# Well-Posed and Ill-Posed Boundary Value Problems for PDE 2013 

Guest Editors: Allaberen Ashyralyev, Sergey Piskarev, Valery Covachev, Ravshan Ashurov, Hasan Ali Yurisever, and Abdullah Said Erdoqan


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## Abstract and Applied Analysis

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## Editorial

# Well-Posed and Ill-Posed Boundary Value Problems for PDE 2013 

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The studies of well-posed and ill-posed boundary value problems for partial differential equations are driven not only by a theoretical interest but also by the fact of several phenomena in engineering and various fields of physics and applied sciences. The present special issue is devoted to the publication of high-quality research papers in the fields of the study of analytic and numerical methods for solutions of well-posed and ill-posed boundary value problems for partial differential equations.

The issue covers a wide variety of problems for different classes of partial differential equations. The topics discussed in the contributed papers are traditional for qualitative theory of differential equations. The issue contains papers on the existence, uniqueness, and asymptotic behavior of a classical solution to the initial and Neumann boundary value problem for a class of nonlinear parabolic equations of Monge-Ampere type and on the blow-up phenomena for a modified twocomponent Dullin-Gottwald-Holm shallow water system. Some new blow-up criteria of strong solutions involving the density and suitable integral form of the momentum are established. Furthermore, an analytical solution for effect of magnetic field and initial stress on an infinite generalized thermoelastic rotating nonhomogeneous diffusion in a medium subjected to certain boundary conditions is studied.

The chemical potential is also assumed to be a known function of time at the boundary of the cavity. The analytical expressions for the displacements, stresses, temperature, concentration, and chemical potential are obtained. Comparison was made with the results obtained in the presence and absence of diffusion. The results indicate that the effects of nonhomogeneity, rotation, magnetic field, relaxation time and diffusion are very pronounced.

A number of papers are concerned with well-posedness of difference schemes for approximate solutions of partial differential equations. Interesting stability and coercive stability estimates are established for solutions of the first and second order of accuracy difference schemes for the inverse problem of the multidimensional elliptic equation with overdetermination. The algorithm for approximate solution is tested in a two-dimensional inverse elliptic problem. Moreover, stability estimates are established for the solution of the first order of accuracy difference scheme for the approximate solution of the determination of a control parameter problem for Schrodinger equations. One paper collected in this special issue addresses construction and investigation of a third order of accuracy absolutely stable difference schemes for the nonlocal boundary value hyperbolic problem. The stability estimates for the solution of this difference scheme are
established. Two authors deal with analysis of the block-grid method for the solution of Laplace's equation on polygons with a slit. The error estimates obtained for solving Laplace's boundary value problem on polygons by the block-grid method contain constants that are difficult to calculate accurately. Therefore, the experimental analysis of the method could be essential. The real characteristics of the block-grid method for solving Laplace's equation on polygons with a slit are analysed by experimental investigations. The numerical results obtained show that the order of convergence of the approximate solution is the same as in the case of a smooth solution. To illustrate the singular behaviour around the singular point, the shape of the highly accurate approximate solution and the figures of its partial derivatives up to second order are given in the singular part of the domain. Finally a highly accurate formula is given to calculate the stress intensity factor, which is an important quantity in fracture mechanics.

The issue contains papers on the spectrum of differential operators and its applications. The nature of the spectrum of the periodic problem for the heat equation with a lowerorder term and with a deviating argument is investigated. A significant influence of the lower-order term on the correct solvability of this problem is obtained. A criterion for the strong solvability of the above-mentioned problem is obtained. One paper deals with a Dirac system with transmission condition and eigenparameter in boundary condition. Some spectral properties of the problem are studied. Finally, spectral properties of Sturm-Liouville type problems with interior singularities are investigated. Special solutions of the homogeneous equation are presented.

Finally, the theory of contrasting structures in singularly perturbed boundary problems for nonlinear parabolic partial differential equations is applied to the research of formation of steady state distributions of power within the nonlinear power-society model. The interpretations of the solutions to the equation are presented in terms of applied model. The possibility theorem for the problem of getting the solution having some preassigned properties by means of parametric control is proved.

The volume is a collection of 12 accepted manuscripts by 23 authors. The selection of the papers included in this volume was based on an international peer review procedure. The accepted manuscripts examine wide ranging and cutting edge developments in various areas of well-posed and illposed boundary value problems for partial differential equations. The papers give a taste of current research. We feel the variety of topics will be of interest to both graduate students and researchers.

Further, we are very grateful to all authors for sending their valuable papers for the publication in the present special issue.

Allaberen Ashyralyev<br>Sergey Piskarev<br>Valery Covachev<br>Ravshan Ashurov<br>Hasan Ali Yurtsever Abdullah Said Erdogan

# An Analytical Solution for Effect of Magnetic Field and Initial Stress on an Infinite Generalized Thermoelastic Rotating Nonhomogeneous Diffusion Medium 

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#### Abstract

The problem of generalized magneto-thermoelastic diffusion in an infinite rotating nonhomogeneity medium subjected to certain boundary conditions is studied. The chemical potential is also assumed to be a known function of time at the boundary of the cavity. The analytical expressions for the displacements, stresses, temperature, concentration, and chemical potential are obtained. Comparison was made between the results obtained in the presence and absence of diffusion. The results indicate that the effect of nonhomogeneity, rotation, magnetic field, relaxation time, and diffusion is very pronounced.


## 1. Introduction

Diffusion can be defined as the spontaneous migration of substances from regions of high concentration to regions of low concentration. There is now a great deal of interest in the study of this phenomenon due to its many applications in geophysics and industrial applications. Thermodiffusion in the solids is one of the transport processes which has great practical importance. Thermodiffusion in an elastic solid is due to the coupling of the fields of temperature, mass diffusion, and that of strain. This mater has attracted the attention of many researchers such as [1-5]. Wave propagation in rotating and nonhomogeneous media was studied by Abd-Alla et al. [6-8]. The extended thermoelasticity theory, introducing one relaxation time in the thermoelastic process, was proposed by Lord and Shulman [9]. In this theory, a modified law of heat conduction including both the heat flux and its time derivative replaces conventional Fourier's law. The heat equation associated with this is a hyperbolic one and hence automatically eliminates the paradox of infinite speeds of propagation inherent in the coupled theory of thermoelasticity. This theory was extended by Dhaliwal and Sherief [10] to include the anisotropic case. Abd-Alla and

Mahmoud [11] investigated the magneto-thermoelastic problem in rotating nonhomogeneous orthotropic hollow cylinder under the hyperbolic heat conduction model. Mahmoud [12] investigated wave propagation in cylindrical poroelastic dry bones.

Kumar and Devi [13] studied deformation in porous thermoelastic material with temperature dependent properties. Othman et al. [14] presented the study of the two-dimensional problems of generalized thermoelasticity with one relaxation time with the modulus of elasticity being dependent on the reference temperature for nonrotating and rotating medium, respectively. Kumar and Gupta [15] investigated deformation due to inclined load in an orthotropic micropolar thermoelastic medium with two relaxation times. The temperaturerate dependent theory of thermoelasticity, which takes into account two relaxation times, was developed by Green and Lindsay [16]. Abd-Alla et al. [17, 18] investigated radial vibrations in a nonhomogeneous orthotropic elastic medium subjected to rotation and gravity field. Sherief et al. [19] developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows the finite speed of propagation waves. Sherief and Saleh [20] investigated the problem of a thermoelastic half-space in the context
of the theory of generalized thermoelastic diffusion with one relaxation time. The reflection of SV waves from the free surface of an elastic solid in generalized thermoelastic diffusion was discussed by Singh [21]. Kumar and Kansal [22] discussed the propagation of Lamb waves in transversely isotropic thermoelastic diffusive plates. Thermomechanical response of generalized thermoelastic diffusion with one relaxation time due to time harmonic sources was discussed by Ram et al. [23]. Aouadi [24] examined the thermoelastic diffusion problem for an infinite elastic body with spherical cavity. Abd-Alla and Mahmoud [25] investigated analytical solution of wave propagation in nonhomogeneous orthotropic rotating elastic media. Othman et al. [26] discussed the effect of diffusion in a two-dimensional problem of generalized thermoelasticity with Green-Naghdi theory. Xia et al. [27] studied the influence of diffusion on generalized thermoelastic problems of infinite body with a cylindrical cavity. Deswal and Kalkal [28] studied the two-dimensional generalized electromagneto-thermoviscoelastic problem for a half-space with diffusion. Abd-Alla and Abo-Dahab [29] found the time-harmonic sources in a generalized magneto-thermo-viscoelastic continuum with and without energy dissipation. Mahmoud [30] discussed influence of rotation and generalized magnetothermoelastic on Rayleigh waves in a granular medium under effect of initial stress and gravity field. Abd-Alla et al. [31,32] studied the generalized magnetothermoelastic Rayleigh waves in a granular medium under the influence of a gravity field and initial stress.

In the present investigation, the temperature, displacements, stresses, diffusion, and concentration as well as chemical potential are obtained in the physical domain using the harmonic vibrations. Also, study of the interaction between the processes of elasticity, nonhomogeneity, rotation, magnetic field, initial stress, heat, and diffusion in an infinite elastic solid with a spherical cavity in the context of the theory of generalized thermoelastic diffusion is presented.

## 2. Formulation of the Problem

Consider a perfect electric conductor and linearized Maxwell equations governing the electromagnetic field in the absence of the displacement current (SI) in the form as in Kraus [33]. Applying an initial magnetic field vector $\vec{H}\left(0,0, H_{0}\right)$ in spherical coordinates $(r, \theta, \phi), \vec{u}=(u(r, t), 0,0)$. One will consider a nonhomogeneous, isotropic medium, occupying the region $a \leq r \leq b$, where a is the radius of the spherical cavity. The strain tensor has the following components:

$$
\begin{gather*}
e_{i j}=\frac{1}{2}\left(u_{i, j}+u_{j, i}\right),  \tag{1a}\\
\omega_{i j}=\frac{1}{2}\left(u_{j, i}-u_{i, j}\right),  \tag{lb}\\
e_{k k}=\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right), \quad i, j, k=1,2,3,  \tag{1c}\\
e_{r r}=\frac{\partial u}{\partial r}, \quad e_{\theta \theta}=e_{\phi \phi}=\frac{u}{r} . \tag{1d}
\end{gather*}
$$

The cubical dilatation is given by $\left(1 / r^{2}\right)(\partial / \partial r)\left(r^{2} u\right)$, where the nonvanishing displacement component is the radial one $u_{r}=u(r, t)$. The elastic medium is rotating uniformly with an angular velocity $\vec{\Omega}=\Omega \vec{n}$, where $\vec{n}$ is a unit vector representing the direction of the axis of rotation. The displacement equation of motion in the rotating frame has two additional terms: $\vec{\Omega} \times(\vec{\Omega} \times \vec{u})$ which is the centripetal acceleration due to time varying motion only, and $2 \vec{\Omega} \times \overrightarrow{\dot{u}}$ is the Coriolis acceleration, where $\vec{\Omega}=(0, \Omega, 0)$. Following Sherief's theory of generalized thermoelastic diffusion [19] and Sherief and Saleh [20], one is going to study an isotropic nonhomogeneous elastic medium which suffers thermal shock. Due to spherical symmetry, the stress-displacement-temperature-diffusion relation or constitutive equations are given by

$$
\begin{align*}
\sigma_{r r}= & \left(2 \mu+P_{1}\right) \frac{\partial u}{\partial r}+\left(\lambda+P_{1}\right) \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)  \tag{2a}\\
& -\beta_{1}\left(\theta-\theta_{0}\right)-\beta_{2} C, \\
\sigma_{\phi \phi}=\sigma_{\theta \theta}= & \left(2 \mu+P_{1}\right) \frac{u}{r}+\left(\lambda+P_{1}\right) \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)  \tag{2b}\\
& -\beta_{1}\left(\theta-\theta_{0}\right)-\beta_{2} C .
\end{align*}
$$

The chemical-displacement-temperature-diffusion relation is given by

$$
\begin{equation*}
P=-\beta_{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+b C-c\left(\theta-\theta_{0}\right) . \tag{3}
\end{equation*}
$$

The governing equation for an isotropic nonhomogeneous elastic solid with generalized magneto-thermoelastic diffusion under effect of rotation is given by

$$
\begin{align*}
& \frac{\partial\left(1 / r^{2}\right)(\partial / \partial r)\left(r^{2} u\right)}{\partial r}-\frac{\beta_{2}}{h_{0}} \frac{\partial C}{\partial r}-\frac{\beta_{1}}{h_{0}} \frac{\partial\left(\theta-\theta_{0}\right)}{\partial r}+F_{r}  \tag{4}\\
& =\frac{\rho}{h_{0}}\left[\frac{\partial^{2} u}{\partial t^{2}}-\Omega^{2} u-2 \Omega \frac{\partial u}{\partial t}\right],
\end{align*}
$$

where $\lambda$ and $\mu$ are Lame's elastic constants, $\delta_{i j}$ is Kronecker's delta, $P_{1}$ is the initial stress, $\rho$ is the density of the medium, and $\vec{F}$ is defined as Lorentz's force which may be written as

$$
\begin{equation*}
\vec{F}=\mu_{e}(\vec{J} \times \vec{H})=\left(\mu_{e} H_{0}^{2} \frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{2 u}{r}\right), 0,0\right) \tag{5}
\end{equation*}
$$

where $\mu_{e}$ is the magnetic permeability, $\vec{H}$ is the magnetic field vector, $\vec{J}$ is the electric current density, $\vec{u}$ is the displacement vector, and $t$ is the time.

Equation of heat conduction is given by

$$
\begin{equation*}
K \nabla^{2} \theta=\left(\frac{\partial}{\partial t}+\tau \frac{\partial^{2}}{\partial t^{2}}\right)\left(\rho c_{v} \theta+\theta_{0} \beta_{1} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+c \theta_{0} C\right), \tag{6}
\end{equation*}
$$

where the Laplacian operator $\nabla^{2}$ is given by $\nabla^{2}=\left(\partial^{2} / \partial r^{2}\right)+$ $(2 / r)(\partial / \partial r)$ and $h_{0}=2 \mu+\lambda+\left(P_{1} / 2\right)+\mu_{e} H_{0}^{2}, h_{0}$ is the
coefficient of linear diffusion expansion, $K$ is the thermal conductivity, $\theta$ is the absolute temperature, $\theta_{0}$ is the initial uniform temperature, and $\left|\left(\theta-\theta_{0}\right) / \theta_{0}\right| \ll 1$.

Equation of conservation of mass diffusion may be written as

$$
\begin{equation*}
D \beta_{2} \nabla^{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+D c \nabla^{2} \theta+\left(\frac{\partial}{\partial t}+\tau \frac{\partial^{2}}{\partial t^{2}}\right) C=D b \nabla^{2} C, \tag{7}
\end{equation*}
$$

where $\tau$ is the diffusion relaxation time, $\tau_{0}$ is the thermal relaxation time, $\alpha_{t}$ is the coefficient of linear thermal expansion, and $\theta_{0}$ is constant, where $\beta_{1}=(3 \lambda+2 \mu) \alpha_{t}, \beta_{2}=$ $(3 \lambda+2 \mu) \alpha_{c}, \sigma_{i j}$ are the components of the stress tensor, $\tau_{i j}$ are the components of stress tensor, $b$ and $c$ are the measures of thermodiffusion and diffusive effects, $C$ is the concentration, $C_{v}$ is the specific heat at constant strain, $D$ is the diffusive coefficient, and $e_{i j}$ are the components of the strain tensor. The thermal relaxation time $\tau_{0}$ will ensure that the heat conduction equation will predict finite speed of heat propagation. The diffusion relaxation time $\tau$, which will ensure the equation satisfied by the concentration $C$, will also predict finite speed of propagation of matter from one medium to the other.

## 3. Dimensionless Quantities

Introduce the following nondimensional parameters:

$$
\begin{gather*}
r^{*}=c_{1} \eta_{0} r, \quad u^{*}=c_{1} \eta_{0} u, \quad T=\frac{\beta_{1}\left(\theta-\theta_{0}\right)}{h_{0}}, \\
C^{*}=\frac{\beta_{2} C}{h_{0}}, \quad \Omega^{*}=\frac{\Omega}{c_{1}^{2} \eta_{0}}, \quad \sigma_{i j}^{*}=\frac{\sigma_{i j}}{h_{0}}  \tag{8}\\
P^{*}=\frac{P}{\beta_{2}}, \quad t^{*}=c_{1}^{2} \eta_{0} t, \quad \tau_{0}^{*}=c_{1}^{2} \eta_{0} \tau_{0} \\
\tau^{*}=c_{1}^{2} \eta_{0} \tau, \quad \eta_{0}=\frac{\rho c_{v}}{K}, \quad c_{1}^{2}=\frac{h_{0}}{\rho}
\end{gather*}
$$

The elastic constants $\lambda, \mu$ and the density $\rho$ of nonhomogeneous material in form [32] are as follows:

$$
\begin{gather*}
\lambda=r^{2 m} \lambda_{0}, \quad \mu=r^{2 m} \mu_{0}, \quad \mu_{h}=r^{2 m} \mu_{0} \\
\rho=r^{2 m} \rho_{0}, \quad p^{*}=p_{0}^{*} r^{2 m} . \tag{9}
\end{gather*}
$$

Using the above non-dimensional parameters and (9) in (10)-(14), the non-dimensional system becomes

$$
\begin{align*}
& \frac{\partial\left(1 / r^{2}\right)(\partial / \partial r)\left(r^{2} u\right)}{\partial r}-\frac{\partial C}{\partial r}-\frac{\partial T}{\partial r}=\frac{\partial^{2} u}{\partial t^{2}}-\Omega^{2} u-2 \Omega \frac{\partial u}{\partial t},  \tag{10}\\
& \nabla^{2} T=\left(\frac{\partial}{\partial t}+\tau_{0} \frac{\partial^{2}}{\partial t^{2}}\right)\left(T+\varepsilon_{1} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+\varepsilon_{2} C\right),  \tag{11}\\
& \nabla^{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+h_{4} \nabla^{2} T+h_{6}\left(\frac{\partial}{\partial t}+\tau \frac{\partial^{2}}{\partial t^{2}}\right) C=h_{5} \nabla^{2} C, \tag{12}
\end{align*}
$$

$$
\begin{align*}
\sigma_{r r} & =h_{1} \frac{\partial u}{\partial r}+h_{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)-T-C  \tag{13a}\\
\sigma_{\theta \theta} & =h_{1} \frac{u}{r}+h_{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)-T-C  \tag{13b}\\
\sigma_{\phi \phi} & =h_{1} \frac{u}{r}+h_{2} \frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)-T-C  \tag{13c}\\
P & =-\frac{1}{r^{2}} \frac{\partial}{\partial r}\left(r^{2} u\right)+h_{3} C-h_{4} T \tag{14}
\end{align*}
$$

where

$$
\begin{gather*}
\varepsilon_{1}=\frac{\beta_{1}^{2} T_{0} m}{h_{0} \rho c_{v}}, \quad \varepsilon_{2}=\frac{\beta_{1} c T_{0} h_{0}(m+2)}{\beta_{2}}, \quad h_{1}=\frac{2 \mu}{h_{0}}, \\
h_{2}=\frac{\lambda}{h_{0}}, \quad h_{3}=\frac{m b h_{0}}{\beta_{2}^{2}}, \quad h_{4}=\frac{c h_{0}}{\beta_{1} \beta_{2}}, \\
h_{5}=\frac{D b h_{0}}{\beta_{2}}, \quad h_{6}=\frac{2 m+h_{0}}{\beta_{2}^{2} D \eta_{0}} . \tag{15}
\end{gather*}
$$

## 4. Boundary Conditions

The nonhomogeneous initial conditions are supplemented by the following boundary conditions. The cavity surface is traction free:

$$
\begin{equation*}
\sigma_{r r}(r, t)+\bar{\tau}_{r r}(r, t)=0, \quad r=a \tag{16a}
\end{equation*}
$$

The cavity surface is subjected to a thermal shock

$$
\begin{equation*}
T(a, t)=T_{0} H(t) \tag{16b}
\end{equation*}
$$

where $H(t)$ is the Heaviside unit step function. The chemical potential is also assumed to be a known function of time at the cavity surface:

$$
\begin{equation*}
P(a, t)=P_{0} H(t), \quad P_{0} \text { is real constant. } \tag{16c}
\end{equation*}
$$

The displacement function is as follows:

$$
\begin{equation*}
u(r, t)=0, \quad r=a . \tag{16d}
\end{equation*}
$$

## 5. Solution of the Problem

In this section, one obtains the analytical solution of the problem for a spherical region with boundary conditions by taking the harmonic vibrations. One assumes that the solution of (10)-(12) as follows:

$$
\begin{array}{ll}
C(r, t)=C^{\prime}(r) e^{i \omega t}, & T(r, t)=T^{\prime}(r) e^{i \omega t} \\
u(r, t)=u^{\prime}(r) e^{i \omega t} . & e(r, t)=E^{\prime}(r) e^{i \omega t} \tag{17b}
\end{array}
$$



$$
\begin{aligned}
& t=0.3, \Omega=1.2, \tau_{0}=0.4 \\
& m=0.4 \\
& m=m=0.8 \\
& \cdots \cdots \quad m=1.5
\end{aligned}
$$



$$
\begin{aligned}
& t=0.3, m=0.8, \tau_{0}=0.4 \\
& -\quad m=0.6 \\
& --\quad m=1.2 \\
& \cdots \cdots \quad m=1.8
\end{aligned}
$$

(a)
(b)

Figure 1: Variation of chemical potential $P$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).


$$
\begin{aligned}
& t=0.3, \Omega=1.2, \tau_{0}=0.4 \\
& -m=0.4 \\
& -m=0.8 \\
& \cdots \cdots m=1.5
\end{aligned}
$$

(a)


$$
\begin{aligned}
& t=0.3, m=0.8, \tau_{0}=0.4 \\
& -\quad \Omega=0.6 \\
& --\quad \Omega=1.2 \\
& \cdots \cdots=1.8
\end{aligned}
$$

(b)

FIgURe 2: Variation of concentration $C$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).
where $e=(\partial u / \partial r)+(2 u / r)=\left(1 / r^{2}\right)(\partial / \partial r)\left(r^{2} u\right)$.
Substituting (17a) and (17b) into (10)-(12) yields

$$
\begin{gather*}
\frac{\partial E^{\prime}}{\partial r}-\frac{\partial C^{\prime}}{\partial r}-\frac{\partial T^{\prime}}{\partial r}=\beta u^{\prime}  \tag{18}\\
\nabla^{2} T^{\prime}=k_{1}\left(T^{\prime}+\varepsilon_{1} E^{\prime}+\varepsilon_{2} C^{\prime}\right),  \tag{19}\\
\nabla^{2} E^{\prime}+h_{4} \nabla^{2} T^{\prime}+h_{6} k_{2} G=h_{5} \nabla^{2} C^{\prime} . \tag{20}
\end{gather*}
$$

Applying the operator Laplacian operator $\nabla^{2}$ to (18), we obtain

$$
\begin{equation*}
\left(\nabla^{2}-\beta\right) E^{\prime}=\nabla^{2} C^{\prime}+\nabla^{2} T^{\prime} \tag{21}
\end{equation*}
$$

From (19)-(21), we obtain

$$
\begin{equation*}
\left(\nabla^{6}+b_{1} \nabla^{4}+b_{2} \nabla^{2}+b_{3}\right)\left(E^{\prime}, T^{\prime}, C^{\prime}\right)=0 \tag{22}
\end{equation*}
$$

where

$$
\begin{align*}
b_{1}= & \frac{-1}{\left(h_{5}-1\right)} \\
& \times\left[k_{2} h_{6}+k_{1}\left(h_{5}+\varepsilon_{2} h_{4}\right)+\beta h_{5}+k_{1}\left(\varepsilon_{1} h_{5}+\varepsilon_{2}\right)\right], \\
b_{2}= & \frac{1}{\left(h_{5}-1\right)} \\
& \times\left[k_{1} k_{2} h_{6}+\beta\left(k_{2} h_{6}+k_{1}\left(h_{5}+\varepsilon_{2} h_{4}\right)\right)+k_{1} k_{2} \varepsilon_{1} h_{6}\right], \\
& b_{3}=\frac{-\beta k_{1} k_{2} h_{6}}{\left(h_{5}-1\right)}, \quad k_{1}=i \omega\left(1+i \omega \tau_{0}\right), \\
k_{2}= & i \omega(1+i \omega \tau), \quad \beta=-\left(\omega^{2}+\Omega^{2}+2 i \omega \Omega\right) . \tag{23}
\end{align*}
$$



FIGURE 3: Variation of temperature $\theta$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).

$t=0.3, \Omega=1.2, \tau_{0}=0.4$

- $m=0.4$
--- $m=0.8$
…. $m=1.5$


$$
\begin{aligned}
& t=0.3, m=0.8, \tau_{0}=0.4 \\
& -\quad m=0.6 \\
& --m=1.2 \\
& \cdots \cdots \quad m=1.8
\end{aligned}
$$

(b)

Figure 4: Variation of displacement $u$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).

Equation (22) can be factorized as

$$
\begin{equation*}
\left(\nabla^{2}+Q_{1}^{2}\right)\left(\nabla^{2}+Q_{2}^{2}\right)\left(\nabla^{2}+Q_{3}^{2}\right)\left(E^{\prime}, T^{\prime}, C^{\prime}\right)=0 \tag{24}
\end{equation*}
$$

where $Q_{1}^{2}, Q_{2}^{2}$, and $Q_{3}^{2}$ are the roots of the characteristic equation

$$
\begin{equation*}
Q^{6}+b_{1} Q^{4}+b_{2} Q^{2}+b_{3}=0 \tag{25}
\end{equation*}
$$

The solution of (24) which is bounded at infinity is given by

$$
\begin{align*}
& T^{\prime}(r, \omega)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}(\omega) K_{1 / 2}\left(Q_{j} r\right), \\
& E^{\prime}(r, \omega)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}^{\prime}(\omega) K_{1 / 2}\left(Q_{j} r\right),  \tag{26}\\
& C^{\prime}(r, \omega)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}^{\prime \prime}(\omega) K_{1 / 2}\left(Q_{j} r\right)
\end{align*}
$$

where $B_{j}, B_{j}^{\prime}$, and $B_{j}^{\prime \prime}$ are parameters depending only on $\omega$ and $K_{1 / 2}$ is the modified spherical Bessel function of the second


Figure 5: Variation of radial stress $\sigma_{r r}$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).


Figure 6: Variation of tangential stress $\sigma_{\phi \phi}$ with radius $r$ (thermoelastic diffusion nonhomogeneity medium).
kind of order 1/2. Compatibility between (26) along with (19) and (20) will give rise to

Substituting (26) into (17a) and (17b), we obtain

$$
\begin{array}{cl}
B_{j}^{\prime}(\omega)=\frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)} B_{j}(\omega), & e(r, t)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}(\omega) K_{1 / 2}\left(Q_{j} r\right) e^{i \omega t}, \\
B_{j}^{\prime \prime}(\omega)=\frac{\varepsilon_{1} k_{1} h_{4} Q_{j}^{2}+Q_{j}^{2}\left(Q_{j}^{2}-k_{1}\right)}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)} B_{j}(\omega) . & C(r, t)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}^{\prime \prime}(\omega) K_{1 / 2}\left(Q_{j} r\right) e^{i \omega t},
\end{array}
$$



Figure 7: Variation of displacement $u$ with radius $r$ (thermoelastic nonhomogeneity medium).


Figure 8: Variation of temperature $\theta$ with radius $r$ (thermoelastic nonhomogeneity medium).

Integrating both sides of (29) from $r$ to infinity and assuming that $u(r, t)$ vanishes at infinity, we obtain

$$
\begin{gathered}
u(r, t)=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} \frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)} \\
\times B_{j}(\omega) K_{3 / 2}\left(Q_{j} r\right) e^{i \omega t}
\end{gathered}
$$



Figure 9: Variation of radial stress $\sigma_{r r}$ with radius $r$ (thermoelastic nonhomogeneity medium).


Figure 10: Variation of tangential stress $\sigma_{\phi \phi}$ with radius $r$ (thermoelastic nonhomogeneity medium).

From (13a), (13b), and (13c)-(14), we get

$$
\begin{aligned}
& \sigma_{r r}=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}(\omega) \frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)} \\
& \times\left\{\left[\left(h_{1}+h_{2}\right)-\frac{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}\right]\right. \\
&\left.\times K_{1 / 2}\left(Q_{j} r\right)-\frac{h_{1}}{r} K_{3 / 2}\left(Q_{j} r\right)\right\} e^{i \omega t},
\end{aligned}
$$

$$
\begin{align*}
& \sigma_{\phi \phi}=\sigma_{\theta \theta} \\
& \begin{aligned}
&=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}(\omega) \\
& \times\left\{\frac{h_{1}}{r} \frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}\right. \\
& \times K_{3 / 2}\left(Q_{j} r\right) \\
&+\left[h_{2} \frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}\right. \\
&\left.-1-\frac{\varepsilon_{1} k_{1} h_{4} Q_{j}^{2}+Q_{j}^{2}\left(Q_{j}^{2}-k_{1}\right)}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}\right] \\
&\left.\times K_{1 / 2}\left(Q_{j} r\right)\right\} e^{i \omega t}
\end{aligned}
\end{align*}
$$

$$
\begin{aligned}
& \times K_{1 / 2}\left(Q_{j} a\right)-\left[\frac{h_{1}}{a}+\mu_{e} H_{\phi}^{2}\left(\frac{3}{a^{2}}-Q_{i}^{2}\right)\right] \\
& \times K_{3 / 2}\left(Q_{j} a\right), \\
& W_{j}=\left\{-\frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}\right. \\
&\left.+h_{3} \frac{\varepsilon_{1} k_{1} h_{4} Q_{j}^{2}+Q_{j}^{2}\left(Q_{j}^{2}-k_{1}\right)}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}-h_{4}\right\} \\
& \times K_{1 / 2}\left(Q_{j} a\right), \\
& M=\{ N_{1}\left[W_{3} K_{1 / 2}\left(a Q_{2}\right)-W_{2} K_{1 / 2}\left(a Q_{3}\right)\right] \\
&-N_{2}\left[W_{3} K_{1 / 2}\left(a Q_{1}\right)-W_{1} K_{1 / 2}\left(a Q_{1}\right)\right] \\
&\left.+N_{3}\left[W_{2} K_{1 / 2}\left(a Q_{1}\right)-W_{1} K_{1 / 2}\left(a Q_{2}\right)\right]\right\}, \\
& j=1,2,3 .
\end{aligned}
$$

$$
P=\frac{1}{\sqrt{r}} \sum_{j=1}^{3} B_{j}(\omega)\left\{-\frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}\right.
$$

$$
\left.+h_{3} \frac{\varepsilon_{1} k_{1} h_{4} Q_{j}^{2}+Q_{j}^{2}\left(Q_{j}^{2}-k_{1}\right)}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}-h_{4}\right\} \begin{aligned}
& \text { If we neglect the initial stress and diffusion effects by } \\
& \text { eliminating (3) and (8) and putting } P_{1}=\beta_{2}=C=0 \text { in (4) } \\
& \text { and (6), we get }(T, e), u(r, t), \sigma_{r r}, \sigma_{\phi \phi} \text {, and } \sigma_{\theta \theta}:
\end{aligned}
$$

$$
\begin{equation*}
\times K_{1 / 2}\left(Q_{j} r\right) e^{i \omega t} \tag{32}
\end{equation*}
$$

Using the boundary conditions, we get
where

$$
\begin{aligned}
& B_{1}(\omega)=\left(-N_{2}\left[\sqrt{a} \theta_{0} W_{3}-K_{1 / 2}\left(a Q_{3}\right) p_{0}\right]\right. \\
& \left.+N_{3}\left[\sqrt{a} \theta_{0} W_{2}-K_{1 / 2}\left(a Q_{2}\right) p_{0}\right]\right) \times(M)^{-1}, \\
& B_{2}(\omega)=\left(\sqrt{a} N_{1}\left[\theta_{0} W_{3}-K_{1 / 2}\left(a Q_{3}\right) p_{0}\right]\right. \\
& \left.+\sqrt{a} N_{3}\left[K_{1 / 2}\left(a Q_{1}\right) p_{0}-\theta_{0} W_{1}\right]\right) \times(M)^{-1}, \\
& B_{3}(\omega)=\left(\sqrt{a} N_{1}\left[K_{1 / 2}\left(a Q_{2}\right) p_{0}-\theta_{0} W_{2}\right]\right. \\
& \left.-\sqrt{a} N_{2}\left[K_{1 / 2}\left(a Q_{1}\right) p_{0}-\theta_{0} W_{1}\right]\right) \times(M)^{-1}, \\
& N_{j}=\frac{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)} \\
& \times\left[\left(h_{1}+h_{2}\right)\right. \\
& \left.-\frac{Q_{j}^{2} \varepsilon_{2} k_{1}+k_{1} \varepsilon_{1}\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)}{\left(Q_{j}^{2}-k_{1}\right)\left(h_{5} Q_{j}^{2}-h_{6} k_{2}\right)-\varepsilon_{2} k_{1} h_{4} Q_{j}^{2}}\right] \\
& \eta_{1}=-\left(\ell_{1}+\beta\right), \quad \eta_{2}=\ell_{1}\left(\beta-\varepsilon_{1}\right), \\
& \left(\lambda_{1}^{2}, \lambda_{2}^{2}\right)=\frac{1}{2}\left[\eta_{1} \pm \sqrt{\eta_{1}^{2}-4 \eta_{2}}\right], \\
& u(r, t)=-\left[\frac{r^{2} \lambda_{1}^{2}+2 r \lambda_{1}+2}{r^{2} \lambda_{1}^{3}} A(\omega) e^{-\lambda_{1} r}\right. \\
& \left.+\frac{r^{2} \lambda_{2}^{2}+2 r \lambda_{2}+2}{r^{2} \lambda_{2}^{3}} B(\omega) e^{-\lambda_{2} r}+C(\omega)\right] e^{i \omega t}, \\
& \sigma_{r r}=\left[\alpha_{1}\left(\frac{4+4 r \lambda_{1}+2 r^{2} \lambda_{1}^{2}+r^{3} \lambda_{1}^{3}}{r^{3} \lambda_{1}^{3}}+\left(\alpha_{2}-1\right)\right) A(\omega) e^{-\lambda_{1} r}\right. \\
& +\alpha_{1}\left(\frac{4+4 r \lambda_{2}+2 r^{2} \lambda_{2}^{2}+r^{3} \lambda_{2}^{3}}{r^{3} \lambda_{2}^{3}}+\left(\alpha_{2}-1\right)\right) \\
& \left.\times B(\omega) e^{-\lambda_{2} r}\right] e^{i \omega t},
\end{aligned}
$$

$$
\begin{align*}
\sigma_{\phi \phi}=\sigma_{\theta \theta}=-\{ & \left(\alpha_{1}\left(\frac{r^{2} \lambda_{1}^{2}+2 r \lambda_{1}+2}{r^{3} \lambda_{1}^{3}}\right)-\left(\alpha_{2}-1\right)\right) \\
& \times A(\omega) e^{-\lambda_{1} r} \\
& +\left[\alpha_{1}\left(\frac{r^{2} \lambda_{2}^{2}+2 r \lambda_{2}+2}{r^{3} \lambda_{2}^{3}}\right)-\left(\alpha_{2}-1\right)\right] \\
& \left.\times B(\omega) e^{-\lambda_{2} r}\right\} e^{i \omega t} \tag{35}
\end{align*}
$$

Using the boundary conditions, we obtain

$$
\begin{gather*}
A(\omega)=\frac{-h_{2} T_{0}}{h_{1} e^{-\lambda_{2} a}-h_{2} e^{-\lambda_{1} a}}, \quad B(\omega)=\frac{h_{1} T_{0}}{h_{1} e^{-\lambda_{2} a}-h_{2} e^{-\lambda_{1} a}}, \\
C(\omega)=\frac{\left(h_{2} h_{3}-h_{1} h_{4}\right) \theta_{0}}{h_{1} e^{-\lambda_{2} a}-h_{2} e^{-\lambda_{1} a}} \\
h_{1}=\alpha_{1}\left(\frac{4+4 a \lambda_{1}+2 a^{2} \lambda_{1}^{2}+a^{3} \lambda_{1}^{3}}{a^{3} \lambda_{1}^{3}}\right. \\
\left.+\left(\alpha_{2}-1-\mu_{e} H_{\phi}^{2} \lambda_{1}\right)\right) e^{-\lambda_{1} a}, \\
h_{2}=\alpha_{1}\left(\frac{4+4 a \lambda_{2}+2 a^{2} \lambda_{2}^{2}+a^{3} \lambda_{2}^{3}}{a^{3} \lambda_{2}^{3}}\right. \\
\left.+\left(\alpha_{2}-1-\mu_{e} H_{\phi}^{2} \lambda_{2}\right)\right) e^{-\lambda_{2} a}, \\
h_{3}=\left(\frac{a^{2} \lambda_{1}^{2}+2 a \lambda_{1}+2}{a^{2} \lambda_{1}^{3}}\right) e^{-\lambda_{1} a}, \\
h_{4}=\left(\frac{a^{2} \lambda_{2}^{2}+2 a \lambda_{2}+2}{a^{2} \lambda_{2}^{3}}\right) e^{-\lambda_{2} a} . \tag{36}
\end{gather*}
$$

## 7. Numerical Results and Discussion

For the purposes of numerical evaluations. The copper material was chosen. The constants of the problem given by Aouadi [24], Sokolnikoff [34] and Thomas [35] are

$$
\begin{gather*}
\mu=3.86 \times 10^{10} \mathrm{~kg} / \mathrm{ms}^{3}, \quad \lambda=7.76 \times 10^{10} \mathrm{~kg} / \mathrm{ms}^{3}, \\
\rho=8954 \mathrm{~kg} / \mathrm{m}^{3}, \quad c_{v}=383.1 \mathrm{~J} / \mathrm{kg} \cdot \mathrm{~K}, \\
\alpha_{t}=1.78 \times 10^{-5} \mathrm{~K}^{-1}, \quad \alpha_{c}=1.98 \times 10^{-4} \mathrm{~m}^{3} / \mathrm{kg} \\
k=386 \mathrm{~W} / \mathrm{mK}, \quad D=0.85 \times 10^{8} \mathrm{~kg} \cdot \mathrm{~s} / \mathrm{m}^{3}  \tag{37}\\
T_{0}=293 \mathrm{~K}, \quad c=1.2 \times 10^{4} \mathrm{~m}^{2} / \mathrm{s}^{2} \mathrm{~K} \\
b=0.9 \times 10^{6} \mathrm{~m}^{5} / \mathrm{s}^{2} \mathrm{~kg}, \quad \eta_{0}=8886.73 \mathrm{~s} / \mathrm{m}^{2}
\end{gather*}
$$

Using the above values, we get $\mu_{e}=1, \theta_{0}=1, P_{0}=1$, $a=2, \omega=9.5, H=0.7 \times 10^{-5}, \tau_{0}=0.1$, and $\tau=0.2$.

The values of radial displacement $u$, temperature distribution $\theta$, concentration $C$, stresses $\sigma_{r r}, \sigma_{\phi \phi}$, and chemical potential distribution $P$ for thermoelastic diffusion and thermoelasticity are studied for force thermal source and chemical potential source. The output is plotted in Figures 1-10. Figure 1 shows that the values of chemical potential distribution $P$ have oscillatory behavior with diffusion in the whole range of radius $r$. The effects of nonhomogeneity $m$, rotation $\Omega$, time $t$ and relaxation time $\tau_{0}$ on chemical potential distribution is shifting from the positive into the negative gradually with the radius $r$. Figure 2 shows that the value of concentration distribution $C$ has oscillatory behavior for diffusion in the whole range of radius $r$ under the effects of nonhomogeneity, rotation, and relaxation time, while it is decreasing with an increase of nonhomogeneity $m$. In these figures, it is clear that the distribution has a nonzero value only in the bounded region of space for $t=0.15$ where the infinite speed of propagation is inherent. The effects of nonhomogeneity, rotation $\Omega$, time $t$, and relaxation time $\tau_{0}$ on concentration distribution is shifting from the positive into the negative gradually. This indicates that the equations are satisfied by the concentration $C$ which predict a finite speed of propagation of matter from first medium to another one. Figure 3 shows that the value of temperature distribution $\theta$ has an oscillatory behavior for thermoelastic diffusion in the whole range of the radius $r$, while the solution is notably different inside the sphere. This is due to the fact that, the thermal waves in the coupled theory travel with an infinite speed of propagation as opposed to finite speed in the generalized case. The effects of nonhomogeneity, rotation $\Omega$, time $t$ and relaxation time $\tau_{0}$ on temperature distribution shift from the positive into the negative gradually. This indicates that the heat propagates as a wave with finite velocity. Figure 4 shows that the value of radial displacement $u$ has oscillatory behavior with diffusion in the whole range of radius $r$. These figures indicate that the medium along $r$ undergoes expansion deformation due to the thermal shock, while the other one shows the compressive deformation. The effect of nonhomogeneity, rotation $\Omega$, and relaxation time $\tau_{0}$ on radial displacement becomes large. Increasing the nonhomogeneity, the radial displacement is shifted upward from negative values to positive values. At a given instant, the radial displacement is finite which is due to the effect of nonhomogeneity, rotation, time, and relaxation time. Figures 5 and 6 show the variations of the radial stress $\sigma_{r r}$ and tangential stress $\sigma_{\phi \phi}$ with respect to the radius $r$, respectively. The values of radial stress and tangential stress are increased and decreased due to the diffusion in a nonuniform behavior for all values of the radius $r$. For the values of $\sigma_{r r}$ and $\sigma_{\phi \phi}$, depicting the effect of nonhomogeneity, diffusion, rotation, and relaxation time, it is shown that the radial stress is compressive in its nature.

Figure 7 shows the values of radial displacement $u$ in thermoelastic medium without diffusion. This figure indicates clearly that the radial displacement at the cavity surface tends to zero which agrees with the boundary conditions prescribed. This coincides with the mechanical boundary condition of the cavity, in case of fixed surface. Figure 8 shows the values of temperature distribution $\theta$ without diffusion in the whole range of radius $r$. It was found that the values
of $\theta$ under effect of nonhomogeneity and rotation $\Omega$ are increase with an increase of nonhomogeneity and rotation $\Omega$ but are decreasing with the increase of the values of $m$. Figures 9 and 10 show the values of radial stress $\sigma_{r r}$ and the tangential stress $\sigma_{\phi \phi}$ without diffusion in the whole range of radius $r$, respectively. It was found that the values of $\sigma_{r r}$ under the effects of nonhomogeneity $m$ and rotation $\Omega$ are increasing with an increase of the values of nonhomogeneity $m$ and rotation $\Omega$, while the values of $\sigma_{r r}$ are decreasing with an increase of nonhomogeneity $m$, while the tangential stress $\sigma_{\phi \phi}$ is decreasing with the increase of the values of nonhomogeneity $m$ and rotation $\Omega$, but the values of $\sigma_{\phi \phi}$ are increasing with an increase of $m$. Due to the complicated nature of the governing equations of the generalized magneto-thermoelastic diffusion theory, the done works in this field are unfortunately limited. The method used in this study provides quite a success in dealing with such problems. This method gives exact solutions in the elastic medium without any restrictions on the actual physical quantities that appear in the governing equations of the considered problem.

## 8. Conclusions

The results presented in this paper will be very helpful for researchers concerned with material science, designers of new materials, and low-temperature physicists, as well as for those working on the development of a theory of hyperbolic propagation of hyperbolic thermodiffusion. Study of the phenomenon of nonhomogeneity, rotation, magnetic field, and diffusion is also used to improve the conditions of oil extractions. It was found that, for values of rotation and nonhomogeneity, the coupled theory and the generalization give close results. The case is quite different when we consider small value of rotation and nonhomogeneity. Comparing Figures 1-6 in case of thermoelastic diffusion medium with the Figures 7-10 in case of thermoelastic medium, it was found that $u, \sigma_{r r}, \sigma_{\phi \phi}, C$, and $P$ have the same behavior in both media. But with the passage of nonhomogeneity and rotation, the numerical values of $u, \sigma_{r r}, \sigma_{\phi \phi}, C$, and $P$ in thermoelastic diffusion medium are large in comparison with those in thermoelastic medium due to the influences of nonhomogeneity, magnetic field, rotation, and mass diffusion.

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## Research Article

# On Stability of a Third Order of Accuracy Difference Scheme for Hyperbolic Nonlocal BVP with Self-Adjoint Operator 

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A third order of accuracy absolutely stable difference schemes is presented for nonlocal boundary value hyperbolic problem of the differential equations in a Hilbert space $H$ with self-adjoint positive definite operator $A$. Stability estimates for solution of the difference scheme are established. In practice, one-dimensional hyperbolic equation with nonlocal boundary conditions is considered.

## 1. Introduction

In modeling several phenomena of physics, biology, and ecology mathematically, there often arise problems with nonlocal boundary conditions (see [1-5] and the references given therein). Nonlocal boundary value problems have been a major research area in the case when it is impossible to determine the boundary conditions of the unknown function. Over the last few decades, the study of nonlocal boundary value problems is of substantial contemporary interest (see, e.g., [6-14] and the references given therein).

We consider the nonlocal boundary value problem

$$
\begin{gather*}
\frac{d^{2} u(t)}{d t^{2}}+A u(t)=f(t), \quad 0<t<1, \\
u(0)=\alpha u(1)+\varphi,  \tag{1}\\
u^{\prime}(0)=\beta u^{\prime}(1)+\psi,
\end{gather*}
$$

for hyperbolic equations in a Hilbert space $H$ with selfadjoint positive definite linear operator $A$ with domain $D(A)$.

A function $u(t)$ is called a solution of problem (1) if the following conditions are satisfied.
(i) $u(t)$ is twice continuously differentiable on the segment $[0,1]$. The derivatives at the endpoints of the
segment are understood as the appropriate unilateral derivatives.
(ii) The element $u(t)$ belongs to $D(A)$ for all $t \in[0,1]$ and the function $A u(t)$ is continuous on the segment $[0,1]$.
(iii) $u(t)$ satisfies the equations and the nonlocal boundary conditions (1).

Here, $\varphi(x), \psi(x)(x \in[0,1])$ and $f(t, x)(t, x \in[0,1])$ are smooth functions.

In the study of numerical methods for solving PDEs, stability is an important research area (see [6-27]). Many scientists work on difference schemes for hyperbolic partial differential equations, in which stability was established under the assumption that the magnitudes of the grid steps $\tau$ and $h$ with respect to the time and space variables are connected. This particularly means that $\tau\left\|A_{h}\right\| \rightarrow 0$ when $\tau \rightarrow 0$.

We are interested in studying high order of accuracy unconditionally stable difference schemes for hyperbolic PDEs.

In the present paper, third order of accuracy difference scheme generated by integer power of $A$ for approximately solving nonlocal boundary value problem (1) is presented.

The stability estimates for solution of the difference scheme are established.

In [8], some results of this paper, without proof, were presented.

The well posedness of nonlocal boundary value problems for parabolic equations, elliptic equations, and equations of mixed types have been studied extensively by many scientists (see, e.g., [11-14, 19-32] and the references therein).

## 2. Third Order of Accuracy Difference Scheme Subject to Nonlocal Conditions

In this section, we obtain stability estimates for the solution of third order of accuracy difference scheme

$$
\begin{gather*}
\tau^{-2}\left(u_{k+1}-2 u_{k}+u_{k-1}\right)+\frac{2}{3} A u_{k}+\frac{1}{6} A\left(u_{k+1}+u_{k-1}\right) \\
+ \\
+\frac{1}{12} \tau^{2} A^{2} u_{k+1}=f_{k}, \\
f_{k}= \\
\frac{2}{3} f\left(t_{k}\right)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right) \\
 \tag{2}\\
-\frac{1}{12} \tau^{2}\left(-A f\left(t_{k+1}\right)+f^{\prime \prime}\left(t_{k+1}\right)\right), \\
t_{k}=k \tau, \quad 1 \leq k \leq N-1, N \tau=1, \\
\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right) \tau^{-1}\left(u_{1}-u_{0}\right)+\frac{\tau}{2} A u_{0}-\tau f_{1,1} \\
= \\
\beta\left(I-\frac{\tau^{2} A}{12}\right) \\
\quad \times\left(\frac{7 u_{N}-8 u_{N-1}+u_{N-2}}{6 \tau}+\frac{\tau}{3}\left(f_{N}-A u_{N}\right)\right) \\
+ \\
\quad\left(I-\frac{\tau^{2} A}{12}\right) \psi
\end{gather*}
$$

for numerical solution of nonlocal boundary value problem (1). Here,

$$
\begin{equation*}
f_{1,1}=f(0)+\left(-f(0)+\tau f^{\prime}(0)\right) \frac{1}{2}-2 f^{\prime}(0) \frac{\tau}{6} . \tag{3}
\end{equation*}
$$

We study the stability of solutions of difference scheme (2) under the following assumption:

$$
\begin{equation*}
|\alpha|+2|\beta|+2|\alpha||\beta|<1 \tag{4}
\end{equation*}
$$

We give a lemma that will be needed in the sequel which was presented in [18]. First, let us present the following operators:

$$
\begin{aligned}
R= & \left(I-\frac{1}{3} \tau^{2} A+i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right) \\
& \times\left(I+\frac{1}{6} \tau^{2} A+\frac{1}{12} \tau^{4} A^{2}\right)^{-1}
\end{aligned}
$$

and its conjugate $\widetilde{R}$,

$$
\begin{align*}
R_{1}=(- & \frac{5 \tau^{4}}{144} A^{2}+\frac{7 \tau^{6}}{216} A^{3}-i \tau A^{1 / 2} \\
& \left.\times\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right) \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right) \\
& \times\left(-i \tau A^{1 / 2}\left(\sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)\right.  \tag{6}\\
& \left.\times\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right)\right)^{-1},
\end{align*}
$$

and its conjugate $\widetilde{R}_{1}$,

$$
\begin{align*}
R_{2}= & \left(I-\frac{\tau^{2}}{12} A\right)\left(I+\frac{\tau^{2}}{6} A+\frac{\tau^{4}}{12} A^{2}\right) \\
& \times\left(-i A^{1 / 2}\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right) \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)^{-1} \\
R_{3}= & \left(I+\frac{\tau^{2}}{6} A+\frac{\tau^{4}}{12} A^{2}\right) \\
& \times\left(\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right)\left(-i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)\right)^{-1} \\
R_{4}= & \left(I+\frac{\tau^{2}}{3} A+\frac{\tau^{4}}{9} A^{2}+\frac{\tau^{6}}{72} A^{3}\right) \\
& \times\left(-i A^{1 / 2}\left(\sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)\left(I+\frac{\tau^{2}}{6} A+\frac{\tau^{4}}{12} A^{2}\right)\right. \\
R_{5}= & \left(-\frac{\tau^{2}}{2} A-\frac{\tau^{4}}{12} A^{2}+i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right) \\
& \left.\times\left(I+\frac{\tau^{2}}{6} A\right)\right)^{-1}, \\
& \times\left(I+\frac{\tau^{4}}{12} A^{2}\right), \tag{7}
\end{align*}
$$

and its conjugate $\widetilde{R}_{5}$, and

$$
\begin{align*}
R_{6}= & \left(I-\frac{1}{3} \tau^{2} A+i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right) \\
& \times\left(\frac{\tau^{2}}{2} A+\frac{\tau^{4}}{12} A^{2}-i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)^{-1} \tag{8}
\end{align*}
$$

and its conjugate $\widetilde{R}_{6}$.

We consider the following operators:

$$
\begin{align*}
R_{7}= & \frac{(7 R-I)}{6 \tau} \\
= & \left(I-\frac{5}{12} \tau^{2} A+\frac{1}{72} \tau^{4} A^{2}+\frac{7}{6} i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right)  \tag{9}\\
& \times \tau^{-1}\left(I+\frac{1}{6} \tau^{2} A+\frac{1}{12} \tau^{4} A^{2}\right)^{-1}
\end{align*}
$$

and its conjugate $\widetilde{R}_{7}$,

$$
\begin{align*}
\widetilde{R}_{7}= & \frac{(7 \widetilde{R}-I)}{6 \tau} \\
= & \left(I-\frac{5}{12} \tau^{2} A+\frac{1}{72} \tau^{4} A^{2}-\frac{7}{6} i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\right) \\
& \times \tau^{-1}\left(I+\frac{1}{6} \tau^{2} A+\frac{1}{12} \tau^{4} A^{2}\right)^{-1} \\
R_{8}= & \left(\frac{7 I-2 \tau^{2} A}{6 \tau}\right)\left(I+\frac{\tau^{2} A}{3}+\frac{\tau^{4} A^{2}}{9}+\frac{\tau^{6} A^{3}}{72}\right) \\
& \times \tau^{-1}\left(I+\frac{\tau^{2} A}{6}\right)^{-1}\left(I+\frac{\tau^{2}}{6} A+\frac{\tau^{4}}{12} A^{2}\right)^{-2}, \\
R_{9}= & \left(I-\frac{5}{3} \tau^{2} A+\frac{\tau^{4} A^{2}}{9}\right)\left(I+\frac{\tau^{2} A}{3}+\frac{\tau^{4} A^{2}}{9}+\frac{\tau^{6} A^{3}}{72}\right) \\
& \times \tau^{-1}\left(I+\frac{\tau^{2} A}{6}\right)^{-1}\left(I+\frac{1}{6} \tau^{2} A+\frac{1}{12} \tau^{4} A^{2}\right)^{-3}, \\
R_{10}= & I+\left(\frac{5}{144} \tau^{4} A^{2}-\frac{9}{288} \tau^{6} A^{3}+\frac{9}{1728} \tau^{8} A^{4}\right) \\
& \times\left(i \tau A^{1 / 2} \sqrt{I+\frac{1}{72} \tau^{4} A^{2}}\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right)\right), \tag{10}
\end{align*}
$$

and its conjugate $\widetilde{R}_{10}$.
Lemma 1. The following estimates hold:

$$
\begin{gather*}
\|R\|_{H \rightarrow H} \leq 1, \quad\|\widetilde{R}\|_{H \rightarrow H} \leq 1 \\
\left\|R_{1}\right\|_{H \rightarrow H} \leq 1, \quad\left\|\widetilde{R}_{1}\right\|_{H \rightarrow H} \leq 1 \\
\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \leq 1, \quad\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \leq 1 \\
\left\|A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \leq 1, \quad\left\|A^{-1 / 2} R_{5}\right\|_{H \rightarrow H} \leq \tau  \tag{11}\\
\left\|A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H} \leq \tau, \quad\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H} \leq 1 \\
\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H} \leq 1
\end{gather*}
$$

Now let us give, without proof, the second lemma.

Lemma 2. The following estimates hold:

$$
\begin{gathered}
\left\|\left(I+i \tau A^{1 / 2}\right) R\right\|_{H \rightarrow H} \leq 2 \\
\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}\right\|_{H \rightarrow H} \leq 2, \\
\left\|\tau R_{7}\right\|_{H \rightarrow H} \leq 1, \quad\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H} \leq 1 \\
\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H} \leq 1, \quad\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H} \leq 1 \\
\left\|\tau R_{8}\right\|_{H \rightarrow H} \leq \frac{7}{6}, \quad\left\|\tau R_{9}\right\|_{H \rightarrow H} \leq 1 \\
\left\|R_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H} \leq 2 \\
\left\|\widetilde{R}_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H} \leq 2
\end{gathered}
$$

Throughout the section, for simplicity, we denote

$$
\begin{align*}
B_{\tau}= & \beta \frac{1}{2} R_{2}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2} \\
& +\beta \frac{1}{2} R_{2}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2} \\
& -\alpha \frac{1}{2}\left[\widetilde{R}_{1} R^{N}-R_{1} \widetilde{R}^{N}\right] \\
& +\alpha \beta \frac{1}{4} \widetilde{R}_{1} R_{2}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) R^{N} \widetilde{R}^{N-2}  \tag{13}\\
& +\alpha \beta \frac{1}{4} R_{1} R_{2}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \widetilde{R}^{N} R^{N-2} \\
& -\alpha \beta \frac{1}{4} \widetilde{R}_{1} R_{2}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \widetilde{R}^{N} R^{N-2} \\
& -\alpha \beta \frac{1}{4} R_{1} R_{2}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) R^{N} \widetilde{R}^{N-2} .
\end{align*}
$$

Lemma 3. Suppose that assumption (4) holds. Then, the operator $I-B_{\tau}$ has an inverse $T_{\tau}=\left(I-B_{\tau}\right)^{-1}$. From symmetry and positivity properties of operator $A$, the following estimate is satisfied:

$$
\begin{equation*}
\left\|T_{\tau}\right\|_{H \mapsto H} \leq \frac{1}{1-|\alpha|-2|\beta|-2|\alpha||\beta|} \tag{14}
\end{equation*}
$$

Proof. Using the definitions of $B_{\tau}, R, \widetilde{R}$, estimates (11), and the following simple estimates,

$$
\begin{align*}
& \left\|\tau A^{1 / 2}\left(I-\frac{\tau^{2}}{12} A\right)\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right)^{-1}\right\|_{H \rightarrow H} \leq 12 \\
& \left\|\tau A^{1 / 2}\left(I+\frac{1}{12} \tau^{2} A+\frac{1}{144} \tau^{4} A^{2}\right)^{-1}\right\|_{H \rightarrow H} \leq \frac{12 \sqrt{11}}{12+\sqrt{11}} \tag{15}
\end{align*}
$$

and the triangle inequality, we get

$$
\begin{equation*}
\leq q, \tag{16}
\end{equation*}
$$

where

$$
\begin{equation*}
q=|\alpha|+2|\beta|+2|\alpha||\beta| . \tag{17}
\end{equation*}
$$

Since $q<1$, the operator $I-B_{\tau}$ has a bounded inverse and

$$
\begin{equation*}
\left\|\left(I-B_{\tau}\right)^{-1}\right\|_{H \rightarrow H} \leq \frac{1}{1-q}=\frac{1}{1-|\alpha|-2|\beta|-2|\alpha||\beta|} \tag{18}
\end{equation*}
$$

Lemma 3 is proved.

$$
\begin{aligned}
& B_{\tau} \leq|\beta| \frac{1}{2}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& +|\beta| \frac{1}{2}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \times\left\|R^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& +\alpha \frac{1}{2}\left[\left\|\widetilde{R}_{1}\right\|_{H \rightarrow H}\left\|R^{N}\right\|_{H \rightarrow H}+\left\|R_{1}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N}\right\|_{H \rightarrow H}\right] \\
& +|\alpha||\beta| \frac{1}{4}\left\|\widetilde{R}_{1}\right\|_{H \rightarrow H}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left\|R^{N}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& +|\alpha||\beta| \frac{1}{4}\left\|R_{1}\right\|_{H \rightarrow H}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left\|\widetilde{R}^{N}\right\|_{H \rightarrow H}\left\|R^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& +|\alpha||\beta| \frac{1}{4}\left\|\widetilde{R}_{1}\right\|_{H \rightarrow H}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left\|\widetilde{R}^{N}\right\|_{H \rightarrow H}\left\|R^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& +|\alpha||\beta| \frac{1}{4}\left\|R_{1}\right\|_{H \rightarrow H}\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left\|R^{N}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right)
\end{aligned}
$$

Now, let us obtain formula for the solution of problem (2). Using the results of [18], one can obtain the following formula:

$$
\begin{gather*}
u_{0}=\mu \\
u_{1}=\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right)^{-1} \\
\times\left(\left(I-\frac{5}{12} \tau^{2} A+\frac{\tau^{4}}{144} A^{2}\right) \mu\right. \\
\left.+\tau\left(I-\frac{\tau^{2}}{12} A\right) \omega+\tau^{2} f_{1,1}\right), \\
u_{k}=\frac{1}{2}\left[\widetilde{R}_{10} R^{k}-R_{10} \widetilde{R}^{k}\right] \mu+\frac{1}{2}\left[\widetilde{R}^{k}-R^{k}\right] R_{2} \omega \\
+\frac{1}{2}\left[\widetilde{R}^{k}-R^{k}\right] R_{3} \tau^{2} f_{1,1}+\frac{1}{2} R_{4} \sum_{s=1}^{k-1}\left[\widetilde{R}^{k-s}-R^{k-s}\right] f_{s} \tau^{2} \tag{19}
\end{gather*}
$$

for the solution of difference scheme

$$
\begin{gathered}
\tau^{-2}\left(u_{k+1}-2 u_{k}+u_{k-1}\right)+\frac{2}{3} A u_{k}+\frac{1}{6} A\left(u_{k+1}+u_{k-1}\right) \\
+ \\
+\frac{1}{12} \tau^{2} A^{2} u_{k+1}=f_{k} \\
f_{k}= \\
\frac{2}{3} f\left(t_{k}\right)+\frac{1}{6}\left(f\left(t_{k+1}\right)+f\left(t_{k-1}\right)\right) \\
\quad-\frac{1}{12} \tau^{2}\left(-A f\left(t_{k+1}\right)+f^{\prime \prime}\left(t_{k+1}\right)\right), \\
t_{k}=k \tau, \quad 1 \leq k \leq N-1, N \tau=1, \\
\left(I+\frac{\tau^{2}}{12} A+\frac{\tau^{4}}{144} A^{2}\right) \tau^{-1}\left(u_{1}-u_{0}\right)+\frac{\tau}{2} A u_{0}-\tau f_{1,1} \\
=\left(I-\frac{\tau^{2} A}{12}\right) \omega .
\end{gathered}
$$

Applying formula (19) and nonlocal boundary conditions

$$
\begin{gather*}
u_{0}=\alpha u_{N}+\varphi, \\
\omega=\beta\left(\frac{7 u_{N}-8 u_{N-1}+u_{N-2}}{6 \tau}+\frac{\tau}{3}\left(f_{N}-A u_{N}\right)\right)+\psi \tag{21}
\end{gather*}
$$

one can write

$$
\begin{align*}
& \mu=\alpha\{ \frac{1}{2}\left[\widetilde{R}_{10} R^{N}-R_{10} \widetilde{R}^{N}\right] \mu+\frac{1}{2}\left[\widetilde{R}^{N}-R^{N}\right] R_{2} \omega \\
&+\frac{1}{2}\left[\widetilde{R}^{N}-R^{N}\right] R_{3} \tau^{2} f_{1,1} \\
&\left.+\frac{1}{2} R_{4} \sum_{s=1}^{N-1}\left[\widetilde{R}^{N-s}-R^{N-s}\right] f_{s} \tau^{2}\right\}+\varphi \\
& \omega=\beta\left\{\frac { 1 } { 2 } \left[\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \widetilde{R}_{10} R^{N-2}\right.\right. \\
&\left.\quad-\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) R_{10} \widetilde{R}^{N-2}\right] \mu \\
&+\frac{1}{2}\left[\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right. \\
&\left.\quad-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right] R_{2} \omega \\
&+\frac{1}{2}\left[\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right. \\
&\left.\quad-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right] R_{3} \tau^{2} f_{1,1} \\
&+ \frac{\tau}{3} f_{N}+\frac{1}{2} R_{8} f_{N-1} \tau^{2}+\frac{1}{2} R_{9} f_{N-2} \tau^{2} \\
&+ \frac{1}{2} R_{4} \tau \sum_{s=1}^{N-3}\left[\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2-s}\right. \\
&\left.\left.\quad-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2-s}\right] f_{s} \tau\right\}+\psi \tag{22}
\end{align*}
$$

Using formulas in (22), we obtain

$$
\begin{aligned}
\mu= & T_{\tau}\left\{\left[\alpha \left(\frac{1}{2}\left(\widetilde{R}^{N}-R^{N}\right) R_{3} \tau^{2} f_{1,1}\right.\right.\right. \\
& \left.\left.+\frac{1}{2} R_{4} \tau \sum_{s=1}^{N-1}\left(\widetilde{R}^{N-s}-R^{N-s}\right) f_{s} \tau\right)+\varphi\right] \\
& \times\left[I-\frac{1}{2}\left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right. \\
& \left.\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right) R_{2}\right]+\left[\alpha \frac{1}{2}\left(\widetilde{R}^{N}-R^{N}\right) R_{2}\right] \\
& \times\left[\beta \frac { 1 } { 2 } \left\{\left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right.\right. \\
& \left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right) \times R_{3} \tau^{2} f_{1,1}+\frac{2 \tau}{3} f_{N}+R_{8} f_{N-1} \tau^{2} \\
& +R_{9} f_{N-2} \tau^{2}+R_{4} \tau \sum_{s=1}^{N-3}\left[\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2-s}\right. \\
& \left.\left.\left.\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \times R^{N-2-s}\right] f_{s} \tau\right\}+\psi\right]\right\}
\end{aligned}
$$

$$
\begin{align*}
& \omega=T_{\tau}\left\{\left[I-\alpha \frac{1}{2}\left(\widetilde{R}_{10} R^{N}-R_{10} \widetilde{R}^{N}\right)\right]\right. \\
& \times\left[\beta \frac { 1 } { 2 } \left\{\left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right.\right. \\
& \left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right) \times R_{3} \tau^{2} f_{1,1}+\frac{2 \tau}{3} f_{N}+R_{8} f_{N-1} \tau^{2} \\
& +R_{9} f_{N-2} \tau^{2}+R_{4} \tau \times \sum_{s=1}^{N-3}\left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2-s}\right. \\
& \begin{array}{r}
\left.\left.\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \times R^{N-2-s}\right) f_{s} \tau\right\}+\psi\right] \\
+\frac{1}{2}\left[\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \widetilde{R}_{10} R^{N-2}+\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) R_{10} \widetilde{R}^{N-2}\right] \\
\times\left[\alpha \left(\frac{1}{2}\left(\widetilde{R}^{N}-R^{N}\right) R_{3} \tau^{2} f_{1,1}\right.\right. \\
\\
\left.\left.\left.+\frac{1}{2} R_{4} \tau \sum_{s=1}^{N-1}\left(\widetilde{R}^{N-s}-R^{N-s}\right) f_{s} \tau\right)+\varphi\right]\right\} .
\end{array}
\end{align*}
$$

So, formulas (19) and (23) give a solution of problem (2). Unfortunately, the estimates for $\max _{1 \leq k \leq N}\left\|u_{k}\right\|_{H}$, $\max _{1 \leq k \leq N}\left\|A^{1 / 2} u_{k}\right\|_{H}$, and $\max _{1 \leq k \leq N}\left\|A u_{k}\right\|_{H} \quad$ cannot be obtained under the conditions
$\max _{1 \leq k \leq N}\left\|u_{k}\right\|_{H}$

$$
\begin{aligned}
& \leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau+\|\varphi\|_{H}+\left\|A^{-1 / 2} \psi\right\|_{H}\right. \\
& \left.\quad+\tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H}\right\}
\end{aligned}
$$

$\max _{1 \leq k \leq N}\left\|A^{1 / 2} u_{k}\right\|_{H}$
$\leq M\left\{\sum_{s=1}^{N-1}\left\|f_{s}\right\|_{H} \tau+\left\|A^{1 / 2} \varphi\right\|_{H}+\|\psi\|_{H}+\tau\left\|f_{1,1}\right\|_{H}\right\}$,

$$
\begin{gathered}
\leq M\left\{\sum_{s=2}^{N-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{1}\right\|_{H}+\|A \varphi\|_{H}\right. \\
\left.+\left\|A^{1 / 2} \psi\right\|_{H}+\tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}\right\}
\end{gathered}
$$

Nevertheless, we have the following theorem.

