Journal of Function Spaces

Fixed-Point Techniques and Applications to Real World Problems

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Journal of Function Spaces
Volume 2022, Article ID 9815927, 1 page
https://doi.org/10.1155/2022/9815927

Corrigendum

Corrigendum to "Strong Convergence of a New Hybrid Iterative Scheme for Nonexpensive Mappings and Applications"

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Received 31 May 2022; Accepted 31 May 2022; Published 4 July 2022

In the article titled "Strong Convergence of a New Hybrid Iterative Scheme for Nonexpensive Mappings and Applications" [1], affiliation 3 was incorrect. The correct affiliation is "Department of Mechanical Engineering, Sejong University, Seoul 05006, South Korea," and it is corrected above.

References

[1] J. Jia, K. Shabbir, K. Ahmad, N. A. Shah, and T. Botmart, "Strong Convergence of a New Hybrid Iterative Scheme for Nonexpensive Mappings and Applications," *Journal of Function Spaces*, vol. 2022, Article ID 4855173, 11 pages, 2022.

Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4527439, 8 pages
https://doi.org/10.1155/2022/4527439

Research Article

Existence of Solutions for Nonlinear Integral Equations in Tempered Sequence Spaces via Generalized Darbo-Type Theorem

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Received 28 February 2022; Revised 12 April 2022; Accepted 15 April 2022; Published 19 May 2022

Academic Editor: Santosh Kumar

Two concepts—one of Darbo-type theorem and the other of Banach sequence spaces—play a very important and active role in ongoing research on existence problems. We first demonstrate the generalized Darbo-type fixed point theorems involving the concept of continuous functions. Keeping one of these theorems into our account, we study the existence of solutions of system of nonlinear integral equations in the setting of tempered sequence space. Moreover, a very interesting and illustrative example is designed to visualize our findings.

1. Introduction and Preliminaries

Darbo [1] constructed the fixed point theorem, and later, researchers called this widely studied theorem by his name, that is, "Darbo fixed point theorem" wherein he enforced the technique of measure of noncompactness (shortly, MNC) while Kuratowski [2] was the first who described the idea of MNC. Many researchers are employing Darbo's theorem to demonstrate the existence or solvability of several functional equations (linear or nonlinear) in conjunction with different kind of Banach sequence spaces or simply called Banach spaces. Recently, the infinite system of several kinds of differential equations was considered by Banas and Lecko [3], Mursaleen et al. [4, 5], and Mohiuddine et al. [6] to obtain the existence of solutions in the framework of Banach spaces, namely, the spaces c_0, c, ℓ_p, and ℓ_1 of null, convergent, absolutely p-summable, and absolutely summable sequences in conjunction with the Dorbo-type theorem. The reader can refer to the recent monographs [7, 8] on the normed/paranormed sequence spaces and related topics.

The integral equations play a significant contribution in diverse branches of science and engineering as well as this theory is applicable in several real life problems such as gas kinetic theory, neutron transportation, and radiation [9]. Most recently, the researchers used different kinds of integral equations (infinite system) (see [10–12]) to demonstrate existence of solutions by means of the notion of MNC, i.e., in ℓ_p [13] and in Banach space [14–16].

Suppose that $\mathfrak{E}$ is a Banach space, and suppose also that $\mathbb{B}(\theta, \hat{r}) = \{x \in \mathfrak{E} : \|x - \theta\| \leq \hat{r}\}$ is a closed ball. If $\mathfrak{X}(\neq \emptyset) \subseteq \mathfrak{E}$, then its closure and convex closure, respectively, will write by symbols $\bar{\mathfrak{X}}$ and Conv$\mathfrak{X}$. Further, $\mathfrak{M}_{\mathfrak{E}}$ will be used to denote the family of bounded (nonempty) subsets of $\mathfrak{E}$ as well as its subfamily, $\mathfrak{N}_{\mathfrak{E}}$, which consists of all relatively compact sets. The MNC is defined in [17] (see also [18]) as follows.

Definition 1. A mapping $\mathscr{G} : \mathfrak{M}_{\mathfrak{E}} \longrightarrow \mathbb{R}_+(= [0,\infty))$ is called MNC in $\mathfrak{E}$ if

(i) $\mathcal{J} \in \mathfrak{M}_{\mathfrak{E}}$, which implies $\mathcal{G}(\mathcal{J}) = 0$ gives $\mathcal{J}$ be relatively compact

(ii) $\ker\mathcal{G} = \{\mathcal{J} \in \mathfrak{M}_{\mathfrak{E}} : \mathcal{G}(\mathcal{J}) = 0\} \neq \emptyset$. Also, $\ker\mathcal{G} \subset \mathfrak{N}_{\mathfrak{E}}$

(iii) $\mathcal{J} \subseteq \mathcal{J}_1\mathcal{G}(\mathcal{J}) \leq \mathcal{G}(\mathcal{J}_1)$

(iv) $\mathcal{G}(\bar{\mathcal{J}}) = \mathcal{G}(\mathcal{J})$

(v) $\mathcal{G}(\mathrm{Conv}\mathcal{J}) = \mathcal{G}(\mathcal{J})$

(vi) $\mathcal{G}(\varsigma\mathcal{J} + (1-\varsigma)\mathcal{J}_1) \leq \varsigma\mathcal{G}(\mathcal{J}) + (1-\varsigma)\mathcal{G}(\mathcal{J}_1), \varsigma \in [0, 1]$

(vii) $\mathcal{J}_j \in \mathfrak{M}_{\mathfrak{E}}, \mathcal{J}_j = \bar{\mathcal{J}}_j, \mathcal{J}_{j+1} \subset \mathcal{J}_j$ for $j \in \mathbb{N}$ and $\lim_{j\longrightarrow\infty} \mathcal{G}(\mathcal{J}_j) = 0$, and then, $\bigcap_{j=1}^{\infty}\mathcal{J}_j \neq \emptyset$

Note that $\mathcal{J}_\infty = \bigcap_{j=1}^{\infty}\mathcal{J}_j \in \ker\mathcal{G}$. Since $\mathcal{G}(\mathcal{J}_\infty) \leq \mathcal{G}(\mathcal{J}_j)$ for any j, we infer that $\mathcal{G}(\mathcal{J}_\infty) = 0$.

Banas and Krajewska [19] proposed the generalization of classical spaces c_0, c, and ℓ_∞ with the help of tempering sequence $\alpha = (\alpha_i)_{i=1}^{\infty}$ while the tempering sequence means that α_i is positive for any $i \in \mathbb{N}$ and (α_i) is nonincreasing, and they defined c_0^α, c^α, and ℓ_∞^α which are called the tempered sequence space. Inspired by these constructions, very recently, Rebbani et al. [20] defined the tempered space ℓ_p^α as follows:

$$\mathbb{L} = \left\{\rho = (\rho_n)_{n=1}^{\infty} \in w : \sum_{n=1}^{\infty} \alpha_n^p |\rho_n|^p < \infty \text{for } 1 \leq p < \infty\right\}, \quad (1)$$

where w is the space of real or complex sequences, or simply, we shall write $\mathbb{L} :\equiv \ell_p^\alpha$. Clearly, ℓ_p^α is a Banach space endowed with

$$\|\rho\|_{\ell_p^\alpha} = \left(\sum_{n=1}^{\infty} \alpha_n^p |\rho_n|^p\right)^{1/p}. \quad (2)$$

In case of $\alpha_n = 1$ for all $n \in \mathbb{N}$, the tempered space ℓ_p^α coincides with ℓ_p, and, in addition, if $p = 1$, ℓ_p^α coincides with ℓ_1. In the same paper, they gave the Hausdorff MNC $\chi_{\ell_p^\alpha}$ for a nonempty bounded set B^α of ℓ_p^α $(1 \leq p < \infty)$ by

$$\chi_{\ell_p^\alpha}(B^\alpha) = \lim_{n\longrightarrow\infty}\left[\sup_{y\in B^\alpha}\left(\sum_{k\geq n} \alpha_k^p |y_k|^p\right)^{1/p}\right]. \quad (3)$$

We will use $C(I, \ell_p^\alpha)$ to denote the collection of all continuous mappings from $I = [0, a]$ $(a > 0)$ to ℓ_p^α, and $C(I, \ell_p^\alpha)$ is a Banach space with the norm

$$\|\rho\|_{C(I,\ell_p^\alpha)} = \sup_{s\in I}\|\rho(s)\|_{\ell_p^\alpha}, \quad (4)$$

where $\rho(s) = (\rho_n(s))_{n=1}^{\infty} \in C(I, \ell_p^\alpha)$. For any nonempty bounded set E^α of $C(I, \ell_p^\alpha)$ and for $s \in I$, one defines $E^\alpha(s) = \{\rho(s): \rho(s) \in E^\alpha\}$ and hence, its MNC is given by

$$\chi_{C(I,\ell_p^\alpha)}(E^\alpha) = \sup_{s\in I}\chi_{\ell_p^\alpha}(E^\alpha(s)). \quad (5)$$

Recall the theorem given in [1] as follows:

Theorem 2. *Suppose that $\mathcal{J}$ is a nonempty, closed, bounded, and convex subset of $\mathbb{E}$, and suppose also that $\mathfrak{S} : \mathcal{J} \longrightarrow \mathcal{J}$ is a continuous mapping, and there exists $\kappa \in [0, 1)$ satisfying*

$$\mathcal{G}(\mathfrak{S}\Lambda) \leq \kappa\mathcal{G}(\Lambda), \Lambda \subseteq \mathcal{J}. \quad (6)$$

Then, $\mathfrak{S}$ has a fixed point.

2. Dorbo-Type Fixed Point Theorems

In order to discuss our Dorbo-type theorems, we first recall the set of functions which has been recently used in [13] as follows: Consider the function $M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ such that

(1) $\max\{\vartheta_1, \vartheta_2\} \leq M(\vartheta_1, \vartheta_2)$ for $\vartheta_1, \vartheta_2 \geq 0$

(2) M is continuous and nondecreasing

(3) $M(\vartheta_1 + \vartheta_1, \nu_1 + \nu_2) \leq M(\vartheta_1, \nu_1) + M(\vartheta_2, \nu_2)$

hold. We will denote the collection of such functions by $\mathbb{M}$. The example of aforesaid kind of function is $M(\vartheta_1, \vartheta_2) = \vartheta_1 + \vartheta_2$.

Theorem 3. *Consider a Banach space $\mathbb{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathbb{E}$, and an arbitrary MNC $\mathcal{G}$. Also, consider a continuous mapping $\mathbb{T} : D \longrightarrow D$ satisfying the inequality*

$$\begin{aligned}&\alpha[M(\mathcal{G}(\mathbb{T}\mathbb{X}), \gamma(\mathcal{G}(\mathbb{T}\mathbb{X})))]\\ &\quad\leq \alpha[M(\mathcal{G}(\mathbb{X}), \gamma(\mathcal{G}(\mathbb{X})))] - \beta[M(\mathcal{G}(\mathbb{X}), \gamma(\mathcal{G}(\mathbb{X})))],\end{aligned} \quad (7)$$

for any $\mathbb{X}(\neq\emptyset) \subseteq D$, where $M \in \mathbb{M}$ and $\alpha, \beta, \gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are functions such that α, γ are continuous on $\mathbb{R}_+$ and β is lower semicontinuous which satisfies the relations

$$\beta(0) = 0 \text{ and } \beta(x) > 0\,(x > 0). \quad (8)$$

Then,

$$\mathbb{T} \text{ has at least one fixed point in } D. \quad (9)$$

Proof. Consider a sequence $\{D_n\}_{n=1}^{\infty}$ such that $D_1 = D$ and $D_{n+1} = \mathrm{Conv}(\mathbb{T}D_n)$ for $n \in \mathbb{N}$. One can find that $\mathbb{T}D_1 = \mathbb{T}D \subseteq D = D_1, D_2 = \mathrm{Conv}(\mathbb{T}D_1) \subseteq D = D_1$. We obtain in a similar way that $D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots \supseteq D_n \supseteq D_{n+1} \supseteq \cdots$. If there exists $n_0 \in \mathbb{N}$ satisfying $\mathcal{G}(D_{n_0}) = 0$, then D_{n_0} is a compact set. With a view of Schauder theorem [21], $\mathbb{T}$ has a fixed point in $D \subseteq \mathbb{E}$.

Further, assume that $\mathcal{G}(D_n) > 0$ for $n \in \mathbb{N}$. Clearly, $\{\mathcal{G}(D_n)\}_{n=1}^{\infty}$ is nonnegative, decreasing, and bounded below sequence. Therefore, $\{\mathcal{G}(D_n)\}_{n=1}^{\infty}$ is convergent and

$$\lim_{n\longrightarrow\infty} \mathcal{G}(D_n) = r \geq 0, say. \tag{10}$$

Inequality (7) gives

$$\begin{aligned}\alpha[M(\mathcal{G}(D_{n+1}), \gamma(\mathcal{G}(D_{n+1})))] &= \alpha[M(\mathcal{G}(\mathrm{Conv}\mathbb{T}D_n), \gamma(\mathcal{G}(\mathrm{Conv}\mathbb{T}D_n)))] \\ &= \alpha[M(\mathcal{G}(\mathbb{T}D_n), \gamma(\mathcal{G}(\mathbb{T}D_n)))] \\ &\leq \alpha[M(\mathcal{G}(D_n), \gamma(\mathcal{G}(D_n)))] - \beta[M(\mathcal{G}(D_n), \gamma(\mathcal{G}(D_n)))].\end{aligned} \tag{11}$$

If possible, assume $r > 0$. Letting $\limsup_{n\longrightarrow\infty}$ in the last inequality, one obtains

$$\begin{aligned}\limsup_{n\longrightarrow\infty} \alpha[M(\mathcal{G}(D_{n+1}), \gamma(\mathcal{G}(D_{n+1})))] &\leq \limsup_{n\longrightarrow\infty} \alpha[M(\mathcal{G}(D_n), \gamma(\mathcal{G}(D_n)))] \\ &\quad - \limsup_{n\longrightarrow\infty} \beta[M(\mathcal{G}(D_n), \gamma(\mathcal{G}(D_n)))],\end{aligned} \tag{12}$$

which yields

$$\alpha[M(r, \gamma(r))] \leq \alpha[M(r, \gamma(r))] - \beta[M(r, \gamma(r))]. \tag{13}$$

It follows from the inequality (13) that

$$\beta[M(r, \gamma(r))] \leq 0. \tag{14}$$

Consequently, we get $\beta[M(r, \gamma(r))] = 0$. So, $\gamma(r) = r = 0$. Therefore, we have

$$\lim_{n\longrightarrow\infty} \mathcal{G}(D_n) = 0. \tag{15}$$

Using the fact $D_n \supseteq D_{n+1}$ and Definition 1, we fairly have

$$D_\infty = \bigcap_{j=1}^{\infty} D_n \subseteq D, \tag{16}$$

which is nonempty, convex, closed subset of D and D_∞ is $\mathbb{T}$ invariant. By taking into account Schauder theorem [21], we conclude that (9) holds. □

Theorem 4. *Consider a Banach space $\mathbb{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathbb{E}$, and an arbitrary MNC $\mathcal{G}$. Also, consider a continuous mapping $\mathbb{T} : D \longrightarrow D$ satisfying the inequality*

$$\begin{aligned}\alpha[\mathcal{G}(\mathbb{T}\mathbb{X}) + \gamma(\mathcal{G}(\mathbb{T}\mathbb{X}))] \leq \alpha[\mathcal{G}(\mathbb{X}) + \gamma(\mathcal{G}(\mathbb{X}))] \\ - \beta[\mathcal{G}(\mathbb{X}) + \gamma(\mathcal{G}(\mathbb{X}))],\end{aligned} \tag{17}$$

for $\mathbb{X}(\neq \emptyset) \subseteq D$, where $\alpha, \beta, \gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are functions such that α, γ are continuous on $\mathbb{R}_+$ and β is lower semicontinuous satisfies relation (8). Then, (9) holds.

Proof. This result can be obtained by considering the function $M(v_1, v_2) = v_1 + v_2$ in Theorem 3. □

Theorem 5. *Consider a Banach space $\mathbb{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathbb{E}$, and an arbitrary MNC $\mathcal{G}$. Also, consider a continuous mapping $\mathbb{T} : D \longrightarrow D$ satisfies the inequality*

$$\begin{aligned}&M(\mathcal{G}(\mathbb{T}\mathbb{X}), \gamma(\mathcal{G}(\mathbb{T}\mathbb{X}))) \\ &\quad \leq \eta[M(\mathcal{G}(\mathbb{X}), \gamma(\mathcal{G}(\mathbb{X})))] \quad (\mathbb{X}(\neq \emptyset) \subseteq D, M \in \mathbb{M}),\end{aligned} \tag{18}$$

where $\gamma, \eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ are two functions such that γ is continuous and η is nondecreasing satisfying

$$\lim_{n\longrightarrow\infty} \eta^n(x) = 0 \quad (x \geq 0). \tag{19}$$

Then, (9) holds.

Proof. Consider $\{D_n\}_{n=1}^{\infty}$ such that

$$D_1 = D \text{ and } D_{n+1} = \mathrm{Conv}(\mathbb{T}D_n) \quad (n \in \mathbb{N}). \tag{20}$$

Then, we see that

$$\mathbb{T}D_1 = \mathbb{T}D \subseteq D = D_1 \text{ and } D_2 = \mathrm{Conv}(\mathbb{T}D_1) \subseteq D = D_1. \tag{21}$$

Continuing in this way, we obtain

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots \supseteq D_n \supseteq D_{n+1} \supseteq \cdots. \tag{22}$$

If there exists $n_0 \in \mathbb{N}$ satisfying the condition $\mathcal{G}(D_{n_0}) = 0$, then the set D_{n_0} is compact. By taking into account Schauder theorem [21], we conclude that (9) holds. □

We now assume $\mathcal{G}(D_n) > 0$ $(n \in \mathbb{N})$. Consequently, a sequence $\{\mathcal{G}(D_n)\}_{n=1}^{\infty}$ is decreasing and bounded below. Thus, $\{\mathcal{G}(D_n)\}_{n=1}^{\infty}$ is convergent and so

$$\lim_{n\longrightarrow\infty} \mathcal{G}(D_n) = r \geq 0, say. \tag{23}$$

With a view of (18), one writes

$$\begin{aligned}M(\mathcal{G}(D_{n+1}), \gamma(\mathcal{G}(D_{n+1}))) &= M(\mathcal{G}(\mathrm{Conv}\mathbb{T}D_n), \gamma(\mathcal{G}(\mathrm{Conv}\mathbb{T}D_n))) \\ &= M(\mathcal{G}(\mathbb{T}D_n), \gamma(\mathcal{G}(\mathbb{T}D_n))) \\ &\leq \eta[M(\mathcal{G}(D_n), \gamma(\mathcal{G}(D_n)))] \\ &\leq \eta^2[M(\mathcal{G}(D_{n-1}), \gamma(\mathcal{G}(D_{n-1})))] \cdots \cdots \\ &\leq \eta^n[M(\mathcal{G}(D), \gamma(\mathcal{G}(D)))].\end{aligned} \tag{24}$$

Suppose that $r > 0$ (if possible). We obtain by letting $n \longrightarrow \infty$ together with (19) and (23) in the inequality (24) that

$$\eta^n[M(\mathscr{G}(D), \gamma(\mathscr{G}(D)))] \longrightarrow 0, \tag{25}$$

which yields

$$M(r, \gamma(r)) = 0. \tag{26}$$

We therefore have $\gamma(r) = r = 0$, so $\lim_{n \longrightarrow \infty} \mathscr{G}(D_n) = 0$. With the help of (22), we obtain nonempty, convex, closed set $D_\infty \subseteq D$ which is $\mathbb{T}$ invariant. Hence, by Schauder theorem [21], we reach to the desired result.

Theorem 6. *Consider a Banach space $\mathbb{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathbb{E}$, and an arbitrary MNC $\mathscr{G}$. Also, consider a continuous mapping $\mathbb{T} : D \longrightarrow D$ satisfies the inequality*

$$M(\mathscr{G}(\mathbb{T}\mathbb{X}), \gamma(\mathscr{G}(\mathbb{T}\mathbb{X}))) \leq kM(\mathscr{G}(\mathbb{X}), \gamma(\mathscr{G}(\mathbb{X}))) \quad (0 \leq k < 1, \mathbb{X}(\neq \varnothing) \subseteq D, M \in \mathbb{M}), \tag{27}$$

where a function $\gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous. Then, (9) holds.

Proof. This can be easily obtained by considering

$$\eta(\tau) = k\tau \quad (0 \leq k < 1, \forall \tau \geq 0). \tag{28}$$

in Theorem 5, above. □

Theorem 7. *Consider a Banach space $\mathbb{E}$, a nonempty, closed, bounded convex set $D \subseteq \mathbb{E}$, and an arbitrary MNC $\mathscr{G}$. Also, consider a continuous mapping $\mathbb{T} : D \longrightarrow D$ having the property*

$$\begin{aligned} &\mathscr{G}(\mathbb{T}\mathbb{X}) + \gamma(\mathscr{G}(\mathbb{T}\mathbb{X})) \\ &\quad \leq k(\mathscr{G}(\mathbb{X}) + \gamma(\mathscr{G}(\mathbb{X}))) \quad (0 \leq k < 1, \mathbb{X}(\neq \varnothing) \subseteq D), \end{aligned} \tag{29}$$

where γ is a continuous function. Then, (9) holds.

Proof. By using the function $M(x, y) = x + y$, the proof is obtained as an immediate consequence of Theorem 6. □

3. Existence of Solutions for Integral Equation

We are studying the existence of solutions for an infinite system of the nonlinear integral equation which is considered as follows:

$$\Omega_n(\xi) = \mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right) (n \in \mathbb{N}), \tag{30}$$

where $\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty$, $\xi \in I = [0, a]$, $a > 0$.

To discuss the result of this section, our assumptions are as below:

(1) For $n \in \mathbb{N}$, the functions $\mathbb{F}_n : I \times C(I, \ell_p^\alpha) \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous with

$$\sum_{n \geq 1} \alpha_n^p \left|\mathbb{F}_n\left(\xi, \Omega^0, 0\right)\right|^p \longrightarrow 0 \quad (\forall \xi \in I), \tag{31}$$

where

$$\begin{aligned} &\Omega^0 = \left(\Omega_n^0(\xi)\right)_{n=1}^\infty \in C\left(I, \ell_p^\alpha\right) \quad \text{and} \\ &\Omega_n^0(\xi) = 0 \quad (\forall n \in \mathbb{N}, t \in I). \end{aligned} \tag{32}$$

Moreover, these exist continuous functions $A_n, B_n : I \longrightarrow \mathbb{R}_+$ such that the inequality

$$\begin{aligned} &\left|\mathbb{F}_n(\xi, \Omega(\xi), p) - \mathbb{F}_n\left(\xi, \bar{\Omega}(\xi), \bar{p}\right)\right|^p \\ &\quad \leq A_n(\xi)\left|\Omega_n(\xi) - \bar{\Omega}_n(\xi)\right|^p + B_n(\xi)|p - \bar{p}|^p \end{aligned} \tag{33}$$

holds, where $\bar{\Omega}(\xi) = (\bar{\Omega}_n(\xi))_{n=1}^\infty \in C(I, \ell_p^\alpha)$.

(2) For $n \in \mathbb{N}$, the functions $\mathbb{G}_n : I \times I \times C(I, \ell_p^\alpha) \longrightarrow \mathbb{R}$ are continuous. Also, there exists L_k satisfying

$$L_k = \sup\left\{\sum_{n \geq k}\left[\alpha_n^p B_n(\xi)\left|\int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right|^p\right] : \xi \in I\right\}. \tag{34}$$

Further,

$$\sup_{n \in \mathbb{N}} L_n = L \quad \text{and} \quad \lim_{n \longrightarrow \infty} L_k = 0 \tag{35}$$

(3) Define an operator H on $I \times C(I, \ell_p^\alpha)$ to $C(I, \ell_p^\alpha)$ as follows

$$\begin{aligned} &(\xi, \Omega(\xi)) \longrightarrow (H\Omega)(\xi) \\ &\quad = \left(\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right)_{n=1}^\infty. \end{aligned} \tag{36}$$

(4) Let

$$\sup\{A_n(\xi): \xi \in I, n \in \mathbb{N}\} = \widehat{A}, \tag{37}$$

such that $0 < 2\widehat{A}^{1/p} < 1$ and

$$\widehat{B} = \sup\left\{\sum_{n \geq 1} \alpha_n^p B_n(\xi): \xi \in I\right\}. \tag{38}$$

Theorem 8. *Under assumptions (1)–(4), the system*

$$\Omega_n(\xi) = \mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right) \quad (39)$$

has at least one solution in $C(I, \ell_p^\alpha)$, *where*

$$\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty, \quad \xi \in I = [0, a], \quad a > 0. \quad (40)$$

Proof. For arbitrary fixed $\xi \in I$,

$$\begin{aligned}
&\|\Omega(\xi)\|_{\ell_p^\alpha}^p \\
&= \sum_{n\geq 1} \alpha_n^p \left|\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right|^p \\
&\leq 2^p \sum_{n\geq 1} \alpha_n^p \left|\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right. \\
&\quad \left. - \mathbb{F}_n\left(\xi, \Omega^0, 0\right)\right|^p + 2^p \sum_{n\geq 1} \left|\mathbb{F}_n\left(\xi, \Omega^0, 0\right)\right|^p \\
&\leq 2^p \sum_{n\geq 1} \alpha_n^p \left\{ A_n(\xi)|\Omega_n(\xi)|^p + B_n(\xi)\left|\int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right|^p \right\} \\
&\leq 2^p \widehat{A} \|\Omega(\xi)\|_{\ell_p^\alpha}^p + 2^p L,
\end{aligned} \quad (41)$$

which yields

$$\|\Omega(\xi)\|_{\ell_p^\alpha}^p \leq \frac{2^p L}{1 - 2^p \widehat{A}} = r^p, say. \quad (42)$$

It follows from (42) that

$$\|\Omega(\xi)\|_{\ell_p^\alpha} \leq r, \quad (43)$$

and hence,

$$\|\Omega\|_{C\left(I, \ell_p^\alpha\right)} \leq r. \quad (44)$$

Let us define nonempty set

$$B = \left\{\Omega(\xi) \in C\left(I, \ell_p^\alpha\right) : \|\Omega\|_{C\left(I, \ell_p^\alpha\right)} \leq r, \xi \in I\right\}, \quad (45)$$

which is closed, bounded, and convex subset of $C(I, \ell_p^\alpha)$. By assumption (3) and for arbitrary fixed $\xi \in I$, we write

$$\begin{aligned}
(H\Omega)(\xi) &= \{(H_n\Omega)(\xi)\}_{n=1}^\infty \\
&= \left\{\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right\}_{n=1}^\infty \quad (\Omega(\xi) \in B).
\end{aligned} \quad (46)$$

Also,

$$\begin{aligned}
&\sum_{n=1}^\infty \alpha_n^p |(H_n\Omega)(\xi)|^p \\
&= \sum_{n\geq 1} \alpha_n^p \left|\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right|^p < \infty.
\end{aligned} \quad (47)$$

Hence, $(H\Omega)(\xi) \in \ell_p^\alpha$.
Since

$$\|(H\Omega)(\xi)\|_{\ell_p^\alpha} \leq r \Rightarrow \|H\Omega\|_{C\left(I, \ell_p^\alpha\right)} \leq r, \quad (48)$$

we have that H maps B into B. We are now claiming that H is continuous on B. For this, suppose $\epsilon > 0$ and $\Omega(\xi) = (\Omega_n(\xi))_{n=1}^\infty, \bar{\Omega}(\xi) = (\bar{\Omega}_n(\xi))_{n=1}^\infty \in B$ satisfying

$$\|\Omega - \bar{\Omega}\|_{C\left(I, \ell_p^\alpha\right)} < \frac{\epsilon}{\left(2\widehat{A}\right)^{1/p}} = \delta. \quad (49)$$

For arbitrary fixed $\xi \in I$,

$$\begin{aligned}
&\left|(H_n\Omega)(\xi) - \left(H_n\bar{\Omega}\right)(\xi)\right|^p \\
&= \left|\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right. \\
&\quad \left. - \mathbb{F}_n\left(\xi, \bar{\Omega}(\xi), \int_0^\xi \mathbb{G}_n\left(\xi, s, \bar{\Omega}(s)\right)ds\right)\right|^p \\
&\leq \widehat{A}\left|\Omega_n(\xi) - \bar{\Omega}_n(\xi)\right|^p + B_n(\xi) \\
&\quad \cdot \left|\int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds - \int_0^\xi \mathbb{G}_n\left(\xi, s, \bar{\Omega}(s)\right)ds\right|^p \\
&\leq \widehat{A}\left|\Omega_n(\xi) - \bar{\Omega}_n(\xi)\right|^p + B_n(\xi) \\
&\quad \cdot \left[\int_0^\xi \left|\mathbb{G}_n(\xi, s, \Omega(s)) - \mathbb{G}_n\left(\xi, s, \bar{\Omega}(s)\right)\right| ds\right]^p.
\end{aligned} \quad (50)$$

Considering the fact $\|\Omega - \bar{\Omega}\|_{C(I, \ell_p^\alpha)} < \delta$ and $\mathbb{G}_n$ is continuous, we get

$$\left|\mathbb{G}_n(\xi, s, \Omega(s)) - \mathbb{G}_n\left(\xi, s, \bar{\Omega}(s)\right)\right| < \frac{\varepsilon}{2^{1/p}\left(\widehat{B} + 1\right)^{1/p} a}, \quad (51)$$

and so,

$$\begin{aligned}
&\int_0^\xi \left|\mathbb{G}_n(\xi, s, \Omega(s)) - \mathbb{G}_n\left(\xi, s, \bar{\Omega}(s)\right)\right| ds \\
&< \frac{\epsilon}{2^{1/p}\left(\widehat{B} + 1\right)^{1/p} a} \int_0^\xi ds < \frac{\epsilon a}{2^{1/p}\left(\widehat{B} + 1\right)^{1/p} a} \\
&= \frac{\epsilon}{2^{1/p}\left(\widehat{B} + 1\right)^{1/p}}.
\end{aligned} \quad (52)$$

It follows from (50) and (52) that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \alpha_n^p \left|(H_n\Omega)(\xi) - \left(H_n\bar{\Omega}\right)(\xi)\right|^p \\
&\quad \le \widehat{A} \sum_{n=1}^{\infty} \alpha_n^p \left|\Omega_n(\xi) - \bar{\Omega}_n(\xi)\right|^p + \frac{\epsilon^p}{2\left(\widehat{B}+1\right)} \sum_{n=1}^{\infty} \alpha_n^p B_n(\xi) \\
&\quad < \frac{\widehat{A}\epsilon^p}{2\widehat{A}} + \frac{\epsilon^p \widehat{B}}{2\left(\widehat{B}+1\right)} < \epsilon^p,
\end{aligned} \tag{53}$$

which gives

$$\left\|(H\Omega)(\xi) - \left(H\bar{\Omega}\right)(\xi)\right\|_{\ell_p^\alpha}^p < \epsilon^p. \tag{54}$$

Therefore,

$$\left\|H\Omega - H\bar{\Omega}\right\|_{C\left(I,\ell_p^\alpha\right)} < \epsilon \text{ whenever } \left\|\Omega - \bar{\Omega}\right\|_{C\left(I,\ell_p^\alpha\right)} < \delta. \tag{55}$$

Hence, H is continuous on B.

Now, for arbitrary fixed $\xi \in I$ and $B(\Omega) = \{\Omega(\xi): \Omega(\xi) \in B\}$, we write

$$\begin{aligned}
&\chi_{\ell_p^\alpha}(H(B(\xi))) \\
&\quad = \lim_{n\longrightarrow\infty} \sup_{\Omega(\xi)\in B(\xi)} \left(\sum_{k=n}^{\infty} \alpha_k^p |\Omega_k(\xi)|^p\right)^{1/p} \\
&\quad = \lim_{n\longrightarrow\infty} \sup_{\Omega(\xi)\in B(\xi)} \left[\sum_{k=n}^{\infty} \alpha_k^p \left|\mathbb{F}_n\left(\xi, \Omega(\xi), \int_0^\xi \mathbb{G}_n(\xi, s, \Omega(s))ds\right)\right|^p\right]^{1/p} \\
&\quad \le \lim_{n\longrightarrow\infty} \sup_{\Omega(\xi)\in B(\xi)} \left[2^p \sum_{k=n}^{\infty} \alpha_k^p \left\{\widehat{A}|\Omega_k(\xi)|^p + L_k\right\}\right]^{1/p},
\end{aligned} \tag{56}$$

or

$$\chi_{\ell_p^\alpha}(H(B(\xi))) \le 2\widehat{A}^{1/p} \chi_{\ell_p^\alpha}(B(\xi)). \tag{57}$$

Operating $\sup_{\xi\in I}$ on both sides of (57), we obtain

$$\sup_{\xi\in I} \chi_{\ell_p^\alpha}(H(B(\xi))) \le 2\widehat{A}^{1/p} \sup_{\xi\in I} \chi_{\ell_p^\alpha}(B(\xi)). \tag{58}$$

We thus have

$$\chi_{C\left(I,\ell_p^\alpha\right)}(H(B)) \le 2\widehat{A}^{1/p} \chi_{C\left(I,\ell_p^\alpha\right)}(B). \tag{59}$$

As $0 < 2\widehat{A}^{1/p} < 1$, so applying Theorem 7 for $\gamma \equiv 0$ gives that H has at least one fixed point on $B \subseteq C(I, \ell_p^\alpha)$, i.e., the considered system admits a solution in $C(I, \ell_p^\alpha)$. □

Example 1. In order to demonstrate Theorem 8, we consider an infinite system of integral equation as follows:

$$\Omega_n(\xi) = \frac{\xi^3 \Omega_n(\xi)}{6(1+\xi)n^2} + \frac{1}{n^3} \int_0^\xi \frac{\cos(\Omega_n(s))}{5 + \sin\left(\sum_{j=1}^n \Omega_j(s)\right)} ds, \tag{60}$$

for $\xi \in [0, 1] = I$ and $n \in \mathbb{N}$. For this demonstration, write

$$\begin{aligned}
\mathbb{F}_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) &= \frac{\xi^3 \Omega_n(\xi)}{6(1+\xi)n^2} + \frac{p_n(\Omega(\xi))}{n^3}, \\
p_n(\Omega(\xi)) &= \int_0^\xi \frac{\cos(\Omega_n(s))}{5 + \sin\left(\sum_{j=1}^n \Omega_j(s)\right)} ds, \\
\mathbb{G}_n(\xi, s, \Omega(\xi)) &= \frac{\cos(\Omega_n(s))}{5 + \sin\left(\sum_{j=1}^n \Omega_j(s)\right)}.
\end{aligned} \tag{61}$$

Further, take $a = 1$ and let $\alpha_n = 1/n$, $n \in \mathbb{N}$. If $\Omega(\xi) \in \ell_p^\alpha$ for some fixed $\xi \in I$, then

$$\begin{aligned}
&\sum_{n\ge1} \alpha_n^p |\mathbb{F}_n(\xi, \Omega(\xi), p_n(\Omega(\xi)))|^p \\
&\quad = \sum_{n\ge1} \frac{1}{n^p} \left|\frac{\xi^3 \Omega_n(\xi)}{6(1+\xi)n^2} + \frac{1}{n^3} \int_0^\xi \frac{\cos(\Omega_n(s))}{5 + \sin\left(\sum_{j=1}^n \Omega_j(s)\right)} ds\right|^p \\
&\quad \le 2^p \sum_{n\ge1} \frac{1}{6^p n^{3p}} |\Omega_n(\xi)|^p + 2^p \sum_{n\ge1} \frac{1}{n^{4p}} \\
&\quad = \frac{1}{3^p} \sum_{n\ge1} \frac{1}{n^{3p}} |\Omega_n(\xi)|^p + 2^p \sum_{n\ge1} \frac{1}{n^{4p}} \\
&\quad \le \frac{1}{3^p} \sum_{n\ge1} \frac{1}{n^p} |\Omega_n(\xi)|^p + 2^p \sum_{n\ge1} \frac{1}{n^{4p}} \\
&\quad = \frac{1}{3^p} \|\Omega(\xi)\|_{\ell_p^\alpha}^p + 2^p \sum_{n\ge1} \frac{1}{n^{4p}} < \infty,
\end{aligned} \tag{62}$$

as the series

$$\sum_{n\ge1} \frac{1}{n^{4p}} \text{ converges for } p \ge 1. \tag{63}$$

Therefore, for arbitrary fixed $\xi \in I$, one has

$$\{\mathbb{F}_n(\xi, \Omega(\xi), p_n(\Omega(\xi)))\}_{n=1}^\infty \in \ell_p^\alpha, \tag{64}$$

and hence,

$$\{\mathbb{F}_n(\xi, \Omega(\xi), p_n(\Omega(\xi)))\}_{n=1}^\infty \in C\left(I, \ell_p^\alpha\right). \tag{65}$$

Let $\bar{\Omega}(\xi) = (\bar{\Omega}_n(\xi))_{n=1}^{\infty} \in C(I, \ell_p^{\alpha})$. Then,

$$\left|\mathbb{F}_n(\xi, \Omega(\xi), p_n(\Omega(\xi))) - \mathbb{F}_n\left(\xi, \bar{\Omega}(\xi), p_n\left(\bar{\Omega}(\xi)\right)\right)\right|^p \leq \frac{1}{3^p}\left|\Omega_n(\xi) - \bar{\Omega}_n(\xi)\right|^p + 2^p\left|p_n(\Omega(\xi)) - p_n\left(\bar{\Omega}(\xi)\right)\right|^p. \tag{66}$$

Here

$$A_n(\xi) = \frac{1}{3^p} \text{ and } B_n(\xi) = 2^p. \tag{67}$$

Also,

$$\begin{aligned} \widehat{A} &= \frac{1}{3^p} \Rightarrow 2\widehat{A}^{1/p} = \frac{2}{3} < 1, \\ \sum_{n\geq 1} \alpha_n^p &\left|F_n\left(\xi, \Omega^0, 0\right)\right|^p \longrightarrow 0 \quad (\forall \xi \in I). \end{aligned} \tag{68}$$

Again,

$$\begin{aligned} \sum_{n\geq k} \alpha_n^p B_n(\xi)|P_n(\Omega(\xi))|^p &\leq 2^p \sum_{n\geq k} \frac{1}{n^p}, \\ L_k &\leq \sup\left\{2^p \sum_{n\geq k} \frac{1}{n^p} : \xi \in I\right\}. \end{aligned} \tag{69}$$

We can find that

$$L_k \longrightarrow 0 (k \longrightarrow \infty) \text{ and } L = 2^p \sum_{n\geq 1} \frac{1}{n^p}. \tag{70}$$

Moreover, we have

$$\begin{aligned} \sum_{n\geq 1} \alpha_n^p B_n(\xi) &= 2^p \sum_{n\geq 1} \frac{1}{n^p},, \\ \widehat{B} = \sup\left\{2^p \sum_{n\geq 1} \frac{1}{n^p} : \xi \in I\right\} &= 2^p \sum_{n\geq 1} \frac{1}{n^p} < \infty. \end{aligned} \tag{71}$$

The functions F_n and G_n are continuous for all $n \in \mathbb{N}$ as well as the conditions (1)–(4) are fulfilled so with a view of Theorem 8, we reach to our conclusion that the considered system (60) admits a solution in $C(I, \ell_p^{\alpha})$.

4. Concluding Remarks

In this work, we linked three different disciplines such as the concept of measure of noncompactness (MNC), the theory of existence of solutions for functional equations, and the Banach space theory, particularly, in tempered sequence spaces. We first discussed some generalized Dorbo-type fixed point theorems by considering the arbitrary MNC and then discussed the existence of solutions for nonlinear integral equation (infinite system) by taking aforesaid newly investigated Dorbo-type theorem in tempered sequence spaces. Finally, we constructed an illustrative example by taking an integral equation to validate our result.

It is worth noting to the reader that one can obtain the results of Section 2 by taking into account another suitable function instead of $M : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and consider two dimensional integral (or fraction integral) equation to extend the results of Section 3.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing this paper. All authors read and approved the manuscript.

Acknowledgments

This project was funded by the Deanship of Scientific Research (DSR) at King Abdulaziz University, Jeddah, under grant no. (G: 735-156-1441). The authors, therefore, acknowledge with thanks DSR for technical and financial support.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2592573, 13 pages
https://doi.org/10.1155/2022/2592573

Research Article

Julia and Mandelbrot Sets of Transcendental Function via Fibonacci-Mann Iteration

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Received 22 February 2022; Revised 9 April 2022; Accepted 12 April 2022; Published 16 May 2022

Academic Editor: Jia-Bao Liu

In this paper, utilizing the Fibonacci-Mann iteration process, we explore Julia and Mandelbrot sets by establishing the escape criteria of a transcendental function, $\sin(z^n) + az + c$, $n \geq 2$; here, z is a complex variable, and a and c are complex numbers. Also, we explore the effect of involved parameters on the deviance of color, appearance, and dynamics of generated fractals. It is well known that fractal geometry portrays the complexity of numerous complicated shapes in our surroundings. In fact, fractals can illustrate shapes and surfaces which cannot be described by the traditional Euclidean geometry.

1. Introduction and Preliminaries

Let us consider the well-known Fibonacci sequence $\{f(n)\}$ defined recursively by

$$f(n+1) = f(n) + f(n-1), n \geq 1, \tag{1}$$

with the initial conditions $f(0) = f(1) = 1$. Recently, a novel iteration process, Fibonacci-Mann iteration, is introduced as

$$z_{n+1} = t_n T^{f(n)}(z_n) + (1 - t_n) z_n, \tag{2}$$

where $t_n \in [0, 1]$ and $n \in \mathbb{N}$ (see [1] for more details). It is worth mentioning here that a fixed point iteration performs a significant role in the generation of geometrical pictures of classical Julia and Mandelbrot sets (for instance, see [2–4], and the references therein). In [2], by establishing the escape criteria for a complex function

$$T(z) = \sin(z^n) + az + c, (n \geq 2), \tag{3}$$

where z is a complex variable and a and c are complex numbers; new Julia sets were studied by providing new algorithms for exploring Julia sets utilizing four distinct iterations (the Picard iteration [5], the Mann iteration [6], the Ishikawa iteration [7], and the Noor-iteration [8]). Also, the effects of change in values of parameters on the deviance of color appearance and dynamics of fractals were investigated in the sequel.

Motivated by these recent studies, our aim in this paper is to develop escape criteria for a function of the form (3) using a new algorithm via the Fibonacci-Mann iteration process (2) for visualizing the stunning fractals. It is well known that the escape criterion [9] is indispensable for exploring the Mandelbrot and Julia sets. We furnish some graphical illustrations of the generated complex fractals using the MATLAB software, algorithm, and colormap to demonstrate the variation in images and explore the effect of the involved parameters on the deviance of color, appearance, and dynamics of generated fractals. Also, we observe that as we zoom in at the edges of the petals of the Mandelbrot set, we come across the Julia set meaning thereby each point of the Mandelbrot set includes massive image data of a Julia set.

A filled Julia set is the set of complex numbers so that the orbits do not converge to a point at infinity ([10, 11]). For the polynomial $T : \mathbb{C} \longrightarrow \mathbb{C}$ of degree ≥2, we denote it by F_T, that is,

$$F_T = \left\{z \in \mathbb{C} : \{|T(z_k)|\}_{k=0}^{\infty} \text{is bounded}\right\}. \tag{4}$$

The boundary of J_T is the Julia set; that is, $J_T = \partial F_T$.

The set of parameters $c \in \mathbb{C}$ so that the filled Julia set J_{T_c} of the polynomial $T_c(z) = z^2 + c$ is connected is known as the Mandelbrot set ([12, 13]), that is,

$$M = \left\{c \in \mathbb{C} : J_{T_c} \text{is connected}\right\}, \tag{5}$$

or

$$M = \{c \in \mathbb{C} : \{|T_c(z_k)|\} \nrightarrow \infty \text{ as } k \longrightarrow \infty\}. \tag{6}$$

2. An Escape Criteria via Fibonacci-Mann Iteration Process

In this section, we establish an escape criterion for the complex transcendental function (3). We take $x_0 = x$, $y_0 = y$, $z_0 = z$, and $T(z)$ as $T_{a,c}(z)$. Suppose that

$$\begin{aligned} \left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \cdots\right| &\geq |u_1|, \\ \left|1 - \frac{y^{2n}}{3!} + \frac{y^{4n}}{5!} - \cdots\right| &\geq |u_2|, \\ \left|1 - \frac{x^{2n}}{3!} + \frac{x^{4n}}{5!} - \cdots\right| &\geq |u_3|, \end{aligned} \tag{7}$$

where $|u_i| \in (0, 1]$, $1 \leq i \leq 3$ except for the values of x, y, and z so that $|u_1| = |u_2| = |u_3| = 0$. Then, we have

$$|\sin(z^n)| = \left|z^n - \frac{z^{3n}}{3!} + \frac{z^{5n}}{5!} - \cdots\right| = |z^n|\left|1 - \frac{z^{2n}}{3!} + \frac{z^{4n}}{5!} - \cdots\right|, \tag{8}$$

and so

$$|\sin(z^n)| \geq |z^n||u_1|, \tag{9}$$

$z \in \mathbb{C}$ except for the value of z so that $|u_1| = 0$, $|u_1| \in (0, 1]$.

Theorem 1. *Let $T_{a,c}(z) = \sin(z^n) + az + c, n \geq 2, \ a, c \in \mathbb{C}$, and the sequence of iterates $\{z_k\}_{k\in\mathbb{N}}$ be the Fibonacci-Mann iteration. Suppose $\mathfrak{t} = \inf\{t_n\} > 0$ and*

$$|z| \geq |c| > \left(\frac{2(\mathfrak{T}|a| + 1)}{\mathfrak{t}|u_1|}\right)^{1/(n-1)}, \tag{10}$$

where $\mathfrak{T} = \sup\{t_n\}$. Then, we have $|z_k| \longrightarrow \infty$ as $k \longrightarrow \infty$.

Proof. Let $z_0 = z$, $T_{a,c}(z) = \sin(z^n) + az + c$. Now

$$|z_{k+1}| = \left|t_k T_{a,c}^{f(k)}(z_k) + (1 - t_k)z_k\right|. \tag{11}$$

For $k = 0$, since we have $|z| \geq |c|$ and $f(0) = 1$, considering inequality (9), we get

$$\begin{aligned} |z_1| &= \left|t_0 T_{a,c}^{f(0)}(z) + (1 - t_0)z\right| = |t_0[\sin(z^n) + az + c] \\ &\quad + (1 - t_0)z| \geq t_0|\sin(z^n)| - t_0|az| - t_0|c| - (1 - t_0)|z| \\ &= t_0|\sin(z^n)| - t_0|a||z| - t_0|c| - (1 - t_0)|z| \geq t_0|u_1||z^n| \\ &\quad - t_0|a||z| - t_0|z| - (1 - t_0)|z| = t_0|u_1||z^n| - |z|(t_0|a| \\ &\quad + t_0 + (1 - t_0)) = t_0|u_1||z^n| - |z|(t_0|a| + 1) \geq \mathfrak{t}|u_1||z^n| \\ &\quad - |z|(\mathfrak{T}|a| + 1) \geq |z|(\mathfrak{T}|a| + 1)\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right). \end{aligned} \tag{12}$$

Hence, we obtain

$$|z_1| \geq \frac{|z_1|}{\mathfrak{T}|a| + 1} \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right). \tag{13}$$

Let $k = 1$. Since $f(1) = 1$, following similar steps and using the inequality (13), we obtain

$$\begin{aligned} |z_2| &\geq |z_1|\left(\frac{\mathfrak{t}|u_1||z_1^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right) \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right) \\ &\quad \cdot \left(\frac{\mathfrak{t}|u_1||z_1^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right) \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right)^2, \end{aligned} \tag{14}$$

and so

$$|z_2| \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right)^2. \tag{15}$$

Because, by inequality (13) and the fact that $|z| \geq |c| > (2(\mathfrak{T}|a| + 1)/\mathfrak{t}|u_1|)^{1/(n-1)}$, it is easy to see that $|z_1| \geq |z|$, and this implies

$$|z_1|\left(\frac{\mathfrak{t}|u_1||z_1^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right) \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right). \tag{16}$$

Again, using the inequality $|z_1| \geq |z| \geq |c| > (2(\mathfrak{T}|a| + 1)/\mathfrak{t}|u_1|)^{1/(n-1)}$ and (14), we find $|z_2| \geq |z_1|$.

Let $k = 2$ and set $\omega_1 = T_{a,c}(z_2)$. By inequality (10), it is easy to see that

$$|z_2^{n-1}||u_1| \geq |a| + 2. \tag{17}$$

Input: $T(z) = \sin(z^n) + az + c$, where $a, c \in \mathbb{C}$ and $n = 2, 3, \cdots$; $A \subset \mathbb{C}-$ area; $K-$ maximum number of iterations; $t_n, u_1 \in (0, 1]-$ Parameters of the generalized Fibonacci-Mann iteration; *colormap*$[0..C-1]$ -color map with C colors.
Output: Julia set for area A.

for $z \in A$ **do**
  $R_1 = (2(\mathfrak{T}|a|+1)/\mathfrak{t}|u_1|)^{1/n-1}$
  $R = \max(|c|, R_1)$
  $n \geq 1$
  $z = 0$
  while $n \leq K$ **do**
    $f(0) = 1$
    $f(1) = 1$
    $f(n+1) = f(n) + f(n-1)$
    $z_{n+1} = t_n T^{f(n)}(z_n) + (1-t_n)z_n$
    if $|z_{n+1}| > R$ **then**
      break
    end if
    $n = n + 1$
  end while
  $i = \lfloor (C-1)(n/K) \rfloor$
  color z with *colormap*$[i]$
end for

Algorithm 1:Geometry of Julia set.

Input: $T(z) = \sin(z^n) + az + c$, where $a, c \in \mathbb{C}$ and $n = 2, 3, \cdots$; $A \subset \mathbb{C}-$ area; $K-$ maximum number of iterations; $t_n, u_1 \in (0, 1]-$ Parameters of the generalized Fibonacci-Mann iteration; *colormap*$[0..C-1]$ -color map with C colors.
Output: Mandelbrot set for area A.

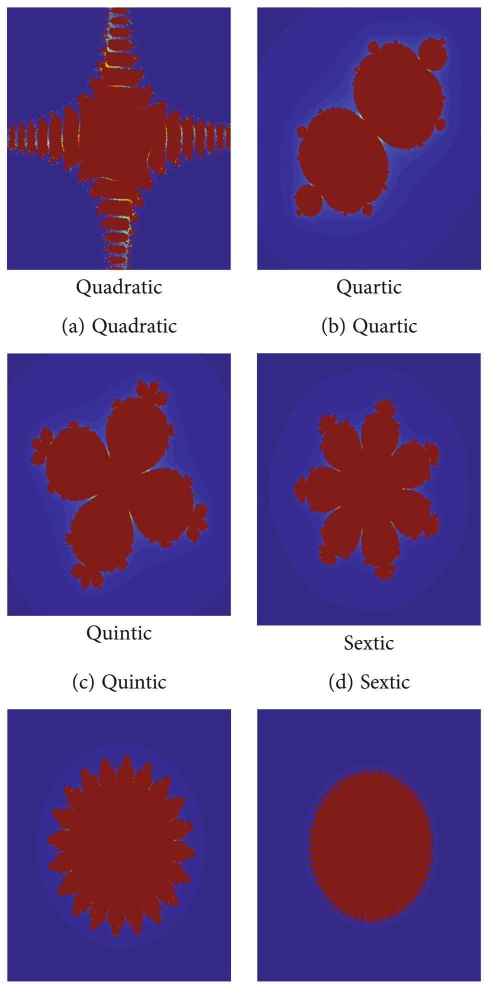
for $c \in A$ **do**
  $R_1 = (2(\mathfrak{T}|a|+1)/\mathfrak{t}|u_1|)^{1/n-1}$
  $R = \max(|c|, R_1)$
  $n \geq 1$
  while $n \leq K$ **do**
    $f(0) = 1$
    $f(1) = 1$
    $f(n+1) = f(n) + f(n-1)$
    $z_{n+1} = t_n T^{f(n)}(z_n) + (1-t_n)z_n$
    if $|z_{n+1}| > R$ **then**
      break
    end if
    $n = n + 1$
  end while
  $i = \lfloor (C-1)(n/K) \rfloor$
  color z with *colormap*$[i]$
end for

Algorithm 2: Geometry of Mandelbrot set.

Figure 1: Colormap used in the graphical examples.

Table 1: Parameters for generation of Julia set for different values of n.

	a	c	t	T	t	u_1	n
(i)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	2
(ii)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	4
(iii)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	5
(iv)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	6
(v)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	8
(vi)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	10



(a) Quadratic (b) Quartic (c) Quintic (d) Sextic (e) Octic (f) Decic

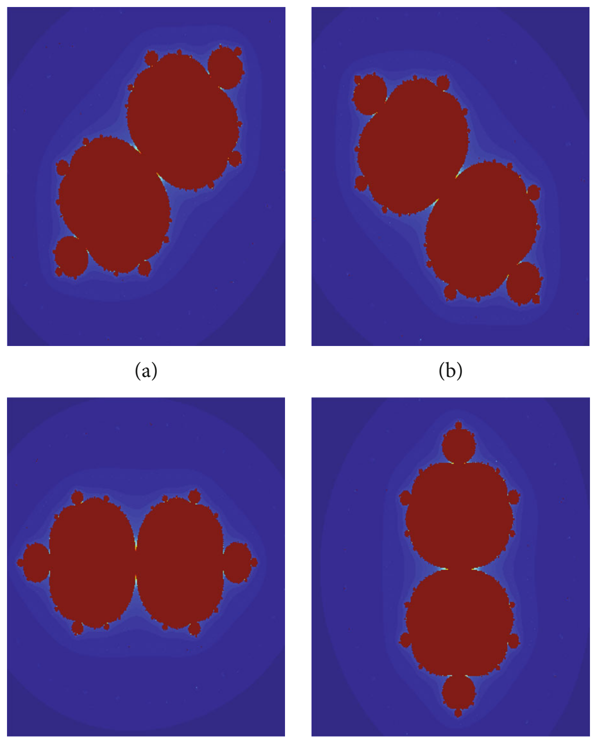
Figure 2: Effect of n on Julia set.

Using this last inequality and inequality (9), we get

$$\frac{|\omega_1|}{|z_2|}=\frac{|\sin(z_2^n)+az_2+c|}{|z_2|}\geq\frac{|\sin(z_2^n)|-|a||z_2|-|c|}{|z_2|}\geq\frac{|z_2^n||u_1|-|a||z_2|-|z_2|}{|z_2|}\geq|z_2^{n-1}||u_1|-|a|-1\geq 1, \tag{18}$$

Table 2: Parameters for generation of quartic Julia set for different values of a.

	a	c	t	T	t	u_1	n
(i)	$19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	4
(ii)	$-19i$	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	4
(iii)	-19	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	4
(iv)	19	$-0.835-0.2321i$	0.0009	0.1	0.1	0.2	4



(a) (b) (c) (d)

Figure 3: Effect of change in sign in the real and complex parameter a of quartic Julia set.

Table 3: Parameters for generation of quadratic Julia set for different values of a.

	a	c	t	T	t	u_1	n
(i)	10	3.14	0.00029901	0.0105	0.0105	0.9	2
(ii)	20	3.14	0.00029901	0.0105	0.0105	0.9	2
(iii)	$-10+50i$	3.14	0.00029901	0.0105	0.0105	0.9	2
(iv)	$50-50i$	3.14	0.00029901	0.0105	0.0105	0.9	2

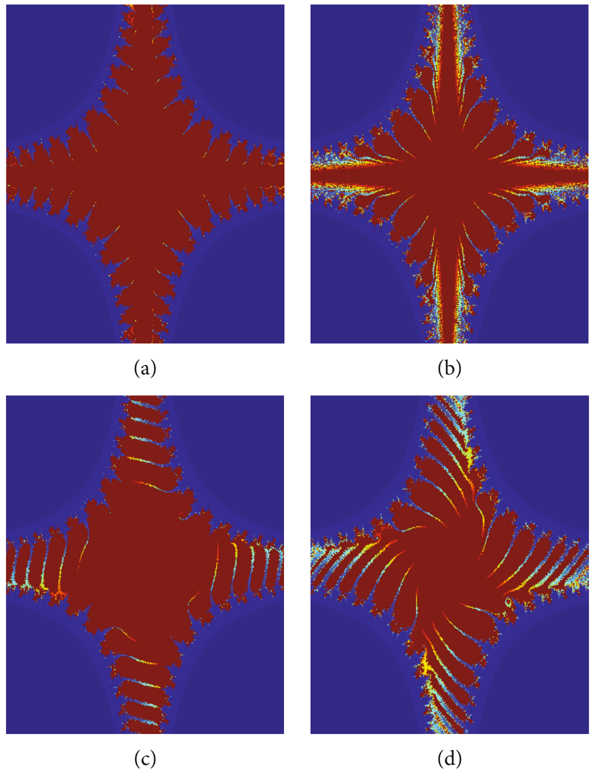


Figure 4: Effect of increase in the absolute value of a on quadratic Julia set.

Table 4: Parameters for generation of cubic Julia set for different values of a and c.

	a	c	t	T	t	u_1	n
(i)	$40 - 40i$	$-3.25 + 3.50i$	0.0019990914	0.0191	0.0191	0.012	3
(ii)	5.7	7.5	0.0019990914	0.0191	0.0191	0.012	3
(iii)	1.8	2.718	0.0019990914	0.0191	0.0191	0.012	3

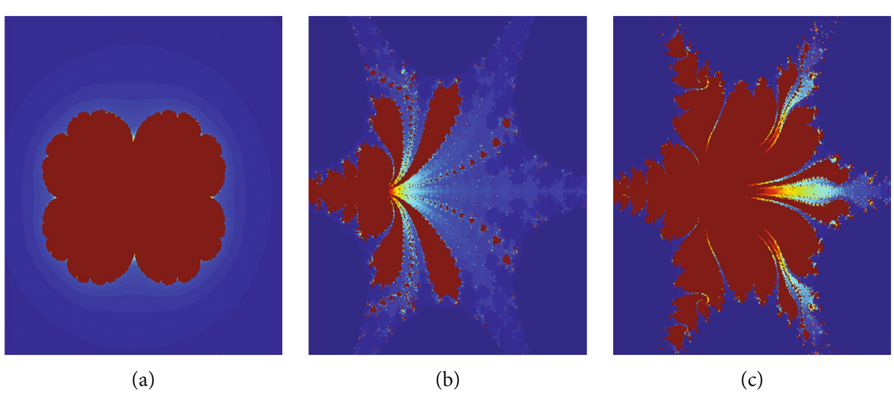
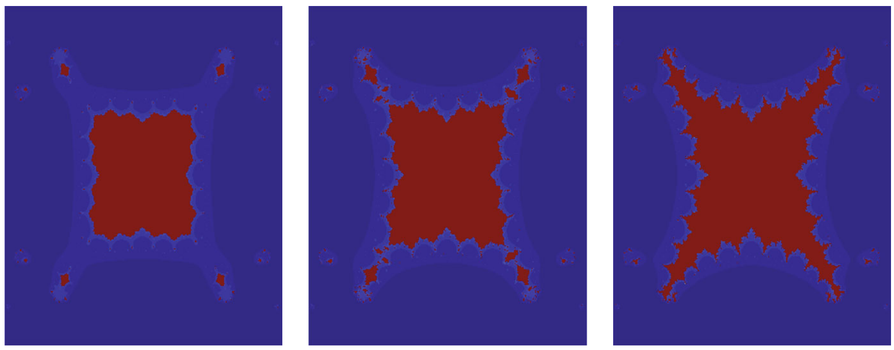


Figure 5: Effect of decrease in the absolute value of parameters a and c simultaneously on cubic Julia set.

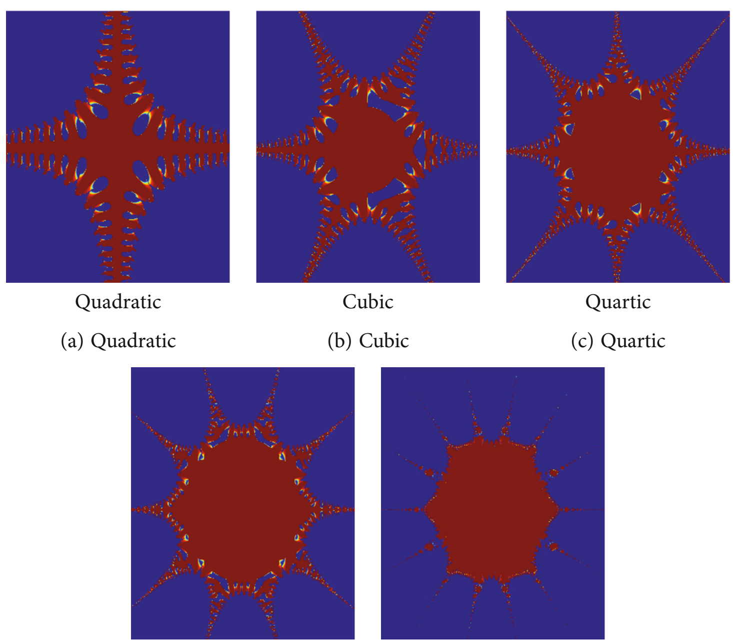
Table 5: Parameters for generation of quintic Julia set for different values of t.

	a	c	t	T	t	u_1	n
(i)	2.2	0.0035	0.35	0.115025	0.115025	0.92	5
(ii)	2.2	0.0035	0.25	0.115025	0.115025	0.92	5
(iii)	2.2	0.0035	0.20	0.115025	0.115025	0.92	5



(a) (b) (c)

Figure 6: Effect of decrease in parameter t on quintic Julia set.



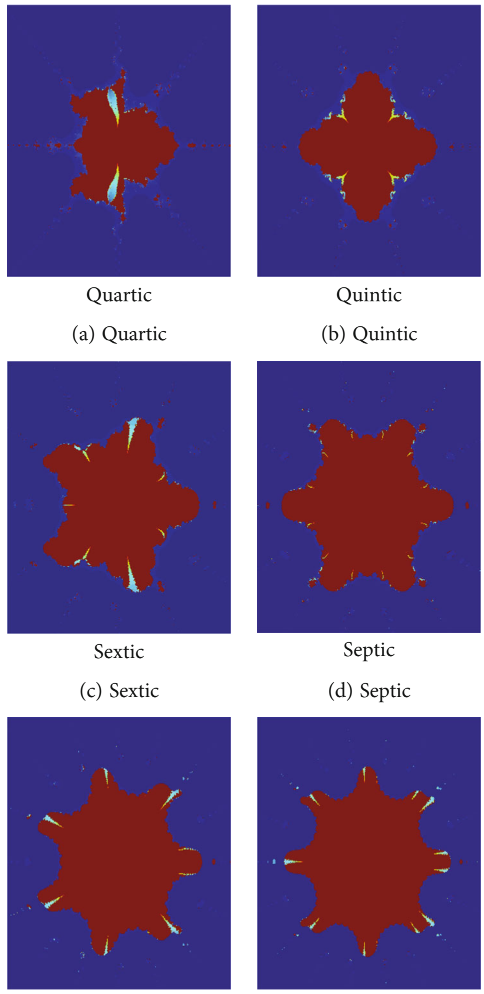
Quadratic Cubic Quartic

(a) Quadratic (b) Cubic (c) Quartic

Quintic Septic

(d) Quintic (e) Septic

Figure 7: Effect of n on Mandelbrot set.



Quartic

(a) Quartic

Quintic

(b) Quintic

Sextic

(c) Sextic

Septic

(d) Septic

Octic

(e) Octic

Nonic

(f) Nonic

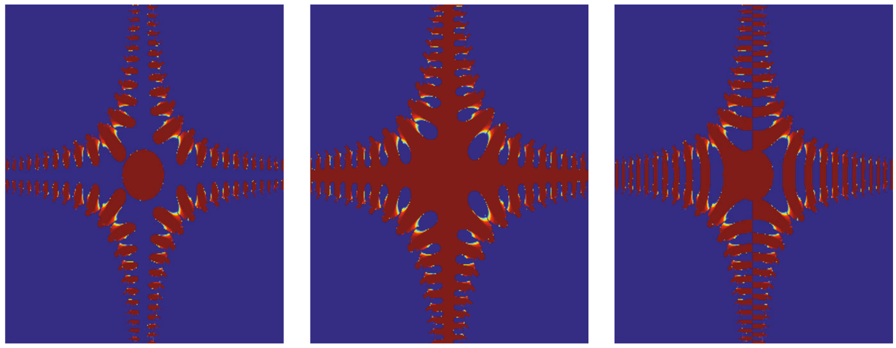
Figure 8: Effect of change in n on Mandelbrot set.

Table 6: Parameters for generation of Mandelbrot set for different values of n.

	a	t	T	t	u_1	n
(i)	-1.87897	0.000026	0.2105	0.2105	0.0932	2
(ii)	-1.87897	0.000026	0.2105	0.2105	0.0932	3
(iii)	-1.87897	0.000026	0.2105	0.2105	0.0932	4
(iv)	-1.87897	0.000026	0.2105	0.2105	0.0932	5
(v)	-1.87897	0.000026	0.2105	0.2105	0.0932	7

Table 7: Parameters for generation of Mandelbrot set for different values of n.

	a	t	T	t	u_1	n
(i)	-2.2	0.1593911	0.115025	0.115025	0.92	4
(ii)	-2.2	0.1593911	0.115025	0.115025	0.92	5
(iii)	-2.2	0.1593911	0.115025	0.115025	0.92	6
(iv)	-2.2	0.1593911	0.115025	0.115025	0.92	7
(v)	-2.2	0.1593911	0.115025	0.115025	0.92	8
(vi)	-2.2	0.1593911	0.115025	0.115025	0.92	9



(a) (b) (c)

Figure 9: Effect of change in sign as well change in real to complex parameter a on quadratic Mandelbrot set.

Table 8: Parameters for generation of quadratic Mandelbrot set for different values of a.

	a	t	T	t	u_1	n
(i)	1.87897	0.000026	0.2105	0.2105	0.0932	2
(ii)	-1.87897	0.000026	0.2105	0.2105	0.0932	2
(iii)	1.87897i	0.000026	0.2105	0.2105	0.0932	2

Table 9: Parameters for generation of cubic Mandelbrot set for different values of t.

	a	t	T	t	u_1	n
(i)	-2.2	0.059	0.115025	0.115025	0.92	3
(ii)	-2.2	0.1593911	0.115025	0.115025	0.92	3
(iii)	-2.2	0.91	0.115025	0.115025	0.92	3

and this implies

$$|\omega_1| \geq |z_2|. \tag{19}$$

Since $f(2) = 2$, we have

$$\begin{aligned} |z_3| &= \left| t_2 T_{a,c}^{f(2)}(z_2) + (1-t_2)z_2 \right| = |t_2[\sin(\omega_1^n) + a\omega_1 + c] \\ &\quad + (1-t_2)z_2| \geq t_2|\sin(\omega_1^n)| - t_2|a\omega_1| - t_2|c| - (1-t_2)|z_2| \\ &= t_2|\sin(\omega_1^n)| - t_2|a||\omega_1| - t_2|c| - (1-t_2)|z_2| \geq t_2|u_1||\omega_1^n| \\ &\quad - t_2|a||\omega_1| - t_2|\omega_1| - (1-t_2)|\omega_1| \geq t_2|u_1||\omega_1^n| \\ &\quad - |\omega_1|(t_2|a| + 1) \geq \mathfrak{t}|u_1||\omega_1^n| \\ &\quad - |\omega_1|(\mathfrak{T}|a| + 1) \geq |\omega_1|(\mathfrak{T}|a| + 1)\left(\frac{\mathfrak{t}|u_1||\omega_1^{n-1}|}{(\mathfrak{T}|a| + 1)} - 1\right), \end{aligned} \tag{20}$$

and hence,

$$|z_3| \geq \frac{|z_3|}{\mathfrak{T}|a| + 1} \geq |\omega_1|\left(\frac{\mathfrak{t}|u_1||\omega_1^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right). \tag{21}$$

Similarly, by inequalities (15), (19), and (21), we get

$$|z_3| \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right)^3. \tag{22}$$

Repeating this process till kth term, we find

$$|z_k| \geq |z|\left(\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1\right)^k. \tag{23}$$

Then, because of inequality (10), we have

$$\frac{\mathfrak{t}|u_1||z^{n-1}|}{\mathfrak{T}|a| + 1} - 1 > 1, \tag{24}$$

where $|u_1| \in (0, 1]$. This implies that the orbit of z tends to infinity; that is, we find $|z_k| \longrightarrow \infty$ as $k \longrightarrow \infty$. □

Corollary 2. *If we consider* $|c| > (2(\mathfrak{T}|a| + 1)/\mathfrak{t}|u_1|)^{1/(n-1)}$, *then the Fibonacci-Mann orbit escapes to infinity.*

Remark 3. The motivation for choosing the Fibonacci-Mann iteration method in the generation of Julia and Mandelbrot fractal sets is the fact that for $t_n \in (0, .5]$, both Mann

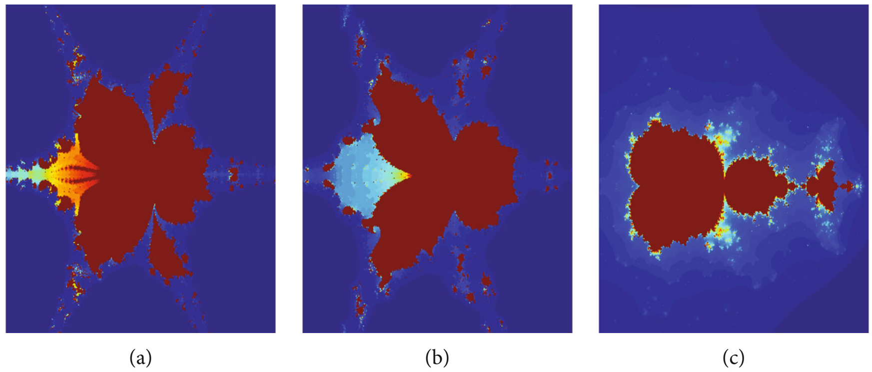


FIGURE 10: Effect of change in parameter t on cubic Mandelbrot set.

