

Fractional Difference and Differential Operators and their Applications in Nonlinear Systems

Lead Guest Editor: Qasem M. Al-Mdallal

Guest Editors: Thabet Abdeljawad and Fahd Jarad





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Discrete Dynamics in Nature and Society

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



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Research Article

Analysis of a Fractional Reaction-Diffusion HBV Model with Cure of Infected Cells

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In this paper, we propose a fractional reaction-diffusion model in order to better understand the mechanisms and dynamics of hepatitis B virus (HBV) infection in human body. The infection transmission is modeled by Hattaf–Yousfi functional response, and the fractional derivative is in the sense of Caputo. The global stability of the model equilibria is analyzed by means of Lyapunov functionals. Finally, numerical simulations are presented to support our analytical results.

1. Introduction

In the recent years, fractional calculus has attracted the attention of many researchers. Hattaf [1] proposed a new fractional derivative with nonsingular kernel which generalizes many forms existing in the literature such as the Caputo–Fabrizio and Atangana–Baleanu fractional derivatives. Furthermore, there are some new methods used to solve numerically fractional models considered to explain deeper investigations of real-world problems [2–6].

On the other hand, hepatitis B is a dangerous infectious disease caused by the hepatitis B virus (HBV). It affects the lives of 257 million people and responsible for the deaths of 56000 people every year according to World Health Organization (WHO) estimates [7]. Therefore, several mathematical models have been proposed and developed to describe the dynamics of HBV infection. For instance, Manna and Chakrabarty [8] proposed and analyzed the dynamics of HBV infection by taking into account the spacial mobility of both HBV DNA-containing capsids and viruses. Their work was an extension of that presented in [9]. Guo et al. [10] studied a nonlinear system of partial differential

equations (PDEs) for HBV infection with three time delays, general incidence rate, and spatial diffusion only in the viruses. Hattaf and Yousfi [11] developed a mathematical HBV infection model with two modes of transmission, which allows the movement of HBV DNA-containing capsids and viruses, and three distributed delays. Since fractional-order models possess property of memory, Bachraoui et al. [12] proposed a mathematical model governed by fractional differential equations (FDEs) to more explore the dynamic characteristics of the HBV infection. They have improved and generalized the ordinary differential equation (ODE) models [9, 13] and also the FDE models [14–16] by using Hattaf's incidence rate [17] that includes the common types such as the bilinear incidence rate, the saturated incidence rate, and the Beddington–DeAngelis functional response [18, 19].

In this study, we present an extension of our model presented in [12] by considering the mobility of capsids and viruses. So, we propose the following mathematical model formulated by fractional partial differential equations (FPDEs) to better describe the dynamics of HBV infection under the effects of diffusion and memory:

$$\begin{cases} \partial_t^\alpha U = \sigma - \delta U(x, t) - F(U(x, t), V(x, t))V(x, t) + \varepsilon I(x, t), \\ \partial_t^\alpha I = F(U(x, t), V(x, t))V(x, t) - (\rho + \varepsilon)I(x, t), \\ \partial_t^\alpha C = d_C \Delta C + \kappa I(x, t) - (\rho + \eta)C(x, t), \\ \partial_t^\alpha V = d_V \Delta V + \eta C(x, t) - \nu V(x, t). \end{cases} \quad (1)$$

The state variables $U(x, t)$, $I(x, t)$, $C(x, t)$, and $V(x, t)$ are, respectively, the concentrations of uninfected liver cells, infected liver cells, and HBV DNA-containing capsids and virions at location x and time t . Uninfected liver cells are produced at constant rate σ , die at rate δU , and become infected by virus at rate $F(U, V)V$. The parameters ε , ρ , κ , η , and ν are, respectively, the cure rate of infected liver cells, the death rate of infected liver cells and capsids, the production rate of capsids from infected liver cells, the rate at which the capsids are converted to virions, and the viral clearance rate. The positive constants d_C and d_V are the diffusion coefficients of capsids and virus. $\Delta = \sum_{i=1}^n \partial^2 / \partial x_i^2$ is the Laplacian operator. The incidence function of (1) is described by Hattaf–Yousfi functional response [17] of the form $F(U, V) = ((bU)/(\alpha_0 + \alpha_1 U + \alpha_2 V + \alpha_3 UV))$, where the nonnegative constants α_i , $i = 0, 1, 2, 3$, measure the saturation, inhibitory, or psychological effects and the positive constant b is the infection rate. This functional response covers several specific cases available in the literature such as the bilinear and saturation incidences, the Beddington–DeAngelis and Crowley–Martin functional responses, and the specific functional response introduced by Hattaf et al. [20]. Finally, ∂_t^α is the Caputo fractional derivative of order $\alpha \in (0, 1]$. The choice of this type of fractional derivative is motivated by the fact that the fractional derivative of a constant is equal to zero, and α was chosen in the interval $(0, 1]$ to have the same initial conditions as those of the PDE systems. Furthermore, a recent study in [21] has shown that the fractional-order model gives better predictions than that of the integer model about the plasma viral load of the patients.

Throughout this paper, we consider system (1) with initial conditions

$$\begin{aligned} U(x, 0) &= U_0(x) \geq 0, \\ I(x, 0) &= I_0(x) \geq 0, \\ V(x, 0) &= V_0(x) \geq 0, \\ C(x, 0) &= C_0(x) \geq 0, \end{aligned} \quad (2)$$

$$\forall x \in \overline{\Omega},$$

and zero-flux boundary conditions

$$\frac{\partial C}{\partial \vec{n}} = \frac{\partial V}{\partial \vec{n}} = 0, \quad \text{on } \partial\Omega \times (0, +\infty), \quad (3)$$

where Ω is a bounded domain in \mathbb{R}^n with smooth boundary $\partial\Omega$ and $\partial/\partial \vec{n}$ denotes the outward normal derivative on $\partial\Omega$. From the biological point of view, these boundary conditions indicate that the capsids and free viral particles do not move across the boundary $\partial\Omega$.

The rest of the paper is organized as follows. The following section is devoted to the calculations of the basic reproduction number and steady states of model (1). The global dynamics of the FPDE model is analyzed in Section 3. To support the analytical results, we present some numerical simulations in Section 4. Finally, we end the paper with biological and mathematical conclusions in Section 5.

2. Equilibria of the FPDE Model

It is easy to verify that the only infection-free steady state of the FPDE model (1) is $P_0(U^0, 0, 0, 0)$, where $U^0 = (\sigma/\delta)$. Then, the basic reproduction number of (1) is given by

$$\mathcal{R}_0 = \frac{\kappa \eta F(U^0, 0)}{\nu(\eta + \rho)(\varepsilon + \rho)}. \quad (4)$$

The other steady states verify the following system:

$$\sigma - \delta U - F(U, V)V + \varepsilon I = 0, \quad (5)$$

$$F(U, V)V - (\varepsilon + \rho)I = 0, \quad (6)$$

$$\kappa I - (\eta + \rho)C = 0, \quad (7)$$

$$\eta C - \nu V = 0. \quad (8)$$

From (5)–(8), we obtain $I = ((\sigma - \delta U)/\rho)$, $C = ((\kappa(\sigma - \delta U))/(\rho(\eta + \rho)))$, $V = ((\eta\kappa(\sigma - \delta U))/(\nu\rho(\eta + \rho)))$, and

$$F\left(U, \frac{\eta\kappa(\sigma - \delta U)}{\nu\rho(\eta + \rho)}\right) = \frac{\nu(\eta + \rho)(\rho + \varepsilon)}{\kappa\eta}. \quad (9)$$

$I = ((\sigma - \delta U)/\rho) \geq 0$ implies that $U \leq (\sigma/\delta)$. So, we consider the function G defined on interval $[0, (\sigma/\delta)]$ by

$$G(U) = F\left(U, \frac{\eta\kappa(\sigma - \delta U)}{\nu\rho(\eta + \rho)}\right) - \frac{\nu(\eta + \rho)(\rho + \varepsilon)}{\kappa\eta}. \quad (10)$$

We have $G(0) = -((\nu(\eta + \rho)(\rho + \varepsilon))/\kappa\eta) < 0$, $G(\sigma/\delta) = ((\nu(\eta + \rho)(\rho + \varepsilon))/\kappa\eta)(\mathcal{R}_0 - 1)$, and

$$G'(U) = \frac{\partial F}{\partial U} - \frac{\kappa\eta\delta}{\nu\rho(\eta + \rho)} \frac{\partial F}{\partial V} > 0. \quad (11)$$

If $\mathcal{R}_0 > 1$, we deduce that system (1) admits a unique infection equilibrium $P_1(U_1, I_1, C_1, V_1)$ with $U_1 \in (0, (\sigma/\delta))$, $I_1 = ((\sigma - \delta U_1)/\rho)$, $C_1 = (\kappa(\sigma - \delta U_1))/(\rho(\eta + \rho))$, and $V_1 = (\eta\kappa(\sigma - \delta U_1))/(\nu\rho(\eta + \rho))$.

We summarize the above discussions in the following result.

Theorem 1

- (i) When $\mathcal{R}_0 \leq 1$, the FPDE model (1) has one infection-free steady state $P_0(U^0, 0, 0, 0)$, where $U^0 = (\sigma/\delta)$
- (ii) When $\mathcal{R}_0 > 1$, the FPDE model (1) has uniquely one chronic infection steady state $P_1(U_1, I_1, C_1, V_1)$, where $U_1 \in (0, (\sigma/\delta))$, $I_1 = ((\sigma - \delta U_1)/\rho)$, $C_1 = (\kappa(\sigma - \delta U_1))/(\rho(\eta + \rho))$, and $V_1 = (\eta\kappa(\sigma - \delta U_1))/(\nu\rho(\eta + \rho))$

3. Global Dynamics

This section analyzes the global dynamics of the FPDE model (1).

Theorem 2. *The infection-free steady state P_0 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$.*

Proof. Let

$$L_0(t) = \int_{\Omega} \left[\frac{\alpha_0}{\alpha_0 + \alpha_1 U^0} U^0 \Phi\left(\frac{U}{U^0}\right) + \frac{\alpha_0 \varepsilon (U(x, t) - U^0 + I(x, t))^2}{2(\rho + \delta)(\alpha_0 + \alpha_1 U^0)U^0} + I(x, t) + \frac{\varepsilon + \rho}{\kappa} C(x, t) + \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa \eta} V(x, t) \right] dx, \quad (12)$$

where $\Phi(x) = x - 1 - \ln(x)$ for $x > 0$. According to [22], we obtain

$$D^\alpha L_0(t) \leq \int_{\Omega} \left[\frac{\alpha_0}{\alpha_0 + \alpha_1 U^0} \left(1 - \frac{U^0}{U}\right) \partial_t^\alpha U + \partial_t^\alpha I + \frac{\alpha_0 \varepsilon (U - U^0 + I)(\partial_t^\alpha U + \partial_t^\alpha I)}{(\rho + \delta)(\alpha_0 + \alpha_1 U^0)U^0} \frac{\varepsilon + \rho}{\kappa} \partial_t^\alpha C + \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa \eta} \partial_t^\alpha V \right] dx. \quad (13)$$

By $\sigma = \delta U^0$, we have

$$\begin{aligned} D^\alpha L_0(t) &\leq \int_{\Omega} \left[\left(\frac{1}{U} + \frac{\varepsilon}{(\rho + \delta)U^0} \right) \frac{\alpha_0 \delta (U - U^0)^2}{\alpha_0 + \alpha_1 U^0} - \frac{\alpha_0 \varepsilon \rho I^2}{(\rho + \delta)(\alpha_0 + \alpha_1 U^0)U^0} \right. \\ &\quad \left. - \frac{\alpha_0 \varepsilon I (U - U^0)^2}{(\alpha_0 + \alpha_1 U^0)U^0 U} + \frac{\nu(\varepsilon + \rho)(\eta + \rho)}{\kappa \eta} \left(\mathcal{R}_0 \frac{F(U, V)}{F(U^0, 0)} - 1 \right) V \right] dx, \\ &\leq - \int_{\Omega} \left[\left(\frac{1}{U} + \frac{\varepsilon}{(\rho + \delta)U^0} \right) \frac{\alpha_0 \delta (U - U^0)^2}{\alpha_0 + \alpha_1 U^0} + \frac{\alpha_0 \varepsilon \rho I^2}{(\rho + \delta)(\alpha_0 + \alpha_1 U^0)U^0} + \frac{\alpha_0 \varepsilon I (U - U^0)^2}{(\alpha_0 + \alpha_1 U^0)U^0 U} \right. \\ &\quad \left. - \frac{\nu(\varepsilon + \rho)(\eta + \rho)}{\kappa \eta} (\mathcal{R}_0 - 1) V \right] dx. \end{aligned} \quad (14)$$

Then, $D^\alpha L_0(t) \leq 0$ when $\mathcal{R}_0 \leq 1$. In addition, $\{P_0\}$ is the largest invariant set in $\{(U, I, C, V) | D^\alpha L_0(t) = 0\}$. By LaSalle's invariance principle [23], P_0 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$. \square

$$\mathcal{R}_0 \leq 1 + \frac{[\delta \rho \nu (\eta + \rho) + \alpha_2 \delta \sigma \kappa \eta] (\varepsilon + \rho) + \alpha_3 \varepsilon \eta \kappa \sigma^2}{\varepsilon \nu \rho (\varepsilon + \rho) (\alpha_0 \delta + \alpha_1 \sigma)}. \quad (15)$$

Proof. Let

Theorem 3. *The chronic infection steady state P_1 is globally asymptotically stable when $\mathcal{R}_0 > 1$ and*

$$\begin{aligned} L_1(t) &= \int_{\Omega} \left[\frac{\alpha_0 + \alpha_2 V_1}{\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1} U_1 \Phi\left(\frac{U}{U_1}\right) + \frac{\varepsilon (\alpha_0 + \alpha_2 V_1) (U(x, t) - U_1 + I(x, t) - I_1)^2}{2(\rho + \delta) (\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1) U_1} \right. \\ &\quad \left. + I_1 \Phi\left(\frac{I}{I_1}\right) + \frac{\varepsilon + \rho}{\kappa} C_1 \Phi\left(\frac{C}{C_1}\right) + \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa \eta} V_1 \Phi\left(\frac{V}{V_1}\right) \right] dx. \end{aligned} \quad (16)$$

Then,

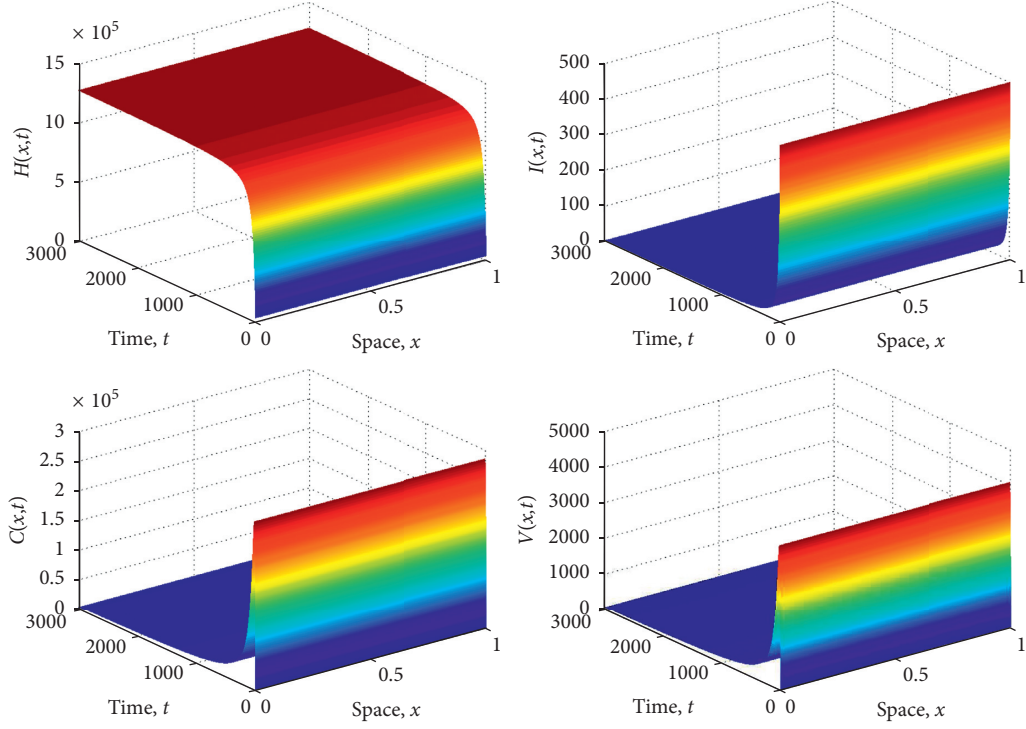
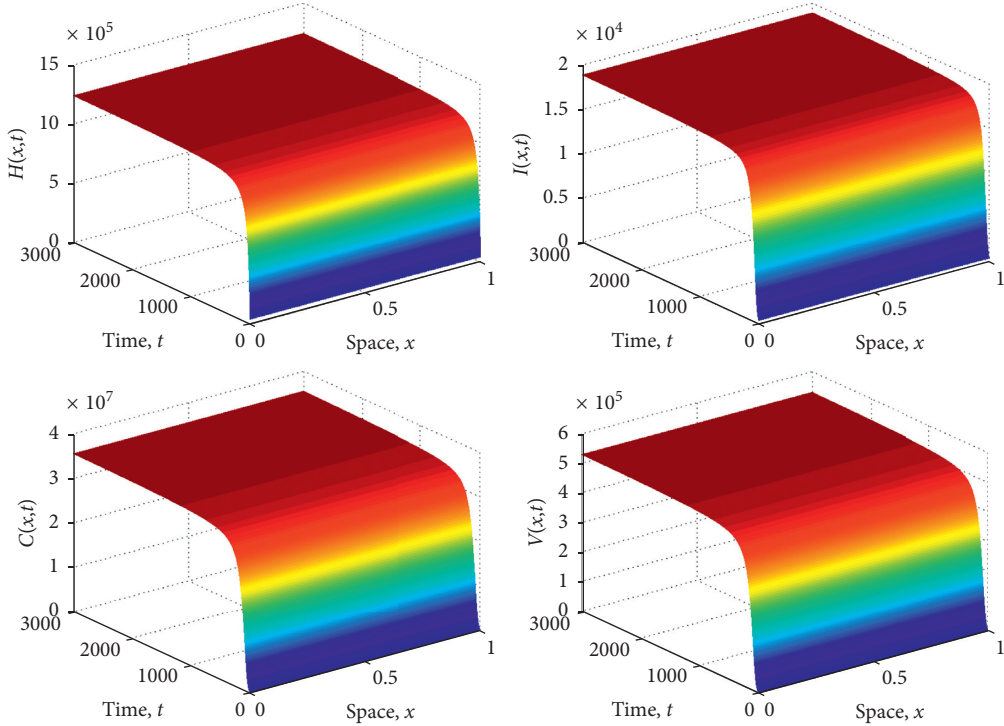
$$\begin{aligned}
 D^\alpha L_1(t) \leq & \int_{\Omega} \left[\frac{\alpha_0 + \alpha_2 V_1}{\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1} \left(1 - \frac{U_1}{U} \right) \partial_t^\alpha U \right. \\
 & + \frac{\varepsilon(\alpha_0 + \alpha_2 V_1)(U - U_1 + I - I_1)(\partial_t^\alpha U + \partial_t^\alpha I)}{(\rho + \delta)(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)U_1} + \left(1 - \frac{I_1}{I} \right) \partial_t^\alpha I \\
 & \left. + \frac{\varepsilon + \rho}{\kappa} \left(1 - \frac{C_1}{C} \right) \partial_t^\alpha C + \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa\eta} \left(1 - \frac{V_1}{V} \right) \partial_t^\alpha V \right] dx.
 \end{aligned} \tag{17}$$

Since $\sigma = \delta U_1 + \rho I_1$, $F(U_1, V_1)V_1 = (\varepsilon + \rho)I_1$, $\kappa I_1 = (\eta + \rho)C_1$, and $1 - ((F(U_1, V_1))/(F(U, V_1))) = ((\alpha_0 + \alpha_2 V_1)/(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1))(1 - (U_1/U))$, we have

$$\begin{aligned}
 D^\alpha L_1(t) \leq & \int_{\Omega} \left[\frac{-\delta(\alpha_0 + \alpha_2 V_1)(U - U_1)^2}{(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)U} + \frac{\varepsilon(\alpha_0 + \alpha_2 V_1)(U - U_1)(I - I_1)}{(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)U} \right. \\
 & + F(U, V_1)V_1 \left(1 - \frac{F(U_1, V_1)}{F(U, V_1)} + \frac{F(U, V)V_1}{F(U, V_1)V_1} \right) + F(U_1, V_1)V_1 \left(1 - \frac{IC_1}{I_1 C} \right) \\
 & + F(U_1, V_1)V_1 \left(1 - \frac{F(U, V)VI_1}{F(U_1, V_1)IV_1} \right) + F(U_1, V_1)V_1 \left(1 - \frac{V}{V_1} - \frac{CV_1}{C_1 V} \right) \\
 & \left. - \frac{\varepsilon(\alpha_0 + \alpha_2 V_1)[\delta(U - U_1)^2 + \rho(I - I_1)^2 + (\delta + \rho)(U - U_1)(I - I_1)]}{(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)(\delta + \rho)U_1} \right] dx \\
 & - \frac{\varepsilon + \rho}{\kappa} d_C C_1 \int_{\Omega} \frac{\|\nabla C\|^2}{C^2} dx - \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa\eta} d_V V_1 \int_{\Omega} \frac{\|\nabla V\|^2}{V^2} dx.
 \end{aligned} \tag{18}$$

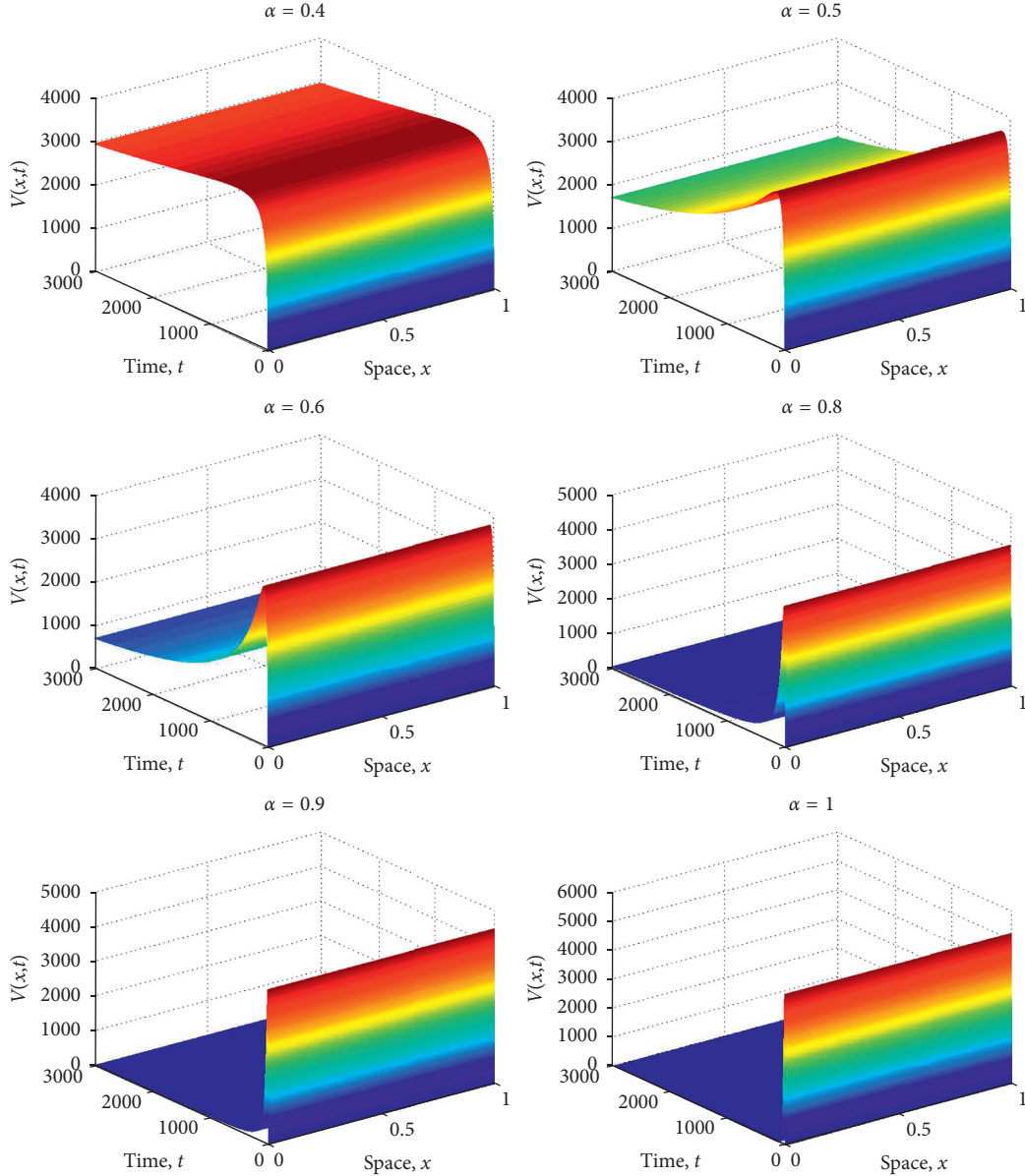
Hence,

$$\begin{aligned}
 D^\alpha L_1(t) \leq & - \int_{\Omega} \left[\frac{(\alpha_0 + \alpha_2 V_1)(U - U_1)^2}{(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)U U_1} \left((\delta U_1 - \varepsilon I_1) + \frac{\delta \varepsilon U}{\rho + \delta} + \varepsilon I \right) \right. \\
 & + \frac{\varepsilon \rho (\alpha_0 + \alpha_2 V_1)(I - I_1)^2}{(\alpha_0 + \alpha_1 U_1 + \alpha_2 V_1 + \alpha_3 U_1 V_1)(\rho + \delta)U_1} \\
 & - F(U_1, V_1)V_1 \left(5 - \frac{F(U_1, V_1)}{F(U, V_1)} - \frac{C_1 I}{C I_1} - \frac{F(U, V)}{F(U_1, V_1)} \frac{V I_1}{V_1 I} - \frac{C V_1}{C_1 V} - \frac{F(U, V_1)}{F(U, V)} \right) \\
 & \left. + \frac{F(U_1, V_1)V_1(\alpha_0 + \alpha_1 U)(\alpha_2 + \alpha_3 U)(V - V_1)^2}{(\alpha_0 + \alpha_1 U + \alpha_2 V_1 + \alpha_3 U V_1)(\alpha_0 + \alpha_1 U + \alpha_2 V + \alpha_3 U V)V_1} \right] dx \\
 & - \frac{\varepsilon + \rho}{\kappa} d_C C_1 \int_{\Omega} \frac{\|\nabla C\|^2}{C^2} dx - \frac{(\varepsilon + \rho)(\eta + \rho)}{\kappa\eta} d_V V_1 \int_{\Omega} \frac{\|\nabla V\|^2}{V^2} dx.
 \end{aligned} \tag{19}$$

FIGURE 1: Stability of the infection-free steady state P_0 .FIGURE 2: Stability of the chronic infection steady state P_1 .

Since $5 - ((F(U_1, V_1))/(F(U, V_1))) - (C_1 I / C I_1) - ((F(U, V))/(F(U_1, V_1)))(V I_1 / V_1 I) - (C V_1 / C_1 V) - ((F(U, V_1))/(F(U, V))) \leq 0$, we have $D^\alpha L_1(t) \leq 0$ if $\mathcal{R}_0 > 1$ and $\varepsilon I_1 \leq \delta U_1$. The last condition is equivalent to

$$\mathcal{R}_0 \leq 1 + \frac{[\delta \rho \nu (\eta + \rho) + \alpha_2 \delta \sigma \kappa \eta] (\varepsilon + \rho) + \alpha_3 \varepsilon \eta \kappa \sigma^2}{\varepsilon \nu \rho (\varepsilon + \rho) (\alpha_0 \delta + \alpha_1 \sigma)}. \quad (20)$$

FIGURE 3: The state variable $V(x, t)$ with different values of α .

Also, $\{P_1\}$ is the largest invariant set in $\{(U, I, C, V) | D^\alpha L_1(t) = 0\}$. According to LaSalle's invariance principle, we deduce that P_1 is globally asymptotically stable. This completes the proof. \square

4. Numerical Simulations

In this section, we present some numerical illustrations to support the obtained analytical results.

Let Δt be the time step size, $\Omega = [x_{\min}, x_{\max}]$, and $\Delta x = (x_{\max} - x_{\min})/N$ be the space step size, where N is a positive integer. The grid points for the space are $x_i = x_{\min} + i\Delta x$ for $i \in \{0, \dots, N\}$ and for time are $t_m = m\Delta t$ for $m \in \mathbb{IN}$. From the Grünwald–Letnikov method [24], the Caputo fractional derivative is approximated as follows:

$${}^C\partial_t^\alpha l(x_i, t_m) \approx \frac{1}{\Delta t^\alpha} \sum_{j=0}^m \beta_j^\alpha l(x_i, t_{m-j}) - \tilde{l}_m, \quad (21)$$

where $\tilde{l}_m = ((l(x_i, 0)t_m^{-\alpha})/(\Gamma(1-\alpha)))$ and β_j^α are the fractional binomial coefficients $\binom{\alpha}{j}$ with the recursion formula:

$$\beta_j^\alpha = \left(1 - \frac{1+\alpha}{j}\right) \beta_{j-1}^\alpha, \quad (22)$$

$$\beta_0^\alpha = 1.$$

Let $(U_i^m, I_i^m, C_i^m, V_i^m)$ be the approximations of the solution (U, I, C, V) of (1) at the discretized point (x_i, t_m) . Then, by applying (21), we obtain

$$\begin{aligned}
\frac{1}{\Delta t^\alpha} \left(U_i^{m+1} + \sum_{j=1}^{m+1} \beta_j^\alpha U_i^{m+1-j} \right) - \tilde{U}_i^{m+1} &= \sigma - \delta U_i^m - F(U_i^m, V_i^m) V_i^m + \varepsilon I_i^m, \\
\frac{1}{\Delta t^\alpha} \left(I_i^{m+1} + \sum_{j=1}^{m+1} \beta_j^\alpha I_i^{m+1-j} \right) - \tilde{I}_i^{m+1} &= F(U_i^m, V_i^m) V_i^m - (\varepsilon + \rho) I_i^m, \\
\frac{1}{\Delta t^\alpha} \left(C_i^{m+1} + \sum_{j=1}^{m+1} \beta_j^\alpha C_i^{m+1-j} \right) - \tilde{C}_i^{m+1} &= d_C \frac{C_{i+1}^m - 2C_i^m + C_{i-1}^m}{\Delta x^2} + \kappa I_i^m - (\eta + \rho) C_i^m, \\
\frac{1}{\Delta t^\alpha} \left(V_i^{m+1} + \sum_{j=1}^{m+1} \beta_j^\alpha V_i^{m+1-j} \right) - \tilde{V}_i^{m+1} &= d_V \frac{V_{i+1}^m - 2V_i^m + V_{i-1}^m}{\Delta x^2} + \eta C_i^m - \nu V_i^m.
\end{aligned} \tag{23}$$

Hence,

$$\begin{aligned}
U_i^{m+1} &= - \sum_{j=1}^{m+1} \beta_j^\alpha U_i^{m+1-j} + \Delta t^\alpha \left[\tilde{U}_i^{m+1} + \sigma - \delta U_i^m - F(U_i^m, V_i^m) V_i^m + \varepsilon I_i^m \right], \\
I_i^{m+1} &= - \sum_{j=1}^{m+1} \beta_j^\alpha I_i^{m+1-j} + \Delta t^\alpha \left[\tilde{I}_i^{m+1} + F(U_i^m, V_i^m) V_i^m - (\varepsilon + \rho) I_i^m \right], \\
C_i^{m+1} &= - \sum_{j=1}^{m+1} \beta_j^\alpha C_i^{m+1-j} + \Delta t^\alpha \left[\tilde{C}_i^{m+1} + d_C \frac{C_{i+1}^m - 2C_i^m + C_{i-1}^m}{\Delta x^2} + \kappa I_i^m - (\eta + \rho) C_i^m \right], \\
V_i^{m+1} &= - \sum_{j=1}^{m+1} \beta_j^\alpha V_i^{m+1-j} + \Delta t^\alpha \left[\tilde{V}_i^{m+1} + d_V \frac{V_{i+1}^m - 2V_i^m + V_{i-1}^m}{\Delta x^2} + \eta C_i^m - \nu V_i^m \right].
\end{aligned} \tag{24}$$

For numerical simulations, we choose $\Omega = [0, 1]$, $d_C = 0.1$, $d_V = 0.1$, $\sigma = 50400$, $b = 3.6 \times 10^{-6}$, $\delta = 0.039$, $\rho = 0.0693$, $\kappa = 150$, $\eta = 0.01$, $\varepsilon = 0.01$, $\alpha = 0.8$, $\alpha_0 = 0.1$, $\alpha_1 = 0.1$, $\alpha_2 = 0.01$, and $\alpha_3 = 0.000001$. By simple calculation, we find $\mathcal{R}_0 = 0.0128 < 1$. According to Theorem 2, the infection-free steady state $P_0 (1.2923 \times 10^6, 0, 0, 0)$ is globally asymptotically stable which means that the virus will disappear and the patient will be completely cured. Figure 1 confirms this result.

To numerically illustrate the global stability of the second steady state of model (1), we take $b = 0.0018$ without changing the values of the other parameters. In this case, $\mathcal{R}_0 = 6.4083 > 1$. From Theorem 1, FPDE model (1) has the unique chronic infection steady state $P_1 (1.244 \times 10^6, 1.891 \times 10^4, 3.562 \times 10^7, 5.314 \times 10^5)$. In addition, we have

$$1 + \frac{[\delta \rho \nu (\eta + \rho) + \alpha_2 \delta \sigma \kappa \eta] (\varepsilon + \rho) + \alpha_3 \varepsilon \eta \kappa \sigma^2}{\varepsilon \nu \rho (\varepsilon + \rho) (\alpha_0 \delta + \alpha_1 \sigma)} = 218.9236, \tag{25}$$

which implies that (15) holds. By Theorem 3, P_1 is globally asymptotically stable. Figure 2 validates this result.

5. Conclusions

In this article, we have presented a fractional reaction-diffusion HBV model that takes into account the HBV DNA-containing capsids and the cure of infected liver cells. The spatial diffusion is considered in capsids and virions, and the incidence of infection is described by Hattaf-Yousfi functional response that includes various forms existing in the literature. We have shown that the global dynamics of the FPDE model is fully determined by a threshold parameter called the basic reproduction number and labeled by \mathcal{R}_0 . More concretely, the infection-free steady state P_0 is globally asymptotically stable if $\mathcal{R}_0 \leq 1$, which biologically means that the HBV is cleared. However, the chronic infection steady state P_1 is globally asymptotically stable when $\mathcal{R}_0 > 1$ and condition (15) holds. In this case, the HBV persists in the liver and the infection becomes chronic.

According to the above analytic results and the numerical simulations, we deduce that the diffusion and the order of fractional derivative in sense of Caputo have no effects on the stability of both steady states, but they can affect the time for arriving to these steady states. For

example, the trajectories quickly converge towards the equilibria of the model for higher values of the fractional derivative order (see, Figure 3). On the other hand, the models and results presented in [8, 9, 12–16] are improved and extended.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Monotone Iterative Technique for Conformable Fractional Differential Equations with Deviating Arguments

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This paper is concerned with the existence of extremal solutions for periodic boundary value problems for conformable fractional differential equations with deviating arguments. We first build two comparison principles for the corresponding linear equation with deviating arguments. With the help of new comparison principles, some sufficient conditions for the existence of extremal solutions are established by combining the method of lower and upper solutions and the monotone iterative technique. As an application, an example is presented to enrich the main results of this article.

1. Introduction

In recent years, people have been paying attention to the progress of the fractional differential equations. In fact, it is the generalization of the ordinary differential equations to a noninteger order. Significantly, fractional differential equations appear more frequently in different fields of science and engineering, such as viscoelasticity, circuit, and neuron modeling [1–3]. Gradually, fractional differential equations are increasingly regarded as effective assistants. We have observed that many papers are exploring the existence of solutions of boundary value problems for fractional differential equations by using nonlinear functional analysis methods such as fixed point theorems, fixed point index on cone, variational methods and critical point theory, the theory of Mawhin coincidence degree, and the upper and lower solution method; see the monographs of Kilbas et al. [1], Podlubny [2], Diethelm [3], the papers [4–26], and the references therein. Among them, the monotone iterative technique is an ingenious and effective method that offers theoretical, as well constructive existence results for

nonlinear problems via linear iterates [9–15, 17, 23, 26]. It yields monotone sequences that converge to the extremal solutions in a sector generated by the upper and lower solutions. For example, the authors of [22] adopted the method of monotone iteration combined with the method of upper and lower solutions to consider the following system of nonlinear fractional differential equations:

$$\begin{cases} D^\alpha v(t) = f(t, v(t), w(t)), & t \in (0, T), \\ D^\alpha w(t) = g(t, w(t), v(t)), & t \in (0, T), \\ t^{1-\alpha} v(t)|_{t=0} = x_0, \\ t^{1-\alpha} w(t)|_{t=0} = y_0, \end{cases} \quad (1)$$

where $0 < T < \infty$, $f, g \in C([0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $x_0, y_0 \in \mathbb{R}$, and $x_0 \leq y_0$. In addition, [15, 24] used these methods to study the initial value problems for nonlinear fractional differential equations with no deviating arguments. On the basis of [22], Jian et al. [13] successfully investigated the following nonlinear fractional order differential systems with deviating arguments:

$$\begin{cases} D^\alpha v(t) = f(t, v(t), v(\theta(t)), w(t), w(\theta(t))), & t \in (0, 1], \\ D^\alpha w(t) = g(t, w(t), w(\theta(t)), v(t), v(\theta(t))), & t \in (0, 1], \\ t^{1-\alpha} v(t)|_{t=0} = x_0, \\ t^{1-\alpha} w(t)|_{t=0} = y_0, \end{cases} \quad (2)$$

where $\theta \in C([0, 1], [0, 1])$. They introduce two well-defined monotone sequences that converge to the solution of the system and, then, establish the existence and uniqueness of the solution of the system. Finally, a numerical iterative scheme is introduced to obtain an accurate approximate solution for the systems.

Motivated by the abovementioned papers, in this paper, we devote ourselves to the existence of solutions to the following boundary value problems with deviation arguments:

$$\begin{cases} \mathcal{D}^\delta \phi(t) = f(t, \phi(t), \phi(\theta(t))), & t \in [0, T], \\ \phi(0) = \phi(T), \end{cases} \quad (3)$$

where $\delta \in (0, 1]$, $\theta \in C([0, 1], [0, 1])$, and $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and $\mathcal{D}^\delta \phi$ is the conformable fractional derivative of order δ . The conformable fractional calculus which was introduced in the work of Khalil et al. [27], then developed by Abdeljawad [28], have been receiving a lot of attention due to the wide application in physics and engineering [29, 30]. The reader is referred to [14, 16, 17, 27–33] and references therein for some recent advances in conformable fractional calculus and its applications.

In this paper, by establishing two comparison results and using the monotone iterative technique combined with the method of upper and lower solutions, some sufficient conditions are presented for the existence of extremal solutions for periodic boundary value problem (3).

2. Preliminaries

Definition 1 (See [27]). Let $f: [0, +\infty \rightarrow \mathbb{R}$ and $t > 0$, and the conformable fractional derivative of order $0 < \alpha \leq 1$ is defined by

$$D_\alpha f(t) = \lim_{\rho \rightarrow 0} \frac{f(t + \rho t^{1-\alpha}) - f(t)}{\rho}, \quad (4)$$

for $t > 0$, and the conformable fractional derivative at 0 is defined as $D_\alpha f(0) = \lim_{t \rightarrow 0^+} (D_\alpha f)(t)$. If f is differentiable, then $D_\alpha f(t) = t^{1-\alpha} f'(t)$.

Definition 2 (See [27]). Let $\alpha \in (0, 1]$. The conformable fractional integral of a function $f: [0, +\infty \rightarrow \mathbb{R}$ of order α is denoted as

$$I_\alpha f(t) = \int_0^t s^{\alpha-1} f(s) ds. \quad (5)$$

Lemma 1 (See [32]). Let $T > 0$. Assume that $f \in C[0, T]$ and $D_\alpha f \in C(0, T) \cap L(0, T)$ with $0 < \alpha \leq 1$. Then, we have

$$I_\alpha D_\alpha f(t) = f(t) - f(0). \quad (6)$$

Lemma 2 (See [27]). Let $\alpha \in (0, 1]$, $l_1, l_2, q, K \in \mathbb{R}$, and the functions f, h be α -differentiable on $[0, +\infty)$. Then,

- (a) $D_\alpha K = 0$ for all constant functions $f(t) = K$
- (b) $D_\alpha(l_1 f + l_2 f) = l_1 D_\alpha f(t) + l_2 D_\alpha h(t)$
- (c) $D_\alpha t^q = q t^{q-\alpha}$
- (d) $D_\alpha(fh) = f(t)D_\alpha h(t) + h(t)D_\alpha f(t)$
- (e) $D_\alpha(f/h) = ((hD_\alpha f - fD_\alpha h)/h^2)$ when $h(t) \neq 0$

Lemma 3 (See [34]). Let $\mathcal{A}: X \rightarrow X$ linear operator, $r(\mathcal{A})$ be the spectral radius of \mathcal{A} , and $\|\mathcal{A}\| = \max_{\|\phi\|=1} \|\mathcal{A}\phi\|$. Then,

- (1) $r(\mathcal{A}) \leq \|\mathcal{A}\|$
- (2) if $r(\mathcal{A}) < 1$, then $(\mathcal{I} - \mathcal{A})^{-1}$ exists and $(\mathcal{I} - \mathcal{A})^{-1} = \sum_{n=0}^{\infty} \mathcal{A}^n$, where \mathcal{I} stands for the identity operator

It is given that $T > 0$. Let $E = C[0, T]$; then, E is a Banach space with the norm $\|x\| = \max_{t \in [0, T]} |x(t)|$.

Let us introduce the following values and functions which will be used in the rest paper.

$$K_1 = \frac{K}{\delta},$$

$$l = e^{-K_1 T^\delta},$$

$$M = \frac{e^{K_1 T^\delta}}{e^{K_1 T^\delta} - 1},$$

$$\Psi_1(t) \equiv Ml,$$

$$\Psi_2(t) \equiv M,$$

$$t \in [0, T],$$

(7)

$$\bar{M} = Ml + \frac{\delta N^2 M^2 l^2 T^\delta}{K(\delta^2 - N^2 M^2 l^2 T^{2\delta})}$$

$$- \frac{\delta^2 NM}{K(\delta^2 - N^2 M^2 l^2 T^{2\delta})},$$

$$\tilde{M} = Ml - \frac{M^2 NT^\delta}{\delta} + \frac{\delta N^2 M^2 l^2 T^\delta}{K(\delta^2 - N^2 M^2 l^2 T^{2\delta})}$$

$$- \frac{N^3 M^3 T^{2\delta}}{K(\delta^2 - N^2 M^2 l^2 T^{2\delta})}.$$

For the forthcoming analysis, we first consider the following two boundary value problems for a linear differential fractional equations:

$$\begin{cases} \mathcal{D}^\delta \phi(t) + K\phi(t) = h(t), & t \in [0, T], \\ \phi(0) = \phi(T) + a, \end{cases} \quad (8)$$

$$\begin{cases} \mathcal{D}^\delta \phi(t) + K\phi(t) + N\phi(\theta(t)) = h(t), & t \in [0, T], \\ \phi(0) = \phi(T) + a. \end{cases} \quad (9)$$

Lemma 4. Let $K > 0$, $a \in \mathbb{R}$, and $h \in E$. Then, problem (8) has the unique solution:

$$\phi(t) = \int_0^T G(t, s)h(s)ds + a\Psi(t), \quad (10)$$

where $\Psi(t) = (1/(1 - e^{-K_1 T^\delta}))e^{-K_1 t^\delta}$ and

$$G(t, s) = \begin{cases} \frac{e^{K_1 T^\delta}}{e^{K_1 T^\delta} - 1} e^{-K_1(t^\delta - s^\delta)} s^{\delta-1}, & 0 < s \leq t \leq T, \\ \frac{1}{e^{K_1 T^\delta} - 1} e^{-K_1(t^\delta - s^\delta)} s^{\delta-1}, & 0 \leq t < s \leq T. \end{cases} \quad (11)$$

Proof. Multiply both sides of the first equation of (8) by $e^{K_1 t^\delta}$, namely,

$$e^{K_1 t^\delta} \mathcal{D}^\delta \phi(t) + K e^{K_1 t^\delta} \phi(t) = e^{K_1 t^\delta} h(t). \quad (12)$$

By using Lemma 2 (d), equation (12) is equivalent to

$$\mathcal{D}^\delta \left[e^{K_1 t^\delta} \phi(t) \right] = e^{K_1 t^\delta} h(t). \quad (13)$$

In view of Lemma 1 and Definition 2, we get

$$e^{K_1 t^\delta} \phi(t) - \phi(0) = \int_0^t s^{\delta-1} e^{K_1 s^\delta} h(s)ds, \quad (14)$$

so

$$\phi(t) = e^{-K_1 t^\delta} \left[\phi(0) + \int_0^t s^{\delta-1} e^{K_1 s^\delta} h(s)ds \right]. \quad (15)$$

The boundary condition $\phi(0) = \phi(T) + a$ leads to

$$\phi(0) = \phi(T) + a = e^{-K_1 T^\delta} \left[\phi(0) + \int_0^T s^{\delta-1} e^{K_1 s^\delta} h(s)ds \right] + a. \quad (16)$$

Clearly,

$$\phi(0) = \frac{1}{e^{K_1 T^\delta} - 1} \int_0^T s^{\delta-1} e^{K_1 s^\delta} h(s)ds + \frac{a}{1 - e^{-K_1 T^\delta}}. \quad (17)$$

Substituting (17) into (15), it follows that linear problem (8) has the following integral representation of the solution:

$$\begin{aligned} \phi(t) &= e^{-K_1 t^\delta} \left[\frac{1}{e^{K_1 T^\delta} - 1} \int_0^T s^{\delta-1} e^{K_1 s^\delta} h(s)ds + \int_0^t s^{\delta-1} e^{K_1 s^\delta} h(s)ds \right] \\ &\quad + \frac{a}{1 - e^{-K_1 T^\delta}} e^{-K_1 t^\delta}, \\ &= \frac{e^{K_1 T^\delta}}{e^{K_1 T^\delta} - 1} \int_0^t s^{\delta-1} e^{-K_1(t^\delta - s^\delta)} h(s)ds \\ &\quad + \frac{1}{e^{K_1 T^\delta} - 1} \int_t^T s^{\delta-1} e^{-K_1(t^\delta - s^\delta)} h(s)ds + \frac{a}{1 - e^{-K_1 T^\delta}} e^{-K_1 t^\delta}, \\ &= \int_0^T G(t, s)h(s)ds + a\Psi(t). \end{aligned} \quad (18)$$

This completes the proof.

For all $0 < \delta \leq 1$, Green's function G admits the following properties:

$$\frac{1}{e^{K_1 T^\delta} - 1} s^{\delta-1} \leq G(t, s) \leq \frac{e^{K_1 T^\delta}}{e^{K_1 T^\delta} - 1} s^{\delta-1}, \quad t \in [0, T], s \in (0, T]. \quad (19)$$

Namely,

$$M s^{\delta-1} \leq G(t, s) \leq M s^{\delta-1}, \quad t \in [0, T], s \in (0, T]. \quad (20)$$

In addition, for Ψ given in Lemma 4, we can get

$$\Psi_1(t) = \frac{e^{-K_1 T^\delta}}{1 - e^{-K_1 T^\delta}} \leq \Psi(t) \leq \frac{1}{1 - e^{-K_1 T^\delta}} = \Psi_2(t). \quad (21)$$

We define the operator \mathcal{A} on E by

$$(\mathcal{A}h)(t) = \int_0^T G(t, s)h(s)ds, \quad h \in E. \quad (22)$$

It is easy to see that $\mathcal{A}: E \rightarrow E$ is a positive linear continuous operator. \square

Lemma 5. $\|\mathcal{A}\| = (1/K)$.

Proof. By direct computation, one has

$$\begin{aligned} \int_0^T G(t, s)ds &= \frac{e^{K_1 T^\delta}}{e^{K_1 T^\delta} - 1} \int_0^t s^{\delta-1} e^{-K_1(t^\delta - s^\delta)} ds + \frac{1}{e^{K_1 T^\delta} - 1} \\ &\quad \int_t^T s^{\delta-1} e^{-K_1(t^\delta - s^\delta)} ds, \\ &= \frac{e^{K_1 T^\delta}}{\delta K_1 (e^{K_1 T^\delta} - 1)} \left(1 - e^{-K_1 t^\delta} \right) + \frac{1}{\delta K_1 (e^{K_1 T^\delta} - 1)} \\ &\quad \left(e^{K_1 T^\delta - K_1 t^\delta} - 1 \right), \\ &= \frac{1}{\delta K_1} = \frac{1}{K}. \end{aligned} \quad (23)$$

Then, for any $h \in E$, we have

$$\|\mathcal{A}h\| = \max_{t \in [0, T]} |(\mathcal{A}h)(t)| \leq \max_{t \in [0, T]} \int_0^T G(t, s) ds \cdot \|h\| = \frac{1}{K} \|h\|, \quad (24)$$

which implies that $\|\mathcal{A}\| \leq (1/K)$. On the other hand, take $h_0(t) \equiv 1$, then $h_0 \in E$, $\|h_0\| = 1$, and

$$\|\mathcal{A}h_0\| = \max_{t \in [0, T]} |(\mathcal{A}h)(t)| = \int_0^T G(t, s) ds = \frac{1}{K} \|h_0\|. \quad (25)$$

This yields $\|\mathcal{A}\| \geq (1/K)$. Therefore, $\|\mathcal{A}\| = (1/K)$. This completes the proof.

We recall that $l = e^{-K_1 T^\delta}$. Then, $l \in (0, 1)$. For $\forall h \in C([0, T], [0, +\infty))$, it follows from (20) that

$$\begin{aligned} (\mathcal{A}h)(t) &= \int_0^T G(t, s)h(s)ds \leq M \int_0^T s^{\delta-1}h(s)ds, \quad t \in [0, T], \\ (\mathcal{A}h)(t) &= \int_0^T G(t, s)h(s)ds \geq Ml \int_0^T s^{\delta-1}h(s)ds, \quad t \in [0, T]. \end{aligned} \quad (26)$$

The abovementioned two inequalities show that

$$(\mathcal{A}h)(t) \geq l(\mathcal{A}h)(s), \quad \forall t, s \in [0, T], \quad \forall h \in C([0, T], [0, +\infty)). \quad (27)$$

Based on the above analysis, we have the following result on (9). \square

Lemma 6. Let $K > 0$, $0 \leq N < K$, $a \in \mathbb{R}$, $\theta \in C([0, T], [0, T])$, and $h \in E$. Then, problem (9) has a unique solution.

Proof. From Lemma 4, it follows that $\phi \in E$ is a solution of (9) if and only if

$$\phi(t) = \int_0^1 G(t, s)[-N\phi(\theta(s)) + h(s)]ds + a\Psi(t). \quad (28)$$

Now, we introduce an operator $\mathcal{B}: E \rightarrow E$ as follows:

$$(\mathcal{B}\phi)(t) = N\phi(\theta(t)), \quad t \in [0, T]. \quad (29)$$

It is easy to see that \mathcal{B} is a positive linear operator with $\|\mathcal{B}\| = N$. Thus, (28) reduces to

$$(I + \mathcal{A}\mathcal{B})\phi(t) = \mathcal{A}h(t) + a\Psi(t). \quad (30)$$

Note from Lemma 5 that $\|\mathcal{A}\mathcal{B}\| \leq \|\mathcal{A}\| \cdot \|\mathcal{B}\| = (N/K) < 1$. Thus, it follows from Lemma 3 that $(I + \mathcal{A}\mathcal{B})^{-1}$ exists and

$$\begin{aligned} (I + \mathcal{A}\mathcal{B})^{-1} &= \sum_{i=0}^{\infty} (-1)^i (\mathcal{A}\mathcal{B})^i = I - \mathcal{A}\mathcal{B} + (\mathcal{A}\mathcal{B})^2 - \dots \\ &\quad + (-1)^n (\mathcal{A}\mathcal{B})^n + \dots \end{aligned} \quad (31)$$

Therefore, the unique solution of (9) is given by

$$\phi(t) = \sum_{i=0}^{\infty} (-1)^i (\mathcal{A}\mathcal{B})^i \mathcal{A}h(t) + a \sum_{i=0}^{\infty} (-1)^i (\mathcal{A}\mathcal{B})^i \Psi(t). \quad (32)$$

The proof is complete.

Now, we present two comparison results. \square

Lemma 7. Let $K > 0$, $0 \leq N \leq Kl^2$, $a \in \mathbb{R}$, and $\theta \in C([0, T], [0, T])$. Assume that $\phi \in E$ satisfies $\mathcal{D}^\delta \phi \in E$ and

$$\begin{cases} \mathcal{D}^\delta \phi(t) \leq -K\phi(t) - N\phi(\theta(t)), & t \in [0, T], \\ \phi(0) \leq \phi(T). \end{cases} \quad (33)$$

Then, $\phi(t) \leq 0$ for all $t \in [0, T]$.

Proof. Take $h(t) = \mathcal{D}^\delta \phi(t) + K\phi(t) + N\phi(\theta(t))$, $a = \phi(0) - \phi(T)$. Then,

$$\begin{aligned} h(t) &\leq 0, \\ a &\leq 0. \end{aligned} \quad (34)$$

Applying Lemma 6, (32) holds, and (32) can be expressed by

$$\phi(t) = \sum_{i=0}^{\infty} (\mathcal{A}\mathcal{B})^{2i} (I - \mathcal{A}\mathcal{B})\mathcal{A}h(t) + a \sum_{i=0}^{\infty} (\mathcal{A}\mathcal{B})^{2i} (I - \mathcal{A}\mathcal{B})\Psi(t). \quad (35)$$

Since $h \leq 0$, it implies that $h_0(t) \equiv -(\mathcal{A}h)(0) \geq 0$. Thus, from (27), we obtain

$$\begin{aligned} -\mathcal{A}h &\geq lh_0, \\ -\mathcal{A}h &\leq \frac{1}{l}h_0. \end{aligned} \quad (36)$$

With the help of positivity of operator $\mathcal{A}\mathcal{B}$, the definition of operator \mathcal{B} , and (23), we have

$$-(\mathcal{A}\mathcal{B})\mathcal{A}h \leq \frac{1}{l}(\mathcal{A}\mathcal{B})h_0 = \frac{N}{lK}h_0. \quad (37)$$

Consequently, we conclude that

$$(I - \mathcal{A}\mathcal{B})\mathcal{A}h \leq -lh_0 + \frac{N}{lK}h_0 = -\left(l - \frac{N}{lK}\right)h_0 \leq 0. \quad (38)$$

On the other hand, by (21), we infer that

$$\begin{aligned} (I - \mathcal{A}\mathcal{B})\Psi(t) &= \frac{e^{-K_1 t^\delta}}{1 - e^{-K_1 T^\delta}} - N \int_0^T G(t, s) \frac{e^{-K_1 (\theta(s))^\delta}}{1 - e^{-K_1 T^\delta}} ds \\ &\geq \frac{e^{-K_1 T^\delta}}{1 - e^{-K_1 T^\delta}} - \frac{N}{1 - e^{-K_1 T^\delta}} \int_0^T G(t, s) ds, \\ &= \frac{1}{1 - e^{-K_1 T^\delta}} \left(l - \frac{N}{K} \right) \geq 0. \end{aligned} \quad (39)$$

Hence, $\phi(t) \leq 0$ holds for all $t \in [0, T]$ that follow from $a \leq 0$ and (35). This completes the proof. \square

Lemma 8. Let $K > 0$, $0 \leq NMT^\delta < \delta$, $0 \leq N < K$, $\tilde{M} > 0$, $\bar{M} > 0$, and $\theta \in C([0, T], [0, T])$. Assume that $\phi \in E$ satisfies $\mathcal{D}^\delta \phi \in E$ and (33). Then, $\phi(t) \leq 0$ for all $t \in [0, T]$.

Proof. Take again $h(t) = \mathcal{D}^\delta \phi(t) + K\phi(t) + N\phi(\theta(t))$, $a = \phi(0) - \phi(T)$. Then,

$$\begin{aligned} h(t) &\leq 0, \\ a &\leq 0. \end{aligned} \quad (40)$$

Applying Lemma 6, (32) holds, and (32) can be expressed by

$$\begin{aligned} \phi(t) &= \sum_{i=0}^{\infty} (\mathcal{AB})^{2i} \mathcal{A}h(t) - \sum_{i=0}^{\infty} (\mathcal{AB})^{2i+1} \mathcal{A}h(t) \\ &\quad + a \left[\sum_{i=0}^{\infty} (\mathcal{AB})^i \Psi(t) - \sum_{i=0}^{\infty} (\mathcal{AB})^{2i+1} \Psi(t) \right]. \end{aligned} \quad (41)$$

Taking notice of the fact that $h(t) \leq 0$, by (20), we have

$$(\mathcal{A}h)(t) = \int_0^T G(t, s)h(s)ds \leq Ml \int_0^T s^{\delta-1} h(s)ds, \quad (42)$$

and for $n \geq 1$,

$$\begin{aligned} (\mathcal{AB})^{2n}(\mathcal{A}h)(t) &= N^{2n} \int_0^T G(t, s) \int_0^T G(\theta(s), \tau_{2n-1}) \\ &\quad \int_0^T G(\theta(\tau_{2n-1}), \tau_{2n-2}) \cdots \int_0^T G(\theta(\tau_2), \tau_1) \\ &\quad \int_0^T G(\theta(\tau_1), \tau_0) h(\tau_0) d\tau_0 d\tau_1, \dots, d\tau_{2n-1} ds \\ &\leq N^{2n} \int_0^T G(t, s) ds \left(Ml \int_0^T s^{\delta-1} ds \right)^{2n-1} \\ &\quad \cdot \int_0^T Ml \tau_0^{\delta-1} h(\tau_0) d\tau_0, \\ &= \frac{N^{2n} M^{2n} l^{2n}}{K} \left(\frac{T^\delta}{\delta} \right)^{2n-1} \int_0^T \tau_0^{\delta-1} h(\tau_0) d\tau_0, \end{aligned} \quad (43)$$

and for $n \geq 1$,

$$\begin{aligned} (\mathcal{AB})^{2n+1}(\mathcal{A}h)(t) &= N^{2n+1} \int_0^T G(t, s) \int_0^T G(\theta(s), \tau_{2n}) \\ &\quad \int_0^T G(\theta(\tau_{2n}), \tau_{2n-1}) \cdots \int_0^T G(\theta(\tau_2), \tau_1) \\ &\quad \int_0^T G(\theta(\tau_1), \tau_0) h(\tau_0) d\tau_0 d\tau_1, \dots, d\tau_{2n} ds \\ &\leq N^{2n+1} \int_0^T G(t, s) ds \left(Ml \int_0^T s^{\delta-1} ds \right)^{2n} \\ &\quad \cdot \int_0^T Ml \tau_0^{\delta-1} h(\tau_0) d\tau_0, \\ &= \frac{N^{2n+1} M^{2n+1}}{K} \left(\frac{T^\delta}{\delta} \right)^{2n} \int_0^T \tau_0^{\delta-1} h(\tau_0) d\tau_0. \end{aligned} \quad (44)$$

These lead us to

$$\begin{aligned} &\sum_{i=0}^{\infty} (\mathcal{AB})^{2i} \mathcal{A}h(t) - \sum_{i=0}^{\infty} (\mathcal{AB})^{2i+1} \mathcal{A}h(t) \\ &\leq \left[Ml + \sum_{i=1}^{\infty} \frac{N^{2i} M^{2i} l^{2i}}{K} \left(\frac{T^\delta}{\delta} \right)^{2i-1} - \sum_{i=0}^{\infty} \frac{N^{2i+1} M^{2i+1}}{K} \left(\frac{T^\delta}{\delta} \right)^{2i} \right] \\ &\quad \int_0^T s^{\delta-1} h(s) ds = \overline{M} \int_0^T s^{\delta-1} h(s) ds \leq 0. \end{aligned} \quad (45)$$

By (20)–(23) and the positivity of operator \mathcal{AB} , we have

$$\Psi(t) \geq \frac{e^{-K_1 T^\delta}}{1 - e^{-K_1 T^\delta}} = Ml,$$

$$(\mathcal{AB})\Psi(t) \leq (\mathcal{AB})\Psi_2(t) = N \int_0^T G(t, s)\Psi_2(\theta(s))ds,$$

$$\leq M^2 N \int_0^T s^{\delta-1} ds = \frac{M^2 N T^\delta}{\delta}, \quad (46)$$

and for $n \geq 1$,

$$\begin{aligned} (\mathcal{AB})^{2n}\Psi(t) &\geq (\mathcal{AB})^{2n}\Psi_1(t) \\ &= N^{2n} \int_0^T G(t, s) \int_0^T G(\theta(s), \tau_{2n-1}) \\ &\quad \cdot \int_0^T G(\theta(\tau_{2n-1}), \tau_{2n-2}) \\ &\quad \cdots \int_0^T G(\theta(\tau_2), \tau_1) \Psi_1(\theta(\tau_1)) d\tau_1 d\tau_2, \dots, d\tau_{2n-1} ds \\ &\geq N^{2n} \Psi_1(t) \int_0^T G(t, s) ds \left(\int_0^T Ml s^{\delta-1} ds \right)^{2n-1}, \\ &= \frac{N^{2n}}{K} Ml \left(\frac{Ml T^\delta}{\delta} \right)^{2n-1} = \frac{N^{2n} M^{2n} l^{2n} T^{(2n-1)\delta}}{K \delta^{2n-1}}, \end{aligned} \quad (47)$$

and for $n \geq 1$,

$$\begin{aligned} (\mathcal{AB})^{2n+1}\Psi(t) &\leq (\mathcal{AB})^{2n+1}\Psi_2(t) \\ &= N^{2n+1} \int_0^T G(t, s) \int_0^T G(\theta(s), \tau_{2n}) \\ &\quad \cdot \int_0^T G(\theta(\tau_{2n}), \tau_{2n-1}) \\ &\quad \cdots \int_0^T G(\theta(\tau_2), \tau_1) \Psi_2(\theta(\tau_1)) d\tau_1 d\tau_2, \dots, d\tau_{2n} ds \\ &\leq N^{2n+1} \Psi_2(t) \int_0^T G(t, s) ds \left(\int_0^T Ml s^{\delta-1} ds \right)^{2n}, \\ &= \frac{N^{2n+1}}{K} Ml \left(\frac{Ml T^\delta}{\delta} \right)^{2n} = \frac{N^{2n+1} M^{2n+1}}{K \delta^{2n}} T^{2n\delta}. \end{aligned} \quad (48)$$

These, together with the fact that $a \leq 0$, ensure that

$$\begin{aligned}
& a \left[\sum_{i=0}^{\infty} (\mathcal{A}\mathcal{B})^i \Psi(t) - \sum_{i=0}^{\infty} (\mathcal{A}\mathcal{B})^{2i+1} \Psi(t) \right] \\
&= a \left[\Psi(t) - (\mathcal{A}\mathcal{B})\Psi(t) + \sum_{i=1}^{\infty} (\mathcal{A}\mathcal{B})^i \Psi(t) - \sum_{i=1}^{\infty} (\mathcal{A}\mathcal{B})^{2i+1} \Psi(t) \right] \\
&\leq a \left[Ml - \frac{M^2 NT^{\delta}}{\delta} + \sum_{i=1}^{\infty} \frac{N^{2i} M^{2i} l^{2i} T^{(2i-1)\delta}}{K \delta^{2i-1}} - \sum_{i=1}^{\infty} \frac{N^{2i+1} M^{2i+1}}{K \delta^{2i}} T^{2i\delta} \right], \\
&= a \tilde{M} \leq 0.
\end{aligned} \tag{49}$$

Thus, by (41), (45), and (49), we have that $\phi(t) \leq 0$ for all $t \in [0, T]$, and the lemma is proved. \square

3. Main Results

Now, we are in the position to prove the existence of extremal solutions of (3) by using the monotone iterative method of lower and upper solutions. To this end, we define the lower and upper solutions of (3).

Definition 3. A function $u_0 \in E$ satisfying $\mathcal{D}^{\delta} u_0 \in E$ is called a lower solution of problem (3) if it satisfies

$$\begin{cases} \mathcal{D}^{\delta} u_0(t) \leq f(t, u_0(t), u_0(\theta(t))), & t \in [0, T], \\ u_0(0) \leq u_0(T). \end{cases} \tag{50}$$

Analogously, a function $w_0 \in E$ satisfying $\mathcal{D}^{\delta} w_0 \in E$ is called an upper solution of (3) if the inequalities

$$\begin{cases} \mathcal{D}^{\delta} w_0(t) \geq f(t, w_0(t), w_0(\theta(t))), & t \in [0, T], \\ w_0(0) \geq w_0(T), \end{cases} \tag{51}$$

hold.

Theorem 1. Assume that the following conditions hold:

(H₁) $\theta \in C([0, T], [0, T])$

(H₂): the functions u_0 and w_0 are lower and upper solutions of problem (3), respectively, such that $u_0(t) \leq w_0(t)$ on $[0, T]$

(H₃) $f \in C([0, T] \times \mathbb{R}^2, \mathbb{R})$ and there exist $K > 0$, $N \geq 0$ such that

$$f(t, x, z) - f(t, \bar{x}, \bar{z}) \geq -K(x - \bar{x}) - N(z - \bar{z}), \tag{52}$$

for all $t \in [0, T]$, $u_0(t) \leq \bar{x} \leq x \leq w_0(t)$, $u_0(t) \leq \bar{z} \leq z \leq w_0(t)$

(H₄): the inequality $N \leq Kl^2$ holds or the inequalities $NMT^{\delta} < \delta$, $N < K$, $\tilde{M} > 0$, $\overline{M} > 0$ hold

Then, (3) has minimal and maximal solution u, w in the sector $[u_0, w_0]$, which can be obtained by monotone iterative sequences starting from u_0 and w_0 , where $[u_0, w_0] = \{z \in E: u_0(t) \leq z(t) \leq w_0(t), t \in [0, T]\}$.

Proof. For $k = 1, 2, \dots$, let us define

$$\begin{cases} \mathcal{D}^{\delta} u_k(t) + Ku_k(t) + Nu_k(\theta(t)) = f(t, u_{k-1}(t), u_{k-1}(\theta(t))) + Ku_{k-1}(t) + Nu_{k-1}(\theta(t)), & t \in [0, T], \\ u_k(0) = u_k(T), \end{cases} \tag{53}$$

$$\begin{cases} \mathcal{D}^{\delta} w_k(t) + Kw_k(t) + Nw_k(\theta(t)) = f(t, w_{k-1}(t), w_{k-1}(\theta(t))) + Kw_{k-1}(t) + Nw_{k-1}(\theta(t)), & t \in [0, T], \\ w_k(0) = w_k(T). \end{cases} \tag{54}$$

By Lemma 6, for any $k = 1, 2, \dots$, we know that linear problems (53) and (54) have a unique solution $u_k(t)$, $w_k(t)$, respectively, which implies that the sequences $\{u_k(t)\}$, $\{w_k(t)\}$ are well defined. Furthermore, $u_k(t)$, $w_k(t)$ can be expressed as

$$\begin{aligned} u_k(t) &= (I + \mathcal{A}\mathcal{B})^{-1} \mathcal{A}\mathcal{F}u_{k-1}(t), \\ w_k(t) &= (I + \mathcal{A}\mathcal{B})^{-1} \mathcal{A}\mathcal{F}w_{k-1}(t), \end{aligned} \tag{55}$$

where $\mathcal{F}: E \rightarrow E$ is a bounded operator defined by

$$(\mathcal{F}u)(t) = f(t, u(t), u(\theta(t))) + Ku(t) + Nu(t), \quad u \in E. \tag{56}$$

By the integral expression of operator \mathcal{A} , it is easy to see that \mathcal{A} is completely continuous. Hence, $(I + \mathcal{A}\mathcal{B})^{-1} \mathcal{A}\mathcal{F}$ is completely continuous.

Firstly, let us prove that

$$u_0 \leq u_1 \leq w_1 \leq w_0. \tag{57}$$

To do this, let $v(t) = u_0(t) - u_1(t)$. By the definition of the lower solution, we get

$$\begin{aligned} \mathcal{D}^{\delta} v(t) &= \mathcal{D}^{\delta} u_0(t) - \mathcal{D}^{\delta} u_1(t) \\ &\leq f(t, u_0(t), u_0(\theta(t))) - f(t, u_0(t), u_0(\theta(t))) \\ &\quad + K(u_1(t) - u_0(t)) \\ &\quad + N(u_1(\theta(t)) - u_0(\theta(t))), \\ &= -Kv(t) - Nv(\theta(t)), \quad t \in [0, T], \\ v(0) &= u_0(0) - u_1(0) \leq u_0(T) - u_1(T) = v(T). \end{aligned} \tag{58}$$

This shows, by Lemma 7 or Lemma 8, that $v(t) \leq 0$ on $[0, T]$, and hence, $u_0 \leq u_1$. Similarly, we can deduce that $w_1 \leq w_0$.

Now, let $v(t) = u_1(t) - w_1(t)$; by (H_2) and (H_3) , we obtain

$$\begin{aligned}
 \mathcal{D}^\delta v(t) &= \mathcal{D}^\delta u_1(t) - \mathcal{D}^\delta w_1(t), \\
 &= f(t, u_0(t), u_0(\theta(t))) - K(u_1(t) - u_0(t)) \\
 &\quad - N(u_1(\theta(t)) - u_0(\theta(t))) \\
 &\quad - f(t, w_0(t), w_0(\theta(t))) + K(w_1(t) - w_0(t)) \\
 &\quad + N(w_1(\theta(t)) - w_0(\theta(t))) \\
 &\leq -K[u_0(t) - w_0(t)] - N[u_0(\theta(t)) - w_0(\theta(t))] \\
 &\quad - K[u_1(t) - u_0(t) - w_1(t) + w_0(t)] \\
 &\quad + N[-u_1(\theta(t)) + u_0 + w_1(\theta(t)) - w_0(\theta(t))], \\
 &= -Kv(t) - Nv(\theta(t)), \quad t \in [0, T], \\
 v(0) &= u_1(0) - w_1(0) = u_1(T) - w_1(T) = v(T).
 \end{aligned} \tag{59}$$

Then, from Lemma 7 or Lemma 8, we get $v(t) \leq 0$, which yields $u_1 \leq w_1$.

Secondly, we need to show that u_1 and w_1 are the lower and upper solutions of problem (3), respectively. In fact, it follows from (H_2) and (H_3) that

$$\begin{cases}
 \mathcal{D}^\delta u_1(t) = f(t, u_0(t), u_0(\theta(t))) - K(u_1(t) - u_0(t)) - N(u_1(\theta(t)) - u_0(\theta(t))) \\
 \quad - f(t, u_1(t), u_1(\theta(t))) + f(t, u_1(t), u_1(\theta(t))) \\
 \leq -K[u_0(t) - u_1(t)] - N[u_0(\theta(t)) - u_1(\theta(t))] - K(u_1(t) - u_0(t)) \\
 \quad - N(u_1(\theta(t)) - u_0(\theta(t))) + f(t, u_1(t), u_1(\theta(t))), \\
 = f(t, u_1(t), u_1(\theta(t))), \\
 u_1(0) = u_1(T),
 \end{cases} \tag{60}$$

which show that v_1 is a lower solution of problem (3). Similarly, we can conclude that w_1 is an upper solution of problem (3).

Repeating the foregoing arguments, we can prove that the sequences $\{u_k(t)\}$, $\{w_k(t)\}$ are lower and upper solutions of problem (3), respectively, and satisfy the following inequality:

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq w_k \leq \dots \leq w_1 \leq w_0. \tag{61}$$

Obviously, the sequences $\{u_k(t)\}$, $\{w_k(t)\}$ are uniformly bounded in E and by (55) and the complete continuity of operator $(I + \mathcal{AB})^{-1}\mathcal{AF}$, and it follows that $\{u_k(t)\}$, $\{w_k(t)\}$ are relatively compact. This, together with the monotonicity of the sequences $\{u_k(t)\}$, $\{w_k(t)\}$, guarantees that the sequences $\{u_k(t)\}$, $\{w_k(t)\}$ converge uniformly to u, w , respectively, and that $u, w \in [v_0, w_0]$ are solutions of (3).

Finally, we prove the minimal and maximal property of u and w on $[v_0, w_0]$. We assume that $z \in [v_0, w_0]$ is any solution of (3) and there exists a positive integer k such that $u_k(t) \leq z(t) \leq w_k(t)$ for $t \in [0, T]$.

Let $v(t) = u_k(t) - z(t)$, then

$$\begin{cases}
 \mathcal{D}^\delta v(t) = \mathcal{D}^\delta u_k(t) - \mathcal{D}^\delta z(t), \\
 = f(t, u_{k-1}(t), u_{k-1}(\theta(t))) - K(u_k(t) - u_{k-1}(t)) \\
 \quad - N(u_k(\theta(t)) - u_{k-1}(\theta(t))) - f(t, z(t), z(\theta(t))) \\
 \leq -K[u_{k-1}(t) - z(t)] - N[u_{k-1}(\theta(t)) - z(\theta(t))] \\
 \quad - K(u_k(t) - u_{k-1}(t)) - N(u_k(\theta(t)) - u_{k-1}(\theta(t))), \\
 = -Kv(t) - Nv(\theta(t)), \\
 v(0) = u_k(0) - z(0) = u_k(T) - z(T) = v(T),
 \end{cases} \tag{62}$$

undoubtedly, $a(t) \leq 0$, namely, $u_k(t) \leq z(t)$. By a similar method, we can show that $z(t) \leq w_k(t)$. Thus, $u_k \leq z \leq w_k$, $k = 1, 2, \dots$. It is easy to find that $u(t) \leq z \leq w(t)$ when $k \rightarrow \infty$. That is u, w are minimal and maximal solutions of (1) in the sector $[u_0, w_0]$. The proof is completed.

Then, by applying Lemma 7 or Lemma 8, we get $v(t) \leq 0$, that is $u_k(t) \leq z(t)$ on $[0, T]$. Similarly, we can show that $z(t) \leq w_k(t)$ on $[0, T]$. Notice that $u_0(t) \leq z(t) \leq w_0(t)$ on $[0, T]$. So, $u_k(t) \leq z(t) \leq w_k(t)$ hold for every k from mathematical induction. Hence, by taking $k \rightarrow +\infty$, we have $u(t) \leq z(t) \leq w(t)$ on $[0, T]$. The proof is complete. \square

Example 1. We consider the following BVP:

$$\begin{cases} \mathcal{D}^{(1/2)}\phi(t) = -\frac{1}{3(1+\sqrt{2\pi})^2}(1+\phi(t))^3 + \frac{\sqrt{2}}{60\pi}\cos\frac{\phi^2(t^2)}{4}, & t \in [0, 1], \\ \phi(0) = \phi(1). \end{cases} \quad (63)$$

Obviously, $\delta = (1/2)$, $T = 1$, $\theta(t) = t^2$, and

$$f(t, u, v) = -\frac{1}{3(1+\sqrt{2\pi})^2}(1+u)^3 + \frac{\sqrt{2}}{60\pi}\cos\frac{v^2}{4}. \quad (64)$$

Let $u_0(t) = -1$, $w_0(t) = \sqrt{2\pi}$; then,

$$\begin{cases} \mathcal{D}^{(1/2)}u_0(t) = 0 < \frac{\sqrt{2}}{60\pi}\cos\frac{1}{4} = f(t, u_0(t), u_0(\theta(t))), & t \in [0, 1], \\ u_0(0) = u_0(1) = -1, \\ \mathcal{D}^{(1/2)}w_0(t) = 0 > -\frac{1+\sqrt{2\pi}}{3} = f(t, w_0(t), w_0(\theta(t))), & t \in [0, 1], \\ w_0(0) = w_0(1) = \sqrt{2\pi}. \end{cases} \quad (65)$$

This shows that u_0, w_0 are lower and upper solutions of (63). On the other hand, it is easy to verify that (H_3) holds for $K = 1$ and $N = (1/60)$. Furthermore, we have

$$K_1 = 2,$$

$$l = e^{-2} \in (0, 1),$$

$$M = \frac{1}{1-e^{-2}} \approx 1.1565,$$

$$N = \frac{1}{60} < 1 = K, \quad (66)$$

$$NMT^\delta \approx 0.0193 < \frac{1}{2} = \delta,$$

$$\overline{M} \approx 0.1388 > 0,$$

$$\tilde{M} \approx 0.1119 > 0.$$

Hence, all conditions of Theorem 1 hold. Therefore, equation (63) has the extremal solution in $[v_0, w_0]$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence and Uniqueness of Positive Solutions for a New System of Fractional Differential Equations

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By virtue of a recent existing fixed point theorem of increasing $\varphi - (h, e)$ -concave operator by Zhai and Wang, we consider the existence and uniqueness of positive solutions for a new system of Caputo-type fractional differential equations with Riemann–Stieltjes integral boundary conditions.

1. Introduction

In this paper, we consider the following nonlinear Caputo-type fractional system:

$$\begin{cases} {}^c D_{0+}^{\theta_1} x(t) + f_1(t, x(t), y(t)) = a_1(t), & t \in [0, 1], \\ {}^c D_{0+}^{\theta_2} y(t) + f_2(t, x(t), y(t)) = a_2(t), & t \in [0, 1], \\ x(0) = x''(0) = 0, \\ x(1) = \int_0^1 x(t) dA_1(t), \\ y(0) = y''(0) = 0, \\ y(1) = \int_0^1 y(t) dA_2(t), \end{cases} \quad (1)$$

where $\theta_1 \in (2, 3)$, $\theta_2 \in (2, 3)$; ${}^c D_{0+}^{\theta_1}$ and ${}^c D_{0+}^{\theta_2}$ are the Caputo fractional derivative; f_1 and $f_2: [0, 1] \times (-\infty, +\infty) \times (-\infty, +\infty) \rightarrow (-\infty, +\infty)$ are continuous; $a_1, a_2: [0, 1] \rightarrow [0, +\infty)$ are continuous; and A_1 and A_2 are bounded variation functions with positive measures with $B_1 = \int_0^1 t dA_1(t) < 1$, $B_2 = \int_0^1 t dA_2(t) < 1$.

In recent decades, fractional-order calculus has been widely used in engineering, biology, physics, and so on. Based on it, many scholars have been interested in the study of the existence of nontrivial solutions for various fractional boundary value problems. For some recent works, we can

refer to [1–30] and the references therein. For example, in [10], by virtue of Guo–Krasnosel'skii fixed point theorem, Ma and Cui studied the following fractional boundary value problem:

$$\begin{cases} {}^c D_{0+}^{\theta} p(y) + \mu f(y, p(y)) = 0, & y \in [0, 1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(y) dA(y), \end{cases} \quad (2)$$

where $f \in C([0, 1] \times [0, +\infty), (0, +\infty))$, ${}^c D_{0+}^{\theta}$ is the Caputo fractional derivative, $\theta \in (2, 3)$, and $\mu > 0$ is a parameter. In the case that the parameter μ satisfied some conditions, the existence of positive solutions for the boundary value problem (2) was proved. Meanwhile, many scholars considered some various fractional systems, such as [16–30] and the reference therein. For instance, in [16], the authors obtained the existence of two positive solutions for a nonlinear Caputo-type fractional system by virtue of fixed point index theory. In [17], by using monotone iterative approach, the authors investigated the iterative positive solutions of a system of fractional Riemann–Liouville-type equations with four-point boundary conditions. In [19], by using Banach's contraction principle, the authors studied the uniqueness of solution for a system of Hadamard-type fractional differential equations with integral boundary

conditions. In [21], by using Guo–Krasnosel'skii fixed point theorem, the authors studied the existence of positive solutions for an infinite system of fractional Caputo-type differential equations.

In [22], Zhai and Wang introduced a new concept of $\varphi - (h, e)$ -concave operator and obtained fixed point theorems of increasing $\varphi - (h, e)$ -concave operator. In recent years, by using the fixed point theorems of $\varphi - (h, e)$ -concave operator, Zhai and Jiang investigated the uniqueness of positive solutions for a Riemann–Liouville-type fractional system with integral boundary conditions in [23]; Zhai and Wang considered the uniqueness of positive solutions for a system of Hadamard fractional differential equations with integral equations in [24]; Zhai and Zhu considered the uniqueness of positive solutions for a system of Riemann–Liouville fractional differential equations in [25].

Inspired by [10, 22–25], we introduce a new system of nonlinear Caputo-type fractional differential equations (1). There are few papers about the application of $\varphi - (h, e)$ -concave operator in nonlinear Caputo-type fractional boundary value problems. So, in this paper, we use the recent fixed point theorems of $\varphi - (h, e)$ -concave operator by Zhai and Wang to study system (1). The result of the existence and uniqueness of positive solutions for system (1) is obtained.

2. Preliminaries

In this section, we briefly introduce Caputo's fractional derivative and the fixed point theorem of $\varphi - (h, e)$ -concave

operator. For details, we can refer to the literature [1, 22]. And we give some lemmas about the relevant Green's functions.

Definition 1 (see [1]). For a function $x \in C^n[0, +\infty)$, we define Caputo's fractional derivative of order $\theta > 0$ as follows:

$${}^c D_{0+}^\theta x(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} x^{(n)}(s) ds, \quad n-1 < \theta < n, \quad (3)$$

where n is the smallest integer greater than or equal to θ .

By [10], the following lemmas are listed.

Lemma 1 (see [10]). Let $u \in C[0, 1]$ and $\theta_1, \theta_2 \in (2, 3)$. Then, the linear Caputo fractional differential equation

$$\begin{cases} {}^c D_{0+}^{\theta_i} p(t) + u(t) = 0, & t \in [0, 1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(t) dA_i(t) \end{cases} \quad (4)$$

has a unique solution

$$p(t) = \int_0^1 K_i(t, s) u(s) ds, \quad (5)$$

where

$$K_i(t, s) = \frac{1}{\Gamma(\theta_i)} \begin{cases} \frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t) \right] - (t-s)^{\theta_i-1}, & 0 \leq s \leq t \leq 1, \\ \frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t) \right], & 0 \leq t \leq s \leq 1, \end{cases} \quad (6)$$

and $B_i = \int_0^1 t dA_i(t) < 1$, $i = 1, 2$.

Lemma 2. The above Green's function $K_i(t, s)$ ($i = 1, 2$) has the following properties:

- (i) $K_i(t, s) \geq 0$ and $K_i(t, s)$ is continuous on $[0, 1] \times [0, 1]$.
- (ii) $((t(1-s)^{\theta_i-1})/(\Gamma(\theta_i)(1-B_i))) \int_0^1 (t-t^{\theta_i-1}) dA_i(t) \leq K_i(t, s) \leq ((t(1-s)^{\theta_i-1})/(\Gamma(\theta_i)(1-B_i))), t, s \in [0, 1]$.

Proof. By (6), we know that $K_i(t, s) \geq 0$ and $K_i(t, s)$ is continuous on $[0, 1] \times [0, 1]$.

By (6), we easily know that

$$K_i(t, s) \leq \frac{t(1-s)^{\theta_i-1}}{\Gamma(\theta_i)(1-B_i)}. \quad (7)$$

From (6), when $0 \leq s \leq t \leq 1$, we have

$$\begin{aligned} & \frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t) \right] - (t-s)^{\theta_i-1} \\ & \geq \frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-ts)^{\theta_i-1} dA_i(t) \right] - (t-ts)^{\theta_i-1} \end{aligned}$$

$$\begin{aligned}
&= \frac{t(1-B_i+B_i)}{1-B_i}(1-s)^{\theta_i-1} - \frac{t}{1-B_i} \int_s^1 (t-ts)^{\theta_i-1} dA_i(t) - t^{\theta_i-1}(1-s)^{\theta_i-1} \\
&= (1-s)^{\theta_i-1}(t-t^{\theta_i-1}) + \frac{t}{1-B_i} \left((1-s)^{\theta_i-1} \int_0^1 t dA_i(t) - \int_s^1 (t-ts)^{\theta_i-1} dA_i(t) \right) \\
&\geq (1-s)^{\theta_i-1}(t-t^{\theta_i-1}) + \frac{t}{1-B_i} (1-s)^{\theta_i-1} \int_0^1 (t-t^{\theta_i-1}) dA_i(t) \\
&\geq \frac{t}{1-B_i} (1-s)^{\theta_i-1} \int_0^1 (t-t^{\theta_i-1}) dA_i(t).
\end{aligned} \tag{8}$$

When $t \leq s$, we have

$$\begin{aligned}
&\frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t) \right] \\
&\geq \frac{t}{1-B_i} \left[(1-s)^{\theta_i-1} - \int_s^1 (t-ts)^{\theta_i-1} dA_i(t) \right] \\
&\geq \frac{t}{1-B_i} (1-s)^{\theta_i-1} \left[1 - \int_s^1 t^{\theta_i-1} dA_i(t) \right] \\
&\geq \frac{t}{1-B_i} (1-s)^{\theta_i-1} \left[1 - \int_0^1 t^{\theta_i-1} dA_i(t) \right] \\
&\geq \frac{t}{1-B_i} (1-s)^{\theta_i-1} \int_0^1 (t-t^{\theta_i-1}) dA_i(t).
\end{aligned} \tag{9}$$

By (8) and (9), we have

$$K_i(t, s) \geq \left(\frac{t(1-s)^{\theta_i-1}}{\Gamma(\theta_i)(1-B_i)} \right) \int_0^1 (t-t^{\theta_i-1}) dA_i(t). \tag{10}$$

Let E be a real Banach space and $P \subset E$ be a cone. A partial order on E is induced by P . For any $x, y \in E$, the notation $x \sim y$ denotes that there are $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. Take $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$); let

$P_h = \{x \in E \mid x \sim h\}$; then obviously, $P_h \subset P$. Choose $e \in P$ with $\theta \leq e \leq h$, and let $P_{h,e} = \{x \in E \mid x + e \in P_h\}$. \square

Definition 2 (see [22]). Let $T: P_{h,e} \rightarrow E$ be an operator. If for any $x \in P_{h,e}$ and $\eta \in (0, 1)$, there exists $\varphi(\eta) > \eta$ such that

$$T(\eta x + (\eta - 1)e) \geq \varphi(\eta)Tx + (\varphi(\eta) - 1)e. \tag{11}$$

Then, T is called a $\varphi - (h, e)$ -concave operator.

Lemma 3 (see [22]). Let P be a normal cone. Suppose that T is an increasing $\varphi - (h, e)$ -concave operator and $Th \in P_{h,e}$; then, T has a unique fixed point $u \in P_{h,e}$. For any $u_0 \in P_{h,e}$, the sequence $u_n = Tu_{n-1}$, $n = 1, 2, \dots$, and then $\|u_n - u\| \rightarrow 0$ as $n \rightarrow \infty$.

3. Main Results

Let $\bar{E} = C[0, 1]$ with the norm $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$. Let $E = \bar{E} \times \bar{E}$ with the norm $\|(x, y)\|_E = \max\{\|x\|, \|y\|\}$. Let $\bar{P} = \{x \in \bar{E} : x(t) \geq 0, t \in [0, 1]\}$. Then, \bar{P} is a normal cone. Let $P = \bar{P} \times \bar{P}$. It is obvious that P is a normal cone of E . We define the following partial order on space E : $(x_1, y_1) \leq (x_2, y_2) \iff x_1(t) \leq x_2(t), y_1(t) \leq y_2(t), t \in [0, 1]$. For the detailed knowledge about the cone, we can refer to [31].

Define the following operators $T_1: E \rightarrow \bar{E}$, $T_2: E \rightarrow \bar{E}$ and $T: E \rightarrow E$:

$$\begin{aligned}
T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds - \int_0^1 K_1(t, s) a_1(s) ds, \quad t \in [0, 1], \\
T_2(x, y)(t) &= \int_0^1 K_2(t, s) f_2(s, x(s), y(s)) ds - \int_0^1 K_2(t, s) a_2(s) ds, \quad t \in [0, 1], \\
T(x, y)(t) &= (T_1(x, y), T_2(x, y))(t), \quad (x, y) \in E, t \in [0, 1],
\end{aligned} \tag{12}$$

where $K_i(t, s)$ ($i = 1, 2$) is defined by (6).

By [10], we easily know that fixed points of the operator T are solutions of system (1).

Let

$$e_1(t) = \int_0^1 K_1(t, s) a_1(s) ds,$$

$$\begin{aligned}
e_2(t) &= \int_0^1 K_2(t, s) a_2(s) ds, \quad t \in [0, 1], \\
h_1(t) &= C_1 t, \\
h_2(t) &= C_2 t, \quad t \in [0, 1],
\end{aligned} \tag{13}$$

where

$$C_1 > \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} a_1(s) ds, \quad (14)$$

$$C_2 > \frac{1}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1} a_2(s) ds.$$

Let

$$\begin{aligned} \bar{P}_{h_1} &= \{x \in \bar{E}: x \sim h_1\}, \\ \bar{P}_{h_2} &= \{x \in \bar{E}: x \sim h_2\}, \\ \bar{P}_{h_1, e_1} &= \{x \in \bar{E}: x + e_1 \sim h_1\}, \\ \bar{P}_{h_2, e_2} &= \{x \in \bar{E}: x + e_2 \sim h_2\}. \end{aligned} \quad (15)$$

Set $h(t) = (h_1(t), h_2(t))$ and $e(t) = (e_1(t), e_2(t))$. By [29], we have

$$\begin{aligned} P_h &= \bar{P}_{h_1} \times \bar{P}_{h_2}, \\ P_{h,e} &= \bar{P}_{h_1, e_1} \times \bar{P}_{h_2, e_2}. \end{aligned} \quad (16)$$

Let $M_1 = \max_{t \in [0,1]} e_1(t)$, $M_2 = \max_{t \in [0,1]} e_2(t)$.

For convenience, the following conditions are given.

- (i) (S_1) $f_1: [0, 1] \times [-M_1, +\infty) \times [-M_2, +\infty) \rightarrow (-\infty, +\infty)$ is increasing about the second and third variables; $f_2: [0, 1] \times [-M_1, +\infty) \times [-M_2, +\infty) \rightarrow (-\infty, +\infty)$ is increasing about the second and third variables.

