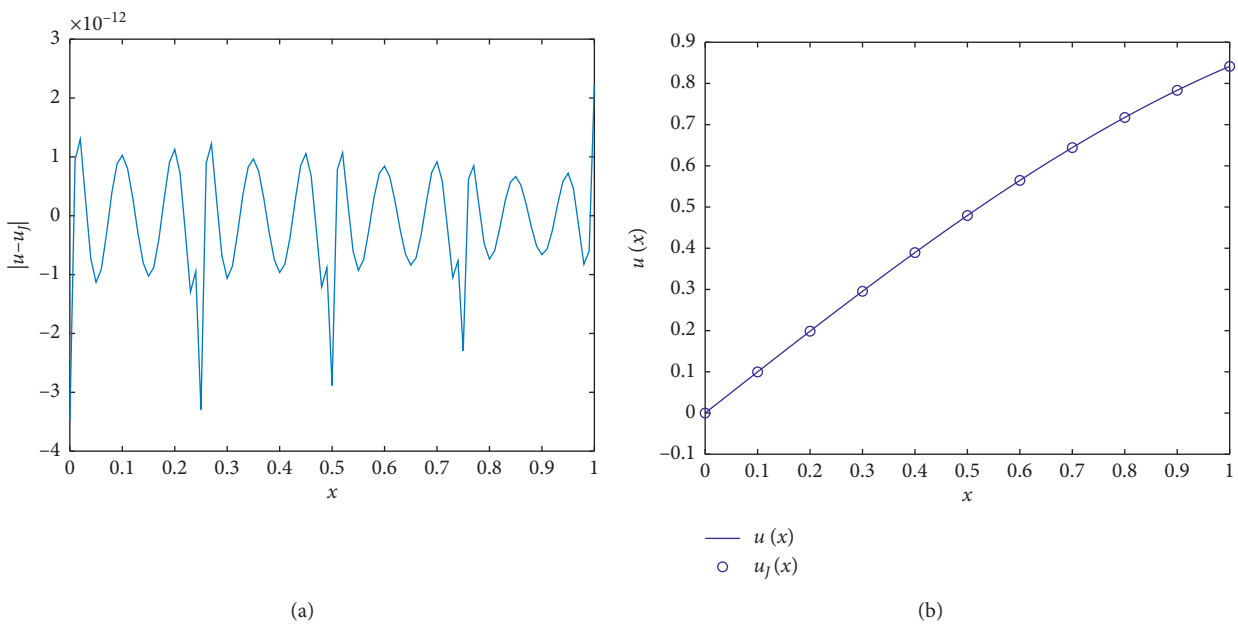

Figure 6: Error of Example 3.

Table 3: Effects of parameters r, J, and ε on sparsity and L_2-error for Example 2.

r	J	Without thresholding		$\varepsilon = 10^{-6}$		$\varepsilon = 10^{-4}$		$\varepsilon = 10^{-2}$	
		S_ε	L_2-error	S_ε	L_2-error	S_ε	L_2-error	S_ε	L_2-error
5	2	0	$1.73e-8$	22.2	$1.00e-6$	51.5	$1.74e-4$	79.2	$5.15e-3$
	3	0	$5.43e-10$	58.1	$1.18e-6$	76.2	$1.74e-4$	91.9	$5.15e-3$
6	2	0	$1.81e-10$	32.5	$5.24e-10$	59.5	$7.81e-5$	82.1	$1.37e-2$
	3	0	$2.84e-12$	65.3	$1.02e-6$	81.7	$7.80e-5$	93.7	$1.37e-2$
7	2	0	$1.62e-12$	39.4	$6.78e-12$	65.1	$9.95e-5$	84.8	$1.84e-2$
	3	0	$1.27e-14$	71.3	$2.74e-7$	84.7	$9.95e-5$	95	$1.84e-2$