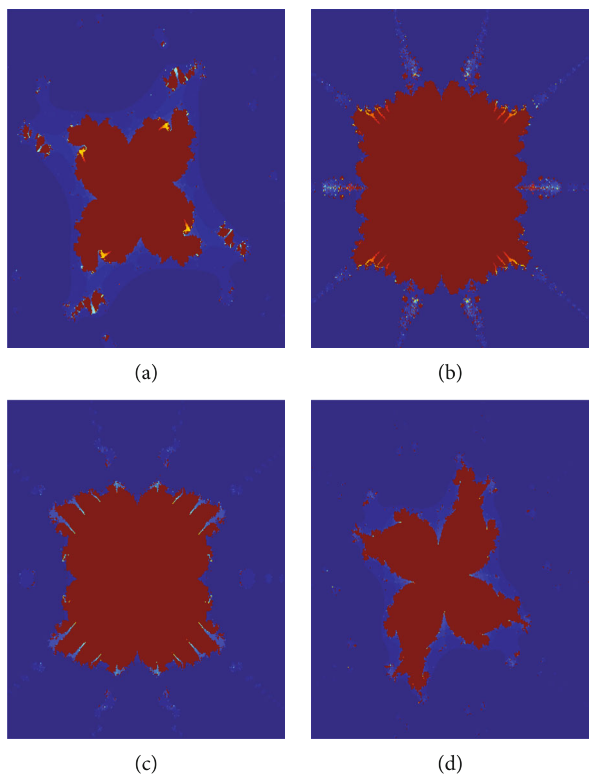
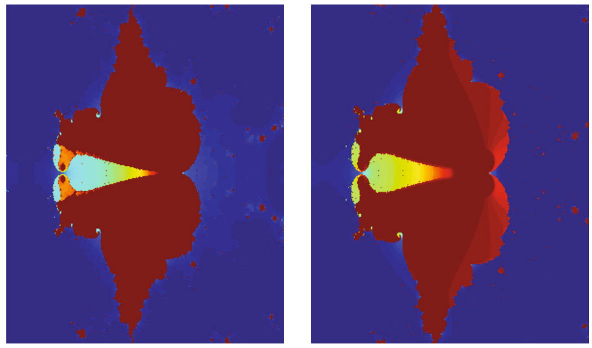


FIGURE 11: Effect of change in parameters a and t simultaneously on quintic Mandelbrot set.

TABLE 10: Parameters for generation of quintic Mandelbrot set for different values of a and t.

	a	t	T	t	u_1	n
(i)	$-2i$	0.13	0.9025	0.9025	0.92	5
(ii)	-0.5	0.1593911	0.9025	0.9025	0.92	5
(iii)	0	0.91	0.9025	0.9025	0.92	5
(iv)	$-2.2i$	0.031	0.9025	0.9025	0.92	5

iteration, as well as Fibonacci-Mann iteration, converge to a fixed point. However, the Fibonacci-Mann iteration converges faster than the Mann iteration. But for $t_n \in (0.5,1)$, Mann iteration needs not converge to a fixed point; however, the Fibonacci-Mann iteration converges for all the initial values. By taking $f(n) = 1$ in inequality (2), we get the Mann iteration [6]. Also, for $f(n) = 1$ and $t_n = 1$, we get the Picard iteration [5]. It neither reduces to Ishikawa-iteration [7], nor



(a) (b)

Figure 12: Effect of change in parameters $\mathfrak{T}$, $\mathfrak{t}$, and u_1 simultaneously on cubic Mandelbrot set.

Table 11: Parameters for generation of cubic Mandelbrot set for different values of $\mathfrak{T}$, $\mathfrak{t}$, and u_1.

	a	t	T	t	u_1	n
(i)	-5	0.0525	0.2	0.01	0.05	3
(ii)	-5	0.0525	0.3	0.05	0.005	3

to Noor-iteration [8] since Ishikawa-iteration is a two-step process and Noor-iteration is a three-step process. On the other hand, Antal et al. [2] used the Picard iteration, the Mann iteration, the Ishikawa iteration, and the Noor-iteration to explore and compare the fractals as Julia sets. It is well known that Banach [14] utilized Picard iteration [5] to approximate a fixed point for underlying contraction mapping. But when we use slightly weaker mapping, then Picard iteration needs not converge. Consequently, Mann iteration [6], Ishikawa iteration [7], Krasnosel'ski iteration [15], modified Mann iteration [16], and so on have been introduced by distinct researchers to solve this issue for different contractions.

Remark 4. In Theorem 1, we proved the conclusion by symmetry by starting with taking $k = 0$, then $k = 1, k = 2$, and repeating the process till the k^{th} term. The parameters selected have not been studied in this point of view till now and are new. We refer the interested reader to [17, 18] for a detailed information about the Fibonacci sequence. It is well-known that the golden ratio and the Fibonacci sequence have numerous applications which range from the description of plant growth, the crystallographic structure of certain solids to music, and the development of computer algorithms for searching data bases. This fascinating sequence of numbers is named after the Italian mathematician Leonardo of Pisa, later known as Fibonacci, who introduced the sequence to Western European mathematics in his 1202 book Liber Abaci. It is interesting to recall that the Fibonacci sequence is initially explored by an ancient Indian mathematician and poet Acharya Pingala (450 BC-200 BC), the author of the Chandaśāstra (the earliest known treatise on Sanskrit prosody).

3. Generation of Julia and Mandelbrot Sets

We use MATLAB 8.5.0 (R2015a) for developing fractals for transcendental complex sine function (3) via the Fibonacci-Mann iteration (2) process. We develop Algorithms 1 and 2 to explore the geometry of Julia and Mandelbrot sets, respectively. It is interesting to notice that the structure of the fractals is very much dependent on the selection of iterative processes. During the simulation process, we have obtained and analyzed many fractals but included a limited number of fractals to discuss the behavior for the different parameter values associated with it. The parameters $a, c, n, u_1, t, \mathfrak{t}$, and $\mathfrak{T}$ perform a very significant role in giving vibrant colors and exploring the characteristics of the associated Julia sets and Mandelbrot sets. Throughout the paper, we use the standard "jet" colormap (as shown in Figure 1).

3.1. Julia Set. As we change the value of n (see Table 1), keeping other parameters fixed, we get amazing fractals, which are visible in Figures 2(a)–2(f). As the value of n increases, the fractal takes a circular shape. For $n = 10$, we obtain a Julia set that is similar to a circular saw or colorful teething ring (Figure 2(f)).

The parameter a gives rotational symmetry when it is purely real (imaginary) and changes the sign. For the same set of parameters and only changing the sign of real and complex parameter a as in Table 2, the resultant fractals can be seen in Figures 3(a)–3(d).

The parameter a also adds beauty to the fractals. As the absolute value of a increases keeping other parameters the same (as in Table 3), the more aesthetic fractals can be seen (Figures 4(a)–4(d)).

The impact of change in the values of parameters a and c simultaneously (see Table 4) on the cubic Julia set can be seen in Figures 5(a)–5(c). Noticeably, cubic Julia set in Figure 5(a) is symmetrical about both the axes; however, in Figures 5(b) and 5(c), it is symmetrical only about x-axis. Changes in the values of a and c from complex to real as well as a decrease in absolute value add beauty to resulting fractals.

The parameter t is responsible for the volume of the fractal (see Table 5). Even a slight decrease in t from 0.35 to 0.20 expands the quintic Julia set which are symmetrical about x -axis as shown in Figures 6(a)–6(c).

3.2. Mandelbrot Set. Like Julia set, Mandelbrot also becomes rounded (see Figures 7 and 8) as n increases (Tables 6 and 7). Noticeably, the number of branches in Figures 7(a)–7(e) is $2n$ while the number of branches in Figures 8(a)–8(f) is $(n-1)$ (unlike Figure 7).

Figure 9 demonstrates the effect of change in sign as well change in real to the complex value of parameter a on quadratic Mandelbrot set (see Table 8).

Lower values (Table 9) of t give more beautiful, artistic, and larger fractals which are symmetrical about x-axis (Figures 10(a)–10(c)).

Figure 11 demonstrates the effect of change in parameters a and t simultaneously on the quintic Mandelbrot set (see Table 10).

Figure 12 demonstrates the effect of change in parameters $\mathfrak{T}, \mathfrak{t}$, and u_1 simultaneously on the cubic Mandelbrot set (see Table 11). Figures 12(a) and 12(b) appear like a pair of duck which are mirror images of each other.

Remark 5.

(i) During the generation of fractals, it is surprising to see that, for the same parameter set values, the effect of even minor changes in one parameter causes a major impact on the appearance of the resultant fractal. Consequently, it is significant to select appropriate parameters to obtain the desired fractal pattern.

(ii) The majority of Julia and Mandelbrot sets generated by the sine function are symmetrical about the x -axis except Figures 2(a)–2(c) and Figures 3(a) and 3(b).

(iii) The change in the sign of the value of parameters a leads to reflexive and rotational symmetry.

(iv) The Julia and Mandelbrot fractals explored in this work are aesthetic, novel, and pleasing because the complex sine function $T(z) = \sin(z^n) + az + c$ contains a lot of attributes in it. The motivation behind this is the fact that on altering the iteration process, the dynamics and behavior of the fractals are also altered, which are significant from the graphical as well as applications viewpoint.

(v) We have displayed just the zoomed kind of fractals since the transcendental function $\sin(z)$ is unbounded so that the fractals which occupy the infinite area may lie in. But due to the unboundedness of $\sin(z)$ only on a real and imaginary axis, it can be observable.

(vi) Almost all the fractals occupy the area from $[-0.1,0.1] \times [-0.1,0.1]$ to $[-10, 10] \times [-10, 10]$.

4. Conclusion

We have generated Mandelbrot and Julia sets of various transcendental complex sine functions to demonstrate the significance of the newly developed Fibonacci-Mann iteration process. We have analyzed the behavior of variants of the Julia and Mandelbrot sets for different parameter values after obtaining fascinating nonclassical variants of classical Mandelbrot and Julia fractals using the MATLAB software. We have noticed that the role of each parameter is distinct. Therefore, we have restricted our discussion to a limited type of combination of parameters. However, we have tried to cover the maximum possible combination of parameters involved in developing the algorithm (escape criterion) in the Corollary 2. Also, we have observed that as we zoom in on the edges of the petals of the Mandelbrot set, we come across the Julia set meaning thereby each point of the Mandelbrot set includes massive image data of a Julia set. Also, the size of fractals relies on the value of parameter n. As the value of n parameter increases, the area captured by the fractals decreases, and its shape becomes circular. On the other hand, the shape as well as the symmetry of each fractal relies on the values of parameters a and c. We have explored a new technique via Fibonacci-Mann iteration for visualizing the filled-in Julia and Mandelbrot sets.

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5863328, 14 pages
https://doi.org/10.1155/2022/5863328

Research Article

A Note on Orthogonal Fuzzy Metric Space, Its Properties, and Fixed Point Theorems

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Received 8 January 2022; Revised 3 March 2022; Accepted 11 April 2022; Published 11 May 2022

Academic Editor: Anita Tomar

The article generalizes the notion of orthogonal fuzzy metric space into a broader term, named as orthogonal picture fuzzy metric space. The obtained results improve and extend the idea of the orthogonal fuzzy metric space and its related results. However, this article outstretches the above-mentioned notion further into a newly defined concept, named as orthogonal picture fuzzy metric space. A detailed insight is given into the topic by presenting some fixed point results in the frame of the newly defined structure. To elaborate the results more precisely, some concrete examples are given.

1. Introduction

In 2013, Cuong [2] proposed a new concept named picture fuzzy sets (PFS), which is an extension of fuzzy sets and intuitionistic fuzzy sets. In a picture fuzzy set, each element is specified by the degree of membership, the degree of nonmembership, and degree of neutrality together with the condition that the sum of these grades should be less or equal to 1.

In this regard, Phong et al. [7] studied some compositions of picture fuzzy relations. Cuong and Hai [8] investigated main fuzzy logic operators: negations, conjunctions, disjunctions, and implications on picture fuzzy sets, and constructed the main operations for fuzzy inference processes in picture fuzzy systems. Singh [9] studied the correlation coefficients of picture fuzzy sets. Cuong et al. [10] then investigated the classification of representable picture t-norms and picture t-conorms operators for picture fuzzy sets.

Eshaghi et al. [4] presented a new generalization of the Banach fixed point theorem (BFPT) by defining the notion of orthogonal sets. The orthogonal set is a non-empty set equipped with a binary relation (called orthogonal relation) having a special structure (see [4]). The metric defined on the orthogonal set is called orthogonal metric space. The orthogonal metric space contains partially ordered metric space and graphical metric space. Hezarjaribi [5] further extended the results of [4] to orthogonal fuzzy metric space. Also, Ishtiaq et al. [6] extended the results of [4] to orthogonal neutrosophic metric space. Some more details about generalized orthogonal metric spaces have been provided by Javed et al. [11], Uddin et al. [12, 13], and Senapati et al. [14].

In this paper, we introduce orthogonal picture fuzzy metric space which generalize picture fuzzy metric space and orthogonal fuzzy metric spaces. We show that every picture fuzzy metric space is an orthogonal picture fuzzy metric space but not conversely. We investigate different conditions on the picture fuzzy to show the existence of fixed points in various types of contractions. We also present some examples in support of the obtained results. The authors intend to further widen the interesting idea of orthogonality to the intuitionistic fuzzy metric space and spherical fuzzy metric spaces. Some interesting results on the same two topics can be read in the articles [15, 16] and [17], respectively.

2. Preliminaries

Definition 1 (see [1]). A fuzzy set is a pair (W, f), where W is a non-empty set, $f : W \longrightarrow [0, 1]$ is a membership function and for each $\mathfrak{I} \in W, f(\mathfrak{I})$ is called the grade of membership of $\mathfrak{I}$ in (W, f).

Definition 2 (see [2]). A picture fuzzy set A on the universe set W is an object of the form where $Y(\partial) \in [0,1]$ is called the degree of positive membership of ∂ in A, $\mathscr{M}(\partial) \in [0,1]$ is called the "degree of neutral membership of ∂ in A," and $D(\partial) \in [0,1]$ is called the degree of negative membership of ∂ in A, and $Y(\partial), \mathscr{M}(\partial), D(\partial)$ satisfy

$$Y(\partial) + \mathscr{M}(\partial) + \mathrm{D}(\partial) \leq 1, \tag{1}$$

for all $\partial \in A$. Then,

$$\text{for all } \partial \in \mathrm{A},\ 1 - (Y(\partial) + \mathscr{M}(\partial) + \mathrm{D}(\partial)), \tag{2}$$

is called the degree of refusal membership of ∂ in A.

Definition 3 (see [3]). Suppose $W \neq \varnothing$ is an arbitrary set, assume a five tuple $(W, Y, \mathscr{M}, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, and $Y, \mathscr{M}$ are FSs on $W \times W \times (0,\infty)$. If $(W, Y, \mathscr{M}, *, \Delta)$ meet the following circumstances for all $\mathfrak{I}, \hbar, \partial \in W$ and $\pi, \wp 0$:

(B1) $Y(\mathfrak{I}, \hbar, \wp) + \mathscr{M}(\mathfrak{I}, \hbar, \wp) \leq 1$,
(B2) $Y(\mathfrak{I}, \hbar, \wp) > 0$,
(B3) $Y(\mathfrak{I}, \hbar, \wp) = 1 \Longleftrightarrow \mathfrak{I} = \hbar$,
(B4) $Y(\mathfrak{I}, \hbar, \wp) = Y(\hbar, \mathfrak{I}, \wp)$,
(B5) $Y(\mathfrak{I}, \partial, (\wp+\pi)) \geq Y(\mathfrak{I}, \hbar, \wp) * Y(\hbar, \partial, \pi)$,
(B6) $Y(\mathfrak{I}, \hbar, \bullet)$ is non decreasing (ND) function of $\mathbb{R}^+$ and $\wp Y(\mathfrak{I}, \hbar, \wp) = 1$,
(B7) $\mathscr{M}(\mathfrak{I}, \hbar, \wp) > 0$,
(B8) $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = 0 \Longleftrightarrow \mathfrak{I} = \hbar$,
(B9) $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = \mathscr{M}(\hbar, \mathfrak{I}, \wp)$,
(B10) $\mathscr{M}(\mathfrak{I}, \partial, (\wp+\pi)) \leq \mathscr{M}(\mathfrak{I}, \hbar, \wp) \Delta \mathscr{M}(\hbar, \partial, \pi)$,
(B11) $\mathscr{M}(\mathfrak{I}, \hbar, \bullet)$ is non increasing (NI) function of $\mathbb{R}^+$ and $\lim_{\wp \longrightarrow \infty} \mathscr{M}(\mathfrak{I}, \hbar, \wp) = 0$.

Then, $(W, Y, \mathscr{M}, *, \Delta)$ is an IFMS.

Definition 4. Suppose $W \neq \varnothing$, assume five tuples $(W, Y, \mathscr{M}, D, *, \Delta)$ where $*$ is a CTN, Δ is a CTCN, and $Y, \mathscr{M}, D$ are picture fuzzy set on $W \times W \times \mathbb{R}^+$. If $(W, Y, \mathscr{M}, D, *, \Delta)$ meet the following circumstances for all $\mathfrak{I}, \hbar, \partial \in W$ and $\pi, \wp > 0$:

(P1) $Y(\mathfrak{I}, \hbar, \wp) + \mathscr{M}(\mathfrak{I}, \hbar, \wp) + \mathrm{D}(\mathfrak{I}, \hbar, \wp) \leq 1$,
(P2) $0 \leq Y(\mathfrak{I}, \hbar, \wp) \leq 1$,
(P3) $Y(\mathfrak{I}, \hbar, \wp) = 1 \Longleftrightarrow \mathfrak{I} = \hbar$,
(P4) $Y(\mathfrak{I}, \hbar, \wp) = Y(\hbar, \mathfrak{I}, \wp)$,
(P5) $Y(\mathfrak{I}, \partial, \wp+\pi) \geq Y(\mathfrak{I}, \hbar, \wp) * Y(\hbar, \partial, \pi)$,
(P6) $Y(\mathfrak{I}, \hbar, \bullet)$ is non decreasing (ND) function of $\mathbb{R}^+$ and $\lim_{\wp \longrightarrow \infty} Y(\mathfrak{I}, \hbar, \wp) = 1$,
(P7) $0 \leq \mathscr{M}(\mathfrak{I}, \hbar, \wp) \leq 1$,
(P8) $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = 0 \Longleftrightarrow \mathfrak{I} = \hbar$,
(P9) $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = \mathscr{M}(\hbar, \mathfrak{I}, \wp)$,
(P10) $\mathscr{M}(\mathfrak{I}, \partial, \wp+\pi) \leq \mathscr{M}(\mathfrak{I}, \hbar, \wp) \Delta S(\hbar, \partial, \pi)$,
(P11) $\mathscr{M}(\mathfrak{I}, \hbar, \bullet)$ is non increasing (NI) function of $\mathbb{R}^+$ and $\lim_{\wp \longrightarrow \infty} \mathscr{M}(\mathfrak{I}, \hbar, \wp) = 0$,
(P12) $0 \leq \mathrm{D}(\mathfrak{I}, \hbar, \wp) \leq 1$,
(P13) $D(\mathfrak{I}, \hbar, \wp) = 0 \Longleftrightarrow \mathfrak{I} = \hbar$,
(P14) $D(\mathfrak{I}, \hbar, \wp) = D(\hbar, \mathfrak{I}, \wp)$,
(P15) $\mathrm{D}(\mathfrak{I}, \partial, \wp+\pi) \leq \mathrm{D}(\mathfrak{I}, \hbar, \wp) \Delta v(\hbar, \partial, \pi)$,
(P16) $D(\mathfrak{I}, \hbar, \bullet)$ is non increasing (NI) function of $\mathbb{R}^+$ and $\lim_{\wp \longrightarrow \infty} D(\mathfrak{I}, \hbar, \wp) = 0$,
(P17) If $\wp \leq 0$,then $Y(\mathfrak{I}, \hbar, \wp) = 0, \mathscr{M}(\mathfrak{I}, \hbar, \wp) = 1$ and $D(\mathfrak{I}, \hbar, \wp) = 1$.

Then, $(W, Y, \mathscr{M}, D, *, \Delta)$ is a PFMS.

Definition 5 (see [4]). Assume $W \neq \Phi$ and $\vdash \in W \times W$ is a binary relation. Assume there exists $\mathfrak{I}_0 \in W$ such that $\mathfrak{I}_0 \vdash \mathfrak{I}$ or $\mathfrak{I} \vdash \mathfrak{I}_0$ for all $\mathfrak{I} \in W$. Thus, W is said to be an OS. Furthermore, we denote OS by $(W, \vdash)$.

Definition 6 (see [4]). Suppose that $(W, \vdash)$ is an OS. A sequence $\{\mathfrak{I}_n\}$ for $n \in \mathbb{N}$ is called an (OS) if for all n, $\mathfrak{I}_n \vdash \mathfrak{I}_{n+1}$ or for all n, $\mathfrak{I}_{n+1} \vdash \mathfrak{I}_n$.

2.1. Orthogonal Picture Fuzzy Metric Space

Definition 7. Let $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ be called an OPFMS if W is a non-empty OS, $*$ is a CTN, Δ is a CTCN, and $Y, \mathscr{M}, D$ are pfs on $W \times W \times \mathbb{R}^+$ if the following condition are satisfied for all $\mathfrak{I}, \hbar, \partial \in W$ with either ($\mathfrak{I} \vdash \hbar$ or $\hbar \vdash \mathfrak{I}$), either ($\mathfrak{I} \vdash \hbar$ or $\delta \vdash \mathfrak{I}$), and either ($\mathfrak{I} \vdash \hbar \vee \hbar \vdash \mathfrak{I}$):

P_1) $Y(\mathfrak{I}, \hbar, \wp) + \mathscr{M}(\mathfrak{I}, \hbar, \wp) + D(\mathfrak{I}, \hbar, \wp) \leq 1$,
$P_\vdash 2)$ $0 \leq Y(\mathfrak{I}, \hbar, \wp) \leq 1$,
$P_\vdash 3)$ $Y(\mathfrak{I}, \hbar, \wp) = 1$ if and only if $\mathfrak{I} = \hbar$,
$P_\vdash 4)$ $Y(\mathfrak{I}, \hbar, \wp) = Y(\hbar, \mathfrak{I}, \wp)$,
$P_\vdash 5)$ $Y(\mathfrak{I}, \partial, \wp+\pi) \geq Y(\mathfrak{I}, \hbar, \lambda) * Y(\hbar, e, \pi)$,
$P_\vdash 6)$ $Y(\mathfrak{I}, \hbar, \bullet)$: $(0,\infty) \longrightarrow [0,1]$ is continuous,
$P_\vdash 7)$ $0 \leq \mathscr{M}(\mathfrak{I}, \hbar, \wp) \leq 1$,
$P_\vdash 8)$ $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = 1$ if and only if $\mathfrak{I} = \hbar$,
$P_\vdash 9)$ $\mathscr{M}(\mathfrak{I}, \hbar, \wp) = \mathscr{M}(\hbar, \mathfrak{I}, \wp)$,
$P_\vdash 10)$ $\mathscr{M}(\mathfrak{I}, \partial, \wp+\pi) \leq \mathscr{M}(\mathfrak{I}, \hbar, \wp) \Delta \mathscr{M}(\hbar, \partial, \pi)$,
$P_\vdash 11)$ $\mathscr{M}(\mathfrak{I}, \hbar, \bullet)$: $(0,\infty) \longrightarrow [0,1]$ is continuous,
$P_\vdash 12)$ $0 \leq D(\mathfrak{I}, \hbar, \wp) \leq 1$,
$P_\vdash 13)$ $D(\mathfrak{I}, \hbar, \wp) = 1$ if and only if $\mathfrak{I} = \hbar$,
$P_\vdash 14)$ $D(\mathfrak{I}, \hbar, \wp) = D(\hbar, \mathfrak{I}, \wp)$,
$P_\vdash 15)$ $D(\mathfrak{I}, \partial, \wp+\pi) \leq D(\mathfrak{I}, \hbar, \wp) \Delta D(\hbar, \partial, \pi)$,
$P_\vdash 16)$ $D(\mathfrak{I}, \hbar, \bullet)$: $(0,\infty) \longrightarrow [0,1]$ is continuous,
$P_\vdash 17)$ If $\wp \leq 0$ then $Y(\mathfrak{I}, \hbar, \wp) = 0, \mathscr{M}(\mathfrak{I}, \hbar, \wp) = 1$ and $D(\mathfrak{I}, \hbar, \wp) = 1$,

Then, $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ is called OPFMS.

Remark 8. Every PFMS is an OPFMS but the converse is not true.

Example 1. Let $W = [-7, 7]$ and define a CTN as $a * b = ab$, CTCN as $a \Delta b = \max \{a, b\}$ and define a binary relation $\vdash$ by $\mathfrak{I} \vdash \hbar$ iff $\mathfrak{I} + \hbar \geq 0$. Take

$$Y(\mathfrak{J},\hbar,\wp)=\begin{cases}1 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\wp}{\wp+\max\{\mathfrak{J},\hbar\}} & \text{if} \quad \text{otherwise},\end{cases}$$
$$\mathscr{M}(\mathfrak{J},\hbar,\wp)=\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\max\{\mathfrak{J},\hbar\}}{\wp+\max\{\mathfrak{J},\hbar\}} & \text{if} \quad \text{otherwise},\end{cases} \tag{3}$$
$$D(\mathfrak{J},\hbar,\wp)=\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\max\{\mathfrak{J},\hbar\}}{\wp} & \text{if} \quad \text{otherwise},\end{cases}$$

for all $\mathfrak{J},\hbar\in W$, $\wp>0$, then it is OPFS, but not an PFMS.

It is easy to see that for $\pi=\wp=1$, $\mathfrak{J}=-1$, $\hbar=-1/2$, $\partial=-2$. $(P_{\vdash}5)$, $(P_{\vdash}10)$, and $(P_{\vdash}15)$ fails.

Remark 9. The above example is also OPFMS if we take

$$\mathscr{M}(\mathfrak{J},\hbar,\wp)=\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar,\\ 1-\dfrac{\wp}{\wp+\max\{\mathfrak{J},\hbar\}} & \text{if} \quad \text{otherwise}.\end{cases} \tag{4}$$

Definition 10. An OS $\{\mathfrak{J}_n\}$ in an OPFMS $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ is said to be orthogonal convergent (O-C) to $\mathfrak{J}\in W$ if

$$\lim_{n\longrightarrow\infty} Y(\mathfrak{J}_n,\mathfrak{J},\wp)=1, \forall\wp>0,$$
$$\lim_{n\longrightarrow\infty} \mathscr{M}(\mathfrak{J}_n,\mathfrak{J},\wp)=0, \forall\wp>0, \tag{5}$$
$$\lim_{n\longrightarrow\infty} \mathrm{D}(\mathfrak{J}_n,\mathfrak{J},\wp)=0, \forall\wp>0.$$

Definition 11. An OS $\{\mathfrak{J}_n\}$ in an OPFMS $(W, Y, \mathscr{M}, *, \Delta, \vdash)$ is said to be Orthogonal Cauchy (O-CS) if there exists $n\in\mathbb{N}$ such that

$$\lim_{n\longrightarrow\infty} Y\left(\mathfrak{J}_n,\mathfrak{J}_{n+p},\wp\right)=1,$$
$$\lim_{n\longrightarrow\infty} \mathscr{M}\left(\mathfrak{J}_n,\mathfrak{J}_{n+p},\wp\right)=0, \tag{6}$$
$$\lim_{n\longrightarrow\infty} \mathrm{D}\left(\mathfrak{J}_n,\mathfrak{J}_{n+p},\wp\right)=0,$$

for all $\wp\geq0$, $p\geq1$.

Definition 12. $\Omega: W\longrightarrow W$ is OC at $\mathfrak{J}\in W$ in an OPFMS $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$, whenever for each OS $\{\mathfrak{J}_n\}$ for all $n\in\mathbb{N}$ in W if $\lim_{n\longrightarrow\infty} R(\mathfrak{J}_n,\mathfrak{J},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{J}_n,\mathfrak{J},\wp)=0$, and $\lim_{n\longrightarrow\infty} D(\mathfrak{J}_n,\mathfrak{J},\wp)=0$ for all $\wp>0$, then $\lim_{n\longrightarrow\infty} Y(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=0$, and $\lim_{n\longrightarrow\infty} D(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=0$ for all $\wp>0$.

Definition 13. An OPFMS $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ is said to be orthogonally complete (O-complete) if every O-CS is convergent.

Example 2. Assume OPFMS as given in Example 1 and define a sequence $\{\mathfrak{J}_n\}$ in W by $\mathfrak{J}_n=1-1/n, \forall n\in\mathbb{N}$ such that $(\forall n; \mathfrak{J}_n\vdash\mathfrak{J}_{n+1})$ or $(\forall n; \mathfrak{J}_{n+1}\vdash\mathfrak{J}_n)$. Define a CTN as $a*b=ab$, CTCN as $a\Delta b=\max\{a,b\}$, and define a binary relation $\vdash$ by $\mathfrak{J}\vdash\hbar$ iff $\mathfrak{J}+\hbar\geq0$. Take

$$\lim_{n\longrightarrow\infty} Y(\mathfrak{J}_n,\mathfrak{J},\wp)=\lim_{n\longrightarrow\infty}\begin{cases}1 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\wp}{\wp+\max\{\mathfrak{J}_n,\mathfrak{J}\}} & \text{if} \quad \text{otherwise},\end{cases}=\begin{cases}1 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\wp}{\wp+\max\{\mathfrak{J},\mathfrak{J}\}} & \text{if} \quad \text{otherwise},\end{cases}$$
$$\lim_{n\longrightarrow\infty} \mathscr{M}(\mathfrak{J}_n,\mathfrak{J},\wp)=\lim_{n\longrightarrow\infty}\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\max\{\mathfrak{J}_n,\mathfrak{J}\}}{\wp+\max\{\mathfrak{J}_n,\mathfrak{J}\}} & \text{if} \quad \text{otherwise},\end{cases}=\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar\\ \dfrac{\max\{\mathfrak{J},\mathfrak{J}\}}{\wp+\max\{\mathfrak{J},\mathfrak{J}\}} & \text{if} \quad \text{otherwise},\end{cases} \tag{7}$$
$$\lim_{n\longrightarrow\infty} \mathrm{D}(\mathfrak{J}_n,\mathfrak{J},\wp)=\lim_{n\longrightarrow\infty}\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar\\ \dfrac{\max\{\mathfrak{J}_j,\mathfrak{J}\}}{\wp} & \text{if} \quad \text{otherwise}\end{cases}=\begin{cases}0 & \text{if} \quad \mathfrak{J}=\hbar,\\ \dfrac{\max\{\mathfrak{J},\mathfrak{J}\}}{\wp} & \text{if} \quad \text{otherwise}.\end{cases}$$

Example 3. From proof of Example 2, $\mathfrak{I}_n = 1 - 1/n, \forall n \in \mathbb{N}$ is a O-CS in an OPFMS.

$$\lim_{n\longrightarrow\infty} Y(\mathfrak{I}_n, \mathfrak{I}_{n+p},\wp) = \lim_{n\longrightarrow\infty}\begin{cases} 1 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\wp}{\wp + \max\{\mathfrak{I}_n, \mathfrak{I}_{n+p}\}} & \text{if otherwise,}\end{cases} = \begin{cases} 1 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\wp}{\wp + \max\{\mathfrak{I}, \mathfrak{I}\}} & \text{if otherwise,}\end{cases}$$
$$\lim_{n\longrightarrow\infty} \mathscr{M}(\mathfrak{I}_n, \mathfrak{I}_{n+p},\wp) = \lim_{n\longrightarrow\infty}\begin{cases} 0 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\max\{\mathfrak{I}_n, \mathfrak{I}_{n+p}\}}{\wp + \max\{\mathfrak{I}_n, \mathfrak{I}_{n+p}\}} & \text{if otherwise,}\end{cases} = \begin{cases} 0 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\max\{\mathfrak{I},\mathfrak{I}\}}{\wp + \max\{\mathfrak{I}, \mathfrak{I}\}} & \text{if otherwise,}\end{cases} \quad (8)$$
$$\lim_{n\longrightarrow\infty} \mathrm{D}(\mathfrak{I}_n, \mathfrak{I}_{n+p},\wp) = \lim_{n\longrightarrow\infty}\begin{cases} 0 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\max\{\mathfrak{I}_n, \mathfrak{I}_{n+p}\}}{\wp} & \text{if otherwise}\end{cases} = \begin{cases} 0 & \text{if } \mathfrak{I} = \hbar, \\ \dfrac{\max\{\mathfrak{I}, \mathfrak{I}\}}{\wp} & \text{if otherwise,}\end{cases}$$

for all $\wp \geq 0, p \geq 1$.

Lemma 14. *If for some $\nu \in (0,1)$ and $\mathfrak{I}, \hbar \in W$,*

$$Y(\mathfrak{I}, \hbar,\wp) \geq Y\left(\mathfrak{I}, \hbar, \frac{\wp}{\nu}\right), \wp > 0,$$
$$\mathscr{M}(\mathfrak{I}, \hbar,\wp) \leq \mathscr{M}\left(\mathfrak{I}, \hbar, \frac{\wp}{\nu}\right), \wp > 0 \quad (9)$$
$$\mathrm{D}(\mathfrak{I}, \hbar,\wp) \leq \mathrm{D}\left(\mathfrak{I}, \hbar, \frac{\wp}{\nu}\right), \wp > 0$$

then $\mathfrak{I} = \hbar$.

Definition 15. Let $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ be an OPFMS. A map $\Omega : W \longrightarrow W$ is an orthogonal contraction if there exists $\theta \in (0,1)$ such that for every $\wp > 0$ and $\mathfrak{I}, \delta \in W$ with $\mathfrak{I} \vdash \hbar$, we have

$$Y(\psi\mathfrak{I}, \Omega\hbar, \theta\lambda) \geq Y(\mathfrak{I}, \hbar,\wp), \quad (10)$$

$$\mathscr{M}(\Omega\mathfrak{I}, \Omega\hbar, \theta\wp) \leq \mathscr{M}(\mathfrak{I}, \hbar,\wp), \quad (11)$$

$$\mathrm{D}(\Omega\mathfrak{I}, \Omega\hbar, \theta\wp) \leq \mathrm{D}(\mathfrak{I}, \hbar,\wp). \quad (12)$$

Theorem 16. *Let $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS such that*

$$\lim_{\wp\longrightarrow\infty} Y(\mathfrak{I}, \hbar,\wp) = 1, \lim_{\wp\longrightarrow\infty} \mathscr{M}(\mathfrak{I}, \hbar,\wp) = 0, \text{and} \lim_{\wp\longrightarrow\infty} \mathrm{D}(\mathfrak{I}, \hbar,\wp) = 0, \forall \mathfrak{I}, \hbar \in \mathrm{W}. \quad (13)$$

Let $\Omega : W \longrightarrow W$ be an OC, O-CON and OPR. Thus, Ω has a unique FP, say $\mathfrak{I}_* \in W$. Furthermore,

$$\lim_{n\longrightarrow\infty} Y(\Omega^n\mathfrak{I}, \mathfrak{I}_*,\wp) = 1, \lim_{n\longrightarrow\infty} \mathscr{M}(\Omega^n\mathfrak{I}, \mathfrak{I}_*,\wp) = 0, \quad (14)$$
$$\text{and} \lim_{n\longrightarrow\infty} \mathrm{D}(\Omega^n\mathfrak{I}, \mathfrak{I}_*,\wp) = 0 \forall \mathfrak{I} \in \mathrm{W} \text{ and } \wp > 0.$$

Proof. Since $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ is an O-complete PFMS, there exists $\mathfrak{I}_0 \in W$ such that

$$\mathfrak{I}_0 \vdash \hbar \text{ for all } \hbar \in W. \quad (15)$$

That is, $\mathfrak{I}_0 \vdash \Omega\mathfrak{I}_0$. Take

$$\mathfrak{I}_n = \Omega^n\mathfrak{I}_0 = \Omega\mathfrak{I}_{n-1} \text{ for all } n \in \mathscr{M}. \quad (16)$$

Since Ω is OPR, $\{\mathfrak{I}_n\}$ is an OS. Now, since Ω is an O-CON, we get

$$Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) = Y(\Omega\mathfrak{I}_n, \Omega\mathfrak{I}_{n-1}, \theta\wp) \geq Y(\mathfrak{I}_n, \mathfrak{I}_{n-1},\wp), \quad (17)$$

for all $n \in \mathscr{M}$ and $\wp > 0$. Note that Y is nondecreasing on $(0, \infty)$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned} Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n,\wp) &\geq Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) = Y(\Omega\mathfrak{I}_n, \Omega\mathfrak{I}_{n-1}, \theta\wp) \\ &\geq Y(\mathfrak{I}_n, \mathfrak{I}_{n-1},\wp) = Y(\Omega\mathfrak{I}_{n-1}, \Omega\mathfrak{I}_{n-2},\wp) \\ &\geq Y\left(\mathfrak{I}_{n-1}, \mathfrak{I}_{n-2}, \frac{\wp}{\theta}\right) \geq \cdots \geq Y\left(\mathfrak{I}_1, \mathfrak{I}_0, \frac{\wp}{\theta^n}\right), \end{aligned} \quad (18)$$

for all $n \in \mathscr{M}$ and $\wp > 0$. Thus, from (15) and $(P_{\vdash}5)$, we have

$$Y(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)\geq Y\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)*Y\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2}\right)\geq Y\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)*Y\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+2},\frac{\wp}{2^2}\right)*Y\left(\mathfrak{F}_{n+2},\mathfrak{F}_{n+3},\frac{\wp}{2^3}\right)*$$
$$\vdots$$
$$*Y\left(\mathfrak{F}_{n+\alpha-1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2^{n+\alpha}}\right)\geq Y\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2\theta^n}\right)*Y\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^2\theta^n}\right)* \tag{19}$$
$$\vdots$$
$$*Y\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^{n+\alpha}\theta^n}\right).$$

We know that $\lim_{\wp\longrightarrow\infty} Y(\mathfrak{F},\hbar,\wp)=1$, for all $\mathfrak{F},\hbar\in W$ and $\wp>0$. So, from (19) we get,

$$\lim_{n\longrightarrow\infty} Y(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)\geq 1*1*\cdots*1=1.$$
$$\mathcal{M}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\theta\wp)=\mathcal{M}(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_{n-1},\theta\wp)\leq\mathcal{M}(\mathfrak{F}_n,\mathfrak{F}_{n-1},\wp), \tag{20}$$

for all $n\in\hbar$ and $\wp>0$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned}\mathcal{M}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\wp)&\leq\mathcal{M}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\theta\wp)=\mathcal{M}(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_{n-1},\theta\wp)\\&\leq\mathcal{M}(\mathfrak{F}_n,\mathfrak{F}_{n-1},\wp)=\mathcal{M}(\Omega\mathfrak{F}_{n-1},\psi\mathfrak{F}_{n-2},\wp)\\&\leq\mathcal{M}\left(\mathfrak{F}_{n-1},\mathfrak{F}_{n-2},\frac{\wp}{\theta}\right)\leq\cdots\leq\mathcal{M}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{\theta^n}\right),\end{aligned} \tag{21}$$

for all $n\in\hbar$ and $\wp>0$. Thus, from (21) and $(P_\perp 10)$, we have

$$\begin{aligned}\mathcal{M}(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)&\leq\mathcal{M}\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathcal{M}\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2}\right)\\&\leq\mathcal{M}\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathcal{M}\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+2},\frac{\wp}{2^2}\right)\\&\quad\Delta\,\mathcal{M}\left(\mathfrak{F}_{n+2},\mathfrak{F},\frac{\wp}{2^3}\right)\Delta\cdots\Delta\,\mathcal{M}\left(\mathfrak{F}_{n+\alpha-1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2^{n+\alpha}}\right)\\&\leq\mathcal{M}\left(\mathfrak{F},\mathfrak{F}_0,\frac{\wp}{2\theta^n}\right)\Delta\,\mathcal{M}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^2\theta^n}\right)\Delta\cdots\\&\quad\Delta\,\mathcal{M}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^{n+\alpha}\theta^n}\right).\end{aligned} \tag{22}$$

We know that $\lim_{\wp\longrightarrow\infty}\mathcal{M}(\mathfrak{F},\hbar,\wp)=0$, for all $\mathfrak{F},\hbar\in W$ and $\wp>0$. So, from (22) we get,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\mathcal{M}(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)\leq 0\,\Delta\,0\,\Delta\cdots\Delta\,0=0,\\&\mathrm{D}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\theta\wp)=\mathrm{D}(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_{n-1},\theta\wp)\\&\qquad\leq\mathrm{D}(\mathfrak{F}_n,\mathfrak{F}_{n-1},\wp),\end{aligned} \tag{23}$$

for all $n\in\mathcal{M}$ and $\wp>0$. Therefore, by applying the above expression, we can deduce

$$\begin{aligned}\mathrm{D}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\wp)&\leq\mathrm{D}(\mathfrak{F}_{n+1},\mathfrak{F}_n,\theta\wp)=\mathrm{D}(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_{n-1},\theta\wp)\\&\leq\mathrm{D}(\mathfrak{F}_n,\mathfrak{F}_{n-1},\wp)=\mathrm{D}(\Omega\mathfrak{F}_{n-1},\Omega\mathfrak{F}_{n-2},\wp)\\&\leq\mathrm{D}\left(\mathfrak{F}_{n-1},\mathfrak{F}_{n-2},\frac{\wp}{\theta}\right)\leq\cdots\leq\mathrm{D}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{\theta^n}\right),\end{aligned} \tag{24}$$

for all $n\in\mathcal{M}$ and $\wp>0$. Thus, from (24) and $(P_\perp 15)$, we have

$$\begin{aligned}\mathrm{D}(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)&\leq\mathrm{D}\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2}\right)\\&\leq\mathrm{D}\left(\mathfrak{F}_n,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{F}_{n+1},\mathfrak{F}_{n+2},\frac{\wp}{2^2}\right)\\&\quad\Delta\,\mathrm{D}\left(\mathfrak{F}_{n+2},\mathfrak{F}_{n+3},\frac{\wp}{2^3}\right)\Delta..\Delta\,\mathrm{D}\left(\mathfrak{F}_{n+\alpha-1},\mathfrak{F}_{n+\alpha},\frac{\wp}{2^{n+\alpha}}\right)\\&\leq\mathrm{D}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2\theta^n}\right)\Delta\,\mathrm{D}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^2\theta^n}\right)\Delta\cdots\Delta\,\mathrm{D}\left(\mathfrak{F}_1,\mathfrak{F}_0,\frac{\wp}{2^{n+\alpha}\theta^n}\right),\end{aligned} \tag{25}$$

We know that $\lim_{\wp\longrightarrow\infty}D(\mathfrak{F},\hbar,\wp)=0$, for all $\mathfrak{F},\hbar\in W$ and $\wp>0$. So, from (25) we get,

$$\lim_{n\longrightarrow\infty}\mathrm{D}(\mathfrak{F}_n,\mathfrak{F}_{n+\alpha},\wp)\leq 0\,\Delta\,0\,\Delta\cdots\Delta\,0=0. \tag{26}$$

So, $\{\mathfrak{F}_n\}$ is a O-CS. The O-completeness of the PFMS $(W,Y,\mathcal{M},D,*,\Delta,\vdash)$ ensures that there exist $\mathfrak{F}_*\in W$ such that $Y(\mathfrak{F}_n,\mathfrak{F}_*,\wp)\longrightarrow 1$, $\mathcal{M}(\mathfrak{F}_n,\mathfrak{F}_*,\wp)\longrightarrow 0$, and $D(\mathfrak{F}_n,\mathfrak{F}_*,\wp)\longrightarrow 0$ as $n\longrightarrow+\infty$ for all $\wp>0$. Now, since Ω is an OC, $Y(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\wp)=Y(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_*,\wp)\longrightarrow 1$, $\mathcal{M}(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\wp)=\mathcal{M}(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_*,\wp)\longrightarrow 0$, and $D(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\wp)=D(\Omega\mathfrak{F}_n,\Omega\mathfrak{F}_*,\wp)\longrightarrow 0$ as $n\longrightarrow+\infty$. Now, we have

$$\begin{aligned}&Y(\mathfrak{F}_*,\psi\mathfrak{F}_*,\wp)\geq Y\left(\mathfrak{F}_*,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)*Y\left(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\frac{\wp}{2}\right),\\&\mathcal{M}(\mathfrak{F}_*,\Omega\mathfrak{F}_*,\wp)\leq\mathcal{M}\left(\mathfrak{F}_*,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathcal{M}\left(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\frac{\wp}{2}\right),\\&\mathrm{D}(\mathfrak{F}_*,\Omega\mathfrak{F}_*,\wp)\leq\mathrm{D}\left(\mathfrak{F}_*,\mathfrak{F}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{F}_{n+1},\Omega\mathfrak{F}_*,\frac{\wp}{2}\right).\end{aligned} \tag{27}$$

Taking limit as $n\longrightarrow+\infty$, we get $Y(\mathfrak{F}_*,\Omega\mathfrak{F}_*,\wp)=1*1=1$, $\mathcal{M}(\mathfrak{F}_*,\Omega\mathfrak{F}_*,\wp)=0\,\Delta\,0=0$, and $D(\mathfrak{F}_*,\Omega\mathfrak{F}_*,\wp)=0\,\Delta\,0=0$ and hence $\Omega\mathfrak{F}_*=\mathfrak{F}_*$.

Now, we show the uniqueness of the FP of the mapping Ω. Assume that $\mathfrak{F}_*$ and $\hbar_*$ are two FPs of Ω such that $\mathfrak{F}_*\neq\delta_*$. We can get

$$\mathfrak{F}_0\vdash\mathfrak{F}_*\text{ and }\mathfrak{F}_0\vdash\hbar_*. \tag{28}$$

Since D is OPR, one writes

$$\Omega^n\mathfrak{J}_0 \vdash \Omega^n\mathfrak{J}_* \text{ and } \Omega^n\mathfrak{J}_0 \vdash \Omega^n\hbar_*, \tag{29}$$

for all $n \in \mathscr{M}$. So from (10), we can derive

$$\begin{aligned} Y(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\geq Y(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \geq Y\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{\theta^n}\right), \\ Y(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\geq Y(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \geq Y\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{\theta^n}\right). \end{aligned} \tag{30}$$

Therefore,

$$\begin{aligned} Y(\mathfrak{J}_*, \hbar_*, \wp) = Y(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) &\geq Y\left(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \frac{\wp}{2}\right) \\ &* Y\left(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \frac{\wp}{2}\right) \geq Y\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{2\theta^n}\right) \\ &* Y\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 1 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{31}$$

So from (11), we can derive

$$\begin{aligned} \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\leq \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \leq \mathscr{M}\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{\theta^n}\right), \\ \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\leq \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \leq \mathscr{M}\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{\theta^n}\right). \end{aligned} \tag{32}$$

Therefore,

$$\begin{aligned} \mathscr{M}(\mathfrak{J}_*, \hbar_*, \wp) = \mathscr{M}(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) &\leq \mathscr{M}\left(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \frac{\wp}{2}\right) \Delta \mathscr{M}\left(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \frac{\wp}{2}\right) \\ &\leq \mathscr{M}\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{2\theta^n}\right) \Delta \mathscr{M}\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{33}$$

Similarly, from (12), we can derive

$$\begin{aligned} \mathrm{D}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\leq \mathrm{D}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \leq \mathrm{D}\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{\theta^n}\right), \\ \mathrm{D}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\leq \mathrm{D}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \leq \mathrm{D}\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{\theta^n}\right). \end{aligned} \tag{34}$$

Therefore,

$$\begin{aligned} \mathrm{D}(\mathfrak{J}_*, \hbar_*, \wp) &= \mathrm{D}(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) \\ &\leq \mathrm{D}\left(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \frac{\wp}{2}\right) \Delta \mathrm{D}\left(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \frac{\wp}{2}\right) \\ &\leq D\left(\mathfrak{J}_0, \mathfrak{J}_*, \frac{\wp}{2\theta^n}\right) \Delta D\left(\mathfrak{J}_0, \hbar_*, \frac{\wp}{2\theta^n}\right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \end{aligned} \tag{35}$$

So, $\mathfrak{J}_* = \hbar$; hence, $\mathfrak{J}_*$ is the unique FP. □

Corollary 17. *Assume $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS. Assume $\Omega : W \longrightarrow W$ be O-CON and OPR and if $\{\mathfrak{J}_n\}$ is an OS with $\mathfrak{J}_n \longrightarrow \mathfrak{J} \in W$, then $\mathfrak{J} \vdash \mathfrak{J}_n$ for all $n \in \mathbb{N}$. Then, Ω has a unique FP, say $\mathfrak{J}_* \in W$.*

Proof. We can similarly derive as in the proof of Theorem 16 that $\{\mathfrak{J}_n\}$ is a O-CS and so it converges to $\mathfrak{J}_* \in W$. Hence, $\mathfrak{J}_* \vdash \mathfrak{J}_n$ for all $n \in \mathbb{N}$. from (10), we can get

$$\begin{aligned} Y(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = Y(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp) \geq Y(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp\theta) \geq Y(\mathfrak{J}_*, \mathfrak{J}_n, \wp), \\ \lim_{n \longrightarrow \infty} Y(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = 1. \end{aligned} \tag{36}$$

Then, we can write

$$Y(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) \geq Y\left(\mathfrak{J}_*, \mathfrak{J}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{J}_{n+1}, \Omega\mathfrak{J}_*, \frac{\wp}{2}\right). \tag{37}$$

Taking limit as $n \longrightarrow +\infty$, we get $Y(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) = 1 * 1 = 1$ and from (11), we can get

$$\begin{aligned} \mathscr{M}(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = \mathscr{M}(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp) \leq \mathscr{M}(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp\theta) \leq \mathscr{M}(\mathfrak{J}_*, \mathfrak{J}_n, \wp), \\ \lim_{n \longrightarrow \infty} \mathscr{M}(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = 0. \end{aligned} \tag{38}$$

Then, we can write

$$\mathscr{M}(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) \leq \mathscr{M}\left(\mathfrak{J}_*, \mathfrak{J}_{n+1}, \frac{\wp}{2}\right) \Delta \mathscr{M}\left(\mathfrak{J}_{n+1}, \Omega\mathfrak{J}_*, \frac{\wp}{2}\right). \tag{39}$$

Taking limit as $n \longrightarrow +\infty$, we get

$$\mathscr{M}(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) = 0 \, \Delta \, 0 = 0, \tag{40}$$

and from (12), we can get

$$\begin{aligned} \mathrm{D}(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = \mathrm{D}(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp) \leq \mathrm{D}(\Omega\mathfrak{J}_*, \Omega\mathfrak{J}_n, \wp\theta) \leq \mathrm{D}(\mathfrak{J}_*, \mathfrak{J}_n, \wp), \\ \lim_{n \longrightarrow \infty} \mathrm{D}(\Omega\mathfrak{J}_*, \mathfrak{J}_{n+1}, \wp) = 0. \end{aligned} \tag{41}$$

Then, we can write

$$\mathrm{D}(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) \leq \mathrm{D}\left(\mathfrak{J}_*, \mathfrak{J}_{n+1}, \frac{\wp}{2}\right) \Delta \mathrm{D}\left(\mathfrak{J}_{n+1}, \Omega\mathfrak{J}_*, \frac{\wp}{2}\right). \tag{42}$$

Taking limit as $n \longrightarrow +\infty$, we get

$$\mathrm{D}(\mathfrak{J}_*, \Omega\mathfrak{J}_*, \wp) = 0 \, \Delta \, 0 = 0, \tag{43}$$

so $\Omega\mathfrak{J}_* = \mathfrak{J}_*$. Next proof is similar as in Theorem 16. □

Example 4. Let $W = [-3, 3]$. We define a binary relation $\vdash$ by $\mathfrak{J} \vdash \hbar \Longleftrightarrow \mathfrak{J} + \hbar \geq 0$.

Define an OPFMS as in Example 1 by

$$Y(\mathfrak{I},\hbar,\wp)=\begin{cases}1 & \text{if } \mathfrak{I}=\hbar,\\ \dfrac{\wp}{\wp+\max\{\mathfrak{I},\hbar\}} & \text{if otherwise,}\end{cases}$$
$$\mathscr{M}(\mathfrak{I},\hbar,\wp)=\begin{cases}0 & \text{if } \mathfrak{I}=\hbar,\\ \dfrac{\max\{\mathfrak{I},\hbar\}}{\wp+\max\{\mathfrak{I},\hbar\}} & \text{if otherwise,}\end{cases} \tag{44}$$
$$D(\mathfrak{I},\hbar,\wp)=\begin{cases}0 & \text{if } \mathfrak{I}=\hbar,\\ \dfrac{\max\{\mathfrak{I},\hbar\}}{\wp} & \text{if otherwise,}\end{cases}$$

for all $\mathfrak{I},\delta\in W,\wp>0$, with the CTN $a*b=a\bullet b$ and CTCN $a\Delta b=\max\{a,b\}$. Then, $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ is an O-complete PFMS. Define $\Omega: W\longrightarrow W$ by

$$\Omega\mathfrak{I}=\begin{cases}\dfrac{\mathfrak{I}}{4}, & \mathfrak{I}\in[-3,0]\\ 0, & \mathfrak{I}\in(0,3]\end{cases}. \tag{45}$$

Then, the following cases are satisfied:

(1) If $\mathfrak{I}\in[-3,0]$ and $\hbar\in(0,3]$, then $\Omega\mathfrak{I}=\mathfrak{I}/4$ and $\psi\hbar=0$

(2) If $\mathfrak{I},\hbar\in[-3,0]$, then $\Omega\mathfrak{I}=\mathfrak{I}/4$ and $\Omega\hbar=\hbar/4$

(3) If $\mathfrak{I},\hbar\in(0,3]$, then $\Omega\mathfrak{I}=0$ and $\Omega\hbar=0$

(4) If $\mathfrak{I}\in(0,3]$ and $\hbar\in[-3,0]$, then $\Omega\mathfrak{I}=0$ and $\Omega\hbar=\hbar/4$

This clearly implies that $\Omega\mathfrak{I}+\Omega\hbar\geq 0$. Hence, Ω is OPR. We can easily see that if $\lim_{n\longrightarrow\infty}Y(\mathfrak{I}_n,\mathfrak{I},\wp)=1$, then $\lim_{n\longrightarrow\infty}Y(\Omega\mathfrak{I}_j,\Omega\mathfrak{I},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{I}_n,\mathfrak{I},\wp)=0$, then $\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega\mathfrak{I}_n,\Omega\mathfrak{I},\wp)=0$ and $\lim_{n\longrightarrow\infty}D(\mathfrak{I}_n,\mathfrak{I},\wp)=0$, then $\lim_{n\longrightarrow\infty}D(\Omega\mathfrak{I}_n,\Omega\mathfrak{I},\wp)=0$ for all $\mathfrak{I}\in W$ and $\wp>0$. Hence, Ω is OC.

The above four cases for $\theta\in[1/2,1)$ satisfies the below contractive conditions:

$$\begin{aligned}Y(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\geq Y(\mathfrak{I},\hbar,\wp),\\ \mathscr{M}(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\leq\mathscr{M}(\mathfrak{I},\hbar,\wp),\\ \mathrm{D}(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\leq\mathrm{D}(\mathfrak{I},\hbar,\wp).\end{aligned} \tag{46}$$

All conditions of Theorem 16 are satisfied. Also, 0 is FP of Ω.

Theorem 18. *Let $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ be an O-complete PFMS such that*

$$\lim_{\wp\longrightarrow\infty}Y(\mathfrak{I},\hbar,\wp)=1 \text{ and } \lim_{\wp\longrightarrow\infty}\mathscr{M}(\mathfrak{I},\hbar,\wp)=0,\forall\mathfrak{I},\hbar\in W \text{ and } \wp>0. \tag{47}$$

Let $\Omega: W\longrightarrow W$ be OC,O-CON and OPR. Assume that there exist $\theta\in(0,1)$ and $\wp>0$ such that

$$\begin{aligned}Y(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\geq\min\{Y(\Omega\mathfrak{I},\mathfrak{I},\wp),Y(\Omega\hbar,\hbar,\wp)\},\\ \mathscr{M}(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\leq\min\{\mathscr{M}(\Omega\mathfrak{I},\mathfrak{I},\wp),\mathscr{M}(\Omega\hbar,\hbar,\wp)\},\\ \mathrm{D}(\Omega\mathfrak{I},\Omega\hbar,\theta\wp)&\leq\min\{\mathrm{D}(\Omega\mathfrak{I},\mathfrak{I},\wp),\mathrm{D}(\Omega\hbar,\hbar,\wp)\},\end{aligned} \tag{48}$$

for all $\mathfrak{I},\hbar\in W,\wp>0$. Then, Ω has a unique FP, so $\mathfrak{I}_*\in W$. Furthermore, $\lim_{n\longrightarrow\infty}Y(\Omega^n\mathfrak{I},\mathfrak{I}_*,\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega^n\mathfrak{I},\mathfrak{I}_*,\wp)=0$, and $\lim_{n\longrightarrow\infty}\mathrm{D}(\Omega^n\mathfrak{I},\mathfrak{I}_*,\wp)=0$ for all $\mathfrak{I}\in W$ and $\wp>0$.

Proof. Since $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ is an O-complete PFMS, there exists $\mathfrak{I}_0\in W$ such that

$$\mathfrak{I}_0\vdash\hbar,\forall\delta\in W. \tag{49}$$

Thus, $\mathfrak{I}\vdash h$.Consider

$$\mathfrak{I}_n=\Omega^n\mathfrak{I}_0=\Omega\mathfrak{I}_{n-1},\forall n\in S. \tag{50}$$

Since Ω is OPR, $\{\mathfrak{I}_n\}$ is an OS. We can get

$$\begin{aligned}Y(\mathfrak{I}_{n+1},\mathfrak{I}_n,\wp)&\geq Y(\mathfrak{I}_{n+1},\mathfrak{I}_n,\theta\wp)=Y(\Omega\mathfrak{I}_n,\Omega\mathfrak{I}_{n-1},\theta\wp)\geq\min\{Y(\Omega\mathfrak{I}_n,\mathfrak{I}_n,\wp),Y(\Omega\mathfrak{I}_{n-1},\mathfrak{I}_{n-1},\wp)\},\\ \mathscr{M}(\mathfrak{I}_{n+1},\mathfrak{I}_n,\wp)&\leq\mathscr{M}(\mathfrak{I}_{n+1},\mathfrak{I}_n,\theta\wp)=\mathscr{M}(\Omega\mathfrak{I}_n,\Omega\mathfrak{I}_{n-1},\theta\wp)\leq\min\{\mathscr{M}(\Omega\mathfrak{I}_n,\mathfrak{I}_n,\wp),\mathscr{M}(\Omega\mathfrak{I}_{n-1},\mathfrak{I}_{n-1},\wp)\},\\ \mathrm{D}(\mathfrak{I}_{n+1},\mathfrak{I}_n,\wp)&\leq\mathrm{D}(\mathfrak{I}_{n+1},\mathfrak{I}_n,\theta\wp)=\mathrm{D}(\Omega\mathfrak{I}_n,\Omega\mathfrak{I}_{n-1},\theta\wp)\leq\min\{\mathrm{D}(\Omega\mathfrak{I}_n,\mathfrak{I}_n,\wp),\mathrm{D}(\Omega\mathfrak{I}_{n-1},\mathfrak{I}_{n-1},\wp)\}.\end{aligned} \tag{51}$$

Two cases arise:

Case1: If $Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \geq Y(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp)$, then

$$Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \geq Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \geq Y(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp) = Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp), \\ \mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathscr{M}(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp). \tag{52}$$

Then,

$$\mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \leq \mathscr{M}(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp) = \mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp), \\ \mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathrm{D}(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp). \tag{53}$$

Then,

$$\mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \leq \mathrm{D}(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \wp) \\ = \mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp). \tag{54}$$

Then by Lemma 14, $\mathfrak{I}_n = \mathfrak{I}_{n+1}$ for all $n \in \mathbb{N}$.

Case2: If $Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \geq Y(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp)$, then

$$Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \geq Y(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \geq Y(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp) \geq Y(\mathfrak{I}_n, \mathfrak{I}_{n-1}, \wp), \\ \mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathscr{M}(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp). \tag{55}$$

Then,

$$\mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathscr{M}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \leq \mathscr{M}(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp) \leq \mathscr{M}(\mathfrak{I}_n, \mathfrak{I}_{n-1}, \wp), \tag{56}$$

and

$$\mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathrm{D}(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp). \tag{57}$$

Then,

$$\mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \wp) \leq \mathrm{D}(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \theta\wp) \leq \mathrm{D}(\Omega\mathfrak{I}_{n-1}, \mathfrak{I}_{n-1}, \wp) \\ \leq \mathrm{D}(\mathfrak{I}_n, \mathfrak{I}_{n-1}, \wp), \tag{58}$$

for all $n \in \mathbb{N}$ and $\wp>0$. Then by Theorem 16, we have a OCS. By completeness of $(W, Y, \mathscr{M}, D, *, \Delta, \vdash)$, there exists $\mathfrak{I}_* \in W$ such that $\lim_{n\longrightarrow\infty} R(\mathfrak{I}_n, \mathfrak{I}_*, \wp) = 1$, $\lim_{n\longrightarrow\infty} \mathscr{M}(\mathfrak{I}_n, \mathfrak{I}_*, \wp) = 0$, and $\lim_{n\longrightarrow\infty} D(\mathfrak{I}_n, \mathfrak{I}_*, \wp) = 0$, for all $\wp>0$.

We know that Ω is an OC, then

$$\lim_{n\longrightarrow\infty} Y(\mathfrak{I}_{n+1}, \Omega\mathfrak{I}_*, \wp) = \lim_{n\longrightarrow\infty} Y(\Omega\mathfrak{I}_n, \Omega\mathfrak{I}_*, \wp) = 1, \\ \lim_{n\longrightarrow\infty} \mathscr{M}(\mathfrak{I}_{n+1}, \Omega\mathfrak{I}_*, \wp) = \lim_{n\longrightarrow\infty} \mathscr{M}(\Omega\mathfrak{I}_n, \Omega\mathfrak{I}_*, \wp) = 0, \\ \lim_{n\longrightarrow\infty} \mathrm{D}(\mathfrak{I}_{n+1}, \Omega\mathfrak{I}_*, \wp) = \lim_{n\longrightarrow\infty} \mathrm{D}(\Omega\mathfrak{I}_j, \Omega\mathfrak{I}_*, \wp) = 0. \tag{59}$$

Now, we prove that $\mathfrak{I}_*$ is a FP of Ω. Let $\wp_1 \in (\theta, 1)$ and $\wp_2 = 1 - \wp_1$. Then,

$$\begin{aligned} Y(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) &\geq Y\left(\Omega\mathfrak{I}_*, \mathfrak{I}_{n+1}, \frac{\wp\wp_1}{2}\right) * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &= Y\left(\Omega\mathfrak{I}_*, \Omega\mathfrak{I}_n, \frac{\wp\wp_1}{2}\right) * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &\geq \min\left\{Y\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), Y\left(\psi\mathfrak{I}_n, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{Y\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right). \end{aligned} \tag{60}$$

Taking $n \longrightarrow \infty$, we get

$$\begin{aligned} &Y(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \geq \min\left\{Y\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), 1\right\} * 1, \\ &Y(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \geq Y\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp}{v}\right), \wp > 0, \\ &\mathscr{M}(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \leq \mathscr{M}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_{n+1}, \frac{\wp\wp_1}{2}\right) \Delta \mathscr{M} \\ &\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) = \mathscr{M}\left(\Omega\mathfrak{I}_*, \Omega\mathfrak{I}_n, \frac{\wp\wp_1}{2}\right) \Delta \mathscr{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &\leq \min\left\{\mathscr{M}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), \mathscr{M}\left(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathscr{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{\mathscr{M}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), \mathscr{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathscr{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right). \end{aligned} \tag{61}$$

Taking $n \longrightarrow \infty$, we get

$$\begin{aligned} &\mathscr{M}(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \leq \min\left\{\mathscr{M}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), 0\right\} \Delta 0, \\ &\mathscr{M}(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \leq \mathscr{M}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp}{v}\right), \wp > 0, \\ &\mathrm{D}(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \wp) \leq \mathrm{D}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_{n+1}, \frac{\wp\wp_1}{2}\right) \Delta \mathrm{D}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) = \mathrm{D}\left(\Omega\mathfrak{I}_*, \Omega\mathfrak{I}_n, \frac{\wp\wp_1}{2}\right) \Delta \mathrm{D}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right) \\ &\leq \min\left\{\mathrm{D}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), \mathrm{D}\left(\Omega\mathfrak{I}_n, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathrm{D}\left(\mathfrak{I}_{n+1}, \mathfrak{I}, \frac{\wp\wp_2}{2}\right) \\ &= \min\left\{\mathrm{D}\left(\Omega\mathfrak{I}_*, \mathfrak{I}_*, \frac{\wp\wp_1}{2\theta}\right), \mathrm{D}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_n, \frac{\wp\wp_1}{2\theta}\right)\right\} \Delta \mathrm{D}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_*, \frac{\wp\wp_2}{2}\right). \end{aligned} \tag{62}$$

Taking $n \longrightarrow \infty$, we get

$$\begin{aligned} D(\Omega\mathfrak{J}_*, \mathfrak{J}_*, \wp) &\leq \min \left\{ D\left(\Omega\mathfrak{J}_*, \mathfrak{J}_*, \frac{\wp\wp_1}{2\theta}\right), 0 \right\} \Delta 0, \\ D(\Omega\mathfrak{J}_*, \mathfrak{J}_*, \wp) &\leq D\left(\Omega\mathfrak{J}_*, \mathfrak{J}_*, \frac{\wp}{\nu}\right), \wp > 0. \end{aligned} \tag{63}$$

Here, $\nu = 2\theta/\wp_1 \in (0, 1)$, from Lemma 14, we have $\Omega\mathfrak{J}_* = \mathfrak{J}_*$. Suppose ν_* and $\hbar_*$ the FPs of Ω. We have

$$\mathfrak{J}_0 \vdash \mathfrak{J}_* \text{ and } \mathfrak{J}_0 \vdash \mathfrak{J}_*. \tag{64}$$

Because Ω is an OPR, so we can write

$$\Omega^n\mathfrak{J}_0 \vdash \Omega^n\mathfrak{J}_* \text{ and } \Omega^n\mathfrak{J}_0 \vdash \Omega^n\mathfrak{J}_* \text{ for all } n \in \mathbb{N}. \tag{65}$$

We can write

$$\begin{aligned} Y(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\geq Y(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \geq \min \{ Y(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), Y(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \wp) \}, \\ Y(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\geq Y(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \geq \min \{ Y(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), Y(\Omega^n\hbar_*, \hbar_*, \wp) \}. \end{aligned} \tag{66}$$

Hence, we write that

$$\begin{aligned} Y(\mathfrak{J}_*, \hbar_*, \wp) &= Y(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) \geq \min \left\{ Y\left(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \frac{\wp}{\theta}\right), Y\left(\Omega^n\hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{1, 1\} = 1, \\ \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\leq \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \leq \min \{ \mathscr{M}(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), \mathscr{M}(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \wp) \}, \\ \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\leq \mathscr{M}(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \leq \min \{ \mathscr{M}(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), \mathscr{M}(\Omega^n\hbar_*, \hbar_*, \wp) \}. \end{aligned} \tag{67}$$

Hence, we write that

$$\begin{aligned} \mathscr{M}(\mathfrak{J}_*, \hbar_*, \wp) &= \mathscr{M}(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) \leq \min \left\{ \mathscr{M}\left(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \frac{\wp}{\theta}\right), \mathscr{M}\left(\Omega^n\hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{0, 0\} = 0, \\ D(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \wp) &\leq D(\Omega^n\mathfrak{J}_0, \Omega^n\mathfrak{J}_*, \theta\wp) \leq \min \{ D(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), D(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \wp) \}, \\ D(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \wp) &\leq D(\Omega^n\mathfrak{J}_0, \Omega^n\hbar_*, \theta\wp) \leq \min \{ D(\Omega^n\mathfrak{J}_0, \mathfrak{J}_0, \wp), D(\Omega^n\hbar_*, \hbar, \wp) \}. \end{aligned} \tag{68}$$

Hence, we write that

$$D(\mathfrak{J}_*, \hbar_*, \wp) = D(\Omega^n\mathfrak{J}_*, \Omega^n\hbar_*, \wp) \leq \min \left\{ D\left(\Omega^n\mathfrak{J}_*, \mathfrak{J}_*, \frac{\wp}{\theta}\right), D\left(\Omega^n\hbar_*, \hbar_*, \frac{\wp}{\theta}\right) \right\} = \min \{0, 0\} = 0, \tag{69}$$

for all $\wp > 0$. Hence, $\mathfrak{J}_* = \hbar_*$. □

Corollary 19. *Let $(W, Y, \mathscr{M}, *, \Delta, \vdash)$ be an O-complete PFMS and $\Omega : W \longrightarrow W$ be an OC and OPR. Then $\theta \in (0, 1)$, we get $\wp > 0$,*

$$\begin{aligned} Y(\Omega\mathfrak{J}, \Omega\hbar, \theta\wp) &\geq \min \{ Y(\Omega\mathfrak{J}, \mathfrak{J}, \wp), Y(\Omega\hbar, \hbar, \wp), Y(\mathfrak{J}, \hbar, \wp) \}, \\ \mathscr{M}(\Omega\mathfrak{J}, \Omega\hbar, \theta\wp) &\leq \min \{ \mathscr{M}(\Omega\mathfrak{J}, \mathfrak{J}, \wp), \mathscr{M}(\Omega\hbar, \hbar, \wp), \mathscr{M}(\mathfrak{J}, \hbar, \wp) \}, \\ D(\Omega\mathfrak{J}, \Omega\hbar, \theta\wp) &\leq \min \{ D(\Omega\mathfrak{J}, \mathfrak{J}, \wp), D(\Omega\hbar, \hbar, \wp), D(\mathfrak{J}, \hbar, \wp) \}, \end{aligned} \tag{70}$$

Then, Ω has a unique FP.