- (ii) (S_2) For $\eta \in (0, 1)$, there exists $\varphi(\eta) > \eta$ such that

$$\begin{aligned} f_1(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) &\geq \varphi(\eta) f_1(t, u_1, u_2), \\ f_2(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) &\geq \varphi(\eta) f_2(t, u_1, u_2), \end{aligned} \quad (17)$$

where

$$t \in [0, 1], u_1, u_2 \in (-\infty, +\infty), v_1 \in [0, M_1], v_2 \in [0, M_2]. \quad (18)$$

$$\begin{aligned} \text{(iii) } (S_3) \quad & f_1(t, 0, 0) \geq 0, \quad f_2(t, 0, 0) \geq 0 \quad \text{and} \\ & f_1(t, 0, 0) \equiv 0, \quad f_2(t, 0, 0) \equiv 0, \quad \forall t \in [0, 1]. \end{aligned}$$

Theorem 1. Suppose that conditions (S_1) , (S_2) , and (S_3) are satisfied. Then, system (1) has a unique solution $(x^*, y^*) \in P_{h,e}$, and for any given $(x_0, y_0) \in P_{h,e}$, we have $x_{n+1}(t) \rightarrow x^*(t) (n \rightarrow \infty)$, $y_{n+1}(t) \rightarrow y^*(t) (n \rightarrow \infty)$, where the sequences:

$$\begin{aligned} x_{n+1}(t) &= \int_0^1 K_1(t, s) f_1(s, x_n(s), y_n(s)) ds \\ &\quad - \int_0^1 K_1(t, s) a_1(s) ds, \\ y_{n+1}(t) &= \int_0^1 K_2(t, s) f_2(s, x_n(s), y_n(s)) ds \\ &\quad - \int_0^1 K_2(t, s) a_2(s) ds, \end{aligned} \quad (19)$$

where $n = 0, 1, 2, \dots$

Proof. By Lemma 2, we know that $K_i(t, s) \geq 0$, $i = 1, 2$. So,

$$\begin{aligned} e_1(t) &= \int_0^1 K_1(t, s) a_1(s) ds \geq 0, \quad t \in [0, 1], \\ e_2(t) &= \int_0^1 K_2(t, s) a_2(s) ds \geq 0, \quad t \in [0, 1]. \end{aligned} \quad (20)$$

By Lemma 2, we have

$$\begin{aligned} \frac{t(1-s)^{\theta_i-1}}{\Gamma(\theta_i)(1-B_i)} \int_0^1 (t-t^{\theta_i-1}) dA_i(t) &\leq K_i(t, s) \\ &\leq \frac{t(1-s)^{\theta_i-1}}{\Gamma(\theta_i)(1-B_i)}, \quad t, s \in [0, 1], i = 1, 2. \end{aligned} \quad (21)$$

From (21), for $t \in [0, 1]$, we have

$$\begin{aligned} e_1(t) &= \int_0^1 K_1(t, s) a_1(s) ds \leq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} a_1(s) ds \leq C_1 t = h_1(t), \\ e_2(t) &= \int_0^1 K_2(t, s) a_2(s) ds \leq \frac{t}{\Gamma(\theta_2)(1-B_2)} \int_0^1 (1-s)^{\theta_2-1} a_2(s) ds \leq C_2 t = h_2(t). \end{aligned} \quad (22)$$

Thus, we obtain that

$$\begin{aligned} 0 &\leq e_1 \leq h_1, \\ 0 &\leq e_2 \leq h_2. \end{aligned} \quad (23)$$

In the following, we divide three parts to prove this theorem. Firstly, we prove that $T: P_{h,e} \rightarrow E$ is a $\varphi - (h, e)$ concave operator. By (S_2) , for $\eta \in (0, 1)$ and $\eta \in (0, 1)$, we have

$$\begin{aligned}
& T_1(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \\
&= \int_0^1 K_1(t, s) f_1(s, \eta x(s) + (\eta - 1)e_1(s), \eta y(s) + (\eta - 1)e_2(s)) ds - \int_0^1 K_1(t, s) a_1(s) ds \\
&\geq \int_0^1 K_1(t, s) \varphi(\eta) f_1(s, x(s), y(s)) ds - \int_0^1 K_1(t, s) a_1(s) ds \\
&= \varphi(\eta) \int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds - e_1(t) \\
&= \varphi(\eta) \left[\int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds - \int_0^1 K_1(t, s) a_1(s) ds \right] + (\varphi(\eta) - 1)e_1(t) \\
&= \varphi(\eta) T_1(x, y)(t) + (\varphi(\eta) - 1)e_1(t).
\end{aligned} \tag{24}$$

Similarly, by (S_2) , for $\eta \in (0, 1)$ and $\eta \in (0, 1)$, we obtain

$$\begin{aligned}
& T_2(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \\
&= \int_0^1 K_2(t, s) f_2(s, \eta x(s) + (\eta - 1)e_1(s), \eta y(s) + (\eta - 1)e_2(s)) ds - \int_0^1 K_2(t, s) a_2(s) ds \\
&\geq \int_0^1 K_2(t, s) \varphi(\eta) f_2(s, x(s), y(s)) ds - \int_0^1 K_2(t, s) a_2(s) ds \\
&= \varphi(\eta) \int_0^1 K_2(t, s) f_2(s, x(s), y(s)) ds - e_2(t) \\
&= \varphi(\eta) \left[\int_0^1 K_2(t, s) f_2(s, x(s), y(s)) ds - \int_0^1 K_2(t, s) a_2(s) ds \right] + (\varphi(\eta) - 1)e_2(t) \\
&= \varphi(\eta) T_2(x, y)(t) + (\varphi(\eta) - 1)e_2(t).
\end{aligned} \tag{25}$$

By (24) and (25), for $(x, y) \in P_{h,e}$, $\eta \in (0, 1)$, and $t \in [0, 1]$, we have

$$\begin{aligned}
& T(\eta(x, y) + (\eta - 1)e)(t) \\
&= T(\eta(x, y) + (\eta - 1)(e_1, e_2))(t) \\
&= T(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t) \\
&= (T_1(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t), T_2(\eta x + (\eta - 1)e_1, \eta y + (\eta - 1)e_2)(t)) \\
&\geq (\varphi(\eta) T_1(x, y)(t) + (\varphi(\eta) - 1)e_1(t), \varphi(\eta) T_2(x, y)(t) + (\varphi(\eta) - 1)e_2(t)) \\
&= \varphi(\eta) (T_1(x, y)(t), T_2(x, y)(t)) + (\varphi(\eta) - 1)(e_1(t), e_2(t)) \\
&= \varphi(\eta) T(x, y)(t) + (\varphi(\eta) - 1)e(t).
\end{aligned} \tag{26}$$

Namely, for $(x, y) \in P_{h,e}$ and $\eta \in (0, 1)$, we obtain that

$$T(\eta(x, y) + (\eta - 1)e)(t) \geq \varphi(\eta) T(x, y) + (\varphi(\eta) - 1)e. \tag{27}$$

Secondly, we show that $T: P_{h,e} \longrightarrow E$ is an increasing operator. By the definition of $P_{h,e}$, for $(x, y) \in P_{h,e}$, we get that $(x, y) + e \in P_h$, i.e., $(x + e_1, y + e_2) \in \bar{P}_{h_1} \times \bar{P}_{h_2}$. By the definitions of \bar{P}_{h_1} and \bar{P}_{h_2} , we obtain that there exist $\eta_1 > 0$ and $\eta_2 > 0$ such that

$$\begin{aligned}
& x(t) + e_1(t) \geq \eta_1 h_1(t), \\
& y(t) + e_2(t) \geq \eta_2 h_2(t), \quad t \in [0, 1].
\end{aligned} \tag{28}$$

So,

$$\begin{aligned}
& x(t) \geq \eta_1 h_1(t) - e_1(t) \geq -e_1(t) \geq -M_1, \\
& y(t) \geq \eta_2 h_2(t) - e_2(t) \geq -e_2(t) \geq -M_2, \quad t \in [0, 1].
\end{aligned} \tag{29}$$

From (S_1) , we easily know that the operators T_1 and T_2 are increasing, so it is obvious that $T: P_{h,e} \rightarrow E$ is increasing.

In the end, we prove $Th \in P_{h,e}$. By [22], we know that $P_{h,e} = \bar{P}_{h_1,e_1} \times \bar{P}_{h_2,e_2}$. Since $(Th)(t) = T(h_1, h_2)(t) =$

$(T_1(h_1, h_2)(t), T_2(h_1, h_2)(t))$, we need to prove $T_1(h_1, h_2) \in \bar{P}_{h_1,e_1}$, $T_2(h_1, h_2) \in \bar{P}_{h_2,e_2}$. In the following, by the definitions of \bar{P}_{h_1,e_1} and \bar{P}_{h_2,e_2} , we prove $T_1(h_1, h_2) + e_1 \in \bar{P}_{h_1}$, $T_2(h_1, h_2) + e_2 \in \bar{P}_{h_2}$, respectively. From (S_1) and (S_3) , combine (21), and we have

$$\begin{aligned}
 & T_1(h_1, h_2)(t) + e_1(t) \\
 &= \int_0^1 K_1(t, s) f_1(s, h_1(s), h_2(s)) ds \\
 &\leq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, h_1(s), h_2(s)) ds \\
 &\leq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, C_1, C_2) ds \\
 &= \frac{h_1(t)}{C_1 \Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, C_1, C_2) ds,
 \end{aligned} \tag{30}$$

$$\begin{aligned}
 & T_1(h_1, h_2)(t) + e_1(t) \\
 &= \int_0^1 K_1(t, s) f_1(s, h_1(s), h_2(s)) ds \\
 &\geq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, h_1(s), h_2(s)) ds \int_0^1 (t-t^{\theta_1-1}) dA_1(t) \\
 &\geq \frac{t}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, 0, 0) ds \int_0^1 (t-t^{\theta_1-1}) dA_1(t) \\
 &= \frac{h_1(t)}{C_1 \Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, 0, 0) ds \int_0^1 (t-t^{\theta_1-1}) dA_1(t).
 \end{aligned}$$

Let

$$\begin{aligned}
 \lambda_1 &= \frac{1}{C_1 \Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, 0, 0) ds \\
 &\quad \cdot \int_0^1 (t-t^{\theta_1-1}) dA_1(t), \\
 \lambda_2 &= \frac{1}{C_1 \Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s, C_1, C_2) ds.
 \end{aligned} \tag{31}$$

Obviously, $\lambda_1 > 0$, $\lambda_2 > 0$. So, we have $\lambda_1 h_1(t) \leq T_1(h_1, h_2)(t) + e_1(t) \leq \lambda_2 h_2(t)$. And $T_1(h_1, h_2) + e_1 \in \bar{P}_{h_1}$. Similarly, we have $T_2(h_1, h_2) + e_2 \in \bar{P}_{h_2}$. So, $Th \in P_{h,e}$ is proved.

Therefore, by Lemma 3, the operator T has a unique fixed point $(x^*, y^*) \in P_{h,e}$. For any given $(x_0, y_0) \in P_{h,e}$, define the sequences:

$$\begin{aligned}
 x_{n+1}(t) &= \int_0^1 K_1(t, s) f_1(s, x_n(s), y_n(s)) ds \\
 &\quad - \int_0^1 K_1(t, s) a_1(s) ds, \\
 y_{n+1}(t) &= \int_0^1 K_2(t, s) f_2(s, x_n(s), y_n(s)) ds \\
 &\quad - \int_0^1 K_2(t, s) a_2(s) ds,
 \end{aligned} \tag{32}$$

where $n = 0, 1, 2, \dots$

□

4. Application

Example 1. We study the following fractional system with integral boundary conditions:

$$\left\{ \begin{array}{l} {}^c D_{0+}^{(5/2)} x(t) + \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} \\ + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} = t, \quad t \in [0, 1], \\ {}^c D_{0+}^{(5/2)} y(t) + \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/3)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} \\ + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/3)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} = 2t, \quad t \in [0, 1], \\ x(0) = x''(0) = 0, \\ x(1) = \frac{1}{2} \int_0^1 x(t) dt, \\ y(0) = y''(0) = 0, \\ y(1) = \frac{1}{2} \int_0^1 y(t) dt, \end{array} \right. \quad (33)$$

where $\theta_1 = \theta_2 = (5/2)$; $A_1(t) = A_2(t) = (1/2)t$; $B_1 = B_2 = (1/4)$; $a_1(t) = t$; $a_2(t) = 2t$; and

$$f_1(t, x, y) = \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)}, \quad (34)$$

$$f_2(t, x, y) = \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/3)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/3)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)}. \quad (35)$$

Obviously,

$$\begin{aligned} f_1(t, 0, 0) &= \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} + \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)}, \\ f_2(t, 0, 0) &= \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} + \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)}, \end{aligned} \quad (36)$$

and $f_1(t, 0, 0) \equiv 0, f_2(t, 0, 0) \geq 0, \quad f_1(t, 0, 0) \geq 0,$
 $f_2(t, 0, 0) \geq 0.$

Green's functions are as follows:

$$K_1(t-s) = K_2(t-s) = K(t-s) = \frac{1}{\Gamma(5/2)} \begin{cases} \frac{4t}{3} \left[(1-s)^{(3/2)} - \frac{1}{5}(1-s)^{(5/2)} \right] - (t-s)^{(3/2)}, & 0 \leq s \leq t \leq 1, \\ \frac{4t}{3} \left[(1-s)^{(3/2)} - \frac{1}{5}(1-s)^{(5/2)} \right], & 0 \leq s \leq t \leq 1. \end{cases} \quad (37)$$

Then, we have

$$\begin{aligned} e_1(t) &= \int_0^1 K_1(t, s) a_1(s) ds \\ &= \frac{4t}{3\Gamma(5/2)} \left[\int_0^t (1-s)^{(3/2)} s ds - \frac{1}{5} \int_0^t (1-s)^{(5/2)} s ds \right] - \frac{1}{\Gamma(5/2)} \int_0^t (t-s)^{(3/2)} s ds \\ &\quad + \frac{4t}{3\Gamma(5/2)} \left[\int_t^1 (1-s)^{(3/2)} s ds - \frac{1}{5} \int_t^1 (1-s)^{(5/2)} s ds \right] \\ &= \frac{4t}{3\Gamma(5/2)} \left[-\frac{2}{5} t(1-t)^{(5/2)} - \frac{4}{35} (1-t)^{(7/2)} + \frac{4}{35} + \frac{2}{35} t(1-t)^{(7/2)} + \frac{4}{315} (1-t)^{(9/2)} - \frac{4}{315} \right] + \frac{1}{\Gamma(5/2)} \frac{4}{35} t^{(7/2)} \\ &\quad + \frac{4t}{3\Gamma(5/2)} \left[\frac{2}{5} t(1-t)^{(5/2)} + \frac{4}{35} (1-t)^{(7/2)} - \frac{2}{35} t(1-t)^{(7/2)} - \frac{4}{315} (1-t)^{(9/2)} \right] \\ &= \frac{16t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right), \\ e_2(t) &= \int_0^1 K_2(t, s) a_2(s) ds = \frac{8t}{3\Gamma(5/2)} \left[\int_0^t (1-s)^{(3/2)} s ds - \frac{1}{5} \int_0^t (1-s)^{(5/2)} s ds \right] - \frac{2}{\Gamma(5/2)} \int_0^t (t-s)^{(3/2)} s ds \\ &\quad + \frac{8t}{3\Gamma(5/2)} \left[\int_t^1 (1-s)^{(3/2)} s ds - \frac{1}{5} \int_t^1 (1-s)^{(5/2)} s ds \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{8t}{3\Gamma(5/2)} \left[-\frac{2}{5} t(1-t)^{(5/2)} - \frac{4}{35} (1-t)^{(7/2)} + \frac{4}{35} + \frac{2}{35} t(1-t)^{(7/2)} + \frac{4}{315} (1-t)^{(9/2)} - \frac{4}{315} \right] + \frac{2}{\Gamma(5/2)} \frac{4}{35} t^{(7/2)} \\
&\quad + \frac{8t}{3\Gamma(5/2)} \left[\frac{2}{5} t(1-t)^{(5/2)} + \frac{4}{35} (1-t)^{(7/2)} - \frac{2}{35} t(1-t)^{(7/2)} - \frac{4}{315} (1-t)^{(9/2)} \right] \\
&= \frac{32t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right), \tag{38}
\end{aligned}$$

$$M_1 = \max_{t \in [0,1]} e_1(t) = \max_{t \in [0,1]} \frac{16t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right) = \frac{944}{2835\sqrt{\pi}},$$

$$M_2 = \max_{t \in [0,1]} e_2(t) = \max_{t \in [0,1]} \frac{32t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right) = \frac{1888}{2835\sqrt{\pi}}.$$

Let $h_1(t) = C_1 t$, $h_2(t) = C_2 t$, where

$$\begin{aligned}
C_1 &> \frac{4}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds = \frac{4}{3\Gamma(5/2)} \frac{4}{35} = \frac{16}{105\Gamma(5/2)}, \\
C_2 &> \frac{8}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds = \frac{8}{3\Gamma(5/2)} \frac{4}{35} = \frac{32}{105\Gamma(5/2)}. \tag{39}
\end{aligned}$$

We have

$$\begin{aligned}
e_1(t) &= \frac{16t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right) \leq \frac{4t}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds \\
&= \frac{16t}{105\Gamma(5/2)} \leq C_1 t = h_1(t), \\
e_2(t) &= \frac{32t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right) \leq \frac{8t}{3\Gamma(5/2)} \int_0^1 (1-s)^{(3/2)} s ds \\
&= \frac{32t}{105\Gamma(5/2)} \leq C_2 t = h_2(t). \tag{40}
\end{aligned}$$

By (34) and (35), we have

$$\begin{aligned}
f_1(t, x, y) &= \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} t^{(1/5)} \\
&= \left(\frac{2835\sqrt{\pi}}{944} x + 1 \right)^{(1/5)} (e_1(t))^{(1/5)} + \left(\frac{2835\sqrt{\pi}}{1888} y + 1 \right)^{(1/5)} (e_2(t))^{(1/5)} \\
&= \left(\frac{2835\sqrt{\pi}}{944} x e_1(t) + e_1(t) \right)^{(1/5)} + \left(\frac{2835\sqrt{\pi}}{1888} y e_2(t) + e_2(t) \right)^{(1/5)},
\end{aligned}$$

$$\begin{aligned}
f_2(t, x, y) &= \left(\frac{2835\sqrt{\pi}}{944}x + 1 \right)^{(1/3)} \left[\frac{16}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} + \left(\frac{2835\sqrt{\pi}}{1888}y + 1 \right)^{(1/3)} \left[\frac{32}{105\sqrt{\pi}} \left(t^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} t^{(1/3)} \\
&= \left(\frac{2835\sqrt{\pi}}{944}x + 1 \right)^{(1/3)} (e_1(t))^{(1/3)} + \left(\frac{2835\sqrt{\pi}}{1888}y + 1 \right)^{(1/3)} (e_2(t))^{(1/3)} \\
&= \left(\frac{2835\sqrt{\pi}}{944}xe_1(t) + e_1(t) \right)^{(1/3)} + \left(\frac{2835\sqrt{\pi}}{1888}ye_2(t) + e_2(t) \right)^{(1/3)}.
\end{aligned} \tag{41}$$

Take $\varphi(\eta) = \eta^{(1/3)}$; then, $\varphi(\eta) > \eta, \eta \in (0, 1)$. For $\eta \in (0, 1)$, $u_1, u_2 \in (-\infty, +\infty)$, $v_1 \in [0, M_1]$, and $v_2 \in [0, M_2]$, we have

$$\begin{aligned}
f_1(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) &= \left[\frac{2835\sqrt{\pi}}{944}e_1(t)[\eta u_1 + (\eta - 1)v_1] + e_1(t) \right]^{(1/5)} \\
&\quad + \left[\frac{2835\sqrt{\pi}}{1888}e_2(t)[\eta u_2 + (\eta - 1)v_2] + e_2(t) \right]^{(1/5)} \\
&= \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{944}e_1(t) \left[u_1 + \left(1 - \frac{1}{\eta} \right) v_1 \right] + \frac{1}{\eta}e_1(t) \right]^{(1/5)} \\
&\quad + \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{1888}e_2(t) \left[u_2 + \left(1 - \frac{1}{\eta} \right) v_2 \right] + \frac{1}{\eta}e_2(t) \right]^{(1/5)} \\
&= \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{944}e_1(t)u_1 + \frac{2835\sqrt{\pi}}{944} \left(1 - \frac{1}{\eta} \right) e_1(t)v_1 + \frac{1}{\eta}e_1(t) \right]^{(1/5)} \\
&\quad + \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{1888}e_2(t)u_2 + \frac{2835\sqrt{\pi}}{1888} \left(1 - \frac{1}{\eta} \right) e_2(t)v_2 + \frac{1}{\eta}e_2(t) \right]^{(1/5)} \\
&\geq \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{944}e_1(t)u_1 + \left(1 - \frac{1}{\eta} \right) e_1(t) + \frac{1}{\eta}e_1(t) \right]^{(1/5)} \\
&\quad + \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{1888}e_2(t)u_2 + \left(1 - \frac{1}{\eta} \right) e_2(t) + \frac{1}{\eta}e_2(t) \right]^{(1/5)} \\
&= \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{944}e_1(t)u_1 + e_1(t) \right]^{(1/5)} + \eta^{(1/5)} \left[\frac{2835\sqrt{\pi}}{1888}e_2(t)u_2 + e_2(t) \right]^{(1/5)}
\end{aligned}$$

$$\begin{aligned}
&= \eta^{(1/5)} f_1(t, u_1, u_2) \geq \varphi(\eta) f_1(t, u_1, u_2), \\
f_2(t, \eta u_1 + (\eta - 1)v_1, \eta u_2 + (\eta - 1)v_2) &= \left[\frac{2835\sqrt{\pi}}{944} e_1(t) [\eta u_1 + (\eta - 1)v_1] + e_1(t) \right]^{(1/3)} \\
&\quad + \left[\frac{2835\sqrt{\pi}}{1888} e_2(t) [\eta u_2 + (\eta - 1)v_2] + e_2(t) \right]^{(1/3)} \\
&= \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{944} e_1(t) \left[u_1 + \left(1 - \frac{1}{\eta}\right) v_1 \right] + \frac{1}{\eta} e_1(t) \right]^{(1/3)} \\
&\quad + \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{1888} e_2(t) \left[u_2 + \left(1 - \frac{1}{\eta}\right) v_2 \right] + \frac{1}{\eta} e_2(t) \right]^{(1/3)} \\
&= \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{944} e_1(t) u_1 + \frac{2835\sqrt{\pi}}{944} \left(1 - \frac{1}{\eta}\right) e_1(t) v_1 + \frac{1}{\eta} e_1(t) \right]^{(1/3)} \\
&\quad + \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{1888} e_2(t) u_2 + \frac{2835\sqrt{\pi}}{1888} \left(1 - \frac{1}{\eta}\right) e_2(t) v_2 + \frac{1}{\eta} e_2(t) \right]^{(1/3)} \\
&\geq \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{944} e_1(t) u_1 + \left(1 - \frac{1}{\eta}\right) e_1(t) + \frac{1}{\eta} e_1(t) \right]^{(1/3)} \\
&\quad + \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{1888} e_2(t) u_2 + \left(1 - \frac{1}{\eta}\right) e_2(t) + \frac{1}{\eta} e_2(t) \right]^{(1/3)} \\
&= \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{944} e_1(t) u_1 + e_1(t) \right]^{(1/3)} + \eta^{(1/3)} \left[\frac{2835\sqrt{\pi}}{1888} e_2(t) u_2 + e_2(t) \right]^{(1/3)} \\
&= \eta^{(1/3)} f_2(t, u_1, u_2) \geq \varphi(\eta) f_2(t, u_1, u_2).
\end{aligned} \tag{42}$$

Therefore, from Theorem 1, we know that system (33) has a unique solution $(x^*, y^*) \in P_{h,e}$, where

$$e(t) = \left(\frac{16t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right), \frac{32t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right) \right), \quad t \in [0, 1],$$

$$h(t) = (C_1 t, C_2 t), \quad t \in [0, 1].$$

(43)

For any given $(x_0, y_0) \in P_{h,e}$, we obtain the following sequences:

$$\begin{aligned}
x_{n+1}(t) &= \int_0^1 K(t, s) \left(\left[\frac{2835\sqrt{\pi}}{944} x_n(s) + 1 \right]^{(1/5)} \left[\frac{16}{105\sqrt{\pi}} \left(s^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} s^{(1/5)} \right. \\
&\quad \left. + \left[\frac{2835\sqrt{\pi}}{1888} y_n(s) + 1 \right]^{(1/5)} \left[\frac{32}{105\sqrt{\pi}} \left(s^{(5/2)} + \frac{32}{27} \right) \right]^{(1/5)} s^{(1/5)} \right) ds - \frac{16t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right), \\
y_{n+1}(t) &= \int_0^1 K(t, s) \left(\left[\frac{2835\sqrt{\pi}}{944} x_n(s) + 1 \right]^{(1/3)} \left[\frac{16}{105\sqrt{\pi}} \left(s^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} s^{(1/3)} \right. \\
&\quad \left. + \left[\frac{2835\sqrt{\pi}}{1888} y_n(s) + 1 \right]^{(1/3)} \left[\frac{32}{105\sqrt{\pi}} \left(s^{(5/2)} + \frac{32}{27} \right) \right]^{(1/3)} s^{(1/3)} \right) ds - \frac{32t}{105\sqrt{\pi}} \left(\frac{32}{27} + t^{(5/2)} \right),
\end{aligned} \tag{44}$$

where $n = 0, 1, 2, \dots$, $K(t, s)$ is defined by (37), $x_{n+1}(t) \rightarrow x^*(n \rightarrow \infty)$, and $y_{n+1}(t) \rightarrow y^*(n \rightarrow \infty)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

Novel Approaches for Getting the Solution of the Fractional Black–Scholes Equation Described by Mittag-Leffler Fractional Derivative

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The value of an option plays an important role in finance. In this paper, we use the Black–Scholes equation, which is described by the nonsingular fractional-order derivative, to determine the value of an option. We propose both a numerical scheme and an analytical solution. Recent studies in fractional calculus have included new fractional derivatives with exponential kernels and Mittag-Leffler kernels. These derivatives have been found to be applicable in many real-world problems. As fractional derivatives without nonsingular kernels, we use a Caputo–Fabrizio fractional derivative and a Mittag-Leffler fractional derivative. Furthermore, we use the Adams–Bashforth numerical scheme and fractional integration to obtain the numerical scheme and the analytical solution, and we provide graphical representations to illustrate these methods. The graphical representations prove that the Adams–Bashforth approach is helpful in getting the approximate solution for the fractional Black–Scholes equation. Finally, we investigate the volatility of the proposed model and discuss the use of the model in finance. We mainly notice in our results that the fractional-order derivative plays a regulator role in the diffusion process of the Black–Scholes equation.

1. Introduction

There are many mathematical models [1–3] used in finance to predict the values of cost, revenue, and options. In this paper, we address the application of fractional calculus in economics and finance. Fractional derivatives occupy an important place in fractional calculus, so this paper investigates the use of fractional derivatives for modeling financial and economic models. The Black–Scholes model is an important tool used in finance to predict the value of an option [2]. There are many styles of options: European options [3, 4], American options [5, 6], and Asian options. The pricing of options is a subject that has been very intensely

debated in economics. So far, the economic and financial literature continues to take an interest in this subject. Today, the questions arise concerning alternative methods for pricing European options, which are derivatives that can only be exercised at maturity. The literature is also interested in the valuation of American options (there is not yet a consensus model). American options are derivatives that can be exercised at any time. The methodology proposed in this paper, therefore, makes it possible to price a European option. It could be extended to American and Asian options and knock-in and knock-out barrier options. Another possible application is the determination of the default risk of bonds listed on the financial market. The model could

possibly be applied to determine the unknowns of the Merton–Black–Scholes-based model. Note that predicting the exact value of the American option is an ongoing problem that is generally not common knowledge. For European options, the model proposed by Black and Scholes [2] has an exact analytical solution and a numerical scheme and that analytical solution uses the normal distribution. The use of the normal distribution in the formula for an option is not always suitable; thus, our work proposes an analytical solution to this problem using recursive approximation that avoids the need for a normal distribution in the formula for the value of a European option.

Some studies in the context of fractional calculus have investigated the use of the Black and Scholes equation for European options. Much of this research has involved the use of the Caputo–Liouville and the Riemann–Liouville derivatives. In [4], Fall et al. offered a new work on fractional Black–Scholes equations described by the generalized fractional derivative. In [4], the authors present a new procedure to obtain the analytical solutions of the fractional Black–Scholes model, which is called the homotopy perturbation method. Sawangston et al. proposed an analytical solution for the fractional Black–Scholes equation with the Caputo–Liouville derivative in [7]. The research in [8] offered an analytical solution for the fractional Black–Scholes described by the conformable derivative. For more numerical schemes proposed for the Black–Scholes equation, see [1, 6, 9–13]. For other numerical procedures in other differential equations, the readers can refer to [14–20].

Modeling the physical phenomena using the fractional-order derivative has many advantages. First of all, it permits us to study the differential models with arbitrary noninteger-order derivatives. Second, it permits us to take into account the memory effect; that is, the next behavior of the dynamic is explained by the past behavior of the dynamic. As we will observe in this paper, the fractional-order derivative can play a regulator role in differential dynamics. In this paper, we investigate and introduce the fractional Black–Scholes equation described by a new fractional-order derivative: the Mittag-Leffler fractional derivative [21]. We prove this can be used in finance to evaluate the value of an option. Note that the new model uses the Atangana–Baleanu derivative and takes into account the integer-order time derivative. We propose a numerical scheme of the introduced model using the Adams–Bashforth method and an analytical solution using the recursive procedure proposed by Liao in [22]. Finally, we use the analytical solution to analyse the volatility of the option to a change in the price of the underlying security.

This manuscript is structured as follows: In Section 2, we define the fractional derivatives with Mittag-Leffler kernels. In Section 3, we introduce and discuss the fractional Black–Scholes equation described by the Mittag-Leffler fractional derivative. In Section 4, we prove the existence and uniqueness of our introduced model. In Section 4, we describe the Adams–Bashforth method. In Section 5, we propose the numerical discretization of the fractional Black–Scholes equation. In Section 6, we propose an analytical solution for the Black–Scholes equation described by the Atangana–Baleanu fractional derivative. In Section 7, we analyse the volatility of

the fractional Black–Scholes equation, and in Section 8, we present graphical representations and a discussion. Section 9, includes our final remarks and conclusions.

2. Fractional Operators without Singular Kernels

In this section, we recall the definitions of fractional derivatives with nonsingular kernels. Fractional calculus began with the Riemann–Liouville fractional derivative and the Caputo–Liouville fractional derivative; for definitions, refer to [23–25]. These two derivatives were proposed in response to Leibniz’s question in 1695. Many other fractional derivatives were introduced in the literature after this date and have been applied in physics, mechanics, and mathematical modeling [25–31]. Some properties were lost in modeling real-world phenomena, specifically the physical aspects. Recently, Caputo and Fabrizio proposed a new fractional derivative with an exponential kernel [32] and gave it an associated fractional integral. The main advantage of this fractional derivative is its lack of singularity, and it can be used in modeling many physical phenomena [26, 33]. The definitions are given in the following.

Definition 1 (see [32]). The Caputo–Fabrizio fractional derivative of the function $u: \mathbb{R} \times [0, +\infty[\rightarrow \mathbb{R}$ of order α is defined in the form

$$D_{\alpha}^{\text{CF}} u(y, t) = \frac{M(\alpha)}{1 - \alpha} \int_0^t u'(y(s), s) \exp\left(-\frac{\alpha}{1 - \alpha}(t - s)\right) ds, \quad (1)$$

where $t > 0$ and the order $\alpha \in (0, 1)$, with the normalization term satisfying $M(0) = M(1) = 1$.

Definition 2 (see [32]). The Caputo–Fabrizio integral for a given function $u: \mathbb{R} \times [0, +\infty[\rightarrow \mathbb{R}$, of order $\alpha \in (0, 1)$, is defined in the form

$$I_{\alpha}^{\text{CF}} u(y, t) = \frac{2(1 - \alpha)}{(2 - \alpha)M(\alpha)} u(y, t) + \frac{2\alpha}{(2 - \alpha)M(\alpha)} \int_0^t u(y(s), s) ds, \quad (2)$$

for all $t > 0$ and the order $\alpha \in (0, 1)$, with the normalization term satisfying $M(0) = M(1) = 1$.

The exponential form is a particular case of the Mittag-Leffler function. Motivated by the fact that the Cauchy problem with the Caputo–Fabrizio derivative generates a solution with an exponential function, Atangana and Baleanu proposed another fractional derivative with a Mittag-Leffler kernel in 2016 [21]. In other words, we can translate the Cauchy equation using Caputo–Fabrizio as the first-order equation with an integer-order derivative.

Definition 3 (see [21]). The Atangana–Baleanu–Caputo derivative for a function $u: \mathbb{R} \times [0, +\infty[\rightarrow \mathbb{R}$, of order $\alpha \in (0, 1)$, is defined in the form

$$D_{\alpha}^{\text{ABC}} u(y, t) = \frac{B(\alpha)}{1-\alpha} \int_0^t u'(y(s), s) E_{\alpha} \left(-\frac{\alpha}{1-\alpha} (t-s)^{\alpha} \right) ds, \quad (3)$$

for all $t > 0$, where the function $\Gamma(\dots)$ is the Euler gamma function and $E_{\alpha}(\cdot)$ denotes the Mittag-Leffler function, with the normalization term satisfying $B(0) = B(1) = 1$.

Definition 4 (see [21]). The Atangana–Baleanu integral for a given function $u: \mathbb{R} \times [0, +\infty[\rightarrow \mathbb{R}$, of order $\alpha \in (0, 1]$, is defined as the form

$$I_{\alpha}^{\text{AB}} u(y, t) = \frac{1-\alpha}{B(\alpha)} u(y, t) + \frac{\alpha}{B(\alpha)} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(y(s), s) ds, \quad (4)$$

for all $t > 0$, where the function $\Gamma(\dots)$ is the Euler gamma function and $E_{\alpha}(\cdot)$ denotes the Mittag-Leffler function, with the normalization term satisfying $B(0) = B(1) = 1$.

We were motivated to consider these two fractional derivatives because of their successful application in modeling real-life phenomena. In this paper, we apply the Atangana–Baleanu fractional derivative in modeling the value of options and investigate the fractional Black–Scholes equation described by the Atangana–Baleanu–Caputo fractional derivative.

3. Black–Scholes in the Context of Mittag–Leffler Fractional Derivative

In this section, we introduce the Black–Scholes equation in the context of the Atangana–Baleanu fractional derivative. We begin by recalling the classical model proposed by Black and Scholes [2]. Let the asset price be S at time, the constant volatility of an underlining asset be represented by the parameter σ , and μ be the expected rate of return. Myron and Fischer stipulated that the stock price follows a Brownian motion denoted by the parameter w ; thus, we have the following:

$$dS = \mu S dt + \sigma S dw. \quad (5)$$

Note that equation (5) describes the asset price S as Brownian motion and represents a particular case of Ito's lemma. There are two types of derivations in the literature for pricing options: in this paper, we use Fisher and Myron's derivation. Black and Scholes expressed the value of the portfolio denoted by P in the following form:

$$dP = dV - \frac{\partial V}{\partial S} dS, \quad (6)$$

where V represents the value of an option. We have decomposed the portfolio into the value of the option and the asset price. $\partial V / \partial S dS$ denotes the asset price obtained per year. Using Ito's lemma, equation (6) can be represented in the following form:

$$dP = \left[\frac{\partial V}{\partial \tau} + \mu S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{\partial V}{\partial S} S \mu \right] dt + \left[\sigma S \frac{\partial V}{\partial S} - \sigma \frac{\partial V}{\partial S} S \right] dw. \quad (7)$$

Taking into account the interest rate r into the value of the portfolio, we express equation (6) in the following form:

$$dP = rV dt - r \frac{\partial V}{\partial S} S dt. \quad (8)$$

Combining equations (7) and (8), the differential equation which calculates the value of a European option is given by the following equation:

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (9)$$

We made the following assumptions related to equation (9): it considers a European option, the risk-less interest rate r is constant, and there are no transaction costs, and we authorize the possibility to buy and to sell any number of stocks with no restriction to short selling at the last moment. The boundary conditions for equation (9) are defined as $V(0, t) = 0$, $V(S, T) \approx S$ as $S \rightarrow \infty$, and the terminal condition is given by

$$V(S, T) = \max(S - E, 0), \quad (10)$$

where the parameter E denotes the strike price of the underlying stock and T represents the expiration time. For the European option, we have the possibility to buy and to exercise the option with no obligation at time T . That means we can sell the risky asset to a seller at a strike price E . Equation (10) can be explained as follows: we exercise the option at time T when the condition $E < S$ is held. That is, $V(S, T) = \max(S - E, 0) = S - E$; in other words, the buyer receives the payoff $S - E$. The benefit is in selling the asset to the seller of the contract rather than on the financial market. When the condition $S < E$ is held, the contract is not good for the buyer, and the buyer can sell the risky asset for a larger price on the financial market.

The use of equation (9) for the analytical solution or the numerical scheme is not trivial, but equation (9) is a diffusion equation and can be rewritten more simply. We use the following changes to the variables described by the relationships:

$$\begin{aligned} S &= Ee^x, \\ t &= T - \frac{2\tau}{\sigma^2}, \\ V &= Eu(x, t). \end{aligned} \quad (11)$$

From which it follows the classical Euler equation given by

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (k-1) \frac{\partial u}{\partial x} - ku, \quad (12)$$

with the initial boundary condition defined by

$$u(x, 0) = \max(e^x - 1, 0), \quad (13)$$

where $k = 2r/\sigma^2$ denotes the balance between the free interest rate and the volatility of the stocks. The difference between equation (9) and equation (12) shows the transition between the finance model and the physical model. Note that it was provided in fractional calculus; many models described by the classical derivative as in equation (9) cannot describe the real behavior of the modeled phenomena. It has been proved that many real-world problems follow fractional phenomena. Equation (12) is a diffusion equation, and there are many diffusion processes in physics, such as subdiffusion, superdiffusion, ballistic diffusion, and hyperdiffusion, which equation (12) does not take into account. This problem requires the introduction of a fractional derivative that takes into account all types of diffusion processes. In this paper, we replace the ordinary time derivative with the fractional-order time derivative, and we describe the fractional differential equation which we consider using the following fractional differential equation:

$$D_\alpha^{ABC} u = \frac{\partial^2 u}{\partial x^2} + (k-1) \frac{\partial u}{\partial x} - ku, \quad (14)$$

with the initial boundary condition defined by

$$u(x, 0) = \max(e^x - 1, 0). \quad (15)$$

In the next section, we try to prove our new model is well defined, admit a unique solution, and use numerical and analytical methods to approach it. For the readers and more understanding of the paper, we summarize the description of the parameters used in this paper in Table 1.

4. Adams–Bashforth Numerical Approach

In this section, we describe the procedure of discretization used in this paper. The method is called the Adams–Bashforth numerical scheme and was introduced in fractional calculus by Atangana in [34]. Adams–Bashforth is a useful method that involves the following fundamental theorem [34].

Theorem 1 (see [34]). *The solution of the fractional differential equation described by $D_\alpha^{ABC} v = f(v, t)$ with initial boundary condition $v(0)$ satisfies the following relationship:*

$$v(t) - v(0) = \frac{1-\alpha}{B(\alpha)} f(v, t) + \frac{\alpha}{B(\alpha)} \int_0^t (t-s)^{\alpha-1} f(v, s) ds. \quad (16)$$

The approximation of the function f uses a Lagrange polynomial, which is the main novelty of Atangana's proposed numerical approximation. The following relationship describes the Lagrange polynomial:

$$p(t) = \frac{t-t_{n-1}}{t_n-t_{n-1}} f(v_n, t_n) + \frac{t-t_n}{t_{n-1}-t_n} f(v_{n-1}, t_{n-1}). \quad (17)$$

Using equation (17), the discretized approximation of equation (16) at time t_{n+1} and t_n , considering $h = t_n - t_{n-1}$ as described in [34], is obtained by the following equation:

$$v(t_{n+1}) - v(t_n) = \theta(\alpha, 1) + \theta(\alpha, 2), \quad (18)$$

where the function $\theta(\alpha, 1)$ is given by the expression described in the following equation:

$$\theta(\alpha, 1) = f(v_n, t_n) \left\{ \frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)\Gamma(\alpha)h} \left(\frac{2ht_{n+1}^{\alpha+1}}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{\alpha+1} \right) - \frac{\alpha}{B(\alpha)\Gamma(\alpha)h} \left(\frac{ht_n^\alpha}{\alpha} - \frac{t^{\alpha+1}}{\alpha+1} \right) \right\}, \quad (19)$$

and where the function $\theta(\alpha, 2)$ is given by the expression described in the following equation:

$$\theta(\alpha, 2) = f(v_{n-1}, t_{n-1}) \left\{ \frac{\alpha-1}{B(\alpha)} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)h} \left(\frac{ht_{n+1}^\alpha}{\alpha} - \frac{t_{n+1}^{\alpha+1}}{1+\alpha} + \frac{t^{\alpha+1}}{B(\alpha)\Gamma(\alpha)h} \right) \right\}. \quad (20)$$

The above discretization proposed by Atangana in [34] is very useful in numerical solutions for fractional differential equations described by certain fractional derivatives, such as the Caputo fractional derivative and the fractional derivative with the Mittag-Leffler kernel. The application of the Adams–Bashforth scheme uses the discretization of the terms $f(v_n, t_n)$ and $f(v_{n-1}, t_{n-1})$. We can do the standard discretization procedures to discretize them. Here, we use the central difference schemes for the second-order space derivative and numerical approximation of the first-order space derivative. Before moving on, we look at the stability of the method used and see that stability is obtained when the

function f is Lipschitzian. Using equations (19) and (20), the following relationship is obtained:

$$v(t_{n+1}) - v(t_n) = \frac{1-\alpha}{B(\alpha)} [f(v_n, t_n) - f(v_{n-1}, t_{n-1})] + \Omega(\alpha, 1) - \Omega(\alpha, 2), \quad (21)$$

where the function $\Omega(\alpha, 1)$ is given by the relationship

$$\Omega(\alpha, 1) = \frac{\alpha}{B(\alpha)} \int_0^{t_{n+1}} (t_{n+1}-s)^{\alpha-1} f(v, s) ds, \quad (22)$$

TABLE 1: Parameters of the Black-Scholes equation.

Parameters	Description of the parameters.
V and T	The value of an option and the expiration time, respectively.
σ	The volatility of the underlining stock.
k	The balance between the free interest rate and the volatility of the stock.
r	The risk-less interest rate.
E and S	The strike price of the underlying stock and the asset price, respectively.

and the function $\Omega(\alpha, 2)$ is given by the relationship

$$\Omega(\alpha, 2) = \frac{\alpha}{B(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(v, s) ds. \quad (23)$$

We will find a threshold for the function $\Omega(\alpha, 1) - \Omega(\alpha, 2)$ by applying the Euclidean norm:

$$\begin{aligned}
\|\Omega(\alpha, 1) - \Omega(\alpha, 2)\| &\leq \left\| \frac{\alpha}{B(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(v, s) ds - \frac{\alpha}{B(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(v, s) ds \right\| \\
&\leq \left\| \frac{\alpha}{B(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} f(v, s) ds \right\| + \left\| \frac{\alpha}{B(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} f(v, s) ds \right\| \\
&\leq \frac{\alpha}{B(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} \|f(v, s)\| ds + \frac{\alpha}{B(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} \|f(v, s)\| ds \\
&\leq \frac{\alpha M}{B(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - s)^{\alpha-1} ds + \frac{\alpha M}{B(\alpha)} \int_0^{t_n} (t_n - s)^{\alpha-1} ds \\
&\leq \frac{\alpha M}{B(\alpha)\Gamma(\alpha+1)} [t_{n+1}^\alpha + t_n^\alpha] = \frac{\alpha M h^\alpha}{B(\alpha)\Gamma(\alpha+1)} [(n+1)^\alpha + (n)^\alpha].
\end{aligned} \quad (24)$$

Using the assumption $t_n = nh$, we get the following relationship:

$$\|\Omega(\alpha, 1) - \Omega(\alpha, 2)\| \leq \frac{\alpha M h^\alpha}{B(\alpha)\Gamma(\alpha+1)} [(n+1)^\alpha + (n)^\alpha]. \quad (25)$$

Applying the norm to both sides of equation (21), we obtain the following relationships:

$$\begin{aligned}
\|v(t_{n+1}) - v(t_n)\| &\leq \frac{1-\alpha}{B(\alpha)} \| [f(v_n, t_n) - f(v_{n-1}, t_{n-1})] \| \\
&\quad + \|\Omega(\alpha, 1) - \Omega(\alpha, 2)\|, \\
&\leq \frac{1-\alpha}{B(\alpha)} \| [f(v_n, t_n) - f(v_{n-1}, t_{n-1})] \| \\
&\quad + \frac{\alpha M h^\alpha}{B(\alpha)\Gamma(\alpha+1)} [(n+1)^\alpha + (n)^\alpha].
\end{aligned} \quad (26)$$

From equation (26), it can be seen that when the function f is locally Lipschitz and h converges to zero, we obtain the following relationship:

$$\|v(t_{n+1}) - v(t_n)\| \longrightarrow 0. \quad (27)$$

We can conclude the Adams-Bashforth numerical scheme is unconditionally stable. In the next section, we apply the Adams-Bashforth numerical scheme to the numerical approximation of the fractional Black-Scholes equation described by the fractional derivative with Mittag-Leffler.

5. Numerical Approach for Fractional Black-Scholes Equation

In this section, we describe the Adams-Bashforth numerical scheme for the fractional Black-Scholes equation represented by the Atangana-Baleanu fractional derivative. Let us begin the numerical approximation of the fractional Black-Scholes equation. Let $t_n = nh$. The Adams-Bashforth numerical scheme for the fractional Black-Scholes equation described by the Atangana-Baleanu fractional derivative takes the following form:

$$u(t_{n+1}) - u(t_n) = \theta(\alpha, 1) - \theta(\alpha, 2), \quad (28)$$

where the function $\theta(\alpha, 1)$ is given by the expression described in the following equation:

$$\theta(\alpha, 1) = f(u_n, t_n) \left\{ \frac{1-\alpha}{B(\alpha)} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right] - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{(n)^\alpha}{\alpha} - \frac{(n)^{\alpha+1}}{\alpha+1} \right] \right\}, \quad (29)$$

and where the function $\theta(\alpha, 2)$ is given by the expression described in the following equation:

$$\theta(\alpha, 2) = f(u_{n-1}, t_{n-1}) \left\{ \frac{1-\alpha}{B(\alpha)} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} + \frac{n^{\alpha+1}}{B(\alpha)\Gamma(\alpha)h} \right] \right\}. \quad (30)$$

The next step consists of finding the discretization of the functions $f(v_n, t_n)$ and $f(v_{n-1}, t_{n-1})$. In the Black-Scholes equation, the function f is given by

$$f(u, t) = \frac{\partial^2 u}{\partial x^2} + (k-1) \frac{\partial u}{\partial x} - ku. \quad (31)$$

Using the central difference approximation for the second-order derivative with respect to the space coordinate and the numerical approximation for the space derivative, we obtain the following discretization at the points u_n and t_n for the function f :

$$f(u_n, t_n) = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + (k-1) \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - ku_j^n + O(\Delta x). \quad (32)$$

Using the central difference approximation again for the second-order derivative with respect to the space coordinate and the numerical approximation for the space derivative,

we obtain the following discretization at the points u_{n-1} and t_{n-1} for the function f :

$$f(u_{n-1}, t_{n-1}) = \frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2} + (k-1) \frac{u_{j+1}^{n-1} - u_{j-1}^{n-1}}{2\Delta x} - ku_j^{n-1} + O(\Delta x). \quad (33)$$

Numerical discretization using the Adams-Bashforth method for the Black-Scholes equation is obtained by combining equations (28)–(33). That is,

$$u_j^{n+1} = u_j^n + \theta(\alpha, 1) - \theta(\alpha, 2). \quad (34)$$

For the computation of our numerical schemes, we make some changes to the variables, such that the terms depending on n , h , and α are constants. Thus, we rewrite $\theta(\alpha, 1)$ and $\theta(\alpha, 2)$, respectively, as in the following relationships:

$$H(n, h, \alpha, 1) = \left\{ \frac{1-\alpha}{B(\alpha)} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{2(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} \right] - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{(n)^\alpha}{\alpha} - \frac{(n)^{\alpha+1}}{\alpha+1} \right] \right\}, \quad (35)$$

$$H(n, h, \alpha, 2) = \left\{ \frac{1-\alpha}{B(\alpha)} - \frac{\alpha}{B(\alpha)\Gamma(\alpha)} h^\alpha \left[\frac{(n+1)^\alpha}{\alpha} - \frac{(n+1)^{\alpha+1}}{\alpha+1} + \frac{n^{\alpha+1}}{B(\alpha)\Gamma(\alpha)h} \right] \right\}. \quad (36)$$

Finally, the numerical scheme using the Adams-Bashforth method for the fractional Black-Scholes equation is given by

$$u_j^{n+1} = u_j^n + H(n, h, \alpha, 1) \left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} + (k-1) \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - ku_j^n \right] - H(n, h, \alpha, 2) \left[\frac{u_{j+1}^{n-1} - 2u_j^{n-1} + u_{j-1}^{n-1}}{\Delta x^2} + (k-1) \frac{u_{j+1}^{n-1} - u_{j-1}^{n-1}}{2\Delta x} - ku_j^{n-1} \right]. \quad (37)$$

To complete the numerical discretization given in equation (37), we recall the discretized form of the initial boundary condition described by the following expression:

$$u_0^n(x) = \max(e^x - 1, 0). \quad (38)$$

6. Analytical Solution for Fractional Black–Scholes Equation

In this section, we use the fractional integrator to propose the analytical solution of the fractional Black–Scholes equation described by the Atangana–Baleanu fractional derivative. The method is described in the following theorem.

Theorem 2. *The solution of the fractional differential equation described by $D_{\alpha}^{ABC}v = f(v, \tau)$ with initial boundary condition $v(0)$ satisfies the following relationship:*

$$v_{n+1}(\tau) - v_{n+1}(0) = \frac{1-\alpha}{B(\alpha)} f(v_n, \tau) + \frac{\alpha}{B(\alpha)} \int_0^{\tau} f(s, v_n) ds, \quad (39)$$

where $n = 0, 1, 2, \dots$. Furthermore, the solution is given by

$$v(x, \tau) = v_0(0) + v_1(\tau) + v_2(\tau) + \dots \quad (40)$$

Consider the fractional Black–Scholes equation (14). Using Theorem 2, with assumption $u_1(x, 0) = 0$, we have the following solution, which is the first step:

$$\begin{aligned} u_1(x, \tau) &= u_1(x, 0) + \frac{1-\alpha}{B(\alpha)} k(\max(e^x, 0) - \max(e^x - 1, 0)) + \frac{k(\max(e^x, 0) - \max(e^x - 1, 0))\tau^{\alpha}}{B(\alpha)\Gamma(1+\alpha)} \\ &= \frac{1-\alpha}{B(\alpha)} k(\max(e^x, 0) - \max(e^x - 1, 0)) + \frac{\alpha k(\max(e^x, 0) - \max(e^x - 1, 0))\tau^{\alpha}}{B(\alpha)\Gamma(1+\alpha)} \\ &= [k(\max(e^x, 0) - \max(e^x - 1, 0))] \left\{ \frac{1-\alpha}{B(\alpha)} + \frac{\alpha\tau^{\alpha}}{B(\alpha)\Gamma(1+\alpha)} \right\}. \end{aligned} \quad (41)$$

Under assumption $u_2(x, 0) = 0$, again using equation (39), the second step gives the following solution:

$$\begin{aligned} u_2(x, \tau) &= [k^2(\max(e^x, 0) - \max(e^x - 1, 0))] (1-\alpha) \left\{ \frac{1-\alpha}{B(\alpha)} + \frac{\alpha\tau^{\alpha}}{B(\alpha)\Gamma(1+\alpha)} \right\} \\ &\quad - [k^2(\max(e^x, 0) - \max(e^x - 1, 0))] (\alpha) \left\{ \frac{(1-\alpha)\tau^{\alpha}}{B(\alpha)\Gamma(1+\alpha)} + \frac{\alpha\tau^{2\alpha}}{B(\alpha)\Gamma(1+2\alpha)} \right\}. \end{aligned} \quad (42)$$

We adopt the procedure described in Theorem 2 for the rest of the steps. The following expression gives the approximate solution of the fractional Black–Scholes equation in the context of the Mittag-Leffler fractional derivative:

$$u(x, \tau) = u(x, 0) + u_1(x, 0) + u_2(x, 0) + \dots \quad (43)$$

We recover the approximate solution of the classical Black–Scholes equation when $\alpha = 1$. In the context of the Mittag-Leffler fractional derivative, when we suppose $\alpha = 1$, we get the following form:

$$\begin{aligned} u(x, \tau) &= u(x, 0) + u_1(x, 0) + u_2(x, 0) + \dots \\ &= \max(e^x - 1, 0) + [k(\max(e^x, 0) - \max(e^x - 1, 0))]\tau \\ &\quad - [k^2(\max(e^x, 0) - \max(e^x - 1, 0))]\tau^2 + \dots \\ &= \max(e^x - 1, 0)(1 - e^{-k\tau}) + \max(e^x, 0)e^{-k\tau}. \end{aligned} \quad (44)$$

Due to space limitation, all term are not written. Finally, we describe the analytical solution for the classical Black–Scholes equation (12) in the following form:

$$u(x, \tau) = \max(e^x, 0)(1 - e^{-k\tau}) + \max(e^x - 1, 0)e^{-k\tau}. \quad (45)$$

The method adopted here gives a solution which is in good agreement with the classical solution of the Black–Scholes equation (12). Let us prove our solution (45) is another representation for the traditional solution of the Black–Scholes equation using the following:

$$V(S, E) = S\mathcal{N}(d_1) - Ee^{-k\tau}\mathcal{N}(d_2), \quad (46)$$

where \mathcal{N} designs the normal distribution function, $d_1 = (\log(S/E) + (r + \sigma^2/2)(T - t))/\sigma\sqrt{T - t}$, and $d_2 = d_1 - \sigma\sqrt{T - t}$. Note in the case “in the money,” that is, $E < S$, we have high volatility; in other words, we have the following equation:

$$\mathcal{N}(d_1) = \mathcal{N}(d_2) = 1. \quad (47)$$

From which, we rewrite equation (46) as the following form:

$$V(S, E) = S - Ee^{-k\tau}. \quad (48)$$

Equation (48) represents the value of the option “in the money.” We recover this value with the solution proposed in

equation (45) using the following reasoning. Note that in the case “in the money,” there exists the following relationship:

$$\begin{aligned}\max(e^x, 0) &= e^x, \\ \max(e^x - 1, 0) &= e^x - 1.\end{aligned}\quad (49)$$

Now, we replace $e^x = S/E$ into equation (45), and using the assumption posed in equation (11), we obtain the following relationship:

$$\begin{aligned}u(x, \tau) &= \frac{S}{E}(1 - e^{-k\tau}) - \left(\frac{S}{E} - 1\right)e^{-k\tau} \\ &= \frac{S}{E} - e^{-k\tau}.\end{aligned}\quad (50)$$

Multiplying the function u by E as in equation (11), we obtain

$$V(S, E) = S - Ee^{-k\tau}.\quad (51)$$

We can see that the solution represented in equation (48) and the solution in equation (51) are the same. Thus, our analytical solution can be used in finance to determine the value of an option, satisfying the Black–Scholes equation.

Note that, in the money “call” is when the price of the underlying asset is higher than the strike price. It is in the interest of the holder of the option to exercise it. He/she has made good anticipations ($S > E$); in this case, the value of the option is given by the formula $V = \max(S - E, 0) = S - E$. Out of the money “call” is when the price of the underlying asset is lower than the strike price. It is not in the interest of the option holder to exercise the option. He/she has made wrong expectations ($S < E$); in this case, the value of the option is given by $V = \max(S - E, 0) = 0$. In the money “put” is when the price of the underlying asset is lower than the strike price. It is in the interest of the option holder to exercise the option. He/she has made good anticipations ($S < E$); in this case, the value of the option is given by $V = \max(E - S, 0) = E - S$. Out of the money “put” is when the price of the underlying asset is higher than the strike price. It is not in the interest of the option holder to exercise the option. He/she has made wrong anticipations ($S > E$); in this case, the value of the option is given by $V = \max(E - S, 0) = 0$.

7. The Volatility of the Fractional Black–Scholes Equation

In this section, we analyse the volatility of the fractional Black–Scholes equation using an analytical solution. In the market, we can buy the call and put with different strike prices and maturity. The volatility analyses the liquidity of the cost of the call and the put in the market. Many types of volatility can be generated by the Black–Scholes equation. In general, volatility measures risk in the financial markets. Volatility is used to control both upward and downward movements. It is calculated from log returns. The delta is part of the Greek letters of options. It is the derivative of the price of the call or put option in relation to the price of the underlying asset. It is used for trading, arbitrage, or hedging

operations on options. It is an important indicator of market risk management. The Basel Committee on International Banking Regulation recommends that banks use delta for exposures to options. In reality delta is a sensitivity factor.

Here, we recall the formula of the volatility delta of the fractional Black–Scholes equation. The delta measures the sensitivity of the option price to a change in the price of the underlying security. Under variable changes, this is expressed as follows:

$$\delta = \frac{\partial u}{\partial x}.\quad (52)$$

Using the approximate solution (52), it is clear that the sensitivity of the option price to a change in the price of the underlying security does not depend on the fractional-order and is given by

$$\delta = \max(e^x, 0).\quad (53)$$

Given the conditions in equation (11), the volatility of the option price to a change in the price of the underlying security is given by

$$\delta = \max\left(\frac{S}{E}, 0\right).\quad (54)$$

Earlier, we stated that the volatility studied in this paper does not depend on the order of the fractional derivative or the balance between the free interest rate and the volatility of the stocks k . This remark can be explained simply from the fact that, except for the first term $u(x, 0)$, we have the following relationship:

$$\frac{\partial u_i}{\partial x} = 0,\quad (55)$$

for all $i = 1, 2, 3, \dots$. Figure 1 shows the volatility surface of the option price to a change in the price of the underlying security in Figure 1.

The volatility surface gives the sensitivity of the option price when the asset price S and the strike price of the underlying stock E both vary in time. Let the asset price S be fixed, and let the strike price of the underlying stock E vary in time. The behavior of the volatility can be seen in Figure 2. In other words, we can see that when $S < E$, the volatility decreases rapidly and converges to zero. Let the underlying stock E and the asset price S vary in time. The behavior of the volatility can be seen in Figure 2. Thus, we can see in Figure 2 that when $E < S$, the volatility quickly increases linearly and converges to infinity. We note high volatility.

8. Graphical Representations and Discussion

In this section, we illustrate our results graphically. Specifically, we depict the behavior of the solutions of the fractional Black–Scholes equation obtained with numerical schemes and the recursive method. We begin with the approximate solution generated by the recursive method previously described in Section 7. In this context, we consider the solution with three iterations given by equation (43). Figure 3 shows the behavior of the approximate

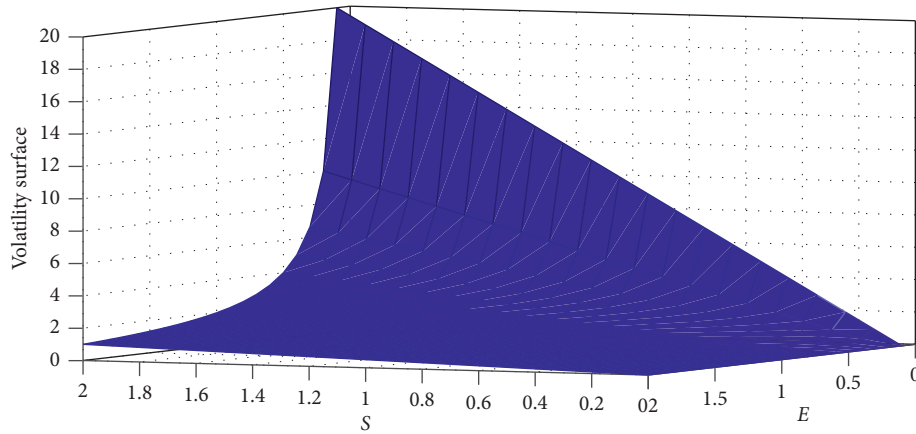
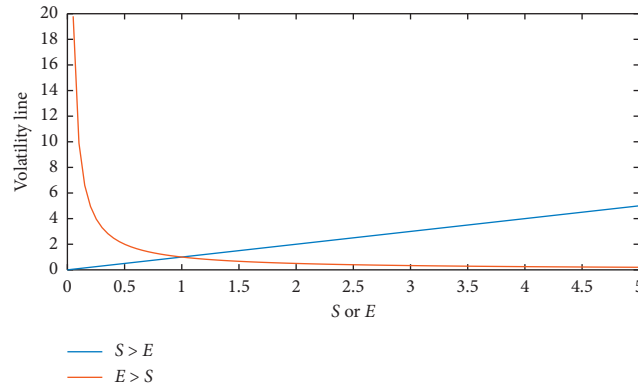
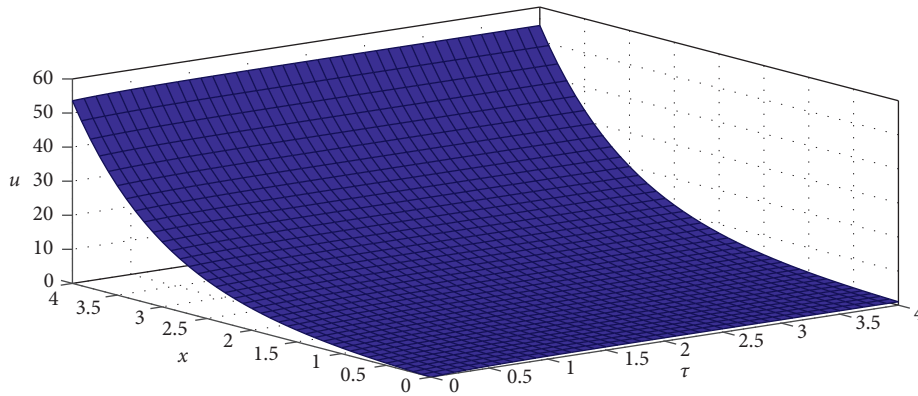


FIGURE 1: Volatility surface of the Black-Scholes equation.

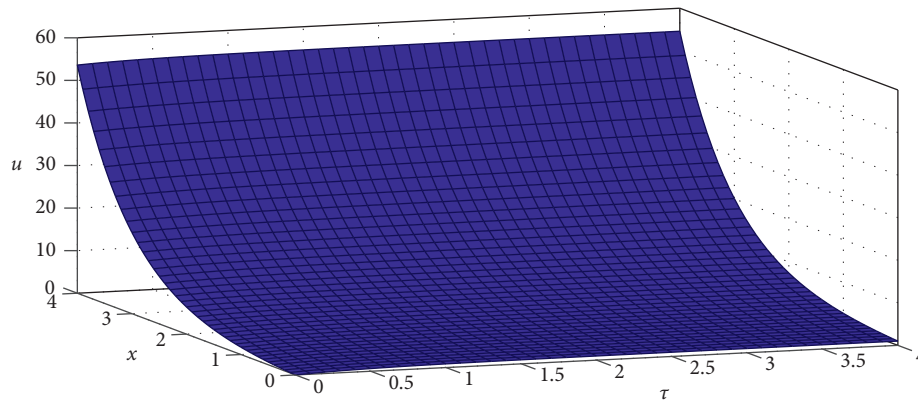
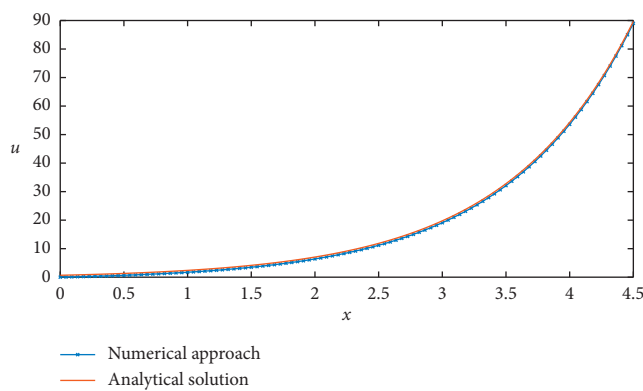
FIGURE 2: Volatility when $S < E$ and $E < S$.FIGURE 3: Behavior of the Black-Scholes equation with $\alpha = 0.5$.

solution for the fractional Black-Scholes equation with the Atangana–Baleanu fractional derivative with order $\alpha = 0.5$ and the balance between the free interest rate and the volatility of the stocks $k = 2$. The constant k is obtained with the risk-free interest rate to expiration $r = 0.04$ and the volatility of the stocks $\sigma = 0.2$ [35].

Figure 4 shows the behavior of the approximate solution for the fractional Black-Scholes equation with the Atangana–Baleanu fractional derivative with order $\alpha = 1$ and the balance between the free interest rate and the volatility of the stocks $k = 2$.

Figure 5 depicts the analytical solution with the recursive method and the approximate solution obtained with the Adams–Bashforth numerical scheme. We suppose $t = 1$ and $\alpha = 0.5$ and observed the numerical solution, and the approximate analytical solution is in good agreement.

The main question now is how to find the optimal order of α . In our previous example, we depicted the figures by choosing the order $\alpha = 0.5$ or $\alpha = 1$. A complicated method is required to find the optimal order α in fractional calculus. First, we must collect the data in the considered market.

FIGURE 4: Behavior of the Black-Scholes equation with $\alpha = 1$.FIGURE 5: Analytical solution vs numerical approximation with $\alpha = 0.5$.TABLE 2: Option values when $E < S$.

x	τ	$u_{\text{Numerical}}$	u_{Homotopy}	$u_{\text{classic}}, \alpha = 1$
0.1541	0.01	0.3240	0.3242	1.1467
0.4677	0.03	0.8455	0.8456	1.5381
0.2401	0.05	0.5745	0.5747	1.1762
0.1697	0.02	0.4924	0.4925	1.1457

Second, we create a figure generated by the obtained data, and finally, we proceed by interpolation to find an order α .

To support the numerical discretization, we describe in Table 2 the different values of the options generated by our numerical scheme, and we compare them with values obtained with the homotopy perturbation method. In other words, the robustness of the used numerical scheme is authenticated by comparing the numerical and analytical results in Table 2. In Table 2, we consider $\alpha = 0.5$, the risk-free interest rate to expiration $\alpha = 0.5$, the volatility of the stocks $\sigma = 0.2$, the volatility of the stocks $k = 2$ [35], and the strike price of the underlying stock $E = 10$.

We mainly observe that the results in Table 2 are in good agreement with the results on the Black-Scholes equation studied in terms of the Caputo derivative in [35]. In general, we also notice that the order of the fractional derivative has a significant impact on the value of the options. In this investigation, the order α has a regulator impact, and we notice

in Table 2 that the values obtained using the fractional order are more beneficial rather than the values obtained with the classical derivative in the financial market.

9. Conclusion

In this paper, we have discussed the numerical scheme and the analytical solution for the fractional Black-Scholes equation described by the Atangana-Baleanu derivative. As observed, the analytical solution of the fractional Black-Scholes equation with the Atangana-Baleanu fractional derivative is not trivial. We have used the Adams-Bashforth numerical scheme to approach the solution, as it is useful and straightforward for proposing the approximate solutions of the fractional Black-Scholes equation. We have also considered the liquidity of the cost of the call and the put in the market, namely, the volatility. The graphical representations have proved the good agreements between the analytical solution and the numerical solutions for the fractional Black-Scholes equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Convergence of Antiperiodic Boundary Value Problems for First-Order Integro-Differential Equations

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In this paper, we investigate the convergence of approximate solutions for a class of first-order integro-differential equations with antiperiodic boundary value conditions. By introducing the definitions of the coupled lower and upper solutions which are different from the former ones and establishing some new comparison principles, the results of the existence and uniqueness of solutions of the problem are given. Finally, we obtain the uniform and rapid convergence of the iterative sequences of approximate solutions via the coupled lower and upper solutions and quasilinearization method. In addition, an example is given to illustrate the feasibility of the method.

1. Introduction

In recent decades, the integro-differential equations have developed rapidly because the models described by various of integro-differential equations have appeared in a number of fields such as fluid dynamics, biology, economics, and control theory, for details and examples, we can refer to references [1–7] and cited therein. Meanwhile, the qualitative theory of integral differential equations creates an branch of nonlinear analysis. Boundary value problems for various first-order integral differential equations have been studied by several researchers, and there are some results on the existence of solutions and extremal solutions, the controllability problem controllability of integral boundary value conditions, and antiperiodic boundary value conditions, such as ordinary differential equations [8–12], difference equations [13, 14], fractional differential equations [15–20], impulsive differential equations [9, 14, 21, 22], integro-differential equations, and impulsive functional differential equations [18, 23–26].

However, we found that most of these known results concerned with the existence and uniformly convergence results of solutions and extremal solutions via the method of upper and lower solutions coupled with the monotone

iterative technique (see [27]). It is well known that the method of quasilinearization (QSL) provides a powerful tool for obtaining convergence of approximate solutions of nonlinear problems [28, 29]. The technique of upper and lower solutions coupled with the QSL have been applied successfully to obtain monotone sequences of approximate solutions converging uniformly and quadratically to the unique solution of integro-differential equations with antiperiodic boundary value conditions [30–32]. In terms of applications, it is important to pay attention to the high-order convergence of sequences of approximate solutions. The high-order convergence results of various differential equations can be found in [33–39].

In this paper, we consider the following first-order integro-differential equations with antiperiodic boundary value conditions (APBVP):

$$\begin{cases} x'(t) = f(t, x(t), (Tx)(t)), & t \in J, \\ x(0) = -x(T), \end{cases} \quad (1)$$

where $f \in C(J \times R^2, R)$, $J = [0, T]$, $(Tx)(t) = \int_0^t k(t, s)x(s)ds$, $k \in C(D, R_+)$, $k_0 = \sup\{k(t, s)\}$, $D = \{(t, s) \in J \times J : t \geq s\}$.

The aim of this paper is to investigate the convergence of approximate solutions of the problem. We give the particular definitions of the coupled lower and upper related solutions which are new and establish some new comparison principles in order to discuss the existence and uniqueness of the solutions. Then, by using the method of quasilinearization, we obtain the two monotone sequences of approximate solutions converging to the unique solution of the problem with rate of convergence of order k . Finally, we give an example to illustrate our main results.

2. Comparison Theorems

In this section, we begin with some comparison principles that will be useful in later discussions.

Lemma 1. Assume that there exist constants $M > 0$, $N > 0$, and $\gamma \geq 0$, such that

$$(M + k_0 NT)T \leq 1. \quad (2)$$

If there exists a function $p \in C^1[J, R]$, such that

$$\begin{cases} p'(t) \leq -M(p(t) + \gamma) - N(Tp)(t), & t \in J, \\ p(0) \leq p(T), \end{cases} \quad (3)$$

then $p(t) \leq 0$ on J .

Proof. Suppose the conclusion is not true, and we consider the following two cases, where $p(0) \leq 0$ and $p(0) > 0$, respectively. \square

Case 1. When $p(0) \leq 0$, there exists a $t^* \in (0, T]$, such that $p(t^*) > 0$. Let $t_* \in [0, t^*]$, such that $p(t_*) = \inf p(t) = -b$, $b \geq 0$. By equation (3), we have

$$p'(t) \leq b[M + k_0 TN]. \quad (4)$$

Integrating inequality (4) from t_* to t^* , we obtain

$$0 < p(t^*) \leq p(t_*) + b \int_{t_*}^{t^*} \{M + k_0 TN\} dt. \quad (5)$$

Thus,

$$b < b \int_0^T \{M + k_0 TN\} dt, \quad (6)$$

which contradicts (2), therefore $p(t) \leq 0$.

Case 2. When $p(0) > 0$, there are two cases: $p(t) > 0$ for $t \in J$ or there exist \bar{t}, \underline{t} , such that $p(\underline{t}) \leq 0$ and $p(\bar{t}) > 0$ for $\bar{t}, \underline{t} \in J$.

Case 3. When $p(t) > 0$, by equation (3), we have $p'(t) \leq 0$, which contradicts the condition of equation (3).

Case 4. If there exist \bar{t} and \underline{t} , such that $p(\underline{t}) \leq 0$ and $p(\bar{t}) > 0$, we have $q(\bar{t}_*) = \inf p(t) = -b$, where $\bar{t}_* \in (0, T)$, $b \geq 0$. Then, equation (4) holds. Integrating inequality (4) from \bar{t}_* to T , we have

$$0 < p(\bar{t}) \leq p(\bar{t}_*) + b \int_{\bar{t}_*}^T \{M + k_0 TN\} dt, \quad (7)$$

which is also a contradiction. The proof of Lemma 1 is completed.

Next, consider the linear APBVP:

$$\begin{cases} x'(t) + M(x(t) + \gamma) + N(Tx)(t) = 0, & t \in J, \\ x(0) = -x(T). \end{cases} \quad (8)$$

Corollary 1. Assume that $M > N > 0$ and $(M + k_0 NT)T \leq 1$, then APBVP (8) has at most one solution.

Proof. Let x_1, x_2 be any solution of APBVP (8), $x_1 \geq x_2$, and $y(t) = x_1(t) - x_2(t)$, then

$$\begin{cases} y'(t) + My(t) + N(Ty)(t) = 0, \\ y(0) = -y(T). \end{cases} \quad (9)$$

If $y(T) > 0$, then it follows from (9) that $y(0) < 0$. By Lemma 1, we have $y(t) \leq 0$, that is a contradiction. On the contrary, if $y(T) < 0$, we have $y(0) > 0$. By the proof of Lemma 1, we have $y(t) \leq 0$, that is also a contradiction. Therefore, we have $y(T) = y(0) = 0$. Furthermore, by Lemma 1, we have $y(t) \leq 0$, that is, $y(t) = 0$ for $t \in J$. The proof of Corollary 1 is completed.

Similar to the proof of Lemma 1, we have the following lemma. \square

Lemma 2. Assume that there exist integrable functions $\phi_i(t) < 0$, $i = 1, 2$, such that

$$\int_0^T \{\phi_1(t) + \phi_2(t)k_0T\} dt \geq -1. \quad (10)$$

If there exist functions $p_i \in C^1[J, R]$, $i = 1, 2$, such that

$$\begin{aligned} p'_1(t) &\leq \phi_1(t)(p_1(t) + \gamma_1) + \phi_2(t)(Tp_1)(t), & \text{for } t \in J, \quad p_1(0) \leq p_2(T), \\ p'_2(t) &\leq \phi_1(t)(p_2(t) + \gamma_2) + \phi_2(t)(Tp_2)(t), & \text{for } t \in J, \quad p_2(0) \leq p_1(T), \end{aligned} \quad (11)$$

where

$$\gamma_1 = \frac{p_1(T) - p_2(0)}{1 - e \int_0^T \phi_1(t) dt}, \quad (12)$$

$$\gamma_2 = \frac{p_2(T) - p_1(0)}{1 - e \int_0^T \phi_2(t) dt}.$$

Then, $p_1(t) \leq 0$ and $p_2(t) \leq 0$ on J .

Proof. We just prove that the case of $p_1(t) \leq 0$. Suppose that the conclusion is not true, we can consider the following two cases, where $p_1(0) \leq 0$ and $p_1(0) > 0$, respectively.

Case 5. When $p_1(0) \leq 0$, by the proof of Lemma 1, we have $\int_0^T \{\phi_1(t) + \phi_2(t)k_0T\}dt < -1$, which contradicts (10).

Case 6. When $p_1(0) > 0$, there are two cases: $p_1(t) > 0$ for $t \in J$ or there exist \bar{t} and \underline{t} , such that $p_1(\underline{t}) \leq 0$ and $p_1(\bar{t}) > 0$ for $\bar{t}, \underline{t} \in J$.

Case 7. When $p_1(t) > 0$, $t \in J$, if $p_2(0) \leq 0$, by the proof of Lemma 1, we have $p_2(t) \leq 0$, that implies $p_1(0) \leq p_2(T) \leq 0$, which is a contradiction.

If $p_2(0) > 0$, we have $p_2(T) \geq p_1(0) > 0$. Then, there are two cases: $p_2(t) > 0$ for $t \in J$ and there exist \bar{t} and \hat{t} , such that $p_2(\bar{t}) \leq 0$ and $p_2(\hat{t}) > 0$, respectively.

Case 8. When $p_2(t) > 0$ for all $t \in J$, we have $p_2'(t) < 0$; hence, $p_2(t)$ is decreasing. By $p(t) > 0$ and equation (11), imply $p_1'(t) < 0$; then, $p_1(t)$ is decreasing and $p_2(0) > p_2(T) \geq p_1(0) > p_1(T) \geq p_2(0)$, which is a contradiction.

Case 9. For another case, we have $p_2(\tilde{t}_*) = \inf p_2(t) = -b$, where $\tilde{t}_* \in (0, T)$, $b \geq 0$. Equation (11) implies that

$$p_2'(t) \leq (-b)[\phi_1(t) + k_0T\phi_2(t)]. \quad (13)$$

Integrating inequality (13) from \tilde{t}_* to T , we have

$$0 < p_2(T) \leq p_2(\tilde{t}_*) - b \int_{\tilde{t}_*}^T \{\phi_1(t) + k_0T\phi_2(t)\}dt. \quad (14)$$

Thus,

$$b < -b \int_0^T \{\phi_1(t) + k_0T\phi_2(t)\}dt, \quad (15)$$

which is also a contradiction.

The proof of Case 4 is analogous to the proof of Lemma 1, and we omit its details here. This completes the proof of Lemma 2.

Remark 1. When $\gamma = 0$ and $\gamma_i = 0$, $i = 1, 2$, respectively, the conclusion of Lemmas 1 and 2 is also true.

3. Linear APBVP

In this section, we consider the linear APBVP:

$$\begin{cases} x'(t) + Mx(t) + N(Tx)(t) = \sigma(t), & t \in J, \\ x(0) + x(T) = 0. \end{cases} \quad (16)$$

We can get the result of the existence and unique solution of equation (16).

Theorem 1. Assume that $M, N > 0$, $M > 2NT$, and $(M + k_0NT)T \leq 1$. Then, APBVP (16) possesses a unique solution.

Proof. For any $x \in C(J, R)$, denoting $\|x\| = \max_{t \in J} |x(t)|$. Let

$$\begin{aligned} \omega_0 &= \max_{t \in J} |\sigma(t)|, \\ \omega_1 &= \frac{2(1 - e^{-MT})\omega_0}{M - 2NT}. \end{aligned} \quad (17)$$

We define an operator $S: E \longrightarrow C(J, R)$ as follows:

$$\begin{aligned} (Sx)(t) &= \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (\sigma(s) - N(Tx)(s))e^{Ms}ds \\ &\quad + e^{-Mt} \int_0^t (\sigma(s) - N(Tx)(s))e^{Ms}ds, \end{aligned} \quad (18)$$

where $E = \{x \in C(J, R): |x| \leq \omega_1, x(0) - x(T) = 0\}$.

It is easy to see that E is a closed, bounded, and convex set. Furthermore, for any $x \in E$, we have

$$\begin{aligned} |(Sx)(t)| &\leq \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| + N|(Tx)(s)|)e^{Ms}ds \\ &\quad + e^{-Mt} \int_0^t (|\sigma(s)| + N|(Tx)(s)|)e^{Ms}ds \\ &\leq \frac{e^{-M(t+T)}}{e^{-MT} + 1} \int_0^T (\omega_0 + NT\omega_1)e^{Ms}ds \\ &\quad + e^{-Mt} \int_0^t (\omega_0 + NT\omega_1)e^{Ms}ds \\ &\leq \frac{2(\omega_0 + N\omega_1T)(1 - e^{-MT})}{M} = \omega_1, \end{aligned} \quad (19)$$

which implies that $\|S(x)\| \leq \omega_1$, that is, $S(E) \subset E$ and S is uniformly bounded. Furthermore, for any $t_1, t_2 \in J$, we have

$$\begin{aligned}
|(Sx)(t_1) - (Sx)(t_2)| &= \left| \frac{e^{-M(t_1+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| - N|(Tx)(s)|) e^{Ms} ds \right. \\
&\quad - \frac{e^{-M(t_2+T)}}{e^{-MT} + 1} \int_0^T (|\sigma(s)| - N|(Tx)(s)|) e^{Ms} ds \\
&\quad + e^{-Mt_1} \int_0^{t_1} (|\sigma(s)| - N|(Tx)(s)|) e^{Ms} ds \\
&\quad \left. - e^{-Mt_2} \int_0^{t_2} (|\sigma(s)| - N|(Tx)(s)|) e^{Ms} ds \right| \\
&\leq \frac{|e^{-M(t_1+T)} - e^{-M(t_2+T)}|}{e^{-MT} + 1} \int_0^T (|\sigma(s)| + N|(Tx)(s)|) e^{Ms} ds \\
&\quad + |e^{-Mt_1} - e^{-Mt_2}| \int_0^{t_1} (|\sigma(s)| + N|(Tx)(s)|) e^{Ms} ds \\
&\quad + e^{-Mt_2} \int_{\min\{t_1, t_2\}}^{\max\{t_1, t_2\}} (|\sigma(s)| + N|(Tx)(s)|) e^{Ms} ds.
\end{aligned} \tag{20}$$

Since σ and x are bounded, thus S is uniformly continuous. According to Ascoli–Arzela’s theorem, there exists the subsequences $\{Sx_n\}$ converging uniformly on J to the continuous functions Sx and $Sx \in E$, then, we can see that S is compact. Therefore, there exists a solution of APBVP (16) by Schauder’s fixed point theorem. The uniqueness of solutions of APBVP (16) follows from Corollary 1. The proof is completed.

4. Nonlinear APBVP

In this section, we give the existence and uniqueness of the solutions of APBVP (1).

Definition 1. The functions $v, w \in C'(J, R)$ are said to be a pair of coupled lower and upper solutions for APBVP (1) if the following inequalities

$$\begin{cases} v'(t) \leq f(t, v(t), (Tv)(t)) - M\gamma_1, \\ v(0) \leq -w(T), \end{cases} \tag{21}$$

$$\begin{cases} w'(t) \geq f(t, w(t), (Tw)(t)) + M\gamma_2, \\ w(0) \geq -v(T), \end{cases} \tag{22}$$

hold, where $M > 0$, $\gamma_1 = (v(T) + w(0))/(1 - e^{-MT})$, and $\gamma_2 = (-w(T) - v(0))/(1 - e^{-MT})$.

Theorem 2. Assume that the following conditions hold.

(H_1) $v, w \in C'(J, R)$ are a pair of coupled lower and upper solutions of APBVP (1) such that $v \leq w$ on J ;

(H_2) There exist constants $M > 0$ and $N > 0$ such that $M > 2NT \geq 0$, $(M + k_0NT)T \leq 1$, and

$$|f(t, \bar{u}, T\bar{u}) - f(t, u, Tu)| \leq M(\bar{u} - u) + N(T\bar{u} - Tu), \tag{23}$$

while $v \leq u \leq \bar{u} \leq w$ and $Tv \leq Tu \leq T\bar{u} \leq Tw$ for $t \in J$.

Then, APBVP (1) has a unique solution $x \in [v, w]$.

Proof. We construct iterative sequences $\{v_n\}, \{w_n\} \subset C'(J, R)$ as follows, $v_1 = v$ and $w_1 = w$ on J , and for $n > 1$, v_n and w_n are the solutions of

$$\begin{cases} v'_n(t) = f(t, v_{n-1}(t), (Tv_{n-1})(t)) - M[v_n(t) - v_{n-1}(t)] - N[(Tv_n)(t) - (Tv_{n-1})(t)], \\ v_n(0) = -w_n(T), \end{cases} \tag{24}$$

$$\begin{cases} w'_n(t) = f(t, w_{n-1}(t), (Tw_{n-1})(t)) - M[w_n(t) - w_{n-1}(t)] - N[(Tw_n)(t) - (Tw_{n-1})(t)], \\ w_n(0) = -v_n(T). \end{cases} \tag{25}$$

The existence and uniqueness of the solution can be obtained by standard arguments for IVP (24) and (25).

We next prove that $v_1 \leq v_2 \leq w_2 \leq w_1$.

$$\begin{cases} p'_1(t) \leq -M[v_1(t) - v_2(t) + \gamma_1] - N[(Tv_1)(t) - (Tv_2)(t)] = -M(p_1(t) + \gamma_1) - N(Tp_1)(t), \\ p'_2(t) \leq -M[w_2(t) - w_1(t) + \gamma_2] - N[(Tw_2)(t) - (Tw_1)(t)] = -M(p_2(t) + \gamma_2) - N(Tp_2)(t), \end{cases} \quad (26)$$

where $\gamma_1 = (p_1(T) - p_2(0))/(1 - e^{-MT})$ and $\gamma_2 = (p_2(T) - p_1(0))/(1 - e^{-MT})$.

By Lemma 2, we have $p_1 = v_1 - v_2 \leq 0$ and $p_2 = w_2 - w_1 \leq 0$ on J .

Let $p = v_2 - w_2$, by the condition of (H_2) , and we have

$$p'(t) \leq -Mp(t) - N(Tp)(t), \quad p(0) = p(T). \quad (27)$$

By using similar arguments of Lemma 1, we have $p = v_2 - w_2 \leq 0$. Therefore, it is easy to see that these sequences satisfy

$$v_n \leq v_{n+1} \leq w_{n+1} \leq w_n, \quad n \geq 1. \quad (28)$$

Then, we have two monotone sequences which are bounded, and there exist ρ and μ , which satisfy $\lim_{n \rightarrow \infty} v_n = \rho$, $\lim_{n \rightarrow \infty} w_n = \mu$, and $\rho \leq \mu$. Moreover, the convergence is uniform on J .

Set $p = \mu - \rho$, then we obtain

$$\begin{cases} p'(t) \leq -Mp(t) - N(Tp)(t), \\ p(0) = p(T). \end{cases} \quad (29)$$

By Lemma 1, we have $p(t) \leq 0$ for $t \in J$. Hence, $\rho \equiv \mu$ for $t \in J$, and we can conclude $\rho \equiv \mu \equiv x$, in which x is the solution of APBVP (1). The proof of Theorem 2 is completed.

5. Quasilinearization

In this section, we apply the quasilinearization method in order to obtain the result on convergence of the iterative sequences of approximate solutions for APBVP (1).

Consider the Banach space $C(J, R)$ with the usual maximum norm $\|x\|_1 = \max_{t \in J} |x(t)|$. For any $x \in C(J, R)$, we call that a given sequence $\{x_n\}$ converges to x with order of convergence k , if $\{x_n\}$ converges to x in $C(J, R)$ and there exist $n_0 \in \mathbb{N}$ and $k > 0$ such that $\|x_{m+1} - x\|_1 \leq \|x_m - x\|_1^k$ for all $m \geq n_0$.

Theorem 3. Assume that the conditions of $(H_1) - (H_2)$ hold.

(H_3) $(\partial^i f / \partial x^i)$ and $\partial^i f / \partial (Tx)^i$ exist and are continuous for $i = 0, 1, \dots, k$, and

Let $p_1 = v_1 - v_2$ and $p_2 = w_2 - w_1$, by the condition of (H_2) , and we have $p_1(0) \leq p_2(T)$, $p_2(0) \leq p_1(T)$, and

$$\begin{aligned} \sum_{i=1}^k i M_i |w - v|^{i-1} &\leq M, \\ \sum_{i=1}^k i N_i |w - v|^{i-1} &\leq N, \end{aligned} \quad (30)$$

where M_i and N_i are constants with

$$\begin{aligned} \left\| \frac{\partial^i f}{\partial x^i}(t, u, Tu) \right\| &\leq (i!) M_i, \\ \left\| \frac{\partial^i f}{\partial (Tx)^i}(t, u, Tu) \right\| &\leq (i!) N_i, \end{aligned} \quad (31)$$

where $(t, u, Tu) \in \Omega = \{(t, u, Tu): v \leq u \leq w\}$.

Then, there exist monotone sequences $\{v_n\}$, $\{w_n\}$ of approximate solutions converging to the unique solution of (1) with rate of convergence of order k .

Proof. Let the function

$$\begin{aligned} f(t, u, Tu) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \frac{(u - \alpha)^i}{i!} + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \frac{(u - \alpha)^k}{k!} \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \frac{T(u - \alpha)^i}{i!} \\ &\quad + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \frac{T(u - \alpha)^k}{k!} \\ &\equiv g(t, u, Tu; \alpha, T\alpha), \end{aligned} \quad (32)$$

where $v \leq \alpha \leq u \leq w$, $\chi \in [\alpha, u]$. Consider the following linear equation:

$$\begin{cases} u'(t) = g(t, u, Tu; \alpha, T\alpha), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (33)$$

Setting $v_0 = v$, by (H_3) , we have

$$\begin{cases} v'_0(t) \leq f(t, v_0(t), (Tv_0)(t)) - M\gamma_1 \equiv g(t, v_0(t), (Tv_0)(t); v_0(t), (Tv_0)(t)) - M\gamma_1, \\ v_0(0) \leq -w_0(T). \end{cases} \quad (34)$$

Similarly, setting $w_0 = w$, we obtain

$$\begin{cases} w'_0(t) \geq f(t, w_0(t), (Tw_0)(t)) + M\gamma_2 \equiv g(t, w_0(t), (Tw_0)(t); w_0(t), (Tw_0)(t)) + M\gamma_2, \\ w_0(0) \geq -v_0(T). \end{cases} \quad (35)$$

Then, v_0 and w_0 are lower and upper solutions of equation (33), respectively. Furthermore, for $t \in J$ and $v \leq x \leq y \leq w$, we have

$$\begin{aligned} & |g(t, x, Tx; \alpha, T\alpha) - g(t, y, Ty; \alpha, T\alpha)| \\ &= \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \left(\frac{(x-\alpha)^i - (y-\alpha)^i}{i!} \right) \right. \\ &\quad + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \left(\frac{(x-\alpha)^k - (y-\alpha)^k}{k!} \right) \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \left(\frac{T(x-\alpha)^i - T(y-\alpha)^i}{i!} \right) \\ &\quad \left. + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \left(\frac{T(x-\alpha)^k - T(y-\alpha)^k}{k!} \right) \right| \\ &= |x-y| \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, \alpha, T\alpha) \left(\frac{1}{i!} \sum_{j=0}^{i-1} (x-\alpha)^{i-1-j} (y-\alpha)^j \right) \right. \\ &\quad + \frac{\partial^k f}{\partial x^k}(t, \chi, T\chi) \left(\frac{(x-\alpha)^{i-1-j} (y-\alpha)^j}{k!} \right) \\ &\quad + T|x-y| \left| \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, \alpha, T\alpha) \left(\frac{1}{i!} \sum_{j=0}^{i-1} T(x-\alpha)^{i-1-j} T(y-\alpha)^j \right) \right. \\ &\quad \left. + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi, T\chi) \left(\frac{T(x-\alpha)^{i-1-j} T(y-\alpha)^j}{k!} \right) \right| \\ &\leq |x-y| \sum_{i=1}^k M_i \left(\sum_{j=0}^{i-1} (y-x)^{i-1-j} \right) + T|x-y| \sum_{i=1}^k N_i \left(\sum_{j=0}^{i-1} T(y-x)^{i-1-j} \right) \\ &\leq M|x-y| + NT|x-y|. \end{aligned} \quad (36)$$

Using Theorem 2, we know that problem (33) has a solution in $[v, w]$.

Let v_1 be a solution of the mentioned problem, with $v_1 \in [v, w]$, and we suppose $v = v_0 \leq v_1 \leq \dots \leq v_n \leq w$, where $v_n \in [v_{n-1}, w]$ is the solution of

$$\begin{cases} u'(t) = g(t, u, Tu; v_{n-1}, Tv_{n-1}), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (37)$$

where v_{n-1} and w are lower and upper solutions, respectively, for the following problem:

$$\begin{cases} u'(t) = g(t, u, Tu; v_n, Tv_n), & t \in J, \\ u(0) = -u(T). \end{cases} \quad (38)$$

Similarly, we know that $g(t, u, Tu; v_n, Tv_n)$ satisfies the conditions of Theorem 2, then problem (33) has a solution in $[v_n, w]$. Let v_{n+1} be a solution of the mentioned problem, with $v_{n+1} \in [v_n, w]$. Hence, the constructed sequence $\{v_n\}$ is nondecreasing and bounded. In the same way, we can construct the sequence $\{w_n\}$ which is nonincreasing and bounded. Therefore, we obtain the two monotone sequences converge uniformly.