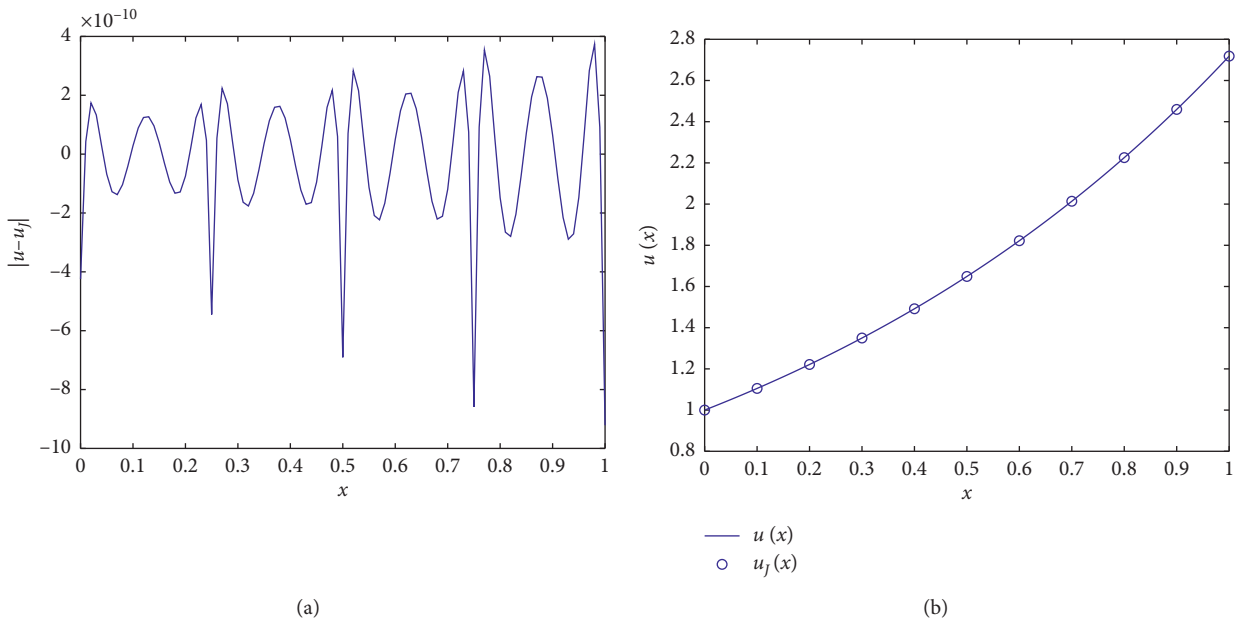

Figure 7: Error of Example 4.

Table 4: Comparison of the error for Example 2.

r	Presented method	Lagrange collocation	Taylor method
2	$4.17e-3$	$1.06e-2$	$4.51e-2$
5	$1.73e-8$	$1.12e-6$	$2.98e-4$
8	$1.27e-14$	$9.35e-7$	$6.14e-7$

Table 5: Effect of the refinement level J and multiplicity parameter r on L_2-error for Example 4.

r	$J=2$	$J=3$	$J=4$	$J=5$	$J=6$	$J=7$
2	$4.18e-3$	$1.04e-3$	$2.61e-4$	$6.53e-5$	$1.68e-5$	$4.15e-6$
3	$8.77e-5$	$1.10e-5$	$1.67e-6$	$1.71e-7$	$2.19e-8$	$2.68e-9$
4	$1.38e-6$	$8.92e-8$	$5.09e-9$	$3.31e-10$	$2.11e-11$	$1.32e-12$
5	$1.73e-8$	$5.71e-10$	$1.79e-11$	$5.28e-13$	$1.61e-14$	$5.16e-16$
6	$1.81e-10$	$2.23e-12$	$4.64e-14$	$6.95e-16$	$1.15e-17$	$1.69e-19$

The exact solution is $u(x)=e^x$.

Table 5 is reported to show the efficiency and accuracy of the proposed method. We observe when the refinement level J and multiplicity parameter r increase, the L^2-errors decrease. Figure 7 shows the error of proposed method on taking $r=6$ and $J=2$.

5. Conclusion

We have employed the multiwavelet Galerkin method to solve the Volterra–Fredholm integral equations. To this end, the Volterra and Fredholm operators are represented in multiwavelet bases. Applying this method leads to a linear or nonlinear system of algebraic equations. In the linear type, we obtain a new sparse system using thresholding due to the decay in the wavelet coefficients. The convergence analysis is investigated, and one can show that the rate of convergence is $O(2^{-Jr})$. The numerical examples illustrate the efficiency and accuracy of the method.

Data Availability

The raw/processed data required to reproduce these findings cannot be shared at this time due to legal or ethical reasons.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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Research Article

A Short Note on Wavelet Frames Based on FMRA on Local Fields

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The concept of frame multiresolution analysis (FMRA) on local fields of positive characteristic was given by Shah in his paper, Frame Multiresolution Analysis on Local Fields published by Journal of Operators. The author has studied the concept of minimum-energy wavelet frames on these prime characteristic fields. We continued the studies based on frame multiresolution analysis and minimum-energy wavelet frames on local fields of positive characteristic. In this paper, we introduce the notion of the construction of minimum-energy wavelet frames based on FMRA on local fields of positive characteristic. We provide a constructive algorithm for the existence of the minimum-energy wavelet frame on the local field of positive characteristic. An explicit construction of the frames and bases is given. In the end, we exhibit an example to illustrate our algorithm.