Proof. It follows from Theorem 16 and Theorem 18. □

Example 5. Let $W=[-2,2]$ and define a binary relation $\perp$ by

$$\Im \vdash \hbar \Longleftrightarrow \Im + \hbar \geq 0. \tag{71}$$

Define $Y$and$\mathscr{M}$by

$$Y(\Im,\hbar,\wp)=\begin{cases}1 & \text{if} \quad \Im=\hbar\\ \dfrac{\wp}{\wp+\max\{\Im,\hbar\}} & \text{otherwise,}\end{cases}$$
$$\mathscr{M}(\Im,\hbar,\wp)=\begin{cases}0 & \text{if} \quad \Im=\hbar\\ \dfrac{\max\{\Im,\hbar\}}{\lambda+\max\{\Im,\hbar\}} & \text{otherwise,}\end{cases} \tag{72}$$
$$\mathrm{D}(\Im,\hbar,\wp)=\begin{cases}0 & \text{if} \quad \Im=\hbar,\\ \dfrac{\max\{\Im,\hbar\}}{\lambda} & \text{otherwise,}\end{cases}$$

for all $\Im,\hbar\in W$ and $\wp>0$, with the CTN and CTCN, respectively; $a*b=a.b$, $a\Delta b=\max\{a,b\}$ then $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ is an O-complete PFMS. Note that $\lim_{\wp\longrightarrow\infty}Y(\Im,\hbar,\wp)=1$, $\lim_{\wp\longrightarrow\infty}\mathscr{M}(\Im,\hbar,\wp)=0$, and $\lim_{\wp\longrightarrow\infty}D(\Im,\hbar,\wp)=0$ $\forall$ $\Im,\hbar\in E$. Define $\Omega: W\longrightarrow W$ by

$$\Omega\Im=\begin{cases}\dfrac{\Im}{4}, & \Im\in\left[-2,\dfrac{2}{3}\right]\\ 1-\Im, & \Im\in\left(\dfrac{2}{3},1\right]\\ \Im-\dfrac{1}{2}, & \Im\in(1,2]\end{cases}. \tag{73}$$

We have the following cases:

(1) If $\Im,\hbar\in[-2,2/3]$, then $\Omega\Im=\Im/4$ and $\Omega\hbar=\hbar/4$

(2) If $\Im,\hbar\in(2/3,1]$, then $\Omega\Im=1-\Im$ and $\Omega\hbar=1-\hbar$

(3) If $\Im,\hbar\in(1,2]$, then $\Omega\Im=\Im-1/2$ and $\Omega\hbar=\hbar-1/2$

(4) If $\Im\in[-2,2/3]$ and $\hbar\in(2/3,1]$, then $\Omega\Im=\Im/4$ and $\Omega\hbar=1-\hbar$

(5) If $\Im\in[-2,2/3]$ and $\hbar\in(1,2]$, then $\Omega\Im=\Im/4$ and $\Omega\hbar=\hbar-1/2$

(6) If $\Im\in(2/3,1]$ and $\hbar\in(1,2]$, then $\Omega\Im=1-\Im$ and $\Omega\hbar=\hbar-1/2$

(7) If $\Im\in(1,2]$ and $\hbar\in(2/3,1]$, then $\Omega\Im=\Im-1/2$ and $\Omega\hbar=1-\hbar$

(8) If $\Im\in(1,2]$ and $\hbar\in[-2,2/3]$, then $\Omega\Im=\Im-1/2$ and $\Omega\hbar=\hbar/4$

(9) If $\Im\in(2/3,1]$ and $\hbar\in[-2,2/3]$, then $\Omega\Im=1-\Im$ and $\Omega\hbar=\hbar/4$

Because $\Im\vdash\hbar\Longleftrightarrow\Im+\hbar\geq 0$, it is clearly implying that $\Omega\Im+\Omega\hbar\geq 0$. Hence, Ω is OPR. Let $\{\Im_n\}$ be an arbitrary OS in W that converges to $\Im\in W$. We have

$$\lim_{n\longrightarrow\infty}Y(\Im_n,\Im,\wp)=\lim_{n\longrightarrow\infty}\begin{cases}1 & \text{if} \quad \Im=\hbar,\\ \dfrac{\wp}{\wp+\max\{\Im_n,\Im\}} & \text{otherwise}=1,\end{cases}$$
$$\lim_{n\longrightarrow\infty}\mathscr{M}(\Im_n,\Im,\wp)=\lim_{n\longrightarrow\infty}\begin{cases}1 & \text{if} \quad \Im=\hbar,\\ \dfrac{\max\{\Im_n,\Im\}}{\wp+\max\{\Im_n,\Im\}} & \text{otherwise}=0,\end{cases}$$
$$\lim_{n\longrightarrow\infty}\mathrm{D}(\Im_n,\Im,\wp)=\lim_{n\longrightarrow\infty}\begin{cases}1 & \text{if} \quad \Im=\hbar,\\ \dfrac{\max\{\Im_n,\Im\}}{\wp} & \text{otherwise}=0.\end{cases} \tag{74}$$

Note that if $\lim_{n\longrightarrow\infty}Y(\Im_n,\Im,\lambda)=1$, $\lim_{n\longrightarrow\infty}S(\Im_n,\Im,\wp)=0$, and $\lim_{n\longrightarrow\infty}D(\Im_n,\Im,\wp)=0$, then $\lim_{n\longrightarrow\infty}Y(\Omega\Im_n,\Omega\Im,\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega\Im_n,\Omega\Im,\wp)=0$, and $\lim_{n\longrightarrow\infty}D(\Omega\Im_n,\Omega\Im,\wp)=0$ for all $\Im\in W$ and $\wp>0$. Hence, Ω is OC. The case $\Im=\hbar$ is clear. Let $\Im\neq\hbar$. We have

$$Y(\Omega\Im,\Omega\hbar,\theta\wp)\geq\min\{Y(\Omega\Im,\Im,\wp),Y(\Omega\hbar,\hbar,\wp)\},$$
$$\mathscr{M}(\Omega\Im,\Omega\hbar,\theta\wp)\leq\min\{\mathscr{M}(\Omega\Im,\Im,\wp),\mathscr{M}(\Omega\hbar,\hbar,\wp)\},$$
$$\mathrm{D}(\Omega\Im,\Omega\hbar,\theta\wp)\leq\min\{\mathrm{D}(\Omega\Im,\Im,\wp),\mathrm{D}(\Omega\hbar,\hbar,\wp)\}. \tag{75}$$

Indeed, it is satisfied for all above 9 cases. But, Ω is not a contraction. Assume

$$\min\{Y(\Omega\Im,\Im,\wp),Y(\Omega\hbar,\hbar,\wp)\}=Y(\Omega\Im,\Im,\wp),$$
$$\min\{\mathscr{M}(\Omega\Im,\Im,\wp),\mathscr{M}(\Omega\hbar,\hbar,\wp)\}=\mathscr{M}(\Omega\Im,\Im,\wp),$$
$$\min\{\mathrm{D}(\Omega\Im,\Im,\wp),T(\Omega\hbar,\hbar,\wp)\}=\mathrm{D}(\Omega\Im,\Im,\wp), \tag{76}$$

then for $\Im=-1,\hbar=-2$, we have

$$Y(\Omega\Im,\Omega\hbar,\theta\wp)=\frac{\theta\wp}{\theta\wp+\max\{(\Im/4),(\delta/4)\}}=\frac{4\theta\wp}{4\theta\wp-1}\geq 1,$$
$$\mathscr{M}(\Omega\Im,\Omega\hbar,\theta\wp)=\frac{\max\{(\Im/4),(\hbar/4)\}}{\theta\lambda+\max\{(\Im/4),(\hbar/4)\}}=\frac{-1}{4\theta\wp-1}\leq 0,$$
$$\mathscr{M}(\Omega\Im,\Omega\hbar,\theta\wp)=\frac{\max\{(\Im/4),(\hbar/4)\}}{\theta\lambda}=\frac{-1}{4\theta\wp}\leq 0. \tag{77}$$

It is a contraction. Hence, all the conditions of Theorem 18 are satisfied and 0 is the unique FP of ψ.

Definition 20. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an OPFMS. A mapping $\Omega : W \longrightarrow W$ is named to be an PF $\vdash$-contractive if $\exists\, \theta \in (0,1)$ so that

$$\frac{1}{Y(\Omega\mathfrak{I}, \Omega\hbar, \lambda)} - 1 \le \theta\left[\frac{1}{Y(\mathfrak{I}, \hbar, \lambda)} - 1\right], \quad (78)$$

$$\mathcal{M}(\Omega\mathfrak{I}, \Omega\hbar, \wp) \le \theta\mathcal{M}(\mathfrak{I}, \hbar, \wp) \text{ and } \mathrm{D}(\Omega\mathfrak{I}, \Omega\hbar, \wp) \le \theta\mathrm{D}(\mathfrak{I}, \hbar, \wp), \quad (79)$$

for all $\mathfrak{I}, \hbar \in W$ and $\wp>0$. Here, θ is called the PFS $\vdash$-contractive constant of Ω.

Theorem 21. *Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS such that*

$$\lim_{\wp \longrightarrow \infty} Y(\mathfrak{I}, \hbar, \wp) = 1, \lim_{\wp \longrightarrow \infty} \mathcal{M}(\mathfrak{I}, \hbar, \wp) = 0, \text{and} \lim_{\wp \longrightarrow \infty} \mathrm{D}(\mathfrak{I}, \hbar, \wp) = 0, \forall \mathfrak{I}, \hbar \in \mathrm{W}. \quad (80)$$

Let $\Omega : W \longrightarrow W$ be an OC,$\vdash$ -contraction and OPR. Thus, Ω has a FP, say $\upsilon \in W$, $Y(\upsilon, \upsilon, \wp) = 1$, $\mathcal{M}(\upsilon, \upsilon, \wp) = 0$ and $D(\upsilon, \upsilon, \wp) = 0$ for all $\wp>0$.

Proof. Let $(W, Y, \mathcal{M}, D, *, \Delta, \vdash)$ be an O-complete PFMS. For an arbitrary $\mathfrak{I}_0 \in W$,

$$\mathfrak{I}_0 \vdash \hbar, \forall \hbar \in W. \quad (81)$$

That is, $\mathfrak{I}_0 \vdash \Omega\mathfrak{I}_0$. Consider

$$\mathfrak{I}_n = \Omega^n \mathfrak{I}_0 = \Omega\mathfrak{I}_{n-1} \text{ for all } n \in \mathbb{N}. \quad (82)$$

Since Ω is OPR, $\{\mathfrak{I}_n\}$ is an OS. If $\mathfrak{I}_n = \mathfrak{I}_{n-1}$ for some $n \in \mathbb{N}$, then $\mathfrak{I}_n$ is a FP of Ω. We assume that $\mathfrak{I}_n \ne \mathfrak{I}_{n-1}$ for all $n \in \mathbb{N}$. For all $\wp>0$ and $n \in \mathbb{N}$, we get from (12),

$$\begin{aligned}
&\frac{1}{Y(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp)} - 1 = \frac{1}{Y(\Omega\mathfrak{I}_{n-1}, \Omega\mathfrak{I}_n, \wp)} - 1 \le \theta\left[\frac{1}{Y(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp)} - 1\right],\\
&\mathcal{M}(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp) = \mathcal{M}(\Omega\mathfrak{I}_{n-1}, \Omega\mathfrak{I}_n, \wp) \le \theta\mathcal{M}(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp),\\
&\mathrm{D}(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp) = \mathrm{D}(\Omega\mathfrak{I}_{n-1}, \Omega\mathfrak{I}_n, \wp) \le \theta\mathrm{D}(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp).
\end{aligned} \quad (83)$$

We have $\forall \wp > 0$

$$\frac{1}{Y(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp)} \le \frac{\theta}{Y(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp)} + (1-\theta), \quad (84)$$

Implying that

$$\frac{\theta}{Y(\Omega\mathfrak{I}_{n-2}, \Omega\mathfrak{I}_{n-1}, \wp)} + (1-\theta) \le \frac{\theta^2}{Y(\mathfrak{I}_{n-2}, \mathfrak{I}_{n-1}, \wp)} + \theta(1-\theta) + (1-\theta). \quad (85)$$

Continuing in this way, we get

$$\begin{aligned}
\frac{1}{Y(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp)} &\le \frac{\theta^n}{Y(\mathfrak{I}_0, \mathfrak{I}_1, \wp)} + \theta^{n-1}(1-\theta) + \theta^{n-2}(1-\theta)\\
&+ \cdots + \theta(1-\theta) + (1-\theta) \le \frac{\theta^n}{Y(\mathfrak{I}_0, \mathfrak{I}_1, \wp)}\\
&+ \left(\theta^{n-1} + \theta^{n-2} + \cdots + 1\right)(1-\theta) \le \frac{\theta^n}{Y(\mathfrak{I}_0, \mathfrak{I}_1, \wp)} + (1-\theta^n).
\end{aligned} \quad (86)$$

We have

$$\frac{1}{(\theta^n / Y(\mathfrak{I}_0, \mathfrak{I}_1, \wp)) + (1-\theta^n)} \le Y(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp), \forall \wp > 0, n \in \mathbb{N}, \quad (87)$$

$$\begin{aligned}
\mathcal{M}(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp) &= \mathcal{M}(\Omega\mathfrak{I}_{n-1}, \Omega\mathfrak{I}_n, \wp) \le \theta\mathcal{M}(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp)\\
&= \theta\mathcal{M}(\Omega\mathfrak{I}_{n-2}, \Omega\mathfrak{I}_{n-1}, \wp) \le \theta^2\mathcal{M}(\mathfrak{I}_{n-2}, \mathfrak{I}_{n-1}, \wp)\\
&\le \cdots \le \theta^n\mathcal{M}(\mathfrak{I}_0, \mathfrak{I}_1, \wp). \forall \wp > 0, n \in \mathbb{N},
\end{aligned} \quad (88)$$

$$\begin{aligned}
\mathrm{D}(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \wp) &= \mathrm{D}(\psi\mathfrak{I}_{n-1}, \psi\mathfrak{I}_n, \wp) \le \theta\mathrm{D}(\mathfrak{I}_{n-1}, \mathfrak{I}_n, \wp)\\
&= \theta\mathrm{D}(\psi\mathfrak{I}_{n-2}, \psi\mathfrak{I}_{n-1}, \wp) \le \theta^2\mathrm{D}(\beta_{n-2}, \mathfrak{I}_{n-1}, \wp)\\
&\le \cdots \le \theta^n\mathrm{D}(\mathfrak{I}_0, \mathfrak{I}_1, \wp). \forall \wp > 0, n \in \mathbb{N}.
\end{aligned} \quad (89)$$

Now, for $m \ge 1$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
Y(\mathfrak{I}_n, \mathfrak{I}_{n+m}, \wp) &\ge Y\left(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_{n+m}, \frac{\wp}{2}\right)\\
&\ge Y\left(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_{n+2}, \frac{\wp}{2^2}\right)\\
&* Y\left(\mathfrak{I}_{n+2}, \mathfrak{I}_{n+m}, \frac{\wp}{2^2}\right).
\end{aligned} \quad (90)$$

Again, continuing in this way, we get

$$\begin{aligned}
&Y(\mathfrak{I}_n, \mathfrak{I}_{n+m}, \wp) \ge Y\left(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \frac{\wp}{2}\right) * Y\left(\mathfrak{I}_{n+1}, \mathfrak{I}_{n+2}, \frac{\wp}{2^2}\right) * \cdots * Y\left(\mathfrak{I}_{n+m-1}, \mathfrak{I}_{n+m}, \frac{\wp}{2^{m-1}}\right),\\
&\mathcal{M}(\mathfrak{I}_n, \mathfrak{I}_{n+p}, \wp) \le \mathcal{M}\left(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_{n+p}, \frac{\wp}{2}\right) \le \mathcal{M}\left(\mathfrak{I}_n, \mathfrak{I}_{n+1}, \frac{\wp}{2}\right) \Delta \mathcal{M}\left(\mathfrak{I}_{n+1}, \mathfrak{I}_{n+2}, \frac{\wp}{2^2}\right) \Delta \mathcal{M}\left(\mathfrak{I}_{n+2}, \mathfrak{I}_{n+p}, \frac{\wp}{2^2}\right).
\end{aligned} \quad (91)$$

Continuing in this way, we get

$$\begin{gathered}\mathscr{M}(\mathfrak{I}_n,\mathfrak{I}_{n+p},\wp)\le\mathscr{M}\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathscr{M}\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+2},\frac{\wp}{2^2}\right)\Delta\cdots\Delta\,\mathscr{M}\left(\mathfrak{I}_{n+p-1},\mathfrak{I}_{n+p},\frac{\wp}{2^{p-1}}\right),\\ \mathrm{D}(\mathfrak{I}_n,\mathfrak{I}_{n+p},\wp)\le\mathrm{D}\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+p},\frac{\wp}{2}\right)\le\mathrm{D}\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+2},\frac{\wp}{2^2}\right)\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+2},\mathfrak{I}_{n+p},\frac{\wp}{2^2}\right).\end{gathered}\tag{92}$$

Continuing in this way, we get

$$\mathrm{D}(\mathfrak{I}_n,\mathfrak{I}_{n+p},\wp)\le\mathrm{D}\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+2},\frac{\wp}{2^2}\right)\Delta\cdots\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+p-1},\mathfrak{I}_{n+p},\frac{\wp}{2^{p-1}}\right).\tag{93}$$

By using (87) in the above inequality, we have

$$\begin{gathered}Y(\mathfrak{I}_n,\mathfrak{I}_{n+m},\wp)\ge\frac{1}{(\theta^n/Y(\mathfrak{I}_0,\mathfrak{I}_1,(\wp/2)))+(1-\theta^n)}*\frac{1}{\left(\theta^{n+1}/Y\left(\mathfrak{I}_0,\mathfrak{I}_1,\left(\wp/2^2\right)\right)\right)+\left(1-\theta^{n+1}\right)}*\cdots*\\ \cdot\frac{1}{\left(\theta^{n+m-1}/Y\left(\mathfrak{I}_0,\mathfrak{I}_1,\left(\wp/2^{m-1}\right)\right)\right)+\left(1-\theta^{n+m-1}\right)},\ge\frac{1}{(\theta^n/Y(\mathfrak{I}_0,\mathfrak{I}_1,(\wp/2)))+1}\\ *\frac{1}{\left(\theta^{n+1}/Y\left(\mathfrak{I}_0,\mathfrak{I}_1,\left(\wp/2^2\right)\right)\right)+1}*\cdots*\frac{1}{\left(\theta^{n+m-1}/Y\left(\mathfrak{I}_0,\mathfrak{I}_1,\left(\wp/2^{m-1}\right)\right)\right)+1},\end{gathered}\tag{94}$$

using (88),

$$\begin{aligned}S(\mathfrak{I}_n,\mathfrak{I}_{n+p},\wp)\le{}&S\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,S\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+2},\frac{\wp}{2^2}\right)\\ &\Delta\cdots\Delta\,S\left(\mathfrak{I}_{n+p-1},\mathfrak{I}_{n+p},\frac{\wp}{2^{p-1}}\right),\end{aligned}\tag{95}$$

and using (89)

$$\begin{aligned}\mathrm{D}(\mathfrak{I}_n,\mathfrak{I}_{n+p},\wp)\le{}&\mathrm{D}\left(\mathfrak{I}_n,\mathfrak{I}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+1},\mathfrak{I}_{n+2},\frac{\wp}{2^2}\right)\\ &\Delta\cdots\Delta\,\mathrm{D}\left(\mathfrak{I}_{n+p-1},\mathfrak{I}_{n+p},\frac{\wp}{2^{p-1}}\right),\end{aligned}\tag{96}$$

$\theta\in(0,1)$ we deduce from the above expression that $\lim_{n\longrightarrow\infty}Y(\mathfrak{I}_n,\mathfrak{I}_{n+m},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{I}_n,\mathfrak{I}_{n+m},\wp)=0$, and $\lim_{n\longrightarrow\infty}D(\mathfrak{I}_n,\mathfrak{I}_{n+m},\wp)=0$ for all $\wp>0$, $m\ge1$.

Therefore, $\{\mathfrak{I}_n\}$ is a O-CS in $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$. By the completeness of $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$, we know that Ω is an OC and there exists $\upsilon\in W$ such that

$$\lim_{n\longrightarrow\infty}Y(\mathfrak{I}_{n+1},\upsilon,\wp)=\lim_{n\longrightarrow\infty}Y(\Omega\mathfrak{I}_n,\Omega\upsilon,\wp)=1,\forall\wp>0,\tag{97}$$

$$\lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{I}_{n+1},\upsilon,\wp)=\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega\mathfrak{I}_n,\Omega\upsilon,\wp)=0\forall\wp>0,\tag{98}$$

$$\lim_{n\longrightarrow\infty}\mathrm{D}(\mathfrak{I}_{n+1},\upsilon,\wp)=\lim_{n\longrightarrow\infty}\mathrm{D}(\Omega\mathfrak{I}_n,\Omega\upsilon,\wp)=0\forall\wp>0.\tag{99}$$

Now, we prove that υ is a FP of Ω. For this, we obtain from (78) that

$$\frac{1}{Y(\Omega\mathfrak{I}_n,\Omega\upsilon,\wp)}-1\le\theta\left[\frac{1}{Y(\mathfrak{I}_n,\upsilon,\wp)}-1\right]=\frac{\theta}{Y(\mathfrak{I}_n,\upsilon,\wp)}-\theta.\tag{100}$$

That is,

$$\frac{1}{(\theta/Y(\mathfrak{I}_n,\upsilon,\wp))+1-\theta}\le Y(\Omega\mathfrak{I}_n,\Omega\upsilon,\wp).\tag{101}$$

Using the above inequality, we obtain

$$Y(\upsilon,\Omega\upsilon,\wp)\geq Y\left(\upsilon,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)*Y\left(\mathfrak{J}_{n+1},\Omega\upsilon,\frac{\wp}{2}\right)=Y\left(\upsilon,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)*Y(\Omega\mathfrak{J}_n,\Omega\upsilon,\wp)\geq Y\left(\upsilon,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)*\frac{1}{(\theta/Y(\mathfrak{J}_n,\upsilon,(\wp/2)))+1-\theta},$$
$$\mathscr{M}(w,v,\wp)=\mathscr{M}(\Omega w,\Omega v,\wp)\leq\theta\mathscr{M}(\mathrm{w},v,\wp)<\mathscr{M}(w,v,\wp),=\mathscr{M}\left(w,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathscr{M}(\Omega\mathfrak{J}_n,\Omega w,\wp)\leq\mathscr{M}\left(w,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)\Delta\theta\mathscr{M}\left(\mathfrak{J}_n,w,\frac{\wp}{2}\right), \quad (102)$$
$$\mathrm{D}(w,v,\wp)=\mathrm{D}(\Omega w,\Omega v,\wp)\leq\theta\mathrm{D}(w,v,\wp)<\mathrm{D}(w,v,\wp),=\mathrm{D}\left(w,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)\Delta\,\mathrm{D}\left(\Omega\mathfrak{J}_n,\Omega w,\frac{\wp}{2}\right)\leq\mathrm{D}\left(w,\mathfrak{J}_{n+1},\frac{\wp}{2}\right)\Delta\theta\mathrm{D}\left(\mathfrak{J}_n,w,\frac{\wp}{2}\right).$$

Taking limit as $n\longrightarrow\infty$ and using (97), (98), and (99) in the above expression, we get that $Y(\upsilon,\Omega\upsilon,\wp)=1$, $\mathscr{M}(\upsilon,\Omega\upsilon,\wp)=0$, and $D(\upsilon,\Omega\upsilon,\wp)=0$, that is, $\Omega\upsilon=\upsilon$. Therefore, υ is a FP of Ω, $Y(\upsilon,\upsilon,\wp)=1$, $\mathscr{M}(\upsilon,\upsilon,\wp)=0$, and $D(\upsilon,\upsilon,\wp)=0$ for all $\wp>0$. □

Corollary 22. *Let* $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ *be a O-complete PFMS and* $\Omega:W\longrightarrow W$ *satisfy*

$$\begin{gathered}\frac{1}{Y(\Omega^n\mathfrak{J},\Omega^n\hbar,\wp)}-1\leq\theta\left[\frac{1}{Y(\mathfrak{J},\hbar,\wp)}-1\right],\\ \mathscr{M}(\Omega^n\mathfrak{J},\Omega^n\hbar,\wp)\leq\theta\mathscr{M}(\mathfrak{J},\hbar,\wp),\\ \mathrm{D}(\Omega^n\mathfrak{J},\Omega^n\hbar,\wp)\leq\theta\mathrm{D}(\mathfrak{J},\hbar,\wp),\end{gathered}\quad (103)$$

for all $n\in\mathbb{N},\mathfrak{J},\hbar\in W,\wp>0$*, where* $0<\theta<1$*. Then,* Ω *has a FP.*

Proof. $\upsilon\in W$ is the unique FP of Ω^n by using Theorem 21, and $Y(\upsilon,\upsilon,\wp)=1,\mathscr{M}(\upsilon,\upsilon,\wp)=0,D(\upsilon,\upsilon,\wp)=0,\forall\wp>0.\Omega\upsilon$ is also a FP of Ω^n as $\Omega^n(\Omega\upsilon)=\Omega\upsilon$. From Theorem 21, $\Omega\upsilon=\upsilon,\upsilon$ is a FP since the FP of Ω is also a FP of Ω^n. □

Example 6. Let $W=[-1,2]$ and define a binary relation $\vdash$ by

$$\mathfrak{J}\vdash\hbar\Longleftrightarrow\mathfrak{J}+\hbar\geq 0.\quad (104)$$

Define $Y,\mathscr{M},D$ by

$$\begin{gathered}Y(\mathfrak{J},\hbar,\wp)=\begin{cases}1 & \text{if}\quad \mathfrak{J}=\hbar,\\ \dfrac{\wp}{\wp+\max\{\mathfrak{J},\hbar\}} & \text{if}\quad \text{otherwise},\end{cases}\\ \mathscr{M}(\mathfrak{J},\hbar,\wp)=\begin{cases}0 & \text{if}\quad \mathfrak{J}=\hbar,\\ 1-\dfrac{\wp}{\wp+\max\{\mathfrak{J},\hbar\}} & \text{if}\quad \text{otherwise},\end{cases}\\ \mathrm{D}(\mathfrak{J},\hbar,\wp)=\begin{cases}0 & \text{if}\quad \mathfrak{J}=\hbar,\\ \dfrac{\max\{\mathfrak{J},\hbar\}}{\wp} & \text{if}\quad \text{otherwise}.\end{cases}\end{gathered}\quad (105)$$

With CTN $a*b=a.b$ and CTCN $a\Delta b=\max\{a,b\}$ then $(W,Y,\mathscr{M},D,*,\Delta,\vdash)$ is an O-complete PFMS. Also observe that $\lim_{\wp\longrightarrow\infty}Y(\mathfrak{J},\hbar,\wp)=1$, $\lim_{\wp\longrightarrow\infty}S(\mathfrak{J},\hbar,\wp)=0$, and $\lim_{\wp\longrightarrow\infty}D(\mathfrak{J},\hbar,\wp)=0$, $\forall\ \mathfrak{J},\hbar\in W$.

Define $\Omega:W\longrightarrow W$ by

$$\Omega\mathfrak{J}=\begin{cases}2-\mathfrak{J}, & \mathfrak{J}\in[-1,1)\\ 1, & \mathfrak{J}\in[1,2]\end{cases}.\quad (106)$$

Therefore, it will satisfy the following cases:

(1) If $\mathfrak{J},\hbar\in[-1,1)$, then $\Omega\mathfrak{J}=2-\mathfrak{J}$ and $\Omega\hbar=2-\hbar$

(2) If $\mathfrak{J},\hbar\in[1,2]$, then $\Omega\mathfrak{J}=\Omega\hbar=1$

(3) If $\mathfrak{J}\in[-1,1)$ and $\hbar\in[1,2]$, then $\Omega\mathfrak{J}=2-\mathfrak{J}$ and $\Omega\hbar=1$

(4) If $\mathfrak{J}\in[1,2]$ and $\hbar\in[-1,1)$, then $\Omega\mathfrak{J}=1$ and $\Omega\hbar=2-\hbar$

Because $\mathfrak{J}\vdash\hbar\Longleftrightarrow\mathfrak{J}+\hbar\geq 0$, it is clearly implying that $\Omega\mathfrak{J}+\Omega\hbar\geq 0$. Hence, Ω is OPR. Let $\{\mathfrak{J}_n\}$ be an arbitrary OS in W that $\{\mathfrak{J}_n\}$ converges to $\mathfrak{J}\in W$.

$$\begin{gathered}\lim_{n\longrightarrow\infty}Y(\mathfrak{J}_n,\mathfrak{J},\wp)=1,\\ \lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{J}_n,\mathfrak{J},\wp)=0,\\ \lim_{n\longrightarrow\infty}\mathrm{D}(\mathfrak{J}_n,\mathfrak{J},\wp)=0,\end{gathered}\quad (107)$$

as $\{\mathfrak{J}_n\}$ converges to $\mathfrak{J}$. We can easily see that if $\lim_{n\longrightarrow\infty}Y(\mathfrak{J}_n,\mathfrak{J},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\mathfrak{J}_n,\mathfrak{J},\wp)=0$, and $\lim_{n\longrightarrow\infty}D(\mathfrak{J}_n,\mathfrak{J},\wp)=0$, then clearly $\lim_{n\longrightarrow\infty}Y(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=1$, $\lim_{n\longrightarrow\infty}\mathscr{M}(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=0$, and $\lim_{n\longrightarrow\infty}D(\Omega\mathfrak{J}_n,\Omega\mathfrak{J},\wp)=0$ for all $\mathfrak{J}\in W$ and $\wp>0$. Hence, Ω is OC. Also above all cases satisfied PFS $\vdash$-contractive mapping

$$\begin{gathered}\frac{1}{Y(\Omega\mathfrak{J},\Omega\hbar,\wp)}-1\leq\theta\left[\frac{1}{Y(\mathfrak{J},\hbar,\wp)}-1\right],\\ \mathscr{M}(\Omega\mathfrak{J},\Omega\hbar,\wp)\leq\theta\mathscr{M}(\mathfrak{J},\hbar,\wp),\\ \mathrm{D}(\Omega\mathfrak{J},\Omega\hbar,\wp)\leq\theta\mathrm{D}(\mathfrak{J},\hbar,\wp).\end{gathered}\quad (108)$$

All conditions of Theorem 21 are satisfied and 1 is a FP of Ω.

3. Conclusions

A picture fuzzy set is more proficient and more capable than an intuitionistic fuzzy set and fuzzy to cope with uncertain and unpredictable information in realistic issues. Herein, we have introduced the notion of orthogonal picture fuzzy metric space and investigated some new type of fixed point theorems in this new setting. Moreover, we have provided nontrivial examples to demonstrate the viability of the proposed results. Since our structure is more general than the class of picture fuzzy metric spaces, our results and notions expand and generalize several previous results. This work can be easily extended in the structure of orthogonal picture fuzzy cone metric spaces, and orthogonal picture fuzzy bipolar metric spaces.

Data Availability

No data was used during this research

Conflicts of Interest

The authors declare that they have no competing interests.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 6161839, 8 pages
https://doi.org/10.1155/2022/6161839

Research Article

Approximating Fixed Points of Enriched Nonexpansive Mappings in Geodesic Spaces

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Received 22 February 2022; Accepted 12 April 2022; Published 3 May 2022

Academic Editor: Santosh Kumar

In this paper, we consider the class of enriched nonexpansive mappings in the setting of geodesic spaces. We obtain a number of fixed point theorems for these mappings in geodesic spaces. Further, we employ the SP iterative method and present some new convergence theorems for the class of enriched nonexpansive mappings under different assumptions. We present some results concerning Δ and strong convergence.

1. Introduction

Nonexpansive mappings are those class of nonlinear mappings which have Lipschitz constant equal to one. A nonexpansive mapping needs not to admit a fixed point in a complete space. However, Browder [1], Göhde [2], and Kirk [3] independently ensured the existence of fixed points of nonexpansive mappings in Banach spaces under certain geometric assumptions. Many mathematicians have generalized and extended these results and considered a number of nonlinear mappings, see [4–9] (see also the references therein).

In 2019, Berinde [10] considered a new class of nonlinear mappings by enriching nonexpansive mappings, known as enriched nonexpansive mappings. He obtained some fixed point theorems for these classes of mappings in Hilbert spaces. It was observed in [10, 11] that class of enriched nonexpansive mappings has strong relations with averaged and nonexpansive mappings.

On the other hand, in 1970, Takahashi [12] considered the structure of convexity outside linear spaces. These spaces are fruitful in the context of fixed point theory. Goebel and Kirk [13] employed Krasnosel'skiĭ-Mann iterative method to find fixed points of nonexpansive mappings in hyperbolic type spaces. In the recent years, a number of papers have appeared in the literature dealing with the fixed point theorems in nonlinear spaces, see [14–24].

The class of enriched nonexpansive mappings has been studied only in linear spaces. Now it is natural to extend this class of mappings outside of linear spaces (or in nonlinear spaces) and ensure the existence of fixed points. The aim of this paper is to study the class of enriched nonexpansive mappings in geodesic spaces. We observe that for every b-enriched nonexpansive mapping, one can define a nonexpansive mapping, and the set of fixed points of both the mappings remains the same. Therefore, the existence of fixed points for b-enriched nonexpansive mappings is equivalent to existence of fixed points for nonexpansive mappings. However, the convergence of fixed points for b-enriched nonexpansive mappings is slightly different than the convergence of fixed points for nonexpansive mappings. We prove that Krasnosel'skiĭ method converges to fixed point of mapping. Further, we use SP iterative method to reckon fixed points of b-enriched nonexpansive mappings under certain assumptions. These results are new even in Hilbert spaces. Our results extend, complement, and generalize some results from [10, 11, 16, 19, 25–27].

2. Preliminaries

Let (Γ, ρ) be a metric space and $[0, c] \subset \mathbb{R}$. A mapping $g : [0, c] \longrightarrow \Gamma$ is called as geodesic path from ζ to ξ if

$$\begin{gathered} g(0) = \zeta, g(c) = \xi, \\ \rho\left(g(s), g\left(s'\right)\right) = |s - s'|, \end{gathered} \tag{1}$$

for all $s, s' \in [0, c]$. The image $g([0, c])$ of g forms a geodesic joining ζ and ξ. It is noted that the geodesic segment joining ζ and ξ is not unique, in general. For more details of geodesic spaces, see [14, 21].

Definition 1 (see [28]). A triplet (Γ, ρ, Ω) is called as a hyperbolic metric space if (Γ, ρ) is a metric space, and function $\Omega : \Gamma \times \Gamma \times [0, 1] \longrightarrow \Gamma$ satisfies the following assumptions for all $\zeta, \xi, v, w \in \Gamma$ and $\mu, \theta \in [0, 1]$

(W1) $\rho(v, \Omega(\zeta, \xi, \mu)) \le (1 - \mu)\rho(v, \zeta) + \mu\rho(v, \xi)$

(W2) $\rho(\Omega(\zeta, \xi, \mu), \Omega(\zeta, \xi, \theta)) = |\mu - \theta|\rho(\zeta, \xi)$

(W3) $\Omega(\zeta, \xi, \mu) = \Omega(\xi, \zeta, 1 - \mu)$

(W4) $\rho(\Omega(\zeta, v, \mu), \Omega(\xi, w, \mu)) \le (1 - \mu)\rho(\zeta, \xi) + \mu\rho(v, w)$

Remark 2. If $\Omega(\zeta, \xi, \mu) = (1 - \mu)\zeta + \mu\xi$ for all $\zeta, \xi \in \Gamma, \mu \in [0, 1]$, then it can be seen that all normed linear spaces are hyperbolic metric space.

Remark 3. If conditions (W1)–(W3) are satisfied, then (Γ, ρ, Ω) is hyperbolic type space considered by Goebel and Kirk [13]. Reich and Shafrir [22] also obtained some important results in hyperbolic metric spaces.

We shall write

$$\Omega(\zeta, \xi, \mu) := (1 - \mu)\zeta \oplus \mu\xi, \tag{2}$$

to denote a point $\Omega(\zeta, \xi, \mu)$ of (Γ, ρ, Ω) space. For $\zeta, \xi \in \Gamma$,

$$[\zeta, \xi] = \{(1 - \mu)\zeta \oplus \mu\xi : \mu \in [0, 1]\}, \tag{3}$$

indicates geodesic segments. A subset $\mathscr{Z}$ of hyperbolic metric space (or hyperbolic space) (Γ, ρ, Ω) is called convex if $[\zeta, \xi] \subset \mathscr{Z}$ whenever $\zeta, \xi \in \mathscr{Z}$.

Remark 4. Leustean [20] proved that the class of CAT(0) spaces is the class of complete uniformly convex hyperbolic spaces (in short, complete UCΩ-hyperbolic space), see the definition of UCΩ-hyperbolic space in [19].

If (Γ, ρ, Ω) is a Busemann space, then there is a unique convexity mapping Ω in such a way that (Γ, ρ, Ω) is Ω-hyperbolic space with unique geodesics. In other words, for all $\zeta \ne \xi \in \Gamma$ and any $\mu \in [0, 1]$, there is an element $v \in \Gamma$ which is unique (say $v = \Omega(\zeta, \xi, \mu)$) in such a way

$$\rho(\zeta, v) = \mu\rho(\zeta, \xi) \text{ and } \rho(\xi, v) = (1 - \mu)\rho(\zeta, \xi). \tag{4}$$

Let ζ, ξ, v be three points in metric space (Γ, ρ); the point ξ is said to lie between ζ and v if

$$\rho(\zeta, v) = \rho(\zeta, \xi) + \rho(\xi, v), \tag{5}$$

and these points are distinct pairwise. Thus, if ξ lies between ζ and v, then ξ lies between v and ζ.

Lemma 5 (see [14]). *Let Γ be a uniquely geodesic space. Let $\zeta, \xi, v \in \Gamma$ be pairwise distinct points. A point ξ lies between ζ and v if and only if $\xi \in [\zeta, v]$.*

Proposition 6 (see [14]). *Let Γ be a metric space and $\zeta, \xi, v, w \in \Gamma$ be pairwise distinct points. The following are equivalent:*

(a) ξ lies between ζ and v, and v lies between ζ and w

(b) ξ lies between ζ and w, and v lies between ξ and w

Let $\{\zeta_n\}$ be a bounded sequence in a hyperbolic space (Γ, ρ, Ω) and $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \ne \varnothing$. A functional $r(.,\{\zeta_n\})$: $\Gamma \longrightarrow [0,\infty)$ can be defined as follows:

$$r(\xi, \{\zeta_n\}) = \limsup_{n \longrightarrow \infty} \rho(\xi, \zeta_n). \tag{6}$$

The asymptotic radius of $\{\zeta_n\}$ with respect to (in short, wrt) $\mathscr{Z}$ is defined as

$$r(\mathscr{Z}, \{\zeta_n\}) = \inf \{r(\xi, \{\zeta_n\}) | \xi \in \mathscr{Z}\}. \tag{7}$$

A point ζ in $\mathscr{Z}$ is called an asymptotic center of $\{\zeta_n\}$ wrt $\mathscr{Z}$ if

$$r(\zeta, \{\zeta_n\}) = r(\mathscr{Z}, \{\zeta_n\}). \tag{8}$$

$A(\mathscr{Z}, \{\zeta_n\})$ is denoted as set of all asymptotic centers of $\{\zeta_n\}$ wrt $\mathscr{Z}$. A bounded sequence $\{\zeta_n\}$ in a hyperbolic space (Γ, ρ, Ω) is said to Δ-converge to ζ if ζ is the unique asymptotic center for every subsequence $\{u_n\}$ of $\{\zeta_n\}$. A sequence $\{\zeta_n\} \subseteq \Gamma$ is called Fejér monotone wrt $\mathscr{Z}$ if for all $\zeta^\dagger \in \mathscr{Z}$

$$\rho\left(\zeta^\dagger, \zeta_{n+1}\right) \le \rho\left(\zeta^\dagger, \zeta_n\right), \tag{9}$$

for all $n \ge 0$.

Definition 7 (see [29]). A mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ is called quasi-nonexpansive if

$$\begin{gathered} \rho\left(F(\zeta), \zeta^\dagger\right) \le \rho\left(\zeta, \zeta^\dagger\right) \forall \zeta \in \mathscr{Z}, \\ \zeta^\dagger \in Fix(F) \ne \varnothing, \end{gathered} \tag{10}$$

where $\mathrm{Fix}(F) = \{\zeta^\dagger \in \mathscr{Z} | F(\zeta^\dagger) = \zeta^\dagger\}$.

Definition 8 (see [30]). The mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ with Fix $(F) \neq \emptyset$ is said to have Condition (I) if the following assumptions are satisfied:

(a) $\exists$ a function $f : [0,\infty) \longrightarrow [0,\infty)$ which is nondecreasing

(b) For $r \in (0,\infty)$, $f(r) > 0$ and $f(0) = 0$

(c) For all $\zeta \in \mathscr{Z}$, $\rho(\zeta, F(\zeta)) \geq f(\rho(\zeta, \mathrm{Fix}(F)))$

where $\rho(\zeta, \mathrm{Fix}(F)) = \inf \{\rho(\zeta, \xi) : \xi \in \mathrm{Fix}(F)\}$.

Definition 9. Let (Γ, ρ) be a metric space and $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \neq \emptyset$. A mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ is called as compact if $F(\mathscr{Z})$ has a compact closure.

Proposition 10 (see [20]). *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space, $\mathscr{Z} \subseteq \Gamma$ with $\mathscr{Z} \neq \emptyset$. Suppose that $\mathscr{Z}$ is convex and closed, and $\{\zeta_n\}$ is bounded sequence in Γ. Then, $\{\zeta_n\}$ has a unique asymptotic center with respect to $\mathscr{Z}$.*

Lemma 11 (see [17]). *Let (Γ, ρ, Ω) be same as in Proposition 10. Let $w \in \Gamma$ and $\{\omega_n\}$ be a sequence with $\{\omega_n\} \subseteq [a, b] \subseteq (0, 1)$. For some $r \geq 0$, if $\{\zeta_n\}$ and $\{\xi_n\}$ are sequences in Γ with $\limsup_{n\longrightarrow\infty} \rho(\zeta_n, w) \leq r$, $\limsup_{n\longrightarrow\infty} \rho(\xi_n, w) \leq r$, and $\lim_{n\longrightarrow\infty} \rho(\omega_n \xi_n \oplus (1 - \omega_n)\zeta_n, w) = r$. Then, $\lim_{n\longrightarrow\infty} \rho(\xi_n, \zeta_n) = 0$.*

Lemma 12. *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Proposition 10. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a mapping. For $\lambda \in (0, 1)$, consider $\Psi : \mathscr{Z} \longrightarrow \mathscr{Z}$ as follows:*

$$\Psi(\zeta) = (1 - \lambda)\zeta \oplus \lambda F(\zeta), \tag{11}$$

for all $\zeta \in \mathscr{Z}$. Then, $Fix(\Psi) = Fix(F)$.

Lemma 13 (see [20]). *Let $\{\zeta_n\}$ be a bounded sequence in Γ and $A(\mathscr{Z}, \{\zeta_n\}) = \{\upsilon\}$. Let $\{\kappa_n\}$ and $\{\nu_n\}$ be two sequences in $\mathbb{R}$ with for all $n \in \mathbb{N}, \kappa_n \in [0,\infty)$, $\limsup \kappa_n \leq 1$ and $\limsup \nu_n \leq 0$. Suppose that $\xi \in \mathscr{Z}$ and there exists m, $q \in \mathbb{N}$ such that*

$$\rho(\xi, \zeta_{n+m}) \leq \kappa_n \rho(\upsilon, \zeta_n) + \nu_n \forall n \geq q. \tag{12}$$

Then, $\xi = \upsilon$.

Lemma 14 (see [14]). *Let (Γ, ρ, Ω) be a metric space, $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$. If $\{\zeta_n\}$ is Fejér monotone wrt $\mathscr{Z}$, $A(\mathscr{Z}, \{\zeta_n\}) = \{\zeta\}$ and $A(\Gamma, \{u_n\}) \subseteq \mathscr{Z}$ for every subsequence $\{u_n\}$ of $\{\zeta_n\}$. Then, the sequence $\{\zeta_n\}$ Δ-converges to $\zeta \in \mathscr{Z}$.*

Lemma 15 (see [16]). *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space and $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$ and $\mathscr{Z}$ is closed convex. Let $\{\zeta_n\}$ be a bounded sequence in Γ and $\tau : \mathscr{Z} \longrightarrow [0,\infty)$ a function defined as follows:*

$$\tau(\zeta) = \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \zeta), \tag{13}$$

for any $\zeta \in \mathscr{Z}$. τ is called as type function, and it is unique.

Then, there is a minimum point (unique) $w \in \mathscr{Z}$ and $\tau(w) = \inf \{\tau(\zeta) : \zeta \in \mathscr{Z}\}$.

Proposition 16 (see [13]). *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Lemma 15 with $\mathscr{Z}$ is bounded. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a nonexpansive mapping. Let $\zeta_0 \in \mathscr{Z}$ and $\vartheta \in (0, 1)$. Define a sequence $\{\zeta_n\}$ in $\mathscr{Z}$ by Krasnosel'skiĭ iterative method* [31].

$$\zeta_{n+1} = (1 - \vartheta)\zeta_n \oplus \vartheta F(\zeta_n), n \in \mathbb{N} \cup \{0\}. \tag{14}$$

Then, $\lim_{n\longrightarrow\infty} \rho(\zeta_n, F(\zeta_n)) = 0$.

The proof of the following theorem is motivated from [16].

Theorem 17. *Let (Γ, ρ, Ω), $\mathscr{Z}$, and F be same as in Proposition 16. Then, $Fix(F) \neq \emptyset$.*

Proof. For a given $\zeta_0 \in \mathscr{Z}$ and for any $\omega \in (0, 1)$, a sequence can be defined:

$$\zeta_{n+1} = (1 - \omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{15}$$

From Proposition 16, it implies that

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{16}$$

From Lemma 15, there is a minimum point (unique) $\upsilon^\dagger \in \mathscr{Z}$ in such a way that

$$\tau(\upsilon^\dagger) = \inf \{\tau(\omega) : \omega \in \mathscr{Z}\}. \tag{17}$$

From the definition of mapping Ψ,

$$\begin{aligned} \tau(\Psi(\upsilon^\dagger)) &= \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\upsilon^\dagger)) \\ &\leq \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) + \limsup_{n\longrightarrow\infty} \rho(\Psi(\zeta_n), \Psi(\upsilon^\dagger)) \\ &\leq \limsup_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon^\dagger). \end{aligned} \tag{18}$$

Then, $\Psi(\upsilon^\dagger) = \upsilon^\dagger$. □

3. Main Results

In 2019, Berinde [10] considered a new class of mappings which is defined below.

Definition 18. Let $(\Gamma, \|.\|)$ be a Banach space and $F : \Gamma \longrightarrow \Gamma$ a mapping. The mapping F is called b-enriched nonexpansive if $\exists b \in [0,\infty)$ in such a way that

$$\|b(\zeta - \xi) + F(\zeta) - F(\xi)\| \leq (b+1)\|\zeta - \xi\|, \tag{19}$$

for all $\zeta, \xi \in \Gamma$.

It can be noted that 0-enriched mapping is nonexpansive mapping. Even both the class of mappings, that is, quasi-nonexpansive and b-enriched nonexpansive, are independent in nature, cf. [27].

Remark 19. Take $b \neq 0$, and it is straight forward from (19) that

$$\begin{aligned}
&\left\|\frac{b}{b+1}(\zeta - \xi) + \frac{1}{b+1}(F(\zeta) - F(\xi))\right\| \\
&\quad \leq \|\zeta - \xi\| \Leftrightarrow \left\|\left(1 - \frac{1}{b+1}\right)(\zeta - \xi) + \frac{1}{b+1}(F(\zeta) - F(\xi))\right\| \\
&\quad \leq \|\zeta - \xi\| \Leftrightarrow \left\|\left(1 - \frac{1}{b+1}\right)\zeta + \frac{1}{b+1}F(\zeta)\right. \\
&\qquad \left. - \left\{\left(1 - \frac{1}{b+1}\right)\xi + \frac{1}{b+1}F(\xi)\right\}\right\| \\
&\quad \leq \|\zeta - \xi\|.
\end{aligned} \tag{20}$$

Take, $\lambda_b = 1/(b+1) \in (0,1)$ then

$$\|(1-\lambda_b)\zeta + \lambda_b F(\zeta) - \{(1-\lambda_b)\xi + \lambda_b F(\xi)\}\| \leq \|\zeta - \xi\|. \tag{21}$$

From the above inequality, we can take convex combination of F and the identity mappings.

In view of Remark 19, we consider Definition 18 in Ω-hyperbolic spaces.

Definition 20. Let (Γ, ρ, Ω) be a Ω-hyperbolic space, $\mathscr{X}$ a subset of Γ such that $\mathscr{X} \neq \emptyset$, and $F : \mathscr{X} \longrightarrow \mathscr{X}$ a mapping. The mapping F is called b-enriched nonexpansive if $\exists b \in [0,\infty)$ in such a way that

$$\rho((1-\lambda_b)\zeta \oplus \lambda_b F(\zeta), (1-\lambda_b)\xi \oplus \lambda_b F(\xi)) \leq \rho(\zeta, \xi), \tag{22}$$

for all $\zeta, \xi \in \mathscr{X}$, where $\lambda_b = 1/(b+1)$.

We prove the following important lemma which will be utilized throughout this paper.

Lemma 21. *Let (Γ, ρ, Ω) be a uniquely geodesic space. For some $\lambda, \omega \in (0,1)$, let $\zeta, \xi, \upsilon \in \Gamma$ be pairwise distinct points with $\xi = (1-\lambda)\zeta \oplus \lambda\upsilon$ and $\upsilon = (1-\omega)\zeta \oplus \omega w$. Then*

$$\xi = (1-\vartheta)\zeta \oplus \vartheta w, \tag{23}$$

where $\vartheta = \omega\lambda$.

Proof. From Lemma 5, ξ lies between ζ and υ. And υ lies between ζ and w. From Proposition 6, ξ lies between ζ and w. Thus, $\xi \in [\zeta, w]$ and

$$\xi = (1-\vartheta)\zeta \oplus \vartheta w, \tag{24}$$

for some $\vartheta \in (0,1)$. Since Γ is uniquely geodesic space, we have

$$\rho(\zeta, \xi) = \vartheta\rho(\zeta, w), \tag{25}$$

$$\rho(\xi, w) = (1-\vartheta)\rho(\zeta, w). \tag{26}$$

Since $\xi = (1-\lambda)\zeta \oplus \lambda\upsilon$, we have

$$\rho(\zeta, \xi) = \lambda\rho(\zeta, \upsilon). \tag{27}$$

Again, since $\upsilon = (1-\omega)\zeta \oplus \omega w$, we have

$$\rho(\zeta, \upsilon) = \omega\rho(\zeta, w). \tag{28}$$

From (25), (27), and (28), one can conclude

$$\rho(\zeta, \xi) = \lambda\rho(\zeta, \upsilon) = \omega\lambda\rho(\zeta, w) = \frac{\omega\lambda}{\vartheta}\rho(\zeta, \xi). \tag{29}$$

Therefore, $\omega\lambda/\vartheta = 1$, and $\vartheta = \omega\lambda$. □

Theorem 22. *Let (Γ, ρ, Ω) be a complete UCΩ-hyperbolic space and $\mathscr{X} \subseteq \Gamma$ such that $\mathscr{X} \neq \emptyset$. Assume that $\mathscr{X}$ is closed, bounded, and convex. Let $F : \mathscr{X} \longrightarrow \mathscr{X}$ be a b-enriched nonexpansive mapping. Then, $Fix(F) \neq \emptyset$. Moreover, for given $\zeta_0 \in \mathscr{X}$, any $\omega \in (0,1)$, there exists $\omega_b = \omega/(b+1)$ such that the sequence $\{\zeta_n\}$ generated by (Krasnosel'skiĭ method)*

$$\zeta_{n+1} = (1-\omega_b)\zeta_n + \omega_b F(\zeta_n) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{30}$$

Δ-converges to an element of $Fix(F)$.

Proof. By the definition of mapping F, we get

$$\rho((1-\lambda_b)\zeta \oplus \lambda_b F(\zeta), (1-\lambda_b)\xi \oplus \lambda_b F(\xi)) \leq \rho(\zeta, \xi), \tag{31}$$

for all $\zeta, \xi \in \mathscr{X}$ and $\lambda_b = 1/(b+1)$. Set the mapping Ψ as follows:

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta) \text{ for all } \zeta \in \mathscr{X}. \tag{32}$$

Thus, from (31), we get, for all $\zeta, \xi \in \mathscr{X}$,

$$\rho(\Psi(\zeta), \Psi(\xi)) \leq \rho(\zeta, \xi), \tag{33}$$

and Ψ is a nonexpansive mapping. For any $\omega \in (0,1)$ and a given $\zeta_0 \in \mathscr{X}$, we can define a sequence

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{34}$$

From Proposition 16, it follows that

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{35}$$

From Theorem 17, $\mathrm{Fix}(\Psi) \neq \emptyset$; thus, from Lemma 12, $\mathrm{Fix}(\Psi) = \mathrm{Fix}(F) \neq \emptyset$. Further, for any $v^\dagger \in \mathrm{Fix}(\Psi)$,

$$\rho(\Psi(\zeta_n), v^\dagger) \leq \rho(\zeta_n, v^\dagger) \text{ for all } n \geq 0. \tag{36}$$

Thus, from (W1)

$$\begin{aligned}\rho(\zeta_{n+1}, v^\dagger) &= \rho((1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n), v^\dagger) \\ &\leq (1-\omega)\rho(\zeta_n, v^\dagger) + \omega\rho(\Psi(\zeta_n), v^\dagger) \\ &\leq \rho(\zeta_n, v^\dagger).\end{aligned} \tag{37}$$

Hence, the sequence $\{\rho(\zeta_n, v^\dagger)\}$ is monotone nonincreasing. It implies that $\{\zeta_n\}$ is Fejér monotone sequence wrt $\mathrm{Fix}(F)$. In view of Proposition 10, the sequence $\{\zeta_n\}$ has unique asymptotic center $w^\dagger$ wrt $\mathrm{Fix}(F)$. Suppose $\{u_n\}$ is a subsequence of $\{\zeta_n\}$ and $u^\dagger$ is unique asymptotic center of $\{u_n\}$ wrt $\mathrm{Fix}(F)$. Now,

$$\begin{aligned}\rho(u_n, \Psi(u^\dagger)) &\leq \rho(\Psi(u_n), \Psi(u^\dagger)) + \rho(\Psi(u_n), u_n) \\ &\leq \rho(u_n, u^\dagger) + \rho(\Psi(u_n), u_n).\end{aligned} \tag{38}$$

From (35) and Lemma 13, it follows that $\Psi(u^\dagger) = u^\dagger$. From Lemma 14, the sequence $\{\zeta_n\}$$\Delta$-converges to an element of $\mathrm{Fix}(F)$. From Lemma 21 with $v = \Psi(\zeta)$ and $w = F(\zeta)$, we have

$$(1-\omega)\zeta \oplus \omega\Psi(\zeta) = (1-\omega\lambda_b)\zeta \oplus \omega\lambda_b F(\zeta), \tag{39}$$

for all $\zeta \in \mathscr{Z}$ since $\omega \in (0,1)$ and $\lambda_b = 1/(b+1)$. It follows that $\omega\lambda_b \in (0, 1/(b+1))$. Thus, for any $\omega_b = \omega\lambda_b \in (0, 1/(b+1))$, the sequence $\{\zeta_n\}$ defined by (30) Δ-converges to a point in $\mathrm{Fix}(F)$. □

Remark 23. It can be seen that Theorem 22 generalizes the results in [10] (Theorem 3.3) from Hilbert spaces to hyperbolic spaces.

Theorem 24. *Let (Γ, ρ, Ω) and F be same as in Theorem 22. Suppose $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \emptyset$, and $\mathscr{Z}$ is closed and convex. Assume F satisfies Condition (I) with $\mathrm{Fix}(F) \neq \emptyset$. For fixed $\zeta_0 \in \mathscr{Z}$ and any $\omega \in (0,1)$, there exists $\omega_b = \omega/(b+1)$ such that the sequence $\{\zeta_n\}$ generated by (Krasnosel'skiĭ method)*

$$\zeta_{n+1} = (1-\omega_b)\zeta_n + \omega_b F(\zeta_n) \forall n \in \mathbb{N} \cup \{0\}, \tag{40}$$

strongly converges to an element of $\mathrm{Fix}(F)$.

Proof. By the similar technique in proof of Theorem 22, one can set a mapping Ψ as in (32), and Ψ is nonexpansive. Let $\omega \in (0,1)$ and define

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{41}$$

For all $v^\dagger \in \mathrm{Fix}(\Psi)$

$$\rho(\Psi(\zeta_n), v^\dagger) \leq \rho(\zeta_n, v^\dagger) \text{ for all } n \geq 1. \tag{42}$$

From (41), we have

$$\rho(\zeta_{n+1}, v^\dagger) \leq \rho(\zeta_n, v^\dagger). \tag{43}$$

Thus, $\{\rho(\zeta_n, v^\dagger)\}$ and $\{\rho(\zeta_n, \mathrm{Fix}(\Psi))\}$ are monotone nonincreasing sequences and $\lim_{n\longrightarrow\infty}\rho(\zeta_n, v^\dagger)$ and $\lim_{n\longrightarrow\infty}\rho(\zeta_n, \mathrm{Fix}(\Psi))$ exist. Let

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, v^\dagger) = r. \tag{44}$$

From (42)

$$\lim_{n\longrightarrow\infty}\rho(\Psi(\zeta_n), v^\dagger) \leq r. \tag{45}$$

By (44), we have

$$r = \lim_{n\longrightarrow\infty}\rho(\zeta_{n+1}, v^\dagger) = \lim_{n\longrightarrow\infty}\rho((1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n), v^\dagger). \tag{46}$$

In view of (44), (45), (46), and Lemma 11, it implies:

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{47}$$

Since

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta), \tag{48}$$

we have

$$\rho(\zeta, \Psi(\zeta)) = \frac{1}{b+1}\rho(\zeta, F(\zeta)) \text{ for } all\, \zeta \in \mathscr{Z}. \tag{49}$$

Since F satisfies Condition (I) and (49), we obtain

$$\begin{aligned}(b+1)\rho(\zeta_n, \Psi(\zeta_n)) = \rho(\zeta_n, F(\zeta_n)) &\geq f(\rho(\zeta_n, \mathrm{Fix}(F))) \\ &= f(\rho(\zeta_n, \mathrm{Fix}(\Psi))).\end{aligned} \tag{50}$$

By (47), $\lim_{n\longrightarrow\infty} f(\rho(\zeta_n, \mathrm{Fix}(\Psi))) = 0$ and

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \mathrm{Fix}(\Psi)) = 0. \tag{51}$$

One can easily show that $\{\zeta_n\}$ is a Cauchy sequence. For the sake of completeness, we prove this claim. From (51), for given $\varepsilon > 0$, $\exists$ a $n_0 \in \mathbb{N}$ in such a way that

$$\rho(\zeta_n, \mathrm{Fix}(\Psi)) < \frac{\varepsilon}{4}, \tag{52}$$

for all $n \geq n_0$. Hence,

$$\inf \left\{ \rho\left(\zeta_{n_0}, \upsilon^\dagger\right) : \upsilon^\dagger \in \mathrm{Fix}(\Psi) \right\} < \frac{\varepsilon}{4}, \tag{53}$$

so there is $\upsilon^\dagger \in \mathrm{Fix}(\Psi)$ in such a way

$$\rho\left(\zeta_{n_0}, \upsilon^\dagger\right) < \frac{\varepsilon}{2}. \tag{54}$$

Therefore, for all $m, n \geq n_0$,

$$\rho(\zeta_{n+m}, \zeta_n) \leq \rho\left(\zeta_{n+m}, \upsilon^\dagger\right) + \rho\left(\upsilon^\dagger, \zeta_n\right) \leq 2\rho\left(\zeta_{n_0}, \upsilon^\dagger\right) < 2\frac{\varepsilon}{2} = \varepsilon, \tag{55}$$

and $\{\zeta_n\}$ is a Cauchy sequence. By the closedness of $\mathscr{Z}$ in Γ, $\{\zeta_n\}$ converges to a point $\zeta^\dagger \in \mathscr{Z}$. Now

$$\begin{aligned} \rho\left(\zeta^\dagger, \Psi\left(\zeta^\dagger\right)\right) &\leq \rho\left(\zeta^\dagger, \zeta_n\right) + \rho(\zeta_n, \Psi(\zeta_n)) + \rho\left(\Psi(\zeta_n), \Psi\left(\zeta^\dagger\right)\right) \\ &\leq 2\rho\left(\zeta^\dagger, \zeta_n\right) + \rho(\zeta_n, \Psi(\zeta_n)). \end{aligned} \tag{56}$$

From (47), $\zeta^\dagger = \Psi(\zeta^\dagger)$. Therefore, the sequence $\{\zeta_n\}$ strongly converges to a point in $\mathrm{Fix}(F)$. Further,

$$(1-\omega)\zeta \oplus \omega\Psi(\zeta) = (1-\omega_b)\zeta \oplus \omega_b F(\zeta), \tag{57}$$

for all $\zeta \in \mathscr{Z}$ with $\omega_b = \omega/(b+1)$. □

Remark 25. Theorem 24 generalizes the results in [11] (Theorem 3.2) from Hilbert spaces to hyperbolic spaces.

Theorem 26. *Let (Γ, ρ, Ω) and F be same as in Theorem 22. Let $\mathscr{Z} \subseteq \Gamma$ such that $\mathscr{Z} \neq \varnothing$ and $\mathscr{Z}$ be a closed and convex. Suppose that F is compact mapping with $\mathrm{Fix}(F) \neq \varnothing$. For fixed $\lambda \in (0, 1/(b+1))$, $\{\zeta_n\}$ is a sequence generated as follows:*

$$\zeta_{n+1} = (1-\lambda)\zeta_n \oplus \lambda F(\zeta_n), \tag{58}$$

strongly converges to an element of $\mathrm{Fix}(F)$.

Proof. We set the nonexpansive mapping Ψ as in the proof of Theorem 22. For given $\zeta_0 \in \mathscr{Z}$ and for any $\omega \in (0, 1)$, define a sequence

$$\zeta_{n+1} = (1-\omega)\zeta_n \oplus \omega\Psi(\zeta_n). \tag{59}$$

Following largely as in Theorem 24 and from Lemma 11

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{60}$$

Since

$$\Psi(\zeta) = \left(1 - \frac{1}{b+1}\right)\zeta \oplus \frac{1}{b+1}F(\zeta), \tag{61}$$

we get

$$\rho(\zeta, \Psi(\zeta)) = \frac{1}{b+1}\rho(\zeta, F(\zeta)) \text{ for all } \zeta \in \mathscr{Z}. \tag{62}$$

From the above equation and (60)

$$\lim_{n\longrightarrow\infty} \rho(\zeta_n, F(\zeta_n)) = 0. \tag{63}$$

Since the range of $\mathscr{Z}$ under F is subset of a compact set, there is a subsequence $\{F(\zeta_{n_j})\}$ of $\{F(\zeta_n)\}$ strongly converges to $\zeta^\dagger \in \mathscr{Z}$. By (63), the subsequence $\{\zeta_{n_j}\}$ strongly converges to $\zeta^\dagger$. Since Ψ is nonexpansive mapping and by the triangle inequality, we obtain

$$\begin{aligned} \rho\left(\zeta_{n_j}, \Psi\left(\zeta^\dagger\right)\right) &\leq \rho\left(\zeta_{n_j}, \Psi\left(\zeta_{n_j}\right)\right) + \rho\left(\Psi\left(\zeta_{n_j}\right), \Psi\left(\zeta^\dagger\right)\right) \\ &\leq \rho\left(\zeta_{n_j}, \Psi\left(\zeta_{n_j}\right)\right) + \rho\left(\zeta_{n_j}, \zeta^\dagger\right). \end{aligned} \tag{64}$$

Thus, subsequence $\{\zeta_{n_j}\}$ strongly converges to $\Psi(\zeta^\dagger)$ and $\Psi(\zeta^\dagger) = \zeta^\dagger$. Since $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \zeta^\dagger)$ exists, it follows that the sequence $\{\zeta_n\}$ strongly converges to an element of $\mathrm{Fix}(F)$. □

4. SP Iterative Method

In this section, we present some convergence results for SP iterative process. For a fix $\zeta_0 \in \mathscr{Z}$ and the mapping $F : \mathscr{Z} \longrightarrow \mathscr{Z}$, the SP iterative method in the setting of hyperbolic metric spaces can be defined as follows [26]:

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n F(\zeta_n), \\ \xi_n = (1-\vartheta_n)w_n \oplus \vartheta_n F(w_n), \\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n F(\xi_n), \end{cases} \tag{65}$$

where $\{\sigma_n\}$, $\{\vartheta_n\}$, and $\{\omega_n\}$ are sequences in $[0, 1]$.