Theorem 4. Suppose that assumption (4) holds and $\varphi \in$ $D\left(A^{3 / 2}\right), \psi \in D\left(A^{1 / 2}\right)$. Then, for solution of difference scheme (2), the following stability estimates hold:
$\max _{1 \leq k \leq N}\left\|u_{k}\right\|_{H}$

$$
\begin{gathered}
\leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
\left.+\left\|A^{-1 / 2} \psi\right\|_{H}+\tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H}\right\}
\end{gathered}
$$

$\max _{1 \leq k \leq N}\left\|A^{1 / 2} u_{k}\right\|_{H}$

$$
\leq M\left\{\sum_{s=1}^{N-1}\left\|f_{s}\right\|_{H} \tau+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right.
$$

$$
\left.+\|\psi\|_{H}+\tau\left\|f_{1,1}\right\|_{H}\right\}
$$

$\max _{1 \leq k \leq N}\left\|A u_{k}\right\|_{H}$

$$
\begin{align*}
& \leq M\left\{\sum_{s=2}^{N-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{1}\right\|_{H}+\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\left\|A^{1 / 2} \psi\right\|_{H}+\tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}\right\} \tag{25}
\end{align*}
$$

where $M$ does not depend on $\tau, \varphi, \psi, f_{1,1}(x)$, and $f_{s}(x), 1 \leq$ $s \leq N-1$.

Proof. Using formulas in (23) and estimates (11), (12), and (14), we obtain

$$
\begin{aligned}
& \left\|\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[| \alpha | \left(\frac { 1 } { 2 } \left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.\quad+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H} \\
& \quad+\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \quad \times \sum_{s=1}^{N-1}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right. \\
& \left.\quad+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\quad \times\left\|A^{-1 / 2} f_{s}\right\|_{H \rightarrow H} \tau\right)+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right]
\end{aligned}
$$

$$
\begin{aligned}
& \times\left[1+\frac{1}{2}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\right] \\
& +|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \\
& \times\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H} \\
& +\frac{2 \tau}{3}\left\|A^{-1 / 2} f_{N}\right\|_{H} \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|A^{-1 / 2} f_{N-1}\right\|_{H} \tau \\
& +\left\|\tau R_{9}\right\|_{H \rightarrow H}\left\|A^{-1 / 2} f_{N-2}\right\|_{H} \tau \\
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-3}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\right.\right. \\
& \times\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H} \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2-s}\right\|_{H \rightarrow H}
\end{aligned}
$$

$$
\begin{align*}
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\right. \\
& \left.\times\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau\right\}+\left\|A^{-1 / 2} \psi\right\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\left\|A^{-1 / 2} \psi\right\|_{H}+\tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H}\right\}, \\
& \left\|A^{-1 / 2} \omega\right\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[1+|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} \widetilde{R}_{10}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H} \\
& +\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} R_{10}\right\|_{H \rightarrow H} \\
& \left.\left.\times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right)\right] \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H} \mid \tau^{-1} A^{-1 / 2} \widetilde{R}_{5} \|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|A^{-1 / 2} f_{1,1}\right\|_{H} \tau \\
& +\frac{2}{3}\left\|A^{-1 / 2} f_{N}\right\|_{H} \tau \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|A^{-1 / 2} f_{N-1}\right\|_{H} \tau  \tag{26}\\
& +\left\|\tau R_{9}\right\|_{H \rightarrow H}\left\|A^{-1 / 2} f_{N-2}\right\|_{H} \tau \\
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-3}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\right.\right. \\
& \left.\times\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2-s}\right\|_{H \rightarrow H}+\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\right. \\
& \left.\times\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\times\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau\right\}+\left\|A^{-1 / 2} \psi\right\|_{H}\right] \\
& +\frac{1}{2}\left[\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} \widetilde{R}_{10}\right\|_{H \rightarrow H}\left\|R^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} R_{10}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H}\right] \\
& \times\left[| \alpha | \frac { 1 } { 2 } \left(\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H} \\
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-1}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau\right)+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=1}^{k-1}\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\left\|A^{-1 / 2} \psi\right\|_{H}+\tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H}\right\} . \\
& \text { Applying } A^{1 / 2} \text { to formulas in (23), we get } \\
& \left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[| \alpha | \left(\frac { 1 } { 2 } \left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right)
\end{align*}
$$

$$
\begin{aligned}
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|f_{1,1}\right\|_{H} \\
& +\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-1}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\times\left\|f_{s}\right\|_{H \rightarrow H} \tau\right)+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right] \\
& \times\left[1+\frac{1}{2}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H}+\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.\left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right)\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\right] \\
& +|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \times\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H}+\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.\left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right)\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|f_{1,1}\right\|_{H}+\frac{2 \tau}{3}\left\|f_{N}\right\|_{H} \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|f_{N-1}\right\|_{H} \tau+\left\|\tau R_{9}\right\|_{H \rightarrow H} \\
& \times\left\|f_{N-2}\right\|_{H} \tau+\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-3}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2-s}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|f_{s}\right\|_{H} \tau\right\}+\|\psi\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=1}^{k-1}\left\|f_{s}\right\|_{H} \tau+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\|\psi\|_{H}+\tau\left\|f_{1,1}\right\|_{H}\right\}, \\
& \|\omega\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[1+|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} \widetilde{R}_{10}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H} \\
& +\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} R_{10}\right\|_{H \rightarrow H} \\
& \left.\left.\times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right)\right] \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|f_{1,1}\right\|_{H} \tau+\frac{2}{3}\left\|f_{N}\right\|_{H} \tau \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|f_{N-1}\right\|_{H} \tau \\
& +\left\|\tau R_{9}\right\|_{H \rightarrow H}\left\|f_{N-2}\right\|_{H} \tau+\left\|A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-3}\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-2-s}\right\|_{H \rightarrow H}+\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\right. \\
& \left.\times\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right)
\end{aligned}
$$

$$
\times\left(f_{s}-f_{s-1}\right)+\left(\widetilde{R}_{6}-R_{6}\right) f_{N-1}
$$

$$
\left.\left.\left.-\left(\widetilde{R}_{6} \widetilde{R}^{N-1}-R_{6} R^{N-1}\right) f_{1}\right)\right)+\varphi\right]
$$

$$
\times\left[I-\frac{1}{2}\left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right.
$$

$$
\left.\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right) R_{2}\right]
$$

$$
+\alpha \frac{1}{2}\left(\widetilde{R}^{N}-R^{N}\right) R_{2}
$$

$$
\times\left[\beta \left\{\frac { 1 } { 2 } \left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right.\right.
$$

$$
\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right)
$$

$$
\times R_{3} \tau^{2} f_{1,1}+\frac{\tau}{3} f_{N}+\frac{1}{2} R_{8} f_{N-1} \tau^{2}+\frac{1}{2} R_{9} f_{N-2} \tau^{2}
$$

$$
+R_{4} \frac{1}{2} \tau^{2}\left(\sum _ { s = 2 } ^ { N - 3 } \left(R_{6}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2-s}\right.\right.
$$

$$
\left.-\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2-s}\right)
$$

$$
\times\left(f_{s}-f_{s-1}\right)
$$

$$
+\left(\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right)\right.
$$

$$
\left.-R_{6}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right)\right) f_{N-3}
$$

$$
-\left(\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-3}\right.
$$

$$
\left.-R_{6}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-3}\right)
$$

$$
\left.\left.\left.\left.\times f_{1}\right)\right\}+\psi\right]\right\}
$$

$$
\omega=T_{\tau}\left\{\left[I-\alpha \frac{1}{2}\left(\widetilde{R}_{1} R^{N}-R_{1} \widetilde{R}^{N}\right)\right]\right.
$$

$$
\begin{equation*}
\times\left[\beta \left\{\frac { 1 } { 2 } \left(\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2}\right.\right.\right. \tag{27}
\end{equation*}
$$

$$
\begin{gathered}
\left.-\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) R^{N-2}\right) \\
\times R_{3} \tau^{2} f_{1,1}+\frac{\tau}{3} f_{N}+\frac{1}{2} R_{8} f_{N-1} \tau^{2} \\
+\frac{1}{2} R_{9} f_{N-2} \tau^{2} \\
+R_{4} \frac{1}{2} \tau^{2}\left(\sum _ { s = 2 } ^ { N - 3 } \left(R_{6}\left(\frac{R_{7} R_{5}}{\tau}-\frac{\tau A}{3} R^{2}\right) R^{N-2-s}\right.\right. \\
\\
\left.-\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-2-s}\right)
\end{gathered}
$$

$$
\begin{align*}
& \left.\left.\left.\times\left\|R^{N-2-s}\right\|_{H \rightarrow H}\right)\left\|f_{s}\right\|_{H} \tau\right\}+\|\psi\|_{H}\right] \\
& +\frac{1}{2}\left[\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} \widetilde{R}_{10}\right\|_{H \rightarrow H}\left\|R^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} R_{10}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H}\right] \\
& \times\left[| \alpha | \left(\frac { 1 } { 2 } \left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|f_{1,1}\right\|_{H} \\
& +\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times \sum_{s=1}^{N-1}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-s}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|f_{s}\right\|_{H}^{\tau}\right)+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=1}^{k-1}\left\|f_{s}\right\|_{H} \tau+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right.  \tag{28}\\
& \left.+\|\psi\|_{H}+\tau\left\|f_{1,1}\right\|_{H}\right\} .
\end{align*}
$$

$$
\begin{array}{r}
\times\left(f_{s}-f_{s-1}\right) \\
+\left(\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right)\right. \\
\left.-R_{6}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right)\right) f_{N-3} \\
-\left(\widetilde{R}_{6}\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) \widetilde{R}^{N-3}\right. \\
-R_{6}\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \\
\left.\left.\left.\left.\times R^{N-3}\right) f_{1}\right)\right\}+\psi\right] \\
+\frac{1}{2}\left[\left(R_{7} R_{5}-\frac{\tau A}{3} R^{2}\right) \widetilde{R}_{1} R^{N-2}\right. \\
\left.-\left(\widetilde{R}_{7} \widetilde{R}_{5}-\frac{\tau A}{3} \widetilde{R}^{2}\right) R_{1} \widetilde{R}^{N-2}\right] \\
\times\left[\alpha \left(\frac{1}{2}\left(\widetilde{R}^{N}-R^{N}\right) R_{3} \tau^{2} f_{1,1}\right.\right. \\
+\frac{1}{2} R_{4} \tau^{2}\left(\sum_{s=2}^{N-1}\left(R_{6} R^{N-s}-\widetilde{R}_{6} \widetilde{R}^{N-s}\right)\right. \\
\times\left(f_{s}-f_{s-1}\right) \\
+\left(\widetilde{R}_{6}-R_{6}\right) f_{N-1} \\
-\left(\widetilde{R}_{6} \widetilde{R}^{N-1}-R_{6} R^{N-1}\right) \\
\left.\left.\left.\left.\times f_{1}\right)\right)+\varphi\right]\right\} . \tag{29}
\end{array}
$$

Next, let us obtain the estimates for $\left\|A\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H}$ and $\left\|A^{1 / 2} \omega\right\|_{H}$. First, applying $A$ to formula (28) and using estimates (11), (12), and (14) and the triangle inequality, one can obtain

$$
\begin{aligned}
& \left\|A\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[| \alpha | \left(\frac { 1 } { 2 } \left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|A^{1 / 2} f_{1,1}\right\|_{H} \tau \\
& +\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \quad \times\left(\sum _ { s = 2 } ^ { N - 1 } \left(\left\|\tau A^{1 / 2} R_{\sigma}\right\|_{H \rightarrow H}\right.\right. \\
& \quad \times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-s}\right\|_{H \rightarrow H}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H} \\
& \left.\times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right) \\
& \times\left\|f_{s}-f_{s-1}\right\|_{H} \\
& +\left(\left\|\tau A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \widetilde{R}_{6}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\tau A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) R_{6}\right\|_{H \rightarrow H}\right) \times\left\|f_{N-1}\right\|_{H} \\
& +\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\right. \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-1}\right\|_{H \rightarrow H} \\
& +\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H} \\
& \left.\times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-1}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|f_{1}\right\|_{H}\right)\right)+\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right] \\
& \times\left[1+\frac{1}{2}\left(\left(\left\|\tau \widetilde{\tau}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right)\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right)\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\right] \\
& +|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H}\right)\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H} \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right)\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \\
& \times \tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}+\frac{2 \tau}{3}\left\|A^{1 / 2} f_{N}\right\|_{H} \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|A^{1 / 2} f_{N-1}\right\|_{H} \tau \\
& +\left\|\tau R_{9}\right\|_{H \rightarrow H}\left\|A^{1 / 2} f_{N-2}\right\|_{H} \tau
\end{aligned}
$$

$$
\begin{align*}
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times\left(\sum _ { s = 2 } ^ { N - 3 } \left(\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\right.\right. \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\right. \\
& \times\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H} \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|R^{N-2-s}\right\|_{H \rightarrow H} \\
& +\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}  \tag{30}\\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\right. \\
& \times\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H} \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|\widetilde{R}^{N-2-s}\right\|_{H \rightarrow H}\right)\left\|f_{s}-f_{s-1}\right\|_{H} \\
& +\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\right. \\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\right. \\
& \times\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H} \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& +\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\right. \\
& \times\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H} \\
& \left.\left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right)\right) \\
& \times\left\|f_{N-3}\right\|_{H} \\
& +\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\right. \\
& \times\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}^{N-3}\right\|_{H \rightarrow H}+\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H} \\
& \times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\left.\left.\times\left\|R^{N-3}\right\|_{H \rightarrow H}\right)\left\|f_{1}\right\|_{H}\right)\right\} \\
& \left.\left.+\left\|A^{1 / 2} \psi\right\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=2}^{N-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{1}\right\|_{H}+\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\left\|A^{1 / 2} \psi\right\|_{H}+\tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}\right\} . \\
& \text { Second, applying } A^{1 / 2} \text { to formula (29) and using estimates } \\
& \text { (11), (12), and (14) and the triangle inequality, we get } \\
& \left\|A^{1 / 2} \omega\right\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left\{\left[1+|\alpha| \frac{1}{2}\left(\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} \widetilde{R}_{10}\right\|_{H \rightarrow H}\right.\right.\right. \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N}\right\|_{H \rightarrow H} \\
& +\left\|\left(I+i \tau A^{1 / 2}\right)^{-1} R_{10}\right\|_{H \rightarrow H} \\
& \left.\left.\times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N}\right\|_{H \rightarrow H}\right)\right] \\
& \times\left[| \beta | \frac { 1 } { 2 } \left\{\left(\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right.\right.\right.\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right)\left\|\widetilde{R}^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau R_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} R_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|R^{N-2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|A^{1 / 2} f_{1,1}\right\|_{H} \\
& +\frac{2}{3} \tau\left\|A^{1 / 2} f_{N}\right\|_{H} \\
& +\left\|\tau R_{8}\right\|_{H \rightarrow H}\left\|A^{1 / 2} f_{N-1}\right\|_{H} \tau \\
& +\left\|\tau R_{9}\right\|_{H \rightarrow H}\left\|A^{1 / 2} f_{N-2}\right\|_{H} \tau \\
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \tau \\
& \times\left(\sum _ { s = 2 } ^ { N - 3 } \left(\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\right.\right.
\end{align*}
$$

$$
\left.\begin{array}{c}
\times\left(\left\|\tau R_{7}\right\|_{H \rightarrow H} \|_{\tau^{-1} A^{-1 / 2} R_{5} \|_{H \rightarrow H}}\right. \\
\left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
\times\left\|R^{N-2-s}\right\|_{H \rightarrow H}+\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H} \\
\times \\
\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right. \\
\left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
\times
\end{array}\right)
$$

$$
\begin{align*}
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} R^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|\widetilde{R}_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H} \\
& \times\left\|\left(I+i \tau A^{1 / 2}\right) R^{N-2}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau \widetilde{R}_{7}\right\|_{H \rightarrow H}\left\|\tau^{-1} A^{-1 / 2} \widetilde{R}_{5}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\frac{1}{3} \tau A^{1 / 2} \widetilde{R}^{2}\right\|_{H \rightarrow H}\right) \\
& \times\left\|R_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H} \\
& \left.\times\left\|\left(I+i \tau A^{1 / 2}\right) \widetilde{R}^{N-2}\right\|_{H \rightarrow H}\right] \\
& \times\left[|\alpha| \frac{1}{2}\left(\left\|\widetilde{R}^{N}\right\|_{H \rightarrow H}+\left\|R^{N}\right\|_{H \rightarrow H}\right)\right. \\
& \times\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H} \tau\left\|A^{1 / 2} f_{1,1}\right\|_{H} \\
& +\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times\left(\sum _ { s = 2 } ^ { N - 1 } \left(\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\left\|R^{N-s}\right\|_{H \rightarrow H}\right.\right. \\
& \left.+\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-s}\right\|_{H \rightarrow H}\right) \\
& \times\left\|f_{s}-f_{s-1}\right\|_{H} \\
& +\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}+\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\right) \\
& \times\left\|f_{N-1}\right\|_{H \rightarrow H} \\
& +\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{N-1}\right\|_{H \rightarrow H}\right. \\
& \left.+\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\left\|R^{N-1}\right\|_{H \rightarrow H}\right) \\
& \left.\times\left\|f_{1}\right\|_{H}\right) \\
& \left.\left.+\left\|\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right]\right\} \\
& \leq M\left\{\sum_{s=2}^{N-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{1}\right\|_{H}+\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.+\left\|A^{1 / 2} \psi\right\|_{H}+\tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}\right\} . \tag{31}
\end{align*}
$$

Now, we will prove estimates (25). Using formula (19), estimates (11), (12), (26), and (27), and the triangle inequality, we obtain

$$
\begin{aligned}
\left\|u_{k}\right\|_{H} \leq & \frac{1}{2}\left(\left\|\widetilde{R}_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right. \\
\quad & \left.\quad\left\|R_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right) \\
\times & \left\|\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H}
\end{aligned}
$$

$$
\begin{align*}
& +\frac{1}{2}\left(\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
& \left.\quad+\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right) \\
& \times\left\|A^{-1 / 2} \omega\right\|_{H} \\
& +\frac{1}{2}\left(\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
& +\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \sum_{s=1}^{k-1}\left[\left\|\widetilde{R}^{k-s}\right\|_{H \rightarrow H}+\left\|R^{k-s}\right\|_{H \rightarrow H}\right] \\
& \left.\times\left\|A^{-1 / 2} R_{3}\right\|_{s \rightarrow H} \tau R_{H}^{k} \|_{H \rightarrow H}\right) \tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H} \\
& \leq M\left\{\sum_{s=1}^{N-1}\left\|A^{-1 / 2} f_{s}\right\|_{H} \tau+\left\|\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.\quad+\left\|A^{-1 / 2} \psi\right\|_{H}+\tau\left\|A^{-1 / 2} f_{1,1}\right\|_{H}\right\}
\end{align*}
$$

for any $k \geq 2$. Applying $A^{1 / 2}$ to (19), we get

$$
\begin{align*}
& \left\|A^{1 / 2} u_{k}\right\|_{H} \\
& \leq \frac{1}{2}\left(\left\|\widetilde{R}_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right. \\
& \left.\quad+\left\|R_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right) \\
& \quad \times\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H} \\
& \quad+\frac{1}{2}\left(\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
& \left.\quad+\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right) \times\|\omega\|_{H} \\
& \quad+\frac{1}{2}\left(\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
& \left.\quad+\left\|\tau A^{1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right) \times \tau\left\|f_{1,1}\right\|_{H} \\
& \quad+\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \sum_{s=1}^{k-1}\left(\left\|\widetilde{R}^{k-s}\right\|_{H \rightarrow H}+\left\|R^{k-s}\right\|_{H \rightarrow H}\right) \\
& \quad \times\left\|f_{s}\right\|_{H} \tau \leq M\left\{\sum_{s=1}^{N-1}\left\|f_{s}\right\|_{H} \tau+\left\|A^{1 / 2}\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H}\right. \\
& \left.\quad+\|\psi\|_{H}+\tau\left\|f_{1,1}\right\|_{H}\right\} \tag{33}
\end{align*}
$$

for $k \geq 2$. Now, applying Abel's formula to (19), we have

$$
\begin{align*}
u_{k}= & \frac{1}{2}\left[\widetilde{R}_{1} R^{k}-R_{1} \widetilde{R}^{k}\right] \mu+\frac{1}{2}\left[\widetilde{R}^{k}-R^{k}\right] R_{2} \omega \\
& +\frac{1}{2}\left[\widetilde{R}^{k}-R^{k}\right] R_{3} \tau^{2} f_{1,1} \\
& +\tau^{2} R_{4} \frac{1}{2}\left(\sum_{s=2}^{k-1}\left[R_{6} R^{k-s}-\widetilde{R}_{6} \widetilde{R}^{k-s}\right]\left(f_{s}-f_{s-1}\right)\right. \\
& +\left(\widetilde{R}_{6}-R_{6}\right) f_{k-1} \\
& \left.-\left[\widetilde{R}_{6} \widetilde{R}^{k-1}-R_{6} R^{k-1}\right] f_{1}\right), \quad 2 \leq k \leq N \tag{34}
\end{align*}
$$

Applying $A$ to formula (34) and using estimates (11) and (12) and the triangle inequality, we obtain

$$
\begin{aligned}
&\left\|A u_{k}\right\|_{H} \leq \frac{1}{2}\left(\left\|\widetilde{R}_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right. \\
&\left.+\left\|R_{10}\left(I+i \tau A^{1 / 2}\right)^{-1}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right) \\
& \times\left\|A\left(I+i \tau A^{1 / 2}\right) \mu\right\|_{H} \\
&+ \frac{1}{2}\left(\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
&\left.+\left\|A^{1 / 2} R_{2}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right) \\
& \times\left\|A^{1 / 2} \omega\right\|_{H}+\frac{1}{2}\left(\left\|\tau A^{-1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k}\right\|_{H \rightarrow H}\right. \\
&\left.\quad+\left\|\tau A^{-1 / 2} R_{3}\right\|_{H \rightarrow H}\left\|R^{k}\right\|_{H \rightarrow H}\right) \\
& \times \tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}+\frac{1}{2}\left\|\tau A^{1 / 2} R_{4}\right\|_{H \rightarrow H} \\
& \times\left(\sum _ { s = 2 } ^ { k - 1 } \left[\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\left\|R^{k-s}\right\|_{H \rightarrow H}\right.\right. \\
&\left.\quad+\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k-s}\right\|_{H \rightarrow H}\right] \\
& \times\left\|f_{s}-f_{s-1}\right\|_{H} \\
&+\left(\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}+\left\|\tau A^{1 / 2} R_{\sigma}\right\|_{H \rightarrow H}\right)\left\|f_{k-1}\right\|_{H} \\
& \quad+\left[\left\|\tau A^{1 / 2} \widetilde{R}_{6}\right\|_{H \rightarrow H}\left\|\widetilde{R}^{k-1}\right\|_{H \rightarrow H}\right. \\
&\left.\left.\quad+\left\|\tau A^{1 / 2} R_{6}\right\|_{H \rightarrow H}\left\|R^{k-1}\right\|_{H \rightarrow H}\right]\left\|f_{1}\right\|_{H}\right)
\end{aligned}
$$

$$
\begin{align*}
\leq M\{ & \sum_{s=2}^{N-1}\left\|f_{s}-f_{s-1}\right\|_{H}+\left\|f_{1}\right\|_{H}+\left\|A\left(I+i \tau A^{1 / 2}\right) \varphi\right\|_{H} \\
& \left.+\left\|A^{1 / 2} \psi\right\|_{H}+\tau\left\|A^{1 / 2} f_{1,1}\right\|_{H}\right\} \tag{35}
\end{align*}
$$

for $k \geq 2$. Theorem 4 is proved.
Note that the stability estimates obtained previously permit us to get the convergence estimate of difference scheme (2) under the smoothness property of solution (1). Actually, under the condition $u(t) \in C([0,1], H)$, we can obtain the third order of accuracy for the error of difference scheme (2). Since $u^{(6)}(t)=-A^{3} u(t)+A^{2} f(t)-A f^{\prime \prime}(t)+f^{(4)}(t)$, this condition is satisfied under the given data $\varphi \in D\left(A^{3}\right)$, $\psi \in D\left(A^{5 / 2}\right), f^{\prime}(t) \in D\left(A^{2}\right)$, and $f(0) \in D\left(A^{3}\right)$.

Now, let us give application of this abstract result for nonlocal boundary value problem

$$
\begin{gather*}
u_{t t}-\left(a(x) u_{x}\right)_{x}+\delta u=f(t, x), \quad 0<t<1,0<x<1, \\
u(0, x)=\alpha u(1, x)+\varphi(x), \quad 0 \leq x \leq 1, \\
u_{t}(0, x)=\beta u_{t}(1, x)+\psi(x), \quad 0 \leq x \leq 1, \\
u(t, 0)=u(t, 1), \quad u_{x}(t, 0)=u_{x}(t, 1), \quad 0 \leq t \leq 1 \tag{36}
\end{gather*}
$$

for hyperbolic equation. Problem (36) has a unique smooth solution $u(t, x), \delta>0$ and the smooth functions $a(x) \geq$ $a>0(a(0)=a(1), x \in(0,1)), \varphi(x), \psi(x)(x \in[0,1])$, and $f(t, x)(t, x \in[0,1])$. This allows us to reduce mixed problem (36) to nonlocal boundary value problem (1) in a Hilbert space $H=L_{2}[0,1]$ with a self-adjoint positive definite operator $A^{x}$ defined by (36).

The discretization of problem (36) is carried out in two steps. In the first step, let us define the grid space

$$
\begin{equation*}
[0,1]_{h}=\left\{x: x_{r}=r h, 0 \leq r \leq K, K h=1\right\} . \tag{37}
\end{equation*}
$$

We introduce Hilbert space $L_{2 h}=L_{2}\left([0,1]_{h}\right), W_{2 h}^{1}=$ $W_{2 h}^{1}\left([0,1]_{h}\right)$, and $W_{2 h}^{2}=W_{2 h}^{2}\left([0,1]_{h}\right)$ of the grid functions $\varphi^{h}(x)=\left\{\varphi_{r}\right\}_{1}^{K-1}$ defined on $[0,1]_{h}$, and we assign the difference operator $A_{h}^{x}$ by the formula

$$
\begin{equation*}
A_{h}^{x} \varphi^{h}(x)=\left\{-\left(a(x) \varphi_{\bar{x}}\right)_{x, r}+\delta \varphi_{r}\right\}_{1}^{K-1} \tag{38}
\end{equation*}
$$

acting in the space of grid functions $\varphi^{h}(x)=\left\{\varphi_{r}\right\}_{0}^{K}$ satisfying the conditions $\varphi_{0}=\varphi_{K}, \varphi_{1}-\varphi_{0}=\varphi_{K}-\varphi_{K-1}$.

With the help of $A_{h}^{x}$, we arrive at the nonlocal boundary value problem

$$
\begin{gather*}
\frac{d^{2} v^{h}(t, x)}{d t^{2}}+A_{h}^{x} v^{h}(t, x)=f^{h}(t, x), \\
0<t<1, x \in[0,1]_{h}, \\
v^{h}(0, x)=\alpha v^{h}(1, x)+\varphi^{h}(x), \quad x \in[0,1]_{h}, \\
v_{t}^{h}(0, x)=\beta v_{t}^{h}(1, x)+\psi^{h}(x), \quad x \in[0,1]_{h} \tag{39}
\end{gather*}
$$

for a system of ordinary differential equations.

In the second step, we replace problem (2) with difference scheme (40)

$$
\begin{align*}
& \tau^{-2}\left(u_{k+1}^{h}(x)-2 u_{k}^{h}(x)+u_{k-1}^{h}(x)\right)+\frac{2}{3} A_{h}^{x} u_{k}^{h}(x) \\
&+\frac{1}{6} A_{h}^{x}\left(u_{k+1}^{h}(x)+u_{k-1}^{h}(x)\right) \\
&+\frac{1}{12} \tau^{2}\left(A_{h}^{x}\right)^{2} u_{k+1}^{h}(x)=f_{k}^{h}(x), \\
& f_{k}^{h}(x)=\frac{2}{3} f^{h}\left(t_{k}, x\right) \\
&+\frac{1}{6}\left(f^{h}\left(t_{k+1}, x\right)+f^{h}\left(t_{k-1}, x\right)\right) \\
&-\frac{1}{12} \tau^{2}\left(-A f^{h}\left(t_{k+1}, x\right)+f_{t t}^{h}\left(t_{k+1}, x\right)\right), \quad x \in[0,1]_{h}, \\
& t_{k}=k \tau, \quad N \tau=1, \quad 1 \leq k \leq N-1, \\
& u_{0}^{h}(x)=\alpha u_{N}^{h}(x)+\varphi^{h}(x), \quad x \in[0,1]_{h}, \\
&\left(I+\frac{\tau^{2}}{12}\left(A_{h}^{x}\right)+\frac{\tau^{4}}{144}\left(A_{h}^{x}\right)^{2}\right) \tau^{-1}\left(u_{1}^{h}(x)-u_{0}^{h}(x)\right) \\
&+\frac{\tau}{2}\left(A_{h}^{x}\right) \varphi^{h}(x)-\tau f_{1,1}^{h}(x) \\
&= \beta\left(I-\frac{\tau^{2}}{12}\left(A_{h}^{x}\right)\right) \\
& \times\left(\frac{1}{6 \tau}\left(7 u_{N}^{h}(x)-8 u_{N-1}^{h}(x)+u_{N-2}^{h}(x)\right)\right. \\
&\left.+\frac{\tau}{3}\left(f_{N}^{h}(x)-A u_{N}^{h}(x)\right)\right) \\
&+\left(I-\frac{\tau^{2}}{12}\left(A_{h}^{x}\right)\right) \psi^{h}(x), \quad x \in[0,1]_{h}, \\
& f_{1,1}^{h}(x)=\frac{1}{2} f^{h}(0, x)+\frac{\tau}{6} f_{t}^{h}(0, x) \tag{40}
\end{align*}
$$

Theorem 5. Let $\tau$ and $h$ be sufficiently small numbers. Then, the solution of difference scheme (40) satisfies the following stability estimates:

$$
\begin{align*}
& \max _{0 \leq k \leq N}\left\|u_{k}^{h}\right\|_{L_{2 h}}+\max _{0 \leq k \leq N}\left\|u_{k}^{h}\right\|_{W_{2 h}^{1}} \\
& \leq M_{1} \max _{1 \leq k \leq N-1}\left\|f_{k}^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}} \\
&\left.\quad\left\|\varphi^{h}\right\|_{W_{2 h}^{1}}+\tau\left\|\varphi^{h}\right\|_{W_{2 h}^{2}}+\tau\left\|f_{1,1}^{h}\right\|_{L_{2 h}}\right] \\
& \max _{1 \leq k \leq N-1}\left\|\tau^{-2}\left(u_{k+1}^{h}-2 u_{k}^{h}+u_{k-1}^{h}\right)\right\|_{L_{2 h}}+\max _{0 \leq k \leq N}\left\|u_{k}^{h}\right\|_{W_{2 h}^{2}}  \tag{41}\\
& \leq M_{1}\left[\left\|f_{1}^{h}\right\|_{L_{2 h}}+\max _{2 \leq k \leq N-1}\left\|\tau^{-1}\left(f_{k}^{h}-f_{k-1}^{h}\right)\right\|_{L_{2 h}}\right. \\
&+\left\|\psi^{h}\right\|_{W_{2 h}^{1}}+\left\|\varphi^{h}\right\|_{W_{2 h}^{2}} \\
&\left.+\tau\left\|\varphi^{h}\right\|_{W_{2 h}^{3}}+\tau\left\|f_{1,1}^{h}\right\|_{W_{2 h}^{1}}\right] .
\end{align*}
$$

Here, $M_{1}$ does not depend on $\tau, h, \varphi^{h}(x), \psi^{h}(x), f_{1,1}^{h}(x)$, and $f_{k}^{h}(x), 1 \leq k<N$.

The proof of Theorem 5 is based on the proof of abstract Theorem 4 and the symmetry property of operator $A_{h}^{x}$ defined by (38).

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## Research Article

# Determination of a Control Parameter for the Difference Schrödinger Equation 

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#### Abstract

The first order of accuracy difference scheme for the numerical solution of the boundary value problem for the differential equation with parameter $p, i(d u(t) / d t)+A u(t)+i u(t)=f(t)+p, 0<t<T, u(0)=\varphi, u(T)=\psi$, in a Hilbert space $H$ with self-adjoint positive definite operator $A$ is constructed. The well-posedness of this difference scheme is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained.


## 1. Introduction: Difference Scheme

The theory and applications of well-posedness of inverse problems for partial differential equations have been studied extensively in a large cycle of papers (see, e.g., $[1-24]$ and the references therein).

Our goal in this paper is to investigate Schrödinger equations with parameter. In the paper [25], the boundary value problem for the differential equation with parameter $p$

$$
\begin{gather*}
i \frac{d u(t)}{d t}+A u(t)+i u(t)=f(t)+p, \quad 0<t<T  \tag{1}\\
u(0)=\varphi, \quad u(T)=\psi
\end{gather*}
$$

in a Hilbert space $H$ with self-adjoint positive definite operator $A$ was studied. The well-posedness of this problem was established. The stability inequalities for the solution of three determinations of control parameter problems for the Schrödinger equation were obtained. In the present paper, the first order of accuracy Rothe difference scheme

$$
\begin{gather*}
i \tau^{-1}\left(u_{k}-u_{k-1}\right)+A u_{k}+i u_{k}=\varphi_{k}+p, \quad \varphi_{k}=f\left(t_{k}\right), \\
t_{k}=k \tau, \quad 1 \leq k \leq N, \quad N \tau=T  \tag{2}\\
u_{0}=\varphi, \quad u_{N}=\psi
\end{gather*}
$$

for the approximate solution of the boundary value problem (1) for the differential equation with parameter $p$ is presented. It is easy to see that

$$
\begin{align*}
& u_{k}=v_{k}+(A+i I)^{-1} p  \tag{3}\\
& p=(A+i I)\left(\psi-v_{N}\right)
\end{align*}
$$

where $\left\{v_{k}\right\}_{k=0}^{N}$ is the solution of the following single-step difference scheme:

$$
\begin{gather*}
i \tau^{-1}\left(v_{k}-v_{k-1}\right)+A v_{k}+i v_{k}=\varphi_{k}, \quad \varphi_{k}=f\left(t_{k}\right) \\
t_{k}=k \tau, \quad 1 \leq k \leq N, \quad N \tau=T \tag{4}
\end{gather*}
$$

$$
v_{0}-v_{N}=\varphi-\psi
$$

The theorem on well-posedness of difference problem (2) is proved. In practice, the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control parameter problems are obtained.

The paper is organized as follows. Section 1 is the introduction. In Section 2, the main theorem on stability of difference problem (2) is established. In Section 3, theorems on the stability inequalities for the solution of difference schemes for the approximate solution of three different types of control
parameter problems are obtained. In Section 4, numerical results are given. Finally, Section 5 is the conclusion.

## 2. The Main Theorem on Stability

In this section, we will study the stability of difference scheme (2).

Let $[0, T]_{\tau}=\left\{t_{k}=k \tau, k=1, \ldots, N, N \tau=T\right\}$ be the uniform grid space with step size $\tau>0$, where $N$ is a fixed positive integer. Throughout the present paper, $\mathscr{F}\left([0, T]_{\tau}, H\right)$ denotes the linear space of grid functions $\varphi^{\tau}=\left\{\varphi_{k}\right\}_{1}^{N}$ with values in the Hilbert space $H$. Let $\mathscr{C}_{\tau}(H)=\mathscr{C}\left([0, T]_{\tau}, H\right)$ be the Banach space of bounded grid functions with the norm

$$
\begin{equation*}
\left\|\varphi^{\tau}\right\|_{\mathscr{C}_{\tau}(H)}=\max _{1 \leq k \leq N}\left\|\varphi_{k}\right\|_{H} \tag{5}
\end{equation*}
$$

Let us start with a lemma we need below. We denote that $R=$ $((1+\tau) I-i \tau A)^{-1}$ is the step operator of problem (2).

Lemma 1. Assume that $A$ is a positive definite self-adjoint operator. The operator $I-R^{N}$ has an inverse $T_{\tau}=\left(I-R^{N}\right)^{-1}$ and the following estimate is satisfied:

$$
\begin{equation*}
\left\|T_{\tau}\right\|_{H \rightarrow H} \leq M(\delta) \tag{6}
\end{equation*}
$$

Proof. The proof of estimate (6) is based on the triangle inequality and the estimate

$$
\begin{align*}
\left\|\left(I-R^{N}\right)^{-1}\right\|_{H \rightarrow H} & \leq \sup _{\delta \leq \mu} \frac{1}{1-\left|(1+\tau(1-i \mu))^{-N}\right|} \\
& \leq \frac{1}{1-\left((1+\tau)^{2}+(\tau \delta)^{2}\right)^{-N / 2}} \leq \mu(\delta) . \tag{7}
\end{align*}
$$

Now, let us obtain the formula for the solution of problem (2). It is clear that the first order of accuracy difference scheme

$$
\begin{gather*}
i \tau^{-1}\left(u_{k}-u_{k-1}\right)+A u_{k}+i u_{k}=p+\varphi_{k}, \quad \varphi_{k}=f\left(t_{k}\right), \\
t_{k}=k \tau, \quad 1 \leq k \leq N, \quad N \tau=T, \quad u_{0}=\varphi \tag{8}
\end{gather*}
$$

has a solution and the following formula

$$
\begin{equation*}
u_{k}=R^{k} \varphi-i \sum_{j=1}^{k} R^{k-j+1}\left(p+\varphi_{j}\right) \tau, \quad 1 \leq k \leq N \tag{9}
\end{equation*}
$$

is satisfied. Applying formula (9) and the boundary condition

$$
\begin{equation*}
u_{N}=\psi \tag{10}
\end{equation*}
$$

we can write

$$
\begin{equation*}
\psi=R^{N} \varphi-i \sum_{j=1}^{N} R^{N-j+1} \varphi_{j} \tau-i \sum_{j=1}^{N} R^{N-j+1} \tau p . \tag{11}
\end{equation*}
$$

Since

$$
\begin{align*}
-i \sum_{j=1}^{N} R^{N-j+1} \tau & =-i(I-i A)^{-1}(I-R) \sum_{j=1}^{N} R^{N-j}  \tag{12}\\
& =-i(I-i A)^{-1}\left(I-R^{N}\right)
\end{align*}
$$

we have that

$$
\begin{equation*}
\psi=R^{N} \varphi-i \sum_{j=1}^{N} R^{N-j+1} \varphi_{j} \tau-i(I-i A)^{-1}\left(I-R^{N}\right) p . \tag{13}
\end{equation*}
$$

By Lemma 1, we get

$$
\begin{align*}
p=T_{\tau}( & (I-i A) \psi-(I-i A) R^{N} \varphi  \tag{14}\\
& \left.-\sum_{j=1}^{N}(I-i A) R^{N-j+1} \varphi_{j} \tau\right) .
\end{align*}
$$

Using (9) and (14), we get

$$
\begin{align*}
u_{k}= & R^{k} \varphi-i \sum_{j=1}^{k} R^{k-j+1} \varphi_{j} \tau \\
& +\sum_{j=1}^{k} R^{k-j+1} \tau T_{\tau}\left((I-i A) \psi-(I-i A) R^{N} \varphi\right.  \tag{15}\\
& \left.-\sum_{j=1}^{N}(I-i A) R^{N-j+1} \varphi_{j} \tau\right), \\
& 1 \leq k \leq N .
\end{align*}
$$

Since

$$
\begin{align*}
\sum_{j=1}^{k} R^{k-j+1} \tau & =(I-i A)^{-1}(I-R) \sum_{j=1}^{k} R^{k-j}  \tag{16}\\
& =(I-i A)^{-1}\left(I-R^{k}\right)
\end{align*}
$$

we have that

$$
\begin{align*}
u_{k}= & R^{k} \varphi+\sum_{j=1}^{k} R^{k-j+1} \varphi_{j} \tau \\
& +\left(I-R^{k}\right) T_{\tau}\left(\psi-R^{N} \varphi-\sum_{j=1}^{N} R^{N-j+1} \varphi_{j} \tau\right), \tag{17}
\end{align*}
$$

Hence, difference scheme (2) is uniquely solvable and for the solution, formulas (14) and (17) hold.

Theorem 2. Suppose that the assumption of Lemma 1 holds. Let $\varphi, \psi \in D(A)$. Then, for the solution $\left(\left\{u_{k}\right\}_{k=1}^{N}, p\right)$ of difference scheme (2) in $C_{\tau}(H) \times H$, the estimates

$$
\begin{gather*}
\|p\|_{H} \leq M\left[\|A \varphi\|_{H}+\|A \psi\|_{H}\right. \\
\left.+\left\|\varphi_{1}\right\|_{H}+\max _{2 \leq k \leq N}\left\|\frac{\varphi_{k}-\varphi_{k-1}}{\tau}\right\|_{H}\right]  \tag{18}\\
\left\|\left\{u_{k}\right\}_{k=1}^{N}\right\|_{C_{\tau}(H)} \leq \tag{19}
\end{gather*}
$$

hold, where $M$ is independent of $\tau, \varphi, \psi$, and $\left\{\varphi_{k}\right\}_{k=1}^{N}$.
Proof. From formulas (9) and (14), it follows that

$$
\begin{align*}
p=T_{\tau}[ & A \psi-R^{N} A \varphi-\varphi_{N}+R^{N} \varphi_{1} \\
& \left.-\sum_{j=2}^{N} R^{N-j+1}\left(\varphi_{j-1}-\varphi_{j}\right)\right] . \tag{20}
\end{align*}
$$

Using this formula, the triangle inequality, and estimate (6), we obtain

$$
\begin{align*}
&\|p\|_{H} \leq\left\|T_{\tau}\right\|_{H \rightarrow H}\left(\|A \psi\|_{H}+\left\|R^{N}\right\|_{H \rightarrow H}\|A \varphi\|_{H}\right. \\
&+\sum_{j=2}^{N}\left\|R^{N-j+1}\right\|_{H \rightarrow H}\left\|\varphi_{j}-\varphi_{j-1}\right\|_{H}  \tag{21}\\
&\left.+\left\|\varphi_{N}\right\|+\left\|R^{N}\right\|_{H \rightarrow H}\left\|\varphi_{1}\right\|_{H}\right) \\
& \leq M\left[\|A \varphi\|_{H}+\|A \psi\|_{H}+\left\|\left\{\varphi_{k}\right\}_{k=1}^{N}\right\|_{C_{\tau}^{(1)}(H)}\right] .
\end{align*}
$$

Estimate (18) is proved. Using formula (17), the triangle inequality, and estimate (6), we obtain

$$
\begin{align*}
\left\|u_{k}\right\|_{H} \leq & {\left[\left\|R^{k}\right\|_{H \rightarrow H}\|\varphi\|_{H}+\sum_{j=1}^{k}\left\|R^{k-j+1}\right\|_{H \rightarrow H}\left\|\varphi_{j}\right\|_{H} \tau\right.} \\
& +\left(1+\left\|R^{k}\right\|_{H \rightarrow H}\right)\left\|T_{\tau}\right\|_{H \rightarrow H} \\
& \times\left(\|\psi\|_{H}+\left\|R^{N}\right\|_{H \rightarrow H}\|\varphi\|_{H}\right.  \tag{22}\\
& \left.\left.+\sum_{j=1}^{N}\left\|R^{N-j+1}\right\|_{H \rightarrow H}\left\|\varphi_{j}\right\|_{H} \tau\right)\right] \\
\leq & M\left[\|\varphi\|_{H}+\|\psi\|_{H}+\left\|\left\{\varphi_{k}\right\}_{k=1}^{N}\right\|_{C_{\tau}(H)}\right]
\end{align*}
$$

for any $k$. From that, it follows estimate (19). This completes the proof of Theorem 2.

## 3. Applications

Now, we consider the simple applications of main Theorem 2.
First, the boundary value problem for the Schrödinger equation

$$
\begin{gather*}
i u_{t}-\left(a(x) u_{x}\right)_{x}+\delta u+i u=p(x)+f(t, x), \\
0<t<T, \quad 0<x<1, \\
u(0, x)=\varphi(x), \quad u(T, x)=\psi(x), \quad 0 \leq x \leq 1,  \tag{23}\\
u(t, 0)=u(t, 1), \quad u_{x}(t, 0)=u_{x}(t, 1), \quad 0 \leq t \leq T
\end{gather*}
$$

is considered. Problem (23) has a unique smooth solution ( $u(t, x), p(x)$ ) for the smooth functions $a(x) \geq a>0, x \in$ $(0,1), \delta>0, a(1)=a(0), \varphi(x), \psi(x)(x \in[0,1])$, and $f(t, x)$ $(t \in(0, T), x \in(0,1))$. This allows us to reduce the boundary value problem (23) to the boundary value problem (1) in a Hilbert space $H=L_{2}[0,1]$ with a self-adjoint positive definite operator $A^{x}$ defined by formula

$$
\begin{equation*}
A^{x} u(x)=-\left(a(x) u_{x}\right)_{x}+\delta u \tag{24}
\end{equation*}
$$

with domain

$$
\begin{gather*}
D\left(A^{x}\right)=\left\{u(x): u(x), u_{x}(x),\left(a(x) u_{x}\right)_{x} \in L_{2}[0,1]\right. \\
\left.u(1)=u(0), u_{x}(1)=u_{x}(0)\right\} \tag{25}
\end{gather*}
$$

The discretization of problem (23) is carried out in two steps. In the first step, we define the grid space

$$
\begin{equation*}
[0,1]_{h}=\left\{x=x_{n}: x_{n}=n h, 0 \leq n \leq M, M h=1\right\} \tag{26}
\end{equation*}
$$

Let us introduce the Hilbert space $L_{2 h}=L_{2}\left([0,1]_{h}\right)$ of the grid functions

$$
\begin{equation*}
\varphi^{h}(x)=\left\{\varphi_{n}\right\}_{1}^{M-1} \tag{27}
\end{equation*}
$$

defined on $[0,1]_{h}$, equipped with the norm

$$
\begin{equation*}
\left\|\varphi^{h}\right\|_{L_{2 h}}=\left(\sum_{x \in[0,1]_{h}}|\varphi(x)|^{2} h\right)^{1 / 2} \tag{28}
\end{equation*}
$$

To the differential operator $A^{x}$ defined by formula (24), we assign the difference operator $A_{h}^{x}$ by the formula

$$
\begin{equation*}
A_{h}^{x} \varphi^{h}(x)=\left\{-\left(a(x) \varphi_{\bar{x}}\right)_{x, n}+\delta \varphi_{n}\right\}_{1}^{M-1} \tag{29}
\end{equation*}
$$

acting in the space of grid functions $\varphi^{h}(x)=\left\{\varphi_{n}\right\}_{1}^{M-1}$ satisfying the conditions $\varphi_{0}=\varphi_{M}, \varphi_{1}-\varphi_{0}=\varphi_{M}-\varphi_{M-1}$. It is well known that $A_{h}^{x}$ is a self-adjoint positive definite operator in $L_{2 h}$. With the help of $A_{h}^{x}$, we reach the boundary value problem

$$
\begin{array}{r}
i \frac{d u^{h}(t, x)}{d t}+A_{h}^{x} u^{h}(t, x)+i u^{h}(t, x)=p^{h}(x)+f^{h}(t, x) \\
0<t<T, \quad x \in[0,1]_{h} \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in[0,1]_{h} . \tag{30}
\end{array}
$$

In the second step, we replace (30) with the difference scheme (2)

$$
\begin{gather*}
i \frac{u_{k}^{h}(x)-u_{k-1}^{h}(x)}{\tau}+A_{h}^{x} u_{k}^{h}(x)+i u_{k}^{h}(x)=p^{h}(x)+f_{k}^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \quad N \tau=T, \\
1 \leq k \leq N, \quad x \in[0,1]_{h}, \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in[0,1]_{h} . \tag{3}
\end{gather*}
$$

Theorem 3. The solution pairs $\left(\left\{u_{k}^{h}(x)\right\}_{0}^{N}, p^{h}(x)\right)$ of problem (31) satisfy the stability estimates

$$
\begin{align*}
&\left\|p^{h}\right\|_{L_{2 h}} \leq M_{1}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|A_{h}^{x} \varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}\right. \\
&\left.+\left\|A_{h}^{x} \psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}^{(1)}\left(L_{2 h}\right)}\right] \\
&\left\|\left\{u u_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)} \leq M_{2}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}\right] \tag{32}
\end{align*}
$$

where $M_{1}$ and $M_{2}$ do not depend on $\varphi^{h}, \psi^{h}$, and $f_{k}^{h}, 1 \leq k \leq N$. Here, $C_{\tau}^{(1)}\left(L_{2 h}\right)$ is the grid space of grid functions $\left\{f_{k}^{h}\right\}_{1}^{N}$ defined on $[0, T]_{\tau} \times[0,1]_{h}$ with norm

$$
\begin{gather*}
\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}^{(1)}\left(L_{2 h}\right)}=\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}+\sup _{1 \leq k<k+r \leq N} \frac{\left\|f_{k+r}^{h}-f_{k}^{h}\right\|_{L_{2 h}}}{r \tau}, \\
\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}=\max _{1 \leq k \leq N}\left\|f_{k}^{h}\right\|_{L_{2 h}} . \tag{33}
\end{gather*}
$$

The proof of Theorem 3 is based on formulas for $p^{h}(x)$ and $\left\{u_{k}^{h}(x)\right\}_{1}^{N}$ and the symmetry property of operator $A_{h}^{x}$.

Second, let $\Omega=\left(x=\left(x_{1}, \ldots, x_{n}\right): 0<x_{k}<1, k=1, \ldots\right.$, $n$ ) be the unit open cube in the $n$-dimensional Euclidean space $\mathbb{R}^{n}$ with boundary $S, \bar{\Omega}=\Omega \cup S$. In $[0, T] \times \Omega$, the boundary value problem for the multidimensional Schrödinger equation

$$
\begin{gather*}
i \frac{\partial u(t, x)}{\partial t}-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}}+i u=p(x)+f(t, x) \\
x=\left(x_{1}, \ldots, x_{n}\right) \in \Omega, \quad 0<t<T  \tag{34}\\
u(0, x)=\varphi(x), \quad u(T, x)=\psi(x), \quad x \in \bar{\Omega} \\
u(t, x)=0, \quad x \in S, 0 \leq t \leq T
\end{gather*}
$$

is considered. Here, $a_{r}(x) \geq a>0(x \in \Omega), f(t, x)(t \in(0, T)$, $x \in \Omega)$, and $\varphi(x), \psi(x)(x \in \bar{\Omega})$ are given smooth functions.

We consider the Hilbert space $L_{2}(\bar{\Omega})$ of all square integrable functions defined on $\bar{\Omega}$, equipped with the norm

$$
\begin{equation*}
\|f\|_{L_{2}(\bar{\Omega})}=\left(\int \cdots \int_{x \in \bar{\Omega}}|f(x)|^{2} d x_{1} \cdots d x_{n}\right)^{1 / 2} . \tag{35}
\end{equation*}
$$

Problem (34) has a unique smooth solution $(u(t, x), p(x))$ for the smooth functions $\varphi(x), \psi(x), a_{r}(x)$, and $f(t, x)$. This allows us to reduce the problem (34) to the boundary value problem (1) in the Hilbert space $H=L_{2}(\bar{\Omega})$ with a selfadjoint positive definite operator $A^{x}$ defined by the formula

$$
\begin{equation*}
A^{x} u(x)=-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}} \tag{36}
\end{equation*}
$$

with domain

$$
\begin{gather*}
D\left(A^{x}\right)=\left\{u(x): u(x), u_{x_{r}}(x),\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}} \in L_{2}(\bar{\Omega})\right. \\
1 \leq r \leq n, u(x)=0, x \in S\} \tag{37}
\end{gather*}
$$

The discretization of problem (34) is carried out in two steps. In the first step, we define the grid space

$$
\begin{gather*}
\bar{\Omega}_{h}=\left\{x=x_{r}=\left(h_{1} j_{1}, \ldots, h_{n} j_{n}\right),\right. \\
j=\left(j_{1}, \ldots, j_{n}\right), 0 \leq j_{r} \leq N_{r}, \\
\left.N_{r} h_{r}=1, r=1, \ldots, n\right\},  \tag{38}\\
\Omega_{h}=\bar{\Omega}_{h} \cap \Omega, \quad S_{h}=\bar{\Omega}_{h} \cap S
\end{gather*}
$$

and introduce the Hilbert space $L_{2 h}=L_{2}\left(\bar{\Omega}_{h}\right)$ of the grid functions

$$
\begin{equation*}
\varphi^{h}(x)=\left\{\varphi\left(h_{1} j_{1}, \ldots, h_{n} j_{n}\right)\right\} \tag{39}
\end{equation*}
$$

defined on $\bar{\Omega}_{h}$, equipped with the norm

$$
\begin{equation*}
\left\|\varphi^{h}\right\|_{L_{2 h}}=\left(\sum_{x \in \Omega_{h}}\left|\varphi^{h}(x)\right|^{2} h_{1} \cdots h_{n}\right)^{1 / 2} \tag{40}
\end{equation*}
$$

To the differential operator $A^{x}$ defined by formula (36), we assign the difference operator $A_{h}^{x}$ by the formula

$$
\begin{equation*}
A_{h}^{x} u^{h}=-\sum_{r=1}^{n}\left(\alpha_{r}(x) u_{x_{r}}^{h}\right)_{x_{r}, j_{r}} \tag{41}
\end{equation*}
$$

where $A_{h}^{x}$ is known as self-adjoint positive definite operator in $L_{2 h}$, acting in the space of grid functions $u^{h}(x)$ satisfying the conditions $u^{h}(x)=0$ for all $x \in S_{h}$. With the help of the difference operator $A_{h}^{x}$, we arrive to the following boundary value problem:

$$
\begin{array}{r}
i u_{t}^{h}(t, x)+A_{h}^{x} u^{h}(t, x)+i u^{h}(t, x)=p^{h}(x)+f^{h}(t, x), \\
0<t<T, \quad x \in \Omega_{h} \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in \Omega_{h} \tag{42}
\end{array}
$$

for an infinite system of ordinary differential equations.

The first order of accuracy difference scheme for the solution of problem (42) is

$$
\begin{gather*}
i \frac{u_{k}^{h}(x)-u_{k-1}^{h}(x)}{\tau}+A_{h}^{x} u_{k}^{h}(x)+i u_{k}^{h}(x) \\
=p^{h}(x)+f_{k}^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \quad N \tau=T,  \tag{43}\\
1 \leq k \leq N, \quad x \in \Omega_{h}, \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in \Omega_{h} .
\end{gather*}
$$

Theorem 4. The solution pairs $\left(\left\{u_{k}^{h}(x)\right\}_{0}^{N}, p^{h}(x)\right)$ of problem (43) satisfy the stability estimates

$$
\begin{gather*}
\left\|p^{h}\right\|_{L_{2 h}} \leq M_{1}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|A_{h}^{x} \varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}\right. \\
\left.+\left\|A_{h}^{x} \psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}^{(1)}\left(L_{2 h}\right)}\right] \\
\left\|\left\{u_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)} \leq M_{2}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}\right] \tag{44}
\end{gather*}
$$

where $M_{1}$ and $M_{2}$ do not depend on $\varphi^{h}, \psi^{h}$, and $f_{k}^{h}, 1 \leq k \leq N$. Here, $C_{\tau}^{(1)}\left(L_{2 h}\right)$ is the grid space of grid functions $\left\{f_{k}^{h}\right\}_{1}^{N}$ defined on $[0, T]_{\tau} \times \overline{\Omega_{h}}$ with norm

$$
\begin{gather*}
\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}^{(1)}\left(L_{2 h}\right)}=\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}+\sup _{1 \leq k<k+r \leq N} \frac{\left\|f_{k+r}^{h}-f_{k}^{h}\right\|_{L_{2 h}}}{r \tau}, \\
\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}=\max _{1 \leq k \leq N}\left\|f_{k}^{h}\right\|_{L_{2 h}} . \tag{45}
\end{gather*}
$$

The proof of Theorem 4 is based on Theorem 3 and the symmetry property of the operator $A_{h}^{x}$ is defined by formula (34) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in $L_{2 h}$.

Theorem 5. For the solutions of the elliptic difference problem [26]

$$
\begin{gather*}
A_{h}^{x} u^{h}(x)=\omega^{h}(x), \quad x \in \Omega_{h},  \tag{46}\\
u^{h}(x)=0, \quad x \in S_{h}
\end{gather*}
$$

the following coercivity inequality holds:

$$
\begin{equation*}
\sum_{r=1}^{n}\left\|u_{x_{r} x_{\bar{r}}}^{h}\right\|_{L_{2 h}} \leq M\left\|\omega^{h}\right\|_{L_{2 h}} \tag{47}
\end{equation*}
$$

where $M$ does not depend on $h$ and $\omega^{h}$.

Third, in $[0, T] \times \Omega$, the boundary value problem for the multidimensional Schrödinger equation

$$
\begin{gather*}
i \frac{\partial u(t, x)}{\partial t}-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}}+\delta u+i u=p(x)+f(t, x) \\
x=\left(x_{1}, \ldots, x_{n}\right) \in \Omega, \quad 0<t<T \\
u(0, x)=\varphi(x), \quad u(T, x)=\psi(x), \quad x \in \bar{\Omega} \\
\frac{\partial u(t, x)}{\partial \vec{n}}=0, \quad x \in S, \quad 0 \leq t \leq T \tag{48}
\end{gather*}
$$

with the Neumann condition is considered. Here, $\vec{n}$ is the normal vector to $S, \delta>0$, and $a_{r}(x) \geq a>0(x \in \Omega), f(t, x)$ $(t \in(0, T), x \in \Omega)$, and $\varphi(x), \psi(x)(x \in \bar{\Omega})$ are given smooth functions.

Problem (48) has a unique smooth solution $(u(t, x), p(x))$ for the smooth functions $\varphi(x), \psi(x), a_{r}(x)$, and $f(t, x)$. This allows us to reduce the problem (48) to the boundary value problem (1) in the Hilbert space $H=L_{2}(\bar{\Omega})$ with a selfadjoint positive definite operator $A^{x}$ defined by formula

$$
\begin{equation*}
A^{x} u(x)=-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}}+\delta u \tag{49}
\end{equation*}
$$

with domain

$$
\begin{gather*}
D\left(A^{x}\right)=\left\{u(x): u(x), u_{x_{r}}(x),\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}} \in L_{2}(\bar{\Omega})\right. \\
\left.1 \leq r \leq n, \frac{\partial u(x)}{\partial \vec{n}}=0, x \in S\right\} \tag{50}
\end{gather*}
$$

The discretization of problem (48) is carried out in two steps. In the first step, we define the difference operator $A_{h}^{x}$ by the formula

$$
\begin{equation*}
A_{h}^{x} u^{h}=-\sum_{r=1}^{n}\left(\alpha_{r}(x) u_{x_{r}}^{h}\right)_{x_{r}, j_{r}}+\delta u^{h} \tag{51}
\end{equation*}
$$

where $A_{h}^{x}$ is known as self-adjoint positive definite operator in $L_{2 h}$, acting in the space of grid functions $u^{h}(x)$ satisfying the conditions $D^{h} u^{h}(x)=0$ for all $x \in S_{h}$. Here, $D^{h}$ is the approximation of the operator $\partial \cdot / \partial \vec{n}$. With the help of the difference operator $A_{h}^{x}$, we arrive to the following boundary value problem:

$$
\begin{array}{r}
i u_{t}^{h}(t, x)+A_{h}^{x} u^{h}(t, x)+i u^{h}(t, x)=p^{h}(x)+f^{h}(t, x), \\
0<t<T, \quad x \in \Omega_{h}, \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in \Omega_{h} \tag{52}
\end{array}
$$

for an infinite system of ordinary differential equations.

The first order of accuracy difference scheme for the solution of problem (52) is

$$
\begin{gather*}
i \frac{u_{k}^{h}(x)-u_{k-1}^{h}(x)}{\tau}+A_{h}^{x} u_{k}^{h}(x)+i u_{k}^{h}(x)=p^{h}(x)+f_{k}^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \quad N \tau=T, \\
1 \leq k \leq N, \quad x \in \Omega_{h}, \\
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x), \quad x \in \Omega_{h} . \tag{53}
\end{gather*}
$$

Theorem 6. The solution pairs $\left(\left\{u_{k}^{h}(x)\right\}_{0}^{N}, p^{h}(x)\right)$ of problem (53) satisfy the stability estimates

$$
\begin{gather*}
\left\|p^{h}\right\|_{L_{2 h}} \leq M_{1}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|A_{h}^{x} \varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}\right. \\
\left.+\left\|A_{h}^{x} \psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}^{(1)}\left(L_{2 h}\right)}\right] \\
\left\|\left\{u_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)} \leq M_{2}\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}+\left\|\left\{f_{k}^{h}\right\}_{1}^{N}\right\|_{C_{\tau}\left(L_{2 h}\right)}\right] \tag{54}
\end{gather*}
$$

where $M_{1}$ and $M_{2}$ do not depend on $\varphi^{h}, \psi^{h}$, and $f_{k}^{h}, 1 \leq k \leq N$.
The proof of Theorem 6 is based on Theorem 2 and the symmetry property of the operator $A_{h}^{x}$ is defined by formula (51) and the following theorem on the coercivity inequality for the solution of the elliptic difference problem in $L_{2 h}$.