1. Introduction

Let K be a field and a topological space. Then, K is called a local field if both K^+ and K^* are locally compact abelian groups, where K^+ and K^* denote the additive and multiplicative groups of K, respectively. If K is any field and is endowed with the discrete topology, then K is a local field. Furthermore, if K is connected, then K is either $\mathbb{R}$ or $\mathbb{C}$. If K is not connected, then it is totally disconnected. Hence, by a local field, we mean a field K which is locally compact, nondiscrete, and totally disconnected. The p-adic fields are examples of local fields. For more details, refer [1]. In the rest of this paper, we use the symbols $\mathbb{N}, \mathbb{N}_0$ and $\mathbb{Z}$ to denote the sets of natural and nonnegative integers and integers, respectively.

Let K be a local field. Let $\mathrm{d}x$ be the Haar measure on the locally compact abelian group K^+. If $\alpha \in K$ and $\alpha \neq 0$, then $\mathrm{d}(\alpha x)$ is also a Haar measure. Let $\mathrm{d}(\alpha x) = |\alpha|\mathrm{d}x$. We call $|\alpha|$ the absolute value of α. Moreover, the map $x \longrightarrow |x|$ has the following properties: (a) $|x| = 0$ if and only if $x = 0$; (b) $|xy| = |x||y|$ for all $x, y \in K$; and (c) $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$. Property (c) is called the ultrametric inequality. The set $\mathfrak{D} = \{x \in K: |x| \leq 1\}$ is called the ring of integers in K. Define $\mathfrak{B} = \{x \in K: |x| < 1\}$. The set $\mathfrak{B}$ is called the prime ideal in K. The prime ideal in K is the unique maximal ideal in $\mathfrak{D}$, and therefore $\mathfrak{B}$ is principal ideal as well as prime ideal. Since the local field K is totally disconnected, there exists an element of $\mathfrak{B}$ of maximal absolute value. Let $\mathfrak{P}$ be a fixed element of the maximum absolute value in $\mathfrak{B}$. Such an element is called a prime element of K. Therefore, for such ideal $\mathfrak{B}$ in $\mathfrak{D}$, we have $\mathfrak{B} = \langle \mathfrak{P} \rangle = \mathfrak{P}\mathfrak{D}$. As it was proved in [1], the set $\mathfrak{D}$ is compact and open. Hence, $\mathfrak{B}$ is compact and open. Therefore, the residue space $\mathfrak{D}/\mathfrak{B}$ is isomorphic to a finite field $\mathrm{GF}(q)$, where $q = p^k$ for some prime p and $k \in \mathbb{N}$.

Let $\mathfrak{D}^* = \mathfrak{D} \setminus \mathfrak{B} = \{x \in K: |x| = 1\}$. Then, it can be proved that $\mathfrak{D}^*$ is a group of units in K^*, and if $x \neq 0$, then we may write $x = \mathfrak{P}^k x\prime, x\prime \in \mathfrak{D}^*$. For a proof of this fact, refer [1]. Moreover, each $\mathfrak{B}^k = \mathfrak{P}^k\mathfrak{D} = \{x \in K: |x| < q^{-k}\}$ is a compact subgroup of K^+ and usually known as the fractional ideals of K^+. Let $\mathcal{U} = \{a_i\}_{i=0}^{q-1}$ be any fixed full set of coset representatives of $\mathfrak{B}$ in $\mathfrak{D}$; then, every element $x \in K$ can be expressed uniquely as $x = \sum_{\ell=k}^{\infty} c_\ell \mathfrak{P}^\ell$ with $c_\ell \in \mathcal{U}$. Let χ be a fixed character on K^+ that is trivial on $\mathfrak{D}$ but is nontrivial on $\mathfrak{B}^{-1}$. Therefore, χ is a constant on cosets of $\mathfrak{D}$, so if $y \in \mathfrak{B}^k$, then $\chi_y(x) = \chi(yx)$, $x \in K$. Suppose that χ_u is any character on K^+; then, clearly, the restriction $\chi_u|\mathfrak{D}$ is also a character on $\mathfrak{D}$. Therefore, if $\{u(n): n \in \mathbb{N}_0\}$ is a complete list of the distinct coset representative of $\mathfrak{D}$ in K^+, then, as it

was proved in [1], the set $\{\chi_{u(n)}: n \in \mathbb{N}_0\}$ of distinct characters on $\mathfrak{D}$ is a complete orthonormal system on $\mathfrak{D}$.