Let $\lim v_n = \rho$ and $\lim w_n = \mu$, and we can get $\rho \equiv \mu \equiv u$ by using Theorem 2, in which u is the solution of (1).

Now, we show that the convergence of $\{v_n\}$, $\{w_n\}$ to u is of order k . Let

$$\begin{aligned} p_n &= u - v_n \geq 0, \\ q_n &= w_n - u \geq 0. \end{aligned} \quad (39)$$

Firstly, we note that

$$\begin{aligned} u'(t) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \frac{(u - v_n)^i}{i!} + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \frac{(u - v_n)^k}{k!} \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \frac{T(u - v_n)^i}{i!} \\ &\quad + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \frac{T(u - v_n)^k}{k!}, \end{aligned} \quad (40)$$

where $\chi_n \in [v_n, u]$. In sequence,

$$\begin{aligned} p'_{n+1}(t) &= \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{p_n^i - (v_{n+1} - v_n)^i}{i!} \right) + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \left(\frac{p_n^k - (v_{n+1} - v_n)^k}{k!} \right) \\ &\quad + \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{Tp_n^i - T(v_{n+1} - v_n)^i}{i!} \right) + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \left(\frac{Tp_n^k - T(v_{n+1} - v_n)^k}{k!} \right) \\ &= p_{n+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_n^{i-1-j} (v_{n+1} - v_n)^j \right) + \frac{\partial^k f}{\partial x^k}(t, \chi_n, T\chi_n) \left(\frac{p_n^k - (v_{n+1} - v_n)^k}{k!} \right) \\ &\quad + Tp_{n+1} \sum_{i=1}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} (Tp_n^{i-1-j} T(v_{n+1} - v_n)^j) \right) + \frac{\partial^k f}{\partial (Tx)^k}(t, \chi_n, T\chi_n) \left(\frac{Tp_n^k - T(v_{n+1} - v_n)^k}{k!} \right). \end{aligned} \quad (41)$$

In view of condition (H_3) , by the continuity of $\partial^i f / \partial x^i$ and $\partial^i f / \partial (Tx)^i$ in Ω , we have

$$p'_{n+1}(t) \leq Ap_{n+1}(t) + BTp_{n+1}(t) + Cp_n^k(t) + DTP_n^k(t), \quad (42)$$

where $C = 2M_k$, $D = 2N_k$, and

$$\begin{aligned} \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial x^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} p_n^{i-1-j} (v_{n+1} - v_n)^j \right) &\leq A, \\ \sum_{i=0}^{k-1} \frac{\partial^i f}{\partial (Tx)^i}(t, v_n, Tv_n) \left(\frac{1}{i!} \sum_{j=0}^{i-1} Tp_n^{i-1-j} T(v_{n+1} - v_n)^j \right) &\leq B. \end{aligned} \quad (43)$$

Similarly, we have

$$q'_{n+1}(t) \leq Aq_{n+1}(t) + BTq_{n+1}(t) + Cq_n^k(t) + DTq_n^k(t). \quad (44)$$

The boundary conditions are

$$\begin{aligned} p_n(0) &= q_n(T), \\ q_n(0) &= p_n(T). \end{aligned} \quad (45)$$

Let $R_n = p_n + q_n$, and we obtain

$$\begin{cases} R'_{n+1}(t) \leq AR_{n+1}(t) + BTR_{n+1}(t) + C(p_n^k(t) + q_n^k(t)) + DT(p_n^k(t) + q_n^k(t)) \leq (A + k_0B)R_{n+1}(t) + (C + k_0D)(p_n^k(t) + q_n^k(t)), \\ R_{n+1}(0) = R_{n+1}(T). \end{cases} \quad (46)$$

Using Gronwall's inequality for (46), we have

$$R_{n+1}(t) \leq e^{(A+k_0B)t} R_{n+1}(0) + \int_0^t e^{(A+k_0B)(t-s)} (C + k_0D)(p_n^k(s) + q_n^k(s)) ds. \quad (47)$$

Let $t = T$, and we have

$$R_{n+1}(0) \leq e^{(A+k_0B)T} R_{n+1}(0) + \frac{T}{A+k_0B} e^{(A+k_0B)T} \cdot (C + k_0D) \max_{t \in J} (p_n^k(t) + q_n^k(t)), \quad (48)$$

which implies

$$R_{n+1}(0) \leq \frac{e^{(A+k_0B)T} (C + k_0D) \max_{t \in J} (p_n^k(t) + q_n^k(t))}{T(1 - e^{(A+k_0B)T})}, \quad (49)$$

that is,

$$\max_{t \in J} R_{n+1}(t) \leq K (\max_{t \in J} p_n^k(t) + \max_{t \in J} q_n^k(t)), \quad (50)$$

where $K = e^{(A+k_0B)T} (e^{(A+k_0B)T} T(C + k_0D) \max_{t \in J} R_n^k(t)) / (1 - e^{(A+k_0B)T}) + T / (A + k_0B) e^{(A+k_0B)T} (C + k_0D)$.

Since

$\max_{t \in J} p_n(t) \leq \max_{t \in J} R_n(t)$ and $\max_{t \in J} q_n(t) \leq \max_{t \in J} R_n(t)$, we get the desired convergence. The proof is completed.

6. An Example

In this section, we will provide an example which demonstrates the application of Theorem 3.

Example 1. Consider the following APBVP:

$$\begin{cases} x'(t) = \frac{1}{20} t(1 + x^2(t)) - \frac{1}{10} x - \frac{1}{20} \int_0^t x(s) \cos s ds, & t \in [0, 1], \\ x(0) = -x(1). \end{cases} \quad (51)$$

It is easy to check that $v_0 = -1$ and $w_0 = 1$ are lower and upper solutions of (51), respectively, which satisfies condition (H_1) of Theorem 3. And we can show that

$$\begin{aligned} |f_x| &\leq \frac{1}{5}, \\ |f_y| &\leq \frac{1}{20}, \\ |f_{xx}| &\leq \frac{1}{10}, \\ f_{yy} &= 0, \quad \frac{\partial^i f}{\partial x^i} = 0, \\ \frac{\partial^i f}{\partial (Tx)^i} &= 0, \\ i &= 3, 4, \dots, k. \end{aligned} \quad (52)$$

Setting $M_1 = 1/5$, $M_2 = 1/10$, $N_1 = 1/20$, $N_2 = 0$, $M_i = 0$, $N_i = 0$, $i = 3, 4, \dots, k$, $M = 3/5$, and $N = 1/20$, satisfy conditions (H_2) and (H_3) of Theorem 3. Then, convergence of the iterative sequences of approximate solutions for APBVP (51) are of order $k \geq 2$.

7. Conclusion

In this paper, we discussed the problem of rapid convergence for the first-order integro-differential equations with anti-periodic boundary value conditions. By using the particular definitions of the coupled lower and upper related solutions, which are new and some new comparison principles, we obtained the existence and uniqueness of solution of the problems. Meanwhile, by using the method of quasilinearization, we obtained the monotone sequences of approximate solutions, converging to the unique solution of such problems with the rate of convergence of order k . Finally, we give an example to illustrate our main results.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no competing interest.

Authors' Contributions

All authors read and approved the final manuscript.

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Research Article

Some Fractional Operators with the Generalized Bessel–Maitland Function

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In this paper, we aim to determine some results of the generalized Bessel–Maitland function in the field of fractional calculus. Here, some relations of the generalized Bessel–Maitland functions and the Mittag–Leffler functions are considered. We develop Saigo and Riemann–Liouville fractional integral operators by using the generalized Bessel–Maitland function, and results can be seen in the form of Fox–Wright functions. We establish a new operator $\mathcal{I}_{\nu, \eta, \rho, \epsilon, w, a^+}^{\mu, \xi, m, \sigma} \phi$ and its inverse operator $D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} \phi$, involving the generalized Bessel–Maitland function as its kernel, and also discuss its convergence and boundedness. Moreover, the Riemann–Liouville operator and the integral transform (Laplace) of the new operator have been developed.

1. Introduction

During the last few years, many types of research studies developed the class of generalized fractional integrals containing a variety of special functions [1–5]. Watson [6] discussed applications of the Bessel function with some fields of applied sciences, biology, chemistry, physical sciences, and engineering. The generalization and extensions of the Bessel–Maitland function [7–14] dealt with special cases that gave useful results in different areas of mathematics. The recent work in the field of fractional calculus theory, differential equations of the Mittag–Leffler function, Sturm–Liouville problems in theoretical sense, Gronwall’s inequality, and exponential kernels of the differential operator [15–18] have found many applications in various subfields of mathematical analysis.

The series representation of the Bessel–Maitland function [19] is defined as

$$J_{\beta}^{\alpha}(s) = \sum_{n=0}^{\infty} \frac{(-s)^n}{n! \Gamma(\alpha n + \beta + 1)} = \phi(\alpha, \beta + 1; -s). \quad (1)$$

The generalization of the Bessel–Maitland function introduced by Singh et al. [7] is

$$J_{\beta, q}^{\alpha, \gamma}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (-s)^n}{\Gamma(\alpha n + \beta + 1) n!}, \quad (2)$$

where $\alpha, \gamma, \beta \in \mathbb{C}$, $\Re(\alpha) \geq 0$, $\Re(\beta) \geq -1$, $\Re(\gamma) \geq 0$, and $q \in (0, 1) \cup \mathbb{N}$.

The extended Bessel–Maitland function investigated in [20] is

$$J_{\beta, \gamma, \delta}^{\alpha, q, p}(s) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn} (-s)^n}{\Gamma(\alpha n + \beta + 1) (\delta)_{pn}}, \quad (3)$$

where $\alpha, \gamma, \beta, \delta \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > -1$, $\Re(\gamma) > 0$, $\Re(\delta) \geq 0$, $p, q > 0$, and $q < \Re(\alpha) + p$.

The Saigo fractional integral operators are defined [21] for $s > 0$, $a, c, d \in \mathbb{C}$, and $\Re(a) > 0$:

$$(\mathcal{F}_{0*+}^{a,c,d} g)(s) = \frac{s^{-a-c}}{\Gamma(a)} \int_0^s (s-\tau)^{a-1} \cdot {}_2R_1\left(a+c, -d; a; \left(1-\frac{\tau}{s}\right)\right) g(\tau) d\tau, \quad (4)$$

$$(\mathcal{F}_{0-}^{a,c,d} g)(s) = \frac{1}{\Gamma(a)} \int_s^\infty (\tau-s)^{a-1} \tau^{-a-c} \cdot {}_2R_1\left(a+c, -d; a; \left(1-\frac{s}{\tau}\right)\right) g(\tau) d\tau. \quad (5)$$

Samko et al. [22] defined the Riemann–Liouville fractional operators for $\Re(a) > 0$ and $n = [\Re(a)] + 1$ as

$$(\mathcal{F}_{0+}^a g)(s) = \frac{1}{\Gamma(a)} \int_0^s (s-\tau)^{a-1} g(\tau) d\tau, \quad (6)$$

$$(\mathcal{D}_{0+}^a \phi)(s) = \left(\frac{d}{ds}\right)^n (\mathcal{F}_{0+}^{n-a} \phi)(s). \quad (7)$$

The Gauss hypergeometric function defined by Saigo [21] for all $a, c, d \in \mathbb{C}$, $a \neq 0$, and $|s| < 1$ is

$${}_2R_1(b, -d; a; s) = \sum_{n=0}^{\infty} \frac{(b)_n (-d)_n s^n}{(a)_n n!}, \quad (8)$$

where $(b)_n$, $(-d)_n$, and $(a)_n$ are Pochhammer's symbols.

Pochhammer's symbols defined by Petojevic [23] are

$$(s)_n = \begin{cases} s(s+1)(s+2)\cdots(s+n-1), & \text{for } n \geq 1, \\ 1, & \text{for } n = 0, s \neq 0, \end{cases} \quad (9)$$

where $s \in \mathbb{C}$ and $n \in \mathbb{N}$, and in gamma form, they can be written as

$$(s)_n = \frac{\Gamma(s+n)}{\Gamma(s)}. \quad (10)$$

The beta function is defined as given in [23], for $\Re(y) > 0$ and $\Re(z) > 0$, and also expressed in the gamma form, respectively:

$$\beta(y, z) = \int_0^1 g^{y-1} (1-g)^{z-1} dg, \quad (11)$$

$$\beta(y, z) = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}. \quad (12)$$

The gamma function is defined [23] for $\Re(u) > 0$ as

$$\Gamma(u) = \int_0^\infty g^{u-1} e^{-g} dg. \quad (13)$$

The generalized hypergeometric function is defined by Rainville [24]:

$${}_kR_r(c_1, \dots, c_k, q_1, \dots, q_r; s) = \sum_{n=0}^{\infty} \frac{(c_1)_n \cdots (c_k)_n}{(q_1)_n \cdots (q_r)_n} \frac{s^n}{n!}, \quad (14)$$

where $c_i, q_j \in \mathbb{C}$, $q_j \neq 0, -1, \dots$, $(i = -1, -2, \dots, k; j = -1, -2, \dots, r)$.

The generalized Fox–Wright function is defined as [25]

$${}_r\psi_s(g) = {}_r\psi_s \left[\begin{matrix} (y_j, p_j)_{1,r} \\ (z_i, q_i)_{1,s} \end{matrix} \middle| g \right] \equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^r \Gamma(y_j + p_j n)}{\prod_{i=1}^s \Gamma(z_i + q_i n)} \frac{g^n}{n!}, \quad (15)$$

where $g \in \mathbb{C}$, $y_j, z_i \in \mathbb{C}$, and $p_j, q_i \in \mathbb{R}$ ($j = 1, 2, \dots, r; i = 1, 2, \dots, s$).

The Gauss hypergeometric function in the gamma form can be written as

$${}_2R_1(a, s; d; 1) = \frac{\Gamma(d)\Gamma(d-a-s)}{\Gamma(d-a)\Gamma(d-s)}, \quad \Re(d-a-s) > 0. \quad (16)$$

The Laplace transform of function $f(z)$ is defined as

$$\mathcal{L}[f(t)] = f(s) = \int_0^\infty e^{-st} f(t) dt. \quad (17)$$

Dirichlet formula (Fubini's theorem) [22] is given by

$$\int_d^c dy \int_y^c g(y, \tau) d\tau = \int_d^c d\tau \int_d^\tau g(y, \tau) dy. \quad (18)$$

Definition 1. The generalization of the generalized Bessel–Maitland function is defined and investigated as

$$\mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(s) = \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-s)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}}, \quad (19)$$

where $\mu, \nu, \eta, \rho, \gamma \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$.

The following notation is used in our results:

$${}_{\mu, \nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} = \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-1)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}}. \quad (20)$$

Definition 2. The extension of generalized Bessel–Maitland function (19) in multivariable function can be defined for $\mu_j, \nu, \eta_j, \rho_j, \gamma_j \in \mathbb{C}$, $\Re(\mu_j) > 0$, $\Re(\nu) > -1$, $\Re(\eta_j) > 0$, $\Re(\rho_j) > 0$, $\Re(\gamma_j) > 0$, $\xi_j, m_j, \sigma_j \geq 0$, and $m_j, \xi_j > \Re(\mu_j) + \sigma_j$, $j = 1, \dots, r$, as

$$\begin{aligned} J_{\nu, \eta_j, \rho_j, \gamma_j}^{\mu_j, \xi_j, m_j, \sigma_j}(s_1 \dots s_r) &= J_{\nu, \eta_1, \dots, \eta_r; \rho_1, \dots, \rho_r; \gamma_1, \dots, \gamma_r}^{\mu_1, \dots, \mu_r; \xi_1, \dots, \xi_r; m_1, \dots, m_r; \sigma_1, \dots, \sigma_r}(s_1 \dots s_r) \\ &= \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(\eta_1)_{\xi_1 n_1} \cdots (\eta_r)_{\xi_r n_r} (\gamma_1)_{\sigma_1 n_1} \cdots (\gamma_r)_{\sigma_r n_r} (-s_1)^{n_1} \cdots (-s_r)^{n_r}}{\Gamma(\nu + 1 + \sum_{j=1}^r n_j \mu_j) (\rho_1)_{m_1 n_1} \cdots (\rho_r)_{m_r n_r}}. \end{aligned} \quad (21)$$

Remark 1. On setting $j = 1$ in equation (21), we get generalized Bessel–Maitland function (19).

Definition 3. An integral operator which involves generalized Bessel–Maitland function (19) as its kernel is defined for $\mu, \nu, \eta, w, \gamma, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$ as follows:

$$\left(\mathcal{J}_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} \phi \right)(s) = \int_a^s (s - \tau)^\nu \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} [w(s - \tau)^\mu] \phi(\tau) d\tau. \quad (22)$$

Remark 2. If we put $w = 0$ and replace ν by $\nu - 1$, then it will become a left-sided Riemann–Liouville fractional integral operator.

The new fractional operator (22) can be discussed to improve the results of some inequalities such as Polya–Szegő inequality, Chebyshev inequality, and Hadamard inequality in the field of analysis.

Definition 4. The left inverse operator of integral operator (22), for $\mu, \nu, \eta, w, \gamma, \rho \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$, $\Re(\gamma) > 0$, $\Re(\rho) > 0$, $\xi, m, \sigma \geq 0$, $m, \xi > \Re(\mu) + \sigma$, and $n = [\nu]$ as $n - \nu > 0$ is defined as follows:

$$\begin{aligned} \left(D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} \phi \right)(s) &= \left(\frac{d^n}{ds^n} \left(\mathcal{J}_{\nu-n, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} \phi \right) \right)(s) \\ &= \frac{d^n}{ds^n} \int_a^s (s - \tau)^{n-\nu} \mathcal{J}_{n-\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} [w(s - \tau)^\mu] \\ &\quad \cdot \phi(\tau) d\tau. \end{aligned} \quad (23)$$

Remark 3. If we put $w = 0$ and replace ν by $\nu - 1$, then equation (23) becomes the Riemann–Liouville fractional differential operator.

Remark 4. If we replace $\sigma = 0$, $\eta = -\eta$, $\xi = \rho = m = 1$, and $\nu = \nu - 1$ in equation (23), we get

$$\left(D_{\nu-1, -\eta, 1, \gamma, w, a^+}^{\mu, 1, 1, 0} \phi \right)(s) = \left(D_{\mu, \gamma, w, a^+}^\eta \phi \right)(s), \quad (24)$$

where the inverse operator $(D_{\mu, \gamma, w, a^+}^\eta \phi)(s)$ is described and discussed by Polito and Tomovski in [26].

2. Relation with the Bessel–Maitland and the Mittag-Leffler Functions

In this section, we discuss some special cases of the generalized Bessel–Maitland function and developed its relations with generalized Mittag-Leffler functions:

(i) On replacing $\sigma = 0$ in equation (19), we obtain the relation

$$\mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, 0}(s) = J_{\nu, \eta, \rho}^{\mu, \xi, m}(s), \quad (25)$$

where $J_{\nu, \eta, \rho}^{\mu, \xi, m}(z)$ is the generalized Bessel–Maitland function investigated in [20].

(ii) On replacing $\sigma = 0$ and $m = \rho = 1$ in equation (19), we obtain the relation

$$\mathcal{J}_{\nu, \eta, 1, 1}^{\mu, \xi, 1, 0}(s) = J_{\nu, \eta}^{\mu, \xi}(s), \quad (26)$$

where $J_{\nu, \eta}^{\mu, \xi}(s)$ is the generalization of Bessel–Maitland function defined by Singh et al. [7].

(iii) On replacing $\sigma = \xi = 0$ and $m = \rho = 1$ in equation (19), we obtain the relation [19]

$$\mathcal{J}_{\nu, \eta, 1, \gamma}^{\mu, 0, 1, 0}(s) = J_{\nu}^{\mu}(s). \quad (27)$$

(iv) On setting $\sigma = 0$ and replacing ν by $\nu - 1$ in equations (19) and (22), then we have

$$\mathcal{J}_{\nu-1, \eta, \rho, \gamma}^{\mu, \xi, m, 0}(-s) = \mathcal{E}_{\mu, \nu, m}^{\eta, \rho, \xi}(s), \quad (28)$$

$$\left(\mathcal{J}_{\nu-1, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, 0} \phi \right)(s) = \left(\mathcal{E}_{\mu, \nu, m, w, a^+}^{\eta, \rho, \xi} \phi \right)(w), \quad (29)$$

where $\mathcal{E}_{\mu, \nu, m}^{\eta, \rho, \xi}(s)$ is the Mittag-Leffler function investigated by Salim and Faraj [27].

(v) On setting $\sigma = 0$ and $\rho = m = 1$ and replacing ν by $\nu - 1$ in equation (19), then we have

$$\mathcal{J}_{\nu-1, \eta, 1, \gamma}^{\mu, \xi, 1, 0}(-s) = \mathcal{E}_{\mu, \nu}^{\eta, \xi}(s), \quad (30)$$

$$\left(\mathcal{J}_{\nu-1, \eta, 1, \gamma, w, a^+}^{\mu, \xi, 1, 0} \phi \right)(s) = \mathcal{E}_{a^+; \mu, \nu}^{w; \eta, \xi}(s),$$

where $\mathcal{E}_{\mu, \nu}^{\eta, \rho}(s)$ is the Mittag-Leffler function defined by Shukla and Prajapati [28] and $\mathcal{E}_{a^+; \mu, \nu}^{w; \eta, \xi}(s)$ is described by Srivastava and Tomovski in [29].

(vi) On setting $\sigma = 0$ and $\xi = m = \rho = 1$ and replacing ν by $\nu - 1$ in equations (19) and (22), then we have

$$\begin{aligned} \mathcal{J}_{\nu-1, \eta, 1, \gamma}^{\mu, 1, 1, 0}(-s) &= \mathcal{E}_{\mu, \nu}^\eta(s), \\ \left(\mathcal{J}_{\nu-1, \eta, 1, \gamma, w, a^+}^{\mu, 1, 1, 0} \phi \right)(s) &= \mathcal{F}^*(\mu, \nu; \eta; w) \phi(s), \end{aligned} \quad (31)$$

where $\mathcal{E}_{\mu, \nu}^\eta(s)$ is defined by Prabhakar in [30] and described the integral

$$\mathcal{F}^*(\mu, \nu; \eta; w) \phi(s). \quad (32)$$

(vii) On setting $\sigma = 0$ and $\xi = m = \eta = \rho = \gamma = 1$ and replacing ν by $\nu - 1$ in equation (19), then we have

$$\mathcal{J}_{\nu-1, 1, 1, 1, \gamma}^{\mu, 1, 1, 0}(-s) = \mathcal{E}_{\mu, \nu}(s), \quad (33)$$

where $\mathcal{E}_{\mu, \nu}(s)$ is the Mittag-Leffler function discussed by Wiman [31].

(viii) On setting $\sigma = 0$, $\xi = m = \eta = \rho = \gamma = 1$, and $\nu = 0$ in equation (19), then we have

$$\mathcal{J}_{0, 1, 1, 1, \gamma}^{\mu, 1, 1, 0}(-s) = \mathcal{E}_\mu(s), \quad (34)$$

where $\mathcal{E}_\mu(s)$ is the Mittag-Leffler function introduced in [32].

3. Convergence and Boundedness of the New Fractional Integral Operator

In this section, we discuss the convergence and boundedness of the fractional integral operator involving the generalized Bessel–Maitland function as its kernel in the form of a theorem.

Theorem 1. Let the operator $(\mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi)(s)$ be bounded on $L(a, c)$ with $\mu, \nu, \eta, w, \gamma \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\left\| \left(\mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi \right) \right\|_c \leq B \|\phi\|_c, \quad (35)$$

where

$$B = (c-a)^{\Re(\nu)} \sum_{n=0}^{\infty} \frac{|(\eta)_{\xi n}| |(\gamma)_{\sigma n}|}{|(\rho)_{mn}|} \cdot \frac{|(-w(c-a)^{\Re(\mu)})^n|}{|\Gamma(\mu n + \nu + 1)| |\Re(\mu)n + \Re(\nu) + 1|}. \quad (36)$$

Proof. Let K_n denote the n th term of (36); then,

$$\begin{aligned} \left| \frac{K_{n+1}}{K_n} \right| &= \left| \frac{(\eta)_{\xi n + \xi}}{(\eta)_{\xi n}} \right| \left| \frac{(\gamma)_{\sigma n + \sigma}}{(\gamma)_{\sigma n}} \right| \left| \frac{(\rho)_{mn}}{(\rho)_{mn+m}} \right| \left| \frac{\Gamma(\mu n + \nu + 1)}{\Gamma(\mu n + \mu + \nu + 1)} \right| \\ &\quad \times \left| \frac{\Re(\mu)n + \Re(\nu) + 1}{\Re(\mu)(n+1) + \Re(\nu) + 1} \right| \left| \frac{(-1)^{n+1}}{(-1)^n} \right| |w(c-a)^{\Re(\mu)}| \\ &\approx \frac{(\xi n)^\xi (\sigma n)^\sigma |w(c-a)^{\Re(\mu)}|}{(\rho n)^\rho |n+1| (|\mu n|^{\Re(\mu)})}, \quad \text{as } n \rightarrow \infty. \end{aligned} \quad (37)$$

Hence, $|K_{n+1}/K_n| \rightarrow 0$ as $n \rightarrow \infty$, and $\xi, \sigma < \rho + \Re(\mu)$ which means that the right-hand side of (36) is convergent and finite under the given condition. The condition of boundedness of the integral operator $(\mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi)(x)$ is discussed in the space of Lebesgue measure $L(a, c)$ of a continuous function on (a, c) , where $c > a$:

$$L(a, c) = \left\{ g(x) : \|g\|_c = \int_a^c |g(x)| dx < \infty \right\}. \quad (38)$$

According to equations (19) and (22), we have

$$\begin{aligned} \left\| \mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi \right\|_c &= \int_a^c \left| \int_a^s (s-t)^\nu \left| \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} [w(s-t)^\mu] \right| \phi(t) dt \right| ds \\ &\leq \int_a^c \left| \int_t^c (s-t)^\nu \left| \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} [w(s-t)^\mu] \right| ds \right| |\phi(t)| dt. \end{aligned} \quad (39)$$

By putting the values $s-t = y \Rightarrow ds = dy$, $s=c \Rightarrow y=c-t$, and $s=t \Rightarrow y=0$ in equation (39), we have

$$\begin{aligned} \left\| \mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi \right\|_c &= \int_a^c \left| \int_0^{c-t} y^{\Re(\nu)} \left| \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} [wy^\mu] \right| dy \right| |\phi(t)| dt \\ &\leq \int_a^c \left| \int_0^{c-a} y^{\Re(\nu)} \left| \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} (wy^\mu) \right| dy \right| |\phi(t)| dt, \\ B &= \int_0^{c-a} y^{\Re(\nu)} \left| \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} (wy^\mu) \right| dy, \\ B &= \sum_{n=0}^{\infty} \frac{|(\eta)_{\xi n}| |(\gamma)_{\sigma n}| |(-w)^n|}{|(\rho)_{mn}| |\Gamma(\mu n + \nu + 1)|} \int_0^{c-a} y^{\Re(\mu)n + \Re(\nu)} dy \\ &= \sum_{n=0}^{\infty} \frac{(c-a)^{\Re(\nu)} |(\eta)_{\xi n}| |(\gamma)_{\sigma n}| |(-w(c-a)^{\Re(\mu)})^n|}{|(\rho)_{mn}| |\Gamma(\mu n + \nu + 1)| |\Re(\mu)n + \Re(\nu) + 1|}. \end{aligned} \quad (40)$$

Therefore,

$$\begin{aligned} \left\| \mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi \right\|_c &\leq \int_a^c B |\phi(t)| dt \leq B \|\phi\|_c \\ &\Rightarrow \left\| \mathcal{I}_{\nu,\eta,\rho,\gamma,w,a^+}^{\mu,\xi,m,\sigma} \phi \right\|_c \leq B \|\phi\|_c. \end{aligned} \quad (41)$$

□

4. The Generalized Bessel–Maitland Function with Some Fractional Integral Operators

In this section, we derive some results of Saigo fractional integral operators with the generalized Bessel–Maitland function, and these results are established in terms of the Fox–Wright function. Also, we develop the composition of Riemann–Liouville operators with the generalized Bessel–Maitland function.

Theorem 2. Let $a, c, d, \mu, \nu, \eta, \rho, \gamma \in \mathbb{C}$ with $\Re(a) > 0$, $\rho > \max[0, \Re(c-d)]$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\begin{aligned} \mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma} (\tau^\delta) \right] (s) &= \frac{s^{\rho-c} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \\ &\quad \times {}_5\Psi_4 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(\rho+1, \delta)(\rho+d-c+1, \delta)(1, 1) \\ (\nu+1, \mu)(\rho, m)(\rho-c+1, \delta)(a+\rho+d+1, \delta) \end{matrix} \middle| -s^\delta \right]. \end{aligned} \quad (42)$$

Proof. Consider the left-sided Saigo fractional integral operator (4), in which using the power function with generalized Bessel–Maitland function (19), we get

$$\begin{aligned} & \mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] (s) \\ &= \frac{s^{-a-c}}{\Gamma(a)} \int_0^s (s-\tau)^{a-1} {}_2R_1 \left(a+c, -d; a; \left(1-\frac{\tau}{s} \right) \right) \\ & \cdot \left[\tau^\rho \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-\tau^\delta)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \right] d\tau. \end{aligned} \quad (43)$$

By using equation (8) in equation (43), we have

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) \\ &= \frac{s^{-c-1} \left[\begin{smallmatrix} \gamma, \sigma \\ \mu, \nu \end{smallmatrix} Q_{\rho,m;n}^{\eta,\xi} \right]}{\Gamma(a)} \sum_{p=0}^{\infty} \frac{(a+c)_p (-d)_p}{(a)_p p!} \\ & \cdot \int_0^s \left(1-\frac{\tau}{s} \right)^{a+p-1} (\tau)^{\rho+\delta n} d\tau. \end{aligned} \quad (44)$$

By putting the values $(\tau/s) = u \Rightarrow d\tau = sdu$, $\tau = s \Rightarrow u = 1$, and $\tau = 0 \Rightarrow u = 0$ in equation (45), we obtain

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) \\ &= s^{\rho-c} \left[\begin{smallmatrix} \gamma, \sigma \\ \mu, \nu \end{smallmatrix} Q_{\rho,m;n}^{\eta,\xi} (s^\delta)^n \right] \sum_{p=0}^{\infty} \frac{(a+c)_p (-d)_p}{\Gamma(a) (a)_p p!} \\ & \cdot \int_0^1 (1-u)^{a+p-1} u^{\rho+\delta n} du. \end{aligned} \quad (45)$$

By using equations (11) and (12) in equation (46), we get

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) \\ &= s^{\rho-c} \left[\begin{smallmatrix} \gamma, \sigma \\ \mu, \nu \end{smallmatrix} Q_{\rho,m;n}^{\eta,\xi} \right] \sum_{p=0}^{\infty} \frac{(a+c)_p (-d)_p}{(a)_p p!} \\ & \cdot \frac{(s^\delta)^n \Gamma(\rho + \delta n + 1) \Gamma(a+p)}{\Gamma(a) \Gamma(\rho + \delta n + a + p + 1)}. \end{aligned} \quad (46)$$

By using equations (10) and (16) in equation (47), we have

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) \\ &= s^{\rho-c} \left[\begin{smallmatrix} \gamma, \sigma \\ \mu, \nu \end{smallmatrix} Q_{\rho,m;n}^{\eta,\xi} \right] (s^\delta)^n \\ & \cdot \frac{\Gamma(\rho + \delta n + d - c + 1) \Gamma(\rho + \delta n + 1)}{\Gamma(\rho + \delta n - c + 1) \Gamma(\rho + \delta n + a + d + 1)}. \end{aligned} \quad (47)$$

By using equations (10) and (20) in equation (51), we get

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) \\ &= \sum_{n=0}^{\infty} \frac{s^{\rho-c} \Gamma(\rho) \Gamma(\eta + \xi n)}{\Gamma(\eta) \Gamma(\mu n + \nu + 1)} \\ & \cdot \frac{\Gamma(\rho + \delta n + d - c + 1) \Gamma(\rho + \delta n + 1) \Gamma(\gamma + \sigma n) (-s^\delta)^n}{\Gamma(\gamma) \Gamma(\rho + mn) \Gamma(\rho + \delta n - c + 1) \Gamma(\rho + \delta n + a + d + 1)}. \end{aligned} \quad (48)$$

Hence, we attain the required result:

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{a,c,d} \left[\tau^\rho \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^\delta) \right] \right) (s) = \frac{s^{\rho-c} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \\ & \times {}_5\Psi_4 \left[\begin{matrix} (\eta, \xi) (\gamma, \sigma) (\rho + 1, \delta) (\rho + d - c + 1, \delta) (1, 1) \\ (\nu + 1, \mu) (\rho, m) (\rho - c + 1, \delta) (a + \rho + d + 1, \delta) \end{matrix} \middle| -s^\delta \right]. \end{aligned} \quad (49)$$

Theorem 3. Let $a, c, d, \mu, \nu, \eta, \rho, \gamma \in \mathbb{C}$ with $\Re(a) > 0$, $\rho > \max[0, \Re(c-d)]$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$,

$\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\begin{aligned} & \left(\mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] \right) (s) = \frac{s^{-c-\rho} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \\ & \times {}_5\Psi_4 \left[\begin{matrix} (\eta, \xi) (\gamma, \sigma) (\rho + d, \delta) (\rho + c, \delta) (1, 1) \\ (\nu + 1, \mu) (\rho, m) (\rho, \delta) (a + \rho + d + c, \delta) \end{matrix} \middle| -s^{-\delta} \right]. \end{aligned} \quad (50)$$

Proof. Consider the right-sided Sagio fractional integral operator (5), in which using the power function with generalized Bessel–Maitland function (19), we get

$$\mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) = \frac{1}{\Gamma(a)} \int_s^\infty \frac{(\tau-s)^{a-1}}{\tau^{a+c}} {}_2R_1 \left(a+c, -d; a; \left(1 - \frac{s}{\tau} \right) \right) \left[\sum_{n=0}^\infty \frac{\tau^{-\rho} (\eta)_{\xi n} (\gamma)_{\sigma n} (-\tau^{-\delta})^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \right] d\tau. \quad (51)$$

By using equation (8) in equation (52), we have

$$\begin{aligned} \mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) &= \left[\mathcal{Q}_{\rho,m;n}^{\eta,\xi} \right] \int_s^\infty \left(1 - \frac{s}{\tau} \right)^{a-1} \sum_{p=0}^\infty \frac{(a+c)_p (-d)_p (1 - (s/\tau))^p}{\tau^{c+1} \Gamma(a) (a)_p p! \tau^\rho (\tau^\delta)^n} d\tau \\ &= \left[\mathcal{Q}_{\rho,m;n}^{\eta,\xi} \right] \sum_{p=0}^\infty \frac{(a+c)_p (-d)_p}{\Gamma(a) (a)_p p!} \int_s^\infty \frac{(1 - (s/\tau))^{a+p-1}}{(\tau)^{\rho+\delta n}} \tau^{-c-1} d\tau. \end{aligned} \quad (52)$$

By putting the values $(s/\tau) = u \Rightarrow d\tau = (-s/u^2)du$, $\tau = s \Rightarrow u = 1$, and $\tau = \infty \Rightarrow u = 0$ in equation (53), we obtain

$$\begin{aligned} \mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) &= \frac{[\mathcal{Q}_{\rho,m;n}^{\eta,\xi}]}{\Gamma(a)} \sum_{p=0}^\infty \frac{(a+c)_p (-d)_p}{(a)_p p!} \int_1^0 \frac{(1-u)^{a+p-1}}{(s/u)^{\rho+\delta n+c+1}} \left(\frac{-s}{u^2} \right) du \\ &= \left[\mathcal{Q}_{\rho,m;n}^{\eta,\xi} \right] s^{-c} \sum_{p=0}^\infty \frac{(a+c)_p (-d)_p}{s^\rho \Gamma(a) (a)_p p! s^{\delta n}} \int_0^1 \frac{(1-u)^{a+p-1}}{u^{-\rho-\delta n-c+1}} du. \end{aligned} \quad (53)$$

By using equations (11) and (12) in equation (54), we have

$$\begin{aligned} \mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) &= \left[\mathcal{Q}_{\rho,m;n}^{\eta,\xi} s^{-\delta n} \right] s^{-\rho-c} \sum_{p=0}^\infty \frac{(a+c)_p (-d)_p \Gamma(\rho + \delta n + c) \Gamma(a+p)}{\Gamma(a) (a)_p p! \Gamma(\rho + \delta n + a + p + c)}. \end{aligned} \quad (54)$$

Now, by using equations (10) and (16) in equation (56), we get

$$\begin{aligned} \mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) &= \frac{[\mathcal{Q}_{\rho,m;n}^{\eta,\xi} s^{-\delta n}]}{s^{\rho+c} \Gamma(\rho + \delta n) \Gamma(\rho + \delta n + a + d + c)} \Gamma(\rho + \delta n + d) \Gamma(\rho + \delta n + c) \end{aligned} \quad (55)$$

By using equations (10) and (20) in (57), we have the result:

$$\begin{aligned} \mathcal{F}_{0-}^{a,c,d} \left[\tau^{-\rho} \mathcal{J}_{\nu,\eta,\rho,\gamma}^{\mu,\xi,m,\sigma}(\tau^{-\delta}) \right] (s) &= \sum_{n=0}^\infty \frac{s^{-\rho-c} \Gamma(\eta + \xi n)}{\Gamma(\mu n + \nu + 1)} \frac{\Gamma(\rho) \Gamma(\gamma + \sigma n) \Gamma(\rho + \delta n + d) \Gamma(\rho + \delta n + c) (-s^{-\delta})^n}{\Gamma(\eta) \Gamma(\gamma) \Gamma(\rho + mn) \Gamma(\rho + \delta n) \Gamma(\rho + \delta n + a + d + c)} \\ &= \frac{s^{-c-\rho} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} {}_5\Psi_4 \left[\begin{matrix} (\eta, \xi) (\gamma, \sigma) (\rho + d, \delta) (\rho + c, \delta) (1, 1) \\ (\nu + 1, \mu) (\rho, m) (\rho, \delta) (a + \rho + d + c, \delta) \end{matrix} \middle| -s^{-\delta} \right]. \end{aligned} \quad (56)$$

□

Theorem 4. Let $\Re(\lambda) > 0$, $\mu, \nu, \eta, \rho, \gamma, w \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\mathcal{F}_{a+}^{\lambda} \left[\frac{\mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(\tau - a)^{\mu})}{(\tau - a)^{-\nu}} \right] (s - a) = \frac{\mathcal{J}_{\nu + \lambda, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(s - a)^{\mu})}{(s - a)^{-\lambda - \nu}}. \quad (57)$$

Proof. Consider the left-sided Riemann–Liouville fractional integral operator (6), in which using the power function with generalized Bessel–Maitland function (19), we get

$$\begin{aligned} & \mathcal{F}_{a+}^{\lambda} \left[(\tau - a)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(\tau - a)^{\mu}) \right] (s - a) \\ &= \left[\mathcal{Q}_{\rho, m; n}^{\eta, \xi} \right]_{\mu, \nu} \int_a^s \frac{(\tau - a)^{\nu} (w(\tau - a)^{\mu})^n}{\Gamma(\lambda) (s - \tau)^{-\lambda + 1}} d\tau \\ &= \left[\mathcal{Q}_{\rho, m; n}^{\eta, \xi} \right]_{\mu, \nu} \int_a^s (s - \tau)^{\lambda - 1} (\tau - a)^{\nu + \mu n} d\tau. \end{aligned} \quad (58)$$

By putting the values $((\tau - a)/(s - a)) = u \Rightarrow d\tau = (s - a)du$, $\tau = s \Rightarrow u = 1$, and $\tau = a \Rightarrow u = 0$ in equation (59), we obtain

$$\begin{aligned} & \mathcal{F}_{a+}^{\lambda} \left[(\tau - a)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(\tau - a)^{\mu}) \right] (s - a) \\ &= \frac{\left[\mathcal{Q}_{\rho, m; n}^{\eta, \xi} \right]_{\mu, \nu} (w)^n}{\Gamma(\lambda)} \int_0^1 \frac{(s - (s - a)u - a)^{\lambda - 1}}{((s - a)u)^{-\mu n - \nu} (s - a)^{-1}} du \\ &= \frac{(s - a)^{\lambda + \nu}}{(w)^n \Gamma(\lambda)} \left[\mathcal{Q}_{\rho, m; n}^{\eta, \xi} \right]_{\mu, \nu} (s - a)^{\mu n} \int_0^1 (1 - u)^{\lambda - 1} u^{\mu n + \nu} du. \end{aligned} \quad (59)$$

By using equations (11) and (12) in equation (60), we have

$$\begin{aligned} & \mathcal{F}_{a+}^{\lambda} \left[(\tau - a)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(\tau - a)^{\mu}) \right] (s - a) \\ &= (s - a)^{\lambda + \nu} \left[\mathcal{Q}_{\rho, m; n}^{\eta, \xi} \right]_{\mu, \nu} (w)^n (s - a)^{\mu n} \frac{\Gamma(\mu n + \nu + 1)}{\Gamma(\lambda + \mu n + \nu + 1)}. \end{aligned} \quad (60)$$

Now, by using equation (20) in equation (61), then the required result is obtained:

$$\begin{aligned} & \mathcal{F}_{a+}^{\lambda} \left[(\tau - a)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(\tau - a)^{\mu}) \right] (s - a) \\ &= (s - a)^{\lambda + \nu} \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w(s - a)^{\mu})^n}{\Gamma(\lambda + \mu n + \nu + 1) (\rho)_{mn}} \\ &= (s - a)^{\lambda + \nu} \mathcal{J}_{\nu + \lambda, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(s - a)^{\mu}). \end{aligned} \quad (61)$$

□

5. Riemann–Liouville Fractional Operators and Laplace Transform of the New Operator

In this section, we discuss the Riemann–Liouville fractional integral and differential operators with the fractional integral

operator. Also, we developed a result which deals with the Laplace transform of the new fractional integral operator.

Theorem 5. Let $w, \lambda, \mu, \nu, \eta, w, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\lambda) > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, $\Re(\rho) > 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) = \left(\mathcal{L}_{\nu + \lambda, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s). \quad (62)$$

Proof. Consider the left-sided Riemann–Liouville integral operator (6) involving new fractional integral operator (22) as

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^s (s - u)^{\lambda - 1} \int_0^u (u - \tau)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(u - \tau)^{\mu}) \phi(\tau) d\tau du. \end{aligned} \quad (63)$$

By using equation (18) in equation (63), we have

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) \\ &= \frac{1}{\Gamma(\lambda)} \int_0^s \int_{\tau}^s (s - u)^{\lambda - 1} (u - \tau)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(u - \tau)^{\mu}) du \phi(\tau) d\tau. \end{aligned} \quad (64)$$

By putting the values $t = u - \tau \Rightarrow dt = du$, $u = s \Rightarrow t = s - \tau$, and $u = \tau \Rightarrow t = 0$ in equation (64), we get

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) = \frac{1}{\Gamma(\lambda)} \int_0^s \int_0^{s - \tau} (t)^{\nu} \frac{\mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(t)^{\mu})}{(s - \tau - t)^{1 - \lambda}} dt \phi(\tau) d\tau \\ &= \int_0^s \frac{1}{\Gamma(\lambda)} \int_0^{s - \tau} (t)^{\nu} \frac{\mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(t)^{\mu})}{(s - \tau - t)^{1 - \lambda}} dt \phi(\tau) d\tau. \end{aligned} \quad (65)$$

By using equation (6) in equation (65), we have

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) = \int_0^s \left[\mathcal{F}_{0+}^{\lambda} (t)^{\nu} \mathcal{J}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(t)^{\mu}) \right] \\ & \quad \cdot (s - \tau) \phi(\tau) d\tau. \end{aligned} \quad (66)$$

By using equation (58) in equation (66), we obtain

$$\begin{aligned} & \left(\mathcal{F}_{0+}^{\lambda} \mathcal{L}_{\nu, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s) = \int_0^s \frac{\mathcal{J}_{\nu + \lambda, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} (w(s - \tau)^{\mu})}{(s - \tau)^{-\nu - \lambda}} \phi(\tau) d\tau \\ &= \left(\mathcal{L}_{\nu + \lambda, \eta, \rho, \gamma, w, 0+}^{\mu, \xi, m, \sigma} \phi \right) (s). \end{aligned} \quad (67)$$

□

Theorem 6. Let $\lambda, \mu, \nu, \eta, w, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\lambda) > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$,

$\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, $\Re(\rho) > 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) = \left(\mathcal{I}_{\nu-\lambda, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s). \quad (68)$$

Proof. Consider the left-sided Riemann–Liouville differential operator (7) involving new fractional integral operator (22); then,

$$\begin{aligned} & \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) \\ &= \frac{1}{\Gamma(m-\lambda)} \left(\frac{d}{ds} \right)^m \int_0^s (s-u)^{m-\lambda-1} \\ & \quad \cdot \int_0^u \frac{\mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(u-\tau)^\mu)}{(u-\tau)^{-\nu}} \phi(\tau) d\tau du. \end{aligned} \quad (69)$$

By using equation (18) in equation (69), we have

$$\begin{aligned} & \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) \\ &= \frac{1}{\Gamma(m-\lambda)} \left(\frac{d}{ds} \right)^m \int_0^s \int_\tau^s (\tau-u)^\nu \frac{\mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(u-\tau)^\mu)}{(s-u)^{\lambda-m+1}} du \phi(\tau) d\tau. \end{aligned} \quad (70)$$

By putting the values $t = u - \tau \Rightarrow dt = du$, $u = s \Rightarrow t = s - \tau$, and $u = \tau \Rightarrow t = 0$ in equation (70), we get

$$\begin{aligned} \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) &= \sum_{n=0}^{\infty} \frac{(-w)^n (\mu n + m + \nu - \lambda)}{\Gamma(\mu n + m + \nu - \lambda + 1)} \times \frac{(\eta)_{\xi n} (\gamma)_{\sigma n}}{(w)_{mn}} \left(\frac{d}{ds} \right)^{m-1} \int_0^s (s-\tau)^{-\lambda+\mu n+m+\nu-1} \phi(\tau) d\tau \\ &= \left(\frac{d}{ds} \right)^{m-1} \int_0^s \frac{\mathcal{I}_{\nu+m-\lambda-1, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(s-\tau)^\mu)}{(s-\tau)^{-m-\nu+\lambda+1}} \phi(\tau) d\tau. \end{aligned} \quad (74)$$

Now, taking the $(m-1)$ derivative of equation (74), we get

$$\begin{aligned} \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) &= \int_0^s \frac{\mathcal{I}_{\nu-\lambda, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(s-\tau)^\mu)}{(s-\tau)^{\lambda-\nu}} \phi(\tau) d\tau \\ &= \left(\mathcal{I}_{\nu-\lambda, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s). \end{aligned} \quad (75)$$

Theorem 7. Let $w, \acute{\gamma}, \acute{\eta}, \acute{\gamma}, \acute{\mu}, \nu, \eta, \gamma \in \mathbb{C} \in \mathbb{C}$, $\Re(\acute{\eta}) > 0$, $\Re(\gamma) > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(w) > 0$, and $\Re(\gamma) > 0$; then, the following relation holds:

$$\mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s) = \mathcal{I}_{\nu+\nu+1, \eta+\eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s). \quad (76)$$

$$\begin{aligned} & \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) \\ &= \frac{1}{\Gamma(m-\lambda)} \left(\frac{d}{ds} \right)^m \int_0^s \int_0^{s-\tau} \frac{(t)^\nu \mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(t)^\mu)}{(s-\tau-t)^{-m+\lambda+1}} dt \phi(\tau) d\tau \\ &= \int_0^s \left(\frac{d}{dx} \right)^m \frac{1}{\Gamma(m-\lambda)} \int_0^{s-\tau} \frac{(t)^\nu \mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(t)^\mu)}{(s-\tau-t)^{-m+\lambda+1}} dt \phi(\tau) d\tau. \end{aligned} \quad (71)$$

Now, by using equation (6) in equation (71), we have

$$\begin{aligned} & \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) \\ &= \int_0^s \left(\frac{d}{ds} \right)^m \mathcal{I}_{0^+}^{m-\lambda} \left[(t)^\nu \mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(t)^\mu) \right] (s-\tau) \phi(\tau) d\tau. \end{aligned} \quad (72)$$

By using equation (58) in equation (72), we obtain

$$\begin{aligned} & \left(\mathcal{D}_{0^+}^\lambda \mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right)(s) \\ &= \int_0^s \left(\frac{d}{ds} \right)^m \frac{\mathcal{I}_{m+\nu-\lambda, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma}(w(s-\tau)^\mu)}{(s-\tau)^{\lambda-m-\nu}} \phi(\tau) d\tau. \end{aligned} \quad (73)$$

By using equation (19) in equation (73) and then taking one-time derivative, we have

Proof. Consider the new fractional integral operator (22):

$$\begin{aligned} & \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s) \\ &= \int_a^s \frac{\mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, 0}(w(s-u)^\mu)}{(s-u)^{-\nu}} du \int_a^u \frac{\mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, 0}(w(u-t)^\mu)}{(u-t)^{-\nu}} \phi(t) dt \\ &= \int_a^s \phi(t) dt \int_t^s \frac{\mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, 0}(w(s-u)^\mu) \mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, 0}(w(u-t)^\mu)}{(s-u)^{-\nu} (u-t)^{-\nu}} du. \end{aligned} \quad (77)$$

By reversing the order of integration and putting $\tau = ((s-u)/(s-t))$, the inner integral is

$$\begin{aligned}
& \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s) \\
&= \int_0^1 \frac{\mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, 0}(w(s-t)\tau)^\mu \mathcal{I}_{\nu, \eta, 1, \gamma}^{\mu, 1, 1, \sigma}(w(s-(s-t)\tau-t)^\mu)}{((s-t)\tau)^{-\nu}(s-(s-t)\tau-t)^{-\nu}(s-t)^{-1}} d\tau \\
&= \sum_{p=0}^{\infty} \frac{(\eta)_p (-w)^p (s-t)^{p\mu}}{\Gamma(\mu p + \nu + 1) p!} \sum_{n=0}^{\infty} \frac{(\eta)_n (-w)^n (s-t)^{\mu n}}{\Gamma(\mu n + \nu + 1) n!} \\
&\quad \times (s-t)^{\nu+\mu p} \int_0^1 \tau^{\nu+\mu p} (1-\tau)^{\nu+\mu n} d\tau \\
&= (s-t)^{\nu+\mu+1} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{(\eta)_p (-w)^{p+n} (s-t)^{\mu(p+n)} (\eta)_n}{\Gamma(\mu p + \mu n + \nu + \mu + 1) p! n!} \\
&= (s-t)^{\nu+\mu+1} \sum_{p=0}^{\infty} \frac{(-w)^p (s-t)^{p\mu}}{\Gamma(\mu p + \nu + \mu + 1)} \sum_{n=0}^{\infty} \frac{(\eta)_{p-n} (\eta)_n}{(p-n)! n!}.
\end{aligned} \tag{78}$$

This implies that

$$\begin{aligned}
& \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \mathcal{I}_{\nu, \eta, 1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s) \\
&= \int_a^s (s-t)^{\nu+\mu+1} \mathcal{I}_{\nu+\mu+1, \eta+\mu+1, \gamma}^{\mu, 1, 1, 0}(w(s-t)^\mu) \phi(t) dt \quad (79) \\
&= \mathcal{I}_{\nu+\mu+1, \eta+\mu+1, \gamma, w, 0^+}^{\mu, 1, 1, 0} \phi(s).
\end{aligned}$$

□

Theorem 8. Let $\mu, \nu, \eta, \rho, \gamma, w \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\begin{aligned}
& \mathcal{L} \left[\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right] \\
&= \frac{s^{-\nu-1} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)^4 \Psi_2} \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(\nu, \mu)(1, 1) \\ (\nu+1, \mu)(\rho, m) \end{matrix} \middle| -\left(\frac{w}{s}\right)^\mu \right]. \quad (80)
\end{aligned}$$

Proof. Consider fractional integral operator (22):

$$\mathcal{L} \left[\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right] = \int_0^\infty e^{-st} \left[\int_0^t (t-u)^\nu \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n (t-u)^{\mu n}}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \phi(u) du \right] dt. \quad (81)$$

Now, after changing the order of integration, we obtain

$$\begin{aligned}
\mathcal{L} \left[\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right] &= \int_0^\infty \int_u^\infty \frac{(t-u)^\nu}{e^{st}} \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n (t-u)^{\mu n}}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} dt \phi(u) du \\
&= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \int_0^\infty \int_u^\infty \frac{(t-u)^{\mu n + \nu}}{e^{st}} dt \phi(u) du.
\end{aligned} \tag{82}$$

By putting $t-u = \tau$, then

$$\begin{aligned}
\mathcal{L} \left[\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \phi \right] &= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \int_0^\infty \frac{\phi(u)}{e^{su}} \int_0^\infty e^{-s\tau} \tau^{\mu n + \nu} d\tau du \\
&= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \frac{\Gamma(\mu n + \nu)}{s^{\mu n + \nu + 1}} \phi(s) \\
&= \frac{s^{-\nu-1} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta + \xi n) \Gamma(\gamma + \sigma n) (-ws^{-\mu})^n \Gamma(\mu n + \nu)}{\Gamma(\mu n + \nu + 1) \Gamma(\rho + mn)} \\
&= \frac{s^{-\nu-1} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)^4 \Psi_2} \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(\nu, \mu)(1, 1) \\ (\nu+1, \mu)(\rho, m) \end{matrix} \middle| -\left(\frac{w}{s}\right)^\mu \right].
\end{aligned} \tag{83}$$

□

Theorem 9. Let $\lambda, \delta, \chi, \mu, \nu, \eta, \rho, \gamma, w \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\left(\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} t^{((\lambda+\delta)/\chi)-1} \right)(s) = \frac{s^{\nu+((\lambda+\delta)/\chi)} \Gamma(\rho) \Gamma((\lambda+\delta)/\chi)}{\Gamma(\eta) \Gamma(\gamma)} {}_3\Psi_2 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(1, 1) \\ \left(\nu + \frac{\lambda+\delta}{\chi} + 1, \mu \right)(\rho, m) \end{matrix} \middle| -ws^\mu \right]. \quad (84)$$

Proof. Consider the new fractional integral operator (22):

$$\begin{aligned} \left(\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} t^{((\lambda+\delta)/\chi)-1} \right)(s) &= \int_0^s (s-\tau)^\nu \mathcal{I}_{\nu, \eta, \rho, \gamma}^{\mu, \xi, m, \sigma} [w(s-\tau)^\mu] \tau^{((\lambda+\delta)/\chi)-1} d\tau \\ &= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n s^{\mu n + \nu}}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \int_0^s \left(1 - \frac{\tau}{s}\right)^{\mu n + \nu} \tau^{((\lambda+\delta)/\chi)-1} d\tau. \end{aligned} \quad (85)$$

By putting $(\tau/s) = u$, then we have

$$\begin{aligned} \left(\mathcal{I}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} t^{((\lambda+\delta)/\chi)-1} \right)(s) &= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n s^{\mu n + \nu}}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \int_0^s (1-u)^{\mu n + \nu} (su)^{((\lambda+\delta)/\chi)-1} s du \\ &= \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n s^{\mu n + \nu + ((\lambda+\delta)/\chi)}}{\Gamma(\mu n + \nu + 1) (\rho)_{mn}} \frac{\Gamma(\mu n + \nu + 1) \Gamma((\lambda+\delta)/\chi)}{\Gamma(\mu n + \nu + ((\lambda+\delta)/\chi) + 1)} \\ &= \frac{s^{\nu+((\lambda+\delta)/\chi)} \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \sum_{n=0}^{\infty} \frac{\Gamma(\eta + \xi n) \Gamma(\gamma + \sigma n) (-ws^\mu)^n \Gamma((\lambda+\delta)/\chi)}{\Gamma(\mu n + \nu + ((\lambda+\delta)/\chi) + 1) \Gamma(\rho + mn)} \\ &= \frac{s^{\nu+((\lambda+\delta)/\chi)} \Gamma(\rho) \Gamma((\lambda+\delta)/\chi)}{\Gamma(\eta) \Gamma(\gamma)} {}_3\Psi_2 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(1, 1) \\ \left(\nu + \frac{\lambda+\delta}{\chi} + 1, \mu \right)(\rho, m) \end{matrix} \middle| -ws^\mu \right]. \end{aligned} \quad (86)$$

6. Inverse Operator with Some Special Functions

In this section, we discuss some applications of the inverse fractional operator. We derive some results of the inverse fractional operator with the Mittag-Leffler function and

Bessel–Maitland function, and results can be seen in the form of Wright functions.

Theorem 10. Consider $\delta, \alpha, \beta, \mu, \nu, \eta, \rho, \gamma, \lambda \in \mathbb{C}$ with $\min\{\Re(\delta), \Re(\alpha), \Re(\beta)\} > 0$, $p, q > 0$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\begin{aligned} \left[D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta}^{\gamma, \delta, q} (\tau - a)^\lambda \right](s) &= \frac{\Gamma(\rho) \Gamma(\delta)}{\Gamma(\gamma) \Gamma(\eta) \Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma + qm)}{\Gamma(\alpha m + \beta)} \\ &\quad \times (s-a)^{\lambda m + \rho - \nu} {}_3\Psi_2 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(1, 1) \\ (\rho, m)(\lambda m + \rho - \nu + 1, \mu) \end{matrix} \middle| -w(s-a)^\mu \right]. \end{aligned} \quad (87)$$

Proof. Consider inverse fractional integral operator (23) with Mittag-Leffler function (28); then, the following results hold:

$$\begin{aligned}
 & \left[D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta, p}^{\gamma, \delta, q} (\tau - a)^\lambda \right] (s) \\
 &= \left(\frac{d}{ds} \right)^p \int_a^s (s - \tau)^{p-\nu} \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w(s - \tau)^\mu)^n}{(\rho)_{mn} \Gamma(\mu n + p - \nu + 1)} (\tau - a)^{\rho-1} \sum_{m=0}^{\infty} \frac{(\gamma)_{qm} (\tau - a)^{\lambda m}}{\Gamma(\alpha m + \beta) (\delta)_{pm}} d\tau \\
 &= \left(\frac{d^p}{ds^p} \right)^{\gamma, \sigma}_{\mu, p-\nu} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm}}{\Gamma(\alpha m + \beta) (\delta)_{pm}} \int_a^s (s - \tau)^{\mu n + p - \nu} (\tau - a)^{\rho + \lambda m - 1} d\tau.
 \end{aligned} \tag{88}$$

Substituting $u = ((s - \tau)/(s - a))$ in equation (88), we obtain

$$\begin{aligned}
 & \left[D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta, p}^{\gamma, \delta, q} (\tau - a)^\lambda \right] (s) \\
 &= \left(\frac{d^p}{ds^p} \right)^{\gamma, \sigma}_{\mu, p-\nu} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm} (s - a)^{p-\nu+\mu n+\lambda m+\rho}}{\Gamma(\alpha m + \beta) (\delta)_{pm}} \int_0^1 (1 - u)^{\mu n + p - \nu} (u)^{\rho + \lambda m - 1} du.
 \end{aligned} \tag{89}$$

By using equations (11) and (12), we get

$$\begin{aligned}
 & \left[D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta, p}^{\gamma, \delta, q} (\tau - a)^\lambda \right] (s) \\
 &= \left(\frac{d^p}{ds^p} \right)^{\gamma, \sigma}_{\mu, p-\nu} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{(\gamma)_{qm} (s - a)^{p-\nu+\mu n+\lambda m+\rho}}{\Gamma(\alpha m + \beta) (\delta)_{pm}} \frac{\Gamma(\mu n + p - \nu + 1) \Gamma(\rho + \lambda m)}{\Gamma(\mu n + p - \nu + \rho + \lambda m + 1)}.
 \end{aligned} \tag{90}$$

Now, back-substituting $\mathcal{Q}_{\mu, p-\nu}^{\gamma, \sigma}_{\rho, m; n}$ in equation (90), we have

$$\begin{aligned}
 & \left[D_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta, p}^{\gamma, \delta, q} (\tau - a)^\lambda \right] (s) \\
 &= \sum_{m, n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (d^p/ds^p) (-w)^n (s - a)^{p-\nu+\mu n+\lambda m+\rho}}{(\rho)_{mn} \Gamma(\mu n + \rho + \lambda m + p - \nu + 1)} \frac{(\gamma)_{qm} \Gamma(\rho + \lambda m)}{\Gamma(\alpha m + \beta) (\delta)_{pm}} \\
 &= \sum_{m, n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (s - a)^{-\nu+\mu n+\lambda m+\rho}}{(\rho)_{mn} \Gamma(\mu n + \rho + \lambda m - \nu + 1)} \frac{(\gamma)_{qm} \Gamma(\rho + \lambda m)}{\Gamma(\alpha m + \beta) (\delta)_{pm}} \\
 &= \frac{\Gamma(\rho) \Gamma(\delta)}{\Gamma(\gamma) \Gamma(\eta) \Gamma(\gamma)} \sum_{m=0}^{\infty} \frac{\Gamma(\gamma + qm) (s - a)^{\lambda m + \rho - \nu}}{\Gamma(\alpha m + \beta)} \\
 &\quad \times {}_3\psi_2 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(1, 1) \\ (\rho, m)(\lambda m + \rho - \nu + 1, \mu) \end{matrix} \middle| -w(s - a)^\mu \right].
 \end{aligned} \tag{91}$$

□

Corollary 1. On setting $\nu = -\nu$ in Theorem 10, we obtain the result in the form of the differential operator:

$$\left[\mathcal{L}_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} (\tau - a)^{\rho-1} E_{\alpha, \beta, p}^{\gamma, \delta, q} (\tau - a)^\lambda \right] (s) = \frac{\Gamma(\rho)\Gamma(\delta)}{\Gamma(\gamma)\Gamma(\eta)\Gamma(\dot{\gamma})} \sum_{m=0}^{\infty} \frac{\Gamma(\dot{\gamma} + qm)}{\Gamma(\alpha m + \beta)} \\ \times (s - a)^{\lambda m + \rho + \nu} {}_3\Psi_2 \left[\begin{matrix} (\eta, \xi)(\gamma, \sigma)(1, 1) \\ (\rho, m)(\lambda m + \rho + \nu + 1, \mu) \end{matrix} \middle| -w(s - a)^\mu \right]. \quad (92)$$

Theorem 11. Consider $\alpha, \beta, \mu, \nu, \eta, \rho, \gamma, \dot{\gamma}, \lambda \in \mathbb{C}$ with $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $\Re(\gamma) > 0$, $\Re(\alpha) > 0$, $\Re(\beta) > -1$, $q > 0$, $\Re(\mu) > 0$, $m, \xi > \Re(\mu) + \sigma$; then, the following relation holds:

$$\left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_\beta^\alpha(\tau) J_{\beta, q}^{\alpha, \dot{\gamma}}(\tau^{-\lambda}) \right] (s) = \frac{\Gamma(\rho)[\Gamma(\eta)]^{-1}}{\Gamma(\gamma)\Gamma(\dot{\gamma})} \sum_{m, n=0}^{\infty} \frac{\Gamma(\eta + \xi n)\Gamma(\gamma + \sigma n)}{\Gamma(\rho + mn)} \\ \times \frac{(-ws^\mu)^n (-s)^m s^{-\nu}}{\Gamma(\alpha m + \beta + 1)m!} {}_2\Psi_2 \left[\begin{matrix} (\dot{\gamma}, q)(m, -\lambda) \\ (\beta + 1, \alpha)(\mu n + m - \nu + 1, -\lambda) \end{matrix} \middle| -s^{-\lambda} \right]. \quad (93)$$

Proof. Consider inverse fractional integral operator (23) with the product of two Bessel–Maitland functions (1) and (2); then, the following results hold:

$$\left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_\beta^\alpha(\tau) J_{\beta, q}^{\alpha, \dot{\gamma}}(\tau^{-\lambda}) \right] (s) \\ = \left(\frac{d^p}{ds^p} \right) \int_0^s (s - \tau)^{p-\nu} \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w(s - \tau)^\mu)^n}{(\rho)_{mn} \Gamma(\mu n + p - \nu + 1)} \times \sum_{m=0}^{\infty} \frac{\tau^{-1} (-\tau)^m}{m! \Gamma(\alpha m + \beta + 1)} \sum_{w=0}^{\infty} \frac{(\dot{\gamma})_{qw} (-1)^w (\tau)^{-\lambda w}}{w! \Gamma(\alpha w + \beta + 1)} d\tau \\ = \left(\frac{d^p}{ds^p} \right)_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} \sum_{m, w=0}^{\infty} \frac{(-1)^{m+w} s^{p-\nu+\mu n}}{m! \Gamma(\alpha m + \beta + 1)} \\ \times \frac{(\dot{\gamma})_{qw} (w)^n}{w! \Gamma(\alpha w + \beta + 1)} \int_0^s \left(1 - \frac{\tau}{s} \right)^{p-\nu+\mu n} \tau^{-1+m-\lambda w} d\tau. \quad (94)$$

By substituting $u = (\tau/s)$ in equation (94), we have

$$\left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_\beta^\alpha(\tau) J_{\beta, q}^{\alpha, \dot{\gamma}}(\tau^{-\lambda}) \right] (s) \\ = \left(\frac{d^p}{ds^p} \right)_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} \sum_{m, w=0}^{\infty} \frac{(w)^n (-1)^{m+w} s^{\mu n}}{m! \Gamma(\alpha m + \beta + 1)} \frac{(\dot{\gamma})_{qw} s^{m-\lambda w + p-\nu}}{w! \Gamma(\alpha w + \beta + 1)} \int_0^1 (1 - u)^{p-\nu+\mu n} u^{-1+m-\lambda w} du. \quad (95)$$

By using beta-gamma relations (11) and (12), we obtain

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_{\beta}^{\alpha}(\tau) J_{\beta, q}^{\alpha, \gamma}(\tau^{-\lambda}) \right] (s) \\ &= \frac{\gamma, \sigma}{\mu, p-\gamma} Q_{\rho, m; n}^{\eta, \xi} \sum_{m, w=0}^{\infty} \frac{(w)^n (-1)^{m+w} s^{\mu n}}{m! \Gamma(\alpha m + \beta + 1)} \frac{(\gamma)_{qw} s^{m-\lambda w + p-\gamma}}{w! \Gamma(\alpha w + \beta + 1)} \frac{\Gamma(p - \gamma + \mu n + 1) \Gamma(m - \lambda w)}{\Gamma(p - \gamma + \mu n + m - \lambda w + 1)}. \end{aligned} \quad (96)$$

Putting the value of $Q_{\rho, m; n}^{\eta, \xi}$ in equation (96), we have

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_{\beta}^{\alpha}(\tau) J_{\beta, q}^{\alpha, \gamma}(\tau^{-\lambda}) \right] (s) \\ &= \sum_{m, n, w=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n (d^p/ds^p)(s)^{p-\gamma+\mu n+\lambda m+\rho}}{(\rho)_{mn} \Gamma(\mu n + \rho + \lambda m + p - \gamma + 1)} \frac{(w)^n (-1)^{m+w}}{m! \Gamma(\alpha m + \beta + 1)} \frac{(\gamma)_{qw} \Gamma(m - \lambda w)}{w! \Gamma(\alpha w + \beta + 1)} \\ &= \sum_{m, n, w=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n (s)^{-\gamma+\mu n+\lambda m+\rho}}{(\rho)_{mn} \Gamma(\mu n + \rho + \lambda m - \gamma + 1)} \frac{(w)^n (-1)^{m+w}}{m! \Gamma(\alpha m + \beta + 1)} \frac{(\gamma)_{qw} \Gamma(m - \lambda w)}{w! \Gamma(\alpha w + \beta + 1)} \\ &= \frac{\Gamma(\rho) [\Gamma(\eta)]^{-1}}{\Gamma(\gamma) \Gamma(\gamma)} \sum_{m, n=0}^{\infty} \frac{\Gamma(\eta + \xi n) \Gamma(\gamma + \sigma n)}{\Gamma(\rho + mn)} \frac{(-ws^{\mu})^n (-s)^m s^{-\gamma}}{\Gamma(\alpha m + \beta + 1) m!} \\ & \quad \times {}_2\Psi_2 \left[\begin{matrix} (\gamma, q)(m, -\lambda) \\ (\beta + 1, \alpha)(\mu n + m - \gamma + 1, -\lambda) \end{matrix} \middle| -s^{-\lambda} \right]. \end{aligned} \quad (97)$$

Corollary 2. If we replace $\nu = -\nu$ in Theorem 11, we have the result in the sense of the left inverse operator:

$$\begin{aligned} & \left[\mathcal{E}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{-1} J_{\beta}^{\alpha}(\tau) J_{\beta, q}^{\alpha, \gamma}(\tau^{-\lambda}) \right] (s) = \frac{\Gamma(\rho) [\Gamma(\eta)]^{-1}}{\Gamma(\gamma) \Gamma(\gamma)} \sum_{m, n=0}^{\infty} \frac{\Gamma(\eta + \xi n) \Gamma(\gamma + \sigma n)}{\Gamma(\rho + mn)} \\ & \quad \times \frac{(-ws^{\mu})^n (-s)^m s^{\gamma}}{\Gamma(\alpha m + \beta + 1) m!^2} {}_2\Psi_2 \left[\begin{matrix} (\gamma, q)(m, -\lambda) \\ (\beta + 1, \alpha)(\mu n + m + \gamma + 1, -\lambda) \end{matrix} \middle| -s^{-\lambda} \right]. \end{aligned} \quad (98)$$

Theorem 12. Let $\alpha, \beta, \mu, \nu, \eta, \rho, \gamma \in \mathbb{C}$, $\Re(\mu) > 0$, $\Re(\nu) \geq -1$, $\Re(\eta) > 0$, $\Re(\rho) > 0$, $\Re(\gamma) > 0$, $\xi, m, \sigma \geq 0$, and $m, \xi > \Re(\mu) + \sigma$; then, there exists the following relation:

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{(\alpha/\beta)-1} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) \\ &= \sum_{m, n=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m}{m! (-\gamma + \mu n + (\alpha/\beta) + 1)_m} \times \frac{\Gamma(\eta + \xi n) \Gamma(\gamma + \sigma n) s^{-\gamma + m + (\alpha/\beta)}}{\Gamma(-\gamma + \mu n + (\alpha/\beta) + 1) \Gamma(\rho + mn)} \frac{(-ws^{\mu})^n \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)}. \end{aligned} \quad (99)$$

Proof. Consider fractional integral operator (22) with Gauss hypergeometric function (8); then, the following results hold:

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \frac{\tau^{\alpha/\beta}}{\tau} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) \\ &= \left(\frac{d^p}{ds^p} \right) \int_0^s (s-\tau)^{p-\nu} \sum_{n=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w(s-\tau)^{\mu})^n}{(\rho)_{mn} \Gamma(\mu n + p - \nu + 1)} \tau^{(\alpha/\beta)-1} \sum_{m=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m}{(\alpha/\beta)_m m!} (\tau)^m d\tau \\ &= \left(\frac{d^p}{ds^p} \right)_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m s^{p-\nu+\mu n}}{(\alpha/\beta)_m m!} \int_0^s \left(\frac{1-\tau}{s} \right)^{p-\nu+\mu n} \tau^{m+(\alpha/\beta)-1} d\tau. \end{aligned} \quad (100)$$

Putting $u = (\tau/s)$ in equation (100), we get

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \frac{\tau^{\alpha/\beta}}{\tau} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) \\ &= \left(\frac{d^p}{ds^p} \right)_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m s^{p-\nu+\mu n}}{(\alpha/\beta)_m m! s^{-m-(\alpha/\beta)}} \int_0^1 (1-u)^{p-\nu+\mu n} u^{m+(\alpha/\beta)-1} du. \end{aligned} \quad (101)$$

Using equations (11) and (12), we have

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \frac{\tau^{\alpha/\beta}}{\tau} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) \\ &= \left(\frac{d^p}{ds^p} \right)_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi} (w)^n \sum_{m=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m s^{p-\nu+\mu n}}{(\alpha/\beta)_m m! s^{-m-(\alpha/\beta)}} \frac{\Gamma(p-\nu+\mu n+1) \Gamma(m+(\alpha/\beta))}{\Gamma(p-\nu+\mu n+m+(\alpha/\beta)+1)}. \end{aligned} \quad (102)$$

By substituting $_{\mu, p-\nu}^{\gamma, \sigma} \mathcal{Q}_{\rho, m; n}^{\eta, \xi}$ in equation (102), we get the required result:

$$\begin{aligned} & \left[D_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{(\alpha/\beta)-1} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) \\ &= \sum_{n, m=0}^{\infty} \frac{(\eta)_{\xi n} (\gamma)_{\sigma n} (-w)^n \Gamma(m+(\alpha/\beta)) (-\eta)_m ((\alpha/\beta) + \mu)_m}{(\rho)_{mn} \Gamma(p-\nu+\mu n+m+(\alpha/\beta)+1)} \frac{d^p}{ds^p} s^{p-\nu+\mu n+m+(\alpha/\beta)} \\ &= \sum_{m, n=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m}{m! (-\nu+\mu n+(\alpha/\beta)+1)_m} \frac{(-ws^{\mu})^n \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)} \frac{\Gamma(\gamma+\sigma n)}{\Gamma(\rho+mn)} \frac{\Gamma(\eta+\xi n) s^{-\nu+m+(\alpha/\beta)}}{\Gamma(-\nu+\mu n+(\alpha/\beta)+1)}. \end{aligned} \quad (103)$$

□

Corollary 3. By putting $\nu = -\nu$ in Theorem 12, we get the following form:

$$\begin{aligned} & \left[\mathcal{Z}_{\nu, \eta, \rho, \gamma, w, 0^+}^{\mu, \xi, m, \sigma} \tau^{(\alpha/\beta)-1} {}_2R_1 \left(\frac{\alpha}{\beta} + \mu, -\eta; \frac{\alpha}{\beta}; \tau \right) \right] (s) = \sum_{m, n=0}^{\infty} \frac{((\alpha/\beta) + \mu)_m (-\eta)_m}{m! (\nu+\mu n+(\alpha/\beta)+1)_m} \\ & \quad \times \frac{\Gamma(\eta+\xi n) \Gamma(\gamma+\sigma n) s^{\nu+m+(\alpha/\beta)}}{\Gamma(\nu+\mu n+(\alpha/\beta)+1) \Gamma(\rho+mn)} \frac{(-ws^{\mu})^n \Gamma(\rho)}{\Gamma(\eta) \Gamma(\gamma)}. \end{aligned} \quad (104)$$

7. Conclusion

In this paper, we discussed some relations of generalized Bessel–Maitland functions and the Mittag–Leffler functions and also developed Saigo and Riemann–Liouville fractional integral operators with the generalized Bessel–Maitland function, and results can be seen in the form of Fox–Wright functions. Also, we established a new operator $\mathcal{I}_{\nu, \eta, \rho, \gamma, w, a^+}^{\mu, \xi, m, \sigma} \phi$ and also discussed its convergence and boundedness. Moreover, we discussed the Riemann–Liouville fractional operator and the integral transform (Laplace) of the new operator and also developed some important applications of the left inverse operator.

Data Availability

No data were used to support this study since they are more of simulation.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Fractional Hybrid Differential Equations and Coupled Fixed-Point Results for α -Admissible $F(\psi_1, \psi_2)$ – Contractions in M – Metric Spaces

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In this paper, we investigate the existence of a unique coupled fixed point for α -admissible mapping which is of $F(\psi_1, \psi_2)$ -contraction in the context of M -metric space. We have also shown that the results presented in this paper would extend many recent results appearing in the literature. Furthermore, we apply our results to develop sufficient conditions for the existence and uniqueness of a solution for a coupled system of fractional hybrid differential equations with linear perturbations of second type and with three-point boundary conditions.

1. Introduction

Fixed-point theory is an outstanding source which gives responsible techniques for the existence of fixed points for self-mappings under different conditions. One of the newest branches of fixed-point theory concerned with the study of coupled fixed points, brought by Guo and Lakshmikantham [1]. In [2], Bhaskar and Lakshmikantham established some fixed and coupled fixed-point theorems for contractions in two variables defined on partially ordered metric spaces with applications to ordinary differential equations. Thereafter, these results were extended by several authors (see [3–6]).

Inspired by the notion of partial metric (or, p -metric) which is one of the vital generalizations of the standard metric, Asadi et al. [7] proposed the concept of M -metric which refines the p -metric and produces useful basic topological concepts. For some fixed-point results and various contractive definitions that have been employed in M -metric space, we refer the reader to [8–12].

In [13] (see also, [14–16]), Monfared et al. established some fixed-point results for α -admissible mappings which are $F(\psi, \varphi)$ -contractions in complete M -metric spaces. Now, we state one of their main results.

Theorem 1. Let (X, μ) be a complete M -metric space and $T: X \rightarrow X$ be an α -admissible mapping. Suppose that the following condition is satisfied:

$$[\psi(\mu(Tx, Ty)) + l]^{\alpha(x, Tx)\alpha(y, Ty)} \leq F(\psi(\mu(x, y)), \varphi(\mu(x, y))) + l, \quad (1)$$

for all $x, y \in X$ and $l \geq 1$, where $F \in \mathcal{C}$, ψ is an altering distance function, and φ is an ultra-altering distance function. Suppose that either

- (a) T is continuous
- (b) If $\{x_n\}$ is a sequence in X such that $\{x_n\} \rightarrow x$, $\alpha(x_n, x_{n+1}) \geq 1$, $\forall n$, then $\alpha(x, Tx) \geq 1$

If there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$, then T has a fixed point.

Hybrid differential equations have been of great interest as they include several dynamic systems as special cases. The papers [17, 18] discussed the existence and uniqueness results and some fundamental differential inequalities for first-order hybrid differential equations with perturbations of 1st and 2nd type, respectively.

Fractional calculus is a field of mathematics that deals with the derivatives and integrals of arbitrary order. Indeed, it is found to be more realistic in describing and modeling several natural phenomena than the classical one. In fact, fractional differential equations (FDEs) play a major role in modeling many real-life problems such as physical phenomena, computer networking, medicine (the modeling of human tissue), mechanics (theory of viscoelasticity), electrical engineering (transmission of ultrasound waves) and many others (see [19–21]).

Fractional hybrid differential equations (FHDEs) can be employed in modeling and describing nonhomogenous physical phenomena that take place in their form. FHDEs have been studied using a Riemann–Liouville differential operator of order $\alpha > 0$ in many literature studies (see [22–26]).

In [27], Shaob et al. used Bashiri fixed-point theorem [22] to prove the existence only of a solution to a three-point boundary value problem for a coupled system of FHDEs in Banach spaces.

In line with the above studies, our purpose in this paper is to introduce the notion of α -admissible mapping with two variables and generalize Theorem 1 to coupled fixed-point version. Then, we apply our main results to prove the existence and uniqueness of a solution to the following system of FHDEs involving Riemann–Liouville fractional derivative:

$$\begin{aligned} \mathcal{D}^\alpha [x(t) - f(t, x(t))] &= g(t, y(t), I^\beta y(t)), \\ x^{(i)}(0) &= \frac{\partial^i f(t, x(t))}{\partial t^i} \Big|_{t=0} = 0, \end{aligned} \quad (2)$$

$$x(\tau) = \delta x(\eta),$$

$$\begin{aligned} \mathcal{D}^\alpha [y(t) - f(t, y(t))] &= g(t, x(t), I^\beta x(t)), \\ y^{(i)}(0) &= \frac{\partial^i f(t, y(t))}{\partial t^i} \Big|_{t=0} = 0, \\ y(\tau) &= \delta y(\eta), \end{aligned} \quad (3)$$

for all $i = 0, 1, \dots, n-2$, $t \in J = [0, \tau]$, $\tau > 0$, $\alpha \in (n-1, n]$, $\beta > 0$, $0 < \eta < \tau$, $\delta \neq (\tau/\eta)^{\alpha-1}$, $f \in C(J \times \mathbb{R})$, and $g \in C(J \times \mathbb{R}^2)$.

2. Preliminaries

In 1994, Matthews [28] introduced the notion of a p -metric space as a part of the study of denotational semantics of

dataflow networks. In p -metric spaces, self-distance of an arbitrary point need not be equal to zero.

Definition 1 (see [28]). A p -metric on a nonempty set X is a mapping $p: X \times X \rightarrow [0, \infty)$ such that, for all $x, y, z \in X$,

$$\begin{aligned} (p_1) \quad & p(x, x) = p(y, y) = p(x, y) \Leftrightarrow x = y \\ (p_2) \quad & p_{(x,x)} \leq p(x, y) \\ (p_3) \quad & p(x, y) = p(y, x) \\ (p_4) \quad & p(x, y) \leq p(x, z) + p(z, y) - p_{(z,z)} \end{aligned}$$

Then, (X, p) is called a p -metric space.

Notice that, every metric space can be defined to be p -metric space with zero self-distance. After that, Asadi et al. generalized the above definition by relaxing the axiom (p_2) as follows.

Definition 2 (see [7]). For a nonempty set X , a function $\mu: X \times X \rightarrow [0, \infty)$ is called an M -metric if it fulfils the following:

$$\begin{aligned} (m_1) \quad & \mu(x, x) = \mu(y, y) = \mu(x, y) \Leftrightarrow x = y \\ (m_2) \quad & m_{xy} \leq \mu(x, y), \text{ where } m_{xy} = \min\{\mu(x, x), \mu(y, y)\} \\ (m_3) \quad & \mu(x, y) = \mu(y, x) \\ (m_4) \quad & (\mu(x, y) - m_{xy}) \leq (\mu(x, z) - m_{xz}) + (\mu(z, y) - m_{zy}) \end{aligned}$$

Then, the pair (X, μ) is called an M -metric space.

Lemma 1 (see [7]). Every p -metric is an M -metric.

Here, we give an example to show that the converse might not be held.

Example 1 (see [7]). Let $X = \{1, 2, 3\}$ and define

$$\begin{aligned} \mu(1, 2) &= \mu(2, 1) = 10, \\ \mu(1, 1) &= 1, \\ \mu(2, 2) &= 9, \\ \mu(1, 3) &= \mu(3, 1) = \mu(3, 2) = \mu(2, 3) = 7, \\ \mu(3, 3) &= 5. \end{aligned} \quad (4)$$

So μ is M -metric but it is not p -metric for $\mu(2, 2) \not\leq \mu(2, 3)$. Also, μ is not metric for self-distances are not zero.

Thus, the class of M -metric spaces is effectively larger than that of both ordinary metric and p -metric spaces.

Notation 1. Let (X, μ) be an M -metric space; then define

$$\begin{aligned} \mu^w(x, y) &= \mu(x, y) - 2m_{xy} + M_{xy}, \\ \text{where } M_{xy} &= \max\{\mu(x, x), \mu(y, y)\}. \end{aligned} \quad (5)$$

Hence, μ^w is an ordinary metric induced by the M -metric μ .

Each M -metric μ on X generates a T_0 topology τ_μ on X formed by the set

$$\{B_\mu(x, \varepsilon): x \in X, \varepsilon > 0\}, \quad (6)$$

where

$$B_\mu(x, \varepsilon) = \{y \in X: \mu(x, y) < m_{xy} + \varepsilon\}. \quad (7)$$

The notions of convergent sequence, Cauchy sequence, and complete M -metric space (X, μ) are given as follows:

- (1) A sequence $\{x_n\}$ in (X, μ) converges to a point $x \in X$ if

$$\lim_{n \rightarrow \infty} (\mu(x_n, x) - m_{x_n x}) = 0. \quad (8)$$

- (2) A sequence $\{x_n\}$ in (X, μ) is called μ -Cauchy if

$$\begin{aligned} \lim_{n, m \rightarrow \infty} (\mu(x_n, x_m) - m_{x_n x_m}), \\ \lim_{n, m \rightarrow \infty} (M_{x_n x_m} - m_{x_n x_m}) \end{aligned} \quad (9)$$

exist and are finite.

- (3) (X, μ) is said to be complete if every μ -Cauchy sequence $\{x_n\}$ in it converges, with respect to τ_μ , to a point $x \in X$, and

$$\lim_{n, m \rightarrow \infty} (\mu(x_n, x_m) - m_{x_n x_m}) = \lim_{n, m \rightarrow \infty} (M_{x_n x_m} - m_{x_n x_m}) = 0. \quad (10)$$

Lemma 2 (see [7]). Let (X, μ) be an M -metric space; then,

- (1) $\{x_n\}$ is a μ -Cauchy sequence in (X, μ) if and only if it is Cauchy sequence in the metric space (X, μ^w) .
 (2) (X, μ) is complete if and only if (X, μ^w) is complete. Furthermore,

$$\begin{aligned} \lim_{n \rightarrow \infty} \mu^w(x_n, x) = 0 &\iff \lim_{n \rightarrow \infty} (\mu(x_n, x) - m_{x_n x}) \\ &= \lim_{n \rightarrow \infty} (M_{x_n x} - m_{x_n x}) = 0. \end{aligned} \quad (11)$$

Lemma 3 (see [7]). Assume that $x_n \rightarrow x$ and $y_n \rightarrow y$ in an M -metric space (X, μ) ; then,

$$\lim_{n \rightarrow \infty} (\mu(x_n, y_n) - m_{x_n y_n}) = (\mu(x, y) - m_{xy}). \quad (12)$$

As a consequence of Lemma 3, we have

$$\begin{aligned} x_n \rightarrow x, \quad \text{in } (X, \mu) &\implies \lim_{n \rightarrow \infty} (\mu(x_n, y) - m_{x_n y}) = (\mu(x, y) - m_{xy}), \\ x_n \rightarrow x \text{ and } x_n \rightarrow y, \quad \text{in } (X, \mu) &\implies \lim_{n \rightarrow \infty} (\mu(x_n, x_n) - m_{x_n x_n}) = (\mu(x, y) - m_{xy}). \end{aligned} \quad (13)$$

Definition 3 (see [29]). A mapping $F: [0, \infty)^2 \rightarrow \mathbb{R}$ is called a C -class function if it is continuous and satisfies the following axioms:

- (1) $F(s, t) \leq s$
 (2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $t \in [0, \infty)$

Let \mathcal{C} denote the C -class functions.

Definition 4 (see [20, 21]). The fractional integral of order $\alpha > 0$ of a function $x: [0, \infty) \rightarrow \mathbb{R}$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad (14)$$

provided that the right side is pointwise defined on $[0, \infty)$.

Definition 5 (see [20, 21]). The fractional derivative of order $\alpha > 0$ of a continuous function $x: [0, \infty) \rightarrow \mathbb{R}$ is given by

$$\mathcal{D}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}} ds, \quad (15)$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $[0, \infty)$.

Lemma 4 (see [30]). Riemann-Liouville fractional integral and derivative have the following properties:

- (1) $I^\alpha I^\beta x(t) = I^{\alpha+\beta} x(t)$ and $\mathcal{D}^\alpha I^\beta x(t) = I^{\beta-\alpha} x(t)$, for all $\beta \geq \alpha > 0$, $x \in L[0, 1]$
 (2) $I^\alpha \mathcal{D}^\alpha x(t) = x(t) + c_1 t^{\alpha-1} + \dots + c_n t^{\alpha-n}$, where $n = [\alpha] + 1$ and $x, \mathcal{D}^\alpha x \in C[0, 1] \cap L[0, 1]$
 (3) $I^\alpha: C[0, 1] \rightarrow C[0, 1]$, $\alpha > 0$

3. Fixed-Point Results

First, we introduce the following concepts that generalize the corresponding ones used in [13] and will be beneficial in the sequel.

Definition 6 Let $T: X \times X \rightarrow X$ and $\alpha: X \times X \rightarrow [0, \infty)$; then, T is called an α -admissible mapping if

$$\begin{aligned} \alpha(x, u) &\geq 1, \\ \alpha(y, v) \geq 1 &\implies \alpha(T(x, y), T(u, v)) \geq 1, \quad \forall (x, y), (u, v) \in X^2. \end{aligned} \quad (16)$$

Note that, if equation (16) holds, then we have $\alpha(T(y, x), T(v, u)) \geq 1$ too. Consider the following classes of functions:

$$\begin{aligned}
\Psi_1 &= \{\psi: [0, \infty)^2 \longrightarrow [0, \infty), \psi \text{ is continuous, strictly increasing and } \psi(t_1, t_2) = 0 \implies t_1 = t_2 = 0\}, \\
\Psi_2 &= \{\psi: [0, \infty) \times [0, \infty) \longrightarrow [0, \infty), \psi \text{ is continuous and } \psi(t_1, t_2) = 0 \implies t_1 = t_2 = 0\}, \\
\Phi &= \left\{ \varphi: [0, \infty) \longrightarrow [0, \infty), \varphi(s+t) \leq \varphi(s) + \varphi(t) \text{ and } \varphi\left(\frac{t}{2}\right) \leq \frac{\varphi(t)}{2} \forall s, t \geq 0 \right\}.
\end{aligned} \tag{17}$$

Theorem 2. Let (X, μ) be a complete M -metric space and $T: X \times X \longrightarrow X$ be an α -admissible mapping for which there exist $F \in \mathcal{C}$, $\phi \in \Phi$, $\psi_1 \in \Psi_1$, and $\psi_2 \in \Psi_2$ such that

$\psi_1(t, t) \leq \phi(t)$ and for all $(x, y), (u, v) \in X^2$ with $\alpha(x, u) \geq 1, \alpha(y, v) \geq 1$; we have

$$[\phi(\mu(T(x, y), T(u, v))) + l]^{\max\{\alpha(x, u), \alpha(y, v)\}} \leq F\left(\psi_1\left(\frac{K(x, u) + K(y, v)}{2}\right), \psi_2\left(\frac{K(x, u) + K(y, v)}{2}\right)\right) + l, \tag{18}$$

where

$$\begin{aligned}
K(x, u) &= \left(\frac{\mu(u, T(u, v))[1 + \mu(x, T(x, y))]}{1 + \mu(x, u)}, \mu(x, u) \right), \\
K(y, v) &= \left(\frac{\mu(v, T(v, u))[1 + \mu(y, T(y, x))]}{1 + \mu(y, v)}, \mu(y, v) \right).
\end{aligned} \tag{19}$$

Suppose that either

(a) T is continuous.

(b) For a convergent sequence $\{x_n\}$ in (X, μ) , we have

$$\begin{aligned}
\{x_n\} &\longrightarrow x, \quad \alpha(x_n, x_{n+1}) \geq 1 \implies \alpha(x_n, x) \geq 1, \forall n, \\
x_n &\longrightarrow x, \quad x_n \longrightarrow y \implies \alpha(x, y) \geq 1.
\end{aligned} \tag{20}$$

If there exist $x_0, y_0 \in X$ such that $\alpha(x_0, T(x_0, y_0)) \geq 1$ and $\alpha(y_0, T(y_0, x_0)) \geq 1$, then T has a coupled fixed point.