Theorem 7. For the solution of the elliptic difference problem [26]

$$
\begin{gather*}
A_{h}^{x} u^{h}(x)=\omega^{h}(x), \quad x \in \Omega_{h},  \tag{55}\\
D^{h} u^{h}(x)=0, \quad x \in S_{h}
\end{gather*}
$$

the following coercivity inequality holds:

$$
\begin{equation*}
\sum_{r=1}^{n}\left\|u_{x_{r} x_{\bar{r}}}^{h}\right\|_{L_{2 h}} \leq M\left\|\omega^{h}\right\|_{L_{2 h}} \tag{56}
\end{equation*}
$$

where $M$ does not depend on $h$ and $\omega^{h}$.

## 4. Numerical Results

In present section, for numerical analysis, the following boundary value problem

$$
\begin{gathered}
i \frac{\partial u(t, x)}{\partial t}-\frac{\partial^{2} u(t, x)}{\partial x^{2}}+i u(t, x)=p(x)+f(t, x) \\
x \in(0, \pi), \quad t \in(0,1) \\
u(0, x)=\sin x, \quad u(1, x)=e^{-1} \sin x, \quad x \in[0, \pi] \\
u(t, 0)=u(t, \pi)=0, \quad t \in[0,1]
\end{gathered}
$$

is considered. The exact solution of problem (57) is $u(t, x)=$ $e^{-t} \sin x$ and $p(x)=\sin x$.

The first order of accuracy difference scheme

$$
\begin{gather*}
i \frac{u_{n}^{k}-u_{n}^{k-1}}{\tau}-\frac{u_{n+1}^{k}-2 u_{n}^{k}+u_{n-1}^{k}}{h^{2}}+i u_{n}^{k}=\varphi_{n}^{k}+p\left(x_{n}\right), \\
1 \leq k \leq N, \quad 1 \leq n \leq M-1, \\
\varphi_{n}^{k}=f\left(t_{k}, x_{n}\right)=\left(e^{-t_{k}}-1\right) \sin x_{n}, \\
t_{k}=k \tau, \quad 0 \leq k \leq N, \quad N \tau=1, \\
x_{n}=n h, \quad 1 \leq n \leq M-1, \quad M h=\pi, \\
u_{n}^{0}=\sin \left(x_{n}\right), \quad u_{n}^{N}=e^{-1} \sin \left(x_{n}\right), \quad x_{n}=n h, 0 \leq n \leq M, \\
u_{0}^{k}=u_{M}^{k}=0, \quad 0 \leq k \leq N \tag{58}
\end{gather*}
$$

for the numerical solution of problem (57) is constructed.
For obtaining the values of $p\left(x_{n}\right)$ at the grid points, we will use the following equation:

$$
\begin{array}{r}
p\left(x_{n}\right)=-e^{-1} \frac{\sin \left(x_{n+1}\right)-2 \sin \left(x_{n}\right)+\sin \left(x_{n-1}\right)}{h^{2}} \\
+i e^{-1} \sin \left(x_{n}\right)+\frac{v_{n+1}^{N}-2 v_{n}^{N}+v_{n-1}^{N}}{h^{2}}-i v_{n}^{N}  \tag{59}\\
x_{n}=n h, \quad 1 \leq n \leq M-1,
\end{array}
$$

where $v_{s}^{k}, s=n \pm 1$, and $n$ is the solution of the first order of accuracy difference scheme

$$
\begin{gather*}
i \frac{v_{n}^{k}-v_{n}^{k-1}}{\tau}-\frac{v_{n+1}^{k}-2 v_{n}^{k}+v_{n-1}^{k}}{h^{2}}+i v_{n}^{k}=\varphi_{n}^{k}, \\
\varphi_{n}^{k}=f\left(t_{k}, x_{n}\right), \quad t_{k}=k \tau, 1 \leq k \leq N, N \tau=1, \\
x_{n}=n h, \quad 1 \leq n \leq M-1, \quad M h=\pi,  \tag{60}\\
v_{n}^{N}-v_{n}^{0}=\left(e^{-1}-1\right) \sin \left(x_{n}\right), \quad x_{n}=n h, 0 \leq n \leq M, \\
v_{0}^{k}=v_{M}^{k}=0, \quad 0 \leq k \leq N
\end{gather*}
$$

generated by difference scheme (58).
Using the difference scheme (60), we obtain $(N+1) \times$ $(M+1)$ system of linear equations and we can write them in the matrix form as

$$
\begin{gather*}
A v_{n+1}+B v_{n}+C v_{n-1}=R \varphi_{n}, \quad 1 \leq n \leq M-1, \\
v_{0}=v_{M}=\widetilde{0}, \tag{61}
\end{gather*}
$$

where

$$
\begin{align*}
C=A & =\left[\begin{array}{cccccccc}
0 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\
0 & x & 0 & \cdot & 0 & 0 & 0 & 0 \\
0 & 0 & x & \cdot & 0 & 0 & 0 & 0 \\
. & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & 0 & 0 & x & 0 \\
0 & 0 & 0 & \cdot & 0 & 0 & 0 & x
\end{array}\right]_{(N+1) \times(N+1)} \\
B & =\left[\begin{array}{ccccccc}
-1 & 0 & 0 & \cdot & 0 & 0 & 1 \\
y & z & 0 & \cdot & 0 & 0 & 0 \\
0 & y & z & \cdot & 0 & 0 & 0 \\
. & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & y & z & 0 \\
0 & 0 & 0 & \cdot & 0 & y & z
\end{array}\right]_{(N+1) \times(N+1)} \tag{62}
\end{align*}
$$

Here,

$$
\begin{gather*}
x=-\frac{1}{h^{2}}, \quad y=-\frac{i}{\tau}, \quad z=\frac{i}{\tau}+\frac{2}{h^{2}}+i, \\
v_{s}=\left[\begin{array}{c}
v_{s}^{0} \\
\vdots \\
v_{s}^{N}
\end{array}\right]_{(N+1) \times 1} \quad \text { for } s=n+1, n, n-1,  \tag{63}\\
\varphi_{n}=\left[\begin{array}{c}
\left(e^{-1}-1\right) \sin x_{n} \\
\varphi_{n}^{1} \\
\vdots \\
\varphi_{n}^{N-1} \\
\varphi_{n}^{N}
\end{array}\right]_{(N+1) \times 1}
\end{gather*}
$$

So, we have the second-order difference equation with respect to $n$ with matrix coefficients. Using the modified Gauss elimination method, we can obtain $v_{n}^{k}, 0 \leq k \leq N, 0 \leq n \leq M$.

For the solution of the matrix equations, we seek the solution of the form

$$
\begin{gather*}
v_{n}=\alpha_{n+1} v_{n+1}+\beta_{n+1}, \quad n=M-1, \ldots, 2,1 \\
v_{M}=\widetilde{0} \tag{64}
\end{gather*}
$$

where $\alpha_{j}$ and $\beta_{j}, j=1, \ldots, M$, are calculated as

$$
\begin{gather*}
\alpha_{n+1}=-\left(B+C \alpha_{n}\right)^{-1}(A), \\
\beta_{n+1}=\left(B+C \alpha_{n}\right)^{-1}\left(D \varphi_{n}-C \beta_{n}\right), \tag{65}
\end{gather*}
$$

where $\alpha_{1}$ is $(N+1) \times(N+1)$ and $\beta_{1}$ is $(N+1) \times 1$ zero matrix.
Then, using (59), values of $p\left(x_{n}\right)$ at grid points are obtained. Replacing $p\left(x_{n}\right)$ in (58), we get $(N+1) \times(M+1)$ system of linear equations and it can be written in the matrix form

$$
\begin{gather*}
A_{2} u_{n+1}+B_{2} u_{n}+C_{2} u_{n-1}=R \theta_{n}, \quad 1 \leq n \leq M-1 \\
1 u_{0}=u_{M}=\widetilde{0} \tag{66}
\end{gather*}
$$

Table 1: Error analysis for the exact solution $u(t, x)$.

| Method | $N=M=20$ | $N=M=40$ | $N=M=80$ |
| :--- | :---: | :---: | :---: |
| 1st order of <br> accuracy d.s. | 0.0024 | 0.0012 | $6.0463 \times 10^{-4}$ |

where

$$
\begin{gather*}
C_{2}=C, \\
B_{2}=\left[\begin{array}{ccccccc}
1 & 0 & 0 & \cdot & 0 & 0 & 0 \\
y & z & 0 & \cdot & 0 & 0 & 0 \\
0 & y & z & \cdot & 0 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & \cdot & y & z & 0 \\
0 & 0 & 0 & \cdot & 0 & y & z
\end{array}\right]_{(N+1) \times(N+1)} \tag{67}
\end{gather*}
$$

Here,

$$
\begin{gather*}
y=-\frac{i}{\tau}, \quad z=\frac{i}{\tau}+\frac{2}{h^{2}}+i \\
u_{s}=\left[\begin{array}{c}
u_{s}^{0} \\
\vdots \\
u_{s}^{N}
\end{array}\right]_{(N+1) \times 1} \quad \text { for } s=n+1, n, n-1  \tag{68}\\
\theta_{n}=\left[\begin{array}{c}
\sin x_{n} \\
\varphi_{n}^{1}+p\left(x_{n}\right) \\
\vdots \\
\varphi_{n}^{N-1}+p\left(x_{n}\right) \\
\varphi_{n}^{N}+p\left(x_{n}\right)
\end{array}\right]_{(N+1) \times 1}
\end{gather*}
$$

Using the modified Gauss elimination method again, we can obtain $u_{n}^{k}, 0 \leq k \leq N, 0 \leq n \leq M$.

We will give the results of the numerical analysis. The numerical solutions are recorded for different values of $N$ and $M$ and $u_{n}^{k}$ represents the numerical solutions of the difference scheme at $\left(t_{k}, x_{n}\right)$. Table 1 is constructed for $N=M=20$, 40 , and 80 , respectively and the errors are computed by the following formula:

$$
\begin{equation*}
E=\max _{1 \leq k \leq N}\left\{\sum_{n=1}^{M}\left|u\left(t_{k}, x_{n}\right)-u_{n}^{k}\right|^{2} h\right\}^{1 / 2} . \tag{69}
\end{equation*}
$$

For their comparison, Table 2 is constructed when errors are computed by

$$
\begin{equation*}
E=\max _{\substack{1 \leq k \leq N \\ 1 \leq n \leq M}}\left|u\left(t_{k}, x_{n}\right)-u_{n}^{k}\right| . \tag{70}
\end{equation*}
$$

Table 3 is constructed for the error of $p(x)$ at the nodes in maximum norm.

## 5. Conclusion

In the present study, the well-posedness of difference problem for the approximate solution of determination of a control

Table 2: Error analysis for the exact solution $u(t, x)$.

| Method | $N=M=20$ | $N=M=40$ | $N=M=80$ |
| :--- | :---: | :---: | :---: |
| lst order of <br> accuracy d.s. | 0.0019 | $9.5692 \times 10^{-4}$ | $4.8241 \times 10^{-4}$ |

TABLE 3: Error analysis for $p(x)$.

| Method | $N=20$ | $N=40$ | $N=80$ |
| :--- | :---: | :---: | :---: |
| 1st order of accuracy d.s. | 0.0145 | 0.0072 | 0.0036 |

parameter for the Schrödinger equation is established. In practice, the stability inequalities for the solution of difference schemes of the approximate solution of three different types of control parameter problems are obtained. The well-posedness of the boundary value problem (1) is established. The stability inequalities for the solution of difference schemes for three different types of control parameter problems for the Schrödinger equation are obtained. Moreover, applying the result of the monograph [15], the high order of accuracy single-step difference schemes for the numerical solution of the boundary value problem (1) can be presented. Of course, the stability inequalities for the solution of these difference schemes have been established without any assumptions about the grid steps.

## Conflict of Interests

The authors declare that they have no conflict of interests.

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## Research Article

# Green's Function Method for Self-Adjoint Realization of Boundary-Value Problems with Interior Singularities 

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#### Abstract

The purpose of this paper is to investigate some spectral properties of Sturm-Liouville type problems with interior singularities. Some of the mathematical aspects necessary for developing our own technique are presented. By applying this technique we construct some special solutions of the homogeneous equation and present a formula and the existence conditions of Green's function. Furthermore, based on these results and introducing operator treatment in adequate Hilbert space, we derive the resolvent operator and prove self-adjointness of the considered problem.


## 1. Introduction

For inhomogeneous linear systems, the basic superposition principle says that the response to a combination of external forces is the self-same combination of responses to the individual forces. In a finite-dimensional system, any forcing function can be decomposed into a linear combination of unit impulse forces, each applied to a single component of the system, and so the full solution can be written as a linear combination of the solutions to the impulse problems. This simple idea will be adapted to boundary value problems governed by differential equations, where the response of the system to a concentrated impulse force is known as Green's function. With Green's function in hand, the solution to the inhomogeneous system with a general forcing function can be reconstructed by superimposing the effects of suitably scaled impulses. Green's function method provides a powerful tool to solve linear problems consisting of a differential equation (partial or ordinary, with, possibly, an inhomogeneous term) and enough initial and/or boundary conditions (also possibly inhomogeneous) so that this problem has a unique solution. The history of Green's function dates back to 1828, when Green [1] published work in which he sought solutions of Poisson's equation $\nabla^{2} u=f$
for the electric potential $u$ defined inside a bounded volume with specified boundary conditions on the surface of the volume. He introduced a function now identified as what Riemann later coined Green's function. In 1877, Neumann [2] embraced the concept of Green's function in his study of Laplace's equation, particularly in the plane. He found that the two-dimensional equivalent of Green's function was not described by singularity of the form $1 /\left|r-r_{0}\right|$ as in the three-dimensional case but by a singularity of the form $\log \left(1 /\left|r-r_{0}\right|\right)$. With the function's success in solving Laplace's equation, other equations began to be solved using Green's function. The heat equation and Green's function have a long association with each other. After discussing heat conduction in free space, the classic solutions of the heat equation in rectangular, cylindrical, and spherical coordinates are offered. In the case of the heat equation, Hobson [3] derived the freespace Green's function for one, two and three dimensions, and the French mathematician Appell [4] recognized that there was a formula similar to Green's for the one-dimensional heat equation. Green's function is particularly well suited for wave problems with the detailed analysis of electromagnetic waves in surface wave guides and water waves. The leading figure in the development of Green's function for the wave equation was Kirchhoff [5], who used it during his study
of the three-dimensional wave. Starting with Green's second formula, he was able to show that the three-dimensional Green's function is

$$
\begin{equation*}
g(x, y, z, t \mid \xi, \eta, \varsigma, \tau)=\frac{\delta(t-\tau-R / c)}{4 \pi R} \tag{1}
\end{equation*}
$$

where $R=\sqrt{(x-\xi)^{2}+(y-\eta)^{2}+(z-\varsigma)^{2}}$.
The application of Green's function to ordinary differential equations involving boundary-value problems began with the work of Burkhardt [6]. Determination of Green's function is also possible using Sturm-Liouville theory. This leads to series representation of Green's function. SturmLiouville problems which contained spectral parameter in boundary conditions form an important part of the spectral theory of boundary value problems. This type of problems has a lot of applications in mechanics and physics (see [79] and references cited therein). In the recent years, there has been increasing interest in this kind of problems which also may have discontinuities in the solution or its derivative at interior points (see [10-18]). In this study, we will investigate some spectral properties of the Sturm-Liouville differential equation on two intervals:

$$
\begin{array}{r}
\mathscr{L} y:=-y^{\prime \prime}(x)+q(x) y(x)=\lambda y(x),  \tag{2}\\
x \in[a, c) \cup(c, b]
\end{array}
$$

on $[a, c) \cup(c, b]$, with eigenparameter-dependent boundary conditions at the end points $x=a$ and $x=b$. One has,

$$
\begin{gather*}
\tau_{1}(y):=\alpha_{10} y(a)+\alpha_{11} y^{\prime}(a)=0,  \tag{3}\\
\tau_{2}(y):=\alpha_{20} y(b)-\alpha_{21} y^{\prime}(b)+\lambda\left(\alpha_{20}^{\prime} y(b)-\alpha_{21}^{\prime} y^{\prime}(b)\right)=0 \tag{4}
\end{gather*}
$$

and the transmission conditions at the singular interior point $x=c$

$$
\begin{align*}
\tau_{3}(y):= & \beta_{11}^{-} y^{\prime}(c-)+\beta_{10}^{-} y(c-) \\
& +\beta_{11}^{+} y^{\prime}(c+)+\beta_{10}^{+} y(c+)=0 \\
\tau_{4}(y):= & \beta_{21}^{-} y^{\prime}(c-)+\beta_{20}^{-} y(c-)  \tag{5}\\
& +\beta_{21}^{+} y^{\prime}(c+)+\beta_{20}^{+} y(c+)=0
\end{align*}
$$

where the potential $q(x)$ is real continuous function in each of the intervals $[a, c)$ and $(c, b]$ and has finite limits $q(c \mp 0)$, $\lambda$ is a complex spectral parameter, $\alpha_{i j}, \beta_{i j}^{ \pm},(i=1,2$ and $j=$ $0,1)$, and $\alpha_{i j}^{\prime}(i=2$ and $j=0,1)$ are real numbers.

Our problem differs from the usual regular SturmLiouville problem in the sense that the eigenvalue parameter $\lambda$ is contained in both differential equation and boundary conditions, and two supplementary transmission conditions at one interior point are added to boundary conditions. Such problems are connected with discontinuous material properties, such as heat and mass transfer, vibrating string problems when the string loaded additionally with points masses, diffraction problems [8,9], and varied assortment of
physical transfer problems. We develop our own technique for the investigation of some spectral properties of this problem. In particular, we construct the Green's function and adequate Hilbert space for self-adjoint realization of the considered problem.

## 2. Some Basic Solutions and Green's Function

Denote the determinant of the $k$ th and $j$ th columns of the matrix

$$
T=\left[\begin{array}{llll}
\beta_{10}^{+} & \beta_{11}^{+} & \beta_{10}^{-} & \beta_{11}^{-}  \tag{6}\\
\beta_{20}^{+} & \beta_{21}^{+} & \beta_{20}^{-} & \beta_{21}^{-}
\end{array}\right]
$$

by $\Delta_{k j}(1 \leq k<j \leq 4)$. For self-adjoint realization in adequate Hilbert space, everywhere below we will assume that

$$
\begin{equation*}
\Delta_{12}>0, \quad \Delta_{34}>0 \tag{7}
\end{equation*}
$$

With a view to construct the Green's function we will define two special solutions of (2) by our own technique as follows. At first, consider the next initial-value problem on the left interval [a, c)

$$
\begin{gather*}
-y^{\prime \prime}+q(x) y=\lambda y \\
y(a)=\alpha_{11}, \quad y^{\prime}(a)=-\alpha_{10} \tag{8}
\end{gather*}
$$

It is known that this problem has an unique solution $u=$ $\varphi^{-}(x, \lambda)$ which is an entire function of $\lambda \in \mathbb{C}$ for each fixed $x \in[a, c)$ (see, e.g., [19]). By applying the similar method of [13], we can prove that (2) on the right interval ( $c, b$ ] has an unique solution $u=\varphi^{+}(x, \lambda)$ satisfying the equalities

$$
\begin{align*}
& \varphi^{+}(c+, \lambda)=\frac{1}{\Delta_{12}}\left(\Delta_{23} \varphi^{-}(c-, \lambda)+\Delta_{24} \frac{\partial \varphi^{-}(c-, \lambda)}{\partial x}\right),  \tag{9}\\
& \frac{\partial \varphi^{+}(c+, \lambda)}{\partial x}=\frac{-1}{\Delta_{12}}\left(\Delta_{13} \varphi^{-}(c-, \lambda)+\Delta_{14} \frac{\partial \varphi^{-}(c-, \lambda)}{\partial x}\right), \tag{10}
\end{align*}
$$

which is also an entire function of the parameter $\lambda$ for each fixed $x \in[c, b]$. Consequently, the solution $u=\varphi(x, \lambda)$ defined by

$$
\varphi(x, \lambda)= \begin{cases}\varphi^{-}(x, \lambda), & x \in[a, c)  \tag{11}\\ \varphi^{+}(x, \lambda), & x \in(c, b]\end{cases}
$$

satisfies (2) on whole $[a, c) \cup(c, b]$, the first boundary condition of (3), and both transmission conditions (5).

By the same technique, we can define the solution by

$$
\psi(x, \lambda)= \begin{cases}\psi^{-}(x, \lambda), & x \in[a, c)  \tag{12}\\ \psi^{+}(x, \lambda), & x \in(c, b]\end{cases}
$$

so that

$$
\begin{align*}
& \psi^{+}(b, \lambda)=\alpha_{21}+\lambda \alpha_{21}^{\prime}, \quad \frac{\partial \psi^{+}(b, \lambda)}{\partial x}=\alpha_{20}+\lambda \alpha_{20}^{\prime},  \tag{13}\\
& \psi^{-}(c-, \lambda)=\frac{-1}{\Delta_{34}}\left(\Delta_{14} \psi^{+}(c+, \lambda)+\Delta_{24} \frac{\partial \psi^{+}(c+, \lambda)}{\partial x}\right),  \tag{14}\\
& \frac{\partial \psi^{-}(c-, \lambda)}{\partial x}=\frac{1}{\Delta_{34}}\left(\Delta_{13} \psi^{+}(c+, \lambda)+\Delta_{23} \frac{\partial \psi^{+}(c+, \lambda)}{\partial x}\right) . \tag{15}
\end{align*}
$$

Consequently, $\psi(x, \lambda)$ satisfies (2) on whole $[a, c) \cup(c, b]$, the second boundary condition (4), and both transmission condition (5). By using (9), (10), (14), and (15) and the well-known fact that the Wronskians $\omega^{-}(\lambda):=W\left[\varphi^{-}(x, \lambda), \psi^{-}(x, \lambda)\right]$ and $w^{+}(\lambda):=W\left[\varphi^{+}(x, \lambda), \psi^{+}(x, \lambda)\right]$ are independent of variable $x$, it is easy to show that $\Delta_{12} w^{+}(\lambda)=\Delta_{34} w^{-}(\lambda)$. We will introduce the characteristic function for the problems (2)-(5) as

$$
\begin{equation*}
w(\lambda):=\Delta_{34} w^{-}(\lambda)=\Delta_{12} w^{+}(\lambda) \tag{16}
\end{equation*}
$$

Similar to [13], we can prove that there are infinitely many eigenvalues $\lambda_{n}, n=1,2, \ldots$ of the BVTP (2)-(5) which coincide with the zeros of characteristic function $w(\lambda)$.

Now, let us consider the nonhomogenous differential equation

$$
\begin{equation*}
y^{\prime \prime}+(\lambda-q(x)) y=f(x) \tag{17}
\end{equation*}
$$

on $[a, c) \cup(c, b]$ together with the same boundary and transmission conditions (2)-(5), when $w(\lambda) \neq 0$. We will search the solution of this problem in the form (see, for example, [13]):

$$
Y(x, \lambda)=\left\{\begin{array}{ll}
\frac{\Delta_{34} \psi^{-}(x, \lambda)}{\omega(\lambda)}  \tag{18}\\
& \times \int_{a}^{x} \varphi^{-}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{34} \varphi^{-}(x, \lambda)}{\omega \lambda} \\
& \times \int_{x}^{c-} \psi^{-}(y, \lambda) f(y) d y \\
& +d_{11} \varphi^{-}(x, \lambda) \\
& +d_{12} \psi^{-}(x, \lambda), \\
\frac{\Delta_{12} \psi^{+}(x, \lambda)}{\omega(\lambda)} & \text { for } x \in[a, c) \\
& \times \int_{c+}^{x} \varphi^{+}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{12} \varphi^{+}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{x}^{b} \psi^{+}(y, \lambda) f(y) d y \\
& +d_{21} \varphi^{+}(x, \lambda) \\
& +d_{22} \psi^{+}(x, \lambda)
\end{array} \quad \text { for } x \in(c, b]\right.
$$

where $d_{i j}(i, j=1,2)$ are arbitrary constants. Putting in (3)(5) we have $d_{12}=0, d_{21}=0$,

$$
\begin{align*}
& d_{11}=\frac{\Delta_{12}}{w(\lambda)} \int_{c+}^{b} f(y) \psi^{+}(y, \lambda) d y \\
& d_{22}=\frac{\Delta_{34}}{w(\lambda)} \int_{a}^{c-} f(y) \varphi^{-}(y, \lambda) d y \tag{19}
\end{align*}
$$

Now, by substituting these equalities in (18), the following formula is obtained for the solution $Y=Y_{0}(x, \lambda)$ of (17) under boundary and transmission conditions (3)-(5):

$$
Y_{0}(x, \lambda)=\left\{\begin{align*}
& \frac{\Delta_{34} \psi^{-}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{a}^{x} \varphi^{-}(y, \lambda) f(y) d y \\
&+\frac{\Delta_{34} \varphi^{-}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{x}^{c-} \psi^{-}(y, \lambda) f(y) d y \\
&+\frac{\Delta_{12} \varphi^{-}(x, \lambda)}{w(\lambda)} \\
& \times \int_{c_{c+}}^{b} f(y) \psi^{+}(y, \lambda) d y, \quad \text { for } x \in[a, c), \\
& \frac{\Delta_{12} \psi^{+}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{c+}^{x} \varphi^{+}(y, \lambda) f(y) d y \\
&+\frac{\Delta_{12} \varphi^{+}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{x}^{b} \psi^{+}(y, \lambda) f(y) d y  \tag{20}\\
&+\frac{\Delta_{34} \psi^{+}(x, \lambda)}{w(\lambda)} \\
& \times \int_{a}^{c-} f(y) \varphi^{-}(y, \lambda) d y \quad \text { for } x \in(c, b]
\end{align*}\right.
$$

From this formula, we find that the Green's function of the problem (2)-(5) has the form:

$$
G_{0}(x, y ; \lambda)=\left\{\begin{array}{rr}
\frac{\varphi(y, \lambda) \psi(x, \lambda)}{\omega(\lambda)}, & \text { for } a \leq y \leq x \leq b  \tag{21}\\
\frac{\varphi(x, \lambda) \psi(y, \lambda)}{\omega(\lambda)}, & \text { for } a \leq x \leq y \leq b \\
x, y \neq c
\end{array}\right.
$$

and the solution (20) can be rewritten in the terms of this Green's function as

$$
\begin{align*}
Y_{0}(x, \lambda)= & \Delta_{34} \int_{a}^{c-} G_{0}(x, y ; \lambda) f(y) d y \\
& +\Delta_{12} \int_{c+}^{b} G_{0}(x, y ; \lambda) f(y) d y \tag{22}
\end{align*}
$$

## 3. Construction of the Resolvent Operator by means of Green's Function in the Adequate Hilbert Space

In this section, we define a linear operator $A$ in suitable Hilbert space in such a way that the considered problem can be interpreted as the eigenvalue problem of this operator. For this, we assume that $\Delta_{0}:=\alpha_{21} \alpha_{20}^{\prime}-\alpha_{20} \alpha_{21}^{\prime}>0$ and introduce a new inner product in the Hilbert space $H=\left(L_{2}[a, c) \oplus\right.$ $\left.L_{2}(c, b]\right) \oplus \mathbb{C}$ by

$$
\begin{align*}
\langle F, G\rangle_{1}:= & \Delta_{34} \int_{a}^{c-} f(x) \overline{g(x)} d x \\
& +\Delta_{12} \int_{c+}^{b} f(x) \overline{g(x)} d x+\frac{\Delta_{12}}{\Delta_{0}} f_{1} \overline{g_{1}} \tag{23}
\end{align*}
$$

for $F=\left(f(x), f_{1}\right), G=\left(g(x), g_{1}\right) \in H$.
Remark 1. Note that this modified inner product is equivalent to standard inner product of $\left(L_{2}[a, c) \oplus L_{2}(c, b]\right) \oplus \mathbb{C}$; so $H_{1}=$ $\left(L_{2}[a, c) \oplus L_{2}(c, b] \oplus \mathbb{C},\langle\cdot, \cdot\rangle_{1}\right)$ is also Hilbert space.

For convenience, denote

$$
\begin{align*}
& T_{b}(f):=\alpha_{20} f(b)-\alpha_{21} f^{\prime}(b) \\
& T_{b}^{\prime}(f):=\alpha_{20}^{\prime} f(b)-\alpha_{21}^{\prime} f^{\prime}(b) \tag{24}
\end{align*}
$$

and define a linear operator

$$
\begin{equation*}
A\left(\mathscr{L} f(x), T_{b}^{\prime}(f)\right)=\left(\mathscr{L} f,-T_{b}(f)\right) \tag{25}
\end{equation*}
$$

with the domain $D(A)$ consisting of all elements $\left(f(x), f_{1}\right) \in$ $H_{1}$ such that $f(x)$ and $f^{\prime}(x)$ are absolutely continuous in each interval $[a, c)$ and $(c, b]$ and has a finite limit $f(c \mp 0)$ and $f_{1}^{\prime}(c \mp 0), \mathscr{L} f \in L_{2}[a, b], \tau_{1} f=\tau_{3} f=\tau_{4} f=0$ and $f_{1}=$ $T_{b}^{\prime}(f)$.

Consequently the problems (2)-(5) can be written in the operator form as

$$
\begin{gather*}
A F=\lambda F, \\
F=\left(f(x), T_{b}^{\prime}(f)\right) \in D(A) \tag{26}
\end{gather*}
$$

in the Hilbert space $H_{1}$. It is easy to see that the operator $A$ is well defined in $H_{1}$. Let $A$ be defined as above and let $\lambda$ not be an eigenvalue of this operator. For construction of the resolvent operator $R(\lambda, A):=(\lambda-A)^{-1}$, we will solve the operator equation

$$
\begin{equation*}
(\lambda-A) Y=F \tag{27}
\end{equation*}
$$

for $F \in H_{1}$. This operator equation is equivalent to the nonhomogeneous differential equation

$$
\begin{equation*}
y^{\prime \prime}+(\lambda-q(x)) y=f(x) \tag{28}
\end{equation*}
$$

on $[a, c) \cup(c, b]$ subject to nonhomogeneous boundary conditions and homogeneous transmission conditions

$$
\begin{equation*}
\tau_{1}(y)=\tau_{3}(y)=\tau_{4}(y)=0, \quad \tau_{2}(y)=-f_{1} \tag{29}
\end{equation*}
$$

Let $\operatorname{Im} \lambda \neq 0$. We already know that the general solution of (28) has the form (18). Putting this general solution in (29) yields

$$
\begin{gather*}
d_{11}=\frac{\Delta_{12}}{\omega(\lambda)} \int_{c+}^{b} \psi^{+}(y, \lambda) f(y) d y+\frac{\Delta_{12} f_{1}}{\omega(\lambda)} \\
d_{12}=0, \quad d_{21}=\frac{\Delta_{12} f_{1}}{\omega(\lambda)}  \tag{30}\\
d_{22}=\frac{\Delta_{34}}{\omega(\lambda)} \int_{a}^{c-} \varphi^{-}(y, \lambda) f(y) d y
\end{gather*}
$$

Thus, the problems (28)-(29) have a unique solution

$$
Y(x, \lambda)=\left\{\begin{array}{ll}
\frac{\Delta_{34} \psi^{-}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{a}^{x} \varphi^{-}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{34} \varphi^{-}(x, \lambda)}{\omega(\lambda)} \\
& \times \int_{x}^{c-} \psi^{-}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{12} \varphi^{-}(x, \lambda)}{\omega(\lambda)} \\
& \times\left(\int_{c+}^{b} \psi^{+}(y, \lambda) f(y) d y\right. \\
& \left.+f_{1}\right) \\
& \quad \int_{12} \psi^{+}(x, \lambda) \\
& +\frac{\Delta_{12}^{x+} \varphi^{+}(x)}{\omega(\lambda)} \varphi^{+}(y, \lambda) f(y) d y \\
& \times \int_{x}^{b} \psi^{+}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{34} \psi^{+}(x, \lambda)}{\omega(\lambda)}  \tag{31}\\
& \times \int_{a}^{c-} \varphi^{-}(y, \lambda) f(y) d y \\
& +\frac{\Delta_{12} f_{1} \varphi^{+}(x, \lambda)}{\omega(\lambda)},
\end{array} \quad \text { for } x \in[a, c)\right.
$$

Consequently,

$$
\begin{equation*}
Y(x, \lambda)=Y_{0}(x, \lambda)+f_{1} \Delta_{12} \frac{\varphi(x, \lambda)}{\omega(\lambda)} \tag{32}
\end{equation*}
$$

where $G_{0}(x, \lambda)$ and $Y_{0}(x, \lambda)$ are the same with (21) and (22), respectively. From the equalities (13) and (21), it follows that

$$
\begin{equation*}
\left(G_{0}(x, ; \lambda)\right)_{\beta}^{\prime}=\frac{\varphi(x, \lambda)}{\omega(\lambda)} \tag{33}
\end{equation*}
$$

By using (22), (31), and (33), we deduce that

$$
\begin{align*}
Y(x, \lambda)= & \Delta_{34} \int_{a}^{c-} G_{0}(x, y ; \lambda) f(y) d y \\
& +\Delta_{12} \int_{c+}^{b} G_{0}(x, y ; \lambda) f(y) d y  \tag{34}\\
& +f_{1} \Delta_{12}\left(G_{0}(x, \cdot ; \lambda)\right)_{\beta}^{\prime} .
\end{align*}
$$

Consequently, the solution $Y(F, \lambda)$ of the operator equation (27) has the form:

$$
\begin{equation*}
Y(F, \lambda)=\left(Y(x, \lambda),(Y(\cdot, \lambda))_{\beta}^{\prime}\right) . \tag{35}
\end{equation*}
$$

From (34) and (35), it follows that

$$
\begin{equation*}
Y(F, \lambda)=\left(\left\langle G_{x, \lambda}, \bar{F}\right\rangle_{1},\left(\left\langle G_{x, \lambda}, \bar{F}\right\rangle_{1}\right)_{\beta}^{\prime}\right) \tag{36}
\end{equation*}
$$

where under Green's vector $G_{x, \lambda}$ we mean

$$
\begin{equation*}
G_{x, \lambda}:=\left(G_{0}(x, \cdot ; \lambda),\left(G_{0}(x, \cdot ; \lambda)\right)_{\beta}^{\prime}\right) \tag{37}
\end{equation*}
$$

Now, making use of (21), (34), (35), (36), and (37), we see that if $\lambda$ not an eigenvalue of operator $A$, then

$$
\begin{equation*}
Y(F, \lambda) \in D(A), \quad \text { for } F \in H_{1} \tag{38}
\end{equation*}
$$

$$
Y((\lambda-A) F, \lambda)=F, \quad \text { for } \in D(A),
$$

$$
\begin{equation*}
\|Y(F, \lambda)\| \leq|\operatorname{Im} \lambda|^{-1}\|F\|, \quad \text { for } F \in H_{1}, \operatorname{Im} \lambda \neq 0 \tag{39}
\end{equation*}
$$

Hence, each nonreal $\lambda \in \mathbb{C}$ is a regular point of an operator $A$ and

$$
\begin{equation*}
R(\lambda, A) F=\left(\left\langle G_{x, \lambda}, \bar{F}\right\rangle_{1},\left(\left\langle G_{x, \lambda}, \bar{F}\right\rangle_{1}\right)_{\beta}^{\prime}\right), \quad \text { for } F \in H_{1} \tag{40}
\end{equation*}
$$

Because of (38) and (40),

$$
\begin{equation*}
(\lambda-A) D(A)=(\bar{\lambda}-A) D(A)=H_{1}, \quad \text { for } \operatorname{Im} \lambda \neq 0 \tag{41}
\end{equation*}
$$

Theorem 2. The Resolvent operator $R(\lambda, A)$ is compact in the Hilbert space $H_{1}$.

Proof. Let us define the operators $\mathbb{B}_{\lambda}: L_{2}[a, c) \oplus L_{2}(c, b] \rightarrow$ $L_{2}[a, c) \oplus L_{2}(c, b], \widetilde{\mathbb{B}_{\lambda}}: H_{1} \rightarrow H_{1}$ and $\mathscr{C}_{\lambda}: H_{1} \rightarrow H_{1}$ by

$$
\begin{align*}
& \mathbb{B}_{\lambda} f:= \Delta_{34} \int_{a}^{c-} G_{0}(x, y ; \lambda) f(y) d y \\
&+\Delta_{12} \int_{c+}^{b} G_{0}(x, y ; \lambda) f(y) d y  \tag{42}\\
& \widetilde{\mathbb{B}_{\lambda}} F:=\left(\mathbb{B}_{\lambda} f,\left(\mathbb{B}_{\lambda} f\right)_{b}^{\prime}\right) \\
& \mathscr{C}_{\lambda} F:=\left(f_{1} \Delta_{12} \frac{\varphi(x, \lambda)}{\omega(\lambda)}, f_{1} \Delta_{12} \frac{(\varphi(\cdot, \lambda))_{b}^{\prime}}{\omega(\lambda)}\right),
\end{align*}
$$

respectively. Then we can expressed the resolvent operator $R(\lambda, A)$ as $R(\lambda, A)=\widetilde{\mathbb{B}_{\lambda}}+\mathscr{C}_{\lambda}$. Since the linear operator $\mathbb{B}_{\lambda}$ is compact in the Hilbert space $L_{2}[a, c) \oplus L_{2}(c, b]$, the linear operator $\widetilde{\mathbb{B}_{\lambda}}$ is compact in the Hilbert space $H_{1}$. Compactness $\mathscr{C}_{\lambda}$ in $H_{1}$ is obvious. Therefore, the resolvent operator $R(\lambda, A)$ is also compact in $H_{1}$.

## 4. Self-Adjoint Realization of the Problem

At first, we will prove the following lemmas.
Lemma 3. The domain $D(A)$ is dense in $H_{1}$.
Proof. Suppose that the element $G=\left(g(x), g_{1}\right) \in H_{1}$ is orthogonal to $D(A)$. Denote by $C_{0}^{\infty}[a, c) \oplus C_{0}^{\infty}(c, b]$ the set of infinitely differentiable functions on $[a, c) \cup(c, b]$, each of which vanishes on some neighborhoods of the end-points $x=a, x=c$, and $x=b$. Since $(f(\cdot), 0) \in D(A)$ for $f \in C_{0}^{\infty}[a, c) \oplus C_{0}^{\infty}(c, b]$, we have

$$
\begin{equation*}
\Delta_{34} \int_{a}^{c-} f(x) \overline{g(x)} d x+\Delta_{12} \int_{c+}^{b} f(x) \overline{g(x)} d x=0 \tag{43}
\end{equation*}
$$

for all $f \in C_{0}^{\infty}[a, c) \oplus C_{0}^{\infty}(c, b]$. Since $f$ is arbitrary,

$$
\begin{align*}
& \int_{a}^{c-} f_{1}(x) \overline{g(x)} d x=0 \\
& \int_{c+}^{b} f_{2}(x) \overline{g(x)} d x=0 \tag{44}
\end{align*}
$$

for all $f_{1} \in C_{0}^{\infty}[a, c]$ and $f_{2} \in C_{0}^{\infty}[c, b]$, respectively. Taking into account that $C_{0}^{\infty}[a, c]$ and $C_{0}^{\infty}[c, b]$ are dense in $L_{2}[a, c]$ and $L_{2}[c, b]$, respectively, we get that the function $g(x)$ is equal to zero as element of $L_{2}[a, c] \oplus L_{2}[c, b]$. By choosing an element $F_{0}=\left(f_{0}(x), T_{b}^{\prime} f_{0}\right)$ such that $T_{b}^{\prime} f_{0}=1$ and putting in $\left\langle F_{0}, G\right\rangle_{1}=0$, we have $g_{1}=0$. So $G=(0,0)$. The proof is completed.

Lemma 4. The linear operator $A$ is symmetric in the Hilbert space $H_{1}$.

Proof. Let $F=\left(f(x), T_{b}^{\prime}(f)\right), G=\left(G_{1}(x), T_{b}^{\prime}(f)\right) \in D(A)$. By partial integration, we get

$$
\begin{align*}
\langle A F, G\rangle_{1}= & \Delta_{34} \int_{a}^{c-}(\mathscr{L} f)(x) \overline{g(x)} d x \\
& +\Delta_{12} \int_{c+}^{b}(\mathscr{L} f)(x) \overline{g(x)} d x+\frac{\Delta_{12}}{\Delta_{0}} T_{b}(f) \overline{T_{b}^{\prime}(g)} \\
= & \langle F, A G\rangle_{1}+\Delta_{34} W(f, \bar{g} ; c-0)-\Delta_{34} W(f, \bar{g} ; a) \\
& +\Delta_{12} W(f, \bar{g} ; b)-\Delta_{12} W(f, \bar{g} ; c+0) \\
& +\frac{\Delta_{12}}{\Delta_{0}}\left(T_{b}^{\prime}(f) \overline{T_{b}(g)}-T_{b}(f) \overline{T_{b}^{\prime}(g)}\right) . \tag{45}
\end{align*}
$$

From the definition of domain $D(A)$, we see easily that $W(f, \bar{g} ; a)=0$. The direct calculation gives

$$
\begin{gather*}
T_{b}^{\prime}(f) \overline{T_{b}(g)}-T_{b}(f) \overline{T_{b}^{\prime}(g)}=-\Delta_{0} W(f, \bar{g} ; b) \\
W(f, \bar{g} ; c-0)=\frac{\Delta_{12}}{\Delta_{34}} W(f, \bar{g} ; c+0) \tag{46}
\end{gather*}
$$

Substituting these equalities in (45), we have

$$
\begin{equation*}
\langle A F, G\rangle_{1}=\langle F, A G\rangle_{1}, \quad \text { for every } F, G \in D(A) ; \tag{47}
\end{equation*}
$$

so the operator $A$ is symmetric in $H$. The proof is completed.

Remark 5. By Lemma 4, all eigenvalues of the problems (2)-(5) are real. Therefore, it is enough to investigate only real-valued eigenfunctions. Taking in view this fact, we can assume that the eigenfunctions are real-valued.

Corollary 6. If $\lambda_{n}$ and $\lambda_{m}$ are distinct eigenvalues of the problems (2)-(5), then the corresponding eigenfunctions $u_{n}(x)$ and $u_{m}(x)$ are orthogonal in the sense of the following equality:

$$
\begin{align*}
\Delta_{34} & \int_{a}^{c-} u(x) v(x) d x+\Delta_{12} \int_{c+}^{b} u(x) v(x) d x  \tag{48}\\
& +\frac{\Delta_{12}}{\Delta_{0}} T_{b}^{\prime}(u) T_{b}^{\prime}(v)=0
\end{align*}
$$

Proof. The proof is immediate from the fact that the eigenelements $\left(u(x), T_{b}^{\prime}(u)\right)$ and $\left(v(x), T_{b}^{\prime}(v)\right)$ of the symmetric linear operator $A$ are orthogonal in the Hilbert space $H_{1}$.

## Theorem 7. The operator $A$ is self-adjoint in $H_{1}$.

Proof. It is clear that the symmetry of a densely defined $A$ is equivalent to the condition $\langle A F, G\rangle_{1}=\langle F, A G\rangle_{1}$ for all $F, G \in$ $D(A)$. Notice that this implies that $A^{*} \supset A$. If, in addition, we also have that $D\left(A^{*}\right)=D(A)$, then $A$ is self-adjoint. Let $U \in D\left(A^{*}\right)$. Then, by definition of $A^{*}$,

$$
\begin{equation*}
\langle A V, U\rangle_{1}=\left\langle V, A^{*} U\right\rangle_{1}, \quad \forall V \in D(A) . \tag{49}
\end{equation*}
$$

Let $\lambda_{0}$ be any complex number for which $\operatorname{Im} \lambda_{0} \neq 0$. From this, it follows that

$$
\begin{equation*}
\left\langle\left(\lambda_{0} I-A\right) V, U\right\rangle_{1}=\left\langle V,\left(\overline{\lambda_{0}} I-A^{*}\right) U\right\rangle_{1} . \tag{50}
\end{equation*}
$$

Since any nonreal complex number is a regular point of $A$, we can define the vector $U_{0} \in D(A)$ as

$$
\begin{equation*}
U_{0}=R\left(\overline{\lambda_{0}}, A\right)\left(\overline{\lambda_{0}} U-A^{*} U\right) \tag{51}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
\left(\overline{\lambda_{0}} I-A\right) U_{0}=\overline{\lambda_{0}} U-A^{*} U . \tag{52}
\end{equation*}
$$

Inserting this in (50) and recalling that $A$ is symmetric and $U_{0} \in D(A)$, we have

$$
\begin{aligned}
\left\langle\left(\lambda_{0} I-A\right) V, U\right\rangle_{1} & =\left\langle V,\left(\overline{\lambda_{0}} I-A\right) U_{0}\right\rangle_{1} \\
& =\left\langle V, \overline{\lambda_{0}} U_{0}\right\rangle_{1}-\left\langle V, A U_{0}\right\rangle_{1} \\
& =\left\langle\lambda_{0} V, U_{0}\right\rangle_{1}-\left\langle A V, U_{0}\right\rangle_{1} \\
& =\left\langle\left(\lambda_{0} I-A\right) V, U_{0}\right\rangle_{1} .
\end{aligned}
$$

Consequently,

$$
\begin{equation*}
\left\langle\left(\lambda_{0} I-A\right) V, U-U_{0}\right\rangle_{1}=0 \quad \forall V \in H_{1} . \tag{54}
\end{equation*}
$$

Since $\lambda_{0}$ is regular point of $A$, we can choose $V=$ $R\left(\lambda_{0}, A\right)\left(U-U_{0}\right)$. Inserting this in the last equality yields $\left\|U-U_{0}\right\|_{1}=0$, and so $U=U_{0}$, and therefore, $U \in D(A)$. The proof is completed.

Remark 8. The main results of this study are derived in modified Hilbert space under simple condition (7). We can show that these conditions cannot be omitted. Indeed, let us consider the next special case of the problems (2)-(5):

$$
\begin{gather*}
-y^{\prime \prime}(x)=\lambda y(x), \quad x \in[-1,0) \cup(0,1] \\
y(-1)=0 \\
(\lambda-1) y^{\prime}(-1)+\lambda y(1)=0  \tag{55}\\
y(0-)=y(0+) \\
y^{\prime}(0-)=-y^{\prime}(0+)
\end{gather*}
$$

for which the condition (7) is not valid $\left(\Delta_{12}<0\right)$. It is easy to verify that the operator $A$ corresponding to this problem is not symmetric in the classic Hilbert space $L_{2}[a, c) \oplus L_{2}(c, b] \oplus$ $\mathbb{C}$ under standard inner-product. Consider the following:

$$
\begin{equation*}
\langle F, G\rangle_{1}:=\int_{a}^{c} f(x) \overline{g(x)} d x+\int_{c}^{b} f(x) \overline{g(x)} d x+\frac{\Delta_{12}}{\Delta_{0}} f_{1} \overline{g_{1}} . \tag{56}
\end{equation*}
$$

Moreover, it is well known that the standard Sturm-liouville problems have infinitely many real eigenvalues. But it can be shown by direct calculation that the problem (55) has only one eigenvalue $\lambda=1$.

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## Research Article

# Nonstationary Fronts in the Singularly Perturbed Power-Society Model 

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The theory of contrasting structures in singularly perturbed boundary problems for nonlinear parabolic partial differential equations is applied to the research of formation of steady state distributions of power within the nonlinear "power-society" model. The interpretations of the solutions to the equation are presented in terms of applied model. The possibility theorem for the problem of getting the solution having some preassigned properties by means of parametric control is proved.

## 1. Introduction

Since the work [1], the theory of contrasting structures has become one of the most booming areas of research of the singularly perturbed differential equations [2-4].

The contrasting structures having the form of nonstationary fronts for parabolic partial differential equations were studied in [5]. The theory was applied to propagation of magnetic fronts in spiral galaxies [6-8]. Here we consider the nonstationary fronts in the Mikhailov "power-society" model [9-12] and the possibility to control them.

In the most general case the "power-society" model has the form of a Neumann boundary value problem for nonlinear parabolic integrodifferential equation. In the absence of some political mechanisms the model is reduced to singularly perturbed parabolic Neumann boundary value problem:

$$
\begin{gather*}
\frac{\partial p}{\partial t}=\varepsilon^{2} \frac{\partial^{2} p}{\partial x^{2}}+F(p, x) \\
\left.\frac{\partial p}{\partial x}\right|_{x=0}=\left.\frac{\partial p}{\partial x}\right|_{x=1}=0  \tag{1}\\
p(x, 0)=p^{0}(x)
\end{gather*}
$$

Here $\varepsilon \ll 1$ is a small positive parameter. Within the "powersociety" model this parameter is small if the hierarchy is long or if society is strong.

The steady-state problem has the form

$$
\begin{equation*}
\varepsilon^{2} \frac{\partial^{2} p}{\partial x^{2}}+F(p, x)=0,\left.\quad \frac{\partial p}{\partial x}\right|_{x=0}=\left.\frac{\partial p}{\partial x}\right|_{x=1}=0 \tag{2}
\end{equation*}
$$

Let the following conditions hold [2-4].
(1) The function $F(p, x)$ has continuous partial derivatives for $0 \leq x \leq 1$ and $p \in(-\infty,+\infty)$.
(2) The degenerate equation $F(p, x)=0$ has three roots $p=\varphi_{1}(x), p=\varphi_{2}(x)$, and $p=\varphi_{3}(x)$ such that $\varphi_{1}(x)<\varphi_{2}(x)<\varphi_{3}(x), 0 \leq x \leq 1$.
(3) The following inequalities

$$
\begin{equation*}
\left.\frac{\partial F}{\partial p}\right|_{p=\varphi_{1}(x)}<0,\left.\quad \frac{\partial F}{\partial p}\right|_{p=\varphi_{2}(x)}>0,\left.\quad \frac{\partial F}{\partial p}\right|_{p=\varphi_{3}(x)}<0 \tag{3}
\end{equation*}
$$

take place.
(4) The equation

$$
\begin{equation*}
\Phi(x)=\int_{\varphi_{1}(x)}^{\varphi_{3}(x)} F(p, x) d p=0 \tag{4}
\end{equation*}
$$

has isolated root $x=x_{0}$ on the interval $0<x<1$.
Under these conditions,
(i) if $\Phi^{\prime}\left(x_{0}\right)<0$, then the solution $p(x, \varepsilon)$ of the problem (2) exists such that

$$
\lim _{\varepsilon \rightarrow 0} p(x, \varepsilon)= \begin{cases}\varphi_{3}(x), & 0<x<x_{0}  \tag{5}\\ \varphi_{1}(x), & x_{0}<x<1\end{cases}
$$

and it is an asymptotically stable stationary solution of problem (1);
(ii) if $\Phi^{\prime}\left(x_{0}\right)>0$, solution $p(x, \varepsilon)$ of problem (2) exists such that

$$
\lim _{\varepsilon \rightarrow 0} p(x, \varepsilon)= \begin{cases}\varphi_{1}(x), & 0<x<x_{0}  \tag{6}\\ \varphi_{3}(x), & x_{0}<x<1\end{cases}
$$

and it is an asymptotically stable stationary solution of problem (1).

The solutions that satisfy (5) or (6) are called the steplike contrasting structures or stationary fronts. There are also other stable stationary solutions of the problem (1). In particular, under Conditions 1-3 the existence of two more solutions, one of which is close to $\varphi_{1}(x)$ :

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} p(x, \varepsilon)=\varphi_{1}(x), \quad 0<x<x_{0} \tag{7}
\end{equation*}
$$

and the other one is close to $\varphi_{3}(x)$ :

$$
\begin{equation*}
\lim _{\varepsilon \rightarrow 0} p(x, \varepsilon)=\varphi_{3}(x), \quad 0<x<x_{0} \tag{8}
\end{equation*}
$$

is guaranteed.
There is an important problem of correspondence between a set of initial functions and a set of steady stationary solutions: given initial function $p^{0}(x)$, what steady-state solution will we have at $t \rightarrow+\infty$ ? And there is the inverse problem: if one of the steady states is more desirable than others, which conditions on $p^{0}(x)$ guarantee approach to this desirable steady state?

At last, when studying mathematical models of particular processes there is the following question which arises: if the existing $p^{0}(x)$ does not correspond to the desirably steady state, is it possible to change the right-hand part of (1) so that the solution would evolve to the desirable steady state?

This work is aimed at considering these problems for the "power-society" model which describes the dynamics of the power distribution in a hierarchy.

We base our study on the theory of contrasting structures [2-5], especially on the Butuzov-Nedelko theorem [13]. Some other issues related to nonstationary fronts were studied in [14-16].

## 2. Nonstationary Fronts and Interpretation in the Nonlinear Singularly Perturbed "Power-Society" Model

This section deals with mathematical modeling of the processes of power dynamics in the hierarchical structures. The model was firstly introduced by Mikhailov, 1994, and the books by Samarskii and Mikhailov 1997 and Mikhailov 2005 should also be mentioned.

Here the hierarchy is a ranked set of instances. Each instance has a particular set of powers. The amount of powers changes with time, and we call such variability the power dynamics. We suppose that there exists a numerical variable which specifies the amount of powers of a particular instance. The power dynamics appear through (a) the self-streamlining of the hierarchy and (b) the influence of the society.

Let us denote the rank of the instance in the hierarchy by $x$ so that $x=0$ at the top of the hierarchy and $x=1$ at the bottom. Denote by $p(x, t)$ the amount of powers of instance at time $t$.

The equation of the "power-society" model $[1,2]$ has the form (1), and $F(p, x)$ is called the reaction of a civil society. The paper [1] has shown that if $F(p, x)=-k_{1}\left(p-p_{0}(x)\right)$ (where $k_{1}=$ const $>0$ and the function $p_{0}(x)$ is the attractive power profile), then the solution $p=p_{0}(x)$ of the stationary degenerated equation $F(p, x)=0$ is stable. This means that the solution $p(x, t, \varepsilon)$ of (1) tends to $p_{0}(x)$ when $t \rightarrow+\infty$, $0<x<1$. So for sufficiently large values of $t$ the power profile is close to $p_{0}(x)$.

It was very important in the paper [1] that only one attractive profile is supposed to exist. Here we consider the case of two stable power profiles $\varphi_{1}(x)$ and $\varphi_{3}(x)$, and each of them is attractive. We call $\varphi_{1}(x)$ the participatory profile and $\varphi_{3}(x)$ the iron-hand profile. Both of them are stable due to inequalities (3).

Henceforth we consider the function $F(p, x)$ having the cubic nonlinearity. So we consider the equation

$$
\begin{equation*}
\frac{\partial p}{\partial t}=\varepsilon^{2} \frac{\partial^{2} p}{\partial x^{2}}-k_{1}(x)\left(p-\varphi_{1}(x)\right)\left(p-\varphi_{2}(x)\right)\left(p-\varphi_{3}(x)\right) \tag{9}
\end{equation*}
$$

with boundary value conditions

$$
\begin{equation*}
\left.\frac{\partial p}{\partial x}\right|_{x=0}=\left.\frac{\partial p}{\partial x}\right|_{x=1}=0 \tag{10}
\end{equation*}
$$

The following conditions are supposed to hold.
Condition 1. The functions $k_{1}(x), \varphi_{i}(x), i=1,2,3$, have the continuous derivatives for $0 \leq x \leq 1$.

Condition 2. Inequalities $k_{1}(x)>0, \varphi_{1}(x)<\varphi_{2}(x)<\varphi_{3}(x)$ hold true for $0 \leq x \leq 1$.

We also notice that though $\varphi_{1}(x)>0$ because of the politological meaning of the function $\varphi_{i}(x)$, this condition must not be required from the mathematical point of view.

Making the substitution (see [17])

$$
\begin{equation*}
p=q \frac{\varphi_{3}(x)-\varphi_{1}(x)}{2}+\frac{\varphi_{3}(x)+\varphi_{1}(x)}{2} \tag{11}
\end{equation*}
$$

we obtain the equation for the function $u(x, t, \varepsilon)$ :

$$
\begin{align*}
\frac{\partial q}{\partial t}= & \varepsilon^{2} \frac{\partial^{2} q}{\partial x^{2}}-\gamma(x)\left(q^{2}-1\right)(q-\varphi(x))  \tag{12}\\
& +\varepsilon^{2} \frac{\partial q}{\partial x} \Phi_{1}(x)+\varepsilon^{2} \Phi_{2}(q, x)
\end{align*}
$$

Here

$$
\begin{gather*}
\gamma(x)=k_{1}(x)\left(\frac{\varphi_{3}(x)-\varphi_{1}(x)}{2}\right)^{2}>0  \tag{13}\\
\varphi(x)=\frac{2 \varphi_{2}(x)-\varphi_{3}(x)-\varphi_{1}(x)}{\varphi_{3}(x)-\varphi_{1}(x)}
\end{gather*}
$$

(notice that $-1<\varphi(x)<1$ ),

$$
\begin{gather*}
\Phi_{1}(x)=\frac{2\left(\varphi_{3}^{\prime}(x)-\varphi_{1}^{\prime}(x)\right)}{\varphi_{3}(x)-\varphi_{1}(x)}  \tag{14}\\
\Phi_{2}(q, x)=q \frac{\varphi_{3}^{\prime \prime}(x)-\varphi_{1}^{\prime \prime}(x)}{\varphi_{3}(x)-\varphi_{1}(x)}+\frac{\varphi_{3}^{\prime \prime}(x)+\varphi_{1}^{\prime \prime}(x)}{\varphi_{3}(x)-\varphi_{1}(x)} .
\end{gather*}
$$

Function $u(x, t, \varepsilon)$ satisfies boundary conditions

$$
\begin{align*}
& {\left.\left[\varphi_{3}(0)-\varphi_{1}(0)\right] \frac{\partial q}{\partial x}\right|_{x=0}} \\
& \quad+\left.\left[\varphi_{3}^{\prime}(0)-\varphi_{1}^{\prime}(0)\right] q\right|_{x=0}+\varphi_{3}^{\prime}(0)+\varphi_{1}^{\prime}(0)=0 \\
& {\left.\left[\varphi_{3}(1)-\varphi_{1}(1)\right] \frac{\partial q}{\partial x}\right|_{x=1}}  \tag{15}\\
& \quad+\left.\left[\varphi_{3}^{\prime}(1)-\varphi_{1}^{\prime}(1)\right] q\right|_{x=1}+\varphi_{3}^{\prime}(1)+\varphi_{1}^{\prime}(1)=0
\end{align*}
$$

Consider stationary $(\partial / \partial t=0)$ equation related to $(12)$ :

$$
\begin{align*}
\varepsilon^{2} q^{\prime \prime}= & \gamma(x)\left(q^{2}-1\right)(q-\varphi(x)) \\
& -\varepsilon^{2} \frac{\partial q}{\partial x} \Phi_{1}(x)-\varepsilon^{2} \Phi_{2}(q, x) \tag{16}
\end{align*}
$$

Using the boundary functions method [18] we construct the asymptotic contrast solution of the problem (16) and (15). The first-order asymptotic expansion has the form

$$
\begin{align*}
q(x, t, \varepsilon)= & \bar{q}_{0}(x)+\varepsilon \bar{q}_{1}(x)+\Pi_{0} q(\tau) \\
& +\varepsilon \Pi_{1} q(\tau)+\varepsilon Q q\left(\tau_{0}, \tau_{1}\right) \tag{17}
\end{align*}
$$

where $\bar{q}_{0}(x)$ and $\bar{q}_{1}(x)$ are the regular terms of asymptotic expansion, $\Pi_{0} q(\tau)$ and $\Pi_{1} q(\tau)$ are zero- and first-order transition layer functions, $\tau=\left(x-x_{*}\right) / \varepsilon$ is a stretched variable, $x_{*}=x_{*}(\varepsilon)$ is a transition point in a small vicinity of which the transition layer is localized, and $Q q\left(\tau_{0}, \tau_{1}\right)$ is function describing the boundary layers near the points $x=0, x=1$
and $\tau_{0}=x / \varepsilon, \tau_{1}=(1-x) / \varepsilon$. The transition point has the following asymptotic form:

$$
\begin{equation*}
x_{*}=x_{0}+\varepsilon x_{1}+\varepsilon^{2} x_{2}+\cdots \tag{18}
\end{equation*}
$$

Using the boundary functions method procedure $[3,4]$ we obtain that the principal term $x_{0}$ of the expansion (18) can be found from the equation

$$
\begin{equation*}
\varphi\left(x_{0}\right)=0 . \tag{19}
\end{equation*}
$$

The full principal order function $\widetilde{q}(\tau)=\bar{q}_{0}\left(x_{0}\right)+\Pi_{0} q(\tau)$ can be found from equation

$$
\begin{gather*}
\tilde{q}^{\prime \prime}=\gamma\left(x_{0}\right) \tilde{q}\left(\tilde{q}^{2}-1\right),  \tag{20}\\
\widetilde{q}(-\infty)=1, \quad \widetilde{q}(+\infty)=-1 \tag{21}
\end{gather*}
$$

From (20) and (21) we have

$$
\begin{equation*}
\widetilde{q}=\frac{1-\exp \left[\sqrt{2 \gamma\left(x_{0}\right)} \tau\right]}{1+\exp \left[\sqrt{2 \gamma\left(x_{0}\right)} \tau\right]} \tag{22}
\end{equation*}
$$

So the principal term of the stationary power profile has the form

$$
\begin{align*}
p_{\mathrm{st}}(x, \varepsilon)= & \frac{1-\exp \left[\sqrt{2 \gamma\left(x_{0}\right)}\left(x-x_{0}\right) / \varepsilon\right]}{1+\exp \left[\sqrt{2 \gamma\left(x_{0}\right)}\left(x-x_{0}\right) / \varepsilon\right]}  \tag{23}\\
& \times \frac{\varphi_{3}(x)-\varphi_{1}(x)}{2}+\frac{\varphi_{3}(x)+\varphi_{1}(x)}{2}
\end{align*}
$$

The power profile $p_{\text {st }}(x, \varepsilon)$ is close to the iron-hand profile $\varphi_{3}(x)$ when $0 \leq x<x_{0}$ and to the participatory profile $\varphi_{1}(x)$ when $x_{0}<x \leq 1$. In the vicinity of the transition point $x_{0}$ we have $\partial p / \partial x \cong \varepsilon^{-1}$. We call such power profiles the contrast power profiles.

Equation (19) can be written in the form

$$
\begin{equation*}
\varphi_{2}\left(x_{0}\right)=\frac{\varphi_{3}\left(x_{0}\right)+\varphi_{1}\left(x_{0}\right)}{2} \tag{24}
\end{equation*}
$$

We call the function $h_{1}(x)=\varphi_{2}(x)-\varphi_{1}(x)$ the participatory domain's width and function $h_{3}(x)=\varphi_{3}(x)-\varphi_{2}(x)$ the iron-hand domain's width. Then (20) can be interpreted in the following way: at the transition point $x_{0}$ of the stationary contrast power profile (SCPP) the participatory domain's width is equal to the iron-hand domain's width $h_{1}\left(x_{0}\right)=h_{3}\left(x_{0}\right)$.

The stability of contrast structures of (9) was investigated by Bozhevol'nov and Nefédov [5] and Vasil'eva et al. [6]. In terms of the "power-society" model the stability result can be interpreted as follows.

SCPP, which are close to the iron-hand profile at the top ranks of the hierarchy ( $p \approx \varphi_{3}(x)$ when $0 \leq x<x_{0}$ ) and to the participatory profile at the bottom ranks ( $p \approx \varphi_{1}(x)$ when $x_{0}<x \leq 1$ ), are stable if the iron-hand domain's width is greater than the participatory domain's width at the top ranks of the hierarchy $\left(h_{3}(x)>h_{1}(x)\right.$ when $\left.0 \leq x<x_{0}\right)$ and less at the bottom ranks $\left(h_{3}(x)<h_{1}(x)\right.$ when $\left.x_{0}<x \leq 1\right)$. If $h_{3}(x)<h_{1}(x)$ when $0 \leq x<x_{0}$ and $h_{3}(x)>h_{1}(x)$ when $x_{0}<x \leq 1$ then the SCPP is unstable.

Remark 1. Similar statement holds for the so-called confederative SCPP which are close to the participatory profile at the top ranks of the hierarchy and to the iron-hand profile at the bottom ranks ( $p \approx \varphi_{1}(x)$ when $0 \leq x<x_{0}$ and $p \approx \varphi_{3}(x)$ when $\left.x_{0}<x \leq 1\right)$. They are stable if $h_{3}(x)<h_{1}(x)$ when $0 \leq x<x_{0}$ and $h_{3}(x)>h_{1}(x)$ when $x_{0}<x \leq 1$ and unstable if $h_{3}(x)>h_{1}(x)$ when $0 \leq x<x_{0}$ and $h_{3}(x)<h_{1}(x)$ when $x_{0}<x \leq 1$.