Similar to [25] (Lemma 4), we model the following lemma.

Lemma 27. *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Theorem 24. Let $\Psi : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a nonexpansive mapping with $\mathrm{Fix}(F) \neq \varnothing$. For fixed $\zeta_0 \in \mathscr{Z}$ and for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega, \vartheta]$ with $\omega, \vartheta \in (0, 1)$, the sequence $\{\zeta_n\}$ is defined by*

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n \Psi(\zeta_n), \\ \xi_n = (1-\vartheta_n)w_n \oplus \vartheta_n \Psi(w_n), \\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n \Psi(\xi_n). \end{cases} \tag{66}$$

Then, the following holds:

(1) $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \upsilon)$ exists $\forall \upsilon \in \mathrm{Fix}(\Psi)$

(2) $\lim_{n\longrightarrow\infty} \rho(\zeta_n, \Psi(\zeta_n)) = 0$

Proof. From (W1), we get

$$\begin{aligned}\rho(w_n, \upsilon) &\leq (1-\sigma_n)\rho(\zeta_n, \upsilon) + \sigma_n\rho(\Psi(\zeta_n), \upsilon)\\ &\leq (1-\sigma_n)\rho(\zeta_n, \upsilon) + \sigma_n\rho(\zeta_n, \upsilon) = \rho(\zeta_n, \upsilon),\end{aligned} \tag{67}$$

$$\begin{aligned}\rho(\xi_n, \upsilon) &\leq (1-\vartheta_n)\rho(w_n, \upsilon) + \vartheta_n\rho(\Psi(w_n), \upsilon)\\ &\leq (1-\vartheta_n)\rho(w_n, \upsilon) + \vartheta_n\rho(w_n, \upsilon) = \rho(w_n, \upsilon).\end{aligned} \tag{68}$$

Further, from (67) and (68), we get

$$\begin{aligned}\rho(\zeta_{n+1}, \upsilon) &\leq (1-\omega_n)\rho(\xi_n, \upsilon) + \omega_n\rho(\Psi(\xi_n), \upsilon)\\ &\leq (1-\omega_n)\rho(\xi_n, \upsilon) + \omega_n\rho(\xi_n, \upsilon)\\ &= \rho(\xi_n, \upsilon) \leq \rho(w_n, \upsilon) \leq \rho(\zeta_n, \upsilon).\end{aligned} \tag{69}$$

Thus, $\{\rho(\zeta_n, \upsilon)\}$ is a monotone nonincreasing sequence. Hence, $\lim_{n\longrightarrow\infty}\rho(\zeta_n, \upsilon)$ exists. Let

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \upsilon) = r > 0. \tag{70}$$

From (69) and (70), we have

$$\begin{aligned} r &\leq \liminf_{n\longrightarrow\infty}\rho(w_n, \upsilon),\\ \limsup_{n\longrightarrow\infty}\rho(w_n, \upsilon) &\leq r.\end{aligned} \tag{71}$$

Thus

$$\lim_{n\longrightarrow\infty}\rho(w_n, \upsilon) = r. \tag{72}$$

Since the mapping Ψ is nonexpansive

$$\limsup_{n\longrightarrow\infty}\rho(\Psi(\zeta_n), \upsilon) \leq \lim_{n\longrightarrow\infty}\rho(\zeta_n, \upsilon) = r, \tag{73}$$

and from (72)

$$\lim_{n\longrightarrow\infty}\rho((1-\sigma_n)\zeta_n \oplus \sigma_n\Psi(\zeta_n), \upsilon) = \lim_{n\longrightarrow\infty}\rho(w_n, \upsilon) = r. \tag{74}$$

From (70), (73), (74), and Lemma 11, it follows:

$$\lim_{n\longrightarrow\infty}\rho(\zeta_n, \Psi(\zeta_n)) = 0. \tag{75}$$

□

Theorem 28. *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Theorem 24. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a b-enriched nonexpansive mapping with $Fix(F) \neq \varnothing$. For fixed $\zeta_0 \in \mathscr{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, ω_n, $\vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0, 1)$, the sequence $\{\zeta_n\}$ generated by (65) Δ-converges to an element of $Fix(F)$.*

Proof. For given $\zeta_0 \in \mathscr{Z}$ and for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n^b, \vartheta_n^b, \sigma_n^b \in [\omega, \vartheta]$ with $\omega, \vartheta \in (0, 1)$, we can consider a sequence $\{\zeta_n\}$:

$$\begin{cases} w_n = \left(1-\sigma_n^b\right)\zeta_n \oplus \sigma_n^b\Psi(\zeta_n),\\ \xi_n = \left(1-\vartheta_n^b\right)w_n \oplus \vartheta_n^b\Psi(w_n),\\ \zeta_{n+1} = \left(1-\omega_n^b\right)\xi_n \oplus \omega_n^b\Psi(\xi_n),\end{cases} \tag{76}$$

where Ψ is a mapping defined as in (32). Using Lemma 21, we have

$$\begin{cases} w_n = (1-\sigma_n)\zeta_n \oplus \sigma_n F(\zeta_n),\\ \xi_n = (1-\vartheta_n)w_n \oplus \vartheta_n F(w_n),\\ \zeta_{n+1} = (1-\omega_n)\xi_n \oplus \omega_n F(\xi_n),\end{cases} \tag{77}$$

where $\sigma_n = \sigma_n^b/(b+1)$, $\vartheta_n = \vartheta_n^b/(b+1)$, and $\omega_n = \omega_n^b/(b+1)$. By Lemma 27 and repeating the technique of proof of Theorem 22, one can complete the proof. □

Theorem 29. *Let (Γ, ρ, Ω), $\mathscr{Z}$, and F be same as in Theorem 24. For fixed $\zeta_0 \in \mathscr{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0, 1)$, the sequence $\{\zeta_n\}$ generated by (65) strongly converges to an element of $Fix(F)$.*

Proof. Using proof of Theorem 28, Lemma 27, and Theorem 24, one can complete the proof. □

Theorem 30. *Let (Γ, ρ, Ω) and $\mathscr{Z}$ be same as in Theorem 24. Let $F : \mathscr{Z} \longrightarrow \mathscr{Z}$ be a compact b-enriched nonexpansive mapping with $Fix(F) \neq \varnothing$. For fixed $\zeta_0 \in \mathscr{Z}$, for all $n \in \mathbb{N} \cup \{0\}$, $\omega_n, \vartheta_n, \sigma_n \in [\omega/(b+1), \vartheta/(b+1)]$ with $\omega, \vartheta \in (0, 1)$, the sequence $\{\zeta_n\}$ generated by (65) strongly converges to a point in $Fix(F)$.*

Proof. Using proof of Theorem 28, Lemma 27, and Theorem 26, one can complete the proof. □

Remark 31. Theorems 28–30 are new even in Hilbert spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to this work.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7078707, 5 pages
https://doi.org/10.1155/2022/7078707

Research Article

A Self-Adaptive Technique for Solving Variational Inequalities: A New Approach to the Problem

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Received 25 January 2022; Accepted 24 March 2022; Published 26 April 2022

Academic Editor: Rale M. Nikoli

Variational inequalities are considered the most significant field in applied mathematics and optimization because of their massive and vast applications. The current study proposed a novel iterative scheme developed through a fixed-point scheme and formulation for solving variational inequalities. Modification is done by using the self-adaptive technique that provides the basis for predicting a new predictor-corrector self-adaptive for solving nonlinear variational inequalities. The motivation of the presented study is to provide a meaningful extension to existing knowledge through convergence at mild conditions. The numerical interpretation provided a significant boost to the results.

1. Introduction

Earlier, most of the equilibrium-related queries were resolved by variational inequalities that are a mathematical theory. In this regard, Stampacchia [1] is considered a pioneer who initially introduced variational inequalities in 1964. At the end of 1964, Stampacchia extended his work by introducing partial differential equations. Since then, this field has become the most emerging and demanding with extensive applications in optimization and control, economics, movements, engineering sciences, and equilibrium problems. Massive utilization of variational inequalities in applied sciences made it branched and more generalized to interact with other fields [2–5], hence proved the novelty and productivity of variational inequalities. Most of the profound task for researchers is to work on extensions and generalized inequalities regarding their applications; consequently, it gives rise to pure and applied mathematics problems. Modifications in variational inequalities produced advances in numerical methods [6–10], sensitivity analysis, and the dynamical system that are efficient in solving mathematics-related problems. Theory and algorithmic advancements meet in the theory of variational inequalities, opening up a brand-new field of application [7, 11, 12]. These issues necessitate a combination of convex, functional, and numerical analysis techniques. There are numerous exciting applications for this fascinating section of applied mathematics in the fields of business, finance, economics, and the social, as well as the pure and applied sciences (see [3, 9, 13, 14] and the references therein for applications and numerical approaches). Such extraordinary progress is based on the most basic and unidirectional linear and nonlinear approaches.

A fundamental problem associated with variational inequalities is the establishment of fast numerical methods. A projection-type method and its variant solve many optimization problems and are also related to variational inequalities. Variational inequalities and fixed-point issues with equivalent effects utilizing projection techniques have grown in popularity in recent years as a study focus. To prove the convergence of fixed-point iterative methods, quantitative knowledge of pseudocontractive and nonlinear monotone (accretive) operators combined with Lipschitz type conditions is required (see [15–17]). The phenomena of variational inequalities have a significant contribution to solving the Wiener-Hopf equations. Salient features of Wiener-Hopf equations and optimization problems in the presence of variational inequalities are addressed by Shi [17]. Together with the Wiener-Hopf equation, the projection method is considered an important technique for approximating the solution of variational inequality problems. Constructing an equivalence between fixed-point problems and variational inequalities is made easier with the concept of the projection method. Utilizing variational problems, several conventional improved ways to establish solutions for open, moving boundary value problems, asymmetric obstacle, unilateral, even-order, and odd-order problems could be developed (see [4–7, 11, 15] and the references therein). An investigation into a new predictor-corrector self-adaptive strategy for solving nonlinear variational inequalities under known assumptions is suggested in the proposed study. It was possible to arrive at this fixed-point formulation using projection, variational inequalities, and Wiener-Hopf equations. Additionally, the convergence of the proposed method is discussed.

2. Formulation and Basic Results

A convex set is denoted by K in H (Hilbert space). We denote norm and inner by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. We consider a variational inequality: for general operator T, find $y \in K$ such that

$$\langle Ty, x-y\rangle \geq 0, \forall x \in K. \tag{1}$$

The inequality (1) is called the variational inequality (VI) introduced by Stampacchia [1]. A large number of problems related to equilibrium, nonsymmetric, physical sciences, engineering, moving boundary value problem, unified, obstacle, unilateral contact, and applied sciences can be discussed via the inequalities (1) [1, 6, 7, 12, 13].

Lemma 1. *[13].*

For $z \in H, y \in K$ holds for the inequality

$$\langle y-z, x-y\rangle \geq 0, \forall x \in K, \tag{2}$$

if and only if

$$y = P_K z, \tag{3}$$

where P_K is the projection of H onto K(convex set).

It is also known that the P_K is called projection operator, which is also nonexpansive and holds for the inequality.

$$\|P_K z - y\| \leq \|z-y\| - \|z - P_K z\|. \tag{4}$$

Lemma 2. *If y is a solution of VI (1), then $y \in K$ satisfies the relation*

$$y = P_K[y - \rho Ty], \tag{5}$$

where $\rho \geq 0$ is taken as constant and P_K is considered the projection operator H onto K.

From Lemma 2, it is obvious that y is a solution of VI (1), if and only if y satisfies the residue vector $r(y,\rho)$ defined by

$$r(y,\rho) = y - P_K[y - \rho Ty]. \tag{6}$$

Related to the original inequality (1), we see the Wiener-Hopf equations (WHE) problem. To be more precise, let $Q_K = I - P_K$, where P_K is the projection operator and I is the identity operator. For the operator $T : H \longrightarrow H$, then for finding $z \in H$, we have

$$\rho T P_K z + Q_K z = 0. \tag{7}$$

Here, Equation (7) is the Wiener-Hopf equation (WHE), investigated by Shi [17]. This WHE (7) is considered more general and gives a unified framework to establish the various powerful and efficient iterative methods and numerical techniques (for the application of the WHE (7), see [17, 18]).

Lemma 3. *The inequality (1) has a unique solution $y \in K$, if and only if $z \in H$ satisfies the WHE (7), provided*

$$y = P_K z, \tag{8}$$

$$z = y - \rho Ty. \tag{9}$$

Lemma 3 implies that the VI (1) is equivalent to WHE (7). Noor et al. [8, 18] considers this fixed-point formulation to establish various iterative schemes for solving the VI and other optimization and related problems.

3. Main Results and Algorithm

To solve the variational inequality (1), we will use an iterative approach that we are developing in this study. The relevant results, algorithm, and theory will be established to make an iterative process for solving the inequality. The convergence of the new technique will also be provided.

We use the fixed-point formulation and suggest a predictor-corrector technique for upgrading the solution for VI.

$$w = P_K[y - \gamma Ty], \text{for } \gamma > 0, \tag{10}$$

$$y = P_K[w - \rho Tw] = P_K[P_K[y - \gamma Ty] - \rho T P_K[y - \gamma Ty]]. \tag{11}$$

Using (6), (8), and (10), the WHE (7) can be written in the form

$$\begin{aligned} 0 &= y - P_K[y - \rho Ty] - \rho Ty + \rho T P_K[y - \rho Ty] \\ &= r(y,\rho) - \rho Ty + \rho T P_K[y - \rho Ty]. \end{aligned} \tag{12}$$

We define the relation

$$D(y,\rho) = r(y,\rho) - \rho T y + \rho T P_K[y - \rho T y]. \tag{13}$$

It is obvious that $y \in K$ is a solution of the VI if and only if $y \in K$ is satisfied with Equation (13).

$$D(y,\rho) = 0. \tag{14}$$

Using (10) and (13), we can rewrite as

$$w = P_K[y - \gamma D(y,\rho) - \gamma T y], \tag{15}$$

This fact has motivated us to establish the new predictor-corrector self-adaptive iterative method for solving the VI (1).

Algorithm 1. Step 1: Give $\epsilon > 0, \gamma > 0, \delta \in (0,1), \delta_0 \in (0,1)$, $\mu \in (0,1), \rho > 0$, and $y^* \in H$ set $n = 0$

Step 2: Set $\rho_n = \rho$; if $\|r(y^n,\rho)\| < \epsilon$, then computation stops; otherwise, the iteration will continue to find the m_n nonnegative integer, and take $\rho_n = \rho\mu^{m_n}$ which satisfies the inequality

$$\|\rho_n(T(y^n) - T(w^n))\| \le \delta \|r(y^n,\rho_n)\|, \tag{16}$$

where

$$w^n = P_K[y^n - \gamma_n D(y^n,\rho_n) - \gamma_n T y^n] \tag{17}$$

Step 3: Compute

$$d(y^n,\rho_n) = r(y^n,\rho_n) - \rho_n T(y^n) + \rho_n T(P_K[y^n - \rho_n T y^n]), \tag{18}$$

where

$$r(y^n,\rho_n) = y^n - P_K[y^n - \rho T y^n] \tag{19}$$

Step 4: Get the next iterate

$$w^n = P_K[y^n - \gamma D(y^n,\rho_n) - \gamma T(y^n)], \tag{20}$$

$$y^{n+1} = P_K[w^n - \rho T w^n], \tag{21}$$

and then set $\rho = \rho_n/\mu$, else set $\rho = \rho_n . n = n+1$, and go to Step 2

We observe that Algorithm 1 is refinement and addition of the standard procedure. Here, we consider $-\gamma D(y^n,\rho_n) - \gamma T(y^n)$, the self-adaptive technique, or we can say the step-size. This technique and procedure are closely related to the projection residue.

The convergence of the newly established result of Algorithm 1 is the important part to consider under some suitable and mild conditions, which is the paper's main target and motivation.

Theorem 4. *Let real Hilbert space be denoted by H and $T: K \longrightarrow H$; we take α as strongly monotone, where β is Lipschitz continuous mapping on a convex subset K of H. Let $y^* \in K$ be a solution of VI (1) and let the sequences $\{y^n\}$ be generated by Algorithm 1. If $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}(1 + \gamma\beta) < 1$, then the sequences $\{y^n\}$ converges to y^*, for*

$$0 < \rho < \frac{2\alpha}{\beta^2}. \tag{22}$$

Proof. Since y^* is a solution of NVI (1), from Lemma 1, we have

$$w^* = P_K[y^* - y^* T y^*], \text{for } \gamma > 0, \tag{23}$$

$$y^* = P_K[w^* - \rho T w^*], \text{for } \rho > 0. \tag{24}$$

□

Applying Algorithm 1, from (19) and (24), we know that P_K is nonexpansive:

$$\begin{aligned} \left\|y^{n+1} - y^*\right\| &= \|P_K[w^n - \rho T w^n] - P_K[w^* - \rho T w^*]\| \\ &\le \|w^n - w^* - \rho T w^n + \rho T w^*]\|. \end{aligned} \tag{25}$$

Since T is considered as strongly monotone and Lipschitz continuous with constant α and β. From (25), we have

$$\begin{aligned} &\|w^n - w^* - \rho(Tw^n - Tw^*)\|^2 \\ &= \|w^n - w^*\|^2 - 2\rho\langle Tw^n - Tw^*, w^n - w^*\rangle \\ &\quad + \rho^2\|Tw^n - Tw^*\|^2 \le \|w^n - w^*\|^2 \\ &\quad - 2\rho\alpha\|w^n - w^*\|^2 + \rho^2\beta^2\|w^n - w^*\|^2 \\ &= \left(1 - 2\rho\alpha + \rho^2\beta^2\right)\|w^n - w^*\|^2. \end{aligned} \tag{26}$$

From (25) and (22), we get

$$\left\|y^{n+1} - y^*\right\| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|w^n - w^*\|. \tag{27}$$

From (18) and (22), we get

$$\begin{aligned} \|w^n - w^*\| &= \|P_K[y^n - \gamma D(y^n,\rho_n) - \gamma T y^n] - P_K[y^* - \gamma T y^*]\| \\ &\le \|y^n - \gamma D(y^n,\rho_n) - \gamma T y^n - y^* + \gamma T y^*\| \\ &\le \|y^n - y^* - \gamma D(y^n,\rho_n)\| + \gamma\|Ty^n - Ty^*\| \\ &\le \|y^n - y^* - \gamma D(y^n,\rho_n)\| + \gamma\beta\|y^n - y^*\|. \end{aligned} \tag{28}$$

Consider

$$\begin{aligned} &\|y^n - y^* - \gamma D(y^n,\rho_n)\|^2 \\ &= \|y^n - y^*\|^2 - 2\gamma\langle y^n - y^*, D(y^n,\rho_n)\rangle + \gamma^2\|D(y^n,\rho_n)\|^2. \end{aligned} \tag{29}$$

Table 1: For Algorithm 1 (numerical results).

Order of matrix n	Numerical results Algorithm 1 No. It.	Numerical results [8] No. It.
100	44	44
200	55	55
300	48	48
500	31	31
700	43	43

We use the definition of $D(y^n, \rho_n)$, and we obtain

$$\|y^n - y^* - \gamma D(y^n, \rho_n)\| \le \|y^n - y^*\|. \tag{30}$$

From (28) and (30), we have

$$\|w^n - w^*\| \le (1 + \gamma\beta)\|y^n - y^*\|. \tag{31}$$

From (27) and (31), we get

$$\|y^{n+1} - y^*\| \le \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}(1 + \gamma\beta)\|y^n - y^*\| = \theta\|y^n - y^*\|, \tag{32}$$

where $\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}(1 + \gamma\beta)$, since $0 < \theta < , \sum_{n=0}^{\infty}\theta^n = \infty$, thus from (32) and Algorithm 1 for an arbitrarily chosen and consider initial points y_0 and y^n obtained from Algorithm 1,which converge strongly to y^*.

4. Numerical Example

Example 1. We take the nonlinear complementarity problems: for finding $y \in R^n$, we have

$$y \ge 0, T(y) \ge 0, \langle y, T(y)\rangle = 0. \tag{33}$$

Here, $T(y) = D_1(y) + D_2(y) + q$,we consider$D_1(y)$as nonlinear part, and$D_2(y) + q$is taken as a linear part, and in ((33)), we take a special case of the VI (1). The matrix $D_2 = B^tB + C$, where B is $n \times n$ matrix whose entries we generate randomly in the interval $(-5,+5)$, and skew-symmetric matrix C is considered in the same way. The vector is denoted by q and is obtained in the interval $(-500,+500)$. This is distributed uniformly. For easy problems, we take $(-500,+500)$ and $(-500,0)$ considered for the hard problem. In $D_1(y)$, the nonlinear part of $T(y)$, the components are $D_j(y) = d_j *$ arctan (y_j), and d_j is a random variable generated in $(0, 1)$.

For the output of the result, we consider, $\mu = 2/3, \delta = 0.95$, $\delta_0 = 0.95, \rho > 0$ and $\gamma = 1.95$; the initial guess $y_0 = (0, 0, 0, \cdots, 0)^T$. The computation starts with $\rho_0 = 1$ and stops as soon as $\|r(y_n, \rho_n)\| \le 10^{-7}$. MATLAB is used for all codes. Table 1 represents the outcomes of Algorithm 1.

5. Conclusion

We have considered the new technique for solving inequality (1). We have applied the self-adaptive technique to control the step size under some mild conditions. Results have been compared with the published paper. It has been observed that the number of iterations is reduced by applying the new suggested method. This is an extension of the previously known results. This work can be enhanced further when the operator is pseudomonotone which is considered a weaker condition when the operator is strongly monotonicity. The numerical results reflect the output of our newly established algorithms well for the considered problems.

Data Availability

The manuscript included all required data and information for its implementation.

Conflicts of Interest

All authors declare no conflicts of interest in this paper.

Acknowledgments

We are thankful to the anonymous referees for their valuable comments which helped us to improve the quality of the paper. The authors wish to express their gratitude to Prince Sultan University for facilitating the publication of this article through the research lab Theoretical and Applied Sciences Lab. The authors would like to acknowledge the support of Prince Sultan University for paying the article processing charges (APC) of this publication.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 7075669, 7 pages
https://doi.org/10.1155/2022/7075669

Research Article

Fuzzy Fixed Point Results in Convex C^*-Algebra-Valued Metric Spaces

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Received 2 February 2022; Accepted 25 March 2022; Published 25 April 2022

Academic Editor: Santosh Kumar

The purpose of this note is to come up with some new directions in fuzzy fixed point theory. To this effect, notions of a C^*-algebra-valued fuzzy λ-contraction and related concepts in a convex C^*-algebra-valued metric space (C^*-AVMS) are set-up. In line with the view of a Hausdorff distance function, an idea of a distance between two approximate quantities is proposed. Consequently, two fixed point results of a C^*-algebra-valued fuzzy mapping (C^*-AVFM) for the new type of contractions are established using Mann and Ishikawa iterative schemes. For some future investigations of our results, two open problems are noted concerning sufficient criteria guaranteeing the existence of fixed points of a C^*-algebra-valued fuzzy λ-contraction and whether or not the Picard iteration for a C^*-algebra-valued fuzzy λ-contraction converges.

1. Introduction and Preliminaries

We begin this section with specific notions of C^*-algebras as follows.

Definition 1 (see [1]). Let $\mathscr{A}$ be a unital algebra with the unit $I_{\mathscr{A}}$. An involution on $\mathscr{A}$ is a conjugate linear map $j \mapsto j^*$ such that $j^{**} = j$ and $(j\ell)^* = \ell^* j^*$, for all $j, \ell \in \mathscr{A}$. The pair $(\mathscr{A}, *)$ is called a $*$-algebra. A Banach $*$-algebra is a $*$-algebra $\mathscr{A}$ together with a submultiplicative norm such that $\|j^*\| = \|j\|$, for all $j \in \mathscr{A}$; where a norm $\|.\|$ on an algebra $\mathscr{A}$ is said to be submultiplicative if $\|j\ell\| \leq \|j\| \|\ell\|$, for all $j, \ell \in \mathscr{A}$. A C*-algebra is a Banach $*$-algebra such that $\|j^* j\| = \|j\|^2$, for all $j \in \mathscr{A}$.

Throughout this paper, $\mathscr{A}$ represents a unital C^*-algebra with a unit $I_{\mathscr{A}}$. Also, we take $\mathscr{A}_a = \{j \in \mathscr{A} : j = j^*\}$ and denote the zero element in $\mathscr{A}$ by $0_{\mathscr{A}}$. An element $j \in \mathscr{A}$ is called positive, written $j \pm 0_{\mathscr{A}}$, if $j \in \mathscr{A}_a$ and $\sigma(j) \subseteq \mathbb{R}_+ = [0, \infty)$, where $\sigma(j) = \{\lambda \in \mathbb{C} : \lambda I - j \text{ is not invertible}\}$ is the spectrum of j. Availing positive elements, we set up a partial ordering $^\circ$ on $\mathscr{A}_a$ as follows: $j^\circ \ell$ if and only if $\ell - j \pm 0_{\mathscr{A}}$. Hereafter, by $\mathscr{A}_+$, we mean the set $\{j \in \mathscr{A} : j \pm 0_{\mathscr{A}}\}$ and $|j| = (j^* j)^{1/2}$ (cf. [2]).

Remark 2. When $\mathscr{A}$ is a unital C^*-algebra, then for any $j \in \mathscr{A}_+$, we have $j^\circ I_{\mathscr{A}}$ if and only if $\|j\| \leq 1$ (cf. [1]).

With the aid of positive elements in $\mathscr{A}$, Ma et al. [2] launched the concept of C^*-AVMS in the following manner.

Definition 3 (see [2]). Let $\mho$ be a nonempty set. Suppose that the mapping $\sigma : \mho^2 \longrightarrow \mathscr{A}$ satisfies the following conditions:

(c1) $0_{\mathscr{A}}{}^\circ\sigma(j, \ell)$ and $\sigma(j, \ell) = 0_{\mathscr{A}}$ if and only if $j = \ell$

(c2) $\sigma(j, \ell) = \sigma(\ell, j)$, for all $j, \ell \in \mho$

(c3) $\sigma(j, \ell)^\circ\sigma(j, z) + \sigma(z, \ell)$, for all $j, \ell, z \in \mho$

Then, σ is called a C^*-algebra-valued metric, and $(\mho, \mathscr{A}, \sigma)$ is known as a C^*-AVMS.

It is clear that a C^*-AVMS generalizes the idea of a metric space, by replacing the set of real numbers with $\mathscr{A}_+$. For some recent fixed point results in C^*-AVMS, one can consult [3, 4] and the references therein.

Definition 4 (see [2]). Given a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$. Suppose that the sequence $\{j_n\}_{n\in\mathbb{N}} \subset \mho$ and $u \in \mho$. If for any $\wp>0$, there exists $\zeta \in \mathbb{N}$ such that for all $n > \zeta$, $\|\sigma(j_n, u)\| \leq \wp$, then $\{j_n\}_{n\in\mathbb{N}}$ is said to be convergent to u with respect to $\mathscr{A}$. In this case, we write $\lim_{n\longrightarrow\infty} j_n = u$.

If for any $\wp>0$, there is $\zeta \in \mathbb{N}$ such that $n, m > \zeta$, $\|\sigma(j_n, j_m)\| \leq \wp$, then the sequence $\{j_n\}_{n\in\mathbb{N}}$ is said to be Cauchy with respect to $\mathscr{A}$. We say that $(\mho, \mathscr{A}, \sigma)$ is a complete C^*-AVMS, if every Cauchy sequence in $\mho$ is convergent with respect to $\mathscr{A}$.

Definition 5 (see [2]). Given a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$, a mapping $Y : \mho \longrightarrow \mho$ is called a C^*-algebra-valued contractive mapping on $\mho$, if there exists $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,

$$\sigma(Yj, Y\ell)^\circ\lambda^*\sigma(j, \ell)\lambda. \tag{1}$$

The following lemma is useful in discussing our main results.

Lemma 6 [1]. *Let $\mathscr{A}$ be a C^*-algebra. Then:*

(i) For any $\tau \in \mathscr{A}$, if $p, q \in \mathscr{A}_a$ with $p^\circ q$, then $r^*pr^\circ r^*qr$

(ii) For any $p, q \in \mathscr{A}_a$, if $0_{\mathscr{A}}{}^\circ p^\circ q$, then $0 \leq \|p\| \leq \|q\|$

In 1970, Takahashi [5] initiated the concept of convexity in metric spaces in the following fashion.

Definition 7 (see [5]). Let $(\mho, \sigma)$ be a metric space and $D = [0, 1]$. A mapping $\Psi : \mho\times \mho\times D \longrightarrow \mho$ is called a convex structure on $\mho$, if for all $j, \ell, a \in \mho$ and $t \in D$,

$$\sigma(a, \Psi(j, \ell, t)) \leq t\sigma(a, j) + (1 - t)\sigma(a, \ell). \tag{2}$$

Using the notion of convexity, Takahashi [5] complemented some FP results originally obtained in Banach spaces. Following [5], several investigators have come up with FP notions in convex metric spaces; for such results, we can refer [6–9] and the references therein. Using the idea of a convex metric space and with the aid of positive elements in a C^*-algebra, Ghanifard et al. [10] brought up the next definition.

Definition 8 (see [10]). Let $(\mho, \mathscr{A}, \sigma)$ be a C^*-AVMS. A mapping $\Psi : \mho\times \mho\times D \longrightarrow \mho$ is called a convex structure on $\mho$ if for all $j, \ell, a \in \mho$, it satisfies the following:

$$\sigma(a, \Psi(j, \ell, t))^\circ t\sigma(a, j) + (1 - t)\sigma(a, \ell). \tag{3}$$

A C^*-AVMS equipped with a convex structure is said to be a convex C^*-AVMS, denoted by $(\mho, \mathscr{A}, \Psi, \sigma)$. A subset Θ of $\mho$ is called convex if for all $j, \ell \in \Theta$ and $t \in D$, $\Psi(j, \ell, t) \in \Theta$.

As an attempt at reducing uncertainties in dealing with practical problems for which conventional mathematics cannot cope effectively, the evolvement of fuzzy mathematics started with the introduction of the concepts of fuzzy sets by Zadeh [11] in 1965. Fuzzy set theory is now well-known as one of the mathematical tools for handling information with nonstatistical uncertainty. As a result, the theory of fuzzy sets has gained greater applications in diverse domains such as management sciences, engineering, environmental sciences, medical sciences, and in other emerging fields. In the meantime, the basic notions of fuzzy sets have been modified and improved in various settings; for example, see [12–15]. Along the lane, Heilpern [16] employed the concept of fuzzy sets to come up with the notion of fuzzy mappings and established a FP result for fuzzy contraction mappings which is a fuzzy version of FP theorems established by Nadler [17] and Banach [18].

Let $\mho$ be a universal set. A fuzzy set in $\mho$ is a map with domain $\mho$ and range set D. Let $I^{\mho}$ be the collection of all fuzzy sets in $\mho$. If ∇ is a fuzzy set in $\mho$, then the function value $\nabla(j)$ is called the grade of membership of j in ∇. The α-level set of a fuzzy set ∇ is denoted by $[\nabla]_\alpha$ and is defined as follows:

$$[\nabla]_\alpha = \begin{pmatrix} \{j \in \mho : \overline{\nabla}(j) > 0\}, & \text{if } \alpha = 0, \\ \{j \in \mho : \nabla(j) \geq \alpha\}, & \text{if } \alpha \in D \setminus \{0\}, \end{pmatrix} \tag{4}$$

where by $\bar{P}$, we mean the closure of the crisp set P.

Definition 9 (see [16]). Let $\mho$ be an arbitrary set and Y be a metric space. A mapping $Y : \mho \longrightarrow I^{\mho}$ is called a fuzzy mapping. A fuzzy mapping Y is a fuzzy subset of $\mho\times Y$ with membership function $Y(j)(\ell)$. The function value $Y(j)(\ell)$ is called the grade of membership of ℓ in $Y(j)$.

Definition 10 (see [16]). Let $\mho$ be a nonempty set and $Y : \mho \longrightarrow I^{\mho}$ be a fuzzy mapping. A point $u \in \mho$ is said to be a fuzzy FP of Y if there exists an $\alpha \in D \setminus \{0\}$ such that $u \in [Yu]_{\alpha(u)}$.

Hereafter, $\mathrm{Fix}(Y) = \{u \in \mho : u \in [Yu]_\alpha \text{ for some } \alpha \in D\}$.

Motivated by the ideas of fuzzy mappings and C^*-AVMSs due to Heilpern [16] and Ma et al. [2], respectively, the aim of this research is to initiate the study of fuzzy FP results in convex C^*-AVMSs. To this effect, some new concepts of C^*-algebra-valued fuzzy contractions in convex

C^*-AVMSs are proposed, and related fuzzy FP theorems are established. The notions put forward herein are not only novel, but complement and unify a few corresponding results in the existing literature.

2. Main Results

In this section, we introduce notions of C^*-algebra-valued fuzzy contractions and some corresponding fixed point results. First, a few requisite auxiliary concepts are initiated as follows.

Definition 11. A fuzzy set Ω in a C*-AVMS $(\mho, \mathscr{A}, \sigma)$ is said to be convex if for all $j, \ell \in \mho$ and $t \in D \setminus \{0, 1\}$, min $\{\Omega(j), \Omega(\ell)\}^\circ\Omega(tj + (1-t)\ell)$. A fuzzy set Ω in $\mho$ is called an approximate quantity if its α-level set is a compact convex subset of $\mho$ for each $\alpha \in D$ and $\sup_{j\in\mho_{\mathscr{A}}}\Omega(j) = 1$.

Throughout, the collection of all approximate quantities in $(\mho, \mathscr{A}, \sigma)$ is denoted by $W_{\mathscr{A}}(\mho)$. We define a distance function between two approximate quantities in $W_{\mathscr{A}}(\mho)$ as follows.

Definition 12. Let $F, G \in W_{\mathscr{A}}(\mho)$ and $\alpha \in D$. Then, we define:

$$\begin{aligned} D^*_\alpha(F, G) &= \aleph\left([F]_\alpha, [G]_\alpha\right), \\ \sigma_\infty(F, G) &= \sup_\alpha D^*_\alpha(F, G), \end{aligned} \tag{5}$$

where the Hausdorff distance function $\aleph : W_{\mathscr{A}} \times W_{\mathscr{A}} \longrightarrow \mathscr{A}$ is set-up as follows:

$$\begin{aligned} \aleph\left([F]_\alpha, [G]_\alpha\right) &= \left(\max\left\{\sup_{j\in[F]_\alpha}\left\|\sigma\left(j, [G]_\alpha\right)\right\|, \sup_{\ell\in[F]_\alpha}\left\|\sigma\left([F]_\alpha, \ell\right)\right\|\right\}\right)I_{\mathscr{A}}, \\ \sigma(a, \varpi) &= (\inf\{\|\sigma(a, \rho)\| : \rho \in \varpi\})I_{\mathscr{A}}. \end{aligned} \tag{6}$$

Consistent with Heilpern [16], we call the function D^*_α an $(\alpha, *)$-distance and σ_∞ a distance between two approximate quantities in $W_{\mathscr{A}}(\mho)$.

We say that a subset Ψ of a C^*-AVMS $(\mho, \mathscr{A}, \sigma)$ is bounded if $\sup_{j,\ell\in\mathscr{A}}\{\|\sigma(j, \ell)\|\} < \infty$. The collection of all closed and bounded subsets of $(\mho, \mathscr{A}, \sigma)$ is represented by $B_{\mathscr{A}}(\mho)$.

Note that σ_∞ is a C^*-algebra-valued metric on $B_{\mathscr{A}}(\mho)$ (induced by the Hausdorff metric $\aleph$), and the completeness of $(\mho, \mathscr{A}, \sigma_\infty)$ implies the completeness of the corresponding C^*-AVMS $(K_\Omega(\mho), \mathscr{A}, \sigma_\infty)$. Moreover, $(\mho, \mathscr{A}, \sigma_\infty) \mapsto (B_{\mathscr{A}}(\mho), \mathscr{A}, \sigma_\infty) \mapsto (K_\Omega(\mho), \mathscr{A}, \sigma_\infty)$ are isometric embedding via the relation $j \mapsto \{j\}$ (crisp set) and $\sqsubseteq \mapsto \chi_\sqsubseteq$, respectively, where $\chi_\sqsubseteq$ is the characteristic function of $\sqsubseteq$, and

$$K_\Omega(\mho) = \left\{\Omega \in I^\mho : [\Omega]_\alpha \in B_{\mathscr{A}}(\mho), \text{for each } \alpha \in D\right\}. \tag{7}$$

Similarly,

$$K_{(\Omega_1,\Omega_2)}(\mho) = \left\{\Omega_1, \Omega_2 \in I^\mho : [\Omega_1]_\alpha, [\Omega_2]_\alpha \in B_{\mathscr{A}}(\mho), \text{for each } \alpha \in D\right\}. \tag{8}$$

We now define the idea of a C^*-AVFM in the following manner.

Definition 13. Let $\mho$ be an arbitrary set and $(\Upsilon, \mathscr{A}, \sigma)$ be a C^*-AVMS. A mapping $Y : \mho \longrightarrow W_{\mathscr{A}}(Y)$ is called a C^*-AVFM.

In line with the idea of fuzzy λ-contraction due to Heilpern [16], we introduce the next concept.

Definition 14. Let $(\mho, \mathscr{A}, \sigma)$ be a C^*-AVMS. A C^*-AVFM $Y : \mho \longrightarrow W_{\mathscr{A}}(\mho)$ is called a C^*-algebra-valued fuzzy λ-contraction, if there exists $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,

$$\sigma_\infty(Yj, Y\ell)^\circ\lambda^*\sigma(j, \ell)\lambda. \tag{9}$$

Example 15. Let $\mho = \mathbb{R}$ and $\mathscr{A} = M_2(\mathbb{R})$ (the collection all 2×2 matrices with real entries) with the norm $\|\Phi\| = \max_{p',q'}|\varsigma_{p'q'}|$, where $\varsigma_{p'q'}$ are the entries of the matrix $\Phi \in M_2(\mathbb{R})$ and the involution given by $\Phi^* = \Phi^T$. Define $\sigma : \mho \times \mho \longrightarrow \mathscr{A}$ by $\sigma(j, \ell) = \text{diag}\,(|j-\ell|, |j-\ell|)$. Obviously, $(\mho, \mathscr{A}, \sigma)$ is a C^*-AVMS. We define a partial ordering on $\mathscr{A}$ as follows:

$$\begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix} = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix} \Leftrightarrow p_i \preccurlyeq \lambda^* q_i \lambda, \tag{10}$$

for $i = 1, 2, 3, 4$ and for some $\lambda \in \mathscr{A}$. Let $\beta \in (0, 1]$ and for each $j \in \mho$, define a fuzzy mapping $Y(j)$: $\mho \longrightarrow [0, 1]$ as follows:

$$Y(j)(t) = \begin{cases} \beta, & \text{if } \dfrac{j}{7} \le t \le \dfrac{j}{5} \\ \dfrac{\beta}{6}, & \text{otherwise.} \end{cases} \tag{11}$$

If we take the mapping $\alpha : \mho \longrightarrow (0, 1]$ as $\alpha(j) = \beta$ for all $j \in \mho$, then

$$[Yj]_{\alpha(j)} = \{t \in \mho : Y(j)(t) \ge \beta\} = \left[\frac{j}{7}, \frac{j}{5}\right]. \tag{12}$$

Obviously, $[Yj]_{\alpha(j)} \in B_{\mathscr{A}}(\mho)$, for each $j \in \mho$. We see that for all $j, \ell \in \mho$,

$$\begin{aligned}\sigma_\infty(Yj, Y\ell) &= \sup_\beta D_\beta(Yj, Y\ell) = \aleph\left([Yj]_\beta, [Y\ell]_\beta\right)\\ &= \left(\max\left\{\sup_{a\in[Yj]_\beta}\left\|\sigma\left(a, [Y\ell]_\beta\right)\right\|, \sup_{b\in[Y\ell]}\left\|\sigma\left([Yj]_\beta, b\right)\right\|\right\}\right)I_{\mathscr{A}}\\ &= \left(\max\left\{\sup\inf_{a\in[Yj]_\beta,\ell\in[Y\ell]_\beta}|a-b|, \sup\inf_{b\in[Y\ell]_\beta,a\in[Yj]_\beta}|a-b|\right\}\right)\\ &\quad\cdot I_{\mathscr{A}}\left(\max\left\{\left|\frac{j}{7}-\frac{\ell}{7}\right|, \left|\frac{j}{5}-\frac{\ell}{5}\right|\right\}\right)I_{\mathscr{A}} = \left(\left|\frac{j}{5}-\frac{\ell}{5}\right|\right)I_{\mathscr{A}}\\ &= \operatorname{diag}\left(\left|\frac{j}{5}-\frac{\ell}{5}\right|\right).\end{aligned} \tag{13}$$

It follows that Y is a C^*-algebra-valued fuzzy λ-contraction with $\lambda = \operatorname{diag}(1/\sqrt{4}, 1/\sqrt{4})$. Clearly, $\|\lambda\| < 1$.

Definition 16. Let $\mho$ be a nonempty set. A point $u \in \mho$ is called a stationary point of a fuzzy mapping $Y : \mho \longrightarrow I^{\mho}$, if there exists an $\alpha \in D\setminus\{0\}$ such that $[Yu]_\alpha = \{u\}$. We say that u is a common stationary point of any two fuzzy mappings $Y, \Lambda : \mho \longrightarrow I^{\mho}$ if $[Yu]_\alpha = \{u\} = [\Lambda u]_\alpha$, for some $\alpha \in D\setminus\{0\}$.

Our main result is provided hereunder.

Theorem 17. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS. Suppose that the mapping $Y : \mho \longrightarrow K_Y(\mho)$ is a C^*-algebra-valued fuzzy λ-contraction such that $Fix(Y) \neq \varnothing$ and every $a \in Fix(Y)$ is a stationary point of Y. Let $\{j_n\}$ be the Mann iteration scheme given by*

$$j_{n+1} = \Psi(\ell_n, j_n, \eta_n), \tag{14}$$

where $\ell_n \in [Yj_n]_{\alpha(j_n)}$ and $\eta_n \in D$. Then, $\{j_n\}$ converges to a fuzzy FP of Y, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, Fix(Y)) = 0_{\mathscr{A}}$.

Proof. Let $a \in \operatorname{Fix}(Y)$ and $\alpha(j) \in D\setminus\{0\}$ for each $j \in \mho$. Then, we have

$$\sigma(j_{n+1}, a) = \sigma(\Psi(\ell_n, j_n, \eta_n), a) \preccurlyeq \eta_n\sigma(\ell_n, a) + (1-\eta_n)\sigma(j_n, a). \tag{15}$$

From (15),

$$\begin{aligned}\|\sigma(j_{n+1}, a)\| &\le \eta_n\|\sigma(\ell_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\left(\sup_{\ell_n\in[Yj_n]_{\alpha(j_n)}, a\in[Ya]_{\alpha(a)}}\|\sigma(\ell_n, a)\|\right)\\ &\quad + (1-\eta_n)\|\sigma(j_n, a)\|\\ &= \eta_n\left\|\sigma\left([Yj_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\right\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\left\|\aleph\left([Yj_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\right\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\|\sigma_\infty(Yj_n, Ya)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &\le \eta_n\|\lambda^*\|\|\sigma(j_n, a)\|\|\lambda\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &= \eta_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\|\\ &< \eta_n\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.\end{aligned} \tag{16}$$

We observe that the strict inequality in (16) is valid whenever $j_n \neq a$, for each $n \in \mathbb{N}$. Indeed, if we take $j_\kappa = a$ for a finite $\kappa \in \mathbb{N}$, then $j_n = a$ for each $n \ge \kappa$, from which it yields that $\{j_n\}_{n\in\mathbb{N}}$ converges to a for finite number of iterations, and hence, we obtain the conclusion of our result.

Now, we prove that the sequence $\{j_n\}_{n\in\mathbb{N}}$ is Cauchy with respect to $\mathscr{A}$. Since $\lim_{n\longrightarrow\infty}\sigma(j_n, \operatorname{Fix}(Y)) = 0_{\mathscr{A}}$, for each $\wp>0$, there exists $m(\wp) \in \mathbb{N}$ such that for all $n \ge m(\wp)$,

$$\|\sigma(j_n, \operatorname{Fix}(Y))\| \le \frac{\wp}{7}. \tag{17}$$

By (17), there exists $r_1 \in \operatorname{Fix}(Y)$ such that for all $n \ge m(\wp)$,

$$\|\sigma(j_n, r_1)\| \le \frac{\wp}{2}. \tag{18}$$

Using the triangle inequality in $(\mho, \mathscr{A}, \sigma)$,

$$\sigma(j_{n+k}, j_n) \preccurlyeq \sigma(j_{n+k}, r_1) + \sigma(r_1, j_n). \tag{19}$$

Therefore, taking (18) into consideration, we get

$$\begin{aligned}\|\sigma(j_{n+k}, j_n)\| &\le \|\sigma(j_{n+k}, r_1)\| + \|\sigma(r_1, j_n)\|\\ &< \|\sigma(j_n, r_1)\| + \|\sigma(r_1, j_n)\| \le \frac{\wp}{2} + \frac{\wp}{2} = \wp,\end{aligned} \tag{20}$$

for $j_n \neq r_1$. This proves that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\mathscr{A}$. The completeness of $\mho_{\mathscr{A}}$ implies that there exists $a^* \in \mho_{\mathscr{A}}$ such that $\lim_{n\longrightarrow\infty} j_n = a^*$. Next, we establish that a^* is a fuzzy FP of Y. For this, take $\wp' > 0$. Since $j_n \longrightarrow a^*$ as $n \longrightarrow \infty$, there exists $m(\wp') \in \mathbb{N}$ such that for all $n \ge m(\wp')$,

$$\|\sigma(j_n, a^*)\| \le \frac{\wp'}{4}. \tag{21}$$

Moreover, $\lim_{n\longrightarrow\infty}\sigma(j_n, \operatorname{Fix}(Y)) = 0_{\mathscr{A}}$ yields that there exists $m'(\wp) \ge m(\wp')$ such that for all $m(\wp) \ge m'(\wp)$,

$$\|\sigma(j_n, \operatorname{Fix}(Y))\| \le \frac{\wp'}{10}. \tag{22}$$

Hence, there exists $r_2 \in \operatorname{Fix}(Y)$ such that for all $m(\wp) \ge m'(\wp)$,

$$\|\sigma(j_n, r_2)\| \le \frac{\wp'}{12}. \tag{23}$$

By triangle inequality in $\mho_{\mathscr{A}}$, there exists write

$$\begin{aligned}\sigma\left([Ya^*]_{\alpha(a^*)}, a^*\right) &\preccurlyeq \sigma\left([Ya^*]_{\alpha(a^*)}, r_2\right) + \sigma\left(r_2, [Yj_3]_{\alpha(j_3)}\right)\\ &\quad + \sigma\left([Yj_3]_{\alpha(j_3)}, r_2\right) + \sigma\left(r_2, j_{n_3}\right)\\ &\quad + \sigma\left(j_{n_3}, a^*\right).\end{aligned} \tag{24}$$

Consequently,

$$\begin{aligned}
\|\sigma\left([Ya^*]_{\alpha(a^*)}, a^*\right)\| &\leq \|\sigma\left([Ya^*]_{\alpha(a^*)}, r_2\right)\| + \|\sigma\left(r_2, \left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}\right)\| \\
&+ \|\sigma\left(\left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}, r_2\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\
&+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\aleph\left([Ya^*]_{\alpha(a^*)}, [Yr_2]_{\alpha(r_2)}\right)\| \\
&+ 2\|\aleph\left([Yr_2]_{\alpha(r_2)}, \left[Yj_{n_3}\right]_{\alpha\left(j_{n_3}\right)}\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\
&+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\sigma_\infty(Ya^*, Yr_2)\| \\
&+ 2\|\sigma_\infty\left(Yr_2, Yj_{n_3}\right)\| + \|\sigma\left(r_2, j_{n_3}\right)\| \\
&+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\lambda^*\|\|\sigma(a^*, r_2)\|\|\lambda\| \\
&+ 2\|\lambda^*\|\|\sigma\left(r_2, j_{n_3}\right)\|\|\lambda\| + \|\sigma(j_n, a^*)\| \\
&+ \|\sigma\left(r_2, j_{n_3}\right)\| + \|\sigma\left(j_{n_3}, a^*\right)\| \\
&= \|\lambda\|^2\|\sigma(a^*, r_2)\| + 2\|\lambda\|^2\|\sigma\left(r_2, j_{n_3}\right)\| + \\
&\|\sigma\left(j_{n_3}, a^*\right)\| < \|\sigma(a^*, r_2)\| + 2\|\sigma\left(r_2, j_{n_3}\right)\| \\
&+ \|\sigma\left(j_{n_3}, a^*\right)\| \leq \|\sigma\left(a^*, j_{n_3}\right)\| + \|\sigma\left(j_{n_3}, r_2\right)\| \\
&+ \|\sigma\left(j_{n_3}, r_2\right)\| + 2\|\sigma\left(j_{n_3}, r_2\right)\| + 2\|\sigma\left(j_{n_3}, r_2\right)\| \\
&+ \|\sigma\left(j_{n_3}, r_2\right)\| + \|\sigma\left(j_{n_3}, a^*\right)\| = 2\|\sigma\left(j_{n_3}, a^*\right)\| \\
&+ 6\|\sigma\left(j_{n_3}, r_2\right)\| \leq 2\left(\frac{\wp'}{4}\right) + 6\left(\frac{\wp'}{12}\right) = \wp',
\end{aligned} \tag{25}$$

whenever $j_{n_3} \neq r_2$. It follows that $\sigma([Ya^*]_{\alpha(a^*)}, a^*) = 0_{\mathscr{A}}$, and thus, $a^* \in [Ya^*]_{\alpha(a^*)}$, for some $\alpha(a^*) \in D \setminus \{0\}$. □

In what follows, we present a fuzzy coincidence theorem for two C^*-algebra fuzzy mappings using Ishikawa iterative scheme.

Theorem 18. *Let $(\mho, \mathscr{A}, \sigma)$ be a complete convex C^*-AVMS and $\Lambda, Y : \mho \longrightarrow K_{(Y,\Lambda)}(\mho)$ be any two C^*-AVFMs satisfying:*

$$\sigma_\infty(Yj, \Lambda\ell) \leq \lambda^*\sigma(j, \ell)\lambda, \tag{26}$$

for all $j, \ell \in \mho$ with $\lambda \in \mathscr{A}$ such that $\|\lambda\|<1$. Assume further that $C_{(Y,\Lambda)} \coloneqq Fix(Y) \cap Fix(\Lambda) \neq \varnothing$, and every $a \in C_{(Y,\Lambda)}$ is a common stationary point of Y and Λ. Then, the sequence of Ishikawa iterative scheme set-up by

$$j_{n+1} = \Psi(z_n, j_n, \eta_n), \ell_n = \Psi\left(z', j_n, \xi_n\right), \tag{27}$$

where $z_n \in [\Lambda j_n]_{\alpha(j_n)}, z'_n \in [Yj_n]_{\alpha(j_n)}$, and $\eta_n, \xi_n \in D$, converges to an element of $C_{(Y,\Lambda)}$, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, C_{(Y,\Lambda)}) = 0_{\mathscr{A}}$.

Proof. Let $a \in C_{(Y,\Lambda)}$. Assume that $j_n \neq a$ for all $n \in \mathbb{N}$. Then

$$\sigma(\ell_n, a) = \sigma\left(\Psi\left(z'_n\right), j_n, \xi_n\right) a \preccurlyeq \xi_n\sigma\left(z'_n, a\right) + (1-\xi_n)\sigma(j_n, a), \tag{28}$$

from which it follows that

$$\begin{aligned}
\|\sigma(\ell_n, a)\| &\leq \xi_n\|\sigma\left(z'_n, a\right)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\
&\leq \xi_n\|\aleph\left([Yj_n]_{\alpha(j_n)}, [\Lambda a]_{\alpha(a)}\right)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\
&\leq \xi_n\|\sigma_\infty(Yj_n, \Lambda a)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\
&\leq \|\lambda^*\sigma(\ell_n, a)\lambda\| + (1-\xi_n)\|\sigma(j_n, a)\| \\
&\leq \xi_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\xi_n)\|\sigma(j_n, a)\| \\
&< \xi_n\|\sigma(j_n, a)\| + (1-\xi_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.
\end{aligned} \tag{29}$$

Similarly,

$$\sigma(j_{n+1}, a) = \sigma(\Psi(z_n, j_n, \eta_n), a) \preccurlyeq \eta_n\sigma(z_n, a) + (1-\eta_n)\sigma(j_n, a), \tag{30}$$

which gives

$$\begin{aligned}
\|\sigma(j_{n+1}, a)\| &\leq \eta_n\|\sigma(z_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\
&\leq \eta_n\|\aleph\left([\Lambda j_n]_{\alpha(j_n)}, [Ya]_{\alpha(a)}\right)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\
&\leq \eta_n\|\lambda\|^2\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| \\
&< \eta_n\|\sigma(j_n, a)\| + (1-\eta_n)\|\sigma(j_n, a)\| = \|\sigma(j_n, a)\|.
\end{aligned} \tag{31}$$

Hence, in line with the proof of Theorem 17, we can prove that $\{j_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to $\mathscr{A}$, and the completeness of $(\mho, \mathscr{A}, \Psi, \sigma)$ implies that $\{j_n\}_{n\in\mathbb{N}}$ converges to some $u^* \in \mho$. Thus, consistent with Theorem 17, we obtain that $u^* \in \mathrm{Fix}(Y) \cap \mathrm{Fix}(\Lambda)$. □

As some consequences of Theorem 17 and Theorem 18, the following two results, using C^*-algebra-valued Hausdorff distance function, can be deduced easily.

Corollary 19. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS and $Y : \mho \longrightarrow K_Y(\mho)$ be a C^*-AVFM. Suppose that $Fix(Y) \neq \varnothing$ and every $a \in Fix(Y)$ is a stationary point of Y. Let $\{j_n\}_{n\in\mathbb{N}}$ be the Mann iterative scheme set-up by (14). If there exist $\alpha \in D \setminus \{0\}$ and $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,*

$$D^*_\alpha(Yj, Y\ell) \leq \lambda^*\sigma(j, \ell)\lambda, \tag{32}$$

then $\{j_n\}_{n\in\mathbb{N}}$ converges to a fuzzy FP of Y, provided $\lim_{n\longrightarrow\infty}\sigma(j_n, Fix(Y)) = 0_{\mathscr{A}}$.

Proof. Since for all $j, \ell \in \mho$ and $\alpha \in D \setminus \{0\}$,

$$\|D_\alpha^*(Yj, Y\ell)\| \le \|\sigma_\infty(Yj, Y\ell)\|. \tag{33}$$

Theorem 17 can be followed to complete the proof. □

On the same steps in deriving Corollary 19, we can also deduce the following result.

Corollary 20. *Let $(\mho, \mathscr{A}, \Psi, \sigma)$ be a complete convex C^*-AVMS and $Y, \Lambda : \mho \longrightarrow K_{(Y,\Lambda)}(\mho)$ be any two C^*-AVFMs. Assume further that $C_{(Y,\Lambda)} = Fix(Y) \cap Fix(\Lambda) \neq \varnothing$ and every $a \in Fix(Y)$ is a common stationary point of Y and Λ. Let $\{j_n\}_{n\in\mathbb{N}}$ be the Ishikawa iterative scheme set-up by (23). If there exist $\alpha \in D \setminus \{0\}$ and $\lambda \in \mathscr{A}$ with $\|\lambda\|<1$ such that for all $j, \ell \in \mho$,*

$$D_\alpha^*(Yj, \Lambda\ell) \le \lambda^* \sigma(j, \ell)\lambda, \tag{34}$$

then $\{j_n\}_{n\in\mathbb{N}}$ converges to a common fuzzy FP of Y and Λ, provided

$$\lim_{n\longrightarrow\infty} \sigma\left(j_n, C_{(Y,\Lambda)}\right) = 0_{\mathscr{A}}. \tag{35}$$

3. Open Problems

For some future examinations of our main results, the following two problems are highlighted:

(P1) It is well-known that the importance of contractive mapping is to guarantee the existence and uniqueness of a fixed point of certain self-mappings in complete spaces. On this note, following Theorem 17 and Theorem 18, sufficient criteria guaranteeing the existence of fixed points of C^*-algebra-valued fuzzy λ-contractions is still a gap that needed to be filled.

(P2) In this article, Mann and Ishikawa iterations are used to develop the ideas of C^*-algebra-valued fuzzy λ-contractions and associated fixed point theorems. Hence, it is natural to ask whether Picard iteration for C^*-algebra-valued fuzzy λ-contraction mapping converges or not.

4. Conclusions

Based on the ideas of fuzzy mappings and C^*-AVMSs in the sense of Heilpern [16] and Ma et al. [2], respectively, analogue notions of C^*-algebra-valued fuzzy contractions in convex C^*-AVMSs and associated FP theorems are established. The obtained fuzzy FP results are analysed using Mann and Ishikawa iterative schemes. It is pertinent to note that the ideas of this paper being discussed in fuzzy setting are very fundamental. Hence, it can be improved upon when presented in the framework of some generalized fuzzy mappings such as L-fuzzy, intuitionistic fuzzy, and soft set-valued mappings. The underlying space can also be fine-tuned in some other pseudo or quasi metric spaces. For some future considerations of our results, two open problems are posed regarding sufficient conditions under which C^*-algebra-valued fuzzy λ-contraction has a fixed point and whether or not the Picard iteration for C^*-algebra-valued fuzzy λ-contraction converges.

Data Availability

No data were used to support this study.

Disclosure

The statements made and views expressed are solely the responsibility of the author.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

The authors contributed equally to this work.

Acknowledgments

The fourth author would like to acknowledge that this publication was made possible by a grant from Carnegie Corporation of New York (provided through the African Institute for Mathematical Sciences).

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 2195754, 6 pages
https://doi.org/10.1155/2022/2195754

Research Article

Hyers–Ulam Stability Results for a Functional Inequality of (s, t)-Type in Banach Spaces

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Received 20 December 2021; Accepted 9 March 2022; Published 11 April 2022

Academic Editor: Juan Martinez-Moreno

We introduce an additive (s, t)-functional inequality where s and t are nonzero complex numbers with $\sqrt{2}|s| + |t| < 1$. Using the direct method and the fixed point method, we give the Hyers–Ulam stability of such functional inequality in Banach spaces.

1. Introduction and Preliminaries

A problem regarding the stability of homomorphisms was mentioned by Ulam [1] in 1940. The first answer was then found by Hyers in [2] which motivating the study of the stability problems of functional equations. We may roughly say that a given functional equation is stable on a class of functions A when any function in A approximately satisfies such equation. One of the well-known functional equations is the (additive) Cauchy functional equation $f(a+b) = f(a) + f(b)$ which is a useful tool in natural and social sciences. The stability of functional equations has been widely acknowledged as Hyers–Ulam stability. It was notably weakened by Rassias in [3] by making use of a direct method. The result was later extended in [4] which uses a general control function instead of the unbounded Cauchy difference. The concept of stability has been also developed for functional inequalities. Recently, Park introduced additive ρ-functional inequalities (s-type functional inequalities) and investigated the Hyers–Ulam stability in [5, 6]. Over the last decades, stability of functional equations and functional inequalities have been extensively studied, see [7–13], for example.

Not only the direct method, the fixed point method is also one of the most popular methods of proving the stability of functional equations and functional inequalities. Applications of stability of functional equations in a fixed point theory and in nonlinear analysis were introduced in [14]. It was known that Hyers–Ulam stability results can be derived using fixed point theorems while the latter can often be obtained from the former, see [15–20] and there references.

The Hyers–Ulam stability concept is very useful in many applications (i.e., optimization, numerical analysis, biology, and economics), since it can be very difficult to find the exact solutions for those physical problems. It is remarkably used in the field of differential equations. For some recent works, see [21–23] (and references therein) where the Hyers–Ulam stability results concerning (fractional) stochastic functional differential equations were given.

We denote $\mathbb{C}$, $\mathbb{N}$, and $\mathbb{R}^+$ the set of complex numbers, the set of positive integers and the set of positive real numbers, respectively, and let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$.

Now, let $s, t \in \mathbb{C} \setminus \{0\}$ such that $\sqrt{2}|s| + |t| < 1$. Yun and Shin [24] investigated the additive (s, t)-functional inequality:

$$\begin{aligned}\left\| 2f\left(\frac{a+b}{2}\right) - f(a) - f(b) \right\| &\le \| s(f(a+b) + f(a-b) - 2f(a)) \| \\ &\quad + \| t(f(a+b) - f(a) - f(b)) \|,\end{aligned} \tag{1}$$

while Park [25] proposed the additive (s, t)-functional inequality:

$$\begin{aligned}\|f(a+b)-f(a)-f(b)\| &\leq \|s(f(\mathrm{a}+b)+f(a-b)-2f(a))\| \\ &\quad + \left\|t\left(2f\left(\frac{a+b}{2}\right)-f(a)-f(b)\right)\right\|,\end{aligned} \tag{2}$$

and provided the Hyers–Ulam stability results in a Banach space.

In this article, motivated by those (s, t)-type inequalities mentioned above, we introduce the additive (s, t)-functional inequality:

$$\begin{aligned}\left\|2f\left(\frac{a+b}{2}\right)+f(a-b)-2f(a)\right\| &\leq \|s(f(a+b)+f(a-b)-2f(a))\| \\ &\quad + \|t(f(a+b)-f(a)-f(b))\|.\end{aligned} \tag{3}$$

We first investigate the Hyers–Ulam stability of such functional inequality using the direct method in Section 2. Then, in Section 3, we use the fixed point method to prove the Hyers–Ulam stability of such inequality. We also include some example and remarks in the last section. Note that, since (s, t)-type functional inequalities generalize s-type functional inequalities, our results simply extend existing Hyers–Ulam stability results for functional inequalities of s-type in the literature. These results span alongside those regarding other (s, t)-type functional inequalities.

Throughout this article, let X and B be a normed space and a Banach space, respectively, and let $s, t \in \mathbb{C} \setminus \{0\}$ such that $\sqrt{2}|s| + |t| < 1$. For convenience, we also require the following classes of mappings:

$$\begin{aligned}\mathscr{F}_0(X, B) &:= \{g : X \longrightarrow B : g(0) = 0\}, \\ \mathscr{A}(X, B) &:= \{g : X \longrightarrow B : g \text{ satisfies } (1.1)\}, \\ \mathscr{A}_0(X, B) &:= \mathscr{F}_0(X, B) \cap \mathscr{A}(X, B).\end{aligned} \tag{4}$$

2. Stability Results: Direct Method

In this section, the stability results of the additive (s, t)-functional inequality (3) are proposed by using the direct method. We begin with the lemma showing that any map g in $\mathscr{A}(X, B)$ is additive.

Lemma 1. *If $g \in \mathscr{A}(X, B)$, then g is additive.*

Proof. Taking $a = b = 0$ into (3), we obtain that $(1 - |t|)\|g(0)\| \leq 0$. However, $|t| < 1$ implies that $g(0) = 0$. Also, if we let $b = 0$ in (3), then

$$g(a) = 2g\left(\frac{a}{2}\right), \tag{5}$$

for all $a \in X$. From (3) and (5),

$$\begin{aligned}(1-|s|)\|g(a+b)+g(a-b)-2g(a)\| \\ \leq |t|\|g(a+b)-g(a)-g(b)\|,\end{aligned} \tag{6}$$

for all $a, b \in X$. Next, taking $c = a + b$ and $d = a - b$ in (3), we have that

$$\begin{aligned}(1-|s|)\left\|g(c)+g(d)-2g\left(\frac{c+d}{2}\right)\right\| \\ \leq |t|\left\|g(c)-g\left(\frac{c+d}{2}\right)-g\left(\frac{c-d}{2}\right)\right\|.\end{aligned} \tag{7}$$

Then, from (5),

$$\begin{aligned}(1-|s|)\|g(c+d)-g(c)-f(d)\| \\ \leq \frac{|t|}{2}\|g(c+d)+g(c-d)-2g(c)\|,\end{aligned} \tag{8}$$

for all $c, d \in X$. Applying (6) and (8),

$$\begin{aligned}(1-|s|)^2\|g(a+b)-g(a)-g(b)\| \\ \leq \frac{|t|^2}{2}\|g(a+b)-g(a)-g(b)\|,\end{aligned} \tag{9}$$

for all $a, b \in X$. Finally, since $\sqrt{2}|s| + |t| < 1$, we obtain that g is additive. □

We are now ready to present the main result.

Theorem 2. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a map such that*

$$\Phi(a, b) := \sum_{j=0}^{\infty} 2^j \varphi\left(2^{-j}a, 2^{-j}b\right) < \infty, \tag{10}$$

for all $a, b \in X$. For any $f \in \mathscr{F}_0(X, B)$ satisfying

$$\begin{aligned}\left\|2f\left(\frac{a+b}{2}\right)+f(a-b)-2f(a)\right\| &\leq \|s(f(a+b)+f(a-b)-2f(a))\| \\ &\quad + \|t(f(a+b)-f(a)-f(b))\| + \varphi(a, b),\end{aligned} \tag{11}$$

for all $a, b \in X$, there exists a unique $F \in \mathscr{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \Phi(a, 0), \tag{12}$$

for all $a \in X$.

Proof. We first let $b = 0$ in (11). This implies that

$$\left\|2f\left(\frac{a}{2}\right) - f(a)\right\| \leq \varphi(a, 0), \tag{13}$$

for all $a \in X$. It follows that for any $m, l \in \mathbb{N}_0$ with $m > l$,

$$\begin{aligned}
&\left\|2^l f\left(2^{-l}a\right) - 2^m f(2^{-m}a)\right\| \\
&\quad \leq \sum_{j=l}^{m-1} \left\|2^j f(2^{-j}a) - 2^{j+1} f\left(2^{-(j+1)}a\right)\right\| \\
&\quad \leq \sum_{j=l}^{m-1} 2^j \varphi(2^{-j}a, 0),
\end{aligned} \tag{14}$$

for all $a \in X$. The completeness of B confirms that the Cauchy sequence $\{2^k f(2^{-k}a)\}$ is convergent for any $a \in X$. Define $F : X \longrightarrow B$ by

$$F(a) = \lim_{k\longrightarrow\infty} 2^k f\left(2^{-k}a\right), \tag{15}$$

for all $a \in X$. Clearly, $F \in \mathcal{F}_0(X, B)$. Next, choosing $l = 0$ and letting $m \longrightarrow \infty$ in (14), we have that F satisfies (12). Then, from (10) and (11),

$$\begin{aligned}
&\left\|2F\left(\frac{a+b}{2}\right) + F(a-b) - 2F(a)\right\| \\
&\quad = \lim_{n\longrightarrow\infty} 2^n \left\|2f\left(2^{-(n+1)}(a+b)\right) + f(2^{-n}(a-b)) - 2f(2^{-n}a)\right\| \\
&\quad \leq |s| \lim_{n\longrightarrow\infty} 2^n \|f(2^{-n}(a+b)) + f(2^{-n}(a-b)) - 2f(2^{-n}a)\| \\
&\qquad + |t| \lim_{n\longrightarrow\infty} 2^n \|f(2^{-n}(a+b)) - f(2^{-n}a) - f(2^{-n}b)\| \\
&\qquad + \lim_{n\longrightarrow\infty} 2^n \varphi(2^{-n}a, 2^{-n}b) = \|s(F(a+b) + F(a-b) - 2F(a))\| \\
&\qquad + \|t(F(a+b) - F(a) - F(b))\|,
\end{aligned} \tag{16}$$

for all $a, b \in X$. By Lemma 1, $F \in \mathcal{A}_0(X, B)$. Finally, let G be another map in $\mathcal{A}_0(X, B)$ satisfying (12). Then, for any $a \in X$,

$$\begin{aligned}
\|F(a) - G(a)\| &= \|2^p F(2^{-p}a) - 2^p G(2^{-p}a)\| \\
&\leq \|2^p F(2^{-p}a) - 2^p f(2^{-p}a)\| \\
&\quad + \|2^p G(2^{-p}a) - 2^p f(2^{-p}a)\| \\
&\leq 2^{p+1} \Phi(2^{-p}a, 0).
\end{aligned} \tag{17}$$

Therefore, $\|F(a) - G(a)\| \longrightarrow 0$ as $p \longrightarrow \infty$. The uniqueness of F follows. □

Corollary 3. *For $r, \vartheta \in \mathbb{R}_0^+$ with $r > 1$, if $f \in \mathcal{F}_0(X, B)$ satisfying*

$$\begin{aligned}
&\left\|2f\left(\frac{a+b}{2}\right) + f(a-b) - 2f(a)\right\| \\
&\quad \leq \|s(f(a+b) + f(a-b) - 2f(a))\| \\
&\qquad + \|t(f(a+b) - f(a) - f(b))\| + \vartheta(\|a\|^r + \|b\|^r),
\end{aligned} \tag{18}$$

for all $a, b \in X$, then there exists a unique $F \in \mathcal{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \frac{2^r \vartheta}{2^r - 2} \|a\|^r, \tag{19}$$

for all $a \in X$.