Proof. Starting with $x_0, y_0 \in X$, define the sequences $\{x_n\}, \{y_n\} \subset X$ by

$$\begin{aligned}
x_{n+1} &= T(x_n, y_n), \\
y_{n+1} &= T(y_n, x_n), \\
\forall n &\in \mathbb{N}_0.
\end{aligned} \tag{21}$$

By induction methodology for $n \in \mathbb{N}_0$, we shall prove that

$$\begin{aligned}
\alpha(x_n, x_{n+1}) &\geq 1, \\
\alpha(y_n, y_{n+1}) &\geq 1, \forall n.
\end{aligned} \tag{22}$$

Indeed, we have $\alpha(x_0, x_1) \geq 1$ and $\alpha(y_0, y_1) \geq 1$. Suppose that (22) holds for some n and we are going to prove it for $n+1$. Since T is α -admissible mapping, then by (21), we obtain $\alpha(x_{n+1}, x_{n+2}) \geq 1$ and $\alpha(y_{n+1}, y_{n+2}) \geq 1$. Thus, (22) holds for all n . From (18)–(22), we have

$$\begin{aligned}
\phi(\mu(x_n, x_{n+1})) + l &\leq [\phi(\mu(T(x_{n-1}, y_{n-1}), T(x_n, y_n))) + l]^{\max\{\alpha(x_{n-1}, x_n), \alpha(y_{n-1}, y_n)\}} \\
&\leq F\left(\psi_1\left(\frac{K(x_{n-1}, x_n) + K(y_{n-1}, y_n)}{2}\right), \psi_2\left(\frac{K(x_{n-1}, x_n) + K(y_{n-1}, y_n)}{2}\right)\right) + l,
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
K(x_{n-1}, x_n) &= \left(\frac{\mu(x_n, T(x_n, y_n))[1 + \mu(x_{n-1}, T(x_{n-1}, y_{n-1}))]}{1 + \mu(x_{n-1}, x_n)}, \mu(x_{n-1}, x_n) \right) \\
&= (\mu(x_n, x_{n+1}), \mu(x_{n-1}, x_n)), \\
K(y_{n-1}, y_n) &= (\mu(y_n, y_{n+1}), \mu(y_{n-1}, y_n)).
\end{aligned} \tag{24}$$

Hence,

$$\phi(\mu(x_n, x_{n+1})) \leq F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right), \quad (25)$$

where $w_n = \mu(x_n, x_{n+1}) + \mu(y_n, y_{n+1})$. Similarly, we have

$$\phi(\mu(y_n, y_{n+1})) \leq F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right). \quad (26)$$

Adding (25) and (26) and using properties on F and ϕ , we obtain

$$\begin{aligned} \psi_1\left(\frac{w_n}{2}, \frac{w_n}{2}\right) &\leq \phi\left(\frac{w_n}{2}\right) \leq F\left(\psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right), \psi_2\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right)\right) \\ &\leq \psi_1\left(\frac{w_n}{2}, \frac{w_{n-1}}{2}\right). \end{aligned} \quad (27)$$

Since ψ_1 is strictly increasing, then $w_n \leq w_{n-1}$, $\forall n$. Hence, the sequence $\{w_n\}$ is monotone decreasing and bounded as follows. Therefore, there exist some $w \geq 0$ such that

$$\lim_{n \rightarrow \infty} w_n = w. \quad (28)$$

Now, we shall prove that $w = 0$. Assume that $w > 0$. Using the properties of ψ_1 , ψ_2 , and F and letting $n \rightarrow \infty$ in (27) yield that

$$\psi_1\left(\frac{w}{2}, \frac{w}{2}\right) \leq F\left(\psi_1\left(\frac{w}{2}, \frac{w}{2}\right), \psi_2\left(\frac{w}{2}, \frac{w}{2}\right)\right) < \psi_1\left(\frac{w}{2}, \frac{w}{2}\right), \quad (29)$$

which is contradiction. Thus, $w = 0$ and

$$\lim_{n \rightarrow \infty} \mu(x_n, x_{n+1}) = \lim_{n \rightarrow \infty} \mu(y_n, y_{n+1}) = 0. \quad (30)$$

In what follows, we prove that $\{x_n\}$ and $\{y_n\}$ are μ -Cauchy sequences in (X, μ) . Since we have

$$\begin{aligned} 0 &\leq m_{x_n, x_{n+1}} \leq \mu(x_n, x_{n+1}) \rightarrow 0, \quad \text{as } n \rightarrow \infty \\ &\Rightarrow \lim_{n \rightarrow \infty} m_{x_n, x_{n+1}} = \min\left\{\lim_{n \rightarrow \infty} \mu(x_n, x_n), \lim_{n \rightarrow \infty} \mu(x_{n+1}, x_{n+1})\right\} = 0 \\ &\Rightarrow \lim_{n \rightarrow \infty} \mu(x_n, x_n) = 0, \end{aligned} \quad (31)$$

then

$$\begin{aligned} \lim_{n, m \rightarrow \infty} m_{x_n, x_m} &= \min\left\{\lim_{n \rightarrow \infty} \mu(x_n, x_n), \lim_{m \rightarrow \infty} \mu(x_m, x_m)\right\} = 0, \\ \lim_{n, m \rightarrow \infty} M_{x_n, x_m} &= \max\left\{\lim_{n \rightarrow \infty} \mu(x_n, x_n), \lim_{m \rightarrow \infty} \mu(x_m, x_m)\right\} = 0. \end{aligned} \quad (32)$$

That is,

$$\lim_{n \rightarrow \infty} (M_{x_n, x_m} - m_{x_n, x_m}) = 0. \quad (33)$$

On the other hand, we have

$$\mu(x_n, x_m) - m_{x_n, x_m} \leq \mu(x_n, x_{n+1}) - m_{x_n, x_{n+1}} + \mu(x_{n+1}, x_{n+2}) - m_{x_{n+1}, x_{n+2}} + \cdots + \mu(x_{m-1}, x_m) - m_{x_{m-1}, x_m} \rightarrow 0, \quad \text{as } n, m \rightarrow \infty. \quad (34)$$

Therefore, (33) and (34) imply that $\{x_n\}$ is an μ -Cauchy sequence. In a similar way, we can show that $\{y_n\}$ is also a μ -Cauchy sequence. By the completeness of the space (X, μ) , there exist $x, y \in X$ such that

$$\begin{aligned} \lim_{n \rightarrow \infty} (\mu(x_n, x) - m_{x_n, x}) &= \lim_{n \rightarrow \infty} (\mu(y_n, y) - m_{y_n, y}) = 0, \\ \lim_{n \rightarrow \infty} (M_{x_n, x} - m_{x_n, x}) &= \lim_{n \rightarrow \infty} (M_{y_n, y} - m_{y_n, y}) = 0. \end{aligned} \quad (35)$$

With respect to the sequence $\{x_n\}$, we obtain

$$\begin{aligned} \mu(x_n, x_n) &\rightarrow 0 \Rightarrow m_{x_n, x} \rightarrow 0 \Rightarrow \mu(x_n, x), \\ M_{x_n, x} &\rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (36)$$

but

$$M_{x_n, x} = \max\{\mu(x_n, x_n), \mu(x, x)\} \rightarrow \mu(x, x). \quad (37)$$

Thus, the uniqueness of the limit implies that

$$\mu(x, x) = 0. \quad (38)$$

Now, suppose that (a) holds. According to Lemma 2, since $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in a complete M -metric space (X, μ) , then they converge to some x, y in the metric space (X, μ^w) . Also, as F is continuous, $F(x_n, y_n)$ converges to $F(x, y)$ in (X, μ^w) , that is, $\lim_{n \rightarrow \infty} \mu^w(F(x_n, y_n), F(x, y)) = 0$ which is equivalent to

$$\mu(F(x_n, y_n), F(x, y)) - m_{F(x_n, y_n), F(x, y)} \rightarrow 0,$$

$$M_{F(x_n, y_n), F(x, y)} - m_{F(x_n, y_n), F(x, y)} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (39)$$

Also, we have

$$\begin{aligned} \mu(x_{n+1}, x_{n+1}) &\rightarrow 0 \Rightarrow m_{F(x_n, y_n), F(x, y)} \\ &\rightarrow 0 \Rightarrow M_{F(x_n, y_n), F(x, y)} \rightarrow 0, \quad \text{as } n \rightarrow \infty, \end{aligned} \quad (40)$$

but

$$\begin{aligned} M_{F(x_n, y_n), F(x, y)} &= \max\{\mu(x_{n+1}, x_{n+1}), \mu(F(x, y), F(x, y))\} \\ &\longrightarrow \mu(F(x, y), F(x, y)). \end{aligned} \quad (41)$$

Thus, the uniqueness of the limit implies that

$$\mu(F(x, y), F(x, y)) = 0. \quad (42)$$

By Lemma 3, we obtain

$$\begin{aligned} \mu(x_{n+1}, F(x, y)) - m_{x_{n+1}, F(x, y)} &\longrightarrow \mu(x, F(x, y)) - m_{x, F(x, y)} \\ &= \mu(x, F(x, y)). \end{aligned} \quad (43)$$

Compared with (39), we obtain

$$\mu(x, F(x, y)) = 0. \quad (44)$$

From (38), (42), and (44), we obtain

$$x = F(x, y). \quad (45)$$

Proceeding as above, one can obtain

$$y = F(y, x). \quad (46)$$

Suppose that (b) holds, then $\alpha(x_n, x) \geq 1$ and $\alpha(y_n, y) \geq 1$. Setting $(x, y) = (x_n, y_n)$ and $(u, v) = (x, y)$ in (18), we obtain

$$[\phi(\mu(T(x_n, y_n), T(x, y))) + l]^{\max\{\alpha(x_n, x), \alpha(y_n, y)\}} \leq F\left(\psi_1\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right), \psi_2\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right)\right) + l, \quad (47)$$

where

$$\begin{aligned} K(x_n, x) &= \left(\frac{\mu(x, T(x, y))[1 + \mu(x_n, x_{n+1})]}{1 + \mu(x_n, x)}, \mu(x_n, x)\right) \longrightarrow (\mu(x, T(x, y)), 0), \\ K(y_n, y) &= \left(\frac{\mu(y, T(y, x))[1 + \mu(y_n, y_{n+1})]}{1 + \mu(y_n, y)}, \mu(y_n, y)\right) \longrightarrow (\mu(y, T(y, x)), 0), \quad \text{as } n \longrightarrow \infty. \end{aligned} \quad (48)$$

That is,

$$\phi(\mu(x_{n+1}, T(x, y))) \leq F\left(\psi_1\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right), \psi_2\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right)\right). \quad (49)$$

In a similar way, one can obtain

$$\phi(\mu(y_{n+1}, T(y, x))) \leq F\left(\psi_1\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right), \psi_2\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right)\right). \quad (50)$$

Adding (49) and (50) and using properties on F and ϕ , we obtain

$$\psi_1\left(\frac{\mu(x_{n+1}, T(x, y)) + \mu(y_{n+1}, T(y, x))}{2}, \frac{\mu(x_{n+1}, T(x, y)) + \mu(y_{n+1}, T(y, x))}{2}\right) \leq \psi_1\left(\frac{K(x_n, x) + K(y_n, y)}{2}\right). \quad (51)$$

Taking limits at $n \longrightarrow \infty$ yields

$$\psi_1\left(\frac{\mu(x, T(x, y)) + \mu(y, T(y, x))}{2}, \frac{\mu(x, T(x, y)) + \mu(y, T(y, x))}{2}\right) \leq \psi_1\left(\frac{\mu(x, T(x, y)) + \mu(y, T(y, x))}{2}, 0\right). \quad (52)$$

Therefore, we have $\mu(x, T(x, y)) + \mu(y, T(y, x)) = 0$. Again from (18) and taking into account that $\alpha(x, x), \alpha(y, y) \geq 1$, we obtain that $\mu(T(x, y), T(x, y)) = \mu(T(y, x), T(y, x)) = 0$. Consequently, $x = T(x, y)$ and $y = T(y, x)$.

For the uniqueness of the coupled fixed point in Theorem 2, we consider the following condition:

if (x, y) and (u, v) are two coupled fixed points of T ,
 then $\alpha(x, u) \geq 1$,
 or $\alpha(y, v) \geq 1$. (53)

□

$$\begin{aligned} \psi_1\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right) &\leq \phi\left(\frac{\mu(x, x) + \mu(y, y)}{2}\right) \\ &\leq F\left(\psi_1\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right), \psi_2\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right)\right) \\ &\leq \psi_1\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right). \end{aligned} \quad (54)$$

Hence, we have

$$\begin{aligned} F\left(\psi_1\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right), \psi_2\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right)\right) \\ = \psi_1\left(\frac{\mu(x, x) + \mu(y, y)}{2}, \frac{\mu(x, x) + \mu(y, y)}{2}\right) \implies \mu(x, x) = \mu(y, y) = 0. \end{aligned} \quad (55)$$

Similarly, we have

$$\begin{aligned} \mu(u, u) &= \mu(v, v) = 0, \\ \mu(x, u) &= \mu(y, v) = 0. \end{aligned} \quad (56)$$

Hence, by (m_1) $x = u$ and $y = v$, i.e., (x, y) is the unique coupled fixed point of T .

If we define $F(s, t) = s - t$ and

$$\alpha(x, y) = \begin{cases} 1, & x < y; \\ 0, & \text{otherwise,} \end{cases} \quad (57)$$

then we get the following corollary which is a generalization of the main results in [31]. □

Corollary 1. Let (X, μ) be an ordered complete M -metric space and $T: X \times X \rightarrow X$ be an increasing mapping for which there exist $\phi \in \Phi$, $\psi_1 \in \Psi_1$, and $\psi_2 \in \Psi_2$ such that $\psi_1(t, t) \leq \phi(t)$ and for all $(x, y), (u, v) \in X^2$ with $x < u$ and $y < v$; we have

Theorem 3. Adding condition (53) to the hypotheses of Theorem 2, we obtain that T has a unique coupled fixed point.

Proof. Theorem 2 asserts that T has at least one coupled fixed point. Assume that (x, y) and (u, v) are two coupled fixed points of T , then $\alpha(x, u) \geq 1$ or $\alpha(y, v) \geq 1$. Now, we apply (18) and use the properties of ϕ, ψ_1, ψ_2 , and F to obtain

$$\begin{aligned} \phi(\mu(T(x, y), T(u, v))) &\leq \psi_1\left(\frac{K(x, u) + K(y, v)}{2}\right) \\ &\quad - \psi_2\left(\frac{K(x, u) + K(y, v)}{2}\right), \end{aligned} \quad (58)$$

where

$$\begin{aligned} K(x, u) &= \left(\frac{\mu(u, T(u, v))[1 + \mu(x, T(x, y))]}{1 + \mu(x, u)}, \mu(x, u)\right), \\ K(y, v) &= \left(\frac{\mu(v, T(v, u))[1 + \mu(y, T(y, x))]}{1 + \mu(y, v)}, \mu(y, v)\right). \end{aligned} \quad (59)$$

Suppose that either

(a) T is continuous.

(b) For a convergent sequence $\{x_n\}$ in (X, μ) , we have

$$\begin{aligned}
\{x_n\} &\longrightarrow x, \\
x_n < x_{n+1} &\implies x_n < x, \quad \forall n, \\
x_n &\longrightarrow x, \\
x_n &\longrightarrow y \implies x < y.
\end{aligned} \tag{60}$$

If there exist $x_0, y_0 \in X$ such that $x_0 < T(x_0, y_0)$ and $y_0 < T(y_0, x_0)$, then T has a coupled fixed point. Now, we introduce the following classes of functions Ψ and Φ by

$$\begin{aligned}
\Psi &= \{\psi: [0, \infty) \longrightarrow [0, \infty), \psi \text{ is continuous, strictly increasing and } \psi(t) > 0 \text{ for } t > 0\}, \\
\Phi &= \{\varphi: [0, \infty) \longrightarrow [0, \infty), \varphi \text{ is continuous and } \varphi(t) > 0 \text{ for } t > 0\}.
\end{aligned} \tag{61}$$

If we consider $\phi(t) = \psi(t)$, $\psi_1(s, t) = \psi(t)$ for some $\psi \in \Psi$ and $\psi_2(s, t) = \varphi(t)$ for some $\varphi \in \Phi$, then we obtain an extension of the main result in [13].

Corollary 2. Let (X, μ) be a complete M -metric space and $T: X \times X \longrightarrow X$ be an α -admissible mapping such that

$$[\psi(\mu(T(x, y), T(u, v))) + l]^{\max\{\alpha(x, u), \alpha(y, v)\}} \leq F\left(\psi\left(\frac{\mu(x, u) + \mu(y, v)}{2}\right), \varphi\left(\frac{\mu(x, u) + \mu(y, v)}{2}\right)\right) + l, \tag{62}$$

for all $(x, y), (u, v) \in X^2$ with $\alpha(x, u) \geq 1, \alpha(y, v) \geq 1$, where $F \in \mathcal{C}$, $\psi \in \Psi$, and $\varphi \in \Phi$. Suppose that either

- (a) T is continuous.
- (b) For a convergent sequence $\{x_n\}$ in (X, μ) , we have

$$\begin{aligned}
\{x_n\} &\longrightarrow x, \\
\alpha(x_n, x_{n+1}) &\geq 1 \implies \alpha(x_n, x) \geq 1, \quad \forall n, \\
x_n &\longrightarrow x, \\
x_n &\longrightarrow y \implies \alpha(x, y) \geq 1.
\end{aligned} \tag{63}$$

If there exist $x_0, y_0 \in X$ such that $\alpha(x_0, T(x_0, y_0)) \geq 1$ and $\alpha(y_0, T(y_0, x_0)) \geq 1$, then T has a coupled fixed point.

Remark 1. Notice that in [32, 33], it was shown that each coupled fixed-point theorem can be observed from the analogue of single/standard fixed-point theorems. On the other hand, for the usage of it in application, the coupled

fixed-point theorem can be used to handle the problem. Therefore, in this paper, we consider the coupled fixed-point results, Theorem 2 and Theorem 3.

4. Fractional Differential Equations

In this section, we present sufficient conditions for the existence and uniqueness of the solution of coupled systems (2) and (3). Before starting and proving the main results, we need to fix the analytical framework of our considered problem.

Consider the complete M -metric space (X, μ) , where $X = C(J, \mathbb{R})$ and μ is defined by

$$\mu(x, y) = \sup_{t \in J} |x(t) - y(t)|, \quad \forall x, y \in X. \tag{64}$$

In addition, define the operator $T: X \times X \longrightarrow X$ as

$$T(x, y) = Ax(t) + By(t), \tag{65}$$

where

$$\begin{aligned}
Ax(t) &= f(t, x(t)) + t^{\alpha-1} \frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}}, \\
By(t) &= I^\alpha g(t, y(t), I^\beta y(t)) + t^{\alpha-1} \frac{\delta I^\alpha g(\eta, y(\eta), I^\beta y(\eta)) - I^\alpha g(\tau, y(\tau), I^\beta y(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}}.
\end{aligned} \tag{66}$$

Now, we claim that whenever $(x, y) \in X^2$ is a coupled fixed point of the operator T , it follows that $x(t)$ and $y(t)$ solve (2) and (3).

Lemma 5. Let $n-1 < \alpha \leq n$, $0 < \eta < \tau$, $\delta \neq (\tau/\eta)^{\alpha-1}$, and $h \in L(0, \tau)$; then, the boundary value problem

$$\mathcal{D}^\alpha [x(t) - f(t, x(t))] = h(t), \quad \forall t \in J,$$

$$x^{(i)}(0) = \frac{\partial^i f(t, x(t))}{\partial t^i} \Big|_{t=0} = 0, \quad x(\tau) = \delta x(\eta), \quad \forall i = 0, 1, \dots, n-2, \tag{67}$$

has the integral representation of solution

$$x(t) = f(t, x(t)) + t^{\alpha-1} \frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^\alpha h(t) + t^{\alpha-1} \frac{\delta I^\alpha h(\eta) - I^\alpha h(\tau)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \quad (68)$$

Proof. Applying the operator I^α on both sides of (67) and using Lemma 4, we obtain

$$x(t) - f(t, x(t)) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n} = I^\alpha h(t), \quad \text{at } t = 0 \implies x(0) = 0, f(0, 0) = 0 \implies c_n = 0. \quad (69)$$

Also, we have

$$\begin{aligned} & \dot{x}(t) - \frac{df(t, x(t))}{dt} + c_1(\alpha-1)t^{\alpha-2} + c_2(\alpha-2)t^{\alpha-3} + \dots + c_{n-1}(\alpha-n+1)t^{\alpha-n} \\ &= I^{\alpha-1}h(t), \quad \text{at } t = 0 \implies \dot{x}(0) = 0, \left. \frac{df(t, x(t))}{dt} \right|_{t=0} = \left. \frac{\partial f(t, x(t))}{\partial t} \right|_{t=0} = 0 \implies c_{n-1} = 0, \\ & \vdots \end{aligned} \quad (70)$$

$$\begin{aligned} & x^{(n-2)}(t) - \frac{d^{n-2}f(t, x(t))}{dt^{n-2}} + c_1(\alpha-1)\dots(\alpha-n+2)t^{\alpha-n+1} + c_2(\alpha-2)\dots(\alpha-n+1)t^{\alpha-n} \\ &= I^{\alpha-n+2}h(t), \quad \text{at } t = 0 \implies x^{(n-2)}(0) = 0, \left. \frac{d^{n-2}f(t, x(t))}{dt^{n-2}} \right|_{t=0} = \left. \frac{\partial^{n-2}f(t, x(t))}{\partial t^{n-2}} \right|_{t=0} = 0 \implies c_2 = 0. \end{aligned}$$

Hence, we obtain

$$x(t) - f(t, x(t)) + c_1 t^{\alpha-1} = I^\alpha h(t). \quad (71)$$

At $t = \tau$ and η , we have

$$x(\tau) - f(\tau, x(\tau)) + c_1 \tau^{\alpha-1} = I^\alpha h(\tau), \quad (72)$$

$$\delta x(\eta) - \delta f(\eta, x(\eta)) + c_1 \delta\eta^{\alpha-1} = \delta I^\alpha h(\eta). \quad (73)$$

By subtracting (73) from (72), we obtain

$$c_1 = \frac{f(\tau, x(\tau)) + I^\alpha h(\tau) - \delta[f(\eta, x(\eta)) + I^\alpha h(\eta)]}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \quad (74)$$

Consequently, the general solution of (67) is

$$\begin{aligned} x(t) = & f(t, x(t)) + t^{\alpha-1} \frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} \\ & + I^\alpha h(t) + t^{\alpha-1} \frac{\delta I^\alpha h(\eta) - I^\alpha h(\tau)}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \end{aligned} \quad (75)$$

Consider the following coupled system of fractional hybrid integral equations (in short, FHIE):

$$x(t) = f(t, x(t)) + t^{\alpha-1} \frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^\alpha g(t, y(t), I^\beta y(t)) + t^{\alpha-1} \frac{\delta I^\alpha g(\eta, y(\eta), I^\beta y(\eta)) - I^\alpha g(\tau, y(\tau), I^\beta y(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}, \quad (76)$$

$$y(t) = f(t, y(t)) + t^{\alpha-1} \frac{\delta f(\eta, y(\eta)) - f(\tau, y(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}} + I^\alpha g(t, x(t), I^\beta x(t)) + t^{\alpha-1} \frac{\delta I^\alpha g(\eta, x(\eta), I^\beta x(\eta)) - I^\alpha g(\tau, x(\tau), I^\beta x(\tau))}{\tau^{\alpha-1} - \delta\eta^{\alpha-1}}. \quad (77)$$

Lemma 6. Assume that the function $\rho: J \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by $\rho(t, x) = x(t) - f(t, x(t))$ satisfies the following:

$$\mathcal{D}^i \rho(t, x) \Big|_{t=0} = 0 \implies x = 0, \quad \forall i = 0, 1, \dots, n-2. \quad (78)$$

Then, $(x, y) \in C^2(J, \mathbb{R})$ is a solution of FHDE systems (2) and (3) if and only if (x, y) is a solution of FHIE systems (76) and (77).

Proof. Let x and y be a solution of (2) and (3). Then, by Lemma 5, we gain that the general solution of (2) has the integral form presented in (76) and the solution of (3) has

the form presented in (77). Thus, x and y satisfy (76) and (77).

Conversely, let x and y fulfill (76) and (77). Then, applying \mathcal{D}^α on both sides of (76) and using the relation $\mathcal{D}^\alpha t^\lambda = ((\Gamma(\lambda + 1))/(\Gamma(\lambda - \alpha + 1)))t^{\lambda - \alpha}$ if $\lambda > -1, \lambda \geq \alpha > 0$, and $\mathcal{D}^\alpha t^\lambda = 0$ if $\lambda < \alpha$ (Remark 2.1 in [34]) yield

$$\begin{aligned} \mathcal{D}^\alpha [x(t) - f(t, x(t))] &= \mathcal{D}^\alpha t^{\alpha-1} \frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} \\ &\quad + \mathcal{D}^\alpha I^\alpha g(t, y(t), I^\beta y(t)) + \mathcal{D}^\alpha t^{\alpha-1} \frac{\delta I^\alpha g(\eta, y(\eta), I^\beta y(\eta)) - I^\alpha g(\tau, y(\tau), I^\beta y(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} \\ \implies \mathcal{D}^\alpha [x(t) - f(t, x(t))] &= g(t, y(t), I^\alpha y(t)). \end{aligned} \quad (79)$$

So $x(t)$ satisfies the differential equation in (2). To see that it also satisfies the boundary conditions in the same equation, fix $i = 0, 1, \dots, n-2$ and apply \mathcal{D}^i in (76):

$$\begin{aligned} \mathcal{D}^i x(t) &= \mathcal{D}^i f(t, x(t)) + I^{\alpha-i} g(t, y(t), I^\beta y(t)) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-i)} t^{\alpha-1-i} \\ &\quad \cdot \left[\frac{\delta f(\eta, x(\eta)) - f(\tau, x(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} + \frac{\delta I^\alpha g(\eta, y(\eta), I^\beta y(\eta)) - I^\alpha g(\tau, y(\tau), I^\beta y(\tau))}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} \right]. \end{aligned} \quad (80)$$

Substituting $t = 0$ in (80) and taking into account that $\alpha - 1 - i > 0$ yield

$$x^{(i)}(t) \Big|_{t=0} - \frac{d^i f(t, x(t))}{dt^i} \Big|_{t=0} = 0 \implies x^{(i)}(0) = \frac{\partial^i f(t, x(t))}{\partial t^i} \Big|_{t=0} = 0. \quad (81)$$

Again, putting $t = \tau$ and $t = \eta$ in (76) implies

$$x(\tau) - \delta x(\eta) = 0. \quad (82)$$

Thus, $x(t)$ satisfies (2). A completely dual calculation reveals that $y(t)$ also satisfies (3).

As a consequence of Lemma 6, the coupled fixed point of the operator T coincides with the solution of (76) and (77) and then with the solution of (2) and (3). \square

Theorem 4. Assume that $f: J \times \mathbb{R} \longrightarrow \mathbb{R}$ and $g: J \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are continuous functions and there exist two functions $\varphi_0, \varphi_1: J \longrightarrow \mathbb{R}$ with bounds $\|\varphi_0\|$ and $\|\varphi_1\|$, respectively, such that

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq \varphi_0(t)|x - y|, \\ |g(t, x, u) - g(t, y, v)| &\leq \varphi_1(t)(|x - y| + |u - v|). \end{aligned} \quad (83)$$

Moreover, define these functions $\phi \in \Phi$, $\psi_1 \in \Psi_1$, $\psi_2 \in \Psi_2$, $F \in \mathcal{C}$, and $\alpha: X^2 \longrightarrow [0, \infty)$ as

$$\begin{aligned} \phi(s) &= s, \\ \psi_1(s, t) &= (\rho + 1)t, \\ \psi_2(s, t) &= t, \\ F(s, t) &= s - t, \\ \alpha(x, y) &= 1, \\ \forall s, t &\in [0, \infty), \\ x, y &\in X, \end{aligned} \quad (84)$$

where

$$\rho = 2 \max \left\{ \|\varphi_0\| \left[1 + \frac{\delta + 1}{|1 - \delta(\eta/\tau)^{\alpha-1}|} \right], \frac{\|\varphi_1\| \tau^\alpha}{\Gamma(\alpha + 1)} \left[1 + \frac{\tau^\beta}{\Gamma(\beta + 1)} \right] \left[1 + \frac{\delta + 1}{|1 - \delta(\eta/\tau)^{\alpha-1}|} \right] \right\} > 0. \quad (85)$$

Then, problems (2) and (3) have a unique solution.

Proof. We check that the hypothesis of Theorems 2 and Theorem 3 is satisfied. For $(x, y), (u, v) \in X^2$, we have

$$\begin{aligned}
 \mu(T(x, y), T(u, v)) &= \sup_{t \in J} |T(x, y)(t) - T(u, v)(t)| \\
 &\leq \sup_{t \in J} \left[\varphi_0(t) \left(|x(t) - u(t)| + \left| \frac{t^{\alpha-1}}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} [\delta |x(\eta) - u(\eta)| + |u(\tau) - x(\tau)|] \right) \right. \right. \\
 &\quad \left. \left. + I^\alpha \left[\varphi_1(t) \left(|y(t) - v(t)| + |I^\beta y(t) - I^\beta v(t)| \right) \right] + \left| \frac{t^{\alpha-1}}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} \right| \right. \right. \\
 &\quad \left. \left. \cdot \left[\delta I^\alpha \left[\varphi_1(t) \left(|y(\eta) - v(\eta)| + |I^\beta y(\eta) - I^\beta v(\eta)| \right) \right] + I^\alpha \left[\varphi_1(t) \left(|y(\tau) - v(\tau)| + |I^\beta y(\tau) - I^\beta v(\tau)| \right) \right] \right] \right] \right] \\
 &\leq \|\varphi_0\| \left(\mu(x, u) + \left| \frac{t^{\alpha-1}}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} [\delta \mu(x, u) + \mu(x, u)] \right) \right) + \|\varphi_1\| \left(\frac{t^\alpha}{\Gamma(\alpha+1)} \left[\mu(y, v) + \frac{t^\beta}{\Gamma(\beta+1)} \mu(y, v) \right] \right. \\
 &\quad \left. + \left| \frac{t^{\alpha-1}}{\tau^{\alpha-1} - \delta \eta^{\alpha-1}} \left[\delta \frac{\eta^\alpha}{\Gamma(\alpha+1)} \left[\mu(y, v) + \frac{\eta^\beta}{\Gamma(\beta+1)} \mu(y, v) \right] + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \left[\mu(y, v) + \frac{\tau^\beta}{\Gamma(\beta+1)} \mu(y, v) \right] \right] \right) \right) \\
 &\leq \|\varphi_0\| \left[1 + \frac{\delta+1}{|1 - \delta(\eta/\tau)^{\alpha-1}|} \right] \mu(x, u) + \frac{\|\varphi_1\| \tau^\alpha}{\Gamma(\alpha+1)} \left[1 + \frac{\tau^\beta}{\Gamma(\beta+1)} \right] \left[1 + \frac{\delta+1}{|1 - \delta(\eta/\tau)^{\alpha-1}|} \right] \mu(y, v) \\
 &\leq (\rho+1) \left[\frac{\mu(x, u) + \mu(y, v)}{2} \right] - \left[\frac{\mu(x, u) + \mu(y, v)}{2} \right].
 \end{aligned} \tag{86}$$

Thus, for any $s \geq 0$, we obtain

$$\begin{aligned}
 \phi(\mu(T(x, y), T(u, v))) &\leq F \left(\psi_1 \left(s, \frac{\mu(x, u) + \mu(y, v)}{2} \right), \right. \\
 &\quad \left. \cdot \psi_2 \left(s, \frac{\mu(x, u) + \mu(y, v)}{2} \right) \right).
 \end{aligned} \tag{87}$$

Therefore, the operator T satisfies condition (18) of Theorem 2. With simple calculations, we can derive that the other hypothesis of Theorems 2 and Theorem 3 holds. So, the operator T has a unique fixed point, or equivalently, systems (2) and (3) have a unique solution in X^2 .

Now, we present an illustrated example to justify our results. \square

Example 2. Consider the following system of two FHDEs with three-point boundary conditions:

$$\begin{aligned}
 \mathcal{D}^\alpha [x(t) - f(t, x(t))] &= g(t, y(t), I^\beta y(t)), \\
 x^{(i)}(0) &= \frac{\partial^i f(t, x(t))}{\partial t^i} \Big|_{t=0} = 0,
 \end{aligned} \tag{88}$$

$$x(\tau) = \delta x(\eta),$$

$$\begin{aligned}
 \mathcal{D}^\alpha [y(t) - f(t, y(t))] &= g(t, x(t), I^\beta x(t)), \\
 y^{(i)}(0) &= \frac{\partial^i f(t, y(t))}{\partial t^i} \Big|_{t=0} = 0, \\
 y(\tau) &= \delta y(\eta),
 \end{aligned} \tag{89}$$

where

$$\begin{aligned}
 \alpha &= \frac{5}{2}, \\
 \beta &= \frac{9}{2}, \\
 \eta &= \frac{1}{2}, \\
 \tau &= \delta = 1,
 \end{aligned} \tag{90}$$

$$f(t, x) = x + \sqrt{x^2 + 1} - e^{t|x|},$$

$$g(t, x, u) = x + \sin u.$$

By computation, we can show that

$$\begin{aligned}
 |f(t, x) - f(t, y)| &= \left| x + \sqrt{x^2 + 1} - e^{t|x|} - (y + \sqrt{y^2 + 1} - e^{t|y|}) \right| \\
 &\leq |x - y| + \left| \sqrt{x^2 + 1} - \sqrt{y^2 + 1} \right| \\
 &\leq 2|x - y|, \\
 |g(t, x, u) - g(t, y, v)| &= |x + \sin u - y + \sin v| \\
 &\leq |x - y| + |u - v|.
 \end{aligned} \tag{91}$$

Applying Theorem 4, we conclude that problem (89) has one solution.

5. Concluding Remarks

In this work, we proved some coupled fixed-point results for α -admissible mappings which are $F(\psi_1, \psi_2)$ -contractions in a larger structure such as M -metric spaces. Furthermore, we applied aforesaid fixed-point results to investigate the existence of a unique solution for a coupled system of higher-order fractional hybrid differential equations which are equipped with three-point boundary conditions. The respective results have been verified by providing a suitable example.

In fact, the results dealing with solutions of the general systems of fractional differential equations are useful in applications to various problems which are simply modelled by means of these systems.

It is believed that several recent studies (see, for example, [35–42]) on fractional calculus and its widespread applications will possibly motivate further research studies on mathematical modeling and analysis of applied problems along the lines which we have developed in this article.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have read and agreed to the published version of the manuscript.

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Research Article

A Novel 2-Stage Fractional Runge–Kutta Method for a Time-Fractional Logistic Growth Model

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In this paper, the fractional Euler method has been studied, and the derivation of the novel 2-stage fractional Runge–Kutta (FRK) method has been presented. The proposed fractional numerical method has been implemented to find the solution of fractional differential equations. The proposed novel method will be helpful to derive the higher-order family of fractional Runge–Kutta methods. The nonlinear fractional Logistic Growth Model is solved and analyzed. The numerical results and graphs of the examples demonstrate the effectiveness of the method.

1. Introduction

In the 20th century, important research in fractional calculus was published in the engineering and science literature. Progress of fractional calculus is reported in various applications in the field of integral equations, fluid mechanics, viscoelastic models, biological models, and electrochemistry [1–3]. Undoubtedly, fractional calculus is an efficient mathematical tool to solve various problems in mathematics, engineering, and sciences. To get more attention in this field and to validate its effectiveness, this paper contributes the solution of new and recent applications of fractional calculus in biological and engineering sciences [4, 5]. Recently, the tool of fractional calculus has been used to analyze the nonlinear dynamics of different problems [6–8].

Mostly, the analytical solutions cannot be obtained for fractional differential equations, so that there is a need of semianalytical and numerical methods to understand the effects of the solutions to the nonlinear problems [9]. In the recent decades, different methods have been implemented to solve the linear as well as the nonlinear dynamical systems, such as the

Adomian decomposition method (ADM) [10], variational iteration method (VIM) [11], Homotopy perturbation method (HPM) [12], Homotopy perturbation method in association with the Laplace transform method [6], Homotopy analysis method (HAM) [13], and Homotopy analysis transform method (HATM) [7]. In the recent years, the novel numerical techniques have also been applied on a two-dimensional telegraph equation on arbitrary domains and modified diffusion equations with nonlinear source terms [14–16].

In the recent past, many numerical methods have been used just for linear equations or often more smaller classes. The generalization of the classical Adams–Bashforth–Moulton method has been introduced for the numerical solutions of nonlinear fractional differential equations [17]. Odibat and Momani also develop the new method with the connection of fractional Euler method and modified Trapezoidal rule by using the generalized Taylor series expansion [18].

Moreover, scientists have been actively worked on logistic growth that is typically the common model of population growth. A biological population with a lot of food, space to grow and no threats from predators, and trends to

grow at a rate that is proportional to the population in each unit of time is a certain percentage of the individuals who produce new individuals [19–21].

In this paper, we derived the 2-stage fractional Runge–Kutta method by using the generalized Taylor series expansion in Section 2. Afterwards, we applied the proposed numerical method on different nonlinear fractional differential equations and present the numerical results in Section 3. More specifically, we have used the fractional Runge–Kutta Method to solve the fractional logistic growth model. The conclusion is drawn in Section 4.

2. Method Description

In order to study the fractional differential equation, we will consider Caputo's fractional order derivative. Caputo's fractional order derivative is the modified form of the Riemann–Liouville definition and beneficial in dealing with the initial value problem more efficiently. Generalized Taylor's formula is defined as follows.

2.1. Generalized Taylor's Formula. Here, we are defining generalized Taylor's formula as given in [18], i.e., suppose that $D^{k\alpha}\phi(x) \in C(0, a]$ for $k = 0, 1, 2, \dots, n+1$, where $0 < \alpha \leq 1$. We have

$$u(x) = \sum_{i=0}^n \frac{x^{i\alpha}}{\Gamma(i\alpha+1)} (D^{i\alpha}\phi)(0) + \frac{(D^{(n+1)\alpha}\phi)(\zeta)}{\Gamma((n+1)\alpha+1)} x^{(n+1)\alpha}, \quad (1)$$

with $0 \leq \zeta \leq x$, $\forall x \in (0, a]$.

2.2. Fractional Euler Method. In order to derive the fractional Euler's method to find the numerical solution of initial value problem with time-fractional derivative in Caputo's sense, we consider the initial value problem of the form

$$\begin{aligned} \frac{d^\alpha}{dt^\alpha} u(t) &= \phi(t, u(t)); \\ u(0) &= u_0, \end{aligned} \quad (2)$$

$$\alpha \in (0, 1],$$

where D^α represents the Caputo fractional differential operator [22]. Consider the initial value problem. Let $[0, a]$ be an interval for which we are finding the solution of the problem in equation (2). The collection of points $(t_j, u(t_j))$ are used to find the approximation. The interval $[0, a]$ is subdivided into r subintervals $[t_j, t_{j+1}]$ of equal step size $h = (a/r)$ using the nodal points $t_j = jh$ for $j = 0, 1, 2, \dots, r$. Suppose that $u(t)$, $D^\alpha u(t)$, and $D^{2\alpha} u(t)$ are continuous functions on the interval $[0, a]$, and applying Taylor's formula involving fractional derivatives, we have

$$u(t+h) = u(t) + \frac{h^\alpha}{\Gamma(\alpha+1)} D^\alpha u(t) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} u(t) + \dots \quad (3)$$

For the very small step size, we neglect the higher terms involving $h^{2\alpha}$ or higher, and substituting the value of $D^\alpha u(t)$ from equation (2), we obtain

$$u(t+h) = u(t) + \frac{h^\alpha}{\Gamma(\alpha+1)} \phi(t, u(t)). \quad (4)$$

By using the abovementioned equation, we can obtain the following iterative formula.

$$u_{n+1} = u_n + \frac{h^\alpha}{\Gamma(\alpha+1)} \phi(t_n, u_n). \quad (5)$$

It is worth mentioning here that if $\alpha = 1$, then fractional Euler's method 2.3 reduced to classical Euler's method. This is the generalization of classical Euler's method.

2.3. Fractional Runge–Kutta Method. This method is the generalization of the Runge–Kutta (RK) method of order 2. Consider fractional order initial value problem (2). The generalized Taylor expansion is

$$u(t+h) = u(t) + \frac{h^\alpha}{\Gamma(\alpha+1)} D^\alpha u(t) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D^{2\alpha} u(t) + \dots, \quad (6)$$

and using the formula $D_t^{2\alpha} u = D_t^\alpha \phi(t, u) + \phi(t, u) D_u^\alpha \phi(t, u)$ in equation (6) gives

$$\begin{aligned} u(t+h) &= u(t) + \frac{h^\alpha}{\Gamma(\alpha+1)} \phi(t, u(t)) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad \cdot \{D_t^\alpha \phi(t, u) + \phi(t, u) D_u^\alpha \phi(t, u)\} + \dots. \end{aligned} \quad (7)$$

Rearranging the abovementioned equation, we have

$$\begin{aligned} u(t+h) &= u(t) + \frac{h^\alpha}{2\Gamma(\alpha+1)} \phi(t, u(t)) + \frac{h^\alpha}{2\Gamma(\alpha+1)} \\ &\quad \cdot \left\{ \phi(t, u(t)) + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} D_t^\alpha \phi(t, u) + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \right. \\ &\quad \cdot \phi(t, u) D_u^\alpha \phi(t, u) \left. \right\} + \dots. \end{aligned} \quad (8)$$

It can also be written as

$$\begin{aligned} u(t+h) &= u(t) + \frac{h^\alpha}{2\Gamma(\alpha+1)} \phi(t, u(t)) + \frac{h^\alpha}{2\Gamma(\alpha+1)} \\ &\quad \cdot \phi\left(t + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}, u(t) + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \phi(t, u)\right). \end{aligned} \quad (9)$$

In view of the abovementioned expression, the following formula is the 2-stage fractional Runge–Kutta method.

$$u_{n+1} = u_n + \frac{h^\alpha}{2\Gamma(\alpha+1)} \{K_1 + K_2\}, \quad (10)$$

where

$$\begin{aligned}
K_1 &= \phi(t_n, u_n), \\
K_2 &= \phi\left(t_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}, u_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} \phi(t_n, u_n)\right).
\end{aligned}
\tag{11}$$

One can easily verify that if $\alpha = 1$, then the fractional order Runge–Kutta method 2.5 reduced to the classical Runge–Kutta method of order 2.

3. Numerical Examples

To understand the methodology to apply the fractional Runge–Kutta method, we have solved three examples and also made a comparison with the exact solution.

Example 1. In the first example, we consider the inhomogeneous linear fractional differential equation

$$D^\alpha u(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} t^{1-\alpha} - u(t) + t^2 - t, \tag{12}$$

$$\begin{aligned}
K_1 &= \frac{2}{\Gamma(3-\alpha)} (t_n)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} (t_n)^{1-\alpha} - u_n + t_n^2 - t_n, \\
K_2 &= \frac{2}{\Gamma(3-\alpha)} \left(t_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^{2-\alpha} - \frac{1}{\Gamma(2-\alpha)} \left(t_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^{1-\alpha} \\
&\quad - \left(u_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} K_1\right) + \left(t_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right)^2 - \left(t_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)}\right).
\end{aligned}
\tag{16}$$

Figure 1 expresses the numerical solutions of equation (12) for different values of α using the fractional Runge–Kutta method. Here, we can easily visualize in Table 1 that when we put $\alpha = 1$ the approximate solution coincide with the exact solution $u(t) = t^2 - t$. In Table 2, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, in Figure 2, hidden effects are visible by changing the values of α which cannot be obtained by using integer order derivative. Accuracy will be improved by using the small mesh size.

Example 2. Consider the nonlinear fractional differential equation

$$D^\alpha u(t) = (u(t))^2 - \frac{2}{(t+1)^2}, \tag{17}$$

along with the conditions

$$\begin{aligned}
u(0) &= -2; \\
0 < \alpha &\leq 1; \\
t &> 0.
\end{aligned}
\tag{18}$$

The exact solution of equation (17) for $\alpha = 1$ is given by

subject to the conditions

$$\begin{aligned}
u(0) &= 0; \\
0 < \alpha &\leq 1; \\
t &> 0;
\end{aligned}
\tag{13}$$

with the exact solution

$$u(t) = t^2 - t. \tag{14}$$

By using the fractional R–K Method, we obtain the iterative relation for equation (12).

$$u_{n+1} = u_n + \frac{h^\alpha}{2\Gamma(\alpha+1)} (K_1 + K_2), \tag{15}$$

where

$$u(t) = -\frac{2}{t+1}. \tag{19}$$

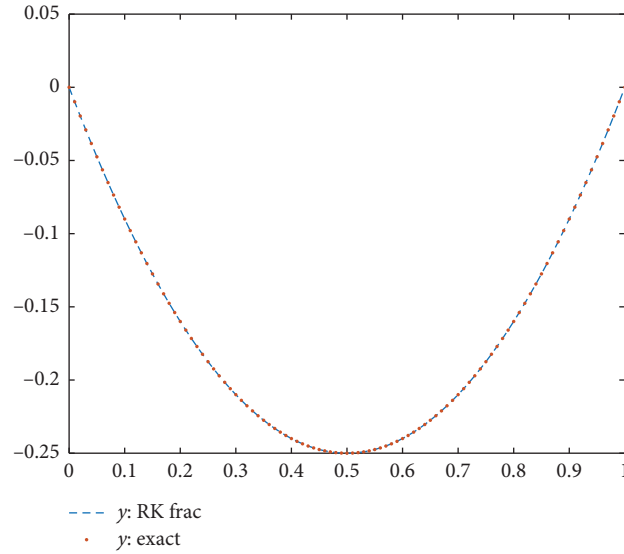
By using the fractional R–K method, we get the iterative relation for equation (12).

$$u_{n+1} = u_n + \frac{h^\alpha}{2\Gamma(\alpha+1)} (K_1 + K_2), \tag{20}$$

where

$$\begin{aligned}
K_1 &= u_n^2 - \frac{2}{(t_n+1)^2}, \\
K_2 &= \left(u_n + \frac{2h^\alpha \Gamma(\alpha+1)}{\Gamma(2\alpha+1)} K_1\right)^2 \\
&\quad - \frac{2}{(t_n + (2h^\alpha \Gamma(\alpha+1)/\Gamma(2\alpha+1)) + 1)^2}.
\end{aligned}
\tag{21}$$

Figures 3 and 4 show numerical solutions of equation (17) for different values of α using the fractional Runge–Kutta method. We can see in Table 3 that when $\alpha = 1$, the approximate solution has excellent agreement with the exact solution $u(t) = -(2/t+1)$. In Table 4, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, the

FIGURE 1: Numerical results of Example 1 for $\alpha = 1$ having discretization $h = 0.01$, respectively.TABLE 1: Numerical results of Example 1 for $\alpha = 1$, with discretization $h = 0.01$.

T	y_{exact}	y_{approx}	AbsError($\alpha = 1$)
0.0	0	0	0
0.1	-0.0900	-0.0900	$4.7820e-06$
0.2	-0.1600	-0.1600	$9.1089e-06$
0.3	-0.2100	-0.2100	$1.3024e-05$
0.4	-0.2400	-0.2400	$1.6567e-05$
0.5	-0.2500	-0.2500	$1.9772e-05$
0.6	-0.2400	-0.2400	$2.2673e-05$
0.7	-0.2100	-0.2100	$2.5297e-05$
0.8	-0.1600	-0.1600	$2.7672e-05$
0.9	-0.0900	-0.0900	$2.9820e-05$
1.0	0	$3.1765e-05$	$3.1765e-05$

TABLE 2: Numerical results of Example 1 for $\alpha = 0.96$, with discretization $h = 0.01$.

T	y_{exact}	y_{approx}	AbsError($\alpha = 0.96$)
0.0	0	0	0
0.1	-0.0900	-0.0912	0.0012
0.2	-0.1600	-0.1700	0.0100
0.3	-0.2100	-0.2274	0.0174
0.4	-0.2400	-0.2626	0.0226
0.5	-0.2500	-0.2752	0.0252
0.6	-0.2400	-0.2650	0.0250
0.7	-0.2100	-0.2322	0.0222
0.8	-0.1600	-0.1766	0.0166
0.9	-0.0900	-0.0985	0.0085
1.0	0	0.0021	0.0021

hidden nonlinearity effects are also visible in Table 2 by changing the value of α . Accuracy will be improved by using the small mesh size.

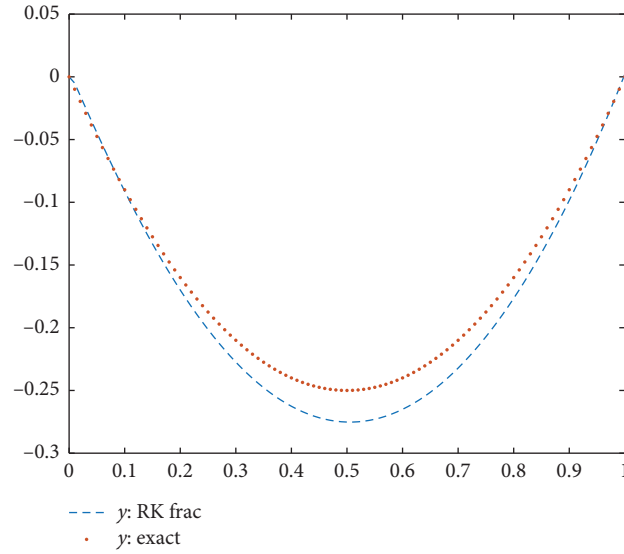
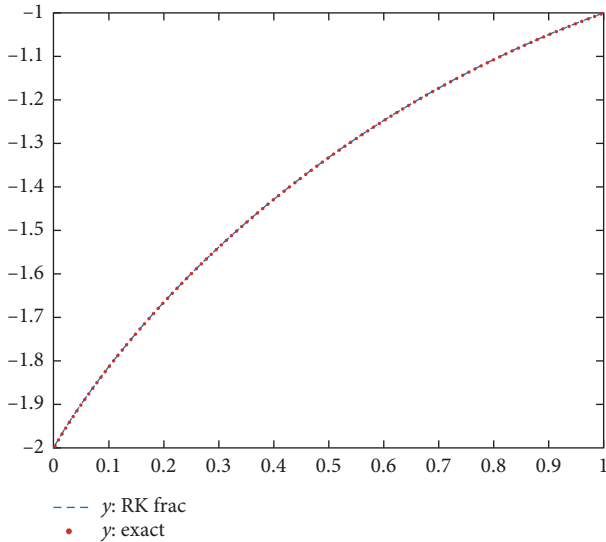
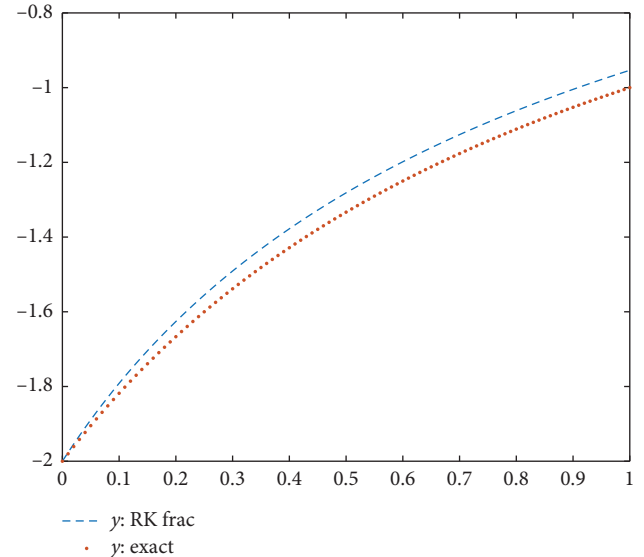
Example 3. Time-Fractional Logistic Growth Model

We consider the time-fractional logistic growth model represented by the equation

$$\frac{d^\alpha P}{dt^\alpha} = rP \left(1 - \frac{P}{M}\right); \quad (22)$$

$$P(t_0) = P_0,$$

where P_0 is the initial density of the population, r is intrinsic growth rate of the population, and M is the

FIGURE 2: Numerical results of Example 1 for $\alpha = 0.96$ having discretization $h = 0.01$, respectively.FIGURE 3: Numerical results of Example 2 for $\alpha = 1$, with mesh size $h = 0.01$.FIGURE 4: Numerical results of Example 2 for $\alpha = 0.96$, with mesh size $h = 0.01$.

carrying capacity. The analytical solution of equation (22) is given by

$$P = \frac{MP_0}{P_0 + (M - P_0)e^{-rt}}. \quad (23)$$

In the review of the fractional Runge–Kutta method, we have

$$P_{n+1} = P_n + \frac{h^\alpha}{2\Gamma(\alpha + 1)} \{K_1 + K_2 d\}, \quad (24)$$

where

TABLE 3: Numerical results of Example 2 for $\alpha = 1$, with discretization $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError ($\alpha = 1$)
0.0	-2	-2	0
0.1	-1.8182	-1.8182	$2.0884e-05$
0.2	-1.6667	-1.6667	$2.9468e-05$
0.3	-1.5385	-1.5385	$3.2069e-05$
0.4	-1.4286	-1.4286	$3.1769e-05$
0.5	-1.3333	-1.3334	$3.0117e-05$
0.6	-1.2500	-1.2500	$2.7902e-05$
0.7	-1.1765	-1.1765	$2.5530e-05$
0.8	-1.1111	-1.1111	$2.3203e-05$
0.9	-1.0526	-1.0527	$2.1018e-05$
1.0	-1	-1	$1.9014e-05$

TABLE 4: Numerical results of Example 2 for $\alpha = 0.96$, with discretization $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError ($\alpha = 0.96$)
0.0	-2	-2	0
0.1	-1.8182	-1.7914	0.0268
0.2	-1.6667	-1.6260	0.0406
0.3	-1.5385	-1.4909	0.0476
0.4	-1.4286	-1.3778	0.0507
0.5	-1.3333	-1.2816	0.0518
0.6	-1.2500	-1.1984	0.0516
0.7	-1.1765	-1.1258	0.0507
0.8	-1.1111	-1.0617	0.0494
0.9	-1.0526	-1.0046	0.0480
1.0	-1	-0.9535	0.0465

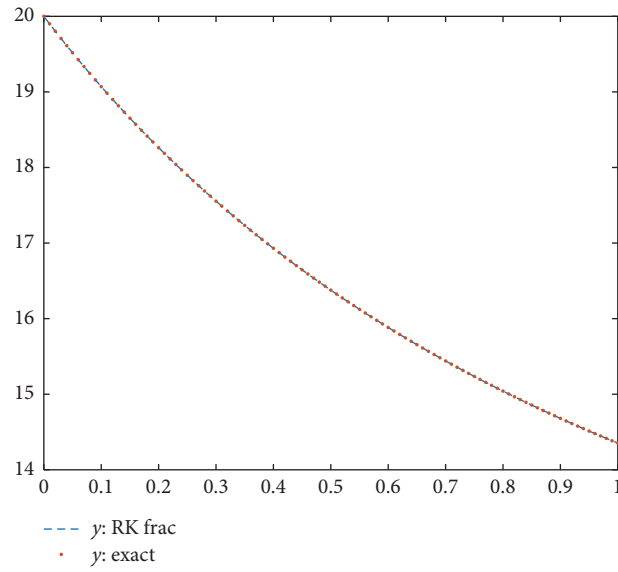
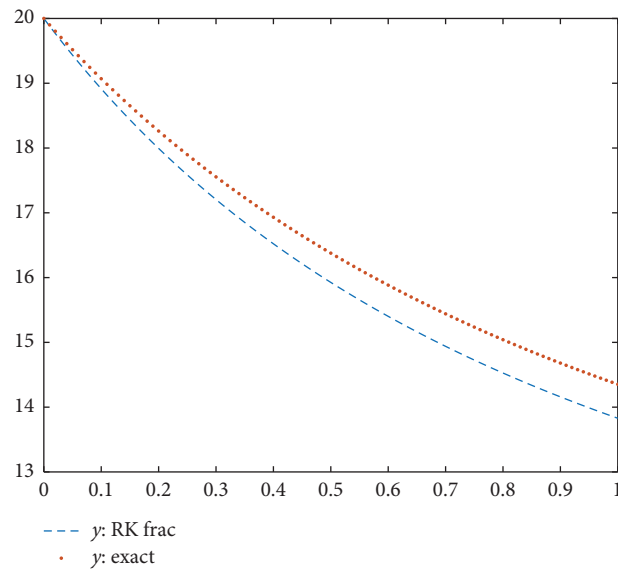
FIGURE 5: Numerical results of fractional logistic growth model for $\alpha = 1$; $r = 0.5$; and $M = 10$ with mesh size $h = 0.01$.FIGURE 6: Numerical results of fractional logistic growth model for $\alpha = 0.96$; $r = 0.5$; and $M = 10$ with mesh size $h = 0.01$.

TABLE 5: Numerical results of Example 3 for $\alpha = 1$, with mesh size $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError($\alpha = 1$)
0.0	20	20	0
0.1	19.0699	19.0700	$2.3897e-05$
0.2	18.2621	18.2622	$3.9362e-05$
0.3	17.5548	17.5548	$4.9190e-05$
0.4	16.9309	16.9310	$5.5198e-05$
0.5	16.3773	16.3774	$5.8593e-05$
0.6	15.8833	15.8834	$6.0190e-05$
0.7	15.4403	15.4404	$6.0546e-05$
0.8	15.0412	15.0413	$6.0048e-05$
0.9	14.6803	14.6803	$5.8968e-05$
1.0	14.3527	14.3527	$5.7498e-05$

TABLE 6: Numerical results of Example 3 for $\alpha = 0.96$, with mesh size $h = 0.01$.

t	y_{exact}	y_{approx}	AbsError($\alpha = 0.96$)
0.0	20	20	0
0.1	19.0699	18.9146	0.1553
0.2	18.2621	17.9941	0.2680
0.3	17.5548	17.2048	0.3500
0.4	16.9309	16.5217	0.4093
0.5	16.3773	15.9256	0.4518
0.6	15.8833	15.4018	0.4816
0.7	15.4403	14.9386	0.5017
0.8	15.0412	14.5269	0.5144
0.9	14.6803	14.1590	0.5213
1.0	14.3527	13.8288	0.5238

$$K_1 = rP_n \left(1 - \frac{P_n}{M} \right),$$

$$K_2 = \frac{rP_n}{M} \left\{ 1 + \frac{2rh^\alpha \Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} - \frac{2rP_n h^\alpha \Gamma(\alpha + 1)}{M\Gamma(2\alpha + 1)} \right\} \cdot \left\{ M - P_n \left\{ 1 + \frac{2rh^\alpha \Gamma(\alpha + 1)}{\Gamma(2\alpha + 1)} - \frac{2rP_n h^\alpha \Gamma(\alpha + 1)}{M\Gamma(2\alpha + 1)} \right\} \right\}. \quad (25)$$

Figures 5 and 6 demonstrate the approximate solutions of fractional Logistic Growth Model represented by equation (22) for different values of α using the fractional Runge–Kutta method.

Table 5 shows that when we put $\alpha = 1$, the approximate solution has excellent agreement with the exact solution given in equation (23). In Table 6, we can further analyze the solutions of the problem for $\alpha = 0.96$. Moreover, we can get better accuracy by using the small mesh size.

4. Conclusions

The fundamental objective of this research is to construct the numerical scheme to solve fractional differential equations. The objective has been achieved by implementing the fractional numerical method (fractional Runge–Kutta method). The derivation of the method is also presented. The method is a new contribution and is reliable to find the

solutions of problems which arise in applied sciences. The comparison of numerical results has been made with exact solutions. The proposed method is useful to derive the higher order family of fractional Runge Kutta Methods. Finally, the recent development in the field of fractional differential equations in applied mathematics makes it needed to implement on such equations to get the numerical solutions. We are hoping that this work is the active contribution in this direction.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

Point-Symmetric Extension-Based Interval Shannon-Cosine Spectral Method for Fractional PDEs

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The approximation accuracy of the wavelet spectral method for the fractional PDEs is sensitive to the order of the fractional derivative and the boundary condition of the PDEs. In order to overcome the shortcoming, an interval Shannon-Cosine wavelet based on the point-symmetric extension is constructed, and the corresponding spectral method on the fractional PDEs is proposed. In the research, a power function of cosine function is introduced to modulate Shannon function, which takes full advantage of the waveform of the *Shannon* function to ensure that many excellent properties can be satisfied such as the partition of unity, smoothness, and compact support. And the interpolative property of Shannon wavelet is held at the same time. Then, based on the point-symmetric extension and the general variational theory, an interval Shannon-Cosine wavelet is constructed. It is proved that the first derivative of the approximated function with this interval wavelet function is continuous. At last, the wavelet spectral method for the fractional PDEs is given by means of the interval Shannon-Cosine wavelet. By means of it, the condition number of the discrete matrix can be suppressed effectively. Compared with Shannon and Shannon-Gabor wavelet quasi-spectral methods, the novel scheme has stronger applicability to the shockwave appeared in the solution besides the higher numerical accuracy and efficiency.

1. Introduction

In recent years, fractional calculus has been attracting more and more researchers in different fields of science and engineering and has been theoretically developed quickly over the last two decades [1–4]. It has been proved that the fractional-order differential equation models are more consistent with the biological phenomena [5] and hydrodynamics [6–8] than those of integer-orders. The Caputo and Riemann–Liouville fractional derivatives are the classical definition, and both of them have a kernel with singularity. To solve the problem of singular kernel, Caputo and Fabrizio proposed a derivative with fractional order based on the exponential function; their derivative in fact does not have singular kernel. In order to overcome the shortcoming of the nonlocal property of the exponential function, Ravichandran et al. [9–12] proposed two generalized fractional derivatives in Caputo and Riemann–Liouville sense [13, 14].

Despite a few special fraction PDEs having analytical solution [15], most of them should be solved by the

numerical method. The solution of the fractional PDEs is sensitive to the iterative step, and so it is disabled to be solved by the traditional numerical method directly. Al-Mdallal has made outstanding contributions in this field, who has proposed many effective algorithms for solving the fractional PDEs, such as the fractional-order Legendre-collocation method [16] and fractional-Legendre spectral Galerkin method [17]. This inspired this work to try to construct a numerical method by means of the Shannon wavelet theory.

Sinc is a famous sampling function, but Shannon's reconstruction formula is rarely used in practice because of the slow decay of the *Shannon* function [18]. Taking the window functions to modulate the sinc function is the common method to improve its decay rate. Many windows have been proposed such as rectangular window, Bartlett window, Hanning window, Hamming window, and Blackman window.

In recent years, wavelet analysis theory has been developed to be a powerful tool to solve the fractional partial

differential equations in recent years [19–21]. Shannon wavelets have been constructed based on the sinc function. A complex Shannon wavelet is defined by

$$\psi(x) = \sqrt{f_b} \sin c(f_b x) e^{2i\pi f_c x}, \quad (1)$$

where f_b is the bandwidth and f_c is the wavelet center frequency. Hoffman et al. [22] have presented the Shannon-Gabor wavelet as follows:

$$\phi_G(x) = \frac{\sin(\pi x)}{\pi x} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \sigma > 0, \quad (2)$$

where σ is the width parameter (or called window size).

Both of the two Shannon-type wavelets are obtained by taking the Gaussian window to modulate the sinc function. The presence of the Gaussian window destroys the normative property possessed by the Shannon wavelet, that is,

$$\hat{\phi}_G(0) = \int_{-\infty}^{\infty} \phi_G(x) dx = \operatorname{erf}\left(\frac{\pi\sigma}{\sqrt{2}}\right), \quad (3)$$

where $\operatorname{erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$ is the error function. Obviously, $\hat{\phi}_G(0)$ is always less than unity except at the limit of $\sigma \rightarrow \infty$. This is the reason why the windowed *Shannon* wavelet is not recommended. They fail to satisfy the partition of unity; this has the disturbing consequence that the reconstruction error will not vanish as the sampling step tends to zero.

Similar to Haar wavelet [23–25], B-Spline wavelet [26, 27], and Legendre wavelet [28, 29], *Shannon* wavelet [30, 31] possesses almost all the excellent numerical properties such as interpolative, relative sparse, and orthogonal properties besides the compact support property. Therefore, it is necessary to construct a novel window for sinc function, which can satisfy the partition of unity, so that it can be utilized to solve fractional PDEs efficiently [32]. In order to overcome the disadvantages in the Shannon-Gabor wavelet, a Shannon-Cosine wavelet is constructed by Mei et al. [32, 33], in which the waveform is used to meet the requirement of the partition of unity. Unfortunately, there are too many parameters in the Shannon-Cosine wavelet function to be identified, and this makes the expression to be very complex. In this paper, a simplified Shannon-Cosine wavelet function is proposed and the corresponding interval wavelet is constructed based on the point-symmetric extension [34, 35]. And then, the interval Shannon-Cosine wavelet is employed to construct a wavelet spectral method for the fractional PDEs.

2. Simplified Shannon-Cosine Scaling Function

As a basis function $\phi(x) = \sin c(x)$ defined in Hilbert space, it could form a basic approximation space V as

$$V(\phi) = \left\{ s(x) = \sum_{k \in \mathbb{Z}} c(k) \phi(x - k), c \in l_2 \right\}. \quad (4)$$

This means that any function $s(x) \in V(\phi)$ is characterized by a sequence of coefficients $c(k)$. The coefficients $c(k)$ are the

samples of the signal, and that $\phi(x)$ is a kind of weight function. Therefore, it should meet the requirements as follows:

- ① The coefficients sequence should be square-summable: $c \in l_2$.
- ② The family of functions $\{\phi_k = \phi(x - k)\}_{k \in \mathbb{Z}}$ should form a Riesz basis of $V(\phi)$. This ensures that the representation is stable and unambiguously defined. The Riesz basis requirement has an equivalent expression in the Fourier domain:

$$A \leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + 2k\pi)|^2 \leq B, \quad (5)$$

where $\hat{\phi}(\omega) = \int \phi(x) e^{-j\omega x} dx$ is the Fourier transform of $\phi(x)$.

- ③ $\phi(x)$ should satisfy the partition of unity condition $\int_{-\infty}^{\infty} \phi(x) dx = 1$.

Sinc is a famous sampling function [20] based on approximating the Dirac delta function as a band-limited function and is given by

$$\phi(x) = \frac{\sin(\pi x)}{\pi x}. \quad (6)$$

In order to overcome the shortcomings of the Shannon-Gabor function, we introduce the parametric cosine function instead of the exponential function to modulate Shannon function as follows:

$$S_C(x) = \frac{\sin(\pi x)}{\pi x} \cos^{2m}\left(\frac{\pi x}{N}\right) \left[\chi\left(x + \frac{N}{2}\right) - \chi\left(x - \frac{N}{2}\right) \right], \quad (7)$$

where N is a constant related to the support domain. $\chi(x)$ is the Heaviside function defined as follows:

$$\chi(x) = \begin{cases} 0, & x < 0, \\ \text{undefined}, & x = 0, \\ 1, & x > 0. \end{cases} \quad (8)$$

This ensures that new modulate Shannon function (7) is a real compact support function, and the support domain is $[-N/2, N/2]$. The function $S_C(x)$ is named as Shannon-Cosine scaling function.

The comparison between the sinc and the Shannon-Cosine scaling function is shown in Figure 1.

It is easy to prove that $S_C(x)$ has the interpolative property; the coefficients $c(k)$ in equation (4) are the samples of the signal. Therefore, $c(k)$ obtained from any signal $s \in L_2$ satisfies the first requirement.

Based on the Parseval equation, it is easy to understand that any basis function with $\int_{-\infty}^{\infty} \phi(x) dx = 1$ meets the second requirement. This means that the third requirement puts the strongest constraint of the selection on an admissible generating function $\phi(x)$. It is well known that the sinc function satisfies the partition of unity condition, and this results in the Gaussian-windowed sinc function are not meeting this requirement. In the next section, we are going to prove that it can be satisfied by choosing the support domain parameter N .

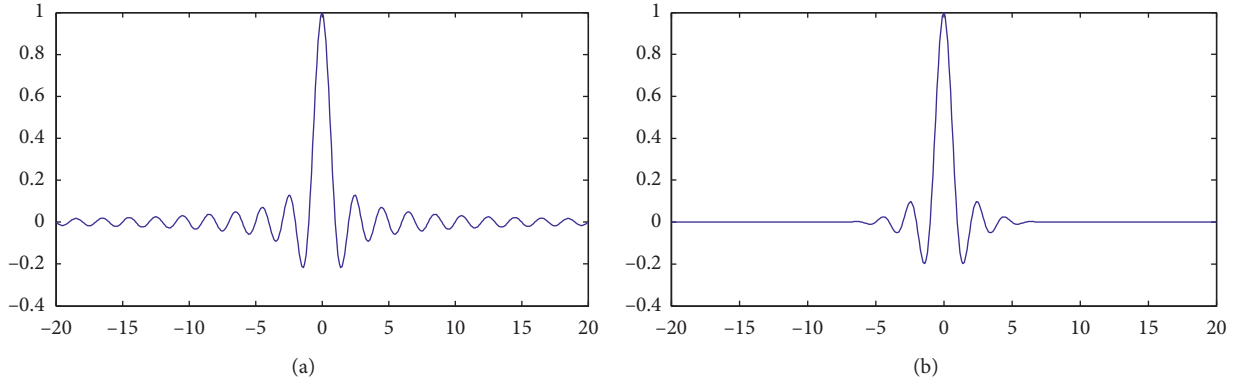


FIGURE 1: Comparison between Shannon and Shannon-Cosine scaling functions. (a) Shannon function. (b) Shannon-Cosine scaling function.

3. Normalization and Choice of the Support Domain Parameter N

Shannon function has the waveform shown in Figure 2; its support domain is $(-\infty, \infty)$ and it meets the normalization condition $\int_{-\infty}^{\infty} \phi(x) dx = 1$. It is easy to notice that the integration value of Shannon function in domain I_1 is larger than 1 and smaller than 1 in domain I_2 . This reminds us that the reasonable choice of the support domain can ensure that the Shannon-Cosine scaling function satisfied the partition of unity condition. Theorem 1 reveals the relation between the support domain parameter N and the partition of unity condition.

Theorem 1. *The Shannon-Cosine scaling function ($m=3$) satisfies the normalization condition as follows:*

$$\int_{-\infty}^{\infty} S_C(x) dx = 1. \quad (9)$$

Proof. The Fourier transform of $S_C(x)$ is given by

$$\begin{aligned} \widehat{S}_C(\omega) &= \int_{-\infty}^{\infty} S_C(x) e^{-i\omega x} dx \\ &= \frac{1}{2\pi} \sum_{n=0}^3 a_n \left[\text{Si}\left(\frac{1}{2}N \cdot (\pi + \omega) + n\pi\right) \right. \\ &\quad \left. + \text{Si}\left(\frac{1}{2}N \cdot (\pi + \omega) - n\pi\right) + \text{Si}\left(\frac{1}{2}N \cdot (\pi - \omega) + n\pi\right) \right. \\ &\quad \left. + \text{Si}\left(\frac{1}{2}N \cdot (\pi - \omega) - n\pi\right) \right]. \end{aligned} \quad (10)$$

Let $m=3$. Substituting $\omega = 0$ and equation (6) into equation (10), we obtain

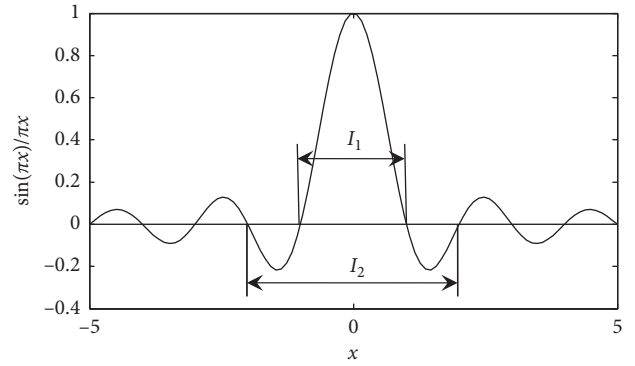


FIGURE 2: Shannon function.

$$\begin{aligned} \widehat{S}_C(0) &= \int_{-\infty}^{\infty} S_C(x) dx \widehat{S}_C(0) \\ &= \frac{1}{\pi} \left[\frac{5}{16} \left(2\text{Si}\left(\frac{1}{2}N\pi\right) \right) + \frac{15}{32} \left(\text{Si}\left(\frac{1}{2}N\pi + \pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - \pi\right) \right) + \frac{3}{16} \left(\text{Si}\left(\frac{1}{2}N\pi + 2\pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - 2\pi\right) \right) + \frac{1}{32} \left(\text{Si}\left(\frac{1}{2}N\pi + 3\pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - 3\pi\right) \right) \right]. \end{aligned} \quad (11)$$

In fact, $\widehat{S}_C(0)$ can be viewed as a continuous function with respect to the real number “ N ,” which is a parameter related to the length of the support domain. Let

$$\begin{aligned}
P(N) &= \widehat{S}_C(0) \\
&= \frac{1}{\pi} \left[\frac{5}{16} \left(2Si\left(\frac{1}{2}N\pi\right) \right) + \frac{15}{32} \left(Si\left(\frac{1}{2}N\pi + \pi\right) \right. \right. \\
&\quad \left. \left. + Si\left(\frac{1}{2}N\pi - \pi\right) \right) + \frac{3}{16} \left(Si\left(\frac{1}{2}N\pi + 2\pi\right) \right) \right. \\
&\quad \left. + Si\left(\frac{1}{2}N\pi - 2\pi\right) \right) + \frac{1}{32} \left(Si\left(\frac{1}{2}N\pi + 3\pi\right) \right. \\
&\quad \left. \left. + Si\left(\frac{1}{2}N\pi - 3\pi\right) \right) \right]. \quad (12)
\end{aligned}$$

Based on the definition of $Si(x)$, we know that $P(N)$ is an oscillator function around the constant 1, that is,

$$\begin{aligned}
P(N) &\begin{cases} >1, & \frac{N}{2} = 2n, \quad n \geq 2, \\ <1, & \frac{N}{2} = 2n+1, \quad n \geq 2, \end{cases} \\
P'(N) &= \frac{dP(N)}{dN} \\
&= -\frac{1440 \sin(N\pi/2)}{\pi N(N^2 - 4)(N^2 - 16)(N^2 - 36)}. \quad (13)
\end{aligned}$$

Obviously,

$$\frac{dP(N)}{dN} = P'(N) \begin{cases} <0, & \frac{N}{2} \in (2n, 2n+1), \quad n > 3, \\ =0, & \frac{N}{2} = 2n, \quad n \in \mathbb{Z}, n \geq 3, \\ >0, & \frac{N}{2} \in (2n+1, 2(n+1)), \quad n > 3. \end{cases} \quad (14)$$

That is, $N = 4n, n \in \mathbb{Z}, n \geq 3$, is in correspondence with the extreme point of the function $P(N)$, which is a

TABLE 1: Solution of $P(N) = 1$.

7.437330346546332	9.323969185188616	11.261199765911442
13.219709116325248	15.189881678437814	17.167295939754695
19.149560460820794	21.135248098522425	23.123447719961405
25.113547757267952	27.105121575295925	29.097861737012863
31.091541469097137	33.085988700389862	35.081073641777039
37.076687097549438	39.072752714157104	41.069202184677124
43.065983057022095	45.063048839569092	47.060366630554199
49.057900428771973	51.055631637573242	53.053528785705566
55.051587104797363	57.049772262573242	59.048089981079102
61.046504974365234	63.045036315917969	65.043693542480469
67.042350769042969	69.041099548339844	

monotone function on the intervals $(2n, 2n+1)$ and $(2n+1, 2n+2)$. Based on the mean value theorem, there must exist unique points $N/2 \in (2n, 2n+1)$ and $N/2 \in (2n, 2n+2)$ so as to $P(N) = 1$, i.e.,

$$P(N) = \widehat{S}_C(0) = \int_{-\infty}^{\infty} S_C(x) dx = 1. \quad (15)$$

This completes the proof. \square

By means of the interval bisection method, it is easy to obtain the value of N which is in correspondence with $P(N) = 1$ as in Table 1.

It is easy to prove that the simplified Shannon-Cosine wavelet is equivalent to the Shannon-Cosine wavelet proposed in [36] using the parameter $m \leq 3$, but only the parameter N should be chosen in applications. So, it is convenient for solving the fractional derivative PDEs.

4. Point-Symmetric Interval Wavelet

In order to eliminate the boundary effect introduced by the wavelet transform, the point-symmetric extension and the general variational theory are employed to construct the interval wavelet. Compared to other extension methods such as the zero extension, symmetric extension, and periodic extension, the point-symmetric extension can ensure that the function is smooth at the endpoint.

Theorem 2. *If the continuous function $f(x)$ is symmetric on point $(x_0, f(x_0))$, then the first-order derivative of the function $f(x)$ is continuous at this point.*

Proof. As the continuous function $f(x)$ is symmetric on point $(x_0, f(x_0))$, let $\Delta x > 0$; we have

$$\begin{aligned}
f(x_0 - \Delta x) &= 2f(x_0) - f(x_0 + \Delta x), \\
\lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{x_0 - \Delta x - x_0} &= \lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} \\
&= \lim_{x_0 - \Delta x \rightarrow x_0} \frac{2f(x_0) - f(x_0 + \Delta x) - f(x_0)}{-\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \\
\lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0} &= \lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.
\end{aligned} \tag{16}$$

Therefore, we have

$$\lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{x_0 - \Delta x - x_0} = \lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0}. \tag{17}$$

This illustrates that the first derivative of the function $f(x)$ continues to reach the point $(x_0, f(x_0))$. This completes the proof.

For convenience, the values of $f(x)$ at $x_{-n}, x_{-n+1}, \dots, x_0, x_1, \dots, x_{2^j+1}$ are denoted as $f_{-n}, f_{-n+1}, \dots, f_0, f_1, \dots, f_{2^j+1}$. Based on the general variational theory, we can deduce the format of the interval wavelet function as follows:

① As $x = -n, -n+1, \dots, -1$,

$$f_{-1} = (2f_0 - f_1)\cos^2(\pi/2n)$$

$$f_{-2} = (2f_0 - f_2)\cos^2(2\pi/2n)$$

\dots

$$f_{-k} = (2f_0 - f_k)\cos^2(k\pi/2n)$$

\dots

$$f_{-N} = (2f_0 - f_N)\cos^2(\pi/2)$$

② As $x = 2^j + 1, 2^j + 2, \dots, 2^j + n$,

$$f_{2^j+1} = (2f_{2^j} - f_{2^j-1})\cos^2(\pi/2n)$$

$$f_{2^j+2} = (2f_{2^j} - f_{2^j-2})\cos^2(2\pi/2n)$$

\dots

$$f_{2^j+k} = (2f_{2^j} - f_{2^j-k})\cos^2(k\pi/2n)$$

\dots

$$f_{2^j+n} = (2f_{2^j} - f_{2^j-n})\cos^2(n\pi/2n)$$

And so, the function $f(x)$ can be expressed as

$$\begin{aligned}
f(x) &= \sum_{k=-n}^{2^j+n} \phi(2^j x - k) f_k^j \\
&= \sum_{k=-n}^{-1} \phi(2^j x - k) (2f_0 - f_{-k}^j) \cos^2\left(\frac{k\pi}{2n}\right) + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j + \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) (2f_{2^j} - f_{2^j-k}^j) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) \\
&= - \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_{-k}^j + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_0 + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\
&\quad + 2 \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j} - \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j+1-k}^j \\
&= - \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_k^j + 2 \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_0 + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\
&\quad + 2 \sum_{k=-2^j-n}^{-2^j-1} \phi(2^j x + k) \cos^2\left(\frac{(k+2^j)\pi}{2n}\right) f_{2^j} - \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j+1-k}^j.
\end{aligned} \tag{18}$$

Let $m = 2^{j+1} - k$; we have

$$\begin{aligned} f(x) = & - \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_k^j + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_0^j + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\ & + 2 \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) f_{2^j}^j - \sum_{m=2^j-n}^{2^j-1} \phi(2^j x - 2^{j+1} + m) \cos^2\left(\frac{(2^j - m)\pi}{2n}\right) f_m^j, \end{aligned} \quad (19)$$

where

So, we have

$$\begin{aligned} \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j &= \phi(2^j x) f_0^j + \sum_{k=1}^n \phi(2^j x - k) f_k^j \\ &+ \sum_{k=n+1}^{2^j-n-1} \phi(2^j x - k) f_k^j + \sum_{k=2^j-n}^{2^j-1} \phi(2^j x - k) f_k^j + \phi(2^j x - 2^j) f_{2^j}^j. \end{aligned} \quad (20)$$

$$\begin{aligned} f(x) = & \left(\phi(2^j x) + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) \right) f_0^j + \sum_{k=1}^n \left(\phi(2^j x - k) - \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) \right) f_k^j \\ & + \sum_{k=N+1}^{2^j-N-1} \phi(2^j x - k) f_k^j + \sum_{k=2^j-N}^{2^j-1} \left(\phi(2^j x - k) - \phi(2^j x - 2^{j+1} + k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) \right) f_k^j \\ & + \left(\phi(2^j x - 2^j) + 2 \sum_{k=2^j+1}^{2^j+N} \phi(2^j x - k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) \right) f_{2^j}^j. \end{aligned} \quad (21)$$

Therefore, the interval interpolative wavelet function can be expressed as

$$w(2^j \mathbf{x} - \mathbf{k}) = w_{j,k} = \begin{cases} \phi(2^j \mathbf{x} - \mathbf{k}) + 2 \sum_{k=1}^P \phi(2^j \mathbf{x} + \mathbf{k}) \cos^2\left(\frac{k\pi}{2P}\right), & \mathbf{k} = 0, \\ \phi(2^j \mathbf{x} - \mathbf{k}) - \phi(2^j \mathbf{x} + \mathbf{k}) \cos^2\left(\frac{k\pi}{2P}\right), & \mathbf{k} = 1, 2, \dots, P, \\ \phi(2^j \mathbf{x} - \mathbf{k}), & \mathbf{k} = N + 1, \dots, 2^j - P - 1, \\ \phi(2^j \mathbf{x} - \mathbf{k}) - \phi(2^j \mathbf{x} - 2^{j+1} + \mathbf{k}) \cos^2\left(\frac{(2^j - k)\pi}{2P}\right), & \mathbf{k} = 2^j - P, \dots, 2^j - 1, \\ \phi(2^j \mathbf{x} - 2^j) + 2 \sum_{k=2^j+1}^{2^j+N} \phi(2^j \mathbf{x} - \mathbf{k}) \cos^2\left(\frac{(2^j - k)\pi}{2P}\right), & \mathbf{k} = 2^j. \end{cases} \quad (22)$$

According to theory 2, it is easy to understand that equation (22) is able to reduce the boundary effect efficiently. In theory, the proposed interval wavelet can prevent the steep shock wave appearing near the boundary, which can introduce the large condition number of the discrete matrix and result in the decrease of the numerical precision.

5. Application in Solving Fractional PDEs

5.1. Interval Wavelet Spectral Method for Fractional Fokker–Planck Equation. The fractional Fokker–Planck equation is a typical fractional PDE, which is often used to describe a subdiffusive behavior of a particle under the combined influence of external nonlinear force field and a Boltzmann thermal heat bath. In the presence of an external force field $F(\mathbf{x}) = -\nu'(x)$, the evolution of a test particle is usually described in terms of the Fokker–Planck equation (FPE):

$$\frac{\partial u(x, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial}{\partial x} \frac{\nu'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad (23)$$

$$a \leq x \leq b, 0 \leq t \leq T,$$

which defines the probability $u(x, t)$ of finding the particle at a certain position x at a given time t . m denotes the mass of the diffusing particle, $K_\alpha > 0$ denotes the generalized diffusion coefficient with dimension $[K_\alpha] = \text{cm}^2 \text{sec}^{-\alpha}$, and η_α is the generalized friction coefficient with dimension $[\eta_\alpha] = \text{sec}^{\alpha-2}$. The corresponding initial condition is

$$u(x, 0) = \varphi(x), \quad a \leq x \leq b, \quad (24)$$

and the boundary conditions are

$$\begin{aligned} u(a, t) &= p_1(t), \\ u(b, t) &= p_2(t), \end{aligned} \quad (25)$$

$$0 < t \leq T.$$

Equation (2) uses the Riemann–Liouville fractional derivative of order $1 - \alpha$, defined by

$${}_0D_t^{1-\alpha} u(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^1 \frac{u(x, \eta)}{(t - \eta)^{1-\alpha}} d\eta, \quad 0 \leq \alpha < 1, \quad (26)$$

where $\Gamma(\alpha)$ is the gamma function.

According to the properties of the Riemann–Liouville fractional derivative, it is easy to know that, if $(x, t) \in C_{x,t}^{2,1}([a, b] \times [0, T])$, equation (2) can be rewritten as follows:

$$D_t^\alpha u(x, t) - \frac{u(x, 0)t^{-\alpha}}{\Gamma(1-\alpha)} = \left[\frac{\partial}{\partial x} \frac{\nu'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad (27)$$

$$a \leq x \leq b, 0 \leq t \leq T.$$

According to the wavelet spectral method, the fractional Fokker–Planck equation can be approximately represented as

$$\begin{aligned} & \tau^{-\alpha} \left[u_j(x_i, t_n) + \sum_{k=1}^{n-1} g_k u_j(x_i, t_{n-k}) - \sum_{k=0}^{n-1} g_k u_j(x_i, t_0) \right] \\ &= f'(x_i) u_j(x_i, t_n) + \sum_{m=0}^{2^j} u_j(x_m, t_n) [w'(x_i - x_m) \\ & \quad + K_\alpha w''(x_i - x_m)], \end{aligned} \quad (28)$$

where $i = 0, 1, 2, \dots, 2^j$. Let

$$\begin{aligned} V_j^n &= (u_j(x_0, t_n), u_j(x_1, t_n), \dots, u_j(x_{2^j}, t_n))^T, \\ F &= \text{diag}(f'(x_0), f'(x_1), \dots, f'(x_{2^j})), \\ W_1 &= \begin{bmatrix} w'(x_0 - x_0) & w'(x_0 - x_1) & \cdots & w'(x_0 - x_{2^j}) \\ w'(x_1 - x_0) & w'(x_1 - x_1) & \cdots & w'(x_1 - x_{2^j}) \\ \vdots & \vdots & \ddots & \vdots \\ w'(x_{2^j} - x_0) & w'(x_{2^j} - x_1) & \cdots & w'(x_{2^j} - x_{2^j}) \end{bmatrix}, \\ W_2 &= \begin{bmatrix} w''(x_0 - x_0) & w''(x_0 - x_1) & \cdots & w''(x_0 - x_{2^j}) \\ w''(x_1 - x_0) & w''(x_1 - x_1) & \cdots & w''(x_1 - x_{2^j}) \\ \vdots & \vdots & \ddots & \vdots \\ w''(x_{2^j} - x_0) & w''(x_{2^j} - x_1) & \cdots & w''(x_{2^j} - x_{2^j}) \end{bmatrix}. \end{aligned} \quad (29)$$

Then, the system of (28) can be expressed in the matrix format:

$$(W_1 + K_\alpha W_2 + F - \tau^{-\alpha} I) V_j^n = \sum_{k=1}^{n-1} g_k V_j^{n-k} - \sum_{k=0}^{n-1} g_k V_j^0. \quad (30)$$

5.2. Numerical Experiments. In this section, a simple Fokker–Planck equation with the exact analytical solution is taken as the example to illustrate the effectiveness. Consider the Fokker–Planck equation as follows:

$$\frac{\partial u(x, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial}{\partial x} (-1) + \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad 0 \leq x \leq 1, t > 0. \quad (31)$$

With the initial condition

$$u(x, 0) = x(1 - x), \quad 0 \leq x \leq 1, \quad (32)$$

the boundary conditions are

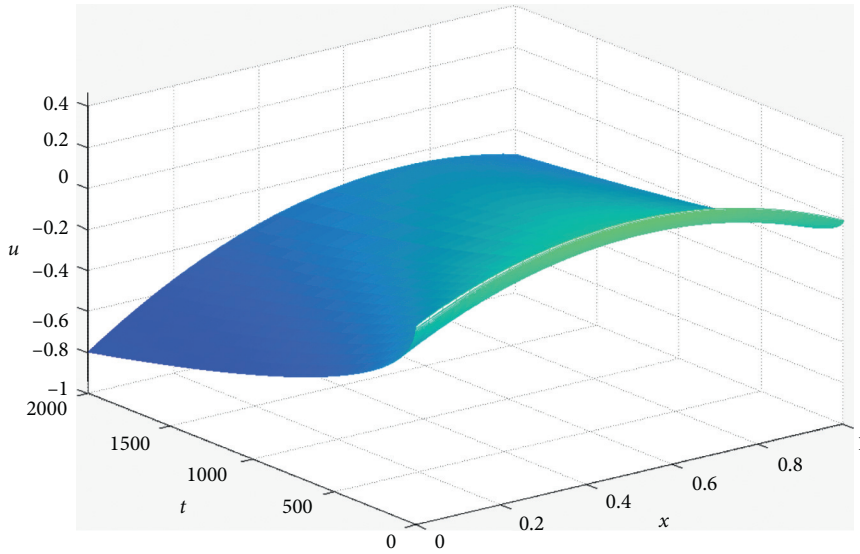
$$\begin{aligned} u(0, t) &= -\frac{3t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad t > 0, \\ u(1, t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad t > 0. \end{aligned} \quad (33)$$

The exact analytic solution is

$$u(x, t) = x(1 - x) + (2x - 3) \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (34)$$

TABLE 2: Influence of α on the numerical precision ($t = 0.0001$, $T = 0.1$).

j	α	e_1			e_2	
		Point-symmetric interval WPM	Lagrange interval WPM ($L = 2$)	Dynamic interval WPM	Lagrange interval WPM ($L = 2$)	Dynamic interval WPM
6	0.8	0.0012	6.2776×10^{-6}	3.5310×10^{-4}	9.5851×10^{-4}	4.9985×10^{-6}
	0.6	0.0079	9.6280×10^{179}	0.0096	0.0066	0.0076
	0.4	0.0509	0	0.0588	0.0433	0.0499
	0.2	0.3488	0	0.3962	0.2909	0.3284
7	0.8	0.0012	1.9290×10^{155}	0.0014	9.9232×10^{-4}	inf ⁴
	0.6	0.008	0	0.0092	0.0067	0.0077
	0.4	0.0511	0	0.0586	0.0437	0.0502
	0.2	0.3503	0	0.3932	0.2936	0.3282

FIGURE 3: Trend of the solution with t .

All the comparisons in this section are made qualitatively by comparing the calculation precision in the same time step and space mesh grid size. The first measure of error e_1 is given by

$$e_1 = \|V_j^n - V_{\text{exact}}^n\|_{\infty}. \quad (35)$$

It provides a measure of the accuracy of the solution near the boundary. The second measure of error e_2 is given by

$$e_2 = \sqrt{\frac{1}{2^j + 1} \sum_{i=0}^{2^j} (u(x_i) - u_{\text{exact}}(x_i))^2}. \quad (36)$$

It provides a general measure of the accuracy of the solution over the main body of the distribution and was often used to investigate the accuracy of the FEM.

In [24], the dynamic interval wavelet spectral method (WPM) is employed to solve the fractional Fokker–Planck equation, in which the Lagrange interpolation-based interval wavelet spectral method is taken to compare with their method. Compared to these two methods, as shown in Table 2, the proposed method is more robust and

insensitive to the parameters j and α . With the increase in the discrete point amount and decrease in the parameter α , the numerical precision of the proposed method is better than of the dynamic interval WPM.

It should be noticed that the numerical precision of the dynamic interval WPM is better than the proposed method as $j = 6$ and $\alpha = 0.8$. The amount of the discrete points is 2^j . This denotes that the numerical precision obtained by fewer discrete points is better than that obtained by more points. This is obviously unreasonable.

In fact, the dynamic interval wavelet function is constructed based on the Lagrange interpolation. The condition number of the discrete matrix is becoming very large with the increase in the Lagrange polynomial degree [36]. This is the primary reason why the numerical precision with fewer discrete points is better than more points by the dynamic interval WPM. Theorem 2 illustrates that the point-symmetric interval wavelet method can overcome this shortcoming of the dynamic interval wavelet as it does not have the steep wave close to the boundary point introduced by the Lagrange interpolation.