Consider again nonstationary equation (9). Suppose that at time $t=t_{0}$ contrasting structure has appeared with the transition layer at the vicinity of the point $x=\xi$. Then for $t>t_{0}$ the solution is a nonstationary contrast structure: $p \approx$ $\varphi_{3}(x)$ when $x<R(t, \varepsilon)$ and $p \approx \varphi_{1}(x)$ when $x>R(t, \varepsilon)$, where the transition point $R(t, \varepsilon)$ depends on time. We call such power profile the nonstationary contrast power profile (NCPP).

Let us construct the asymptotic NCPP.
Like in Section 3, make the substitution (6) and consider (7). It was shown in [7] that the principal term of the nonstationary contrast structure looks similar to one of the stationary contrast structure (18):

$$
\begin{equation*}
\widehat{q}=\frac{1-\exp [\sqrt{2 \gamma(R)}(x-R) / \varepsilon]}{1+\exp [\sqrt{2 \gamma(R)}(x-R) / \varepsilon]} \tag{25}
\end{equation*}
$$

where the function $R=R(t, \varepsilon)$ can be found from the equation

$$
\begin{equation*}
\frac{d R}{d t}=-\varepsilon \sqrt{2 \gamma(R)} \varphi(R) \tag{26}
\end{equation*}
$$

So the principal term of NCPP has the form

$$
\begin{align*}
p(x, t, \varepsilon)= & \frac{1-\exp [\sqrt{2 \gamma(R)}(x-R) / \varepsilon]}{1+\exp [\sqrt{2 \gamma(R)}(x-R) / \varepsilon]}  \tag{27}\\
& \times \frac{\varphi_{3}(x)-\varphi_{1}(x)}{2}+\frac{\varphi_{3}(x)+\varphi_{1}(x)}{2}
\end{align*}
$$

Power profile $p(x, t, \varepsilon)$ is close to the iron-hand profile $\varphi_{3}(x)$ when $0 \leq x<R$ and to the participatory profile $\varphi_{1}(x)$ when $R<x \leq 1$. The value of $d R / d t$ represents the speed of the transition layer. In terms of the "power-society" model the expression for $d R / d t$ has the form

$$
\begin{equation*}
\frac{d R}{d t}=-\varepsilon \sqrt{\frac{k_{1}(R)}{2}}\left[2 \varphi_{2}(R)-\varphi_{3}(R)-\varphi_{1}(R)\right] \tag{28}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{d R}{d t}=\varepsilon \sqrt{\frac{k_{1}(R)}{2}}\left[h_{3}(R)-h_{1}(R)\right] . \tag{29}
\end{equation*}
$$

Consider now some important cases of using formula (29); see also [17].
2.1. Attraction to the "Iron-Hand" Profile (1). Let the "ironhand" domain's width be larger than participatory domain's
width: $h_{3}(x)>h_{1}(x)$ for any $x \in[0,1]$. This means that the iron-hand profile looks more attractive from the society's point of view. Then SCPP do not exist because (24) has no roots. After appearing at time $t_{0}$ the contrast structure begins to move according to formula (29). Evidently $d R / d t>0$, and after small time of order $\varepsilon^{-1}$ transition point $R(t, \varepsilon)$ comes to the right end of the segment $[0,1]$. So the power profile appears close to the iron-hand profile for any $x \in[0,1]$. Notice that if at time $t=0$ function $p(x, 0, \varepsilon)$ is entirely in the participatory domain then for any $t$ the power profile is close to the participatory profile even if $h_{3}(x)>h_{1}(x)$. For appearing the power profile close to the iron-hand profile function $p(x, 0, \varepsilon)$ must be located in the iron-hand domain on at least one point in the interval $(0,1)$. This statement is based on the theorem proved by Bozhevol'nov and Nefedov [5].
2.2. Attraction to the "Iron-Hand" Profile (2). Let point $x_{0} \in$ $(0,1)$ exist such that $h_{1}\left(x_{0}\right)=h_{3}\left(x_{0}\right), h_{1}(x)>h_{3}(x)$ when $x<x_{0}$ and $h_{1}(x)<h_{3}(x)$ when $x>x_{0}$. Then unstable SCPP exist having transition layer in the vicinity of the point $x_{0}$. Let function $p(x, 0, \varepsilon)$ be in the iron-hand domain for $x<\xi$ and in the participatory domain for $x>\xi$ where $x_{0}<\xi<1$. Then the power profile $p(x, t, \varepsilon)$ is attracted to the iron-hand profile for $x<\xi$ and to the participatory profile for $x>\xi$. After appearing at time $t_{0}$ the contrast structure begins to move according to formula (29) and initial condition $R(0, \varepsilon)=\xi$. As $\xi>x_{0}$ then $d R / d t>0$. So after small time of order $\varepsilon^{-1}$ transition point $R(t, \varepsilon)$ comes to the right end of the segment $[0,1]$. So the power profile appears close to the iron-hand profile for any $x \in[0,1]$.
2.3. Attraction to the Participatory Profile (1). Let $h_{3}(x)<$ $h_{1}(x)$ for any $x \in[0,1]$. This means that the participatory profile looks more attractive from the society's point of view. Then SCPP do not exist because (24) has no roots. After appearing at time $t_{0}$ the contrast structure begins to move according to formula (29). Evidently $d R / d t<0$ and after small time of order $\varepsilon^{-1}$ transition point $R(t, \varepsilon)$ comes to the left end of the segment $[0,1]$. So the power profile appears close to the participatory profile for any $x \in[0,1]$. Notice that if at time $t=0$ function $p(x, 0, \varepsilon)$ is entirely in the iron-hand domain then for any $t$ the power profile is close to the ironhand profile even if $h_{3}(x)>h_{1}(x)$. For appearing the power profile close to the participatory profile function $p(x, 0, \varepsilon)$ must be smooth and located in the participatory domain on at least one point in the interval $(0,1)$. This statement is based on the theorem proved by Bozhevol'nov and Nefedov [5].
2.4. Attraction to the Participatory Profile (2). Let point $x_{0} \in$ $(0,1)$ exist such that $h_{1}\left(x_{0}\right)=h_{3}\left(x_{0}\right), h_{1}(x)>h_{3}(x)$ when $x<x_{0}$ and $h_{1}(x)<h_{3}(x)$ when $x>x_{0}$. Then unstable SCPP exist having transition layer in the vicinity of the point $x_{0}$. Let function $p(x, 0, \varepsilon)$ be in the iron-hand domain for $x<\xi$ and in the participatory domain for $x>\xi$ where $x_{0}<\xi<1$. Then according to (3) power profile $p(x, t, \varepsilon)$ is attracted to the iron-hand profile for $x<\xi$ and to the
participatory profile for $x>\xi$. After appearing at time $t_{0}$ the contrast structure begins to move according to formula (26) and initial condition $R(0, \varepsilon)=\xi$. As $\xi<x_{0}$ then $d R / d t<0$. So after small time of order $\varepsilon^{-1}$ transition point $R(t, \varepsilon)$ comes to the left end of the segment $[0,1]$. So the power profile appears close to the participatory profile.
2.5. Attraction to SCPP. Let point $x_{0} \in(0,1)$ exist such that $h_{1}\left(x_{0}\right)=h_{3}\left(x_{0}\right), h_{1}(x)<h_{3}(x)$ when $x<x_{0}$ and $h_{1}(x)>$ $h_{3}(x)$ if $x>x_{0}$. Then the stable SCPP exist having transition layer in the vicinity of the point $x_{0}$. If there is $\xi \in(0,1)$ such that the initial function $p^{0}(x)$ satisfies $p^{0}(x)>\varphi_{2}(x)$, $0<x<\xi$, and $p^{0}(x)<\varphi_{2}(x), \xi<x<1$, then the contrasting structure appears after a short time $t_{0}$ such that $p(x) \approx \varphi_{3}(x)$ for $0<x<\xi$ and $p(x) \approx \varphi_{1}(x)$ for $\xi<x<1$. Then the contrasting structure begins to move according to formula (29) and initial condition $R\left(t_{0}, \varepsilon\right)=\xi$. So if $\xi<x_{0}$ then $d R / d t>0$ and if $\xi>x_{0}$ then $d R / d t<0$. So when $t \rightarrow \infty$, the transition point $R(t, \varepsilon)$ tends to the stationary transition point $x_{0}$. Thus the SCPP having transition layer in the vicinity of point $x_{0}$ appears.

## 3. Parametric Optimization

The total amount of power of the hierarchy is $\widetilde{P}(t, \varepsilon)=$ $\int_{0}^{1} p(x, t, \varepsilon)$. It was shown in [19] that there exists the optimal value $P_{0}$ of the total power which provides a maximum of steady-state consumption per capita (in frame of the "power-society-economics" model [19]). So we should introduce the control parameter into the "power-society" model to make it controllable. So the problem would be to find the value of the control parameter under which $\widetilde{P}(t, \varepsilon) \rightarrow P_{0}$, when $t \rightarrow \infty$, $\varepsilon \rightarrow 0$.

Generally speaking, the model could be formulated such that the control is considered to be a function of time or $x$. In any case, the control describes the exogenous impact on the political system, such as a political pressure through media and political institutions. We restrict ourselves to the parametric control.

Definition 2. The value $P_{0}$ is called the asymptotically achievable amount of the total power if there exists an admissible value of control parameter $u$ such that the steady-state total power $P_{u}(\varepsilon)$ satisfies $P_{u}(\varepsilon) \rightarrow P_{0}$ when $t \rightarrow \infty, \varepsilon \rightarrow 0$.

So consider the "power-society" model with nonlinear reaction of civil society:

$$
\begin{align*}
f(p, x, t, u)= & -k_{1}(x)\left(p-\varphi_{1}(x)\right)  \tag{30}\\
& \times\left(p-\left(\varphi_{2}(x)+\gamma u\right)\right)\left(p-\varphi_{3}(x)\right) .
\end{align*}
$$

Here $k_{1}(x)>0$, the functions $k_{1}(x), \varphi_{1}(x), \varphi_{2}(x)$, and $\varphi_{3}(x)$ have continuous derivatives, and $\gamma$ is a constant. Thus the lowest and the biggest roots $\varphi_{1}(x), \varphi_{3}(x)$ of the degenerate equation $F(p, x, u)=0$ do not depend on the control, but
there is an impact from the control to the "middle" $\operatorname{root} \varphi_{2}(x)$. So the model has the form

$$
\begin{align*}
\frac{\partial p}{\partial t}= & \varepsilon^{2} \frac{\partial^{2} p}{\partial x^{2}}-k_{1}(x)\left(p-\varphi_{1}(x)\right)  \tag{31}\\
& \times\left(p-\left(\varphi_{2}(x)+\gamma u\right)\right)\left(p-\varphi_{3}(x)\right) \\
\left.\frac{\partial p}{\partial x}\right|_{x=0} & =\left.\frac{\partial p}{\partial x}\right|_{x=1}=0,\left.\quad p\right|_{t=0}=p^{0}(x) \tag{32}
\end{align*}
$$

The initial function $p^{0}(x)$ is supposed to be smooth and satisfying $\varphi_{1}(x)<p^{0}(x)<\varphi_{3}(x)$. In other words, the initial distribution of power is between the iron-hand and participatory profiles.

The steady-state equation for (31) has the form

$$
\begin{gather*}
\varepsilon^{2} \frac{\partial^{2} p}{\partial x^{2}}=k_{1}(x)\left(p-\varphi_{1}(x)\right)\left(p-\left(\varphi_{2}(x)+\gamma u\right)\right)\left(p-\varphi_{3}(x)\right) \\
\left.\frac{\partial p}{\partial x}\right|_{x=0}=\left.\frac{\partial p}{\partial x}\right|_{x=1}=0 \tag{33}
\end{gather*}
$$

Let us stress here that the control influences the relation between the width of the iron-hand domain and the width of the participatory domain.

Let the following conditions be fulfilled.

## Condition 3. We have $-1 \leq u \leq 1$.

Condition 4. We have $\varphi_{1}(x)<\varphi_{2}(x)-\gamma<\varphi_{2}(x)+\gamma<\varphi_{3}(x)$ for $0 \leq x \leq 1$.

Condition 5. We have $H^{\prime}(x)<0$ for $0 \leq x \leq 1$, and here $H(x)=\varphi_{1}(x)+\varphi_{3}(x)-2 \varphi_{2}(x)$.

Conditions 3 and 4 introduce the normalization of the control such that for any admissible control $u \in[-1 ; 1]$ the root $\varphi_{2}(x)+\gamma u$ is between the $\varphi_{1}(x)$ and $\varphi_{3}(x)$. The steadystate solution has no more than one transition point due to Condition 5.

If the control parameter $u$ is increased, the root of the equation $\varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)+\gamma u\right)=0$ will move to the left. This means greater support to the participation ideas. If it exists. Analogically, the less the value of $u$ is, the more to the right the root of this equation is.

Consider the following problem. Let the desirable (optimal) value of total power be $P_{0}$. Is there a value of control parameter $u$, under which the steady-state solution (33) is such that $P_{u}(\varepsilon)=\int_{0}^{1} p(x, \varepsilon) d x \rightarrow P_{0}$ when $\varepsilon \rightarrow 0$ ?

From the practical point of view, such a formulation of the problem can be justified in the following way. We know from the "power-society-economics" model that the optimal value of the total power is some $P_{0}$, so we should try to tune the political system to provide this optimal value of power for the steady-state regime.

Several cases should be distinguished.
Let us start our consideration from the situation in which both equations

$$
\begin{align*}
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)+\gamma\right)=0  \tag{34}\\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)-\gamma\right)=0 \tag{35}
\end{align*}
$$

have roots in the interval $(0 ; 1)$. Let us denote these roots by $a$ and $b$, respectively. Here we have $a<b$ in view of Condition 5.

The points $x=a$ and $x=b$ are the main asymptotic terms for the boundaries of the range within which the transition point of the stationary front is located.

Therefore, the value

$$
\begin{equation*}
P_{a s}(u)=\lim _{\varepsilon \rightarrow 0} P(u, \varepsilon)=\lim _{\varepsilon \rightarrow 0} \int_{0}^{1} p(x, u, \varepsilon) d x \tag{36}
\end{equation*}
$$

satisfies the inequality

$$
\begin{align*}
\int_{0}^{a} \varphi_{3}(x) d x+\int_{a}^{1} \varphi_{1}(x) d x & \leq P_{a s}(u) \\
& \leq \int_{0}^{b} \varphi_{3}(x) d x+\int_{b}^{1} \varphi_{1}(x) d x \tag{37}
\end{align*}
$$

Thus, in this case, $P_{0}$ is asymptotically achievable, if inequality

$$
\begin{align*}
\int_{0}^{a} \varphi_{3}(x) d x+\int_{a}^{1} \varphi_{1}(x) d x & \leq P_{0} \\
& \leq \int_{0}^{b} \varphi_{3}(x) d x+\int_{b}^{1} \varphi_{1}(x) d x \tag{38}
\end{align*}
$$

holds true.
The steady-state problem (32) and (33) has also solutions without transition layers: the iron-hand profile and the participatory one. So the values of total power

$$
\begin{equation*}
P_{0}=\int_{0}^{1} \varphi_{3}(x) d x, \quad P_{0}=\int_{0}^{1} \varphi_{1}(x) d x \tag{39}
\end{equation*}
$$

are also asymptotically achievable.
So, if both (34) and (35) have roots in the interval $(0 ; 1)$ then the set of asymptotically achievable values comprises the closed interval (38) and two isolated values (39): one of them is to the left of this closed interval, and the other one is to the right of it.

Now let us consider the situation in which (34) has a root $x=a \in(0 ; 1)$ and (35) has no roots on $(0 ; 1)$.

In other words, $\varphi_{2}(x)-\gamma<\left(\varphi_{1}(x)+\varphi_{3}(x)\right) / 2$ for any $x$. So the values of $P_{0}$ in the closed interval

$$
\begin{equation*}
\int_{0}^{a} \varphi_{3}(x) d x+\int_{a}^{1} \varphi_{1}(x) d x \leq P_{0} \leq \int_{0}^{1} \varphi_{3}(x) d x \tag{40}
\end{equation*}
$$

are asymptotically achievable.

Let now (35) have the root $x=b \in(0 ; 1)$, and let (34) have no roots in the interval. That is, at $u=1$, for any $x$, the middle root $\varphi_{2}(x)+\gamma$ is larger than half-sum of $\varphi_{1}(x)$ and $\varphi_{3}(x)$. Then the asymptotically achievable values are given by the inequality

$$
\begin{equation*}
\int_{0}^{1} \varphi_{1}(x) d x \leq P_{0} \leq \int_{0}^{b} \varphi_{3}(x) d x+\int_{b}^{1} \varphi_{1}(x) d x \tag{41}
\end{equation*}
$$

At last, consider the case in which neither of the equations has a root in the interval $(0 ; 1)$. Then the following three subcases are possible:

$$
\begin{align*}
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)+\gamma\right)<0 \\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)-\gamma\right)<0 \quad \text { for any } x  \tag{42}\\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)+\gamma\right)>0  \tag{43}\\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)-\gamma\right)>0 \quad \text { for any } x \\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)+\gamma\right)>0 \\
& \varphi_{1}(x)+\varphi_{3}(x)-2\left(\varphi_{2}(x)-\gamma\right)<0 \quad \text { for any } x \tag{44}
\end{align*}
$$

It can be easily shown (see [2], e.g.) that in subcases (42) and (43) a steady-state front does not exist for any control parameter. So only the values of $P_{0}$ given by (39) are asymptotically achievable.

In the subcase (44) for any given $x_{0} \in(0 ; 1)$, such $u$ exists that the problem (32) and (33) has the stationary front with the transition point in the $\varepsilon$-vicinity of $x_{0}$. Therefore, any $P_{0}$ from closed interval

$$
\begin{equation*}
\int_{0}^{1} \varphi_{1}(x) d x \leq P_{0} \leq \int_{0}^{1} \varphi_{3}(x) d x \tag{45}
\end{equation*}
$$

is asymptotically achievable.
The above speculations can be summarized as follows.
Theorem 3. Consider the problem (32) and (33) with parametric control. Let the following conditions be satisfied:
(1) $k_{1}(x), \varphi_{1}(x), \varphi_{2}(x), \varphi_{3}(x) \in C^{1}[0,1]$,
(2) $-1 \leq u \leq 1$,
(3) $\varphi_{1}(x)<\varphi_{2}(x)-\gamma<\varphi_{2}(x)+\gamma<\varphi_{3}(x)$,
(4) $H^{\prime}(x)<0$, where $H(x)=\varphi_{1}(x)+\varphi_{3}(x)-2 \varphi_{2}(x)$.

Then the set of asymptotically achievable values is not empty.
After the provided analysis of a steady-state problem (32) and (33), we go back to the initial parabolic partial problem (31) and (32).

Let some value $P_{0}$ be asymptotically achievable in the corresponding stationary problem. It means that there is a value of parametric control $u$, at which the problem (31), (32) has the steady-state solution for which the total power $P_{u}(\varepsilon)$ of the hierarchy asymptotically tends to $P_{0}$ when $\varepsilon \rightarrow 0$. However for the $\widetilde{P}_{u}(t, \varepsilon)=\int_{0}^{1} p_{u}(x, t, \varepsilon) d x$ we have $\widetilde{P}_{u}(t, \varepsilon) \rightarrow$ $P_{u}(\varepsilon)$ just for some class of initial functions $p^{0}(x)$.

Thus, there is a problem to determine the class of initial power distributions for which, under the found value parametric control, the solution of the parabolic partial problem converges to the proper steady-state solution at $t \rightarrow \infty$.

The answer is given by the following theorem.
Theorem 4. (1) Let all the conditions of Theorem 3 be satisfied;
(2) let $P_{0}$ be an asymptotically achievable value of total power (denote by $u=u_{0}$ the corresponding value of parametric control);
(3) let the point $x_{0} \in(0 ; 1)$ exist such that

$$
\begin{equation*}
P_{0}=\int_{0}^{x_{0}} \varphi_{3}(x) d x+\int_{x_{0}}^{1} \varphi_{1}(x) d x \tag{46}
\end{equation*}
$$

(4) let the points $x_{1} \in\left(0 ; x_{0}\right), x_{2} \in\left(x_{0} ; 1\right)$ exist such that

$$
\begin{equation*}
p^{0}\left(x_{1}\right)>\varphi_{2}\left(x_{1}\right)+\gamma u_{0}, \quad p^{0}\left(x_{2}\right)<\varphi_{2}\left(x_{2}\right)+\gamma u_{0} . \tag{47}
\end{equation*}
$$

Then the solution $p_{u_{0}}(x, t, \varepsilon)$ of the parabolic partial problem (31) and (32) is such that the total power

$$
\begin{equation*}
Q_{u_{0}}(t, \varepsilon)=\int_{0}^{1} p_{u_{0}}(x, t, \varepsilon) d x \tag{48}
\end{equation*}
$$

converges to

$$
\begin{equation*}
P_{0}: \lim _{\varepsilon \rightarrow \infty} \lim _{t \rightarrow \infty} P\left(t, u_{0}, \varepsilon\right)=P_{0} \tag{49}
\end{equation*}
$$

Proof. It is easy to see that under these conditions the Butuzov-Nedelko theorem is fulfilled [13]. Therefore, at the chosen value of control $u=u_{0}$, the solution $p_{u_{0}}(x, t, \varepsilon)$ of the problem (31) and (32) has a passage to the limit

$$
\begin{equation*}
\lim _{t \rightarrow \infty} p_{u_{0}}(x, t, \varepsilon)=p_{s t, u_{0}}(x, \varepsilon) \tag{50}
\end{equation*}
$$

where $p_{\mathrm{st}}(x, \varepsilon)$ is the steady state solution for which

$$
\lim _{t \rightarrow \infty} p_{s t, u_{0}}(x, \varepsilon)= \begin{cases}\varphi_{3}(x), & x<x_{0}  \tag{51}\\ \varphi_{1}(x), & x>x_{0}\end{cases}
$$

By integrating (50) from $x=0$ to $x=1$ we get

$$
\begin{equation*}
\lim _{t \rightarrow \infty} \int_{0}^{1} p_{u_{0}}(x, t, \varepsilon) d x=\int_{0}^{1} p_{s t, u_{0}}(x, \varepsilon) d x \tag{52}
\end{equation*}
$$

Passing to a limit $\varepsilon \rightarrow 0$, we get

$$
\begin{align*}
\lim _{\varepsilon \rightarrow 0} \lim _{t \rightarrow \infty} \int_{0}^{1} p_{u_{0}}(x, t, \varepsilon) d x & =\lim _{\varepsilon \rightarrow 0} \int_{0}^{1} p_{\mathrm{st}, u_{0}}(x, \varepsilon) d x \\
& =\int_{0}^{x_{0}} \varphi_{3}(x) d x+\int_{x_{0}}^{1} \varphi_{1}(x) d x \\
& =P_{0} \tag{53}
\end{align*}
$$

Thus $\lim _{\varepsilon \rightarrow 0} \lim _{t \rightarrow \infty} P\left(t, u_{0}, \varepsilon\right)=P_{0}$. Theorem 4 is proved.

## 4. Conclusion

It is shown that the theory of contrasting structures in singularly perturbed boundary value problems allows for investigating the properties of nonstationary fronts in the singularly perturbed "power-society" model. Depending on the initial condition, these fronts evolve to one of the asymptotically stable steady-state distributions of power within a government hierarchy.

There are some reasons to introduce a concept of desirable steady-state total amount of power of the hierarchy. The possibility theorem is proved for the problem of getting this amount by means of parametric control. The results can be used in investigating governing hierarchical systems.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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## Research Article

# Approximate Solution of Inverse Problem for Elliptic Equation with Overdetermination 

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#### Abstract

A finite difference method for the approximate solution of the inverse problem for the multidimensional elliptic equation with overdetermination is applied. Stability and coercive stability estimates of the first and second orders of accuracy difference schemes for this problem are established. The algorithm for approximate solution is tested in a two-dimensional inverse problem.


## 1. Introduction

It is well known that inverse problems arise in various branches of science (see [1, 2]). The theory and applications of well-posedness of inverse problems for partial differential equations have been studied extensively by many researchers (see, e.g., [3-17] and the references therein). One of the effective approaches for solving inverse problem is reduction to nonlocal boundary value problem (see, e.g., $[6,8,11]$ ). Well-posedness of the nonlocal boundary value problems of elliptic type equations was investigated in [18-25] (see also the references therein).

In [4], Orlovsky proved existence and uniqueness theorems for the inverse problem of finding a function $u$ and an element $p$ for the elliptic equation in an arbitrary Hilbert space $H$ with the self-adjoint positive definite operator $A$ :

$$
\begin{gather*}
-u_{t t}(t)+A u(t)=f(t)+p, \quad 0<t<T, \\
u(0)=\varphi, \quad u(T)=\psi, \quad u(\lambda)=\xi, \quad 0<\lambda<T . \tag{1}
\end{gather*}
$$

In [11], the authors established stability estimates for this problem and studied inverse problem for multidimensional elliptic equation with overdetermination in which the Dirichlet condition is required on the boundary.

In present work, we study inverse problem for multidimensional elliptic equation with Dirichlet-Neumann boundary conditions.

Let $\Omega=(0, \ell) \times \cdots \times(0, \ell)$ be the open cube in the $n$ dimensional Euclidean space with boundary $S$ and $\bar{\Omega}=\Omega \cup$ $S$. In $[0, T] \times \Omega$, we consider the inverse problem of finding functions $u(t, x)$ and $p(x)$ for the multidimensional elliptic equation

$$
\begin{gather*}
-u_{t t}(t, x)-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}}+\delta u(t, x)=f(t, x)+p(x), \\
x=\left(x_{1}, \ldots, x_{n}\right) \in \Omega, \quad 0<t<T, \\
u(0, x)=\varphi(x), \quad u(T, x)=\psi(x), \quad u(\lambda, x)=\xi(x), \\
\\
x \in \bar{\Omega},  \tag{2}\\
\frac{\partial u(t, x)}{\partial \vec{n}}=0, \quad x \in S, 0 \leq t \leq T .
\end{gather*}
$$

Here, $0<\lambda<T$ and $\delta>0$ are known numbers, $a_{r}(x)(x \in \Omega), \varphi(x), \psi(x), \xi(x)(x \in \bar{\Omega})$, and $\quad f(t, x)(t \in$ $(0, T), x \in \Omega)$ are given smooth functions, and also $a_{r}(x) \geq$ $a>0(x \in \Omega)$.

The aim of this paper is to investigate inverse problem (2) for multidimensional elliptic equation with DirichletNeumann boundary conditions. We obtain well-posedness of problem (2). For the approximate solution of problem (2), we construct first and second order of accuracy in
$t$ and difference schemes with second order of accuracy in space variables. Stability and coercive stability estimates for these difference schemes are established by applying operator approach. The modified Gauss elimination method is applied for solving these difference schemes in a two-dimensional case.

The remainder of this paper is organized as follows. In Section 2, we obtain stability and coercive stability estimates for problem (2). In Section 3, we construct the difference schemes for (2) and establish their well-posedness. In Section 4, the numerical results in a two-dimensional case are presented. Section 5 is conclusion.

## 2. Well-Posedness of Inverse Problem with Overdetermination

It is known that the differential expression [26]

$$
\begin{equation*}
A^{x} u(x)=-\sum_{r=1}^{n}\left(a_{r}(x) u_{x_{r}}\right)_{x_{r}}+\delta u(x) \tag{3}
\end{equation*}
$$

defines a self-adjoint positive definite operator $A^{x}$ acting on $L_{2}(\bar{\Omega})$ with the domain $D\left(A^{x}\right)=\left\{u(x) \in W_{2}^{2}(\bar{\Omega}), \partial u / \partial \vec{n}=\right.$ 0 on $S\}$.

Let $H$ be the Hilbert space $L_{2}(\bar{\Omega})$. By using abstract Theorems 2.1 and 2.2 of paper [11], we get the following theorems about well-posedness of problem (2).

Theorem 1. Assume that $A^{x}$ is defined by formula (3), $\varphi, \xi, \psi \in$ $D\left(A^{x}\right)$. Then, for the solutions $(u, p)$ of inverse boundary value problem (2), the stability estimates are satisfied:

$$
\begin{align*}
& \|u\|_{C\left(L_{2}(\bar{\Omega})\right)} \\
& \leq\left[M\|\varphi\|_{L_{2}(\bar{\Omega})}+\|\psi\|_{L_{2}(\bar{\Omega})}+\|\xi\|_{L_{2}(\bar{\Omega})}+\|f\|_{C\left(L_{2}(\bar{\Omega})\right)}\right], \\
& \left\|\left(A^{x}\right)^{-1} p\right\|_{L_{2}(\bar{\Omega})} \leq M\left[\|\varphi\|_{L_{2}(\bar{\Omega})}\right. \\
& \left.\quad+\|\xi\|_{L_{2}(\bar{\Omega})}+\|f\|_{C\left(L_{2}(\bar{\Omega})\right)}\right], \\
& \|p\|_{L_{2}(\bar{\Omega})} \leq M\left[\left\|A^{x} \varphi\right\|_{L_{2}(\bar{\Omega})}+\left\|A^{x} \psi\right\|_{L_{2}(\bar{\Omega})}\right. \\
& \left.\quad+\left\|A^{x} \xi\right\|_{L_{2}(\bar{\Omega})}+\frac{1}{\alpha(1-\alpha)}\|f\|_{\delta_{0 r}^{\alpha x \alpha}\left(L_{2}(\bar{\Omega})\right)}\right], \tag{4}
\end{align*}
$$

where $M$ is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t, x)$.
Here, $\mathscr{C}_{0 T}^{\alpha, \alpha}\left(L_{2}(\bar{\Omega})\right)$ is the space obtained by completion of the space of all smooth $L_{2}(\bar{\Omega})$-valued functions $\rho$ on $[0, T]$ with the norm

$$
\begin{aligned}
&\|\rho\|_{\mathscr{C}_{0 T}^{\alpha, \alpha}\left(L_{2}(\bar{\Omega})\right)} \\
&=\|\rho\|_{\mathscr{C}\left(L_{2}(\bar{\Omega})\right)} \\
& \quad+\sup _{0 \leq t<t+\tau \leq T} \frac{(t+\tau)^{\alpha}(T-t)^{\alpha}\|\rho(t+\tau)-\rho(t)\|_{L_{2}(\bar{\Omega})}}{\tau^{\alpha}} .
\end{aligned}
$$

Theorem 2. Assume that $A^{x}$ is defined by formula (3), $\varphi, \psi, \xi \in D\left(A^{x}\right)$. Then, for the solution of inverse boundary value problem (2), coercive stability estimate

$$
\begin{align*}
& \left\|u^{\prime \prime}\right\|_{\mathscr{C}_{0 T}^{\alpha, \alpha}\left(L_{2}(\bar{\Omega})\right)}+\|u\|_{\mathscr{C}_{0 T}^{\alpha, \alpha}\left(W_{2}^{2}(\bar{\Omega})\right)}+\|p\|_{L_{2}(\bar{\Omega})} \\
& \leq M\left[\frac{1}{\alpha(1-\alpha)}\|f\|_{\mathscr{C}_{0 T}^{\alpha, \alpha}\left(L_{2}(\bar{\Omega})\right)}+\|\varphi\|_{W_{2}^{2}(\bar{\Omega})}\right.  \tag{6}\\
& \left.\quad+\|\psi\|_{W_{2}^{2}(\bar{\Omega})}+\|\xi\|_{W_{2}^{2}(\bar{\Omega})}\right]
\end{align*}
$$

holds, where $M$ is independent of $\alpha, \varphi(x), \xi(x), \psi(x)$, and $f(t, x)$.

## 3. Difference Schemes and Their Well-Posedness

Suppose that $A^{x}$ is defined by formula (3). Then (see [26]), $C=(1 / 2)\left(\tau A^{x}+\sqrt{4 A^{x}+\tau^{2}\left(A^{x}\right)^{2}}\right)$ is a self-adjoint positive definite operator and $R=(I+\tau C)^{-1}$ which is defined on the whole space $H=L_{2}(\bar{\Omega})$ is a bounded operator. Here, $I$ is the identity operator.

Now we present the following lemmas, which will be used later.

Lemma 3. The following estimates are satisfied (see [27]):

$$
\begin{gather*}
\left\|R^{k}\right\|_{H \rightarrow H} \leq M\left(1+\delta^{1 / 2} \tau\right)^{-k}, \quad \delta>0 \\
\left\|C R^{k}\right\|_{H \rightarrow H} \leq \frac{M}{k \tau}, \quad k \geq 1  \tag{7}\\
\left\|\left(I-R^{2 N}\right)^{-1}\right\|_{H \rightarrow H} \leq M
\end{gather*}
$$

Lemma 4. For $1 \leq l \leq N-1$ and for the operator $S=R^{2 N}+$ $R^{l}-R^{2 N-l}+R^{N-l}-R^{N+l}$, the operator $I-S$ has an inverse $G=(I-S)^{-1}$ and the estimate

$$
\begin{equation*}
\|G\|_{H \rightarrow H} \leq M \tag{8}
\end{equation*}
$$

is satisfied, where $M$ does not depend on $\tau$.
Proof of Lemma 4 is based on Lemma 3 and representation

$$
\begin{align*}
Q & =I-R^{2 N}-R^{l}+R^{2 N-l}-R^{N-l}+R^{N+l} \\
& =\left(I-R^{N-l}\right)\left(I-R^{N}\right)\left(I-R^{l}\right) . \tag{9}
\end{align*}
$$

Lemma 5. For $1 \leq l \leq N-1$ and for the operator

$$
\begin{align*}
S_{1}= & R^{2 N}-\left(\frac{\lambda}{\tau}-l-1\right)\left(R^{l}-R^{2 N-l}+R^{N-l}-R^{N+l}\right) \\
& +\left(\frac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2 N-l-1}+R^{N-l-1}-R^{N+l+1}\right) \tag{10}
\end{align*}
$$

the operator $I-S_{1}$ has an inverse

$$
\begin{align*}
G_{1}=( & I-R^{2 N}+\left(\frac{\lambda}{\tau}-l-1\right) \\
& \times\left(R^{l}-R^{2 N-l}+R^{N-l}-R^{N+l}\right)  \tag{11}\\
& \left.-\left(\frac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2 N-l-1}+R^{N-l-1}-R^{N+l+1}\right)\right)^{-1}
\end{align*}
$$

and the estimate

$$
\begin{equation*}
\left\|G_{1}\right\|_{H \rightarrow H} \leq M \tag{12}
\end{equation*}
$$

is valid, where $M$ is independent of $\tau$.
Proof. We have that

$$
\begin{equation*}
G_{1}-G=G_{1} G K \tag{13}
\end{equation*}
$$

where

$$
\begin{align*}
K= & -\left(\frac{\lambda}{\tau}-l\right)\left(R^{l}-R^{2 N-l}+R^{N-l}-R^{N+l}\right)  \tag{14}\\
& +\left(\frac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2 N-l-1}+R^{N-l-1}-R^{N+l+1}\right)
\end{align*}
$$

By using estimates of Lemma 3, we have that

$$
\begin{array}{r}
\|K\|_{H \rightarrow H}=\|-\left(\frac{\lambda}{\tau}-l\right)\left(R^{l}-R^{2 N-l}+R^{N-l}-R^{N+l}\right) \\
+\left(\frac{\lambda}{\tau}-l\right)\left(R^{l+1}-R^{2 N-l-1}+R^{N-l-1}\right.  \tag{15}\\
\left.-R^{N+l+1}\right) \|_{H \rightarrow H} \leq M_{1} \tau
\end{array}
$$

where $M_{1}$ is independent of $\tau$. Using the triangle inequality, formula (13), and estimates (8) and (15), we obtain

$$
\begin{align*}
\left\|G_{1}\right\|_{H \rightarrow H} & =\|G\|_{H \rightarrow H}+\left\|G_{1}\right\|_{H \rightarrow H}\|G\|_{H \rightarrow H}  \tag{16}\\
& \leq M+\left\|G_{1}\right\|_{H \rightarrow H} M M_{1} \tau
\end{align*}
$$

for sufficiently small positive $\tau$. From that it follows estimate (11). Lemma 5 is proved.

Further, we discretize problem (2) in two steps. In the first step, we define the grid spaces

$$
\begin{gather*}
\widetilde{\Omega}_{h}=\left\{x=x_{m}=\left(h_{1} m_{1}, \ldots, h_{n} m_{n}\right) ; m=\left(m_{1}, \ldots, m_{n}\right),\right. \\
\left.m_{r}=0, \ldots, M_{r}, h_{r} M_{r}=\ell, r=1, \ldots, n\right\},  \tag{17}\\
\Omega_{h}=\widetilde{\Omega}_{h} \cap \Omega, \quad S_{h}=\widetilde{\Omega}_{h} \cap S .
\end{gather*}
$$

Introduce the Hilbert space $L_{2 h}=L_{2}\left(\widetilde{\Omega}_{h}\right)$ and $W_{2 h}^{2}=$ $W_{2}^{2}\left(\widetilde{\Omega}_{h}\right)$ of grid functions $\rho^{h}(x)=\left\{\rho\left(h_{1} m_{1}, \ldots, h_{n} m_{n}\right)\right\}$, defined on $\widetilde{\Omega}_{h}$, equipped with the norms

$$
\begin{gather*}
\left\|\rho^{h}\right\|_{L_{2 h}}=\left(\sum_{x \in \widetilde{\Omega}_{h}}\left|\rho^{h}(x)\right|^{2} h_{1}, \ldots, h_{n}\right)^{1 / 2}, \\
\left\|\rho^{h}\right\|_{W_{2 h}^{2}}=\left\|\rho^{h}\right\|_{L_{2 h}}+\left(\sum_{x \in \widetilde{\Omega}_{h}} \sum_{r=1}^{n}\left|\left(\rho^{h}\right)_{x_{r}}\right|^{2} h_{1}, \ldots, h_{n}\right)^{1 / 2}  \tag{18}\\
+\left(\sum_{x \in \widetilde{\Omega}_{h}} \sum_{r=1}^{n}\left|\left(\rho^{h}(x)\right)_{x_{r} \overline{x_{r}}, m_{r}}\right|^{2} h_{1}, \ldots, h_{n}\right)^{1 / 2},
\end{gather*}
$$

respectively.
To the differential operator $A^{x}$ generated by problem (2) we assign the difference operator $A_{h}^{x}$ defined by formula (3), acting in the space of grid functions $u^{h}(x)$, satisfying the condition $D^{h} u^{h}(x)=0$ for all $x \in S_{h}$. Here, $D^{h} u^{h}(x)$ is an approximation of $\partial u / \partial \vec{n}$.

By using $A_{h}^{x}$, for obtaining $u^{h}(t, x)$ functions, we arrive at problem

$$
\begin{aligned}
-\frac{d^{2} u^{h}(t, x)}{d t^{2}}+A_{h}^{x} u^{h}(t, x)= & f^{h}(t, x)+p^{h}(x) \\
& 0<t<T, x \in \Omega_{h}
\end{aligned}
$$

$$
u^{h}(0, x)=\varphi^{h}(x), \quad u^{h}(\lambda, x)=\xi^{h}(x), \quad u^{h}(T, x)=\psi^{h}(x)
$$

$$
x \in \widetilde{\Omega}_{h}
$$

For finding a solution $u^{h}(t, x)$ of problem (19) we apply the substitution

$$
\begin{equation*}
u^{h}(t, x)=v^{h}(t, x)+\left(A_{h}^{x}\right)^{-1} p^{h}(x), \tag{20}
\end{equation*}
$$

where $v^{h}(t, x)$ is the solution of nonlocal boundary value problem; a system of ordinary differential equations

$$
\begin{gather*}
-\frac{d^{2} v^{h}(t, x)}{d t^{2}}+A_{h}^{x} v^{h}(t, x)=f^{h}(t, x), \\
0<t<T, \quad x \in \Omega_{h},  \tag{21}\\
v^{h}(0, x)-v^{h}(\lambda, x)=\varphi^{h}(x)-\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, \\
v^{h}(T, x)-v^{h}(\lambda, x)=\psi^{h}(x)-\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h}
\end{gather*}
$$

and unknown function $p^{h}(x)$ is defined by formula

$$
\begin{equation*}
p^{h}(x)=A_{h}^{x} \varphi^{h}(x)-A_{h}^{x} v^{h}(0, x), \quad x \in \widetilde{\Omega}_{h} . \tag{22}
\end{equation*}
$$

Thus, we consider the algorithm for solving problem (19) which includes three stages. In the first stage, we get
the nonlocal boundary value problem (21) and obtain $v^{h}(t, x)$. In the second stage, we put $t=0$ and find $v^{h}(0, x)$. Then, using (22), we obtain $p^{h}(x)$. Finally, in the third stage, we use formula (20) for obtaining the solution $u^{h}(t, x)$ of problem (19).

In the second step, we approximate (19) in variable $t$. Let $[0, T]_{\tau}=\left\{t_{k}=k \tau, k=1, \ldots, N, N \tau=T\right\}$ be the uniform grid space with step size $\tau>0$, where $N$ is a fixed positive integer. Applying the approximate formulas

$$
\begin{align*}
u^{h}(\lambda, x) & =u^{h}\left(\left[\frac{\lambda}{\tau}\right] \tau, x\right)+o(\tau), \quad x \in \Omega_{h} \\
u^{h}(\lambda, x)= & u^{h}\left(\left[\frac{\lambda}{\tau}\right] \tau, x\right)+\left(\frac{\lambda}{\tau}-\left[\frac{\lambda}{\tau}\right]\right)  \tag{23}\\
& \times\left(u^{h}\left(\left[\frac{\lambda}{\tau}\right] \tau+\tau, x\right)-u^{h}\left(\left[\frac{\lambda}{\tau}\right] \tau, x\right)\right) \\
& +o\left(\tau^{2}\right), \quad x \in \Omega_{h}
\end{align*}
$$

for $u^{h}(\lambda, x)=\xi^{h}(x)$, problem (19) is replaced by first order of accuracy difference scheme

$$
\begin{gather*}
-\frac{u_{k+1}^{h}(x)-2 u_{k}^{h}(x)+u_{k-1}^{h}(x)}{\tau^{2}}+A_{h}^{x} u_{k}^{h}(x) \\
=f_{k}^{h}(x)+p^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \\
1 \leq k \leq N-1, \quad x \in \Omega_{h},  \tag{24}\\
u_{0}^{h}(x)=\varphi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, \\
u_{l}^{h}(x)=\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, \\
u_{N}^{h}(x)=\psi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, l=\left[\frac{\lambda}{\tau}\right]
\end{gather*}
$$

and second order of accuracy difference scheme

$$
\begin{gathered}
-\frac{u_{k+1}^{h}(x)-2 u_{k}^{h}(x)+u_{k-1}^{h}(x)}{\tau^{2}}+A_{h}^{x} u_{k}^{h}(x) \\
=f_{k}^{h}(x)+p^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \\
1 \leq k \leq N-1, \quad x \in \Omega_{h}, \\
u_{0}^{h}(x)=\varphi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, \\
u_{l}^{h}(x)+\left(\frac{\lambda}{\tau}-l\right)\left(u_{l+1}^{h}(x)-u_{l}^{h}(x)\right)=\xi^{h}(x), \\
x \in \widetilde{\Omega}_{h}, \\
u_{N}^{h}(x)=\psi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, l=\left[\frac{\lambda}{\tau}\right] .
\end{gathered}
$$

For approximate solution of nonlocal problem (21), we have first order of accuracy difference scheme

$$
\begin{gather*}
-\frac{v_{k+1}^{h}(x)-2 v_{k}^{h}(x)+v_{k-1}^{h}(x)}{\tau^{2}}+A_{h}^{x} v_{k}^{h}(x)=f_{k}^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \quad t_{k}=k \tau, \\
1 \leq k \leq N-1, \quad x \in \Omega_{h}, \\
v_{0}^{h}(x)-v_{l}^{h}(x)=\varphi^{h}(x)-\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h}, \\
v_{N}^{h}(x)-v_{l}^{h}(x)=\psi^{h}(x)-\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h} \tag{26}
\end{gather*}
$$

and second order of accuracy difference scheme

$$
\begin{gather*}
-\frac{v_{k+1}^{h}(x)-2 v_{k}^{h}(x)+v_{k-1}^{h}(x)}{\tau^{2}}+A_{h}^{x} v_{k}^{h}(x)=f_{k}^{h}(x), \\
f_{k}^{h}(x)=f^{h}\left(t_{k}, x\right), \\
t_{k}=k \tau, \quad 1 \leq k \leq N, \quad x \in \Omega_{h}, \\
v_{0}^{h}(x)-\left(\frac{\lambda}{\tau}-l\right) v_{l+1}^{h}(x)+\left(\frac{\lambda}{\tau}-l-1\right) v_{l}^{h}(x) \\
=\varphi^{h}(x)-\xi^{h}(x), \\
v_{N}^{h}(x)-\left(\frac{\lambda}{\tau}-l\right) v_{l+1}^{h}(x)+\left(\frac{\lambda}{\tau}-l-1\right) v_{l}^{h}(x) \\
=\psi^{h}(x)-\xi^{h}(x), \quad x \in \widetilde{\Omega}_{h} \tag{27}
\end{gather*}
$$

respectively.
Theorem 6. Let $\tau$ and $|h|=\sqrt{h_{1}^{2}+\cdots+h_{n}^{2}}$ be sufficiently small positive numbers. Then, for the solutions $\left(\left\{u_{k}^{h}\right\}_{K-1}^{N-1}, p^{h}\right)$ of difference schemes (24) and (25) the stability estimates

$$
\begin{gathered}
\left\|\left\{u_{k}^{h}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(L_{2 h}\right)} \leq M\left[\left\|\varphi^{h}\right\|_{L_{2 h}}+\left\|\psi^{h}\right\|_{L_{2 h}}+\left\|\xi^{h}\right\|_{L_{2 h}}\right. \\
\left.+\left\|\left\{f_{k}^{h}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(L_{2 h}\right)}\right] \\
\left\|p^{h}\right\|_{L_{2 h}} \leq M\left[\left\|\varphi^{h}\right\|_{W_{2 h}^{2}}+\left\|\psi^{h}\right\|_{W_{2 h}^{2}}+\left\|\xi^{h}\right\|_{W_{2 h}^{2}}\right. \\
\left.+\frac{1}{\alpha(1-\alpha)}\left\|\left\{f_{k}^{h}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(L_{2 h}\right)}\right]
\end{gathered}
$$

hold, where $M$ is independent of $\tau, \alpha, h, \varphi^{h}(x), \psi^{h}(x), \xi^{h}(x)$, and $\left\{f_{k}^{h}(x)\right\}_{1}^{N-1}$.

Theorem 7. Let $\tau$ and $|h|=\sqrt{h_{1}^{2}+\cdots+h_{n}^{2}}$ be sufficiently small positive numbers. Then, for the solutions of difference schemes (24) and (25) the following almost coercive stability estimate

$$
\begin{gather*}
\left\|\left\{\frac{\left.u_{k+1}^{h}-2 u_{k}^{h}+u_{k-1}^{k}\right)}{\tau^{2}}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(L_{2 h}\right)} \\
+\left\|\left\{u_{k}^{h}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(W_{2 h}^{2}\right)}+\left\|p^{h}\right\|_{L_{2 h}}  \tag{29}\\
\leq M\left[\left\|\varphi^{h}\right\|_{W_{2 h}^{2}}+\left\|\psi^{h}\right\|_{W_{2 h}^{2}}+\left\|\xi^{h}\right\|_{W_{2 h}^{2}}\right. \\
\\
\left.\quad+\ln \left(\frac{1}{\tau+h}\right)\left\|\left\{f_{k}^{h}\right\}_{1}^{N-1}\right\|_{\mathscr{C}_{\tau}\left(L_{2 h}\right)}\right]
\end{gather*}
$$

holds, where $M$ is independent of $\tau, \alpha, h, \varphi^{h}(x), \psi^{h}(x), \xi^{h}(x)$, and $\left\{f_{k}^{h}(x)\right\}_{1}^{N-1}$.

Proofs of Theorems 6 and 7 are based on the symmetry property of operator $A^{x}$, on Lemmas 3-5, the formulas

$$
\begin{aligned}
& u_{k}^{h}(x)=(I-\left.R^{2 N}\right)^{-1} \\
& \times {\left[\left(\left(R^{k}-R^{2 N-k}\right) v_{0}^{h}(x)\right.\right.} \\
&\left.+\left(R^{N-k}-R^{N+k}\right) v_{N}^{h}(x)\right) \\
& \quad\left(R^{N-k}-R^{N+k}\right)(I+\tau C)(2 I+\tau C)^{-1} \\
&\left.\times C^{-1} \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right) f_{i}^{h}(x) \tau\right] \\
&+(I+ \\
&+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right) \\
& \times f_{i}^{h}(x) \tau+\varphi^{h}(x)-v_{0}^{h}(x) \\
& p^{h}(x)=A_{h}^{x} \varphi^{h}(x)-A_{h}^{x} v_{0}^{h}(x) \\
& v_{N}^{h}(x)=v_{0}^{h}(x)+\psi^{h}(x)-\varphi^{h}(x)
\end{aligned}
$$

$$
\begin{align*}
v_{0}^{h}(x)= & -G\left(R^{N-l}-R^{N+l}\right) \\
& \times(I+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right) f_{i}^{h}(x) \tau \\
& +G\left(I-R^{2 N}\right)(I+\tau C) \\
& \times(2 I+\tau C)^{-1} C^{-1} \sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right) f_{i}^{h}(x) \tau \\
& +G\left(I-R^{2 N}\right)\left(\varphi^{h}(x)-\xi^{h}(x)\right) \\
& +G\left(R^{N-l}-R^{N+l}\right)\left(\psi^{h}(x)-\varphi^{h}(x)\right), \tag{30}
\end{align*}
$$

for difference scheme (24),

$$
\begin{aligned}
& v_{0}^{h}(x)=\left(\frac{\lambda}{\tau}-l-1\right) G_{1}\left(R^{N-l}-R^{N+l}\right) \\
& \times(I+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right) f_{i}^{h}(x) \tau \\
& -\left(\frac{\lambda}{\tau}-l-1\right) G_{1}\left(I-R^{2 N}\right) \\
& \times(I+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{|l-i|}-R^{l+i}\right) f_{i}^{h}(x) \tau \\
& -\left(\frac{\lambda}{\tau}-l\right) G_{1}\left(R^{N-l-1}-R^{N+l+1}\right) \\
& \times(I+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right) f_{i}^{h}(x) \tau \\
& +\left(\frac{\lambda}{\tau}-l\right) G_{1}\left(I-R^{2 N}\right) \\
& \times(I+\tau C)(2 I+\tau C)^{-1} C^{-1} \\
& \times \sum_{i=1}^{N-1}\left(R^{|l+1-i|}-R^{l+1+i}\right) f_{i}^{h}(x) \tau \\
& +G_{1}\left(I-R^{2 N}\right)\left(\varphi^{h}(x)-\xi^{h}(x)\right)
\end{aligned}
$$

$$
\begin{align*}
& +\left(\left(\frac{\lambda}{\tau}-l-1\right) G_{1}\left(R^{N-l}-R^{N+l}\right)\right. \\
& \left.\quad+\left(\frac{\lambda}{\tau}-l\right) G_{1}\left(R^{N-l-1}-R^{N+l+1}\right)\right) \\
& \times\left(\psi^{h}(x)-\varphi^{h}(x)\right), \tag{31}
\end{align*}
$$

for difference scheme (25), and on the following theorem on the coercivity inequality for the solution of the elliptic difference problem in $L_{2 h}$.

Theorem 8 (see [28]). For the solution of the elliptic difference problem

$$
\begin{gather*}
A_{h}^{x} u^{h}(x)=\omega^{h}(x), \quad x \in \widetilde{\Omega}_{h} \\
D^{h} u^{h}(x)=0, \quad x \in S_{h} \tag{32}
\end{gather*}
$$

the following coercivity inequality holds:

$$
\begin{equation*}
\sum_{r=1}^{n}\left\|\left(u_{k}^{h}\right)_{\bar{x}_{r} \bar{x}_{r}, j_{r}}\right\|_{L_{2 h}} \leq M\left\|\omega^{h}\right\|_{L_{2 h}} \tag{33}
\end{equation*}
$$

where $M$ does not depend on $h$ and $\omega^{h}$.

## 4. Numerical Results

We have not been able to obtain a sharp estimate for the constants figuring in the stability estimates. Therefore, we will give the following results of numerical experiments of the inverse problem for the two-dimensional elliptic equation with Dirichlet-Neumann boundary conditions

$$
\begin{gather*}
-\frac{\partial^{2} u(t, x)}{\partial t^{2}}-\frac{\partial}{\partial x}\left((2+\cos x) \frac{\partial u(t, x)}{\partial x}\right)+u(t, x) \\
=f(t, x)+p(x), \quad 0<x<\pi, 0<t<T, \\
f(t, x)=-\exp (-t) \cos (x) \\
\quad+(\exp (-t)+t)(3 \cos (x)+\cos (2 x)), \\
u(0, x)=2 \cos (x), \quad 0 \leq x \leq \pi, \\
\begin{array}{r}
u(T, x)= \\
u(\lambda, x)= \\
(\exp (-T)+T+1) \cos (x), \quad 0 \leq x \leq \pi
\end{array}, \\
u_{x}(t, 0)=u_{x}(t, \pi)=0, \quad 0 \leq t \leq T, \quad \lambda=\frac{3 T}{5} .
\end{gather*}
$$

It is clear that $u(t, x)=(\exp (-t)+t+1) \cos (x)$ and $p(x)=$ $\sin (x)+(x+2) \cos (x)$ are the exact solutions of (34).

We can obtain $u(t, x)$ by formula $u(t, x)=v(t, x)+w(t, x)$, where $v(t, x)$ is the solution of the nonlocal boundary value problem

$$
\begin{gather*}
-\frac{d^{2} v(t, x)}{d t^{2}}-\frac{\partial}{\partial x}\left((2+\cos x) \frac{\partial v(t, x)}{\partial x}\right)+v(t, x) \\
=f(t, x), \quad 0<x<\pi, 0<t<T \\
v(0, x)-v(\lambda, x)=(1-\exp (-\lambda)-\lambda) \cos (x) \\
0 \leq x \leq \pi \\
=(\exp (-T)-\exp (-\lambda)+T-\lambda) \cos (x), \\
\begin{array}{r}
v(T, x)-v(\lambda, x) \\
0 \leq x \leq \pi
\end{array} \\
v_{x}(t, 0)=v_{x}(t, \pi)=0, \quad 0 \leq t \leq T
\end{gather*}
$$

and $w(t, x)$ is the solution of the boundary value problem

$$
\begin{gather*}
-\frac{d^{2} w(t, x)}{d t^{2}}-\frac{\partial}{\partial x}\left((2+\cos x) \frac{\partial w(t, x)}{\partial x}\right)+w(t, x) \\
=p(x), \quad 0<x<\pi, 0<t<T \\
w(0, x)=(\exp (-\lambda)+\lambda+1) \cos (x)-v(\lambda, x) \\
0 \leq x \leq \pi \\
w(T, x)=(\exp (-\lambda)+\lambda+1) \cos (x)-v(\lambda, x) \\
0 \leq x \leq \pi
\end{gather*}
$$

Introduce small parameters $\tau$ and $h$ such that $N \tau=$ $T, M h=\pi$. For approximate solution of nonlocal boundary value problem (35) we consider the set $[0, T]_{\tau} \times[0, \pi]_{h}$ of a family of grid points

$$
\begin{align*}
& {[0, T]_{\tau} \times[0, \pi]_{h}} \\
& =\left\{\left(t_{k}, x_{n}\right): t_{k}=k \tau\right.  \tag{37}\\
& \left.\quad k=0, \ldots, N, x_{n}=n h, n=0, \ldots, M\right\}
\end{align*}
$$

Applying (21), we obtain difference schemes of the first order of accuracy in $t$ and the second order of accuracy in $x$

$$
\begin{align*}
& \frac{v_{n}^{k+1}-2 v_{n}^{k}+v_{n}^{k-1}}{\tau^{2}}+\left(2+\cos \left(x_{n}\right)\right) \frac{v_{n+1}^{k}-2 v_{n}^{k}+v_{n-1}^{k}}{h^{2}} \\
& -\sin \left(x_{n}\right) \frac{v_{n+1}^{k}-v_{n-1}^{k}}{2 h}-v_{n}^{k}=\theta_{n}^{k}, \\
& \theta_{n}^{k}=-f\left(t_{k}, x_{n}\right), \quad k=1, \ldots, N-1, n=1, \ldots, M-1, \\
& v_{0}^{k}-v_{1}^{k}=v_{M}^{k}-v_{M-1}^{k}=0, \quad k=0, \ldots, N, \\
& v_{n}^{0}-v_{n}^{l}=(1-\exp (-\lambda)-\lambda) \cos \left(x_{n}\right), \quad n=0, \ldots, M, \\
& v_{n}^{N}-v_{n}^{l}=\left(\exp \left(-t_{N}\right)-\exp (-\lambda)+t_{N}-\lambda\right) \cos \left(x_{n}\right), \\
& n=0, \ldots, M, \quad l=\left[\frac{\lambda}{\tau}\right], \tag{38}
\end{align*}
$$

for the approximate solutions of the nonlocal boundary value problem (35), and

$$
\begin{align*}
& \frac{w_{n}^{k+1}-2 w_{n}^{k}+w_{n}^{k-1}}{\tau^{2}}+\left(2+\cos \left(x_{n}\right)\right) \frac{w_{n+1}^{k}-2 w_{n}^{k}+w_{n-1}^{k}}{h^{2}} \\
& -\sin \left(x_{n}\right) \frac{w_{n+1}^{k}-w_{n-1}^{k}}{2 h}-w_{n}^{k} \\
& =-p_{n}, \quad k=1, \ldots, N-1, \\
& p_{n}=p\left(x_{n}\right), n=1, \ldots, M-1, \\
& w_{0}^{k}-w_{1}^{k}=w_{M}^{k}-w_{M-1}^{k}=0, \quad k=0, \ldots, N, \\
& w_{n}^{0}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right)-v_{n}^{l}, \\
& n=0, \ldots, M, l=\left[\frac{\lambda}{\tau}\right], \\
& w_{n}^{N}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right)-v_{n}^{l}, \\
& n=0, \ldots, M, \tag{39}
\end{align*}
$$

for the approximate solutions of the boundary value problem (36).

By using (22) and second order of accuracy in $x$ approximation of $A$, we get the following values of $p$ in grid points:

$$
\begin{aligned}
p_{n}= & -\frac{\left(2+\cos \left(x_{n}\right)\right)}{h^{2}}\left(\left(\varphi_{n+1}-v_{n+1}^{0}\right)-2\left(\varphi_{n}-v_{n}^{0}\right)\right. \\
& \left.+\left(\varphi_{n-1}-v_{n-1}^{0}\right)\right)+\frac{\sin \left(x_{n}\right)}{2 h} \\
& \times\left(\left(\varphi_{n+1}-v_{n+1}^{0}\right)-\left(\varphi_{n-1}-v_{n-1}^{0}\right)\right) \\
& +\left(\varphi_{n}-v_{n}^{0}\right), \quad n=1, \ldots, M-1 .
\end{aligned}
$$

We can rewrite difference scheme (38) in the matrix form

$$
\begin{gather*}
A_{n} v_{n+1}+B_{n} v_{n}+C_{n} v_{n-1}=I \theta_{n}^{k}, \quad n=1, \ldots, M-1,  \tag{41}\\
v_{0}=v_{1}, \quad v_{M}=v_{M-1}
\end{gather*}
$$

Here, $I$ is the $(N+1) \times(N+1)$ identity matrix, $A_{n}, B_{n}, C_{n}$ are $(N+1) \times(N+1)$ square matrices, and $\theta_{n}$ is a $(N+1) \times 1$ column matrix which are defined by

$$
\begin{align*}
& A_{n}=\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & a_{n} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & a_{n} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{n} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & a_{n} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & a_{n} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & a_{n} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{array}\right],  \tag{42}\\
& B_{n}=\left[\begin{array}{ccccccccccc}
1 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 0 \\
d & b_{n} & d & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & d & b_{n} & d & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & d & b_{n} & \cdots & 0 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & b_{n} & d & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & d & b_{n} & d & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & d & b_{n} & d \\
0 & 0 & 0 & 0 & \cdots & -1 & \cdots & 0 & 0 & 0 & 1
\end{array}\right], \\
& C_{n}=\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & c_{n} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & c_{n} & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & c_{n} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & c_{n} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & c_{n} & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & c_{n} & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0
\end{array}\right],  \tag{43}\\
& a_{n}=\frac{2+\cos \left(x_{n}\right)}{h^{2}}-\frac{\sin \left(x_{n}\right)}{2 h}, \\
& b_{n}=-\frac{2}{\tau^{2}}-\frac{2\left(2+\cos \left(x_{n}\right)\right)}{h^{2}}-1,  \tag{44}\\
& c_{n}=\frac{2+\cos \left(x_{n}\right)}{h^{2}}+\frac{\sin \left(x_{n}\right)}{2 h}, \quad d=\frac{1}{\tau^{2}},
\end{align*}
$$

$$
\begin{gather*}
\theta_{n}=\left[\begin{array}{c}
\theta_{n}^{0} \\
\vdots \\
\theta_{n}^{N}
\end{array}\right], \\
\theta_{n}^{0}=(1-\exp (-\lambda)-\lambda) \cos \left(x_{n}\right), \\
\theta_{n}^{N}=\left(\exp \left(-t_{N}\right)-\exp (-\lambda)+t_{N}-\lambda\right) \cos \left(x_{n}\right), \\
n=1, \ldots, M-1, \\
\theta_{n}^{k}=-f\left(t_{k}, x_{n}\right), \quad k=1, \ldots, N-1, n=1, \ldots, M-1, \\
v_{s}=\left[\begin{array}{c}
v_{s}^{0} \\
\vdots \\
v_{s}^{N}
\end{array}\right]_{(N+1) \times 1}, \quad s=n-1, n, n+1 . \tag{45}
\end{gather*}
$$

For solving (41) we use the modified Gauss elimination method (see [29]). Namely, we seek solution of (41) by the formula

$$
\begin{equation*}
v_{n}=\alpha_{n+1} v_{n+1}+\beta_{n+1}, \quad n=M-1, \ldots, 1, \tag{46}
\end{equation*}
$$

where $v_{M}=\overrightarrow{0}, \alpha_{n}(n=1, \ldots, M-1)$ are $(N+1) \times(N+1)$ square matrices and $\beta_{n}(n=1, \ldots, M-1)$ are $(N+1) \times 1$ column matrices. For $\alpha_{n+1}, \beta_{n+1}$, we get formulas

$$
\begin{gather*}
\alpha_{n+1}=-\left(B_{n}+C_{n} \alpha_{n}\right)^{-1} A_{n} \\
\beta_{n+1}=-\left(B_{n}+C_{n} \alpha_{n}\right)^{-1}\left(I \theta_{n}-C_{n} \beta_{n}\right), \quad n=1, \ldots, M-1, \tag{47}
\end{gather*}
$$

where $\alpha_{1}$ is the $(N+1) \times(N+1)$ identity matrix and $\beta_{1}$ is the $(N+1) \times 1$ zero column vector.