The Fourier transform $\widehat{f}$ of a function $f \in L^1(K) \cap L^2(K)$ is defined by

$$\widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\mathrm{d}x. \tag{1}$$

It is noted that

$$\widehat{f}(\xi) = \int_K f(x)\overline{\chi_\xi(x)}\mathrm{d}x = \int_K f(x)\chi(-\xi x)\mathrm{d}x. \tag{2}$$

Furthermore, the properties of the Fourier transform on local field K are much similar to those on the real line. In particular, Fourier transform is unitary on $L^2(K)$.

We now impose a natural order on the sequence $\{u(n)\}_{n=0}^{\infty}$. We have $\mathfrak{D}/\mathfrak{B} \cong \mathrm{GF}(q)$, where $\mathrm{GF}(q)$ is a c-dimensional vector space over the field $\mathrm{GF}(p)$. We choose a set $\{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathfrak{D}^*$ such that span $\{\zeta_j\}_{j=0}^{c-1} \cong \mathrm{GF}(q)$. For $n \in \mathbb{N}_0$ satisfying

$$\begin{gathered} 0 \le n < q,\ n = a_0 + a_1 p + \cdots + a_{c-1}p^{c-1}, \\ 0 \le a_k < p,\ k = 0, 1, \ldots, c-1, \end{gathered} \tag{3}$$

we define

$$u(n) = (a_0 + a_1\zeta_1 + \cdots + a_{c-1}\zeta_{c-1})\mathfrak{P}^{-1}. \tag{4}$$

Also, for $n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s$, $n \in \mathbb{N}_0$, $0 \le b_k < q$, $k = 0, 1, 2, \ldots, s$, we set

$$u(n) = u(b_0) + u(b_1)\mathfrak{P}^{-1} + \cdots + u(b_s)\mathfrak{P}^{-s}. \tag{5}$$

This defines $u(n)$ for all $n \in \mathbb{N}_0$. In general, it is not true that $u(m+n) = u(m) + u(n)$. However, if $r, k \in \mathbb{N}_0$ and $0 \le s < q^k$, then $u(rq^k + s) = u(r)\mathfrak{P}^{-k} + u(s)$. Furthermore, it is also easy to verify that $u(n) = 0$ if and only if $n = 0$ and $\{u(\ell) + u(k): k \in \mathbb{N}_0\} = \{u(k): k \in \mathbb{N}_0\}$ for a fixed $\ell \in \mathbb{N}_0$. Hereafter, we use the notation $\chi_n = \chi_{u(n)}, n \ge 0$.

Let the local field K be of characteristic $t > 0$ and $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}$ be as above. We define a character χ on K as follows:

$$\chi\left(\zeta_\mu \mathfrak{P}^{-j}\right) = \begin{cases} \exp\left(\dfrac{2\pi i}{j}\right), & \mu = 0 \text{ and } j = 1, \\ 1, & \mu = 1, \ldots, c-1 \text{ or } j \ne 1. \end{cases} \tag{6}$$

In 2015, Shah [2] introduced the concept of frame multiresolution analysis (FMRA) on local fields, which can be sought as an extension of multiresolution analysis (MRA) on local fields of positive characteristic. First of all, let us recall the definition of FMRA as given by Shah. Let K be a local field of positive characteristic $p > 0$ and $\mathfrak{P}$ be a prime element of K. A frame multiresolution analysis (FMRA) of $L^2(K)$ is a sequence of closed subspaces $\{V_j: j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$, for all $j \in \mathbb{Z}$
(b) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(K)$
(c) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$
(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{P}^{-1}\cdot) \in V_{j+1}$, for all $j \in \mathbb{Z}$
(e) There is a function $\varphi \in V_0$ such that $\{\varphi(\cdot - u(k)): k \in \mathbb{N}_0\}$ forms a frame in V_0

The function φ is called a frame refinable function. It is noted that the shifts of φ form a tight frame in the above FMRA. Replacing "a tight frame" in the above by "an orthonormal or a Riesz base" will arrive on the definition of a MRA on local fields of positive characteristic.