Proof. Let $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 2. The result immediately follows. □

Theorem 4. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a map satisfying*

$$\Psi(a, b) := \sum_{j=1}^{\infty} 2^{-j} \varphi\left(2^j a, 2^j b\right) < \infty, \tag{20}$$

for all $a, b \in X$, and let $f \in \mathcal{F}_0(X, B)$ satisfy (11). Then, there exists a unique $F \in \mathcal{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \Psi(a, 0), \tag{21}$$

for all $a \in X$.

Proof. It follows from (13) that $\|f(a) - (1/2)f(2a)\| \leq (1/2)\varphi(2a, 0)$ for all $a \in X$. Then, for $m, l \in \mathbb{N}_0$ with $m > l$,

$$\begin{aligned}
&\left\|2^{-l} f\left(2^l a\right) - 2^{-m} f(2^m a)\right\| \\
&\quad \leq \sum_{j=l}^{m-1} \left\|2^{-j} f(2^j a) - 2^{-(j+1)} f\left(2^{j+1} a\right)\right\| \\
&\quad \leq \sum_{j=l+1}^{m} 2^{-j} \varphi\left(2^j a, 0\right),
\end{aligned} \tag{22}$$

for all $a \in X$. Now, let $a \in X$. It follows from the completeness of B that $\{2^{-n} f(2^n a)\}$ is convergent in X. Next, define a map $F : X \longrightarrow B$ by

$$F(a) = \lim_{n\longrightarrow\infty} 2^{-n} f(2^n a), \tag{23}$$

for all $a \in X$. Choosing $l = 0$ and taking $m \longrightarrow \infty$ in (22), we have that F satisfies (21). The rest is similar to the Proof of Theorem 2. □

Let $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$. The following result is straightforward.

Corollary 5. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r < 1$. If $f \in \mathcal{F}_0(X, B)$ satisfies (18), then there exists a unique $F \in \mathcal{A}_0(X, B)$ such that*

$$\|f(a) - F(a)\| \leq \frac{2^r \vartheta}{2 - 2^r} \|a\|^r, \tag{24}$$

for all $a \in X$.

3. Stability Results: Fixed Point Method

In this section, we apply the fixed point method to present the Hyers–Ulam stability of the functional inequality (3).

We first state a useful tool in the field of fixed point theory.

Proposition 6. *[26, 27]. Let (X, d) be a complete generalized metric space, and let $\mathscr{L} : X \longrightarrow X$ be a strict Lipschitz contraction with the Lipschitz constant $\alpha < 1$. Then, for $a \in X$, either*

(a) $d(\mathscr{L}^n a, \mathscr{L}^{n+1} a) = \infty$ for all $n \in \mathbb{N}_0$ or

(b) $d(\mathscr{L}^n a, \mathscr{L}^{n+1} a) < \infty$ for all $n \geq n_0$ for some $n_0 \in \mathbb{N}$; $\mathscr{L}^n a \longrightarrow b^*$ where b^* is a unique fixed point of $\mathscr{L}$ in $X_0 := \{b \in X \mid d(\mathscr{L}^{n_0} a, b) < \infty\}$ and $d(b, b^*) \leq (1/(1-\alpha)) d(b, \mathscr{L} b)$ for all $b \in X_0$

Theorem 7. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a function such that*

$$\varphi\left(\frac{a}{2}, \frac{b}{2}\right) \leq \frac{L}{2} \varphi(a, b), \tag{25}$$

for all $a, b \in X$ for some $L \in \mathbb{R}_0^+$ with $L < 1$. Then, for $f \in \mathscr{F}_0(X, B)$ satisfying (11), there exists a unique $F \in \mathscr{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \frac{1}{1-L} \varphi(a, 0), \tag{26}$$

for all $a \in X$.

Proof. Firstly, let us equip $\mathscr{F}_0(X, B)$ with the generalized metric d defined by

$$d(g, h) = \inf \{\mu \in \mathbb{R}^+ : \|g(a) - h(a)\| \leq \mu\varphi(a, 0), \text{for all } a \in X\}. \tag{27}$$

□

Then, from [28], $(\mathscr{F}_0(X, B), d)$ is complete. Next, define a map $\mathscr{J} : \mathscr{F}_0(X, B) \longrightarrow \mathscr{F}_0(X, B)$ by

$$\mathscr{J} g(a) = 2g\left(\frac{a}{2}\right), \tag{28}$$

for all $a \in X$. Let $g, h \in \mathscr{F}_0(X, B)$ where $d(g, h) = \varepsilon$. Then,

$$\|g(a) - h(a)\| \leq \varepsilon\varphi(a, 0), \tag{29}$$

for all $a \in X$. Consequently,

$$\begin{aligned} \|\mathscr{J} g(a) - \mathscr{J} h(a)\| &= \left\|2g\left(\frac{a}{2}\right) - 2h\left(\frac{a}{2}\right)\right\| \leq 2\varepsilon\varphi\left(\frac{a}{2}, 0\right) \\ &\leq 2\varepsilon \frac{L}{2} \varphi(a, 0) = L\varepsilon\varphi(a, 0), \end{aligned} \tag{30}$$

for all $a \in X$. Then, $d(\mathscr{J} g, \mathscr{J} h) \leq L\varepsilon$ which means that

$$d(\mathscr{J} g, \mathscr{J} h) \leq L d(g, h), \tag{31}$$

for all $g, h \in \mathscr{F}_0(X, B)$. By (13), we have that $d(f, \mathscr{J} f) \leq 1$.

Now, let $a \in X$. From Proposition 6, there exists $F : X \longrightarrow B$ satisfying the following:

(i) F is a unique fixed point of $\mathscr{J}$, i.e., $F(a) = 2F(a/2)$ for all $a \in X$

(ii) $d(\mathscr{J}^l f, F) \longrightarrow 0$ as $l \longrightarrow \infty$

$$d(f, F) \leq \frac{1}{1-L} d(f, \mathscr{J} f). \tag{32}$$

It follows that

$$\|f(a) - F(a)\| \leq \mu\varphi(a, 0) \tag{33}$$

(a) $\lim_{l \longrightarrow \infty} 2^l f(2^{-l} a) = F(a)$ and

$$\|f(a) - F(a)\| \leq \frac{1}{1-L} \varphi(a, 0) \tag{34}$$

Using the same method as in the proof of Theorem 2, we can prove that $F \in \mathscr{A}_0(X, B)$.

Corollary 8. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r > 1$. If $f \in \mathscr{F}_0(X, B)$ satisfies (18), then there exists a unique $F \in \mathscr{A}_0(X, B)$ such that*

$$\|f(a) - F(a)\| \leq \frac{2^r \vartheta}{2^r - 2} \|a\|^r, \tag{35}$$

for all $a \in X$.

Proof. By taking $L = 2^{1-r}$ and $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 7, the result follows. □

Theorem 9. *Let $\varphi : X \times X \longrightarrow \mathbb{R}_0^+$ be a map such that*

$$\varphi(a, b) \leq 2L\varphi\left(\frac{a}{2}, \frac{b}{2}\right), \tag{36}$$

for all $a, b \in X$, for some $L \in \mathbb{R}_0^+$ with $L < 1$. Then, for any $f \in \mathscr{F}_0(X, B)$ satisfying (11), there exists a unique $F \in \mathscr{A}_0(X, B)$ such that

$$\|f(a) - F(a)\| \leq \frac{L}{1-L} \varphi(a, 0), \tag{37}$$

for all $a \in X$.

Proof. We first consider the complete metric space $(\mathscr{F}_0(X, B), d)$ given as in the proof of Theorem 7. Define a mapping $\mathscr{J} : \mathscr{F}_0(X, B) \longrightarrow \mathscr{F}_0(X, B)$ by

$$\mathscr{J} g(a) = \frac{1}{2} g(2a), \tag{38}$$

for all $a \in X$. It follows from (13) and (36) that

$$\left\| f(a) - \frac{1}{2} f(2a) \right\| \le \frac{1}{2} \varphi(2a, 0) \le L\varphi(a, 0), \tag{39}$$

for all $a \in X$. As in the proof of Theorem 2 and Theorem 7, there exists a unique $F \in \mathscr{A}_0(X, B)$ satisfying (37). □

Corollary 10. *Let $r, \vartheta \in \mathbb{R}_0^+$ with $r < 1$, and let $f \in \mathscr{F}_0(X, B)$ be a map satisfying (18). Then, there exists a unique $F \in \mathscr{A}_0(X, B)$ such that*

$$\| f(a) - F(a) \| \le \frac{2^r \vartheta}{2 - 2^r} \| a \|^r, \tag{40}$$

for all $a \in X$.

Proof. Taking $L = 2^{r-1}$ and $\varphi(a, b) = \vartheta(\|a\|^r + \|b\|^r)$ for all $a, b \in X$ in Theorem 9, the result follows. □

4. Conclusions and Final Remarks

We have obtained several Hyers–Ulam stability results for the functional inequality (3) using the direct method and the fixed point method. We now discuss some example for Theorem 2 (via Corollary 3). Consider the sequence space l_2 equipped with the 2-norm. Define $f : l_2 \longrightarrow l_2$ by

$$f(a) = (a_1 + a_2, a_1 - a_2, 2a_3, 0, 0, 0, \cdots), \tag{41}$$

for all $a = (a_1, a_2, a_3, \cdots) \in l_2$. Let $\vartheta = r = 2$. Then, $f \in \mathscr{F}_0(X, B)$ satisfies (18). By Corollary 3, there exists a unique $F \in \mathscr{A}_0(X, B)$ such that $\|f(a) - F(a)\|_2 \le 4\|a\|_2^2$ for all $a \in l_2$. This example is also valid for the other corollaries in the paper.

There could also be other (s, t)-type functional inequalities to be investigated, and thus, of course, their stability results to be examined. Moreover, these functional inequalities can still be possibly generalized in several ways.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This research was partially supported by the Chiang Mai University and by the Faculty of Science, Chiang Mai University, Thailand.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5800790, 8 pages
https://doi.org/10.1155/2022/5800790

Research Article

New Fixed Point Theorems for Admissible Hybrid Maps

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Received 30 October 2021; Accepted 18 March 2022; Published 4 April 2022

Academic Editor: Santosh Kumar

The aim of this work is to investigate the concept of a new hybrid Suzuki contractive by using the Rus-Reich-Ćirić-type interpolative mappings in b-metric spaces. We seek the presence and uniqueness of a fixed point of such new contraction type mappings and prove some related results. We further give an application of Ulam-Hyers-type stability to show the well-posedness of our results.

1. Introduction and Preliminaries

Fixed point hypothesis has been a considerable area of research for mathematics and other sciences for the last century. It is the basis of functional analysis in mathematics, which is one of the critical topics of mathematics. The first concept of fixed point theory is knowing to appear in the work of Liouville in 1837 and Picard in 1890. But the main fixed point theorem was introduced by Banach [1]. The theorem is named after Banach. There are many generalizations of Banach theorem in the literature. In 1968, one of the most famous generalizations due to know, Kannan [2] introduced a new and useful contraction using Banach's theorem. Suzuki [3] introduced important extensions of Banach's main theorem, which we refer to [4–6]. In one of these studies [7], the researchers investigated a new extensive result by using simulation function. On the other hand, in [8], by using other auxiliary functions, called the Wardowski functions, they observed a contraction that combines both linear and nonlinear contractions. We also mention that in [9], the author obtained a fixed point theorem without the Picard operator. For more interesting results, see, e.g., [10–19]. In addition, Banach's fixed point theorem is a significant mean in the theory of metric spaces. The metric concept has been generalized from different angles. One of the significant generalizes is defined b-metric which was defined as follows.

Definition 1 (see [20, 21]). Let $\mathscr{L}$ be a (nonempty) set and $s \geq 1$ a real number. A function $b : \mathscr{L} \times \mathscr{L} \longrightarrow [0,\infty)$ is a b-metric space on $\mathscr{L}$ if following conditions are satisfied:

(i) $b(r, v) = 0$, if $r = v$

(ii) $b(r, v) = b(v, r)$

(iii) $b(r, v) \leq s[b(r, q) + b(q, v)]$, for every $r, v, q \in \mathscr{L}$

In this case, the pair $(\mathscr{L}, b, s)$ is called a b-metric space.

We recollect some basic notions that are used in our study.

A map $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is defined as a comparison function if it is increasing and $\varphi^{q}(z) \longrightarrow 0, q \longrightarrow \infty$, for any $z \in [0, \infty)$. We state by Φ the class of all the comparison functions $\varphi : [0,\infty) \longrightarrow [0,\infty)$, see, e.g., [22–24]. Defined by $\Psi = \{\psi : [0,\infty) \longrightarrow [0,\infty) \,|\psi \text{ is the } b\text{-comparison function}\}$.

Lemma 2 (see [22, 23]). *For a comparison function,* $\varphi : [0, \infty) \longrightarrow [0,\infty)$ *satisfying the below statements take*

(1) every iterate φ^{l} *of* $\varphi, l \geq 1$ *is a comparison function*

(2) φ *is continuous*

(3) $\varphi(z) < z$, for any $z > 0$

Lemma 3 (see [25]). *If $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is a b-comparison function, then,*

(1) the series $\sum_{l=0}^{\infty} s^l \varphi^l(z)$ converges for any $z \in [0,\infty)$

(2) the function $b_s : [0,\infty) \longrightarrow [0,\infty)$ defined by $b_s(z) = \sum_{l=0}^{\infty} s^l \varphi^l(z), z \in [0,\infty)$ is increasing and continuous at 0.

We state that any b-comparison function is a comparison function because of Lemma 2.3, and thus, in Lemma 2.2 any b-comparison function ψ satisfies $\psi(z) < z$.

Karapinar [26] introduced interpolation Kannan-type contraction generalized from the famous Kannan fixed point theorem by using interpolative operator. In the following, the common fixed point theorem for this contraction was obtained [27]. In [28], the authors extended the results in [26] by introducing the interpolative Reich-Rus-Ćirić contractive in a general framework, in the setting of partial metric space. In addition, the interpolative Hardy-Rogers-type contractive was defined and discussed in [28]. The contraction, defined in [29], was generalized in [30] by involving the admissibility into the contraction inequality. Furthermore, in [31], hybrid contractions were considered. Indeed, the notion of hybrid contraction here refers to combination of interpolative (nonlinear) contraction and linear contraction. For more interesting papers, see [32–34].

In 2019, inspired by interpolative contraction, researchers [35] obtained and published a hybrid contractive that integrates Reich-Rus-Ćirić-type contractive and interpolative-type mappings. In particular, this approach was applied for Pata-Suzuki-type contraction in [36]. On the other hand, by using hybrid contraction, a solution for a Volterra fractional integral equation was proposed in [37]. Furthermore, the hybrid contractions were discussed in a distinct abstract space, namely, Branciari-type distance space, in [38]. Another advance was recorded in [39] where the authors investigated the Ulam-type stability of this consideration. In addition, new hybrid contractions were developed in b-metric spaces [40]. As a result, as can be seen in the literature review, many papers were published on the subject of interpolative contraction and hybrid contraction inspired by it. The contractions are a current study topic for fixed point theory. Therefore, the results of the study contribute to the existing literature.

Now we give the idea of α-admissibility defined by Samet et al. [41] and Karapnar and Samet [42].

Definition 4. A mapping $\mathrm{M} : \mathscr{L} \longrightarrow \mathscr{L}$ is defined α-admissible if for each $r, v \in \mathscr{L}$ we have

$$\alpha(r, v) \geq 1 \Rightarrow \alpha(\mathrm{M}r, \mathrm{M}v) \geq 1, \tag{1}$$

where $\alpha : \mathscr{L} \times \mathscr{L} \longrightarrow [0,\infty)$ is a given function.

The mapping of w-orbital admissibility was presented by Popescu [43] as a modification of α-admissibility as follows:

Definition 5. Let $w : \mathscr{L} \times \mathscr{L} \longrightarrow [0,\infty)$ be a mapping and $\mathscr{L} \neq \varnothing$. A map $\mathrm{M} : \mathscr{L} \longrightarrow \mathscr{L}$ is defined w-orbital admissible if for every $r \in \mathscr{L}$, we get

$$w(r, \mathrm{M}r) \geq 1 \Rightarrow w\left(\mathrm{M}r, \mathrm{M}^2 r\right) \geq 1. \tag{2}$$

The following condition has often been considered on account of refraining from the continuity of the concerned contractive mappings.

Definition 6. A space $(\mathscr{L}, \mathrm{b}, s)$ is defined w-regular, if $\{r_{\mathrm{q}}\}$ is a sequence in $\mathscr{L}$ such that $\alpha(r_{\mathrm{q}}, r_{\mathrm{q}+1}) \geq 1$ for all $\mathrm{q} \in \mathbb{N}$ and $r_{\mathrm{q}} \longrightarrow r \in \mathscr{L}$ as $\mathrm{q} \longrightarrow \infty$; then, there exists a subsequence $\{r_{\mathrm{q}(p)}\}$ of $\{r_{\mathrm{q}}\}$ such that $w(r_{\mathrm{q}(p)}, r) \geq 1$ for all p.

The framework of this study is organized into four sections. After the first introduction section, in Section 2, we introduced the definitions, theorems, and some results on the Ćirić-Rus-Reich-Suzuki-type hybrid. In Section 3, we give an application Ulam-Hyers-type stability to show the well-posedness for our fixed point theorem. Finally, in the last section, the conclusions are drawn.

2. Main Results

We begin with the definition of the Ćirić-Rus-Reich-Suzuki-type hybrid contraction:

Definition 7. Let $(\mathscr{L}, \mathrm{b}, s)$ be a b-metric space and $w : \mathscr{L} \times \mathscr{L} \longrightarrow [0,\infty)$ be a function. A map $\mathrm{M} : \mathscr{L} \longrightarrow \mathscr{L}$ is a Ćirić-Rus-Reich-Suzuki-type hybrid contraction (CRRS-type hybrid contraction) if there exist $\psi \in \Psi$ such that

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \Rightarrow w(r, v) b(Mr, Mv) \leq \psi(\chi_M^a(r, v)), \tag{3}$$

for each $r, v \in \mathscr{L}$, where $a \geq 0$ and $\rho_i \geq 0, i = 1,2,3$, such that $\rho_1 + \rho_2 + \rho_3 = 1$,

$$\chi_{\mathrm{M}}^a(r, v) = \begin{cases} \left[\varrho_1 (b(r, v))^a + \varrho_2 (b(r, \mathrm{M}r))^a + \varrho_3 (b(v, \mathrm{M}v))^a\right]^{1/a}, \text{for } a > 0, r \neq v \\ (b(r, v))^{\rho_1} (b(r, \mathrm{M}r))^{\rho_2} (b(v, \mathrm{M}v))^{\rho_3}, \text{for } a = 0, r, v \in \mathscr{L} \setminus Fix(M). \end{cases} \tag{4}$$

Theorem 8. *Let $(\mathscr{L}, b, s)$ be a complete b-metric space and w -orbital admissible map also $w(r_0, Mr_0) \geq 1$ for some $r_0 \in \mathscr{L}$. Given that $M : \mathscr{L} \longrightarrow \mathscr{L}$ be a CRRS-type hybrid contraction satisfying one of the following conditions:*

(h_1) $(\mathscr{L}, b, s)$ is w-regular

(h_2) M is continuous

(h_3) M^2 is continuous and $w(r, Mr) \geq 1$, where $r \in Fix(M^2)$.

Thereupon, M admits a fixed point in $\mathscr{L}$.

Proof. We install an iterative sequence $\{r_q\}$ of points such that $M^q(r_0)=r_q$ for $q=0,1,2,\cdots$ and $r_0\in\mathscr{L}$ with $w(r_0,Mr_0)\geq 1$. If $r_{q_0}=r_{q_{0+1}}$ for some integers q_0, then $r_{q_0}=Mr_{q_0}$. Thus, suppose that $r_q\neq r_{q+1}$, as M is w-orbital admissible, then $w(r_0,Mr_0)=w(r_0,r_1)\geq 1$ implies that $w(r_1,Mr_1)=w(r_1,r_2)\geq 1$. Continuing this process, we get

$$w(r_q,r_{q+1})\geq 1. \tag{5}$$

Condition 1: $a>0$, by taking $\chi_M^a(r,v)$ choosing $r=r_{q-1}$ and $v=Mr_{q-1}=r_q$ in (3) we get

$$\frac{1}{2s}b(r_{q-1},Mr_{q-1})=\frac{1}{2s}b(r_{q-1},r_q)\leq b(r_{q-1},r_q)\Rightarrow, \tag{6}$$

$$w(r_{q-1},r_q)b(Mr_{q-1},Mr_q)\leq\psi(\chi_M^a(r_{q-1},r_q)), \tag{7}$$

where

$$\begin{aligned}\chi_M^a(r_{q-1},Mr_{q-1})&=\left[\varrho_1(b(r_{q-1},Mr_{q-1}))^a+\varrho_2(b(r_{q-1},Mr_{q-1}))^a\right.\\&\quad\left.+\varrho_3(b(Mr_{q-1},M^2r_{q-1}))^a\right]^{1/a}=\left[\varrho_1(b(r_{q-1},r_q))^a\right.\\&\quad\left.+\varrho_2(b(r_{q-1},r_q))^a+\varrho_3(b(r_q,r_{q+1}))^a\right]^{1/a}.\end{aligned} \tag{8}$$

Whereupon, we deduce that

$$\begin{aligned}b(r_q,r_{q+1})&\leq w(r_{q-1},r_q)b(Mr_{q-1},Mr_q)\leq\psi(\chi_M^a(r_{q-1},r_q))\\&=\psi\left(\left[\varrho_1(b(r_{q-1},r_q))^a+\varrho_2(b(r_{q-1},r_q))^a+\varrho_3(b(r_q,r_{q+1}))^a\right]^{1/a}\right)\\&=\psi\left(\left[(\varrho_1+\varrho_2)(b(r_{q-1},r_q))^a+\varrho_3(b(r_q,r_{q+1}))^a\right]^{1/a}\right).\end{aligned} \tag{9}$$

If we have given that $b(r_q,r_{q+1})\geq b(r_{q-1},r_q)$, then, accompanying that ψ is nondecreasing with (9), we get

$$\begin{aligned}b(r_q,r_{q+1})&\leq\psi\left(\left[(\varrho_1+\varrho_2)(b(r_{q-1},r_q))^a+\varrho_3(b(r_q,r_{q+1}))^a\right]^{1/a}\right)\\&\leq\psi\left(\left[(\varrho_1+\varrho_2)(b(r_q,r_{q+1}))^a+\varrho_3(b(r_q,r_{q+1}))^a\right]^{1/a}\right)\\&=\psi\left(\left[(\varrho_1+\varrho_2+\varrho_3)(b(r_q,r_{q+1}))^a\right]^{1/a}\right)\\&=\psi\left(\left[(b(r_q,r_{q+1}))^a\right]^{1/a}\right)=\psi(b(r_q,r_{q+1}))<b(r_q,r_{q+1}),\end{aligned} \tag{10}$$

which is a contradiction. Thus, we obtain

$$b(r_q,r_{q+1})<b(r_{q-1},r_q). \tag{11}$$

As a result, from (9), we will turn into

$$b(r_q,r_{q+1})\leq\psi(b(r_{q-1},r_q))<b(r_{q-1},r_q), \tag{12}$$

and by similarly this process, we obtain that

$$b(r_q,r_{q+1})\leq\psi^q(b(r_0,r_1)). \tag{13}$$

for any $q\in\mathbb{N}$.

We argue that $\{r_q\}$ is a fundamental sequence in $(\mathscr{L},b,s)$. Then, let $q,l\in\mathbb{N}$ such that $l>q$ and using the triangle inequality with (13), we take

$$\begin{aligned}b(r_q,r_l)&\leq sb(r_q,r_{q+1})+s^2b(r_{q+1},r_{q+2})+\cdots+s^{l-q}b(r_{l-1},r_l)\\&\leq s\psi^q(b(r_0,r_1))+s^2\psi^{q+1}(b(r_0,r_1))+\cdots+s^{l-q}\psi^{l-1}(b(r_0,r_1))\\&=\frac{1}{s^{q-1}}\left(s^q\psi^q(b(r_0,r_1))+s^{q+1}\psi^{q+1}(b(r_0,r_1))+\cdots+s^{l-1}\psi^{l-1}(b(r_0,r_1))\right)\\&=\frac{1}{s^{q-1}}\sum_{q=q}^{l-1}s^q\psi^q(b(r_1,r_0)).\end{aligned} \tag{14}$$

By using Lemma 3, the series $\sum_{q=0}^{\infty}s^q\psi^q(b(r_1,r_0))$ is convergent where $H_t=\sum_{q=0}^{t}s^q\psi^q(b(r_0,r_1))$, the above inequality finds

$$b(r_q,r_l)\leq\frac{1}{s^{q-1}}(H_{l-1}-H_{q-1}) \tag{15}$$

and $q,l\longrightarrow\infty$, we obtain

$$b(r_q,r_l)\longrightarrow 0. \tag{16}$$

Thus, $\{r_q\}$ is a fundamental sequence. Accompanying this together with the fact that the space $(\mathscr{L},b,s)$ is complete, it will imply that there exists $p\in\mathscr{L}$ such that

$$\lim_{q\longrightarrow\infty}b(r_q,p)=0. \tag{17}$$

We argue that p is a fixed point of M.

If the suppose (h_1) takes, we get $w(r_q,p)\geq 1$, and we assert that

$$\text{either}\frac{1}{2s}b(r_q,Mr_q)\leq b(r_q,p)\text{or}\frac{1}{2s}b(Mr_q,M(Mr_q))\leq b(Mr_q,p), \tag{18}$$

for every $q\in\mathbb{N}$. Since, if we have given that

$$\frac{1}{2s}b(r_q,Mr_q)>b(r_q,p)\text{and}\frac{1}{2s}b(Mr_q,M(Mr_q))>b(Mr_q,p), \tag{19}$$

then, by using conditions of b-metric spaces $(\mathscr{L},b,s)$, since the sequence $\{b(r_q,r_{q+1})\}$ is decreasing, we write that

$$\begin{aligned}b(r_q,r_{q+1})&=b(r_q,Mr_q)\leq s(b(r_q,p)+b(p,Mr_q))<\frac{1}{2}b(r_q,Mr_q)+\frac{1}{2}b(Mr_q,M(Mr_q))\\&=\frac{1}{2}b(r_q,r_{q+1})+\frac{1}{2}b(r_{q+1},r_{q+2})<\frac{1}{2}b(r_q,r_{q+1})+\frac{1}{2}b(r_q,r_{q+1})=b(r_q,r_{q+1})\end{aligned} \tag{20}$$

a contradiction. Therefore, for all $q \in \mathbb{N}$, either

$$\frac{1}{2s} b\left(r_q, Mr_q\right) \leq b\left(r_q, p\right), \tag{21}$$

or

$$\frac{1}{2s} b\left(Mr_q, M\left(Mr_q\right)\right) \leq b\left(Mr_q, p\right) \tag{22}$$

provides. In the condition that (21) takes, then by (3), we get

$$\begin{aligned} b\left(r_{q+1}, Mp\right) &\leq w\left(r_q, p\right) b\left(Mr_q, Mp\right) \\ &\leq \psi\left[\varrho_1\left(b\left(r_q, p\right)\right)^a + \varrho_2\left(b\left(r_q, Mr_q\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &< \left[\varrho_1\left(b\left(r_q, p\right)\right)^a + \varrho_2\left(b\left(r_q, Mr_q\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &= \left[\varrho_1\left(b\left(r_q, p\right)\right)^a + \varrho_2\left(b\left(r_q, r_{q+1}\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a}. \end{aligned} \tag{23}$$

If the second condition, (22) holds, we obtain

$$\begin{aligned} b\left(r_{q+2}, Mp\right) &\leq w\left(r_{q+1}, p\right) b\left(M^2 r_q, Mp\right) \\ &\leq \psi\left[\varrho_1\left(b\left(Mr_q, p\right)\right)^a + \varrho_2\left(b\left(Mr_q, M^2 r_q\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &< \left[\varrho_1\left(b\left(Mr_q, p\right)\right)^a + \varrho_2\left(b\left(Mr_q, M^2 r_q\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &= \left[\varrho_1\left(b\left(r_{q+1}, p\right)\right)^a + \varrho_2\left(b\left(r_{q+1}, r_{q+2}\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a}. \end{aligned} \tag{24}$$

Thereupon, taking $q \longrightarrow \infty$ in (23) and (24),

$$b(p, Mp) < \varrho_3^{1/a} b(p, Mp) \leq b(p, Mp) \tag{25}$$

which is contraction. Therefore, we get that $b(p, Mp) = 0$ that is $p = Mp$.

If the presume (h_2) is correct, and the map M is continuous, we get

$$Mp = \lim_{q \longrightarrow \infty} Mr_q = \lim_{q \longrightarrow \infty} r_{q+1} = p. \tag{26}$$

In case that last supposition, (h_3) holds, from above, we write $M^2 p = \lim_{q \longrightarrow \infty} M^2 r_q = \lim_{q \longrightarrow \infty} r_{q+2} = p$, we want to show that $Mp = p$. Let us pretend otherwise, that is, $p \neq Mp$ from

$$\frac{1}{2s} b\left(Mp, M^2 p\right) = \frac{1}{2s} b(Mp, p) \leq b(Mp, p) \tag{27}$$

using (3) we obtain that

$$\begin{aligned} b(p, Mp) &= b\left(M^2 p, Mp\right) \leq w(Mp, p) b\left(M^2 p, Mp\right) \\ &\leq \psi\left[\varrho_1(b(Mp, p))^a + \varrho_2\left(b\left(Mp, M^2 p\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &< \left[\varrho_1(b(Mp, p))^a + \varrho_2\left(b\left(Mp, M^2 p\right)\right)^a + \varrho_3(b(p, Mp))^a\right]^{1/a} \\ &= \left[(\varrho_1 + \varrho_2 + \varrho_3)(b(p, Mp))^a\right]^{1/a} = b(p, Mp) \end{aligned} \tag{28}$$

a contradiction. Eventually, $p = Mp$.

Condition 2: if $a = 0$, in the equation $\chi_M^a(r, v)$ taking $r = r_{q-1}$ and $v = Mr_{q-1} = r_q$ in (3) we write

$$\frac{1}{2s} b\left(r_{q-1}, Mr_{q-1}\right) = \frac{1}{2s} b\left(r_{q-1}, r_q\right) \leq b\left(r_{q-1}, r_q\right) \Rightarrow, \tag{29}$$

$$\begin{aligned} b\left(r_q, r_{q+1}\right) &\leq w\left(r_{q-1}, r_q\right) b\left(Mr_{q-1}, Mr_q\right) \leq \psi\left(\chi_M^a\left(r_{q-1}, r_q\right)\right) \\ &= \psi\left(\left[b\left(r_{q-1}, r_q\right)\right]^{\rho_1} \cdot \left[b\left(r_{q-1}, Mr_{q-1}\right)\right]^{\rho_2} \cdot \left[b\left(r_q, Mr_q\right)\right]^{\rho_3}\right) \\ &< \left[b\left(r_{q-1}, r_q\right)\right]^{\rho_1} \cdot \left[b\left(r_{q-1}, r_q\right)\right]^{\rho_2} \cdot \left[b\left(r_q, r_{q+1}\right)\right]^{\rho_3}. \end{aligned} \tag{30}$$

From (30), we find

$$\left(b\left(r_q, r_{q+1}\right)\right)^{1-\varrho_3} < \left(b\left(r_{q-1}, r_q\right)\right)^{\varrho_1+\varrho_2} \tag{31}$$

and from $\varrho_1 + \varrho_2 + \varrho_3 = 1$, we attain that $b(r_q, r_{q+1}) < b(r_{q-1}, r_q)$ for every $q \in \mathbb{N}$. Using (30), we take

$$b\left(r_q, r_{q+1}\right) \leq \psi\left(b\left(r_{q-1}, r_q\right)\right) \tag{32}$$

and as in condition 1, we can prove that

$$b\left(r_q, r_{q+1}\right) \leq \psi^q\left(b\left(r_0, r_1\right)\right). \tag{33}$$

Since the equal methods as in the case of $a > 0$, we clearly prove that $\{r_q\}$ is a fundamental sequence in a complete b-metric space. Additionally, for $p \in \mathscr{L}$ so, $\lim_{q \longrightarrow \infty} b(r_q, p) = 0$ also we assert that $p = Mp$. In the meanwhile, $(\mathscr{L}, b, s)$ is w-regular; thus, as $\{r_q\}$ confirm (5), and $w(r_q, r_{q+1}) \geq 1$ for each $q \in \mathbb{N}$, we obtain $w(r_q, p) \geq 1$. Moreover, as in the proof of condition 1, we know that either

$$\frac{1}{2s} b\left(r_q, Mr_q\right) \leq b\left(r_q, p\right), \tag{34}$$

or

$$\frac{1}{2s} b\left(Mr_q, M\left(Mr_q\right)\right) \leq b\left(Mr_q, p\right) \tag{35}$$

holds, for each $q \in \mathbb{N}$. If (34) is taken, we conclude that

$$\begin{aligned} b\left(r_{q+1}, Mp\right) &\leq w\left(r_q, p\right) b\left(Mr_q, Mp\right) \leq \psi\left(\left[b\left(r_q, p\right)\right]^{\varrho_1} \cdot \left[b\left(r_q, Mr_q\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}\right), \\ &= \psi\left(\left[b\left(r_q, p\right)\right]^{\varrho_1} \cdot \left[b\left(r_q, r_{q+1}\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}\right), \\ &< \left[b\left(r_q, p\right)\right]^{\varrho_1} \cdot \left[b\left(r_q, r_{q+1}\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}, \end{aligned} \tag{36}$$

Let us assume that inequality (35) is satisfied, then

$$\begin{aligned} b\left(r_{q+2}, Mp\right) &\leq w\left(r_{q+1}, p\right) b\left(M^2 r_q, Mp\right) \\ &\leq \psi\left(\left[b\left(Mr_q, p\right)\right]^{\varrho_1} \cdot \left[b\left(Mr_q, M^2 r_q\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}\right), \\ &= \psi\left(\left[b\left(r_{q+1}, p\right)\right]^{\varrho_1} \cdot \left[b\left(r_{q+1}, r_{q+2}\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}\right), \\ &< \left[b\left(r_{q+1}, p\right)\right]^{\varrho_1} \cdot \left[b\left(r_{q+1}, r_{q+2}\right)\right]^{\varrho_2} \cdot [b(p, Mp)]^{\varrho_3}, \end{aligned} \tag{37}$$

Then, getting to the limit, we conclude that $b(p, Mp) = 0$, and $p = Mp$. Now, the continuity of M implies $p = Mp$ (from condition 1). Therefore, supposition (h_3) lead to $M^2 p = \lim_{q \longrightarrow \infty} M^2 r_q = \lim_{q \longrightarrow \infty} r_{q+2} = p$. We will prove that $Mp = p$. Let's presume otherwise, that is, $p \neq Mp$

$$\frac{1}{2s} b\left(Mp, M^2 p\right) = \frac{1}{2s} b(Mp, p) \leq b(Mp, p) \tag{38}$$

using (3) we find that

$$\begin{aligned} b(p, Mp) = b\left(M^2 p, Mp\right) &\leq w(Mp, p) b\left(M^2 p, Mp\right) \\ &\leq \psi\left(\left[b(Mp, p)\right]^{\varrho_1} \cdot \left[b\left(Mp, M^2 p\right)\right]^{\varrho_2} \cdot \left[b(p, Mp)^{\varrho_3}\right]\right) \\ &< \left[b(Mp, p)\right]^{\varrho_1} \cdot \left[b(Mp, p)\right]^{\varrho_2} \cdot \left[b(p, Mp)^{\varrho_3}\right] = b(Mp, p), \end{aligned} \tag{39}$$

a contradiction. Consequently, $p = Mp$. Thus, the proof of the Theorem is completed. □

Theorem 9. *Adding $w(p, p^*) \geq 1$ for any $p, p^* \in Fix_M(\mathscr{L})$ and if supplying to all the hypothesis of Theorem 8, we prove the uniqueness of fixed point.*

Proof. Supposing that different p^* is fixed point of M, that is $Mp^* = p^*$ with $p \neq p^*$.In the case that $a > 0$, then, from (3) we have

$$\frac{1}{2s} b(p, Mp) = 0 \leq b(p, p^*) \text{implies} \tag{40}$$

$$\begin{aligned} b(p, p^*) &\leq w(p, p^*) b(Mp, Mp^*) \leq \psi(\chi_M^a(p, p^*)) < \chi_M^a(p, p^*) \\ &= \left[\varrho_1 (d(p, p^*))^a + \varrho_2 (b(p, Mp))^a + \varrho_3 (b(p^*, Mp^*))^a\right]^{1/a}. \end{aligned} \tag{41}$$

Thus,

$$b(p, p^*) < (\varrho_1)^{1/a} b(p, p^*) \leq b(p, p^*) \tag{42}$$

which is contradiction. In the case that $a = 0$, then, from (4) we get that

$$0 < b(p, p^*) < 0, \tag{43}$$

a contradiction. Eventually, $p = p^*$, so p is a unique fixed point of M. □

Example 1. Let $b : \mathscr{L} \times \mathscr{L} \longrightarrow [0,+\infty)$, $b(r, v) = |r - v|^2$ for every $r, v \in \mathscr{L}$ with $s = 2$ and

$$w(r, v) = \begin{cases} 4, & \text{if} r, v \in [0, 1] \\ 1, & \text{if} r = 0, v = 2 \\ 0, & \text{otherwise} \end{cases} \tag{44}$$

also, the function $\psi \in \Psi$ with $\psi(t) = t/4$. Define a mapping $M : \mathscr{L} \longrightarrow \mathscr{L}$ as

$$Mr = \begin{cases} \dfrac{1}{5}, & if r \in [0, 1] \\ \dfrac{r}{5}, & if r \in (1, 2] \end{cases} \tag{45}$$

also, $M^2 = r/10$, we get that M^2 is continuous but M is not continuous, where $\mathscr{L} = [0, 2]$.

We choose $a = 2$ and $\varrho_1 = \varrho_2 = \varrho_3 = 1/3$, then we obtain the following conditions:

(a) : For $r, v \in [0, 1]$ we get $b(Mr, Mv) = 0$, then, (3) holds

(b) : If $r = 0$ and $v = 2$

$$\frac{1}{2s} b(0, M0) = \frac{1}{100} < 4 = b(0, 2) \Rightarrow . \tag{46}$$

$$\begin{aligned} w(0, 2) b(M0, M2) = 0.04 \leq 0.3708099243547831 &= \frac{1}{4} \sqrt{\frac{1}{3}(2)^2 + \frac{1}{3}\left(\frac{1}{5}\right)^2 + \frac{1}{3}\left(2 - \frac{2}{5}\right)^2} \\ &= \psi \sqrt{\rho_1((b(r, v))^2 + \rho_2(b(r, Mr))^2 + \rho_3(b(v, Mv))^2}. \end{aligned} \tag{47}$$

Other conditions are confirmed, from $w(r, v) = 0$. Consequently, the assumptions of Theorem 8, being supplied, M has a fixed point ($r = 1/5$).

Corollary 10. *Let $(\mathscr{L}, b, s)$ be a complete b-metric space and let $M : \mathscr{L} \longrightarrow \mathscr{L}$ a continuous map satisfying the following inequality:*

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \text{ implies } b(Mr, Mv) \leq \psi(\chi_M^a(r, v)), \tag{48}$$

where $\chi_M^a(r, v)$ is defined by (4), $\psi \in \Psi$ and for all $r, v \in \mathscr{L}$, where $a \geq 0$ and $\varrho_i \geq 0, i = 1, 2, 3$ with $\varrho_1 + \varrho_2 + \varrho_3 = 1$. In the case of M or M^2 functions continuity, M admits a fixed point in $\mathscr{L}$.

Proof. It is sufficient to get $w(r, v) = 1$ for $r, v \in \mathscr{L}$ in Theorem 8. □

Corollary 11. *Let $(\mathscr{L}, b, s)$ be a complete b-metric space and let $M : \mathscr{L} \longrightarrow \mathscr{L}$ a continuous map satisfying the following inequality*

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \text{ implies } b(Mr, Mv) \leq \eta(\chi_M^a(r, v)), \tag{49}$$

where $\chi_M^a(r, v)$ is defined by (4), $\eta \in [0, 1)$ and for each $r, v \in \mathscr{L}$ where $a \geq 0$ and $\rho_i \geq 0, i = 1, 2, 3$ with $\rho_1 + \rho_2 + \rho_3 = 1$. In the event of M or M^2 functions continuity, M admits a fixed point in $\mathscr{L}$.

Proof. It is adequate get $\psi(v) = \eta v$ for any $v \geq 0$ in Corollary 10. □

Corollary 12. *Let* $(\mathscr{L}, b, s)$ *be a complete b-metric space and* $M : \mathscr{L} \longrightarrow \mathscr{L}$ *a continuous map. If there exist* $\eta \in [0, 1)$ *such that*

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \text{implies} \tag{50}$$

$$b(Mr, Mv) \leq \eta \sqrt[3]{b(r, v) b(r, Mr) b(v, Mv)} \tag{51}$$

for each $r, v \in \mathscr{L} \setminus Fix(M)$, *in the case of* M *or* M^2 *functions continuity,* M *admits a fixed point in* $\mathscr{L}$.

Proof. If $a = 0$, using Corollary 11, getting $\rho_1 = \rho_2 = \rho_3 = 1/3$. □

Corollary 13. *Let* $(\mathscr{L}, b, s)$ *be a complete b-metric space and* $M : \mathscr{L} \longrightarrow \mathscr{L}$ *a continuous map. If there exist* $\eta \in [0, 1)$ *such that*

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \text{implies} \tag{52}$$

$$b(Mr, Mv) \leq \frac{\eta}{3} (b(r, v) + b(r, Mr) + b(v, Mv)) \tag{53}$$

for each $r, v \in \mathscr{L} \setminus Fix(M)$, *in the case of* M *or* M^2 *functions continuity,* M *admits a fixed point in* $\mathscr{L}$.

Proof. By using Corollary 11, letting $\varrho_1 = \varrho_2 = \varrho_3 = 1/3$ and $a = 1$. □

Corollary 14. *Let* $(\mathscr{L}, b, s)$ *be a complete b-metric space and* $M : \mathscr{L} \longrightarrow \mathscr{L}$ *a continuous map. If there exist* $\eta \in [0, 1)$ *such that*

$$\frac{1}{2s} b(r, Mr) \leq b(r, v) \text{implies} \tag{54}$$

$$b(Mr, Mv) \leq \frac{\eta}{\sqrt{3}} \left(\sqrt{(b(r, v))^2 + (b(r, Mr))^2 + (b(v, Mv))^2} \right. \tag{55}$$

for each $r, v \in \mathscr{L} \setminus Fix(M)$, *in the case of* M *or* M^2 *functions continuity,* M *admits a fixed point in* $\mathscr{L}$.

Proof. By using Corollary 11, taking $\varrho_1 = \varrho_2 = \varrho_3 = 1/3$ and $a = 2$. □

3. An Application: Ulam-Hyers-Type Stability

The stability of the solution is a considerable important subject of nonlinear analysis. Recently, Ulam stability [44, 45] results in fixed point theory have been investigated heavily. In what follows. we investigate the Ulam stability of our main theorem.

Consider the following function:

$$\Upsilon : \{\gamma : [0,\infty) \longrightarrow [0,\infty) \textit{such that } \gamma \text{ is } \textit{continuous at zero with} \tag{56}$$

$$\gamma(0) = 0 \text{ and increasing}\} \tag{57}$$

Assume that $M : \mathscr{L} \longrightarrow \mathscr{L}$ is a map on a b-metric spaces $(\mathscr{L}, b, s)$. The fixed point problem of M is to notice an $r \in \mathscr{L}$ such that

$$r = Mr. \tag{58}$$

Equality (58) is also known as fixed point implication. The fixed point implication is called to be general Ulam-Hyers stable if and only if there exists a function $\gamma \in \Upsilon$ so that for all $\varepsilon > 0$ also for every $v_* \in \mathscr{L}$ which satisfies the following inequality,

$$b(v_*, Mv_*) \leq \varepsilon \tag{59}$$

there exists $u \in \mathscr{L}$ providing the equation (58) such that

$$b(u, v_*) \leq \gamma(\varepsilon). \tag{60}$$

Moreover, if there exists a $P > 0$ such that $\gamma(t) = Pt$ for all $t \in \mathbb{R}^+$, then the fixed point equation (58) is said to be Ulam-Hyers stable. On the b-metric spaces $(\mathscr{L}, b, s)$, fixed point problem (58) and $M : \mathscr{L} \longrightarrow \mathscr{L}$ are defined to be well-known if the following suppositions are satisfy:

(l_1) M has a unique fixed point $u \in \mathscr{L}$

(l_2) $\lim_{q \longrightarrow \infty} b(u, r_q) = 0$ for every sequence $r_q \in \mathscr{L}$ such that

$$\lim_{q \longrightarrow \infty} b(r_q, Mr_q) = 0 \tag{61}$$

Theorem 15. *Let* $(\mathscr{L}, b, s)$ *be a complete b-metric space. If we joint the condition* $a > 0$ *and* $e(a) s^a \rho_1 < 1$, *where* $e(a) = \max\{1, 2^{a-1}\}$ *and* $s^a \rho_1 + e(a) s^a (\rho_i + 1) < 1$, *where* $i = 1$ *or* $i = 2$ *or* $i = 3$, *also suppositions of Theorem 9, thus the following conditions hold:*

(a) *the fixed point problem (58) is Ulam-Hyers stable, if* $w(n, m) \geq 1$ *for any* n, m *satisfying the condition (59)*

(b) *the fixed point problem (58) is well-known, if* $w(r_q, u) \geq 1$ *for any* $\{r_q\}$ *in* $\mathscr{L}$ *such that* $\lim_{q \longrightarrow \infty} b(Mr_q, r_q) = 0$ *and* $Fix_M(\mathscr{L}) = u$.

Proof.

(a) Taking into account Theorem 9, we consider that there is a unique u in $\mathscr{L}$ such that $Mu = u$. Assume that v_* is a solution of (59), that is $b(v_*, Mv_*) \leq \varepsilon$ for $\varepsilon > 0$. Clearly, u holds (59), then we get that $w(u, v_*) \geq 1$ and using triangular inequality satisfies

$$b(u, v_*) = b(Mu, v_*) \leq s[b(Mu, Mv_*) + b(Mv_*, v_*)] \tag{62}$$

Since M is CRRS-type hybrid contraction, we obtain

$$\frac{1}{2s}b(u, Mu) = 0 \le b(u, v_*) implies \tag{63}$$

$$\begin{aligned} b(u, v_*) &\le s[b(Mu, Mv_*) + b(Mv_*, v_*)] \\ &\le s[w(u, v_*)b(Mu, Mv_*) + b(Mv_*, v_*)] \\ &\le s[\psi(\chi_M^a(u, v_*)) + b(Mv_*, v_*)] \\ &< s[\chi_M^a(u, v_*) + b(Mv_*, v_*)] \\ &\le s[\rho_1(b(u, v_*))^a + \rho_2(d(u, Mu))^a + \rho_3(b(Mv_*, v_*))^a]^{1/a} \\ &\quad + sb(Mv_*, v_*) \le s[\rho_1(b(u, v_*))^a + \rho_3\varepsilon^a]^{1/a} + s\varepsilon \end{aligned} \tag{64}$$

Thus, we get

$$(b(u, v_*))^a \le e(a)[s^a\varrho_1(b(u, v_*))^a + s^a\varrho_3\varepsilon^a + s^a\varepsilon^a] \tag{65}$$

then,

$$(b(u, v_*))^a \le \frac{(1 + \rho_3)e(a)s^a}{1 - \rho_1 e(a)s^a}\varepsilon^a \tag{66}$$

$$b(u, v_*) \le n\varepsilon \tag{67}$$

where $n = [(1 + \varrho_3)e(a)s^a/1 - \varrho_1 e(a)s^a]^{1/a}$ for any $a > 0$ and $\varrho_1 \in [0, 1)$ such that $\varrho_1 < 1/e(a)s^a$..

(b) The Picard iterations is M-stable, that is, let $r_q \in \mathscr{L}$ such that $\lim_{q \longrightarrow \infty} b(r_{q+1}, Mr_q) = 0$ and $Fix_M(\mathscr{L}) = u$. From the triangular inequality, we can write

$$b(r_q, u) \le s[b(r_q, Mr_q) + b(Mr_q, Mu)]. \tag{68}$$

Thus, M is a CRRS contraction, we have

$$\frac{1}{2s}b(r_q, Mr_q) \le b(r_q, u)\text{implies} \tag{69}$$

$$\begin{aligned} b(r_q, u) &\le s[b(r_q, Mr_q) + b(Mr_q, Mu)] \\ &\le s[b(r_q, Mr_q) + w(r_q, u)b(Mr_q, Mu)] \\ &\le s[\psi(\chi_M^a(r_q, u)) + b(r_q, Mr_q)] < s[\chi_M^a(r_q, u) + b(r_q, Mr_q)] \\ &\le s[\rho_1(b(r_q, u))^a + \rho_2(b(r_q, Mr_q))^a + \rho_3(b(Mu, u))^a]^{1/a} + sb(r_q, Mr_q) \\ &\le s[\rho_1(b(r_q, u))^a + \rho_2(b(r_q, Mr_q))^a]^{1/a} + sb(r_q, Mr_q). \end{aligned} \tag{70}$$

Then, we calculate process

$$(b(r_q, u))^a \le e(a)[s^a\varrho_1(b(r_q, u))^a + s^a\varrho_2(b(r_q, Mr_q))^a + s^a(b(r_q, Mr_q))^a] \tag{71}$$

then,

$$(b(r_q, u))^a \le \frac{(1 + \varrho_2)e(a)s^a}{1 - \varrho_1 e(a)s^a}(b(r_q, Mr_q))^a. \tag{72}$$

Taking $q \longrightarrow \infty$ in the above inequality and using $\lim_{q \longrightarrow \infty} b(r_q, Mr_q) = 0$, we obtain $\lim_{q \longrightarrow \infty} b(r_q, u) = 0$ the fixed point equation (58) is well posed. □

4. Conclusion

In this study, we present new hybrid fixed point theorems in b-metric spaces. We obtain the extended results of the interpolative Reich-Rus-Ćirić fixed point theorem by using w-orbital admissible and Suzuki-type mapping. We also offer an example to show the availability of introduced results. Further, we obtain Ulam-Hyers-type stability of the fixed point theorem which is the application of our study.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

The authors contributed equally to this manuscript. All authors have read and agreed to the published version of the manuscript.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 4855173, 11 pages
https://doi.org/10.1155/2022/4855173

Research Article

Strong Convergence of a New Hybrid Iterative Scheme for Nonexpensive Mappings and Applications

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Received 27 October 2021; Revised 2 January 2022; Accepted 1 February 2022; Published 19 March 2022

Academic Editor: Santosh Kumar

In the article, we have proposed a new type of hybrid iterative scheme which is a hybrid of Picard and Thakur et al. repetitive schemes. This new hybrid iterative scheme converges faster than all leading schemes like Picard-S* hybrid, Picard-S, Picard-Ishikawa hybrid, Picard-Mann hybrid, Thakur et al. and Abbas and Nazir, S-iterative, Ishikawa and Mann iterative schemes for contraction mapping. By using the Picard-Thakur hybrid iterative scheme, we can find the solution of delay differential equations and also prove some convergence results for nonexpansive mapping in a uniformly convex Banach space.

1. Introduction

In this article, the set of all positive integers is denoted by I^+. Let N denote the nonempty convex subset of a normed space and S be its convex subset, and $\mathcal{V} : S \longrightarrow S$ is called contraction mapping if $\|\mathcal{V}_j - \mathcal{V}_k\| \leq \delta \|j - k\|$ for all $j, k \in S$ and $\delta \in (0, 1)$. If $\delta = 1$, then, the mapping $\mathcal{V}$ is called nonexpansive mapping. An element $j \in S$ is said to be a fixed point of $\mathcal{V}$ if $\mathcal{V} j = j$, and the set of fixed points of $\mathcal{V}$ is denoted by $F(\mathcal{V})$.

In 1890, Picard [1] presented an iterative scheme for approximating the fixed point which is defined by the sequence $\{j_n\}$ as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathcal{V} j_n, \end{cases} \quad n \in I^+. \tag{1}$$

The Krasnoselskii [2] iterative sequence $\{u_n\}$ is defined as

$$\begin{cases} u_1 = u \in S, \\ u_{n+1} = (1-\mu)u_n + \mu \mathcal{V} u_n, \end{cases} \quad n \in I^+, \tag{2}$$

where $\mu \in (0, 1)$.

In 1953, Mann [3] proposed an iterative scheme which is defined as

$$\begin{cases} v_1 = v \in S, \\ v_{n+1} = (1-\theta_n)v_n + \theta_n \mathcal{V} v_n, \end{cases} \quad n \in I^+, \tag{3}$$

where $\{\theta_n\} \in (0, 1)$.

In 1974, Ishikawa [4] gave the concept of the two-step iterative scheme and the sequence $\{w_n\}$ of this iterative is defined as

$$\begin{cases} w_1 = w \in S, \\ w_{n+1} = (1-\theta_n)w_n + \theta_n \mathscr{V} t_n, n \in I^+, \\ t_n = (1-\vartheta_n)w_n + \vartheta_n \mathscr{V} w_n, \end{cases} \tag{4}$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

In 2007, Agarwal et al. [5] introduced a more generalized form of the Ishikawa iterative scheme and they called it the S -iterative scheme and the sequence $\{p_n\}$ of the iterative scheme is defined as

$$\begin{cases} p_1 = p \in S, \\ p_{n+1} = (1-\theta_n)\mathscr{V} p_n + \theta_n \mathscr{V} q_n, n \in I^+, \\ q_n = (1-\vartheta_n)p_n + \vartheta_n \mathscr{V} p_n, \end{cases} \tag{5}$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

In 2016, Sahu et al. [6] and Thakur et al. [7] proposed a new scheme which converges faster than all the existing schemes. The iterative sequence $\{k_n\}$ of this scheme is defined as

$$\begin{cases} k_1 = k \in S, \\ k_{n+1} = (1-\theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n, \\ l_n = (1-\vartheta_n)m_n + \vartheta_n \mathscr{V} m_n, \\ m_n = (1-\sigma_n)k_n + \sigma_n \mathscr{V} k_n, \end{cases} n \in I^+, \tag{6}$$

where $\{\theta_n\}, \{\vartheta_n\}$, and $\{\sigma_n\} \in (0,1)$.

Thakur et al. [7] proposed another iterative scheme which converges faster than all the above schemes and the iterative sequence $\{j_n\}$ of Thakur et al. is defined as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = \mathscr{V}((1-\theta_n)j_n + \theta_n \mathscr{V} l_n), \\ l_n = (1-\vartheta_n)j_n + \vartheta_n \mathscr{V} j_n), \end{cases} n \in I^+, \tag{7}$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

Recently, Lamba and Panwar [8] introduced a new three-step iteration process for Susuzki's nonexpansive mapping and called it the Ap iterative scheme whose rate of convergence is faster than those of the leading schemes. The sequence of the Ap iterative scheme is defined as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = \mathscr{V}((1-\theta_n)\mathscr{V} j_n + \theta_n \mathscr{V} l_n), \\ l_n = \mathscr{V}((1-\vartheta_n)j_n + \vartheta_n \mathscr{V} j_n), \end{cases} n \in I^+, \tag{8}$$

where $\{\theta_n\}, \{\vartheta_n\} \in (0,1)$.

Many physical problems of engineering and applied sciences are mostly constructed in the form of fixed point equations. In the existence of a fixed point equation involving an operator, $\mathscr{V}$ is guaranteed but the exact solution is not possible. We can only approximate the solution which becomes very relevant and this necessitated various iterative schemes [9–14]. Also, the iterative schemes are used for solving different problems like minimization, equilibrium, viscosity approximation, and many more problems in different spaces [15–18].

The Picard iterative scheme is the simplest iteration to estimate the solution of a fixed point equation. Chidume [19] introduced some basic results on this iterative scheme. Chidume generalized and improved the existing results of the fixed point equation in [20]. Okeke and Abbas [21] proved the convergence and almost $\mathscr{V}$-stability of Mann-type and Ishikawa-type random iterative schemes.

In 2013, Khan [22] proposed the Picard-Mann hybrid iterative scheme. The sequence $\{r_n\}$ of this scheme is defined as

$$\begin{cases} r_1 = r \in S, \\ r_{n+1} = \mathscr{V} s_n, \\ s_n = (1-\theta_n)r_n + \theta_n \mathscr{V} r_n, \end{cases} n \in I^+, \tag{9}$$

where $\{\theta_n\} \in (0,1)$.

In 2017, Okeke and Abbas [23] proposed the Picard-Krasnoselskii hybrid iterative scheme and the sequence $\{r_n\}$ of this iterative scheme is defined as

$$\begin{cases} r_1 = r \in S, \\ r_{n+1} = \mathscr{V} s_n, \\ s_n = (1-\nu)r_n + \nu \mathscr{V} r_n, \end{cases} n \in I^+, \tag{10}$$

where $\nu \in (0,1)$.

In 2019, Okeke [24] proposed the Picard-Ishikawa hybrid iterative scheme and the sequence $\{f_n\}$ of this iteration defined as

$$\begin{cases} f_1 = f \in S, \\ f_{n+1} = \mathscr{V} g_n, \\ g_n = (1-\theta_n) f_n + \theta_n \mathscr{V} h_n, \\ h_n = (1-\vartheta_n) f_n + \vartheta_n \mathscr{V} f_n, \end{cases} n \in I^+, \quad (11)$$

where $\{\theta_n\}$ and $\{\vartheta_n\} \in (0,1)$.

Recently, Srivastava [25] introduced a new type of hybrid iterative scheme from Picard and S-iteration (Picars-S hybrid iterative scheme). The sequence $\{a_n\}$ of the scheme is defined as

$$\begin{cases} a_1 = a \in S, \\ a_{n+1} = \mathscr{V} b_n, \\ b_n = (1-\theta_n) \mathscr{V} a_n + \theta_n \mathscr{V} c_n, \\ c_n = (1-\vartheta_n) a_n + \vartheta_n \mathscr{V} a_n, \end{cases} n \in I^+, \quad (12)$$

where $\{\theta_n\}$ and $\{\vartheta_n\} \in (0,1)$.

Also Lamba and Panwar [26] introduced another hybrid scheme from Picard and S^*-iteration (Picard-S^* hybrid iterative scheme) and the sequence $\{a_n\}$ of the scheme is defined as

$$\begin{cases} a_1 = a \in S, \\ a_{n+1} = \mathscr{V} b_n, \\ b_n = (1-\theta_n) \mathscr{V} a_n + \theta_n \mathscr{V} c_n, n \in I^+, \\ c_n = (1-\vartheta_n) \mathscr{V} a_n + \vartheta_n \mathscr{V} d_n, \\ d_n = (1-\sigma_n) a_n + \sigma_n \mathscr{V} a_n, \end{cases} \quad (13)$$

where $\{\theta_n\}$, $\{\vartheta_n\}$, and $\{\sigma_n\} \in (0,1)$.

With the motivation towards the usage of hybridization of iterative schemes, we proposed another type of hybrid scheme which is the Picard-Thakur hybrid iterative scheme, defined by the sequence $\{j_n\}$ as

$$\begin{cases} j_1 = j \in S, \\ j_{n+1} = \mathscr{V} k_n, \\ k_n = (1-\theta_n) \mathscr{V} m_n + \theta_n \mathscr{V} l_n, n \in I^+, \\ l_n = (1-\vartheta_n) m_n + \vartheta_n \mathscr{V} m_n, \\ m_n = (1-\sigma_n) j_n + \sigma_n \mathscr{V} j_n, \end{cases} \quad (14)$$

where $\{\theta_n\}$, $\{\vartheta_n\}$ and $\{\sigma_n\} \in (0,1)$.

Rhoades [27] commented on the convergence of two iterative schemes that converges to a certain fixed point is as follows:

Let $\{a_n\}$ and $\{b_n\}$ be the two fixed point iterative schemes that converge to a certain fixed point j^* of a given operator $\mathscr{V}$. The sequence $\{a_n\}$ is better than $\{b_n\}$ if

$$\|a_n - j^*\| \le \|b_n - j^*\| \quad \forall n \in I^+. \quad (15)$$

2. Preliminaries

Berinde and Takens [10] gave the following definitions.

Definition 1 (see [10]). Let $\{t_n\}$ and $\{w_n\}$ be the two sequences of the real number converging to t and w, respectively. Suppose that

$$\lim_{n \longrightarrow \infty} \frac{|t_n - t|}{|w_n - w|} = k. \quad (16)$$

(i) If $k = 0$, then, $\{t_n\} \longrightarrow t$ faster than $\{w_n\} \longrightarrow w$

(ii) If $0 < k < \infty$, then, the rate of convergence of both sequences are the same

Definition 2 (see [10]). Let $\{t_n\}$ and $\{w_n\}$ be the two sequences of a fixed point iterative scheme, both converges to a fixed point ξ for a given operator $\mathscr{V}$ and $\{p_n\}$,$\{q_n\}$ are two sequences of positive numbers. Suppose that the error estimates,

$$\begin{aligned} \|t_n - \xi\| &\le p_n \quad \forall n \in I^+, \\ \|w_n - \xi\| &\le q_n \quad \forall n \in I^+, \end{aligned} \quad (17)$$

are available and $\{p_n\}$,$\{q_n\}$ converge to zero. If $\{p_n\}$ converges faster than $\{q_n\}$, then, $\{t_n\}$ converges faster than $\{w_n\} \longrightarrow \xi$. Most of the literature on the iterative schemes deals with the convergence rate and some analyzes its stability [28–34]. For proving the results, we need the following lemma.

Lemma 3 (see [35]). *Let $\{r_n\} \in \mathbb{R}^+$ be a sequence with $r_{n+1} \le (1-\tau_n) r_n$. If $\{\tau_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} = \infty$, then, $\lim_{n \longrightarrow \infty} r_n = 0$.*

Definition 4 (see [36]). Let S be a subset of Banach space B which is nonempty closed and convex. A mapping $\mathscr{V} : S \longrightarrow S$ is demiclosed w.r.t. $b \in B$, if for each sequence $\{j_n\}$ in S and $a \in S$, $\{j_n\}$ converges weakly at a and $\{\mathscr{V} j_n\}$ converges strongly at $b \Rightarrow \mathscr{V} a = b$.

Definition 5 (see [37]). A Banach space B is said to satisfy Opial's condition if for any sequence $\{j_n\} \in B$,$\{j_n\} \rightharpoonup a$, implies that

$$\liminf_{n \longrightarrow \infty} \|j_n - a\| \le \liminf_{n \longrightarrow \infty} \|j_n - b\|, \quad (18)$$

for all $b \in B$ with $a \ne b$.

Lemma 6 (see [38]). *Let B be a uniformly convex Banach space and $0 < x \le \rho_n \le y < 1 \forall n \in \mathbb{I}^+$. Let $\{j_n\}$, $\{k_n\}$ be the two sequences such that $\limsup_{n\longrightarrow\infty}\|j_n\| \le l$, $\limsup_{n\longrightarrow\infty}\|k_n\| \le l$, and $\limsup_{n\longrightarrow\infty}\|(1-\sigma_n)j_n + \sigma_n k_n\| = l$ hold for some $l \ge 0$, then $\lim_{n\longrightarrow\infty}\|j_n - k_n\| = 0$.*

Lemma 7 (see [36]). *Let $\mathscr{V} : S \longrightarrow S$ be a nonexpansive mapping with Opial's property. If $\{j_n\} \rightharpoonup a$ and $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V} j_n\| = 0$, then, $\mathscr{V} a = a$, i.e., $I - \mathscr{V}$ is demiclosed at zero, where I is the identity mapping on B.*

Proposition 8 (see [39]). *Let S be a subset of Banach space B and $\mathscr{V} : S \longrightarrow S$ a nonexpansive mapping. Then, for all $p, q \in S$*

$$\|p - \mathscr{V} q\| \le 3\|p - \mathscr{V} p\| + \|p - q\|. \tag{19}$$

Senter and Dotson [40] introduced the concept of condition (I) which is defined as

Definition 9. Let $\mathscr{V}$ be a self-mapping on S which is said to satisfy condition (I), if there is an increasing function $Z : [0,\infty) \longrightarrow [0,\infty)$ with $Z(0) = 0$ and $Z(t) > 0$, for all $t > 0$ such that

$$d(j, \mathscr{V}(j)) \ge Z(d(j, F(\mathscr{V}))), \quad \forall j \in S, \tag{20}$$

where $d(j, F(\mathscr{V})) = \inf\{d(j, j^*): j^* \in F(\mathscr{V})\}$.

In this article, we proposed a new hybrid iterative scheme which converges faster than Mann [3], Ishikawa [4], S-iteration [5], Abbas et al. [9], Thakur et al. [7], Picard-Mann hybrid [22], Picard-Krasnoselskii [23], Picard-Ishikawa [24], and Picard-S hybrid iterative schemes [25]. Recently, Srivastava [25] already proved that the Picard-S hybrid iterative scheme converges faster than all of the above iterative schemes. Therefore, we show that our hybrid iterative scheme converges faster than all the leading schemes. We find the solution of delay differential equations using our proposed hybrid iterative scheme while in last section, we prove some results of this scheme for nonexpansive mapping in the uniformly convex Banach space.

3. Convergence Analysis

This section deals with the rate of convergence of the Picard-Thakur hybrid iterative scheme (14) with Picard-S (12), Picard-Ishikawa (11), Picard-Mann (9), and Thakur et al. (6).