Futher, we rewrite difference scheme (39) in the matrix form

$$
\begin{gather*}
A_{n} w_{n+1}+E_{n} w_{n}+C_{n} w_{n-1}=I \eta_{n}^{k} \\
n=1, \ldots, M-1  \tag{48}\\
w_{0}=w_{1}, \quad w_{M}=w_{M-1}
\end{gather*}
$$

Here,

$$
E_{n}=\left[\begin{array}{ccccccccc}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
d & b_{n} & d & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & d & b_{n} & d & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & d & b_{n} & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \cdots & b_{n} & d & 0 & 0 \\
0 & 0 & 0 & 0 & \cdots & d & b_{n} & d & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & d & b_{n} & d \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1
\end{array}\right],
$$

$A_{n}$ and $C_{n}$ are defined by (42) and (43) and $(N+1) \times 1$ column matrix $\eta_{n}$ is defined by

$$
\eta_{n}=\left[\begin{array}{c}
\eta_{n}^{0} \\
\vdots \\
\eta_{n}^{N}
\end{array}\right]
$$

$$
\eta_{n}^{0}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right)-v_{n}^{l},
$$

$$
\begin{gather*}
\eta_{n}^{N}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right)-v_{n}^{l}, \quad n=1, \ldots, M-1, \\
\eta_{n}^{k}=-p_{n}, \quad k=1, \ldots, N-1, n=1, \ldots, M-1, \\
w_{s}=\left[\begin{array}{c}
w_{s}^{0} \\
\vdots \\
w_{s}^{N}
\end{array}\right]_{(N+1) \times 1}, \quad s=n-1, n, n+1 . \tag{50}
\end{gather*}
$$

Now we present second order of accuracy in $t$ and $x$ difference schemes for problems (35) and (36). Applying (27) and formulas for sufficiently smooth function $\rho$

$$
\begin{gather*}
\frac{\rho\left(x_{n+1}\right)-\rho\left(x_{n-1}\right)}{2 h}-\rho^{\prime}\left(x_{n}\right)=O\left(h^{2}\right), \\
\frac{\rho\left(x_{n+1}\right)-2 \rho\left(x_{n}\right)+\rho\left(x_{n-1}\right)}{h^{2}}-\rho^{\prime \prime}\left(x_{n}\right)=O\left(h^{2}\right), \\
\frac{10 \rho(0)-15 \rho(h)+6 \rho(2 h)-\rho(3 h)}{h^{3}}-\rho^{\prime \prime \prime}(0)=O\left(h^{2}\right), \\
\frac{-3 \rho(0)+4 \rho(h)-\rho(2 h)}{2 h}-\rho^{\prime}(0)=O\left(h^{2}\right), \\
\frac{10 \rho(\pi)-15 \rho(\pi-h)+6 \rho(\pi-2 h)-\rho(\pi-3 h)}{h^{3}} \\
\frac{-3 \rho(\pi)+4 \rho(\pi-h)-\rho(\pi-2 h)}{2 h}-\rho^{\prime}(\pi)=O\left(h^{2}\right),
\end{gather*}
$$

we get

$$
\begin{aligned}
& \frac{v_{n}^{k+1}-2 v_{n}^{k}+v_{n}^{k-1}}{\tau^{2}}+\left(2+\cos \left(x_{n}\right)\right) \frac{v_{n+1}^{k}-2 v_{n}^{k}+v_{n-1}^{k}}{h^{2}} \\
& \quad-\sin \left(x_{n}\right) \frac{v_{n+1}^{k}-v_{n-1}^{k}}{2 h}-v_{n}^{k}=\theta_{n}^{k}
\end{aligned}
$$

$$
\begin{align*}
& \theta_{n}^{k}=- f\left(t_{k}, x_{n}\right), \quad k=1, \ldots, N-1, \quad n=1, \ldots, M-1, \\
&- 3 v_{0}^{k}+4 v_{1}^{k}-v_{2}^{k} \\
&=-3 v_{M}^{k}+4 v_{M-1}^{k}-v_{M-2}^{k}=0, \quad k=0, \ldots, N, \\
& 10 v_{0}^{k}-15 v_{1}^{k}+6 v_{2}^{k}-v_{3}^{k} \\
&=10 v_{M}^{k}-15 v_{M-1}^{k}+6 v_{M-2}^{k}-v_{M-3}^{k}=0, \\
& v_{n}^{0}+\left(\frac{\lambda}{\tau}-l-1\right) v_{n}^{l}-\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1} \\
&=(1-\exp (-\lambda)-\lambda) \cos \left(x_{n}\right), \quad n=0, \ldots, M, \\
& v_{n}^{N}+\left(\frac{\lambda}{\tau}-l-1\right) v_{n}^{l}-\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1} \\
&=\left(\exp \left(-t_{N}\right)-\exp (-\lambda)+t_{N}-\lambda\right) \cos \left(x_{n}\right), \\
& n=0, \ldots, M, \tag{52}
\end{align*}
$$

difference scheme for nonlocal problem (35), and

$$
\begin{align*}
& \frac{w_{n}^{k+1}-2 w_{n}^{k}+w_{n}^{k-1}}{\tau^{2}}+\left(2+\cos \left(x_{n}\right)\right) \frac{w_{n+1}^{k}-2 w_{n}^{k}+w_{n-1}^{k}}{h^{2}} \\
& -\sin \left(x_{n}\right) \frac{w_{n+1}^{k}-w_{n-1}^{k}}{2 h}=-p_{n}, \\
& k=1, \ldots, N-1, \quad p_{n}=p\left(x_{n}\right), \quad n=1, \ldots, M-1, \\
& -3 w_{0}^{k}+4 w_{1}^{k}-w_{2}^{k}=-3 w_{M}^{k}+4 w_{M-1}^{k}-w_{M-2}^{k}=0, \\
& \quad k=0, \ldots, N, \\
& \quad=10 w_{M}^{k}-15 w_{M-1}^{k}+6 w_{M-2}^{k}-w_{M-3}^{k}=0, \\
& w_{n}^{0}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}^{k}\right) \\
& \quad+\left(\frac{\lambda}{\tau}-l-1\right) w_{n}^{k}-w_{3}^{k} \quad\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1}, \quad n=0, \ldots, M, \\
& \quad w_{n}^{N}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right) \\
& \quad+\left(\frac{\lambda}{\tau}-l-1\right) v_{n}^{l}-\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1}, \\
& \quad \xi_{n}=\xi\left(x_{n}\right), \quad n=0, \ldots, M,
\end{align*}
$$

difference scheme for boundary value problem (36).

By difference scheme (52), we write in matrix form

$$
\begin{align*}
& A_{n} v_{n+1}+B_{n} v_{n}+C_{n} v_{n-1}=I \theta_{n}^{k}, \quad n=1, \ldots, M-1 \\
&-3 v_{0}+4 v_{1}-v_{2}=0  \tag{54}\\
&-3 v_{M}+4 v_{M-1}-v_{M-2}=0
\end{align*}
$$

where $A_{n}, C_{n}$ are defined by (42), (43), (44), and $B_{n}$ is defined by

\[

\]

$$
\begin{equation*}
d=\frac{1}{\tau^{2}}, \quad y=\left(\frac{\lambda}{\tau}-l-1\right), \quad z=-\left(\frac{\lambda}{\tau}-l\right) \tag{55}
\end{equation*}
$$

We seek solution of (54) by the formula

$$
\begin{equation*}
v_{n}=\alpha_{n} v_{n+1}+\beta_{n} v_{n+2}+\gamma_{n}, \quad n=M-2, \ldots, 0, \tag{56}
\end{equation*}
$$

where $\alpha_{n}, \beta_{n}(n=0, \ldots, M-2)$ are $(N+1) \times(N+1)$ square matrices and $\gamma_{n}(n=0, \ldots, M-2)$ are $(N+1) \times 1$ column matrices. For the solution of difference equation (41) we need to use the following formulas for $\alpha_{n}, \beta_{n}$ :

$$
\begin{gather*}
\alpha_{n}=-\left(B_{n}+C_{n} \alpha_{n-1}\right)^{-1}\left(A_{n}+C_{n} \beta_{n-1}\right), \\
\beta_{n}=0 \\
\gamma_{n}=-\left(B_{n}+C_{n} \alpha_{n-1}\right)^{-1}\left(I \theta_{n}-C_{n} \gamma_{n-1}\right), \quad n=1, \ldots, M-1, \tag{57}
\end{gather*}
$$

where

$$
\begin{array}{ll}
\alpha_{0}=\frac{4}{3} I, & \beta_{0}=-\frac{1}{3} I, \\
\alpha_{1}=\frac{8}{5} I, & \beta_{1}=-\frac{3}{5} I,  \tag{58}\\
\alpha_{M-2}=4 I, & \beta_{M-2}=-3 I \\
\alpha_{M-3}=\frac{8}{3} I, & \beta_{M-3}=-\frac{5}{3} I
\end{array}
$$

and $\gamma_{0}, \gamma_{1}, \gamma_{M-2}, \gamma_{M-3}$ are the $(N+1) \times 1$ zero column vector. For $v_{M}$ and $v_{M-1}$ we have

$$
\begin{gather*}
v_{M}=\left(Q_{11}-Q_{12} Q_{22}^{-1} Q_{21}\right)^{-1}\left(G_{1}-Q_{12} Q_{22}^{-1} G_{2}\right),  \tag{59}\\
v_{M-1}=Q_{22}^{-1}\left(G_{2}-Q_{21} v_{M}\right),
\end{gather*}
$$

where

$$
\begin{gather*}
Q_{11}=-3 A_{M-2}-8 B_{M-2}-8 C_{M-2} \alpha_{M-3}-3 C_{M-2} \beta_{M-3}, \\
Q_{12}=4 A_{M-2}+9 B_{M-2}+9 C_{M-2} \alpha_{M-3}+4 C_{M-2} \beta_{M-3}, \\
Q_{21}=-3 B_{M-1}-8 C_{M-1}, \\
Q_{22}=A_{M-1}+4 B_{M-1}+9 C_{M-1}, \\
G_{1}=I \theta_{M-2}-C_{M-2} \gamma_{M-3}, \quad G_{2}=I \theta_{M-1} . \tag{60}
\end{gather*}
$$

We can rewrite difference scheme (53) in the matrix form

$$
\begin{gather*}
A_{n} w_{n+1}+E_{n} w_{n}+C_{n} w_{n-1}=I \eta_{n}^{k}, \quad n=1, \ldots, M-1 \\
-3 w_{0}+4 w_{1}-w_{2}=0  \tag{61}\\
-3 w_{M}+4 w_{M-1}-w_{M-2}=0
\end{gather*}
$$

where $A_{n}, E_{n}, C_{n}$ are defined by (42), (49), (43), and (44) and $\eta_{n}$ is defined by

$$
\begin{gather*}
\eta_{n}=\left[\begin{array}{c}
\eta_{n}^{0} \\
\vdots \\
\eta_{n}^{N}
\end{array}\right], \\
\eta_{n}^{0}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right) \\
+\left(\frac{\lambda}{\tau}-l-1\right) v_{n}^{l}-\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1},  \tag{62}\\
\eta_{n}^{N}=(\exp (-\lambda)+\lambda+1) \cos \left(x_{n}\right)+\left(\frac{\lambda}{\tau}-l-1\right) v_{n}^{l} \\
-\left(\frac{\lambda}{\tau}-l\right) v_{n}^{l+1}, \quad n=0, \ldots, M, \\
\eta_{n}^{k}=-p_{n}, \quad k=1, \ldots, N-1, \quad n=1, \ldots, M-1 .
\end{gather*}
$$

Now, we give the results of the numerical realization of finite difference method for (34) by using MATLAB programs. The numerical solutions are recorded for $T=$ 2 and different values of $N=M$. Grid functions $v_{n}^{k}$, $u_{n}^{k}$ represent the numerical solutions of difference schemes for auxiliary nonlocal problem (35) and inverse problem (34) at $\left(t_{k}, x_{n}\right)$, respectively. Grid function $p_{n}$ calculated by (40) represents numerical solution at $x_{n}$ for unknown function $p$. The errors are computed by the norms

$$
\begin{gathered}
E v_{M}^{N}=\max _{1 \leq k \leq N-1}\left(\sum_{n=1}^{M-1}\left|v\left(t_{k}, x_{n}\right)-v_{n}^{k}\right|^{2} h\right)^{1 / 2}, \\
E u_{M}^{N}=\max _{1 \leq k \leq N-1}\left(\sum_{n=1}^{M-1}\left|u\left(t_{k}, x_{n}\right)-u_{n}^{k}\right|^{2} h\right)^{1 / 2}, \\
E p_{M}=\left(\sum_{n=1}^{M-1}\left|p\left(x_{n}\right)-p_{n}\right|^{2} h\right)^{1 / 2}
\end{gathered}
$$

TABLE 1: Error analysis for nonlocal problem.

|  | $N=M=$ | $N=M=$ | $N=M=$ | $N=M=$ |
| :--- | :---: | :---: | :---: | :---: |
| 20 | 40 | 80 | 160 |  |
| Difference <br> scheme (38) | 0.30522 | 0.14933 | 0.073953 | 0.036814 |
| Difference <br> scheme (52) | 0.024714 | 0.0031054 | $4.36 \times 10^{-4}$ | $7.52 \times 10^{-5}$ |

Table 2: Error analysis for $p$.

|  | $N=M=$ <br> 20 | $N=M=$ <br> 40 | $N=M=$ <br> 80 | $N=M=$ <br> 160 |
| :--- | :---: | :---: | :---: | :---: |
| Difference <br> scheme (38), <br> $(40)$ | 0.57878 | 0.33755 | 0.20387 | 0.12905 |
| Difference <br> scheme (52), <br> $(40)$ | 0.058201 | 0.010646 | 0.0020228 | $4.03 \times 10^{-4}$ |

Table 3: Error analysis for $u$.

|  | $N=M=$ | $N=M=$ | $N=M=$ | $N=M=$ |
| :--- | :---: | :---: | :---: | :---: |
|  | 20 | 40 | 80 | 160 |
| Difference <br> scheme (38), <br> (40), (39) | 0.088225 | 0.038586 | 0.01815 | 0.008818 |
| Difference <br> lcheme (52), <br> (40), (53) | 0.017034 | 0.0020225 | $2.47 \times 10^{-4}$ | $3.08 \times 10^{-5}$ |

Tables 1-3 present the error between the exact solution and numerical solutions derived by corresponding difference schemes. The results are recorded for $N=M=20,40,80$ and 160 , respectively. The tables show that the second order of accuracy difference scheme is more accurate than the first order of accuracy difference scheme for both auxiliary nonlocal and inverse problems. Table 1 contains error between the exact and approximate solutions $v$ of auxiliary nonlocal boundary value problem (35). Table 2 includes error between the exact and approximate solutions $p$ of inverse problem (34). Table 3 represents error between the exact solution $u$ of inverse problem (34) and approximate solution which is derived by the first and second orders accuracy of difference schemes.

## 5. Conclusion

In this paper, inverse problem for multidimensional elliptic equation with Dirichlet-Neumann conditions is considered. The stability and coercive stability estimates for solution of this problem are established. First and second order of accuracy difference schemes are presented for approximate solutions of inverse problem. Theorems on the stability and coercive stability inequalities for difference schemes are proved. The theoretical statements for the solution of these difference schemes are supported by the results of numerical example in a two-dimensional case. As it can be seen from Tables $1-3$, second order of accuracy difference
scheme is more accurate compared with the first order of accuracy difference scheme. Moreover, applying the result of the monograph [29] the high order of accuracy difference schemes for the numerical solution of the boundary value problem (2) can be presented.

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## Research Article

# About the Nature of the Spectrum of the Periodic Problem for the Heat Equation with a Deviating Argument 

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We study the nature of the spectrum of the periodic problem for the heat equation with a lower-order term and with a deviating argument. A significant influence of the lower-order term on the correct solvability of this problem is found. We obtain a criterion for the strong solvability of the above-mentioned problem.

## 1. Introduction

In paper [1] we constructed a spectral theory of a model differential equation of first order with a deviating argument. The main idea of the article [1] has been further developed in $[2,3]$.

The equation that we studied belongs to a class of functional-differential equations. These equations have only recently become a subject of research of individual authors; particularly, such equations are studied in [4-6]. Functionaldifferential equations have been actively studied recently by some authors in papers $[7,8]$.

In this paper using the methods of paper [1] we have found a solution of the mixed problem for the heat equation with a deviating argument. As a result we got that the classical solvability of the boundary value problem requires a certain smoothness of the right-hand side of the equation (see (11) of Theorem 4) and the strong solvability of the problem is provided by the properties of the coefficient of the lowerorder term of the equation (Theorem 11).

Let $\Omega \subset R^{2}$ be a rectangle bounded by the following segments:

$$
\begin{array}{ll}
A B: 0 \leq t \leq T, x=0, & B C: 0 \leq x \leq l, t=T, \\
C D: 0 \leq t \leq T, x=l, & D A: 0 \leq x \leq l, t=0 . \tag{1}
\end{array}
$$

We denote by $C^{2,1}(\Omega)$ the set of functions $u(x, t)$ that are twice continuously differentiable with respect to the variable
$x$ and once continuously differentiable with respect to the variable $t$. The boundary of the area $\Omega$ is a set of segments $B=A B \cup A D \cup C D$.
Periodic Problem. Find a solution of the equation

$$
\begin{equation*}
L u=u_{t}(x, T-t)+u_{x x}(x, t)+a u_{x}=f(x, t) \tag{2}
\end{equation*}
$$

satisfying the conditions

$$
\begin{gather*}
\left.u\right|_{t=0}=0  \tag{3}\\
\left.u\right|_{x=0}-\left.u\right|_{x=l}=0,\left.\quad u\right|_{x=0}-\left.u_{x}\right|_{x=l}=0, \tag{4}
\end{gather*}
$$

where $f \in L^{2}(\Omega)$ and $a$ is constant.
Further the coefficient $a$ will be called the coefficient of influence.

Definition 1. The function $u(x, t) \in L^{2}(\Omega)$ is called a strong solution of (2)-(4) if there exists a sequence of functions $\left\{u_{n}\right\} \in C^{2,1}(\Omega) \cap C^{1,0}(\bar{\Omega}), n=1,2,3, \ldots$, that satisfies the boundary conditions of the problem and such that $\left\{u_{n}\right\}$ and $\left\{L u_{n}\right\}, n=1,2,3, \ldots$, converge, respectively, to $u$ and $f$.

Definition 2. The boundary value problem (2)-(4) is called strongly solvable if a strong solution of the problem exists for any right-hand side $f(x, t) \in L^{2}(\Omega)$ and this solution is unique.

The aim of this study is to investigate the nature of the spectrum and the influence of the lower-order term on the strong solvability of the problem (2)-(4) in the space $L^{2}(\Omega)$.

Using Fourier method we obtained the conditions of the existence of a strong solution of the boundary value problem (2)-(4) in the space $L^{2}(\Omega)$ in the form of Theorem 4. With the help of the spectral theory of linear operators we have established criteria of strong solvability of this problem, presented in the form of Theorem 11, which is the main result of this paper.

## 2. Results

Problem about Spectrum. Examine the nature of the spectrum of the functional-differential operator

$$
L u=u_{t}(x, T-t)+u_{x x}(x, t)+a u_{x},
$$

$$
\begin{align*}
& D(L)=\left\{u \in C^{2,1}(\Omega) \cap C(\bar{\Omega}),\right. \\
& \left.\left.\quad u\right|_{t=0}=\left.u\right|_{x=0}-\left.u\right|_{x=l}=\left.u_{x}\right|_{x=0}-\left.u_{x}\right|_{x=l}=0\right\} . \tag{5}
\end{align*}
$$

Theorem 3. The spectral problem

$$
\begin{gather*}
L u=u_{t}(x, T-t)+u_{x x}(x, t)+a u_{x}(x, t)=\lambda u(x, t), \\
\left.u\right|_{t=0}=\left.u\right|_{x=0}-\left.u\right|_{x=l}=\left.u_{x}\right|_{x=0}-\left.u_{x}\right|_{x=l}=0 \tag{6}
\end{gather*}
$$

has an infinite number of eigenvalues:

$$
\begin{gather*}
\lambda_{m n}=(-1)^{n}\left(n+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}+\frac{2 m \pi i}{l} a  \tag{7}\\
m=0, \pm 1, \pm 2, \ldots, \quad n=0,1,2, \ldots
\end{gather*}
$$

and the corresponding eigenfunctions

$$
\begin{array}{r}
u_{m n}(x, t)=\frac{2}{\sqrt{T l}} \exp \left(\frac{2 m \pi i}{l} x\right) \cdot \sin \left(n+\frac{1}{2}\right) \frac{\pi t}{T}  \tag{8}\\
m=0, \pm 1, \pm 2, \ldots, \quad n=0,1,2, \ldots
\end{array}
$$

which form an orthonormal basis of the space $L^{2}(\Omega)$, where $\Omega=[0, l] \times[0, T]$.

Proof. By method of separation of variables we get two spectral problems. The first problem is the Sturm-Liouville problem with Dirichlet condition. The second problem is the Cauchy problem for the first-order equation with deviating arguments, which is studied in detail in [1]. The rest is elementary.

Theorem 4. For the existence and uniqueness of a strong solution of the boundary problem (2)-(4), it is necessary and sufficient to fulfill the condition

$$
\begin{align*}
& {\left[(-1)^{n}\left(n+\frac{1}{2}\right) \cdot \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}-\frac{2 m \pi}{l} \cdot \operatorname{Im} a\right]^{2}} \\
& \quad+\left(\frac{2 m \pi}{l} \cdot \operatorname{Re} a\right)^{2} \neq 0, \quad \forall m=0, \pm 1, \pm 2, \ldots, n=0,1,2, \ldots \tag{9}
\end{align*}
$$

When this condition is fulfilled a strong solution of the problem exists and has the form

$$
\begin{equation*}
u(x, t)=\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\left(f, u_{m n}\right)}{\lambda_{m n}} \cdot u_{m n}(x, t) \tag{10}
\end{equation*}
$$

for all $f(x, t) \in L^{2}(\Omega)$ satisfying

$$
\begin{equation*}
\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty}\left|\frac{\left(f, u_{m n}\right)}{\lambda_{m n}}\right|^{2}<+\infty \tag{11}
\end{equation*}
$$

where the eigenvalues $\lambda_{m n}$ and the eigenfunctions $u_{m n}(x, t)$ are given by (7) and (8).

The proof is omitted since this theorem is a simple consequence of the preceding theorem.

Theorem 5. If $a+\bar{a}=0$, then the differential operator (5) is essentially self-adjoint in the space $H=L^{2}(\Omega)$, where $\Omega=$ $[0, l] \times[0, T]$ is a rectangle, lying in the upper half-plane $(x, t) \in$ $R^{2}$ 。

Proof. It easily follows from the symmetry of the operator $L$ and the completeness of its eigenvectors.

From Theorems 4 and 5 there follows the Theorem 6.
Theorem 6. If
(a) $\operatorname{Re} a=0$,
(b) $(-1)^{n}(n+1 / 2) \cdot(\pi / T)-(2 m \pi / l)^{2}-(2 m \pi / l) \cdot \operatorname{Im} a \neq 0$,
then the inverse operator $(\bar{L})^{-1}$ exists, which is also self-adjoint.
Proof. From Theorem 4 there follows the existence of the operator $(\bar{L})^{-1}$; the rest follows from the series of equalities $\left(\bar{L}^{-1}\right)^{*}=\left(\bar{L}^{*}\right)^{-1}=(\bar{L})^{-1}$, in which Theorem 5 and the known equality $\left(A^{-1}\right)^{*}=\left(A^{*}\right)^{-1}$ were used.

Definition 7. A linear operator $A$ (not necessarily bounded) in space $H$ is called normal if it is densely defined and closed and satisfies the condition $A^{*} A=A A^{*}$.

Lemma 8 (see [9]). Let A be a densely defined operator in a Hilbert space H. Then
(a) $A^{*}$ is closed;
(b) A allows a closure if and only if $D\left(A^{*}\right)$ is dense, and in this case $\bar{A}=A^{* *}$;
(c) if A allows a closure, then $(\bar{A})^{*}=A^{*}$.

Lemma 9 (see [10]). Let A be a normal operator in the space H. Then
(a) $D(A)=D\left(A^{*}\right)$;
(b) $\|A x\|=\left\|A^{*} x\right\|$ for $\forall x \in D(A)$;
(c) $A$ is the maximum normal operator.

Using these lemmas we obtain the next theorem.
Theorem 10. If $a+\bar{a} \neq 0$, then the closure of the operator $L$ is a normal operator; that is, the equality $\bar{L}^{*} \bar{L}=\bar{L} \bar{L}^{*}$ is satisfied.

Proof. This formula can be verified directly.

## 3. About the Nature of the Spectrum of the Operator $\bar{L}$

Theorem 11. (a) If $\operatorname{Re} a \neq 0$, then there exists an inverse operator $\bar{L}^{-1}$, which is normal and compact. We have the estimate

$$
\begin{equation*}
\left\|\bar{L}^{-1}\right\| \leq K^{-1}, \quad K=\max \left\{\frac{\pi}{2 T}, \frac{2 \pi}{l}|\operatorname{Re} a|\right\} . \tag{12}
\end{equation*}
$$

The spectrum of the operator $\bar{L}$ is discrete, that is, has no limit points in the finite part of the plane.
(b) If $\operatorname{Re} a=0, \operatorname{Im} a \neq\left((-1)^{n}(n+1 / 2) / 2 m\right) \cdot(l / T)-$ $2 m \pi / l, n=0,1,2, \ldots, m=0, \pm 1, \pm 2, \ldots$, and both values of $2 \pi T / l^{2}$ and $T \cdot \operatorname{Im} a / l$ are rational, then the inverse operator $\bar{L}^{1}$ exists and is bounded but not compact. The operator $\bar{L}$ is self-adjoint; its spectrum consists of an infinite number of eigenvalues, among which there are an infinite number of infinite multiple eigenvalues.
(c) If $\operatorname{Re} a=0, \operatorname{Im} a \neq\left((-1)^{n}(n+1 / 2) / 2 m\right) \cdot(l / T)-2 m \pi / l$, $n=0,1,2, \ldots, m=0, \pm 1, \pm 2, \ldots$ and at least one of the values of $2 \pi T / l^{2}$ and $T \cdot \operatorname{Im} a / l$ is irrational, then the inverse operator $\bar{L}^{-1}$ exists, but is not bounded. The operator $\bar{L}$ is self-adjoint and its spectrum consists of an infinite number of eigenvalues and continuous spectrum filling the entire real axis $(-\infty,+\infty)$. The points of the continuous spectrum are the limit points of eigenvalues.
(d) If $\operatorname{Re} a=0$ and $\operatorname{Im} a=\left(\left((-1)^{n}(n+1 / 2)\right) / 2 m\right) \cdot(l / T)-$ $2 m \pi / l$ for some values $n=0,1,2, \ldots ; m=0, \pm 1, \pm 2, \ldots$, the inverse operator $\bar{L}^{-1}$ does not exist. The operator $\bar{L}$ is selfadjoint. If both values of $2 \pi T / l^{2}$ and $T \cdot \operatorname{Im} a / l$ are rational, the spectrum consists of an infinite number of eigenvalues, among which there are an infinite number of infinite multiple eigenvalues. If at least one of the values of $2 \pi T / l^{2}, T \cdot \operatorname{Im} a / l$ is irrational, then the spectrum of the operator $\bar{L}$ fills the entire real line $(-\infty,+\infty)$.

Proof. In the work [11] Weyl introduced the concept of a sequence uniformly distributed modulo 1 and also proved a criterion of uniform distribution. In the same paper he gave examples of sequences distributed uniformly modulo 1 . The simplest of these sequences is the sequence $1 \xi, 2 \xi, 3 \xi, \ldots$ with some irrational number $\xi$.

Weyl's First Theorem. If $\varphi(z)$ is a polynomial with a constant term $\alpha_{0}$ and not all coefficients of $\varphi(z)-\alpha_{0}$ are rational, then the sequence of numbers $\varphi(1), \varphi(2), \varphi(3), \ldots$ is distributed uniformly dense everywhere.

In particular:
Weyl's Second Theorem. If $\xi$ is some irrational number, the sequence of points $1 \xi, 4 \xi, 9 \xi, 16 \xi, 25 \xi, \ldots$ when winding of a
real axis around a circle of length 1 covers it evenly dense everywhere. The same will hold if the squares of numbers are replaced by their cubes or fourth degrees, and so forth.

Next, we show that the set of eigenvalues $\left\{\lambda_{m n}\right\}$ is compacted with an increase in the indices $n$ and $m$.

The eigenvalues that we found are of the form

$$
\begin{align*}
& \lambda_{m n}=(-1)^{n}\left(n+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}+\frac{2 m \pi i}{l} a,  \tag{13}\\
& a \neq 0, \quad m=0, \pm 1, \pm 2, \ldots, \quad n=0,1,2, \ldots
\end{align*}
$$

Let us consider the neighborhood of the origin. If it was a limit point of the set of eigenvalues, then the inverse operator $(\bar{L})^{-1}$ would be unbounded.

If $\lambda_{m n} \rightarrow 0$ for some subsequence, then

$$
\begin{align*}
\left|\lambda_{m n}\right|^{2}= & \left((-1)^{n}\left(n+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}-\frac{2 m \pi}{l} \cdot \operatorname{Im} a\right)^{2} \\
& +\left(\frac{2 m \pi}{l} \cdot \operatorname{Re} a\right)^{2} \longrightarrow 0 \tag{14}
\end{align*}
$$

This is impossible when $\operatorname{Re} a \neq 0$. If $\operatorname{Re} a \neq 0$, then

$$
\begin{align*}
&\left|\lambda_{m n}\right|^{2} \geq\left(\frac{2 \pi}{l} \operatorname{Re} a\right)^{2}, \quad \forall m=1,2, \ldots, \\
& \Longrightarrow\left|\lambda_{m n}\right| \geq\left(\frac{2 \pi}{l} \operatorname{Re} a\right), \quad \forall m=1,2, \ldots,  \tag{15}\\
&\left|\lambda_{0 n}\right|^{2}=\left[\left(n+\frac{1}{2}\right) \frac{\pi}{T}\right]^{2} \geq\left(\frac{\pi}{2 T}\right)^{2}, \\
&\left|\lambda_{0 n}\right| \geq \frac{\pi}{2 T}, \quad \forall n=0,1,2, \ldots
\end{align*}
$$

Hence,

$$
\begin{array}{r}
\left|\lambda_{m n}\right| \geq \max \left\{\frac{\pi}{2 T}, \frac{2 \pi}{l}|\operatorname{Re} a|\right\}, \quad \forall m=0, \pm 1, \pm 2, \ldots ;  \tag{16}\\
n=0,1,2, \ldots
\end{array}
$$

Thus, when Re $a \neq 0$ the inverse operator $\bar{L}^{-1}$ exists and is bounded. If a subsequence $\left\{\lambda_{k l}\right\}$ of the sequence $\left\{\lambda_{m n}\right\}$ converges to a point $\lambda_{0}$ in the complex plane, then the sequence $\left\{\left|\lambda_{k l}\right|\right\}$ is bounded; therefore, the second index $l$ takes only a finite number of values. Then the first index takes a finite number of values too. We have a contradiction, since assumption on $\left\{\lambda_{k l}\right\}$ is an infinite convergent sequence. Therefore, the sequence $\left\{\lambda_{m n}\right\}$ has no limit points in the finite part of the complex plane $\lambda$, which means that the spectrum of the operator $\bar{L}$ is discrete.

Now we will investigate whether the operator $\bar{L}^{-1}$ is compact. Any subsequence of the sequence $\left\{\lambda_{m n}\right\}, m=$ $0, \pm 1, \pm 2, \ldots, n=0,1,2 \ldots$, tends to infinity. In fact, let $\left\{\lambda_{i j}\right\}$ be an arbitrary infinite subsequence of the sequence $\left\{\lambda_{m n}\right\}$. Then two situations are possible:
(1) either the first index takes an infinite number of values, then

$$
\begin{equation*}
\left|\lambda_{i j}\right|^{2} \geq\left(\frac{2 i \pi}{l} \operatorname{Re} a\right)^{2}, \quad \Longrightarrow\left|\lambda_{i j}\right| \longrightarrow+\infty \tag{17}
\end{equation*}
$$

(2) or the first index takes a finite number of values, while the second index takes an infinite number of values; therefore

$$
\begin{align*}
\left|\lambda_{i j}\right|^{2}= & {\left[(-1)^{j}\left(j+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 i \pi}{l}\right)^{2}\right.} \\
& \left.-\frac{2 i \pi}{l} \cdot \operatorname{Im} a\right]^{2}+\left(\frac{2 i \pi}{l} \cdot \operatorname{Re} a\right)^{2} \\
\geq & {\left[\left(j+\frac{1}{2}\right) \frac{\pi}{T}+(-1)^{j+1}\left(\frac{2 i \pi}{l}\right)^{2}\right.}  \tag{18}\\
& \left.+(-1)^{j+1} \frac{2 i \pi}{l} \cdot \operatorname{Im} a\right]^{2} \longrightarrow \infty
\end{align*}
$$

Due to the fact that $j \rightarrow+\infty$, the second and third terms are the bounded quantities.

Lemma 12 (see [12]). For the complete continuity of the operator of the normal type it is necessary and sufficient to fulfill the condition

$$
\begin{equation*}
\lim _{n \rightarrow+\infty} \lambda_{n}=0 . \tag{19}
\end{equation*}
$$

On the basis of Lemma 12 and (17) and (18) it follows that the inverse operator $\bar{L}^{-1}$ is completely continuous. Therefore its spectrum is discrete.

Now consider the case Re $a=0$.
In this case, the eigenvalues have the form

$$
\begin{equation*}
\lambda_{m n}=(-1)^{n}\left(n+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}-\frac{2 m \pi}{l} \operatorname{Im} a \tag{20}
\end{equation*}
$$

$\operatorname{Im} a \neq 0$.
Suppose that $\lambda_{m n} \neq 0$, that is,

$$
\begin{equation*}
\operatorname{Im} a \neq \frac{(-1)^{n}(n+1 / 2)}{2 m} \cdot \frac{l}{T}-\frac{2 m \pi}{l} \tag{21}
\end{equation*}
$$

If $n=2 k+1$ and $k=0,1,2, \ldots$, then

$$
\begin{equation*}
\lambda_{m, 2 k+1}=-\left(2 k+\frac{3}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}-\frac{2 m \pi}{l} \operatorname{Im} a \longrightarrow+\infty \tag{22}
\end{equation*}
$$

at $m, k \rightarrow \infty$; therefore this subsequence has no limit points.
If $n=2 k, k=0,1,2, \ldots$, then

$$
\begin{array}{r}
\lambda_{m, 2 k}=\left(2 k+\frac{1}{2}\right) \frac{\pi}{T}-\left(\frac{2 m \pi}{l}\right)^{2}-\frac{2 m \pi}{l} \operatorname{Im} a  \tag{23}\\
k=0,1,2, \ldots, \quad m=0, \pm 1, \pm 2, \ldots
\end{array}
$$

Transform this expression to a form convenient for us

$$
\begin{align*}
\lambda_{m, 2 k} & =\frac{2 \pi}{T}\left[k+\frac{1}{4}-\frac{2 m^{2} \pi T}{l^{2}}-\frac{m T}{l} \operatorname{Im} a\right]  \tag{24}\\
& =\frac{2 \pi}{T}\left[k+\frac{1}{4}-\left(m^{2} \frac{2 \pi T}{l^{2}}+m \frac{T \cdot \operatorname{Im} a}{l}\right)\right] .
\end{align*}
$$

For convenience we introduce the following notation:

$$
\begin{equation*}
\varphi(m)=m^{2} \frac{2 \pi T}{l^{2}}+m \frac{T \cdot \operatorname{Im} a}{l} \tag{25}
\end{equation*}
$$

where $[x]$ is the integer part and $(x)$ is the fractional part. Suppose that $k=[\varphi(m)]$, then

$$
\begin{align*}
\lambda_{m, 2 k} & =\frac{2 \pi}{T}\left[\frac{1}{4}+[\varphi(m)]-\varphi(m)\right] \\
& =\frac{2 \pi}{T}\left[\frac{1}{4}-(\varphi(m)-[\varphi(m)])\right]=\frac{2 \pi}{T}\left[\frac{1}{4}-(\varphi(m))\right] . \tag{26}
\end{align*}
$$

Now we use the Weyl's theorem [11], for this we assume that at least one of the values of

$$
\begin{equation*}
\frac{2 \pi T}{l^{2}}, \quad \frac{T \cdot \operatorname{Im} a}{l} \tag{27}
\end{equation*}
$$

is irrational. Then by Weyl's theorem the fractional part $\varphi(m)$, that is, $(\varphi(m))$, fills the interval [ 0,1$]$ uniformly dense when $m=0,1,2, \ldots$. Then the subsequence $\left\{\lambda_{m, 2 k}\right\}, m=$ $0,1,2, \ldots, k=[\varphi(m)]$, is dense everywhere in the interval $[-3 \pi / 2 T, \pi / 2 T]$.

Assuming $k=[\varphi(m)]+1$ and $k=[\varphi(m)]+2, \ldots$, then $k=[\varphi(m)]-1, k=[\varphi(m)]-2, \ldots$, and so on; we obtain that the sequence $\left\{\lambda_{m n}\right\}$ is uniformly dense everywhere; that is, the continuous spectrum of the operator $\bar{L}$ fills the entire real axis from $-\infty$ to $+\infty$. Let now both values of

$$
\begin{equation*}
\frac{2 \pi T}{l^{2}}, \quad \frac{T \cdot \operatorname{Im} a}{l} \tag{28}
\end{equation*}
$$

be rational; then $\varphi(m)$ is always rational. To be specific let

$$
\begin{equation*}
\frac{2 \pi T}{l^{2}}=\frac{p}{q}, \quad \frac{T \cdot \operatorname{Im} a}{l}=\frac{r}{k} \tag{29}
\end{equation*}
$$

Then

$$
\begin{equation*}
\varphi(m)=m^{2} \cdot \frac{p}{q}+m \cdot \frac{r}{k}=\frac{m^{2} p+m r}{q \cdot k}=[\varphi(m)]+(\varphi(m)) . \tag{30}
\end{equation*}
$$

The fractional part $\varphi(m)$ takes only a finite number of values; they are the remainders of the division $m^{2} p+m r$ by $q \cdot k$; that is

$$
\begin{equation*}
0, \frac{1}{q \cdot k}, \frac{2}{q \cdot k}, \ldots, \frac{q \cdot k-1}{q \cdot k} . \tag{31}
\end{equation*}
$$

When $m$ is changing from $-\infty$ to $+\infty$, these values will repeat infinitely many times, at least one or all of them. For us it is important that they do not coincide with $1 / 4$.

Assuming $k=[\varphi(m)], m=0,1,2, \ldots$, we see that

$$
\begin{equation*}
\lambda_{m, 2 k}=\frac{2 \pi}{T}\left[\frac{1}{4}-(\varphi(m))\right], \quad m=0,1,2, \ldots . \tag{32}
\end{equation*}
$$

This infinite sequence is contained in a segment $[-3 \pi / 2 T, \pi / 2 T]$ and consists of a finite number of fractions of the form

$$
\begin{align*}
& \frac{2 \pi}{T} \cdot \frac{1}{4}, \frac{2 \pi}{T} \cdot\left(\frac{1}{4}-\frac{1}{q \cdot k}\right), \frac{2 \pi}{T} \cdot\left(\frac{1}{4}-\frac{2}{q \cdot k}\right), \ldots,  \tag{33}\\
& \frac{2 \pi}{T} \cdot\left(\frac{1}{4}-\frac{q k-1}{q \cdot k}\right)
\end{align*}
$$

so at least one of them, or all, or some of them are repeated infinitely many times. This suggests that some numbers in the segment $[-3 \pi / 2 T, \pi / 2 T]$ are the infinitely multiple eigenvalues.

Continuing this reasoning as in the irrational case, we see that the spectrum of the operator $\bar{L}$ consists of an infinite set of eigenvalues and among the eigenvalues there are the infinite set of the infinitely multiple eigenvalues. By our assumption

$$
\begin{align*}
& \operatorname{Im} a \neq \frac{(-1)^{n}(n+1 / 2)}{2 m} \cdot \frac{l}{T}-\frac{2 m \pi}{l}  \tag{34}\\
& n=0,1,2, \ldots, m=0, \pm 1, \pm 2, \ldots
\end{align*}
$$

therefore the inverse operator exists and is bounded but is not compact in view of the existence of eigenvalues of infinite multiplicity, as the spectrum of the compact operator has a finite multiplicity. If $\operatorname{Re} a=0$ and

$$
\begin{equation*}
\operatorname{Im} a=\frac{(-1)^{n}(n+1 / 2)}{2 m} \cdot \frac{l}{T}-\frac{2 m \pi}{l} \tag{35}
\end{equation*}
$$

for some values $n=0,1,2, \ldots, m=0, \pm 1, \pm 2, \ldots$, the inverse operator $\bar{L}^{1}$ does not exist and zero is an eigenvalue, perhaps, of the infinite multiplicity. In this case, if at least one of the values of

$$
\begin{equation*}
\frac{2 \pi T}{l^{2}}, \quad \frac{T \cdot \operatorname{Im} a}{l} \tag{36}
\end{equation*}
$$

is irrational, then the spectrum of the operator $\bar{L}$ fills the entire real axis. If both of these values are rational, then the spectrum of the operator $\bar{L}$ consists of an infinite number of eigenvalues, among which there are an infinite number of infinite multiple eigenvalues.

Finally, we note that the boundary value problem (2)-(4) is strongly solvable in the cases of (a) and (b) of Theorem 11 but in cases (c) and (d) is not.

## 4. Conclusions

By means of Fourier method, a criterion for strong solvability of the mixed Cauchy problem for the heat equation with a deviating argument was found. The nature of the spectrum of the periodic problem for the heat equation with a deviating
argument was studied in detail, and the norm of the inverse operator was estimated through the influence coefficient. The dependence between the coefficient of influence and the nature of the spectrum of the heat operator with a deviating argument was found.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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## Research Article

# A Dirac System with Transmission Condition and Eigenparameter in Boundary Condition 

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This paper deals with a Dirac system with transmission condition and eigenparameter in boundary condition. We give an operatortheoretic formulation of the problem then investigate the existence of the solution. Some spectral properties of the problem are studied.

## 1. Introduction

After Walter [1] had given an operator-theoretic formulation of eigenvalue problems with eigenvalue parameter in the boundary conditions, Fulton [2,3] has carried over the methods of Titchmarsh [4, chapter 1] to this problem. Then, a large amount of the mathematical literature was devoted to these subjects during the last twenty years. We will mention some of the papers published at least twenty years ago, but of course there are many other interesting and important papers published more recently, which are not referred to here. The existence of solution and some spectral properties of Sturm-Liouville problem with eigenparameter-dependent boundary conditions and also with transmission conditions at one or more inner points of considered finite interval has been studied by Mukhtarov and Tunç [5]; see also [6, 7]. A Dirac system when the eigenparameter appears in boundary conditions has been studied by Kerimov [8]. In [9], an inverse problem for the Dirac system with eigenvaluedependent boundary conditions and transmission condition is investigated.

The aim of the present paper is to study a Dirac system with transmission condition and eigenparameter in boundary condition. For this, we follow the method in [5]. We consider the Dirac system

$$
\begin{equation*}
\ell(u)=A u^{\prime}(x)-P(x) u(x)=\lambda u(x) \tag{1}
\end{equation*}
$$

where

$$
\begin{gather*}
A=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right), \\
P(x)=\left(\begin{array}{cc}
p_{1}(x) & 0 \\
0 & p_{2}(x)
\end{array}\right),  \tag{2}\\
u(x)=\binom{u_{1}(x)}{u_{2}(x)},
\end{gather*}
$$

$$
\begin{align*}
& u_{2}^{\prime}(x)-p_{1}(x) u_{1}(x)=\lambda u_{1}(x)  \tag{3}\\
& u_{1}^{\prime}(x)+p_{2}(x) u_{2}(x)=-\lambda u_{2}(x), \quad x \in[a, c) \cup(c, b]
\end{align*}
$$

with boundary conditions

$$
\begin{gather*}
\sin \alpha u_{1}(a)-\cos \alpha u_{2}(a)=0  \tag{4}\\
b_{1} u_{1}(b)-a_{1} u_{2}(b)+\lambda\left(\sin \beta u_{1}(b)-\cos \beta u_{2}(b)\right)=0 \tag{5}
\end{gather*}
$$

and transmission conditions at the inner point $x=c$

$$
\begin{gather*}
u_{1}(c-0)=\gamma u_{1}(c+0) \\
u_{2}(c-0)=\gamma^{-1} u_{2}(c+0) \tag{6}
\end{gather*}
$$

Here and later on, $\lambda$ is a complex eigenvalue parameter; the functions $p_{i}(x)(i=1,2)$ are continuous on $[a, c) \cup(c, b]$ which have finite limits $p_{i}( \pm c)=\lim _{x \rightarrow \pm c} p_{i}(x)(i=1,2) . a_{1}, b_{1}, \gamma$ are real numbers and $\alpha, \beta \in[0, \pi)$.

## 2. Operator Formulation of the Problem

For convenience, we will assume that $\left|a_{1}\right|+\left|b_{1}\right| \neq 0, \gamma \neq 0$. To formulate a theoretic approach to problem (1)-(6), we define the Hilbert space $\mathbb{H}=L_{2}[a, c) \cup L_{2}(c, b] \oplus \mathbb{C}_{\sigma}$ with an inner product

$$
\begin{equation*}
\langle U, V\rangle_{\mathbb{H}}=\int_{a}^{c} u^{T}(x) \bar{v}(x) d x+\int_{c}^{b} u^{T}(x) \bar{v}(x) d x+\frac{1}{\sigma} \widetilde{u} \overline{\tilde{v}}, \tag{7}
\end{equation*}
$$

where $T$ stands for the transpose and

$$
\begin{array}{cc}
U=\binom{u(x)}{\widetilde{u}}, \quad V=\binom{v(x)}{\widetilde{v}} \in \mathbb{H}, \\
u(x)=\binom{u_{1}(x)}{u_{2}(x)}, \quad v(x)=\binom{v_{1}(x)}{v_{2}(x)} \in H, \tag{8}
\end{array}
$$

$u_{i}(x), v_{i}(x) \in L_{2}[a, c) \cup L_{2}(c, b],(i=1,2), \tilde{u}, \widetilde{v} \in \mathbb{C}$. The constant $\sigma$ is defined by

$$
\sigma:=\operatorname{det}\left(\begin{array}{cc}
b_{1} & a_{1}  \tag{9}\\
\sin \beta & \cos \beta
\end{array}\right)>0
$$

Let $\operatorname{dom}(\mathbf{A}) \subseteq \mathbb{H}$ be set of all $U=\binom{u(x)}{\hat{u}} \in \mathbb{H}$, such that $u_{1}(x), u_{2}(x)$ are absolutely continuous on $[a, c) \cup(c, b], \widehat{u}=$ $\sin \beta u_{1}(b)-\cos \beta u_{2}(b)$ and $\ell(u) \in \mathbb{H}, \sin \alpha u_{1}(a)-\cos \alpha u_{2}(a)=$ $0, u_{1}( \pm c), u_{2}( \pm c)$ have finite limits, $\widetilde{u}=b_{1} u_{1}(b)-a_{1} u_{2}(b)$. Now define the operator $\mathbf{A}: \operatorname{dom}(\mathbf{A}) \rightarrow \mathbb{H}$ by

$$
\begin{equation*}
\mathbf{A}\binom{u(x)}{\widehat{u}}=\binom{\ell(u)}{-\widetilde{u}} . \tag{10}
\end{equation*}
$$

Hence, we can rewrite the problem (1)-(6) in the operator form as

$$
\begin{equation*}
\mathbf{A} U=\lambda U \tag{11}
\end{equation*}
$$

Obviously, the operator A and the Dirac system (1)-(6) have the same eigenvalues. Also the eigenvectors of (1)-(6) coincide with the first two components of the corresponding eigenelement of the operator $\mathbf{A}$.

Lemma 1. The $\operatorname{dom}(\mathbf{A})$ is dense in $\mathbb{H}$.
Proof. It is easily seen that there is no nonzero vector $F=$ $(f(x), \widehat{f}) \in \mathbb{H}$ such that for every $U=(u(x), \widehat{u}) \in \operatorname{dom}(\mathbf{A})$, $\langle F, U\rangle_{\oplus}=0$. This implies $\operatorname{dom}(\mathbf{A})^{\perp}=\{\Theta\}$, where $\Theta=$ $(0,0,0)$. Therefore, $\operatorname{dom}(\mathbf{A})$ is dense in $\mathbb{H}$.

Theorem 2. The operator $\mathbf{A}$ is symmetric.
Proof. For each $U, V \in \operatorname{dom}(\mathbf{A})$ from the inner product (7) and the integration by parts, we have

$$
\langle\mathbf{A} U, V\rangle_{\mathbb{H}}=\int_{a}^{c}\left(u_{2}^{\prime}-p_{1} u_{1}\right) \bar{v}_{1} d x-\int_{a}^{c}\left(u_{1}^{\prime}+p_{2} u_{2}\right) \bar{v}_{2} d x
$$

$$
\begin{align*}
& +\int_{c}^{b}\left(u_{2}^{\prime}-p_{1} u_{1}\right) \bar{v}_{1} d x \\
& -\int_{c}^{b}\left(u_{1}^{\prime}+p_{2} u_{2}\right) \bar{v}_{2} d x-\frac{1}{\sigma} \widetilde{u} \overline{\hat{v}} \\
= & {\left[u_{2} \bar{v}_{1}-u_{1} \bar{v}_{2}\right]_{a}^{c-0}+\left[u_{2} \bar{v}_{1}-u_{1} \bar{v}_{2}\right]_{c+0}^{b} } \\
& -\int_{a}^{c} u_{2} \bar{v}_{1}^{\prime} d x-\int_{a}^{c} p_{1} u_{1} \bar{v}_{1} d x+\int_{a}^{c} u_{1} \bar{v}_{2}^{\prime} d x \\
& -\int_{a}^{c} p_{2} u_{2} \bar{v}_{2} d x-\int_{c}^{b} u_{2} \bar{v}_{1}^{\prime} d x-\int_{c}^{b} p_{1} u_{1} \bar{v}_{1} d x \\
& +\int_{c}^{b} u_{1} \bar{v}_{2}^{\prime} d x-\int_{c}^{b} p_{2} u_{2} \bar{v}_{2} d x-\frac{1}{\sigma} \widetilde{u}^{2} \\
= & {\left[u_{2}(c-0) \bar{v}_{1}(c-0)-u_{1}(c-0) \bar{v}_{2}(c-0)\right] } \\
& -\left[u_{2}(a) \bar{v}_{1}(a)-u_{1}(a) \bar{v}_{2}(a)\right] \\
& +\left[u_{2}(b) \bar{v}_{1}(b)-u_{1}(b) \bar{v}_{2}(b)\right] \\
& -\left[u_{2}(c+0) \bar{v}_{1}(c+0)-u_{1}(c+0) \bar{v}_{2}(c+0)\right] \\
& -\int_{a}^{c} u_{2}\left(\bar{v}_{1}^{\prime}+p_{2} \bar{v}_{2}\right) d x+\int_{a}^{c} u_{1}\left(\bar{v}_{2}^{\prime}-p_{1} \bar{v}_{1}\right) d x \\
& -\int_{c}^{b} u_{2}\left(\bar{v}_{1}^{\prime}+p_{2} \bar{v}_{2}\right) d x+\int_{c}^{b} u_{1}\left(\bar{v}_{2}^{\prime}-p_{1} \bar{v}_{1}\right) d x \\
& -\frac{1}{\sigma}\left(b_{1} u_{1}(b)-a_{1} u_{2}(b)\right) \\
& \times\left(\sin \beta \bar{v}_{1}(b)-\cos \beta \bar{v}_{2}(b)\right) \tag{12}
\end{align*}
$$

Since $U$ and $V$ satisfy the same boundary condition (4) at $x=$ a,

$$
\begin{equation*}
u_{2}(a) \bar{v}_{1}(a)=u_{1}(a) \bar{v}_{2}(a) . \tag{13}
\end{equation*}
$$

From transmission condition (6), it follows that

$$
\begin{align*}
& u_{2}(c-0) \bar{v}_{1}(c-0)=u_{2}(c+0) \bar{v}_{1}(c+0)  \tag{14}\\
& u_{1}(c-0) \bar{v}_{2}(c-0)=u_{1}(c+0) \bar{v}_{2}(c+0)
\end{align*}
$$

Furthermore,

$$
\begin{align*}
& {\left[u_{2}(b) \bar{v}_{1}(b)-u_{1}(b) \bar{v}_{2}(b)\right]-\frac{1}{\sigma}\left(b_{1} u_{1}(b)-a_{1} u_{2}(b)\right)} \\
& \quad \times\left(\sin \beta \bar{v}_{1}(b)-\cos \beta \bar{v}_{2}(b)\right) \\
& \quad=-\frac{1}{\sigma}\left(\sin \beta u_{1}(b)-\cos \beta u_{2}(b)\right)\left(b_{1} \bar{v}_{1}(b)-a_{1} \bar{v}_{2}(b)\right) \\
& \quad=-\frac{1}{\sigma} \widehat{u} \overline{\tilde{v}} . \tag{15}
\end{align*}
$$

Now substituting (13), (14), and (15) in (12), we obtain

$$
\begin{equation*}
\langle\mathbf{A} U, V\rangle_{\mathbb{H}}=\langle U, \mathbf{A} V\rangle_{\mathbb{H}} . \tag{16}
\end{equation*}
$$

Since the operator A is symmetric, the following orthogonality relation is valid.

Corollary 3. All the eigenvalues of the system (1)-(6) are real and to every eigenvalue $\lambda_{n}$, there corresponds a vector-valued eigenfunction $u_{n}^{T}\left(x, \lambda_{n}\right)=\left(u_{1 n}\left(x, \lambda_{n}\right), u_{2 n}\left(x, \lambda_{n}\right)\right)$. Moreover, vector-valued eigenfunctions belonging to different eigenvalues are orthogonal in the sense of

$$
\begin{equation*}
\left\langle u_{n}, u_{m}\right\rangle_{\mathbb{H}}=\int_{a}^{c} u_{n}^{T} \bar{u}_{m} d x+\int_{c}^{b} u_{n}^{T} \bar{u}_{m} d x-\frac{1}{\sigma} \widetilde{u}_{n} \overline{\hat{u}}_{m}=0 . \tag{17}
\end{equation*}
$$

Remark 4. The vector-valued eigenfunctions stated in Corollary 3 are not orthogonal in the usual sense in the Hilbert space $L_{2}[a, b]$.

## 3. Existence of Solutions

In this section, we study the existence of the solution of the Dirac system (1) with boundary conditions (4) and transmission condition (6).

Theorem 5. The Dirac system (1) has a solution $\Phi(x, \lambda)$ on $[a, b]$ satisfying boundary condition (4) and transmission condition (6). For each $x, \Phi(x, \lambda)$ is a vector-valued entire function of $\lambda$.

Proof. From the classical theory of differential equations (see [10]), since the Dirac system

$$
\begin{equation*}
A u^{\prime}(x)-P(x) u(x)=\lambda u(x), \quad x \in[a, c) \tag{18}
\end{equation*}
$$

with the initial conditions

$$
\begin{equation*}
u_{1}(a)=\cos \alpha, \quad u_{2}(a)=\sin \alpha \tag{19}
\end{equation*}
$$

is continuous on the interval $[a, c)$, this system has a unique solution $\Phi_{1}(x, \lambda)=\left(\Phi_{11}(x, \lambda), \Phi_{21}(x, \lambda)\right)^{T}$ which is an entire function of $\lambda$ on $[a, c)$.

Now consider the Dirac system of differential equations

$$
\begin{gather*}
u_{2}^{\prime}(x)-p_{1}(x) u_{1}(x)=\lambda u_{1}(x)  \tag{20}\\
u_{1}^{\prime}(x)+p_{2}(x) u_{2}(x)=-\lambda u_{2}(x), \quad x \in(c, b]
\end{gather*}
$$

and nonstandard initial conditions contain eigenparameter

$$
\begin{gather*}
u_{1}(c+0)=\gamma^{-1} \Phi_{11}(c-0, \lambda) \\
u_{2}(c+0)=\gamma \Phi_{21}(c-0, \lambda) . \tag{21}
\end{gather*}
$$

Let us denote solutions of (20) by $u_{0}(x, \lambda)=\left(u_{10}(x, \lambda)\right.$, $\left.u_{20}(x, \lambda)\right)^{T}$ in the case $p_{1}(x)=p_{2}(x) \equiv 0$. It is clear that the vector-valued function $u_{0}(x, \lambda)$ is written as

$$
\begin{align*}
& u_{10}(x, \lambda)=c_{1} \cos \lambda x+c_{2} \sin \lambda x \\
& u_{20}(x, \lambda)=-c_{1} \sin \lambda x+c_{2} \cos \lambda x \tag{22}
\end{align*}
$$

From the initial conditions (21), we obtain constants $c_{1}$ and $c_{2}$. Then, inserting these values into (22) and using some basic trigonometric identities, we arrive at

$$
\begin{equation*}
u_{0}(x, \lambda)=\binom{u_{10}(x, \lambda)}{u_{20}(x, \lambda)}=\binom{\gamma^{-1} \Phi_{11}(c-0, \lambda) \cos \lambda(x-(c+0))+\gamma \Phi_{21}(c-0, \lambda) \sin \lambda(x-(c+0))}{\gamma_{1}^{-1} \Phi_{11}(c-0, \lambda) \sin \lambda(x-(c+0))+\gamma \Phi_{21}(c-0, \lambda) \cos \lambda(x-(c+0))} \tag{23}
\end{equation*}
$$

By applying the method of variation of the constants as in [11, page 243], we find the following system of integral equations:

$$
\begin{equation*}
u(x, \lambda)=\binom{u_{1}(x, \lambda)}{u_{2}(x, \lambda)}=\binom{u_{10}(x, \lambda)+\int_{c}^{x}\left\{p_{1}(s) u_{1}(x, \lambda) \sin \lambda(s-x)-p_{2}(s) u_{2}(x, \lambda) \cos \lambda(s-x)\right\} d s}{u_{20}(x, \lambda)+\int_{c}^{x}\left\{p_{1}(s) u_{1}(x, \lambda) \cos \lambda(s-x)+p_{2}(s) u_{2}(x, \lambda) \sin \lambda(s-x)\right\} d s} \tag{24}
\end{equation*}
$$

In what follows, we use the method of successive approximations, which is helpful in constructing a solution of
the integral equation system (24). This method requires a sequence of functions $\left\{u_{n}(x, \lambda)\right\}$ for $n=1,2, \ldots$ defined as

$$
\begin{equation*}
u_{n}(x, \lambda)=\binom{u_{1 n}(x, \lambda)}{u_{2 n}(x, \lambda)}=\binom{u_{10}(x, \lambda)+\int_{c}^{x}\left\{p_{1}(s) u_{1 n-1} \sin \lambda(s-x)-p_{2}(s) u_{2 n-1} \cos \lambda(s-x)\right\} d s}{u_{20}(x, \lambda)+\int_{c}^{x}\left\{p_{1}(s) u_{1 n-1} \cos \lambda(s-x)+p_{2}(s) u_{2 n-1} \sin \lambda(s-x)\right\} d s} \tag{25}
\end{equation*}
$$

where $u_{10}(x, \lambda)$ and $u_{20}(x, \lambda)$ are defined in (23). It is obvious that each of $u_{n}(x, \lambda)$ is an entire function of $\lambda$ for every $x \in$ (c, b].