A finite family $\Psi := \{\psi^1, \psi^2, \ldots, \psi^L\}$ generates a wavelet frame for $L^2(K)$ if there exist positive numbers $0 < A \le B < \infty$ such that, for all $f \in L^2(K)$,

$$A\|f\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f, \psi_{j,k}^\ell\rangle\right|^2 \le B\|f\|_2^2, \tag{7}$$

where $\psi_{j,k}^\ell = q^{j/2}\psi^\ell(\mathfrak{P}^{-j}\cdot - u(k))$. The largest constant A and the smallest constant B satisfying the above are called the lower and upper wavelet frame bound, respectively. A wavelet frame is a tight wavelet frame if A and B are chosen so that $A = B$, and then the set $\Psi := \{\psi^1, \psi^2, \ldots, \psi^L\}$ is called a set of generators for the corresponding tight wavelet frame. Furthermore, the wavelet frame is called a Parseval wavelet frame if $A = B = 1$, i.e.,

$$\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\left|\langle f, \psi_{j,k}^\ell\rangle\right|^2 = |f|_2^2, \quad \text{for all } f \in L^2(K), \tag{8}$$

and in this case, every function $f \in L^2(K)$ can be written as

$$f(x) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{N}_0}\langle f, \psi_{j,k}^\ell\rangle\psi_{j,k}^\ell(x). \tag{9}$$

A tight wavelet frame Ψ is called a FMRA tight frame on local fields with frame bound 1 if $\Psi \subset V_1$. Here, in this article, we are concerned with a minimum-energy wavelet frame which is more restrictive than a FMRA tight frame on local fields of positive characteristic. Here, we recall the definition of minimum-energy wavelet frames on local fields of positive characteristic [3].

Definition 1. Let $\varphi \in L^2(K)$ satisfy $\widehat{\varphi} \in L^\infty$, $\widehat{\varphi}$ be continuous at 0, and $\widehat{\varphi}(0) = 1$. Suppose that φ generates the nested closed subspaces $\{V_j: j \in \mathbb{Z}\}$. Then, a finite family $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\} \subset V_1$ is called a minimum-energy wavelet frame associated with φ if

$$\sum_{k\in\mathbb{N}_0}\left|\langle f, \varphi_{1,k}\rangle\right|^2 = \sum_{k\in\mathbb{N}_0}\left|\langle f, \varphi_{0,k}\rangle\right|^2 + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\left|\langle f, \psi_{0,k}^\ell\rangle\right|^2, \quad \text{for all } f \in L^2(K), \tag{10}$$

where $\varphi_{j,k}(\cdot) = q^{j/2}\varphi(\mathfrak{P}^{-1}\cdot - u(k))$. By the Parseval identity, minimum-energy wavelet frame Ψ must be a tight frame for $L^2(K)$ with the frame bound being equal to 1. At the same time, the above equation is equivalent to

$$\sum_{k\in\mathbb{N}_0}\langle f,\varphi_{1,k}\rangle\varphi_{1,k} = \sum_{k\in\mathbb{N}_0}\langle f,\varphi_{0,k}\rangle\varphi_{0,k} + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{N}_0}\langle f,\psi_{0,k}^{\ell}\rangle\psi_{0,k}^{\ell}, \quad \text{for all } f\in L^2(K). \tag{11}$$

Motivated and inspired by various constructions of minimum-energy wavelet frames [4–10] and classical wavelet frames on finite fields [11–13], we, in this paper, discuss some constructions of minimum-energy wavelet frames which are based on the frame multiresolution analysis on local fields of positive characteristic. This paper is organized in the following manner. In Section 2, we present some preliminaries for the FMRA and the minimum-energy wavelet frames on local fields of positive characteristic. In Section 3, we present the main results. Here, we provide a constructive algorithm for the existence of the minimum-energy wavelet frame on the local field of positive characteristic. We also construct an example to illustrate our algorithm.

2. Notations and Preliminaries

Here, we present some preliminaries for the FMRA and the minimum-energy wavelet frames on local fields of positive characteristic.