The trend of the solution with the parameter “ t ” obtained by the proposed method is shown in Figure 3. This illustrates

that the point-symmetric interval WPM is a robust method for the fractional PDEs.

6. Conclusions

By means of the waveform of the sinc function, a family of simplified Shannon-Cosine scaling functions is presented, which can be utilized to construct the wavelet spectral method for solving the fractional PDEs, combining with the point-symmetric extension, the solutions of which are permitted to have different smoothness. Compared with the sinc and Shannon-Gabor functions, the Shannon-Cosine scaling functions possess almost all the excellent numerical properties such as the compact support, interpolation, and derivability. The point-symmetric extension-based interval wavelet is a basis with robust properties, which can prevent the increase in the condition number introduced by the Lagrange interpolation. Besides, the proposed interval wavelet need not choose the smoothness of the approximation function near the boundary points, and this is helpful to improve the efficiency of the algorithm. Based on the choice scheme of the parameters appeared in the simplified Shannon-Cosine wavelet function, the adaptability of the spectral method to the smoothness of the solution can be improved greatly at even fewer collocation points. Compared with the Shannon-Cosine wavelets, there are fewer parameters in the simplified wavelet function, and this brings us a lot of conveniences in solving PDEs.

Data Availability

The MATLAB source code supporting the findings of this study has been deposited in the GitHub repository (<https://github.com/meishuli/meishuli/tree/Matlab-source-code>).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Existence Theory and Novel Iterative Method for Dynamical System of Infectious Diseases

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This manuscript is devoted to investigate qualitative theory of existence and uniqueness of the solution to a dynamical system of an infectious disease known as measles. For the respective theory, we utilize fixed point theory to construct sufficient conditions for existence and uniqueness of the solution. Some results corresponding to Hyers–Ulam stability are also investigated. Furthermore, some semianalytical results are computed for the considered system by using integral transform due to the Laplace and decomposition technique of Adomian. The obtained results are presented by graphs also.

1. Introduction

In recent time, the subject of fractional calculus has got much attention from the researcher. This is due to large numbers of applications in various disciplines of science and engineering where the concept of derivatives and integrals is frequently used. Normally, the integer-order derivative does not explore the dynamics of real-world problems more comprehensively as compared to the fractional derivative. Also, the fractional differential operator is global and poses greater degree of freedom, while the ordinary differential operator is local and often cannot explain the memory and hereditary process of the real-world problem more efficiently. In fact, fractional derivative is definite integral over a domain; therefore, it has been defined in many ways. In this regard, a variety of definitions have been introduced by different researchers in the literature. Some famous definitions are those given by Riemann and Liouville, Caputo, Hadamard, and so on (see [1]). The definitions of Riemann–Liouville and Caputo have been very well used in applied problems. These definitions involve singular kernel which often causes difficulty in dealing some problems. Therefore, Caputo and Fabrizio in 2015 introduced a new concept about fractional-order derivatives based on nonsingular kernel (see [2–4]). Some remarkable merits of

Caputo–Fabrizio fractional derivative and integral and their applications were given by many researchers in previous few years (see, for detail, [5–13]). The concerned fractional integral of a function is the fractional average of the function itself and its fractional integral in Riemann–Liouville sense. Moreover, in some articles, it has been displayed that the derivative has some constructive applications in thermal science, material sciences, and so on (see, for detail, [11, 12, 14, 15]). Since differential operators have greater degree of freedom, therefore, to find the exact solution to each and every problem is quite difficult. In this regard, great motivation has been observed in the last two decades to establish best tools to handle such problems. One of the important techniques is to find analytical approximate solutions to many nonlinear problems of FODEs. For this purpose, the usual decompositions, perturbation, and integral transform methods were greatly utilized to investigate ordinary differential and integral equations. The mentioned techniques have been very well explored for fractional differential equations (see [16–25]). One of the powerful methods which has been used very frequently in the past is due to the Laplace Adomian decomposition method. For usual FODEs, the mentioned method has been used very regularly in the literature. However, to the best of our information, the aforesaid method is very rarely used for FODEs involving nonsingular kernel, see [26, 27].

Here, we remark that mathematical models are the powerful tools to study various dynamical problems of physical and biological sciences. The concept was initiated by Bernoulli in 1776. However, a formal mathematical model of three compartments was constructed in 1927 by McKendrick and his co-author called the SIR mode. Later on, the subject of mathematical modeling was extended to infectious diseases. Because mathematical models of the biological problem have become powerful tools to understand various infectious diseases, the proper method has to be developed to control the disease or minimize its transmission in the society. From ancient time, measles disease is one of the most threatful diseases. It was a big reason for children death in the past. This dangerous disease was caused by germs called morbilliform. Measles disease spreads, when an infected person coughs or sneezes, because its virus can live for up to two hours in an airspace where the infected person coughed or sneezed. Measles-infected individuals can transfer their germs to the other people 4–8 days before and after the skin eruptions start. It transmits a disease in young babies up to 30–40 million every year. Measles appears once and is present for a long time in life for immunity. Symptoms of the disease include runny nose, high temperature, coughing, and spots on the whole body and in highly complicated cases, ear infections, diarrhea, and pneumonia. The vaccination of measles has been used to control disease in kids. Some vaccinated individuals could remain susceptible individuals when vaccination gets failed. Worldwide vaccination reduced 80% death caused by measles between 2000 and 2017. However, measles disease is still familiar disease in highly developing countries of Asia and Africa particularly due to the lack of proper treatment of this infectious disease. For this purpose, a massive research has been carried out to enhance the understanding of the virus of measles dynamics in various areas. For example, the authors in [28–33] discussed the global stability of the model with five compartments as

$$\begin{cases} \frac{du}{dt} = \Lambda(1-q) - \beta u(x+y) + \gamma v - \mu u, \\ \frac{dv}{dt} = \Lambda(q) + \mu u - \gamma x - \mu x - \omega x, \\ \frac{dx}{dt} = \beta u(v+y) - \mu v - \alpha v - \mu_2 v, \\ \frac{dy}{dt} = \alpha v + \delta y - \mu y - \mu_3 y, \\ \frac{dz}{dt} = \mu_2 v + \omega x + \mu_3 x - \mu z, \end{cases} \quad (1)$$

where (u) represents the susceptible, (v) represents the vaccinated, (x) represents the exposed, (y) represents the infected, and (z) represents the recovered individuals. The description of the parameters is given in the analytical section. These models have been investigated corresponding to ordinary and usual fractional-order derivatives. Furthermore, the researchers have given the global and local dynamics by computing the basic reproductive numbers. Here, we investigate the given model under the nonsingular fractional derivative of Caputo and Fabrizio (CFFD) from other perspectives including the qualitative analysis by using fixed point approach. Further stability is a required aspect in the dynamical problem. Since we are going to derive approximate solutions, therefore, in this regard, Hyers–Ulam-type stability results are investigated. The mentioned stability has been very well studied for the general problem of FODEs, see, for detail, [5, 34–36]. Also, the analytical results are investigated through the Laplace Adomian decomposition method. We considered model (1) under the CFFD with fractional order $\eta \in (0, 1]$ as

$$\begin{cases} {}^{CF}D_t^\eta(u)(t) = \Lambda(1-q) - \beta u(v+y) + \gamma v - \mu u, \\ {}^{CF}D_t^\eta(v)(t) = \Lambda(q) + \mu u - \gamma x - \mu x - \omega x, \\ {}^{CF}D_t^\eta(x)(t) = \beta u(v+y) - \mu v - \alpha v - \mu_2 v, \\ {}^{CF}D_t^\eta(y)(t) = \alpha v + \delta y - \mu y - \mu_3 y, \\ {}^{CF}D_t^\eta(z)(t) = \mu_2 v + \omega x + \mu_3 x - \mu z, \end{cases} \quad (2)$$

under the initial conditions

$$\begin{aligned} u(0) &\geq 0, \\ v(0) &\geq 0, \\ x(0) &\geq 0, \\ y(0) &\geq 0, \\ z(0) &\geq 0. \end{aligned} \quad (3)$$

We obtain the solution in the form of series for the considered problem. Also, we display the results against

different values of fractional order $\eta \in (0, 1]$. Also, we provide results about the existence and uniqueness of the solution for the concerned model by using fixed point theorems due to Schauder and Banach. Here, we remark that we use the Laplace Adomian decomposition method because this method is easy and efficient and less expensive. Furthermore, the mentioned method does not require any predefined step size or controlling parameter which are needed by RK methods or the homotopy method, respectively, for detail, see [37–39]. Furthermore, the convergence of the method has been showed in many papers of the proposed method, for instance, see [27].

2. Preliminaries

Definition 1 (see [3]). Let $\varphi \in \mathcal{H}^1(a_1, a_2)$, $a_2 > a_1$, and $\eta \in (0, 1)$; then, the CFFD is recalled as

$${}^{CF}\mathcal{D}_t^\eta(\varphi(t)) = \frac{\mathcal{M}(\eta)}{1-\eta} \int_a^t \varphi'(\xi) \exp\left[-\eta \frac{t-\xi}{1-\eta}\right] d\xi. \quad (4)$$

$\mathcal{M}(\eta)$ is the normalization function defined as $\mathcal{M}(\eta) = (2\eta/(2-\eta))$ and satisfies the conditions $\mathcal{M}(1) = \mathcal{M}(0) = 1$. If the function fails to exist in $\mathcal{H}^1(a_1, a_2)$, then the derivative can be redefined as

$${}^{CF}\mathcal{D}_t^\eta(\varphi(t)) = \frac{\mathcal{M}(\eta)}{1-\eta} \int_a^t (\varphi(t) - \varphi(\xi)) \exp\left[-\eta \frac{t-\xi}{1-\eta}\right] d\xi. \quad (5)$$

Definition 2 (see [3]). The integral of fractional order $\eta \in (0, 1]$ of a function φ is defined by

$${}^{CF}\mathcal{I}_t^\eta[\varphi(t)] = \frac{(1-\eta)}{\mathcal{M}(\eta)} \varphi(t) + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \varphi(\xi) d\xi, \quad t \geq 0. \quad (6)$$

Lemma 1 (see [40]). The solution of the differential equation involving CFFD for $y \in L[0, T]$ as

$$\begin{aligned} {}^{CF}\mathcal{D}_t^\eta(\varphi(t)) &= y(t), \quad 0 < \eta \leq 1, \\ \varphi(0) &= \varphi_0, \varphi \text{ is any real constant,} \end{aligned} \quad (7)$$

is given by

$$\varphi(t) = \varphi_0 + \frac{(1-\eta)}{\mathcal{M}(\eta)} [y(t) - y(0)] + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t y(\xi) d\xi. \quad (8)$$

Definition 3 (see [34, 35]). The solution $\varphi \in C[0, T]$ of the differential equation

$$\begin{aligned} {}^{CF}\mathcal{D}_t^\eta(\varphi(t)) &= y(t), \quad 0 < \eta \leq 1, \\ \varphi(0) &= \varphi_0, \varphi \text{ is any real constant,} \end{aligned} \quad (9)$$

is Hyers–Ulam stable; if there exists $\varepsilon > 0$ such that, for the inequality

$$|{}^{CF}\mathcal{D}_t^\eta(\varphi(t)) - y(t)| \leq \varepsilon, \quad t \in [0, T], \quad (10)$$

there exists a unique solution $\bar{\varphi} \in C[0, T]$ with a constant $C_y > 0$ such that

$$|\varphi(t) - \bar{\varphi}(t)| \leq C_y \varepsilon, \quad t \in [0, T]. \quad (11)$$

Furthermore, if there exists nondecreasing function $g: (0, 1) \rightarrow \mathcal{R}^+$ such that (11) may be written as

$$|\varphi(t) - \bar{\varphi}(t)| \leq C_y g(\varepsilon), \quad \text{with } g(0) = 0, \quad (12)$$

then the concerned solution of problem (9) is generalized Hyers–Ulam stable.

Theorem 1 (see [40]). Let \mathbf{B} be a convex subset of Banach space \mathbf{X} , with operators \mathbf{G} and \mathbf{H} with

- (1) $\mathbf{G}(u) + \mathbf{H}(v) \in \mathbf{B}$ for all $u, v \in \mathbf{B}$
- (2) \mathbf{G} is the condensing operator
- (3) \mathbf{H} is continuous and compact

Then, there exists at least one solution $u \in \mathbf{B}$ such that

$$\mathbf{G}(u) + \mathbf{H}(u) = u. \quad (13)$$

Definition 4 (see [4, 14]). The Laplace transform of CFFD ${}^{CF}\mathcal{D}_t^\eta x(t)$ is given as

$$\mathcal{L}[{}^{CF}\mathcal{D}_t^\eta x(t)] = \frac{s\mathcal{L}[x(t)] - x(0)}{s + \eta(1-s)}, \quad s \geq 0, \eta \in (0, 1]. \quad (14)$$

3. Existence and Stability Results for the Considered Model

In this part of the manuscript, we determine existence results for model (2) using the fixed point theorem due to Banach.

In this regard, we first define the following functions:

$$\begin{aligned} \varphi_1(t, u, v, x, y, z) &= \Lambda(1-q) - \beta u(x+y) + \gamma v - \mu u, \\ \varphi_2(t, u, v, x, y, z) &= \Lambda(q) + \mu u - \gamma x - \mu x - \omega x, \\ \varphi_3(t, u, v, x, y, z) &= \beta u(v+y) - \mu v - \alpha v - \mu_2 v, \\ \varphi_4(t, u, v, x, y, z) &= \alpha v + \delta y - \mu y - \mu_3 y, \\ \varphi_5(t, u, v, x, y, z) &= \mu_2 v + \omega x + \mu_3 x - \mu z. \end{aligned} \quad (15)$$

Then, we write some notions for easiness as

$$\begin{aligned} \mathbf{Y}(t) &= \begin{cases} u(t) \\ v(t) \\ x(t), \\ y(t) \\ z(t) \end{cases} \\ \mathbf{Y}_0 &= \begin{cases} u(0) \\ v(0) \\ x(0), \\ y(0) \\ z(0) \end{cases} \\ \Psi(t, \mathbf{Y}(t)) &= \begin{cases} \varphi_1(t, u, v, x, y, z) \\ \varphi_2(t, u, v, x, y, z) \\ \varphi_3(t, u, v, x, y, z), \\ \varphi_4(t, u, v, x, y, z) \\ \varphi_5(t, u, v, x, y, z) \end{cases} \end{aligned} \quad (16)$$

and

$$\Psi_0 = \begin{cases} \varphi_1(0, u(0), v(0), x(0), y(0), z(0)) \\ \varphi_2(0, u(0), v(0), x(0), y(0), z(0)) \\ \varphi_3(0, u(0), v(0), x(0), y(0), z(0)) \\ \varphi_4(0, u(0), v(0), x(0), y(0), z(0)) \\ \varphi_5(0, u(0), v(0), x(0), y(0), z(0)) \end{cases} \quad (17)$$

Using (16), system (2) can be written as

$${}^{CF}D_t^\eta[\mathbf{Y}(t)] = \Psi(t, \mathbf{Y}(t)), \quad t \in [0, T], \quad \mathbf{Y}(0) = \mathbf{Y}_0. \quad (18)$$

In view of Lemma 1, problem (18) can be converted to the given integral equation as

$$\begin{aligned} \mathbf{Y}(t) = & \mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{1-\eta}{\mathcal{M}(\eta)} \\ & + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi. \end{aligned} \quad (19)$$

Now, let $0 \leq t \leq T < \infty$, and let $\mathcal{J} = [0, T]$; we define the Banach space $\mathbf{X} = C([0, T] \times \mathcal{R}^5, \mathcal{R})$ under the norm

$$\|\mathbf{Y}\| = \sup_{t \in \mathcal{J}} \{\|\mathbf{Y}\| : \mathbf{Y} \in \mathbf{X}\}. \quad (20)$$

The assumptions given in the following hold true:

(\mathcal{A}_1) Under the continuity of $\Psi: \mathcal{J} \times \mathcal{R}^5 \rightarrow \mathcal{R}$, there exists $K_\Psi > 0$ such that

$$|\Psi(t, \mathbf{Y}(t)) - \Psi(t, \bar{\mathbf{Y}}(t))| \leq K_\Psi \|\mathbf{Y}(t) - \bar{\mathbf{Y}}(t)\|. \quad (21)$$

(\mathcal{A}_2) There exists positive constant $C_\Psi, M_\Psi > 0$ such that $|\Psi(t, \mathbf{Y}(t))| \leq C_\Psi \|\mathbf{Y}\| + M_\Psi$.

Theorem 2. Under the assumptions (\mathcal{A}_1), (\mathcal{A}_2), problem (19) has at least one solution if $K_\Psi < \mathcal{M}(\eta)$; consequently, the considered system (2) has at least one solution.

Proof. Let $\mathbf{B} = \{\mathbf{Y} \in X : \|\mathbf{Y}\| \leq \rho, \rho > 0\} \subset X$ be a closed convex set. Now, we define the operators from (19) as

$$\begin{aligned} \mathbf{GY}(t) = & \mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)}, \\ \mathbf{HY}(t) = & \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi. \end{aligned} \quad (22)$$

To derive the required results, we first prove that operator $\mathbf{G}: \mathbf{B} \rightarrow \mathbf{B}$ is a contraction.

For any $\mathbf{Y}, \bar{\mathbf{Y}} \in \mathbf{B}$, one has

$$\begin{aligned} \|\mathbf{GY} - \mathbf{G}\bar{\mathbf{Y}}\| = & \sup_{t \in \mathcal{J}} \left\| \left[\mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} \right] \right. \\ & \left. - \left[\mathbf{Y}_0 + [\Psi(t, \bar{\mathbf{Y}}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} \right] \right\| \\ \leq & \frac{K_\Psi}{\mathcal{M}(\eta)} \|\mathbf{Y} - \bar{\mathbf{Y}}\|. \end{aligned} \quad (23)$$

This shows that \mathbf{G} is a contraction. Now, to show that $\mathbf{H}: \mathbf{B} \rightarrow \mathbf{B}$ is bounded and equicontinuous, the continuity of Ψ implies that \mathbf{H} is continuous. For any $\mathbf{Y} \in \mathbf{B}$, we have

$$\begin{aligned} \|\mathbf{HY}\| = & \sup_{t \in \mathcal{J}} \left| \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right| \\ \leq & \frac{\eta}{\mathcal{M}(\eta)} (C_\Psi \|\mathbf{Y}\| + M_\Psi) \int_0^T d\xi \\ \leq & \frac{T}{\mathcal{M}(\eta)} (C_\Psi \rho + M_\Psi). \end{aligned} \quad (24)$$

This shows that \mathbf{H} is bounded; for equicontinuity, let $t_1 > t_2$, and we have

$$\begin{aligned} |\mathbf{HY}(t_1) - \mathbf{HY}(t_2)| = & \frac{\eta}{\mathcal{M}(\eta)} \left| \int_0^{t_1} \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right. \\ & \left. - \int_0^{t_2} \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right| \\ \leq & \frac{(C_\Psi \rho + M_\Psi)}{\mathcal{M}(\eta)} [t_1 - t_2], \end{aligned} \quad (25)$$

which implies that $|\mathbf{HY}(t_1) - \mathbf{HY}(t_2)| \rightarrow 0$ as $t_1 \rightarrow t_2$. So, \mathbf{H} is uniformly continuous and bounded. Thus, by Arzelà–Ascoli theorem, \mathbf{H} is relatively compact and so is completely continuous. Thus, by Theorem 1, problem (18) has at least one solution. Consequently, the considered model (2) has at least one solution. \square

Next, we define the operator $\mathbf{T}: X \rightarrow X$ by

$$\begin{aligned} \mathbf{T}(\mathbf{Y}) = & \mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{1-\eta}{\mathcal{M}(\eta)} \\ & + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi. \end{aligned} \quad (26)$$

Theorem 3. Under assumption (\mathcal{A}_2), the operator $\mathbf{T}: X \rightarrow X$ as defined in (26) is a contraction; then, problem (18) has a unique solution with the condition $L_\Psi = (K_\Psi(1+T)/\mathcal{M}(\eta)) < 1$, and consequently, our proposed system (2) has a unique solution.

Proof. Let $\mathbf{Y}, \bar{\mathbf{Y}} \in X$; then, from (26), one has

$$\begin{aligned} \|\mathbf{TY} - \mathbf{T}\bar{\mathbf{Y}}\| = & \sup_{t \in \mathcal{J}} \left\| \left(\mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} \right. \right. \\ & \left. \left. + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right) \right. \\ & \left. - \left(\mathbf{Y}_0 + [\Psi(t, \bar{\mathbf{Y}}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} \right. \right. \\ & \left. \left. + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \bar{\mathbf{Y}}(\xi)) d\xi \right) \right\| \\ \leq & L_\Psi \|\mathbf{Y} - \bar{\mathbf{Y}}\|. \end{aligned} \quad (27)$$

This shows that \mathbf{T} is a contraction. Therefore, problem (18) has a unique solution. Hence, our considered system (2) has a unique solution. \square

Remark 1. Next, for stability analysis, we consider a small perturbation θ , such that $\theta(0) = 0$, depends only on the solution.

(i) $|\theta(t)| < \varepsilon$ for $\varepsilon > 0$

(ii) ${}_0^{\text{CF}}\mathbf{D}_t^\eta \mathbf{Y}(t) = \Psi(t, \mathbf{Y}(t)) + \theta(t)$

Lemma 2. *The solution of the perturbed problem*

$$\begin{aligned} {}_0^{\text{CF}}\mathbf{D}_t^\eta \mathbf{Y}(t) &= \Psi(t, \mathbf{Y}(t)) + \theta(t), \\ \mathbf{Y}(0) &= \mathbf{Y}_0, \end{aligned} \quad (28)$$

satisfies the given relation:

$$\left| \mathbf{Y}(t) - \left(\mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right) \right| \leq \Delta \varepsilon, \quad (29)$$

where $((1 + T)/\mathcal{M}(\eta)) = \Delta$.

Proof. In view of Lemma 1, the solution of perturb problem (28) is given by

$$\mathbf{Y}(t) = \mathbf{Y}_0 + \frac{(1-\eta)}{\mathcal{M}(\eta)} [\Psi(t, \mathbf{Y}(t)) - \Psi_0] + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi + \frac{(1-\eta)}{\mathcal{M}(\eta)} \theta(t) + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \theta(\xi) d\xi. \quad (30)$$

From (30), on using Remark 1, we have

$$\begin{aligned} & \left| \mathbf{Y}(t) - \left[\mathbf{Y}_0 + \frac{(1-\eta)}{\mathcal{M}(\eta)} [\Psi(t, \mathbf{Y}(t)) - \Psi_0] + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right] \right| \\ &= \left| \frac{(1-\eta)}{\mathcal{M}(\eta)} \theta(t) + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \theta(\xi) d\xi \right| \\ &\leq \frac{(1-\eta)}{\mathcal{M}(\eta)} |\theta(t)| + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \theta(\xi) d\xi \\ &\leq \Delta \varepsilon. \end{aligned} \quad (31)$$

Theorem 4. *Under assumption \mathcal{A}_2 , together with Lemma 2, problem (18) is Hyers–Ulam stable if $L_\Psi < 1$, which yields that our considered system (2) is Hyers–Ulam stable.*

Proof. Let $\mathbf{Y} \in X$ be any solution, and $\bar{\mathbf{Y}} \in X$ is a unique solution; then, \square

$$\begin{aligned} |\mathbf{Y}(t) - (\bar{\mathbf{Y}})(t)| &= \left| \mathbf{Y}(t) - \left(\mathbf{Y}_0 + [\Psi(t, \bar{\mathbf{Y}}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \bar{\mathbf{Y}}(\xi)) d\xi \right) \right| \\ &\leq \left| \mathbf{Y}(t) - \left(\mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right) \right| \\ &\quad + \left| \left(\mathbf{Y}_0 + [\Psi(t, \mathbf{Y}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \mathbf{Y}(\xi)) d\xi \right) \right. \\ &\quad \left. - \left(\mathbf{Y}_0 + [\Psi(t, \bar{\mathbf{Y}}(t)) - \Psi_0] \frac{(1-\eta)}{\mathcal{M}(\eta)} + \frac{\eta}{\mathcal{M}(\eta)} \int_0^t \Psi(\xi, \bar{\mathbf{Y}}(\xi)) d\xi \right) \right| \\ &\leq \Delta \varepsilon + L_\Psi \|\mathbf{Y} - \bar{\mathbf{Y}}\|. \end{aligned} \quad (32)$$

This implies that

$$\|\mathbf{Y} - \bar{\mathbf{Y}}\| \leq \frac{\Delta}{1 - L_\Psi} \varepsilon. \quad (33)$$

Therefore, the solution of (18) is Hyers–Ulam stable. Hence, the solution of the proposed system (2) is Hyers–Ulam stable. \square

Remark 2. In the same line, we can also develop the results of generalized Hyers–Ulam, Rassias–Hyers–Ulam stability. The aforementioned stability analysis has been studied for simple mathematical models of biology and physics in [41–43].

4. Derivation of the General Semianalytical Solution to the Considered Model (2)

Here, in this section, we are going to compute series solution for the suggested problem. To receive this goal, taking Laplace transform of (16), we have

$$\begin{cases} \mathcal{L}[u(t)] = u(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \beta u(x+y) + \gamma v - \mu u], \\ \mathcal{L}[v(t)] = v(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(q) + \mu u - \gamma x - \mu x - \omega x], \\ \mathcal{L}[x(t)] = x(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}[\beta u(v+y) - \mu v - \alpha v - \mu_2 v], \\ \mathcal{L}[y(t)] = y(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}[\alpha v + \delta y - \mu y - \mu_3 y], \\ \mathcal{L}[z(t)] = z(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}[\mu_2 v + \omega x + \mu_3 x - \mu z]. \end{cases} \quad (34)$$

Now, assuming the solution in the series form,

$$\begin{aligned} u(t) &= \sum_{q=0}^{\infty} u_q(t), \\ v(t) &= \sum_{q=0}^{\infty} v_q(t), \\ x(t) &= \sum_{q=0}^{\infty} x_q(t), \\ y(t) &= \sum_{q=0}^{\infty} y_q(t), \\ z(t) &= \sum_{q=0}^{\infty} z_q(t). \end{aligned} \quad (35)$$

Further decomposing the nonlinear terms $u(t)x(t)$, $u(t)y(t)$, $u(t)v(t)$, etc. in terms of Adomian polynomials,

$$\begin{aligned} u(t)x(t) &= \sum_{q=0}^{\infty} A_q(u, x), \\ u(t)y(t) &= \sum_{q=0}^{\infty} B_q(u, y), \\ u(t)v(t) &= \sum_{q=0}^{\infty} C_q(u, v), \end{aligned} \quad (36)$$

where the Adomian polynomial $A_q(u, x)$ can be defined as

$$A_q(u, x) = \frac{1}{q!} \frac{d^q}{d\lambda^q} \left[\sum_{j=0}^p \lambda^j u_j(t) \sum_{j=0}^p \lambda^j x_j(t) \right] \Big|_{\lambda=0}. \quad (37)$$

In the same way, the other polynomials B_q, C_q can be defined.

Hence, in view of (35) and (36), system (34) becomes

$$\begin{cases} \mathcal{L}\left[\sum_{q=0}^{\infty} u_q(t)\right] = u(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}\left[\Lambda(1-q) - \beta \sum_{q=0}^{\infty} A_q(u, x) - \beta \sum_{q=0}^{\infty} B_q(u, y) + \gamma \sum_{q=0}^{\infty} v_q - \mu \sum_{q=0}^{\infty} u_q\right], \\ \mathcal{L}\left[\sum_{q=0}^{\infty} v_q(t)\right] = v(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}\left[\Lambda(q) + \mu \sum_{q=0}^{\infty} u_q - \gamma \sum_{q=0}^{\infty} x_q - \mu \sum_{q=0}^{\infty} x_q - \omega \sum_{q=0}^{\infty} x_q\right], \\ \mathcal{L}\left[\sum_{q=0}^{\infty} x_q(t)\right] = x(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}\left[\beta \sum_{q=0}^{\infty} C_q(u, v) + \beta \sum_{q=0}^{\infty} C_q(u, y) - \mu \sum_{q=0}^{\infty} v_q - \alpha \sum_{q=0}^{\infty} v_q - \mu_2 \sum_{q=0}^{\infty} v_q\right], \\ \mathcal{L}\left[\sum_{q=0}^{\infty} y_q(t)\right] = y(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}\left[\alpha \sum_{q=0}^{\infty} v_q + \delta \sum_{q=0}^{\infty} y_q - \mu \sum_{q=0}^{\infty} y_q - \mu_3 \sum_{q=0}^{\infty} y_q\right], \\ \mathcal{L}\left[\sum_{q=0}^{\infty} z_q(t)\right] = z(0) + \frac{s + \eta(1-s)}{s} \mathcal{L}\left[\mu_2 \sum_{q=0}^{\infty} v_q + \omega \sum_{q=0}^{\infty} x_q + \mu_3 \sum_{q=0}^{\infty} x_q - \mu \sum_{q=0}^{\infty} z_q\right]. \end{cases} \quad (38)$$

Now, equating terms on both sides of (38), we have

$$\left\{ \begin{array}{l}
 \mathcal{L}[u_0(t)] = u_0, \mathcal{L}[v_0(t)] = v_0, \mathcal{L}[x_0(t)] = x_0, \mathcal{L}[y_0(t)] = y_0, \\
 \mathcal{L}[z_0(t)] = z_0, \mathcal{L}[u_1(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \beta A_0(u, x) - \beta B_0(u, y) + \gamma v_0 - \mu u_0], \\
 \mathcal{L}[v_1(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(q) + \mu u_0 - \gamma x_0 - \mu x_0 - \omega x_0], \\
 \mathcal{L}[x_1(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\beta C_0(u, v) + \beta C_0(u, y) - \mu v_0 - \alpha v_0 - \mu_2 v_0], \\
 \mathcal{L}[y_1(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\alpha v_0 + \delta y_0 - \mu y_0 - \mu_3 y_0], \\
 \mathcal{L}[z_1(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\mu_2 v_0 + \omega x_0 + \mu_3 x_0 - \mu y_0], \\
 \mathcal{L}[u_2(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \beta A_1(u, x) - \beta B_1(u, y) + \gamma v_1 - \mu u_1], \\
 \mathcal{L}[v_2(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(q) + \mu u_1 - \gamma x_1 - \mu x_1 - \omega x_1], \\
 \mathcal{L}[x_2(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\beta C_1(u, v) + \beta C_1(u, y) - \mu v_1 - \alpha v_1 - \mu_2 v_1], \\
 \mathcal{L}[y_2(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\alpha v_1 + \delta y_1 - \mu y_1 - \mu_3 y_1], \\
 \mathcal{L}[z_2(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\mu_2 v_1 + \omega x_1 + \mu_3 x_1 - \mu z_1], \\
 \vdots \\
 \mathcal{L}[u_{q+1}(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(1-q) - \beta A_q(u, x) - \beta B_q(u, y) + \gamma v_q - \mu u_q], \\
 \mathcal{L}[v_{q+1}(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\Lambda(q) + \mu u_q - \gamma x_q - \mu x_q - \omega x_q], \\
 \mathcal{L}[x_{q+1}(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\beta C_q(u, v) + \beta C_q(u, y) - \mu v_q - \alpha v_q - \mu_2 v_q], \\
 \mathcal{L}[y_{q+1}(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\alpha v_q + \delta y_q - \mu y_q - \mu_3 y_q], \\
 \mathcal{L}[z_{q+1}(t)] = \frac{s + \eta(1-s)}{s} \mathcal{L}[\mu_2 v_q + \omega x_q + \mu_3 x_q - \mu z_q], \quad q \geq 0.
 \end{array} \right. \quad (39)$$

Evaluating the Laplace transform in (39), we get

$$\left\{ \begin{array}{l} u_0(t) = u_0, v_0(t) = v_0, x_0(t) = x_0, y_0(t) = y_0, \\ z_0(t) = z_0, u_1(t) = [\Lambda(1-q) - \beta u_0 x_0 - \beta u_0 y_0 + \gamma v_0 - \mu u_0](1 + \eta(t-1)), \\ v_1(t) = 1 + \eta(t-1)[\Lambda(q) + \mu u_0 - \gamma x_0 - \mu x_0 - \omega x_0], \\ x_1(t) = [\beta u_0 v_0 + \beta u_0 y_0 - \mu v_0 - \alpha v_0 - \mu_2 v_0](1 + \eta(t-1)), \\ y_1(t) = [\alpha v_0 + \delta y_0 - \mu y_0 - \mu_3 y_0](1 + \eta(t-1)), \\ z_1(t) = [\mu_2 v_0 + \omega x_0 + \mu_3 x_0 - \mu z_0](1 + \eta(t-1)), \\ u_2(t) = (1 + \eta(t-1))\Lambda(1-q) + [\gamma(\Lambda(q) + \mu u_0 - \gamma x_0 - \mu x_0 - \omega x_0) \\ - \beta u_0(\beta u_0 v_0 + \beta u_0 y_0 - \alpha v_0 - \mu v_0 - \mu_2 v_0 + \alpha v_0 + \eta y_0 - \mu y_0 - \mu_3 y_0) \\ - (\mu + \beta x_0 + \beta y_0)(\Lambda(1-q) - \beta x_0 u_0 - \beta y_0 u_0 + \gamma v_0 - \mu u_0)](1 + \eta^2(t-1)), \\ v_2(t) = (1 + \eta(t-1))\Lambda(q) + [\mu(\Lambda(1-q) - \beta x_0 u_0 - \beta y_0 u_0 - \gamma v_0 - \mu u_0) \\ - (\gamma + \mu + \omega)(\beta v_0 u_0 + \beta y_0 u_0 - \mu u_0 - \alpha v_0 - \mu_2 u_0)](1 + \eta^2(t-1)), \\ x_2(t) = (1 + \eta^2(t-1))[\beta u_0(\Lambda(q) + \mu u_0 - \gamma x_0 - \mu x_0 - \omega x_0 + \alpha v_0 + \eta y_0 - \mu y_0 - \mu_3 y_0) \\ - (\alpha + \mu + \mu_2)(\Lambda q + \mu u_0 - \gamma x_0 - \mu x_0 - \eta u_0) + \beta(v_0 + y_0)(\Lambda(1-q) - \beta u_0 x_0 - \beta u_0 y_0 + \gamma v_0 - \mu u_0)], \\ y_2(t) = (1 + \eta^2(t-1))[\alpha(\Lambda q - (\gamma - \mu - \omega)x_0 + \mu u_0) \\ + (\delta(\delta - \mu - \mu_3) + \alpha v_0(\delta - \mu - \mu_3) - \mu(\delta - \mu - \mu_3) - \delta_3(\delta - \mu - \mu_3))y_0], \\ z_2(t) = 1 + \eta^2(t-1)[\mu_2(\Lambda q + \mu u_0 - \gamma x_0 - \mu x_0 - \omega x_0) \\ + (\omega + \mu_3)(\beta u_0 v_0 + \beta u_0 y_0 - \mu v_0 - \alpha v_0 - \mu_2 v_0) - \mu(\mu_2 v_0 + \omega x_0 + \mu_3 x_0 - \mu z_0)], \end{array} \right. \quad (40)$$

and so on. Therefore, we get the required solution as given by

$$\left\{ \begin{array}{l} u(t) = u_0 + u_1(t) + u_2(t) + u_3(t) + \dots, \\ v(t) = v_0 + v_1(t) + v_2(t) + v_3(t) + \dots, \\ x(t) = x_0 + x_1(t) + x_2(t) + x_3(t) + \dots, \\ y(t) = y_0 + y_1(t) + y_2(t) + y_3(t) + \dots, \\ z(t) = z_0 + z_1(t) + z_2(t) + z_3(t) + \dots. \end{array} \right. \quad (41)$$

Note. For the convergence of the proposed method, see the paper [27].

5. Results and Discussion

This part of the manuscript is related to provide numerical results and some discussion about the approximate solution of the considered problem. For this purpose, we apply the Laplace Adomian decomposition methods for the

solution. We select appropriate values for parameters, as we take [3] $u_0 = 400$, $v_0 = 200$, $x_0 = 80$, $y_0 = 10$, $z_0 = 0$, $\Lambda = 0.005711$, $\beta = 0.0001104$, $\gamma = 0.0007791$; $\alpha = 0.7791$; $\mu = 0.001103$, $\mu_2 = 0.005728$, $\mu_3 = 0.00391$, $\omega = 0.002837$, $q = 0.002807$, and $\delta = 0.007032$. In view of these values, we draw the graph of approximate solutions (40) for first ten terms by using Matlab in Figures 1–5 against various fractional-order derivatives.

From Figures 1–5, we see that the susceptible class population is decreasing with different rates. It decreases with faster speed when the order is smaller as compared with the larger order. Also, the dynamics of the vaccinated class is increasing with different rates due to the fractional order. Also, the exposed population is increasing up, and hence, on using the vaccine, the density of the population of the infected class is decreasing, while the density of the recovered class increases with the same scenario. From all these figures, we concluded that the fractional derivative with exponential kernel can also be used to provide the global dynamics of the considered model.

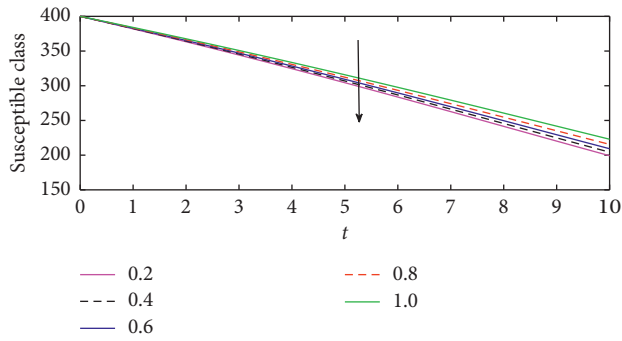


FIGURE 1: Plot of the susceptible class at different fractional values of η .

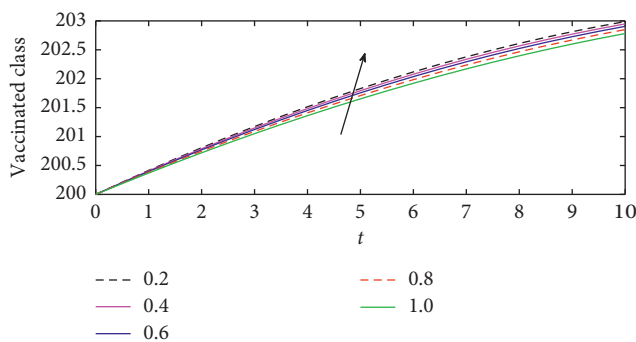


FIGURE 2: Plot of the vaccinated class at different fractional values of η .

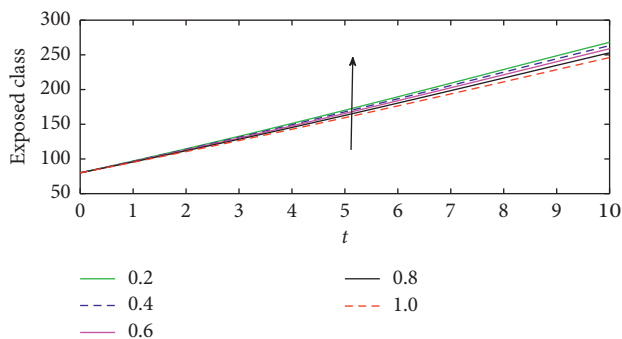


FIGURE 3: Plot of the exposed class at different fractional values of η .

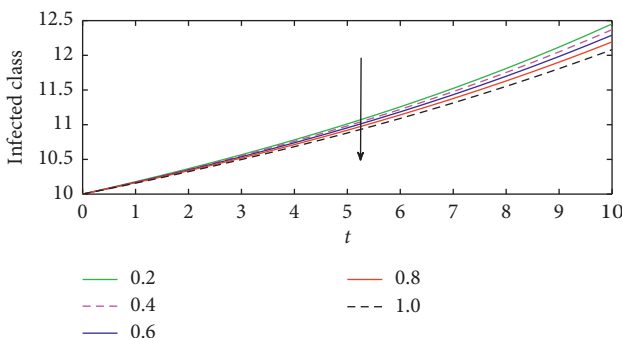


FIGURE 4: Plot of the infected class at different fractional values of η .

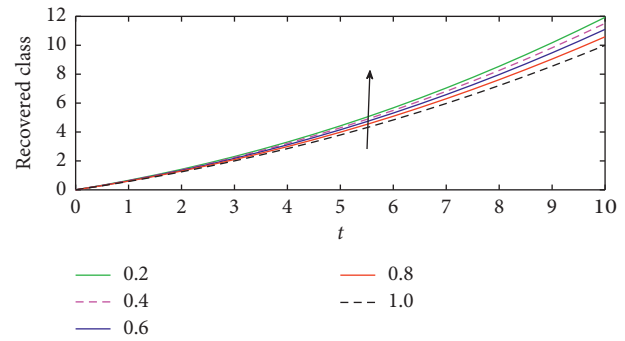


FIGURE 5: Plot of the recovered class at different fractional values of η .

6. Conclusion

By fixed point approach, we have investigated the existence of the considered model under CFFD. Also, by using the tools of nonlinear functional analysis, we have established sufficient conditions for Hyers–Ulam-type stability of the approximate solutions of the considered model. Also, we have provided the semianalytical solution to the considered model by the Laplace Adomian decomposition method. The concerned method needs no discretization of data nor extra axillary parameter as needed by homotopy methods on which these methods depend. The proposed method has been utilized extensively for usual fractional differential equations, but in the case of new fractional differential operators, this method has not been properly used. Hence, we concluded that CFFD can also be used as powerful tools to investigate biological models of infectious diseases. Also, the concerned models involving CFFD can be handled easily by using the Laplace Adomian decomposition method.

Data Availability

No data were used to support this study.

Conflicts of Interest

There exist no conflicts of interest regarding this research work.

Authors' Contributions

All authors have equal contribution to this work.

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Research Article

Global Dynamics of Some Exponential Type Systems

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We explore the boundedness and persistence, existence of an invariant rectangle, local dynamical properties about the unique positive fixed point, global dynamics by the discrete-time Lyapunov function, and the rate of convergence of some (2, 3)-type exponential systems of difference equations. Finally, theoretical results are numerically verified.

1. Introduction

Recently, global dynamical properties of (2, 2) as well as (2, 3)-type exponential difference equations or systems of exponential difference equations have widely been explored. In this regard, Ozturk et al. [1] have explored the global dynamical properties of the following (2, 2)-type exponential difference equation:

$$x_{n+1} = \frac{\alpha_{10} + \alpha_{11}e^{-x_n}}{\alpha_{12} + x_{n-1}}, \quad (1)$$

where α_s ($s = 10, 11, 12$) and x_s ($s = -1, 0$) are the positive real numbers. Equation (1) may be viewed as a model in mathematical biology where α_{10} is the immigration rate, α_{11} is the population growth rate, and α_{12} is the carrying capacity. Bozkurt [2] has explored the global dynamical properties of the following (2, 3)-type exponential difference equation:

$$x_{n+1} = \frac{\alpha_{10}e^{-x_n} + \alpha_{11}e^{-x_{n-1}}}{\alpha_{12} + \alpha_{10}x_n + \alpha_{11}x_{n-1}}, \quad (2)$$

where α_s ($s = 10, 11, 12$) and x_s ($s = -1, 0$) are the positive real numbers. More precisely, Bozkurt [2] has explored the local asymptotic stability of the equilibrium point by linearized stability theorem, gasymptotic stability behavior by Lyapunov function, and semicycle analysis of positive solutions of the exponential difference equation, which is depicted in (2). Finally, theoretical results are verified numerically. Motivated from the aforementioned studies, here our purpose is to explore the global dynamical properties of the following (2, 3)-type exponential systems, that is the extension of the work of Bozkurt [2]:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{10}e^{-y_n} + \alpha_{11}e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{13}e^{-z_n} + \alpha_{14}e^{-z_{n-1}}}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{16}e^{-x_n} + \alpha_{17}e^{-x_{n-1}}}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}, \end{aligned} \quad (3)$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{19}e^{-z_n} + \alpha_{20}e^{-z_{n-1}}}{\alpha_{21} + \alpha_{19}z_n + \alpha_{20}z_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{22}e^{-x_n} + \alpha_{23}e^{-x_{n-1}}}{\alpha_{24} + \alpha_{22}x_n + \alpha_{23}x_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{25}e^{-y_n} + \alpha_{26}e^{-y_{n-1}}}{\alpha_{27} + \alpha_{25}y_n + \alpha_{26}y_{n-1}}, \end{aligned} \quad (4)$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{28}e^{-x_n} + \alpha_{29}e^{-x_{n-1}}}{\alpha_{30} + \alpha_{28}x_n + \alpha_{29}x_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{31}e^{-y_n} + \alpha_{32}e^{-y_{n-1}}}{\alpha_{33} + \alpha_{31}z_n + \alpha_{32}z_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{34}e^{-z_n} + \alpha_{35}e^{-z_{n-1}}}{\alpha_{36} + \alpha_{34}y_n + \alpha_{35}y_{n-1}}, \end{aligned} \quad (5)$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{37}e^{-x_n} + \alpha_{38}e^{-x_{n-1}}}{\alpha_{39} + \alpha_{37}y_n + \alpha_{38}y_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{40}e^{-y_n} + \alpha_{41}e^{-y_{n-1}}}{\alpha_{42} + \alpha_{40}x_n + \alpha_{41}x_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{43}e^{-z_n} + \alpha_{44}e^{-z_{n-1}}}{\alpha_{45} + \alpha_{43}z_n + \alpha_{44}z_{n-1}}, \end{aligned} \quad (6)$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{46}e^{-x_n} + \alpha_{47}e^{-x_{n-1}}}{\alpha_{48} + \alpha_{46}z_n + \alpha_{47}z_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{49}e^{-y_n} + \alpha_{50}e^{-y_{n-1}}}{\alpha_{51} + \alpha_{49}y_n + \alpha_{50}y_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{52}e^{-z_n} + \alpha_{53}e^{-z_{n-1}}}{\alpha_{54} + \alpha_{52}x_n + \alpha_{53}x_{n-1}}, \end{aligned} \quad (7)$$

$$\begin{aligned} x_{n+1} &= \frac{\alpha_{55}e^{-z_n} + \alpha_{56}e^{-z_{n-1}}}{\alpha_{57} + \alpha_{55}x_n + \alpha_{56}x_{n-1}}, \\ y_{n+1} &= \frac{\alpha_{58}e^{-x_n} + \alpha_{59}e^{-x_{n-1}}}{\alpha_{60} + \alpha_{58}y_n + \alpha_{59}y_{n-1}}, \\ z_{n+1} &= \frac{\alpha_{61}e^{-y_n} + \alpha_{62}e^{-y_{n-1}}}{\alpha_{63} + \alpha_{61}z_n + \alpha_{62}z_{n-1}}, \end{aligned} \quad (8)$$

where α_s ($s = 10, \dots, 63$) and x_s, y_s, z_s ($s = -1, 0$) are the positive real numbers.

The rest of the paper is organized as follows: in Section 2, we explore that every positive solution of systems (3)–(8) is bounded and persists, whereas construction of an invariant rectangle is explored in Section 3. In Section 4, we explore the existence as well as uniqueness of the positive equilibrium point of systems (3)–(8). In Section 5, we explore the local dynamical properties about the unique positive equilibrium point of systems (3)–(8). In Section 6, we explore global dynamics about the positive equilibrium by the discrete-time Lyapunov function. We study the rate of convergence in Section 7, whereas discussion along with numerical simulations is presented in Section 8.

2. Boundedness and Persistence of Systems (3)–(8)

Theorem 1. Every positive solution $\{\Omega_n\}_{n=-1}^{\infty}$ of systems (3)–(8) is bounded and persists.

Proof. (i) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (3), then

$$\begin{aligned} x_n &\leq \frac{\alpha_{10} + \alpha_{11}}{\alpha_{12}} = U_1, \\ y_n &\leq \frac{\alpha_{13} + \alpha_{14}}{\alpha_{15}} = U_2, \\ z_n &\leq \frac{\alpha_{16} + \alpha_{17}}{\alpha_{18}} = U_3, \\ n &= 1, 2, \dots \end{aligned} \quad (9)$$

From (3) and (9), one gets

$$\begin{aligned} x_n &\geq \frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})/\alpha_{15})}}{\alpha_{10} + (\alpha_{11} + \alpha_{12})((\alpha_{13} + \alpha_{14})/\alpha_{15})} = L_1, \\ y_n &\geq \frac{(\alpha_{13} + \alpha_{14})e^{-((\alpha_{16} + \alpha_{17})/\alpha_{18})}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})((\alpha_{16} + \alpha_{17})/\alpha_{18})} = L_2, \\ z_n &\geq \frac{(\alpha_{16} + \alpha_{17})e^{-((\alpha_{10} + \alpha_{11})/\alpha_{12})}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})((\alpha_{10} + \alpha_{11})/\alpha_{12})} = L_3, \\ n &= 2, 3, \dots \end{aligned} \quad (10)$$

So, from (9) and (10), one has

$$\begin{aligned} L_1 &\leq x_n \leq U_1, \\ L_2 &\leq y_n \leq U_2, \\ L_3 &\leq z_n \leq U_3, \\ n &= 3, 4, \dots \end{aligned} \quad (11)$$

(ii) If $\{\Omega_n\}_{n=-1}^{\infty}$ is a positive solution of (4), then

$$\begin{aligned} x_n &\leq \frac{\alpha_{19} + \alpha_{20}}{\alpha_{21}} = U_4, \\ y_n &\leq \frac{\alpha_{22} + \alpha_{23}}{\alpha_{24}} = U_5, \\ z_n &\leq \frac{\alpha_{25} + \alpha_{26}}{\alpha_{27}} = U_6, \\ n &= 1, 2, \dots \end{aligned} \quad (12)$$

From (4) and (12), one gets

$$\begin{aligned}
x_n &\geq \frac{(\alpha_{19} + \alpha_{20})e^{-((\alpha_{25} + \alpha_{26})/\alpha_{27})}}{\alpha_{21} + (\alpha_{19} + \alpha_{20})((\alpha_{25} + \alpha_{26})/\alpha_{27})} = L_4, \\
y_n &\geq \frac{(\alpha_{22} + \alpha_{23})e^{-((\alpha_{19} + \alpha_{20})/\alpha_{21})}}{\alpha_{24} + (\alpha_{22} + \alpha_{23})((\alpha_{19} + \alpha_{20})/\alpha_{21})} = L_5, \\
z_n &\geq \frac{(\alpha_{25} + \alpha_{26})e^{-((\alpha_{22} + \alpha_{23})/\alpha_{24})}}{\alpha_{27} + (\alpha_{25} + \alpha_{26})((\alpha_{22} + \alpha_{23})/\alpha_{24})} = L_6, \\
n &= 2, 3, \dots
\end{aligned} \tag{13}$$

So, from (12) and (13), one gets

$$\begin{aligned}
L_4 &\leq x_n \leq U_4, \\
L_5 &\leq y_n \leq U_5, \\
L_6 &\leq z_n \leq U_6, \\
n &= 3, 4, \dots
\end{aligned} \tag{14}$$

(iii) If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution of (5), then

$$\begin{aligned}
x_n &\leq \frac{\alpha_{28} + \alpha_{29}}{\alpha_{30}} = U_7, \\
y_n &\leq \frac{\alpha_{31} + \alpha_{32}}{\alpha_{33}} = U_8, \\
z_n &\leq \frac{\alpha_{34} + \alpha_{35}}{\alpha_{36}} = U_9, \\
n &= 1, 2, \dots
\end{aligned} \tag{15}$$

From (5) and (15), one gets

$$\begin{aligned}
x_n &\geq \frac{(\alpha_{28} + \alpha_{29})e^{-((\alpha_{28} + \alpha_{29})/\alpha_{30})}}{\alpha_{30} + (\alpha_{28} + \alpha_{29})((\alpha_{28} + \alpha_{29})/\alpha_{30})} = L_7, \\
y_n &\geq \frac{(\alpha_{31} + \alpha_{32})e^{-((\alpha_{31} + \alpha_{32})/\alpha_{33})}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})((\alpha_{31} + \alpha_{32})/\alpha_{33})} = L_8, \\
z_n &\geq \frac{(\alpha_{34} + \alpha_{35})e^{-((\alpha_{34} + \alpha_{35})/\alpha_{36})}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})((\alpha_{34} + \alpha_{35})/\alpha_{36})} = L_9, \\
n &= 2, 3, \dots
\end{aligned} \tag{16}$$

So, from (15) and (16), one gets

$$\begin{aligned}
L_7 &\leq x_n \leq U_7, \\
L_8 &\leq y_n \leq U_8, \\
L_9 &\leq z_n \leq U_9, \\
n &= 3, 4, \dots
\end{aligned} \tag{17}$$

(iv) If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution of (6), then

$$\begin{aligned}
x_n &\leq \frac{\alpha_{37} + \alpha_{38}}{\alpha_{39}} = U_{10}, \\
y_n &\leq \frac{\alpha_{40} + \alpha_{41}}{\alpha_{42}} = U_{11}, \\
z_n &\leq \frac{\alpha_{43} + \alpha_{44}}{\alpha_{45}} = U_{12}, \\
n &= 1, 2, \dots
\end{aligned} \tag{18}$$

From (6) and (18), one gets

$$\begin{aligned}
x_n &\geq \frac{(\alpha_{37} + \alpha_{38})e^{-((\alpha_{37} + \alpha_{38})/\alpha_{39})}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})((\alpha_{40} + \alpha_{41})/\alpha_{42})} = L_{10}, \\
y_n &\geq \frac{(\alpha_{40} + \alpha_{41})e^{-((\alpha_{40} + \alpha_{41})/\alpha_{42})}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})((\alpha_{37} + \alpha_{38})/\alpha_{39})} = L_{11}, \\
z_n &\geq \frac{(\alpha_{43} + \alpha_{44})e^{-((\alpha_{43} + \alpha_{44})/\alpha_{45})}}{\alpha_{45} + (\alpha_{43} + \alpha_{44})((\alpha_{43} + \alpha_{44})/\alpha_{45})} = L_{12}, \\
n &= 2, 3, \dots
\end{aligned} \tag{19}$$

So, from (18) and (19), one gets

$$\begin{aligned}
L_{10} &\leq x_n \leq U_{10}, \\
L_{11} &\leq y_n \leq U_{11}, \\
L_{12} &\leq z_n \leq U_{12}, \\
n &= 3, 4, \dots
\end{aligned} \tag{20}$$

(v) If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution of (7), then

$$\begin{aligned}
x_n &\leq \frac{\alpha_{46} + \alpha_{47}}{\alpha_{48}} = U_{13}, \\
y_n &\leq \frac{\alpha_{49} + \alpha_{50}}{\alpha_{51}} = U_{14}, \\
z_n &\leq \frac{\alpha_{52} + \alpha_{53}}{\alpha_{54}} = U_{15}, \\
n &= 1, 2, \dots
\end{aligned} \tag{21}$$

From (7) and (21), one gets

$$\begin{aligned}
x_n &\geq \frac{(\alpha_{46} + \alpha_{47})e^{-((\alpha_{46} + \alpha_{47})/\alpha_{48})}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})((\alpha_{52} + \alpha_{53})/\alpha_{54})} = L_{13}, \\
y_n &\geq \frac{(\alpha_{49} + \alpha_{50})e^{-((\alpha_{49} + \alpha_{50})/\alpha_{51})}}{\alpha_{51} + (\alpha_{49} + \alpha_{50})((\alpha_{49} + \alpha_{50})/\alpha_{51})} = L_{14}, \\
z_n &\geq \frac{(\alpha_{52} + \alpha_{53})e^{-((\alpha_{52} + \alpha_{53})/\alpha_{54})}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})((\alpha_{46} + \alpha_{47})/\alpha_{48})} = L_{15}, \\
n &= 2, 3, \dots
\end{aligned} \tag{22}$$

So, from (21) and (22), one gets

$$\begin{aligned} L_{13} &\leq x_n \leq U_{13}, \\ L_{14} &\leq y_n \leq U_{14}, \\ L_{15} &\leq z_n \leq U_{15}, \\ n &= 3, 4, \dots \end{aligned} \quad (23)$$

(vi) If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution of (8), then

$$\begin{aligned} x_n &\leq \frac{\alpha_{55} + \alpha_{56}}{\alpha_{57}} = U_{16}, \\ y_n &\leq \frac{\alpha_{58} + \alpha_{59}}{\alpha_{60}} = U_{17}, \\ z_n &\leq \frac{\alpha_{61} + \alpha_{62}}{\alpha_{63}} = U_{18}, \\ n &= 1, 2, \dots \end{aligned} \quad (24)$$

From (8) and (24), one gets

$$\begin{aligned} x_n &\geq \frac{(\alpha_{55} + \alpha_{56})e^{-((\alpha_{61} + \alpha_{62})/\alpha_{63})}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})((\alpha_{55} + \alpha_{56})/\alpha_{57})} = L_{16}, \\ y_n &\geq \frac{(\alpha_{58} + \alpha_{59})e^{-((\alpha_{55} + \alpha_{56})/\alpha_{57})}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})((\alpha_{58} + \alpha_{59})/\alpha_{60})} = L_{17}, \\ z_n &\geq \frac{(\alpha_{61} + \alpha_{62})e^{-((\alpha_{58} + \alpha_{59})/\alpha_{60})}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})((\alpha_{61} + \alpha_{62})/\alpha_{63})} = L_{18}, \\ n &= 2, 3, \dots \end{aligned} \quad (25)$$

So, from (24) and (25), one gets

$$\begin{aligned} L_{16} &\leq x_n \leq U_{16}, \\ L_{17} &\leq y_n \leq U_{17}, \\ L_{18} &\leq z_n \leq U_{18}, \\ n &= 3, 4, \dots \end{aligned} \quad (26)$$

□

3. Existence of Invariant Rectangle of Systems (3)–(8)

Theorem 2. If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution of systems (3)–(8), then their corresponding invariant rectangles,

respectively, are $[L_1, U_1] \times [L_2, U_2] \times [L_3, U_3]$, $[L_4, U_4] \times [L_5, U_5] \times [L_6, U_6]$, $[L_7, U_7] \times [L_8, U_8] \times [L_9, U_9]$, $[L_{10}, U_{10}] \times [L_{11}, U_{11}] \times [L_{12}, U_{12}]$, $[L_{13}, U_{13}] \times [L_{14}, U_{14}] \times [L_{15}, U_{15}]$, and $[L_{16}, U_{16}] \times [L_{17}, U_{17}] \times [L_{18}, U_{18}]$.

Proof. If $\{\Omega_n\}_{n=-1}^\infty$ is a positive solution with $x_0, x_{-1} \in [L_1, U_1]$, $y_0, y_{-1} \in [L_2, U_2]$, and $z_0, z_{-1} \in [L_3, U_3]$, then from (3), one has

$$\begin{aligned} x_1 &= \frac{\alpha_{10}e^{-y_0} + \alpha_{11}e^{-y_{-1}}}{\alpha_{12} + \alpha_{10}y_0 + \alpha_{11}y_{-1}} \leq \frac{\alpha_{10} + \alpha_{11}}{\alpha_{12}}, \\ x_1 &= \frac{\alpha_{10}e^{-y_0} + \alpha_{11}e^{-y_{-1}}}{\alpha_{12} + \alpha_{10}y_0 + \alpha_{11}y_{-1}} \geq \frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})/\alpha_{15})}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})((\alpha_{13} + \alpha_{14})/\alpha_{15})}, \\ y_1 &= \frac{\alpha_{13}e^{-z_0} + \alpha_{14}e^{-z_{-1}}}{\alpha_{15} + \alpha_{13}z_0 + \alpha_{14}z_{-1}} \leq \frac{\alpha_{13} + \alpha_{14}}{\alpha_{15}}, \\ y_1 &= \frac{\alpha_{13}e^{-z_0} + \alpha_{14}e^{-z_{-1}}}{\alpha_{15} + \alpha_{13}z_0 + \alpha_{14}z_{-1}} \geq \frac{(\alpha_{13} + \alpha_{14})e^{-((\alpha_{16} + \alpha_{17})/\alpha_{18})}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})((\alpha_{16} + \alpha_{17})/\alpha_{18})}, \\ z_1 &= \frac{\alpha_{16}e^{-x_0} + \alpha_{17}e^{-x_{-1}}}{\alpha_{18} + \alpha_{16}x_0 + \alpha_{17}x_{-1}} \leq \frac{\alpha_{16} + \alpha_{17}}{\alpha_{18}}, \\ z_1 &= \frac{\alpha_{16}e^{-x_0} + \alpha_{17}e^{-x_{-1}}}{\alpha_{18} + \alpha_{16}x_0 + \alpha_{17}x_{-1}} \geq \frac{(\alpha_{16} + \alpha_{17})e^{-((\alpha_{10} + \alpha_{11})/\alpha_{12})}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})((\alpha_{10} + \alpha_{11})/\alpha_{12})}. \end{aligned} \quad (27)$$

Hence (27) then implies that $x_1 \in [L_1, U_1]$, $y_1 \in [L_2, U_2]$, and $z_1 \in [L_3, U_3]$. Finally from (3), it is easy to establish that $x_{k+1} \in [L_1, U_1]$ (resp., $y_{k+1} \in [L_2, U_2]$ and $z_{k+1} \in [L_3, U_3]$) if $x_k \in [L_1, U_1]$ (resp., $y_k \in [L_2, U_2]$ and $z_k \in [L_3, U_3]$). □

Remark 1. In a similar way, one can prove the invariant rectangle for systems (4)–(8).

4. Existence as well as Uniqueness of Positive Fixed Point of Systems (3)–(8)

Existence as well as uniqueness of a positive fixed point of systems (3)–(8) is explored in this section, as follows.

Theorem 3.

(i) System (3) has a unique positive fixed point: $Y_1 = (\bar{x}, \bar{y}, \bar{z}) \in [L_1, U_1] \times [L_2, U_2] \times [L_3, U_3]$ if

$$\begin{aligned} &\frac{(\alpha_{13} + \alpha_{14})e^{-L_1}((U_1 + 1)(\alpha_{13} + \alpha_{14}) + \alpha_{15})(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-U_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1))}}{(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)^2(\alpha_{18} + (\alpha_{16} + \alpha_{17}))^2} \\ &\times \frac{((\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)) + 1) + \alpha_{12}}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)))^2} (\alpha_{16} + \alpha_{17}) \\ &\cdot e^{-((\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1))}/(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)))} \\ &\times ((\alpha_{16} + \alpha_{17})(\Lambda_1 + 1) + \alpha_{18}) < \Lambda_1^2, \end{aligned} \quad (28)$$

where

$$\Lambda_1 = \frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-U_3})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3))}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_3})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3))}; \quad (29)$$

(ii) System (4) has a unique positive fixed point: $Y_2 = (\bar{x}, \bar{y}, \bar{z}) \in [L_4, U_4] \times [L_5, U_5] \times [L_6, U_6]$ if

$$\begin{aligned} & \frac{(\alpha_{19} + \alpha_{20})e^{-L_6}((U_6 + 1)(\alpha_{19} + \alpha_{20}) + \alpha_{21})(\alpha_{22} + \alpha_{23})e^{-((\alpha_{19} + \alpha_{20})e^{-U_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6))}}{(\alpha_{21} + (\alpha_{19} + \alpha_{20})U_6)^2} \\ & \times \frac{((\alpha_{22} + \alpha_{23})(((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)) + 1) + \alpha_{24})(\alpha_{25} + \alpha_{26})e^{-((\alpha_{22} + \alpha_{23})e^{-((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6))})/((\alpha_{24} + (\alpha_{22} + \alpha_{23}))((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)))}}{(\alpha_{24} + (\alpha_{22} + \alpha_{23})(((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)))^2} \\ & \times \frac{((\alpha_{25} + \alpha_{26})(\Lambda_2 + 1) + \alpha_{27})}{(\alpha_{27} + (\alpha_{25} + \alpha_{26}))^2} < \Lambda_2^2, \end{aligned} \quad (30)$$

where

$$\Lambda_2 = \frac{(\alpha_{22} + \alpha_{23})e^{-((\alpha_{19} + \alpha_{20})e^{-U_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6))}}{\alpha_{24} + (\alpha_{22} + \alpha_{23})(((\alpha_{19} + \alpha_{20})e^{-L_6})/(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6))}; \quad (31)$$

(iii) System (5) has a unique positive fixed point: $Y_3 = (\bar{x}, \bar{y}, \bar{z}) \in [L_7, U_7] \times [L_8, U_8] \times [L_9, U_9]$ if

$$\begin{aligned} & \frac{e^{-((e^{-U_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))}e^{-L_8}(U_8 + 1)((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})) + 1)}{((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))^2 U_8^2} \\ & \cdot e^{-((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32}))) / (e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32}))) - (\alpha_{36}/(\alpha_{34} + \alpha_{35}))} < \Lambda_3^2, \end{aligned} \quad (32)$$

where

$$\Lambda_3 = \frac{e^{-((e^{-U_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))}}{((e^{-L_8}/L_8) - (\alpha_{33}/(\alpha_{31} + \alpha_{32})))} - \frac{\alpha_{36}}{\alpha_{34} + \alpha_{35}}; \quad (33)$$

(iv) System (6) has a unique positive fixed point: $Y_4 = (\bar{x}, \bar{y}, \bar{z}) \in [L_{10}, U_{10}] \times [L_{11}, U_{11}] \times [L_{12}, U_{12}]$ if

$$\begin{aligned} & \frac{e^{-((e^{-U_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})))}e^{-L_{10}}(U_{10} + 1)((e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})) + 1)}{((e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})))^2 U_{10}^2} \\ & \cdot e^{-((e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38}))) / (e^{-L_{10}}/L_{10}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38}))) - (\alpha_{42}/(\alpha_{40} + \alpha_{41}))} < \Lambda_4^2, \end{aligned} \quad (34)$$

where

$$\Lambda_4 = \frac{e^{-((e^{-U_{10}/L_{10}}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})))}}{((e^{-L_{10}/L_{10}}) - (\alpha_{39}/(\alpha_{37} + \alpha_{38})))} - \frac{\alpha_{42}}{\alpha_{40} + \alpha_{41}}; \quad (35)$$

(v) System (7) has a unique positive fixed point: $Y_5 = (\bar{x}, \bar{y}, \bar{z}) \in [L_{13}, U_{13}] \times [L_{14}, U_{14}] \times [L_{15}, U_{15}]$ if

$$\frac{e^{-((e^{-U_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})))} e^{-L_{13}} (U_{13} + 1) ((e^{-L_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})) + 1)}{((e^{-L_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})))^2 U_{13}^2} \cdot e^{-((e^{-((e^{-L_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})))}/(e^{-L_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47}))) - (\alpha_{54}/(\alpha_{52} + \alpha_{53})))} < \Lambda_5^2, \quad (36)$$

where

$$\Lambda_5 = \frac{e^{-((e^{-U_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})))}}{((e^{-L_{13}/L_{13}}) - (\alpha_{48}/(\alpha_{46} + \alpha_{47})))} - \frac{\alpha_{54}}{\alpha_{52} + \alpha_{53}}; \quad (37)$$

(vi) System (8) has a unique positive fixed point: $Y_6 = (\bar{x}, \bar{y}, \bar{z}) \in [L_{16}, U_{16}] \times [L_{17}, U_{17}] \times [L_{18}, U_{18}]$ if

$$\begin{aligned} & \frac{(\alpha_{57} + 2U_{16}(\alpha_{55} + \alpha_{56}))(\alpha_{63} + 2U_{16}(\alpha_{61} + \alpha_{62})\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}))))}{(\alpha_{63}\ln(\alpha_{55} + \alpha_{56})/L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}) + (\alpha_{61} + \alpha_{62})(\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}))))^2)} \\ & \times \left(\frac{\alpha_{57} + 2U_{16}(\alpha_{55} + \alpha_{56})(\alpha_{63} + 2(\alpha_{61} + \alpha_{62})\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}))))}{\alpha_{63}\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}))))} + \Lambda_6 \right) \\ & \times \frac{\alpha_{60}(\alpha_{58} + \alpha_{59})}{L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})(\alpha_{60} + (\alpha_{58} + \alpha_{59}))} < \Lambda_6, \end{aligned} \quad (38)$$

where

$$\Lambda_6 = \ln \left(\frac{\alpha_{61} + \alpha_{62}}{\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})))(\alpha_{63} + (\alpha_{61} + \alpha_{62})\ln((\alpha_{55} + \alpha_{56})/(L_{16}(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}))))} \right). \quad (39)$$

Proof. (i) From (3), one has

$$\begin{aligned} x &= \frac{(\alpha_{10} + \alpha_{11})e^{-y}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})y}, \\ y &= \frac{(\alpha_{13} + \alpha_{14})e^{-z}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})z}, \\ z &= \frac{(\alpha_{16} + \alpha_{17})e^{-x}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})x}. \end{aligned} \quad (40)$$

From (40), one gets

$$\begin{aligned} z &= \frac{(\alpha_{16} + \alpha_{17})e^{-x}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})x}, \\ x &= \frac{(\alpha_{10} + \alpha_{11})e^{-y}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})y}, \\ y &= \frac{(\alpha_{13} + \alpha_{14})e^{-z}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})z}. \end{aligned} \quad (41)$$

From (41), setting

$$\begin{aligned} y &= g(z) = \frac{(\alpha_{13} + \alpha_{14})e^{-z}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})z}, \\ x &= f(y) = \frac{(\alpha_{10} + \alpha_{11})e^{-y}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})y}. \end{aligned} \quad (42)$$

Denote

$$F(z) := \frac{(\alpha_{16} + \alpha_{17})e^{-k(z)}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})k(z)} - z, \quad (43)$$

where

$$k(z) := x = f(g(z)),$$

$$\begin{aligned} &= f\left(\frac{(\alpha_{13} + \alpha_{14})e^{-z}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})z}\right), \\ &= \frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))}, \end{aligned} \quad (44)$$

and $z \in [L_3, U_3]$. Here, our finding is that $F(z) = 0$ has a unique solution, where $z \in [L_3, U_3]$. From (43) and (44), one gets

$$F'(z) = -\frac{k'(z)(\alpha_{16} + \alpha_{17})e^{-k(z)}((k(z) + 1)(\alpha_{16} + \alpha_{17}) + \alpha_{18})}{(\alpha_{18} + (\alpha_{16} + \alpha_{17})k(z))^2} - 1, \quad (45)$$

where

$$\begin{aligned} k'(z) &= -\frac{(\alpha_{13} + \alpha_{14})e^{-z}((z + 1)(\alpha_{13} + \alpha_{14}) + \alpha_{15})(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)}}{(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)^2(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)))^2} \\ &\quad \times \left((\alpha_{10} + \alpha_{11}) \left(\frac{(\alpha_{13} + \alpha_{14})e^{-z}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})z} + 1 \right) + \alpha_{12} \right). \end{aligned} \quad (46)$$

Using (44) and (46) in (45), we obtain

$$\begin{aligned} F'(z) &= \frac{(\alpha_{13} + \alpha_{14})e^{-z}((z + 1)(\alpha_{13} + \alpha_{14}) + \alpha_{15})(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)}}{(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)^2(\alpha_{18} + (\alpha_{16} + \alpha_{17})(((\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))})/(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))))^2} \times \\ &\quad \frac{((\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)) + 1) + \alpha_{12}}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)))^2} (\alpha_{16} + \alpha_{17})e^{-((\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))})/(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))))} \times \\ &\quad \cdot \left((\alpha_{16} + \alpha_{17}) \left(\frac{(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z)}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-z})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})z))} + 1 \right) + \alpha_{18} \right) - 1 \\ &\leq \frac{(\alpha_{13} + \alpha_{14})e^{-L_1}((U_1 + 1)(\alpha_{13} + \alpha_{14}) + \alpha_{15})(\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-U_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)}}{(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)^2(\alpha_{18} + (\alpha_{16} + \alpha_{17})\Lambda_1)^2} \times \\ &\quad \frac{((\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)) + 1) + \alpha_{12}}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1)))^2} (\alpha_{16} + \alpha_{17})e^{-((\alpha_{10} + \alpha_{11})e^{-((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1))})/(\alpha_{12} + (\alpha_{10} + \alpha_{11})(((\alpha_{13} + \alpha_{14})e^{-L_1})/(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_1))))} \times \\ &\quad \cdot ((\alpha_{16} + \alpha_{17})(\Lambda_1 + 1) + \alpha_{18}) - 1, \end{aligned} \quad (47)$$

where Λ_1 is depicted in (29). Now, assuming that if (28) along with (29) holds then from (47), one gets $F'(z) < 0$. \square

Remark 2. The proof of (ii)–(vi) is same as the proof of (i). So, it is omitted.

5. Local Dynamics about Unique Positive Fixed Point of Systems (3)–(8)

The local dynamics about $Y_i (i = 1, \dots, 6)$, respectively, of systems (3) to (8) is explored in this section, as follows.

Theorem 4. For Y_1 of (3), the following holds:

(i) Y_1 is a sink if

$$\begin{aligned} & \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ & + \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ & + \frac{(\alpha_{11}\alpha_{13} + \alpha_{10}\alpha_{14})(\alpha_{16} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} < 1; \end{aligned} \quad (48)$$

(ii) Y_1 is a source if

$$\begin{aligned} & \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\ & + \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\ & + \frac{(\alpha_{11}\alpha_{13} + \alpha_{10}\alpha_{14})(\alpha_{16} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} > 1. \end{aligned} \quad (49)$$

Proof. (i) If Y_1 is a fixed point of (3), then

$$\begin{aligned} \bar{x} &= \frac{(\alpha_{10} + \alpha_{11})e^{-\bar{y}}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}}, \\ \bar{y} &= \frac{(\alpha_{13} + \alpha_{14})e^{-\bar{z}}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}}, \\ \bar{z} &= \frac{(\alpha_{16} + \alpha_{17})e^{-\bar{x}}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}}. \end{aligned} \quad (50)$$

Moreover, we have the following map for constructing the corresponding linearized form of (3):

$$(x_{n+1}, x_n, y_{n+1}, y_n, z_{n+1}, z_n) \longrightarrow (f, f_1, g, g_1, h, h_1), \quad (51)$$

where

$$\begin{aligned} f &= \frac{\alpha_{10}e^{-y_n} + \alpha_{11}e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}, \\ f_1 &= x_n, \\ g &= \frac{\alpha_{13}e^{-z_n} + \alpha_{14}e^{-z_{n-1}}}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}, \\ g_1 &= y_n, \\ h &= \frac{\alpha_{16}e^{-x_n} + \alpha_{17}e^{-x_{n-1}}}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}, \\ h_1 &= z_n. \end{aligned} \quad (52)$$

The $J|_{Y_1}$ about Y_1 subject to the map (51) is

$$J|_{Y_1} = \begin{pmatrix} 0 & 0 & b_{13} & b_{14} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{35} & b_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{51} & b_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (53)$$

where

$$\begin{aligned} b_{13} &= -\frac{\alpha_{10}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}}, \\ b_{14} &= -\frac{\alpha_{11}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}}, \\ b_{35} &= -\frac{\alpha_{13}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}}, \\ b_{36} &= -\frac{\alpha_{14}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}}, \\ b_{51} &= -\frac{\alpha_{16}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}}, \\ b_{52} &= -\frac{\alpha_{17}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}}. \end{aligned} \quad (54)$$

The auxiliary equation of $J|_{Y_1}$ about Y_1 is

$$P(\lambda) = \lambda^6 + L_1\lambda^3 + L_2\lambda^2 + L_3\lambda + L_4 = 0, \quad (55)$$

Now,

where

$$\begin{aligned} L_1 &= -b_{13}b_{35}b_{51}, \\ L_2 &= -b_{13}b_{35}b_{52} - b_{13}b_{36}b_{51} - b_{14}b_{35}b_{51}, \\ L_3 &= -b_{13}b_{36}b_{52} - b_{14}b_{35}b_{52} - b_{14}b_{36}b_{51}, \\ L_4 &= -b_{14}b_{36}b_{52}. \end{aligned} \quad (56)$$

$$\begin{aligned} \sum_{i=1}^4 |L_i| &= \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-\bar{x}} + \bar{z})(e^{-\bar{y}} + \bar{x})(e^{-\bar{z}} + \bar{y})}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y})(\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z})(\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x})} \\ &+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-\bar{x}} + \bar{z})(e^{-\bar{y}} + \bar{x})(e^{-\bar{z}} + \bar{y})}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y})(\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z})(\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x})} \\ &+ \frac{\alpha_{11}\alpha_{13}(\alpha_{16} + \alpha_{17})(e^{-\bar{x}} + \bar{z})(e^{-\bar{y}} + \bar{x})(e^{-\bar{z}} + \bar{y})}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y})(\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z})(\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x})} \\ &+ \frac{\alpha_{10}\alpha_{14}(\alpha_{16} + \alpha_{17})(e^{-\bar{x}} + \bar{z})(e^{-\bar{y}} + \bar{x})(e^{-\bar{z}} + \bar{y})}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y})(\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z})(\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x})} \\ &\leq \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &+ \frac{\alpha_{11}\alpha_{13}(\alpha_{16} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &+ \frac{\alpha_{10}\alpha_{14}(\alpha_{16} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &= \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)} \\ &+ \frac{(\alpha_{11}\alpha_{13} + \alpha_{10}\alpha_{14})(\alpha_{16} + \alpha_{17})(e^{-L_1} + L_3)(e^{-L_2} + L_1)(e^{-L_3} + L_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)}. \end{aligned} \quad (57)$$

Assuming that (48) holds, and then from (57), one gets $\sum_{i=1}^4 |L_i| < 1$. Hence, Y_1 of (3) is a sink.

(ii) Using similar manipulation as in the proof of (i), one has

$$\begin{aligned}
\sum_{i=1}^4 |L_i| &\geq \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&+ \frac{\alpha_{11}\alpha_{13}(\alpha_{16} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{16} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&= \frac{\alpha_{10}\alpha_{13}(\alpha_{14} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&+ \frac{\alpha_{10}\alpha_{14}(\alpha_{13} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)} \\
&+ \frac{(\alpha_{11}\alpha_{13} + \alpha_{10}\alpha_{14})(\alpha_{16} + \alpha_{17})(e^{-U_1} + U_3)(e^{-U_2} + U_1)(e^{-U_3} + U_2)}{(\alpha_{12} + (\alpha_{10} + \alpha_{11})U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})U_1)}.
\end{aligned} \tag{58}$$

Assuming that (49) holds, and then from (58), one gets $\sum_{i=1}^4 |L_i| > 1$. Hence, Y_1 of (3) is a source. \square

In similar manner, one can explore the local dynamics about Y_i ($i = 2, \dots, 6$), respectively, of systems (4)–(8), as follows.

Theorem 5.