Set

$$
\begin{equation*}
z_{n}(x, \lambda)=u_{n}(x, \lambda)-u_{n-1}(x, \lambda) \tag{26}
\end{equation*}
$$

where $z_{n}^{T}(x, \lambda)=\left(z_{1 n}(x, \lambda), z_{2 n}(x, \lambda)\right)$, and let $M_{1}=$ $\max _{x \in(c, b]}\left|p_{1}(x)\right|, M_{2}=\max _{x \in(c, b]}\left|p_{2}(x)\right|, M=\max \left(M_{1}, M_{2}\right)$, $N_{1}(\lambda)=\max _{x \in(c, b]}\left|u_{10}(x, \lambda)\right|, N_{2}(\lambda)=\max _{x \in(c, b]}\left|u_{20}(x, \lambda)\right|$. Then,

$$
\begin{align*}
&\left\|z_{1}(x, \lambda)\right\| \leq \int_{c}^{x} \mid p_{1}(s) u_{10} \sin \lambda(s-x) \\
& \quad-p_{2}(s) u_{20} \cos \lambda(s-x) \mid d s \\
&+\int_{c}^{x} \mid p_{1}(s) u_{10} \cos \lambda(s-x)  \tag{27}\\
& \quad+p_{2}(s) u_{20} \sin \lambda(s-x) \mid d s \\
& \leq 2 M\left(N_{1}(\lambda)+N_{2}(\lambda)\right)(x-c)
\end{align*}
$$

where the norm $\|\cdot\|$ can be any convenient norm in $\mathbb{H}$, but for the sake of presentation, we used 1 - norm. Furthermore, let $N_{1}=\max _{|\lambda| \leq R} N_{1}(\lambda), N_{2}=\max _{|\lambda| \leq R} N_{2}(\lambda)$, and $N_{R}=$ $\max \left(N_{1}, N_{2}\right)$ in closed contour $\{\lambda \in \mathbb{C}:|\lambda| \leq R\}$; then

$$
\begin{equation*}
\left\|z_{1}(x, \lambda)\right\| \leq 2 M N_{R}(x-c) \tag{28}
\end{equation*}
$$

Similarly,

$$
\begin{align*}
\left\|z_{2}(x, \lambda)\right\| \leq & \int_{c}^{x} \mid p_{1}(s)\left(u_{11}-u_{10}\right) \sin \lambda(s-x) \\
& \quad-p_{2}(s)\left(u_{21}-u_{20}\right) \cos \lambda(s-x) \mid d s \\
& +\int_{c}^{x} \mid p_{1}(s)\left(u_{11}-u_{10}\right) \cos \lambda(s-x) \\
& \quad+p_{2}(s)\left(u_{21}-u_{20}\right) \sin \lambda(s-x) \mid d s \\
\leq & 2^{2} M^{2} N_{R} \frac{(x-c)^{2}}{2}, \tag{29}
\end{align*}
$$

and so generally,

$$
\begin{equation*}
\left\|z_{n}(x, \lambda)\right\| \leq 2^{n} M^{n} N_{R} \frac{(x-c)^{n}}{n!} \tag{30}
\end{equation*}
$$

Now, consider the infinite series

$$
\begin{equation*}
u_{0}(x, \lambda)+\sum_{k=1}^{\infty} z_{k}(x, \lambda) \tag{31}
\end{equation*}
$$

The $n$th partial sum of this series is $u_{n}(x, \lambda)$; that is,

$$
\begin{equation*}
u_{n}(x, \lambda)=u_{0}(x, \lambda)+\sum_{k=1}^{n} z_{k}(x, \lambda) \tag{32}
\end{equation*}
$$

Therefore, the sequence $\left\{u_{n}(x, \lambda)\right\}$ converges if and only if series (31) does so. In view of (30), it follows that series (31) is uniformly convergent with respect to $x$ on $(c, b]$ and $\lambda$ in the closed contour $\{\lambda \in \mathbb{C}:|\lambda| \leq R\}$. Let the sum of series (31) be $\Phi_{2}(x, \lambda)=\left(\Phi_{12}(x, \lambda), \Phi_{22}(x, \lambda)\right)^{T}$; that is,

$$
\begin{equation*}
\Phi_{2}(x, \lambda)=u_{0}(x, \lambda)+\sum_{k=1}^{\infty} z_{k}(x, \lambda) \tag{33}
\end{equation*}
$$

and so, (32) gives

$$
\begin{equation*}
\lim _{n \rightarrow \infty} u_{n}(x, \lambda)=\Phi_{2}(x, \lambda) \tag{34}
\end{equation*}
$$

Finally, we will show next that the limit function $\Phi_{2}(x, \lambda)$ satisfies (20). For this, we need to find $\Phi_{2}^{\prime}(x, \lambda)$. From (33),

$$
\begin{align*}
\Phi_{2}^{\prime}(x, \lambda) & =\binom{\Phi_{12}^{\prime}(x, \lambda)}{\Phi_{22}^{\prime}(x, \lambda)} \\
& =\binom{u_{11}^{\prime}(x, \lambda)}{u_{21}^{\prime}(x, \lambda)}+\sum_{k=2}^{\infty}\binom{z_{1 k}^{\prime}(x, \lambda)}{z_{2 k}^{\prime}(x, \lambda)} \tag{35}
\end{align*}
$$

For the first term on the right-hand side of (35), if we take $n=1$ in (25), then

$$
\begin{align*}
\binom{u_{11}}{u_{21}}= & \binom{u_{10}}{u_{20}} \\
& +\int_{c}^{x}\left(\begin{array}{cc}
p_{1}(s) \sin \lambda(s-x) & -p_{2}(s) \cos \lambda(s-x) \\
p_{1}(s) \cos \lambda(s-x) & p_{2}(s) \sin \lambda(s-x)
\end{array}\right) \\
& \times\binom{ u_{10}}{u_{20}} d s, \\
\binom{u_{11}^{\prime}}{u_{21}^{\prime}}= & \binom{u_{10}^{\prime}}{u_{20}^{\prime}} \\
& +\int_{c}^{x}\left(\begin{array}{cc}
-\lambda p_{1}(s) \cos \lambda(s-x) & -\lambda p_{2}(s) \sin \lambda(s-x) \\
\lambda p_{1}(s) \sin \lambda(s-x) & -\lambda p_{2}(s) \cos \lambda(s-x)
\end{array}\right) \\
& \times\binom{ u_{10}}{u_{20}} d s \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right)\binom{u_{10}}{u_{20}} ; \tag{36}
\end{align*}
$$

now from (25) and the fact that $\left(u_{10}, u_{20}\right)^{T}$ is a solution of the homogeneous system, we have

$$
\binom{u_{11}^{\prime}}{u_{21}^{\prime}}=\left(\begin{array}{cc}
0 & -\lambda  \tag{37}\\
\lambda & 0
\end{array}\right)\binom{u_{11}}{u_{21}}+\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right)\binom{u_{10}}{u_{20}} .
$$

For the second term on the right-hand side of (35), it follows from (25) and (26) that

$$
\begin{gather*}
\binom{z_{1 k}}{z_{2 k}}=\int_{c}^{x}\left(\begin{array}{cc}
p_{1}(s) \sin \lambda(s-x) & -p_{2}(s) \cos \lambda(s-x) \\
p_{1}(s) \cos \lambda(s-x) & p_{2}(s) \sin \lambda(s-x)
\end{array}\right) \\
\quad \times\binom{ z_{1 k-1}}{z_{2 k-1}} d s \tag{38}
\end{gather*}
$$

and its derivative is

$$
\begin{align*}
\binom{z_{1 k}^{\prime}}{z_{2 k}^{\prime}}= & \int_{c}^{x}\left(\begin{array}{cc}
-\lambda p_{1}(s) \cos \lambda(s-x) & -\lambda p_{2}(s) \sin \lambda(s-x) \\
\lambda p_{1}(s) \sin \lambda(s-x) & -\lambda p_{2}(s) \cos \lambda(s-x)
\end{array}\right) \\
& \times\binom{ z_{1 k-1}}{z_{2 k-1}} d s \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right)\binom{z_{1 k-1}}{z_{2 k-1}} \tag{39}
\end{align*}
$$

In this equation

$$
\begin{align*}
& \int_{c}^{x}\left(\begin{array}{cc}
-\lambda p_{1}(s) \cos \lambda(s-x) & -\lambda p_{2}(s) \sin \lambda(s-x) \\
\lambda p_{1}(s) \sin \lambda(s-x) & -\lambda p_{2}(s) \cos \lambda(s-x)
\end{array}\right)  \tag{40}\\
& \quad \times\binom{ z_{1 k-1}}{z_{2 k-1}} d s=\left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right)\binom{z_{1 k}}{z_{2 k}} .
\end{align*}
$$

By using (39) and (40), the second term on the right-hand side of (35) becomes

$$
\begin{align*}
\sum_{n=2}^{\infty}\binom{z_{1 k}^{\prime}(x, \lambda)}{z_{2 k}^{\prime}(x, \lambda)}= & \left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right) \sum_{k=2}^{\infty}\binom{z_{1 k}}{z_{2 k}} \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right) \sum_{k=2}^{\infty}\binom{z_{1 k-1}}{z_{2 k-1}}  \tag{41}\\
= & \left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right)\left[\sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}}-\binom{z_{11}}{z_{21}}\right] \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right) \sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}}
\end{align*}
$$

Substituting (37) and (41) into (35) gives

$$
\begin{aligned}
\binom{\Phi_{12}^{\prime}(x, \lambda)}{\Phi_{22}^{\prime}(x, \lambda)}= & \left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right)\left[\binom{u_{11}}{u_{21}}-\binom{z_{11}}{z_{21}}\right] \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right)\binom{u_{10}}{u_{20}} \\
& +\left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right) \sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}} \\
& +\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) & 0
\end{array}\right) \sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}} \\
= & \left(\begin{array}{cc}
0 & -\lambda \\
\lambda & 0
\end{array}\right)\binom{u_{10}}{u_{20}}+\left(\begin{array}{cc}
0 & -p_{2}(x) \\
p_{1}(x) \\
0
\end{array}\right)\binom{u_{10}}{u_{20}} \\
& +\left(\begin{array}{cc}
0 & -\lambda-p_{2}(x) \\
\lambda+p_{1}(x) & 0
\end{array}\right) \sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}} \\
= & \left(\begin{array}{cc}
0 & -\lambda-p_{2}(x) \\
\lambda+p_{1}(x) & 0
\end{array}\right)
\end{aligned}
$$

$$
\begin{align*}
& \times\left[\binom{u_{10}}{u_{20}}+\sum_{k=1}^{\infty}\binom{z_{1 k}}{z_{2 k}}\right] \\
= & \left(\begin{array}{cc}
0 & -\lambda-p_{2}(x) \\
\lambda+p_{1}(x) & 0
\end{array}\right)\binom{\Phi_{12}(x, \lambda)}{\Phi_{22}(x, \lambda)} \tag{42}
\end{align*}
$$

so that $\Phi_{2}(x, \lambda)$ satisfies (20) on $(c, b]$. It also clearly satisfies the boundary conditions (21). As a result, the vector-valued function $\Phi(x, \lambda)$ defined by

$$
\Phi(x, \lambda)= \begin{cases}\Phi_{1}^{T}(x, \lambda)=\left(\Phi_{11}, \Phi_{21}\right), & x \in[a, c)  \tag{43}\\ \Phi_{2}^{T}(x, \lambda)=\left(\Phi_{12}, \Phi_{22}\right), & x \in(c, b]\end{cases}
$$

satisfies the Dirac system (1), (4), and (6).
Theorem 6. For any $\lambda \in \mathbb{C}$, the Dirac system

$$
\begin{align*}
& u_{2}^{\prime}(x)-p_{1}(x) u_{1}(x)=\lambda u_{1}(x),  \tag{44}\\
& u_{1}^{\prime}(x)+p_{2}(x) u_{2}(x)=-\lambda u_{2}(x)
\end{align*}
$$

has a solution

$$
\Psi(x, \lambda)= \begin{cases}\Psi_{1}^{T}(x, \lambda)=\left(\Psi_{11}, \Psi_{21}\right), & x \in[a, c)  \tag{45}\\ \Psi_{2}^{T}(x, \lambda)=\left(\Psi_{12}, \Psi_{22}\right), & x \in(c, b]\end{cases}
$$

on $[a, c) \cup(c, b]$ satisfying the boundary condition (5) and transmission condition (6). For each $x \in[a, c) \cup(c, b], \Psi(x, \lambda)$ is a vector-valued entire function of $\lambda$.

Proof. The proof of this theorem is similar to that of Theorem 5 and hence is omitted.

## 4. The Eigenvalues of the Problem

We know from [11, page 194] that the Wronskians $W\left(\Phi_{i}, \Psi_{i}\right)$, $(i=1,2)$ do not depend on $x \in[a, c) \cup(c, b]$. They depend only on $\lambda$, and let $W\left(\Phi_{i}(x, \lambda), \Psi_{i}(x, \lambda)\right)=: \omega_{i}(\lambda)(i=1,2)$. However, it follows from (6) that

$$
\begin{align*}
\omega_{1}(\lambda)= & W\left(\Phi_{1}, \Psi_{1}\right)=\left|\begin{array}{ll}
\Phi_{11}(x, \lambda) & \Phi_{21}(x, \lambda) \\
\Psi_{11}(x, \lambda) & \Psi_{21}(x, \lambda)
\end{array}\right| \\
= & \Phi_{11}(c-0, \lambda) \Psi_{21}(c-0, \lambda) \\
& -\Phi_{21}(c-0, \lambda) \Psi_{11}(c-0, \lambda) \\
= & \gamma^{-1} \Phi_{12}(c+0, \lambda) \gamma \Psi_{22}(c+0, \lambda) \\
& -\gamma \Phi_{22}(c+0, \lambda) \gamma^{-1} \Psi_{12}(c+0, \lambda) \\
= & \left|\begin{array}{ll}
\Phi_{12}(x, \lambda) & \Phi_{22}(x, \lambda) \\
\Psi_{12}(x, \lambda) & \Psi_{22}(x, \lambda)
\end{array}\right|=W\left(\Phi_{2}, \Psi_{2}\right)=\omega_{2}(\lambda) . \tag{46}
\end{align*}
$$

Hence, we get

$$
\begin{equation*}
\omega_{1}(\lambda)=\omega_{2}(\lambda):=\omega(\lambda) \tag{47}
\end{equation*}
$$

Here we defined a function $\omega(\lambda)$.
Let the solutions $\Phi(x, \lambda)$ and $\Psi(x, \lambda)$ of (1)-(6) be defined by the initial conditions for some $\alpha, \beta \in[0, \pi)$

$$
\begin{align*}
\Phi_{11}(a, \lambda)=\cos \alpha, & \Phi_{21}(a, \lambda)=\sin \alpha \\
\Psi_{12}(b, \lambda)=a_{1}+\lambda \cos \beta, & \Psi_{22}(b, \lambda)=b_{1}+\lambda \sin \beta \tag{48}
\end{align*}
$$

Therefore, any solution of (1)-(6) may be represented as

$$
\begin{align*}
& u(x, \lambda) \\
& \quad= \begin{cases}u_{1}^{T}(x, \lambda)=\left(c_{1} \Phi_{11}+c_{2} \Psi_{11}, c_{1} \Phi_{21}+c_{2} \Psi_{21}\right), & x \in[a, c) \\
u_{2}^{T}(x, \lambda)=\left(c_{3} \Phi_{12}+c_{4} \Psi_{12}, c_{3} \Phi_{22}+c_{4} \Psi_{22}\right), & x \in(c, b]\end{cases} \tag{49}
\end{align*}
$$

Applying conditions (4), (5), and (6) to solution (49) and considering the initial values (48), we obtain the following coefficients matrix of linear system equations of the variables $c_{1}, c_{2}, c_{3}, c_{4}$ :

$$
\left[\begin{array}{cccc}
0 & \omega_{1}(\lambda) & 0 & 0  \tag{50}\\
0 & 0 & \omega_{2}(\lambda) & 0 \\
\Phi_{11}(c-0, \lambda) & \Psi_{11}(c-0, \lambda) & -\gamma \Phi_{12}(c+0, \lambda) & -\gamma \Psi_{12}(c+0, \lambda) \\
\Phi_{21}(c-0, \lambda) & \Psi_{21}(c-0, \lambda) & -\gamma^{-1} \Phi_{22}(c+0, \lambda) & -\gamma^{-1} \Psi_{22}(c+0, \lambda)
\end{array}\right]
$$

and let us denote the determinant of this matrix by $W(\lambda)$; then for every $\lambda \in \mathbb{C}$,

$$
\begin{align*}
W(\lambda) & =-\omega_{1}(\lambda) \omega_{2}(\lambda)\left|\begin{array}{cc}
\Phi_{11}(c-0, \lambda) & \Psi_{11}(c-0, \lambda) \\
\Phi_{21}(c-0, \lambda) & \Psi_{21}(c-0, \lambda)
\end{array}\right|  \tag{51}\\
& =-\omega_{1}^{2}(\lambda) \omega_{2}(\lambda)=-\omega^{3}(\lambda) .
\end{align*}
$$

Theorem 7. The eigenvalues of the problem (1)-(6) are the zeros of the function $\omega(\lambda)$.

Proof. Let $\omega\left(\lambda_{n}\right)=0$ for any $\lambda=\lambda_{n}$. Then, it follows from (51) that the Wronskian of $\Phi_{2}\left(x, \lambda_{n}\right)$ and $\Psi_{2}\left(x, \lambda_{n}\right)$ is zero, so that $\Psi_{2}\left(x, \lambda_{n}\right)$ is a constant multiple of $\Phi_{2}\left(x, \lambda_{n}\right)$, say

$$
\begin{equation*}
\Psi_{2}\left(x, \lambda_{n}\right)=k \Phi_{2}\left(x, \lambda_{n}\right), \quad x \in(c, b] . \tag{52}
\end{equation*}
$$

It follows that $\Psi\left(x, \lambda_{n}\right)$ also fulfils the boundary condition (5) and, therefore, is a vector-valued eigenfunction of the problem (1)-(6) for eigenvalue $\lambda_{n}$.

Conversely, let $u_{n}\left(x, \lambda_{n}\right)$ be a vector-valued eigenfunction corresponding to eigenvalue $\lambda_{n}$, but $\omega\left(\lambda_{n}\right) \neq 0$. Then, from (51), at least one of the pair of the functions $\left(\Phi_{1}^{T}, \Phi_{2}^{T}\right)$ and ( $\Psi_{1}^{T}, \Psi_{2}^{T}$ ) would be linearly independent. Therefore, $u_{n}\left(x, \lambda_{n}\right)$ can be expressed as

$$
u_{n}\left(x, \lambda_{n}\right)= \begin{cases}C_{1} \Phi_{1}^{T}\left(x, \lambda_{n}\right)+C_{2} \Psi_{1}^{T}\left(x, \lambda_{n}\right), & x \in[a, c)  \tag{53}\\ D_{1} \Phi_{2}^{T}\left(x, \lambda_{n}\right)+D_{2} \Psi_{2}^{T}\left(x, \lambda_{n}\right), & x \in(c, b]\end{cases}
$$

where at least one of the constants $C_{1}, C_{2}, D_{1}, D_{2}$ is not zero. Since $u_{n}\left(x, \lambda_{n}\right)$ is a vector-valued eigenfunction corresponding to eigenvalue $\lambda_{n}$ by substitution in conditions (4)-(6), we obtain a system of linear, homogeneous equations and the determinant of this system is zero. This means that $W\left(\lambda_{n}\right)=$ 0 , and from (51), $\omega\left(\lambda_{n}\right)=0$ which yields a contradiction to the assumption that $\omega\left(\lambda_{n}\right) \neq 0$. This completes the proof.

Since $\omega(\lambda)$ is an entire function of $\lambda$ and the eigenvalues of the problem (1)-(6) consist of the zeros of $\omega(\lambda)$, we have the next theorem.

Theorem 8. The Dirac system (1)-(6) has at most denumerably many eigenvalues, and these eigenvalues have no finite limit point.

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## Research Article

# The Initial and Neumann Boundary Value Problem for a Class Parabolic Monge-Ampère Equation 

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#### Abstract

We consider the existence, uniqueness, and asymptotic behavior of a classical solution to the initial and Neumann boundary value problem for a class nonlinear parabolic equation of Monge-Ampère type. We show that such solution exists for all times and is unique. It converges eventually to a solution that satisfies a Neumann type problem for nonlinear elliptic equation of MongeAmpère type.


## 1. Introduction

Historically, the study of Monge-Ampère is motivated by the following two problems: Minkowski and Weyl problems. One is of prescribing curvature type, and the other is of embedding type. The development of Monge-Ampère theory in PDE is closely related to that of fully nonlinear equations. Generally speaking, there are two ways to tackle the problems. One is via continuity method involving some appropriate a priori estimates, and the other is weak solution theory. MongeAmpère equations have many applications. In recent years new applications have been found in affine geometry and optimal transportation problem.

Many scholars have studied this kind of equations (see, e.g., $[1-5]$ and the references given therein). Their main work is directed at the first or the third boundary value problem. But concerning Neumann boundary value problem, there is lack of research. In this paper, we consider the existence, uniqueness, and asymptotic behavior of a classical solution to the initial and Neumann boundary value problem for a class parabolic equation of Monge-Ampère type as follows:

$$
\begin{gathered}
\dot{u}=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \quad \text { in } \Omega \times(0, T], \\
u_{v}=\varphi(x, u) \quad \text { on } \partial \Omega \times[0, T], \\
\left.u\right|_{t=0}=u_{0} \quad \text { in } \Omega,
\end{gathered}
$$

where $\dot{u}=\partial u / \partial t$ and $\Omega$ is a bounded, uniformly convex domain in $R^{n}$ with the boundary $\partial \Omega \in C^{4+\alpha}$. $\nu$ denotes the unit inner normal on $\partial \Omega$ which has been extended on $\overline{Q_{T}}$ to become a properly smooth vector field independent of $t$. For some $T_{0}, T_{0} \in(0, T)$, when $t \in\left(0, T_{0}\right], g_{\sigma}(x, u)=g_{1}(x, u)$, and when $t \in\left(T_{0}, T\right], g_{\sigma}(x, u)=g_{2}(x, u)$. The function $g_{\sigma} \in$ $C^{2+\alpha, 2+\alpha}(\bar{\Omega} \times R), \sigma=1,2$. For each $x \in \Omega, \lim _{t \rightarrow T_{0}^{+}} g_{2}(x, u(x, t))=$ $g_{1}\left(x, u\left(x, T_{0}\right)\right)$. Here $\varphi \in C^{3+\alpha, 3+\alpha}(\bar{\Omega} \times R)$ and $u_{0} \in C^{4+\alpha}(\bar{\Omega})$. The initial value $u_{0}$ is a strictly convex function on $\bar{\Omega}$. In the sequel we assume for simplicity that $0 \in \Omega$.

To guarantee the existence of the classical solutions for (1) and convergence to a solution with prescribed curvature, we have to assume several structure conditions analogous to [6]. These are

$$
\begin{gather*}
\varphi_{z} \equiv \frac{\partial \varphi(x, z)}{\partial z} \geq c_{\varphi}>0  \tag{2}\\
g_{\sigma}>0, \quad\left(g_{\sigma}\right)_{z} \equiv \frac{\partial g_{\sigma}(x, z)}{\partial z} \geq 0, \quad \text { for } \sigma=1,2  \tag{3}\\
\operatorname{det}^{1 / n}\left(D_{x}^{2} u_{0}\right)-g\left(x, u_{0}\right) \geq 0 \tag{4}
\end{gather*}
$$

Moreover, we will always assume the following compatibility conditions to be fulfilled on $\partial \Omega \times\{t=0\}$ :

$$
\begin{gather*}
\left(u_{0}\right)_{v}=\varphi\left(x, u_{0}\right) \\
\left(\operatorname{det}^{1 / n}\left(D_{x}^{2} u_{0}\right)-g_{1}\left(x, u_{0}\right)\right)_{v}  \tag{5}\\
=\varphi_{z}\left(x, u_{0}\right)\left(\operatorname{det}^{1 / n}\left(D_{x}^{2} u_{0}\right)-g_{1}\left(x, u_{0}\right)\right)
\end{gather*}
$$

Elliptic equations of Monge-Ampère type have been explored in [7-10] by using the continuity method. Some of the techniques used there will be applied in our paper as well. For the parabolic case, Schnürer and Smoczyk [6] consider the flow of a strictly convex hypersurface driven by the Gauss curvature. For the Neumann boundary value problem and for the second boundary value problem, they show that such a flow exists for all times and converges eventually to a solution of the prescribed Gauss curvature equation. Zhou and Lian [11] proved the existence and uniqueness of classical solutions to the third initial and boundary value problem for equation of parabolic Monge-Ampère type of the form $-\partial u / \partial t+\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)=f(x, t)$. In this paper we will consider more general case than [11] under the structure and compatibility conditions analogous to [6] and extend some results in [7] from elliptic case to parabolic case.

The organization of this paper is as follows. In Section 2, we will review some notations, definitions, and results. In Section 3, we will obtain the uniqueness of the strictly convex classical solutions by the comparison principle. In Section 4, we shall prove uniform estimates for $|\dot{u}|$. This will be used in Section 5 to derive $C^{0}$-estimates. $C^{1}$-estimates then follow from [7]. In Section 6, we shall derive $C^{2}$-estimates and the $C^{2+\beta, 1+\beta / 2}$-estimates. In Section 7, we will give the proof of Theorem 1.

Our main result is as follows.
Theorem 1 (the main theorem). Assume that $\Omega$ is a bounded, uniformly convex domain in $R^{n}$ with the boundary $\partial \Omega \in$ $C^{4+\alpha} . v$ denotes the unit inner normal on $\partial \Omega$ which has been extended on $\overline{Q_{T}}$ to become a properly smooth vector field independent of $t$. Let $g_{\sigma} \in C^{2+\alpha, 2+\alpha}(\bar{\Omega} \times R), \sigma=1,2$, and $\varphi \in C^{3+\alpha, 3+\alpha}(\bar{\Omega} \times R)$ that satisfy (2)-(3). Let $u_{0} \in C^{4+\alpha}(\bar{\Omega})$ be a strictly convex function that satisfies (4). Moreover, the compatibility conditions (5) are fulfilled. Then there exists a unique strictly convex solution of (1) in $K^{4+\alpha}$ for some $\alpha \in$ $(0,1)$, where

$$
\begin{gather*}
K^{4+\alpha}:=\left\{v(x, t) \mid v(x, t) \in C^{2,1}\left(Q_{T}\right) \cap C^{1,0}\left(\overline{Q_{T}}\right)\right. \\
\text { and } v(\cdot, t) \text { is strictly convex for every time } \\
t \in[0, T]\} \cap C^{4+\alpha, 2+\alpha / 2}\left(\overline{Q_{T}}\right)  \tag{6}\\
Q_{T}=\Omega \times(0, T]
\end{gather*}
$$

As $t \rightarrow \infty$, the functions $\left.u\right|_{t}$ converge to a limit function $u^{\infty}$ such that $u^{\infty}$ is of class $C^{4}(\bar{\Omega})$ and satisfies the Neumann boundary value problem

$$
\begin{gather*}
\operatorname{det}^{1 / n}\left(D_{x}^{2} u^{\infty}\right)=g_{2}\left(x, u^{\infty}\right) \quad \text { in } \Omega \\
u_{v}^{\infty}(x)=\varphi\left(x, u^{\infty}\right) \quad \text { on } \partial \Omega, \tag{7}
\end{gather*}
$$

where $\nu$ is the inward ponting unit normal of $\partial \Omega$.

Proof. Uniqueness of the strictly convex classical solution is given by Theorem 5. From the estimates obtained in Sections $4-6$, we get the existence and the asymptotic behavior of the classical solution in Section 7.

## 2. Review of Some Notations, Definitions, and Results

We first review some notations and definitions as follows:
$R^{n}$ is the $n$-dimensional Euclidean space with $n \geq 2$;
$\Omega$ is a bounded, uniformly convex domain in $R^{n}$, and $\partial \Omega$ denotes the boundary of $\Omega$;
$Q_{T}=\Omega \times(0, T]$, and $\partial_{P} Q_{T}$ denotes the parabolic boundary of $Q_{T}, \partial_{P} Q_{T}=\overline{Q_{T}}-Q_{T}$;
$\dot{u}=\partial u / \partial t, \ddot{u}=\partial^{2} u / \partial t^{2} ;$
$u_{i}=D_{i} u=\partial u / \partial x_{i}, D u=\left(D_{1} u, \ldots, D_{n} u\right) ;$
$|D u|^{2}:=\sum_{i=1}^{n}\left|D_{i} u\right|^{2}, D_{i j}:=\partial^{2} / \partial x_{i} x_{j} ;$
$\left(u^{i j}\right)$ denotes the inverse of $\left(u_{i j}\right)$;
$\operatorname{tr}\left(D_{x}^{2} u\right)$ denotes the trace of the Hessian matrix $\left(u_{i j}\right)$; $\operatorname{det}\left(D_{x}^{2} u\right)$ denotes the determinant of the Hessian matrix $\left(u_{i j}\right)$;
$C^{l, k}\left(Q_{T}\right):=\left\{u(x, t) \mid D_{x}^{i} u\right.$ and $D_{t}^{j} u$ are all continuous in $\left.Q_{T}(0 \leq i \leq l, 0 \leq j \leq k)\right\}$;
$C^{2 l+\alpha, l+\alpha / 2}:=\left\{u ; D^{\beta} D_{t}^{r} u \in C^{\alpha, \alpha / 2}\left(\overline{Q_{T}}\right)\right.$, for all $\beta$ and $r$ that satisfy $|\beta|+2 r \leq 2 l\}$ with the norm

$$
\begin{equation*}
|u|_{2 l+\alpha, l+\alpha / 2 ; Q_{T}}=\sum_{|\beta|+2 r \leq 2 l}\left|D^{\beta} D_{t}^{r} u\right|_{\alpha, \alpha / 2 ; Q_{T}} . \tag{8}
\end{equation*}
$$

Indices $z$ and $p_{i}$ denote partial derivatives with respect to the argument used for the function $u$ and for its gradient, respectively. This paper adopts the Einstein summation convention and sums over repeated Latin indices from 1 to $n$. For example, $u_{i} v^{i}$ means $\sum_{i=1}^{n} u_{i} v^{i}$. We will say "a constant $C$ under control" or "a controllable constant $C$ " if the constant $C$ (independent of $T$ ) depends only on the known or estimated quantities, for example, the $C^{4}$ normal of $u_{0}$ and $n$-the dimension of $R^{n}$. We point out that the inequalities remain valid when $C$ is enlarged.

Now, we state existence results.
Lemma 2 (see [11]). Let $F\left(D_{x}^{2} u\right)=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right), F^{i j}\left(D_{x}^{2} u\right)=$ $\partial F\left(D_{x}^{2} u\right) / \partial u_{i j}$; then $F\left(D_{x}^{2} u\right)$ is a concave function, $\left(F^{i j}\left(D_{x}^{2} u\right)\right)$ is a positive matrix, and $\operatorname{tr}\left(F^{i j}\left(D_{x}^{2} u\right)\right)=\sum_{i=1}^{n} F^{i i}\left(D_{x}^{2} u\right) \geq 1$.

Lemma 3 (see [12]). If $f \in C^{2}([0,1])$, then there exists a constant $M$ which is independent of $f$, such that

$$
\begin{equation*}
\left\|f^{\prime}\right\| \leq M\|f\|\left(\|f\|+\left\|f^{\prime \prime}\right\|\right) \tag{9}
\end{equation*}
$$

where $\|f\|=\max \{|f(x)|: 0 \leq x \leq 1\}$.

## 3. Comparison Principle and Uniqueness

This section is concerned with the uniqueness of the strictly convex classical solution for (1). First of all, we will prove a comparison principle as follows.

Lemma 4. Assume that $u, v \in C^{2,1}\left(\overline{Q_{T}}\right)$ and $u(\cdot, t), v(\cdot, t)$ are all convex for every time $t \in(0, T]$. For some $T_{0}, T_{0} \in(0, T)$, when $t \in\left(0, T_{0}\right], g_{\sigma}(x, u)=g_{1}(x, u)$, and when $t \in\left(T_{0}, T\right]$, $g_{\sigma}(x, u)=g_{2}(x, u)$. Let $g_{\sigma} \in C^{2,2}(\bar{\Omega} \times R), \sigma=1,2$, and $\left(g_{\sigma}\right)_{z}=$ $\partial g_{\sigma}(x, z) / \partial z \geq 0$. Moreover, assume that
(1) $-\dot{u}+\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \geq-\dot{v}+\operatorname{det}^{1 / n}\left(D_{x}^{2} v\right)-$ $g_{\sigma}(x, v)$ in $\Omega \times(0, T]$,
(2) if $u>v$, then $u_{v}>v_{v}$ on $\partial \Omega \times[0, T]$,
(3) $u \leq v$ on $\Omega \times\{t=0\}$,
where $\nu$ is the inward pointing unit normal of $\partial \Omega$; then $u \leq v$ in $\overline{Q_{T}}$.

Proof. Consider

$$
\begin{align*}
-\dot{u}+ & \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u)-\left(-\dot{v}+\operatorname{det}^{1 / n}\left(D_{x}^{2} v\right)-g_{\sigma}(x, v)\right) \\
= & -(\dot{u}-\dot{v}) \\
& +\int_{0}^{1} \frac{\partial \operatorname{det}^{1 / n}\left[s D_{x}^{2} u+(1-s) D_{x}^{2} v\right]}{\partial\left(s u_{i j}+(1-s) v_{i j}\right)} d s(u-v)_{i j} \\
& -\int_{0}^{1} \frac{\partial g_{\sigma}(x, s u+(1-s) v)}{\partial(s u+(1-s) v)} d s(u-v) \\
= & -(\dot{u}-\dot{v})+a^{i j}(u-v)_{i j}-b(u-v) \tag{10}
\end{align*}
$$

where $a^{i j}=\int_{0}^{1}\left(\partial \operatorname{det}^{1 / n}\left[s D_{x}^{2} u+(1-s) D_{x}^{2} v\right] / \partial\left(s u_{i j}+(1-\right.\right.$ s) $\left.\left.v_{i j}\right)\right) d s, b=\int_{0}^{1}\left(\partial g_{\sigma}(x, s u+(1-s) v) / \partial(s u+(1-s) v)\right) d s$.

From the assumptions and Lemma 2, we obtain that ( $a^{i j}$ ) is a positive matrix and $b \geq 0$.

Combining (10) with condition (1), we infer that

$$
\begin{equation*}
-(\dot{u}-\dot{v})+a^{i j}(u-v)_{i j}-b(u-v) \geq 0 \quad \text { in } \Omega \times(0, T] ; \tag{11}
\end{equation*}
$$

an application of the weak parabolic maximum principle gives $\max _{\overline{\mathrm{Q}_{T}}}(u-v) \leq \max _{\partial_{P} Q_{T}}(u-v)^{+}$. In addition, from condition (2), $u-v$ cannot admit a positive maximum on $\partial \Omega \times[0, T]$. And from condition (3), $u \leq v$ on $\Omega \times\{t=0\}$. So we obtain that $u \leq v$ in $\overline{Q_{T}}$.

Theorem 5. Under the assumptions of Theorem 1, there exists a unique classical solution of (1).

Proof. Assume that $u, v \in C^{2,1}\left(\overline{Q_{T}}\right)$ are two classical solutions of (1). Then we have

$$
\begin{gather*}
\dot{u}=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \quad \text { in } \Omega \times(0, T] \\
u=u_{0}(x) \quad \text { on } \Omega \times\{t=0\} \tag{12}
\end{gather*}
$$

meanwhile,

$$
\begin{gather*}
\dot{v}=\operatorname{det}^{1 / n}\left(D_{x}^{2} v\right)-g_{\sigma}(x, v) \quad \text { in } \Omega \times(0, T],  \tag{13}\\
v=u_{0}(x) \quad \text { on } \Omega \times\{t=0\} .
\end{gather*}
$$

Thus,

$$
\begin{align*}
& -\dot{u}+\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \\
& =-\dot{v}+\operatorname{det}^{1 / n}\left(D_{x}^{2} v\right)-g_{\sigma}(x, v) \quad \text { in } \Omega \times(0, T]  \tag{14}\\
& u=v \quad \text { on } \Omega \times\{t=0\} .
\end{align*}
$$

It follows that conditions (1) and (3) in Lemma 4 are satisfied.
From $u_{v}=\varphi(x, u)$ on $\partial \Omega \times[0, T]$ and the structure condition (2), we obtain that condition (2) in Lemma 4 is satisfied.

Since $g_{\sigma} \in C^{2+\alpha, 2+\alpha}(\bar{\Omega} \times R)$ and the structure condition (3) is satisfied, we obtained from Lemma 4 that $u=v$ for all $(x, t) \in \overline{Q_{T}}$.

## 4. $\dot{U}$-Estimates

The proof of the $\dot{u}$-estimates can be carried out as in [6]. For a constant $\lambda$ we define the function $r=e^{\lambda t}(\dot{u})^{2}$; thus

$$
\begin{align*}
\dot{r}= & \lambda e^{\lambda t}(\dot{u})^{2}+2 e^{\lambda t} \dot{u} \ddot{u} \\
= & \lambda r+2 e^{\lambda t} \dot{u} \frac{d}{d t}\left(\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u)\right) \\
= & \lambda r+\frac{2}{n} e^{\lambda t} \dot{u} \dot{u}_{i j} u^{i j} \cdot \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-2 r\left(g_{\sigma}\right)_{z} \\
= & \lambda r+\frac{1}{n}\left(r_{i j}-2 e^{\lambda t} \dot{u}_{i} \dot{u}_{j}\right) u^{i j} \cdot \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-2 r\left(g_{\sigma}\right)_{z} \\
= & \frac{1}{n} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j} r_{i j}-\frac{2}{n} e^{\lambda t} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j} \dot{u}_{i} \dot{u}_{j} \\
& +\left(\lambda-2\left(g_{\sigma}\right)_{z}\right) r . \tag{15}
\end{align*}
$$

So (1) implies the following evolution equation for $r$ :

$$
\begin{align*}
\dot{r}= & \frac{1}{n} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j} r_{i j}-\frac{2}{n} e^{\lambda t} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j} \dot{u}_{i} \dot{u}_{j}  \tag{16}\\
& +\left(\lambda-2\left(g_{\sigma}\right)_{z}\right) r .
\end{align*}
$$

Theorem 6. As long as a strictly convex solution of (1) exists, one obtains the estimates

$$
\begin{equation*}
|\dot{u}|_{0, \bar{Q}_{T}} \leq \bar{M}, \tag{17}
\end{equation*}
$$

where $\bar{M}$ is a controllable constant.
Proof. If $(\dot{u})^{2}$ admits a positive local maximum in $x \in \partial \Omega$ for a positive time, then we differentiate the Neumann boundary condition and obtain from (2) that

$$
\begin{equation*}
\left((\dot{u})^{2}\right)_{v}=2 \dot{u}(\dot{u})_{v}=2 \dot{u}\left(\dot{u}_{v}\right)=2(\dot{u})^{2} \varphi_{z}>0 \tag{18}
\end{equation*}
$$

which contradicts the maximality of $(\dot{u})^{2}$ at $x$.

Now we choose $\lambda=0$ in (16) and get

$$
\begin{align*}
\frac{d(\dot{u})^{2}}{d t}= & \frac{1}{n} u^{i j} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)\left((\dot{u})^{2}\right)_{i j} \\
& -\frac{2}{n} u^{i j} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) \dot{u}_{i} \dot{u}_{i}-2\left(g_{\sigma}\right)_{z}(\dot{u})^{2}  \tag{19}\\
\leq & \frac{1}{n} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j}\left((\dot{u})^{2}\right)_{i j}-2\left(g_{\sigma}\right)_{z}(\dot{u})^{2} .
\end{align*}
$$

Since det $D_{x}^{2} u>0,\left(g_{\sigma}\right)_{z} \geq 0$, we obtain from the parabolic maximum principle that

$$
\begin{equation*}
\max _{\overline{Q_{T}}}(\dot{u})^{2} \leq \max _{\partial_{P} Q_{T}}\left((\dot{u})^{2}\right)^{+} . \tag{20}
\end{equation*}
$$

From the aforementioned a positive local maximum of $(\dot{u})^{2}$ cannot occur at a point of $\partial \Omega$ for a positive time, so

$$
\begin{equation*}
(\dot{u})^{2} \leq \max _{t=0}(\dot{u})^{2} \Longrightarrow|\dot{u}| \leq \max _{t=0}|\dot{u}| . \tag{21}
\end{equation*}
$$

From the fact that the solution is smooth up to the initial time $t=0$, we get

$$
\begin{equation*}
\dot{u}=\operatorname{det}^{1 / n}\left(D_{x}^{2} u_{0}\right)-g_{1}\left(x, u_{0}\right) \quad \text { on } \bar{\Omega} \times\{t=0\} \tag{22}
\end{equation*}
$$

By (21) and (22), there exists a controllable constant $\bar{M}$ such that $|\dot{u}|_{0, \bar{Q}_{T}} \leq \bar{M}$. Here we have used the fact that $u_{0} \in C^{4}(\bar{\Omega})$.

Lemma 7. If $0 \leq \dot{u}(x, 0) \not \equiv 0$ for $t=0$, then a solution of (1) satisfies $\dot{u}>0$ or equivalently $\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u)>0$ for $\sigma=1,2$ and $t>0$.

Proof. We use the methods known from [6]. Differentiating the equation

$$
\begin{equation*}
\dot{u}=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \tag{23}
\end{equation*}
$$

yields

$$
\begin{equation*}
\ddot{u}=\frac{1}{n} \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right) u^{i j} \dot{u}_{i j}-\left(g_{\sigma}\right)_{z} \dot{u} . \tag{24}
\end{equation*}
$$

From (24) and parabolic maximum principle, we see that

$$
\begin{equation*}
\inf _{Q_{T}}(\dot{u}) \geq \inf _{\partial_{P} Q_{T}}(\dot{u})_{-}, \tag{25}
\end{equation*}
$$

where $(\dot{u})_{-}=\min \{\dot{u}, 0\}$.
If $\dot{u}$ admits a negative local minimum in $x \in \partial \Omega$ for a positive time, then we differentiate the Neumann boundary condition and get from (2) that

$$
\begin{equation*}
(\dot{u})_{v}=\left(\dot{u}_{v}\right)=\varphi_{z} \dot{u}<0 \tag{26}
\end{equation*}
$$

which contradicts the minimum of $(\dot{u})$ at $x$. Since $0 \leq \dot{u}(x, 0)$, it follows that $\inf _{\partial_{P} Q_{T}}(\dot{u})_{-}=0$. That is,

$$
\begin{equation*}
\inf _{\mathrm{Q}_{T}}(\dot{u}) \geq \inf _{\partial_{P} Q_{T}}(\dot{u})_{-}=0 . \tag{27}
\end{equation*}
$$

So $\dot{u} \geq 0$ or equivalently $\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \geq 0$ for $\sigma=$ 1,2 and $t>0$.

From (24) and the strong parabolic maximum principle [13], we obtain that $\dot{u}$ has to vanish identically if it vanishes in $\left(x_{0}, t_{0}\right) \in \Omega \times(0, T]$, contradicting $\dot{u} \not \equiv 0$ for $t=0$. If $\dot{u}=0$ for $x_{0} \in \partial \Omega$, the Neumann boundary condition implies that

$$
\begin{equation*}
(\dot{u})_{v}=(\dot{u}) \varphi_{z}=0, \tag{28}
\end{equation*}
$$

but this is impossible in view of the Hopf lemma applied to (24).

Consequently, if $0 \leq \dot{u}(x, 0) \not \equiv 0$ for $t=0$, then a solution of (1) satisfies $\dot{u}>0$ or equivalently $\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{\sigma}(x, u)>0$ for $\sigma=1,2$ and $t>0$.

## 5. $C^{0}$ - and $C^{1}$-Estimates

In this section we derive the $C^{0}$ - and $C^{1}$-estimates of the solution to problem (1).

Theorem 8. Let $\Omega$ be a bounded, uniformly convex domain in $R^{n}$. Also, $u \in C^{2,1}\left(Q_{T}\right) \cap C^{1,0}\left(\overline{Q_{T}}\right)$ is a strictly convex solution of (1). Then there exists a controllable constant $M_{0}$, such that $|u|_{0, \overline{Q_{T}}} \leq M_{0}$.

Proof. Since (4) is satisfied, we obtained from Lemma 7 that $\dot{u} \geq 0$ in $\overline{Q_{T}}$. So $u(x, t) \geq u(x, 0)=u_{0}(x)$. As $u_{0}(x) \in C^{4+\alpha}(\bar{\Omega})$, then there exists a controllable constant $N_{1}$ such that

$$
\begin{equation*}
u(x, t) \geq N_{1} \quad \text { in } \overline{Q_{T}} \tag{29}
\end{equation*}
$$

Next we will prove that $u$ is uniformly a priori bounded from above.

At a maximum of $u$, which necessarily occurs on $\partial \Omega$ since $u$ is convex, we have $u_{v} \leq 0$. Since $u_{v}=\varphi(x, u)$ on $\partial \Omega \times[0, T]$, then

$$
\begin{equation*}
\varphi(x, u) \leq 0 \quad(x, t) \in \partial \Omega \times[0, T] \tag{30}
\end{equation*}
$$

From (2) we get that $\varphi(\cdot, z) \rightarrow \infty$ uniformly as $z \rightarrow \infty$. Then we can deduce that $\varphi(x, z)>0$ for all $x \in \partial \Omega$ and $z>$ $N_{2}$, where $N_{2}$ is controllable constant. Combining (30) yields

$$
\begin{equation*}
u \leq N_{2} \tag{31}
\end{equation*}
$$

This completes the proof of the theorem.
Theorem 9. Let $\Omega$ be a bounded, uniformly convex domain in $R^{n}$, and $u \in C^{4,2}\left(Q_{T}\right) \cap C^{1,0}\left(\overline{Q_{T}}\right)$ is a strictly convex solution of (1). Then one has

$$
\begin{equation*}
\sup _{Q_{T}}|D u| \leq M^{*} \tag{32}
\end{equation*}
$$

where $M^{*}$ is a controllable constant.
Proof. For any $t_{0} \in[0, T], u\left(x, t_{0}\right)$ is a continuous differentiable, convex function. From $u_{v}=\varphi(x, u)$ and the $C^{0}$ estimates, we get

$$
\begin{equation*}
u_{v} \geq-M \tag{33}
\end{equation*}
$$

where $M$ is a controllable constant. Then using Theorem 2.2 in [7], we have

$$
\begin{equation*}
\left|D u\left(x, t_{0}\right)\right| \leq M_{1} \quad \text { on } \bar{\Omega} . \tag{34}
\end{equation*}
$$

Since $t_{0}$ is arbitrary, we obtain that

$$
\begin{equation*}
|D u| \leq M^{*}, \tag{35}
\end{equation*}
$$

where $M^{*}$ is a controllable constant. This completes the proof of the theorem.

## 6. $C^{2}$ - and $C^{2+\beta, 1+\beta / 2}$-Estimates

This section is concerned with the $C^{2}$-estimates and the $C^{2+\beta, 1+\beta / 2}$-estimates of the solution to problem (1).

Theorem 10. Assume that $\Omega$ is a $C^{4}$ bounded, uniformly convex domain in $R^{n}$ and $u \in C^{4,2}\left(\overline{Q_{T}}\right)$ is a strictly convex solution of (1). Let $g_{\sigma} \in C^{2,2}(\bar{\Omega} \times R), \sigma=1,2, \varphi \in C^{3,3}(\bar{\Omega} \times R)$. Then one has

$$
\begin{equation*}
\sup _{\mathrm{Q}_{T}}\left|D_{x}^{2} u\right| \leq M^{\prime \prime} \tag{36}
\end{equation*}
$$

where $M^{\prime \prime}$ is a controllable constant.
Proof. Let $\xi \in S^{n-1}$. First we observe that $D_{\xi \xi} u>0$, since $u$ is strictly convex. So we only need to prove the fact that $D_{\xi \xi} u$ is a priori bounded from above.

We define for $(x, t, \xi) \in \bar{\Omega} \times[0, T] \times S^{n-1}$ that

$$
\begin{equation*}
\omega(x, t, \xi)=D_{\xi \xi} u-V(x, t, \xi)+K|x|^{2} \tag{37}
\end{equation*}
$$

where $V(x, \xi, t)$ is given by

$$
\begin{equation*}
V(x, \xi, t)=2\langle\xi, \nu\rangle \xi_{i}^{\prime}\left(D_{i} \varphi-D_{k} u D_{i} v_{k}\right) . \tag{38}
\end{equation*}
$$

Here $\nu$ is a smooth extension of the inner unit normal on $\partial \Omega$ that is independent of $t . \xi^{\prime}$ is given by

$$
\begin{equation*}
\xi^{\prime}=\xi-\langle\xi, v\rangle v \tag{39}
\end{equation*}
$$

$K$ is a constant to be chosen, and $D$ indicates that the chain rule has not yet been applied to the respective terms.

Let

$$
\begin{gather*}
a_{k}=2\langle\xi, \nu\rangle\left(\varphi_{z} \xi_{k}^{\prime}-\xi_{i}^{\prime} D_{i} v_{k}\right),  \tag{40}\\
b=2\langle\xi, \nu\rangle \xi_{i}^{\prime} \varphi_{i}
\end{gather*}
$$

then

$$
\begin{gather*}
V(x, \xi, t)=a_{k} u_{k}+b,  \tag{41}\\
\omega(x, t, \xi)=D_{\xi \xi} u-a_{k} u_{k}-b+K|x|^{2} .
\end{gather*}
$$

We compute that

$$
\begin{align*}
-\frac{\partial \omega}{\partial t}+F^{i j} D_{i j} \omega= & -D_{\xi \xi t} u+a_{k} D_{t k} u+D_{t} a_{k} \cdot u_{k}+D_{t} b \\
& +F^{i j} D_{i j \xi \xi} u-a_{k} F^{i j} \cdot u_{i j k}-2 F^{i j} D_{i} a_{k} \cdot u_{k j} \\
& -F^{i j} \cdot u_{k} D_{i j} a_{k}-F^{i j} D_{i j} b+2 K F^{i j} \delta_{i j} \tag{42}
\end{align*}
$$

Next, we estimate the right-hand side of (42), respectively.
Let $F\left(D_{x}^{2} u\right)=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)$. From Lemma 2, we have that $F\left(D_{x}^{2} u\right)$ is a concave function, $\left(F^{i j}\left(D_{x}^{2} u\right)\right)$ is a positive matrix, and $\operatorname{tr}\left(F^{i j}\left(D_{x}^{2} u\right)\right)=\sum_{i=1}^{n} F^{i i}\left(D_{x}^{2} u\right) \geq 1$.

Differentiating the equation

$$
\begin{equation*}
\dot{u}=F\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \tag{43}
\end{equation*}
$$

twice in the direction $\xi, \xi \in S^{n-1}$, we therefore obtain

$$
\begin{equation*}
-D_{t \xi \xi} u+F^{i j} D_{i j \xi \xi} u+F^{i j, k l} D_{i j \xi} u D_{k l \xi} u=D_{\xi \xi} g_{\sigma}(x, u) \tag{44}
\end{equation*}
$$

Using the concavity of $F$, we have

$$
\begin{equation*}
F^{i j, k l} D_{i j \xi} u D_{k l \xi} u \leq 0 ; \tag{45}
\end{equation*}
$$

then

$$
\begin{equation*}
-D_{t \xi \xi} u+F^{i j} D_{i j \xi \xi} u \geq D_{\xi \xi} g_{\sigma}(x, u) \tag{46}
\end{equation*}
$$

Differentiating the equation

$$
\begin{equation*}
\dot{u}=F\left(D_{x}^{2} u\right)-g_{\sigma}(x, u) \tag{47}
\end{equation*}
$$

in the $k$ th coordinate direction, we obtain

$$
\begin{equation*}
-D_{t k} u+F^{i j} D_{i j k} u=D_{k} g_{\sigma} . \tag{48}
\end{equation*}
$$

From $\left(F^{i j}\right)=\partial F\left(D_{x}^{2} u\right) / \partial u_{i j}=\left(F\left(D_{x}^{2} u\right) / n\right) u^{i j}$, where $\left(u^{i j}\right)$ is the inverse of $\left(u_{i j}\right)$, we have

$$
\begin{equation*}
F^{i j} D_{i} a_{k} \cdot u_{k j}=\frac{F\left(D_{x}^{2} u\right)}{n} u^{i j} u_{k j} \cdot D_{i} a_{k}=\frac{F\left(D_{x}^{2} u\right)}{n} D_{i} a_{i} \tag{49}
\end{equation*}
$$

Using the estimates of $\dot{u}$ and $u$, we obtain that $F\left(D_{x}^{2} u\right)=\dot{u}+g_{\sigma}$ is bounded. From

$$
\begin{align*}
D_{j} a_{j}= & 2 \xi_{l} D_{j} v_{l}\left(\varphi_{z} \xi_{j}^{\prime}-\xi_{i}^{\prime} D_{i} v_{j}\right) \\
& +2\langle\xi, v\rangle\left[\left(\varphi_{z j}+\varphi_{z z} D_{j} u\right) \xi_{j}^{\prime}-\xi_{i}^{\prime} D_{j i} v_{j}\right] \tag{50}
\end{align*}
$$

as well as $C^{0}$ - and $C^{1}$-estimates, it follows that $\left|D_{j} a_{j}\right|$ is bounded. Thus there exists a controllable constant $C$ such that

$$
\begin{equation*}
\left|F^{i j} D_{i} a_{k} \cdot u_{k j}\right| \leq C \tag{51}
\end{equation*}
$$

Since $\left(F^{i j}\right)$ is positive definite, we can get that

$$
\begin{equation*}
\left|F^{i j}\right| \leq \frac{1}{2}\left(F^{i i}+F^{j j}\right) \tag{52}
\end{equation*}
$$

Applying (52), $C^{1}$-estimates, and the following equality:

$$
\begin{align*}
D_{j h} a_{k}= & 2 \xi_{l} D_{j h} \nu_{l}\left(\varphi_{z} \xi_{k}^{\prime}-\xi_{i}^{\prime} D_{i} v_{k}\right) \\
& +2 \xi_{l} D_{j} v_{l}\left[\left(\varphi_{z h}+\varphi_{z z} u_{h}\right) \xi_{k}^{\prime}-\xi_{i}^{\prime} D_{i h} v_{k}\right] \\
& +2 \xi_{s} D_{h} \nu_{s}\left[\left(\varphi_{z j}+\varphi_{z z} u_{j}\right) \xi_{k}^{\prime}-\xi_{i}^{\prime} D_{i j} v_{k}\right] \\
+ & 2\langle\xi, v\rangle\left\{\left[\left(\varphi_{z j h}+\varphi_{z j z} u_{h}\right)+\left(\varphi_{z z h}+\varphi_{z z z} u_{h}\right) u_{j}\right.\right. \\
& \left.\left.\quad+\varphi_{z z} u_{j h}\right] \xi_{k}^{\prime}-\xi_{i}^{\prime} D_{i j h} v_{k}\right\} \tag{53}
\end{align*}
$$

we obtain that

$$
\begin{align*}
\left|F^{h j} \cdot u_{k} D_{h j} a_{k}\right| & \leq\left|u_{k} F^{h j} 2\langle\xi, \nu\rangle \varphi_{z z} u_{j h} \xi_{k}^{\prime}\right|+C\left|F^{h j}\right| \\
& =\left|2 u_{k}\langle\xi, \nu\rangle \varphi_{z z} \xi_{k}^{\prime} \frac{F\left(D_{x}^{2} u\right)}{n} u^{h j} u_{j h}\right|+C\left|F^{h j}\right| \\
& \leq C_{1}+C_{2} \operatorname{tr}\left(F^{i j}\right), \tag{54}
\end{align*}
$$

where $C_{1}$ and $C_{2}$ are positive controllable constants.
From (51), (54), and the estimates like these, it follows that

$$
\begin{align*}
& \left|D_{t} a_{k} \cdot u_{k}+D_{t} b-2 F^{i j} D_{i} a_{k} \cdot u_{k j}-F^{i j} \cdot u_{k} D_{i j} a_{k}-F^{i j} D_{i j} b\right| \\
& \quad \leq c_{1} \operatorname{tr}\left(F^{i j}\right)+c_{2}, \tag{55}
\end{align*}
$$

where $c_{1}$ and $c_{2}$ are positive controllable constants. Then using (46) and (48), we can obtain

$$
\begin{align*}
-\frac{\partial \omega}{\partial t}+F^{i j} D_{i j} \omega \geq & D_{\xi \xi} g_{\sigma}-a_{k} D_{k} g_{\sigma} \\
& -\left(c_{1} \operatorname{tr}\left(F^{i j}\right)+c_{2}\right)+2 K F^{i j} \delta_{i j} \\
= & \left(g_{\sigma}\right)_{\xi \xi}+2\left(g_{\sigma}\right)_{\xi z} u_{\xi}+\left(g_{\sigma}\right)_{z z} u_{\xi} u_{\xi} \\
& +\left(g_{\sigma}\right)_{z} u_{\xi \xi}-a_{k}\left(\left(g_{\sigma}\right)_{k}+\left(g_{\sigma}\right)_{z} u_{k}\right) \\
& -\left(c_{1} \operatorname{tr}\left(F^{i j}\right)+c_{2}\right)+2 K F^{i j} \delta_{i j} \\
\geq & \left(g_{\sigma}\right)_{\xi \xi}+2\left(g_{\sigma}\right)_{\xi z} u_{\xi}+\left(g_{\sigma}\right)_{z z} u_{\xi} u_{\xi} \\
& -a_{k}\left(\left(g_{\sigma}\right)_{k}+\left(g_{\sigma}\right)_{z} u_{k}\right) \\
& -\left(c_{1} \operatorname{tr}\left(F^{i j}\right)+c_{2}\right)+2 K F^{i j} \delta_{i j} \tag{56}
\end{align*}
$$

where we have used the structure condition (3) and the convexity of $u$. Using $C^{0}$ - and $C^{1}$-estimates, there exist positive controllable constants $c_{3}$ and $c_{4}$ such that

$$
\begin{equation*}
-\frac{\partial \omega}{\partial t}+F^{i j} D_{i j} \omega \geq\left(2 K-c_{3}\right) \operatorname{tr}\left(F^{i j}\right)-c_{4} . \tag{57}
\end{equation*}
$$

Since $\operatorname{tr}\left(F^{i j}\right) \geq 1$, we fix $K \geq(1 / 2)\left(c_{3}+c_{4}+1\right)$ and deduce that

$$
\begin{equation*}
-\frac{\partial \omega}{\partial t}+F^{i j} D_{i j} \omega \geq 1 \tag{58}
\end{equation*}
$$

Thus by the parabolic maximum principle, we have

$$
\begin{equation*}
\omega \leq \sup _{\partial_{P} Q_{T}} \omega . \tag{59}
\end{equation*}
$$

As $\omega$ is known on $\bar{\Omega} \times\{t=0\} \times S^{n-1}$, we need only to estimate $\omega$ on $\partial \Omega \times[0, T] \times S^{n-1}$.

The estimation of $\omega$ on $\partial \Omega \times[0, T] \times S^{n-1}$ splits into four stages according to the direction $\xi$. The first three stages: (i) the mixed tangential normal second derivatives of $u$ on $\partial \Omega \times[0, T] \times S^{n-1}$, (ii) $\xi$ tangential, and (iii) $\xi$ nontangential, can be carried out as in [7]. The details of this procedure could be seen in [7]. Stage (i) is readily estimated. Stages (ii) and (iii) are reduced to the purely normal case. So we only give the proof of the fourth stage: (iv) $\xi$ normal. We extend the argument given in [2] and modified for the parabolic case.

Set $h(x, t)=v_{k} D_{k} u-\varphi(x, u)=D_{v} u-\varphi(x, u)$. By (48), a direct calculation yields

$$
\begin{align*}
L h= & -D_{t} h+F^{i j} D_{i j} h \\
= & -v_{k} D_{k t} u+\varphi_{z} D_{t} u+v_{k} F^{i j} D_{i j k} u \\
& +2 F^{i j} D_{i} v_{k} D_{j k} u+F^{i j} D_{i j} v_{k} D_{k} u-F^{i j} D_{i j} \varphi \\
= & v_{k}\left(\left(g_{\sigma}\right)_{k}+\left(g_{\sigma}\right)_{z} u_{k}\right)+\varphi_{z} D_{t} u  \tag{60}\\
& +2 F^{i j} D_{i} v_{k} D_{j k} u+F^{i j} D_{i j} v_{k} D_{k} u \\
& -F^{i j}\left(\varphi_{i j}+2 \varphi_{i z} u_{j}+\varphi_{z z} u_{i} u_{j}+\varphi_{z} u_{i j}\right) .
\end{align*}
$$

Thus, using $\left(F^{i j}\right)=\left(F\left(D_{x}^{2} u\right) / n\right) u^{i j}$, (52), and our a priori estimates, we have

$$
\begin{equation*}
|L h|=\left|-D_{t} h+F^{i j} D_{i j} h\right| \leq C_{0}\left(1+\operatorname{tr}\left(F^{i j}\right)\right) \leq C \operatorname{tr}\left(F^{i j}\right), \tag{61}
\end{equation*}
$$

where $C$ is a controllable constant.
Let $\left(x_{0}, t_{0}\right) \in \partial \Omega \times[0, T]$, and $\left(x_{0}, t_{0}\right)$ is arbitrary. We observe that $\Omega$ is a bounded, uniformly convex domain in $R^{n}$, so there exists a uniformly closed ball $B_{R}\left(x^{*}\right)$ such that

$$
\begin{align*}
\Omega \subset B_{R}\left(x^{*}\right) & \subset R^{n}, \\
\partial B_{R}\left(x^{*}\right) \cap \partial \Omega & =\left\{x_{0}\right\} . \tag{62}
\end{align*}
$$

Meanwhile, we assume that $\left|x-x^{*}\right|>1$ for all $x \in \bar{\Omega}$.
We consider the auxiliary function in $B_{R}\left(x^{*}\right) \times[0, T]$

$$
\begin{equation*}
q(x, t)=e^{K_{1} R^{2}}-e^{K_{1}\left|x-x^{*}\right|^{2}}, \tag{63}
\end{equation*}
$$

where $K_{1}$ is a positive constant to be determined.
If we choose $K_{1}$ sufficiently large, it is easy to see that $q(x, t) \geq h(x, t)$ on $\partial_{P} Q_{T}$. For sufficiently large $K_{1}$, we have

$$
\begin{align*}
-\frac{\partial q}{\partial t}+F^{i j} D_{i j} q= & -2 K_{1} \operatorname{tr}\left(F^{i j}\right) e^{K_{1}\left|x-x^{*}\right|^{2}} \\
& -4 K_{1}^{2} F^{i j}\left(x-x^{*}\right)_{i}\left(x-x^{*}\right)_{j} e^{K_{1}\left|x-x^{*}\right|^{2}} \\
\leq & -2 K_{1} \operatorname{tr}\left(F^{i j}\right) e^{K_{1}\left|x-x^{*}\right|^{2}} \\
\leq & -C \operatorname{tr}\left(F^{i j}\right) \tag{64}
\end{align*}
$$

where we have used the fact that $\left(F^{i j}\right)$ is positive definite.

By (61) and (64), we get

$$
\begin{equation*}
-\frac{\partial(q-h)}{\partial t}+F^{i j} D_{i j}(q-h) \leq 0 \tag{65}
\end{equation*}
$$

thus we obtain $q-h \geq \inf _{\partial_{P} Q_{T}}(q-h) \geq 0$ on $\overline{Q_{T}}$ in view of the parabolic maximum principle. Since $q\left(x_{0}, t_{0}\right)=h\left(x_{0}, t_{0}\right)=0$, it follows that

$$
\begin{equation*}
\frac{(q-h)\left(x_{0}+\rho v\right)-(q-h)\left(x_{0}, t_{0}\right)}{\rho} \geq 0 \tag{66}
\end{equation*}
$$

thus

$$
\begin{equation*}
\lim _{\rho \rightarrow 0^{+}} \frac{(q-h)\left(x_{0}+\rho \nu\right)-(q-h)\left(x_{0}, t_{0}\right)}{\rho} \geq 0 \tag{67}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
D_{v}(q-h)\left(x_{0}, t_{0}\right) \geq 0 . \tag{68}
\end{equation*}
$$

Hence,

$$
\begin{equation*}
h_{\nu}\left(x_{0}, t_{0}\right) \leq q_{v}\left(x_{0}, t_{0}\right) \leq c_{1}, \tag{69}
\end{equation*}
$$

where $c_{1}$ is a controllable constant.
For $-q$, in a similar fashion we can obtain

$$
\begin{equation*}
h_{v}\left(x_{0}, t_{0}\right) \geq-c_{2} \tag{70}
\end{equation*}
$$

where $c_{2}$ is a controllable constant.
Since $\left(x_{0}, t_{0}\right) \in \partial \Omega \times[0, T]$ is arbitrary, we obtain

$$
\begin{equation*}
\sup _{\partial \Omega \times[0, T]} D_{\nu \nu} u \leq C, \tag{71}
\end{equation*}
$$

where $C$ is a controllable constant.
Combining the estimates of the four stages, we obtain that there exists a controllable constant $C$ such that $D_{\xi \xi} u \leq C$ on $\overline{Q_{T}}$.

Since $\xi$ is an arbitrary direction in $S^{n-1}$, now let $\xi=$ $e_{i} \pm e_{j} / 2^{1 / 2}$, where $e_{i}=(0,0 \ldots, 1, \ldots 0)=i$ th standard coordinate vector. Thus we can get the required bounded for $D_{x}^{2} u$ immediately. This completes the proof of the theorem.

From the uniform $C^{0}$-estimates, $\dot{u}$-estimates, and the assumptions on $g_{\sigma}, \sigma=0,1$, we can conclude that $F\left(D^{2} u\right)$ has a priori positive bound from below. And using the uniform $C^{2}$-estimates for $u$, we obtain that (1) is uniformly parabolic. So we can apply the method of [14] to obtain the $C^{2+\beta, 1+\beta / 2}$ interior estimates and the estimates near the bottom. Using the estimates near the side in [15], we can get the Hölder seminorm estimates for $\dot{u}$ and $D_{x}^{2} u$. Thus we have the $C^{2+\beta, 1+\beta / 2}$-estimates.

## 7. The Proof of Theorem 1

In Section 3 we proved the uniqueness of the strictly convex solution for (1). The existence of the strictly convex solution for (1) is obtained by using the continuity method. Applying

Theorem 5.3 in [16], the implicit function theorem, and the Arzela-Ascoli theorem, we can get the desired result. Then the standard regularity of parabolic equation implies that $u \in$ $C^{4+\beta, 2+\beta / 2}\left(\overline{Q_{T}}\right)$. Since there are sufficient a priori estimates, we can extend a solution of (1) on a time interval $[0, T]$ to $[0, T+\epsilon)$ for a small $\epsilon>0$. In this way we obtain existence for all $t \geq 0$ from the a priori estimates. We then need the following lemma to prove the asymptotic behavior of a classical solution of (1).

Lemma 11. If a solution of (1) exists for all $t \geq 0$ and (4) is satisfied, then as $t \rightarrow \infty$, the functions $\left.u\right|_{t}$ converge to a limit function $u^{\infty}(x)$ such that $u^{\infty}(x)$ satisfies the Neumann boundary value problem

$$
\begin{gather*}
\operatorname{det}^{1 / n}\left(D_{x}^{2} u^{\infty}\right)=g_{2}\left(x, u^{\infty}\right) \quad x \in \Omega  \tag{72}\\
u_{v}^{\infty}=\varphi\left(x, u^{\infty}\right) \quad x \in \partial \Omega
\end{gather*}
$$

where $\nu$ is the unit inner normal on $\partial \Omega$. Moreover, $u(x, t) \rightarrow$ $u^{\infty}(x)$ in $C^{3}$-norm.

Proof. We may assume that $\dot{u}(\cdot, 0) \not \equiv 0$ and proceed as in [17]. Integrating the equation

$$
\begin{equation*}
\dot{u}=\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{2}(x, u) \tag{73}
\end{equation*}
$$

with respect to $t$ yields

$$
\begin{align*}
& u(x, t)-u\left(x, T_{0}\right) \\
& \quad=\int_{T_{0}}^{t}\left(\operatorname{det}^{1 / n}\left(D_{x}^{2} u(x, \tau)\right)-g_{2}(x, u(x, \tau))\right) d \tau \tag{74}
\end{align*}
$$

The left-hand side is uniformly bounded in view of the $C^{0}$-estimates. By applying Lemma $7, \operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{2}$ is nonnegative, and we can find that $t_{k}=t_{k}(x) \rightarrow \infty$ such that

$$
\begin{equation*}
\left.\left(\operatorname{det}^{1 / n}\left(D_{x}^{2} u\right)-g_{2}\right)\right|_{t=t_{k}} \longrightarrow 0 \tag{75}
\end{equation*}
$$

On the other hand, $u(x, \cdot)$ is monotone, and therefore

$$
\begin{equation*}
\lim _{t \rightarrow \infty} u(x, t)=: u^{\infty}(x) \tag{76}
\end{equation*}
$$

exists and is of class $C^{4}(\bar{\Omega})$ in view of the a priori estimates.
From differential interpolation inequality in Lemma 3, we can obtain the interpolation inequality of the form

$$
\begin{equation*}
\|D \widetilde{u}\| \leq C\|\widetilde{u}\| \cdot\left(\left\|D_{x}^{2} \widetilde{u}\right\|+\|\widetilde{u}\|\right) \tag{77}
\end{equation*}
$$

for $\tilde{u}=u-u^{\infty}$, where $\|\cdot\|$ denotes the sup-norm.
Dini's theorem and interpolation inequalities of the form (77) yield $u(x, t) \rightarrow u^{\infty}(x)$ in $C^{3}$-norm. We finally, obtain in view of (75) that $u^{\infty}$ is a solution of the problem (72). This complete, the proof of the lemma.

Now we completed the proof of Theorem 1.

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## Research Article

# Blowup Phenomena for a Modified Dullin-Gottwald-Holm Shallow Water System 

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We discuss blowup phenomena for a modified two-component Dullin-Gottwald-Holm shallow water system. In this paper, some new blowup criteria of strong solutions involving the density and suitable integral form of the momentum are established.