Proposition 10. *Assume that S be a nonempty closed convex subset of a normed space N and let $\mathscr{V} : S \longrightarrow S$ be a contraction mapping. Suppose that the iterative schemes (12), (11), (10), (9), and (6) converge to the same fixed point j^* of $\mathscr{V}$ where $\{\theta_n\}, \{\vartheta_n\}$, and $\{\sigma_n\}$ are sequences in $(0, 1)$ such that $0 < \mu \le \{\theta_n\}, \{\vartheta_n\}, \{\sigma_n\} < 1, \forall n \in I^+$, and for some μ and $\delta \in (0, 1)$. Then, the Picard-Thakur hybrid iterative scheme (14) converges faster than all the other schemes.*

Proof. Let j^* be a fixed point of an operator $\mathscr{V}$. Using the definition of contractive mapping and the Thakur et al. iterative scheme (6), we have

$$\begin{aligned}
\|k_{n+1} - j^*\| &= \|(1-\theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n - j^*\| \\
&\le (1-\theta_n)\|\mathscr{V} m_n - j^*\| + \theta_n\|\mathscr{V} l_n - j^*\| \\
&\le (1-\theta_n)\delta\|m_n - j^*\| + \theta_n\delta\|l_n - j^*\| \\
&\le (1-\theta_n)\delta(1-(1-\delta)\sigma_n)\|k_n - j^*\| \\
&\quad + \delta\theta_n(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n)\{1-\theta_n + \vartheta_n(1-(1-\delta)\sigma_n)\}]\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n)(1-(1-(1-\delta)\theta_n\sigma_n)]\|k_n - j^*\| \\
&\le \delta[(1-(1-\delta)\sigma_n - (1-\delta)\theta_n\vartheta_n) \\
&\quad + (1-\delta)^2\theta_n\vartheta_n\sigma_n]\|k_n - j^*\| \le \delta[(1-(1-\delta)\sigma_n \\
&\quad - (1-\delta)\theta_n\vartheta_n) + (1-\delta)\theta_n\vartheta_n\sigma_n]\|k_n - j^*\| \\
&\le \delta(1-(1-\delta)\sigma_n)\|k_n - j^*\|.
\end{aligned} \tag{21}$$

Let

$$a_n = \delta^n(1-(1-\delta)\sigma)^n\|k_1 - j^*\|. \tag{22}$$

Now, for (14),

$$\begin{aligned}
\|m_n - j^*\| &= \|(1-\sigma_n j_n + \mathscr{V} j_n - j^*\| \le (1-\sigma_n)\|j_n - j^*\| \\
&\quad + \sigma_n\delta\|j_n - j^*\| \le (1-(1-\delta)\sigma_n)\|j_n - j^*\|,
\end{aligned}$$

$$\begin{aligned}
\|l_n - j^*\| &= \|(1-\vartheta_n)m_n + \vartheta_n\mathscr{V} m_n - j^*\| \le (1-\vartheta_n)\|m_n - j^*\| \\
&\quad + \vartheta_n\delta\|m_n - j^*\| \le (1-(1-\delta)\vartheta_n)\|m_n - j^*\| \\
&\le (1-(1-\delta)\vartheta_n(1-(1-\delta)\sigma_n)\|j_n - j^*\|,
\end{aligned}$$

$$\begin{aligned}
\|k_n - j^*\| &= \|(1-\theta_n)\mathscr{V} m_n + \theta_n\mathscr{V} l_n - j^*\| \\
&\le \delta(1-\theta_n)\|m_n - j^*\| + \delta\theta_n\|l_n - j^*\| \\
&= \delta((1-\theta_n)(1-(1-\delta)\sigma_n)\|j_n - j^*\| \\
&\quad + \theta_n(1-(1-\delta)\vartheta_n)(1-(1-\delta)\sigma_n)\|j_n - j^*\|) \\
&= \delta(1-(1-\delta)\sigma_n)[1-\theta_n + \theta_n - (1-\delta)\theta_n\vartheta_n] \\
&\quad \cdot \|j_n - j^*\| = \delta(1-(1-\delta)\sigma_n - (1-(1-\delta)\sigma_n) \\
&\quad \cdot ((1-\delta)\theta_n\vartheta_n))\|j_n - j^*\| = \delta(1-(1-\delta)\sigma_n \\
&\quad - (1-\delta)\theta_n\vartheta_n + (1-\delta)^2\theta_n\vartheta_n\sigma_n)\|j_n - j^*\| \\
&\le \delta(1-(1-\delta)\sigma_n - (1-\delta)\theta_n\vartheta_n + (1-\delta)\theta_n\vartheta_n\sigma_n) \\
&\quad \cdot \|j_n - j^*\| \le \delta(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n \\
&\quad - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\|.
\end{aligned} \tag{23}$$

Also,

$$\begin{aligned}
\|j_{n+1} - j^*\| &= \|\mathscr{V} k_n - j^*\| \le \delta\|k_n - j^*\| \\
&\le \delta(\delta(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\| \\
&\le \delta^2(1-(1-\delta)(\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n))\|j_n - j^*\|.
\end{aligned} \tag{24}$$

Let

$$b_n = \delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n \|j_1 - j^*\|. \tag{25}$$

Then,

$$\frac{b_n}{a_n} = \frac{\delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n \|j_1 - j^*\|}{\delta^n(1-(1-\delta)\sigma)^n \|k_1 - j^*\|} = \frac{\delta^{n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n \|j_1 - j^*\|}{(1-(1-\delta)\sigma)^n \|k_1 - j^*\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{26}$$

Thus, $\{j_n\}$ converges faster than $\{k_n\}$, i.e., the Picard-Thakur iterative scheme converges faster than the Thakur iterative scheme. Similarly, the inequality proved in Proposition 3.1 of the Picard-S hybrid iterative scheme [25] is as follows:

$$c_n = \delta^{2n}(1-(1-\delta)\theta\vartheta)^n \|a_1 - j^*\|. \tag{27}$$

Then,

$$\frac{b_n}{a_n} = \frac{\delta^{2n}(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n \|j_1 - j^*\|}{\delta^{2n}(1-(1-\delta)\theta\vartheta)^n \|a_1 - j^*\|} = \frac{(1-(1-\delta)(\sigma+\theta\vartheta-\theta\vartheta\sigma))^n \|j_1 - j^*\|}{(1-(1-\delta)\theta\vartheta)^n \|a_1 - j^*\|} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{28}$$

Thus, $\{j_n\}$ converges faster than $\{a_n\}$., i.e., the Picard-Thakur iterative scheme converges faster than the Picard-S iterative scheme. Similarly, we can show that Picard-Thakur hybrid iterative scheme (14) converges faster than (11), (10), and (9). □

Next, we gave an example to show that the Picard-Thakur hybrid iterative scheme (14) converges faster than the Picard-S hybrid, Picard-Ishikawa hybrid, Picard-Mann hybrid, and Thakur iterative schemes.

Example 11. Let $\mathscr{V} : S \longrightarrow S$ where $S = [0, 2] \subset N = \mathbb{R}$ be an operator defined by

$$\mathscr{V}(j) = \begin{cases} 1, & \text{if } j \in [0, 1], \\ \sqrt{\dfrac{4-j^2}{3}}, & \text{if } j \in [1, 2]. \end{cases} \tag{29}$$

Choose $\theta_n = (n+2)/(n+6)$, $\vartheta_n = (n^2+1)/(n^2+n+1)$, $\sigma_n = \sqrt{(n+1)/(2n+7)}$, for each $n \in I^+$ with an initial value $j_1 = 0.6.\mathscr{V}$ is nonexpansive mapping. All the iterative schemes converge to the fixed point $j^* = 1$. Clearly, in the Table 1 and Figure 1, we can see that the Picard-Thakur hybrid iterative scheme (14) converges faster than the schemes discussed above.

4. Application: Delay Differential Equations

In this section, we can find the solution of the delay differential equation by using our proposed iterative scheme.

Let the space of all continuous real-valued functions be denoted by $C([u, v])$ on closed interval $[u, v]$ endowed with the Chebyshev norm $\|j - m\|_\infty$ and defined as $\|j - m\|_\infty = \sup_{r\in[u,v]} |j(r) - m(r)|$, and it is clear that in [41] that $(C([u, v], \|.\|_\infty))$ is a Banach space. Now, consider the following delay differential equation

$$j'(r) = \psi(r, j(r), j(r-\gamma)), \quad r \in [r_0, v], \tag{30}$$

with initial condition

$$j(r) = \zeta(r), \quad r \in [r_0 - \gamma, r_0]. \tag{31}$$

By the solution of the above delay differential equation, we mean a function $j \in C([r_0 - \gamma, v], \mathbb{R}) \cap C^1([r_0, v], \mathbb{R}))$ satisfying (30) and (31).

Assume that the following conditions are satisfied.

(1) $r_0, v \in \mathbb{R}), \gamma > 0$

(2) $\psi \in C([r_0, v] \times \mathbb{R}^2, \mathbb{R})$

(3) $\zeta \in C([r_0 - \gamma, v], \mathbb{R})$

(4) There exists $L_\psi > 0$ such that

$$|\psi(r, s_1, s_2) - \psi(r, t_1, t_2)| \le L_\psi \Sigma_{i=1}^2 |s_i - t_i|, \quad \forall s_i, t_i \in \mathbb{R}, r \in [r_0, v] \tag{32}$$

(5) $2L_\psi(v - r_0) < 1$

Now, we construct (30) and (31) by the integral equation as

$$j(r) = \begin{cases} \zeta(r), & r \in [r_0 - \gamma, v], \\ \zeta(r_0) + \displaystyle\int_{r_0}^{r} \psi(t, j(t), j(t-\gamma))dt, & r \in [r_0, v]. \end{cases} \tag{33}$$

The following result is the generalization of the result of Coman et al. [42].

Theorem 12. *Let the conditions* $*_1$) *to* $*_5$) *be satisfied. Then, (30) and (31) have unique solution* $j^* \in C([r_0 - \gamma], \mathbb{R}) \cap C^1([r_0 v], \mathbb{R})$ *and*

$$j^* = \lim_{n\longrightarrow\infty} \mathscr{V}^n(j), \quad \textit{for any } j \in C([r_0 - \gamma, v], \mathbb{R}). \tag{34}$$

Now, by using the Picard-Thakur hybrid iterative scheme (14), we prove the following result.

Theorem 13. *Let the conditions $*_1) - *_5)$ be satisfied. Then, (30) and (31) have a unique solution $j^* \in C([r_0 - \gamma], \mathbb{R}) \cap C^1([r_0, v], \mathbb{R})$ and the Picard-Thakurb hybrid iterative scheme (14) converges to j^*.*

Proof. Let $\{j_n\}$ be a sequence generated by the Picard-Thakur hybrid iterative scheme (14) for an operator $\mathscr{V}$ defined by

$$\mathscr{V} j(r) = \begin{cases} \zeta(r), & r \in [r_0 - \gamma, v], \\ \zeta(r_0) + \int_{r_0}^{r} \psi(p, j(p), j(p-\gamma))dp, & r \in [r_0, v]. \end{cases} \tag{35}$$

Let j^* be a fixed point of $\mathscr{V}$. Now, we prove that $j_n \longrightarrow j^*$ as $n \longrightarrow \infty$. It is easy to see that $j_n \longrightarrow j^*$ as $n \longrightarrow \infty$ for each $r \in [r_0 - \gamma, r_0]$.

Now, for each $r \in [r_0, v]$, we have

$$\begin{aligned} \|j_{n+1} - j^*\|_\infty &\le \|\mathscr{V} k_n - j^*\|_\infty \le \sup_{r_0 \in [r_0, v]} |\mathscr{V} k_n - \mathscr{V} j^*| \le \sup_{r_0 \in [r_0, v]} \Big|\zeta(r_0) \\ &\quad + \int_{r_0}^{r} \psi(p, k_n(p), k_n(p-\gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\right)\Big| \\ &\le \sup_{r_0 \in [r_0, v]} \int_{r_0}^{r} |\psi(p, k_n(p), k_n(p-\gamma)) - \psi(p, j^*(p), j^*(p-\gamma))|dp \\ &\le \sup_{r_0 \in [r_0, v]} \int_{r_0}^{r} L_\psi(|k_n(p) - j^*(p)| + |k_n(p-\gamma) - j^*(p-\gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi \sup_{r_0 \in [r_0, v]} (|k_n(p) - j^*(p)| + |k_n(p-\gamma) - j^*(p-\gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\|k_n - j^*\|_\infty + \|k_n - j^*\|_\infty\right)dp \le 2L_\psi(v - r_0)\|k_n - j^*\|_\infty. \end{aligned} \tag{36}$$

Now,

$$\begin{aligned} \|k_n - j^*\|_\infty &= \|(1-\theta_n)\mathscr{V} m_n + \theta_n \mathscr{V} l_n - j^*\|_\infty \\ &\le (1-\theta_n)\|\mathscr{V} m_n - j^*\|_\infty + \theta_n\|\mathscr{V} l_n - j^*\|_\infty, \end{aligned} \tag{37}$$

As

$$\begin{aligned} \|\mathscr{V} l_n - j^*\|_\infty &= \|\mathscr{V} l_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, v]} \\ &\quad \cdot \Big|\zeta(r_0) + \int_{r_0^r} \psi(p, l_n(p), l_n(p-\gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\right)\Big| \le \sup_{r \in [r_0 - \gamma, v]} \\ &\quad \cdot \Big|\int_{r_0}^{r} \psi(p, l_n(p), l_n(p-\gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\Big| \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} |\psi(p, l_n(p), l_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} |\psi(p, l_n(p), l_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} L_\psi(|l_n(p) - j^*(p)| + |l_n(p-\gamma) - j^*(p-\gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\sup_{r \in [r_0 - \gamma, v]} |l_n(p) - j^*(p)| + \sup_{r \in [r_0 - \gamma, v]} |l_n(p-\gamma) - j^*(p-\gamma)|\right)dp \\ &\le \int_{r_0}^{r} L_\psi(\|l_n - j^*\|_\infty + \|l_n - j^*\|_\infty)dp \\ &\le 2L_\psi(r - r_0)\|l_n - j^*\|_\infty \le 2L_\psi(v - r_0)\|l_n - j^*\|_\infty, \end{aligned} \tag{38}$$

$$\begin{aligned} \|l_n - j^*\|_\infty &= \|(1-\vartheta_n)m_n + \vartheta_n \mathscr{V} m_n - j^*\|_\infty \\ &\le (1-\vartheta_n)\|m_n - j^*\|_\infty + \vartheta_n\|\mathscr{V} m_n - j^*\|_\infty. \end{aligned} \tag{39}$$

For

$$\begin{aligned} \|\mathscr{V} m_n - j^*\|_\infty &= \|\mathscr{V} m_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, v]} \\ &\quad \cdot \Big|\zeta(r_0) + \int_{r_0^r} \psi(p, m_n(p), m_n(p-\gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\right)\Big| \\ &\le \sup_{r \in [r_0 - \gamma, v]} \Big|\int_{r_0}^{r} \psi(p, m_n(p), m_n(p-\gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\Big| \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} |\psi(p, m_n(p), m_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} |\psi(p, m_n(p), m_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))|dp \\ &\le \sup_{r \in [r_0 - \gamma, v]} \int_{r_0}^{r} L_\psi(|m_n(p) - j^*(p)| + |m_n(p-\gamma) - j^*(p-\gamma)|)dp \\ &\le \int_{r_0}^{r} L_\psi\left(\sup_{r \in [r_0 - \gamma, v]} |m_n(p) - j^*(p)| + \sup_{r \in [r_0 - \gamma, v]} |m_n(p-\gamma) - j^*(p-\gamma)|\right)dp \\ &\le \int_{r_0}^{r} L_\psi(\|m_n - j^*\|m_n - j^*\|\|_\infty + |m_n - j^*\|m_n - j^*\|\|_\infty)\, dp \\ &\le 2L_\psi(r - r_0)\|m_n - j^*\|_\infty \le 2L_\psi(v - r_0)\|m_n - j^*\|_\infty, \end{aligned} \tag{40}$$

$$\begin{aligned} \|m_n - j^*\|_\infty &= \|(1-\sigma_n)j_n + \sigma_n \mathscr{V} j_n - j^*\|_\infty \\ &\le (1-\sigma_n)\|j_n - j^*\|_\infty + \sigma_n\|\mathscr{V} j_n - j^*\|_\infty, \end{aligned} \tag{41}$$

as

$$\begin{aligned} \|\mathscr{V} j_n - j^*\| &= \|\mathscr{V} j_n - \mathscr{V} j^*\|_\infty \le \sup_{r \in [r_0 - \gamma, v]} \\ &\quad \cdot \Big|\zeta(r_0) + \int_{r_0^r} \psi(p, j_n(p), j_n(p-\gamma))dp \\ &\quad - \left(\zeta(r_0) + \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\right)\Big| \le \sup_{r \in [r_0 - \gamma, v]} \\ &\quad \cdot \Big|\int_{r_0}^{r} \psi(p, j_n(p), j_n(p-\gamma))dp \\ &\quad - \int_{r_0}^{r} \psi(p, j^*(p), j^*(p-\gamma))dp\Big| \end{aligned}$$

Table 1: Convergence behavior of Thakur et al. (7), Ap (8), Picard-S (12), Picard-S^* (13), and Picard-Thakur hybrid Iterative schemes (14).

Steps	Picard-Ishikawa hybrid	Thakur et al.	Ap iterative scheme	Picard-S hybrid	Picard-S^* hybrid	Picard-Thakur hybrid
1	0.6000000000	0.6000000000	0.6000000000	0.6000000000	0.6000000000	0.6000000000
2	1.0172938494	1.0023262974	1.0033992688	1.0028485141	0.9942090597	0.9992233640
3	0.9991010616	0.9999896158	0.9999617294	0.9999670218	0.9998760856	0.9999988412
4	1.0000463544	1.0000000464	1.0000004299	1.0000003808	0.9999973348	0.9999999983
5	0.9999976087	0.9999999997	0.9999999952	0.9999999956	0.9999999427	0.9999999999
6	1.0000001234	1.0000000000	1.0000000001	1.0000000001	0.9999999988	0.9999999999
7	0.9999999936	0.9999999999	0.9999999999	0.9999999999	0.9999999999	1.0000000000
8	1.0000000003	1.0000000000	1.0000000000	1.0000000000	0.9999999999	1.0000000000
9	0.9999999999	1.0000000000	1.0000000000	1.0000000000	0.9999999999	1.0000000000
10	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000

$$\begin{aligned}
&\leq \sup_{r\in[r_0-\gamma,\nu]} \int_{r_0}^{r} |\psi(p, j_n(p), j_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))| dp \\
&\leq \sup_{r\in[r_0-\gamma,\nu]} \int_{r_0}^{r} |\psi(p, j_n(p), j_n(p-\gamma) - \psi(p, j^*(p), j^*(p-\gamma))| dp \\
&\leq \sup_{r\in[r_0-\gamma,\nu]} \int_{r_0}^{r} L_\psi(|j_n(p) - j^*(p)| + |j_n(p-\gamma) - j^*(p-\gamma)|) dp \\
&\leq \int_{r_0}^{r} L_\psi \left(\sup_{r\in[r_0-\gamma,\nu]} |j_n(p) - j^*(p)| + \sup_{r\in[r_0-\gamma,\nu]} |j_n(p-\gamma) - j^*(p-\gamma)| \right) dp \\
&\leq \int_{r_0}^{r} L_\psi(\|j_n - j^*\|_\infty + \|j_n - j^*\|_\infty) dp \leq 2L_\psi(r - r_0)\|j_n - j^*\|_\infty \\
&\leq 2L_\psi(\nu - r_0)\|j_n - j^*\|_\infty.
\end{aligned} \tag{42}$$

Putting (42) in (41), we get

$$\begin{aligned}
\|m_n - j^*\|_\infty &\leq (1-\sigma_n)\|j_n - j^*\|_\infty + \sigma_n 2L_\psi(\nu - r_0)\|j_n - j^*\|_\infty \\
&\leq [1 - (1 - 2L_\psi(\nu - r_0)\sigma_n] \|j_n - j^*\|_\infty.
\end{aligned} \tag{43}$$

Putting (43) in (40), we get

$$\|\mathscr{V} m_n - j^*\|_\infty \leq 2L_\psi(\nu - r_0)[1 - ((11 - 2L_\psi(\nu - r_0)\sigma_n] \|j_n - j^*\|_\infty. \tag{44}$$

Putting (44) and (43) in (39), we get

$$\begin{aligned}
\|l_n - j^*\|_\infty &\leq (1-\vartheta_n)[1 - ((11 - 2L_\psi(\nu - r_0)\sigma_n] \|j_n - j^*\|_\infty \\
&\quad + \vartheta_n 2L_\psi(\nu - r_0)[1 - (1 - 2L_\psi(\nu - r_0)\sigma_n] \|j_n - j^*\|_\infty \\
&\leq (1 - (1 - 2L_\psi(\nu - r_0)\sigma_n)(1 - (1 - (1 - 2L_\psi(\nu - r_0)\vartheta_n) \|j_n - j^*\|_\infty \\
&\leq [1 - (1 - 2L_\psi(\nu - r_0)\sigma_n) - (1 - (1 - 2L_\psi(\nu - r_0)\sigma_n)(1 - 2L_\psi(\nu - r_0)\vartheta_n)] \\
&\quad \cdot \|j_n - j^*\|_\infty \leq [1 - (1 - 2L_\psi(\nu - r_0))\sigma_n - (1 - 2L_\psi(\nu - r_0))\vartheta_n \\
&\quad + [(1 - 2L_\psi(\nu - r_0)]^2 \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty \leq [1 - (1 - 2L_\psi(\nu - r_0))\sigma_n \\
&\quad - (1 - 2L_\psi(\nu - r_0))\vartheta_n + (1 - 2L_\psi(\nu - r_0))\vartheta_n\sigma_n] \|j_n - j^*\|_\infty \\
&\leq [1 - (1 - 2L_\psi(\nu - r_0))(\sigma_n - \vartheta_n + \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty.
\end{aligned} \tag{45}$$

Putting (45) in (38), we get

$$\|\mathscr{V} l_n - j^*\|_\infty \leq 2L_\psi(\nu - r_0)[1 - (1 - 2L_\psi(\nu - r_0))(\sigma_n - \vartheta_n + \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty. \tag{46}$$

Putting (46) and (40) in (37), we get

$$\begin{aligned}
\|k_n - j^*\|_\infty &\leq (1-\theta_n)2L_\psi(\nu - r_0)\|m_n - j^*\|_\infty + \theta_n 2L_\psi(\nu - r_0) \\
&\quad \cdot \|l_n - j^*\|_\infty \leq 2L_\psi(\nu - r_0)[(1-\theta_n)\|m_n - j^*\|_\infty + \theta_n\|l_n - j^*\|_\infty] \\
&\leq 2L_\psi(\nu - r_0)[(1-\theta_n)[1 - (1 - 2L_\psi(\nu - r_0)\sigma_n] \|j_n - j^*\|_\infty \\
&\quad + \theta_n[1 - (1 - 2L_\psi(\nu - r_0))(\sigma_n - \vartheta_n + \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty] \\
&\leq 2L_\psi(\nu - r_0)[1 - \theta_n)(1 - (1 - 2L_\psi(\nu - r_0)\sigma_n) \\
&\quad + \theta_n[1 - (1 - 2L_\psi(\nu - r_0))(\sigma_n - \vartheta_n + \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty \\
&\leq 2L_\psi(\nu - r_0)[1 - \theta_n - (1 - 2L_\psi(\nu - r_0)\sigma_n + (1 - 2L_\psi(\nu - r_0)\theta_n\sigma_n \\
&\quad + \theta_n - (1 - 2L_\psi(\nu - r_0)\theta_n(\sigma_n - \vartheta_n + \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty \\
&\leq 2L_\psi(\nu - r_0)[1 - (1 - 2L_\psi(\nu - r_0)(\sigma_n - \theta_n\sigma_n \\
&\quad + \theta_n(\sigma_n + \vartheta_n - \vartheta_n\sigma_n)] \|j_n - j^*\|_\infty \leq 2L_\psi(\nu - r_0) \\
&\quad \cdot [1 - (1 - 2L_\psi(\nu - r_0)(\sigma_n - \theta_n\sigma_n + \theta_n\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n)] \|j_n \\
&\quad - j^*\|_\infty \leq 2L_\psi(\nu - r_0)[1 - (1 - 2L_\psi(\nu - r_0)(\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n)] \\
&\quad \cdot \|j_n - j^*\|_\infty.
\end{aligned} \tag{47}$$

Let $\sigma_n + \theta_n\vartheta_n - \theta_n\vartheta_n\sigma_n = \rho_n$, and by using condition $*_5$), we have

$$\|k_n - j^*\|_\infty \leq [1 - (1 - 2L_\psi(\nu - r_0)\rho_n] \|j_n - j^*\|_\infty. \tag{48}$$

Putting (48) in (36), we have

$$\|j_{n+1} - j^*\|_\infty \leq 2L_\psi(\nu - r_0)[1 - (1 - 2L_\psi(\nu - r_0)\rho_n] \|j_n - j^*\|_\infty. \tag{49}$$

Again, using condition $*_5$), we get

$$\|j_{n+1} - j^*\|_\infty \leq [1 - (1 - 2L_\psi(\nu - r_0)\rho_n] \|j_n - j^*\|_\infty. \tag{50}$$

Let $(1 - 2L_\psi(\nu - r_0)\rho_n = \tau_n < 1$ and $\|j_n - j^*\|_\infty = r_n$. So, the conditions of Lemma 3 are satisfied. Hence, $\lim_{n\longrightarrow\infty}\| j_n - j^*\| = 0$. □

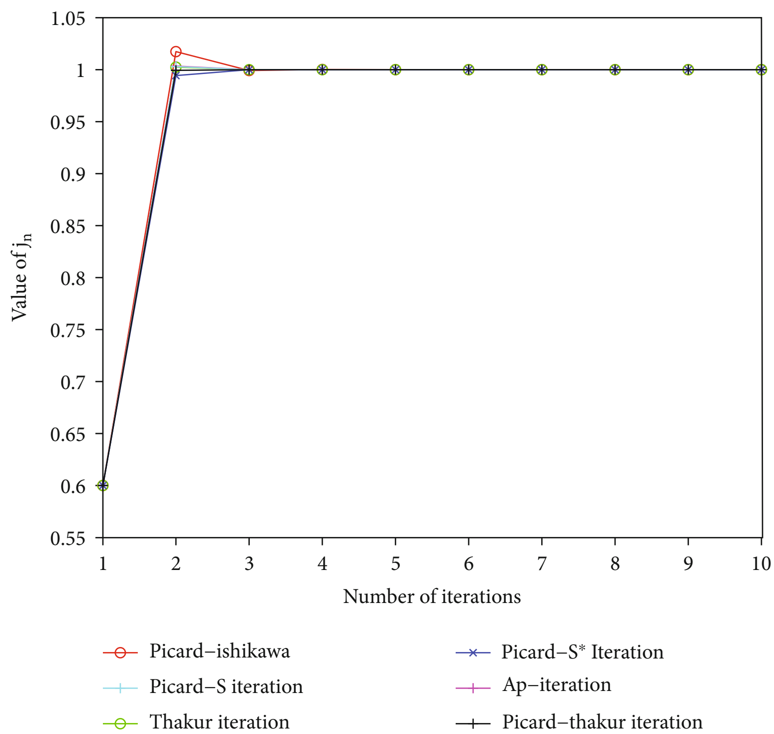

FIGURE 1: Convergence behavior of Thakur et al. (7), Ap (8), Picard-S (12), Picard-S^* (13), and Picard-Thakur hybrid iterative schemes (14).

5. Convergence Results for Nonexpansive Mapping

Lemma 14. *Let S be a nonempty closed and convex subset of uniformly convex Banach space B and $\mathscr{V} : S \longrightarrow S$ be a nonexpansive mapping. If $\{j_n\}$ be a sequence generated by Picard-Thakur hybrid iterative scheme (14) and $F(\mathscr{V}) \neq \varnothing$, then, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists.*

Proof. Let $j^* \in F(\mathscr{V})$, and $\mathscr{V}$ is nonexpansive then

$$\begin{aligned}\|m_n - j^*\| &= \|(1-\sigma_n)j_n + \sigma_n\mathscr{V}j_n - j^*\| \le (1-\sigma_n)\|j_n \\ &\quad - j^*\| + \sigma_n\|\mathscr{V}j_n - j^*\| \le (1-\sigma_n)\|j_n - j^*\| + \sigma_n\|j_n \\ &\quad - j^*\| \le \|j_n - j^*.\end{aligned} \tag{51}$$

Also,

$$\begin{aligned}\|l_n - j^*\| &= \|(1-\vartheta_n)m_n + \vartheta_n\mathscr{V}m_n - j^*\| \le (1-\vartheta_n)\|m_n \\ &\quad - j^*\| + \vartheta_n\|\mathscr{V}m_n - j^*\| \le (1-\vartheta_n)\|m_n - j^*\| + \vartheta_n\|m_n \\ &\quad - j^*\| \le \|m_n - j^*.\end{aligned} \tag{52}$$

Similarly,

$$\begin{aligned}\|k_n - j^*\| &= \|(1-\theta_n)\mathscr{V}m_n + \theta_n\mathscr{V}l_n - j^*\| \le (1-\theta_n)\|\mathscr{V}m_n \\ &\quad - j^*\| + \theta_n\|\mathscr{V}l_n - j^*\| \le (1-\theta_n)\|m_n - j^*\| + \theta_n\|l_n - j^*\| \\ &\quad \le (1-\theta_n)\|m_n - j^*\| + \theta_n\|m_n - j^*\| \le \|m_n - j^*\| \le \|j_n - j^*\|.\end{aligned} \tag{53}$$

Now,

$$\|j_{n+1} - j^*\| = \|\mathscr{V}k_n - j^*\| \le \|k_n - j^*\| \le \|j_n - j^*\|. \tag{54}$$

This shows that $\{\|j_n - j^*\|\}$ is a decreasing sequence and bounded below $\forall j^* \in F(\mathscr{V})$. Hence, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. □

Lemma 15. *Let S and $\mathscr{V} : S \longrightarrow S$ be as in Lemma 14. Let $\{j_n\}$ be a sequence defined by Picard-Thakur hybrid iterative scheme (14) with $F(\mathscr{V}) \neq \varnothing$. Then, $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0$.*

Proof. As from the above Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists for each $j^* \in F(\mathscr{V})$. Suppose that for some $l \ge 0$, we have

$$\lim_{n\longrightarrow\infty}\|j_n - j^*\| = l. \tag{55}$$

As from (53), (52), and (51), we have

$$\|m_n - j^*\| \le \|j_n - j^*\|, \tag{56}$$

$$\|l_n - j^*\| \le \|j_n - j^*\|, \tag{57}$$

$$\|k_n - j^*\| \le \|j_n - j^*\|. \tag{58}$$

Taking lim sup as $n \longrightarrow \infty$ of (58), (57), and (56), we get

$$\limsup_{n\longrightarrow\infty}\|m_n - j^*\| \le l, \tag{59}$$

$$\limsup_{n\longrightarrow\infty}\|l_n - j^*\| \le l, \tag{60}$$

$$\limsup_{n\longrightarrow\infty}\|k_n - j^*\| \le l. \tag{61}$$

Since $\mathscr{V}$ is nonexpansive, we have

$$\limsup_{n\longrightarrow\infty}\|\mathscr{V}j_n - j^*\| \le l, \tag{62}$$

$$l = \liminf_{n\longrightarrow\infty}\|j_{n+1} - j^*\| = \liminf_{n\longrightarrow\infty}\|\mathscr{V}k_n - j^*\| \le \liminf_{n\longrightarrow\infty}\|k_n - j^*\|, \tag{63}$$

From (63) and (61), we get

$$\lim_{n\longrightarrow\infty}\|k_n - j^*\| = l. \tag{64}$$

Now, from (53), we have

$$\|k_n - j^*\| \le \|m_n - j^*\|. \tag{65}$$

Taking liminf as $n \longrightarrow \infty$, we have

$$\liminf_{n\longrightarrow\infty}\|k_n - j^*\| \le \liminf_{n\longrightarrow\infty}\|m_n - j^*\|, \tag{66}$$

$$l \le \liminf_{n\longrightarrow\infty}\|m_n - j^*\|. \tag{67}$$

So, from (67) and (59), we have

$$\begin{aligned} l = \lim_{n\longrightarrow\infty}\|m_n - j^*\| = \lim_{n\longrightarrow\infty}\|(1-\sigma_n)j_n + \sigma_n\mathscr{V}j_n \\ - j^*\| = \lim_{n\longrightarrow\infty}\|(1-\sigma_n)(j_n - j^*) + \sigma_n(\mathscr{V}j_n - \mathscr{V}j^*)\|. \end{aligned} \tag{68}$$

From (68), (62), and (55) and applying Lemma 6, we get

$$\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0. \tag{69}$$

□

Theorem 16. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Let B be the uniformly convex Banach space which satisfies Opial's condition; then, $\{j_n\}$ converges weakly to a fixed point of $\mathscr{V}$.*

Proof. Let $j^* \in F(\mathscr{V})$; then, by Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. Now, we show that $\{j_n\}$ has a unique weak subsequential limit in $F(\mathscr{V})$.

Let $\{a_n\}$ and $\{b_n\}$ be two subsequences of $\{j_n\}$ and a, b be the weak limits of the subsequences of $\{j_n\}$, respectively. From Lemma 15, $\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}(j_n)\| = 0$ and $I - \mathscr{V}$ is demiclosed at zero. By Lemma 7.

Therefore, we get $\mathscr{V}a = a$. For $b \in F(\mathscr{V})$, we follow the same manner.

From Lemma 14, we know that $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists.

For uniqueness, supposing that $a \neq b$, then, by using Opial's condition,

$$\begin{aligned} \lim_{n\longrightarrow\infty}\|j_n - a\| = \lim_{n\longrightarrow\infty}\|a_n - a\| < \lim_{n\longrightarrow\infty}\|a_n - b\| = \lim_{n\longrightarrow\infty}\|j_n \\ - b\| = \lim_{n\longrightarrow\infty}\|b_n - b\| < \lim_{n\longrightarrow\infty}\|b_n - a\| = \lim_{n\longrightarrow\infty}\|j_n - a\|. \end{aligned} \tag{70}$$

This is a contradiction, so $a = b$. Hence, $\{j_n\}$ converges weakly to $F(\mathscr{V})$. □

Theorem 17. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Then, $\{j_n\}$ converges to a point of $F(\mathscr{V})$ if and only if $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ or $\limsup_{n\longrightarrow\infty}(j_n, F(\mathscr{V})) = 0$, where $d(a_n, F(\mathscr{V})) = \inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\}$.*

Proof. If the sequence $\{j_n\} \longrightarrow j^* \in F(\mathscr{V})$, then, it is oblivious that $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ or $\limsup_{n\longrightarrow\infty}(j_n, F(\mathscr{V})) = 0$.

Conversely, assume that $\liminf_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$. From Lemma 14,

$\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists, $\forall j^* \in F(\mathscr{V})$. Therefore, by assumption,

$$\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0. \tag{71}$$

Now, to show, the sequence $\{j_n\}$ is cauchy in S. As $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$, for given $\lambda > 0$, there exists $m_0 \in I^+$ such that $\forall n \ge m_0$,

$$d(j_n, F(\mathscr{V})) < \frac{\lambda}{2} \Rightarrow \inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\} < \frac{\lambda}{2}. \tag{72}$$

Particularly, $\inf\{\|j_n - j^*\| : j^* \in F(\mathscr{V})\} < \lambda/2$. Therefore, there is $j^* \in F(\mathscr{V})$ such that

$$\|j_{m_0} - j^*\| < \frac{\lambda}{2}. \tag{73}$$

Now, for $m, n \ge m_0$,

$$\begin{aligned} \|j_{n+m} - j_n\| \le \|j_{m+n} - j^*\| + \|j_n - j^*\| \le \|j_{m_0} - j^*\| + \|j_{m_0} - j^*\| \\ = 2\|j_{m_0} - j^*\| < \lambda. \end{aligned} \tag{74}$$

This shows that the sequence $\{j_n\}$ is cauchy in S. As $S \subset B$, so, p is a point in S such that $\lim_{n\longrightarrow\infty}j_n = p$. Now, $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0$ gives that $\lim_{n\longrightarrow\infty}d(j_n, F(\mathscr{V})) = 0 \Rightarrow p \in F(\mathscr{V})$. □

Theorem 18. *Let S, $\mathscr{V}$, $\{j_n\}$ be as in Lemma 14. Then, $\{j_n\}$ converges strongly to $F(\mathscr{V}) \neq \varnothing$.*

Proof. By Lemma 15, we have

$$\lim_{n\longrightarrow\infty}\|j_n - \mathscr{V}j_n\| = 0. \tag{75}$$

Since, S is compact, then, let $\{j_{n_k}\}$ be a subsequence of $\{j_n\}$ which converges strongly to j^*, for some $j^* \in S$. By Proposition 8, we have

$$\|j_{n_k} - \mathscr{V} j^*\| \le 3\|j_{n_k} - \mathscr{V} j_{n_k}\| + \|j_{n_k} - j^*\| \quad \forall k \ge 1. \tag{76}$$

Letting $k \longrightarrow \infty$, we get

$$j_{n_k} \longrightarrow \mathscr{V} j^* \Rightarrow \mathscr{V} j^* = j^*, \quad \text{i.e.,} j^* \in F(\mathscr{V}). \tag{77}$$

Also, by Lemma 14, $\lim_{n\longrightarrow\infty}\|j_n - j^*\|$ exists. Thus, $\{j_n\}$ converges strongly to j^*. □

Now, by using condition (I), we prove the strong convergence result.

Theorem 19. *Let S, $\mathscr{V}$ be as in Lemma 14. Let B be a uniformly convex Banach space which is satisfying condition (I). Then, the sequence $\{j_n\}$ defined by the Picard-Thakur hybrid iterative scheme (14) converges strongly to $F(\mathscr{V}) \neq \varnothing$.*

Proof. As by Lemma 15, we have

$$\lim_{n\longrightarrow\infty} \|j_n - \mathscr{V} j_n\| = 0. \tag{78}$$

By condition (I) and (78), we get

$$\begin{aligned} 0 &\le \lim_{n\longrightarrow\infty} Z(d(j_n, F(\mathscr{V}))) \\ &\le \lim_{n\longrightarrow\infty} \|j_n - \mathscr{V} j_n\| \Rightarrow \lim_{n\longrightarrow\infty} Z(d(j_n, F(\mathscr{V}))) = 0. \end{aligned} \tag{79}$$

Since $Z : [0,\infty) \longrightarrow [0,\infty)$ is an increasing function satisfying $Z(0) = 0, Z(t) > 0 \forall t > 0$.

Hence, we have

$$\lim_{n\longrightarrow\infty} d(j_n, F(\mathscr{V})) = 0. \tag{80}$$

Since all the conditions of Theorem 17 are satisfied, therefore, we can say that $\{j_n\}$ converges strongly to $F(\mathscr{V})$. □

6. Conclusion

In this paper, we present a new hybrid scheme of Picard and Thakur et al. We discuss the convergence of this scheme to the iterative scheme of Mann, Ishikawa, Picard-Mann, Picard-Ishikawa, Picard-S, and Thakur et al. We showed the convergence of Picard-Thakur hybrid iterative with other iterative schemes on graphs and gave application to delay differential equations. We also generalize and extend various results for nonexpansive mapping in a uniformly convex Banach space including [7, 24, 25, 43].

Data Availability

All data required for this research is included within this paper.

Conflicts of Interest

The authors declare that they do not have any competing interests.

Authors' Contributions

Jie Jia analyzed the results and used a software to compare the results, Khurram Shabbir proposed the problem and supervised this work, Khushdil Ahmad wrote the first version of this paper, Nehad Ali Shah verified the results and wrote the final version of this paper, and Thongchai Botmart prepared the example sketch and the plots and arranged the funding for this paper. Jie Jia and Nehad Ali Shah are the first co-authors and contributed equally in this work.

Acknowledgments

This research work is supported by the Higher Education Commission (Islamabad) through the National Research Program for Universities, Grant no. 7359/Punjab/NRPU/R $D/HEC/2017.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1803348, 6 pages
https://doi.org/10.1155/2022/1803348

Research Article

Some Fixed Points Results in b-Metric and Quasi b-Metric Spaces

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Received 19 November 2021; Revised 30 December 2021; Accepted 5 January 2022; Published 22 January 2022

Academic Editor: Santosh Kumar

We present a fixed point result in quasi b-metric spaces. Our result generalizes recent fixed point results obtained by Aleksić et al., Dung and Hang, Jovanović et al., Sarwar, and Rahman and classical results obtained by Hardy, Rogers, and Ćirić. Also, we obtain a common fixed point result in b-metric spaces. As a special case, we get a result of Ćirić and Wong.

1. Introduction

The notion of a generalized contraction was presented by Ćirić in his dissertation [1]. In [1], Ćirić proved the first fixed point result for this class of mappings, which was published in [2]. Ćirić also published several papers on generalized contractions, such as for multivalued mappings in [3], on common fixed point of not necessarily commuting mappings in [4], for probabilistic metric spaces in [5, 6] and fixed point result of Meir-Keeler type in [7]. For further historical remarks of the papers of Ćirić, see [8].

In 1973, Hardy and Rogers [9] proved a result of fixed point on metric space, which was extended to common fixed point result by Wong [10].

The results of common fixed points of Wong [10] and Ćirić [4] are independent. More concepts of common fixed points can be seen in [11, 12].

Also, Fréchet in the paper [13] introduced a class of metric spaces which are included in the class b-metric spaces. First, fixed point result in a b-metric space was presented by Bakhtin [14] and Czerwik [15] (for more on b-metric spaces see [16–23]). In the last few decades, many generalizations of a metric space appeared in literature. For some historical aspects of various generalizations of a metric space, the reader may refer to [24].

In this paper, we present a fixed point theorem for a mapping defined on a quasi b-metric space which generalizes recent fixed point results obtained by Aleksić et al. [16], Dung and Hang [18], Jovanović et al. [25], and Sarwar and Rahman [22]. Further, we obtain a result of common fixed point on a b-metric space. Our result generalizes the classical results presented by Ćirić [4] and Wong [10].

2. The Quasi b-Metric Spaces

We start with definition of quasi b-metric spaces, which was introduced by Shah and Hussain [23].

Definition 1. Let X be a nonempty set, $d : X \times X \longrightarrow [0,+\infty)$ and $s \in [0,+\infty)$. Then, (X, d, s) is a quasi b-metric space if

(1) $d(\mu, \nu) = 0$ if and only if $\mu = \nu$

(2) $d(\mu, \xi) \leq s[d(\mu, \nu) + d(\nu, \xi)]$, for all $\mu, \nu, \xi \in X$

Clearly, $(X, d, 1)$ is a quasi metric space.

Remark 2. Let (X, d, s) be a quasi b-metric space and $d(\mu, \nu) = d(\nu, \mu)$ for all $\mu, \nu \in X$. Then, (X, d, s) is a b-metric space.

Lemma 3. *Let (X, d, s) be a quasi b-metric space. Then, $s \geq 1$.*

Proof. Let $\mu, \nu \in X$. Then, $d(\mu, \nu) \leq s[d(\mu, \nu) + d(\nu, \nu)] = sd(\mu, \nu)$. So, $s \geq 1$.□ □

Remark 4. Let (r_n) be a sequence of nonnegative real numbers such that $r_{n+1} \leq r_n$ and $\lim_{n\longrightarrow+\infty} r_n = 0$. A quasi b-metric space is a topological space with $\{B_n(\mu)\}_{n\in\mathbb{N}}$, as a base of neighborhood filter of the point μ where $B_n(\mu) = \{\nu \in X : d(\mu, \nu) < r_n\}$.

Definition 5. Let (X, d, s) be a quasi b-metric space and a sequence $(\mu_n) \subseteq X$.

(1) Sequence (μ_n) is a left Cauchy sequence, if $d(\mu_n, \mu_m) \longrightarrow 0$ as $m, n \longrightarrow +\infty$

(2) A quasi b-metric space (X, d, s) is left complete if every left Cauchy sequence converges to some $\mu \in X$

Definition 6. Let (X, d, s) be a quasi b-metric space and the sequences (μ_n), (ν_n) in X be such that $\lim_{n\longrightarrow+\infty} \mu_n = \mu$ *and* $\lim_{m\longrightarrow+\infty} \nu_n = \nu$. A mapping d is sequentially continuous if $\lim_{n,m\longrightarrow+\infty} d(\mu_n, \nu_m) = d(\mu, \nu)$.

We will use the following lemma in our main results.

Lemma 7 (see [26]). *Let (X, d, s) be a quasi b-metric space and $(\mu_n) \subseteq X$. If there exists $\lambda \in [0, 1)$ such that*

$$d(\mu_n, \mu_{n+1}) \leq \lambda d(\mu_{n-1}, \mu_n), \tag{1}$$

for all $n \in \mathbb{N}$, then (μ_n) is a left Cauchy sequence.

3. A Fixed Point Theorem in Quasi b-Metric Spaces

Let $X \neq \varnothing$ and $f : X \longrightarrow X$ be a given mapping. Then, $\mu^* \in X$ is a fixed point of mapping f if $f(\mu^*) = \mu^*$. Let $\mu_0 \in X$, and consider the sequence (μ_n) defined by $\mu_n = f^n(\mu_0)$, i.e., (μ_n) is a sequence of Picard iterates of mapping f at point μ_0.

Now, we present our first result, which generalizes recent fixed point results obtained in [16, 18, 22, 25] for generalized contractive mappings defined on b-metric spaces.

Theorem 8. *Let (X, d, s) be a left complete quasi b-metric space and a mapping $f : X \longrightarrow X$. If there exist $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma < 1$, $\beta \leq \gamma$ and*

$$d(f\mu, f\nu) \leq \alpha \max\left\{d(\mu, \nu), d(\mu, f\mu), d(\nu, f\nu), \frac{d(\mu, f\nu) + d(f\mu, \nu)}{2s}\right\} + \beta\frac{d(\mu, f\nu)}{s} + \gamma d(f\mu, \nu), \tag{2}$$

for any $\mu, \nu \in X$, then for any $\mu_0 \in X$ sequence of Picard iterates (μ_n) defined by mapping f at μ_0 is left Cauchy sequence. Moreover, if f is sequentially continuous or d is sequentially continuous, then, f has unique fixed point $\mu^* \in X$ and $\mu_n \longrightarrow \mu^*$ as $n \longrightarrow +\infty$.

Proof. Let $\mu_0 \in X$ be arbitrary and (μ_n) sequence of Picard iterates defined by f at μ_0. Then

$$\begin{aligned} d(\mu_{n+1}, \mu_{n+2}) &= d(f\mu_n, f\mu_{n+1}) \\ &\leq \alpha \max\left\{d(\mu_n, \mu_{n+1}), d(\mu_n, f\mu_n), d(\mu_{n+1}, f\mu_{n+2}), \frac{d(\mu_n, f\mu_{n+1}) + d(f\mu_n, \mu_{n+1})}{2s}\right\} \\ &\quad + \beta\frac{d(\mu_n, f\mu_{n+1})}{s} + \gamma d(f\mu_n, \mu_{n+1}) \\ &\leq \alpha \max\left\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2}), \frac{d(\mu_n, \mu_{n+1}) + d(\mu_{n+1}, \mu_{n+2})}{2}\right\} + \beta\frac{d(\mu_n, \mu_{n+2})}{s} \\ &\leq \alpha \max\{d(\mu_n, \mu_{n+1}), d(\mu_{n+1}, \mu_{n+2})\} + \beta d(\mu_n, \mu_{n+1}) + \beta d(\mu_{n+1}, \mu_{n+2}). \end{aligned} \tag{3}$$

If $d(\mu_n, \mu_{n+1}) < d(\mu_{n+1}, \mu_{n+2})$, then,

$$(1 - \alpha - \beta) d(\mu_{n+1}, \mu_{n+2}) < \beta d(\mu_n, \mu_{n+1}), \tag{4}$$

which implies

$$d(\mu_{n+1}, \mu_{n+2}) \leq \frac{\beta}{1 - \alpha - \beta} d(\mu_n, \mu_{n+1}) < d(\mu_n, \mu_{n+1}). \tag{5}$$

So, $d(\mu_n, \mu_{n+1}) \geq d(\mu_{n+1}, \mu_{n+2})$ which implies

$$(1 - \beta) d(\mu_{n+1}, \mu_{n+2}) \leq (\alpha + \beta) d(\mu_n, \mu_{n+1}). \tag{6}$$

Hence, we get that

$$d(\mu_{n+1}, \mu_{n+2}) \leq \frac{\alpha + \beta}{1 - \beta} d(\mu_n, \mu_{n+1}) < \frac{\alpha + \beta}{\alpha + \gamma} d(\mu_n, \mu_{n+1}) = \lambda d(\mu_n, \mu_{n+1}), \tag{7}$$

where $\lambda = \alpha + \beta/\alpha + \gamma < 1$. So, by Lemma 7, we obtain that (μ_n) is left Cauchy sequence. It is convergent because (X, d, s) is left complete. Thus, exists $\mu^* \in X$ such that $\mu^* = \lim \mu_n$.□ □

Case 9. Let a mapping f be a sequentially continuous. Then

$$\mu^* = \lim \mu_n = \lim f\mu_n = f\mu^*. \tag{8}$$

Case 10. Let d be a sequentially continuous. Then

$$d(f\mu_n, f\mu^*) \leq \alpha \max\left\{d(\mu_n, \mu^*), d(\mu_n, \mu_{n+1}), d(\mu^*, f\mu^*), \frac{d(\mu_n, f\mu^*) + d(f\mu_n, \mu^*)}{2s}\right\} + \beta\frac{d(\mu_n, \mu_{n+2})}{2s} + \gamma d(f\mu_n, \mu^*), \tag{9}$$

which implies

$$\lim d(f\mu_n, f\mu^*) \le \lim\left[\alpha\left\{d(\mu_n,\mu^*), d(\mu^*, f\mu^*), \frac{d(\mu_n, f\mu^*) + d(f\mu_n,\mu^*)}{2s}\right\} + \beta\frac{d(\mu_n,\mu_{n+2})}{2s} + \gamma d(f\mu_n,\mu^*)\right]. \tag{10}$$

So, we get that

$$d(\lim\mu_{n+1}, f\mu^*)) \le \lim[\alpha \max\{d(\lim\mu_n,\mu^*), d(\lim\mu_n, \lim\mu_{n+1}), d(\mu^*, f\mu^*), \cdot\frac{d(\lim\mu_n, f\mu^*) + d(\lim\mu_{n+1},\mu^*)}{2s}\right\} + \beta\frac{d(\mu_n,\mu_{n+2})}{2s} + \gamma d(\lim\mu_{n+1},\mu^*)]. \tag{11}$$

Hence,

$$d(\mu^*, f\mu^*) \le \alpha \max\left\{d(\mu^*,\mu^*), d(\mu^*,\mu^*), d(\mu^*, f\mu^*), \frac{d(\mu^*, f\mu^*) + d(\mu^*,\mu^*)}{2s}\right\} + \beta\frac{d(\mu^*,\mu^*)}{2s} + \gamma d(\mu^*,\mu^*) = \alpha d(\mu^*, f\mu^*). \tag{12}$$

It follows that $\mu^* = f(\mu^*)$ because $\alpha \in [0,1)$. Finally, suppose that there are two fixed points of mapping f, i.e., $f\mu^* = \mu^*$, $f\nu^* = \nu^*$. Then, we get

$$d(\mu^*,\nu^*) = d(f\mu^*, f\nu^*) \le \alpha\left\{d(\mu^*,\nu^*), d(\mu^*,\mu^*), d(\nu^*,\nu^*), \frac{d(\mu^*,\nu^*) + d(\mu^*,\nu^*)}{2s}\right\} + \beta\frac{d(\mu^*,\nu^*)}{2s} + \gamma d(\mu^*,\nu^*) \le (\alpha+\beta+\gamma)d(\mu^*,\nu^*). \tag{13}$$

which implies that $\mu^* = \nu^*$.

Corollary 11. *Let (X, d, s) be a left complete quasi b-metric space and a mapping $f : X \longrightarrow X$. If there exist $\alpha \in [0, 1)$ such that*

$$d(f\mu, f\nu) \le \alpha d(\mu,\nu), \tag{14}$$

for any $\mu,\nu \in X$, then for any $\mu_0 \in X$ sequence of Picard iterates (μ_n) defined by mapping f at μ_0 is left Cauchy sequence. Moreover, if f is sequentially continuous or d is sequentially continuous, then, f has unique fixed point $\mu^* \in X$ and $\mu_n \longrightarrow \mu^*$ as $n \longrightarrow +\infty$.

Example 12. Let $X = [0, 1]$ and mapping $f : X \longrightarrow X$ defined by $f\mu = \mu/2$, $\mu \in X$. Let $d : X \times X \longrightarrow [0,+\infty)$ defined by

$$d(\mu,\nu) = \begin{cases} (\mu-\nu)^2, & \mu > \nu, \\ (\mu-\nu)^4, & \mu < \nu, \\ 0, & \mu = \nu. \end{cases} \tag{15}$$

Since, $(a+b)^2 \le 2(a^2+b^2)$ and $(a+b)^4 \le 8(a^4+b^4)$, for all $a, b \in \mathbb{R}$, we obtain that for d holds

$$d(\mu,\xi) \le 8[d(\mu,\nu) + d(\nu,\xi)], \tag{16}$$

for all $\mu,\nu,\xi \in X$. Also, $d(\mu,\nu) = 0$ if and only if $\mu = \nu$. So, $(X, d, 8)$ is a quasi b-metric space. Note that $d(\mu,\nu) = d(\nu, \mu)$ does not hold in the general case. In this case, all the conditions of Corollary 11 are valid, and we conclude that the mapping f has a fixed point.

4. A Common Fixed Point Theorem in b-Metric Spaces

Now we obtain a common fixed point result for mappings defined on b-metric spaces. Our result improves the classical results presented by Ćirić [4] and Wong [10].

Theorem 13. *Let (X, d, s) be a complete b-metric space and the mappings $f, g : X \longrightarrow X$. If there exist $\alpha, \beta \in [0, 1]$ such that $\alpha + 2\beta < 1$ and*

$$d(f\mu, g\nu) \le \alpha \max\left\{d(\mu,\nu), d(\mu, f\mu), d(\nu, g\nu), \frac{d(\mu, g\nu) + d(f\mu,\nu)}{2s}\right\} + \beta\frac{d(\mu, g\nu)}{s} + \beta\frac{d(f\mu,\nu)}{s}, \tag{17}$$

for any $\mu,\nu \in X$, then for any $\mu_0 \in X$ sequence of Picard iterates (μ_n) defined by $g \circ f$ at μ_0 is left Cauchy sequence. If f and g are sequentially continuous or d is sequentially continuous then f and g has unique fixed point which is unique limit of all Picard sequences defined by $g \circ f$.

Proof. Let $\mu_0 \in X$ be arbitrary and (μ_n) sequence defined by $\mu_{2n+1} = f\mu_{2n}$ and $\mu_{2n+2} = g\mu_{2n+1}$. Then

$$\begin{aligned} d(\mu_{2n+1},\mu_{2n+2}) &= d(f\mu_{2n}, g\mu_{2n+1}) \le \alpha \max\{d(\mu_{2n},\mu_{2n+1}), d(\mu_{2n}, f\mu_{2n}), d(\mu_{2n+1}, g\mu_{2n+1}), \\ &\quad \cdot\frac{d(\mu_{2n}, g\mu_{2n+1}) + d(\mu_{2n},\mu_{2n+1})}{2s}\right\} + \beta\frac{d(\mu_{2n}, f\mu_{n+1})}{s} + \beta\frac{d(f\mu_{2n},\mu_{2n+1})}{s} \\ &\le \alpha \max\left\{d(\mu_{2n},\mu_{2n+1}), d(\mu_{2n+1},\mu_{2n+2}), \frac{d(\mu_{2n},\mu_{2n+1}) + d(\mu_{2n+1},\mu_{2n+2})}{2}\right\} \\ &\quad + \beta\frac{d(\mu_{2n},\mu_{n+2})}{s} \le \alpha \max\{d(\mu_{2n},\mu_{2n+1}), d(\mu_{2n+1},\mu_{2n+2})\} \\ &\quad + \beta d(\mu_{2n},\mu_{2n+1}) + \beta d(\mu_{2n+1},\mu_{2n+2}). \end{aligned} \tag{18}$$

If $d(\mu_{2n},\mu_{2n+1}) < d(\mu_{2n+1},\mu_{n+2})$ then

$$(1-\alpha-\beta)d(\mu_{2n+1},\mu_{2n+2}) < \beta d(\mu_{2n},\mu_{2n+1}). \tag{19}$$

So, we get that

$$d(\mu_{2n+1},\mu_{2n+2}) \le \frac{\beta}{1-\alpha-\beta}d(\mu_{2n},\mu_{2n+1}) < d(\mu_{2n},\mu_{2n+1}), \tag{20}$$

therefore, $d(\mu_{2n},\mu_{2n+1}) \ge d(\mu_{2n+1},\mu_{2n+2})$ which implies

$$(1-\beta)d(\mu_{n+1},\mu_{n+2}) \le (\alpha+\beta)d(\mu_{2n},\mu_{2n+1}). \tag{21}$$

Hence, we get that

$$d(\mu_{2n+1}, \mu_{2n+2}) \le \frac{\alpha+\beta}{1-\beta} d(\mu_{2n}, \mu_{2n+1}). \tag{22}$$

So, we obtained

$$d(\mu_{2n+1}, \mu_{2n+2}) \le \lambda d(\mu_{2n}, \mu_{2n+1}), \tag{23}$$

where $\lambda = \alpha + \beta/1 - \beta < 1$. Further, we have

$$\begin{aligned} d(\mu_{2n}, \mu_{2n+1}) &= d(g\mu_{2n-1}, f\mu_{2n}) = d(f\mu_{2n}, g\mu_{2n-1}) \\ &\le \alpha \max \{d(\mu_{2n-1}, \mu_{2n}), d(\mu_{2n}, f\mu_{2n}), d(\mu_{2n-1}, g\mu_{2n-1}), \\ &\quad \cdot \frac{d(\mu_{2n-1}, f\mu_{2n}) + d(\mu_{2n}, g\mu_{2n-1})}{2s}\Big\} + \beta \frac{d(\mu_{2n}, g\mu_{2n-1})}{s} + \beta \frac{d(f\mu_{2n}, \mu_{2n-1})}{s} \\ &\le \alpha \max \left\{d(\mu_{2n-1}, \mu_{2n}), d(\mu_{2n}, \mu_{2n+1}), \frac{d(\mu_{2n}, \mu_{2n+1}) + d(\mu_{2n-1}, \mu_{2n})}{2}\right\} \\ &\quad + \beta \frac{d(\mu_{2n-1}, \mu_{2n+1})}{s} \le \alpha \max \{d(\mu_{2n}, \mu_{2n+1}), d(\mu_{2n+1}, \mu_{2n+2})\} \\ &\quad + \beta d(\mu_{2n-1}, \mu_{2n}) + \beta d(\mu_{2n}, \mu_{2n+1}). \end{aligned} \tag{24}$$

If $d(\mu_{2n-1}, \mu_{2n}) < d(\mu_n, \mu_{n+1})$ then

$$(1-\alpha-\beta) d(\mu_{2n}, \mu_{2n+1}) < \beta d(\mu_{2n-1}, \mu_{2n}), \tag{25}$$

which implies

$$d(\mu_{2n}, \mu_{2n+1}) \le \frac{\beta}{1-\alpha-\beta} d(\mu_{2n}, \mu_{2n-1}) < d(\mu_{2n-1}, \mu_{2n}), \tag{26}$$

therefore, $d(\mu_{2n-1}, \mu_{2n}) \ge d(\mu_{2n}, \mu_{2n+1})$ which implies that

$$d(\mu_{2n}, \mu_{2n+1}) \le \alpha d(\mu_{2n-1}, \mu_{2n}) + \beta d(\mu_{2n-1}, \mu_{2n}) + \beta d(\mu_{2n}, \mu_{2n+1}). \tag{27}$$

Therefore, we obtain

$$(1-\beta) d(\mu_{2n}, \mu_{2n+1}) \le (\alpha+\beta) d(\mu_{2n-1}, \mu_{2n}). \tag{28}$$

It follows

$$d(\mu_{2n}, \mu_{2n+1}) \le \frac{\alpha+\beta}{1-\beta} d(\mu_{2n-1}, \mu_{2n}). \tag{29}$$

So we obtain,

$$d(\mu_{2n}, \mu_{2n+1}) \le \lambda d(\mu_{2n-1}, \mu_{2n}), \tag{30}$$

where $\lambda = \alpha + \beta/1 - \beta < 1$. Hence,

$$d(\mu_n, \mu_{n+1}) \le \lambda d(\mu_{n-1}, \mu_n), \tag{31}$$

for each positive integer n. So, by Lemma 7, we obtain that (μ_n) is a Cauchy sequence. It is convergent because $(X, d, s$ $)$ is complete. Therefore, there exists $\xi \in X$ such that $\xi = \lim \mu_n$.□ □

Case 14. Let f and g be sequentially continuous functions. Then, we have

$$\xi = \lim \mu_n = \lim f\mu_n = f\xi = \lim \mu_n = \lim g\mu_n = g\xi. \tag{32}$$

Case 15. Let d be a sequentially continuous. Then,

$$\begin{aligned} &d(f\xi, g\mu_{2n+1}) \\ &\le \alpha \max \left\{d(\xi, \mu_{2n+1}), d(\xi, f\xi), d(\mu_{2n+1}, g\mu_{2n+1}), \frac{d(\mu_{2n+1}, f\xi) + d(g\mu_{2n+1}, \xi)}{2s}\right\} \\ &\quad + \beta \frac{d(\mu_{2n+1}, f\xi)}{2s} + \beta \frac{d(g\mu_{2n+1}, \xi)}{s}, \end{aligned} \tag{33}$$

which implies

$$\begin{aligned} \lim d(f\xi, g\mu_{2n+1}) \le \lim [&\alpha \max \{d(\xi, \mu_{2n+1}), d(\xi, f\xi), d(\mu_{2n+1}, g\mu_{2n+1}), \\ &\cdot \frac{d(\mu_{2n+1}, f\xi) + d(g\mu_{2n+1}, \xi)}{2s}\Big\} + \beta \frac{d(\mu_{2n+1}, f\xi)}{2s} \\ &+ \beta \frac{d(g\mu_{2n+1}, \xi)}{s}\Big]. \end{aligned} \tag{34}$$

So, we get that

$$\begin{aligned} d(\lim g\mu_{2n+1}, f\xi) \le \alpha \max \{&d(\lim \mu_{2n+1}, \xi) d(\xi, f\xi), d(\lim \mu_{2n+1}, \lim f\mu_{2n+1}), \\ &\cdot \frac{d(\lim \mu_{2n+1}, f\xi) + d(\lim g\mu_{2n+1}, \xi)}{2s}\Big\} + \beta \frac{d(\lim \mu_{2n+1}, f\xi)}{2s} \\ &+ \beta \frac{d(\lim g\mu_{2n+1}, \xi)}{s}\Big]. \end{aligned} \tag{35}$$

Hence,

$$\begin{aligned} d(\xi, f\xi) &\le \alpha \max \left\{d(\xi, \xi), d(\xi, f\xi), d(\xi, \xi), \frac{d(\xi, f\xi) + d(\xi, \xi)}{2s}\right\} + \beta \frac{d(\xi, f\xi)}{2s} + \beta d(\xi, \xi) \\ &< (\alpha+\beta) d(\xi, f\xi). \end{aligned} \tag{36}$$

It follows that $\xi = f\xi$ because $(\alpha + \beta) \in [0, 1)$. Further, we have

$$\begin{aligned} d(f\mu_{2n}, g\xi) \le \alpha \max &\left\{d(\mu_{2n}, \xi), d(\mu_{2n}, f\mu_{2n}), d(\xi, g\xi), \frac{d(\mu_{2n}, g\xi) + d(f\mu_{2n}, \xi)}{2s}\right\} \\ &+ \beta \frac{d(\mu_{2n}, g\xi)}{2s} + \beta \frac{d(f\mu_{2n}, \xi)}{s}, \end{aligned} \tag{37}$$

which implies

$$\begin{aligned} \lim d(f\mu_{2n}, g\xi) \le \lim \Big[&\alpha \max \left\{d(\mu_{2n}, \xi), d(\mu_{2n}, f\mu_{2n}), d(\xi, g\xi), \frac{d(\mu_{2n}, g\xi) + d(f\mu_{2n}, \xi)}{2s}\right\} \\ &+ \beta \frac{d(\mu_{2n}, g\xi)}{2s} + \beta \frac{d(f\mu_{2n}, \xi)}{s}\Big]. \end{aligned} \tag{38}$$

So, we get that

$$d(\lim f\mu_{2n}, g\xi) \leq \alpha \max\Big\{d(\lim\mu_{2n}, \xi), d(\lim\mu_{2n}, \lim f\mu_{2n}), d(\xi, g\xi), \cdot \frac{d(\lim\mu_{2n}, g\xi) + d(\lim f\mu_{2n}, \xi)}{2s}\Big\} + \beta\frac{d(\lim\mu_{2n}, g\xi)}{2s} + \beta\frac{d(\lim f\mu_{2n}, \xi)}{s}\Big]. \tag{39}$$

Hence,

$$d(\xi, g\xi) \leq \alpha \max\left\{d(\xi,\xi), d(\xi,\xi), d(\xi, g\xi), \frac{d(\xi, g\xi) + d(\xi,\xi)}{2s}\right\} + \beta\frac{d(\xi, g\xi)}{2s} + \beta d(\xi,\xi) < (\alpha+\beta) d(\xi, g\xi). \tag{40}$$

It follows that $\xi = g(\xi)$ because $(\alpha+\beta) \in [0, 1)$.

Now, we prove that the fixed point is unique. Suppose that there are ξ and ξ', i.e., $g\xi = f\xi = \xi$ and $g\xi' = f\xi' = \xi'$. Then, we obtain

$$d\left(\xi, \xi'\right) = d\left(f\xi, g\xi'\right) \leq \alpha\left\{d\left(\xi, \xi'\right), d(\xi,\xi) d\left(\xi', \xi'\right), \frac{d\left(\xi,\xi'\right) + d\left(\xi,\xi'\right)}{2s}\right\} + \beta\frac{d\left(\xi,\xi'\right)}{2s} + \beta d\left(\xi,\xi'\right) \leq (\alpha + 2\beta) d\left(\xi,\xi'\right). \tag{41}$$

which implies that $\xi = \xi'$.

Corollary 16. *Let (X, d, s) be a complete b-metric space and mapping $f : X \longrightarrow X$. If there exist $\alpha, \beta, \gamma \in [0, 1]$ such that $\alpha + \beta + \gamma < 1$ and*

$$d(f\mu, f\nu) \leq \alpha \max\left\{d(\mu,\nu), d(\mu, f\mu), d(\nu, f\nu), \frac{d(\mu, f\nu) + d(f\mu, \nu)}{2s}\right\} + \beta\frac{d(\mu, f\nu)}{s} + \gamma\frac{d(f\mu,\nu)}{s}, \tag{42}$$

for any $\mu, \nu \in X$, then for any $\mu_0 \in X$ sequence of Picard iterates (μ_n) defined by f at μ_0 is Cauchy sequence. If f is sequentially continuous or d is sequentially continuous, then, f has unique fixed point which is unique limit of all Picard sequences defined by f.

Proof. From

$$d(f\mu, f\nu) \leq \alpha \max\left\{d(\mu,\nu), d(\mu, f\mu), d(\nu, f\nu), \frac{d(\mu, f\nu) + d(f\mu, \nu)}{2s}\right\} + \beta\frac{d(\mu, f\nu)}{s} + \gamma\frac{d(f\mu,\nu)}{s}, \tag{43}$$

and

$$d(f\nu, f\mu) \leq \alpha \max\left\{d(\nu,\mu), d(\nu, f\nu), d(\mu, f\mu), \frac{d(\nu, f\mu) + d(f\nu, \mu)}{2s}\right\} + \beta\frac{d(\nu, f\mu)}{s} + \gamma\frac{d(f\nu,\mu)}{s}, \tag{44}$$

it follows

$$d(f\mu, f\nu) \leq \alpha \max\left\{d(\mu,\nu), d(\mu, f\mu), d(\nu, f\nu), \frac{d(\mu, f\nu) + d(f\mu, \nu)}{2s}\right\} + \delta\frac{d(\mu, f\nu)}{s} + \delta\frac{d(f\mu,\nu)}{s}, \tag{45}$$

where $\delta = \beta + \gamma/2$.□ □

Example 17. Let $X = [0, 4]$ and $d(\mu, \nu) = (\mu - \nu)^2$, for each $\mu, \nu \in X$. Then $(X, d, 2)$ is a b-metric space. Define a mapping $f : X \longrightarrow X$ by

$$f(t) = \begin{cases} \dfrac{t}{3}, & t \in [0, 3], \\ \dfrac{t}{6}, & t \in (3, 4], \end{cases} \tag{46}$$

for any $t \in X$. For $\mu, \nu \in [0, 3]$, we have

$$d(f\mu, f\nu) = \frac{1}{9}(\mu - \nu)^2. \tag{47}$$

For $\mu, \nu \in (3, 4]$, we have

$$d(f\mu, f\nu) = \frac{1}{36} d(\mu, \nu). \tag{48}$$

For $\mu \in [0, 3]$ and $\nu \in (3, 4]$, we have

$$d(f\mu, f\nu) = \left(\frac{\mu}{3} - \frac{\nu}{6}\right)^2 \leq \frac{4}{9} < \frac{1}{2} d(\nu, f\nu), \tag{49}$$

because $d(\nu, f\nu) = (5\nu/6)^2 > 25 \cdot 9/36$.