(i) For Y_2 of (4), the following holds:

(i.1) Y_2 is a sink if

$$\begin{aligned}
&\frac{\alpha_{19}\alpha_{25}(\alpha_{22} + \alpha_{23})(e^{-L_4} + L_5)(e^{-L_6} + L_4)(e^{-L_4} + L_6)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})L_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})L_4)} \\
&+ \frac{\alpha_{19}\alpha_{26}(\alpha_{22} + \alpha_{23})(e^{-L_4} + L_5)(e^{-L_5} + L_6)(e^{-L_6} + L_4)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})L_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})L_4)} \\
&+ \frac{(\alpha_{20}\alpha_{22} + \alpha_{20}\alpha_{23})(\alpha_{26} + \alpha_{25})(e^{-L_4} + L_5)(e^{-L_5} + L_6)(e^{-L_6} + L_4)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})L_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})L_4)} < 1.
\end{aligned} \tag{59}$$

(i.2) Y_2 is a source if

$$\begin{aligned}
&\frac{\alpha_{19}\alpha_{25}(\alpha_{22} + \alpha_{23})(e^{-U_4} + U_5)(e^{-U_6} + U_4)(e^{-U_4} + U_6)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})U_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})U_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})U_4)} \\
&+ \frac{\alpha_{19}\alpha_{26}(\alpha_{22} + \alpha_{23})(e^{-U_4} + U_5)(e^{-U_5} + U_6)(e^{-U_6} + U_4)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})U_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})U_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})U_4)} \\
&+ \frac{(\alpha_{20}\alpha_{22} + \alpha_{20}\alpha_{23})(\alpha_{26} + \alpha_{25})(e^{-U_4} + U_5)(e^{-U_5} + U_6)(e^{-U_6} + U_4)}{(\alpha_{27} + (\alpha_{25} + \alpha_{26})U_5)(\alpha_{21} + (\alpha_{19} + \alpha_{20})U_6)(\alpha_{24} + (\alpha_{22} + \alpha_{23})U_4)} > 1,
\end{aligned} \tag{60}$$

with

$$J|_{Y_2} = \begin{pmatrix} 0 & 0 & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{61}$$

where

$$\begin{aligned}
b_{15} &= -\frac{\alpha_{19}(e^{-\bar{x}} + \bar{y})}{\alpha_{21} + (\alpha_{19} + \alpha_{20})\bar{x}}, \\
b_{16} &= -\frac{\alpha_{20}(e^{-\bar{x}} + \bar{y})}{\alpha_{21} + (\alpha_{19} + \alpha_{20})\bar{x}}, \\
b_{31} &= -\frac{\alpha_{22}(e^{-\bar{z}} + \bar{x})}{\alpha_{24} + (\alpha_{22} + \alpha_{23})\bar{z}}, \\
b_{32} &= -\frac{\alpha_{23}(e^{-\bar{z}} + \bar{x})}{\alpha_{24} + (\alpha_{22} + \alpha_{23})\bar{z}}, \\
b_{53} &= -\frac{\alpha_{25}(e^{-\bar{y}} + \bar{z})}{\alpha_{27} + (\alpha_{25} + \alpha_{26})\bar{y}}, \\
b_{54} &= -\frac{\alpha_{26}(e^{-\bar{y}} + \bar{z})}{\alpha_{27} + (\alpha_{25} + \alpha_{26})\bar{y}};
\end{aligned} \tag{62}$$

(ii) For Y_3 of (5), the following holds:

(ii.1) Y_3 is a sink if

$$\begin{aligned}
& \frac{(\alpha_{28} + \alpha_{29})(e^{-L_7} + L_7)}{\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7} + \frac{(\alpha_{31} + \alpha_{32})e^{-L_8}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9} + \frac{(\alpha_{34} + \alpha_{35})e^{-L_9}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8} \\
& + \frac{(\alpha_{34}\alpha_{28} + \alpha_{29}\alpha_{34} + \alpha_{35}\alpha_{28} + \alpha_{29}\alpha_{35} + \alpha_{29}\alpha_{32})e^{-L_7-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{31}\alpha_{34} + \alpha_{35}\alpha_{31} + \alpha_{32}\alpha_{35})e^{-L_8-L_9}}{(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{28}\alpha_{34} + \alpha_{29}\alpha_{34} + \alpha_{28}\alpha_{35} + \alpha_{29}\alpha_{32} + \alpha_{29}\alpha_{35})L_7e^{-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{31}\alpha_{34} + \alpha_{32}\alpha_{34} + \alpha_{31}\alpha_{35} + \alpha_{32}\alpha_{35})L_7L_8}{(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})e^{-L_7-L_8-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})e^{-L_7-L_8-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})L_7L_8L_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})L_7L_8L_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})e^{-L_7}L_8L_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})e^{-L_7}L_8L_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})L_7e^{-L_8-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})L_7e^{-L_8-L_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8)} < 1.
\end{aligned} \tag{63}$$

(ii.2) Y_3 is a source if

$$\begin{aligned}
& \frac{(\alpha_{28} + \alpha_{29})(e^{-U_7} + U_7)}{\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7} + \frac{(\alpha_{31} + \alpha_{32})e^{-U_8}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9} + \frac{(\alpha_{34} + \alpha_{35})e^{-U_9}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8} \\
& + \frac{(\alpha_{34}\alpha_{28} + \alpha_{29}\alpha_{34} + \alpha_{35}\alpha_{28} + \alpha_{29}\alpha_{35} + \alpha_{29}\alpha_{32})e^{-U_7-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{31}\alpha_{34} + \alpha_{35}\alpha_{31} + \alpha_{32}\alpha_{35})e^{-U_8-U_9}}{(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{28}\alpha_{34} + \alpha_{29}\alpha_{34} + \alpha_{28}\alpha_{35} + \alpha_{29}\alpha_{32} + \alpha_{29}\alpha_{35})U_7e^{-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{31}\alpha_{34} + \alpha_{32}\alpha_{34} + \alpha_{31}\alpha_{35} + \alpha_{32}\alpha_{35})U_7U_8}{(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})e^{-U_7-U_8-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})e^{-U_7-U_8-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})U_7U_8U_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})U_7U_8U_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})e^{-U_7}U_8U_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})e^{-U_7}U_8U_9}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{28}\alpha_{31}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{34} + \alpha_{28}\alpha_{32}\alpha_{34} + \alpha_{28}\alpha_{35}\alpha_{31})U_7e^{-U_8-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} \\
& + \frac{(\alpha_{29}\alpha_{32}\alpha_{34} + \alpha_{29}\alpha_{31}\alpha_{35} + \alpha_{28}\alpha_{32}\alpha_{35} + \alpha_{29}\alpha_{32}\alpha_{35})U_7e^{-U_8-U_9}}{(\alpha_{30} + (\alpha_{28} + \alpha_{29})U_7)(\alpha_{33} + (\alpha_{31} + \alpha_{32})U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})U_8)} > 1,
\end{aligned} \tag{64}$$

with

$$J|_{Y_3} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & b_{35} & b_{36} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (65)$$

where

$$\begin{aligned} b_{11} &= -\frac{\alpha_{28}(e^{-\bar{x}} + \bar{x})}{\alpha_{30} + (\alpha_{28} + \alpha_{29})\bar{x}}, \\ b_{12} &= -\frac{\alpha_{29}(e^{-\bar{x}} + \bar{x})}{\alpha_{30} + (\alpha_{28} + \alpha_{29})\bar{x}}, \\ b_{33} &= -\frac{\alpha_{31}e^{-\bar{y}}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})\bar{z}}, \\ b_{34} &= -\frac{\alpha_{32}e^{-\bar{y}}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})\bar{z}}, \\ b_{35} &= -\frac{\alpha_{31}\bar{y}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})\bar{z}}, \\ b_{36} &= -\frac{\alpha_{32}\bar{y}}{\alpha_{33} + (\alpha_{31} + \alpha_{32})\bar{z}}, \\ b_{53} &= -\frac{\alpha_{34}\bar{z}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})\bar{y}}, \\ b_{54} &= -\frac{\alpha_{35}\bar{z}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})\bar{y}}, \\ b_{55} &= -\frac{\alpha_{34}e^{-\bar{z}}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})\bar{y}}, \\ b_{56} &= -\frac{\alpha_{35}e^{-\bar{z}}}{\alpha_{36} + (\alpha_{34} + \alpha_{35})\bar{y}}; \end{aligned} \quad (66)$$

(iii) For Y_4 of (6), the following holds:

(iii.1) Y_4 is a sink if

$$\begin{aligned} & \frac{(\alpha_{37} + \alpha_{38})e^{-L_{10}}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11}} + \frac{(\alpha_{40} + \alpha_{41})e^{-L_{11}}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10}} \\ & + \frac{(\alpha_{43} + \alpha_{44})(e^{-L_{12}} + L_{12})}{\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12}} \\ & + \frac{(\alpha_{37}\alpha_{40} + \alpha_{41}\alpha_{37} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41})e^{-L_{10}-L_{11}}}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10})} \\ & + \frac{(\alpha_{37}\alpha_{40} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41})L_{10}L_{11}}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10})} \\ & + \frac{(\alpha_{40}\alpha_{43} + \alpha_{41}\alpha_{44} + \alpha_{41}\alpha_{43} + \alpha_4\alpha_{44})e^{-L_{11}}(e^{-L_{12}} + L_{12})}{(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10})(\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12})} \\ & + \frac{(\alpha_{37}\alpha_{43} + \alpha_{38}\alpha_{43} + \alpha_{38}\alpha_{44} + \alpha_{37}\alpha_{44})e^{-L_{10}}(e^{-L_{12}} + L_{10})}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11})(\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12})} \\ & + [(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} \\ & + \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})e^{-L_{10}-L_{11}-L_{12}} \\ & + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} \\ & + \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})L_{10}L_{11}L_{12} \\ & + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} \\ & + \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})e^{-L_{12}}L_{10}L_{11} \\ & + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{43} \\ & + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})L_{12}e^{-L_{10}-L_{11}}] \\ & \frac{1}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10})(\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12})} \\ & < 1. \end{aligned} \quad (67)$$

(iii.2) Y_4 is a source if

$$\begin{aligned}
& \frac{(\alpha_{37} + \alpha_{38})e^{-U_{10}}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})U_{11}} + \frac{(\alpha_{40} + \alpha_{41})e^{-U_{11}}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})U_{10}} \\
& + \frac{(\alpha_{43} + \alpha_{44})(e^{-U_{12}} + L_{12})}{\alpha_{45} + (\alpha_{43} + \alpha_{44})U_{12}} \\
& + \frac{(\alpha_{37}\alpha_{40} + \alpha_{41}\alpha_{37} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41})e^{-U_{10}-U_{11}}}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})U_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})U_{10})} \\
& + \frac{(\alpha_{37}\alpha_{40} + \alpha_{38}\alpha_{40} + \alpha_{37}\alpha_{41} + \alpha_{38}\alpha_{41})U_{10}U_{11}}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})U_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})U_{10})} \\
& + \frac{(\alpha_{40}\alpha_{43} + \alpha_{41}\alpha_{44} + \alpha_{41}\alpha_{43} + \alpha_{44}\alpha_{44})e^{-U_{11}}(e^{-U_{12}} + U_{12})}{(\alpha_{42} + (\alpha_{40} + \alpha_{41})U_{10})(\alpha_{45} + (\alpha_{43} + \alpha_{44})U_{12})} \\
& + \frac{(\alpha_{37}\alpha_{43} + \alpha_{38}\alpha_{43} + \alpha_{38}\alpha_{44} + \alpha_{37}\alpha_{44})e^{-U_{10}}(e^{-U_{12}} + U_{10})}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})U_{11})(\alpha_{45} + (\alpha_{43} + \alpha_{44})U_{12})} \\
& + [(\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} + \\
& \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})e^{-U_{10}-U_{11}-U_{12}} \\
& + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} \\
& + \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})U_{10}U_{11}U_{12} \\
& + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} \\
& + \alpha_{38}\alpha_{41}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})e^{-U_{12}}U_{10}U_{11} \\
& + (\alpha_{37}\alpha_{40}\alpha_{43} + \alpha_{38}\alpha_{40}\alpha_{43} + \alpha_{37}\alpha_{41}\alpha_{43} + \alpha_{37}\alpha_{40}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{43} \\
& + \alpha_{38}\alpha_{40}\alpha_{44} + \alpha_{37}\alpha_{41}\alpha_{44} + \alpha_{38}\alpha_{41}\alpha_{44})U_{12}e^{-U_{10}-U_{11}}] \\
& \times \frac{1}{(\alpha_{39} + (\alpha_{37} + \alpha_{38})U_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})U_{10})(\alpha_{45} + (\alpha_{43} + \alpha_{44})U_{12})} \\
& > 1,
\end{aligned} \tag{68}$$

with

$$J|_{Y_4} = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{69}$$

where

$$\begin{aligned}
b_{11} &= -\frac{\alpha_{37}e^{-\bar{x}}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})\bar{y}}, \\
b_{12} &= -\frac{\alpha_{38}e^{-\bar{x}}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})\bar{y}}, \\
b_{13} &= -\frac{\alpha_{37}\bar{x}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})\bar{y}}, \\
b_{14} &= -\frac{\alpha_{38}\bar{x}}{\alpha_{39} + (\alpha_{37} + \alpha_{38})\bar{y}}, \\
b_{31} &= -\frac{\alpha_{40}\bar{y}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})\bar{x}}, \\
b_{32} &= -\frac{\alpha_{41}\bar{y}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})\bar{x}}, \\
b_{33} &= -\frac{\alpha_{40}e^{-\bar{y}}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})\bar{x}}, \\
b_{34} &= -\frac{\alpha_{41}e^{-\bar{y}}}{\alpha_{42} + (\alpha_{40} + \alpha_{41})\bar{x}}, \\
b_{55} &= -\frac{\alpha_{43}(e^{-\bar{z}} + \bar{z})}{\alpha_{45} + (\alpha_{43} + \alpha_{44})\bar{z}}, \\
b_{56} &= -\frac{\alpha_{44}(e^{-\bar{z}} + \bar{z})}{\alpha_{45} + (\alpha_{43} + \alpha_{44})\bar{z}}.
\end{aligned} \tag{70}$$

(iv) For Y_5 of (7), the following holds:

(iv.1) Y_5 is a sink if

$$\begin{aligned}
& \frac{(\alpha_{46} + \alpha_{47})e^{-L_{13}}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15}} + \frac{(\alpha_{52} + \alpha_{53})e^{-L_{15}}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13}} + \frac{(\alpha_{49} + \alpha_{50})(e^{-L_{14}} + L_{14})}{\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14}} \\
& + \frac{(\alpha_{46}\alpha_{52} + \alpha_{47}\alpha_{52} + \alpha_{46}\alpha_{53} + \alpha_{47}\alpha_{53})e^{-L_{13}-L_{15}}}{(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15})(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13})} \\
& + \frac{(\alpha_{46}\alpha_{52} + \alpha_{47}\alpha_{52} + \alpha_{46}\alpha_{53} + \alpha_{47}\alpha_{53})L_{13}L_{15}}{(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15})(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13})} \\
& + \frac{(\alpha_{\alpha_{46}}\alpha_{49} + \alpha_{47}\alpha_{47} + \alpha_{46}\alpha_{50} + \alpha_{47}\alpha_{50})e^{-L_{13}}(e^{-L_{14}} + L_{14})}{(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15})(\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14})} \\
& + \frac{(\alpha_{49}\alpha_{52} + \alpha_{50}\alpha_{52} + \alpha_{49}\alpha_{53} + \alpha_{50}\alpha_{53})e^{-L_{15}}(e^{-L_{14}} + L_{14})}{(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13})(\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14})} \\
& + [(\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
& + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})e^{-L_{13}-L_{14}-L_{15}} \\
& + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
& + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})L_{13}L_{14}L_{15} \\
& + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
& + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})e^{-L_{14}}L_{13}L_{15} \\
& + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
& + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})L_{14}e^{-L_{13}-L_{15}}] \\
& \times \frac{1}{(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15})(\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14})(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13})} < 1.
\end{aligned} \tag{71}$$

(iv.2) Υ_5 is a source if

$$\begin{aligned}
 & \frac{(\alpha_{46} + \alpha_{47})e^{-U_{13}}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})U_{15}} + \frac{(\alpha_{52} + \alpha_{53})e^{-U_{15}}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})U_{13}} \\
 & + \frac{(\alpha_{49} + \alpha_{50})(e^{-U_{14}} + U_{14})}{\alpha_{51} + (\alpha_{49} + \alpha_{50})U_{14}} \\
 & \frac{(\alpha_{46}\alpha_{52} + \alpha_{47}\alpha_{52} + \alpha_{46}\alpha_{53} + \alpha_{47}\alpha_{53})e^{-U_{13}-U_{15}}}{(\alpha_{54} + (\alpha_{52} + \alpha_{53})U_{13})(\alpha_{48} + (\alpha_{46} + \alpha_{47})U_{15})} \\
 & + \frac{(\alpha_{46}\alpha_{52} + \alpha_{47}\alpha_{52} + \alpha_{46}\alpha_{53} + \alpha_{47}\alpha_{53})U_{13}U_{15}}{(\alpha_{54} + (\alpha_{52} + \alpha_{53})U_{13})(\alpha_{48} + (\alpha_{46} + \alpha_{47})U_{15})} \\
 & + \frac{(\alpha_{46}\alpha_{49} + \alpha_{47}\alpha_{49} + \alpha_{46}\alpha_{50} + \alpha_{47}\alpha_{50})e^{-U_{13}}(e^{-U_{14}} + U_{14})}{(\alpha_{54} + (\alpha_{52} + \alpha_{53})U_{15})(\alpha_{51} + (\alpha_{49} + \alpha_{48})U_{14})} \\
 & + \frac{(\alpha_{49}\alpha_{52} + \alpha_{50}\alpha_{52} + \alpha_{49}\alpha_{53} + \alpha_{50}\alpha_{53})e^{-U_{15}}(e^{-U_{14}} + U_{14})}{(\alpha_{54} + (\alpha_{52} + \alpha_{53})U_{15})(\alpha_{51} + (\alpha_{49} + \alpha_{48})U_{14})} \\
 & + [(\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} + \\
 & \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})e^{-U_{13}-U_{14}-U_{15}} \\
 & + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
 & + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})U_{13}U_{14}U_{15} \\
 & + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} \\
 & + \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})e^{-U_{14}}U_{13}U_{15} \\
 & + (\alpha_{46}\alpha_{49}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{52} + \alpha_{46}\alpha_{50}\alpha_{52} + \alpha_{46}\alpha_{49}\alpha_{53} + \\
 & \alpha_{47}\alpha_{50}\alpha_{52} + \alpha_{47}\alpha_{49}\alpha_{53} + \alpha_{46}\alpha_{50}\alpha_{53} + \alpha_{47}\alpha_{50}\alpha_{53})U_{14}e^{-U_{13}-U_{15}}] \\
 & \times \frac{1}{(\alpha_{48} + (\alpha_{46} + \alpha_{47})U_{15})(\alpha_{51} + (\alpha_{49} + \alpha_{50})U_{14})(\alpha_{54}(\alpha_{52} + \alpha_{53})U_{13})} \\
 & > 1,
 \end{aligned} \tag{72}$$

with

$$J|_{\Upsilon_5} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{51} & b_{52} & 0 & 0 & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{73}$$

where

$$\begin{aligned}
 b_{11} &= -\frac{\alpha_{46}e^{-\bar{x}}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})\bar{z}}, \\
 b_{12} &= -\frac{\alpha_{47}e^{-\bar{x}}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})\bar{z}}, \\
 b_{15} &= -\frac{\alpha_{46}\bar{x}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})\bar{z}}, \\
 b_{16} &= -\frac{\alpha_{47}\bar{x}}{\alpha_{48} + (\alpha_{46} + \alpha_{47})\bar{z}}, \\
 b_{33} &= -\frac{\alpha_{49}(e^{-\bar{y}} + \bar{y})}{\alpha_{51} + (\alpha_{49} + \alpha_{50})\bar{y}}, \\
 b_{34} &= -\frac{\alpha_{50}(e^{-\bar{y}} + \bar{y})}{\alpha_{51} + (\alpha_{49} + \alpha_{50})\bar{y}}, \\
 b_{51} &= -\frac{\alpha_{52}\bar{z}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})\bar{x}}, \\
 b_{52} &= -\frac{\alpha_{53}\bar{z}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})\bar{x}}, \\
 b_{55} &= -\frac{\alpha_{52}e^{-\bar{z}}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})\bar{x}}, \\
 b_{56} &= -\frac{\alpha_{53}e^{-\bar{z}}}{\alpha_{54} + (\alpha_{52} + \alpha_{53})\bar{x}};
 \end{aligned} \tag{74}$$

(v) For Υ_6 of (8), the following holds:

(v.1) Υ_6 is a sink if

$$\begin{aligned}
& \frac{(\alpha_{55}\alpha_{58} + \alpha_{56}\alpha_{58} + \alpha_{55}\alpha_{59} + \alpha_{56}\alpha_{59})L_{16}L_{17}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})(\alpha_{60} + (\alpha_{58} + \alpha_{59})L_{17})} + \frac{(\alpha_{55} + \alpha_{56})L_{16}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16}} \\
& + \frac{(\alpha_{55}\alpha_{61} + \alpha_{55}\alpha_{62} + \alpha_{56}\alpha_{61} + \alpha_{56}\alpha_{62})L_{16}L_{18}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})(\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18})} + \frac{(\alpha_{58} + \alpha_{59})L_{17}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})L_{17}} \\
& + \frac{(\alpha_{58}\alpha_{62} + \alpha_{59}\alpha_{61} + \alpha_{59}\alpha_{62} + \alpha_{58}\alpha_{61})L_{16}L_{18}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})(\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18})} + \frac{(\alpha_{61} + \alpha_{62})L_{18}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18}} \\
& + [(\alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{62} \\
& + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{56}\alpha_{58}\alpha_{62} + \alpha_{55}\alpha_{58}\alpha_{62} + \alpha_{56}\alpha_{58}\alpha_{62})e^{-L_{16}-L_{17}-L_{18}} \\
& + (\alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{62} + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{59}\alpha_{61} \\
& + \alpha_{56}\alpha_{58}\alpha_{62} + \alpha_{55}\alpha_{59}\alpha_{62} + \alpha_{56}\alpha_{59}\alpha_{61} + \alpha_{56}\alpha_{59}\alpha_{62})L_{16}L_{17}L_{18}] \\
& \times \frac{1}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})L_{16})(\alpha_{60} + (\alpha_{58} + \alpha_{59})L_{17})(\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18})} < 1.
\end{aligned} \tag{75}$$

(v.2) Y_6 is a source if

$$\begin{aligned}
& \frac{(\alpha_{55}\alpha_{58} + \alpha_{56}\alpha_{58} + \alpha_{55}\alpha_{59} + \alpha_{56}\alpha_{59})U_{16}U_{17}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})U_{16})(\alpha_{60} + (\alpha_{58} + \alpha_{59})U_{17})} + \frac{(\alpha_{55} + \alpha_{56})U_{16}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})U_{16}} \\
& + \frac{(\alpha_{55}\alpha_{61} + \alpha_{55}\alpha_{62} + \alpha_{56}\alpha_{61} + \alpha_{56}\alpha_{62})U_{16}U_{18}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})U_{16})(\alpha_{63} + (\alpha_{61} + \alpha_{62})U_{18})} + \frac{(\alpha_{58} + \alpha_{59})U_{17}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})U_{17}} \\
& + \frac{(\alpha_{58}\alpha_{62} + \alpha_{59}\alpha_{61} + \alpha_{59}\alpha_{62} + \alpha_{58}\alpha_{61})U_{16}U_{18}}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})U_{16})(\alpha_{63} + (\alpha_{61} + \alpha_{62})U_{18})} + \frac{(\alpha_{61} + \alpha_{62})U_{18}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})U_{18}} \\
& + [(\alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{62} \\
& + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{56}\alpha_{58}\alpha_{62} + \alpha_{55}\alpha_{58}\alpha_{62} + \alpha_{56}\alpha_{58}\alpha_{62})e^{-U_{16}-U_{17}-U_{18}} \\
& + (\alpha_{55}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{58}\alpha_{62} + \alpha_{56}\alpha_{58}\alpha_{61} + \alpha_{55}\alpha_{59}\alpha_{61} \\
& + \alpha_{56}\alpha_{58}\alpha_{62} + \alpha_{55}\alpha_{59}\alpha_{62} + \alpha_{56}\alpha_{59}\alpha_{61} + \alpha_{56}\alpha_{59}\alpha_{62})U_{16}U_{17}U_{18}] \\
& \times \frac{1}{(\alpha_{57} + (\alpha_{55} + \alpha_{56})U_{16})(\alpha_{60} + (\alpha_{58} + \alpha_{59})U_{17})(\alpha_{63} + (\alpha_{61} + \alpha_{62})U_{18})} > 1,
\end{aligned} \tag{76}$$

with

$$J|_{Y_6} = \begin{pmatrix} b_{11} & b_{12} & 0 & 0 & b_{15} & b_{16} \\ 1 & 0 & 0 & 0 & 0 & 0 \\ b_{31} & b_{32} & b_{33} & b_{34} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & b_{53} & b_{54} & b_{55} & b_{56} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (77)$$

where

$$\begin{aligned} b_{11} &= -\frac{\alpha_{55}\bar{x}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\bar{x}}, \\ b_{12} &= -\frac{\alpha_{56}\bar{x}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\bar{x}}, \\ b_{15} &= -\frac{\alpha_{55}e^{-\bar{z}}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\bar{x}}, \\ b_{16} &= -\frac{\alpha_{56}e^{-\bar{z}}}{\alpha_{57} + (\alpha_{55} + \alpha_{56})\bar{x}}, \\ b_{31} &= -\frac{\alpha_{58}e^{-\bar{x}}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\bar{y}}, \\ b_{32} &= -\frac{\alpha_{59}e^{-\bar{x}}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\bar{y}}, \\ b_{33} &= -\frac{\alpha_{58}\bar{y}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\bar{y}}, \\ b_{34} &= -\frac{\alpha_{59}\bar{y}}{\alpha_{60} + (\alpha_{58} + \alpha_{59})\bar{y}}, \\ b_{53} &= -\frac{\alpha_{61}e^{-\bar{y}}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\bar{z}}, \\ b_{54} &= -\frac{\alpha_{62}e^{-\bar{y}}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\bar{z}}, \\ b_{55} &= -\frac{\alpha_{61}\bar{z}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\bar{z}}, \\ b_{56} &= -\frac{\alpha_{62}\bar{z}}{\alpha_{63} + (\alpha_{61} + \alpha_{62})\bar{z}}. \end{aligned} \quad (78)$$

Proof. It is similar to the proof of Theorem 4. So its proof is omitted. \square

Hereafter, by constructing a Lyapunov function with discrete time motivated from the work of [2], global dynamics about Y_i ($i = 1, \dots, 6$), respectively, of systems (3)–(8) is explored. \square

6. Global Dynamics of Systems (3)–(8)

Theorem 6. For global dynamics about Y_i ($i = 1, \dots, 6$), respectively, of systems (3)–(8), following statements hold:

(i) Y_1 of (3) is global asymptotically stable if

$$\begin{aligned} (\alpha_{10} + \alpha_{11})e^{-L_2} &< (2\bar{x} - U_1)(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2), \\ (\alpha_{13} + \alpha_{14})e^{-L_3} &< (2\bar{y} - U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3), \\ (\alpha_{16} + \alpha_{17})e^{-L_1} &< (2\bar{z} - U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1); \end{aligned} \quad (79)$$

(ii) Y_2 of (4) is global asymptotically stable if

$$\begin{aligned} (\alpha_{19} + \alpha_{20})e^{-L_6} &< (2\bar{x} - U_4)(\alpha_{21} + (\alpha_{19} + \alpha_{20})L_6), \\ (\alpha_{22} + \alpha_{23})e^{-L_4} &< (2\bar{y} - U_5)(\alpha_{24} + (\alpha_{22} + \alpha_{23})L_4), \\ (\alpha_{25} + \alpha_{26})e^{-L_5} &< (2\bar{z} - U_6)(\alpha_{27} + (\alpha_{25} + \alpha_{26})L_5); \end{aligned} \quad (80)$$

(iii) Y_3 of (5) is global asymptotically stable if

$$\begin{aligned} (\alpha_{28} + \alpha_{29})e^{-L_7} &< (2\bar{x} - U_7)(\alpha_{30} + (\alpha_{28} + \alpha_{29})L_7), \\ (\alpha_{31} + \alpha_{32})e^{-L_8} &< (2\bar{y} - U_8)(\alpha_{33} + (\alpha_{31} + \alpha_{32})L_8), \\ (\alpha_{34} + \alpha_{35})e^{-L_9} &< (2\bar{z} - U_9)(\alpha_{36} + (\alpha_{34} + \alpha_{35})L_8); \end{aligned} \quad (81)$$

(iv) Y_4 of (6) is global asymptotically stable if

$$\begin{aligned} (\alpha_{37} + \alpha_{38})e^{-L_{10}} &< (2\bar{x} - U_{10})(\alpha_{39} + (\alpha_{37} + \alpha_{38})L_{11}), \\ (\alpha_{40} + \alpha_{41})e^{-L_{11}} &< (2\bar{y} - U_{11})(\alpha_{42} + (\alpha_{40} + \alpha_{41})L_{10}), \\ (\alpha_{43} + \alpha_{44})e^{-L_{12}} &< (2\bar{z} - U_{12})(\alpha_{45} + (\alpha_{43} + \alpha_{44})L_{12}); \end{aligned} \quad (82)$$

(v) Y_5 of (7) is global asymptotically stable if

$$\begin{aligned} (\alpha_{46} + \alpha_{47})e^{-L_{13}} &< (2\bar{x} - U_{13})(\alpha_{48} + (\alpha_{46} + \alpha_{47})L_{15}), \\ (\alpha_{49} + \alpha_{50})e^{-L_{14}} &< (2\bar{y} - U_{14})(\alpha_{51} + (\alpha_{49} + \alpha_{50})L_{14}), \\ (\alpha_{52} + \alpha_{53})e^{-L_{15}} &< (2\bar{z} - U_{15})(\alpha_{54} + (\alpha_{52} + \alpha_{53})L_{13}); \end{aligned} \quad (83)$$

(vi) Y_6 of (8) is global asymptotically stable if

$$\begin{aligned} (\alpha_{55} + \alpha_{56})e^{-L_{18}} &< (2\bar{x} - U_{16})(\alpha_{57} + (\alpha_{56} + \alpha_{57})L_{16}), \\ (\alpha_{58} + \alpha_{59})e^{-L_{16}} &< (2\bar{y} - U_{17})(\alpha_{60} + (\alpha_{58} + \alpha_{59})L_{17}), \\ (\alpha_{61} + \alpha_{62})e^{-L_{17}} &< (2\bar{z} - U_{18})(\alpha_{63} + (\alpha_{61} + \alpha_{62})L_{18}). \end{aligned} \quad (84)$$

Proof. (i) Consider the following discrete-time Lyapunov function:

Now

$$\Gamma_n = (x_n - \bar{x})^2 + (y_n - \bar{y})^2 + (z_n - \bar{z})^2. \quad (85)$$

$$\begin{aligned}
 \Delta\Gamma_n &= \Gamma_{n+1} - \Gamma_n \\
 &= (x_{n+1} - x_n)(x_{n+1} + x_n - 2\bar{x}) + (y_{n+1} - y_n)(y_{n+1} + y_n - 2\bar{y}) + (z_{n+1} - z_n)(z_{n+1} + z_n - 2\bar{z}), \\
 &= (x_{n+1} - x_n) \left(\frac{\alpha_{10}e^{-y_n} + \alpha_{11}e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} + x_n - 2\bar{x} \right) \\
 &\quad + (y_{n+1} - y_n) \left(\frac{\alpha_{13}e^{-z_n} + \alpha_{14}e^{-z_{n-1}}}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}} + y_n - 2\bar{y} \right) \\
 &\quad + (z_{n+1} - z_n) \left(\frac{\alpha_{16}e^{-x_n} + \alpha_{17}e^{-x_{n-1}}}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}} + z_n - 2\bar{z} \right) \\
 &\leq (U_1 - L_1) \left(\frac{(\alpha_{10} + \alpha_{11})e^{-L_2}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2} + U_1 - 2\bar{x} \right) \\
 &\quad + (U_2 - L_2) \left(\frac{(\alpha_{13} + \alpha_{14})e^{-L_3}}{\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3} + U_2 - 2\bar{y} \right) \\
 &\quad + (U_3 - L_3) \left(\frac{(\alpha_{16} + \alpha_{17})e^{-L_1}}{\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1} + U_3 - 2\bar{z} \right) \\
 &= (U_1 - L_1) \left(\frac{(\alpha_{10} + \alpha_{11})e^{-L_2} - (2\bar{x} - U_1)(\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2)}{\alpha_{12} + (\alpha_{10} + \alpha_{11})L_2} \right) \\
 &\quad + (U_2 - L_2) \left(\frac{(\alpha_{13} + \alpha_{14})e^{-L_3} - (2\bar{y} - U_2)(\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3)}{\alpha_{15} + (\alpha_{13} + \alpha_{14})L_3} \right) \\
 &\quad + (U_3 - L_3) \left(\frac{(\alpha_{16} + \alpha_{17})e^{-L_1} - (2\bar{z} - U_3)(\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1)}{\alpha_{18} + (\alpha_{16} + \alpha_{17})L_1} \right).
 \end{aligned} \quad (86)$$

From (80) and (87), one gets $\Delta\Gamma_n < 0 \forall n \geq 0$. Hence, we obtain that $\lim_{n \rightarrow \infty} (\Gamma_{n+1} - \Gamma_n) = 0$, and thus Y_1 of (3) is global asymptotically stable. \square

where

$$\begin{aligned}
 \bar{x} &\in [L_1, U_1], \\
 \bar{y} &\in [L_2, U_2], \\
 \bar{z} &\in [L_3, U_3],
 \end{aligned} \quad (88)$$

Remark 3. The proof of (ii)–(vi) is same as the proof of (i).

7. Rate of Convergence

then the error vector, i.e.,

We will explore the convergence result about the equilibrium point of systems (3)–(8) motivated from the existing literature [3–5], in this section.

Theorem 7. If the positive solution of (3) is $\{\Omega_n\}_{n=-1}^{\infty}$, s.t.

$$\begin{aligned}
 \lim_{n \rightarrow \infty} x_n &= \bar{x}, \\
 \lim_{n \rightarrow \infty} y_n &= \bar{y}, \\
 \lim_{n \rightarrow \infty} z_n &= \bar{z},
 \end{aligned} \quad (87)$$

$$\theta_n = \begin{pmatrix} \theta_n^1 \\ \theta_{n-1}^1 \\ \theta_n^2 \\ \theta_{n-1}^2 \\ \theta_n^3 \\ \theta_{n-1}^3 \end{pmatrix}, \quad (89)$$

satisfies the following relation:

$$\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} = |\lambda_{1,\dots,6} J|_{Y_1}|,$$

$$\lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} = |\lambda_{1,\dots,6} J|_{Y_1}|,$$

(90)

where $|\lambda_{1,\dots,6} J|_{Y_1}|$ are the roots of $J|_{Y_1}$.

Proof. If the positive solution of (3) is $\{\Omega_n\}_{n=-1}^\infty$, s.t. (88) along with (89) holds. To find the error terms, one has

$$\begin{aligned} x_{n+1} - \bar{x} &= \frac{\alpha_{10}e^{-y_n} + \alpha_{11}e^{-y_{n-1}}}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} - \frac{(\alpha_{10} + \alpha_{11})e^{-\bar{y}}}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}} \\ &= -\frac{\alpha_{10}e^{-y_n}(e^{y_n-\bar{y}} - 1)}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} - \frac{\alpha_{11}e^{-y_{n-1}}(e^{y_{n-1}-\bar{y}} - 1)}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} - \frac{\alpha_{10}\bar{x}(y_n - \bar{y})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} \\ &\quad - \frac{\alpha_{11}\bar{x}(y_{n-1} - \bar{y})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}} \\ &= -\frac{\alpha_{10}(e^{-y_n}(y_n - \bar{y} + O_1((y_n - \bar{y})^2)) + \bar{x}(y_n - \bar{y}))}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_n - \bar{y}) \\ &\quad - \frac{\alpha_{11}(e^{-y_{n-1}}(y_{n-1} - \bar{y} + O_2((y_{n-1} - \bar{y})^2)) + \bar{x}(y_{n-1} - \bar{y}))}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_{n-1} - \bar{y}). \end{aligned} \quad (91)$$

So,

$$\begin{aligned} x_{n+1} - \bar{x} &= -\frac{\alpha_{10}(e^{-y_n} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_n - \bar{y}) - \frac{\alpha_{11}(e^{-y_{n-1}} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_{n-1} - \bar{y}) \\ &\quad + O_1((y_n - \bar{y})^2) + O_2((y_{n-1} - \bar{y})^2). \end{aligned} \quad (92)$$

Similarly,

$$\begin{aligned} y_{n+1} - \bar{y} &= -\frac{\alpha_{13}(e^{-z_n} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}(z_n - \bar{z}) - \frac{\alpha_{14}(e^{-z_{n-1}} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}(z_{n-1} - \bar{z}) \\ &\quad + O_3((z_n - \bar{z})^2) + O_4((z_{n-1} - \bar{z})^2), \\ z_{n+1} - \bar{z} &= -\frac{\alpha_{16}(e^{-x_n} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}(x_n - \bar{x}) - \frac{\alpha_{17}(e^{-x_{n-1}} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}(x_{n-1} - \bar{x}) \\ &\quad + O_5((x_n - \bar{x})^2) + O_6((x_{n-1} - \bar{x})^2). \end{aligned} \quad (93)$$

From (92) and (93), one gets

$$\begin{aligned} x_{n+1} - \bar{x} &\approx -\frac{\alpha_{10}(e^{-y_n} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_n - \bar{y}) - \frac{\alpha_{11}(e^{-y_{n-1}} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}(y_{n-1} - \bar{y}), \\ y_{n+1} - \bar{y} &\approx -\frac{\alpha_{13}(e^{-z_n} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}(z_n - \bar{z}) - \frac{\alpha_{14}(e^{-z_{n-1}} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}(z_{n-1} - \bar{z}), \\ z_{n+1} - \bar{z} &\approx -\frac{\alpha_{16}(e^{-x_n} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}(x_n - \bar{x}) - \frac{\alpha_{17}(e^{-x_{n-1}} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}(x_{n-1} - \bar{x}). \end{aligned} \quad (94)$$

Denote

$$\begin{aligned}\theta_n^1 &= x_n - \bar{x}, \\ \theta_n^2 &= y_n - \bar{y}, \\ \theta_n^3 &= z_n - \bar{z}.\end{aligned}\quad (95)$$

In view of (95), from (94), one gets

$$\begin{aligned}\theta_{n+1}^1 &= \omega_{1n} \theta_n^2 + \omega_{2n} \theta_{n-1}^2, \\ \theta_{n+1}^2 &= \omega_{3n} \theta_n^3 + \omega_{4n} \theta_{n-1}^3, \\ \theta_{n+1}^3 &= \omega_{5n} \theta_n^1 + \omega_{6n} \theta_{n-1}^1,\end{aligned}\quad (96)$$

where

$$\begin{aligned}\omega_{1n} &= -\frac{\alpha_{10}(e^{-y_n} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}, \\ \omega_{2n} &= -\frac{\alpha_{11}(e^{-y_{n-1}} + \bar{x})}{\alpha_{12} + \alpha_{10}y_n + \alpha_{11}y_{n-1}}, \\ \omega_{3n} &= -\frac{\alpha_{13}(e^{-z_n} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}, \\ \omega_{4n} &= -\frac{\alpha_{14}(e^{-z_{n-1}} + \bar{y})}{\alpha_{15} + \alpha_{13}z_n + \alpha_{14}z_{n-1}}, \\ \omega_{5n} &= -\frac{\alpha_{16}(e^{-x_n} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}, \\ \omega_{6n} &= -\frac{\alpha_{17}(e^{-x_{n-1}} + \bar{z})}{\alpha_{18} + \alpha_{16}x_n + \alpha_{17}x_{n-1}}.\end{aligned}\quad (97)$$

From (97), one gets

$$\begin{aligned}\lim_{n \rightarrow \infty} \omega_{1n} &= -\frac{\alpha_{10}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}}, \\ \lim_{n \rightarrow \infty} \omega_{2n} &= -\frac{\alpha_{11}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}}, \\ \lim_{n \rightarrow \infty} \omega_{3n} &= -\frac{\alpha_{13}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}}, \\ \lim_{n \rightarrow \infty} \omega_{4n} &= -\frac{\alpha_{14}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}}, \\ \lim_{n \rightarrow \infty} \omega_{5n} &= -\frac{\alpha_{16}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}}, \\ \lim_{n \rightarrow \infty} \omega_{6n} &= -\frac{\alpha_{17}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}},\end{aligned}\quad (98)$$

that is,

$$\begin{aligned}\omega_{1n} &= -\frac{\alpha_{10}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}} + b_{1n}, \\ \omega_{2n} &= -\frac{\alpha_{11}(e^{-\bar{y}} + \bar{x})}{\alpha_{12} + (\alpha_{10} + \alpha_{11})\bar{y}} + b_{1n-1}, \\ \omega_{3n} &= -\frac{\alpha_{13}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}} + b_{2n}, \\ \omega_{4n} &= -\frac{\alpha_{14}(e^{-\bar{z}} + \bar{y})}{\alpha_{15} + (\alpha_{13} + \alpha_{14})\bar{z}} + b_{2n-1}, \\ \omega_{5n} &= -\frac{\alpha_{16}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}} + b_{3n}, \\ \omega_{6n} &= -\frac{\alpha_{17}(e^{-\bar{x}} + \bar{z})}{\alpha_{18} + (\alpha_{16} + \alpha_{17})\bar{x}} + b_{3n-1},\end{aligned}\quad (99)$$

where $b_{1n}, b_{1n-1}, b_{2n}, b_{2n-1}, b_{3n},$ and $b_{3n-1} \rightarrow 0$ as $n \rightarrow \infty$. Now, we have system 1.10 of [6], where $A = J|_{Y_1}$ and

$$B(n) = \begin{pmatrix} 0 & 0 & b_{1n} & b_{1n-1} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b_{2n} & b_{2n-1} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ b_{3n} & b_{3n-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad (100)$$

such that $\|B(n)\| \rightarrow \infty, n \rightarrow \infty$. So, about Y_1 of (3) the limiting system becomes

$$\begin{pmatrix} \theta_{n+1}^1 \\ \theta_n^1 \\ \theta_{n+1}^2 \\ \theta_n^2 \\ \theta_{n+1}^3 \\ \theta_n^3 \end{pmatrix} = J|_{Y_1} \begin{pmatrix} \theta_n^1 \\ \theta_{n-1}^1 \\ \theta_n^2 \\ \theta_{n-1}^2 \\ \theta_n^3 \\ \theta_{n-1}^3 \end{pmatrix}, \quad (101)$$

which is as $J|_{Y_1}$ about Y_1 . \square

In the following theorem, we will summarize the convergence results for systems (4) to (8).

Theorem 8. (i) If the positive solution of (4) is $\{\Omega_n\}_{n=-1}^{\infty}$, s.t. (87) along with the following relation holds:

$$\begin{aligned}\bar{x} &\in [L_4, U_4], \\ \bar{y} &\in [L_5, U_5], \\ \bar{z} &\in [L_6, U_6],\end{aligned}\quad (102)$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\begin{aligned}\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} &= |\lambda_{1,\dots,6} J|Y_2|, \\ \lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} &= |\lambda_{1,\dots,6} J|Y_2|,\end{aligned}\quad (103)$$

where $|\lambda_{1,\dots,6} J|Y_2|$ are the roots of $J|Y_2|$.

(ii) If the positive solution of (5) is $\{\Omega_n\}_{n=-1}^\infty$, s.t. (87) along with the following relation holds:

$$\begin{aligned}\bar{x} &\in [L_7, U_7], \\ \bar{y} &\in [L_8, U_8], \\ \bar{z} &\in [L_9, U_9],\end{aligned}\quad (104)$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\begin{aligned}\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} &= |\lambda_{1,\dots,6} J|Y_3|, \\ \lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} &= |\lambda_{1,\dots,6} J|Y_3|,\end{aligned}\quad (105)$$

where $|\lambda_{1,\dots,6} J|Y_3|$ are the roots of $J|Y_3|$.

(iii) If the positive solution of (6) is $\{\Omega_n\}_{n=-1}^\infty$, s.t. (87) along with the following relation holds:

$$\begin{aligned}\bar{x} &\in [L_{10}, U_{10}], \\ \bar{y} &\in [L_{11}, U_{11}], \\ \bar{z} &\in [L_{12}, U_{12}],\end{aligned}\quad (106)$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\begin{aligned}\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} &= |\lambda_{1,\dots,6} J|Y_4|, \\ \lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} &= |\lambda_{1,\dots,6} J|Y_4|,\end{aligned}\quad (107)$$

where $|\lambda_{1,\dots,6} J|Y_4|$ are the roots of $J|Y_4|$.

(iv) If the positive solution of (7) is $\{\Omega_n\}_{n=-1}^\infty$, s.t. (87) along with the following relation holds:

$$\begin{aligned}\bar{x} &\in [L_{13}, U_{13}], \\ \bar{y} &\in [L_{14}, U_{14}], \\ \bar{z} &\in [L_{15}, U_{15}],\end{aligned}\quad (108)$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\begin{aligned}\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} &= |\lambda_{1,\dots,6} J|Y_5|, \\ \lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} &= |\lambda_{1,\dots,6} J|Y_5|,\end{aligned}\quad (109)$$

where $|\lambda_{1,\dots,6} J|Y_5|$ are the roots of $J|Y_5|$.

(v) If the positive solution of (8) is $\{\Omega_n\}_{n=-1}^\infty$, s.t. (87) along with the following relation holds:

$$\begin{aligned}\bar{x} &\in [L_{16}, U_{16}], \\ \bar{y} &\in [L_{17}, U_{17}], \\ \bar{z} &\in [L_{18}, U_{18}],\end{aligned}\quad (110)$$

then the error vector, which is depicted in (89), satisfies the following relations:

$$\begin{aligned}\lim_{n \rightarrow \infty} (\|\theta_n\|)^{1/n} &= |\lambda_{1,\dots,6} J|Y_6|, \\ \lim_{n \rightarrow \infty} \frac{\|\theta_{n+1}\|}{\|\theta_n\|} &= |\lambda_{1,\dots,6} J|Y_6|,\end{aligned}\quad (111)$$

where $|\lambda_{1,\dots,6} J|Y_6|$ are the roots of $J|Y_6|$.

Proof. It is similar to Theorem 7, and hence its proof is omitted. \square

8. Discussion and Simulations

In the reported work, we have explored the global dynamics of (2,3)-type exponential systems of difference equations. We have investigated that $\{\Omega_n\}_{n=-1}^\infty$ of systems (3) to (8) is bounded and persists, and the corresponding invariant rectangles, respectively, are $[L_1, U_1] \times [L_2, U_2] \times [L_3, U_3]$, $[L_4, U_4] \times [L_5, U_5] \times [L_6, U_6]$, $[L_7, U_7] \times [L_8, U_8] \times [L_9, U_9]$, $[L_{10}, U_{10}] \times [L_{11}, U_{11}] \times [L_{12}, U_{12}]$, $[L_{13}, U_{13}] \times [L_{14}, U_{14}] \times [L_{15}, U_{15}]$, and $[L_{16}, U_{16}] \times [L_{17}, U_{17}] \times [L_{18}, U_{18}]$. Further, we have explored the existence and uniqueness of the positive equilibrium and the global and local dynamics of systems (3)–(8). We have also investigated the rate of convergence of the positive solution of systems (3)–(8). Finally, some numerical examples are provided to support the theoretical results. For instance, if α_i ($i = 10, \dots, 18$), respectively, are 13, 24, 319, 12, 0.1, 0.2, 1.5, 0.4, and 0.002, then from Figures 1(a)–1(c), the positive fixed point (0.06304125281075143, 0.567941569702224, 0.4220224516533974) of (3) is stable and its corresponding attractor is shown in Figure 1(s). Now, if α_i ($i = 19, \dots, 27$), respectively, are 19, 14, 9, 112, 0.1, 0.2, 15, 14, and 2, then from Figures 1(d)–1(f), the positive equilibrium point (0.6122355979161732, 0.996719892698477, 0.8425637558539959) of (4) is stable and its corresponding attractor is shown in Figure 1(t). For (5), if α_i ($i = 28, \dots, 36$), respectively, are 9, 0.4, 9, 12, 0.1, 0.2, 15, 2, and 15, then from Figures 1(g)–1(i), its unique positive equilibrium point (0.277465951, 0.924713, 0.88573) is stable and its corresponding attractor is shown in Figure 1(u). For (6), if α_i ($i = 37, \dots, 45$),

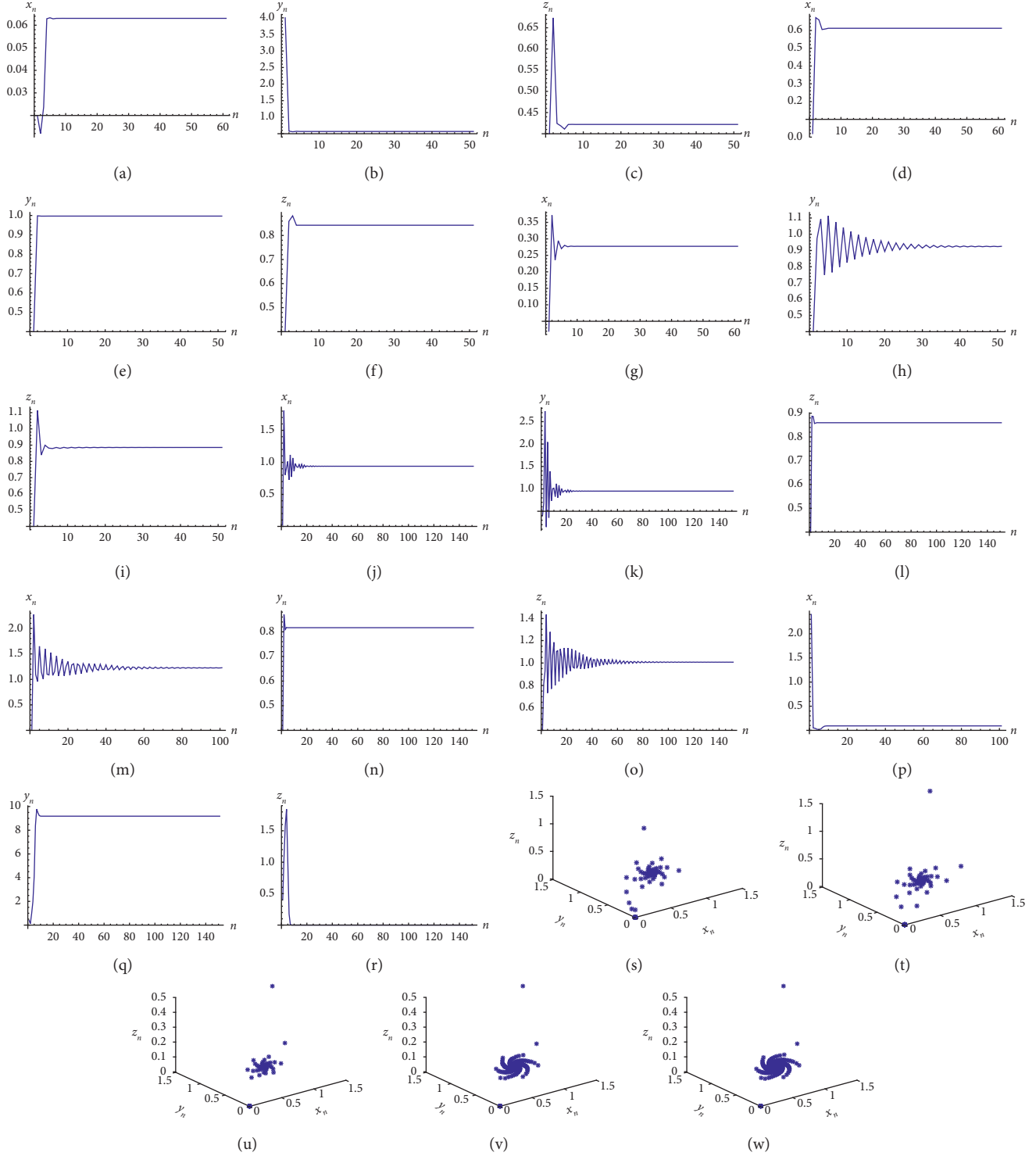


FIGURE 1: Trajectories for (3) to (8) with initial values x_s, y_s, z_s ($s = -1, 0$), respectively, being 0.7, 2.4, 0.9, 0.4, 1.9, and 0.4.

respectively, are 29, 14, 9, 12, 0.1, 0.2, 15, 14, and 2, then from Figures 1(j)–1(l), its unique positive equilibrium point (0.9391896591799592, 0.9495483631730844, 0.8598773120562713) is stable and its corresponding attractor is shown in Figure 1(v). For (7), if α_i ($i = 46, \dots, 54$), respectively, are 29, 14, 1.9, 12, 0.1, 1.2, 15, 14, and 2, then from Figures 1(m)–1(o), its unique positive equilibrium point (1.22564,

0.8167047, 1.007332) is stable and its corresponding attractor is shown in Figure 1(v). Finally, if α_i ($i = 55, \dots, 63$), respectively, are 9, 4, 129, 12, 0.1, 25, 14, 14, and 2, then from Figures 1(p)–1(r), the unique positive equilibrium point (0.09228515208223682, 9.184938317317343, 0.0000975782517529335) of system (8) is stable and its corresponding attractor is shown in Figure 1(w). For more

results on dynamical properties of difference equations, we refer the reader to [7, 8] and the references cited therein.

Data Availability

All the data utilized in this article have been included and the sources from where they were adopted are cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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Research Article

Hartman-Type and Lyapunov-Type Inequalities for a Fractional Differential Equation with Fractional Boundary Conditions

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We prove Hartman-type and Lyapunov-type inequalities for a class of Riemann–Liouville fractional boundary value problems with fractional boundary conditions. Some applications including a lower bound for the corresponding eigenvalue problem are obtained.

1. Introduction

In [1], Lyapunov established the following striking inequality:

Theorem 1. Let $q \in C([a, b], \mathbb{R})$. Assume that the problem

$$\begin{cases} \omega'' + q(x)\omega = 0, & x \in (a, b), \\ \omega(a) = \omega(b) = 0, \end{cases} \quad (1)$$

has a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$(b-a) \int_a^b |q(z)| dz > 4, \quad (2)$$

and constant 4 is the best possible largest number.

It has been shown that this result serves as a good tool in the study of several properties of solutions of differential equations (such as eigenvalue problems and eigenvalue inequalities) (see, for example, [2–5] and the references therein). Many authors have worked on generalizations of classical inequalities (see, for instance, [4–16] and the references therein).

In [11], the authors use the Hahn integral operator to prove a description of new generalization of Minkowski's inequality.

In [5], the authors improve inequality in (2) by proving the following Hartman–Winter inequality:

$$\int_a^b (b-z)(z-a)q^+(z) dz > b-a, \quad (3)$$

where $q^+(z) = \max(q(z), 0)$ is the nonnegative part of $q(z)$.

Inequality (3) is also known as the best Lyapunov inequality.

In [17], Ferreira considered the following fractional differential problem:

$$\begin{cases} D_{a^+}^\alpha \omega + q(x)\omega = 0, & a < x < b, 1 < \alpha \leq 2, \\ \omega(a) = \omega(b) = 0, \end{cases} \quad (4)$$

where $q \in C([a, b], \mathbb{R})$ and $D_{a^+}^\alpha$ denotes the Riemann–Liouville fractional derivative of order α (see Definition 2 in the following).

The author established the following Lyapunov-type inequality for problem (4).

Theorem 2 (see [17]). Assume that problem (4) has a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$\frac{1}{\Gamma(\alpha)} \int_a^b |q(z)| dz > \left(\frac{4}{b-a}\right)^{\alpha-1}. \quad (5)$$

Remark 1. Note that if we let $\alpha = 2$ in (5), one obtains Lyapunov's classical inequality (2).

For the convenience of the reader, we recall the concept of fractional integral and derivative of order $\gamma \geq 0$.

Definition 1 (see [18, 19]). The Riemann–Liouville fractional integral of order $\gamma \geq 0$ for a real-valued function ω is defined by $(I_{a^+}^0 \omega)(x) = \omega(x)$ and

$$(I_{a^+}^\gamma \omega)(x) := \frac{1}{\Gamma(\gamma)} \int_a^x (x-z)^{\gamma-1} \omega(z) dz, \quad \gamma > 0, x \in [a, b], \quad (6)$$

where $\Gamma(\gamma)$ is the Euler gamma function.

Definition 2 (see [18, 19]). The Riemann–Liouville fractional derivative of order $\gamma \geq 0$ for function ω is defined by $(D_{a^+}^0 \omega)(x) = \omega(x)$ and

$$(D_{a^+}^\gamma \omega)(x) := \left(\frac{d}{dx} \right)^n (I_{a^+}^{n-\gamma} \omega)(x), \quad \text{for } \gamma > 0, \quad (7)$$

where $n = [\gamma] + 1$ with $[\gamma]$ the integer part of γ .

The new development in fractional calculus has attracted the attention of researchers of various disciplines. Different mathematical procedures have been considered by several authors through different research-oriented aspects of fractional differential equations (see, for instance, [20–22] and the references therein).

Our goal in this paper is to establish Hartman-type and Lyapunov-type inequalities for the following problem:

$$\begin{cases} D_{a^+}^\alpha \omega + q(x)\omega = 0, x \in (a, b), \\ \omega(a) = D_{a^+}^{\alpha-3} \omega(a) = D_{a^+}^{\alpha-2} \omega(a) = \omega''(b) = 0, \end{cases} \quad (8)$$

where $\alpha \in (3, 4]$ and $q \in C([a, b], \mathbb{R})$. Some applications are given to illustrate our result.

The organization of the paper is as follows. In Section 2, we derive the explicit expression of the Green function corresponding to problem (8) and we establish some properties on it. This allows us to prove Hartman-type and Lyapunov-type inequalities for problem (8). In Section 3, we present some applications including a lower bound for the corresponding eigenvalue problem.

2. Main Results

2.1. Green's Function. First, we recall the following well-known properties (see, for example, [18, 19]).

Lemma 1. Let $\alpha \in (3, 4)$ and $\omega \in C((a, b), \mathbb{R}) \cap L^1((a, b))$. Then,

- (i) For $0 < \gamma < \alpha$, $D_{a^+}^\gamma (I_{a^+}^\alpha \omega) = I_{a^+}^{\alpha-\gamma} \omega$ and $D_{a^+}^\alpha (I_{a^+}^\alpha \omega) = \omega$
- (ii) $D_{a^+}^\alpha \omega(x) = 0$ if and only if $\omega(x) = \sum_{i=1}^4 c_i (x-a)^{\alpha-i}$, where $c_i \in \mathbb{R}$, for $i \in \{1, 2, 3, 4\}$
- (iii) Assume that $D_{a^+}^\alpha \omega \in C((a, b), \mathbb{R}) \cap L^1((a, b))$; then,

$$I_{a^+}^\alpha (D_{a^+}^\alpha \omega)(x) = \omega(x) + \sum_{i=1}^4 c_i (x-a)^{\alpha-i}, \quad (9)$$

where $c_i \in \mathbb{R}$, for $i \in \{1, 2, 3, 4\}$.

Lemma 2. Let $\omega \in C([a, b])$ be a solution of problem (8). Then,

$$\omega(x) = \int_a^b G_\alpha(x, y) q(y) \omega(y) dy, \quad (10)$$

where $G_\alpha(x, y)$ is Green's function of problem (8) given by

$$G_\alpha(x, y) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-1} - (x-y)^{\alpha-1}, & a \leq y \leq x \leq b, \\ \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-1}, & a \leq x \leq y \leq b. \end{cases} \quad (11)$$

Proof. Let ω be such solution. By Lemma 1, we have

$$\omega(x) = \sum_{i=1}^4 c_i (x-a)^{\alpha-i} - I_{a^+}^\alpha (q\omega)(x). \quad (12)$$

Using the fact that $\omega(a) = D_{a^+}^{\alpha-3} \omega(a) = D_{a^+}^{\alpha-2} \omega(a) = \omega''(b) = 0$, we obtain $c_2 = c_3 = c_4 = 0$ and $(b-a)^{\alpha-3} \Gamma(\alpha) c_1 = \int_a^b (b-y)^{\alpha-3} q(y) \omega(y) dy$.

Therefore,

$$\begin{aligned} \omega(x) &= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-1} q(y) \omega(y) dy \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} q(y) \omega(y) dy \\ &= \int_a^b G_\alpha(x, y) q(y) \omega(y) dy. \end{aligned} \quad (13)$$

This ends the proof. \square

To get a quick perspective, in Figure 1, we have the representation of Green's function $G_{7/2}(x, y)$ with the contours and some projections.

One can see from Figure 1 that Green's function $G_{7/2}(x, y) \geq 0$ and it is nondecreasing with respect to the first variable. This important observation will be proved for $G_\alpha(x, y)$ with $\alpha \in (3, 4]$.

Definition 3. Let $f, g: [a, b] \times [a, b] \longrightarrow \mathbb{R}$ with $f, g \geq 0$. We say that

$$f(x, y) \approx g(x, y) \text{ on } [a, b] \times [a, b], \quad (14)$$

if there exists $c > 0$ such that $(1/c)g(x, y) \leq f(x, y) \leq cg(x, y)$ for all $(x, y) \in [a, b] \times [a, b]$.

Remark 2. Let $\tau > 0$ and $x, z \in [0, 1]$. Then,

$$\min(1, \tau)(1 - zx) \leq 1 - zx^\tau \leq \max(1, \tau)(1 - zx). \quad (15)$$

Next, we establish some properties on Green's function $G_\alpha(x, y)$ given by (11).

Proposition 1

(i) On $[a, b] \times [a, b]$,

$$G_\alpha(x, y) \approx (x - a)^{\alpha-2} (b - y)^{\alpha-3} \min(x - a, y - a). \quad (16)$$

(ii) On $[a, b] \times [a, b]$,

$$\frac{\partial}{\partial x} G_\alpha(x, y) \approx (x - a)^{\alpha-3} (b - y)^{\alpha-3} \min(x - a, y - a). \quad (17)$$

(iii) The function G_α satisfies the following property:

$$0 \leq G_\alpha(x, y) \leq G_\alpha(b, y), \quad (x, y) \in [a, b] \times [a, b]. \quad (18)$$

Proof (i) From Lemma 2, for $x, y \in (a, b)$, we have

$$G_\alpha(x, y) = \frac{1}{\Gamma(\alpha)} \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-1} \cdot \left[1 - \left(\frac{b-y}{b-a} \right)^2 \left(\frac{(b-a)(x-y)^+}{(x-a)(b-y)} \right)^{\alpha-1} \right], \quad (19)$$

where $(x-y)^+ = \max((x-y), 0)$.

Now, since $((b-a)(x-y)^+)/((x-a)(b-y)) \in [0, 1]$, for $x, y \in (a, b)$, then by using Remark 2, with $\tau = \alpha - 1$ and $z = ((b-y)^2)/((b-a)^2) \in [0, 1]$, we obtain

$$G_\alpha(x, y) \approx (b-y)^{\alpha-3} (x-a)^{\alpha-2} [(b-a)(x-a) - (b-y)(x-y)^+]. \quad (20)$$

Hence, inequalities in (16) follow by observing that

$$(b-a)(x-a) - (b-y)(x-y)^+ \approx \min((x-a), (y-a)). \quad (21)$$

(ii) We have

$$\frac{\partial}{\partial x} G_\alpha(x, y) = \frac{(\alpha-1)}{\Gamma(\alpha)} \begin{cases} \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2} - (x-y)^{\alpha-2}, & a \leq y \leq x \leq b, \\ \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2}, & a \leq x \leq y \leq b. \end{cases} \quad (22)$$

Similar to case (i), by using the fact that

$$\frac{\partial}{\partial x} G_\alpha(x, y) = \frac{(\alpha-1)}{\Gamma(\alpha)} \left(\frac{b-y}{b-a} \right)^{\alpha-3} (x-a)^{\alpha-2} \left[1 - \left(\frac{b-y}{b-a} \right) \left(\frac{(b-a)(x-y)^+}{(x-a)(b-y)} \right)^{\alpha-2} \right] \quad (23)$$

and applying Remark 2 with $\tau = \alpha - 2$ and $z = (b-y)/(b-a) \in [0, 1]$, we obtain the required result.

(iii) Let $y \in [a, b]$. Since the function $x \longrightarrow (\partial/\partial x)G_\alpha(x, y)$ is nondecreasing on $[a, b]$, we deduce that

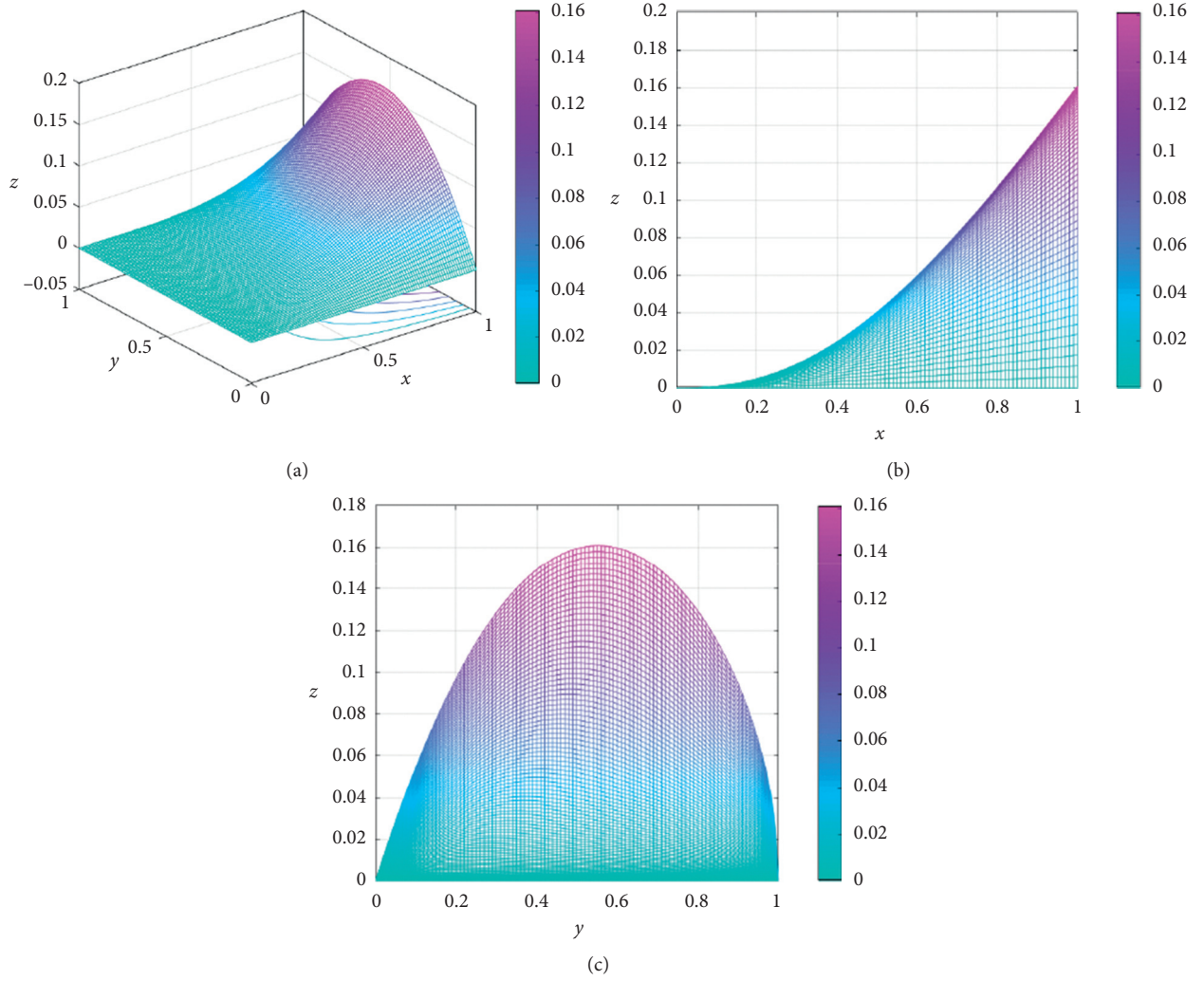


FIGURE 1: $G_\alpha(x, y)$ for $\alpha = 7/2$. (a) $G_\alpha(x, y)$ and contours. (b) Projection on xz . (c) Projection on yz .

$$0 = G_\alpha(a, y) \leq G_\alpha(x, y) \leq G_\alpha(b, y). \quad (24)$$

This completes the proof. \square

2.2. Statements and Proofs of Main Results

Theorem 3 (Hartman–Winter-type inequality)

Let $q \in C([a, b], \mathbb{R})$. Assume that problem (8) has a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$\int_a^b (b-z)^{\alpha-3} (z-a)(2b-a-z) q^+(z) dz \geq \Gamma(\alpha), \quad (25)$$

where $q^+(z) = \max(q(z), 0)$.

Proof. From Lemma 2, we know that

$$\omega(x) = \int_a^b G_\alpha(x, z) q(z) \omega(z) dz, \quad x \in [a, b]. \quad (26)$$

Without loss of generality, we may assume that $\omega(x) > 0$ for $x \in (a, b)$.

Using (26), Proposition 1 (iii) and the fact that $q(z) \leq q^+(z)$, we deduce that

$$\omega(x) \leq \int_a^b G_\alpha(x, z) q^+(z) \omega(z) dz \leq \int_a^b G_\alpha(b, z) q^+(z) \omega(z) dz. \quad (27)$$

Hence,

$$\|\omega\| \leq \int_a^b G_\alpha(b, z) q^+(z) \|\omega\| dz, \quad (28)$$

or equivalently,

$$1 \leq \int_a^b G_\alpha(b, z) q^+(z) dz. \quad (29)$$

Therefore,

$$1 \leq \frac{1}{\Gamma(\alpha)} \int_a^b (b-z)^{\alpha-3} (z-a)(2b-a-z) q^+(z) dz, \quad (30)$$

from which inequality (25) follows. \square

Remark 3. Let $q \in C([a, b], \mathbb{R})$. Under the same conditions as in Theorem 3, we have

$$\int_a^b (b-z)^{\alpha-3} (z-a)(2b-a-z)|q(z)|dz \geq \Gamma(\alpha). \quad (31)$$

By applying the previous theorem with $\alpha = 4$, we obtain the following:

Corollary 1. Let $q \in C([a, b], \mathbb{R})$. Assume that the problems

$$\begin{cases} \omega^{(4)} + q(x)\omega = 0, & x \in (a, b), \\ \omega(a) = \omega'(a) = \omega''(a) = \omega'''(b) = 0, \end{cases} \quad (32)$$

admit a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$\int_a^b (z-a)(b-z)(2b-a-z)q^+(z)dz \geq 6. \quad (33)$$

In particular,

$$\int_a^b (z-a)(b-z)q^+(z)dz \geq \frac{3}{(b-a)}. \quad (34)$$

Corollary 2 (Lyapunov – type inequality)

Under the same conditions as in Theorem 3, we have

$$\int_a^b q^+(z)dz \geq \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-2}}{2(b-a)^{\alpha-1}(\sqrt{(\alpha-1)(\alpha-3)})^{\alpha-3}}. \quad (35)$$

Proof. By Theorem 3, we have

$$\int_a^b f(z)q^+(z)dz \geq \Gamma(\alpha). \quad (36)$$

where $f(z) := (b-z)^{\alpha-3}(z-a)(2b-a-z) \geq 0$.

For $z \in (a, b)$, we have

$$f'(z) = (b-z)^{\alpha-4} [2(b-z)^2 - (\alpha-3)(z-a)(2b-a-z)]. \quad (37)$$

Note that

$$\begin{aligned} f'(z) &= 0 \text{ on } (a, b) \text{ if and only if } z = z^* \\ &:= \frac{1}{\alpha-1} ((\alpha-1)b - (b-a)\sqrt{(\alpha-1)(\alpha-3)}). \end{aligned} \quad (38)$$

Furthermore, $f'(z) > 0$ on (a, z^*) and $f'(z) < 0$ on (z^*, b) .

Hence,

$$\sup_{z \in [a, b]} f(z) = f(z^*) = 2 \frac{(b-a)^{\alpha-1}}{(\alpha-1)^{\alpha-2}} (\sqrt{(\alpha-1)(\alpha-3)})^{\alpha-3}. \quad (39)$$

So Lyapunov-type inequality (35) follows from (36) and (39). \square

Corollary 3. Let $q \in C([a, b], \mathbb{R})$. Assume that the problems

$$\begin{cases} \omega^{(4)} + q(x)\omega = 0, & x \in (a, b), \\ \omega(a) = \omega'(a) = \omega''(a) = \omega'''(b) = 0, \end{cases} \quad (40)$$

admit a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$\int_a^b q^+(z)dz \geq \frac{9\sqrt{3}}{(b-a)^3}. \quad (41)$$

Proof. Inequality (41) follows from (39) with $\alpha = 4$. \square

3. Applications

3.1. Lower Bound for the Eigenvalues. Consider the following eigenvalue problem:

$$\begin{cases} D_{0+}^{\alpha} \omega(x) + \lambda \omega(x) = 0, & x \in (0, 1), 3 < \alpha \leq 4, \\ \omega(0) = D_{0+}^{\alpha-3} \omega(0) = D_{0+}^{\alpha-2} \omega(0) = \omega''(1) = 0. \end{cases} \quad (42)$$

Theorem 4. Assume that eigenvalue problem (42) has a solution $\omega \in C([a, b], \mathbb{R})$ such that $\omega(x) \neq 0$ for $x \in (a, b)$. Then,

$$|\lambda| \geq \frac{(\alpha-2)\Gamma(\alpha+1)}{2}. \quad (43)$$

Proof. By Remark 3 (with $a = 0$ and $b = 1$), we have

$$|\lambda| \int_0^1 (1-z)^{\alpha-3} z(2-z)dz \geq \Gamma(\alpha), \quad (44)$$

from which inequality (43) follows by observing that

$$\int_0^1 (1-z)^{\alpha-3} z(2-z)dz = \frac{2}{\alpha(\alpha-2)}. \quad (45)$$

\square

3.2. Nonexistence Results. Consider the following problem:

$$\begin{cases} D_{0+}^{\alpha} \omega + q(x)\omega = 0, & x \in (0, 1), 3 < \alpha \leq 4, \\ \omega(0) = D_{0+}^{\alpha-3} \omega(0) = D_{0+}^{\alpha-2} \omega(0) = \omega''(1) = 0, \end{cases} \quad (46)$$

where $q \in C([0, 1], \mathbb{R})$. Then, we have the following result.

Theorem 5. Assume that

$$\int_0^1 q^+(z)dz < \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-2}}{2(\sqrt{(\alpha-1)(\alpha-3)})^{\alpha-3}}. \quad (47)$$

Then, problem (46) has no nontrivial solution.

Proof. The assertion follows from Corollary 2. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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Research Article

Study of a Class of Generalized Multiterm Fractional Differential Equations with Generalized Fractional Integral Boundary Conditions

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The aim of this work is to study the new generalized fractional differential equations involving generalized multiterms and equipped with multipoint boundary conditions. The nonlinear term is taken from Orlicz space. The existence and uniqueness results, with the help of contemporary tools of fixed point theory, are investigated. The Ulam stability of our problem is also presented. The obtained results are well illustrated by examples.

1. Introduction

Fractional differential equations (FDEs) have a considerable interest both in mathematics and in applications. They were used in modeling in various disciplines of engineering, physics, chemical technology, population dynamics, biotechnology, economics, etc. (see, for example, [1–10], and the references cited therein). Recently, some authors explored FDEs with various types of conditions as multipoint, nonlocal, antiperiodic, and integral boundary conditions [11–15].

Furthermore, there has been a significant development in fractional derivatives and integrals due to the necessity of a better model for real phenomena. In [16], Katugampola suggested a new fractional integral that combines the Riemann–Liouville and Hadamard integrals into a single integral. In 2015, Caputo and Fabrizio [17] defined the so-called Caputo–Fabrizio fractional derivatives by imposing a nonsingular exponential kernel. Later, the exponential kernel was replaced by the Mittag–Leffler function [18, 19]. Another definition of generalized proportional fractional derivatives-generated Caputo and Riemann–Liouville

involving exponential functions in their kernels was introduced by Jarad et al. [20]. Another new type of new fractional derivative can be found in [21]. An application on such derivatives can be seen in [22]. More recently, generalized fractional derivatives that contain kernels depending on the function on the space of absolutely continuous functions were investigated [23]. Following the above points, many authors established various generalized fractional boundary value problems (GFBVPs) (see, for example, [24–27]). It is worth to mention that some authors worked on the existence of solutions and studied the stability analysis for nonlinear singular fractional differential equations. For more details, we refer to [28–30].

On the other hand, Birnbaum and Orlicz [31] proposed a more general setting of function spaces in 1931, which were called Orlicz spaces. These spaces are the generalization of the classical Lebesgue spaces L^p ($1 < p < \infty$), where the kernel is given by a convex function instead of x^p . Some results that deal with differential equations in the framework of Orlicz spaces can be found in [32, 33].

In this paper, we investigate the boundary value problem as follows:

$$\left\{ \begin{array}{l} {}^C D_{0^+}^{\alpha, \rho} u(t) = f(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)), \quad t \in [0, 1], \rho > 0, \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I_0^{\beta, \rho} u(\xi_i), \quad 0 < \xi_i < 1, \delta = t^{1-\rho} \frac{d}{dt}, \end{array} \right. \quad (1)$$

where ${}^C D_{0^+}^{\alpha, \rho}$ is the generalized fractional derivative of Caputo type of order α , $3 < \alpha \leq 4$, $I_0^{\beta, \rho}$ denotes the Katugampola-type fractional integral of order $0 < \beta < 1$, f is defined on an Orlicz space $L_F([0, 1])$, and $\delta_i \in \mathbb{R}$, $(i = 1, \dots, m)$.

The rest of the paper is organized as follows. In Section 2, we will briefly recall some preliminary materials related to our problem. In Section 3, we discuss the existence and uniqueness results of GFBVP using some fixed point theorems and support the obtained results by using examples to well illustrate. The Ulam stability of our problem is given in Section 4.

2. Preliminaries

For convenience of the reader, we present some basic definitions about generalized fractional calculus theory, which can be found in [16, 34, 35]. Also, we introduce some necessary concepts for Orlicz spaces which are used throughout this paper. For more details about Orlicz spaces, one can see [36, 37].

Definition 1. Let $Q: [0, \infty) \rightarrow [0, \infty)$ be a right continuous, monotone, increasing function with the following:

- (i) $Q(0) = 0$
- (ii) $\lim_{t \rightarrow \infty} Q(t) = \infty$
- (iii) $Q(t) > 0$ whenever $t > 0$

Then, the function defined by

$$F(x) = \int_0^x Q(t) dt, \quad x \geq 0, \quad (2)$$

is called N -function. Alternatively, the function F is an N -function iff F is continuous, even, and convex with the following:

- (i) $\lim_{x \rightarrow 0} (F(x)/x) = 0$
- (ii) $\lim_{x \rightarrow 0} (F(x)/x) = \infty$
- (iii) $F(x) > 0$ if $x > 0$

Definition 2. For an N -function, we define

$$F^*(x) = \int_0^x Q^{-1}(t) dt, \quad x \geq 0, \quad (3)$$

where Q^{-1} is the right inverse of the right derivative of F and is called the complementary of F and it satisfies the following condition:

$$F^*(x) = \sup\{tx - F(t): t \geq 0\}, \quad \forall x \geq 0. \quad (4)$$

Remark 1. Note that the function F^* is also N -function and the complementary pairs F and F^* satisfy the following Young inequality:

$$xt \leq F(x) + F^*(t), \quad \forall x, t \geq 0. \quad (5)$$

Definition 3. A function $F: [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is convex and satisfies the conditions $F(0) = \lim_{x \rightarrow 0^+} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = \infty$.

Definition 4 (Orlicz space). For an N -function F , the Orlicz space $L_F([0, 1])$ is the space of measurable functions $u: [0, 1] \rightarrow \mathbb{R}$ such that $\int_0^1 F(|u(x)|) dx < \infty$. This space endowed with the Luxemburg norm, i.e.,

$$\|u\|_F = \inf \left\{ \lambda > 0: \int_0^1 F\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}, \quad (6)$$

and the pair $(L_F([0, 1]), \|u\|_F)$ is a Banach space.

Remark 2. For an Orlicz space, the Hölder inequality holds, that is,

$$\int_0^1 uv \, dx \leq 2 \|u\|_F \|v\|_{F^*}, \quad (7)$$

where $u \in L_F([0, 1])$ and $v \in L_{F^*}([0, 1])$.

For generalized fractional calculus, Katugampola [16, 34, 38] introduced the following definitions.

Definition 5 (Katugampola generalized fractional integral). The generalized fractional integral of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho > 0$ for $-\infty < a < t < \infty$ is defined by

$$({}^I_{a^+}^{\alpha, \rho} u)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t-s)^{1-\alpha}} u(s) ds, \quad (8)$$

where u belongs to the space $X_c^p(a, b)$, which denotes the space of all complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, \quad 1 \leq p < \infty, c \in \mathbb{R}. \quad (9)$$

Note that integral (8) is called the left-sided generalized fractional integral.

Definition 6 (Katugampola generalized Caputo fractional derivative). The left generalized fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho > 0$ of $u \in AC_\delta^n[a, b]$, where $-\infty < a < t < \infty$, is defined by

$$\begin{aligned} ({}^C D_{a^+}^{\alpha, \rho} u)(t) &= \frac{1}{\Gamma(n - \alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n - \alpha - 1} \frac{(\delta^n u)(s) ds}{s^{1 - \rho}} \\ &= I_{a^+}^{n - \alpha, \rho} (\delta^n u)(s), \end{aligned} \quad (10)$$

$$AC_\delta^n[a, b] = \left\{ f: [a, b] \longrightarrow \mathbb{C} \text{ and } \delta^{n-1} f \in AC[a, b], \delta = t^{1-\rho} \frac{d}{dt} \right\}, \quad (11)$$

$$C_{\delta, \epsilon}^n[a, b] = \left\{ f: [a, b] \longrightarrow \mathbb{C} \text{ and } \delta^{n-1} f \in C[a, b], \delta^n f \in C_{\epsilon, \rho}[a, b], \delta = t^{1-\rho} \frac{d}{dt} \right\}, \quad (12)$$

for $0 \leq \epsilon \leq 1$, endowed with the norms $\|f\|_{C_\delta^n} = \sum_{k=0}^n \|\delta^k f\|_C$ and $\|f\|_{C_{\delta, \epsilon}^n} = \sum_{k=0}^{n-1} \|\delta^k f\|_C + \|\delta^n f\|_{C_{\epsilon, \rho}}$, respectively.

Next, we recall some basic properties of generalized fractional integral and derivative [39].

Lemma 1. Let $\Re(\alpha) \geq 0, n = [\Re(\alpha) + 1]$, and $f \in AC_\delta^n[a, b]$, with $0 < a < b < \infty$. For $k, m \in \mathbb{N}$,

$$\begin{aligned} ({}_a I^{\alpha, \rho})^k ({}_a^C D^{\alpha, \rho})^m f(t) \\ = \frac{({}_a^C D^{\alpha, \rho})^m f(\tau) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad \tau \in (a, t). \end{aligned} \quad (13)$$

Theorem 1. Let $\Re(\alpha) \geq 0, n = [\Re(\alpha) + 1]$, and $f \in AC_\delta^n[a, b]$, where $0 < a < b < \infty$. Then,

- (i) If $\alpha \notin \mathbb{N}_0$, $({}_a^C D^{\alpha, \rho} f)(x) = {}_a I^{n - \alpha, \rho} (\delta^n f)(x)$
- (ii) If $\alpha \in \mathbb{N}$, $({}_a^C D^{\alpha, \rho} f)(x) = (\delta^n f)(x)$
- (iii) ${}_a^C D^{0, \rho} f = f$

Theorem 2. Let $f(x) \in C_\delta^n[a, b]$, $0 < a < b < \infty$, and $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha) \geq 0$ and $\Re(\beta) \geq 0$. Then,

$${}_a^C D^{\alpha, \rho} {}_a I^{\beta, \rho} f(x) = {}_a I^{\beta - \alpha, \rho} f(x). \quad (14)$$

Our main results are based on utilizing the following fixed point theorems.

Theorem 3 (Schaefer's fixed point theorem [40]). Let X be a Banach space. Assume that $\mathcal{L}: X \longrightarrow X$ is a completely continuous operator and the set $T = \{u \in X: u = \lambda \mathcal{L}u, 0 < \lambda < 1\}$ is bounded. Then, \mathcal{L} has a fixed point in X .

Theorem 4 (Krasnoselskii's fixed point theorem [40]). Let \mathcal{N} be a closed, convex, bounded, and nonempty subset of a Banach space X . Let T_1 and T_2 be operators such that

- (i) $T_1(u_1) + T_2(u_2)$ belong to \mathcal{N} whenever $u_1, u_2 \in \mathcal{N}$

where $n = [\alpha] + 1$ and $AC_\delta^n[a, b]$ denotes the space of all absolutely continuous real valued functions on $[a, b]$, where

(ii) T_1 is a compact and continuous and T_2 is a contraction mapping

Then, there exists $u_0 \in \mathcal{N}$ such that $u_0 = T_1(u_0) + T_2(u_0)$.

For computational convenience, we introduce the following notations:

$$\psi = \frac{1}{\rho^3(3\rho + 1)} - \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\beta+3)} \Gamma(4)}{\rho^{\beta+3} \Gamma(\beta + 4)}. \quad (15)$$

Lemma 2. Let $h \in L_F([0, 1])$ and ψ be given by (15). Then, the solution of the GFBVP

$$\begin{cases} {}^C D_{0^+}^{\alpha, \rho} u(t) = h(t), t \in [0, 1], \quad \rho > 0, \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I^{\beta, \rho} u(\xi_i), \quad \xi_i \in (0, 1), \delta = t^{1-\rho} \frac{d}{dt}, \end{cases} \quad (16)$$

is given by

$$u(t) = I^{\alpha, \rho} h(t) + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} h(\xi_i) - \int_0^1 I^{\alpha, \rho} h(s) ds \right]. \quad (17)$$

Proof. By using Lemma 1, we obtain

$$u(t) = I^{\alpha, \rho} h(t) + C_0 + C_1 \frac{t^\rho}{\rho} + C_2 \frac{t^{2\rho}}{\rho^2} + C_3 \frac{t^{3\rho}}{\rho^3}, \quad (18)$$

where $C_i \in \mathbb{R}$, $(i = 0, 1, 2, 3)$ are the arbitrary constants. Using the boundary conditions $u(0) = \delta u(0) = \delta^2 u(0) = 0$, we get $C_0 = C_1 = C_2 = 0$. Then, from $\int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I^{\beta, \rho} u(\xi_i)$, it follows that

$$C_3 = \frac{1}{\psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} h(s) ds \right. \\ \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^\alpha \tau^{\rho-1} h(\tau) d\tau \right) ds \right]. \quad (19)$$

Substituting the values of C_0, C_1, C_2 , and C_3 in (18), we get (17). \square

3. Main Results

In this section, we prove the existence and uniqueness of solutions of GFBVP (1). We assume that f belongs to an

Orlicz space $L_E[0, 1]$. For $3 < \alpha \leq 4$, let $X = \{u: u, {}^C D_{0+}^{\alpha-1, \rho} u, {}^C D_{0+}^{\alpha-2, \rho} u, {}^C D_{0+}^{\alpha-3, \rho} u \in C([0, 1], \mathbb{R})\}$ denote the Banach space of all continuous functions defined on $[0, 1]$ into \mathbb{R} endowed with the norm

$$\|u\| = \sup \left\{ |u(t)| + \left| {}^C D_{0+}^{\alpha-1, \rho} u(t) \right| + \left| {}^C D_{0+}^{\alpha-2, \rho} u(t) \right| + \left| {}^C D_{0+}^{\alpha-3, \rho} u(t) \right|, \quad t \in [0, 1] \right\}. \quad (20)$$

Relative to GFBVP (1), in view of Lemma 2, we define an operator $\mathcal{L}: X \rightarrow X$ as

$$\mathcal{L}u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right. \\ \left. - \int_0^1 t^{\alpha\rho} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right], \quad t \in [0, 1]. \quad (21)$$

Notice that GFBVP (1) has solutions if and only if the operator \mathcal{L} has fixed points.

The following result plays a major role in our analysis.

Lemma 3. *Let F be a Young function which has a Young complement F^* satisfying*

$$\int_0^t F^*(s^{\alpha-1}) ds < \infty, \\ \int_0^t F^*(s^{\alpha+\beta-1}) ds < \infty, \quad (22) \\ t > 0,$$

for $\alpha \in (3, 4]$ and $\beta \in (0, 1)$. Then, the operator \mathcal{L} exists and is well defined.

Proof. Let $u \in X$. Define a function as follows:

$$\varrho_1(s) = \begin{cases} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}, & \text{if } s \in [0, t], t > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$

At the beginning, we will show that $\varrho_1 \in L_{F^*}[0, 1]$. By using appropriate substitution and properties of the Young functions, one obtains

$$\int_0^1 F^*\left(\frac{|\varrho_1(s)|}{\kappa}\right) ds = \int_0^t F^*\left(\frac{(t^\rho - s^\rho)^{\alpha-1}}{\kappa}\right) s^{\rho-1} ds \\ = \frac{1}{\rho} \left(\frac{1}{\kappa}\right)^{1/(1-\alpha)} \int_0^{\kappa^{1/(1-\alpha)} t^\rho} F^*(s^{\alpha-1}) ds, \quad (24)$$

by the assumption of the theorem, and we get $\varrho_1 \in L_{F^*}[0, 1]$. Similarly, setting

$$\varrho_2(s) = \begin{cases} (t^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1}, & \text{if } s \in [0, t], t > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

one can get $\varrho_2 \in L_{F^*}[0, 1]$. Next, we show that \mathcal{L} is well defined, i.e., $\mathcal{L}u(t) \in C([0, 1], \mathbb{R})$. Let $0 \leq \tau < t \leq 1$. Thus, we have

$$\begin{aligned}
|(\mathcal{L}u)(t) - (\mathcal{L}u)(\tau)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^\tau \left| (t^\rho - s^\rho)^{\alpha-1} - (\tau^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad + \int_\tau^t \left| (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| \left\| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right\| ds \Big] \\
&\quad + \frac{(t^{3\rho} - \tau^{3\rho})}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} \left| (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s \left| (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(\tau, u(\tau), {}^C D_{0+}^{\alpha-1, \rho} u(\tau), {}^C D_{0+}^{\alpha-2, \rho} u(\tau), {}^C D_{0+}^{\alpha-3, \rho} u(\tau)\right) \right| d\tau \Big) ds \Big] \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^1 [\mathcal{G}_1(s) + \mathcal{G}_2(s)] \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right. \\
&\quad + \frac{(t^{3\rho} - \tau^{3\rho})}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} \left| (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s \left| (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(\tau, u(\tau), {}^C D_{0+}^{\alpha-1, \rho} u(\tau), {}^C D_{0+}^{\alpha-2, \rho} u(\tau), {}^C D_{0+}^{\alpha-3, \rho} u(\tau)\right) \right| d\tau \Big) ds \Big],
\end{aligned} \tag{26}$$

where

$$\mathcal{G}_1(s) = \begin{cases} \left| (t^\rho - s^\rho)^{\alpha-1} - (\tau^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1}, & \text{if } s \in [0, \tau], \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

$$\mathcal{G}_2(s) = \begin{cases} \left| (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right|, & \text{if } s \in [\tau, t], \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

The functions \mathcal{G}_i , $i = 1, 2$, belong to $L_{F^*}[0, 1]$ with $\|\mathcal{G}_i\|_{F^*} \leq g(|t^\rho - \tau^\rho|)$, $i = 1, 2$, where $g: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and increasing function with $g(0) = 0$. Using the Hölder inequality, we have

$$\begin{aligned}
|(\mathcal{L}u)(t) - (\mathcal{L}u)(\tau)| &\leq \|f\|_F \left\{ \frac{2}{\rho^{\alpha-1} \Gamma(\alpha)} [\|\mathcal{G}_1\|_{F^*} + \|\mathcal{G}_2\|_{F^*}] \right. \\
&\quad \left. + \frac{|t^{3\rho} - \tau^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}.
\end{aligned} \tag{29}$$

Then, $0 < |t^\rho - \tau^\rho| < \epsilon$ and by the continuity of g , we see that $\mathcal{L}u$ is continuous, which completes the proof.

Our first existence result relies on Schaefer's fixed point theorem. \square

Theorem 5. Assume that there exists $\mathcal{M} \in C([0, 1], \mathbb{R})$ such that

$$\begin{aligned} (H_1) \left| f\left(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)\right) \right| \\ \leq \mathcal{M}(t), \quad \text{for } t \in [0, 1], \end{aligned} \quad (30)$$

with $\|\mathcal{M}\| = \max_{t \in [0, 1]} |\mathcal{M}(t)|$. Then, the GFBVP (1) has at least one solution on $[0, 1]$.

Proof. The proof proceeds in few steps as follows:

Step 1: we will prove the operator \mathcal{L} maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$. Let $E \subset X$ be a bounded set. Then, for all $u \in E$, by using the assumption (H_1) , for $t \in [0, 1]$, we are able to obtain

$$\begin{aligned} |\mathcal{L}u(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\ &\quad + \frac{|t^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\ &\quad \cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\ &\quad \left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |\mathcal{M}(s)| ds \\ &\quad + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} |\mathcal{M}(s)| ds \right. \\ &\quad \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |\mathcal{M}(\tau)| d\tau \right) ds \right] \\ &\leq \frac{\rho^{\alpha-1} \|\mathcal{M}\|}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds \\ &\quad + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i (\alpha + \beta) \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} ds \right. \\ &\quad \left. - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right]. \end{aligned} \quad (31)$$

Thus, we obtain

$$\begin{aligned} \|\mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right] \right\} \\ &= \mu_1. \end{aligned} \quad (32)$$

Now, with some efforts of computation, we have

$$\begin{aligned}
& \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u \right)(t) = \left(I^{1, \rho} f \right)(t) \\
& + {}^C D_{0^+}^{\alpha-1, \rho} \left\{ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \right. \\
& \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& \left. \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \right\} \\
& = \int_0^t s^{\rho-1} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& + \frac{6t^{\rho(4-\alpha)}}{\rho^{4-\alpha} \psi \Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
& \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \\
& \leq \int_0^t s^{\rho-1} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& + \frac{6t^{\rho(4-\alpha)}}{\rho^{4-\alpha} \psi \Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
& \cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
& \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right. \right. \\
& \cdot \left. \left. \left| f\left(\tau, u(\tau), {}^C D_{0^+}^{\alpha-1, \rho} u(\tau), {}^C D_{0^+}^{\alpha-2, \rho} u(\tau), {}^C D_{0^+}^{\alpha-3, \rho} u(\tau)\right) \tau^{\rho-1} \right| d\tau \right) ds \right].
\end{aligned} \tag{33}$$

Therefore, we have

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u\| &\leq \|\mathcal{M}\| \int_0^t s^{\rho-1} ds \\
&+ \frac{6t^{\rho(4-\alpha)}}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\|\mathcal{M}\|\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \right. \\
&\cdot \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} ds - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \Big] \\
&\leq \frac{\|\mathcal{M}\|t^\rho}{\rho} + \frac{6t^{\rho(4-\alpha)}}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\|\mathcal{M}\|\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \\
&\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho} + \frac{6}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} = \mu_2.
\end{aligned} \tag{34}$$

In a similar way, we arrive at

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-2} \mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^2 \Gamma(3)} + \frac{6}{|\psi|\rho^{5-\alpha}\Gamma(6-\alpha)} \right. \\
&\cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} \\
&= \mu_3.
\end{aligned} \tag{35}$$

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-3} \mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^3 \Gamma(4)} + \frac{6}{|\psi|\rho^{6-\alpha}\Gamma(7-\alpha)} \right. \\
&\cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} \\
&= \mu_4.
\end{aligned} \tag{36}$$

Step 2: from (32)–(36), we can immediately get the operator $\mathcal{L}: X \rightarrow X$ to map bounded sets into bounded sets. Let $0 < t_1 < t_2 < 1$, and for all $u \in E$, we get

$$\begin{aligned}
|(\mathcal{L}u)(t_2) - (\mathcal{L}u)(t_1)| &= \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right. \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_1^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&+ \frac{t_2^{3\rho} - t_1^{3\rho}}{\psi \rho^3} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \Big| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} [(t_2^\rho - s^\rho)^{\alpha-1} - (t_1^\rho - s^\rho)^{\alpha-1}] s^{\rho-1} \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&+ \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&+ \frac{(t_2^{3\rho} - t_1^{3\rho})}{|\psi|\rho^3} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \right].
\end{aligned} \tag{37}$$

Thus, we have

$$\begin{aligned}
|(\mathcal{L}u)(t_2) - (\mathcal{L}u)(t_1)| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^\alpha \Gamma(\alpha+1)} [2(t_2^\rho - t_1^\rho)^\alpha - (t_2^{\alpha\rho} - t_1^{\alpha\rho})] \right. \\
&+ \frac{(t_2^{3\rho} - t_1^{3\rho})}{|\psi|\rho^3} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}.
\end{aligned} \tag{38}$$

Similarly, it can be easily shown that

$$\left| \left({}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^\rho - t_1^\rho)}{\rho} + \frac{6(t_2^{\rho(4-\alpha)} - t_1^{\rho(4-\alpha)})}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (39)$$

$$\left| \left({}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^{2\rho} - t_1^{2\rho})}{\rho^2\Gamma(3)} + \frac{6(t_2^{\rho(5-\alpha)} - t_1^{\rho(5-\alpha)})}{|\psi|\rho^{5-\alpha}\Gamma(6-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (40)$$

$$\left| \left({}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^{3\rho} - t_1^{3\rho})}{\rho^3\Gamma(4)} + \frac{6(t_2^{\rho(6-\alpha)} - t_1^{\rho(6-\alpha)})}{|\psi|\rho^{6-\alpha}\Gamma(7-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (41)$$

The right side of (38)–(41) goes to zero as $(t_2 - t_1) \rightarrow 0$. In view of steps I and II, it follows that by Arzela–Ascoli theorem, the sets $\{\mathcal{L}(u): u \in E\}$, $\{{}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u: u \in E\}$, $\{{}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u: u \in E\}$, and $\{{}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u: u \in E\}$ are relatively compact in $C([0, 1])$. Therefore, $\mathcal{L}(E)$ is a relatively compact set in X . We consider the set $T = \{u \in X; u = \lambda \mathcal{L}u, 0 < \lambda < 1\}$. Then, T is bounded. Indeed, let $u \in T$. So, $u = \lambda \mathcal{L}u$, $0 < \lambda < 1$, for any $t \in [0, 1]$, and it follows that

$$\|u\| \leq \frac{\| \mathcal{M} \|}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left[\sum_{i=1}^m \delta_i \frac{\| \mathcal{M} \| \xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{\| \mathcal{M} \|}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right]. \quad (42)$$

Thus, all the hypotheses of Theorem 3 are satisfied. Therefore, we can conclude that the operator \mathcal{L} has at least

one fixed point. Hence, the GFBVP (1) has at least one solution on $[0, 1]$.

For computation convenience, we set

$$V = \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)}, \quad (43)$$

$$K = \frac{6}{\rho^{4-\alpha}\psi\Gamma(5-\alpha)}, \quad (44)$$

$$\sigma = \frac{1}{\rho}. \quad (45)$$

Now, we make use of Theorem 4 to prove the existence of solutions of GFBVP (1). \square

Theorem 6. Let $f: [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be a continuous function such that the following assumptions hold:

$$(H_2) |f(t, u, p, q, w) - f(t, v, \tilde{p}, \tilde{q}, \tilde{w})| \leq L(|u - v| + |p - \tilde{p}| + |q - \tilde{q}| + |w - \tilde{w}|),$$

$$\text{for all } t \in [0, 1], u, v, p, q, w, \tilde{p}, \tilde{q}, \tilde{w} \in \mathbb{R} \text{ and } L > 0,$$

$$(H_3) \left| f(t, u(t), {}^C D_{0+}^{\alpha-1,\rho} u(t), {}^C D_{0+}^{\alpha-2,\rho} u(t), {}^C D_{0+}^{\alpha-3,\rho} u(t)) \right| \leq \phi(t), \quad (46)$$

$$\text{for } t \in [0, 1] \text{ and } \phi \in C([0, 1], \mathbb{R}) \text{ with } \|\phi\| = \max_{t \in [0, 1]} |\phi(t)|.$$

Then, GFBVP (1) has at least one solution on $[0, 1]$, provided that

$$LKV < 1. \quad (47)$$

Proof. We define $B_r = \{u \in X: \|u\| \leq r\}$, where $r \geq \|\phi\|(\sigma + KV)$. We split the operator \mathcal{L} defined by (21) on B_r as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where \mathcal{L}_1 and \mathcal{L}_2 are given by

$$\begin{aligned}
(\mathcal{L}_1 u)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds. \\
(\mathcal{L}_2 u)(t) &= \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \cdot f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right].
\end{aligned} \tag{48}$$

For $u, v \in B_r$, we find that

$$\begin{aligned}
\|\mathcal{L}_1(u) + \mathcal{L}_2(u)\| &\leq \max_{t \in [0,1]} \left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad \left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \Big\} \\
&\leq \|\phi\| \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^\rho (\alpha+\beta)}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right) \right] \\
&\leq \|\phi\| (\sigma + KV) \leq r.
\end{aligned} \tag{49}$$

Furthermore,

$$\begin{aligned}
\left\| \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho} + \frac{6}{|\psi| \rho^{4-\alpha} \Gamma(5-\alpha)} \right. \\
&\quad \cdot \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \Bigg] \\
&\leq \|\phi\| (\sigma + KV) \leq r, \\
\left\| \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho^2 \Gamma(3)} + \frac{6}{|\psi| \rho^{5-\alpha} \Gamma(6-\alpha)} \right. \\
&\quad \cdot \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \Bigg] \quad (50) \\
&\leq \|\phi\| (\sigma + KV) \leq r, \\
\left\| \left({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-3, \rho} T_3 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho^3 \Gamma(4)} + \frac{6}{\|\psi\| \rho^{6-\alpha} \Gamma(7-\alpha)} \right. \\
&\quad \cdot \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \Bigg] \\
&\leq \|\phi\| (\sigma + KV) \leq r.
\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{L}_1(u) + \mathcal{L}_2(u)\|_X \leq \|\phi\| (\sigma + KV) \leq r, \quad (51)$$

which shows that $\mathcal{L}_1(u) + \mathcal{L}_2(u) \in B_r$. In what follows, we prove that \mathcal{L}_2 is a contraction. Let $u, v \in B_r$ and for all $t \in [0, 1]$, we get

$$\begin{aligned}
\|\mathcal{L}_2(u) - \mathcal{L}_2(v)\| &= \sup_{t \in [0, 1]} |\mathcal{L}_2(u) - \mathcal{L}_2(v)| \\
&= \sup_{t \in [0, 1]} \left| \frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta+1} s^{p-1} \right. \\
&\quad \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad - \frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta+1} s^{p-1} \\
&\quad \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad \left. + \int_0^1 I^{\alpha, \rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1, \rho} v(s), {}^C D_{0^+}^{\alpha-2, \rho} v(s), {}^C D_{0^+}^{\alpha-3, \rho} v(s)\right) ds \right| \\
&= \sup_{t \in [0, 1]} \left| \frac{t^3 \rho}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta+1} s^{p-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right. \right. \\
& \left. \left. - f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right) ds \\
& - \int_0^1 I^{\alpha, \rho} \left(f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \left. \left. - f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right) ds \right] \\
& \leq \sup_{t \in [0,1]} \left[L \left(|u(t) - v(t)| + \left| {}^C D_{0^+}^{\alpha-1, \rho} u(t) - {}^C D_{0^+}^{\alpha-1, \rho} v(t) \right| \right. \right. \\
& \left. \left. + \left| {}^C D_{0^+}^{\alpha-2, \rho} u(t) - {}^C D_{0^+}^{\alpha-2, \rho} v(t) \right| + \left| {}^C D_{0^+}^{\alpha-3, \rho} u(t) - {}^C D_{0^+}^{\alpha-3, \rho} v(t) \right| \right) \right. \\
& \left. \cdot \left(\frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta+1} s^{\rho-1} ds - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right. \\
& \left. \leq L \left[\frac{1}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \|u - v\| \right. \\
& \left. \leq LKV \|u - v\|. \right. \tag{52}
\end{aligned}$$

With the same process, one has

$$\begin{aligned}
& \left\| {}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|, \\
& \left\| {}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|, \tag{53} \\
& \left\| {}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|.
\end{aligned}$$

Hence, we get

$$\left\| \mathcal{L}_2(u) - \mathcal{L}_2(v) \right\|_X \leq LKV \|u - v\|_X. \tag{54}$$

It remains to show that \mathcal{L}_1 is continuous and compact. The continuity of the function f implies that the operator \mathcal{L}_1 is continuous. To achieve the compactness of the operator \mathcal{L}_1 , we first prove that \mathcal{L}_1 is uniformly bounded on B_r as follows:

$$\begin{aligned}
\left\| \mathcal{L}_1(u) \right\| &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sup_{t \in [0,1]} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\leq \frac{\|\phi\|}{\rho^\alpha \Gamma(\alpha+1)} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^\rho}{\rho} \right) \leq \frac{\|\phi\|}{\rho} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^{2\rho}}{\rho^2 \Gamma(3)} \right) \leq \frac{\|\phi\|}{\rho^2 \Gamma(3)} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^{3\rho}}{\rho^3 \Gamma(4)} \right) \leq \frac{\|\phi\|}{\rho^3 \Gamma(4)} \leq \|\phi\| \sigma.
\end{aligned} \tag{55}$$

So,

$$\|\mathcal{L}_1(u)\|_X \leq \|\phi\|\sigma. \quad (56)$$

Now, let $\sup_{t \in [0,1] \times B_r^4} |f(t, u, p, q, w)| = \eta$. Consequently, for $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$|(\mathcal{L}_1 u)(t_2) - (\mathcal{L}_1 u)(t_1)| \leq \frac{\eta}{\rho} [2(t_2^\rho - t_1^\rho)^\alpha - (t_2^{\alpha\rho} - t_1^{\alpha\rho})]. \quad (57)$$

Furthermore,

$$\begin{aligned} & \left| ({}^C D_{0+}^{\alpha-1, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-1, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^\rho - t_1^\rho|, \\ & \left| ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^{2\rho} - t_1^{2\rho}|, \\ & \left| ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^{3\rho} - t_1^{3\rho}|, \end{aligned} \quad (58)$$

Therefore, as $(t_2 - t_1) \rightarrow 0$, the right-hand sides of the above inequalities tend to zero independently of $u \in B_r$. Thus, \mathcal{L}_1 is equicontinuous and so it is relatively compact on B_r . According to the Arzela–Ascoli theorem, the operator \mathcal{L}_1 is compact. By using Theorem 4, there exists at least one solution of GFBVP (1) on $[0, 1]$.