## 1. Introduction

We consider the following two-component DGH type system:

$$
\begin{gather*}
y_{t}+2 \omega u_{x}+u y_{x}+2 y u_{x}+\gamma u_{x x x}+g \rho \bar{\rho}_{x}=0 \\
y=u-\alpha^{2} u_{x x}  \tag{1}\\
\rho_{t}+(\rho u)_{x}=0
\end{gather*}
$$

where $u=u(x, t),(x, t) \in\left(\mathbb{R}, \mathbb{R}^{+}\right)$denotes the velocity field, $g$ is the downward constant acceleration of gravity in applications to shallow water waves, and $\rho=\left(1-\partial_{x}^{2}\right)(\bar{\rho}-$ $\bar{\rho}_{0}$ ), where $\bar{\rho}_{0}$ is taken to be a constant. It is obvious that if $\rho \equiv 0$, then (1) reduces to the well-known Dullin-GottwaldHolm equation [1] (DGH equation for short). There are some contributions to DGH equation concerning the wellposedness, scattering problem, blowup phenomenon, and so forth; see, for example, [2-5] and references therein. We find that (1) is expressed in terms of an averaged filtered density component $\bar{\rho}$ in analogy to the relation between momentum and velocity by setting $\rho=\left(1-\partial_{x}^{2}\right)\left(\bar{\rho}-\bar{\rho}_{0}\right)$ and the velocity component $u$. The idea is actually from the recent work [6]. Our modification breaks the structure of DGH2 system derived by following Ivanov's approach [7] by the authors in [8]. The motivation of current research is stated as follows. From geometric point of view, (1) is the model for geodesic motion on the semidirect product Lie group of
diffeomorphisms acting on densities, with respect to the $H^{1}$ norm of velocity $u$ and the $H^{1}$-norm on filtered density. From a physical point of view, (1) admits wave breaking phenomena in finite time which attracts researchers' interest. We also find that the $H^{1}$-norm of $(u, \bar{\rho})$ is conserved with respect to time variable. This makes further different discussions on the singularities, unlike those for the DGH2 system or two-component Camassa-Holm system, possible. In the previous works [9-11] on the two-component CamassaHolm equation and its modified version, blowup conditions were established in view of the negativity of initial velocity slope at some point; basically, the initial integral form of momentum is never involved. That is why we consider this kind of blowup condition in this paper. Precisely, we show the solutions blowup in finite time provided that the initial density and momentum satisfy certain sign conditions. To our knowledge, less results exist yet for the formation of singularities of (1) although the approaches we applied here are standard. The methods in previous works cannot be moved to this model parallelly. For convenience, let $v=\bar{\rho}-\bar{\rho}_{0}$ and $\Lambda=\left(1-\alpha^{2} \partial_{x}^{2}\right)^{-1}$; then the operator $\Lambda$ can be expressed by its associated Green's function $G(x)=(1 / 2 \alpha) e^{-|x / \alpha|}$ with

$$
\begin{equation*}
\Lambda f(x)=G * f(x)=\int_{\mathbb{R}} G(x-y) f(y) d y \tag{2}
\end{equation*}
$$

Using this identity, system (1) takes an equivalent form of a
quasilinear evolution equation of hyperbolic type as follows:

$$
\begin{align*}
& u_{t}+u_{x}\left(u-\frac{\gamma}{\alpha^{2}}\right) \\
& =-\partial_{x} G *\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+\left(2 \omega+\frac{\gamma}{\alpha^{2}}\right) u+\frac{g}{2} v^{2}-\frac{g}{2} v_{x}^{2}\right), \\
& v_{t}+u v_{x}=-G *\left(\left(u_{x} v_{x}\right)_{x}+u_{x} v\right) . \tag{3}
\end{align*}
$$

The current paper is based on some results on the Camassa-Holm equation [12-19] and its two-component generalizations [20-27]. We investigate further formation of singularities of solutions to (3) with the case of $g=1$ and $\alpha>$ 0 , just for simplicity mathematically. This paper is organized as follows. In Section 2, we recall some preliminary results on the well-posedness and blowup scenario. In Section 3, the detailed blowup conditions are presented.

## 2. Preliminaries

In this section, for completeness, we recall some elementary results and skip their proofs since they are not the main concern of this work. For convenience, in what follows, we let $\lambda=-\gamma / \alpha^{2}$ and $2 \kappa=2 \omega+\gamma / \alpha^{2}$.

We can apply Kato's theory [28] to establish the following local well-posedness theorem for (3).

Theorem 1. Assume an initial data $\left(u_{0}, v_{0}\right) \in H^{s} \times H^{s-1}, s \geq$ $5 / 2$. Then there exists a maximal $T=T\left(\left\|u_{0}, v_{0}\right\|_{H^{s} \times H^{s-1}}\right)>0$ and a unique solution

$$
\begin{equation*}
(u, v) \in C\left([0, T) ; H^{s} \times H^{s-1}\right) \cap C^{1}\left([0, T) ; H^{s-1} \times H^{s-2}\right) \tag{4}
\end{equation*}
$$

of system (3). Moreover, the solution ( $u, v$ ) depends continuously on the initial value ( $u_{0}, v_{0}$ ), and the maximal time of existence $T>0$ is independent of $s$.

The proof of Theorem 1 is similar to the one in [11]. Moreover, using the techniques in [11], one can get the criterion for finite time wave breaking to (3) as follows.

Theorem 2. Let $\left(u_{0}, v_{0}\right) \in H^{s} \times H^{s-1}$ with $s \geq 5 / 2$, and let $T>0$ be the maximal time of existence of the solution $(u, v)$ to (3) with initial data $\left(u_{0}, v_{0}\right)$. Then the corresponding solution $(u, v)$ blowsup in finite time if and only if

$$
\begin{equation*}
\lim _{t \uparrow T}\left\{\inf _{x \in \mathbb{R}} u_{x}(t, x)\right\}=-\infty \tag{5}
\end{equation*}
$$

Lemma 3 (see [29]). Assume that a differentiable function $y(t)$ satisfies

$$
\begin{equation*}
y^{\prime}(t) \leq-C y^{2}(t)+K \tag{6}
\end{equation*}
$$

with constants $C, K>0$. If the initial datum $y(0)=y_{0}<$ $-\sqrt{K / C}$, then the solution to (9) goes to $-\infty$ before $t$ tends to $1 /\left(-C y_{0}+K / y_{0}\right)$.

Lemma 4 (see [19]). Suppose that $\Psi(t)$ is twice continuously differential satisfying

$$
\begin{gather*}
\Psi^{\prime \prime}(t) \geq C_{0} \Psi^{\prime}(t) \Psi(t), \quad t>0, C_{0}>0 \\
\Psi(t)>0, \quad \Psi^{\prime}(t)>0 \tag{7}
\end{gather*}
$$

Then $\Psi$ blowsup in finite time. Moreover the blowup time can be estimated in terms of the initial datum as

$$
\begin{equation*}
T \leq \max \left\{\frac{2}{C_{0} \Psi(0)}, \frac{\Psi(0)}{\Psi^{\prime}(0)}\right\} . \tag{8}
\end{equation*}
$$

We also need to introduce the standard particle trajectory method for later use. Consider now the following two initial value problems:

$$
\begin{gather*}
q_{1, t}=u\left(t, q_{1}\right)+\lambda, \quad t \in[0, T)  \tag{9}\\
q_{1}(0, x)=x, \quad x \in \mathbb{R} \\
q_{2, t}=u\left(t, q_{2}\right), \quad t \in[0, T)  \tag{10}\\
q_{2}(0, x)=x, \quad x \in \mathbb{R}
\end{gather*}
$$

where $u \in C^{1}\left([0, T), H^{s-1}\right)$ is the first component of the solution $(u, v)$ to system (3) with initial data $\left(u_{0}, v_{0}\right) \in H^{s} \times$ $H^{s-1}(s \geq 5 / 2)$, and $T>0$ is the maximal time of existence. By direct computation, we have

$$
\begin{equation*}
q_{i, t x}(t, x)=u_{x}\left(t, q_{i}(t, x)\right) q_{i, x}(t, x), \quad i=1,2 . \tag{11}
\end{equation*}
$$

Then,

$$
\begin{equation*}
q_{i, x}(t, x)=\exp \left(\int_{0}^{t} u_{x}\left(\tau, q_{i}(\tau, x)\right) d \tau\right)>0, \quad t>0, x \in \mathbb{R} \tag{12}
\end{equation*}
$$

which means that $q_{i}(t, \cdot): \mathbb{R} \rightarrow \mathbb{R}$ is a diffeomorphism of the line for every $t \in[0, T)$. Consequently, the $L^{\infty}$-norm of any function $v(t, \cdot)$ is preserved under the family of the diffeomorphisms $q_{i}(t, \cdot)$; that is,

$$
\begin{array}{r}
\|v(t, \cdot)\|_{L^{\infty}}=\left\|v\left(t, q_{1}(t, \cdot)\right)\right\|_{L^{\infty}}=\left\|v\left(t, q_{2}(t, \cdot)\right)\right\|_{L^{\infty}}, \\
t \in[0, T) . \tag{13}
\end{array}
$$

Similarly,

$$
\begin{align*}
& \inf _{x \in \mathbb{R}} v(t, x)=\inf _{x \in \mathbb{R}} v\left(t, q_{1}(t, x)\right)=\inf _{x \in \mathbb{R}} v\left(t, q_{2}(t, x)\right) \text {, } \\
& t \in[0, T), \\
& \sup _{x \in \mathbb{R}} v(t, x)=\sup _{x \in \mathbb{R}} v\left(t, q_{1}(t, x)\right)=\sup _{x \in \mathbb{R}} v\left(t, q_{2}(t, x)\right),  \tag{14}\\
& t \in[0, T) .
\end{align*}
$$

## 3. Blowup Phenomenon

In this section, we show that blowup phenomenon is the only one way that singularity arises in smooth solutions. We start this section with the following useful lemma.

Lemma 5. Let $X_{0}=\left(u_{0}, v_{0}\right) \in H^{s} \times H^{s-1}, s \geq 2 . T$ is assumed to be the maximal existence time of the solution $X=(u, v)$ to system (3) corresponding to the initial data $X_{0}$. Then for all $t \in[0, T)$, one has the following conservation law:

$$
\begin{equation*}
E(t)=\int_{\mathbb{R}}\left(u^{2}+\alpha^{2} u_{x}^{2}+v^{2}+v_{x}^{2}\right) d x \tag{15}
\end{equation*}
$$

Proof. We will prove that $E(t)$ is a conserved quantity with respect to time variable. Here we use the classical energy method. Multiplying the first equation in (3) by $u(x, t)$ and integrating by parts, we obtain

$$
\begin{equation*}
\int_{\mathbb{R}} u u_{t} d x+\int_{\mathbb{R}} \alpha^{2} u_{x} u_{x t} d x=-\int_{\mathbb{R}} u v_{x}\left(v-v_{x x}\right) d x \tag{16}
\end{equation*}
$$

Similarly, we have the following inequality for the second equation (3):

$$
\begin{equation*}
\int_{\mathbb{R}} v v_{t}+v_{x} v_{x t} d x=\int_{\mathbb{R}} u v_{x}\left(v-v_{x x}\right) d x \tag{17}
\end{equation*}
$$

This implies that

$$
\begin{equation*}
\int_{\mathbb{R}}\left(u u_{t}+\alpha^{2} u_{x} u_{x x}+v v_{t}+v_{x} v_{x x}\right) d x=0 \tag{18}
\end{equation*}
$$

Thus, we have

$$
\begin{align*}
& \frac{d}{d t} \int_{\mathbb{R}}\left(u^{2}+\alpha^{2} u_{x}^{2}+v^{2}+v_{x}^{2}\right) d x  \tag{19}\\
& \quad=2 \int_{\mathbb{R}}\left(u u_{t}+\alpha^{2} u_{x} u_{x x}+v v_{t}+v_{x} v_{x x}\right) d x=0
\end{align*}
$$

This completes the proof.
Using this conservation law, we obtain

$$
\begin{aligned}
& \|u(\cdot, t)\|_{L^{\infty}(\mathbb{R})}^{2}+\|v(\cdot, t)\|_{L^{\infty}(\mathbb{R})}^{2} \\
& \quad \leq \frac{1}{2 \alpha}\|u(\cdot, t)\|_{H_{\alpha}^{1}(\mathbb{R})}^{2}+\frac{1}{2}\|v(\cdot, t)\|_{H^{1}(\mathbb{R})}^{2} \leq C_{1} E(0)
\end{aligned}
$$

where

$$
\begin{equation*}
C_{1}=\max \left\{\frac{1}{2 \alpha}, \frac{1}{2}\right\} \tag{21}
\end{equation*}
$$

Theorem 6. Suppose that $X_{0}=\left(u_{0}, v_{0}\right) \in H^{s} \times H^{s-1}, s \geq$ $5 / 2, \rho_{0}\left(x_{0}\right)=y_{0}\left(x_{0}\right)+\kappa=0$, and the initial data satisfies the following conditions:
(i) $\rho_{0}(x) \geq 0$ on $\left(-\infty, x_{0}\right)$ and $\rho_{0}(x) \leq 0$ on $\left(x_{0}, \infty\right)$,
(ii) $\int_{-\infty}^{x_{0}} e^{\xi / \alpha}\left(y_{0}(\xi)+\kappa\right) d \xi>0$ and $\int_{-\infty}^{x_{0}} e^{-\xi / \alpha}\left(y_{0}(\xi)+\right.$ $\kappa) d \xi<0$,
for some point $x_{0} \in \mathbb{R}$. Then the solution to system (3) with the initial value $X_{0}$ blowsup in finite time.

Proof. Differentiating the first equation of (3) with respect to $x$, we obtain

$$
\begin{align*}
u_{x t}+ & u_{x}^{2}+u u_{x x}+\lambda u_{x x} \\
& +\partial_{x}^{2} G *\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u+\frac{1}{2} v^{2}-\frac{1}{2} v_{x}^{2}\right)=0 . \tag{22}
\end{align*}
$$

Applying the relation $\partial_{x}^{2}(G * f)=\left(1 / \alpha^{2}\right)(G * f-f)$ yields

$$
\begin{align*}
u_{x t}+ & u_{x}^{2}+u u_{x x}+\lambda u_{x x} \\
& +\frac{1}{\alpha^{2}} G *\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u+\frac{1}{2} v^{2}-\frac{1}{2} v_{x}^{2}\right)  \tag{23}\\
& -\frac{1}{\alpha^{2}}\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u+\frac{1}{2} v^{2}-\frac{1}{2} v_{x}^{2}\right)=0 .
\end{align*}
$$

From (23) we have

$$
\begin{align*}
\frac{d}{d t} u_{x} & \left(q_{1}\left(x_{0}, t\right), t\right) \\
= & \left(u_{x t}+u u_{x x}+\lambda u_{x x}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
= & -u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)-\frac{1}{\alpha^{2}} G \\
& *\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u+\frac{1}{2} v^{2}-\frac{1}{2} v_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{\alpha^{2}}\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u+\frac{1}{2} v^{2}-\frac{1}{2} v_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
\leq & -\frac{1}{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)+\frac{1}{2 \alpha^{2}}(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{2 \alpha^{2}}\left(v^{2}-v_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& -\frac{1}{2 \alpha^{2}}\left(G *\left(v^{2}-v_{x}^{2}\right)\right)\left(q_{1}\left(x_{0}, t\right), t\right), \tag{24}
\end{align*}
$$

where we used the fact proved in [30] that

$$
\begin{equation*}
G *\left((u+\kappa)^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}\right) \geq \frac{1}{2}(u+\kappa)^{2} . \tag{25}
\end{equation*}
$$

In order to arrive at our result, we need the following three claims.

Claim 1. $\left.y\left(q_{1}\left(x_{0}, t\right), t\right)\right)+\kappa=0$ for all $t$ in its lifespan; $q_{1}$ is defined in (9).

It is worth noting the equivalent form of the first equation in (3) in what follows:

$$
\begin{equation*}
y_{t}+u y_{x}+2 y u_{x}+\lambda y_{x}+2 \kappa u_{x}+\rho v_{x}=0 \tag{26}
\end{equation*}
$$

From the previous equation, we can get

$$
\begin{align*}
\frac{d}{d t} & \left(\left(y\left(q_{1}(x, t), t\right)+\kappa\right) q_{1, x}^{2}(x, t)\right) \\
& =\left(y_{t}+u y_{x}+2 y u_{x}+\lambda y_{x}+2 \kappa u_{x}\right)\left(q_{1}(x, t), t\right) q_{1, x}^{2}(x, t) \\
& =-\rho\left(q_{1}(x, t), t\right) v_{x}\left(q_{1}(x, t), t\right) q_{1, x}^{2}(x, t) \tag{27}
\end{align*}
$$

Since $q_{2}(x, \cdot)$ defined by (10) is a diffeomorphism of the line for any $t \in[0, T)$, so there exists an $x_{3}(t) \in \mathbb{R}$ such that

$$
\begin{equation*}
q_{2}\left(t, x_{3}(t)\right)=q_{1}\left(t, x_{0}\right), \quad t \in[0, t) \tag{28}
\end{equation*}
$$

When $t=0$, we have

$$
\begin{equation*}
x_{3}(0)=q_{2}\left(0, x_{3}(0)\right)=q_{1}\left(0, x_{0}\right)=x_{0} . \tag{29}
\end{equation*}
$$

Now we prove that $\rho\left(t, q_{1}\left(t, x_{0}\right)\right)=0$. It is easy to get

$$
\begin{equation*}
\frac{d}{d t} \rho\left(t, q_{2}\left(t, x_{3}(t)\right)\right)=-\left(\rho u_{x}\right)\left(t, q_{2}\left(t, x_{3}(t)\right)\right) \tag{30}
\end{equation*}
$$

Since

$$
\begin{equation*}
\rho_{0}\left(x_{0}\right)=0 \tag{31}
\end{equation*}
$$

integrating the previous equation, we can obtain

$$
\begin{align*}
\rho\left(t, q_{2}\left(t, x_{3}(t)\right)\right) & =\rho\left(0, q_{2}\left(0, x_{3}(0)\right)\right) e^{-\int_{0}^{t} u_{x}\left(\tau, q_{2}\left(\tau, x_{3}(\tau)\right)\right) d \tau} \\
& =\rho_{0}\left(x_{0}\right) e^{-\int_{0}^{t} u_{x}\left(\tau, q_{2}\left(\tau, x_{3}(\tau)\right)\right) d \tau}=0 ; \tag{32}
\end{align*}
$$

thus we have

$$
\begin{equation*}
\rho\left(t, q_{1}\left(t, x_{0}\right)\right)=\rho\left(t, q_{2}\left(t, x_{3}(t)\right)\right)=0 \tag{33}
\end{equation*}
$$

So we can get

$$
\begin{align*}
\frac{d}{d t} & \left(\left(y\left(q_{1}\left(x_{0}, t\right), t\right)+\kappa\right) q_{1, x}^{2}\left(x_{0}, t\right)\right)  \tag{34}\\
& =-\rho\left(q_{1}\left(x_{0}, t\right), t\right) v_{x}\left(q_{1}\left(x_{0}, t\right), t\right) q_{1, x}^{2}\left(x_{0}, t\right)=0
\end{align*}
$$

then we have

$$
\begin{equation*}
y\left(q_{1}\left(x_{0}, t\right), t\right)+\kappa=y_{0}\left(x_{0}\right)+\kappa=0 . \tag{35}
\end{equation*}
$$

Our claim is proved.

Claim 2. For any fixed $t, v_{x}^{2}(x, t)-v^{2}(x, t) \leq v_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)-$ $v^{2}\left(q_{1}\left(x_{0}, t\right), t\right)$ for all $x \in \mathbb{R}$. For any fixed $t$, if $x \leq q_{1}\left(x_{0}, t\right)$, then

$$
\begin{align*}
v_{x}^{2}(x, t) & -v^{2}(x, t) \\
= & -\left(\int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi} \rho(\xi, t) d \xi-\int_{x}^{q_{1}\left(x_{0}, t\right)} e^{\xi} \rho(\xi, t) d \xi\right) \\
& \times\left(\int_{q_{1}\left(x_{0}, t\right)}^{\infty} e^{-\xi} \rho(\xi, t) d \xi+\int_{x}^{q_{1}\left(x_{0}, t\right)} e^{-\xi} \rho(\xi, t) d \xi\right) \\
= & v_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)-v^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \\
& -\int_{-\infty}^{x} e^{\xi} \rho(\xi, t) d \xi \int_{x}^{q_{1}\left(x_{0}, t\right)} e^{-\xi} \rho(\xi, t) d \xi \\
& +\int_{x}^{q_{1}\left(x_{0}, t\right)} e^{\xi} \rho(\xi, t) d \xi \int_{q_{1}\left(x_{0}, t\right)}^{\infty} e^{-\xi} \rho(\xi, t) d \xi \\
\leq & v_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)-v^{2}\left(q_{1}\left(x_{0}, t\right), t\right), \tag{36}
\end{align*}
$$

where the condition (i) is used. Similarly, for $x \geq q_{1}\left(x_{0}, t\right)$, we also have

$$
\begin{equation*}
v_{x}^{2}(x, t)-v^{2}(x, t) \leq v_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)-v^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \tag{37}
\end{equation*}
$$

So Claim 2 is proved. Consequently, we can obtain

$$
\begin{align*}
(G * & \left.\left(v^{2}-v_{x}^{2}\right)\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& =\frac{1}{2 \alpha} \int_{\mathbb{R}} e^{-\left|q_{1}\left(x_{0}, t\right)-\xi\right| / \alpha}\left(v^{2}-v_{x}^{2}\right)(\xi, t) d \xi \\
& \geq \frac{1}{2 \alpha} \int_{\mathbb{R}} e^{-\left|q_{1}\left(x_{0}, t\right)-\xi\right| / \alpha}\left(v^{2}-v_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) d \xi  \tag{38}\\
& =\left(v^{2}-v_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) .
\end{align*}
$$

Thus, one can get

$$
\begin{align*}
\frac{d}{d t} u_{x}\left(q_{1}\left(x_{0}, t\right), t\right) \leq & -\frac{1}{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)  \tag{39}\\
& +\frac{1}{2 \alpha^{2}}(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t\right), t\right)
\end{align*}
$$

Claim 3. $(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t\right), t\right)<\alpha^{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)$ for all $t \geq 0$. Furthermore, $u_{x}\left(q_{1}\left(x_{0}, t\right), t\right)<0$ is strictly decreasing.

Suppose that there exists a $t_{0}$ such that $(u+\kappa)^{2}\left(q_{1}\left(x_{0}\right.\right.$, $t), t)<\alpha^{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)$ on $\left[0, t_{0}\right)$ and $(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t_{0}\right), t_{0}\right)=$ $\alpha^{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t_{0}\right), t_{0}\right)$. From the expression of $u(x, t)$ in terms of $y(x, t)$, we can rewrite $u(x, t)+\kappa$ and $u_{x}(x, t)$ as follows:

$$
\begin{align*}
u(x, t)+\kappa= & \frac{1}{2 \alpha} e^{-x / \alpha} \int_{-\infty}^{x} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi \\
& +\frac{1}{2 \alpha} e^{x / \alpha} \int_{x}^{\infty} e^{-\xi / \alpha}(y(\xi, t)+\kappa) d \xi \\
u_{x}(x, t)= & -\frac{1}{2 \alpha^{2}} e^{-x / \alpha} \int_{-\infty}^{x} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi  \tag{40}\\
& +\frac{1}{2 \alpha^{2}} e^{x / \alpha} \int_{x}^{\infty} e^{-\xi / \alpha}(y(\xi, t)+\kappa) d \xi
\end{align*}
$$

Letting

$$
\begin{align*}
& I(t)=e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi \\
& I I(t)=e^{q_{1}\left(x_{0}, t\right) / \alpha} \int_{q_{1}\left(x_{0}, t\right)}^{\infty} e^{-\xi / \alpha}(y(\xi, t)+\kappa) d \xi \tag{41}
\end{align*}
$$

then

$$
\begin{align*}
\frac{d I(t)}{d t}= & -\frac{1}{\alpha}\left(u\left(q_{1}\left(x_{0}, t\right), t\right)+\lambda\right) e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi  \tag{42}\\
& +e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha} y_{t}(\xi, t) d \xi
\end{align*}
$$

Integrating by parts, the first term of (42) yields

$$
\begin{align*}
-\frac{1}{\alpha}(u & \left.\left(q_{1}\left(x_{0}, t\right), t\right)+\lambda\right) e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \quad \times \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi  \tag{43}\\
= & \left(\alpha u u_{x}-u^{2}-\kappa u\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& -\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha} \lambda(y(\xi, t)+\kappa) d \xi
\end{align*}
$$

For the second term of (42), we have the following equation in the view of Claim 1:

$$
\begin{aligned}
& e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha} y_{t}(\xi, t) d \xi \\
& =-e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(((y+\kappa) u)_{x}+\frac{1}{2}\left(u^{2}-\alpha^{2} u_{x}^{2}\right)_{x}\right. \\
& \left.\quad+\lambda(y+\kappa)_{x}+\kappa u_{x}+\rho v_{x}\right) d \xi
\end{aligned}
$$

$$
\begin{align*}
= & \left(\frac{\alpha^{2}}{2} u_{x}^{2}-\frac{1}{2} u^{2}-\kappa u\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& -e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha} \rho v_{x} d \xi \\
& +\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left((y+\kappa) u+\frac{1}{2}\left(u^{2}-\alpha^{2} u_{x}^{2}\right)\right. \\
& +\lambda(y+\kappa)+\kappa u) d \xi \\
= & \left(\frac{\alpha^{2}}{2} u_{x}^{2}-\frac{1}{2} u^{2}-\kappa u+\frac{1}{2}\left(v_{x}^{2}-v^{2}\right)\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{2 \alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(v^{2}-v_{x}^{2}\right) d \xi \\
& +\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(\frac{3}{2} u^{2}-\frac{\alpha^{2}}{2} u_{x}^{2}-\alpha^{2} u u_{x x}\right. \\
& +\lambda(y+\kappa)+2 \kappa u) d \xi \\
= & \left(\frac{\alpha^{2}}{2} u_{x}^{2}-\alpha u u_{x}-\kappa u+\frac{1}{2}\left(v_{x}^{2}-v^{2}\right)\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{2 \alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(v^{2}-v_{x}^{2}\right) d \xi \\
& +\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}\right. \\
& +\lambda(y+\kappa)+2 \kappa u) d \xi . \tag{44}
\end{align*}
$$

Here we have used

$$
\begin{align*}
& -\alpha e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(u u_{x x}\right)(\xi, t) d \xi \\
& \quad=-\alpha\left(u u_{x}\right)\left(q_{1}\left(x_{0}, t\right), t\right)+\frac{1}{2} u^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \\
& \quad+\alpha e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(u_{x}^{2}-\frac{1}{2 \alpha^{2}} u^{2}\right)(\xi, t) d \xi \tag{45}
\end{align*}
$$

Combining the previous equations together, and with the help of (38), (42) reads as

$$
\begin{aligned}
\frac{d I(t)}{d t}= & \left(\frac{\alpha^{2}}{2} u_{x}^{2}-u^{2}-2 \kappa u\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(u^{2}+\frac{\alpha^{2}}{2} u_{x}^{2}+2 \kappa u\right) d \xi \\
& +\frac{1}{2}\left(v_{x}^{2}-v^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)+\frac{1}{2 \alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(v^{2}-v_{x}^{2}\right) d \xi
\end{aligned}
$$

$$
\begin{align*}
= & \left(\frac{\alpha^{2}}{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)+\frac{1}{\alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(\frac{\alpha^{2}}{2} u_{x}^{2}+(u+\kappa)^{2}\right) d \xi \\
& +\frac{1}{2}\left(v_{x}^{2}-v^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)+\frac{1}{2 \alpha} e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}\left(v^{2}-v_{x}^{2}\right) d \xi \\
\geq & \frac{1}{2}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)>0, \quad \text { on }\left[0, t_{0}\right), \tag{46}
\end{align*}
$$

where Claim 2 and the inequality [30]

$$
\begin{equation*}
\int_{-\infty}^{x} e^{\xi / \alpha}\left((u+\kappa)^{2}+\frac{\alpha^{2}}{2} u_{\xi}^{2}\right)(\xi, t) d \xi \geq \frac{\alpha}{2} e^{x / \alpha}(u+\kappa)^{2} \tag{47}
\end{equation*}
$$

have been used. From the continuity property, we have

$$
\begin{align*}
& e^{-q_{1}\left(x_{0}, t_{0}\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t_{0}\right)} e^{\xi / \alpha}\left(y\left(\xi, t_{0}\right)+\kappa\right) d \xi  \tag{48}\\
& \quad>e^{-x_{0} / \alpha} \int_{-\infty}^{x_{0}} e^{\xi / \alpha}\left(y_{0}(\xi)+\kappa\right) d \xi>0
\end{align*}
$$

Similarly,

$$
\begin{equation*}
\frac{d I I(t)}{d t} \leq \frac{1}{2}\left((u+\kappa)^{2}-\alpha^{2} u_{x}^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)<0, \quad \text { on }\left[0, t_{0}\right) . \tag{49}
\end{equation*}
$$

Thus, by continuity property,

$$
\begin{align*}
& e^{q_{1}\left(x_{0}, t_{0}\right) / \alpha} \int_{-\infty}^{q_{1}\left(x_{0}, t_{0}\right)} e^{-\xi / \alpha}\left(y\left(\xi, t_{0}\right)+\kappa\right) d \xi  \tag{50}\\
& \quad<e^{x_{0} / \alpha} \int_{-\infty}^{x_{0}} e^{-\xi / \alpha}\left(y_{0}(\xi)+\kappa\right) d \xi<0 .
\end{align*}
$$

Summarizing (48) and (50), we obtain

$$
\begin{align*}
& \alpha^{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t_{0}\right), t_{0}\right)-(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t_{0}\right), t_{0}\right) \\
& =-\frac{1}{\alpha^{2}} \int_{-\infty}^{q_{1}\left(x_{0}, t_{0}\right)} e^{\xi / \alpha}\left(y\left(\xi, t_{0}\right)+\kappa\right) d \xi \\
& \quad \times \int_{q_{1}\left(x_{0}, t_{0}\right)}^{\infty} e^{-\xi / \alpha}\left(y\left(\xi, t_{0}\right)+\kappa\right) d \xi \\
& >-\frac{1}{\alpha^{2}} \int_{-\infty}^{x_{0}} e^{\xi / \alpha}\left(y_{0}(\xi)+\kappa\right) d \xi \int_{x_{0}}^{\infty} e^{-\xi / \alpha}\left(y_{0}(\xi)+\kappa\right) d \xi \\
& =\alpha^{2} u_{0 x}^{2}\left(x_{0}\right)-\left(u_{0}+\kappa\right)^{2}\left(x_{0}\right)>0 . \tag{51}
\end{align*}
$$

That is a contradiction. On the other hand, from the expression of $u_{x}(x, t)$ in terms of $y(x, t)$, we can easily get that $u_{x}\left(q_{1}\left(x_{0}, t\right), t\right)<0$. So we complete the proof of Claim 3.

Furthermore, due to (46) and (49), we can obtain

$$
\begin{align*}
& \frac{d}{d t}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
&=-\frac{1}{\alpha^{2}} \frac{d}{d t}\left(\int_{-\infty}^{q_{1}\left(x_{0}, t\right)} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi\right. \\
&\left.\times \int_{q_{1}\left(x_{0}, t\right)}^{\infty} e^{-\xi / \alpha}(y(\xi, t)+\kappa) d \xi\right) \\
& \geq-\frac{1}{2 \alpha^{2}}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) e^{q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{-\infty}^{x} e^{-\xi / \alpha}(y(\xi, t)+\kappa) d \xi \\
&+\frac{1}{2 \alpha^{2}}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) e^{-q_{1}\left(x_{0}, t\right) / \alpha} \\
& \times \int_{x}^{\infty} e^{\xi / \alpha}(y(\xi, t)+\kappa) d \xi \\
&=-u_{x}\left(q_{1}\left(x_{0}, t\right), t\right)\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) . \tag{52}
\end{align*}
$$

Integrating (39) and then substituting it into the previous inequality, we have

$$
\begin{align*}
& \frac{d}{d t}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& \quad \geq \frac{1}{2 \alpha^{2}}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right) \\
& \quad \times\left(\int_{0}^{t}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, \tau\right), \tau\right) d \tau\right.  \tag{53}\\
& \left.\quad-2 \alpha^{2} u_{0 x}\left(x_{0}\right)\right)
\end{align*}
$$

Let $\Psi(t)=\int_{0}^{t}\left(\alpha^{2} u_{x}^{2}-(u+\kappa)^{2}\right)\left(q_{1}\left(x_{0}, \tau\right), \tau\right) d \tau-2 \alpha^{2} u_{0 x}\left(x_{0}\right)$; then we can complete the proof with the help of Lemma 4.

Remark 7. We note that if the condition (i) is replaced by the following one:
(i') $\rho_{0}(x) \leq 0$ on $\left(-\infty, x_{0}\right)$ and $\rho_{0}(x) \geq 0$ on $\left(x_{0}, \infty\right)$,
then Claim 2 also holds; that is, the theorem always holds with anyone of (i) and (i').

As a corollary of Theorem 6, we have the following.
Theorem 8. Suppose that $X_{0}=\left(u_{0}, v_{0}\right) \in H^{s} \times H^{s-1}, s \geq 5 / 2$, and the initial data satisfies the following conditions:
(i) $\rho_{0}(x) \geq 0$ on $\left(-\infty, x_{0}\right)$ and $\rho_{0}(x) \leq 0$ on $\left(x_{0}\right.$, $\infty)\left(\right.$ or $\rho_{0}(x) \leq 0$ on $\left(-\infty, x_{0}\right)$ and $\rho_{0}(x) \geq$ 0 on $\left(x_{0}, \infty\right)$ ),
(ii) $u_{0}^{\prime}\left(x_{0}\right) \leq-\left(\sqrt{C_{1} E_{0}}+|\kappa|\right) / \alpha$,
for some point $x_{0} \in \mathbb{R}$. Then the solution to system (3) with the initial value $X_{0}$ blowsup in finite time.

Proof. As shown in Theorem 6, condition (i) guarantees that $v_{x}^{2}(x, t)+v^{2}(x, t) \leq\left(v_{x}^{2}-v^{2}\right)\left(q_{1}\left(x_{0}, t\right), t\right)$ for all $x \in \mathbb{R}$. Then,

$$
\begin{align*}
\frac{d}{d t} u_{x}\left(q_{1}\left(x_{0}, t\right), t\right) \leq & -\frac{1}{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \\
& +\frac{1}{2 \alpha^{2}}(u+\kappa)^{2}\left(q_{1}\left(x_{0}, t\right), t\right) \\
\leq & -\frac{1}{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)  \tag{54}\\
& +\frac{1}{2 \alpha^{2}}\left(\sqrt{C_{1} E_{0}}+|\kappa|\right)^{2} \\
:= & -\frac{1}{2} u_{x}^{2}\left(q_{1}\left(x_{0}, t\right), t\right)+K^{2}
\end{align*}
$$

where $K>0$ is a constant. By setting $\varphi(t)=u_{x}\left(q_{1}\left(x_{0}, t\right), t\right)$, we obtain

$$
\begin{equation*}
\frac{d \varphi}{d t}=-\frac{1}{2} \varphi^{2}+K^{2} \tag{55}
\end{equation*}
$$

Applying Lemma 3, we have

$$
\begin{equation*}
\lim _{t \uparrow T} \varphi(t)=-\infty \quad \text { with } T=\frac{1}{-(1 / 2) \varphi_{0}-\left(K^{2} / \varphi_{0}\right)} \tag{56}
\end{equation*}
$$

when

$$
\begin{equation*}
\varphi_{0}<-\sqrt{2} K=-\frac{\sqrt{C_{1} E_{0}}+|\kappa|}{\alpha} \tag{57}
\end{equation*}
$$

This completes the proof.

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# A Fourth-Order Block-Grid Method for Solving Laplace's Equation on a Staircase Polygon with Boundary Functions in $C^{k, \lambda}$ 

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#### Abstract

The integral representations of the solution around the vertices of the interior reentered angles (on the "singular" parts) are approximated by the composite midpoint rule when the boundary functions are from $C^{4, \lambda}, 0<\lambda<1$. These approximations are connected with the 9 -point approximation of Laplace's equation on each rectangular grid on the "nonsingular" part of the polygon by the fourth-order gluing operator. It is proved that the uniform error is of order $O\left(h^{4}+\varepsilon\right)$, where $\varepsilon>0$ and $h$ is the mesh step. For the $p$-order derivatives ( $p=0,1, \ldots$ ) of the difference between the approximate and the exact solutions, in each " singular" part $O\left(\left(h^{4}+\varepsilon\right) r_{j}^{1 / \alpha_{j}-p}\right)$ order is obtained; here $r_{j}$ is the distance from the current point to the vertex in question and $\alpha_{j} \pi$ is the value of the interior angle of the $j$ th vertex. Numerical results are given in the last section to support the theoretical results.


## 1. Introduction

In the last two decades, among different approaches to solve the elliptic boundary value problems with singularities, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see [1,2], and references therein).

In [2-7], a new combined difference-analytical method, called the block-grid method (BGM), is proposed for the solution of the Laplace equation on polygons, when the boundary functions on the sides causing the singular vertices are given as algebraic polynomials of the arc length. In the BGM, the given polygon is covered by a finite number of overlapping sectors around the singular vertices ("singular" parts) and rectangles for the part of the polygon which lies at a positive distance from these vertices ("nonsingular" part). The special integral representation in each "singular" part is approximated, and they are connected by the appropriate order gluing operator with the finite difference equations used in the "nonsingular" part of the polygon.

In $[8,9]$, the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices in the BGM was removed. It was assumed that the boundary function on each side of the polygon is given from the Hölder classes $C^{k, \lambda}, 0<\lambda<1$, and on the "nonsingular" part the 5-point scheme is used when $k=2$ [8] and the 9 -point scheme is used when $k=6$ [9]. For the 5-point scheme a simple linear interpolation with 4 points is used. For the 9-point scheme an interpolation with 31 points is used to construct a gluing operator connecting the subsystems. Moreover, to connect the quadrature nodes which are at a distance of less than $4 h$ from boundary of the polygon, a special representation of the harmonic function through the integrals of Poisson type for a half plane is used (see [9]).

In this paper the BGM is developed for the Dirichlet problem when the boundary function on each side of the polygon is from $C^{4, \lambda}$, by using the 9 -point scheme on the "nonsingular" part with 16 -point gluing operator for all quadrature nodes, including those near the boundary. The
paper is organized as follows: in Section 2, the boundary value problem and the integral representations of the exact solution in each "singular" part are given. In Section 3, to support the aim of the paper, a Dirichlet problem on the rectangle for the known exact solution from $C^{k, \lambda}, k=3,4$, is solved using the 9 -point scheme and the numerical results are illustrated. In Section 4, the system of block-grid equations and the convergence theorems are given. In Section 5 a highly accurate approximation of the coefficient of the leading singular term of the exact solution (stress intensity factor) is given. In Section 6 the method is illustrated for solving the problem in L-shaped polygon with the corner singularity. The conclusions are summarized in Section 7.

## 2. Dirichlet Problem on a Staircase Polygon

Let $G$ be an open simply connected polygon, $\gamma_{j}, j=$ $1,2, \ldots, N$, its sides, including the ends, enumerated counterclockwise, $\gamma=\gamma_{1} \cup \cdots \cup \gamma_{N}$ the boundary of $G$, and $\alpha_{j} \pi$, ( $\alpha_{j}=1 / 2,1,3 / 2,2$ ), the interior angle formed by the sides $\gamma_{j-1}$ and $\gamma_{j},\left(\gamma_{0}=\gamma_{N}\right)$. Denote by $A_{j}=\gamma_{j-1} \cap \gamma_{j}$ the vertex of the $j$ th angle and by $r_{j}, \theta_{j}$ a polar system of coordinates with a pole in $A_{j}$, where the angle $\theta_{j}$ is taken counterclockwise from the side $\gamma_{j}$.

We consider the boundary value problem

$$
\begin{equation*}
\Delta u=0 \quad \text { on } G, \quad u=\varphi_{j}(s) \quad \text { on } \gamma_{j}, \quad 1 \leq j \leq N \tag{1}
\end{equation*}
$$

where $\Delta \equiv \partial^{2} / \partial x^{2}+\partial^{2} / \partial y^{2}, \varphi_{j}$ is a given function on $\gamma_{j}$ of the arc length $s$ taken along $\gamma$, and $\varphi_{j} \in C^{4, \lambda}\left(\gamma_{j}\right), 0<\lambda<1$; that is, $\varphi_{j}$ has the fourth-order derivative on $\gamma_{j}$, which satisfies a Hölder condition with exponent $\lambda$.

At some vertices $A_{j},\left(s=s_{j}\right)$ for $\alpha_{j}=1 / 2$ the conjugation conditions

$$
\begin{equation*}
\varphi_{j-1}^{(2 q)}\left(s_{j}\right)=(-1)^{q} \varphi_{j}^{(2 q)}\left(s_{j}\right), \quad q=0,1 \tag{2}
\end{equation*}
$$

are fulfilled. For the remaining vertices $A_{j}$, the values of $\varphi_{j-1}$ and $\varphi_{j}$ at $A_{j}$ might be different. Let $E$ be the set of all $j,(1 \leq$ $j \leq N$ ) for which $\alpha_{j} \neq 1 / 2$ or $\alpha_{j}=1 / 2$, but (2) is not fulfilled. In the neighborhood of $A_{j}, j \in E$, we construct two fixed block sectors $T_{j}^{i}=T_{j}\left(r_{j i}\right) \subset G, i=1,2$, where $0<r_{j 2}<r_{j 1}<$ $\min \left\{s_{j+1}-s_{j}, s_{j}-s_{j-1}\right\}, T_{j}(r)=\left\{\left(r_{j}, \theta_{j}\right): 0<r_{j}<r, 0<\theta_{j}<\right.$ $\left.\alpha_{j} \pi\right\}$.

Let (see [10])

$$
\begin{align*}
\varphi_{j 0}(t) & =\varphi_{j}\left(s_{j}+t\right)-\varphi_{j}\left(s_{j}\right) \\
\varphi_{j 1}(t)= & \varphi_{j-1}\left(s_{j}-t\right)-\varphi_{j-1}\left(s_{j}\right)  \tag{3}\\
Q_{j}\left(r_{j}, \theta_{j}\right)= & \varphi_{j}\left(s_{j}\right)+\frac{\left(\varphi_{j-1}\left(s_{j}\right)-\varphi_{j}\left(s_{j}\right)\right) \theta_{j}}{\alpha_{j} \pi} \\
& +\frac{1}{\pi} \sum_{k=0}^{1} \int_{0}^{\sigma_{j k}} \frac{\tilde{y}_{j} \varphi_{j k}\left(t^{\alpha_{j}}\right) d t}{\left(t-(-1)^{k} \widetilde{x}_{j}\right)^{2}+\widetilde{y}_{j}^{2}} \tag{4}
\end{align*}
$$

where

$$
\begin{gather*}
\tilde{x}_{j}=r_{j}^{1 / \alpha_{j}} \cos \left(\frac{\theta_{j}}{\alpha_{j}}\right), \quad \tilde{y}_{j}=r_{j}^{1 / \alpha_{j}} \sin \left(\frac{\theta_{j}}{\alpha_{j}}\right),  \tag{5}\\
\sigma_{j k}=\left|s_{j+1-k}-s_{j-k}\right|^{1 / \alpha_{j}}
\end{gather*}
$$

The function $Q_{j}\left(r_{j}, \theta_{j}\right)$ is harmonic on $T_{j}^{1}$ and satisfies the boundary conditions in (1) on $\gamma_{j-1} \cap \bar{T}_{j}^{1}$ and $\gamma_{j} \cap \bar{T}_{j}^{1}, j \in E$, except for the point $A_{j}$ when $\varphi_{j-1}\left(s_{j}\right) \neq \varphi_{j}\left(s_{j}\right)$.

We formally set the value of $Q_{j}\left(r_{j}, \theta_{j}\right)$ and the solution $u$ of the problem (1) at the vertex $A_{j}$ equal to $\left(\varphi_{j-1}\left(s_{j}\right)+\right.$ $\left.\varphi_{j}\left(s_{j}\right)\right) / 2, j \in E$.

$$
\begin{align*}
& R_{j}(r, \theta, \eta) \\
& \qquad=\frac{1}{\alpha_{j}} \sum_{k=0}^{1}(-1)^{k} R\left(\left(\frac{r}{r_{j 2}}\right)^{1 / \alpha_{j}}, \frac{\theta}{\alpha_{j}},(-1)^{k} \frac{\eta}{\alpha_{j}}\right)  \tag{6}\\
& j \in E
\end{align*}
$$

where

$$
\begin{equation*}
R(r, \theta, \eta)=\frac{1-r^{2}}{2 \pi\left(1-2 r \cos (\theta-\eta)+r^{2}\right)} \tag{7}
\end{equation*}
$$

is the kernel of the Poisson integral for a unit circle.
Lemma 1 (see [10]). The solution $u$ of the boundary value problem (1) can be represented on $\bar{T}_{j}^{2} \backslash V_{j}, j \in E$, in the form

$$
\begin{align*}
u\left(r_{j}, \theta_{j}\right)= & Q_{j}\left(r_{j}, \theta_{j}\right) \\
& +\int_{0}^{\alpha_{j} \pi} R_{j}\left(r_{j}, \theta_{j}, \eta\right)\left(u\left(r_{j 2}, \eta\right)-Q_{j}\left(r_{j 2}, \eta\right)\right) d \eta \tag{8}
\end{align*}
$$

where $V_{j}$ is the curvilinear part of the boundary of $T_{j}^{2}$, and $Q_{j}\left(r_{j}, \theta_{j}\right)$ is the function defined by (4).

## 3. 9-Point Solution on Rectangles

Let $\Pi=\{(x, y): 0<x<a, 0<y<b\}$ be a rectangle, with $a / b$ being rational, $\gamma_{j}, j=1,2,3,4$ the sides, including the ends, enumerated counterclockwise, starting from the left side $\left(\gamma_{0} \equiv \gamma_{4}, \gamma_{5} \equiv \gamma_{1}\right)$, and $\gamma=\cup_{j=1}^{4} \gamma_{j}$ the boundary of $\Pi$.

We consider the boundary value problem

$$
\begin{gather*}
\Delta u=0 \quad \text { on } \Pi  \tag{9}\\
u=\varphi_{j} \quad \text { on } \gamma_{j}, \quad j=1,2,3,4
\end{gather*}
$$

where $\varphi_{j}$ is the given function on $\gamma_{j}$.
Definition 2. One says that the solution $u$ of the problem (9) belongs to $\widetilde{C}^{4, \lambda}(\bar{\Pi})$ if

$$
\begin{equation*}
\varphi_{j} \in C^{4, \lambda}\left(\gamma_{j}\right), \quad 0<\lambda<1, \quad j=1,2,3,4 \tag{10}
\end{equation*}
$$

and at the vertices $A_{j}=\gamma_{j-1} \cap \gamma_{j}$ the conjugation conditions

$$
\begin{equation*}
\varphi_{j}^{(2 q)}=(-1)^{q} \varphi_{j-1}^{(2 q)}, \quad q=0,1 \tag{11}
\end{equation*}
$$

are satisfied.
Remark 3. From Theorem 3.1 in [11] it follows that the class of functions $\widetilde{C}^{4, \lambda}(\bar{\Pi})$ is wider than $C^{4, \lambda}(\bar{\Pi})$.

Let $h>0$, with $a / h \geq 2, b / h \geq 2$ integers. We assign to $\Pi^{h}$ a square net on $\Pi$, with step $h$, obtained with the lines , $y=0, h, 2 h, \ldots$. Let $\gamma_{j}^{h}$ be a set of nodes on the interior of $\gamma_{j}$ and let

$$
\begin{gather*}
\dot{\gamma}_{j}^{h}=\gamma_{j} \cap \gamma_{j+1}, \quad \gamma^{h}=\cup_{j=1}^{4}\left(\gamma_{j}^{h} \cup \dot{\gamma}_{j}^{h}\right), \\
\bar{\Pi}^{h}=\Pi^{h} \cup \gamma^{h} . \tag{12}
\end{gather*}
$$

We consider the system of finite difference equations

$$
\begin{gather*}
u_{h}=B u_{h} \quad \text { on } \Pi^{h},  \tag{13}\\
u_{h}=\varphi_{j} \quad \text { on } \gamma_{j}^{h}, j=1,2,3,4
\end{gather*}
$$

where

$$
\begin{align*}
& B u(x, y) \\
& \begin{array}{l}
\equiv \frac{(u(x+h, y)+u(x, y+h)+u(x-h, y)+u(x, y-h))}{5} \\
\quad+(((u(x+h, y+h)+u(x-h, y+h) \\
\quad+u(x-h, y-h)+u(x+h, y-h))) \\
\left.\quad \times 20^{-1}\right) .
\end{array}
\end{align*}
$$

On the basis of the maximum principle the unique solvability of the system of finite difference equations (13) follows (see [12, Chapter 4]).

Everywhere below we will denote constants which are independent of $h$ and of the cofactors on their right by $c, c_{0}, c_{1}, \ldots$, generally using the same notation for different constants for simplicity.

Theorem 4. Let $u$ be the solution of problem (9). If $u \in$ $\widetilde{C}^{4, \lambda}(\bar{\Pi})$, then

$$
\begin{equation*}
\max _{\bar{\Pi}^{h}}\left|u_{h}-u\right| \leq c h^{4}, \tag{15}
\end{equation*}
$$

where $u_{h}$ is the solution of the system (13).
Proof. For the proof of this theorem see [13].
Let $\Pi^{\prime}=\{(x, y):-0.25<x<0.25,0<y<1\}$ and let $\gamma^{\prime}$ be the boundary of $\Pi^{\prime}$. We consider the Dirichlet problem

$$
\begin{align*}
& \Delta u=0 \quad \text { on } \Pi^{\prime}, \\
& u=v \quad \text { on } \gamma^{\prime}, \tag{16}
\end{align*}
$$



Figure 1: Dependence of the approximate solutions for the boundary functions from $C^{k, \lambda}$.
where $v=r^{k+\lambda} \cos (k+\lambda) \theta, r=\sqrt{x^{2}+y^{2}}, 0<\lambda<1$, is the exact solution of this problem, which is from $C^{k, \lambda}\left(\bar{\Pi}^{\prime}\right)$.

We solve the problem (16) by approximating 9 -point scheme when $k=3,4$ for the different values of $\lambda$.

In Figure 1, the order of numerical convergence

$$
\begin{equation*}
\mathfrak{R}_{\Pi^{h}}^{\varrho}=\frac{\max _{\Pi^{h}}\left|u_{2^{-e}}-u\right|}{\max _{\Pi^{h}}\left|u_{2^{-(e+1)}}-u\right|} \tag{17}
\end{equation*}
$$

of the 9 -point solution $u_{h}$, for different $h=2^{-\varrho}$ and $\varrho=$ $4,5,6,7$, is demonstrated. These results show that the order of numerical convergence, when the exact solution $u \in C^{k, \lambda}(\bar{\Pi})$, depends on $k$ and $\lambda$ and is $O\left(h^{4}\right)$ when $k=4$, which supports estimation (15). Moreover, this dependence also requires the use of fourth-order gluing operator for all quadrature nodes in the construction of the system of block-grid equations, when the given boundary functions are from the Hölder classes $C^{4, \lambda}$.

## 4. System of Block-Grid Equations

In addition to the sectors $T_{j}^{1}$ and $T_{j}^{2}$ (see Section 2) in the neighborhood of each vertex $A_{j}, j \in E$ of the polygon $G$, we construct two more sectors $T_{j}^{3}$ and $T_{j}^{4}$, where $0<r_{j 4}<r_{j 3}<$ $r_{j 2}, r_{j 3}=\left(r_{j 2}+r_{j 4}\right) / 2$ and $T_{k}^{3} \cap T_{l}^{3}=\emptyset, k \neq l, k, l \in E$, and let $G_{T}=G \backslash\left(\cup_{j \in E} T_{j}^{4}\right)$.

We cover the given solution domain (a staircase polygon) by the finite number of sectors $T_{j}^{1}, j \in E$, and rectangles $\Pi_{k} \subset G_{T}, k=1,2, \ldots, M$, as is shown in Figure 2, for the case of $L$-shaped polygon, where $j=1, M=4$ (see also [2]). It is assumed that for the sides $a_{1 k}$ and $a_{2 k}$ of $\Pi_{k}$ the quotient $a_{1 k} / a_{2 k}$ is rational and $G=\left(\cup_{k=1}^{M} \Pi_{k}\right) \cup$ $\left(\cup_{j \in E} T_{j}^{3}\right)$. Let $\eta_{k}$ be the boundary of the rectangle $\Pi_{k}$, let $V_{j}$ be


Figure 2: Description of BGM for the L-shaped domain.
the curvilinear part of the boundary of the sector $T_{j}^{2}$, and let $t_{k j}=\eta_{k} \cap \bar{T}_{j}^{3}$. We choose a natural number $n$ and define the quantities $n(j)=\max \left\{4,\left[\alpha_{j} n\right]\right\}, \beta_{j}=\alpha_{j} \pi / n(j)$, and $\theta_{j}^{m}=(m-1 / 2) \beta_{j}, j \in E, 1 \leq m \leq n(j)$. On the $\operatorname{arc} V_{j}$ we take the points $\left(r_{j 2}, \theta_{j}^{m}\right), 1 \leq m \leq n(j)$, and denote the set of these points by $V_{j}^{n}$. We introduce the parameter $h \in$ $\left(0, \varkappa_{0} / 4\right]$, where $\varkappa_{0}$ is a gluing depth of the rectangles $\Pi_{k}, k=$ $1,2, \ldots, M$, and define a square grid on $\Pi_{k}, k=1,2, \ldots, M$, with maximal possible step $h_{k} \leq \min \left\{h, \min \left\{a_{1 k}, a_{2 k}\right\} / 6\right\}$ such that the boundary $\eta_{k}$ lies entirely on the grid lines. Let $\Pi_{k}^{h}$ be the set of grid nodes on $\Pi_{k}$, let $\eta_{k}^{h}$ be the set of nodes on $\eta_{k}$, and let $\bar{\Pi}_{k}^{h}=\Pi_{k}^{h} \cup \eta_{k}^{h}$. We denote the set of nodes on the closure of $\eta_{k} \cap G_{T}$ by $\eta_{k 0}^{h}$, the set of nodes on $t_{k j}$ by $t_{k j}^{h}$, and the set of remaining nodes on $\eta_{k}$ by $\eta_{k 1}^{h}$.

Let

$$
\begin{align*}
& \omega^{h, n}=\left(\cup_{k=1}^{M} \eta_{k 0}^{h}\right) \cup\left(\cup_{j \in E} V_{j}^{n}\right), \\
& \bar{G}_{T}^{h, n}=\omega^{h, n} \cup\left(\cup_{k=1}^{M} \bar{\Pi}_{k}^{h}\right) \tag{18}
\end{align*}
$$

Let $\varphi=\left\{\varphi_{j}\right\}_{j=1}^{N}$, where $\varphi_{j} \in C^{4, \lambda}\left(\gamma_{j}\right), 0<\lambda<1$, is the given function in (1). We use the matching operator $S^{4}$ at the points of the set $\omega^{h, n}$ constructed in [14]. The value of $S^{4}\left(u_{h}, \varphi\right)$ at the point $P \in \omega^{h, n}$ is expressed linearly in terms of the values of $u_{h}$ at the points $P_{k}$ of the grid constructed on $\Pi_{k(P)},\left(P \in \Pi_{k(P)}\right)$ some part of whose boundary located in $G$ is the maximum distance away from $P$, and in terms of the boundary values of $\varphi^{(m)}, m=0,1,2,3$ at a fixed number of points. Moreover $S^{4}\left(u_{h}, 0\right)$ has the representation

$$
\begin{equation*}
S^{4}\left(u_{h}, 0\right)=\sum_{0 \leq l \leq 15} \xi_{l} u_{h, l}, \tag{19}
\end{equation*}
$$

where $u_{h, k}=u_{h}\left(P_{k}\right)$,

$$
\begin{gather*}
\xi_{l} \geq 0, \quad \sum_{0 \leq l \leq 15} \xi_{l}=1,  \tag{20}\\
u-S^{4}(u, \varphi)=O\left(h^{4}\right) . \tag{21}
\end{gather*}
$$

Let $\omega_{I}^{h, n} \subset \omega^{h, n}$ be the set of such points $P \in \omega^{h, n}$, for which all points $P_{l}$ in expression (19) are in $\cup_{k=1}^{M} \bar{\Pi}_{k}^{h}$. If some points $P_{l}$ in (19) emerge through the side $\gamma_{m}$, then the set of such points $P$ is denoted by $\omega_{D}^{h, n}$. According to the construction of $S^{4}$ in [14], the expression $S^{4}\left(u_{h}, \varphi\right)$ at each point $P \in \omega^{h, n}=\omega_{I}^{h, n} \cup \omega_{D}^{h, n}$ can be expressed as follows:

$$
\begin{align*}
& S^{4}\left(u_{h}, \varphi\right) \\
& \quad= \begin{cases}S^{4} u_{h}, & P \in \omega_{I}^{h, n}, \\
S^{4}\left(u_{h}-\sum_{k=0}^{3} a_{k} \operatorname{Re} z^{k}\right)+\left(\sum_{k=0}^{3} a_{k} \operatorname{Re} z^{k}\right)_{P}, & P \in \omega_{D}^{h, n},\end{cases} \tag{22}
\end{align*}
$$

where

$$
\begin{equation*}
a_{k}=\left.\frac{1}{k!} \frac{d^{k} \varphi_{m}(s)}{d s^{k}}\right|_{s=s_{p}}, \quad k=0,1,2,3 \tag{23}
\end{equation*}
$$

and $s_{P}$ corresponds to such a point $Q \in \gamma_{m}$ for which the line $P Q$ is perpendicular to $\gamma_{m}$.

Let

$$
\begin{equation*}
Q_{j}=Q_{j}\left(r_{j}, \theta_{j}\right), \quad Q_{j 2}^{q}=Q_{j}\left(r_{j 2}, \theta_{j}^{q}\right) \tag{24}
\end{equation*}
$$

The quantities in (24) are given by (4) and (5), which contain integrals that have not been computed exactly in the general case. Assume that approximate values $Q_{j}^{\varepsilon}$ and $Q_{j 2}^{q \varepsilon}$ of the quantities in (24) are known with accuracy $\varepsilon>0$; that is,

$$
\begin{equation*}
\left|Q_{j}^{\varepsilon}-Q_{j}\right| \leq c_{1} \varepsilon, \quad\left|Q_{j 2}^{q \varepsilon}-Q_{j 2}^{q}\right| \leq c_{2} \varepsilon, \tag{25}
\end{equation*}
$$

where $j \in E, 1 \leq q \leq n(j)$, and $c_{1}, c_{2}$ are constants independent of $\varepsilon$.

Consider the system of linear algebraic equations

$$
\begin{align*}
& u_{h}^{\varepsilon}=B u_{h}^{\varepsilon} \quad \text { on } \Pi_{k}^{h}, \\
& u_{h}^{\varepsilon}=\varphi_{m} \quad \text { on } \eta_{k 1}^{h} \cap \gamma_{m}, \\
& u_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right) \\
& =Q_{j}^{\varepsilon}+\beta_{j} \sum_{q=1}^{n(j)}\left(u_{h}^{\varepsilon}\left(r_{j 2}, \theta_{j}^{q}\right)-Q_{j 2}^{q \varepsilon}\right)  \tag{26}\\
& \quad \times R_{j}\left(r_{j}, \theta_{j}, \theta_{j}^{q}\right) \quad \text { on }\left(r_{j}, \theta_{j}\right) \in t_{k j}^{h},
\end{align*}
$$

$$
u_{h}^{\varepsilon}=S^{4} u_{h}^{\varepsilon} \quad \text { on } \omega^{h, n}
$$

where $1 \leq k \leq M, 1 \leq m \leq N$, and $j \in E$.

Table 1: The order of convergence in the "nonsingular" part when $h=2^{-\varrho}$ and $\varepsilon=5 \times 10^{-13}$.

| $\left(2^{-\varrho}, n\right)$ | $\left\\|\zeta_{h}^{\varepsilon}\right\\|_{G^{N S}}$ | $\Re_{G^{N S}}^{\varrho}$ |
| :--- | :---: | :---: |
| $\left(2^{-4}, 60\right)$ | $1.609 \times 10^{-8}$ | 15.577 |
| $\left(2^{-5}, 170\right)$ | $1.033 \times 10^{-9}$ |  |
| $\left(2^{-5}, 130\right)$ | $1.191 \times 10^{-9}$ | 16.690 |
| $\left(2^{-6}, 150\right)$ | $7.136 \times 10^{-11}$ |  |
| $\left(2^{-5}, 140\right)$ | $1.136 \times 10^{-9}$ | 16.259 |
| $\left(2^{-6}, 170\right)$ | $6.991 \times 10^{-11}$ |  |
| $\left(2^{-6}, 100\right)$ | $2.169 \times 10^{-10}$ | 17.096 |
| $\left(2^{-7}, 130\right)$ | $1.269 \times 10^{-11}$ |  |

Table 2: The order of convergence in the "singular" part when $h=$ $2^{-\varrho}$ and $\varepsilon=5 \times 10^{-13}$.

| $\left(2^{-\varrho}, n\right)$ | $\left\\|\zeta_{h}^{\varepsilon}\right\\|_{G^{s}}$ | $\Re_{G^{s}}^{\varrho}$ |
| :--- | :---: | :---: |
| $\left(2^{-4}, 100\right)$ | $1.931 \times 10^{-8}$ | 16.078 |
| $\left(2^{-5}, 150\right)$ | $1.182 \times 10^{-9}$ |  |
| $\left(2^{-5}, 130\right)$ | $1.294 \times 10^{-9}$ | 16.789 |
| $\left(2^{-6}, 150\right)$ | $7.708 \times 10^{-11}$ |  |
| $\left(2^{-5}, 140\right)$ | $1.312 \times 10^{-9}$ | 17.967 |
| $\left(2^{-6}, 170\right)$ | $7.304 \times 10^{-11}$ |  |
| $\left(2^{-6}, 100\right)$ | $2.389 \times 10^{-10}$ | 18.164 |
| $\left(2^{-7}, 130\right)$ | $1.315 \times 10^{-11}$ |  |

Table 3: The minimum errors of the solution over the pairs $\left(h^{-1}, n\right)$ in maximum norm when $\varepsilon=5 \times 10^{-13}$.

| $\left(h^{-1}, n\right)$ | $\left\\|\zeta_{h}^{\varepsilon}\right\\|_{G^{N S}}$ | $\left\\|\zeta_{h}^{\varepsilon}\right\\|_{G^{S}}$ | Iteration |
| :--- | :---: | :---: | :---: |
| $(16,70)$ | $1.139 \times 10^{-8}$ | $1.572 \times 10^{-8}$ | 22 |
| $(32,170)$ | $1.033 \times 10^{-9}$ | $1.184 \times 10^{-9}$ | 23 |
| $(64,170)$ | $6.990 \times 10^{-11}$ | $7.304 \times 10^{-11}$ | 24 |
| $(128,200)$ | $8.628 \times 10^{-12}$ | $8.833 \times 10^{-12}$ | 25 |

Let $T_{j}^{*}=T_{j}\left(r_{j}^{*}\right)$ be the sector, where $r_{j}^{*}=\left(r_{j 2}+r_{j 3}\right) / 2, j \in$ $E$, and let $\mathcal{u}_{h}^{\varepsilon}\left(r_{j 2}, \theta_{j}^{q}\right), 1 \leq q \leq n(j), j \in E$, be the solution values of the system (26) on $V_{j}^{h}$ (at the quadrature nodes). The function

$$
\begin{align*}
U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)= & Q_{j}\left(r_{j}, \theta_{j}\right) \\
& +\beta_{j} \sum_{q=1}^{n(j)} R_{j}\left(r_{j}, \theta_{j}, \theta_{j}^{q}\right)\left(u_{h}^{\varepsilon}\left(r_{j 2}, \theta_{j}^{q}\right)-Q_{j 2}^{q \varepsilon}\right), \tag{27}
\end{align*}
$$

defined on $T_{j}^{*}$, is called an approximate solution of the problem (1) on the closed block $\bar{T}_{j}^{3}, j \in E$.