For $\nu \in [0, 3]$ and $\mu \in (3, 4]$, we have

$$d(f\mu, f\nu) = \left(\frac{\mu}{3} - \frac{\nu}{6}\right)^2 \leq \frac{16}{9} < \frac{3}{4} d(\mu, f\mu), \tag{50}$$

because $d(\mu, f\mu) = (5\mu/6)^2 > 25 \cdot 9/36$.

Since conditions of Corollary 16 is satisfied for $\alpha = 3/4$ and $\beta = \gamma = 0$. So, f has unique fixed point which is unique limit of all Picard sequences defined by f, because d is sequentially continuous.

Data Availability

No data are used.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 1064803, 8 pages
https://doi.org/10.1155/2022/1064803

Research Article

On a Unique Solution of the Stochastic Functional Equation Arising in Gambling Theory and Human Learning Process

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Received 31 August 2021; Revised 16 December 2021; Accepted 20 December 2021; Published 20 January 2022

Academic Editor: Santosh Kumar

The term “learning” is often used to refer to a generally stable behavioral change resulting from practice. However, it is a fundamental biological capacity far more developed in humans than in other living beings. In an animal or human being, the learning phase may often be viewed as a series of choices between multiple possible reactions. Here, we analyze a specific type of human learning process related to gambling in which a subject inserts a poker chip to operate a two-armed bandit device and then presses one of the two keys. Through the use of an electromagnet, one or more poker chips are given to the individual in a container located in the apparatus’s center. If a chip is provided, it is declared a winner; otherwise, it is considered a loser. The goal of this paper is to look at the subject’s actions in such situations and provide a mathematical model that is appropriate for it. The existence of a unique solution to the suggested human learning model is examined using relevant fixed point results.

1. Introduction

Learning is a fundamental biological capacity that is much more evolved in humans than in any other living being. The central topic in learning philosophy is how multiple forms of learning take place in a human brain and body since this was explicitly formulated in the discipline of learning psychology, but with additional feedback from other psychological disciplines and the adjacent areas of sociology, pedagogy, and biology, including contemporary brain science.

In modern mathematical learning experiments, the researchers concluded that a basic learning experiment was compatible with any stochastic process. Thus, it is not a novel concept (for detail, see [1]). However, after 1950, two critical features emerged mainly in the research initiated by Bush, Estes, and Mosteller. Firstly, the learning method egalitarian essence was a core feature of the developed model. Secondly, these frameworks were studied and applied in areas that did not conceal their quantitative aspects.

Several studies on human actions in probability-learning scenarios have produced different results (for the detail, see [2–5]).

In 2019, Turab and Sintunavarat [6, 7] proposed a functional equation to examine the experimental work of Bush and Wilson [8] on a paradise fish. In this experiment, a fish was given two options for swimming. The fish had options to swim on either side (right or left) of the tank’s far end.

In [9], the authors recently addressed a kind of traumatic avoidance learning experiment for normal dogs suggested by Solomon and Wynne [10]. They examined the psychological responses of 30 dogs enclosed in a small steel grid cage and proposed a mathematical model. The suggested avoidance learning model’s existence and uniqueness of a solution result were investigated using the appropriate fixed point method.

For the research in this area, especially related to the two-choice behavior, we refer to [11–13] and the references therein. It is worth noting that most animal behavior studies in a two-choice situation discussed above have focused only on the animals' approach toward an inevitable conclusion. Bush and Wilson [8], on the other hand, divided such responses into four categories depending on the food source and side chosen (right-reward, right nonreward, left reward, and left nonreward).

In this work, by following the work presented by Turab and Sintunavarat [6, 9] and the idea discussed in [8, 14], our aims are to discuss the two-armed bandit experiment proposed by Goodnow and Pettigrew [15] and propose a convenient mathematical model. We evaluate our findings under the experimenter-subject controlled events to see the feasibility of the suggested model. The existence of a unique solution to the proposed model is examined by using the appropriate fixed point theorem. In the end, we raise some open problems for the interested readers.

2. A Two-Armed Bandit Experiment

In [15], Goodnow and Pettigrew presented an experiment related to the gambling theory. This gambling activity involves playing a poker game with chips worth one penny each (see Figure 1). The subject (**S**) is given 200 chips by an experimenter (**E**). He/She inserts into the machine one of these chips and pushes one of two buttons. A chip drops into the payout box with a clatter of noise when the bet is successful. The payoff box has a glass face, and the heap of chips he/she has won can be seen by **S**. The subject is not permitted until the end of the experiment to carry the chips out of this box. Whatever the outcome of the bet, between each test, the machine becomes unusable for several seconds, and **S** wait until two signal lights and a loud buzz appear, indicating that the device is ready to take the next bet. The apparatus is fully programmed such that inserting a chip before the device's ready is useless for **S**.

When the subject **S** implants a chip (upper center light) and clicks a key (left or right lower), the lights on the face of the machine flash on successively (upper outer lights in Figure 1). These lights are parallel to the control machine's lights controlled in an adjacent space by **E**. A master switch to turn the device on or off is also included in the control machine, along with a key that allows the machine to eject a chip into the pay-off box when pushed. The one-way mirror enables **E** from the control room to view **S**'s activities.

2.1. Procedure. The assignment's method and directions were given to **S** and **E**. The **S** was instructed that he/she is playing for cash and that he/she would be paid for the discrepancy between the number of wins and losses. There were 120 trials allowed for every **S**, divided into 12 blocks of 10 trials each. The probability of the above task was 50 : 50, 70 : 30, and 90 : 10. When the experiment is completed, **S** was asked the following questions:

(1) How did you decide which alternative you should choose?

(2) How he/she thought about the strategy of always betting on one key?

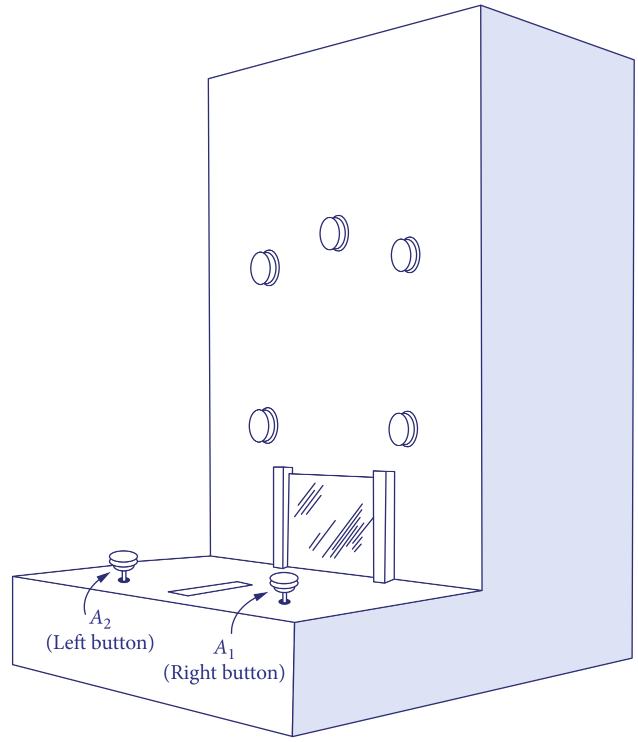

Figure 1: A sketch of a two-armed bandit machine.

2.2. Results. The results were described in terms of the average proportion of choices of one alternative: pushing the 'left button' in the gambling experiment provided the greater likelihood of these alternatives outside the 50 : 50 scenario. In Table 1, the findings are presented.

3. Mathematical Modeling of the Two-Armed Bandit Experiment

In the above experiment, significant interest lies in the behavior of a subject **S**; press right or left button, `$\mathbf{A_1}$' or `$\mathbf{A_2}$,' and get the reward in terms of a poker chip. In our view, if a subject chooses the reward side, there would be an occurrence of alternative $\mathbf{O_1}$, and if the subject made a move to the other side, then there will be an occurrence of alternative $\mathbf{O_2}$. Thus, according to the mathematical point of view, there would be four possibilities of events, depending on the action of the subject and the reward. These events are listed in Table 2.

Depending on the action of the subject and getting the chance of the reward, we have the following four events (see Table 3).

The probability of the outcomes $\mathbf{A_1}$ and $\mathbf{A_2}$ are x and ($1-x$), respectively, where $x \in 0, 1]$. The experimental pattern asks for the outcomes of the responses (whether the subject get the reward or not), trials' fixed proportion of $p \in 0, 1]$. Therefore, we get the event probabilities stated below (see Table 4).

Table 1: Mean proportional choices by group and by blocks of 20 trials of pressing a 'left key' in gambling experiment.

	Group		
Trials	50 : 50	70 : 30	90 : 10
1-20	0.430	0.489	0.765
21-40	0.505	0.664	0.878
41-60	0.550	0.721	0.950
61-80	0.495	0.722	0.954
81-100	0.465	0.782	0.965
101-120	0.515	0.815	0.964
N	10	14*	14*

*Those two groups where **N** was increased to 14 because the data were required for another purpose (an analysis of choice sequences), and the incidence of 100 : 0 choice distributions was cutting down on the amount of data available for such analysis.

Table 2: The possible four responses in two-armed bandit experiment.

Responses	Outcomes
$\mathbf{A_1}$: press right button	$\mathbf{O_1}$: reward (poker chips)
$\mathbf{A_1}$: press right button	$\mathbf{O_2}$: no reward (no poker chips)
$\mathbf{A_2}$: press left button	$\mathbf{O_1}$: reward (poker chips)
$\mathbf{A_2}$: press left button	$\mathbf{O_2}$: no reward (no poker chips)

Table 3: The corresponding events of the subject.

Response	Outcomes	Events
$\mathbf{A_1}$	$\mathbf{O_1}$	$\mathbf{E_1}$
$\mathbf{A_1}$	$\mathbf{O_2}$	$\mathbf{E_2}$
$\mathbf{A_2}$	$\mathbf{O_1}$	$\mathbf{E_3}$
$\mathbf{A_2}$	$\mathbf{O_2}$	$\mathbf{E_4}$

Table 4: Probabilities of the four events.

Event	Probability of occurrence
$\mathbf{E_1}$	px
$\mathbf{E_2}$	$(1-p)x$
$\mathbf{E_3}$	$p(1-x)$
$\mathbf{E_4}$	$(1-p)(1-x)$

We define $\eta_1, \eta_2, \eta_3, \eta_4 \in (0,1)$ as the learning rate parameters and their values can be recognized as a measure of the ineffectiveness of the corresponding events $\mathbf{E_1} - \mathbf{E_4}$ in altering the response probability.

If, on some trial, px is the possibility of response $\mathbf{A_1}$ with outcome $\mathbf{O_1}$ and $\mathbf{A_1}$ is fulfilled, the next possibility of $\mathbf{A_1}$ with outcome $\mathbf{O_1}$ will be $\eta_1 x + (1-\eta_1)$, and if $\mathbf{A_1}$ is achieved with outcome $\mathbf{O_2}$ then the new probability would be $\eta_2 x + (1-\eta_2)$ with the event probability $(1-p)x$. Similarly, if $\mathbf{A_2}$ is performed with outcomes $\mathbf{O_1}$ and $\mathbf{O_2}$, then the new probabilities of $\mathbf{A_2}$ are $\eta_3 x + (1-\eta_3)$ and $\eta_4 x + (1-\eta_4)$, with the event probabilities $p(1-x)$ and $(1-p)(1-x)$, respectively. For the four events $\mathbf{E_1} - \mathbf{E_4}$, we can define the transition operators $\mathcal{Q}_1 - \mathcal{Q}_4 : [0,1] \longrightarrow 0, 1]$ as

$$\begin{cases} \mathcal{Q}_1 x = \eta_1 x + (1-\eta_1), \\ \mathcal{Q}_2 x = \eta_2 x + (1-\eta_2), \\ \mathcal{Q}_3 x = \eta_3 x + (1-\eta_3), \\ \mathcal{Q}_4 x = \eta_4 x + (1-\eta_4), \end{cases} \tag{1}$$

for all $x \in 0, 1]$.

By considering the work presented in [6, 8, 9] and the above transition operators with their corresponding probabilities and events given in Table 4, we introduce the following functional equation, which can discuss all the aspects of the two-armed bandit model.

$$\begin{aligned} \mathcal{Q}(x, \eta_1, \eta_2, \eta_3, \eta_4) = {} & px\mathcal{Q}(\eta_1 x + (1-\eta_1), \eta_1) + (1-p)x\mathcal{Q}(\eta_2 x + (1-\eta_2), \eta_2) \\ & + p(1-x)\mathcal{Q}(\eta_3 x + (1-\eta_3), \eta_3) \\ & + (1-p)(1-x)\mathcal{Q}(\eta_4 x + (1-\eta_4), \eta_4). \end{aligned} \tag{2}$$

Fixed point theory, on the other hand, began in the second half of the nineteenth century as a method of using iterative estimations to demonstrate the existence and uniqueness of solutions to ordinary differential and integral equations. It is a wonderful combination of basic and applied analysis, geometry, and topology. A fixed point theoretic viewpoint can be seen in Picard's work, which is a fundamental notion in the field of metric fixed point theory. Nevertheless, it is credited to the Polish mathematician "Banach," who abstracted the underlying principles into a framework that can be applied to find the existence of a unique solution to the broad range of applications beyond differential and integral equations. It has been extended and generalized in numerous directions (for the detail, see [16–18]). We suggest the reader to see [19–21] for further information on fixed point theory and its applications in various spaces.

The following stated outcome will be required in the progression.

Theorem 1 (see [22]). *Let $(\mathcal{O}, d)$ be a complete metric space and $\mathcal{J} : \mathcal{O} \longrightarrow \mathcal{O}$ be a Banach contraction mapping (shortly, BCM), that is,*

$$d(\mathcal{J}\omega, \mathcal{J}\varpi) \leq \delta d(\omega, \varpi), \tag{3}$$

for some $\delta < 1$ and for all $\omega, \varpi \in \mathcal{O}$. Then, $\mathcal{O}$ has one fixed point. Furthermore, the Picard iteration $\{\omega_n\}$ in $\mathcal{O}$ that can be defined as $\omega_n = \mathcal{O}\omega_{n-1}$ for all $n \in \mathbb{N}$, where $\omega_0 \in \mathcal{O}$, converges to the unique fixed point of $\mathcal{O}$.

4. Existence and Uniqueness Results

We let $\mathcal{O} = [0, 1]$. For the rest of this article, $\mathcal{D}$ represents the class $\mathcal{J} : \mathcal{O} \longrightarrow \mathbb{R}$ with $\mathcal{J}(0) = 0$ consisting of all real-valued continuous functions which satisfy the following relation

$$\sup_{\omega \neq \varpi} \frac{|\mathscr{J}(\omega) - \mathscr{J}(\varpi)|}{|\omega - \varpi|} < \infty. \tag{4}$$

Clearly, $(\mathscr{D}, \|\cdot\|)$ is a Banach space with

$$\|\mathscr{J}\| = \sup_{\omega \neq \varpi} \frac{|\mathscr{J}(\omega) - \mathscr{J}(\varpi)|}{|\omega - \varpi|}, \tag{5}$$

for all $\mathscr{J} \in \mathscr{D}$.

Following that, we can rewrite the functional equation (2) as

$$\begin{aligned} \mathscr{J}(x) &= px\mathscr{J}(\eta_1 x + 1 - \eta_1) + (1-p)x\mathscr{J}(\eta_2 x + 1 - \eta_2) \\ &\quad + (1-x)p\mathscr{J}(\eta_3 x + 1 - \eta_3) + (1-p)(1-x)\mathscr{J}(\eta_4 x + 1 - \eta_4), \end{aligned} \tag{6}$$

where $\mathscr{J} : \mathscr{O} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \eta_1, \eta_2, \eta_3, \eta_4 < 1$.

Theorem 2. *For $0 < \eta_1, \eta_2, \eta_3, \eta_4 < 1$ and $p \in \mathscr{O}$ with $\Theta_1 < 1$, where*

$$\Theta_1 \coloneqq [2p(\eta_1 + \eta_3) + 2(1-p)(\eta_2 + \eta_4) + 2p]. \tag{7}$$

If there is a $\mathscr{C} \subseteq \mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$-invariant, that is, $\mathscr{W}(\mathscr{C}) \subseteq \mathscr{C}$, where $\mathscr{W} : \mathscr{C} \longrightarrow \mathscr{C}$ is defined for each $\mathscr{J} \in \mathscr{C}$ as

$$\begin{aligned} (\mathscr{W}\mathscr{J})(x) &= px\mathscr{J}(\eta_1 x + 1 - \eta_1) + (1-p)x\mathscr{J}(\eta_2 x + 1 - \eta_2) \\ &\quad + (1-x)p\mathscr{J}(\eta_3 x + 1 - \eta_3) + (1-p)(1-x)\mathscr{J}(\eta_4 x + 1 - \eta_4), \end{aligned} \tag{8}$$

for all $x \in \mathscr{O}$, then $\mathscr{W}$ is a BCM.

Proof. Let $\mathscr{J}_1, \mathscr{J}_2 \in \mathscr{C}$. For each distinct points $\omega, \varpi \in \mathscr{O}$, we obtain

$$\begin{aligned} &\frac{|(\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2)(\omega) - (\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2)(\varpi)|}{|\omega - \varpi|} = \bigg| \frac{1}{\omega - \varpi} [p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) + (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) + p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) \\ &\quad + (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) - p\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - (1-p)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) - p(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) \\ &\quad - (1-p)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)] \bigg| = \bigg| \frac{1}{\omega - \varpi} [p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) + (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) \\ &\quad - (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) + p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) + (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) \\ &\quad - (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) + p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - p\varpi(\mathscr{Z}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) + (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) \\ &\quad - (1-p)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) + p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) - p(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) + (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) \\ &\quad - (1-p)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)] \bigg| = \bigg| \frac{1}{\omega - \varpi} [p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)] + \frac{1}{\omega - \varpi}[(1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) \\ &\quad - (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)] + \frac{1}{\omega - \varpi}[p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)] \\ &\quad + \frac{1}{\omega - \varpi}[(1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) - (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)] + \frac{1}{\omega - \varpi}[p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - p\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)] \\ &\quad + \frac{1}{\omega - \varpi}[(1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) - (1-p)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)] + \frac{1}{\omega - \varpi}[p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) - p(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)] \\ &\quad + \frac{1}{\omega - \varpi}[(1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) - (1-p)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)] \bigg| \le \frac{|p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)|}{|\eta_1\omega - \eta_1\varpi|} \\ &\quad \times \frac{|\eta_1\omega - \eta_1\varpi|}{|\omega - \varpi|} + \frac{|(1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) - (1-p)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)|}{|\eta_2\omega - \eta_2\varpi|} \times \frac{|\eta_2\omega - \eta_2\varpi|}{|\omega - \varpi|} \\ &\quad + \frac{|p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)|}{|\eta_3\omega - \eta_3\varpi|} \times \frac{|\eta_3\omega - \eta_3\varpi|}{|\omega - \varpi|} \\ &\quad + \frac{|(1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) - (1-p)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)|}{|\eta_4\omega - \eta_4\varpi|} \times \frac{|\eta_4\omega - \eta_4\varpi|}{|\omega - \varpi|} + |p(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)| \\ &\quad + |(1-p)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)| + |p(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)| + |(1-p)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)|. \end{aligned} \tag{9}$$

By applying the definition of the norm (5), we obtain

$$\begin{aligned}
&\frac{|(\mathscr{W}\mathscr{J}_1-\mathscr{W}\mathscr{J}_2)(\omega)-(\mathscr{W}\mathscr{J}_1-\mathscr{W}\mathscr{J}_2)(\varpi)|}{|\omega-\varpi|} \leq \eta_1 p\omega\|\mathscr{J}_1-\mathscr{J}_2\|+\eta_2(1-p)\omega\|\mathscr{J}_1-\mathscr{J}_2\| \\
&\quad+\eta_3 p(1-\omega)\|\mathscr{J}_1-\mathscr{J}_2\|+\eta_4(1-p)(1-\omega)\|\mathscr{J}_1-\mathscr{J}_2\|+|p(\mathscr{J}_1-\mathscr{J}_2)(\eta_1\varpi+1-\eta_1) \\
&\quad-p(\mathscr{J}_1-\mathscr{J}_2)(0)|+|(1-p)(\mathscr{J}_1-\mathscr{J}_2)(\eta_2\varpi+1-\eta_2)-(1-p)(\mathscr{J}_1-\mathscr{J}_2)(1)| \\
&\quad+|p(\mathscr{J}_1-\mathscr{J}_2)(\eta_3\varpi+1-\eta_3)-p(\mathscr{J}_1-\mathscr{J}_2)(0)|+|(1-p)(\mathscr{J}_1-\mathscr{J}_2)(\eta_4\varpi+1-\eta_4) \\
&\quad-(1-p)(\mathscr{J}_1-\mathscr{J}_2)(1)|=\eta_1 p\omega\|\mathscr{J}_1-\mathscr{J}_2\|+\eta_2(1-p)\omega\|\mathscr{J}_1-\mathscr{J}_2\| \\
&\quad+\eta_3 p(1-\omega)\|\mathscr{J}_1-\mathscr{J}_2\|+\eta_4(1-p)(1-\omega)\|\mathscr{J}_1-\mathscr{J}_2\|+p(\eta_1\varpi+1-\eta_1)\|\mathscr{J}_1-\mathscr{J}_2\| \\
&\quad+(1-p)(\eta_2\varpi-\eta_2)\|\mathscr{W}_1-\mathscr{J}_2\|+p(\eta_3\varpi+1-\eta_3)\|\mathscr{J}_1-\mathscr{J}_2\| \\
&\quad+(1-p)(\eta_4\varpi-\eta_4)\|\mathscr{J}_1-\mathscr{J}_2\| \leq \Theta_1\|\mathscr{J}_1-\mathscr{J}_2\|,
\end{aligned} \tag{10}$$

where Θ_1 is defined in (7). This gives that

$$d(\mathscr{W}\mathscr{J}_1,\mathscr{W}\mathscr{J}_2)=\|\mathscr{W}\mathscr{J}_1-\mathscr{W}\mathscr{J}_2\| \leq \Theta_1\|\mathscr{J}_1-\mathscr{J}_2\|=\Theta_1 d(\mathscr{J}_1,\mathscr{J}_2). \tag{11}$$

As a result of $0<\Theta_1<1$, we can claim that $\mathscr{W}$ is a BCM with the metric d imposed by $\|\cdot\|$.

We get the following conclusion from Theorem 2 about the uniqueness of a functional equation (6)'s solution.□□
□

Theorem 3. *The stochastic equation (6) has a unique solution with $\Theta_1<1$, where Θ_1 is defined in (7). Assume that there is a $\mathscr{C}\subseteq\mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$ -invariant, that is, $\mathscr{W}(\mathscr{C})\subseteq\mathscr{C}$, where $\mathscr{W}:\mathscr{C}\longrightarrow\mathscr{C}$ defined for each $\mathscr{J}\in\mathscr{C}$ as*

$$\begin{aligned}
(\mathscr{W}\mathscr{J})(x)&=px\mathscr{J}(\eta_1 x+1-\eta_1)+(1-p)x\mathscr{J}(\eta_2 x+1-\eta_2) \\
&\quad+p(1-x)\mathscr{J}(\eta_3 x+1-\eta_3)+(1-p)(1-x)\mathscr{J}(\eta_4 x+1-\eta_4),
\end{aligned} \tag{12}$$

for all $x\in\mathscr{O}$. Furthermore, the following iteration $\{\mathscr{J}_n\}$ in $\mathscr{C}$ ($\forall n\in\mathbb{N}$ and $\mathscr{J}_0\in\mathscr{C}$) defined by

$$\begin{aligned}
(\mathscr{J}_n)(x)&=px\mathscr{J}_{n-1}(\eta_1 x+1-\eta_1)+(1-p)x\mathscr{J}_{n-1}(\eta_2 x+1-\eta_2) \\
&\quad+p(1-x)\mathscr{J}_{n-1}(\eta_3 x+1-\eta_3) \\
&\quad+(1-p)(1-x)\mathscr{J}_{n-1}(\eta_4 x+1-\eta_4),
\end{aligned} \tag{13}$$

converges to the unique solution of (12).

Proof. We reach the conclusion of this theorem by combining the Banach fixed point theorem with Theorem 2.□ □

The following corollaries arise from the preceding findings.

Table 5: Four events under conditional probability of occurrence.

Events	Outcomes	Transition operators	Probabilities of occurrence
$\mathbf{A_1}$	$\mathbf{O_1}$	$\mathscr{Q}_1 x=\eta_1 x+1-\eta_1$	$p_1 x$
$\mathbf{A_1}$	$\mathbf{O_2}$	$\mathscr{Q}_2 x=\eta_2 x+1-\eta_2$	$(1-p_1)x$
$\mathbf{A_2}$	$\mathbf{O_1}$	$\mathscr{Q}_3 x=\eta_3 x+1-\eta_3$	$p_2(1-x)$
$\mathbf{A_2}$	$\mathbf{O_2}$	$\mathscr{Q}_4 x=\eta_4 x+1-\eta_4$	$(1-p_2)(1-x)$

Corollary 4. *For $0<\eta_1\leq\eta_2\leq\eta_3\leq\eta_4<1$ and $p\in\mathscr{O}$ with $\tilde{\Theta}_1<1$, where*

$$\tilde{\Theta}_1 := (2(p+\eta_4)). \tag{14}$$

If there is a $\mathscr{C}\subseteq\mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$-invariant, that is, $\mathscr{W}(\mathscr{C})\subseteq\mathscr{C}$, where $\mathscr{W}:\mathscr{C}\longrightarrow\mathscr{C}$ defined for each $\mathscr{J}\in\mathscr{C}$ as

$$\begin{aligned}
(\mathscr{W}\mathscr{J})(x)&=px\mathscr{J}(\eta_1 x+1-\eta_1)+(1-p)x\mathscr{J}(\eta_2 x+1-\eta_2) \\
&\quad+(1-x)p\mathscr{J}(\eta_3 x+1-\eta_3) \\
&\quad+(1-p)(1-x)\mathscr{J}(\eta_4 x+1-\eta_4),
\end{aligned} \tag{15}$$

for all $x\in\mathscr{O}$, then $\mathscr{W}$ is a BCM.

Corollary 5. *The stochastic equation (6) has a unique solution with $\tilde{\Theta}_1<1$, where $\tilde{\Theta}_1$ is defined in (7). Assume that there is a $\mathscr{C}\subseteq\mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$-invariant, that is, $\mathscr{W}(\mathscr{C})\subseteq\mathscr{C}$, where $\mathscr{W}:\mathscr{C}\longrightarrow\mathscr{C}$ defined for each $\mathscr{J}\in\mathscr{C}$ as*

$$\begin{aligned}
(\mathscr{W}\mathscr{J})(x)&=px\mathscr{J}(\eta_1 x+1-\eta_1)+(1-p)x\mathscr{J}(\eta_2 x+1-\eta_2) \\
&\quad+p(1-x)\mathscr{J}(\eta_3 x+1-\eta_3) \\
&\quad+(1-p)(1-x)\mathscr{J}(\eta_4 x+1-\eta_4),
\end{aligned} \tag{16}$$

for all $x\in\mathscr{O}$. Furthermore, the iteration $\{\mathscr{J}_n\}$ in $\mathscr{C}$ ($\forall n\in\mathbb{N}$ and $\mathscr{J}_0\in\mathscr{C}$) defined by

$$\begin{aligned}
(\mathscr{J}_n)(x)&=px\mathscr{J}_{n-1}(\eta_1 x+1-\eta_1)+(1-p)x\mathscr{J}_{n-1}(\eta_2 x+1-\eta_2) \\
&\quad+p(1-x)\mathscr{J}_{n-1}(\eta_3 x+1-\eta_3) \\
&\quad+(1-p)(1-x)\mathscr{J}_{n-1}(\eta_4 x+1-\eta_4),
\end{aligned} \tag{17}$$

converges to the unique solution of (12).

5. A Certain Case with Experimenter-Subject-Controlled Events

It has been highlighted that the examination of any experiment is truly based on suppositions. Therefore, experiments are classified into contingent and noncontingent, based on the occurrences of the results. It has been suggested that the correspondence of contingent experiments is for the events of experimental-subject (contingent) and noncontingent experiments are for the events of experimental control.

In the previous models on imitation problems such as T-maze experiments with fish and dog (see [6, 9]), it was already mentioned that such experiments required a contingent approach; the result of the trials was entirely dependent on the subject's choice. Thus, such types of models required experimenter-subject-controlled events. The two responses $\mathbf{A}_1$ and $\mathbf{A}_2$, along with outcomes $\mathbf{O}_1$ and $\mathbf{O}_2$, are choosing the right or left side or pushing the right or left button, which coincides with rewarding and non-rewarding or correct and incorrect, respectively. Now we define the probabilities p_1 and p_2 which indicate the conditional probability of outcomes $\mathbf{O}_1$ and $\mathbf{O}_2$ of the given alternatives $\mathbf{A}_1$ and $\mathbf{A}_2$, respectively. With such conditions, we have the following Table 5.

We have the following functional equation from the data given above:

$$\begin{aligned}\mathscr{J}(x) = {} & p_1 x\mathscr{J}(\eta_1 x + 1 - \eta_1) + (1-p_1)x\mathscr{J}(\eta_2 x + 1 - \eta_2) \\ & + p_2(1-x)\mathscr{J}(\eta_3 x + 1 - \eta_3) \\ & + (1-p_2)(1-x)\mathscr{J}(\eta_4 x + 1 - \eta_4),\end{aligned} \tag{18}$$

where $\mathscr{J} : \mathcal{O} \longrightarrow \mathbb{R}$ is an unknown function, $0 < \eta_1, \eta_2, \eta_3, \eta_4 < 1$ and $p_1, p_2 \in \mathcal{O}$. We shall begin with the following finding.

Theorem 6. *For $0 < \eta_1, \eta_2, \eta_3, \eta_4 < 1$ and $p_1, p_2 \in \mathcal{O}$ with $\Theta_2 < 1$, where*

$$\Theta_2 \coloneqq [2p_1(\eta_1 - \eta_2) + 2p_2(\eta_3 - \eta_4) + 2(\eta_2 + \eta_4) + (p_1 + p_2)]. \tag{19}$$

Assume that, if there is a $\mathscr{C} \subseteq \mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$-invariant, that is, $\mathscr{W}(\mathscr{C}) \subseteq \mathscr{C}$, where $\mathscr{W} : \mathscr{C} \longrightarrow \mathscr{C}$ defined for each $\mathscr{J} \in \mathscr{C}$ as

$$\begin{aligned}(\mathscr{W}\mathscr{J})(x) = {} & p_1 x\mathscr{J}(\eta_1 x + 1 - \eta_1) + (1-p_1)(1-x)\mathscr{J}(\eta_2 x + 1 - \eta_2) \\ & + p_2(1-x)\mathscr{J}(\eta_3 x + 1 - \eta_3) \\ & + (1-p_2)(1-x)\mathscr{J}(\eta_4 x + 1 - \eta_4),\end{aligned} \tag{20}$$

for all $x \in \mathcal{O}$, then $\mathscr{W}$ is a BCM.

Proof. Let $\mathscr{J}_1, \mathscr{J}_2 \in \mathscr{C}$. For each distinct points $\omega, \varpi \in \mathcal{O}$, we obtain

$$\begin{aligned}
&\frac{|(\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2)(\omega) - (\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{W}_2)(\varpi)|}{|\omega - \varpi|} \\
&\quad = \left|\frac{1}{\omega-\varpi}[p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) + (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2)\right. \\
&\qquad + p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) + (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) \\
&\qquad - p_1\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - (1-p_1)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) \\
&\qquad \left. - p_2(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) - (1-p_2)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)]\right| \\
&\quad = \left|\frac{1}{\omega-\varpi}[p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)\right. \\
&\qquad + (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) - (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) \\
&\qquad + p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) \\
&\qquad + (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) - (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) \\
&\qquad + p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - p_1\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) \\
&\qquad + (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) - (1-p_1)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) \\
&\qquad + p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) - p_2(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)\eta_3\varpi + 1 - \eta_3) \\
&\qquad \left. + (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) - (1-p_2)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)]\right| \\
&\quad = \left|\frac{1}{\omega-\varpi}[p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)]\right. \\
&\qquad + \frac{1}{\omega-\varpi}[(1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) - (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)] \\
&\qquad + \frac{1}{\omega-\varpi}[p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)] \\
&\qquad + \frac{1}{\omega-\varpi}[(1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) \\
&\qquad - (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)] + \frac{1}{\omega-\varpi}[p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) \\
&\qquad - p_1\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)] + \frac{1}{\omega-\varpi}[(1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) \\
&\qquad - (1-p_1)\varpi(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)] + \frac{1}{\omega-\varpi}[p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) \\
&\qquad - p_2(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)] + \frac{1}{\omega-\varpi}[(1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) \\
&\qquad \left. - (1-p_2)(1-\varpi)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)]\right| \\
&\quad \le \frac{|p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\omega + 1 - \eta_1) - p_1\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)|}{|\eta_1\omega - \eta_1\varpi|} \times \frac{|\eta_1\omega - \eta_1\varpi|}{|\omega - \varpi|} \\
&\qquad + \frac{|(1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\omega + 1 - \eta_2) - (1-p_1)\omega(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)|}{|\eta_2\omega - \eta_2\varpi|} \times \frac{|\eta_2\omega - \eta_2\varpi|}{|\omega - \varpi|} \\
&\qquad + \frac{|p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\omega + 1 - \eta_3) - p_2(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)|}{|\eta_3\omega - \eta_3\varpi|} \times \frac{|\eta_3\omega - \eta_3\varpi|}{|\omega - \varpi|} \\
&\qquad + \frac{|(1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\omega + 1 - \eta_4) - (1-p_2)(1-\omega)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)|}{|\eta_4\omega - \eta_4\varpi|} \\
&\qquad \times \frac{|\eta_4\omega - \eta_4\varpi|}{|\omega - \varpi|} + |p_1(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1)| + |(1-p_1)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2)| \\
&\qquad + |p_2(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3)| + |(1-p_2)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4)|.
\end{aligned} \tag{21}$$

By applying the definition of the norm (5), we obtain

$$\begin{aligned}
&\frac{|(\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2)(\omega) - (\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2)(\varpi)|}{|\omega - \varpi|} \\
&\quad \le \eta_1 p_1\omega\|\mathscr{J}_1 - \mathscr{J}_2\| + \eta_2(1-p_1)\omega\|\mathscr{J}_1 - \mathscr{J}_2\| + \eta_3 p_2(1-\omega)\|\mathscr{J}_1 - \mathscr{J}_2\| \\
&\qquad + \eta_4(1-p_2)(1-\omega)\|\mathscr{J}_1 - \mathscr{J}_2\| + |p_1(\mathscr{J}_1 - \mathscr{J}_2)(\eta_1\varpi + 1 - \eta_1) - p_1(\mathscr{J}_1 - \mathscr{J}_2)(0)| \\
&\qquad + |(1-p_1)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_2\varpi + 1 - \eta_2) - (1-p_1)(\mathscr{J}_1 - \mathscr{J}_2)(1)| \\
&\qquad + |p_2(\mathscr{J}_1 - \mathscr{J}_2)(\eta_3\varpi + 1 - \eta_3) - p_2(\mathscr{J}_1 - \mathscr{J}_2)(0)| \\
&\qquad + |(1-p_2)(\mathscr{J}_1 - \mathscr{J}_2)(\eta_4\varpi + 1 - \eta_4) - (1-p_2)(\mathscr{J}_1 - \mathscr{J}_2)(1)| = \eta_1 p_1\omega\|\mathscr{J}_1 - \mathscr{J}_2\| \\
&\qquad + \eta_2(1-p_1)\omega\|\mathscr{J}_1 - \mathscr{J}_2\| + \eta_3 p_2(1-\omega)\|\mathscr{J}_1 - \mathscr{J}_2\| + \eta_4(1-p_2)(1-\omega)\|\mathscr{J}_1 - \mathscr{J}_2\| \\
&\qquad + p_1(\eta_1\varpi + 1 - \eta_1)\|\mathscr{J}_1 - \mathscr{J}_2\| + (1-p_1)(\eta_2\varpi + 1 - \eta_2)\|\mathscr{J}_1 - \mathscr{J}_2\| \\
&\qquad + p_2(\eta_3\varpi + 1 - \eta_3)\|\mathscr{J}_1 - \mathscr{J}_2\| + (1-p_2)(\eta_4\varpi + 1 - \eta_4)\|\mathscr{J}_1 - \mathscr{J}_2\| \le \Theta_2\|\mathscr{J}_1 - \mathscr{J}_2\|,
\end{aligned} \tag{22}$$

where Θ_2 is defined in (19). Thus, we have

$$d(\mathscr{W}\mathscr{J}_1, \mathscr{W}\mathscr{J}_2) = \|\mathscr{W}\mathscr{J}_1 - \mathscr{W}\mathscr{J}_2\| \le \Theta_2\|\mathscr{J}_1 - \mathscr{J}_2\| = \Theta_2 d(\mathscr{J}_1, \mathscr{J}_2). \tag{23}$$

As a result of $0 < \Theta_2 < 1$, one can see that $\mathscr{W}$ is a BCM.□□ □

For the unique solution of (18), we get the subsequent conclusion from Theorem 6.

Theorem 7. *The stochastic equation (18) has a unique solution with $\Theta_2 < 1$. Assume that, there is a $\mathscr{C} \subseteq \mathscr{D}$ such that $\mathscr{C}$ is $\mathscr{W}$ -invariant, that is, $\mathscr{W}(\mathscr{C}) \subseteq \mathscr{C}$, where $\mathscr{W} : \mathscr{C} \longrightarrow \mathscr{C}$ defined for each $\mathscr{J} \in \mathscr{C}$ as*

$$\begin{aligned}(\mathscr{W}\mathscr{J})(x) &= p_1 x \mathscr{J}(\eta_1 x + 1 - \eta_1) + (1 - p_1)(1 - x)\mathscr{J}(\eta_2 x + 1 - \eta_2) \\ &\quad + p_2(1 - x)\mathscr{J}(\eta_3 x + 1 - \eta_3) + (1 - p_2)(1 - x)\mathscr{J}(\eta_4 x + 1 - \eta_4),\end{aligned} \tag{24}$$

for all $x \in \mathcal{O}$. Furthermore, the iteration $\{\mathscr{J}_n\}$ in $\mathscr{C}$ ($\forall n \in \mathbb{N}$ and $\mathscr{J}_0 \in \mathscr{C}$) defined by

$$\begin{aligned}(\mathscr{J}_n)(x) &= p_1 x \mathscr{J}_{n-1}(\eta_1 x + 1 - \eta_1) + (1 - p_1)(1 - x)\mathscr{J}_{n-1}(\eta_2 x + 1 - \eta_2) \\ &\quad + p_2(1 - x)\mathscr{J}_{n-1}(\eta_3 x + 1 - \eta_3) \\ &\quad + (1 - p_2)(1 - x)\mathscr{J}_{n-1}(\eta_4 x + 1 - \eta_4),\end{aligned} \tag{25}$$

converges to the unique solution of (24).

Proof. The conclusion of this theorem can be found by combining Theorem 6 with the Banach fixed point theorem.□□ □

6. Conclusion

In this work, we have discussed a special type of stochastic process related to the two-armed bandit experiment [15] which plays a vital role in observing the subject's behavior in a two-choice situation. We reviewed the operant's responses under such conditions and provided a mathematical model for it. The Banach fixed point theorem was used to determine the existence of a unique solution to the two-armed bandit learning model. We investigated the proposed model's adaptability by subjecting it to some controlled events. Moreover, the presented approach is straightforward and easy to verifiable. Thus, the proposed approach can be used to investigate more psychological learning experiments related to animals and humans in the future.

Now, for the interested readers, we propose the following open problems.

Question 1. Assume that if a subject does not press any button on a specific trial k, how can we describe such an event by a model?

In the end, we also leave the stability problem (for the detail, see [23–27]) of the stochastic equation given below as an open problem:

$$\begin{aligned}\mathscr{J}(x) &= px\mathscr{J}(\eta_1 x + (1 - \eta_1)) + (1 - p)x\mathscr{J}(\eta_2 x + (1 - \eta_2)) \\ &\quad + p(1 - x)\mathscr{J}(\eta_3 x + (1 - \eta_3)) \\ &\quad + (1 - p)(1 - x)\mathscr{J}(\eta_4 x + (1 - \eta_4)),\end{aligned} \tag{26}$$

where $0 < \eta_1, \eta_2, \eta_3, \eta_4 < 1$ and $\mathscr{J} : \mathcal{O} \longrightarrow \mathbb{R}$ is an unknown function.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflict of interest.

Authors' Contributions

All authors contributed equally to the manuscript and typed, read, and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. (UJ-21-DR-93). The authors, therefore, acknowledge with thanks the university technical and financial support.

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Hindawi
Journal of Function Spaces
Volume 2022, Article ID 5862251, 19 pages
https://doi.org/10.1155/2022/5862251

Research Article

An Approach of Integral Equations in Complex-Valued b-Metric Space Using Commuting Self-Maps

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Received 19 September 2021; Accepted 29 November 2021; Published 7 January 2022

Academic Editor: Santosh Kumar

This paper is aimed at establishing some unique common fixed point theorems in complex-valued b-metric space under the rational type contraction conditions for three self-mappings in which the one self-map is continuous. A continuous self-map is commutable with the other two self-mappings. Our results are verified by some suitable examples. Ultimately, our results have been utilized to prove the existing solution to the two Urysohn integral type equations. This application illustrates how complex-valued b-metric space can be used in other types of integral operators.

1. Introduction

In 1922, Banach [1] proved a fixed point theorem (FP-theorem), which is stated as the following: "a single-valued contractive type mapping on a complete metric space has a unique fixed point." After the publication of the Banach FP-theorem, many researchers have contributed their ideas to the theory of FP. Chandok [2, 3], Jungck and Rhoades [4], and Al-Shami and Abo-Tabl [5, 6] proved different contractive types of FPs and a-fixed soft point results in the context of metric spaces.

Bakhtin [7] introduced the idea of b-metric space, while Czerwik [8] proved some fixed point results for nonlinear set-valued contractive type mappings in b-metric spaces. Suzuki in [9] proved basic inequality and some FP-theorems. Jain and Kaur [10] presented a new class of functions to define new contractive maps and established FP-results for these maps. They also extended some results in the framework of b-metric-like spaces. They presented examples and established the application of their main results. They also presented some open problems. Petrusel et al. [11] considered coupled FP-problems for single-valued operators satisfying contraction in said space. They discussed uniqueness, data dependence, and shadowing-property of coupled FP-problem and also established an application for main results. Ameer et al. [12], Boriceanu [13, 14], Bota et al. [15], Czerwik et al. [16, 17], Hussain and Shah [18], Karapinar et al. [19], and Samreen et al. [20] established different contractive type FP and common FP (CFP) results in the context of b-metric spaces.

In 2011, the concept of complex-valued metric space was given by Azam et al. [21], and they proved some CFP-theorems for self-mappings. The notion of said space was proposed by Rouzkard and Imdad [22] which generalizes the results of Azam et al. [21] and established some CFP-results. Abbas et al. [23] presented some generalized CFP-results by using cocyclic mappings in complex-valued metric space. They provided examples to indicate the authenticity of his expressions. Sarwar and Zada [24] used the ideas of $(E.A)$ and (CLR) properties and proved

FP-results for six self-mappings. They showed the existence of their results by establishing some examples. Abbas et al. [25], Nashine et al. [26], Mohanta and Maitra [27], Sintunavarat and Kumam [28], and Verma and Pathak [29] proved some results in the context of complex-valued metric space.

In 2013, Rao et al. [30] introduced the notion of complex-valued b-metric space. Mukheimer et al. [31] established CFP-results on said space by extending and generalizing the results of [30, 31]. In [32], Chantakun et al. extend the work of Dubey et al. [33] by introducing sufficient conditions to prove some CFP-results in complex-valued b-metric space. Yadav et al. [34] used compatible and weakly compatible maps to find CFP-results. They proved the validity of the results by providing some examples and establishing an application. Berrah et al. [35], Hasana [36], Mehmood et al. [37], and Mukheimer [38] established some FP and CFP theorems in complex-valued b-metric spaces.

In this paper, we provide some extended and effective CFP-results for commuting three self-maps on complex-valued b-metric spaces. To verify the validity of our work, we present some illustrative examples in the main section. Further, our results have been utilized to prove the existing solution to the two Urysohn integral type equations. This application is also illustrative of how complex-valued b-metric space can be used in other integral type operators. This paper is organized as follows: In Section 2, we present the preliminary concepts. In Section 3, we establish some extended and modified CFP-results for commuting self-maps in complex-valued b-metric space under the generalized rational type conditions. We also provide authentic examples to indicate the effectiveness of these results. In Section 4, we present an application of the two UITEs to support our main work. Finally, in Section 5, we discuss the conclusion.

2. Preliminaries

Let $\mathbb{C}$ be the set of all complex numbers and $z_i, z_{ii} \in \mathbb{C}$. Define $\leq$ as $z_i \leq z_{ii}$, iff $R_e(z_i) \leq R_e(z_{ii})$ and $I_m(z_i) \leq I_m(z_{ii})$, where R_e denotes the real part and I_m denotes the imaginary part of a complex number. Accordingly, $z_i \leq z_{ii}$, if any one of the following conditions holds:

$(C_1) R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$
$(C_2) R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) = I_m(z_{ii})$
$(C_3) R_e(z_i) = R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$
$(C_4) R_e(z_i) < R_e(z_{ii})$ and $I_m(z_i) < I_m(z_{ii})$

Know that $z_i \lneqq z_{ii}$ if $z_i \neq z_{ii}$ and one of (C_2), (C_3), and (C_4) is satisfied.

Remark 1 (see [31]). We can easily check the following:

(i) If $a_1, a_2 \in \mathbb{R}$ and $a_1 \leq a_2 \Rightarrow a_1 y \leq a_2 y$, $\forall y \in \mathbb{C}$

(ii) $0 \leq z_i \lneqq z_{ii} \Rightarrow |z_i| < |z_{ii}|$

(iii) $z_i \leq z_{ii}$ and $z_{ii} < z_{iii} \Rightarrow z_i < z_{iii}$

Definition 2 (see [8]). Let Ω be a nonempty set and $b \geq 1$ a given real number. A mapping $\eth : \Omega \times \Omega \longrightarrow [0,\infty)$ *is* called a b-metric on Ω if the following conditions are satisfied:

$(b_m 1) \eth(\rho_1, \rho_2) = 0$ if and only if $\rho_1 = \rho_2$
$(b_m 2) \eth(\rho_1, \rho_2) = \eth(\rho_2, \rho_1)$
$(b_m 3) \eth(\rho_1, \rho_2) \leq b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$,

for all $\rho_1, \rho_2, \rho_3 \in \Omega$. Then, $(\Omega, \eth)$ is called a b-metric space.

Definition 3 (see [30]). Let Ω be a nonempty set and $b \geq 1$ a given real number. A mapping $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ *is* called a complex-valued b-metric on Ω if the following conditions are satisfied:

$(Cb_m 1) \eth(\rho_1, \rho_2) \geq 0$ and $\eth(\rho_1, \rho_2) = 0$ if and only if $\rho_1 = \rho_2$
$(Cb_m 2) \eth(\rho_1, \rho_2) = \eth(\rho_2, \rho_1)$
$(Cb_m 3) \eth(\rho_1, \rho_2) \leq b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$,

for all $\rho_1, \rho_2, \rho_3 \in \Omega$. Then, $(\Omega, \eth)$ is called a complex-valued b-metric space.

Example 4. Let $\Omega = R^+$. Define the mapping $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ *by* $\eth(\rho_1, \rho_2) = 7/17|\rho_1 - \rho_2|^2 + i7/17|\rho_1 - \rho_2|^2$, for all $\rho_1, \rho_2 \in \Omega$.

Then, $(\Omega, \eth)$ is a complex-valued b-metric space with $b = 2$.

Definition 5 (see [30, 31]). Let $(\Omega, \eth)$ be a complex-valued b-metric space and $\{\rho_n\}$ a sequence in Ω and $\rho \in \Omega$. Then,

(1) $\{\rho_n\}$ is said to converge to ρ if for every $0 < c^* \in \mathbb{C}$ there exists $N^* \in \mathbb{N}$ such that $\eth(\rho_n, \rho) < c^*$, $\forall n > N^*$. We denote this by $\lim_{n \longrightarrow \infty} \rho_n = \rho$ or $\{\rho_n\} \longrightarrow \rho$ as $n \longrightarrow \infty$

(2) if for every $0 < c^* \in \mathbb{C}$ there exists $N^* \in \mathbb{N}$ such that $\eth(\rho_n, \rho_{n+m}) < c^*$ for all $n > N^*$, $m \in \mathbb{N}$, then $\{\rho_n\}$ is called a Cauchy sequence

(3) if every Cauchy sequence is convergent, then $(\Omega, \eth)$ is called a complete complex-valued b-metric space

Lemma 6 (see [30, 31]). *Let $(\Omega, \eth)$ be a complex-valued b-metric space and let $\{\rho_n\}$ be a sequence in Ω. Then, $\{\rho_n\}$ converges to ρ iff $|\eth(\rho_n, \rho)| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Lemma 7 (see [30, 31]). *Let $(\Omega, \eth)$ be a complex-valued b-metric space and let $\{\rho_n\}$ be a sequence in Ω. Then, $\{\rho_n\}$ is a Cauchy sequence iff $|\eth(\rho_n, \rho_{n+m})| \longrightarrow 0$ as $n \longrightarrow \infty$.*

Definition 8 (see [39]). Let $(\Omega, \eth)$ be a complex-valued b-metric space. The self-mappings f_1 and f_2 are said to be commuting if $f_1 f_2 \rho = f_2 f_1 \rho$ for all $\rho \in \Omega$.

3. Main Result

In this section, we prove some CFP theorems in complex-valued g-metric space under the generalized rational type contraction conditions for three self-mappings in which one is continuous. We present some examples for the validation of our work.

Theorem 9. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\begin{aligned}\eth(f_1\rho_1, f_2\rho_2) &\leq \kappa_1 \eth(f\rho_1, f\rho_2) \\ &+ \kappa_2 \frac{\eth(f\rho_1, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_2, f_2\rho_2)}{1+\eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_2, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}\end{array}\right\},\end{aligned} \tag{1}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$ and $b \geq 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\begin{aligned}\Gamma_{2n} &= f\rho_{2n+1} = f_1\rho_{2n}, \\ \Gamma_{2n+1} &= f\rho_{2n+2} = f_2\rho_{2n+1}, \\ &\forall n \geq 0.\end{aligned} \tag{2}$$

□

Now, by using (1),

$$\begin{aligned}\eth(\Gamma_{2n}, \Gamma_{2n+1}) &= \eth(f_1\rho_{2n}, f_2\rho_{2n+1}) \leq \kappa_1\eth(f\rho_{2n}, f\rho_{2n+1}) \\ &+ \kappa_2 \frac{\eth(f\rho_{2n}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_{2n}, f_1\rho_{2n}), \eth(f\rho_{2n+1}, f_2\rho_{2n+1}), \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(f\rho_{2n}, f\rho_{2n+1})}, \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n+1})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}, \\ \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}\end{array}\right\} \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n}) + \kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n}, \Gamma_{2n+1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n}, \Gamma_{2n+1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}\end{array}\right\}.\end{aligned} \tag{3}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n}, \Gamma_{2n+1})| &\leq \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})| \\ &+ \kappa_2 \frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|)} \\ &+ \kappa_3 \min\left\{\begin{array}{c}|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, |\eth(\Gamma_{2n}, \Gamma_{2n+1})|, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n+1})|}{|1+\eth(\Gamma_{2n-1}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}\end{array}\right\}.\end{aligned} \tag{4}$$

After simplification, we get that

$$|\eth(\Gamma_{2n}, \Gamma_{2n+1})| \leq \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})|. \tag{5}$$

Again, by using (1) and (2),

$$\begin{aligned}\eth(\Gamma_{2n-1}, \Gamma_{2n}) &= \eth(f_2\rho_{2n-1}, f_1\rho_{2n}) = \eth(f_1\rho_{2n}, f_2\rho_{2n-1}) \leq \kappa_1\eth(f\rho_{2n}, f\rho_{2n-1}) \\ &+ \kappa_2 \frac{\eth(f\rho_{2n}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(f\rho_{2n}, f_1\rho_{2n}), \eth(f\rho_{2n-1}, f_2\rho_{2n-1}), \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n-1}, f_2\rho_{2n-1})}{1+\eth(f\rho_{2n}, f\rho_{2n-1})}, \\ \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n-1})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}, \\ \dfrac{\eth(f\rho_{2n-1}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}\end{array}\right\} \\ &= \kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n-2}) + \kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n}))} \\ &+ \kappa_3 \min\left\{\begin{array}{c}\eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n-2}, \Gamma_{2n-1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n-1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n-2})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n-1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-2}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}\end{array}\right\}.\end{aligned} \tag{6}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n-1}, \Gamma_{2n})| &\leq \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n-2})| \\ &+ \kappa_2 \frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|)} \\ &+ \kappa_3 \min\left\{\begin{array}{c}|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, |\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|}{|1+\eth(\Gamma_{2n-1}, \Gamma_{2n-2})|}, \\ \dfrac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}, \\ \dfrac{|\eth(\Gamma_{2n-2}, \Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n-1})|+|\eth(\Gamma_{2n-2}, \Gamma_{2n})|}\end{array}\right\}.\end{aligned} \tag{7}$$

After simplification, we get that

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le \kappa_1|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|. \tag{8}$$

Now, from (8) and (5) and by induction, we have that

$$\begin{aligned}|\eth(\Gamma_{2n},\Gamma_{2n+1})| &\le \kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n})|\\ &\le \kappa_1^2|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \le \cdots \le \kappa_1^{2n}|\eth(\Gamma_0,\Gamma_1)|.\end{aligned} \tag{9}$$

So, for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}|\eth(\Gamma_n,\Gamma_m)| &\le b|\eth(\Gamma_n,\Gamma_{n+1})| + b|\eth(\Gamma_{n+1},\Gamma_m)| \le b|\eth(\Gamma_n,\Gamma_{n+1})|\\ &\quad + b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})|+\cdots+b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)|\\ &\le b\kappa_1^n|\eth(\Gamma_0,\Gamma_1)| + b^2\kappa_1^{n+1}|\eth(\Gamma_0,\Gamma_1)|+\cdots\\ &\quad +b^{m-n}\kappa_1^{m-1}|\eth(\Gamma_0,\Gamma_1)|\\ &\le \left[b\kappa_1^n + b^2\kappa_1^{n+1}+\cdots+b^{m-n}\kappa_1^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)|\\ &= \left[b\kappa_1^n + b^2\kappa_1^{n+1}+\cdots+b^{m-n}\kappa_1^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)|\\ &= b\kappa_1^n\left[1 + b\kappa_1 + b^2\kappa_1^2 \cdots +b^{m-(n+1)}\kappa_1^{m-(n+1)}\right]\\ &\quad \cdot|\eth(\Gamma_0,\Gamma_1)| = b\kappa_1^n\sum_{t=0}^{m-(n+1)} b^t\kappa_1^t|\eth(\Gamma_0,\Gamma_1)|\\ &\le b\kappa_1^n\sum_{t=0}^{\infty} b^t\kappa_1^t|\eth(\Gamma_0,\Gamma_1)|\\ &= \frac{b\kappa_1^n}{1-b\kappa_1}|\eth(\Gamma_0,\Gamma_1)| \longrightarrow 0, \text{as } n \longrightarrow \infty.\end{aligned} \tag{10}$$

Therefore, the sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s \in \Omega$ such that $\Gamma_n \longrightarrow s$, as $n \longrightarrow \infty$, or $\lim_{n\longrightarrow\infty}\Gamma_n = s$, and from (2), we have

$$\begin{aligned}\lim_{n\longrightarrow\infty} f\rho_{2n+1} &= s,\\ \lim_{n\longrightarrow\infty} f_1\rho_{2n} &= s,\\ \lim_{n\longrightarrow\infty} f_2\rho_{2n+1} &= s.\end{aligned} \tag{11}$$

As f is continuous, so

$$\begin{aligned}\lim_{n\longrightarrow\infty} f(f\rho_{2n+1}) &= fs,\\ \lim_{n\longrightarrow\infty} f(f_1\rho_{2n}) &= fs,\\ \lim_{n\longrightarrow\infty} f(f_2\rho_{2n+1}) &= fs.\end{aligned} \tag{12}$$

Since, (f, f_1) and (f, f_2) are commutable pairs, therefore, from (12), we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty} f_1(f\rho_{2n}) &= fs,\\ \lim_{n\longrightarrow\infty} f_2(f\rho_{2n+1}) &= fs.\end{aligned} \tag{13}$$

Now, we have to show that $fs = s$, so by putting $\rho_1 = f\rho_{2n}$ and $\rho_2 = \rho_{2n+1}$, in (1),

$$\begin{aligned}\eth(f_1(f\rho_{2n}), f_2\rho_{2n+1}) &\le \kappa_1\eth(f(f\rho_{2n}), f\rho_{2n+1})\\ &\quad + \kappa_2\frac{\eth(f(f\rho_{2n}), f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n})))}\\ &\quad + \kappa_3\min\left\{\begin{array}{c}\eth(f(f\rho_{2n}), f_1(f\rho_{2n})), \eth(f\rho_{2n+1}, f_2\rho_{2n+1}),\\ \dfrac{\eth(f(f\rho_{2n}), f_1(f\rho_{2n}))\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}), f\rho_{2n+1})},\\ \dfrac{\eth(f(f\rho_{2n}), f_1(f\rho_{2n}))\cdot\eth(f(f\rho_{2n}), f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))},\\ \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}), f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1(f\rho_{2n}))}\end{array}\right\}.\end{aligned} \tag{14}$$

This implies that

$$\begin{aligned}|\eth(f_1(f\rho_{2n}), f_2\rho_{2n+1})| &\le \kappa_1|\eth(f(f\rho_{2n}), f\rho_{2n+1})|\\ &\quad + \kappa_2\frac{|\eth(f(f\rho_{2n}), f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}), f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))|)}\\ &\quad + \kappa_3\min\left\{\begin{array}{c}|\eth(f(f\rho_{2n}), f_1(f\rho_{2n}))|, |\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|,\\ \dfrac{|\eth(f(f\rho_{2n}), f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}), f\rho_{2n+1})|},\\ \dfrac{|\eth(f(f\rho_{2n}), f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}), f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}), f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))|},\\ \dfrac{|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}), f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1(f\rho_{2n}))|}\end{array}\right\}.\end{aligned} \tag{15}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (11), (12), and (13), we get that

$$\begin{aligned}|\eth(fs, s)| &\le \kappa_1|\eth(fs, s)| + \kappa_2\frac{|\eth(fs, s)|\cdot|\eth(s, fs)|}{1/2(|\eth(fs, s)| + |\eth(s, fs)|)}\\ &\quad + \kappa_3\min\left\{\begin{array}{c}|\eth(fs, fs)|, |\eth(s, s)|,\\ \dfrac{|\eth(fs, fs)|\cdot|\eth(s, s)|}{|1+\eth(fs, s)|},\\ \dfrac{|\eth(fs, fs)|\cdot|\eth(fs, s)|}{|\eth(fs, s)| + |\eth(s, fs)|},\\ \dfrac{|\eth(s, s)|\cdot|\eth(s, fs)|}{|\eth(fs, s)|+|\eth(s, fs)|}\end{array}\right\}.\end{aligned} \tag{16}$$

After simplification, we get that

$$|\eth(fs, s)| \le (\kappa_1 + \kappa_2)|\eth(fs, s)| \Rightarrow (1 - \kappa_1 - \kappa_2)|\eth(fs, s)| \le 0. \tag{17}$$

Since $(1-\kappa_1-\kappa_2)\neq 0 \Rightarrow |\eth(fs,s)|=0$, hence, we get that

$$fs=s. \tag{18}$$

Next, we have to show that $f_1 s=s$, by the view of (1),

$$\begin{aligned}\eth(f_1s,f\rho_{2n+2})=\eth(f_1s,f_2\rho_{2n+1})&\le\kappa_1\eth(fs,f\rho_{2n+1})\\&+\kappa_2\frac{\eth(fs,f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{1/2(\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s))}\\&+\kappa_3\min\left\{\begin{array}{c}\eth(fs,f_1s),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\ \dfrac{\eth(fs,f_1s)\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(fs,f\rho_{2n+1})},\\ \dfrac{\eth(fs,f_1s)\cdot\eth(fs,f_2\rho_{2n+1})}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)},\\ \dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)}\end{array}\right\}.\end{aligned}\tag{19}$$

This implies that

$$\begin{aligned}|\eth(f_1s,f\rho_{2n+2})|&\le\kappa_1|\eth(fs,f\rho_{2n+1})|\\&+\kappa_2\frac{|\eth(fs,f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{1/2(|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|)}\\&+\kappa_3\min\left\{\begin{array}{c}|\eth(fs,f_1s)|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(fs,f\rho_{2n+1})|},\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(fs,f_2\rho_{2n+1})|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|}\end{array}\right\}.\end{aligned}\tag{20}$$

Now, again applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (11) and (18), we have that

$$\begin{aligned}|\eth(f_1s,s)|&\le\kappa_1|\eth(s,s)|+\kappa_2\frac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{1/2(|\eth(s,s)|+|\eth(s,f_1s)|)}\\&+\kappa_3\min\left\{\begin{array}{c}|\eth(s,f_1s)|,|\eth(s,s)|,\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|\eth(s,s)|+|\eth(s,f_1s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{|\eth(s,s)|+|\eth(s,f_1s)|}\end{array}\right\}.\end{aligned}\tag{21}$$

This implies that $|\eth(f_1s,s)|\le 0$. Hence,

$$f_1s=s. \tag{22}$$

Now, we have to show that $f_2s=s$, by using (1),

$$\begin{aligned}\eth(f\rho_{2n+1},f_2s)=\eth(f_1\rho_{2n},f_2s)&\le\kappa_1\eth(f\rho_{2n},fs)\\&+\kappa_2\frac{\eth(f\rho_{2n},f_2s)\cdot\eth(fs,f_1\rho_{2n})}{1/2(\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n}))}\\&+\kappa_3\min\left\{\begin{array}{c}\eth(f\rho_{2n},f_1\rho_{2n}),\eth(fs,f_2s),\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(fs,f_2s)}{1+\eth(f\rho_{2n},fs)},\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(f\rho_{2n},f_2s)}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})},\\ \dfrac{\eth(fs,f_2s)\cdot\eth(fs,f_1\rho_{2n})}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})}\end{array}\right\}.\end{aligned}\tag{23}$$

This implies that

$$\begin{aligned}|\eth(f\rho_{2n+1},f_2s)|&\le\kappa_1|\eth(f\rho_{2n},fs)|\\&+\kappa_2\frac{|\eth(f\rho_{2n},f_2s)|\cdot|\eth(fs,f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|)}\\&+\kappa_3\min\left\{\begin{array}{c}|\eth(f\rho_{2n},f_1\rho_{2n})|,|\eth(fs,f_2s)|,\\ \dfrac{|\eth(f\rho_{2n},f_1\rho_{2n})|\cdot|\eth(fs,f_2s)|}{|1+\eth(f\rho_{2n},fs)|},\\ \dfrac{|\eth(f\rho_{2n},f_1\rho_{2n})|\cdot|\eth(f\rho_{2n},f_2s)|}{|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|},\\ \dfrac{|\eth(fs,f_2s)|\cdot|\eth(fs,f_1\rho_{2n})|}{|\eth(f\rho_{2n},f_2s)|+|\eth(fs,f_1\rho_{2n})|}\end{array}\right\}.\end{aligned}\tag{24}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (11) and (18), we get

$$\begin{aligned}|\eth(s,f_2s)|&\le\kappa_1|\eth(s,s)|+\kappa_2\frac{|\eth(s,f_2s)|\cdot|\eth(s,s)|}{1/2(|\eth(s,f_2s)|+|\eth(s,s)|)}\\&+\kappa_3\min\left\{\begin{array}{c}|\eth(s,s)|,|\eth(s,f_2s)|,\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_2s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_2s)|}{|\eth(s,f_2s)|+|\eth(s,s)|},\\ \dfrac{|\eth(s,f_2s)|\cdot|\eth(s,s)|}{|\eth(s,f_2s)|+|\eth(s,s)|}\end{array}\right\}.\end{aligned}\tag{25}$$

This implies that $|\eth(s,f_2s)|\le 0$. Hence,

$$f_2s=s. \tag{26}$$

Thus, from (18), (22), and (26), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs=f_1s=f_2s=s. \tag{27}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{28}$$

Then, from (1), we have that

$$\eth(s, s^*) = \eth(f_1 s, f_2 s^*) \le \kappa_1 \eth(fs, fs^*) + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} + \kappa_3 \min \left\{ \begin{array}{c} \eth(fs, f_1 s), \eth(fs^*, f_2 s^*), \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)}, \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)}, \\ \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right\} = (\kappa_1 + \kappa_2) \eth(s, s^*). \tag{29}$$

This implies that $|\eth(s, s^*)| \le (\kappa_1 + \kappa_2) |\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2) |\eth(s, s^*)| \le 0$. Since $(1 - \kappa_1 - \kappa_2) \ne 0 \Rightarrow |\eth(s, s^*)| = 0 \Rightarrow s = s^*$. Hence, prove that f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_3 = 0$ in Theorem 9, we get the following corollary.

Corollary 10. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1 \rho_1, f_2 \rho_2) \le \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_2 \frac{\eth(f \rho_1, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{1/2(\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1))}, \tag{30}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$ and $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_2 = 0$ in Theorem 9, we can get the following corollary.