In the next result, we prove the existence of solutions for GFBVP (1) by applying Banach's fixed point theorem. \square

Theorem 7. Assume that the condition (H_2) holds. Then, GFBVP (1) has a unique solution on $[0, 1]$, provided that

$$LA < 1, \quad (59)$$

where $A = \sigma + KV$ and V, K , and σ are given by (43)–(45).

Proof. Consider the operator $\mathcal{L}: X \rightarrow X$ defined by

$$\begin{aligned} (\mathcal{L}u)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ &+ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\ &\cdot f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ &\left. - \int_0^1 I^{\alpha, \rho} \left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s) \right) ds \right]. \end{aligned} \quad (60)$$

Set

$$\begin{aligned} \sup_{t \in [0,1]} |f(t, 0, 0, 0, 0)| &= f_0, \\ r &\geq \frac{f_0 A}{1 - LA}. \end{aligned} \quad (61)$$

We shall show that $\mathcal{L}B_r \subset B_r$, where $B_r = \{u \in C([0, 1], \mathbb{R}) : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned} |\mathcal{L}u(t)| &\leq I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| \\ &+ \frac{|t^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| \right. \\ &\left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \\ &\leq I^{\alpha, \rho} \left[\left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right] \end{aligned}$$

$$\begin{aligned}
& + [|f(s, 0, 0, 0, 0)|] + \frac{t^{3\rho}}{\rho^3|\psi|} \left\{ \sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} \right. \\
& \cdot \left[\left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right. \\
& + [|f(s, 0, 0, 0, 0)|] \\
& - \int_0^1 I^{\alpha, \rho} \left[\left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right. \\
& \left. \left. + \{ [|f(s, 0, 0, 0, 0)|] ds \} \right. \right.
\end{aligned} \tag{62}$$

Therefore,

$$\begin{aligned}
|\mathcal{L}u(t)| & \leq \sup_{t \in [0, 1]} \left[L \left(|u(t)| + \left| {}^C D_{0^+}^{\alpha-1, \rho} u(t) \right| + \left| {}^C D_{0^+}^{\alpha-2, \rho} u(t) \right| + \left| {}^C D_{0^+}^{\alpha-3, \rho} u(t) \right| \right) + f_0 \right] \\
& \cdot \left[\frac{t^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{t^{3\rho}}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq (L\|u\| + f_0) \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
& \quad \left. \left. \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r.
\end{aligned} \tag{63}$$

Similarly,

$$\begin{aligned}
\left| \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r, \\
\left| \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r, \\
\left| \left({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r,
\end{aligned} \tag{64}$$

By taking the norm from $t \in [0, 1]$, it yields $\|\mathcal{L}u\| \leq r$, $\|{}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u\| \leq r$, $\|{}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}u\| \leq r$, and $\|{}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}u\| \leq r$. Consequently, we have

$$\|\mathcal{L}u\|_X \leq (Lr + f_0)A \leq r, \tag{65}$$

which shows that \mathcal{L} maps B_r into itself. In order to show that the operator \mathcal{L} is a contraction, let $u, v \in C([0, 1], \mathbb{R})$. Then, for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
\|\mathcal{L}u - \mathcal{L}v\| & = \sup_{t \in [0, 1]} |\mathcal{L}u(t) - \mathcal{L}v(t)| \\
& = \sup_{t \in [0, 1]} \left| I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \quad + \frac{t^{3\rho}}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \quad \left. \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \right|
\end{aligned}$$

$$\begin{aligned}
& -I^{\alpha,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) \\
& -\frac{t^{3\rho}}{\rho^3|\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) \right. \\
& \left. + \int_0^1 I^{\alpha,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) ds \right] \\
& \leq \sup_{t \in [0,1]} \left[L \left(|u(t) - v(t)| + \left| {}^C D_{0^+}^{\alpha-1,\rho} u(t) - {}^C D_{0^+}^{\alpha-1,\rho} v(t) \right| \right. \right. \\
& \quad \left. \left. + \left| {}^C D_{0^+}^{\alpha-2,\rho} u(t) - {}^C D_{0^+}^{\alpha-2,\rho} v(t) \right| + \left| {}^C D_{0^+}^{\alpha-3,\rho} u(t) - {}^C D_{0^+}^{\alpha-3,\rho} v(t) \right| \right) \right. \\
& \quad \left. \cdot \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \right] \\
& \leq L \|u - v\| \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
& \quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq LA \|u - v\|.
\end{aligned} \tag{66}$$

Similarly,

$$\begin{aligned}
& \left\| ({}^C D_{0^+}^{\alpha-1,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-1,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|, \\
& \left\| ({}^C D_{0^+}^{\alpha-2,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-2,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|, \\
& \left\| ({}^C D_{0^+}^{\alpha-3,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-3,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|.
\end{aligned} \tag{67}$$

Consequently,

$$\|(\mathcal{L}u) - (\mathcal{L}v)\|_X \leq LA \|u - v\|_X. \tag{68}$$

Thus, in view of condition (59), it follows that the operator \mathcal{L} is a contraction. Hence, the operator \mathcal{L} has a unique fixed point which corresponds to a unique solution of GFBVP (1).

We conclude this section with some examples showing the applicability of our main results. \square

Example 1. Let us consider the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{(7/2), (1/3)} u(t) = f(t, u(t), {}^C D_{0^+}^{\alpha-1,\rho} u(t), {}^C D_{0^+}^{\alpha-2,\rho} u(t), {}^C D_{0^+}^{\alpha-3,\rho} u(t)), & t \in [0, 1], \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^2 \delta_i I^{(1/2), (1/3)} u(\xi_i), \end{cases} \tag{69}$$

where $\alpha = 7/2, \beta = 1/2, \rho = 1/3, \delta_1 = 2/3, \delta_2 = 7/6, \xi_1 = 1/3, \xi_2 = 5/6$, and

$$\begin{aligned}
f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) &= \frac{1}{9} e^t \sin t + \frac{u(t)}{(t+3)^2} \\
&+ \frac{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)|}{t^2 + 20} + \frac{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)|}{5\sqrt{10t+64}} + \frac{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)|}{11(t+1)}.
\end{aligned} \tag{70}$$

To check that the condition (H_2) holds, let $u, v \in C([0, 1], \mathbb{R})$ and $t \in [0, 1]$; then, we get

$$\begin{aligned} & \left| f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \right. \\ & \quad \left. - f\left(t, v(t), {}^C D_{0^+}^{(5/2), (1/3)} v(t), {}^C D_{0^+}^{(3/2), (1/3)} v(t), {}^C D_{0^+}^{(1/2), (1/3)} v(t)\right) \right| \\ & \leq \frac{1}{9} |u(t) - v(t)| + \frac{1}{20} \left| {}^C D_{0^+}^{(5/2), (1/3)} u(t) - {}^C D_{0^+}^{(5/2), (1/3)} v(t) \right| + \frac{1}{40} \left| {}^C D_{0^+}^{(3/2), (1/3)} u(t) - {}^C D_{0^+}^{(3/2), (1/3)} v(t) \right| \\ & \quad + \frac{1}{11} \left| {}^C D_{0^+}^{(1/2), (1/3)} u(t) - {}^C D_{0^+}^{(1/2), (1/3)} v(t) \right| \leq \frac{1}{9} \|u - v\|_X. \end{aligned} \quad (71)$$

Hence, the condition (H_2) holds with $L = 1/9$ and it follows that $LA = 0.497230703 < 1$, where $A = \sigma + KV$ and V, K and σ are given by (43)–(45). Then, the hypotheses of Theorem 7 are satisfied and problem (69) has a unique solution on $[0, 1]$.

Example 2. We consider the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{(7/2), (1/3)} u(t) = f\left(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)\right), & t \in [0, 1], \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^2 \delta_i I^{(1/2), (1/3)} u(\xi_i). \end{cases} \quad (72)$$

where $\alpha = 7/2, \beta = 1/2, \rho = 1/3, \delta_1 = 2/3, \delta_2 = 7/6, \xi_1 = 1/3$, and $\xi_2 = 5/6$. Choose

$$\begin{aligned} & f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \\ & = \frac{1}{30} e^t + \frac{\tan^{-1} u(t)}{\sqrt{\cos t + 145}} + \frac{t}{8t^3 + 30} \left(\frac{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)| + 1} \right) \\ & \quad + \frac{3}{6(t + 10)} \left(\frac{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)| + 1} \right) + \frac{1}{\sqrt{\sin t + 169}} \left(\frac{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)| + 1} \right). \end{aligned} \quad (73)$$

Set $F(u) = |u|^\alpha / \alpha, \alpha > 1$. In particular, if $\alpha = 2$, then F is an N-function and satisfies

$$\int_0^1 F(|f(u(s))|) ds < \infty, \quad (74)$$

which shows that f belongs to an Orlicz space $L_F([0, 1])$. Furthermore,

$$\begin{aligned} & \left| f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \right| \\ & \leq \mathcal{M}(t), \end{aligned} \quad (75)$$

with $\mathcal{M}(t) = (e^t/30) + (\pi/2\sqrt{\cos t + 145}) + (t/(8t^3 + 30)) + (3/6(t + 10)) + (1/\sqrt{\sin t + 169})$ and $\|\mathcal{M}\| = 1.225$. Thus, the condition (H_1) holds and by the conclusion of Theorem 5, problem (72) has at least one solution on $[0, 1]$.

Moreover,

$$\begin{aligned}
& \left| f\left(t, u(t), {}^C D_{0+}^{(5/2), (1/3)} u(t), {}^C D_{0+}^{(3/2), (1/3)} u(t), {}^C D_{0+}^{(1/2), (1/3)} u(t)\right) \right. \\
& \quad \left. - f\left(t, v(t), {}^C D_{0+}^{(5/2), (1/3)} v(t), {}^C D_{0+}^{(3/2), (1/3)} v(t), {}^C D_{0+}^{(1/2), (1/3)} v(t)\right) \right| \\
& \leq \frac{1}{12} |u(t) - v(t)| + \frac{1}{30} \left| {}^C D_{0+}^{(5/2), (1/3)} u(t) - {}^C D_{0+}^{(5/2), (1/3)} v(t) \right| + \frac{1}{20} \left| {}^C D_{0+}^{(3/2), (1/3)} u(t) - {}^C D_{0+}^{(3/2), (1/3)} v(t) \right| \\
& \quad + \frac{1}{13} \left| {}^C D_{0+}^{(1/2), (1/3)} u(t) - {}^C D_{0+}^{(1/2), (1/3)} v(t) \right| \leq \frac{1}{12} \|u - v\|_X.
\end{aligned} \tag{76}$$

which proves the validity of condition (H_2) with $LA = 0.372923027 < 1$, where $A = \sigma + KV$ and V, K , and σ are given by (43)–(45). Hence, Theorem 7 can be applicable and so problem (72) has a unique solution on $[0, 1]$.

4. Ulam Stability

In this section, we develop the criteria for Ulam stability of GFBVP (1) by means of its equivalent integral equation. For simplicity, we set

$$f\left(s, v(s), {}^C D_{0+}^{\alpha-1, \rho} v(s), {}^C D_{0+}^{\alpha-2, \rho} v(s), {}^C D_{0+}^{\alpha-3, \rho} v(s)\right) = \widehat{f}(s). \tag{77}$$

Then,

$$\begin{aligned}
v(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \widehat{f}(s) ds \\
&+ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \left. \cdot \widehat{f}(s) ds - \int_0^1 I^{\alpha, \rho} \widehat{f}(s) ds \right],
\end{aligned} \tag{78}$$

where $v \in X$ and $f: [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is a continuous function. Next, we define a continuous nonlinear operator $\mathcal{F}: X \rightarrow X$ as

$$\mathcal{F}v(t) = {}^C D_{0+}^{\alpha, \rho} v(t) - \widehat{f}(t). \tag{79}$$

For definitions of Ulam–Hyers, generalized Ulam–Hyers and Ulam–Hyers–Rassias stability, and we refer to [41].

Definition 7. GFBVP (1) is said to be Ulam–Hyers stable if there exists a real number $b > 0$ such that, for each $\epsilon > 0$ and for each solution $v \in X$,

$$\|\mathcal{F}v\| \leq \epsilon, \quad t \in [0, 1], \tag{80}$$

there exists a solution $u \in X$ of (1) satisfying the inequality

$$\|u - v\| \leq b\epsilon_1, \quad t \in [0, 1], \tag{81}$$

where ϵ_1 is a positive real number depending on ϵ .

Definition 8. GFBVP (1) is generalized Ulam–Hyers stable if there exists $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that, for each solution $v \in X$ of (1), there exists a solution $u \in X$ of (1) with

$$|u(t) - v(t)| \leq \Theta(\epsilon), \quad t \in [0, 1]. \tag{82}$$

Definition 9. GFBVP (1) is Ulam–Hyers–Rassias stable with respect to $\Psi \in C([0, 1], \mathbb{R})$ if there exists a real number $b > 0$ such that, for each $\epsilon > 0$ and for each solution $v \in X$ of (1),

$$|\mathcal{F}v(t)| \leq \epsilon \Psi(t), \quad t \in [0, 1], \tag{83}$$

we can find a solution $u \in X$ of (1) satisfying the inequality

$$|u(t) - v(t)| \leq b\epsilon_1 \Psi(t), \quad t \in [0, 1], \tag{84}$$

where ϵ_1 is a positive real number depending on ϵ .

Theorem 8. Assume that conditions (47) and (59) hold. Then, GFBVP (1) satisfies both Ulam–Hyers and generalized Ulam–Hyers stability criteria.

Proof. We know that $u \in X$ is a unique solution of (1) (by Theorem 7). Let $v \in X$ be another solution of (1) satisfying $\|\mathcal{F}v\| \leq \epsilon, t \in [0, 1]$. Observe that the operators \mathcal{F} and $\mathcal{L} - \mathcal{F}$ are equivalent for every solution $v \in X$ of (1). Therefore, by the fixed point property of the operator \mathcal{L} together with $u(t) = \mathcal{L}u(t)$ and $\|\mathcal{F}v\| \leq \epsilon, t \in [0, 1]$, we have

$$\begin{aligned}
|v(t) - u(t)| &= |v(t) - \mathcal{L}v(t) + \mathcal{L}v(t) - \mathcal{L}u(t)| \\
&\leq |\mathcal{L}u(t) - \mathcal{L}v(t)| + |\mathcal{L}v(t) - v(t)| \\
&\leq LA\|u - v\|_X + \epsilon,
\end{aligned} \tag{85}$$

where $\epsilon > 0$ and $LA < 1$. Taking the norm for $t \in [0, 1]$ and solving for $\|u - v\|_X$, we obtain

$$\|u - v\|_X \leq \frac{\epsilon}{1 - LA}. \tag{86}$$

Choosing $\epsilon_1 = \epsilon/(1 - LA)$ and $b = 1$, the Ulam–Hyers stability condition is satisfied. More generally, defining $\Theta(\epsilon) = \epsilon/(1 - LA)$, the generalized Ulam–Hyers stability condition is also satisfied which completes the proof. \square

Theorem 9. Assume that conditions (47) and (59) hold and there exists a function $z \in C([0, 1], \mathbb{R})$ satisfying condition

$\|u(t) - v(t)\| \leq b\epsilon_1 \Psi(t), t \in [0, 1]$. Then, GFBVP (1) is Ulam–Hyers–Rassias stable with respect to Ψ .

Proof. Similar to the discussion in the proof of Theorem 8, we have

$$\|u - v\|_X \leq \epsilon_1 \Psi(t), \quad (87)$$

with $\epsilon_1 = \epsilon/(1 - LA)$, and we get the desired result. \square

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Wafa Shammakh, Hadeel Z. Alzumi, and Zahra Albarqi contributed to the design and implementation of the research, to the analysis of the results, and to the writing of the manuscript.

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Research Article

Monotonicity Analysis of Fractional Proportional Differences

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In this work, the nabla discrete new Riemann–Liouville and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale \mathbb{Z} are formulated. The differences and summations of discrete fractional proportional are detected on \mathbb{Z} , and the fractional proportional sums associated to $({}^R\nabla^{\varepsilon,\rho}\chi)(z)$ with order $0 < \varepsilon < 1$ are defined. The relation between nabla Riemann–Liouville and Caputo fractional proportional differences is derived. The monotonicity results for the nabla Caputo fractional proportional difference are proved; specifically, if $({}^R\nabla^{\varepsilon,\rho}\chi)(z) > 0$ then $\chi(z)$ is $\varepsilon\rho$ -increasing, and if $\chi(z)$ is strictly increasing on \mathbb{N}_c and $\chi(c) > 0$, then $({}^R\nabla^{\varepsilon,\rho}\chi)(z) > 0$. As an application of our findings, a new version of the fractional proportional difference of the mean value theorem (MVT) on \mathbb{Z} is proved.

1. Introduction

Many problems in science, engineering, and media can be formulated using continuous and discrete fractional calculus [1–14]. The fractional sums and differences and their monotonicity properties are deeply studied in [15–25]. In [26], Atangana and Baleanu solved the fractional heat transfer model using new fractional derivatives with exponential kernels, and they presented many applications of the new notations of fractional derivatives. Applications of discrete fractional calculus are successfully discussed by many researchers in the last decade, for example, in [27–29]. Recently, studying the monotonicity for fractional difference operators with non-singular discrete kernels is under focus [30, 31]. Monotonicity results for fractional difference operators with discrete exponential kernels were studied in [32] when the time step $h = 1$. In [3], deep monotonicity analysis is done for nabla h -discrete fractional differences with a discrete Mittag–Leffler kernel in the time scale $h\mathbb{Z}$ with $0 < \varepsilon < 1$ and $0 < h \leq 1$. The results of the research generalized those obtained in [22] where $0 < \varepsilon < 0.5$ and $h = 1$. After that, monotonicity analysis of fractional proportional differences is studied and then the results are prettified by formulating a new version of mean value theorem as an application. In [33], the nabla fractional sums and

differences of order $0 < \varepsilon < 1$ on the time scale $h\mathbb{Z}$ where $0 < h \leq 1$ are formulated, and the monotonicity results for the nabla h -Caputo fractional difference operator were concluded. In this paper, the authors formulated the nabla discrete new Riemann–Liouville (RL) and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale \mathbb{Z} . They also proved a new version of the fractional proportional difference of the mean value theorem (MVT) on \mathbb{Z} .

The article is organized as follows: Section 2 presents the main definitions and needed preliminaries. In Section 3, the monotonicity results for fractional proportional differences are classified. In Section 4, we formulate a new version of the mean value theorem as an application. Finally, we provide the conclusions in Section 5.

2. Definitions and Preliminary Results

Definition 1. The discrete proportional difference of order $0 < \rho \leq 1$ for the function χ is defined by

$$\begin{aligned}\nabla^\rho \chi(z) &= (1 - \rho)\chi(z) + \rho \nabla \chi(z), \\ z \in \mathbb{N}_{c+1} &= \{c + 1, c + 2, c + 3, \dots\},\end{aligned}\tag{1}$$

and $c \geq 0$ is an integer.

Definition 2. Let $z \in \mathbb{N}_c$, $0 < \rho \leq 1$, and $p = (\rho - 1/\rho)$, then $\widehat{e}_p(z, c) = \rho^{z-c}$.

Definition 3. For any real number α , the α rising function is $z^{\overline{\alpha}} = (\Gamma(z + \alpha)/\Gamma(z))$, such that $z \in \mathbb{C} \setminus \{\dots, -2, -1, 0\}$, $0^{\overline{\alpha}} = 0$, where $\Gamma(z)$ is the gamma function.

Definition 4 (nabla fractional proportional sums).

For a function $\chi: \mathbb{N}_c \rightarrow \mathbb{R}$, $\rho > 0$, and $\varepsilon \in \mathbb{C}$, $0 < \operatorname{Re}(\varepsilon) < 1$, the nabla left fractional proportional sum of χ starting at c is defined by

$$\begin{aligned} ({}_c \nabla^{-\varepsilon, \rho} \chi)(z) &= \frac{1}{\Gamma(\varepsilon)} \int_c^z \widehat{e}_p(z-s, 0) (z-\varsigma(s))^{\overline{\varepsilon-1}} \chi(s) \nabla s \\ &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \widehat{e}_p(z-\iota, 0) (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota), \quad z \in \mathbb{N}_c. \end{aligned} \quad (2)$$

For the function $\chi: {}_d \mathbb{N} = \{d, d-1, d-2, \dots\} \rightarrow \mathbb{R}$, the nabla right fractional proportional sum ending at d is defined by

$$\begin{aligned} (\nabla_d^{-\varepsilon, \rho} \chi)(z) &= \frac{1}{\Gamma(\varepsilon)} \int_z^d \widehat{e}_p(s-z, 0) (s-\varsigma(z))^{\overline{\varepsilon-1}} \chi(s) \Delta s \\ &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=z}^{d-1} \widehat{e}_p(\iota-z, 0) (\iota-\varsigma(z))^{\overline{\varepsilon-1}} \chi(\iota), \quad z \in {}_d \mathbb{N}. \end{aligned} \quad (3)$$

We notice that by setting $\rho = 1$, the given definitions of the fractional sums are generalizations of the Riemann fractional sums.

Lemma 1. Let $\chi: \mathbb{N}_c \rightarrow \mathbb{R}$ be a function, $p = (\rho - 1/\rho)$, $0 < \varepsilon < 1$, and $0 < \rho \leq 1$, then

$$\begin{aligned} ({}_c \nabla^{-\varepsilon, \rho} \nabla \chi)(z) &= (\nabla {}_c \nabla^{-\varepsilon, \rho} \chi)(z) - \frac{(z-c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)} \widehat{e}_p(z-1, c) \chi(c). \end{aligned} \quad (4)$$

Proof.

$$\begin{aligned} ({}_c \nabla^{-\varepsilon, \rho} \nabla \chi)(z) &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \widehat{e}_p(z-\iota, 0) (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \nabla \chi(\iota) \\ &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} (\chi(\iota) - \chi(\iota-1)) \\ &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) \\ &\quad - \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota-1) \\ &= \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) \\ &\quad - \frac{1}{\Gamma(\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota-1} (z-\varsigma(\iota+1))^{\overline{\varepsilon-1}} \chi(\iota) \\ &= \frac{1}{\Gamma(\varepsilon)} \left(\sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) - \sum_{\iota=c}^{z-1} \rho^{(z-1)-\iota} ((z-1)-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) \right) \\ &= \frac{1}{\Gamma(\varepsilon)} \left(\sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) - \sum_{\iota=c+1}^{z-1} \rho^{(z-1)-\iota} ((z-1)-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) \right) \\ &= \frac{1}{\Gamma(\varepsilon)} \rho^{(z-1)-c} ((z-1)-\varsigma(c))^{\overline{\varepsilon-1}} \chi(c) \\ &= \frac{1}{\Gamma(\varepsilon)} \nabla \sum_{\iota=c+1}^z \rho^{z-\iota} (z-\varsigma(\iota))^{\overline{\varepsilon-1}} \chi(\iota) - \frac{1}{\Gamma(\varepsilon)} \rho^{(z-1)-c} (z-1-c+1)^{\overline{\varepsilon-1}} \chi(c) \\ &= (\nabla {}_c \nabla^{-\varepsilon, \rho} \chi)(z) - \frac{(z-c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)} \widehat{e}_p(z-1, c) \chi(c). \end{aligned} \quad (5)$$

□

Lemma 2. Let $\chi: \mathbb{N}_c \longrightarrow \mathbb{R}$, $p = (\rho - 1/\rho)$, $0 < \varepsilon < 1$, and $0 < \rho \leq 1$, then

$$({}_c \nabla^{-\varepsilon, \rho} \nabla^\rho \chi)(z) = (\nabla^\rho {}_c \nabla^{-\varepsilon, \rho} \chi)(z) - \frac{\rho}{\Gamma(\varepsilon)} (z - c)^{\overline{\varepsilon-1}} \widehat{e}_p(z-1, c) \chi(c). \quad (6)$$

Proof.

$$\nabla^\rho \chi(z) = (1 - \rho) \chi(z) + \rho \nabla \chi(z), \quad (7)$$

hence,

$$\begin{aligned} ({}_c \nabla^{-\varepsilon, \rho} \nabla^\rho \chi)(z) &= {}_c \nabla^{-\varepsilon, \rho} ((1 - \rho) \chi(z) + \rho \nabla \chi(z)) \\ &= (1 - \rho) ({}_c \nabla^{-\varepsilon, \rho} \chi)(z) + \rho ({}_c \nabla^{-\varepsilon, \rho} \nabla \chi)(z) \text{ using Lemma 1} \\ &= (1 - \rho) ({}_c \nabla^{-\varepsilon, \rho} \chi)(z) + \rho \left((\nabla^\rho {}_c \nabla^{-\varepsilon, \rho} \chi)(z) - \frac{(z - c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)} \widehat{e}_p(z-1, c) \chi(c) \right) \\ &= ((1 - \rho) ({}_c \nabla^{-\varepsilon, \rho} \chi)(z) + \rho \nabla^\rho ({}_c \nabla^{-\varepsilon, \rho} \chi)(z)) - \frac{\rho}{\Gamma(\varepsilon)} (z - c)^{\overline{\varepsilon-1}} \widehat{e}_p(z-1, c) \chi(c) \\ &= (\nabla^\rho {}_c \nabla^{-\varepsilon, \rho} \chi)(z) - \frac{\rho}{\Gamma(\varepsilon)} (z - c)^{\overline{\varepsilon-1}} \widehat{e}_p(z-1, c) \chi(c). \end{aligned} \quad (8)$$

Note that if $\rho = 1$, we get

$$({}_c \nabla^{-\varepsilon} \nabla \chi)(z) = (\nabla {}_c \nabla^{-\varepsilon} \chi)(z) - \frac{1}{\Gamma(\varepsilon)} (z - c)^{\overline{\varepsilon-1}} \chi(c). \quad (9)$$

□

Definition 5 (Riemann–Liouville (RL) fractional proportional differences)

For $0 < \rho \leq 1$, $\varepsilon \in \mathbb{C}$, $0 < \operatorname{Re}(\varepsilon) < 1$, and χ be a function defined on \mathbb{N}_c or on ${}_d\mathbb{N}$, then the left Riemann–Liouville fractional proportional difference starting at c is defined by

$$\begin{aligned} ({}_c^R \nabla^{\varepsilon, \rho} \chi)(z) &= \nabla^\rho {}_c \nabla^{-(1-\varepsilon), \rho} \chi(z) \\ &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \int_c^z \widehat{e}_p(z-s, 0) (z - \varsigma(s))^{\overline{-\varepsilon}} \chi(s) \nabla s \\ &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^z \widehat{e}_p(z-\iota, 0) (z - \varsigma(\iota))^{\overline{-\varepsilon}} \chi(\iota) \\ &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{\overline{-\varepsilon}} \chi(\iota), \end{aligned} \quad (10)$$

and the right Riemann–Liouville fractional proportional difference ending at d is defined by

$$\begin{aligned} ({}_d^R \nabla^{\varepsilon, \rho} \chi)(z) &= -\Delta^\rho \nabla_d^{-(1-\varepsilon), \rho} \chi(z) \\ &= \frac{-\Delta^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota - \varsigma(z))^{\overline{-\varepsilon}} \chi(\iota). \end{aligned} \quad (11)$$

We notice that by setting $\rho = 1$, the given definitions of the fractional differences are generalizations of the Riemann fractional differences.

Definition 6 (Caputo fractional proportional differences)

For $0 < \rho \leq 1$, $\varepsilon \in \mathbb{C}$, $0 < \operatorname{Re}(\varepsilon) < 1$, and χ be a function defined on \mathbb{N}_c or on ${}_d\mathbb{N}$, then the left Caputo fractional proportional difference starting at c is defined by

$$\begin{aligned} ({}_c^C \nabla^{\varepsilon, \rho} \chi)(z) &= {}_c \nabla^{-(1-\varepsilon), \rho} \nabla^\rho \chi(z) \\ &= \frac{1}{\Gamma(1-\varepsilon)} \int_c^z \widehat{e}_p(z-s, 0) (z - \varsigma(s))^{\overline{-\varepsilon}} (\nabla^\rho \chi(s)) \nabla s \\ &= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{\overline{-\varepsilon}} \nabla^\rho \chi(\iota), \end{aligned} \quad (12)$$

and the right Caputo fractional proportional difference ending at d is defined by

$$\begin{aligned} ({}_d^C \nabla^{\varepsilon, \rho} \chi)(z) &= \nabla_d^{-(1-\varepsilon), \rho} (-\Delta^\rho \chi)(z) \\ &= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota - \varsigma(z))^{\overline{-\varepsilon}} (-\Delta^\rho \chi(\iota)). \end{aligned} \quad (13)$$

We notice that by setting $\rho = 1$, the given definitions of the fractional differences are generalizations of the Caputo fractional differences.

Proposition 1 (the relation between nabla RL and Caputo fractional proportional differences)

For any $\varepsilon \in \mathbb{C}$, $0 < \operatorname{Re}(\varepsilon) < 1$, and $0 < \rho \leq 1$, the relation between nabla RL and Caputo fractional proportional differences is given as follows:

- (i) $({}_c^C \nabla^{\varepsilon, \rho} \chi)(z) = ({}_c^R \nabla^{\varepsilon, \rho} \chi)(z) - (z - c)^{-\varepsilon} / \Gamma(1 - \varepsilon) \widehat{e}_p(z, c) \chi(c), \quad \text{Proof.}$
- (ii) $({}_c^C \nabla^{\varepsilon, \rho} \chi)(z) = ({}_c^R \nabla^{\varepsilon, \rho} \chi)(z) - (d - z)^{-\varepsilon} / \Gamma(1 - \varepsilon) \widehat{e}_p(d, z) \chi(d).$
-

$$\begin{aligned}
({}_c^C \nabla^{\varepsilon, \rho} \chi)(z) &= \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \nabla^{\rho} \chi(\iota) \\
&= \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} ((1 - \rho) \chi(\iota) + \rho \nabla \chi(\iota)) \\
&= \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} (1 - \rho) \chi(\iota) \\
&\quad + \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} (\rho \nabla \chi(\iota)) \\
&= \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} (1 - \rho) \chi(\iota) \\
&\quad + \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \rho (\chi(\iota) - \chi(\iota - 1)) \\
&= \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} (1 - \rho) \chi(\iota) \\
&\quad + \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \rho \chi(\iota) \\
&\quad - \frac{1}{\Gamma(1 - \varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota-1} (z - \varsigma(\iota + 1))^{-\varepsilon} \rho \chi(\iota) \\
&= \frac{1 - \rho}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \chi(\iota) \\
&\quad + \frac{\rho}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^{z-1} \rho^{z-1-\iota} (z - 1 - \varsigma(\iota))^{-\varepsilon} \chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1 - \varepsilon)} \rho^{z-1-c} (z - 1 - \varsigma(c))^{-\varepsilon} \chi(c) \\
&= \frac{1 - \rho}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \chi(\iota) + \frac{\rho \nabla}{\Gamma(1 - \varepsilon)} \sum_{\iota=c+1}^z \rho^{z-\iota} (z - \varsigma(\iota))^{-\varepsilon} \chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1 - \varepsilon)} \rho^{z-1-c} (z - 1 - c + 1)^{-\varepsilon} \chi(c)
\end{aligned}$$

$$\begin{aligned}
&= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^z \widehat{e}_\rho(z-\iota, 0) (z-\varsigma(\iota))^{-\varepsilon} \chi(\iota) \\
&\quad - \frac{(z-c)^{-\varepsilon}}{\Gamma(1-\varepsilon)} \rho^{z-c} \chi(c) = ({}_c^C \nabla^{\varepsilon, \rho} \chi)(z) - \frac{(z-c)^{-\varepsilon}}{\Gamma(1-\varepsilon)} \widehat{e}_\rho(z, c) \chi(c).
\end{aligned} \tag{14}$$

$$\begin{aligned}
({}_d^C \nabla^{\varepsilon, \rho} \chi)(z) &= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} (-\Delta^\rho) \chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} ((1-\rho)\chi(\iota) + \rho\Delta\chi(\iota)) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} (1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho(\chi(\iota+1) - \chi(\iota)) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} (1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho\chi(\iota+1) + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho\chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} (1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z+1}^d \rho^{\iota-1-z} (\iota-1-\varsigma(z))^{-\varepsilon} \rho\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho\chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} (1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \rho^{d-1-z} (d-1-\varsigma(z))^{-\varepsilon} \rho\chi(d) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z+1}^{d-1} \rho^{\iota-(z+1)} (\iota-\varsigma(z+1))^{-\varepsilon} \rho\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho\chi(\iota) \\
&= \frac{-(1-\rho)}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \chi(\iota) - \frac{\rho\Delta}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \rho\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \rho^{d-z} (d-1-z+1)^{-\varepsilon} \chi(d) \\
&= \frac{-\Delta^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z} (\iota-\varsigma(z))^{-\varepsilon} \chi(\iota) - \frac{1}{\Gamma(1-\varepsilon)} \rho^{d-z} (d-z)^{-\varepsilon} \chi(d) \\
&= ({}_d^R \nabla^{\varepsilon, \rho} \chi)(z) - \frac{(d-z)^{-\varepsilon}}{\Gamma(1-\varepsilon)} \widehat{e}_\rho(d, z) \chi(d).
\end{aligned} \tag{15}$$

Numerical calculations have been done in order to verify the first equation in Proposition 1. The values used are $c = 2.5$, $\rho = 0.7$, and $\varepsilon = 0.3$. The results are illustrated in Figure 1.

In addition to that, the data are presented in Table 1. \square

Lemma 3. Let $0 < \varepsilon < 1$, $\frac{z}{\varepsilon}, \iota \in \mathbb{N}_c$, and $\iota < z$, then $\nabla(z - \zeta(\iota))^{-\varepsilon} = -\varepsilon(z - \zeta(\iota))^{-\varepsilon-1}$.

Proof.

$$\begin{aligned}
 \nabla(z - \zeta(\iota))^{-\varepsilon} &= (z - \zeta(\iota))^{-\varepsilon} - (z - 1 - \zeta(\iota))^{-\varepsilon} \\
 &= (z - \iota + 1)^{-\varepsilon} - (z - \iota)^{-\varepsilon} \\
 &= \frac{\Gamma(z - \iota - \varepsilon + 1)}{\Gamma(z - \iota + 1)} - \frac{\Gamma(z - \iota - \varepsilon)}{\Gamma(z - \iota)} \\
 &= \frac{\Gamma(z - \iota - \varepsilon)}{\Gamma(z - \iota)} \left(\frac{z - \iota - \varepsilon}{z - \iota} - 1 \right) \\
 &= \frac{\Gamma(z - \iota + 1 - \varepsilon - 1)}{\Gamma(z - \iota)} \left(\frac{-\varepsilon}{z - \iota} \right) \\
 &= -\varepsilon \frac{\Gamma(z - \iota + 1 - \varepsilon - 1)}{\Gamma(z - \iota + 1)} \\
 &= -\varepsilon(z - \iota + 1)^{-\varepsilon-1} = -\varepsilon(z - \zeta(\iota))^{-\varepsilon-1}.
 \end{aligned} \tag{16}$$

\square

3. Monotonicity Results

The following two monotonicity definitions are given in [18].

Definition 7. Let $y: \mathbb{N}_a \rightarrow \mathbb{R}$ be a function satisfying $y(a) \geq 0$, $0 < \alpha < 1$. Then, $y(t)$ is called an α -increasing function on \mathbb{N}_a if $y(t+1) \geq \alpha y(t) \quad \forall t \in \mathbb{N}_a$.

Definition 8. Let $y: \mathbb{N}_a \rightarrow \mathbb{R}$ be a function satisfying $y(a) \leq 0$, $0 < \alpha < 1$. Then, $y(t)$ is called an α -decreasing function on \mathbb{N}_a if $y(t+1) \leq \alpha y(t) \quad \forall t \in \mathbb{N}_a$.

In the following, we report the new proportional monotonicity main results.

Theorem 1. Let $\chi: \mathbb{N}_c \rightarrow \mathbb{R}$ be a function, and suppose that $({}^R_{c-1} \nabla^{\varepsilon, \rho} \chi)(z) \geq 0$ for $0 < \varepsilon < 1$ and $0 < \rho \leq 1$, $z \in \mathbb{N}_{c-1}$. Then, $\chi(z)$ is $\varepsilon\rho$ -increasing.

Proof.

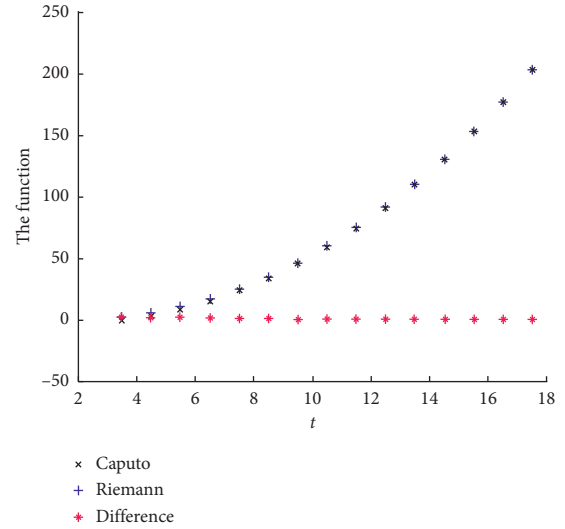


FIGURE 1: The relation between nabla Riemann and Caputo fractional proportional differences.

TABLE 1: The relation between nabla Riemann and Caputo fractional proportional differences.

z	Caputo	Riemann	Difference
3.5	2.181077	-0.415443	2.596520
4.5	5.786530	3.190010	2.596520
5.5	10.787558	8.649248	2.138310
6.5	17.274639	15.611509	1.663130
7.5	25.339691	24.081105	1.258585
8.5	35.058768	34.121524	0.937244
9.5	46.490945	45.800344	0.690601
10.5	59.680789	59.175722	0.505066
11.5	74.661373	74.294052	0.367321
12.5	91.456990	91.190999	0.265991
13.5	110.085354	109.893402	0.191952
14.5	130.559312	130.421178	0.138134
15.5	152.888147	152.788974	0.099173
16.5	177.078543	177.007482	0.071061
17.5	203.135301	203.084469	0.050832

$$\begin{aligned}
 ({}^R_{c-1} \nabla^{\varepsilon, \rho} \chi)(z) &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^z \hat{e}_\rho(z-\iota, 0) (z-\zeta(\iota))^{-\varepsilon} \chi(\iota) \\
 &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^z \rho^{z-\iota} (z-\zeta(\iota))^{-\varepsilon} \chi(\iota).
 \end{aligned} \tag{17}$$

Let

$$S(z) = \sum_{\iota=c}^z \rho^{z-\iota} (z-\zeta(\iota))^{-\varepsilon} \chi(\iota). \tag{18}$$

Then,

$$({}^R_{c-1} \nabla^{\varepsilon, \rho} \chi)(z) = \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} S(z). \tag{19}$$

Hence, from the assumption, we have $\nabla^\rho S(z) \geq 0$. That is,

$$\begin{aligned}
\nabla^\rho S(z) &= (1-\rho)S(z) + \rho \nabla S(z) \\
&= (1-\rho)S(z) + \rho(S(z) - S(z-1)) \\
&= S(z) - \rho S(z) + \rho S(z) - \rho S(z-1) \\
&= S(z) - \rho S(z-1) \\
&= \sum_{l=c}^z \rho^{z-l} (z-\varsigma(l))^{-\varepsilon} \chi(l) \\
&\quad - \rho \sum_{l=c}^{z-1} \rho^{z-1-l} (z-1-\varsigma(l))^{-\varepsilon} \chi(l) \\
&= (z-\varsigma(z))^{-\varepsilon} \chi(z) + \sum_{l=c}^{z-1} \rho^{z-l} (z-\varsigma(l))^{-\varepsilon} \chi(l) \\
&\quad - \sum_{l=c}^{z-1} \rho^{z-1-l} (z-1-\varsigma(l))^{-\varepsilon} \chi(l) \\
&= (z-z+1)^{-\varepsilon} \chi(z) + \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) \\
&\quad \cdot \left((z-\varsigma(l))^{-\varepsilon} - (z-1-\varsigma(l))^{-\varepsilon} \right) \\
&= (1)^{-\varepsilon} \chi(z) + \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) \nabla (z-\varsigma(l))^{-\varepsilon} \\
&= \frac{\Gamma(1-\varepsilon)}{\Gamma(1)} \chi(z) + \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) \left(-\varepsilon (z-\varsigma(l))^{-\varepsilon-1} \right) \\
&= \Gamma(1-\varepsilon) \chi(z) - \varepsilon \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) (z-\varsigma(l))^{-\varepsilon-1} \\
&\geq 0.
\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}
({}_{c-1}^R \nabla^{\varepsilon, \rho} \chi)(z) &= \frac{\nabla^\rho}{\Gamma(1-\varepsilon)} S(z) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \left(\Gamma(1-\varepsilon) \chi(z) - \varepsilon \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) (z-\varsigma(l))^{-\varepsilon-1} \right) \\
&= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) (z-\varsigma(l))^{-\varepsilon-1} \\
&= 0.
\end{aligned} \tag{21}$$

Hence,

$$\chi(z) \geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \rho^{z-l} \chi(l) (z-\varsigma(l))^{-\varepsilon-1}. \tag{22}$$

Clearly, $\chi(c-1) = 0$. So, we can start the induction from the next step. When $z = c$, we get $\chi(c) \geq 0$; also, when $z = c+1$, we have

$$\begin{aligned}
\chi(c+1) &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^c \rho^{c+1-l} \chi(l) (c+1-\varsigma(l))^{-\varepsilon-1} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{c+1-c} \chi(c) (c+1-\varsigma(c))^{-\varepsilon-1} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho \chi(c) (c+1-c+1)^{-\varepsilon-1} \\
&= \frac{\varepsilon \rho}{\Gamma(1-\varepsilon)} \chi(c) \frac{\Gamma(1-\varepsilon)}{\Gamma(2)} \\
&= \varepsilon \rho \chi(c).
\end{aligned} \tag{23}$$

Now for $z+1$, replace z by $z+1$, then we get

$$\begin{aligned}
\chi(z+1) &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^z \rho^{z+1-l} \chi(l) (z+1-\varsigma(l))^{-\varepsilon-1} \\
&\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{z+1-z} \chi(z) (z+1-\varsigma(z))^{-\varepsilon-1} \\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho \chi(z) (z+1-z+1)^{-\varepsilon-1} \\
&= \frac{\varepsilon \rho}{\Gamma(1-\varepsilon)} \chi(z) 2^{-\varepsilon-1} \\
&= \varepsilon \rho \chi(z).
\end{aligned} \tag{24}$$

Hence, $\chi(z)$ is $\varepsilon\rho$ -increasing which completes the proof.

Using Theorem 1 and Proposition 1 we can state the following Caputo fractional proportional difference monotonicity result.

□

Corollary 1. Let $\chi: \mathbb{N}_{c-1} \rightarrow \mathbb{R}$ be a function, and suppose that for $0 < \varepsilon < 1$ and $0 < \rho \leq 1$. Suppose that

$$({}_{c-1}^C \nabla^{\varepsilon, \rho} \chi)(z) \geq \frac{-\hat{e}_p(z, c-1)}{\Gamma(1-\varepsilon)} (z-c+1)^{-\varepsilon} \chi(c-1), \quad z \in \mathbb{N}_{c-1}, \tag{25}$$

then $\chi(z)$ is $\varepsilon\rho$ -increasing.

Proof.

$$\begin{aligned}
({}_{c-1}^C \nabla^{\varepsilon, \rho} \chi)(z) &= ({}_{c-1}^R \nabla^{\varepsilon, \rho} \chi)(z) - \frac{\hat{e}_p(z, c-1)}{\Gamma(1-\varepsilon)} \\
&\quad \cdot (z-c+1)^{-\varepsilon} \chi(c-1), \quad \forall z \in \mathbb{N}_{c-1},
\end{aligned} \tag{26}$$

now, from the assumption we have

$$({}_{c-1}^C \nabla^{\varepsilon, \rho} \chi)(z) \geq \frac{-\hat{e}_p(z, c-1)}{\Gamma(1-\varepsilon)} (z-c+1)^{-\varepsilon} \chi(c-1), \tag{27}$$

$$z \in \mathbb{N}_{c-1}, z \in \mathbb{N}_{c-1},$$

hence,

$$\begin{aligned} ({}^C_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) + \frac{\widehat{e}_\rho(z, c-1)}{\Gamma(1-\varepsilon)}(z-c+1)^{-\varepsilon}\chi(c-1) &\geq 0, \\ z \in \mathbb{N}_{c-1}, z \in \mathbb{N}_{c-1}, \end{aligned} \quad (28)$$

which means that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0$.

Now, from Theorem 1, we conclude that $\chi(z)$ is $\varepsilon\rho$ -increasing. \square

Theorem 2. Let $\chi: \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be a function which satisfies $\chi(c) \geq 0$, and suppose that for $0 < \varepsilon < 1$ and $0 < \rho \leq 1$. If $\chi(z)$ is increasing on \mathbb{N}_c , then we have

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0, \quad \forall z \in \mathbb{N}_{c-1}. \quad (29)$$

Proof. Since

$$\begin{aligned} ({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(i), \\ z \in \mathbb{N}_{c-1}, \end{aligned} \quad (30)$$

when $z = c$, we have from the assumption $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(c) = \chi(c) \geq 0$.

Clearly, $\chi(c-1) = 0$. So, we can start the induction from the next step.

Assume that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(i) \geq 0, \forall i < z$. We shall show that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0$.

Since, from assumption, $\chi(z)$ is increasing, it follows that $\chi(z) \geq \chi(z-1) \geq \chi(c) \geq 0, \forall z \in \mathbb{N}_c$:

$$\begin{aligned} ({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(i) \\ &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \rho^{z-z+1} (z-\varsigma(z-1))^{-\varepsilon-1} \chi(z-1) \\ &\quad - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(i) \\ &= \chi(z) - \varepsilon\rho\chi(z-1) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} \\ &\quad \cdot (z-\varsigma(i))^{-\varepsilon-1} \chi(i) \\ &\quad + \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &\quad - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &= \chi(z) - \varepsilon\rho\chi(z-1) \\ &\quad + \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} (\chi(z-1) - \chi(i)) \end{aligned}$$

$$\begin{aligned} &- \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &\geq \chi(z) - \varepsilon\rho\chi(z-1) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} \\ &\quad \cdot (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} \\ &\quad \cdot (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &= \chi(z) - \chi(z-1) + \chi(z-1) \\ &\quad - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-2} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &\geq \chi(z-1) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} \\ &\quad \cdot (z-\varsigma(i))^{-\varepsilon-1} \chi(z-1) \\ &= \chi(z-1) \left(1 - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} (z-\varsigma(i))^{-\varepsilon-1} \right) \\ &\geq 0. \end{aligned} \quad (31)$$

\square

Theorem 3. Let $\chi: \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be a function which satisfies $\chi(c) > 0$ and be strictly increasing on \mathbb{N}_c , where $0 < \varepsilon < 1$ and $0 < \rho \leq 1$. Then,

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi) > 0, \quad z \in \mathbb{N}_{c-1}. \quad (32)$$

Proof. Since

$$\begin{aligned} ({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{i=c}^{z-1} \rho^{z-i} \\ &\quad \cdot (z-\varsigma(i))^{-\varepsilon-1} \chi(i), \quad z \in \mathbb{N}_{c-1}, \end{aligned} \quad (33)$$

when $z = c$, we have $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(c) = \chi(c) > 0$. Clearly, $\chi(c-1) = 0$, and so we can start the induction from the next step.

Assume that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(i) > 0, \forall i < z$. We shall show that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) > 0$.

Since, from assumption, $\chi(z)$ is increasing it follows that $\chi(z) > \chi(z-1) > \chi(c) > 0, \forall z \in \mathbb{N}_c$:

$$\begin{aligned}
({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \rho^{z-l} (z-\varsigma(l))^{-\varepsilon-1} \chi(l) \\
&> \chi(z) - \chi(z-1) + \chi(z-1) \\
&\quad - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \rho^{z-l} (z-\varsigma(l))^{-\varepsilon-1} \chi(z-1) \\
&> \chi(z-1) - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \\
&\quad \cdot \rho^{z-l} (z-\varsigma(l))^{-\varepsilon-1} \chi(z-1) \\
&= \chi(z-1) \left(1 - \frac{\varepsilon}{\Gamma(1-\varepsilon)} \sum_{l=c}^{z-1} \rho^{z-l} (z-\varsigma(l))^{-\varepsilon-1} \right) > 0.
\end{aligned} \tag{34}$$

□

Theorem 4. Let $\chi: \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be a function, and suppose that $({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \leq 0$ for $0 < \varepsilon < 1$ and $0 < \rho \leq 1, z \in \mathbb{N}_{c-1}$. Then, $\chi(z)$ is $\varepsilon\rho$ -decreasing.

Proof. Let $\theta: \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be a function such that $\theta(z) = -\chi(z)$; hence,

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(z) = ({}^R_{c-1}\nabla^{\varepsilon,\rho}(-\chi))(z) = -({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0. \tag{35}$$

Now by Theorem 1, we conclude that $\theta(z)$ is $\varepsilon\rho$ -increasing.

Hence,

$$\theta(z+1) \geq \varepsilon\rho\theta(z), \tag{36}$$

which is

$$\begin{aligned}
-\chi(z+1) &\geq \varepsilon\rho(-\chi(z)), \\
\chi(z+1) &\leq \varepsilon\rho\chi(z),
\end{aligned} \tag{37}$$

that is to say, $\chi(z)$ is $\varepsilon\rho$ -decreasing.

□

Theorem 5. Let a function $\chi: \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be decreasing on \mathbb{N}_c such that $\chi(c) \leq 0$. Then, for $0 < \varepsilon < 1$ and $0 < \rho \leq 1$, we have

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) \leq 0, \quad \forall z \in \mathbb{N}_{c-1}. \tag{38}$$

Proof. The proof follows by applying Theorem 2 to $\theta(z) = -\chi(z)$.

Using Theorem 4.3 in [4] we can state the following.

□

Theorem 6 (see [4]). For any $0 < \varepsilon < 1$, $0 < \rho \leq 1$, $p = (\rho - 1/\rho)$, and $\chi: \mathbb{N}_{c+1} \longrightarrow \mathbb{R}$, the following equality holds:

$$({}^R_c\nabla^{-\varepsilon,\rho} {}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) = \chi(z) - \frac{\widehat{e}_p(z,c)}{\Gamma(\varepsilon)} (z-c+1)^{\overline{\varepsilon-1}} \chi(c). \tag{39}$$

4. Application: Mean Value Theorem (MVT)

First, for the sake of simplification, depending on Theorem 6, we shall write

$$({}^R_c\nabla^{-\varepsilon,\rho} {}^R_{c-1}\nabla^{\varepsilon,\rho}\chi)(z) = \chi(z) - S(z,c)\chi(c), \tag{40}$$

where

$$S(z,c) = \frac{\widehat{e}_p(z,c)}{\Gamma(\varepsilon)} (z-c+1)^{\overline{\varepsilon-1}}. \tag{41}$$

Theorem 7 (the fractional proportional difference MVT)

Let Θ and θ be functions defined on $\mathbb{N}_c \cap_d \mathbb{N} = \{c, c+1, c+2, \dots, d-2, d-1, d\}$, where $c \equiv d \pmod{1}$. Assume that θ is strictly increasing, $\theta(c) > 0$, and $0 < \varepsilon < 1$ and $0 < \rho \leq 1$. Then, there exist $s_1, s_2 \in \mathbb{N}_c \cap_d \mathbb{N}$ such that

$$\frac{({}^R_{c-1}\nabla^{\varepsilon,\rho}\Theta)(s_1)}{({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(s_1)} \leq \frac{\Theta(d) - S(d,c)\Theta(c)}{\theta(d) - S(d,c)\theta(c)} \leq \frac{({}^R_{c-1}\nabla^{\varepsilon,\rho}\Theta)(s_2)}{({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(s_2)}. \tag{42}$$

Proof. First we need to show that $\theta(d) - S(d,c)\theta(c) > 0$. Since θ is strictly increasing, then by Theorem 3 we have

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(z) > 0, \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}. \tag{43}$$

Applying the fractional sum operator associated to $({}^R_c\nabla^{\varepsilon,\rho}\theta)(z)$ on both sides of the inequality, by means of (40), we get

$$({}^R_c\nabla^{-\varepsilon,\rho} ({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta))(z) > ({}^R_{c-1}\nabla^{\varepsilon,\rho}(0)) \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}, \tag{44}$$

or we have

$$\theta(z) - S(z,c)\theta(c) > 0. \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}. \tag{45}$$

For $z = d$, we get

$$\theta(d) - S(d,c)\theta(c) > 0. \tag{46}$$

To prove the theorem, we use the proof by contradiction. Assume (42) is not true, then either

$$\frac{\Theta(d) - S(d,c)\Theta(c)}{\theta(d) - S(d,c)\theta(c)} < \frac{({}^R_{c-1}\nabla^{\varepsilon,\rho}\Theta)(z)}{({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(z)}, \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}, \tag{47}$$

or

$$\frac{\Theta(d) - S(d,c)\Theta(c)}{\theta(d) - S(d,c)\theta(c)} > \frac{({}^R_{c-1}\nabla^{\varepsilon,\rho}\Theta)(z)}{({}^R_{c-1}\nabla^{\varepsilon,\rho}\theta)(z)}, \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}. \tag{48}$$

Again, since θ is strictly increasing, then by Theorem 3 we conclude that

$$({}^R_{c-1}\nabla^{\varepsilon,\rho}\Theta)(z), \quad \forall z \in \mathbb{N}_c \cap_d \mathbb{N}. \tag{49}$$

Hence, (47) becomes

$$\frac{\Theta(d) - S(d, c)\Theta(c)}{\theta(d) - S(d, c)\theta(c)} ({}^R_{c-1}\nabla^{\varepsilon, \rho}\theta)(z) < ({}^R_{c-1}\nabla^{\varepsilon, \rho}\Theta)(z), \quad (50)$$

$$\forall z \in \mathbb{N}_c \cap {}_d\mathbb{N}.$$

Applying the fractional sum operator on both sides of the inequality at $z = d$ and by making use of (43), we see that

$$\frac{\Theta(d) - S(d, c)\Theta(c)}{\theta(d) - S(d, c)\theta(c)} (\theta(d) - S(d, c)\theta(c)) \quad (51)$$

$$< (\Theta(d) - S(d, c)\Theta(c)),$$

and hence, $\Theta(d) < \Theta(d)$, which is a contradiction. In a similar way, (48) leads to contradiction. \square

5. Conclusions

The conclusions of this article are summarized as follows:

- (1) The summation and difference of a discrete fractional proportional have been detected.
- (2) The nabla discrete new Riemann–Liouville and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale \mathbb{Z} have been formulated.
- (3) The fractional proportional sums associated to $({}^R_{c-1}\nabla^{\varepsilon, \rho}\chi)(z)$ with order $0 < \varepsilon < 1$ have been defined.
- (4) The relation between nabla Riemann–Liouville and Caputo fractional proportional differences has been detected.
- (5) The monotonicity results for the nabla Caputo fractional proportional difference which are if $({}^R_{c-1}\nabla^{\varepsilon, \rho}\chi)(z) > 0$, then $\chi(z)$ is $\varepsilon\rho$ –increasing; if $\chi(z)$ is strictly increasing on \mathbb{N}_c and $\chi(c) > 0$, then $({}^R_{c-1}\nabla^{\varepsilon, \rho}\chi)(z) > 0$ has been proved as well.
- (6) A new version of the fractional proportional difference of the mean value theorem on \mathbb{Z} has been proved as an application.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors participated in obtaining the main results of this manuscript and drafted the manuscript. All authors read and approved the final manuscript.

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Research Article

Numerical Solution of Fractional-Order HIV Model Using Homotopy Method

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In this study, we construct a convergent algorithm for generating an approximate analytic solution for the fractional HIV infection of CD4⁺ T cells with Atangana–Baleanu fractional derivatives in the Caputo sense. We compute the solution by utilizing the fractional homotopy analysis transform method (FHATM) and achieved a convergence region of the solution by employing an auxiliary parameter. Moreover, we apply a numerical scheme proposed by Toufik and Atangana for solving this kind of problem and compared with our results. A good agreement between the new algorithm and the numerical scheme is remarkable. The solution via the present algorithm can be obtained without any linearization or discretization which makes it reliable and easy to apply.

1. Introduction

Fractional calculus has played a significant role within the field of science and engineering, and many mathematicians and scientists have been working in this field lately. In recent decades, fractional calculus has been used in several areas of physics, biology, engineering, and others. Further details about fractional calculus and its applications can be found in the literature [1–9].

Because most nonlinear fractional differential equations cannot be solved exactly, it is necessary to use approximate and numerical methods. Various powerful mathematical techniques such as the Adomian decomposition method (ADM) [10, 11], homotopy analysis method (HAM) [12–15], optimal homotopy asymptotic method (OHAM) [16], homotopy perturbation method (HPM) [17], and variational iterative method (VIM) [18, 19] have been used to obtain an exact and approximate analytical solution.

The HAM was first introduced and employed in 1992 by Liao [20], after which many researchers successfully applied this method to solve linear and nonlinear differential equations. In recent years, many researchers have devoted their attention to obtaining a solution of linear and

nonlinear differential equations using a variety of methods based on Laplace transform such as the Laplace decomposition method (LDM) [21] and the homotopy perturbation transform method (HPTM) [22]. Khan et al. [23] and Kumar et al. [24–26] coupled the HAM with Laplace transform to solve a nonlinear differential equation and Volterra integral equation. The homotopy analysis transform method (HATM) is a combination of HAM with Laplace transformation. The main advantage of this method is its ability to combine two powerful methods to obtain a rapid convergent series for fractional differential equations. This method provides us with a convenient way to control the convergence of the series solution.

Recently, Toufik and Atangana [27] developed a numerical scheme to solve a nonlinear fractional differential equation considering the Atangana–Baleanu fractional derivative. This method is a combination of the fundamental fractional calculus theorem with two-step Lagrange polynomial which is successfully used to solve many real-world problems [28–30].

Since the early 1980s, researchers have made an enormous effort to mathematically model the human immunodeficiency virus (HIV), the virus responsible for causing acquired immune deficiency syndrome (AIDS). In 1989,

Perelson [31] considered the interaction of uninfected (T) and infected $CD4^+$ (I) T cells, and free virus molecules (F) in his model, following which Perelson et al. [32] extended the original model [31]. Culshaw and Ruan [33] reduced the model discussed in [32] as follows:

$$\frac{dT}{dt} = p - \mu_T T + kT \left(1 - \frac{T+I}{T_{\max}}\right) - k_1 FT, \quad (1)$$

$$\frac{dI}{dt} = k_1' FT - \mu_I I, \quad (2)$$

$$\frac{dF}{dt} = M\mu_b I - k_1 FT - \mu_F F, \quad (3)$$

where $T(t)$, $I(t)$, and $F(t)$ represent the concentration of healthy $CD4^+$ T cells at time t , infected $CD4^+$ T cells, and the free HI virus at time t , respectively. Table 1 summarizes the meanings of functions and parameters. Equation (1) describes the rate of change in the uninfected population of $CD4^+$ T cells. The first term is the constant rate at which the body produces $CD4^+$ T cells from precursors in the bone marrow. Because the virus can infect both thymocytes and T cells, as for all cells in the body, these cells have a finite lifetime; thus, the second term describes the decreasing source. The third term describes the logistic growth of the healthy $CD4^+$ T cells, and the proliferation of infected $CD4^+$ T cells is neglected. The last term models the rate at which the free virus infects a $CD4^+$ T cell. Once a T cell has been infected, it becomes an infected cell; therefore, $k_1 FT$ is subtracted from equations (1) and (3) and added to equation (2). Hence, F and T decrease concurrently.

Equation (2) describes the rate of change in the infected population of actively infected T cells. The first term represents the rate of infection of $CD4^+$ T cells by the virus. The second term represents the rate of disappearance of infected cells.

The three terms in equation (3) refer to the rate of production and destruction of the free infection virus. An actively infected $CD4^+$ T cell produces M virus particles; thus, the rate at which the virus is produced is set equal to M times the lytic death rate for the infected cell. A free virus is lost as a result of binding to an uninfected $CD4^+$ T cell at $k_1 FT$. The third term accounts for the loss of viral infectivity, viral death, and/or clearance from the body.

In this study, our approach to solving the fractional HIV model is to determine the order in which the fractional derivative changes by extending the classical HIV model (1)–(3) to the following set of fractional ordinary differential equations of the order α , β , and γ :

$${}_a^{ABC}D_t^\alpha T(t) = p - \mu_T T + kT \left(1 - \frac{T+I}{T_{\max}}\right) - k_1 FT, \quad 0 < \alpha < 1, \quad (4)$$

$${}_a^{ABC}D_t^\beta I(t) = k_1' FT - \mu_I I, \quad 0 < \beta < 1, \quad (5)$$

TABLE 1: List of parameters and functions.

Parameters and functions	Description	Values
$T(t)$	Concentration of uninfected $CD4^+$ T cells	$T(0) = 1000$
$I(t)$	Concentration of infected $CD4^+$ T cells	$I(0) = 0$
$F(t)$	Concentration of HIV RNA	$F(0) = 0.001$
μ_T	Natural death rate of $CD4^+$ T cells (concentration)	0.02
μ_I	Blanket death rate of infected $CD4^+$ T cells	0.26
μ_b	Lytic death rate of infected cells	0.24
μ_F	Death rate of free virus	2.4
k_1	Rate at which $CD4^+$ T cells become infected with the virus	2.4×10^{-5}
k_1'	Rate at which infected cells become active	2×10^{-5}
k	Growth rate of concentration of $CD4^+$ T cells	0.03
M	Number of virion produced by infected $CD4^+$ T cells	500
T_{\max}	Maximal concentration of $CD4^+$ T cells	1500
p	Source term for uninfected $CD4^+$ T cells	10

$$F(t) = M\mu_b I - k_1 FT - \mu_F F, \quad 0 < \gamma < 1, \quad (6)$$

with initial conditions

$$\begin{aligned} T(0) &= T_0(0), \\ I(0) &= I_0(0), \\ F(0) &= F_0(0), \end{aligned} \quad (7)$$

where ${}_a^{ABC}D_t^\alpha$, ${}_a^{ABC}D_t^\beta$, and ${}_a^{ABC}D_t^\gamma$ are the Atangana–Baleanu fractional derivative in the Caputo sense (ABC). To the best of our knowledge, this is the first work that solves fractional HIV infection of the $CD4^+$ T cells model in ABC sense analytically and numerically. To indicate the strength of our proposed method, we compare our findings with the algorithm of Toufik and Atangana [27].

2. Preliminaries and Notations

The Atangana–Baleanu fractional derivative in the Caputo sense (ABC) is defined as [6, 34]

$${}_a^{ABC}D_t^\alpha f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t \frac{d}{ds} f(s) E_\alpha \left(\frac{-\alpha}{1-\alpha} (t-s)^\alpha \right) ds, \quad (8)$$

$$n-1 < \alpha \leq n,$$

where $\alpha \in \mathbb{R}$, $M(\alpha) > 0$ is a normalization function satisfying

$$M(\alpha) = (1-\alpha) + \frac{\alpha}{\Gamma(\alpha)}, \quad (9)$$

with $M(0) = M(1) = 1$, $E_\alpha(\cdot)$ denotes the Mittag-Leffler function, defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad (10)$$

and $\Gamma(\cdot)$ denotes Euler's gamma function defined as

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z). \quad (11)$$

The fractional integral for the ABC, which is newly defined with a nonlocal kernel and does not have singularities at $t = s$, is defined as follows [6]:

$${}_a^{\text{ABC}} I_t^\alpha f(t) = \frac{1-\alpha}{M(\alpha)} f(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)} \int_a^t f(s)(t-s)^{\alpha-1} ds. \quad (12)$$

Here, when α equals zero, the initial function is recovered, and when α equals unity, the classical ordinary integral is obtained.

The Laplace transform of the fractional definitions with ABC is given as follows [6]:

$$\mathcal{L}\{{}_a^{\text{ABC}} D_t^\alpha f(t)\}(s) = \left(\frac{M(\alpha)}{1-\alpha} \frac{s^\alpha \mathcal{L}\{f(t)\}(s) - s^{\alpha-1} f(a)}{s^\alpha + (\alpha/1 - \alpha)} \right). \quad (13)$$

3. Homotopy and Laplace Transform for FHATM

Applying the Laplace transform to equations (4)–(6) and using the formula for the Laplace transform of the ABC and then simplifying these equations, we find that

$$\begin{aligned} \mathcal{L}\{T(t; q)\} &= \frac{T(0)}{s} + \frac{p}{s} \frac{s^\alpha(1-\alpha) + \alpha}{s^\alpha M(\alpha)} + \frac{s^\alpha(1-\alpha) + \alpha}{s^\alpha M(\alpha)} \\ &\quad \cdot \left(\mathcal{L}\{(k - \mu_T)T(t) - \frac{k}{T_{\max}}(T(t))^2 - \frac{k}{T_{\max}}T \right. \\ &\quad \cdot (t)I(t) - k_1 F(t)T(t)\}, \\ \mathcal{L}\{I(t; q)\} &= \frac{I(0)}{s} + \frac{s^\beta(1-\beta) + \beta}{s^\beta M(\beta)} \mathcal{L}\{k_1' F(t)T(t) - \mu_I I(t)\}, \\ \mathcal{L}\{F(t; q)\} &= \frac{F(0)}{s} + \frac{s^\gamma(1-\gamma) + \gamma}{s^\gamma M(\gamma)} \mathcal{L}\{M\mu_b T(t) - k_1 F(t)T(t) \\ &\quad - \mu_F F(t)\}. \end{aligned} \quad (14)$$

Next, defining the nonlinear operators as

$$\begin{aligned} N_T[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)] &= \mathcal{L}[\varphi_1(t; q)] - \frac{T(0)}{s} - \frac{p}{s} \frac{s^\alpha(1-\alpha) - \alpha}{s^\alpha M(\alpha)} - \frac{s^\alpha(1-\alpha) + \alpha}{s^\alpha M(\alpha)} \mathcal{L} \\ &\quad \cdot \left\{ \left((k - \mu_T)\varphi_1(t; q); -\frac{k}{T_{\max}}(\varphi_1(t; q))^2 - \frac{k}{T_{\max}}\varphi_1(t; q)\varphi_2(t; q) - k_1\varphi_3(t; q)\varphi_1(t; q) \right) \right\}, \\ N_I[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)] &= \mathcal{L}[\varphi_2(t; q)] - \frac{I(0)}{s} - \frac{s^\beta(1-\beta) + \beta}{s^\beta M(\beta)} \mathcal{L}\{k_1'\varphi_3(t; q)\varphi_1(t; q) - \mu_I\varphi_2(t; q)\}, \\ N_F[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)] &= \mathcal{L}[\varphi_3(t; q)] - \frac{F(0)}{s} - \frac{s^\gamma(1-\gamma) + \gamma}{s^\gamma M(\gamma)} \mathcal{L}\{M\mu_b\varphi_1(t; q) - k_1\varphi_3(t; q)\varphi_1(t; q) - \mu_F\varphi_3(t; q)\}, \end{aligned} \quad (15)$$

where N_T , N_I , and N_F are the nonlinear operators. Let \hbar be a nonzero auxiliary parameter. Using the embedding parameter $q \in [0, 1]$, we construct the so-called zeroth-order deformation equation:

$$(1-q)\mathcal{L}[\varphi_1(t; q) - T_0(t)] = q\hbar N_1[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)], \quad (16)$$

$$(1-q)\mathcal{L}[\varphi_2(t; q) - I_0(t)] = q\hbar N_2[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)], \quad (17)$$

$$(1-q)\mathcal{L}[\varphi_3(t; q) - F_0(t)] = q\hbar N_3[\varphi_1(t; q), \varphi_2(t; q), \varphi_3(t; q)], \quad (18)$$

where \mathcal{L} is the Laplace operator, subject to the initial conditions

$$\begin{aligned} \varphi_1(0; q) &= T_0(0), \\ \varphi_2(0; q) &= I_0(0), \\ \varphi_3(0; q) &= F_0(0). \end{aligned} \quad (19)$$

Clearly if $q = 0$ and $q = 1$ we obtain

$$\begin{aligned} \varphi_1(t; 0) &= T_0(t), \\ \varphi_1(t; 1) &= T(t), \\ \varphi_2(t; 0) &= I_0(t), \\ \varphi_2(t; 1) &= I(t), \\ \varphi_3(t; 0) &= F_0(t), \\ \varphi_3(t; 1) &= F(t), \end{aligned} \quad (20)$$

when q varies from zero to unity, the solution of the model (4)–(6) will vary from the initial guesses $T_0(t)$, $I_0(t)$, and

$F_0(t)$ to the exact solution $T(t)$, $I(t)$, and $F(t)$ of the model (4)–(6). Expanding $\varphi_i(t; q)$, $i = 1, 2, 3$ by the Taylor series with respect to the embedding parameter q yields

$$\varphi_1(t; q) = T_0(t) + \sum_{m=1}^{\infty} T_m(t)q^m, \quad (21)$$

$$\varphi_2(t; q) = I_0(t) + \sum_{m=1}^{\infty} I_m(t)q^m, \quad (22)$$

$$\varphi_3(t; q) = F_0(t) + \sum_{m=1}^{\infty} F_m(t)q^m, \quad (23)$$

where

$$T_m(t) = \frac{1}{m!} \frac{\partial^m \varphi_1(t; q)}{\partial q^m} \Big|_{q=0},$$

$$I_m(t) = \frac{1}{m!} \frac{\partial^m \varphi_2(t; q)}{\partial q^m} \Big|_{q=0}, \quad (24)$$

$$F_m(t) = \frac{1}{m!} \frac{\partial^m \varphi_3(t; q)}{\partial q^m} \Big|_{q=0}.$$

The convergence of equations (21)–(23) depends on the nonzero auxiliary parameters \hbar [20]. Moreover, if the initial values guessed for $T_0(t)$, $I_0(t)$, and $F_0(t)$ and the auxiliary parameter \hbar are appropriately selected, then at $q = 1$, series (21)–(23) converges

$$\varphi_1(t; 1) = T_0(t) + \sum_{m=1}^{\infty} T_m(t) \text{ i.e. } T(t) = T_0(t) + \sum_{m=1}^{\infty} T_m(t), \quad (25)$$

$$\varphi_2(t; 1) = I_0(t) + \sum_{m=1}^{\infty} I_m(t) \text{ i.e. } I(t) = I_0(t) + \sum_{m=1}^{\infty} I_m(t), \quad (26)$$

$$\varphi_3(t; 1) = F_0(t) + \sum_{m=1}^{\infty} F_m(t) \text{ i.e. } F(t) = F_0(t) + \sum_{m=1}^{\infty} F_m(t), \quad (27)$$

which must be one of the solution of model (4)–(6), as proved by [20]. The equations governing the unknown functions can be deduced from the zeroth-deformation equations (16)–(18). Define the vectors

$$\vec{T}_m(t) = \{T_0(t), T_1(t), \dots, T_m(t)\}, \quad m = 1, 2, \dots, n,$$

$$\vec{I}_m(t) = \{I_0(t), I_1(t), \dots, I_m(t)\}, \quad m = 1, 2, \dots, n,$$

$$\vec{F}_m(t) = \{F_0(t), F_1(t), \dots, F_m(t)\}, \quad m = 1, 2, \dots, n. \quad (28)$$

Differentiating the zeroth-deformation equations (16)–(18) m -times with respect to the embedding parameter q , then setting $q = 0$, and finally dividing them by $m!$, enables the m th-order deformation equations to be obtained:

$$\mathcal{L}[T_m(t) - \chi_m T_{m-1}(t)] = \hbar R_{m,T}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}), \quad m = 1, 2, \dots, n, \quad (29)$$

$$\mathcal{L}[I_m(t) - \chi_m I_{m-1}(t)] = \hbar R_{m,I}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}), \quad m = 1, 2, \dots, n, \quad (30)$$

$$\mathcal{L}[F_m(t) - \chi_m F_{m-1}(t)] = \hbar R_{m,F}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}), \quad m = 1, 2, \dots, n, \quad (31)$$

with initial conditions

$$\begin{aligned} T_m(0) &= 0, \\ I_m(0) &= 0, \\ F_m(0) &= 0. \end{aligned} \quad (32)$$

It should be emphasized that the m th-order deformation equations (29)–(31) are linear; hence, they can be solved by Mathematica or MATLAB. For simplicity, we can specify the auxiliary functions to be equal to unity. Applying the inverse Laplace transform to equations (29)–(31), we obtain

$$\begin{aligned} T_m(t) &= \chi_m T_{m-1}(t) + \hbar \mathcal{L}^{-1} \left(R_{m,T}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}) \right), \quad m = 1, 2, \dots, n, \\ I_m(t) &= \chi_m I_{m-1}(t) + \hbar \mathcal{L}^{-1} \left(R_{m,I}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}) \right), \quad m = 1, 2, \dots, n, \\ F_m(t) &= \chi_m F_{m-1}(t) + \hbar \mathcal{L}^{-1} \left(R_{m,F}(\vec{T}_{m-1}, \vec{I}_{m-1}, \vec{F}_{m-1}) \right), \quad m = 1, 2, \dots, n, \end{aligned} \quad (33)$$

where

$$\begin{aligned}
 R_{m,T}(t) &= \mathcal{L}[T_{m-1}(t)] - (1 - \chi_m) \left(T_0 + \frac{s\alpha t^\alpha}{M(\alpha)\Gamma(\alpha+1)} + \frac{s(1-\alpha)}{M(\alpha)} \right) - \frac{s^\alpha(1-\alpha) + \alpha}{s^\alpha M(\alpha)} \mathcal{L} \\
 &\quad \cdot \left\{ (\mu_T - k)T_{m-1}(t) + \frac{k}{T_{\max}} \sum_{i=0}^{m-1} T_i(t)T_{m-1-i}(t), + \frac{k}{T_{\max}} \sum_{i=0}^{m-1} T_i(t)I_{m-1-i}(t) + k_1 \sum_{i=0}^{m-1} T_i(t)F_{m-1-i}(t) \right\}, \\
 R_{m,I}(t) &= \mathcal{L}[I_{m-1}(t)] - (1 - \chi_m)I_0 - \frac{s^\beta(1-\beta) + \beta}{s^\beta M(\beta)} \mathcal{L} \left(k'_1 \sum_{i=0}^{m-1} F_i(t)T_{m-1-i}(t) - \mu_I I_{m-1}(t) \right), \\
 R_{m,F}(t) &= \mathcal{L}[F_{m-1}(t)] - (1 - \chi_m)F_0 - \frac{s^\gamma(1-\gamma) + \gamma}{s^\gamma M(\gamma)} \mathcal{L} \left\{ k_1 \sum_{i=0}^{m-1} F_i(t)T_{m-1-i}(t) + \mu_F F_{m-1}(t) - M\mu_b I_{m-1}(t) \right\}, \\
 \chi_m &= \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}
 \end{aligned} \tag{34}$$

Next, the solution of the m th-order deformation equations (29)–(31) is given as

$$\begin{aligned}
 T_m(t) &= (\chi_m + \hbar)T_{m-1}(t) - \hbar(1 - \chi_m) \left(T_0 + \frac{s\alpha t^\alpha}{M(\alpha)\Gamma(\alpha+1)} + \frac{s(1-\alpha)}{M(\alpha)} \right) - \hbar \mathcal{L}^{-1} \frac{s^\alpha(1-\alpha) + \alpha}{s^\alpha M(\alpha)} \mathcal{L} \\
 &\quad \cdot \left\{ \left[(\mu_T - k)T_{m-1}(t) + \frac{k}{T_{\max}} \sum_{i=0}^{m-1} T_i(t)T_{m-1-i}(t) + \frac{k}{T_{\max}} \sum_{i=0}^{m-1} T_i(t)I_{m-1-i}(t) + k_1 \sum_{i=0}^{m-1} T_i(t)F_{m-1-i}(t) \right] \right\},
 \end{aligned} \tag{35}$$

$$I_m(t) = (\chi_m + \hbar)I_{m-1}(t) - \hbar(1 - \chi_m)I_0 - \hbar \mathcal{L}^{-1} \left\{ \frac{s^\beta(1-\beta) + \beta}{s^\beta M(\beta)} \mathcal{L} \left[k'_1 \sum_{i=0}^{m-1} F_i(t)T_{m-1-i}(t) - \mu_I I_{m-1}(t) \right] \right\}, \tag{36}$$

$$F_m(t) = (\chi_m + \hbar)F_{m-1}(t) - \hbar(1 - \chi_m)F_0 - \hbar \mathcal{L}^{-1} \left\{ \frac{s^\gamma(1-\gamma) + \gamma}{s^\gamma M(\gamma)} \mathcal{L} \left[k_1 \sum_{i=0}^{m-1} F_i(t)T_{m-1-i}(t) + \mu_F F_{m-1}(t) - M\mu_b I_{m-1}(t) \right] \right\}. \tag{37}$$

Taking initial conditions and equations (35)–(37), we obtain

$$\begin{cases} T_1 = -\frac{\hbar}{M(\alpha)} \left(p + (k - \mu_T)T_0 + kT_0 \left(1 - \frac{T_0 + I_0}{T_{\max}} \right) - k_1 T_0 F_0 \right) \left(1 - \alpha + \frac{\alpha t^\alpha}{\Gamma(1+\alpha)} \right), \\ I_1 = -\frac{\hbar(k'_1 T_0 F_0 - \mu_I I_0)}{M(\beta)} \left(1 - \beta + \frac{\beta t^\beta}{\Gamma(1+\beta)} \right), \\ F_1 = -\frac{\hbar(M\mu_b I_0 - k_1 T_0 F_0 - \mu_F F_0)}{M(\gamma)} \left(1 - \gamma + \frac{\gamma t^\gamma}{\Gamma(1+\gamma)} \right), \end{cases}$$

$$\begin{aligned}
T_2 &= (1 + \hbar)T_1 \\
&+ \frac{\hbar^2 m_1}{M(\alpha)^2} \left(k - \mu_T - \frac{k}{T_{\max}} (2T_0 + I_0) - k_1 F_0 \right) \left(p + (k - \mu_T)T_0 + kT_0 \left(1 - \frac{T_0 + I_0}{T_{\max}} \right) - k_1 T_0 F_0 \right) \\
&- \frac{\hbar^2 k T_0 m_2}{M(\alpha)M(\beta)} \frac{(k'_1 T_0 F_0 - \mu_I I_0)}{T_{\max}} \\
&- \frac{\hbar^2 k_1 T_0 m_3}{M(\alpha)M(\gamma)} (M\mu_b I_0 - k_1 T_0 F_0 - \mu_F F_0), \\
I_2 &= (1 + \hbar)I_1 \\
&- \frac{\hbar^2 \mu_I m_4}{(M(\beta))^2} (k'_1 T_0 F_0 - \mu_I I_0) \\
&+ \frac{\hbar^2 k'_1 F_0 m_2}{M(\alpha)M(\beta)} \left(p + (k - \mu_T)T_0 + kT_0 \left(1 - \frac{T_0 + I_0}{T_{\max}} \right) - k_1 T_0 F_0 \right) \\
&+ \frac{\hbar^2 k'_1 T_0 m_5}{M(\gamma)M(\beta)} (M\mu_b I_0 - k_1 T_0 F_0 - \mu_F F_0), \\
F_2 &= (1 + \hbar)F_1 \\
&- \frac{\hbar^2 (k_1 T_0 + \mu_F) m_6}{(M(\gamma))^2} (M\mu_b I_0 - k_1 T_0 F_0 - \mu_F F_0) \\
&- \frac{\hbar^2 k_1 F_0 m_3}{M(\alpha)M(\gamma)} \left(p + (k - \mu_T)T_0 + kT_0 \left(1 - \frac{T_0 + I_0}{T_{\max}} \right) - k_1 T_0 F_0 \right) \\
&+ \frac{\hbar^2 M\mu_b m_5}{M(\gamma)M(\beta)} (k'_1 T_0 F_0 - \mu_I I_0),
\end{aligned} \tag{38}$$

where $m_i, i = 1, 2, \dots, 6$, are given by

$$\left\{ \begin{aligned} m_1 &= (1 - \alpha)^2 + \frac{2(1 - \alpha)\alpha t^\alpha}{\Gamma(1 + \alpha)} + \frac{(\alpha t^\alpha)^2}{\Gamma(2\alpha + 1)}, \\ m_4 &= (1 - \beta)^2 + \frac{2(1 - \beta)\beta t^\beta}{\Gamma(1 + \beta)} + \frac{(\beta t^\beta)^2}{\Gamma(2\beta + 1)}, \\ m_6 &= (1 - \gamma)^2 + \frac{2(1 - \gamma)\gamma t^\gamma}{\Gamma(1 + \gamma)} + \frac{(\gamma t^\gamma)^2}{\Gamma(2\gamma + 1)}, \\ m_2 &= (1 - \alpha)(1 - \beta) + \frac{(1 - \alpha)\beta t^\beta}{\Gamma(1 + \beta)} + \frac{(1 - \beta)\alpha t^\alpha}{\Gamma(1 + \alpha)} + \frac{\alpha\beta t^{\alpha+\beta}}{\Gamma(\alpha + \beta + 1)}, \\ m_3 &= (1 - \alpha)(1 - \gamma) + \frac{(1 - \alpha)\gamma t^\gamma}{\Gamma(1 + \gamma)} + \frac{(1 - \gamma)\alpha t^\alpha}{\Gamma(1 + \alpha)} + \frac{\alpha\gamma t^{\alpha+\gamma}}{\Gamma(\alpha + \gamma + 1)}, \\ m_5 &= (1 - \beta)(1 - \gamma) + \frac{(1 - \beta)\gamma t^\gamma}{\Gamma(1 + \gamma)} + \frac{(1 - \gamma)\beta t^\beta}{\Gamma(1 + \beta)} + \frac{\beta\gamma t^{\beta+\gamma}}{\Gamma(\beta + \gamma + 1)}. \end{aligned} \right. \tag{39}$$

In a similar way, T_m , I_m , and F_m , for $m \geq 3$ can be obtained. Finally, the solution of model (2) is given by

$$T(t) = \sum_{m=0}^{n-1} T_m(t), \quad (40)$$

$$I(t) = \sum_{m=0}^{n-1} I_m(t), \quad (41)$$

$$F(t) = \sum_{m=0}^{n-1} F_m(t), \quad (42)$$

and by choosing a suitable value for h for the convergence of the series according to [20]. The analysis of the convergence of the HATM can found in the literature [35].

4. Numerical Illustration

According to [36], it should be noted that the solution of the series contain the auxiliary parameter h , which offers an easy way to control the convergence of the solution of the series. Because it is essential to assure that the series equations (25)–(27) is convergent, we plotted the h curve of 6 terms of the FHATM solution for the fractional-time ABC equations in Figures 1–3. Using these h curves, we note that the straight line that parallels the h axis provides the region of convergence. These valid regions are listed in Table 2.

Furthermore, if h is appropriately chosen, equations (25)–(27) may converge fast. To this end, we have to compute the optimal values of the convergence control parameters from the minimum of the averaged residual errors.

Niu and Chun [37] introduced several methods to obtain the optimal value of h . The optimal value of the convergence control parameter is defined by using the concept of the square residual error. An error analysis is presented to determine the optimal values of h . We substitute equations (47)–(49) into equations (4)–(6) and obtain the residual functions as follows:

$$\begin{aligned} E_{m,T}(t; h_1) = & {}_a^{ABC}D_t^\alpha \psi_T(t; h_1) - p + \mu_T \psi_T(t; h_1) - k \psi_T(t; h_1) \\ & \cdot \left(1 - \frac{\psi_T(t; h_1) + \psi_I(t; h_1)}{T_{\max}} \right) \\ & + k_1 \psi_F(t; h_1) \psi_T(t; h_1), \end{aligned} \quad (43)$$

$$\begin{aligned} E_{m,I}(t; h_2) = & {}_a^{ABC}D_t^\beta \psi_I(t; h_2) - k_1' \psi_F(t; h_2) \psi_T(t; h_2) \\ & + \mu_I \psi_I(t; h_2), \end{aligned} \quad (44)$$

$$\begin{aligned} E_{m,F}(t; h_3) = & {}_a^{ABC}D_t^\gamma \psi_F(t; h_3) - M \mu_b \psi_I(t; h_3) \\ & + k_1 \psi_F(t; h_3) \psi_T(t; h_3) + \mu_F \psi_F(t; h_3). \end{aligned} \quad (45)$$

Then, the square residual error for the sixth-order approximation is defined as

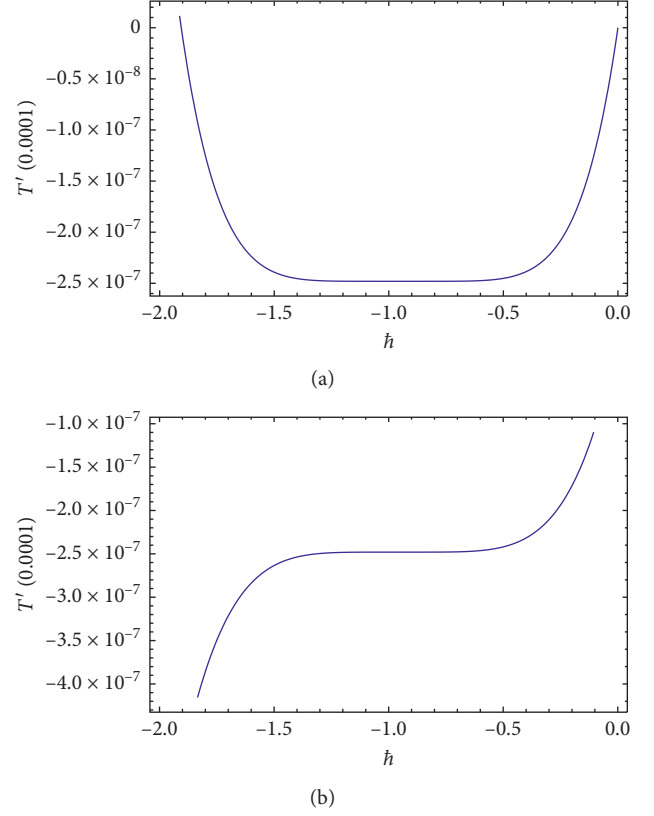


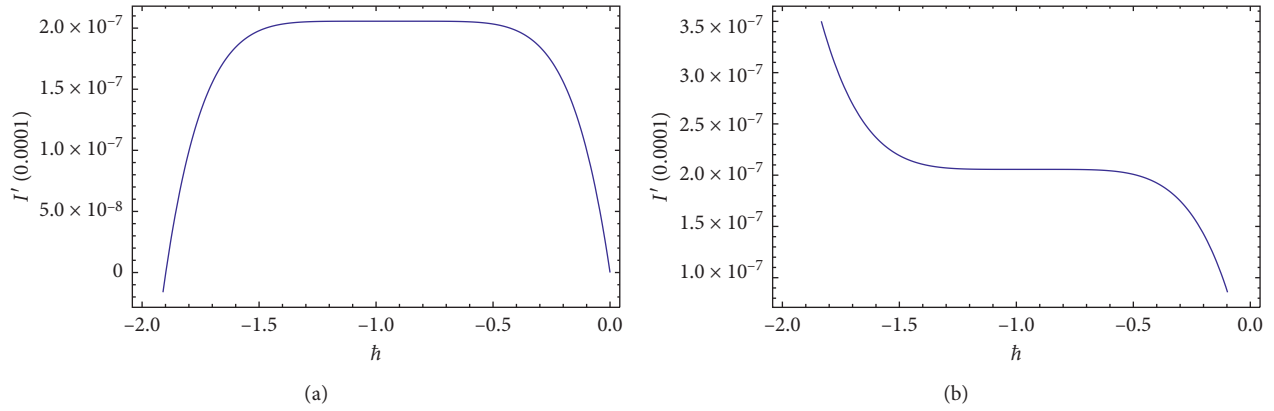
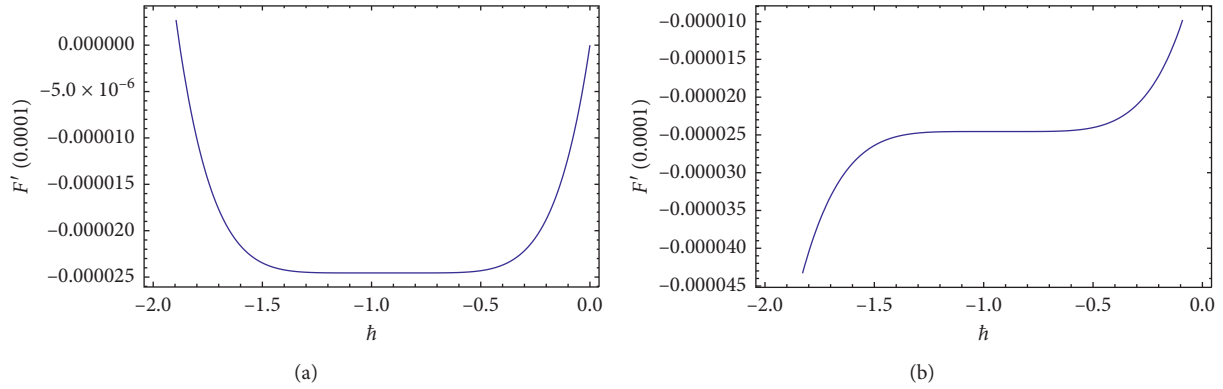
FIGURE 1: h curves obtained by the (a) 6th-order and (b) 5th-order approximation of the FHATM.

$$\begin{aligned} SE_{m,T}(h_1) &= \frac{1}{(N+1)} \sum_{l=0}^N \left[E_{m,T} \left(\sum_{z=1}^m T(l\Delta t) \right) \right], \\ SE_{m,I}(h_2) &= \frac{1}{(N+1)} \sum_{l=0}^N \left[E_{m,I} \left(\sum_{z=1}^m I(l\Delta t) \right) \right], \\ SE_{m,F}(h_3) &= \frac{1}{(N+1)} \sum_{l=0}^N \left[E_{m,F} \left(\sum_{z=1}^m F(l\Delta t) \right) \right]. \end{aligned} \quad (46)$$

The use of the first derivative test enables us to determine the values of the auxiliary parameters h_1 , h_2 , and h_3 for which $SE_{m,T}(h_1)$, $SE_{m,I}(h_2)$, and $SE_{m,F}(h_3)$ are minimized. It should be emphasized that the approximation procedures that are used to select the optimal value of h in FHATM are similar to those of HAM [38].