From the definition of FMRA on local fields, we know that $V_0 \subset V_1 = \overline{\operatorname{span}\{\varphi(\mathfrak{P}^{-1}\cdot - u(k)): k\in\mathbb{N}_0\}}$. Since $\varphi(\cdot)\in V_0$, there exists a sequence $\{h_k\}_{k\in\mathbb{N}_0}\in l^2(\mathbb{N}_0)$ such that

$$\varphi(x) = \sqrt{q}\sum_{k\in\mathbb{N}_0} h_k\varphi\left(\mathfrak{P}^{-1}x - u(k)\right). \tag{12}$$

The Fourier transform of (12) yields

$$\widehat{\varphi}(\xi) = m_0(\mathfrak{P}\xi)\widehat{\varphi}(\mathfrak{P}\xi), \tag{13}$$

where

$$m_0(\xi) = \frac{1}{\sqrt{q}}\sum_{k\in\mathbb{N}_0} h_k\overline{\chi_k(\xi)}, \tag{14}$$

is an integral periodic function in $L^2(\mathfrak{D})$, where $\mathfrak{D} = \{x\in K: |x|\le 1\}$ is the ring of integers in K and is often called the refinement symbol of φ. Observe that $\chi_k(0) = \widehat{\varphi}(0) = 1$. Therefore, by letting $\xi = 0$ in (13) and (14), we obtain $\sum_{k\in\mathbb{N}_0} h_k = 1$.

Consider $\Psi = \{\psi^1,\psi^2,\ldots,\psi^L\}\subset V_1$, with

$$\psi^{\ell}(x) = \sqrt{q}\sum_{k\in\mathbb{N}_0} h_k^{\ell}\varphi\left(\mathfrak{P}^{-1}x - u(k)\right), \quad \ell = 1,2,\ldots,L. \tag{15}$$

Equation (15) can be written in the frequency domain as

$$\widehat{\psi^{\ell}}(\xi) = m_{\ell}(\mathfrak{P}\xi)\widehat{\varphi}(\mathfrak{P}\xi), \tag{16}$$

where

$$m_{\ell}(\xi) = \frac{1}{\sqrt{q}}\sum_{k\in\mathbb{N}_0} h_k^{\ell}\overline{\chi_k(\xi)}, \quad \ell = 1,2,\ldots,L, \tag{17}$$

are the integral periodic function in $L^2(\mathfrak{D})$ and are called the framelet symbols or wavelet masks.

With $m_{\ell}(\xi), \ell = 0,1,\ldots,L$, as framelet symbols, we formulate the $q\times(L+1)$ matrix $\mathscr{M}(\xi)$ as

$$\mathscr{M}(\xi) = \begin{pmatrix} m_0(\xi+\mathfrak{P}u(0)) & m_1(\xi+\mathfrak{P}u(0)) & \cdots & m_L(\xi+\mathfrak{P}u(0)) \\ m_0(\xi+\mathfrak{P}u(1)) & m_1(\xi+\mathfrak{P}u(1)) & \cdots & m_L(\xi+\mathfrak{P}u(1)) \\ \vdots & \vdots & \ddots & \vdots \\ m_0(\xi+\mathfrak{P}u(q-1)) & m_1(\xi+\mathfrak{P}u(q-1)) & \cdots & m_L(\xi+\mathfrak{P}u(q-1)) \end{pmatrix}. \tag{18}$$

Shah and Debnath [3] proved that if $\Psi = \{\psi^1,\psi^2,\ldots,\psi^L\}\subset V_1$ forms a minimum-energy wavelet frame in $L^2(K)$, then the mask matrix $\mathscr{M}(\xi)$ should satisfy certain conditions as follows.

Lemma 1. *Suppose that the refinable function φ and the framelet symbols $m_{\ell}(\xi), \ell = 0,1,\ldots,L$, satisfy (13)–(17). If $\widehat{\varphi}$ is continuous at 0 and $\varphi(x)$ generates a sequence of nested closed subspaces $\{V_j: j\in\mathbb{Z}\}$, then the following statements are equivalent:*

(1) Ψ is a minimum-energy frame associated with φ.

(2) $\mathscr{M}(\xi)\mathscr{M}^*(\xi) = I_q$, I_q is an identity matrix of order q (19).

(3) $\alpha_{r,s} = \sum_{k\in\mathbb{N}_0}\left\{h_{r-qk}h_{s-qk} + \sum_{\ell=1}^{L} h_{r-qk}^{\ell}h_{s-qk}^{\ell}\right\} - q\delta_{r,s} = 0$, $\forall r,s\in\mathbb{N}_0$.

Lemma 1 gives the necessary and sufficient condition for the existence of the minimum-energy wavelet frames associated with refinable function φ. However, it is not a good choice to use this theorem to construct the minimum-energy wavelet frames. For convenience, Shah and Debnath [3] presented some conditions in terms of the framelet symbols.