Definition 5. The system (26) and (27) is called the system of block-grid equations.

Theorem 6. There is a natural number $n_{0}$, such that for all $n \geq n_{0}$ and for any $\varepsilon>0$ the system (26) has a unique solution.

Proof. From the estimation (2.29) in [15] follows the existence of the positive constants $n_{0}$ and $\sigma$, such that for all $n \geq n_{0}$

$$
\begin{equation*}
\max _{\left(r_{j}, \theta_{j}\right) \in \bar{T}_{j}^{3}} \beta_{j} \sum_{q=1}^{n(j)} R_{j}\left(r_{j}, \theta_{j}, \theta_{j}^{q}\right) \leq \sigma<1 . \tag{28}
\end{equation*}
$$

The proof is obtained on the basis of principle of maximum by taking into account (14), (19), (20), and (28).

Theorem 7. There exists a natural number $n_{0}$, such that for all

$$
\begin{equation*}
n \geq \max \left\{n_{0},\left[\ln ^{1+x} h^{-1}\right]+1\right\} \tag{29}
\end{equation*}
$$

where $x>0$ is a fixed number, and for any $\varepsilon>0$ the following inequalities are valid:

$$
\begin{gather*}
\max _{\overline{\mathrm{G}}_{T}^{h, n}}\left|u_{h}^{\varepsilon}-u\right| \leq c\left(h^{4}+\varepsilon\right),  \tag{30}\\
\left|\frac{\partial^{p}}{\partial x^{p-q} \partial y^{q}}\left(U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)-u\left(r_{j}, \theta_{j}\right)\right)\right| \leq c_{p}\left(h^{4}+\varepsilon\right) \quad \text { on } \bar{T}_{j}^{3}, \tag{31}
\end{gather*}
$$

for integer $1 / \alpha_{j}$ when $p \geq 1 / \alpha_{j}$,

$$
\begin{align*}
& \left|\frac{\partial^{p}}{\partial x^{p-q} \partial y^{q}}\left(U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)-u\left(r_{j}, \theta_{j}\right)\right)\right| \\
& \quad \leq \frac{c_{p}\left(h^{4}+\varepsilon\right)}{r^{p-1 / \alpha_{j}}} \text { on } \bar{T}_{j}^{3}, \tag{32}
\end{align*}
$$

for any $1 / \alpha_{j}$, if $0 \leq p<1 / \alpha_{j}$,

$$
\begin{gather*}
\left|\frac{\partial^{p}}{\partial x^{p-q} \partial y^{q}}\left(U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)-u\left(r_{j}, \theta_{j}\right)\right)\right| \\
\quad \leq \frac{c_{p}\left(h^{4}+\varepsilon\right)}{r^{p-1 / \alpha_{j}}} \quad \text { on } \bar{T}_{j}^{3} \backslash A_{j}, \tag{33}
\end{gather*}
$$

for noninteger $1 / \alpha_{j}$, when $p>1 / \alpha_{j}$. Everywhere $0 \leq q \leq p, u$ is the exact solution of the problem (1) and $U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)$ is defined by formula (27).

Proof. Let

$$
\begin{equation*}
\xi_{h}^{\varepsilon}=u_{h}^{\varepsilon}-u, \tag{34}
\end{equation*}
$$

where $u_{h}^{\varepsilon}$ is a solution of system (26) and $u$ is the trace on $\bar{G}_{T}^{h, n}$ of the solution of (1). On the basis of (1), (26), and (34) the error $\xi_{h}^{\varepsilon}$ satisfies the system of difference equations

$$
\begin{align*}
& \xi_{h}^{\varepsilon}=B \xi_{h}^{\varepsilon}+r_{h}^{1} \quad \text { on } \Pi_{k}^{h}, \\
& \begin{aligned}
\xi_{h}^{\varepsilon}=0 & \text { on } \eta_{k 1}^{h},
\end{aligned} \\
& \begin{aligned}
\xi_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right) & =\beta_{j} \sum_{q=1}^{n(j)} \xi_{h}^{\varepsilon}\left(r_{j 2}, \theta_{j}^{q}\right) R_{j}\left(r_{j}, \theta_{j}, \theta_{j}^{q}\right) \\
& +r_{j h}^{2}, \quad\left(r_{j}, \theta_{j}\right) \in t_{k j}^{h},
\end{aligned}  \tag{35}\\
& \xi_{h}^{\varepsilon}=S^{4} \xi_{h}^{\varepsilon}+r_{h}^{3} \quad \text { on } \omega^{h, n},
\end{align*}
$$

Table 4: In $G^{S} \cap r \geq 0.2$, the minimum errors of the derivatives over the pairs $\left(h^{-1}, n\right)$ in maximum norm when $\varepsilon=5 \times 10^{-13}$.

| $\left(h^{-1}, n\right)$ | $\operatorname{Max}_{G^{s} \cap\{r \geq 0.2\}} r^{1 / 3}\left\\|\frac{\partial U_{h}^{\varepsilon}}{\partial x}-\frac{\partial u}{\partial x}\right\\|$ | $\operatorname{Max}_{G^{s} \cap\{r \geq 0.2\}} r^{1 / 3}\left\\|\frac{\partial U_{h}^{e}}{\partial y}-\frac{\partial u}{\partial y}\right\\|$ |
| :--- | :---: | :---: |
| $(16,70)$ | $3.895 \times 10^{-7}$ | $3.895 \times 10^{-7}$ |
| $(32,170)$ | $4.627 \times 10^{-8}$ | $4.627 \times 10^{-8}$ |
| $(64,170)$ | $1.124 \times 10^{-9}$ | $3.125 \times 10^{-9}$ |
| $(128,200)$ | $2.214 \times 10^{-10}$ | $2.233 \times 10^{-10}$ |

Table 5: In $G^{S}$, the minimum errors of the derivatives over the pairs $\left(h^{-1}, n\right)$ in maximum norm when $\varepsilon=5 \times 10^{-13}$.

| $\left(h^{-1}, n\right)$ | $\operatorname{Max}_{G^{S}} r^{1 / 3}\left\\|\frac{\partial U_{h}^{e}}{\partial x}-\frac{\partial u}{\partial x}\right\\|$ | $\operatorname{Max}_{G^{s}} r^{1 / 3}\left\\|\frac{\partial U_{h}^{e}}{\partial y}-\frac{\partial u}{\partial y}\right\\|$ |
| :--- | :---: | :---: |
| $(16,70)$ | $9.663 \times 10^{-6}$ | $9.663 \times 10^{-6}$ |
| $(32,170)$ | $9.653 \times 10^{-6}$ | $9.653 \times 10^{-6}$ |
| $(64,170)$ | $9.649 \times 10^{-6}$ | $9.649 \times 10^{-6}$ |
| $(128,200)$ | $9.648 \times 10^{-6}$ | $9.648 \times 10^{-6}$ |

where $1 \leq k \leq M, j \in E$,

$$
\left.\begin{array}{c}
r_{h}^{1}=B u-u \text { on } \cup_{k=1}^{M} \Pi_{k}^{h}, \\
r_{j h}^{2}=\beta_{j} \sum_{q=1}^{n(j)}\left(u\left(r_{j 2}, \theta_{j}^{q}\right)-Q_{j 2}^{q \varepsilon}\right) R_{j}\left(r_{j}, \theta_{j}, \theta_{j}^{q}\right) \\
\\
-\left(u-Q_{j}^{\varepsilon}\right) \quad \text { on } \cup_{k=1}^{M}\left(\cup_{j \in E} t_{k j}^{h}\right),
\end{array}\right\} \begin{array}{ll}
S_{h}^{4} u-u \\
S^{4}\left(u-\sum_{k=0}^{3} a_{k} \operatorname{Re} z^{k}\right)-\left(u-\sum_{k=0}^{3} a_{k} \operatorname{Re} z^{k}\right)_{P}, & P \in \omega_{D}^{h, n} . \tag{38}
\end{array}
$$

On the basis of estimations (15), (21), (25), and Lemma 1 by analogy to the proof of Theorem 4.3 in [9] the proof of inequality (30) follows.

The function $U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)$ given by formula (27), defined on the closed sector $\bar{T}_{j}^{*}, j \in E$, where $r_{j}^{*}=\left(r_{j 2}+r_{j 3}\right) / 2$, and the integral representation (8) of the exact solution of the problem (1) is given on $\bar{T}_{j}^{2} \backslash V_{j}, j \in E$, and then the difference function $\zeta_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)=U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)-u\left(r_{j}, \theta_{j}\right)$ is defined on $\bar{T}_{j}^{*}, j \in E$ and

$$
\begin{equation*}
\zeta_{h}^{\varepsilon}\left(r_{j}^{*}, 0\right)=\zeta_{h}^{\varepsilon}\left(r_{j}^{*}, \alpha_{j} \pi\right)=0, \quad j \in E \tag{39}
\end{equation*}
$$

On the basis of Lemma 6.11 from [16], (25), and (28), for $n \geq$ $\max \left\{n_{0},\left[\ln ^{1+\varkappa} h^{-1}\right]+1\right\}, \varkappa>0$ is a fixed number, and we obtain

$$
\begin{equation*}
\left|\zeta_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)\right| \leq c\left(h^{4}+\varepsilon\right) \quad \text { on } \bar{T}_{j}^{*}, j \in E \tag{40}
\end{equation*}
$$

Furthermore, the function $\zeta_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)$ continuous on $\bar{T}_{j}^{*}$ is a solution of the following Dirichlet problem:

$$
\begin{gather*}
\Delta \zeta_{h}^{\varepsilon}=0 \quad \text { on } T_{j}^{*}, \\
\zeta_{h}^{\varepsilon}=0 \quad \text { on } \gamma_{m} \cap \bar{T}_{j}^{*}, m=j-1, j,  \tag{41}\\
\zeta_{h}^{\varepsilon}\left(r_{j}^{*}, \theta_{j}\right)=U_{h}^{\varepsilon}\left(r_{j}^{*}, \theta_{j}\right)-u\left(r_{j}^{*}, \theta_{j}\right), \quad 0 \leq \theta_{j} \leq \alpha_{j} \pi .
\end{gather*}
$$

Since $T_{j}^{3} \subset \bar{T}_{j}^{*}$, on the basis of (39) and (40), from Lemma 6.12 in [16], inequalities (31)-(33) of Theorem 7 follow.

## 5. Stress Intensity Factor

Let, in the condition $\varphi_{j} \in C^{4, \lambda}\left(\gamma_{j}\right)$, the exponent $\lambda$ be such that

$$
\begin{equation*}
\left\{\alpha_{j}(4+\lambda)\right\} \neq 0, \quad\left\{2 \alpha_{j}(4+\lambda)\right\} \neq 0 \tag{42}
\end{equation*}
$$

where $\{\cdot\}$ is the symbol of fractional part. These conditions for the given $\alpha_{j}$ can be fulfilled by decreasing $\lambda$.

On the basis of Section 2 of [11], a solution of the problem (1) can be represented in $\bar{T}_{j}^{*}, j \in E$, as follows:

$$
\begin{align*}
u\left(x_{j}, y_{j}\right)= & \tilde{u}\left(x_{j}, y_{j}\right)+\sum_{k=0}^{4} \mu_{k}^{(j)} \operatorname{Im}\left\{z^{k} \ln z\right\}  \tag{43}\\
& +\sum_{k=1}^{n_{\alpha_{j}}} \tau_{k}^{(j)} r_{j}^{k / \alpha_{j}} \sin \frac{k \theta_{j}}{\alpha_{j}},
\end{align*}
$$

where $n_{\alpha_{j}}=\left[\alpha_{j}(4+\lambda)\right],[\cdot]$ is the integer part, $z=x_{j}+i y_{j}, \mu_{k}^{(j)}$ and $\tau_{k}^{(j)}$ are some numbers, and $\tilde{u}\left(x_{j}, y_{j}\right) \in C^{4, \lambda}\left(T_{j}^{2}\right)$ is the harmonic on $T_{j}^{2}$. By taking $\theta_{j}=\alpha_{j} \pi / 2$, from the formula (43), it follows that the coefficient $\tau_{1}^{(j)}$ which is called the stress intensity factor can be represented as

$$
\begin{align*}
\tau_{1}^{(j)}=\lim _{r_{j} \rightarrow 0} \frac{1}{r_{j}^{1 / \alpha_{j}}}( & u\left(x_{j}, y_{j}\right)-\tilde{u}\left(x_{j}, y_{j}\right)  \tag{44}\\
& \left.-\sum_{k=0}^{4} \mu_{k}^{(j)} \operatorname{Im}\left\{z^{k} \ln z\right\}\right) .
\end{align*}
$$

TABLE 6: The stress intensity factor $\tau_{1, n}^{\epsilon}$ for $n=70,170,200$ when $\varepsilon=5 \times 10^{-13}$.

| $h^{-1}$ | $\tau_{1,70}^{\varepsilon}$ | $\tau_{1,170}^{\varepsilon}$ | $\tau_{1,200}^{\varepsilon}$ |
| :--- | :---: | :---: | :---: |
| 16 | 1.000000014856688 | 1.000000017180415 | 1.000000017197438 |
| 32 | 1.000000005800844 | 1.000000001230267 | 1.000000001236709 |
| 64 | 1.000000004138169 | 1.000000000073107 | 1.000000000079938 |
| 128 | 1.000000004053153 | 1.000000000003531 | 1.000000000003267 |



Figure 3: Dependence on $\varepsilon$ for $h^{-1}=16,32$.

From formula (44) it follows that $\tau_{1}^{(j)}$ can be approximated by

$$
\begin{align*}
& \tau_{1, n}^{(j) \varepsilon} \\
& \begin{aligned}
=\lim _{r_{j} \rightarrow 0} \frac{1}{r_{j}^{1 / \alpha_{j}}} & \left(U_{h}^{\varepsilon}\left(r_{j}, \theta_{j}\right)\right. \\
& \left.\quad-\left(\varphi_{j}\left(s_{j}\right)+\left(\varphi_{j-1}\left(s_{j}\right)-\varphi_{j}\left(s_{j}\right)\right) \frac{\theta_{j}}{\alpha_{j} \pi}\right)\right)
\end{aligned}
\end{align*}
$$

Using formula (3), (4), and (27) from (45) for the stress intensity factor (see [17]), we obtain the next formula:

$$
\begin{align*}
\tau_{1, n}^{(j) \varepsilon}= & \frac{1}{\pi} \int_{0}^{\sigma_{j 0}} \frac{\varphi_{j 0}\left(t^{\alpha_{j}}\right) d t}{t^{2}}+\frac{1}{\pi} \int_{0}^{\sigma_{j 1}} \frac{\varphi_{j 1}\left(t^{\alpha_{j}}\right) d t}{t^{2}} \\
& +\frac{2}{n(j) r_{j 2}^{1 / \alpha_{j}}} \sum_{q=1}^{n(j)}\left(u_{h}^{\varepsilon}\left(r_{j 2}, \theta_{j}^{q}\right)-Q_{j 2}^{q \varepsilon}\right) \sin \frac{1}{\alpha_{j}} \theta_{j}^{q} \tag{46}
\end{align*}
$$

This formula is obtained for the second-order BGM in [8].

(a)

(b)

Figure 4: Dependence on $\varepsilon$ for $h^{-1}=64,128$.

## 6. Numerical Results

Let $G$ be L-shaped and defined as follows:

$$
\begin{equation*}
G=\{(x, y):-1<x<1,-1<y<1\} \backslash \Omega, \tag{47}
\end{equation*}
$$

where $\Omega=\{(x, y): 0 \leq x \leq 1,-1 \leq y \leq 0\}$ and $\gamma$ is the boundary of $G$.

We consider the following problem:

$$
\begin{gather*}
\Delta u=0 \quad \text { in } G  \tag{48}\\
u=v(r, \theta) \quad \text { on } \gamma
\end{gather*}
$$

where

$$
\begin{equation*}
v(r, \theta)=r^{2 / 3} \sin \left(\frac{2}{3} \theta\right)+0.0051 r^{16 / 3} \cos \left(\frac{16}{3} \theta\right) \tag{49}
\end{equation*}
$$

is the exact solution of this problem.
We choose a "singular" part of $G$ as

$$
\begin{equation*}
G^{S}=\{(x, y):-0.5<x<0.5,-0.5<y<0.5\} \backslash \Omega_{1}, \tag{50}
\end{equation*}
$$

where $\Omega_{1}=\{(x, y): 0 \leq x \leq 0.5,-0.5 \leq y \leq 0\}$. Then $G^{N S}=G \backslash G^{S}$ is a "nonsingular" part of $G$.

The given domain $G$ is covered by four overlapping rectangles $\Pi_{k}, k=1, \ldots, 4$, and by the block sector $T_{1}^{3}$


Figure 5: Maximum error depending on the number of quadrature nodes $n$.


Figure 6: The approximate solution $U_{h}^{\varepsilon}$ and the exact solution $u$ in the "singular" part for $\varepsilon=5 \times 10^{-13}$.


Figure 7: The error function in "singular" part when $\varepsilon=5 \times 10^{-13}$.


Figure 8: $\partial U_{h}^{\varepsilon} / \partial_{x}$ in the "singular" part.
(see Figure 2). For the boundary of $G^{S}$ on $G$ is the polygonal line $t_{1}=a b c d e$. The radius $r_{12}$ of sector $T_{1}^{2}$ is taken as 0.93 . According to (49), the function $Q(r, \theta)$ in (4) is

$$
\begin{align*}
Q(r, \theta)= & \frac{0.0051}{\pi} \int_{0}^{1} \frac{\tilde{y} t^{8} d t}{(t-\widetilde{x})^{2}+\widetilde{y}^{2}} \\
& +\frac{0.0051}{\pi} \int_{0}^{1} \frac{\widetilde{y} t^{8} d t}{(t-\widetilde{x})^{2}+\widetilde{y}^{2}} \tag{51}
\end{align*}
$$

where $\tilde{x}=r^{2 / 3} \cos (2 \theta / 3)$ and $\tilde{y}=r^{2 / 3} \sin (2 \theta / 3)$. Since we have only one singular point, we omit subindices in (51). We calculate the values $Q^{\varepsilon}\left(r_{12}, \theta^{q}\right)$ and $Q^{\varepsilon}(r, \theta)$ on the grids $t_{1}^{h}$, with an accuracy of $\varepsilon$ using the quadrature formulae proposed in [10].

On the basis of (46) and (51), for the stress intensity factor, we have

$$
\begin{equation*}
\tau_{1, n}^{\varepsilon}=\frac{0.0102}{7 \pi}+\frac{2}{n(0.93)^{2 / 3}} \sum_{q=1}^{n}\left(u_{h}^{\varepsilon}\left(0.93, \theta_{j}^{q}\right)-Q_{j 2}^{q \varepsilon}\right) \sin \frac{2}{3} \theta_{j}^{q} \tag{52}
\end{equation*}
$$

Taking the zero approximation $u_{h}^{\varepsilon(0)}=0$, the results of realization of the Schwarz iteration (see [2]) for the solution of the problem (48) are given in Tables 1, 2, 3, and 4. Tables


Figure 9: $\partial U_{h}^{\varepsilon} / \partial_{y}$ in the "singular" part.


Figure 10: $\mathfrak{R}_{G^{S}}^{\varrho}$ when $\varrho=5$ by fixing $n$ for $h^{-1}=32$ for different $n$ values of $h^{-1}=64$.


Figure 11: $\Re_{G^{N S}}^{\varrho}$ when $\varrho=5$ by fixing $n$ for $h^{-1}=32$ for different $n$ values of $h^{-1}=64$.

1 and 2 represent the order of convergence. Table 6 shows a highly accurate approximation of the stress intensity factor by the proposed fourth order BGM

$$
\begin{equation*}
\mathfrak{R}_{G^{N S}}^{\varrho}=\frac{\max _{G^{N S}}\left|u_{2^{-e}}^{\varepsilon}-u\right|}{\max _{G^{N S}}\left|u_{2^{-(e+1)}}^{\varepsilon}-u\right|} \tag{53}
\end{equation*}
$$

in the "nonsingular" and the order of convergence

$$
\begin{equation*}
\mathfrak{R}_{G^{s}}^{\varrho}=\frac{\max _{G^{s}}\left|U_{2^{-e}}^{\varepsilon}-u\right|}{\max _{G^{s}}\left|U_{2^{-(\varrho+1)}}^{\varepsilon}-u\right|} \tag{54}
\end{equation*}
$$

in the "singular" parts of $G$, respectively, for $\varepsilon=5 \times$ $10^{-13}$, where $\varrho$ is a positive integer. In Table 3, the minimal values over the pairs $\left(h^{-1}, n\right)$ of the errors in maximum norm, of the approximate solution when $\varepsilon=5 \times 10^{-13}$, are presented. The similar values of errors for the first-order derivatives are presented in Table 4, when $\partial Q / \partial x$ and $\partial Q / \partial y$ are approximated by fourth-order central difference formula on $G^{S}$ for $r \geq 0.2$. For $r<0.2$, the order of errors decreases down to $10^{-6}$, which are presented in Table 5. This happens because the integrands in (51) are not sufficiently smooth for fourth-order differentiation formula. The order of accuracy of the derivatives for $r<0.2$ can be increased if we use similar quadrature rules, which we used for the integrals in (51) for the derivatives of integrands also.

Figures 3 and 4 show the dependence on $\varepsilon$ for different mesh steps $h$. Figure 5 demonstrates the convergence of the BGM with respect to the number of quadrature nodes for different mesh steps $h$. The approximate solution and the exact solution in the "singular" part are given in Figure 6, to illustrate the accuracy of the BGM. The error of the block-grid solution, when the function $Q(r, \theta)$ in (51) is calculated with an accuracy of $\varepsilon=5 \times 10^{-13}$, is presented in Figure 7. Figures 8 and 9 show the singular behaviour of the first-order partial derivatives in the "singular" part. The ratios $\Re_{G^{s}}^{\varrho}$ and $\mathfrak{R}_{G^{N S}}^{\varrho}$, when $\varrho=5$ with respect to different $n$ values for $h^{-1}=64$ and for a fixed value of $n$ of $h^{-1}=32$, are illustrated in Figures 10 and 11 , respectively. These ratios show that the order of the convergence in both the "singular" and the "nonsingular" parts is asymptotically equal to 16 when $n$ is kept fixed for $h^{-1}=32$, and it is selected as large as possible $(n>100)$ for $h^{-1}=64$.

## 7. Conclusions

In the block-grid method (BGM) for solving Laplace's equation, the restriction on the boundary functions to be algebraic polynomials on the sides of the polygon causing the singular vertices is removed. This condition is replaced by the functions from the Hölder classes $C^{4, \lambda}, 0<\lambda<1$. In the integral representations around singular vertices (on the "singular" part), which are combined with the 9-point finite difference equations on the "nonsingular" part of the polygon, the boundary conditions are taken into account with the help of integrals of Poisson type for a half-plane. To connect the subsystems, a homogeneous fourth-order gluing operator is used. It is proved that the final uniform error is of order
$O\left(h^{4}+\varepsilon\right)$, where $\varepsilon$ is the error of the approximation of the mentioned integrals and $h$ is the mesh step. For the $p$ order derivatives $(p=0,1, \ldots)$ of the difference between the approximate and the exact solutions, in each "singular" part $O\left(\left(h^{4}+\varepsilon\right) r_{j}^{1 / \alpha_{j}-p}\right)$ order is obtained. The method is illustrated in solving the problem in L-shaped polygon with the corner singularity. Dependence of the approximate solution and its errors on $\varepsilon, h$ and the number of quadrature nodes $n$ are demonstrated. Furthermore, by the constructed approximate solution on the "singular" part of the polygon, a highly accurate formula for the stress intensity factor is given.

From the error estimation formula (33) of Theorem 7 it follows that the error of the approximate solution on the block sectors decreases as $r_{j}^{1 / \alpha_{j}}\left(h^{4}+\varepsilon\right)$, which gives an additional accuracy of the BGM near the singular points.

The method and results of this paper are valid for multiply connected polygons.

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## Research Article

# Analysis of the Block-Grid Method for the Solution of Laplace's Equation on Polygons with a Slit 

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#### Abstract

The error estimates obtained for solving Laplace's boundary value problem on polygons by the block-grid method contain constants that are difficult to calculate accurately. Therefore, the experimental analysis of the method could be essential. The real characteristics of the block-grid method for solving Laplace's equation on polygons with a slit are analysed by experimental investigations. The numerical results obtained show that the order of convergence of the approximate solution is the same as in the case of a smooth solution. To illustrate the singular behaviour around the singular point, the shape of the highly accurate approximate solution and the figures of its partial derivatives up to second order are given in the "singular" part of the domain. Finally a highly accurate formula is given to calculate the stress intensity factor, which is an important quantity in fracture mechanics.


## 1. Introduction

In the past few decades, in order to improve the accuracy and resolve the convergence difficulties that appear in the neighbourhood of singular points, many different methods have been proposed for the numerical solution of plane elliptic boundary value problems with singularities. Among many approaches, a special emphasis has been placed on the construction of combined methods, in which differential properties of the solution in different parts of the domain are used (see [1]).

In [2-6] a new combined difference-analytical method called the block-grid method (BGM) is given for solving the Laplace equation on polygons, when the boundary functions on the sides causing the singular vertices are given as algebraic polynomials of the arclength. This method is a combination of the exponentially convergent block method (see $[7,8]$ ) in the "singular" part, and the finite difference method, which has a simple structure on the "nonsingular" part of the polygon. A $k$ th order gluing operator $S^{k}$ is constructed for gluing together the grids and the blocks. The uniform estimate of the error of the BGM is of order $O\left(h^{k}\right)$ ( $h$ is the mesh step) when the given boundary function on the boundary of the "nonsingular" part might be from the Hölder classes $C^{k, \lambda}, 0<\lambda<1$ (see [2-4]
for $k=6$, [6] for $k=4$, and [5] for $k=2$ ). For the errors of $p$ order derivatives $(p=1,2, \ldots)$ the estimation $O\left(h^{k} / r_{j}^{p-1 / \alpha_{j}}\right)$ is obtained in a finite neighborhood of the vertices, where $r_{j}$ is the distance from the current point to the vertex in question, and $\alpha_{j} \pi$ is the value of the interior angle at the considered vertex. Moreover, BGM can give a simple and highly accurate formula for the stress intensity factor which is an important quantity from an engineering standpoint.

The experimental investigation of the block-grid method is important and numerical results could be interesting to support the theoretical results in [2-6]. The objective of this paper is to analyze the real characteristics of the BGM for solving the Laplace equation on polygons with a slit. For this purpose a slit problem on a square domain whose exact solution is known is considered. The computational algorithm by the BGM with 5-point and 9-point schemes is given and implemented. The obtained numerical results justify the theoretical results given in [2-5]. Moreover, for the approximate solution $U_{h}^{6}$ (by 9-point scheme with $S^{6}$ ) and the error function the graphs are given to demonstrate the high accuracy of the block-grid method. The shapes of the partial derivatives $\partial U_{h}^{6} / \partial x, \partial U_{h}^{6} / \partial y, \partial^{2} U_{h}^{6} / \partial x^{2}, \partial^{2} U_{h}^{6} / \partial y^{2}$, $\partial^{2} U_{h}^{6} / \partial x \partial y$ are given to illustrate the singular behavior in the
"singular" part of the domain. Furthermore, a simple and highly accurate formula is given to calculate the stress intensity factor.

The experimental analyses of the different methods on slit problems were given in many papers (see $[9,10]$ ).

## 2. The Slit Problem and the Integral Representation of the Solution

Let $G$ be an open domain in the plane $x O y$, that is obtained from the unit square $G=\{(x, y):|x|<1,|y|<1\}$ by making a cut $O A$ along the positive semiaxis $O x$ from the center (see Figure 1). Let $\gamma_{j}, j=1(1) 7$, be its sides, including the ends, enumerated counterclockwise, $\gamma=\gamma_{1} \cup \cdots \cup \gamma_{7},\left(\gamma_{0}=\gamma_{7}\right)$, be the boundary of $G, 2 \pi$ is the interior angle formed by the sides $\gamma_{1}$ and $\gamma_{0}$. Denote by $O=\gamma_{0} \cap \gamma_{1}$ the vertex of this angle and let $r, \theta$ be a polar system of coordinates with a pole in $O$, where the angle $\theta$ is taken counterclockwise from the side $\gamma_{1}$.

We consider the boundary value problem

$$
\begin{gather*}
\Delta u=0 \quad \text { on } G  \tag{1}\\
u=\varphi_{j} \quad \text { on } \gamma_{j}, \quad j=1,2, \ldots, 7 \tag{2}
\end{gather*}
$$

where $\Delta \equiv \partial^{2} / \partial x^{2}+\partial^{2} / \partial y^{2}$ and $\varphi_{j}$ is the value of the function $v(r, \theta)=\sqrt{2} r^{1 / 2} \sin (1 / 2) \theta$ on $\gamma_{j}$.

In the neighborhood of $O$, we construct two fixed blocksectors $T^{i}=T\left(r_{i}\right) \subset G, i=1,2$, where $0<r_{2}<r_{1}<1$, $T(\rho)=\{(r, \theta): 0<r<\rho, 0<\theta<2 \pi\} \subset G$.

Let

$$
\begin{equation*}
R_{1}(r, \theta, \eta)=\frac{1}{2} \sum_{k=0}^{1}(-1)^{k} R\left(\left(\frac{r}{r_{2}}\right)^{1 / 2}, \frac{\theta}{2},(-1)^{k} \frac{\eta}{2}\right) \tag{3}
\end{equation*}
$$

where

$$
\begin{equation*}
R(r, \theta, \eta)=\frac{1-r^{2}}{2 \pi\left(1-2 r \cos (\theta-\eta)+r^{2}\right)} \tag{4}
\end{equation*}
$$

is the kernel of the Poisson integral for a unit circle.
Lemma 1. The solution $u$ of the boundary value problem (1), (2) can be represented on $\bar{T}^{2} \backslash V$, in the form

$$
\begin{equation*}
u(r, \theta)=\int_{0}^{2 \pi} R_{1}(r, \theta, \eta) u\left(r_{2}, \eta\right) d \eta \tag{5}
\end{equation*}
$$

where $V$ is the curvilinear part of the boundary of $T^{2}$.
Proof. The proof follows from Theorem 3.1 in [8] by taking into account that $\varphi_{0}=\varphi_{1}=0$.

## 3. The Block-Grid Method for the Slit Problem

The realization of the BGM for the solution of the problem (1), (2) is as follows. Let $T^{2}=T(0.93)$ and $t$ be a polygonal line abcde which lies on $T^{2}$ with a positive distance from the vertex $O$ and from the curvilinear boundary $V=\{(r, \theta)$ :


Figure 1: Covering the square domain with a slit by overlapping rectangles and sector.
$r=0.93,0<\theta<2 \pi\}$ of $T^{2}$. The set of points $T^{2}$ from $O$ up to $t$ is denoted by $G^{\mathrm{S}}$ which is called the "singular" part of $G$ and the set $G^{\text {NS }}=G \backslash G^{S}$ is the "nonsingular" part of $G$. In addition to the sector $T^{2}$ in the neighborhood of the vertex $O$ of the polygon $G$ we construct two more sectors $T^{3}=T(0.85)$ and $T^{4}=T(\sqrt{5})$. Let $G_{T}=G \backslash\left(T^{4}\right)$ and $\Pi_{l} \subset G^{\mathrm{NS}} \subset G_{T}$, $l=1(1) 5$, be fixed open rectangles (see Figure 1). Then the domain $G$ can be represented as $G=\left(\cup_{l=1}^{5} \Pi_{l}\right) \cup\left(T^{3}\right)$. Let $\eta_{l}$ be the boundary of the rectangle $\Pi_{l}$ and $t_{l}=\eta_{l} \cap t$. We define a square grid on $\Pi_{l}, l=1(1) 5$, with step $h$ such that the boundary $\eta_{l}$ lies entirely on the grid lines. $\Pi_{l}^{h}$ denotes the set of grid nodes on $\Pi_{l}, \eta_{l}^{h}$ denotes the set of nodes on $\eta_{l}$ and $\bar{\Pi}_{l}^{h}=\Pi_{l}^{h} \cup \eta_{l}^{h}$. We refer to the set of nodes on the closure of $\eta_{l} \cap G_{T}$ as $\eta_{l 0}^{h}$, the set of nodes on $t_{l}$ as $t_{l}^{h}$ and the set of remaining nodes on $\eta_{l}$ as $\eta_{l 1}^{h}$. We also introduce the natural number $n \geq 4$, and $\theta^{q}=(q-1 / 2) 2 \pi / n, 1 \leq q \leq n$. On the arc $V$, we choose the points $\left(0.93, \theta^{q}\right), 1 \leq q \leq n$, denote the set of these points by $V^{n}$ and let $\bar{G}_{T}^{h, n}=V^{n} \cup\left(\cup_{l=1}^{5} \bar{\Pi}_{l}^{h}\right)$.

Let $\varphi=\left\{\varphi_{j}\right\}_{j=1}^{7}$, where $\varphi_{j}$ is the given function in (2). We introduce a gluing operator $S^{k}, k=2,6$ ([5] for $k=2$ and [24] for $k=6$ ) at the points of the set $V^{n}$. We denote by $u_{h}^{k}\left(U_{h}^{k}\right)$ the approximate solution of the problem (1), (2) obtained by the 5-point scheme with $S^{2}$ for $k=2$, and by the 9 -point scheme with $S^{6}$ for $k=6$, on the "singular" ("nonsingular") part of $G$. The operator $S^{2}$ is defined at each point $P \in V^{n}$ in the following way: we consider the set of all rectangles $\left\{\Pi_{l}\right\}$ in the intersections of which the point $P$ lies, and we choose one of these rectangles $\Pi_{l(P)}$ part of whose boundary situated in $G^{T}$ is furthest away from $P$. The value $S^{2}\left(u_{h}^{2}, \varphi\right)$ at the point $P \in V^{n}$ is computed according to the values of the function at the four vertices $P_{\kappa}, \kappa=1(1) 4$, of the closure of the cell, containing the point $P$ of the grid constructed on $\bar{\Pi}_{l(P)}$ by multilinear interpolation.

The value of $S^{6}\left(u_{h}^{6}, \varphi\right)$ at the point $P \in V^{n}$ is expressed linearly in terms of the values of $u_{h}^{6}$ at the points $P_{\kappa}, \kappa=$ $1(1) 31$, of the grid constructed on $\Pi_{l(P)} \ni P$ some part of whose boundary located in $G$ is the maximum distance away from $P$, and in terms of the boundary values of $\varphi^{(\tau)}, \tau=$ $0,1, \ldots, 5$ at a fixed number of points. Moreover, $S^{k}\left(u_{h}^{k}, 0\right)$ has the representation

$$
S^{k}\left(u_{h}^{k}, 0\right)= \begin{cases}\sum_{1 \leq \kappa \leq 31} \xi_{\kappa} u_{h, \kappa}^{k}, & \text { for } k=6  \tag{6}\\ \sum_{1 \leq \kappa \leq 4} \lambda_{\kappa} u_{h, \kappa}^{k}, & \text { for } k=2\end{cases}
$$

where $u_{h, \kappa}^{k}=u_{h}^{k}\left(P_{\kappa}\right)$,

$$
\begin{equation*}
\xi_{\kappa} \geq 0, \quad \sum_{1 \leq \kappa \leq 31} \xi_{\kappa}=1, \quad \lambda_{\kappa} \geq 0, \quad \sum_{1 \leq \kappa \leq 4} \lambda_{\kappa}=1 \tag{7}
\end{equation*}
$$

and for the exact solution $u$ of the problem (1), (2), we have

$$
\begin{align*}
& u-S^{6}(u, \varphi)=O\left(h^{6}\right)  \tag{8}\\
& u-S^{2}(u, \varphi)=O\left(h^{2}\right)
\end{align*}
$$

Remark 2. Let $V_{I}^{n} \subset V^{n}$ be the set of such points $P \in V^{n}$, for which all points $P_{\kappa}$ in the expression (6) are in $\cup_{l=1}^{5} \bar{\Pi}_{l}^{h}$. If some of the points $P_{\kappa}$ in (6) emerge through the side $\gamma_{j}$ when $u=\varphi_{j}, 1 \leq j \leq 7$, we denote the set of such points $P$ by $V_{D}^{n}$. Then, according to the construction of $S^{6}$ in [4] the expression $S^{6}\left(u_{h}^{6}, \varphi\right)$ at each point $P \in V^{n}=V_{I}^{n} \cup V_{D}^{n}$ can be expressed as follows:

$$
\begin{align*}
& S^{6}\left(u_{h}^{6}, \varphi\right) \\
& \quad= \begin{cases}S^{6} u_{h}^{6}, & P \in V_{I}^{n} \\
S^{6}\left(u_{h}^{6}-\sum_{\tau=0}^{5} a_{\tau} \operatorname{Re} z^{\tau}\right) & \\
+\left(\sum_{\tau=0}^{5} a_{\tau} \operatorname{Re} z^{\tau}\right)_{P}, & P \in V_{D}^{n}\end{cases} \tag{9}
\end{align*}
$$

where

$$
\begin{equation*}
a_{\tau}=\left.\frac{1}{\tau!} \frac{d^{\tau} \varphi_{j}(s)}{d s^{\tau}}\right|_{s=s_{P}}, \quad \tau=0,1, \ldots, 5 \tag{10}
\end{equation*}
$$

$s_{P}$ corresponds to such point $Q \in \gamma_{j}$ for which the line $P Q$ is perpendicular to $\gamma_{j}$.

Consider for each $k=2,6$ the following system of linear algebraic equations:

$$
\begin{gather*}
u_{h}^{k}=B_{k} u_{h}^{k} \quad \text { on } \Pi_{l}^{h}  \tag{11}\\
u_{h}^{k}=\sqrt{2} r^{1 / 2} \sin \frac{1}{2} \theta \quad \text { on } \eta_{l 1}^{h} \cap \gamma_{j}  \tag{12}\\
u_{h}^{k}(r, \theta)=\frac{2 \pi}{n} \sum_{q=1}^{n} u_{h}^{k}\left(0.93, \theta^{q}\right)  \tag{13}\\
\times R_{1}\left(r, \theta, \theta^{q}\right) \quad \text { on }(r, \theta) \in t_{l}^{h} \\
u_{h}^{k}=S^{k} u_{h}^{k} \quad \text { on } V^{n}, 1 \leq l \leq 5,1 \leq j \leq 7 \tag{14}
\end{gather*}
$$

where

$$
\begin{align*}
& B_{6} u(x, y) \equiv(u(x+h, y)+u(x, y+h)+u(x-h, y) \\
&+u(x, y-h)) / 5 \\
&+(u(x+h, y+h) \\
&+u(x-h, y+h)+u(x-h, y-h)  \tag{15}\\
&+u(x+h, y-h)) / 20 \\
& B_{2} u(x, y) \equiv(u(x+h, y)+u(x, y+h) \\
&+u(x-h, y)+u(x, y-h)) / 4 .
\end{align*}
$$

Theorem 3. There is a natural number $n_{0}$ such that for all $n \geq$ $n_{0}$, and for each $k=2,6$, the system (11)-(14) has a unique solution.

Proof. The proof follows when $k=2$ from [5], and when $k=$ 6 from [3, 4].

We consider the sector $T^{*}=T(0.89)$, and let $u_{h}^{k}\left(0.93, \theta^{q}\right)$, $1 \leq q \leq n$, be the values of the solution of the system (11)-(14) on $V^{n}$ (at the quadrature nodes). The function

$$
\begin{equation*}
U_{h}^{k}(r, \theta)=\frac{2 \pi}{n} \sum_{q=1}^{n} R_{1}\left(r, \theta, \theta^{q}\right) u_{h}^{k}\left(0.93, \theta^{q}\right) \tag{16}
\end{equation*}
$$

defined on $T^{*}$ is called an approximate solution of the problem (1), (2) on the closed block $\bar{T}^{3}$.

Everywhere below we will denote constants which are independent of $h$ and of the cofactors on their right by $c, c_{0}, c_{1}$ for simplicity.

Theorem 4. There exists a natural number $n_{0}$ such that for

$$
\begin{equation*}
n \geq \max \left\{n_{0},\left[\ln ^{1+x} h^{-1}\right]+1\right\} \tag{17}
\end{equation*}
$$

where $\varkappa>0$ is a fixed number, the following inequalities are valid:

$$
\begin{gather*}
\max \left|u_{h}^{k}-u\right| \leq c h^{k},  \tag{18}\\
\left|\left(U_{h}^{k}(r, \theta)-u(r, \theta)\right)\right| \leq c_{0} r^{1 / 2} h^{k} \quad \text { on } \bar{T}^{3},  \tag{19}\\
\left|\frac{\partial^{p}}{\partial x^{p-q} \partial y^{q}}\left(U_{h}^{k}(r, \theta)-u(r, \theta)\right)\right|  \tag{20}\\
\leq c_{1} h^{k} / r^{p-(1 / 2)} \quad \text { on } \bar{T}^{3} \backslash O,
\end{gather*}
$$

for all $p=1,2, \ldots$. Everywhere $0 \leq q \leq p, u$ is a solution of the problem (1), (2).

Proof. The proof is carried out analogically to the proof of Theorems 1 and 2 in [3].

## 4. Computational Algorithm

Let $\Pi=\left\{(x, y): a_{1}<x<a_{2}, b_{1}<y<b_{2}\right\}$, where $a_{2}-$ $a_{1}=2^{p} h_{0}, b_{2}-b_{1}=2^{q} h_{0}, h_{0}>0$ is a fixed number, and $p$ and $q$ are integers. We introduce a square grid with the lines $x=a_{1}+i h, y=b_{1}+j h, h=h_{0} 2^{-m}, m \geq 0$ is an integer, $i=0,1, \ldots, 2^{p+m}, j=0,1, \ldots, 2^{q+m}$. Let $\Pi_{h}=\{(x, y): x=$ $\left.x_{i}=a_{1}+i h, 0<i<2^{p+m}, y=y_{j}=b_{1}+j h, 0<j<2^{q+m}\right\}$ and $\Gamma_{h}=\Gamma_{1 h} \cup \Gamma_{2 h} \cup \Gamma_{3 h} \cup \Gamma_{4 h}$ be a set of nodes on $\Gamma$ (the boundary of $\Pi$ ) where

$$
\begin{align*}
& \Gamma_{1 h}=\left\{(x, y): x=a_{1}+i h, 1 \leq i \leq 2^{p+m}, y=b_{1}\right\},  \tag{21}\\
& \Gamma_{2 h}=\left\{(x, y): x=a_{2}, y=b_{1}+j h, 1 \leq j \leq 2^{q+m}\right\},  \tag{22}\\
& \Gamma_{3 h}=\left\{(x, y): x=a_{1}+i h, 1 \leq i \leq 2^{p+m}, y=b_{2}\right\},  \tag{23}\\
& \Gamma_{4 h}=\left\{(x, y): x=a_{1}, y=b_{1}+j h, 1 \leq j \leq 2^{q+m}\right\} . \tag{24}
\end{align*}
$$

We consider for each $k=2,6$ the finite difference problem

$$
\begin{gather*}
u_{h}^{k}=B_{k} u_{h}^{k} \quad \text { on } \Pi_{h}, \\
u_{h}^{k}=\varphi_{j h} \quad \text { on } \Gamma_{j h}, j=1,2,3,4 \tag{25}
\end{gather*}
$$

where $\varphi_{j h}$ is a given function on $\Gamma_{j h}$ that vanishes at the end points.

The solution of the problem (25) can be found using the superposition principle $u_{h}^{k}=u_{1 h}^{k}+u_{2 h}^{k}+u_{3 h}^{k}+u_{4 h}^{k}$ as the sum of solution of four problems of the type

$$
\begin{gather*}
u_{j h}^{k}=B_{k} u_{h} \quad \text { on } \Pi_{h}, \\
u_{j h}^{k}= \begin{cases}\varphi_{j h} & \text { on } \Gamma_{j h}, \\
0 & \text { on } \Gamma_{h} \backslash \Gamma_{j h}, j=1,2,3,4 .\end{cases} \tag{26}
\end{gather*}
$$

The solution of the problem (26), when $j=1$ has the representation

$$
\begin{align*}
& u_{1 h}^{k}(x, y) \\
& \quad=\sum_{n=1}^{2^{p+m}-1} d_{n} \frac{\sinh \left(\beta_{n}^{k}\left(1-y /\left(b_{2}-b_{1}\right)\right)\right)}{\sinh \beta_{n}^{k}} \sin \frac{n \pi x}{a_{2}-a_{1}} \\
& d_{n}=2^{1-p-m} \sum_{r=1}^{2^{p+m}-1} \varphi_{1 h}\left(a_{1}+r h\right) \sin \frac{n \pi\left(a_{1}+r h\right)}{a_{2}-a_{1}}, \tag{27}
\end{align*}
$$

where

$$
\begin{equation*}
\beta_{n}^{2}=\frac{2\left(b_{2}-b_{1}\right)}{h} \sinh ^{-1}\left(\sin \frac{n \pi h}{2\left(a_{2}-a_{1}\right)}\right) \tag{28}
\end{equation*}
$$

for the 5-point approximation [11],

$$
\begin{align*}
\beta_{n}^{6}= & \frac{2\left(b_{2}-b_{1}\right)}{h} \\
& \times \sinh ^{-1}\left(\frac{\sin n \pi h / 2\left(a_{2}-a_{1}\right)}{\sqrt{1-2 \sin ^{2}\left(n \pi h / 2\left(a_{2}-a_{1}\right)\right) / 3}}\right), \tag{29}
\end{align*}
$$

for the 9-point approximation [12].


Figure 2: The errors with respect to number of quadrature nodes $n$, in the "singular" part and in the "nonsingular" part by the BGM when 5-point scheme is used with $S^{2}$ for $h^{-1}=32,64$.



$$
\begin{aligned}
& \stackrel{*}{\stackrel{*}{c}}\left\|\varepsilon_{h}\right\|_{\left(G^{\mathrm{NS}}\right)} \\
& \\
&
\end{aligned}\left\|\varepsilon_{h}\right\|_{\left(G^{\mathrm{S}}\right)}
$$

Figure 3: The errors with respect to number of quadrature nodes $n$, in the "singular" part and in the "nonsingular" part by the BGM when 9-point scheme is used with $S^{6}$ for $h^{-1}=32,64$.

The Discrete Fast Fourier Transform is used for the realization of the finite sums in (27). The solution of the problem (26), for $j=2,3,4$ can be represented analogously.

Now we describe the algorithm of implementing the BGM for the slit problem.

Step 1. Suppose that we have zero approximation $u_{h}^{k(0)}$ to the exact solution $u_{h}^{k}$ of (11)-(14).


Figure 4: The highly accurate approximate solution $U_{h}^{6(M)}$ and the exact solution $u$ in the "singular" part for $h^{-1}=64, n=140$.

TAbLE 1: The errors by BGM when 5-point scheme is used with $S^{2}$ interpolation.

| $\left(h^{-1}, n\right)$ | $\left\\|\varepsilon_{h}\right\\|_{\left(G^{\mathrm{NS})}\right.}$ | $\left\\|\varepsilon_{h}\right\\|_{\left(G^{s}\right)}$ | $\left\\|\varepsilon_{h}^{(1)}\right\\|_{\left(G^{s}\right)}$ | $\left\\|\varepsilon_{h}^{(2)}\right\\|_{\left(G^{s}\right)}$ | iter |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $(16,85)$ | $6.138 \times 10^{-5}$ | $2.3127 \times 10^{-5}$ | $6.432 \times 10^{-5}$ | $4.464 \times 10^{-4}$ | 12 |
| $(16,100)$ | $5.264 \times 10^{-5}$ | $1.393 \times 10^{-5}$ | $5.126 \times 10^{-5}$ | $3.244 \times 10^{-4}$ | 12 |
| $(16,120)$ | $5.599 \times 10^{-5}$ | $1.150 \times 10^{-5}$ | $5.385 \times 10^{-5}$ | $3.430 \times 10^{-4}$ | 12 |
| $(32,85)$ | $1.317 \times 10^{-5}$ | $4.676 \times 10^{-6}$ | $1.129 \times 10^{-5}$ | $2.272 \times 10^{-5}$ | 13 |
| $(32,100)$ | $1.488 \times 10^{-5}$ | $1.889 \times 10^{-6}$ | $2.056 \times 10^{-5}$ | $1.523 \times 10^{-4}$ | 13 |
| $(32,120)$ | $1.491 \times 10^{-5}$ | $1.956 \times 10^{-6}$ | $2.053 \times 10^{-5}$ | $1.740 \times 10^{-4}$ | 13 |
| $(32,130)$ | $1.508 \times 10^{-5}$ | $5.172 \times 10^{-6}$ | $1.728 \times 10^{-5}$ | $4.659 \times 10^{-5}$ | 13 |
| $(32,140)$ | $1.571 \times 10^{-5}$ | $3.319 \times 10^{-6}$ | $2.407 \times 10^{-5}$ | $2.155 \times 10^{-4}$ | 13 |
| $(64,130)$ | $3.720 \times 10^{-6}$ | $7.391 \times 10^{-7}$ | $2.306 \times 10^{-6}$ | $1.941 \times 10^{-5}$ | 14 |
| $(64,140)$ | $3.583 \times 10^{-6}$ | $7.071 \times 10^{-7}$ | $3.852 \times 10^{-6}$ | $3.364 \times 10^{-5}$ | 14 |

Step 2. Finding $u_{h}^{k(1)}$ by the formula (13) on $t_{l}^{h}$ we solve the system (11), (12) on each grid $\bar{\Pi}_{l}^{h}$ by using the representation of finite difference solution described before Step 1.

Step 3. Using (6) we calculate the values $u_{h}^{k(1)}\left(0.93, \theta^{q}\right)$ at the quadrature nodes for each $\theta^{q}=(q-1 / 2) 2 \pi / n, 1 \leq q \leq n$ by the formula (14).

Step 4. Repeating Steps 2 and 3 we have the sequence $u_{h}^{k(1)}, u_{h}^{k(2)}, \ldots$, of Schwarz's iterations defined as follows:

$$
\begin{aligned}
& u_{h}^{k(m)}(r, \theta)= \frac{2 \pi}{n} \sum_{q=1}^{n} \\
& R_{1}\left(r, \theta, \theta^{q}\right) \\
& \times(r, \theta) u_{h}^{k(m-1)}\left(0.93, \theta^{q}\right) \quad \text { on } t_{l}^{h}, \\
& u_{h}^{k(m)}= S^{k} u_{h}^{k(m-1)} \quad \text { on } V^{n},
\end{aligned}
$$

$$
\begin{gather*}
u_{h}^{k(m)}=B_{k} u_{h}^{k(m)} \quad \text { on } \Pi_{l}^{h}, \\
u_{h}^{k(m)}=\varphi \quad \text { on } \eta_{l 1}^{h}, k=2,6,1 \leq l \leq 5, m=1,2, \ldots \tag{30}
\end{gather*}
$$

As a stopping criteria of the Schwarz's iterations (30), we use the inequality $\max _{\eta_{10}, l=1,2, \ldots, 5}\left|u_{h}^{k(m)}-u_{h}^{k(m-1)}\right| \leq \epsilon$ for the prescribed accuracy of $\epsilon>0$.

Step 5. Let $u_{h}^{k(M)}\left(0.93, \theta^{q}\right), \theta^{q}=(q-1 / 2) 2 \pi / n, 1 \leq q \leq n$, in (14) be the values at the quadrature nodes on $V^{n}$ for the final iteration $m=M$. Using these values we can calculate the value of the solution at any point in the singular part by the explicit formula

$$
\begin{equation*}
U_{h}^{k(M)}(r, \theta)=\frac{2 \pi}{n} \sum_{q=1}^{n} R_{1}\left(r, \theta, \theta^{q}\right) u_{h}^{k(M)}\left(0.93, \theta^{q}\right) \tag{31}
\end{equation*}
$$

TABLE 2: The errors by BGM when 9-point scheme is used with $S^{6}$ interpolation.

| $\left(h^{-1}, n\right)$ | $\left\\|\varepsilon_{h}\right\\|_{\left(G^{\mathrm{NS})}\right.}$ | $\left\\|\varepsilon_{h}\right\\|_{\left(G^{\mathrm{S}}\right)}$ | $\left\\|\varepsilon_{h}^{(1)}\right\\|_{\left(G^{s}\right)}$ | $\left\\|\varepsilon_{h}^{(2)}\right\\|_{\left(G^{s}\right)}$ | iter. |
| :--- | :---: | :---: | :---: | :---: | :---: |
| $(16,100)$ | $2.741 \times 10^{-9}$ | $4.895 \times 10^{-10}$ | $6.778 \times 10^{-10}$ | $2.293 \times 10^{-9}$ | 23 |
| $(16,145)$ | $2.789 \times 10^{-9}$ | $3.786 \times 10^{-10}$ | $5.108 \times 10^{-10}$ | $1.859 \times 10^{-9}$ | 23 |
| $(32,100)$ | $4.706 \times 10^{-11}$ | $7.158 \times 10^{-12}$ | $2.593 \times 10^{-10}$ | $4.691 \times 10^{-9}$ | 24 |
| $(32,125)$ | $4.805 \times 10^{-11}$ | $2.694 \times 10^{-12}$ | $6.623 \times 10^{-12}$ | $4.358 \times 10^{-11}$ | 24 |
| $(32,130)$ | $4.838 \times 10^{-11}$ | $1.831 \times 10^{-12}$ | $7.257 \times 10^{-12}$ | $5.186 \times 10^{-11}$ | 24 |
| $(32,145)$ | $4.745 \times 10^{-11}$ | $3.903 \times 10^{-12}$ | $8.808 \times 10^{-12}$ | $5.379 \times 10^{-11}$ | 24 |
| $(64,125)$ | $7.856 \times 10^{-13}$ | $2.220 \times 10^{-14}$ | $1.296 \times 10^{-12}$ | $3.364 \times 10^{-11}$ | 25 |
| $(64,130)$ | $7.545 \times 10^{-13}$ | $4.097 \times 10^{-14}$ | $5.376 \times 10^{-13}$ | $4.157 \times 10^{-11}$ | 25 |
| $(64,145)$ | $7.503 \times 10^{-13}$ | $4.396 \times 10^{-14}$ | $1.614 \times 10^{-13}$ | $1.312 \times 10^{-12}$ | 25 |



Figure 5: The error $\left|U_{h}^{6(M)}-u\right|$ in the "singular" part for $h^{-1}=64$, $n=140$.

TABLE 3: The order of convergence $R_{G^{N S}}^{2, \varrho}$, and $R_{G^{S}}^{2, \varrho}$ when $h=2^{-e}$.

| $\left(2^{-\varrho}, n\right)$ | $\mathfrak{R}_{G^{\text {NS }}}^{2, \varrho}$ | $\mathfrak{R}_{G^{S}}^{2, \varrho}$ |
| :--- | :--- | :---: |
| $\left(2^{-4}, 85\right)$ | 4.659 | 4.9459 |
| $\left(2^{-5}, 85\right)$ |  |  |
| $\left(2^{-4}, 100\right)$ | 3.5376 | 7.3742 |
| $\left(2^{-5}, 100\right)$ |  |  |
| $\left(2^{-4}, 120\right)$ | 3.7551 | 5.8793 |
| $\left(2^{-5}, 120\right)$ |  |  |
| $\left(2^{-5}, 130\right)$ | 4.0756 | 6.9976 |
| $\left(2^{-6}, 130\right)$ |  |  |
| $\left(2^{-5}, 140\right)$ | 4.3845 | 4.6938 |
| $\left(2^{-6}, 140\right)$ |  |  |

## 5. Numerical Results

The computational algorithm in Section 4 is applied and the implementation of the block-grid method is carried out using double precision. Let $\varepsilon_{h}=U_{h}^{k(M)}-u, \varepsilon_{h}^{(1)}=r^{1 / 2}\left(\left(\partial U_{h}^{k(M)} / \partial x\right)-\right.$ $(\partial u / \partial x)), \varepsilon_{h}^{(2)}=r^{3 / 2}\left(\left(\partial^{2} U_{h}^{k(M)} / \partial x^{2}\right)-\left(\partial^{2} u / \partial x^{2}\right)\right)$ be the errors in the "singular" part and $\varepsilon_{h}=u_{h}^{k(M)}-u$ be the error in the "nonsingular" part of the domain $G$.

Table 4: The order of convergence $R_{G^{N S}}^{6, \varrho}$, and $R_{G^{S}}^{6, \varrho}$ when $h=2^{-\varrho}$.

| $\left(2^{-\varrho}, n\right)$ | $\Re_{G^{\text {VS }}}^{6, \varrho}$ | $\Re_{G^{6}}^{6, \varrho}$ |
| :--- | :---: | :---: |
| $\left(2^{-4}, 100\right)$ | 58.253 | 68.386 |
| $\left(2^{-5}, 100\right)$ |  |  |
| $\left(2^{-4}, 145\right)$ | 58.788 | 97.005 |
| $\left(2^{-5}, 145\right)$ |  |  |
| $\left(2^{-5}, 125\right)$ | 61.164 | 65.711 |
| $\left(2^{-6}, 125\right)$ | 64.140 | 74.926 |
| $\left(2^{-5}, 130\right)$ |  |  |
| $\left(2^{-6}, 130\right)$ | 63.241 | 88.92 |
| $\left(2^{-5}, 145\right)$ |  |  |
| $\left(2^{-6}, 145\right)$ |  |  |

In Table 1 the errors are given by the BGM when 5-point scheme with $S^{2}$ interpolation is used, and the iterations are terminated by using $\epsilon=5 \times 10^{-8}$. Table 2 represents the errors by the BGM when 9 -point scheme is used with $S^{6}$ and the stopping criteria for the Schwarz's iterations is taken as $\epsilon=$ $5 \times 10^{-14}$.

The order of convergence in the "nonsingular" part, and the order of convergence in the "singular" part of $G$ are

$$
\begin{align*}
& \boldsymbol{R}_{G^{\mathrm{SS}}}^{k, \varrho}=\frac{\max _{G^{\mathrm{NS}}}\left|u_{2^{-e}}^{k(M)}-u\right|}{\max _{G^{\mathrm{NS}}}\left|u_{2^{(e+1)}}^{k(M)}-u\right|}, \\
& \boldsymbol{R}_{G^{s}}^{k, \varrho}=\frac{\max _{G^{s}}\left|U_{2^{-e}}^{k(M)}-u\right|}{\max _{G^{\mathrm{S}}}\left|U_{2^{-(e+1)}}^{k(M)}-u\right|}, \tag{32}
\end{align*}
$$

respectively, where $\varrho$ is a positive integer, $M$ is the final iteration number (Section 4), $k=2,6$. Taking $h=2^{-\varrho}$, $\varrho=4,5,6$, Tables 3 and 4 represent the order of convergence of the BGM in the "nonsingular" part and the "singular" part of the domain $G$ for $k=2$ and $k=6$, respectively.

The obtained numerical results in Tables 3 and 4 show that the order of convergence of the approximate solution is $O\left(h^{2}\right)$ for the 5-point scheme with $S^{2}$ interpolation $(k=2)$ and it is $O\left(h^{6}\right)$ for the 9-point scheme with $S^{6}$ interpolation $(k=6)$ in the "nonsingular" part. In both tables, the order of convergence in the "singular" part is higher than the order of convergence in the "nonsingular" part of the domain, which


Figure 6: The first partial derivatives $\partial U_{h}^{6(M)} / \partial x$ and $\partial U_{h}^{6(M)} / \partial y$ in the "singular" part for $h^{-1}=64, n=140$.


Figure 7: The second partial derivative $\partial^{2} U_{h}^{6(M)} / \partial x^{2}$ in the "singular" part for $h^{-1}=64, n=140$.
justifies the estimation (19) in Theorem 4. The errors with respect to the number of quadrature nodes $n$ in the "singular" part and in the "nonsingular" part by BGM for $k=2$, and $k=6$ are given in Figures 2 and 3, respectively. These figures demonstrate that the error in the "singular" part is less than the error in the "nonsingular" part for sufficiently large $n$ as it follows from the estimation (19) in Theorem 4. The graphical results in Figures 4-9 are obtained by the BGM when 9-point scheme is used with $S^{6}$ interpolation for $h^{-1}=64, n=140$. In Figure 4, the highly accurate approximate solution $U_{h}^{6(M)}$ and the exact solution $u$ is illustrated. Figure 5 represents the decrease of the error function $\left|U_{h}^{6(M)}-u\right|$ in the "singular" part of the domain as $r$ approaches to zero, which agrees with


Figure 8: The second partial derivative $\partial^{2} U_{h}^{6(M)} / \partial y^{2}$ in the "singular" part for $h^{-1}=64, n=140$.
the estimation (19) in Theorem 4. Moreover, on the "singular" part, up to second order derivatives of the solution at grid points are approximated effectively by a simple differentiation of the function (31). The shapes of the first partial derivatives $\partial U_{h}^{6(M)} / \partial x, \partial U_{h}^{6(M)} / \partial y$ are demonstrated in Figure 6 and the shapes $\left(\partial^{2} U_{h}^{6(M)} / \partial x^{2}\right)\left(\partial^{2} U_{h}^{6(M)} / \partial y^{2}\right),\left(\partial^{2} U_{h}^{6(M)} / \partial x \partial y\right)$ are given in Figures 7, 8, and 9, respectively, to show the singular behaviour of the solution around the singular point.
5.1. Stress Intensity Factor. In engineering problems a very important constant is the so-called stress intensity factor $\sigma$. This constant gives a measure of "the amount of torsion the beam can endure before fracture occurs" $[10,13]$. On the basis


Figure 9: The mixed partial derivative $\partial^{2} U_{h}^{6(M)} / \partial x \partial y$ in the "singular" part for $h^{-1}=64, n=140$.

Table 5: The error of the stress intensity factor for fixed $n=140$.

| $h^{-1}$ | $\left\|\sigma_{2}-\sigma\right\|$ | $\left\|\sigma_{6}-\sigma\right\|$ |
| :--- | :---: | :---: |
| 16 | $3.4791 \times 10^{-6}$ | $3.7655 \times 10^{-10}$ |
| 32 | $3.0364 \times 10^{-6}$ | $1.9451 \times 10^{-12}$ |
| 64 | $1.4899 \times 10^{-8}$ | $5.9952 \times 10^{-15}$ |

of (31) we give a simple and highly accurate formula for the stress intensity factor $\sigma$ denoting by $\sigma_{k}$ for $k=2,6$ :

$$
\begin{align*}
\sigma_{k} & =\lim _{r \rightarrow 0} \frac{U_{h}^{k(M)}(r, \pi)}{r^{1 / 2}} \\
& =\frac{2}{n \sqrt{(0.93)}} \sum_{q=1}^{n} u_{h}^{k(M)}\left(0.93, \theta^{q}\right) \sin \frac{\theta^{q}}{2} \tag{33}
\end{align*}
$$

where $\theta^{q}=(q-1 / 2) 2 \pi / n$ and $M$ is the final iteration number. The exact value of the stress intensity factor $\sigma$ is $\sqrt{2}$. For fixed number of quadrature nodes $n=140$, the second column in Table 5 represents the error of the stress intensity factor when 5-point scheme is used with $S^{2}$ and the last column represents this error when 9-point scheme is used with $S^{6}$.

## 6. Conclusion

For the solution of the Laplace equation on polygons with a slit, the real characteristics of the block-grid method is investigated. The given polygon is decomposed into five overlapping rectangles and one sector. In the sector, we approximate the special integral representation of the solution, which takes into account the behaviour of the exact solution near the end point of the slit. On the rectangles, to approximate Laplace's equation on square grids either 5-point scheme is used which is simpler by means of sparsity, or 9-point scheme is used, which gives a highly accurate approximation. In correspondence with the finite difference scheme used, a gluing together of the subsystems is carried out effectively by a sufficiently simple linear interpolation $S^{2}$, or a highly accurate interpolation $S^{6}$. By choosing the step size $h=2^{-4}, 2^{-5}, 2^{-6}$,
the obtained numerical results show that the order of convergence of the approximate solution is $O\left(h^{2}\right)$ for the 5-point scheme with $S^{2}$ and it is $O\left(h^{6}\right)$ for the 9-point scheme with $S^{6}$ in the "nonsingular" part. The results also show that the order of convergence in the "singular" part is higher than the order of convergence in the "nonsingular" part of the domain. This conclusion justifies the theoretical results obtained in [2-5]. Moreover, the shapes up to the second-order derivatives of the highly accurate solution obtained by the BGM are shown to display the singular behaviour at the end point of the slit. Finally the stress intensity factor is approximated by the given highly accurate formula.

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