In Table 3, the minimum values of the square residual error are given for the optimal values of h_1 , h_2 , and h_3 when $\alpha = \beta = \gamma = 0.99$.

The absolute residual errors that were calculated for various $t \in (0, 1)$ are listed in Table 2. These results show that the FHATM obtains an accurate approximate solution for the fractional HIV model (4)–(6). The residual errors are plotted in Figure 4 for $t \in (0, 1)$ and various values of h . The square residual errors are plotted in Figure 5, and Figure 6 shows the absolute residual

FIGURE 2: h curves obtained by the (a) 6th-order and (b) 5th-order approximation of the FHATM.FIGURE 3: h curves obtained by the (a) 6th-order and (b) 5th-order approximation of the FHATM.TABLE 2: Regions of convergence, optimal values of h , and minimum values when $t = 0.0001$.

h	h^*	Minimum values
$-1.32 \leq h \leq -0.62$	-0.804051	1.84022×10^{-11}
$-1.35 \leq h \leq -0.65$	-0.805328	1.77401×10^{-11}
$-1.35 \leq h \leq -0.65$	-0.808061	5.58464×10^{-7}

TABLE 3: Residual errors $E_{m,T}$, $E_{m,I}$, and $E_{m,F}$ for the optimal h .

t	$E_{m,T}(t; h_1^*)$	$E_{m,I}(t; h_2^*)$	$E_{m,F}(t; h_3^*)$
0	0.0000234312	0.0000194777	0.00234336
0.2	1.47222×10^{-8}	1.17419×10^{-8}	1.24526×10^{-6}
0.4	1.21518×10^{-8}	8.98671×10^{-9}	6.94956×10^{-7}
0.6	1.72222×10^{-8}	1.12419×10^{-8}	2.78878×10^{-7}
0.8	2.67042×10^{-8}	3.1236×10^{-8}	6.87927×10^{-6}
1	1.09289×10^{-8}	1.12475×10^{-8}	2.43936×10^{-6}

functions for the optimal h . As these figures show, the solution obtained by using FHATM provides us with a sufficiently accurate analytical solution that only requires

a few iterative steps. Mathematica software was used to calculate the six-term approximations for T , I , and F , respectively.

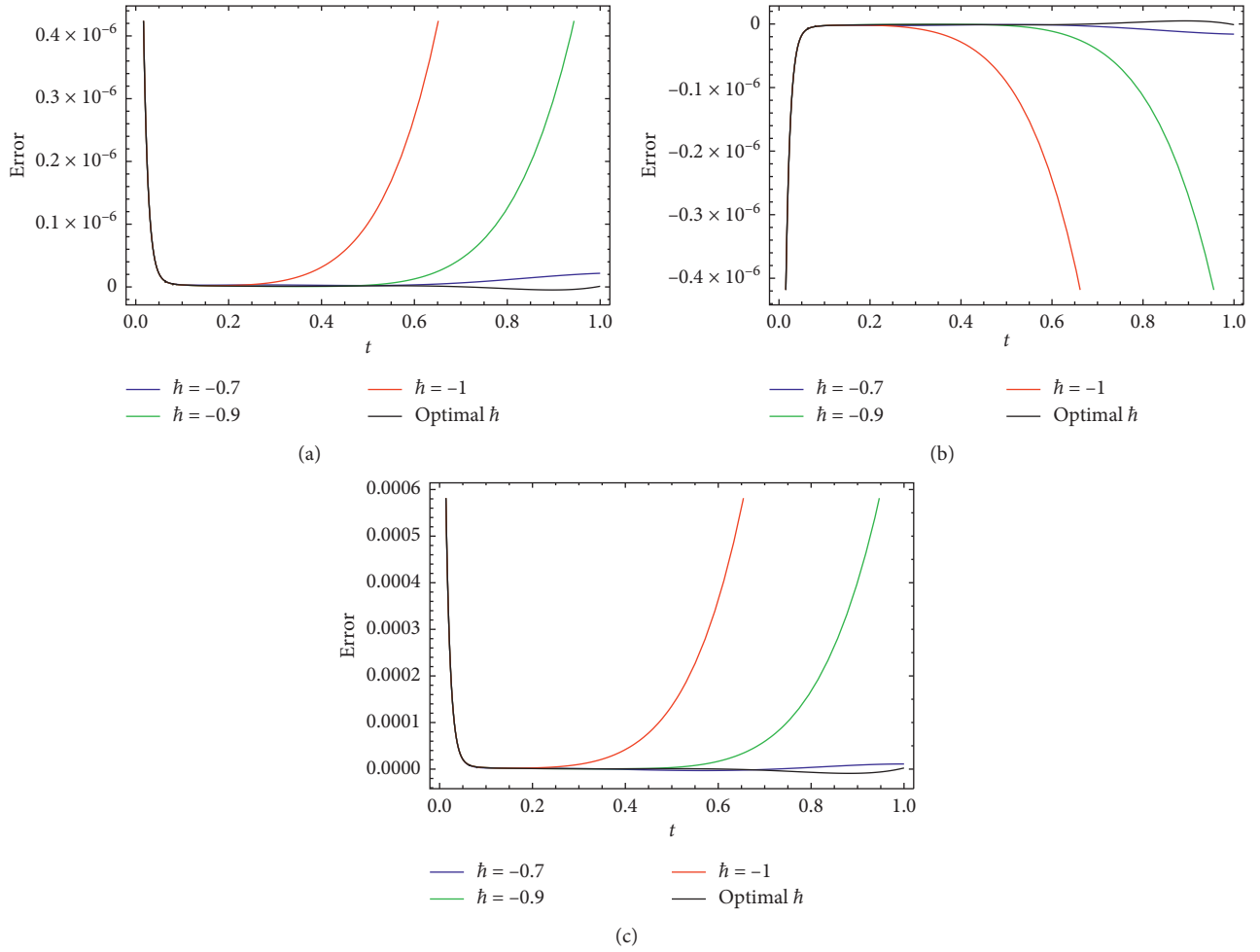


FIGURE 4: Errors of residual functions equations (39)–(41) using the sixth order of the approximation solution for various values of \hbar . (a) Comparison errors of residual function ET for various \hbar , (b) comparison errors of residual function EI for various \hbar , and (c) comparison errors of residual function EF for various \hbar .

$$\begin{aligned}
 Y_T(t; \hbar) = \sum_{m=0}^6 T_m(t) = & 1000 + 1.44837 \times 10^{-6} \hbar + 3.70969 \times 10^{-6} \hbar^2 + 5.06867 \times 10^{-6} \hbar^3 \\
 & + 3.89645 \times 10^{-6} \hbar^4 + 1.59786 \times 10^{-6} \hbar^5 + 2.73079 \times 10^{-7} \hbar^6 + 0.000143992 \hbar t^{0.99} \\
 & + 0.000377629 \hbar^2 t^{0.99} + 0.000528248 \hbar^3 t^{0.99} + 0.000415689 \hbar^4 t^{0.99} + 0.000174472 \hbar^5 t^{0.99} \\
 & + 0.0000305132 \hbar^6 t^{0.99} + \dots,
 \end{aligned} \tag{47}$$

$$\begin{aligned}
 Y_I(t; \hbar) = \sum_{m=0}^6 I_m(t) = & -1.20698 \times 10^{-6} \hbar - 3.0988 \times 10^{-6} \hbar^2 - 4.24412 \times 10^{-6} \hbar^3 - 3.27012 \times 10^{-6} \hbar^4 \\
 & - 1.34403 \times 10^{-6} \hbar^5 - 2.30202 \times 10^{-7} \hbar^6 - 0.000119993 \hbar t^{0.99} - 0.000316179 \hbar^2 t^{0.99} \\
 & - 0.000444257 \hbar^3 t^{0.99} - 0.000351059 \hbar^4 t^{0.99} - 0.000147925 \hbar^5 t^{0.99} - 0.0000259663 \hbar^6 t^{0.99} - \dots,
 \end{aligned} \tag{48}$$

$$\begin{aligned}
 Y_F(t; \hbar) = \sum_{m=0}^6 F_m(t) = & 0.001 + 0.000146285 \hbar + 0.000378272 \hbar^2 + 0.000521647 \hbar^3 + 0.000404615 \hbar^4 \\
 & + 0.000167371 \hbar^5 + 0.0000288456 \hbar^6 + 0.0145431 \hbar t^{0.99} + 0.0388549 \hbar^2 t^{0.99}; \\
 & + 0.0552967 \hbar^3 t^{0.99} + 0.0442161 \hbar^4 t^{0.99} + 0.0188364 \hbar^5 t^{0.99} + 0.00334019 \hbar^6 t^{0.99} + \dots.
 \end{aligned} \tag{49}$$

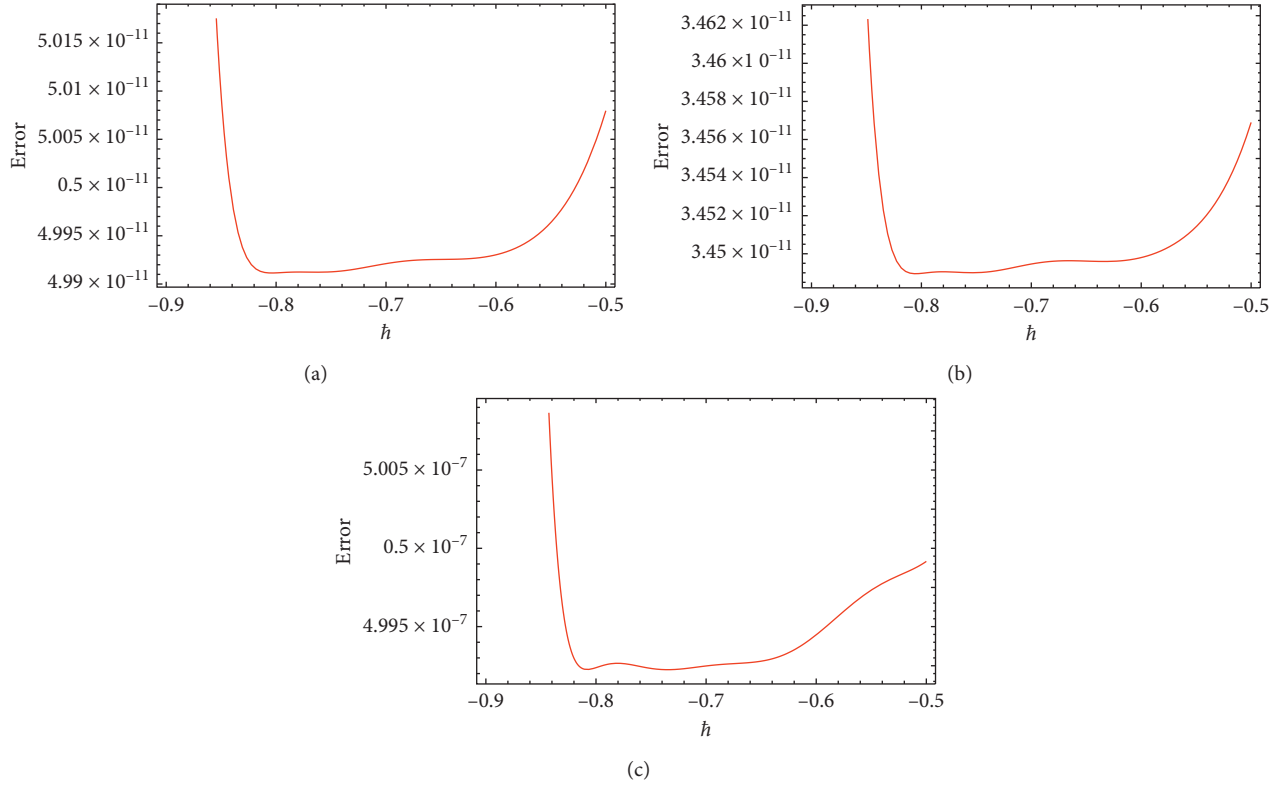


FIGURE 5: Square residual function equations (42)–(44) using the sixth order of the approximation solution for $h \in (-0.9, -0.5)$. (a) Square residual errors of T versus h , (b) square residual errors of I versus h , and (c) square residual errors of F versus h .

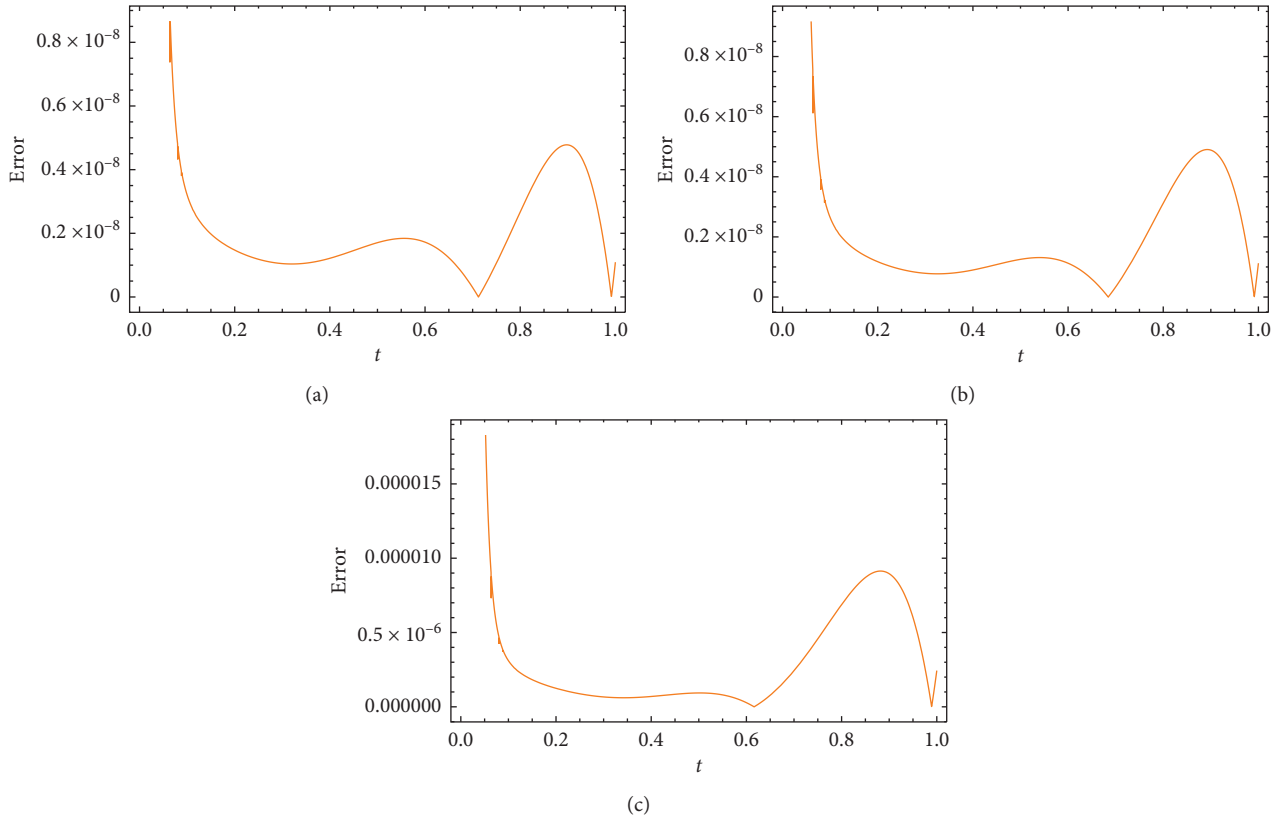


FIGURE 6: Absolute residual function equations (39)–(41) using the sixth order of the approximation solution for h^* . (a) Absolute residual error functions for T and the optimal h , (b) absolute residual error functions for I and the optimal h , and (c) absolute residual error functions for F and the optimal h .

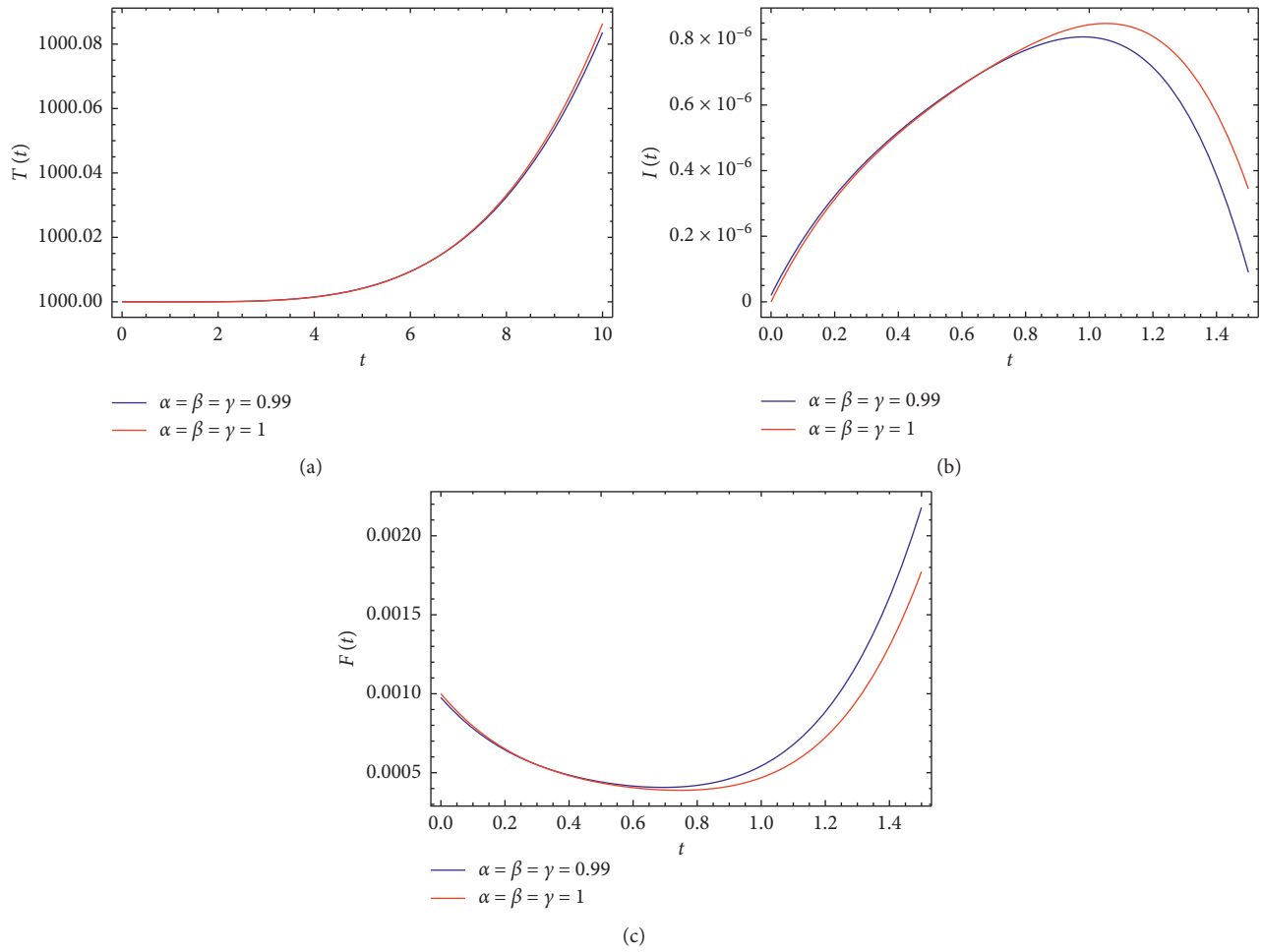


FIGURE 7: Numerical simulation of the concentrations of uninfected CD4⁺ T cells $T(t)$, infected CD4⁺ T cells $I(t)$, and HIV RNA $F(t)$ for different values of α , β , and γ and the optimal values of h^* . (a) Approximate solut[[parms resize(1),pos(50,50),size(200,200),bgcol(156)]], and (c) approximate solutions of $F(t)$.

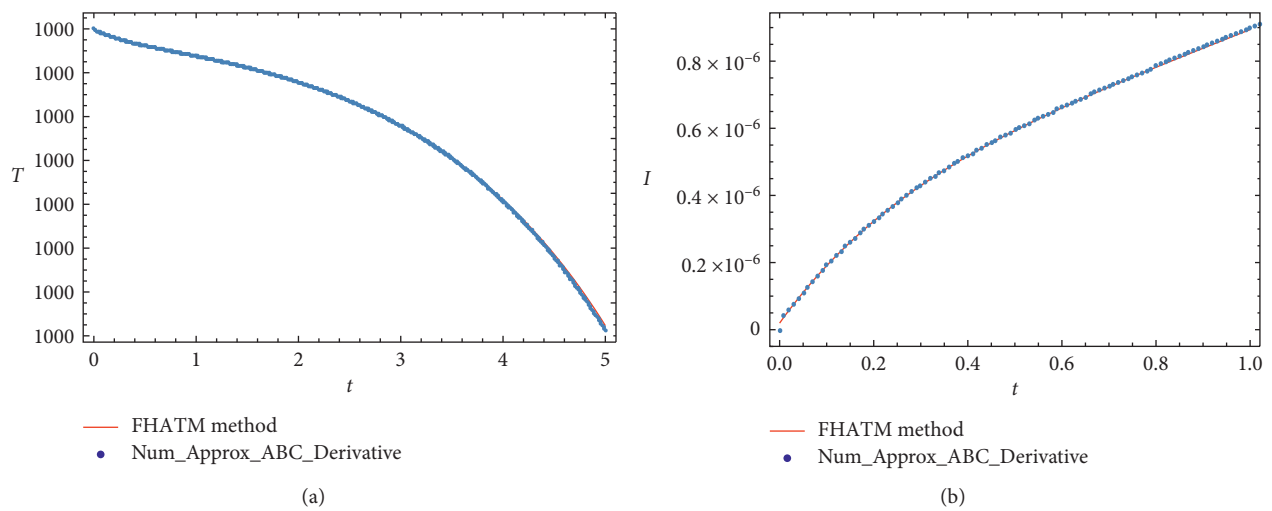


FIGURE 8: Continued.

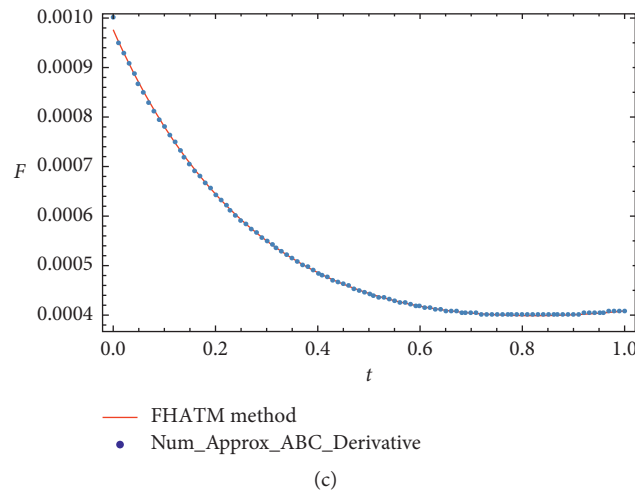


FIGURE 8: Comparison of the numerical solution with that obtained by FHATM for $h = 0.01$. (a) Comparison of the numerical solution and FHATM solution for $\alpha = 0.99$, (b) comparison of the numerical solution and FHATM solution for $\beta = 0.99$, and (c) comparison of the numerical solution and FHATM solution for $\gamma = 0.99$.

Note that, if we set $\alpha = \beta = \gamma = 1$, then the FHATM solution is the same as that obtained with the HAM in [13]. The numerical results are plotted in Figure 7.

5. Numerical Scheme

In this section, we solve fractional a HIV model numerically using the numerical scheme introduced by Toufik and Atangana [27]. The numerical solution and FHATM solution are compared in Figure 8.

6. Conclusion and Further Work

In this study, we successfully solved the fractional HIV infection by using the $CD4^+$ T cells model numerically and analytically, which includes an operator of the type of the Atangana–Baleanu fractional derivative in the Caputo sense (ABC). Analytically approximate solution was obtained for this derivative by incorporating the FHATM in the model of the fractional HIV infection of $CD4^+$ T cells. The solution includes the auxiliary parameter h , which provides an easy way to control the convergence region of the resulting infinite series. The results we obtained show that the FHATM is a successful technique for obtaining an approximate solution of the fractional HIV infection of $CD4^+$ T cells. Moreover, our result agrees strongly with the computation of Toufik and Atangana [27]. Studying the dynamics and stability of the system based on the ABC fractional definition and the FHATM algorithm is an interesting idea for the researchers.

Data Availability

No data were used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The Iterative Positive Solution for a System of Fractional q -Difference Equations with Four-Point Boundary Conditions

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In this work, we investigate the following system of fractional q -difference equations with four-point boundary problems: $\{D_q^\alpha u(t) + f(t, v(t)) = 0, 0 < t < 1; D_q^\beta v(t) + g(t, u(t)) = 0, 0 < t < 1; u(0) = 0, u(1) = c_1 u(\eta_1); \text{ and } v(0) = 0, v(1) = c_2 u(\eta_2)\}$, where D_q^α and D_q^β are the fractional Riemann–Liouville q -derivative of order α and β , respectively, $0 < q < 1$, $1 < \beta \leq \alpha \leq 2$, $0 < \eta_1, \eta_2 < 1$, $0 < \gamma_1 \eta_1^{\alpha-1} < 1$, and $0 < \gamma_2 \eta_2^{\beta-1} < 1$. By virtue of monotone iterative approach, the iterative positive solutions are obtained. An example to illustrate our result is given.

1. Introduction

In [1, 2], Jackson studied the q -difference calculus firstly; since then, many authors have investigated this subject duo to applications of the q -difference calculus in quantum mechanics, particle physics, hypergeometric series, and complex analysis [3, 4]. The extension of q -difference calculus is the fractional q -difference calculus, which was originally investigated by Al-Salam [5] and Agarwal [6]. In the past decade, in many works concerning nonlinear fractional q -difference boundary value problem, the results of the existence and the uniqueness of solutions have been given. In [7], Ferreira considered the existence of positive solutions to the nonlinear fractional q -difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\ u(0) = u(1) = 0. \end{cases} \quad (1)$$

In [8], Ferreira studied the existence of positive solutions to the nonlinear fractional q -difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 2 < \alpha \leq 3, \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \geq 0. \end{cases} \quad (2)$$

By using a fixed-point theorem in partially ordered sets, Garzi and Agheli [9] studied the existence and uniqueness of a positive and nondecreasing solution to the fractional q -difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 3 < \alpha \leq 4, \\ u(0) = D_q u(0) = D_q^2 u(0) = 0, & D_q^2 u(1) = \beta D_q^2 u(\eta), \end{cases} \quad (3)$$

where $0 < \eta < 1$ and $1 - \beta \eta^{\alpha-3} > 0$.

In [10], Guo and Kang obtained the existence and uniqueness of a positive solution for the fractional q -difference equation of the form

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \leq 2, \\ u(0) = 0, & u(1) = \beta u(\eta), \end{cases} \quad (4)$$

by virtue of fixed-point theorems for the mixed monotone operator. Here, $1 < \alpha \leq 2$ and $0 < \beta \eta^{\alpha-1} < 1$.

Recently, by using the monotone iterative approach, in [11], Wang investigated the iterative positive solutions of the following fractional q -difference equations with three-point boundary conditions:

$$\begin{cases} D_q^\alpha u(x) + \lambda h(x)f(u(x)) = 0, & 0 < x < 1, 2 < \alpha \leq 3, \\ u(0) = D_q u(0) = D_q u(1) = 0. \end{cases} \quad (5)$$

It should be noted that the existence of positive solutions of problem (5) had been studied by Li et al. [12] by means of a fixed-point theorem in cones. The novel idea of [11] is to find the positive solution.

Motivated by the above mentioned works, in this paper, we consider the following system of fractional q -difference equations with four-point boundary conditions:

$$\begin{cases} D_q^\alpha u(t) + f(t, v(t)) = 0, & 0 < t < 1, \\ D_q^\beta v(t) + g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \\ u(1) = \gamma_1 u(\eta_1), \\ v(0) = 0, \\ v(1) = \gamma_2 v(\eta_2), \end{cases} \quad (6)$$

where D_q^α and D_q^β are the fractional Riemann–Liouville q -derivative of order α and β , respectively, $0 < q < 1$, $1 < \beta \leq \alpha \leq 2$, $0 < \eta_1, \eta_2 < 1$, $0 < \gamma_1 \eta_1^{\alpha-1} < 1$, and $0 < \gamma_2 \eta_2^{\beta-1} < 1$.

By using the monotone iterative approach, in this paper, we will construct two convergent monotone iterative schemes for seeking one coupled positive solution and obtain the coupled positive solution of problem (6). To the best of our knowledge, there is no paper to study the iterative coupled positive solutions for the coupled system of fractional q -difference boundary value problems. It is noted that we may investigate the approximate solutions of problem (6) by numerical approximation algorithms, which will be presented as another paper. For the latest development of numerical approximation algorithms of some boundary value problems, see [13–17] and the references therein.

2. Preliminaries

Let $q \in (0, 1)$, the q -derivative of a function f is defined by

$$\begin{aligned} (D_q f)(x) &= \frac{f(qx) - f(x)}{(q-1)x}, \\ (D_q f)(0) &= \lim_{x \rightarrow 0} (D_q f)(x), \end{aligned} \quad (7)$$

and q -derivatives of higher order by

$$\begin{aligned} (D_q^0 f)(x) &= f(x), \\ (D_q^n f)(x) &= D_q (D_q^{n-1} f)(x), \quad n \in \mathbb{N}. \end{aligned} \quad (8)$$

The q -integral of a function f defined in the interval $[0, b]$ is given by

$$(I_q f)(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} f(xq^k) q^k, \quad x \in [0, b]. \quad (9)$$

Similar to the derivatives, the operator I_q^n is given by

$$\begin{aligned} (I_q^0 f)(x) &= f(x), \\ (I_q^n f)(x) &= I_q (I_q^{n-1} f)(x), \quad n \in \mathbb{N}. \end{aligned} \quad (10)$$

Define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \quad (11)$$

The q -analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$\begin{aligned} (a-b)^0 &= 1, \\ (a-b)^n &= \prod_{k=0}^{n-1} (a - bq^k), \quad n \in \mathbb{N}, a, b \in \mathbb{R}. \end{aligned} \quad (12)$$

Moreover, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}. \quad (13)$$

Remark 1. If $b = 0$, then $a^{(\alpha)} = a^\alpha$. If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

The q -gamma function [18] is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \setminus \{0, -1, -2, \dots\}, \quad (14)$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

Definition 1. We say (u_*, v_*) is a solution of system (6), if (u_*, v_*) satisfies the first and second equations of (6) and boundary conditions of (6).

Definition 2 (see [19]). Let $\alpha > 0$ and f be a function defined on $[0, 1]$. The fractional q -integral of the Riemann–Liouville type is

$$(I_q^\alpha f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x-qt)^{(\alpha-1)} f(t) d_q t, \quad x \in [0, 1]. \quad (15)$$

Definition 3 (see [19]). The fractional q -derivative of the Riemann–Liouville type is defined by

$$(D_q^\alpha f)(x) = (D_q^n I_q^{n-\alpha} f)(x), \quad \alpha > 0, \quad (16)$$

where n is the smallest integer greater than or equal to α .

Lemma 1 (see [19]). Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, 1]$. Then, the following formulas hold:

$$\begin{aligned} (1) \quad (I_q^\beta I_q^\alpha f)(x) &= (I_q^{\alpha+\beta} f)(x), \\ (2) \quad (D_q^\alpha I_q^\alpha f)(x) &= f(x). \end{aligned}$$

Lemma 2 (see [13]). Let $\alpha > 0$ and n be a positive integer. Then, the following equality holds:

$$(I_q^\alpha D_q^n f)(x) = (D_q^n I_q^\alpha f)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha+k-n+1)} (D_q^k f)(0). \quad (17)$$

By Lemmas 1 and 2, Guo and Kang in [10] obtained the following lemma.

Lemma 3. For any $g \in C[0, 1]$, the boundary value problem

$$\begin{cases} D_q^\alpha u(t) + g(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \gamma_1 u(\eta_1), \end{cases} \quad (18)$$

has a unique solution:

$$u(t) = \int_0^1 G_1(t, qs)g(s)d_qs, \quad (19)$$

where

$$G_1(t, qs) = \begin{cases} \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - t^{\alpha-1}\gamma_1(\eta_1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}(1-\gamma_1\eta_1^{\alpha-1})}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \leq qs \leq t \leq 1, qs \leq \eta_1, \\ \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}(1-\gamma_1\eta_1^{\alpha-1})}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \leq \eta_1 \leq qs \leq t \leq 1, \\ \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - t^{\alpha-1}\gamma_1(\eta_1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \leq t \leq qs \leq 1, \\ \frac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \leq t \leq qs \leq 1, \eta_1 \leq qs, \end{cases} \quad (20)$$

is the Green function of BVP (18).

Similarly, we have the following.

Lemma 4. For any $h \in C[0, 1]$, the boundary value problem

$$\begin{cases} D_q^\beta v(t) + h(t) = 0, & 0 < t < 1, \\ v(0) = 0, & v(1) = \gamma_2 v(\eta_2), \end{cases} \quad (21)$$

has a unique solution:

$$v(t) = \int_0^1 G_2(t, qs)h(s)d_qs, \quad (22)$$

where

$$G_2(t, qs) = \begin{cases} \frac{t^{\beta-1}(1-qs)^{(\beta-1)} - t^{\beta-1}\gamma_2(\eta_2-qs)^{(\beta-1)} - (t-qs)^{(\beta-1)}(1-\gamma_2\eta_2^{\beta-1})}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \leq qs \leq t \leq 1, qs \leq \eta_2, \\ \frac{t^{\beta-1}(1-qs)^{(\beta-1)} - (t-qs)^{(\beta-1)}(1-\gamma_2\eta_2^{\beta-1})}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \leq \eta_2 \leq qs \leq t \leq 1, \\ \frac{t^{\beta-1}(1-qs)^{(\beta-1)} - t^{\beta-1}\gamma_2(\eta_2-qs)^{(\beta-1)}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \leq t \leq qs \leq 1, \\ \frac{t^{\beta-1}(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \leq t \leq qs \leq 1, \eta_2 \leq qs, \end{cases} \quad (23)$$

is the Green function of BVP (21).

Lemma 5. (see [10]). For $G_1(t, qs)$ and $G_2(t, qs)$ defined as in Lemmas 3 and 4, respectively, we have

- (i) $G_1(t, qs)$ and $G_2(t, qs)$ are two continuous functions
- (ii) $(M_1 qs(1-qs)^{(\alpha-1)}/\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1}))t^{\alpha-1} \leq G_1(t, qs) \leq ((1-qs)^{(\alpha-1)}/\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1}))t^{\alpha-1}$, $\forall 0 \leq t, s \leq 1$, where $0 < M_1 = \min\{1-\gamma_1\eta_1^{\alpha-1}, \gamma_1\eta_1^{\alpha-2}(1-\eta_1), \gamma_1\eta_1^{\alpha-1}\} < 1$

$$(iii) (M_2 q s (1 - q s)^{(\beta-1)/\Gamma_q(\beta)} (1 - \gamma_2 \eta_2^{\beta-1})) t^{\beta-1} \leq G_2(t, q s) \leq ((1 - q s)^{(\beta-1)/\Gamma_q(\beta)} (1 - \gamma_2 \eta_2^{\beta-1})) t^{\beta-1}, \forall 0 \leq t, s \leq 1, \text{ where } 0 < M_2 = \min \left\{ 1 - \gamma_2 \eta_2^{\beta-1}, \gamma_2 \eta_2^{\beta-2} (1 - \eta_2), \gamma_2 \eta_2^{\beta-1} \right\} < 1$$

$$P_1 = \{u \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_1 < 1 < b_1, \text{ such that } a_1 t^{\alpha-1} \leq u(t) \leq b_1 t^{\alpha-1}, t \in [0, 1]\},$$

$$P_2 = \{v \in C[0, 1] \mid \text{there exist two positive numbers } 0 < a_2 < 1 < b_2, \text{ such that } a_2 t^{\beta-1} \leq v(t) \leq b_2 t^{\beta-1}, t \in [0, 1]\}.$$

Now, we define the operators $T_i: C[0, 1] \rightarrow C[0, 1]$ ($i = 1, 2$) by

$$T_1 v(t) = \int_0^1 G_1(t, qs) f(s, v(s)) d_q s,$$

$$T_2 u(t) = \int_0^1 G_2(t, qs) g(s, u(s)) d_q s. \quad (25)$$

From Lemmas 3 and 4, BVP (6) can be transformed into the following system of integral equations:

$$\begin{cases} u(t) = \int_0^1 G_1(t, qs) f(s, v(s)) d_q s, \\ v(t) = \int_0^1 G_2(t, qs) g(s, u(s)) d_q s, \end{cases} \quad (26)$$

By (26), we know that (u_*, v_*) is a solution of (6) if and only if $u_* = T_1 v_*$ and $v_* = T_2 u_*$.

In order to facilitate our investigation, we make the following assumptions:

(H1) $f \in C([0, 1] \times [0, \infty), [0, \infty))$ is nondecreasing with respect to v , and there exists a positive constant $\sigma_1 > 1$, such that

$$f(t, rv) \geq r^{\sigma_1} f(t, v), \quad \forall t \in [0, 1], v \in [0, +\infty), r \in (0, 1]. \quad (27)$$

(H2) $g \in C([0, 1] \times [0, \infty), [0, \infty))$ is nondecreasing with respect to u , and there exists a positive constant $0 < \sigma_2 < 1$, such that

$$g(t, ru) \geq r^{\sigma_2} g(t, u), \quad \forall t \in [0, 1], u \in [0, +\infty), r \in (0, 1]. \quad (28)$$

$$(H3) 0 < \int_0^1 (1 - qs)^{(\alpha-1)} f(s, 1) d_q s < +\infty.$$

$$(H4) 0 < \int_0^1 (1 - qs)^{(\beta-1)} g(s, 1) d_q s < +\infty.$$

3. Main Result

In this paper, we will employ the Banach space $C[0, 1]$, equipped with norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$ for each $u \in C[0, 1]$. Define two cones P_1 and P_2 in $C[0, 1]$ as follows:

Remark 2. The conditions (H1) and (H2) imply that, for $\forall r > 1$, we have $f(t, rv) \leq r^{\sigma_1} f(t, v)$ and $g(t, ru) \leq r^{\sigma_2} g(t, u)$.

Theorem 1. Assume that conditions (H1)–(H4) hold and there exist two positive constants R_1 and R_2 such that

$$\frac{1}{\Gamma_q(\alpha)(1 - \gamma_1 \eta_1^{\alpha-1})} \int_0^1 (1 - qs)^{(\alpha-1)} f(s, 1) d_q s \leq R_1^{1-\sigma_1}, \quad (29)$$

$$\frac{1}{\Gamma_q(\beta)(1 - \gamma_2 \eta_2^{\beta-1})} \int_0^1 (1 - qs)^{(\beta-1)} g(s, 1) d_q s \leq R_2^{1-\sigma_2}, \quad (30)$$

then the fractional q -difference system (6) has one positive solution (u^*, v^*) , where $u^* \in P_1$ and $v^* \in P_2$. Moreover, for each $t \in [0, 1]$, there exist constants $0 < m_i < 1 < n_i$ ($i = 1, 2$), such that

$$u^*(t) \in [m_1 t^{\alpha-1}, n_1 t^{\alpha-1}],$$

$$v^*(t) \in [m_2 t^{\beta-1}, n_2 t^{\beta-1}], \quad (31)$$

which can be obtained by monotone iterative schemes $\{u_n\}$ and $\{v_n\}$ generated by

$$u_n(t) = \int_0^1 G_1(t, qs) f(s, v_{n-1}(s)) d_q s,$$

$$v_n(t) = \int_0^1 G_2(t, qs) g(s, u_{n-1}(s)) d_q s. \quad (32)$$

i.e., $\|u_n - u^*\| \rightarrow 0$ and $\|v_n - v^*\| \rightarrow 0$ as $n \rightarrow \infty$.

Proof. For any $v \in P_2$, we know that there exist two constants a_1 and b_1 with $0 < a_1 < 1 < b_1$ such that

$$a_1 t^{\beta-1} \leq v(t) \leq b_1 t^{\beta-1}, \quad t \in [0, 1]. \quad (33)$$

From Lemma 5 and condition (H1), we obtain

$$\begin{aligned}
T_1 v(t) &= \int_0^1 G_1(t, qs) f(s, v(s)) d_q s \\
&\geq \frac{qM_1 t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, v(s)) d_q s \\
&\geq \frac{qM_1 t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, a_1 s^{\beta-1}) d_q s \\
&\geq \frac{qM_1 a_1^{\sigma_1} t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, s^{\beta-1}) d_q s \\
&\geq c_1 t^{\alpha-1}, \\
T_1 v(t) &\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, v(s)) d_q s \\
&\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, b_1 s^{\beta-1}) d_q s \\
&\leq \frac{b_1^{\sigma_1} t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, s^{\beta-1}) d_q s \\
&\leq d_1 t^{\alpha-1},
\end{aligned} \tag{34}$$

where d_1 and c_1 are two positive constants satisfying

$$\begin{aligned}
d_1 &> \max \left\{ 1, \frac{b_1^{\sigma_1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, s^{\beta-1}) d_q s \right\}, \\
0 < c_1 &< \min \left\{ 1, \frac{qM_1 a_1^{\sigma_1}}{\Gamma_q(\alpha)(1-\gamma_1 \eta_1^{\alpha-1})} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, s^{\beta-1}) d_q s \right\}.
\end{aligned} \tag{35}$$

Thus, T_1 maps P_2 into P_1 . For each $u \in P_1$, there exist two constants a_2 and b_2 with $0 < a_2 < 1 < b_2$ such that

$$a_2 t^{\alpha-1} \leq u(t) \leq b_2 t^{\alpha-1}, \quad t \in [0, 1]. \tag{36}$$

Similarly, by Lemma 5 and condition (H2), we can get that

$$c_2 t^{\beta-1} \leq T_2 u(t) \leq d_2 t^{\beta-1}, \tag{37}$$

where d_2 and c_2 are two positive constants satisfying

$$\begin{aligned}
d_2 &> \max \left\{ 1, \frac{b_2^{\sigma_2}}{\Gamma_q(\beta)(1-\gamma_2 \eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)} g(s, s^{\alpha-1}) d_q s \right\}, \\
0 < c_2 &< \min \left\{ 1, \frac{qM_2 a_2^{\sigma_2}}{\Gamma_q(\beta)(1-\gamma_2 \eta_2^{\beta-1})} \int_0^1 s(1-qs)^{(\beta-1)} g(s, s^{\alpha-1}) d_q s \right\},
\end{aligned} \tag{38}$$

which implies that T_2 maps P_1 into P_2 . On the other hand, the proof of completely continuous T_1 and T_2 are as the same as in [12], and we omit it here.

Let $P_i(R) = \{u \mid u \in P_i, \|u\| \leq R\}$ ($i = 1, 2$). In the following, we will prove $T_1(P_2(R_1)) \subset P_1(R_1)$ and $T_2(P_1(R_2)) \subset P_2(R_2)$. In fact, for any $v \in P_2(R_1)$ and $u \in P_1(R_2)$, by conditions (29) and (30), we obtain

$$\begin{aligned}
 T_1 v(t) &\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, v(s)) d_qs \\
 &\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, R_1) d_qs \\
 &\leq \frac{R_1^{\sigma_1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s, 1) d_qs \\
 &\leq R_1, \\
 T_2 u(t) &\leq \frac{t^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)} g(s, u(s)) d_qs \\
 &\leq \frac{t^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)} g(s, R_2) d_qs \\
 &\leq \frac{R_2^{\sigma_2}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)} g(s, 1) d_qs \\
 &\leq R_2,
 \end{aligned} \tag{39}$$

which implies that $\|T_1 v\| \leq R_1$ and $\|T_2 u\| \leq R_2$. So, $T_1(P_2(R_1)) \subset P_1(R_1)$ and $T_2(P_1(R_2)) \subset P_2(R_2)$.

Taking $e_1(t) = t^{\alpha-1}$ and $e_2(t) = t^{\beta-1}$, then $e_1 \in P_1$, $e_2 \in P_2$, $T_1(e_2) \in P_1$, and $T_2(e_1) \in P_2$. Thus, there exist constants $0 < m_i < 1 < n_i$ ($i = 1, 2$) such that

$$\begin{aligned}
 m_1 t^{\alpha-1} &\leq T_1 e_2(t) \leq n_1 t^{\alpha-1}, \\
 m_2 t^{\beta-1} &\leq T_2 e_1(t) \leq n_2 t^{\beta-1}.
 \end{aligned} \tag{40}$$

Let l_1 and l_2 be two positive numbers satisfying $0 < l_1 < l_2 < 1$, $l_2 < l_1^{\sigma_2}$, and

$$\begin{aligned}
 l_1 l_2^{\sigma_1} &\leq m_1, \\
 l_2 l_1^{\sigma_2} &\leq m_2.
 \end{aligned} \tag{41}$$

Set

$$\begin{aligned}
 u_0(t) &= l_1 e_1(t), \\
 v_0(t) &= l_2 e_2(t),
 \end{aligned} \tag{42}$$

$$\begin{aligned}
 u_n &= T_1 v_{n-1}, \\
 v_n &= T_2 u_{n-1}, \\
 n &= 1, 2, \dots
 \end{aligned} \tag{43}$$

Obviously, $u_0(t) \leq v_0(t)$ by $\beta \leq \alpha$ and $0 < l_1 < l_2 < 1$, $u_0 \in P_1(R_2)$ and $v_0 \in P_2(R_1)$. By (H1) and (H2), we have

$$\begin{aligned}
 u_1(t) &= T_1 v_0(t) = \int_0^1 G_1(t, qs) f(s, v_0(s)) d_qs \\
 &= \int_0^1 G_1(t, qs) f(s, l_2 e_2(s)) d_qs \\
 &\geq l_2^{\sigma_1} \int_0^1 G_1(t, qs) f(s, e_2(s)) d_qs \\
 &= l_2^{\sigma_1} T_1 e_2(t) \geq l_2^{\sigma_1} m_1 e_1(t) \geq l_1 e_1(t) = u_0(t), \\
 v_1(t) &= T_2 u_0(t) = \int_0^1 G_2(t, qs) g(s, u_0(s)) d_qs \\
 &= \int_0^1 G_2(t, qs) g(s, l_1 e_1(s)) d_qs \\
 &\geq l_1^{\sigma_2} \int_0^1 G_2(t, qs) g(s, e_1(s)) d_qs \\
 &= l_1^{\sigma_2} T_2 e_1(t) \geq l_1^{\sigma_2} m_2 e_2(t) \geq l_2 e_2(t) = v_0(t).
 \end{aligned} \tag{44}$$

From conditions (H1) and (H2), we know that T_1 and T_2 are two nondecreasing operators. Thus, by induction, we can obtain

$$\begin{aligned}
 u_0 &\leq u_1 \leq \dots \leq u_n \leq \dots, \\
 v_0 &\leq v_1 \leq \dots \leq v_n \leq \dots, \\
 u_n &\in P_1(R_1), \\
 v_n &\in P_2(R_2), \\
 n &= 1, 2, \dots
 \end{aligned} \tag{45}$$

By the compactness of the operators T_1 and T_2 , we have that $\{u_n\}$ and $\{v_n\}$ are two sequentially compact sets. Therefore, there exist $u_* \in P_1(R_1)$ and $v_* \in P_2(R_2)$, such that u_n converges to u_* and v_n converges to v_* as $n \rightarrow \infty$, respectively. Since the operators T_1 and T_2 are continuous, $u_n = T_1 v_{n-1}$ and $v_n = T_2 u_{n-1}$, and we obtain $u_* = T_1 v_*$ and $v_* = T_2 u_*$ as $n \rightarrow \infty$, which implies that system (6) has a positive solution (u^*, v^*) , and $u^* \in [m_1 t^{\alpha-1}, n_1 t^{\alpha-1}]$, $v^* \in [m_2 t^{\beta-1}, n_2 t^{\beta-1}]$, and $\forall t \in [0, 1]$, where m_i and n_i are constants and $0 < m_i < 1 < n_i$ ($i = 1, 2$), which can be achieved by the monotone scheme:

$$\begin{aligned}
 u_n(t) &= \int_0^1 G_1(t, qs) f(s, v_{n-1}(s)) d_qs, \\
 v_n(t) &= \int_0^1 G_2(t, qs) g(s, u_{n-1}(s)) d_qs,
 \end{aligned} \tag{46}$$

with initial values $u_0(t)$ and $v_0(t)$ defined as in (42).

In the following, we give an example to illustrate the existence of positive solutions of BVP (6). \square

Example 1. Consider the following system of fractional q -difference with boundary conditions:

$$\left\{ \begin{array}{l} D_{(1/3)}^{(5/3)} u(t) + \frac{1}{8} t v^{(3/2)}(t) = 0, \\ D_{(1/3)}^{(3/2)} v(t) + \sqrt{t} \left(u^{(1/4)}(t) + \frac{u^{(1/3)}(t)}{1 + u^{(1/4)}(t)} \right) = 0, \\ u(0) = 0, \\ v(0) = 0, \end{array} \right. \quad \begin{array}{l} 0 < t < 1, \\ 0 < t < 1, \\ u(1) = u\left(\frac{3}{4}\right), \\ v(1) = \frac{5}{4} v\left(\frac{1}{2}\right), \end{array} \quad (47)$$

where $q = (1/3)$, $\alpha = (5/3)$, $\beta = (3/2)$, $\eta_1 = (3/4)$, $\eta_2 = (1/2)$, $\gamma_1 = 1$, $\gamma_2 = (5/4)$, and

$$\begin{aligned} f(t, v) &= \frac{1}{8} t v^{(3/2)}, \\ g(t, u) &= \sqrt{t} \left(u^{(1/4)} + \frac{u^{(1/3)}}{1 + u^{(1/4)}} \right), \end{aligned} \quad (48)$$

Obviously, $f(t, v)$ and $g(t, u)$ are nondecreasing with respect to v and u , respectively, and

$$\begin{aligned} 0 < \gamma_1 \eta_1^{\alpha-1} &= \left(\frac{3}{4}\right)^{(2/3)} < 1, \\ 0 < \gamma_2 \eta_2^{\beta-1} &= \frac{5}{4} \left(\frac{1}{2}\right)^{(1/2)} < 1. \end{aligned} \quad (49)$$

Choosing $\sigma_1 = 2 > 1$ and $\sigma_2 = (1/3) < 1$, we have

$$f(t, rv) = \frac{1}{8} t r^{(3/2)} v^{(3/2)} \geq r^2 f(t, v), \quad \forall v \in [0, +\infty), r \in (0, 1],$$

$$\begin{aligned} g(t, ru) &= \sqrt{t} \left(r^{(1/4)} u^{(1/4)} + \frac{r^{(1/3)} u^{(1/3)}}{1 + r^{(1/4)} u^{(1/4)}} \right), \\ &\geq \sqrt{t} \left(r^{(1/3)} u^{(1/4)} + \frac{r^{(1/3)} u^{(1/3)}}{1 + u^{(1/4)}} \right) = r^{(1/3)} g(t, u), \\ &\forall u \in [0, +\infty), r \in (0, 1]. \end{aligned} \quad (50)$$

So, conditions (H1) and (H2) hold. Moreover, we can show that

$$\begin{aligned} 0 < \int_0^1 \left(1 - \frac{1}{3}s\right)^{(2/3)} f(s, 1) d_q s &= \frac{1}{8} \int_0^1 \left(1 - \frac{1}{3}s\right)^{(2/3)} s d_q s < \infty, \\ 0 < \int_0^1 \left(1 - \frac{1}{3}s\right)^{(1/2)} g(s, 1) d_q s &= \frac{3}{2} \int_0^1 \left(1 - \frac{1}{3}s\right)^{(1/2)} \sqrt{s} d_q s < \infty, \end{aligned} \quad (51)$$

which implies that (H3) and (H4) hold. Moreover, we know that there exist two positive constants R_1 and R_2 such that (29) and (30) hold, respectively. Thus, it follows from Theorem 1 that boundary value problem of fractional q -difference system (47) has one iterative positive solution

(u^*, v^*) which can be obtained with the aid of monotone iterative sequences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript.

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Research Article

Multiplicity Solutions for Integral Boundary Value Problem of Fractional Differential Systems

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This paper deals with the existence and multiplicity of solutions for the integral boundary value problem of fractional differential

systems:
$$\begin{cases} D_{0+}^{\alpha_1} u_1(t) = f_1(t, u_1(t), u_2(t)), \\ D_{0+}^{\alpha_2} u_2(t) = f_2(t, u_1(t), u_2(t)), \\ u_1(0) = 0, D_{0+}^{\beta_1} u_1(0) = 0, D_{0+}^{\gamma_1} u_1(1) = \int_0^1 D_{0+}^{\gamma_1} u_1(\eta) dA_1(\eta), \\ u_2(0) = 0, D_{0+}^{\beta_2} u_2(0) = 0, D_{0+}^{\gamma_2} u_2(1) = \int_0^1 D_{0+}^{\gamma_2} u_2(\eta) dA_2(\eta), \end{cases}$$
 where $f_i: [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is continuous and $\alpha_i - 2 < \beta_i \leq 2, \alpha_i - \gamma_i \geq 1, 2 < \alpha_i \leq 3, \gamma_i \geq 1 (i = 1, 2)$. D_{0+}^{α} is the standard Riemann–Liouville's fractional derivative of order α . Our result is based on an extension of the Krasnosel'skii's fixed-point theorem due to Radu Precup and Jorge Rodriguez-Lopez in 2019. The main results are explained by the help of an example in the end of the article.

1. Introduction

With the deepening of people's understanding of mathematics, the knowledge of mathematics is more and more closely related to the way of production and life of human beings. In recent years, fractional calculus is very active in the field of applied mathematics. It can be applied not only in biochemistry, mathematical physics equation, physical science experiment, and other academic fields but also in precision production [1–3].

In many recent papers are researched the fractional differential equations with the existence of the solutions [4–43]. For example, Zhang and Zhong [38] used the fixed-point theorem on cones to find the existence result of at least two positive solutions which are considered the nonlinear fractional differential equations of nonlocal boundary value problems as follows:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t \leq 1, \\ D_{0+}^{\beta} u(1) = \sum_{i=1}^{\infty} \xi_i D_{0+}^{\beta} u(\eta_i), & u(0) = 0, D_{0+}^{\beta} u(0) = 0, \end{cases} \quad (1)$$

where $2 < \alpha \leq 3, 1 \leq \beta \leq 2, \alpha - \beta \geq 1, 0 < \xi_i, \eta_i < 1$ with $\sum_{i=1}^{\infty} \xi_i \eta_i^{\alpha-\beta-1} < 1$.

In [22], the authors obtained the uniqueness results of positive solution by the contraction map principle and obtained some existence results of positive solution through the fixed-point index theory, which is as follows:

$$\begin{cases} D_{0+}^{\alpha} u(t) + f(t, u(t)) = 0, & 0 < t \leq 1, \\ \beta u(\eta) = u(1), & u(0) = 0, \end{cases} \quad (2)$$

where $1 < \alpha \leq 2, 0 < \beta \eta^{\alpha-1} < 1, 0 < \eta < 1$, D_{0+}^{α} is the standard Riemann–Liouville differentiation, and the function f is continuous on $[0, 1] \times [0, \infty)$.

However, in recent years, many scholars have begun to use the fixed-point index to study the existence and multiplicity of operator equation and operator systems [44–51]. For example, in [46], the authors use the fixed-point index in cones to study the existence, localization, and multiplicity of positive solutions to operator systems of the following form:

$$\begin{cases} L_i(u_i) = F_i(u_1, u_2), \\ u_i \in D(L_i), \\ i = 1, 2, \end{cases} \quad (3)$$

where for each i , $L_i: D(L_i) \subset X \rightarrow Y$ is invertible, $F_i: Y \times Y \rightarrow Y$, and $X, Y \subset C[0, 1]$. It should be noted that each component of the fixed-point operator systems may have the same or different behaviors.

In particular, only few papers studied the existence and multiplicity of solutions to specific differential systems [52–58]. Therefore, in this paper, we will apply an extension of the Krasnosel'skiĭ's fixed-point theorem to investigate the existence and multiplicity of solutions for a class of fractional differential systems. More precisely, the following fractional differential systems are studied:

$$\begin{cases} D_{0+}^{\alpha_1} u_1(t) = f_1(t, u_1(t), u_2(t)), \\ D_{0+}^{\alpha_2} u_2(t) = f_2(t, u_1(t), u_2(t)), \\ u_1(0) = 0, \quad D_{0+}^{\beta_1} u_1(0) = 0, \quad D_{0+}^{\gamma_1} u_1(1) = \int_0^1 D_{0+}^{\gamma_1} u_1(\eta) dA_1(\eta), \\ u_2(0) = 0, \quad D_{0+}^{\beta_2} u_2(0) = 0, \quad D_{0+}^{\gamma_2} u_2(1) = \int_0^1 D_{0+}^{\gamma_2} u_2(\eta) dA_2(\eta), \end{cases} \quad (4)$$

where $\alpha_i - 2 < \beta_i \leq 2$, $\alpha_i - \gamma_i \geq 1$, $2 < \alpha_i \leq 3$, $\gamma_i \geq 1$, $f_i \in C([0, 1] \times [0, \infty)^2, [0, \infty))$, and A_i are nondecreasing on $[0, 1]$, left continuous at $t = 1$.

2. Preliminaries

In this part, we first give the basic definitions, lemmas, and theorems related to fractional calculus.

Definition 1 (see [2]). Define the Riemann–Liouville fractional derivative of order $\alpha > 0$ for function σ as

$$D_{0+}^{\alpha} \sigma(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t \frac{\sigma(s)}{(t-s)^{\alpha+1-n}} ds, \quad n = [\alpha] + 1, \quad (5)$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2 (see [2]). Let σ define the Riemann–Liouville fractional integral of order $\alpha > 0$ for σ as

$$I_{0+}^{\alpha} \sigma(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \sigma(s) (t-s)^{\alpha-1} ds, \quad (6)$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Lemma 1 (see [2]). Let $n-1 < \alpha \leq n$ and $u \in C(0, 1) \cap L^1(0, 1)$; then,

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad (7)$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots$

For convenience, we first consider the following linear fractional differential equation:

$$\begin{cases} D_{0+}^{\alpha_1} u_1(t) + \sigma(t) = 0, \quad 0 < t \leq 1, \\ D_{0+}^{\gamma_1} u_1(1) = \int_0^1 D_{0+}^{\gamma_1} u_1(\eta) dA_1(\eta), \\ u_1(0) = 0, \quad D_{0+}^{\beta_1} u_1(0) = 0, \end{cases} \quad (8)$$

where $\alpha_1 - 2 < \beta_1 \leq 2$, $\alpha_1 - \gamma_1 \geq 1$, $2 < \alpha_1 \leq 3$, $\gamma_1 \geq 1$, and $A_1(t)$ is nondecreasing on $[0, 1]$, left continuous at $t = 1$.

Lemma 2. Let $1 - \int_0^1 \eta^{\alpha_1-\gamma_1-1} dA_1(\eta) > 0$ and $\sigma \in C[0, 1]$; then, boundary value problem (8) has a unique solution $u_1(t) = \int_0^1 G_1(t, s) \sigma(s) ds$, where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha_1) p_1(0)} \begin{cases} t^{\alpha_1-1} p_1(s) (1-s)^{\alpha_1-\gamma_1-1}, & 0 \leq t \leq s \leq 1; \\ t^{\alpha_1-1} p_1(s) (1-s)^{\alpha_1-\gamma_1-1} - p_1(0) (t-s)^{\alpha_1-1}, & 0 \leq s \leq t \leq 1, \end{cases} \quad (9)$$

$$p_1(s) = 1 - \int_s^1 \frac{(\eta-s)^{\alpha_1-\gamma_1-1}}{(1-s)^{\alpha_1-\gamma_1-1}} dA_1(\eta).$$

Proof. It follows from Lemma 1 that $u_1(t) = -I_{0+}^{\alpha_1} \sigma(t) + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2} + c_3 t^{\alpha_1-3}$. With consideration of the boundary value conditions $u_1(0) = 0$, we can get $c_3 = 0$. Consequently,

$$u_1(t) = -I_{0+}^{\alpha_1} \sigma_1(t) + c_1 t^{\alpha_1-1} + c_2 t^{\alpha_1-2}. \quad (10)$$

Notice that $D_{0+}^{\beta_1} t^{\alpha-i} = (\Gamma(\alpha-i+1)/\Gamma(\alpha-i-\beta_1+1)) t^{\alpha-i-\beta_1}$ ($i = 1, 2$); we get

$$\begin{aligned} D_{0+}^{\beta_1} u_1(t) &= D_{0+}^{\beta_1} \left(- \int_0^t \frac{1}{\Gamma_1(\alpha_1)} (t-s)^{\alpha_1-1} \sigma(s) ds \right) \\ &\quad + c_1 \frac{\Gamma_1(\alpha_1)}{\Gamma_1(\alpha_1 - \beta_1)} t^{\alpha_1-\beta_1-1} \\ &\quad + c_2 \frac{\Gamma_1(\alpha_1 - 1)}{\Gamma_1(\alpha_1 - \beta_1 - 1)} t^{\alpha_1-\beta_1-2}. \end{aligned} \quad (11)$$

Since $\alpha_1 - 2 < \beta_1$ and $D_{0+}^{\beta_1} u_1(0) = 0$, we conclude that $c_2 = 0$. Therefore, (10) reduces to

$$u_1(t) = -I_{0+}^{\alpha_1} \sigma_1(t) + c_1 t^{\alpha_1-1}. \quad (12)$$

Taking into account that $D_{0+}^{\gamma_1} u_1(1) = \int_0^1 D_{0+}^{\gamma_1} u_1(\eta) dA_1(\eta)$ and $D_{0+}^{\gamma_1} I_{0+}^{\alpha_1} \sigma_1(t) = I_{0+}^{\alpha_1-\gamma_1} \sigma_1(t)$, we have

$$\begin{aligned} D_{0+}^{\gamma_1} u_1(1) &= -\frac{1}{\Gamma(\alpha_1 - \gamma_1)} \int_0^1 (1-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds + c_1 \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1 - \gamma_1)} \\ &= \int_0^1 D_{0+}^{\gamma_1} u_1(\eta) dA_1(\eta) \\ &= \int_0^1 \left[\frac{1}{\Gamma(\alpha_1 - \gamma_1)} \int_0^\eta (\eta-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds \right. \\ &\quad \left. + c_1 \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1 - \gamma_1)} \eta^{\alpha_1-\gamma_1-1} \right] dA_1(\eta) \\ &= -\frac{1}{\Gamma(\alpha_1 - \gamma_1)} \int_0^1 \int_0^\eta (\eta-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds dA_1(\eta) \\ &\quad + c_1 \frac{\Gamma(\alpha_1)}{\Gamma(\alpha_1 - \gamma_1)} \int_0^1 \eta^{\alpha_1-\gamma_1-1} dA_1(\eta). \end{aligned} \quad (13)$$

This yields

$$c_1 = \frac{\int_0^1 (1-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds - \int_0^1 \int_0^\eta (\eta-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds dA_1(\eta)}{\Gamma(\alpha_1) \left(1 - \int_0^1 \eta^{\alpha_1-\gamma_1-1} dA_1(\eta) \right)}. \quad (14)$$

Taking the above equality into (12), we have

$$\begin{aligned} u_1(t) &= -I_{0+}^{\alpha_1} \sigma(t) + c_1 t^{\alpha_1-1} \\ &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \sigma(s) ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1) p(0)} \\ &\quad \cdot \left[\int_0^1 (1-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds \right. \\ &\quad \left. - \int_0^1 \int_0^\eta (\eta-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds dA_1(\eta) \right] \\ &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \sigma(s) ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1) p(0)} \\ &\quad \cdot \left[\int_0^1 (1-s)^{\alpha_1-\gamma_1-1} \sigma(s) ds \right. \\ &\quad \left. \cdot \int_0^1 \int_s^1 (\eta-s)^{\alpha_1-\gamma_1-1} dA_1(\eta) \sigma(s) ds \right] \\ &= -\frac{1}{\Gamma(\alpha_1)} \int_0^t (t-s)^{\alpha_1-1} \sigma(s) ds + \frac{t^{\alpha_1-1}}{\Gamma(\alpha_1) p(0)} \\ &\quad \cdot \int_0^1 (1-s)^{\alpha_1-\gamma_1-1} p(s) \sigma(s) ds \\ &= \int_0^1 G_1(t, s) \sigma(s) ds, \end{aligned} \quad (15)$$

where $G_1(t, s)$ and $p(s)$ are given in Lemma 2. \square

The following proposition of Green's function $G_1(t, s)$ will be used throughout the paper.

Lemma 3. The function $p_1(s) > 0, s \in [0, 1]$, and p_1 is nondecreasing on $[0, 1]$.

Proof. After direct computation, we will get

$$\begin{aligned}
p_1'(s) &= \int_s^1 \frac{(\alpha_1 - \gamma_1 - 1)(\eta - s)^{\alpha_1 - \gamma_1 - 2} (1 - s)^{\alpha_1 - \gamma_1 - 1}}{(1 - s)^2 (\alpha_1 - \gamma_1 - 1)} dA_1(\eta) \\
&\quad - \int_s^1 \frac{(\eta - s)^{\alpha_1 - \gamma_1 - 1} (1 - s)^{\alpha_1 - \gamma_1 - 2} (\alpha_1 - \gamma_1 - 1)}{(1 - s)^2 (\alpha_1 - \gamma_1 - 1)} dA_1(\eta) \\
&= (\alpha_1 - \gamma_1 - 1) \int_s^1 \frac{(\eta - s)^{\alpha_1 - \gamma_1 - 2} - (\eta - s)^{\alpha_1 - \gamma_1 - 1} (1 - s)^{-1}}{(1 - s)^{\alpha_1 - \gamma_1 - 1}} dA_1(\eta) \\
&= (\alpha_1 - \gamma_1 - 1) \int_s^1 \frac{(\eta - s)^{\alpha_1 - \gamma_1 - 2} (1 - (\eta - s/1 - s))}{(1 - s)^{\alpha_1 - \gamma_1 - 1}} dA_1(\eta) \\
&= (\alpha_1 - \gamma_1 - 1) \int_s^1 \frac{(1 - \eta/1 - s)(\eta - s)^{\alpha_1 - \gamma_1 - 2}}{(1 - s)^{\alpha_1 - \gamma_1 - 1}} dA_1(\eta) \\
&\geq 0, s \in [0, 1].
\end{aligned} \tag{16}$$

Then, we conclude that $p_1(s)$ is a nondecreasing function on $[0, 1]$, which implies that $p_1(s) \geq p_1(0) = 1 - \int_0^1 \eta^{\alpha_1 - \gamma_1 - 1} dA_1(\eta) > 0$, $s \in [0, 1]$. This proves the content of lemma. \square

Lemma 4. The function $G_1(t, s)$ has the following properties:

- (i) $G_1(t, s) > 0$, $(\partial/\partial t)G_1(t, s) > 0$, $0 < t, s < 1$
- (ii) $\max_{t \in [0, 1]} G_1(t, s) = G_1(1, s)$, $0 \leq s \leq 1$
- (iii) $G_1(t, s) \geq t^{\alpha_1 - 1} G_1(1, s)$, $0 \leq t, s \leq 1$

Proof. According to Lemma 2, we have learned that Green's function G_1 is divided into two cases, and next, we will prove three properties of G_1 , respectively.

- (i) When $0 \leq t \leq s \leq 1$,

$$G_1(t, s) = \frac{t^{\alpha_1 - 1}}{\Gamma(\alpha_1) p_1(0)} (1 - s)^{\alpha_1 - \gamma_1 - 1} p_1(s) > 0, \tag{17}$$

then by a direct calculation, it is easy to get

$$\frac{\partial}{\partial t} G_1(t, s) = \frac{p_1(s)(1 - s)^{\alpha_1 - \gamma_1 - 1}}{\Gamma(\alpha_1) p_1(0)} (\alpha_1 - 1) t^{\alpha_1 - 2} \tag{18}$$

$$\geq 0.$$

When $0 \leq s \leq t \leq 1$,

$$\begin{aligned}
G_1(t, s) &= \frac{1}{p_1(0)\Gamma(\alpha_1)} [t^{\alpha_1 - 1} p_1(s)(1 - s)^{\alpha_1 - \gamma_1 - 1} - p_1(0)(t - s)^{\alpha_1 - 1}] \\
&\geq \frac{1}{p_1(0)\Gamma(\alpha_1)} [t^{\alpha_1 - 1} p_1(0)(1 - s)^{\alpha_1 - \gamma_1 - 1} - p_1(0)(t - s)^{\alpha_1 - 1}] \\
&= \frac{1}{p_1(0)\Gamma(\alpha_1)} t^{\alpha_1 - 1} \left[p_1(0)(1 - s)^{\alpha_1 - \gamma_1 - 1} - p_1(0) \left(1 - \frac{s}{t}\right)^{\alpha_1 - 1} \right] \\
&= \frac{t^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \left[(1 - s)^{\alpha_1 - \gamma_1 - 1} - \left(1 - \frac{s}{t}\right)^{\alpha_1 - 1} \right] \\
&\geq 0, \\
\frac{\partial}{\partial t} G_1(t, s) &= \frac{1}{p_1(0)\Gamma(\alpha_1)} [(\alpha_1 - 1)t^{\alpha_1 - 2} p_1(s)(1 - s)^{\alpha_1 - \gamma_1 - 1} - (\alpha_1 - 1)(t - s)^{\alpha_1 - 2} p_1(0)] \\
&= \frac{\alpha_1 - 1}{p_1(0)\Gamma(\alpha_1)} [t^{\alpha_1 - 2} p_1(s)(1 - s)^{\alpha_1 - \gamma_1 - 1} - p_1(0)(t - s)^{\alpha_1 - 2}] \\
&= \frac{\alpha_1 - 1}{p_1(0)\Gamma(\alpha_1)} t^{\alpha_1 - 2} \left[p_1(s)(1 - s)^{\alpha_1 - \gamma_1 - 1} - p_1(0) \left(1 - \frac{s}{t}\right)^{\alpha_1 - 2} \right] \\
&\geq \frac{\alpha_1 - 1}{\Gamma(\alpha_1)} t^{\alpha_1 - 2} \left[(1 - s)^{\alpha_1 - \gamma_1 - 1} - \left(1 - \frac{s}{t}\right)^{\alpha_1 - 2} \right] \\
&\geq 0.
\end{aligned} \tag{19}$$

(ii) Based on the property (i), it follows that $G_1(t, s)$ is increasing with respect to t . Obviously, we have

$$\begin{aligned} \max_{t \in [0,1]} G_1(t, s) &= G_1(1, s) \\ &= \frac{1}{p_1(0)\Gamma(\alpha_1)} \left[p_1(s)(1-s)^{\alpha_1-\gamma_1-1} \right. \\ &\quad \left. - p_1(0)(1-s)^{\alpha_1-1} \right]. \end{aligned} \quad (20)$$

(iii) For $(t, s) \in [0, 1] \times [0, 1]$, we discuss two cases.

When $0 \leq t \leq s \leq 1$, $G_1(t, s) = (t^{\alpha_1-1}/p_1(0)\Gamma(\alpha_1))((1-s)^{\alpha_1-\gamma_1-1}p_1(s) - t^{\alpha_1-1}G_1(1, s))$, it is easy to get that $G_1(t, s) \geq t^{\alpha_1-1}G_1(1, s)$.

When $0 \leq s \leq t \leq 1$,

$$\begin{aligned} G_1(t, s) &= \frac{1}{p_1(0)\Gamma(\alpha_1)} \left[t^{\alpha_1-1}p_1(s)(1-s)^{\alpha_1-\gamma_1-1} \right. \\ &\quad \left. - p_1(0)(t-s)^{\alpha_1-1} \right] \\ &= \frac{t^{\alpha_1-1}}{p_1(0)\Gamma(\alpha_1)} \left[p_1(s)(1-s)^{\alpha_1-\gamma_1-1} \right. \\ &\quad \left. - p_1(0)\left(1 - \frac{s}{t}\right)^{\alpha_1-1} \right] \\ &\geq t^{\alpha_1-1}G_1(1, s). \end{aligned} \quad (21)$$

This yields the desired result. \square

The main proof of this research uses the following theorem in [46].

Theorem 1 (see [46]). Let $(X, \|\cdot\|)$ be a Banach space, $K_1, K_2 \subset X$ two cones, and $K = K_1 \times K_2$ the corresponding cone of $X^2 = X \times X$. Let $p_i, q_i > 0$ with $p_i \neq q_i$, $U_{p_i} = \{u \in K_i: \|u\| < p_i\}$. Assume that $N: \overline{W_1} \times \overline{W_2} \rightarrow K$, $T = (T_1, T_2)$ is a compact map (where $W_i = U_{p_i} \cup U_{q_i}$ for $i = 1, 2$) and there exist $\varphi_i \in K_i \setminus \{0\}$, $i = 1, 2$, such that for each $i \in \{1, 2\}$, the following condition is satisfied in $\overline{W_1} \times \overline{W_2}$:

(i) $\lambda x_i \neq T_i x$ for $\|x_i\| = p_i$ and $\lambda \geq 1$

(ii) $x_i \neq T_i x + \mu \varphi_i$ for $\|x_i\| = q_i$ and $\mu \geq 0$

Then,

- (1) T has at least one fixed point in K such that $x_i \in U_{p_i} \setminus \overline{U_{q_i}}$ for $i = 1, 2$ if $p_i > q_i$ for $i = 1, 2$
- (2) T has at least two fixed points located in $(U_{p_1} \setminus \overline{U_{q_1}}) \times U_{p_2}$ and $(U_{p_1} \setminus \overline{U_{q_1}}) \times (U_{q_2} \setminus \overline{U_{p_2}})$ if $q_1 < p_1$ and $q_2 > p_2$
- (3) T has at least two fixed points located in $U_{p_1} \times (U_{p_2} \setminus \overline{U_{q_2}})$ and $(U_{q_1} \setminus \overline{U_{p_1}}) \times (U_{p_2} \setminus \overline{U_{q_2}})$ if $q_1 > p_1$ and $q_2 < p_2$
- (4) T has at least four fixed points located in $U_{p_1} \times U_{p_2}, U_{p_1} \times (U_{q_2} \setminus \overline{U_{p_2}}), (U_{q_1} \setminus \overline{U_{p_1}}) \times U_{p_2}, (U_{q_1} \setminus \overline{U_{p_1}}) \times (U_{q_2} \setminus \overline{U_{p_2}})$ if $p_i < q_i$ for $i = 1, 2$

3. Main Results

Let $X = C[0, 1]$, $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$, $K = \{x \in C[0, 1]: x(t) \geq 0\}$, and $P_i = \{x \in C[0, 1]: x(t) \geq t^{\alpha_i-1}\|x\|_\infty\}$ ($i = 1, 2$). Then, X becomes a real Banach space with the norm $\|\cdot\|_\infty$ and K , and P_i are cones on X . Also the product space $X \times X$ is a Banach space endowed with norm $\|(x, y)\| = \max\{\|x\|_\infty, \|y\|_\infty\}$ and $P_1 \times P_2$ is a cone in $X \times X$.

For convenience, we present some basic conditions as follows which we will be used later:

(H1) $f_i \in C([0, 1] \times [0, +\infty) \times [0, +\infty))$, $[0, +\infty)$ ($i = 1, 2$).

(H2) There exist $r_i, \beta_i \in (0, +\infty)$ and $\delta_i \in (0, 1)$ ($i = 1, 2$) such that

$$\begin{aligned} f_1(t, u_1, u_2) &> N_1 r_1 \quad \text{for } (t, u_1, u_2) \in [\delta_1, 1] \times [h_1 r_1, r_1] \times [0, R_2] \\ f_1(t, u_1, u_2) &< M_1 \beta_1 \quad \text{for } (t, u_1, u_2) \in [0, 1] \times [0, \beta_1] \times [0, R_2] \\ f_2(t, u_1, u_2) &> N_2 r_2 \quad \text{for } (t, u_1, u_2) \in [\delta_2, 1] \times [0, R_1] \times [h_2 r_2, r_2] \\ f_2(t, u_1, u_2) &< M_2 \beta_2 \quad \text{for } (t, u_1, u_2) \in [0, 1] \times [0, R_1] \times [0, \beta_2] \end{aligned}$$

where

$$\begin{aligned} R_i &= \max\{r_i, \beta_i\}, \\ h_i &= \delta_i^{\alpha_i-1}, \\ N_i &= \left(1/h_i \int_{\delta_i}^1 G_i(1, s) ds\right), \\ M_i &= \left(1/\int_0^1 G_i(1, s) ds\right), \end{aligned} \quad (22)$$

$$G_i(t, s) = \frac{1}{p_i(0)\Gamma(\alpha_i)} \begin{cases} t^{\alpha_i-1}(1-s)^{\alpha_i-\gamma_i-1}p_i(s), & 0 \leq t \leq s \leq 1, \\ t^{\alpha_i-1}p_i(s)(1-s)^{\alpha_i-\gamma_i-1} - p_i(0)(t-s)^{\alpha_i-1}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$p_i(s) = 1 - \int_s^1 \frac{(\eta-s)^{\alpha_i-\gamma_i-1}}{(1-s)^{\alpha_i-\gamma_i-1}} dA_i(\eta).$$

$$(H3) \quad 1 - \int_0^1 \eta^{\alpha_i - \gamma_i - 1} dA_i(\eta) > 0, \quad i = 1, 2.$$

Employing Lemma 2 and the condition of (H1), system (4) has the following integral representation:

$$\begin{cases} u_1(t) = \int_0^1 G_1(t, s) f_1(s, u_1(s), u_2(s)) ds, & t \in [0, 1], \\ u_2(t) = \int_0^1 G_2(t, s) f_2(s, u_1(s), u_2(s)) ds, & t \in [0, 1]. \end{cases} \quad (23)$$

Let us define two operators $T_i: K \times K \longrightarrow X$ ($i = 1, 2$) as follows:

$$\begin{aligned} T_1(u_1, u_2)(t) &= \int_0^1 G_1(t, s) f_1(s, u_1(s), u_2(s)) ds, & t \in [0, 1], \\ T_2(u_1, u_2)(t) &= \int_0^1 G_2(t, s) f_2(s, u_1(s), u_2(s)) ds, & t \in [0, 1]. \end{aligned} \quad (24)$$

Then, we can define an operator $T: K \times K \longrightarrow X \times X$ as follows:

$$T(u_1, u_2) = (T_1(u_1, u_2), T_2(u_1, u_2)), \quad (u_1, u_2) \in K \times K. \quad (25)$$

Lemma 5. Assume that (H1), (H2), and (H3) hold. Then, $T: K \times K \longrightarrow P_1 \times P_2$ is completely continuous.

Proof. Firstly, we prove that $T: K \times K \longrightarrow P_1 \times P_2$. In fact, for $(u_1, u_2) \in K \times K$, by (H1), it is obvious that $T_i(u_1, u_2)(t) \geq 0$ for $i = 1, 2$ and $t \in [0, 1]$. In addition, if $(u_1, u_2) \in K \times K$, then

$$\begin{aligned} T_i(u_1, u_2)(t) &= \int_0^1 G_i(t, s) f_i(s, u_1(s), u_2(s)) ds \\ &\leq \int_0^1 G_i(1, s) f_i(s, u_1(s), u_2(s)) ds, & (26) \\ &t \in [0, 1]. \end{aligned}$$

So,

$$\|T_i(u_1, u_2)\|_\infty \leq \int_0^1 G_i(1, s) f_i(s, u_1(s), u_2(s)) ds. \quad (27)$$

On the other hand, for any $(u_1, u_2) \in K \times K$ and any $t \in [0, 1]$, it follows from Lemma 4 that

$$\begin{aligned} T_i(u_1, u_2)(t) &= \int_0^1 G_i(t, s) f_i(s, u_1(s), u_2(s)) ds \\ &\geq t^{\alpha_i - 1} \int_0^1 G_i(1, s) f_i(s, u_1(s), u_2(s)) ds \\ &\geq t^{\alpha_i - 1} \|T_i(u_1, u_2)\|_\infty. \end{aligned} \quad (28)$$

Thus, from the above discussion, we conclude that $T: K \times K \longrightarrow P_1 \times P_2$, and then, it obviously shows that T is well defined. The complete continuity of operator T can be given by a standard argument with the help of the Arzela–Arscoli Theorem. We omit the details. \square

Theorem 2. Assume that (H1), (H2), and (H3) hold. Then, we have

- (i) If $r_1 < \beta_1$ and $r_2 < \beta_2$, then (23) has at least a positive solution located in $(U_{\beta_1} \setminus \overline{U_{r_1}}) \times (U_{\beta_2} \setminus \overline{U_{r_2}})$, where $U_{r_i} = \{u \in P_i: \|u\|_\infty < r_i\}$
- (ii) If $r_1 < \beta_1$ and $r_2 > \beta_2$, then (23) has at least two positive solutions located in $(U_{\beta_1} \setminus \overline{U_{r_1}}) \times U_{\beta_2}$ and $(U_{\beta_1} \setminus \overline{U_{r_1}}) \times (U_{r_2} \setminus \overline{U_{\beta_2}})$
- (iii) If $r_1 > \beta_1$ and $r_2 < \beta_2$, then (23) has at least two positive solutions located in $U_{\beta_1} \times (U_{\beta_2} \setminus \overline{U_{r_2}})$ and $(U_{r_1} \setminus \overline{U_{\beta_1}}) \times (U_{\beta_2} \setminus \overline{U_{r_2}})$
- (iv) If $r_1 > \beta_1$ and $r_2 > \beta_2$, then (23) has at least three positive solutions located in $U_{\beta_1} \times U_{\beta_2}$, $U_{\beta_1} \times (U_{r_2} \setminus \overline{U_{\beta_2}})$, $(U_{r_1} \setminus \overline{U_{\beta_1}}) \times U_{\beta_2}$, $(U_{r_2} \setminus \overline{U_{\beta_2}}) \times (U_{r_1} \setminus \overline{U_{\beta_1}})$

Proof. It follows from Lemma 5 that the existence of a positive solution of problem (23) is equivalent to the existence of a nontrivial fixed point of T in $P_1 \times P_2$. Let $W_i = \{u \in P_i: \|u\|_\infty < R_i\}$.

First, note that if $u = (u_1, u_2) \in \overline{W_1 \times W_2}$ with $\|u_1\|_\infty = r_1$, then $\|u_2\|_\infty \leq R_2$ and by the definition of P_1 ,

$$r_1 t^{\alpha_1 - 1} \leq u_1(t) \leq r_1, \quad 0 \leq u_2(t) \leq R_2, \quad t \in [0, 1]. \quad (29)$$

In the following, we conclude that for $i \in \{1, 2\}$, the following properties hold:

$$\begin{aligned} \lambda u_i &\neq T_i u, & \text{for } \|u_i\|_\infty = \beta_i, \lambda \geq 1; \\ u_i &\neq T_i u + \mu t^{\alpha_i - 1}, & \text{for } \|u_i\|_\infty = r_i, \mu \geq 0, \end{aligned} \quad (30)$$

guaranteeing the validity of Theorem 1.

In fact, if $\|u_1\|_\infty = \beta_1$ and $\lambda u_1 = T_1 u$ for a $\lambda \geq 1$, then by (H2),

$$\begin{aligned} u_1(t) &\leq \lambda u_1(t) = (T_1 u_1)(t) \leq \int_0^1 G_1(1, s) f_1(s, u_1(s), u_2(s)) ds \\ &< M_1 \beta_1 \int_0^1 G_1(1, s) ds = \beta_1, \\ &t \in [0, 1], \end{aligned} \quad (31)$$

whence, in particular, we conclude $\beta_1 < \beta_1$, a contradiction. Now, if $u_1 = T_1 u + \mu_1 t^{\alpha_1 - 1}$ for $\|u_1\|_\infty = r_1$ and $\mu_1 \geq 0$, then by (H2), we obtain

$$\begin{aligned} u_1(t) &= (T_1 u)(t) + \mu_1 t^{\alpha_1 - 1} \geq \int_{\delta_1}^1 G_1(t, s) f_1(s, u_1(s), u_2(s)) ds \\ &\geq \delta_1^{\alpha_1 - 1} \int_{\delta_1}^1 G_1(1, s) f_1(s, u_1(s), u_2(s)) ds \\ &> \delta_1^{\alpha_1 - 1} \int_{\delta_1}^1 G_1(1, s) N_1 r_1 ds = r_1, \end{aligned} \quad (32)$$

for all $t \in [0, 1]$. This yields the contradiction $r_1 < r_1$. Hence, (30) holds for $i = 1$. Similarly, (30) is true for $i = 2$. \square

Example 1. Consider the following integral boundary value problem of fractional differential systems:

$$\begin{cases} D_{0+}^{11/4} u_1(t) = \sqrt{u_1(t)} \arctan(u_1(t) + 1) \left(\frac{1}{(u_2(t) + 1)^2} + 100 \right), \\ D_{0+}^{5/2} u_2(t) = \frac{1}{2} \sin t + \frac{1}{2} \cos u_1(t) u_2(t) + u_2(t) + 3, \\ u_1(0) = 0, \quad D_{0+}^{3/4} u_1(0) = 0, D_{0+}^{5/4} u_1(1) = \int_0^1 D_{0+}^{5/4} u_1(\tau) d\tau, \\ u_2(0) = 0, \quad D_{0+}^{1/2} u_2(0) = 0, D_{0+}^{3/2} u_2(1) = \frac{1}{2} D_{0+}^{3/2} u_2\left(\frac{1}{2}\right). \end{cases} \quad (33)$$

Then, (33) has at least two positive solutions (u_1, v_1) and (u_2, v_2) with $(u_1, v_1) \in (U_{100} \setminus \overline{U_{0.72}}) \times U_5$ and $(u_2, v_2) \in (U_{100} \setminus \overline{U_{0.72}}) \times (U_{7100} \setminus \overline{U_5})$.

To see this, we will apply Theorem 2 with

$$\alpha_1 = \frac{11}{4},$$

$$\gamma_1 = \frac{5}{4},$$

$$\alpha_2 = \frac{5}{2},$$

$$\gamma_2 = \frac{3}{2},$$

$$f_1(t, u_1, u_2) = \sqrt{u_1} \arctan(u_1 + 1) \left(\frac{1}{(u_2 + 1)^2} + 100 \right),$$

$$f_2(t, u_1, u_2) = \frac{1}{2} \sin t + \frac{1}{2} \cos u_1 u_2 + \frac{u_2^2}{10}$$

$$A_1(t) = t,$$

$$A_2(t) = \begin{cases} \frac{1}{2}, & t \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < t \leq 1. \end{cases}$$

(34)

Clearly,

$$1 - \int_0^1 t^{\alpha_1 - \gamma_1 - 1} dA_1(t) = 1 - \int_0^1 t^{1/2} dt = \frac{1}{3} > 0,$$

(35)

$$1 - \int_0^1 t^{\alpha_2 - \gamma_2 - 1} dA_2(t) = 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

Thus, (H3) holds.

Take

$$G_1(t, s) = \frac{1}{\Gamma(11/4)} \begin{cases} t^{7/4} (1-s)^{1/2} (1+2s), & 0 \leq t \leq s \leq 1, \\ t^{7/4} (1-s)^{1/2} (1+2s) - (t-s)^{7/4}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$G_2(t, s) = \frac{1}{\Gamma(5/2)} \begin{cases} 2t^{3/2} p_2(s), & 0 \leq t \leq s \leq 1, \\ 2t^{3/2} p_2(s) - (t-s)^{3/2}, & 0 \leq s \leq t \leq 1, \end{cases}$$

$$p_1(s) = \frac{1+2s}{3},$$

$$p_2(s) = \begin{cases} \frac{1}{2}, & s \leq \frac{1}{2}, \\ 1, & \frac{1}{2} < s \leq 1. \end{cases}$$

(36)

Let $\delta_1 = \delta_2 = (1/4)$. By simple computation, we have

$$h_1 \approx 0.0884,$$

$$h_2 = 0.125,$$

$$M_1 \approx 1.9231,$$

$$M_2 \approx 1.2085,$$

$$N_1 \approx 24.4377,$$

$$N_2 \approx 10.0794.$$

(37)

We choose $r_1 = 0.72$, $\beta_1 = 100$, $r_2 = 7100$, and $\beta_2 = 5$. Then, $R_1 = 100$, $R_2 = 7100$,

$$f_1(t, u_1, u_2) > 25\pi \sqrt{h_1 r_1} > N_1 r_1,$$

$$(t, u_1, u_2) \in [\delta_1, 1] \times [h_1 r_1, r_1] \times [0, R_2],$$

$$f_1(t, u_1, u_2) < \frac{101\pi}{2} \sqrt{\beta_1} < M_1 \beta_1,$$

$$(t, u_1, u_2) \in [0, 1] \times [0, \beta_1] \times [0, R_2],$$

$$f_2(t, u_1, u_2) > \frac{3}{2} + \frac{h_2^2 r_2^2}{10} > N_2 r_2,$$

$$(t, u_1, u_2) \in [\delta_2, 1] \times [0, R_1] \times [h_2 r_2, r_2],$$

$$f_2(t, u_1, u_2) < 3 + \frac{\beta_2^2}{10} < M_2 \beta_2,$$

$$(t, u_1, u_2) \in [0, 1] \times [0, R_1] \times [0, \beta_2].$$

(38)

Consequently, (H2) holds with $r_1 < \beta_1$ and $r_2 > \beta_2$, and our conclusion follows from Theorem 2.

4. Conclusions

In this paper, we investigate the existence and multiplicity of positive solutions for the integral boundary value problem of higher-order fractional differential systems. This result is based on an extension of the Krasnosel'skiĭ's fixed-point theorem due to Radu Precup and Jorge Rodriguez-Lopez in [46]. We rewrite the original fractional differential systems as equivalent fractional integral systems. With the help of properties of Green's function, we obtain some sufficient conditions of existence and multiplicity of positive solutions. Finally, an example is presented to illustrate the effectiveness of the main result. The interesting point is that the integral boundary condition is dependent on the lower-order fractional derivative.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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