Lemma 2. *Let $\varphi \in L^2(K)$ be the refinable function with refinement mask $m_0(\xi)$ such that $\widehat{\varphi}$ is continuous at 0 and $\widehat{\varphi}(0) = 1$. If $\Psi = \{\psi^1, \psi^2, \ldots, \psi^L\}$ is the minimum-energy wavelet frame associated with φ, then*

$$\sum_{r=0}^{q-1} \left| m_0(\mathfrak{P}\xi + \mathfrak{P}u(r)) \right|^2 \le 1, \quad \text{for all } \xi \in K. \tag{19}$$

In this paper, we start with the orthogonal vectors to obtain an explicit construction for the minimum-energy wavelet frame on local fields of positive characteristic.

3. Main Results

From Lemma 1, we know that if we want to obtain a minimum-energy wavelet frame on local fields of positive characteristic, we should find L functions whose masks satisfy (19). Note the correlation of the rows of $\mathscr{M}(\xi)$; we should remove this feature first. For this, we introduce the polyphase decomposition technique. Similar to [3], we write

$$\begin{aligned} m_0(\xi) &= \frac{1}{\sqrt{q}} \left\{ \overline{\chi_{u(0)}(\xi)} f_1(\xi) + \overline{\chi_{u(1)}(\xi)} f_2(\xi) + \cdots + \overline{\chi_{u(q-1)}(\xi)} f_q(\xi) \right\}, \\ m_\ell(\xi) &= \frac{1}{\sqrt{q}} \left\{ \overline{\chi_{u(0)}(\xi)} g^{\ell 1}(\xi) + \overline{\chi_{u(1)}(\xi)} g^{\ell 2}(\xi) + \cdots + \overline{\chi_{u(q-1)}(\xi)} g^{\ell q}(\xi) \right\}, \end{aligned} \tag{20}$$

where $f_r(\xi)$ and $g^{\ell r}, r = 1, 2, \ldots, q$, are the polyphase decompositions of $m_0(\xi)$ and $m_\ell(\xi)$, respectively, and all these functions are $1/q$–periodic. Let

$$\begin{aligned} \mathscr{P}(\xi) &= \begin{pmatrix} f_1(\xi) & g^{11}(\xi) & \cdots & g^{\ell 1}(\xi) \\ f_2(\xi) & g^{12}(\xi) & \cdots & g^{\ell 2}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ f_{q-1}(\xi) & g^{1q-1}(\xi) & \cdots & g^{\ell q-1}(\xi) \end{pmatrix}, \\ \mathscr{S}(\xi) &= \begin{pmatrix} \chi_{u(0)} & \chi_{u(1)} & \cdots & \chi_{u(q-1)} \\ \chi_{u(0)} & \chi_{\mathfrak{P}u(1)+u(1)} & \cdots & \chi_{\mathfrak{P}u(1)+u(q-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \chi_{u(0)} & \chi_{\mathfrak{P}u(q-1)+u(1)} & \cdots & \chi_{\mathfrak{P}u(q-1)+u(q-1)} \end{pmatrix}. \end{aligned} \tag{21}$$

Then, we can easily see that $\mathscr{M}(\xi) = (1/\sqrt{q})\mathscr{S}(\xi)\mathscr{P}(\xi)$, and (19) is equivalent to

$$\mathscr{P}(\xi)\mathscr{P}(\xi)^* = I_q. \tag{22}$$

The difference of (19) and (25) is that the rows of (25) are irrelevant to one another. Since $|m_0(\xi + \mathfrak{P}u(0))|^2 + |m_0(\xi + \mathfrak{P}u(1))|^2 = |f_1(\xi + \mathfrak{P}u(0))|^2 + |f_2(\xi + \mathfrak{P}u(0))|^2$, we have $|f_1(\xi + \mathfrak{P}u(0))|^2 + |f_2(\xi + \mathfrak{P}u(0))|^2 \le 1$.

Riesz lemma tells us that there can exist a function $f_3(\xi + \mathfrak{P}u(0))$ such that $|f_1(\xi + \mathfrak{P}u(0))|^2 + |f_2(\xi + \mathfrak{P}u(0))|^2 + |f_3(\xi + \mathfrak{P}u(0))|^2 = 1$. When $|f_1(\xi + \mathfrak{P}u(0))|^2 + |f_2(\xi + \mathfrak{P}u(0))|^2 = 1$, then $f_3(\xi + \mathfrak{P}u(0)) = 0$, which is the special case of the orthonormal wavelet base on local fields of positive characteristic. With $f_3(\xi + \mathfrak{P}u(0))$ in hand, we can have a vector, in fact, a unit column vector, $\zeta_0 = (f_1(\xi + \mathfrak{P}u(0)), f_2(\xi + \mathfrak{P}u(0)), f_3(\xi + \mathfrak{P}u(0)))^T$. Now, we expect the existence of two more unit column vectors ζ_1 and ζ_2 such that

$$\zeta_0^T \cdot \zeta_1 = \zeta_0^T \cdot \zeta_2 = \zeta_1^T \cdot \zeta_2 = 0. \tag{23}$$

In fact, ζ_1 and ζ_2 form an orthonormal fundamental system of solutions of the linear equation $\zeta_0^T \cdot x = 0$. By straightforward calculation, an orthonormal fundamental system of solutions is

$$\begin{aligned} \zeta_1 &= \left(-\frac{\overline{f_3}}{\Delta}, 0, \frac{\overline{f_1}}{\Delta} \right)^T, \\ \zeta_2 &= \left(-\frac{f_1\overline{f_2}}{\Delta}, \Delta, -\frac{\overline{f_2}f_3}{\Delta} \right)^T. \end{aligned} \tag{24}$$

Here, $\Delta^2 = |f_1|^2 + |f_3|^2 = 1 - |f_2|^2$. Here, we notice that if we choose $\mathscr{P}(\xi)$ as the first two rows of the following matrix

$$(\zeta_0, \zeta_1, \zeta_2) = \begin{pmatrix} f_1 & -\frac{\overline{f_3}}{\Delta} & -\frac{f_1\overline{f_2}}{\Delta} \\ f_2 & 0 & \Delta \\ f_3 & \frac{\overline{f_1}}{\Delta} & -\frac{\overline{f_2}f_3}{\Delta} \end{pmatrix}, \tag{25}$$

then $\mathscr{P}(\xi)$ is a unitary matrix which implies that the corresponding matrix $\mathscr{M}(\xi)$ is a mask matrix. Moreover, the minimum-energy wavelet frame matrix $\mathscr{M}(\xi)$ on local fields has the shape

$$\mathscr{P}(\xi) = \begin{pmatrix} f_1 & -\frac{\overline{f_3}}{\Delta} & -\frac{f_1\overline{f_2}}{\Delta} \\ f_2 & 0 & \Delta \end{pmatrix} \mathscr{N}(\xi), \tag{26}$$

where $\mathscr{N}(\xi)$ is a square matrix which is also unitary.

Given a refinement function $\varphi(x)$, the refinement mask $m_0(\xi + \mathfrak{P}u(0))$ should satisfy $m_0(\xi + \mathfrak{P}u(0)) + m_0(\xi + \mathfrak{P}u(1)) + m_0(\xi + \mathfrak{P}u(2)) = 1$, which is the same as to the first column of (26). So, we can select the orthonormal wavelet masks as those of (26). So, all the orthonormal wavelet masks are of the shape

$$(\zeta_0, \zeta_1, \zeta_2) = \begin{pmatrix} f_1 & -\frac{\overline{f_3}}{\Delta} & -\frac{f_1\overline{f_2}}{\Delta} \\ f_2 & 0 & \Delta \\ f_3 & \frac{\overline{f_1}}{\Delta} & -\frac{\overline{f_2}f_3}{\Delta} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{N}(\xi) \end{pmatrix}. \quad (27)$$

4. Example

Let us consider the case of the Haar wavelet. The refinement function of the Haar wavelet on local fields will be given by $\varphi(x) = \chi_{\mathfrak{D}}(x)$, and its refinement mask will be $m_0(\xi) = (1/q)(1 + \overline{\chi(\xi)})$. It can be observed that $m(\xi)$ satisfies $|m_0(\xi + \mathfrak{P}u(0))|^2 + |m_0(\xi + \mathfrak{P}u(1))|^2 = 1$, and the corresponding polyphase decompositions are $f_1 = f_2 = 1/\sqrt{q}$. Here, we notice that the polyphase decompositions of the orthonormal wavelet masks $m_\ell(\xi)$ are $g^{\ell 1}(\xi) = -1/\sqrt{q}$ and $g^{\ell 2}(\xi) = 1/\sqrt{q}$. Hence, the wavelet masks are given by $m_\ell(\xi) = -(1/\sqrt{q})(1 - \overline{\chi_\ell(\xi)})$.

Data Availability

The data used to support the findings of this study are available upon request to the author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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