Corollary 11. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying:*

$$\eth(f_1 \rho_1, f_2 \rho_2) \le \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_3 \min \left\{ \begin{array}{c} \eth(f \rho_1, f_1 \rho_1), \eth(f \rho_2, f_2 \rho_2), \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_2, f_2 \rho_2)}{1 + \eth(f \rho_1, f \rho_2)}, \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_1, f_2 \rho_2)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)}, \\ \dfrac{\eth(f \rho_2, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)} \end{array} \right\}, \tag{31}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_3 \in [0, 1)$ and $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Theorem 12. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1 \rho_1, f_2 \rho_2) \le \kappa_1 \eth(f \rho_1, f \rho_2) + \kappa_2 \frac{\eth(f \rho_1, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{1/2(\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1))} + \kappa_3 \max \left\{ \begin{array}{c} \eth(f \rho_1, f_1 \rho_1), \eth(f \rho_2, f_2 \rho_2), \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_2, f_2 \rho_2)}{1 + \eth(f \rho_1, f \rho_2)}, \\ \dfrac{\eth(f \rho_1, f_1 \rho_1) \cdot \eth(f \rho_1, f_2 \rho_2)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)}, \\ \dfrac{\eth(f \rho_2, f_2 \rho_2) \cdot \eth(f \rho_2, f_1 \rho_1)}{\eth(f \rho_1, f_2 \rho_2) + \eth(f \rho_2, f_1 \rho_1)} \end{array} \right\}, \tag{32}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\begin{aligned} \Gamma_{2n} &= f \rho_{2n+1} = f_1 \rho_{2n}, \\ \Gamma_{2n+1} &= f \rho_{2n+2} = f_2 \rho_{2n+1}, \\ &\forall n \ge 0. \end{aligned} \tag{33}$$

□

Now, by using (32),

$$\begin{aligned} \eth(\Gamma_{2n}, \Gamma_{2n+1}) = \eth(f_1 \rho_{2n}, f_2 \rho_{2n+1}) &\le \kappa_1 \eth(f \rho_{2n}, f \rho_{2n+1}) + \kappa_2 \frac{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) \cdot \eth(f \rho_{2n+1}, f_1 \rho_{2n})}{1/2(\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n}))} \\ &+ \kappa_3 \max \left\{ \begin{array}{c} \eth(f \rho_{2n}, f_1 \rho_{2n}), \eth(f \rho_{2n+1}, f_2 \rho_{2n+1}), \\ \dfrac{\eth(f \rho_{2n}, f_1 \rho_{2n}) \cdot \eth(f \rho_{2n+1}, f_2 \rho_{2n+1})}{1 + \eth(f \rho_{2n}, f \rho_{2n+1})}, \\ \dfrac{\eth(f \rho_{2n}, f_1 \rho_{2n}) \cdot \eth(f \rho_{2n}, f_2 \rho_{2n+1})}{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n})}, \\ \dfrac{\eth(f \rho_{2n+1}, f_2 \rho_{2n+1}) \cdot \eth(f \rho_{2n+1}, f_1 \rho_{2n})}{\eth(f \rho_{2n}, f_2 \rho_{2n+1}) + \eth(f \rho_{2n+1}, f_1 \rho_{2n})} \end{array} \right\} \\ &= \kappa_1 \eth(\Gamma_{2n-1}, \Gamma_{2n}) + \kappa_2 \frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n}))} \\ &+ \kappa_3 \max \left\{ \begin{array}{c} \eth(\Gamma_{2n-1}, \Gamma_{2n}), \eth(\Gamma_{2n}, \Gamma_{2n+1}), \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n+1})}{1 + \eth(\Gamma_{2n-1}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n}) \cdot \eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})}, \\ \dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1}) \cdot \eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1}) + \eth(\Gamma_{2n}, \Gamma_{2n})} \end{array} \right\}. \end{aligned} \tag{34}$$

This implies that,

$$
\begin{aligned}
|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le{}& \kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n})| \\
&+\kappa_2\frac{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n},\Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1},\Gamma_{2n+1})|+|\eth(\Gamma_{2n},\Gamma_{2n})|)} \\
&+\kappa_3\max\left\{\begin{array}{c}|\eth(\Gamma_{2n-1},\Gamma_{2n})|,|\eth(\Gamma_{2n},\Gamma_{2n+1})|,\\[4pt] \dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n},\Gamma_{2n+1})|}{|1+\eth(\Gamma_{2n-1},\Gamma_{2n})|},\\[8pt] \dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1},\Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})|+|\eth(\Gamma_{2n},\Gamma_{2n})|},\\[8pt] \dfrac{|\eth(\Gamma_{2n},\Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n},\Gamma_{2n})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n+1})|+|\eth(\Gamma_{2n},\Gamma_{2n})|}\end{array}\right\}.
\end{aligned}
\tag{35}
$$

After simplification, we get that

$$
\begin{aligned}
|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le{}& \kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n})| \\
&+\kappa_3\max\{|\eth(\Gamma_{2n-1},\Gamma_{2n})|,|\eth(\Gamma_{2n},\Gamma_{2n+1})|\}.
\end{aligned}
\tag{36}
$$

Now, there are two possibilities:

(i) If $\eth(\Gamma_{2n-1},\Gamma_{2n})$ is a maximum term in $\{|\eth(\Gamma_{2n-1},\Gamma_{2n})|,|\eth(\Gamma_{2n},\Gamma_{2n+1})|\}$, then after simplification, (36) can be written as

$$
|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g_1|\eth(\Gamma_{2n-1},\Gamma_{2n})|, \text{where } g_1=\kappa_1+\kappa_3<1
\tag{37}
$$

(ii) If $\eth(\Gamma_{2n},\Gamma_{2n+1})$ is a maximum term in $\{\eth(\Gamma_{2n-1},\Gamma_{2n})|,|\eth(\Gamma_{2n},\Gamma_{2n+1})|\}$, then after simplification, (36) can be written as

$$
|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g_2|\eth(\Gamma_{2n-1},\Gamma_{2n})|, \text{where } g_2=\frac{\kappa_1}{1-\kappa_3}<1
\tag{38}
$$

Let $g:=\max\{g_1,g_2\}<1$, then from (37) and (38), for all $n\ge 0$, we have

$$
|\eth(\Gamma_{2n},\Gamma_{2n+1})| \le g|\eth(\Gamma_{2n-1},\Gamma_{2n})|.
\tag{39}
$$

Similarly,

$$
|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|.
\tag{40}
$$

Now, from (40) and (39) and by induction, we have that

$$
\begin{aligned}
|\eth(\Gamma_{2n},\Gamma_{2n+1})| &\le g|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g^2|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \\
&\le \cdots \le g^{2n}|\eth(\Gamma_0,\Gamma_1)|.
\end{aligned}
\tag{41}
$$

Now, for $m,n\in\mathbb{N}$ with $m>n$,

$$
\begin{aligned}
|\eth(\Gamma_n,\Gamma_m)| &\le b|\eth(\Gamma_n,\Gamma_{n+1})|+b|\eth(\Gamma_{n+1},\Gamma_m)| \le b|\eth(\Gamma_n,\Gamma_{n+1})| \\
&\quad+b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})|+\cdots+b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)| \\
&\le bg^n|\eth(\Gamma_0,\Gamma_1)|+b^2g^{n+1}|\eth(\Gamma_0,\Gamma_1)|+\cdots+b^{m-n}g^{m-1}|\eth(\Gamma_0,\Gamma_1)| \\
&\le \left[bg^n+b^2g^{n+1}+\cdots+b^{m-n}g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\
&= \left[bg^n+b^2g^{n+1}+\cdots+b^{m-n}g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\
&= bg^n\left[1+bg+b^2g^2\cdots+b^{m-(n+1)}g^{m-(n+1)}\right]|\eth(\Gamma_0,\Gamma_1)| \\
&= bg^n\sum_{t=0}^{m-(n+1)}b^tg^t|\eth(\Gamma_0,\Gamma_1)| \le bg^n\sum_{t=0}^{\infty}b^tg^t|\eth(\Gamma_0,\Gamma_1)| \\
&= \frac{bg^n}{1-bg}|\eth(\Gamma_0,\Gamma_1)| \longrightarrow 0, \quad \text{as } n\longrightarrow\infty.
\end{aligned}
\tag{42}
$$

Therefore, sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s\in\Omega$ such that $\Gamma_n\longrightarrow s$, as $n\longrightarrow\infty$, or $\lim_{n\longrightarrow\infty}\Gamma_n=s$, and from (33), we have

$$
\begin{aligned}
\lim_{n\longrightarrow\infty} f\rho_{2n+1} &= s, \\
\lim_{n\longrightarrow\infty} f_1\rho_{2n} &= s, \\
\lim_{n\longrightarrow\infty} f_2\rho_{2n+1} &= s.
\end{aligned}
\tag{43}
$$

As f is continuous, so

$$
\begin{aligned}
\lim_{n\longrightarrow\infty} f(f\rho_{2n+1}) &= fs, \\
\lim_{n\longrightarrow\infty} f(f_1\rho_{2n}) &= fs, \\
\lim_{n\longrightarrow\infty} f(f_2\rho_{2n+1}) &= fs.
\end{aligned}
\tag{44}
$$

Since, (f,f_1) and (f,f_2) are commutable pairs, therefore, from (44), we have that

$$
\begin{aligned}
\lim_{n\longrightarrow\infty} f_1(f\rho_{2n}) &= fs, \\
\lim_{n\longrightarrow\infty} f_2(f\rho_{2n+1}) &= fs.
\end{aligned}
\tag{45}
$$

Now, we have to show that $fs=s$, so by putting $\rho_1=f\rho_{2n}$ and $\rho_2=\rho_{2n+1}$, in (32):

$$
\begin{aligned}
\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1}) \le{}& \kappa_1\eth(f(f\rho_{2n}),f\rho_{2n+1}) \\
&+\kappa_2\frac{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n})))} \\
&+\kappa_3\max\left\{\begin{array}{c}\eth(f(f\rho_{2n}),f_1(f\rho_{2n})),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\[4pt] \dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}),f\rho_{2n+1})},\\[8pt] \dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f(f\rho_{2n}),f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))},\\[8pt] \dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}\end{array}\right\}.
\end{aligned}
\tag{46}
$$

This implies that

$$\begin{aligned}|\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})| &\le \kappa_1|\eth(f(f\rho_{2n}),f\rho_{2n+1})| \\ &+\kappa_2\frac{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|)} \\ &+\kappa_3\max\left\{\begin{array}{c}|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}),f\rho_{2n+1})|},\\ \dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}\end{array}\right\}.\end{aligned} \tag{47}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (43), (44), and (45), we get that

$$\begin{aligned}|\eth(fs,s)| &\le \kappa_1|\eth(fs,s)|+\kappa_2\frac{|\eth(fs,s)|\cdot|\eth(s,fs)|}{1/2(|\eth(fs,s)|+|\eth(s,fs)|)} \\ &+\kappa_3\max\left\{\begin{array}{c}|\eth(fs,fs)|,|\eth(s,s)|,\dfrac{|\eth(fs,fs)|\cdot|\eth(s,s)|}{|1+\eth(fs,s)|},\\ \dfrac{|\eth(fs,fs)|\cdot|\eth(fs,s)|}{|\eth(fs,s)|+|\eth(s,fs)|},\dfrac{|\eth(s,s)|\cdot|\eth(s,fs)|}{|\eth(fs,s)|+|\eth(s,fs)|}\end{array}\right\}.\end{aligned} \tag{48}$$

After simplification, we get that

$$|\eth(fs,s)|\le(\kappa_1+\kappa_2)|\eth(fs,s)|\Rightarrow(1-\kappa_1-\kappa_2)|\eth(fs,s)|\le 0. \tag{49}$$

Since $(1-\kappa_1-\kappa_2)\ne 0\Rightarrow|\eth(fs,s)|=0$; hence, we get that

$$fs=s. \tag{50}$$

Next, we have to show that $f_1s=s$, by the view of (32),

$$\begin{aligned}\eth(f_1s,f\rho_{2n+2}) &= \eth(f_1s,f_2\rho_{2n+1})\le\kappa_1\eth(fs,f\rho_{2n+1}) \\ &+\kappa_2\frac{\eth(fs,f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{1/2(\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s))} \\ &+\kappa_3\max\left\{\begin{array}{c}\eth(fs,f_1s),\eth(f\rho_{2n+1},f_2\rho_{2n+1}),\\ \dfrac{\eth(fs,f_1s)\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(fs,f\rho_{2n+1})},\\ \dfrac{\eth(fs,f_1s)\cdot\eth(fs,f_2\rho_{2n+1})}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)},\\ \dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1s)}{\eth(fs,f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1s)}\end{array}\right\}.\end{aligned} \tag{51}$$

This implies that

$$\begin{aligned}|\eth(f_1s,f\rho_{2n+2})| &\le \kappa_1|\eth(fs,f\rho_{2n+1})| \\ &+\kappa_2\frac{|\eth(fs,f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{1/2(|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|)} \\ &+\kappa_3\max\left\{\begin{array}{c}|\eth(fs,f_1s)|,|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|,\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(fs,f\rho_{2n+1})|},\\ \dfrac{|\eth(fs,f_1s)|\cdot|\eth(fs,f_2\rho_{2n+1})|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|},\\ \dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1s)|}{|\eth(fs,f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1s)|}\end{array}\right\}.\end{aligned} \tag{52}$$

Now, again applying $\lim_{n\longrightarrow\infty}$ on both sides and by using (43) and (50), we have that

$$\begin{aligned}|\eth(f_1s,s)| &\le \kappa_1|\eth(s,s)|+\kappa_2\frac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{1/2(|\eth(s,s)|+|\eth(s,f_1s)|)} \\ &+\kappa_3\max\left\{\begin{array}{c}|\eth(s,f_1s)|,|\eth(s,s)|,\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|1+\eth(s,s)|},\\ \dfrac{|\eth(s,f_1s)|\cdot|\eth(s,s)|}{|\eth(s,s)|+|\eth(s,f_1s)|},\\ \dfrac{|\eth(s,s)|\cdot|\eth(s,f_1s)|}{|\eth(s,s)|+|\eth(s,f_1s)|}\end{array}\right\}=\kappa_3|\eth(s,f_1s)|.\end{aligned} \tag{53}$$

This implies that $(1-\kappa_3)\,|\,\eth(f_1s,s)\,|\le 0$. Since $(1-\kappa_3)\ne 0\Rightarrow|\eth(f_1s,s)\,|=0$. Hence,

$$f_1s=s. \tag{54}$$

Now, we have to show that $f_2s=s$, by using (32),

$$\begin{aligned}\eth(f\rho_{2n+1},f_2s) &= \eth(f_1\rho_{2n},f_2s)\le\kappa_1\eth(f\rho_{2n},fs) \\ &+\kappa_2\frac{\eth(f\rho_{2n},f_2s)\cdot\eth(fs,f_1\rho_{2n})}{1/2(\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n}))} \\ &+\kappa_3\max\left\{\begin{array}{c}\eth(f\rho_{2n},f_1\rho_{2n}),\eth(fs,f_2s),\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(fs,f_2s)}{1+\eth(f\rho_{2n},fs)},\\ \dfrac{\eth(f\rho_{2n},f_1\rho_{2n})\cdot\eth(f\rho_{2n},f_2s)}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})},\\ \dfrac{\eth(fs,f_2s)\cdot\eth(fs,f_1\rho_{2n})}{\eth(f\rho_{2n},f_2s)+\eth(fs,f_1\rho_{2n})}\end{array}\right\}.\end{aligned} \tag{55}$$

This implies that

$$|\eth(f\rho_{2n+1}, f_2 s)| \le \kappa_1 |\eth(f\rho_{2n}, fs)| + \kappa_2 \frac{|\eth(f\rho_{2n}, f_2 s)| \cdot |\eth(fs, f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|)} + \kappa_3 \max \left\{ \begin{array}{c} |\eth(f\rho_{2n}, f_1\rho_{2n})|, |\eth(fs, f_2 s)|, \\ \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})| \cdot |\eth(fs, f_2 s)|}{|1 + \eth(f\rho_{2n}, fs)|}, \\ \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})| \cdot |\eth(f\rho_{2n}, f_2 s)|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|}, \\ \dfrac{|\eth(fs, f_2 s)| \cdot |\eth(fs, f_1\rho_{2n})|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|} \end{array} \right\}. \tag{56}$$

Taking $\lim_{n \longrightarrow \infty}$ and using (43) and (50), we get

$$|\eth(s, f_2 s)| \le \kappa_1 |\eth(s, s)| + \kappa_2 \frac{|\eth(s, f_2 s)| \cdot |\eth(s, s)|}{1/2(|\eth(s, f_2 s)| + |\eth(s, s)|)} + \kappa_3 \max \left\{ \begin{array}{c} |\eth(s, s)|, |\eth(s, f_2 s)|, \\ \dfrac{|\eth(s, s)| \cdot |\eth(s, f_2 s)|}{|1 + \eth(s, s)|}, \\ \dfrac{|\eth(s, s)| \cdot |\eth(s, f_2 s)|}{|\eth(s, f_2 s)| + |\eth(s, s)|}, \\ \dfrac{|\eth(s, f_2 s)| \cdot |\eth(s, s)|}{|\eth(s, f_2 s)| + |\eth(s, s)|} \end{array} \right\} = \kappa_3 |\eth(s, f_2 s)|. \tag{57}$$

This implies that $(1 - \kappa_3) \mid \eth(s, f_2 s) \mid \le 0$. Since $(1 - \kappa_3) \neq 0 \Rightarrow |\eth(s, f_2 s)| = 0$. Hence,

$$f_2 s = s. \tag{58}$$

Thus, from (50), (54), and (58), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs = f_1 s = f_2 s = s. \tag{59}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{60}$$

Then, from (32), we have that

$$\eth(s, s^*) = \eth(f_1 s, f_2 s^*) \le \kappa_1 \eth(fs, fs^*) + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} + \kappa_3 \max \left\{ \begin{array}{c} \eth(fs, f_1 s), \eth(fs^*, f_2 s^*), \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)}, \\ \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)}, \\ \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right\} = (\kappa_1 + \kappa_2) \eth(s, s^*). \tag{61}$$

This implies that $|\eth(s, s^*) \mid \le (\kappa_1 + \kappa_2) |\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2) |\eth(s, s^*)| \le 0$. Since $(1 - \kappa_1 - \kappa_2) \neq 0 \Rightarrow |\eth(s, s^*) \mid = 0 \Rightarrow s = s^*$, hence proving that f, f_1, and f_2 have a unique CFP in Ω.

If we put $\kappa_2 = 0$ in Theorem 12, we can get the following corollary.

Corollary 13. *Let $(\Omega, \eth)$ be a complete complex-valued b -metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2) + \kappa_3 \max \left\{ \begin{array}{c} \eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right\}, \tag{62}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $\kappa_1/(1 - \kappa_3) < 1$, where $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Corollary 14. *Let $(\Omega, \eth)$ be a complete complex-valued b -metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2) + \kappa_3 \max \left\{ \begin{array}{c} \eth(f\rho_1, f_1\rho_1), \eth(f\rho_2, f_2\rho_2), \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)}, \\ \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)}, \\ \dfrac{\eth(f\rho_2, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right\}, \tag{63}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$ such that $(\kappa_1 + \kappa_2) < 1$, $(\kappa_1 + \kappa_3) < 1$ and $\kappa_1/(1 - \kappa_3) < 1$, where $b \ge 1$. If f is continuous and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Example 15. Let $(\Omega, \eth)$ be a complex-valued b -metric space, where $\Omega = [0, 1]$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ with $\eth(\rho_1, \rho_2) = 4 |\rho_1 - \rho_2|^2/9 + i(4|\rho_1 - \rho_2|^2/9)$, for all $\rho_1, \rho_2 \in \Omega$. Now, we find b,

$$
\begin{aligned}
\eth(\rho_1,\rho_2) &= \frac{4|\rho_1-\rho_2|^2}{9} + i\frac{4|\rho_1-\rho_2|^2}{9} \le \frac{4|(\rho_1-\rho_3)+(\rho_3-\rho_2)|^2}{9} \\
&\quad + i\frac{4|(\rho_1-\rho_3)+(\rho_3-\rho_2)|^2}{9} \\
&\le \left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4}{9}(2|\rho_1-\rho_3||\rho_3-\rho_2|)\right) \\
&\quad + i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4}{9}(2|\rho_1-\rho_3||\rho_3-\rho_2|)\right) \\
&\le \left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&\quad + i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + \frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) + 2i\left(\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2\left(\frac{4|\rho_1-\rho_3|^2}{9} + i\frac{4|\rho_1-\rho_3|^2}{9} + \frac{4|\rho_3-\rho_2|^2}{9} + i\frac{4|\rho_3-\rho_2|^2}{9}\right) \\
&= 2[\eth(\rho_1,\rho_3) + \eth(\rho_3,\rho_2)].
\end{aligned} \tag{64}
$$

That is $\eth(\rho_1,\rho_2) \le b[\eth(\rho_1,\rho_3) + \eth(\rho_3,\rho_2)]$, where $b = 2$. Now, define $f_1, f_2, f : \Omega \longrightarrow \Omega$ as

$$
\begin{aligned}
f_1\rho_1 &= f_2\rho_1 = \frac{3\rho_1}{20}, \\
f\rho_1 &= \frac{\rho_1}{4} \text{ for } \rho_1 \in \Omega.
\end{aligned} \tag{65}
$$

Notice that

$$
\left\{
\begin{array}{c}
|\eth(f\rho_1,f\rho_2)|, \dfrac{|\eth(f\rho_1,f_2\rho_2)|\cdot|\eth(f\rho_2,f_1\rho_1)|}{1/2(|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|)}, \\
\max\left\{
\begin{array}{c}
|\eth(f\rho_1,f_1\rho_1)|, |\eth(f\rho_2,f_2\rho_2)|, \\
\dfrac{|\eth(f\rho_1,f_1\rho_1)|\cdot|\eth(f\rho_2,f_2\rho_2)|}{|1+\eth(f\rho_1,f\rho_2)|}, \\
\dfrac{|\eth(f\rho_1,f_1\rho_1)|\cdot|\eth(f\rho_1,f_2\rho_2)|}{|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|}, \\
\dfrac{|\eth(f\rho_2,f_2\rho_2)|\cdot|\eth(f\rho_2,f_1\rho_1)|}{|\eth(f\rho_1,f_2\rho_2)|+|\eth(f\rho_2,f_1\rho_1)|}
\end{array}
\right\}
\end{array}
\right\} \ge 0. \tag{66}
$$

In all regards, it is enough to show that $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$, for all $\rho_1,\rho_2 \in [0,1]$ and $\kappa_1,\kappa_2,\kappa_3 \in [0,1)$, such that $(\kappa_1+\kappa_2) < 1$ and $(\kappa_1+\kappa_3) < 1$, where $b \ge 1$, we have

$$
\begin{aligned}
\eth(f_1\rho_1,f_2\rho_2) &= \left(\frac{4|f_1\rho_1 - f_2\rho_2|^2}{9} + i\frac{4|f_1\rho_1 - f_2\rho_2|^2}{9}\right) \\
&= \left(\frac{4|3\rho_1/20 - 3\rho_2/20|^2}{9} + i\frac{4|3\rho_1/20 - 3\rho_2/20|^2}{9}\right) \\
&= \left(\frac{3}{5}\right)^2\left(\frac{4|\rho_1/4 - \rho_2/4|^2}{9} + i\frac{4|\rho_1/4 - \rho_2/4|^2}{9}\right) \\
&= \frac{9}{25}\left(\frac{4|\rho_1/4 - \rho_2/4|^2}{9} + i\frac{4|\rho_1/4 - \rho_2/4|^2}{9}\right),
\end{aligned} \tag{67}
$$

$$
\begin{aligned}
\eth(f\rho_1,f\rho_2) &= \left(\frac{4|f\rho_1 - f\rho_2|^2}{9} + i\frac{4|f\rho_1 - f\rho_2|^2}{9}\right) \\
&= \left(\frac{4|\rho_1/4 - \rho_2/4|^2}{9} + i\frac{4|\rho_1/4 - \rho_2/4|^2}{9}\right).
\end{aligned} \tag{68}
$$

For $\rho_1, \rho_2 \in [0,1]$, we discuss different cases with $\kappa_1 = 2/5$, $\kappa_2 = 1/5$, $\kappa_3 = 1/10$, and $b = 2$. Hence,

$$
\begin{aligned}
\kappa_1 + \kappa_2 &= \frac{2}{5} + \frac{1}{5} = \frac{3}{5} < 1, \\
\kappa_1 + \kappa_3 &= \frac{2}{5} + \frac{1}{10} = \frac{1}{2} < 1.
\end{aligned} \tag{69}
$$

Case 1. Let $\rho_1 = 0, \rho_2 = 0$, then from (67) and (68), directly, we get that $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$. Hence, (32) is satisfied with $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b = 2$.

Case 2. Let $\rho_1 = 1, \rho_2 = 0$, then from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$, satisfied with $\kappa_1 = 2/5$, i.e.,

$$
\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/4 - 0/4|^2}{9} + i\frac{4|1/4 - 0/4|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/4 - 0/4|^2}{9} + i\frac{4|1/4 - 0/4|^2}{9}\right)0.0099(1+i) \\
&\quad \le 0.0110(1+i).
\end{aligned} \tag{70}
$$

Thus, (32) is true for $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b = 2$.

Case 3. Let $\rho_1 = 1/2, \rho_2 = 1/4$; then, from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$ is satisfied with $\kappa_1 = 2/5$, i.e.,

$$
\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/8 - 1/16|^2}{9} + i\frac{4|1/8 - 1/16|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/8 - 1/16|^2}{9} + i\frac{4|1/8 - 1/16|^2}{9}\right)0.00061(1+i) \\
&\quad \le 0.00068(1+i).
\end{aligned} \tag{71}
$$

Thus, (32) is true for $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b = 2$.

Case 4. Let $\rho_1 = 1/2, \rho_2 = 1$; then, from (67) and (68), we find $\eth(f_1\rho_1,f_2\rho_2) \le \kappa_1\eth(f\rho_1,f\rho_2)$ is satisfied with $\kappa_1 = 2/5$, i.e.,

$$
\begin{aligned}
&\frac{9}{25}\left(\frac{4|1/8 - 1/4|^2}{9} + i\frac{4|1/8 - 1/4|^2}{9}\right) \\
&\quad \le \kappa_1\left(\frac{4|1/8 - 1/4|^2}{9} + i\frac{4|1/8 - 1/4|^2}{9}\right)0.0024(1+i) \\
&\quad \le 0.0027(1+i).
\end{aligned} \tag{72}
$$

Hence, (32) is satisfied with $\kappa_1 = 2/5, \kappa_2 = 1/5, \kappa_3 = 1/10$, and $b=2$. The pairs of self-mappings (f,f_1) and (f,f_2) are commutable; that is,

$$f_1(f(\rho_1)) = f(f_1(\rho_1)) = \frac{3\rho_1}{80},\\ f_2(f(\rho_1)) = f(f_2(\rho_1)) = \frac{3\rho_1}{80}, \forall \rho_1 \in \Omega. \tag{73}$$

Thus, all the conditions of Theorem 12 are satisfied with noticing that the point $0 \in \Omega$, which remains fixed under mappings f, f_1, and f_2, is indeed unique.

Theorem 16. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\begin{aligned}\eth(f_1\rho_1, f_2\rho_2) &\leq \kappa_1\eth(f\rho_1, f\rho_2) + \kappa_2\frac{\eth(f\rho_1, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1))}\\ &+\kappa_3\begin{pmatrix}\eth(f\rho_1, f_1\rho_1)+\eth(f\rho_2, f_2\rho_2)\\ +\dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_2, f_2\rho_2)}{1+\eth(f\rho_1, f\rho_2)}\\ +\dfrac{\eth(f\rho_1, f_1\rho_1)\cdot\eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}\\ +\dfrac{\eth(f\rho_2, f_2\rho_2)\cdot\eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2)+\eth(f\rho_2, f_1\rho_1)}\end{pmatrix},\end{aligned} \tag{74}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0,1)$, such that $(\kappa_1+\kappa_2)<1$, $(\kappa_1+4\kappa_3)<1$ and $b \geq 1$. If f is a continuous self-map and (f,f_1), (f,f_2) are commutable pairs, then f, f_1, and f_2 have a unique CFP in Ω.

Proof. Fix $\rho_0 \in \Omega$, and define a sequence $\{\rho_n\}$ sequences in Ω such that

$$\Gamma_{2n} = f\rho_{2n+1} = f_1\rho_{2n},\\ \Gamma_{2n+1} = f\rho_{2n+2} = f_2\rho_{2n+1},\\ \forall n \geq 0. \tag{75}$$

□

Now, by the view of (74) and (75),

$$\begin{aligned}\eth(\Gamma_{2n}, \Gamma_{2n+1}) &= \eth(f_1\rho_{2n}, f_2\rho_{2n+1}) \leq \kappa_1\eth(f\rho_{2n}, f\rho_{2n+1})\\ &+\kappa_2\frac{\eth(f\rho_{2n}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n}))}\\ &+\kappa_3\begin{pmatrix}\eth(f\rho_{2n}, f_1\rho_{2n})+\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(f\rho_{2n}, f\rho_{2n+1})}\\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n+1})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}\\ +\dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n+1})+\eth(f\rho_{2n+1}, f_1\rho_{2n})}\end{pmatrix}\\ &=\kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n})+\kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n}))}\\ &+\kappa_3\begin{pmatrix}\eth(\Gamma_{2n-1}, \Gamma_{2n})+\eth(\Gamma_{2n}, \Gamma_{2n+1})\\ +\dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n}, \Gamma_{2n+1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n})}\\ +\dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n+1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}\\ +\dfrac{\eth(\Gamma_{2n}, \Gamma_{2n+1})\cdot\eth(\Gamma_{2n}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n+1})+\eth(\Gamma_{2n}, \Gamma_{2n})}\end{pmatrix}.\end{aligned} \tag{76}$$

This implies that

$$\begin{aligned}|\eth(\Gamma_{2n}, \Gamma_{2n+1})| &\leq \kappa_1|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\\ &+\kappa_2\frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|)}\\ &+\kappa_3\Bigg(|\eth(\Gamma_{2n-1}, \Gamma_{2n})|+|\eth(\Gamma_{2n}, \Gamma_{2n+1})|\\ &+\frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n+1})|}{|1+\eth(\Gamma_{2n-1}, \Gamma_{2n})|}\\ &+\frac{|\eth(\Gamma_{2n-1}, \Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}\\ &+\frac{|\eth(\Gamma_{2n}, \Gamma_{2n+1})|\cdot|\eth(\Gamma_{2n}, \Gamma_{2n})|}{|\eth(\Gamma_{2n-1}, \Gamma_{2n+1})|+|\eth(\Gamma_{2n}, \Gamma_{2n})|}\Bigg).\end{aligned} \tag{77}$$

After simplification, we get that

$$|\eth(\Gamma_{2n}, \Gamma_{2n+1})| \leq g|\eth(\Gamma_{2n-1}, \Gamma_{2n})|, \text{where } g=\frac{\kappa_1+2\kappa_3}{1-2\kappa_3}<1. \tag{78}$$

Again, by the view of (74) and (75),

$$\begin{aligned}\eth(\Gamma_{2n-1}, \Gamma_{2n}) &= \eth(f_2\rho_{2n-1}, f_1\rho_{2n}) = \eth(f_1\rho_{2n}, f_2\rho_{2n-1}) \leq \kappa_1\eth(f\rho_{2n}, f\rho_{2n-1})\\ &+\kappa_2\frac{\eth(f\rho_{2n}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n}))}\\ &+\kappa_3\begin{pmatrix}\eth(f\rho_{2n}, f_1\rho_{2n})+\eth(f\rho_{2n-1}, f_2\rho_{2n-1})\\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n-1}, f_2\rho_{2n-1})}{1+\eth(f\rho_{2n}, f\rho_{2n-1})}\\ +\dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2\rho_{2n-1})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}\\ +\dfrac{\eth(f\rho_{2n-1}, f_2\rho_{2n-1})\cdot\eth(f\rho_{2n-1}, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2\rho_{2n-1})+\eth(f\rho_{2n-1}, f_1\rho_{2n})}\end{pmatrix}\\ &=\kappa_1\eth(\Gamma_{2n-1}, \Gamma_{2n-2})+\kappa_2\frac{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{1/2(\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n}))}\\ &+\kappa_3\begin{pmatrix}\eth(\Gamma_{2n-1}, \Gamma_{2n})+\eth(\Gamma_{2n-2}, \Gamma_{2n-1})\\ +\dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n-1})}{1+\eth(\Gamma_{2n-1}, \Gamma_{2n-2})}\\ +\dfrac{\eth(\Gamma_{2n-1}, \Gamma_{2n})\cdot\eth(\Gamma_{2n-1}, \Gamma_{2n-1})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}\\ +\dfrac{\eth(\Gamma_{2n-2}, \Gamma_{2n-1})\cdot\eth(\Gamma_{2n-2}, \Gamma_{2n})}{\eth(\Gamma_{2n-1}, \Gamma_{2n-1})+\eth(\Gamma_{2n-2}, \Gamma_{2n})}\end{pmatrix}.\end{aligned} \tag{79}$$

This implies that

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le \kappa_1|\eth(\Gamma_{2n-1},\Gamma_{2n-2})| + \kappa_2\frac{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n})|}{1/2(|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|)} + \kappa_3\left(\begin{array}{c}|\eth(\Gamma_{2n-1},\Gamma_{2n})|+|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \\ +\dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|}{|1+\eth(\Gamma_{2n-1},\Gamma_{2n-2})|} \\ +\dfrac{|\eth(\Gamma_{2n-1},\Gamma_{2n})|\cdot|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|} \\ +\dfrac{|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|\cdot|\eth(\Gamma_{2n-2},\Gamma_{2n})|}{|\eth(\Gamma_{2n-1},\Gamma_{2n-1})|+|\eth(\Gamma_{2n-2},\Gamma_{2n})|}\end{array}\right). \tag{80}$$

After simplification, we get that

$$|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g|\eth(\Gamma_{2n-2},\Gamma_{2n-1})|, \text{ since } g=\frac{\kappa_1+2\kappa_3}{1-2\kappa_3}<1. \tag{81}$$

Now, from (81) and (78) and by induction, we have

$$\begin{aligned}|\eth(\Gamma_{2n},\Gamma_{2n+1})| &\le g|\eth(\Gamma_{2n-1},\Gamma_{2n})| \le g^2|\eth(\Gamma_{2n-2},\Gamma_{2n-1})| \\ &\le \cdots \le g^{2n}|\eth(\Gamma_0,\Gamma_1)|.\end{aligned} \tag{82}$$

So, for $m, n \in \mathbb{N}$ with $m > n$,

$$\begin{aligned}|\eth(\Gamma_n,\Gamma_m)| &\le b|\eth(\Gamma_n,\Gamma_{n+1})| + b|\eth(\Gamma_{n+1},\Gamma_m)| \le b|\eth(\Gamma_n,\Gamma_{n+1})| \\ &\quad + b^2|\eth(\Gamma_{n+1},\Gamma_{n+2})| + \cdots + b^{m-n}|\eth(\Gamma_{m-1},\Gamma_m)| \\ &\le bg^n|\eth(\Gamma_0,\Gamma_1)| + b^2g^{n+1}|\eth(\Gamma_0,\Gamma_1)| \\ &\quad + \cdots + b^{m-n}g^{m-1}|\eth(\Gamma_0,\Gamma_1)| \\ &\le \left[bg^n + b^2g^{n+1} + \cdots + b^{m-n}g^{m-1}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n\left[1 + bg + b^2g^2 \cdots + b^{m-(n+1)}g^{m-(n+1)}\right]|\eth(\Gamma_0,\Gamma_1)| \\ &= bg^n\sum_{t=0}^{m-(n+1)} b^t g^t|\eth(\Gamma_0,\Gamma_1)| \le bg^n\sum_{t=0}^{\infty} b^t g^t|\eth(\Gamma_0,\Gamma_1)| \\ &= \frac{bg^n}{1-bg}|\eth(\Gamma_0,\Gamma_1)| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{83}$$

Therefore, sequence $\{\Gamma_n\}$ is Cauchy. Since Ω is complete, there exists $s \in \Omega$ such that $\Gamma_n \longrightarrow s$, as $n \longrightarrow \infty$, and from (75), we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty} f\rho_{2n+1} &= s, \\ \lim_{n\longrightarrow\infty} f_1\rho_{2n} &= s, \\ \lim_{n\longrightarrow\infty} f_2\rho_{2n+1} &= s.\end{aligned} \tag{84}$$

As f is continuous, so

$$\begin{aligned}\lim_{n\longrightarrow\infty} f(f\rho_{2n+1}) &= fs, \\ \lim_{n\longrightarrow\infty} f(f_1\rho_{2n}) &= fs, \\ \lim_{n\longrightarrow\infty} f(f_2\rho_{2n+1}) &= fs.\end{aligned} \tag{85}$$

Since, (f, f_1) and (f, f_2) are commutable pairs, therefore, from (85), we have that

$$\begin{aligned}\lim_{n\longrightarrow\infty} f_1(f\rho_{2n}) &= fs, \\ \lim_{n\longrightarrow\infty} f_2(f\rho_{2n+1}) &= fs.\end{aligned} \tag{86}$$

Now, we prove $fs = s$. So, for this, we put $\rho_1 = f\rho_{2n}$ and $\rho_2 = \rho_{2n+1}$ in (74),

$$\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1}) \le \kappa_1\eth(f(f\rho_{2n}),f\rho_{2n+1}) + \kappa_2\frac{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{1/2(\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n})))} + \kappa_3\left(\begin{array}{c}\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))+\eth(f\rho_{2n+1},f_2\rho_{2n+1}) \\ +\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f\rho_{2n+1},f_2\rho_{2n+1})}{1+\eth(f(f\rho_{2n}),f\rho_{2n+1})} \\ +\dfrac{\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))\cdot\eth(f(f\rho_{2n}),f_2\rho_{2n+1})}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))} \\ +\dfrac{\eth(f\rho_{2n+1},f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}{\eth(f(f\rho_{2n}),f_2\rho_{2n+1})+\eth(f\rho_{2n+1},f_1(f\rho_{2n}))}\end{array}\right). \tag{87}$$

This implies that

$$|\eth(f_1(f\rho_{2n}),f_2\rho_{2n+1})| \le \kappa_1|\eth(f(f\rho_{2n}),f\rho_{2n+1})| + \kappa_2\frac{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{1/2(|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|)} + \kappa_3\left(\begin{array}{c}|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|+|\eth(f\rho_{2n+1},f_2\rho_{2n+1})| \\ +\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|}{|1+\eth(f(f\rho_{2n}),f\rho_{2n+1})|} \\ +\dfrac{|\eth(f(f\rho_{2n}),f_1(f\rho_{2n}))|\cdot|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|} \\ +\dfrac{|\eth(f\rho_{2n+1},f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}{|\eth(f(f\rho_{2n}),f_2\rho_{2n+1})|+|\eth(f\rho_{2n+1},f_1(f\rho_{2n}))|}\end{array}\right). \tag{88}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84), (85), and (86), we get that

$$|\eth(fs,s)| \le \kappa_1|\eth(fs,s)| + \kappa_2\frac{|\eth(fs,s)|\cdot|\eth(s,fs)|}{1/2(|\eth(fs,s)|+|\eth(s,fs)|)} + \kappa_3\left(\begin{array}{c}|\eth(fs,fs)|+|\eth(s,s)| \\ +\dfrac{|\eth(fs,fs)|\cdot|\eth(s,s)|}{|1+\eth(fs,s)|} \\ +\dfrac{|\eth(fs,fs)|\cdot|\eth(fs,s)|}{|\eth(fs,s)|+|\eth(s,fs)|} \\ +\dfrac{|\eth(s,s)|\cdot|\eth(s,fs)|}{|\eth(fs,s)|+|\eth(s,fs)|}\end{array}\right) = (\kappa_1+\kappa_2)|\eth(fs,s)|. \tag{89}$$

This implies that $(1-\kappa_1-\kappa_2)\,|\,\eth(fs,s)\,|\le 0$. Since, $(1-\kappa_1-\kappa_2)\neq 0 \Rightarrow |\eth(fs,s)\,|=0$. Hence,

$$fs = s. \tag{90}$$

Next, we have to show that $f_1 s = s$, by using (74),

$$\eth(f_1 s, f\rho_{2n+2}) = \eth(f_1 s, f_2\rho_{2n+1}) \le \kappa_1 \eth(fs, f\rho_{2n+1}) + \kappa_2 \frac{\eth(fs, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1 s)}{1/2(\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1 s))} + \kappa_3 \begin{pmatrix} \eth(fs, f_1 s) + \eth(f\rho_{2n+1}, f_2\rho_{2n+1}) \\ + \dfrac{\eth(fs, f_1 s)\cdot\eth(f\rho_{2n+1}, f_2\rho_{2n+1})}{1+\eth(fs, f\rho_{2n+1})} \\ + \dfrac{\eth(fs, f_1 s)\cdot\eth(fs, f_2\rho_{2n+1})}{\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1 s)} \\ + \dfrac{\eth(f\rho_{2n+1}, f_2\rho_{2n+1})\cdot\eth(f\rho_{2n+1}, f_1 s)}{\eth(fs, f_2\rho_{2n+1}) + \eth(f\rho_{2n+1}, f_1 s)} \end{pmatrix}. \tag{91}$$

This implies that

$$|\eth(f_1 s, f\rho_{2n+2})| \le \kappa_1 |\eth(fs, f\rho_{2n+1})| + \kappa_2 \frac{|\eth(fs, f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1 s)|}{1/2(|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1 s)|)} + \kappa_3 \begin{pmatrix} |\eth(fs, f_1 s)| + |\eth(f\rho_{2n+1}, f_2\rho_{2n+1})| \\ + \dfrac{|\eth(fs, f_1 s)|\cdot|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|}{|1+\eth(fs, f\rho_{2n+1})|} \\ + \dfrac{|\eth(fs, f_1 s)|\cdot|\eth(fs, f_2\rho_{2n+1})|}{|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1 s)|} \\ + \dfrac{|\eth(f\rho_{2n+1}, f_2\rho_{2n+1})|\cdot|\eth(f\rho_{2n+1}, f_1 s)|}{|\eth(fs, f_2\rho_{2n+1})| + |\eth(f\rho_{2n+1}, f_1 s)|} \end{pmatrix}. \tag{92}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84) and (90), we get

$$|\eth(f_1 s, s)| \le \kappa_1 |\eth(s,s)| + \kappa_2 \frac{|\eth(s,s)|\cdot|\eth(s, f_1 s)|}{1/2(|\eth(s,s)| + |\eth(s, f_1 s)|)} + \kappa_3 \begin{pmatrix} |\eth(s, f_1 s)| + |\eth(s,s)| \\ + \dfrac{|\eth(s, f_1 s)\,|\cdot|\,\eth(s,s)\,|}{|1+\eth(s,s)\,|} \\ + \dfrac{|\eth(s, f_1 s)\,|\cdot|\,\eth(s,s)\,|}{|\eth(s,s)|+|\eth(s, f_1 s)\,|} \\ + \dfrac{|\eth(s,s)|\cdot|\eth(s, f_1 s)|}{|\eth(s,s)| + |\eth(s, f_1 s)|} \end{pmatrix}. \tag{93}$$

Thus, we get that $|\eth(f_1 s, s)\,|\le\kappa_3|\eth(s, f_1 s)| \Rightarrow (1-\kappa_3)|\eth(f_1 s, s)| \le 0$. Since $(1-\kappa_3)\neq 0$, therefore, $|\eth(f_1 s, s)\,|=0$. Hence,

$$f_1 s = s. \tag{94}$$

Now, we have to show that $f_2 s = s$, by using (74),

$$\eth(f\rho_{2n+1}, f_2 s) = \eth(f_1\rho_{2n}, f_2 s) \le \kappa_1 \eth(f\rho_{2n}, fs) + \kappa_2 \frac{\eth(f\rho_{2n}, f_2 s)\cdot\eth(fs, f_1\rho_{2n})}{1/2(\eth(f\rho_{2n}, f_2 s) + \eth(fs, f_1\rho_{2n}))} + \kappa_3 \begin{pmatrix} \eth(f\rho_{2n}, f_1\rho_{2n}) + \eth(fs, f_2 s) \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(fs, f_2 s)}{1+\eth(f\rho_{2n}, fs)} \\ + \dfrac{\eth(f\rho_{2n}, f_1\rho_{2n})\cdot\eth(f\rho_{2n}, f_2 s)}{\eth(f\rho_{2n}, f_2 s) + \eth(fs, f_1\rho_{2n})} \\ + \dfrac{\eth(fs, f_2 s)\cdot\eth(fs, f_1\rho_{2n})}{\eth(f\rho_{2n}, f_2 s) + \eth(fs, f_1\rho_{2n})} \end{pmatrix}. \tag{95}$$

This implies that

$$|\eth(f\rho_{2n+1}, f_2 s)| \le \kappa_1 |\eth(f\rho_{2n}, fs)| + \kappa_2 \frac{|\eth(f\rho_{2n}, f_2 s)|\cdot|\eth(fs, f_1\rho_{2n})|}{1/2(|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|)} + \kappa_3 \begin{pmatrix} |\eth(f\rho_{2n}, f_1\rho_{2n})| + |\eth(fs, f_2 s)| \\ + \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})|\cdot|\eth(fs, f_2 s)|}{|1+\eth(f\rho_{2n}, fs)|} \\ + \dfrac{|\eth(f\rho_{2n}, f_1\rho_{2n})|\cdot|\eth(f\rho_{2n}, f_2 s)|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|} \\ + \dfrac{|\eth(fs, f_2 s)|\cdot|\eth(fs, f_1\rho_{2n})|}{|\eth(f\rho_{2n}, f_2 s)| + |\eth(fs, f_1\rho_{2n})|} \end{pmatrix}. \tag{96}$$

Taking $\lim_{n\longrightarrow\infty}$ and using (84) and (90), we get

$$|\eth(s, f_2 s)| \le \kappa_1 |\eth(s,s)| + \kappa_2 \frac{|\eth(s, f_2 s)|\cdot|\eth(s,s)|}{1/2(|\eth(s, f_2 s)| + |\eth(s,s)|)} + \kappa_3 \begin{pmatrix} |\eth(s,s)| + |\eth(s, f_2 s)| \\ + \dfrac{|\eth(s,s)\,|\cdot|\,\eth(s, f_2 s)\,|}{|1+\eth(s,s)\,|} \\ + \dfrac{|\eth(s,s)\,|\cdot|\,\eth(s, f_2 s)\,|}{|\eth(s, f_2 s)|+|\eth(s,s)\,|} \\ + \dfrac{|\eth(s, f_2 s)\,|\cdot|\,\eth(s,s)\,|}{|\eth(s, f_2 s)|+|\eth(s,s)\,|} \end{pmatrix}. \tag{97}$$

So, we get that $|\eth(s, f_2 s)\,|\le\kappa_3|\eth(s, f_2 s)| \Rightarrow (1-\kappa_3)|\eth(s, f_2 s)| \le 0$. Since $(1-\kappa_3)\neq 0$, therefore, $|\eth(s, f_2 s)\,|=0$. Hence,

$$f_2 s = s. \tag{98}$$

Thus, from (90), (94), and (98), we find that s is a CFP of f, f_1, and f_2, i.e.,

$$fs = f_1 s = f_2 s = s. \tag{99}$$

Uniqueness: suppose that $s^* \in \Omega$ is another CFP of f, f_1, and f_2 such that

$$\begin{aligned} fs &= f_1 s = f_2 s = s, \\ fs^* &= f_1 s^* = f_2 s^* = s^*. \end{aligned} \tag{100}$$

Then, from (74), we have that

$$\begin{aligned} \eth(s, s^*) &= \eth(f_1 s, f_2 s^*) \le \kappa_1 \eth(fs, fs^*) \\ &\quad + \kappa_2 \frac{\eth(fs, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{1/2(\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s))} \\ &\quad + \kappa_3 \left(\begin{array}{c} \eth(fs, f_1 s) + \eth(fs^*, f_2 s^*) \\ + \dfrac{\eth(fs, f_1 s) \cdot \eth(fs^*, f_2 s^*)}{1 + \eth(fs, fs^*)} \\ + \dfrac{\eth(fs, f_1 s) \cdot \eth(fs, f_2 s^*)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \\ + \dfrac{\eth(fs^*, f_2 s^*) \cdot \eth(fs^*, f_1 s)}{\eth(fs, f_2 s^*) + \eth(fs^*, f_1 s)} \end{array} \right) \\ &= (\kappa_1 + \kappa_2) \eth(s, s^*). \end{aligned} \tag{101}$$

This implies that $|\eth(s, s^*)| \le (\kappa_1 + \kappa_2)|\eth(s, s^*)| \Rightarrow (1 - \kappa_1 - \kappa_2)|\eth(s, s^*)| \le 0$. Since $(1 - \kappa_1 - \kappa_2) \ne 0$, therefore, $|\eth(s, s^*)| = 0 \Rightarrow s = s^*$, hence proving that f, f_1, and f_2 have a unique CFP in Ω.

Corollary 17. *Let $(\Omega, \eth)$ be a complete complex-valued b-metric space and $f_1, f_2, f : \Omega \longrightarrow \Omega$ be three self-maps satisfying*

$$\begin{aligned} \eth(f_1\rho_1, f_2\rho_2) &\le \kappa_1 \eth(f\rho_1, f\rho_2) \\ &\quad + \kappa_2 \frac{\eth(f\rho_1, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{1/2(\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1))} \\ &\quad + \kappa_3 \left(\begin{array}{c} \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_2, f_2\rho_2)}{1 + \eth(f\rho_1, f\rho_2)} \\ + \dfrac{\eth(f\rho_1, f_1\rho_1) \cdot \eth(f\rho_1, f_2\rho_2)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \\ + \dfrac{\eth(f\rho_2, f_2\rho_2) \cdot \eth(f\rho_2, f_1\rho_1)}{\eth(f\rho_1, f_2\rho_2) + \eth(f\rho_2, f_1\rho_1)} \end{array} \right), \end{aligned} \tag{102}$$

for all $\rho_1, \rho_2 \in \Omega$, $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$, such that $(\kappa_1 + \kappa_2) < 1$ and $(\kappa_1 + \kappa_3)/(1 - \kappa_3) < 1$, with $b \ge 1$. If f is a continuous self-mapping and (f, f_1), (f, f_2) are commutable pairs, then f, f_1, and f_2 have a unique common fixed point in Ω.

Example 18. Let $\Omega = [0, \infty)$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ be defined as $\eth(\rho_1, \rho_2) = 3|\rho_1 - \rho_2|^2/13 + i3|\rho_1 - \rho_2|^2/13$ for all $\rho_1, \rho_2 \in \Omega$. Then, $(\Omega, \eth)$ is a complex-valued b-metric space. Now, we find b:

$$\begin{aligned} \eth(\rho_1, \rho_2) &= \frac{3|\rho_1 - \rho_2|^2}{13} + i\frac{3|\rho_1 - \rho_2|^2}{13} \le \frac{3|(\rho_1 - \rho_3) + (\rho_3 - \rho_2)|^2}{13} \\ &\quad + i\frac{3|(\rho_1 - \rho_3) + (\rho_3 - \rho_2)|^2}{13} \\ &\le \left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3}{13}(2|\rho_1 - \rho_3||\rho_3 - \rho_2|) \right) \\ &\quad + i\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3}{13}(2|\rho_1 - \rho_3||\rho_3 - \rho_2|) \right) \\ &\le \left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &\quad + i\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + \frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) + i2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2\left(\frac{3|\rho_1 - \rho_3|^2}{13} + i\frac{3|\rho_1 - \rho_3|^2}{13} + \frac{3|\rho_3 - \rho_2|^2}{13} + i\frac{3|\rho_3 - \rho_2|^2}{13} \right) \\ &= 2[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]. \end{aligned} \tag{103}$$

That is $\eth(\rho_1, \rho_2) \le b[\eth(\rho_1, \rho_3) + \eth(\rho_3, \rho_2)]$, where $b = 2$. Now, we define $f_1, f_2, f : \Omega \longrightarrow \Omega$ by

$$\begin{aligned} f_1\rho_1 &= f_2\rho_1 = \ln\left(1 + \frac{\rho_1}{4 + \rho_1}\right), \\ f\rho_1 &= e^{4\rho_1} - 1, \\ &\text{for all } \rho_1 \in \Omega. \end{aligned} \tag{104}$$

Notice that

$$\left\{ \begin{array}{c} |\eth(f\rho_1, f\rho_2)|, \dfrac{|\eth(f\rho_1, f_2\rho_2)| \cdot |\eth(f\rho_2, f_1\rho_1)|}{1/2(|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|)}, \\ \left(\begin{array}{c} |\eth(f\rho_1, f_1\rho_1)| + |\eth(f\rho_2, f_2\rho_2)| \\ + \dfrac{|\eth(f\rho_1, f_1\rho_1)| \cdot |\eth(f\rho_2, f_2\rho_2)|}{|1 + \eth(f\rho_1, f\rho_2)|} \\ + \dfrac{|\eth(f\rho_1, f_1\rho_1)| \cdot |\eth(f\rho_1, f_2\rho_2)|}{|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|} \\ + \dfrac{|\eth(f\rho_2, f_2\rho_2)| \cdot |\eth(f\rho_2, f_1\rho_1)|}{|\eth(f\rho_1, f_2\rho_2)| + |\eth(f\rho_2, f_1\rho_1)|} \end{array} \right) \end{array} \right\} \ge 0. \tag{105}$$

In all regards, it is enough to show that $\eth(f_1\rho_1, f_2\rho_2) \le \kappa_1 \eth(f\rho_1, f\rho_2)$, for all $\rho_1, \rho_2 \in [0, \infty)$ and $\kappa_1, \kappa_2, \kappa_3 \in [0, 1)$, such that $(\kappa_1 + \kappa_2) < 1$ and $(\kappa_1 + 4\kappa_3) < 1$, where $b \ge 1$, we have

$$\begin{aligned}
\eth(f_1\rho_1, f_2\rho_2) &= \left(\frac{3|f_1\rho_1 - f_2\rho_2|^2}{13} + i\frac{3|f_1\rho_1 - f_2\rho_2|^2}{13}\right) \\
&= \left(\frac{3|\ln(1+\rho_1/(4+\rho_1)) - \ln(1+\rho_2/(4+\rho_2))|^2}{13}\right. \\
&\quad \left. + i\frac{3|\ln(1+\rho_1/(4+\rho_1)) - \ln(1+\rho_2/(4+\rho_2))|^2}{13}\right) \\
&\leq \left(\frac{3|\rho_1/(4+\rho_1) - \rho_2/(4+\rho_2)|^2}{13}\right. \\
&\quad \left. + i\frac{3|\rho_1/(4+\rho_1) - \rho_2/(4+\rho_2)|^2}{13}\right) \\
&\leq \left(\frac{3|(4\rho_1 - 4\rho_2)/16|^2}{13} + i\frac{3|(4\rho_1 - 4\rho_2)/16|^2}{13}\right) \\
&= \frac{1}{16^2}\left(\frac{3|4\rho_1 - 4\rho_2|^2}{13} + i\frac{3|4\rho_1 - 4\rho_2|^2}{13}\right) \\
&\leq \frac{1}{256}\left(\frac{3\left|e^{4\rho_1} - e^{4\rho_2}\right|^2}{13} + i\frac{3\left|e^{4\rho_1} - e^{4\rho_2}\right|^2}{13}\right),
\end{aligned} \tag{106}$$

$$\begin{aligned}
\eth(f\rho_1, f\rho_2) &= \left(\frac{3|f\rho_1 - f\rho_2|^2}{13} + i\frac{3|f\rho_1 - f\rho_2|^2}{13}\right) \\
&= \left(\frac{3\left|\left(e^{4\rho_1} - 1\right) - \left(e^{4\rho_2} - 1\right)\right|^2}{13} + i\frac{3\left|\left(e^{4\rho_1} - 1\right) - \left(e^{4\rho_2} - 1\right)\right|^2}{13}\right) \\
&= \left(\frac{3\left|e^{4\rho_1} - e^{4\rho_2}\right|^2}{13} + i\frac{3\left|e^{4\rho_1} - e^{4\rho_2}\right|^2}{13}\right).
\end{aligned} \tag{107}$$

For $\rho_1, \rho_2 \in [0,\infty)$, we discuss different cases with $\kappa_1 = 1/5$, $\kappa_2 = 1/4$, and $\kappa_3 = 1/10$, where $b = 2$. Hence,

$$\begin{aligned}
\kappa_1 + \kappa_2 &= \frac{1}{5} + \frac{1}{4} = \frac{9}{20} < 1, \\
\kappa_1 + 4\kappa_3 &= \frac{1}{5} + \frac{2}{5} = \frac{3}{5} < 1.
\end{aligned} \tag{108}$$

Case 1. Let $\rho_1 = 0, \rho_2 = 0$. Then, from (106) and (107), directly, we get that $\eth(f_1\rho_1, f_2\rho_2) \leq \kappa_1 \eth(f\rho_1, f\rho_2)$. Hence, (74) is satisfied with $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 2. Let $\rho_1 = 0$, $\rho_2 = 1$, then from (106) and (107), we find $\eth(f_1\rho_1, f_2\rho_2) \leq \kappa_1 \eth(f\rho_1, f\rho_2)$ is satisfied with $\kappa_1 = 1/5$, as

$$\begin{aligned}
&\frac{1}{256}\left(\frac{3\left|e^0 - e^4\right|^2}{13} + i\frac{3\left|e^0 - e^4\right|^2}{13}\right) \\
&\quad \leq \kappa_1\left(\frac{3\left|e^0 - e^4\right|^2}{13} + i\frac{3\left|e^0 - e^4\right|^2}{13}\right).
\end{aligned} \tag{109}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}
&\frac{1}{256}\left(\frac{3|-53.5981|^2}{13} + i\frac{3|-53.5981|^2}{13}\right) \\
&\quad \leq \frac{1}{5}\left(\frac{3|-53.5981|^2}{13} + i\frac{3|-53.5981|^2}{13}\right)2.5896(1+i) \\
&\quad \leq 132.5887(1+i).
\end{aligned} \tag{110}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 3. Let $\rho_1 = 1/2$, $\rho_2 = 1/4$, then from (106) and (107), we find $\eth(f_1\rho_1, f_2\rho_2) \leq \kappa_1 \eth(f\rho_1, f\rho_2)$ is true for $\kappa_1 = 1/5$, as

$$\frac{1}{256}\left(\frac{3\left|e^2 - e^1\right|^2}{13} + i\frac{3\left|e^2 - e^1\right|^2}{13}\right) \leq \kappa_1\left(\frac{3\left|e^2 - e^1\right|^2}{13} + i\frac{3\left|e^2 - e^1\right|^2}{13}\right). \tag{111}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}
&\frac{1}{256}\left(\frac{3|4.6708|^2}{13} + i\frac{3|4.6708|^2}{13}\right) \leq \frac{1}{5}\left(\frac{3|4.6708|^2}{13} + i\frac{3|4.6708|^2}{13}\right) \\
&\quad \cdot 0.0196(1+i) \leq 1.0069(1+i).
\end{aligned} \tag{112}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

Case 4. Let $\rho_1 = 1/2, \rho_2 = 1$, then from (106) and (107), we get that $\eth(f_1\rho_1, f_2\rho_2) \leq \kappa_1 \eth(f\rho_1, f\rho_2)$ is true for $\kappa_1 = 1/5$, as

$$\frac{1}{256}\left(\frac{3\left|e^2 - e^4\right|^2}{13} + i\frac{3\left|e^2 - e^4\right|^2}{13}\right) \leq \kappa_1\left(\frac{3\left|e^2 - e^4\right|^2}{13} + i\frac{3\left|e^2 - e^4\right|^2}{13}\right). \tag{113}$$

By using $\kappa_1 = 1/5$ and after simplifying, we get that

$$\begin{aligned}
&\frac{1}{256}\left(\frac{3|-51.8799|^2}{13} + i\frac{3|-51.8799|^2}{13}\right) \\
&\quad \leq \frac{1}{5}\left(\frac{3|-51.8799|^2}{13} + i\frac{3|-51.8799|^2}{13}\right)2.4262(1+i) \\
&\quad \leq 124.2241(1+i).
\end{aligned} \tag{114}$$

Thus, (74) is true for $\kappa_1 = 1/5, \kappa_2 = 1/4, \kappa_3 = 1/10$, and $b = 2$.

So, all conditions of Theorem 16 are satisfied to get a unique CFP, that is "0" of the mappings f, f_1, and f_2.

4. Applications

Here, we provide an application to support our main result. To do this, we take a couple of UITEs to obtain the existing result of a common solution to check the effectiveness of our result. Let the set $\Omega = C([k_1, k_2], \mathbb{R})$ contain real-valued continuous functions defined on $[k_1, k_2]$. In the following, we use Theorem 9 to obtain the existing result of a common solution. This enables us to establish a theorem based on UITEs to attain the existing result of a common solution.

Theorem 19 (see [28]). *Let $\Omega = C([k_1, k_2], \mathbb{R})$, where $[k_1, k_2] \subseteq \mathbb{R}$ and $\eth : \Omega \times \Omega \longrightarrow \mathbb{C}$ is defined as*

$$\eth(\rho_1, \rho_2) = \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \tag{115}$$

for all $\rho_1, \rho_2 \in \Omega$ and $y \in [k_1, k_2]$. Consider that the UITEs are

$$\begin{aligned} \rho_1(y) &= \int_{k_1}^{k_2} Q_1(y, r, \rho_1(r)) dr + \Gamma_1(y), \\ \rho_2(y) &= \int_{k_1}^{k_2} Q_2(y, r, \rho_2(r)) dr + \Gamma_2(y), \end{aligned} \tag{116}$$

where $r \in [k_1, k_2]$. Let $Q_1, Q_2 : [k_1, k_2] \times [k_1, k_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ be such that $D_{\rho_1}, E_{\rho_2} \in \Omega$ for every $\rho_1, \rho_2 \in \Omega$, we have that

$$\begin{aligned} D_{\rho_1}(y) &= \int_{k_1}^{k_2} Q_1(y, r, \rho_1(r)) dr, \\ E_{\rho_2}(y) &= \int_{k_1}^{k_2} Q_2(y, r, \rho_2(r)) dr. \end{aligned} \tag{117}$$

If there exists $\mu \in (0, 1)$ such that, for all $\rho_1, \rho_2 \in \Omega$,

$$\begin{aligned} &\left\| D_{\rho_1}(y) - E_{\rho_2}(y) + \Gamma_1(y) - \Gamma_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1} \\ &\leq \mu M(\rho_1, \rho_2), \end{aligned} \tag{118}$$

where

$$M(\rho_1, \rho_2) = \max \{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}, \tag{119}$$

with

$$\begin{aligned} A_1(\rho_1, \rho_2)(y) &= \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \\ A_2(\rho_1, \rho_2)(y) &= \frac{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \left(\sqrt{1 + k_1^2} e^{i \cot k_1} \right)}{1/2 \left(\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \right)}, \end{aligned} \tag{120}$$

$$\begin{aligned} A_3(\rho_1, \rho_2)(y) = \min \{ &a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), \\ &a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y) \}, \end{aligned} \tag{121}$$

where

$$\begin{aligned} a_1(\rho_1, \rho_2)(y) &= \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \\ a_2(\rho_1, \rho_2)(y) &= \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \\ a_3(\rho_1, \rho_2)(y) &= \frac{\left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \left(\sqrt{1 + k_1^2} e^{i \cot k_1} \right)^2}{1 + \|\rho_1(y) - \rho_2(y)\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}, \\ a_4(\rho_1, \rho_2)(y) &= \frac{\left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_1(y) \right\|^2 \left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2}, \\ a_5(\rho_1, \rho_2)(y) &= \frac{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_2(y) \right\|^2 \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}}{\left\| E_{\rho_2}(y) + \Gamma_2(y) - \rho_1(y) \right\|^2 + \left\| D_{\rho_1}(y) + \Gamma_1(y) - \rho_2(y) \right\|^2}. \end{aligned} \tag{122}$$

Then, the two UITEs, i.e., (41), have a unique common solution.

Proof. Define $f_1, f_2, f : \Omega \longrightarrow \Omega$ as

$$\begin{aligned} f_1 \rho_1 = f_1 \rho_1(y) &= D_{\rho_1}(y) + \Gamma_1(y) = D_{\rho_1} + \Gamma_1, \\ f \rho_1 &= f \rho_1(y) = \rho_1(y) = \rho_1, \\ f_2 \rho_2 = f_2 \rho_2(y) &= E_{\rho_2}(y) + \Gamma_2(y) = E_{\rho_2} + \Gamma_2, \\ f \rho_2 &= f \rho_2(y) = \rho_2(y) = \rho_2. \end{aligned} \tag{123}$$

□

Then, we have the following three cases:

(1) If $A_1(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (118), (119), and (123), we have that

$$\eth(f_1 \rho_1, f_2 \rho_2) \leq \mu \|\rho_1 - \rho_2\|^2 \sqrt{1 + k_1^2} e^{i \cot k_1}, \tag{124}$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, f_1, f_2, and f satisfy all conditions of Theorem 9 with $\mu = \kappa_1$ and $\kappa_2 = \kappa_3 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

(2) If $A_2(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (118), (119), and (123), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right|^2\right|\left|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2 \left(\sqrt{1+k_1^2}e^{i \cot k_1}\right)}{1/2\left(\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2\right)}, \tag{125}$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, f_1, f_2, and f satisfy all conditions of Theorem 9 with $\mu = \kappa_2$ and $\kappa_1 = \kappa_3 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

(3) If $A_3(\rho_1, \rho_2)(y)$ is the maximum term in $\{A_1(\rho_1, \rho_2)(y), A_2(\rho_1, \rho_2)(y), A_3(\rho_1, \rho_2)(y)\}$, then from (119), we have that

$$M(\rho_1, \rho_2) = A_3(\rho_1, \rho_2)(y). \tag{126}$$

Then, there are furthermore five subcases arising:

(i) If $a_1(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu\left\|D_{\rho_1} + \Gamma_1 - \rho_1\right\|^2 \sqrt{1+k_1^2}e^{i \cot k_1}, \tag{127}$$

for all $\rho_1, \rho_2 \in \Omega$

(ii) If $a_2(\rho_1, \rho_2)(y)$is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu\left\|E_{\rho_2} + \Gamma_2 - \rho_2\right\|^2 \sqrt{1+k_1^2}e^{i \cot k_1}, \tag{128}$$

for all $\rho_1, \rho_2 \in \Omega$

(iii) If $a_3(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|D_{\rho_1} + \Gamma_1 - \rho_1\right|^2\right|\left|E_{\rho_2} + \Gamma_2 - \rho_2\right\|^2 \left(\sqrt{1+k_1^2}e^{i \cot k_1}\right)^2}{1 + \|\rho_1 - \rho_2\|^2 \sqrt{1+k_1^2}e^{i \cot k_1}}, \tag{129}$$

for all $\rho_1, \rho_2 \in \Omega$

(iv) If $a_4(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|D_{\rho_1} + \Gamma_1 - \rho_1\right|^2\right|\left|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 \sqrt{1+k_1^2}e^{i \cot k_1}}{\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2}, \tag{130}$$

for all $\rho_1, \rho_2 \in \Omega$

(v) If $a_5(\rho_1, \rho_2)(y)$ is the minimum term in $\{a_1(\rho_1, \rho_2)(y), a_2(\rho_1, \rho_2)(y), a_3(\rho_1, \rho_2)(y), a_4(\rho_1, \rho_2)(y), a_5(\rho_1, \rho_2)(y)\}$. Then from (118), (121), (123), and (126), we have that

$$\eth(f_1\rho_1, f_2\rho_2) \le \mu \frac{\left\|\left\|E_{\rho_2} + \Gamma_2 - \rho_2\right|^2\right|\left|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2 \sqrt{1+k_1^2}e^{i \cot k_1}}{\left\|E_{\rho_2} + \Gamma_2 - \rho_1\right\|^2 + \left\|D_{\rho_1} + \Gamma_1 - \rho_2\right\|^2}, \tag{131}$$

for all $\rho_1, \rho_2 \in \Omega$. Thus, the subcases of Case 3 (Case (i-v)) for the mappings f_1, f_2, and f satisfy all the conditions of Theorem 9 with $\mu = \kappa_3$ and $\kappa_1 = \kappa_2 = 0$ in (1). Then, two UITEs, i.e., (116), have a unique common solution in Ω.

5. Conclusions

We have established some unique CFP-results in complex-valued b-metric space by using rational contraction conditions for three self-mappings in which one self-map is continuous and commutable with the other two self-mappings. In our main work, we have generalized the results (e.g., see [28, 37, 38]). To show the authenticity of our results, we have given some useful examples in the main section. We have also provided an application for our main result to indicate its utility. In this direction, many results can be contributed to the said space by applying different contractions with different types of integral operators.

Data Availability

Data sharing is not applicable to this article as no data set was generated or analyzed during the current study.

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

Acknowledgments

This work was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah, under Grant No. FP-043-43.

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Hindawi
Journal of Function Spaces
Volume 2021, Article ID 4995428, 16 pages
https://doi.org/10.1155/2021/4995428

Research Article

Some Fixed Point Results in Premodular Special Space of Sequences and Their Associated Pre-Quasi-Operator Ideal

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Received 24 August 2021; Accepted 3 December 2021; Published 23 December 2021

Academic Editor: Anita Tomar

A weighted Nakano sequence space and the s-numbers it contains are the subject of this article, which explains the concept of the pre-quasi-norm and its operator ideal. We show that both Kannan contraction and nonexpansive mappings acting on these spaces have a fixed point. A slew of numerical experiments back up our findings. The presence of summable equations' solutions is shown to be useful in a number of ways. Weight and power of the weighted Nakano sequence space are used to define the parameters for this technique, resulting in customizable solutions.

1. Introduction

The spaces of all, bounded, r-absolutely summable, and null sequences of real numbers will be denoted throughout the article by $\mathbb{R}^{\mathbb{N}_0}$, ℓ_∞, ℓ_r, and c_0, respectively, where $\mathbb{N}_0$ is the set of nonnegative integers. $e_y = \{0, 0, \cdots, 1, 0, 0, \cdots\}$, while 1 lies in the y^{th} place, with $y \in \mathbb{N}_0$.

Definition 1 (see [1]). A function $s : \mathscr{L}(\mathscr{E}, \mathscr{H}) \longrightarrow [0,\infty)^{\mathbb{N}_0}$, where $\mathscr{L}(\mathscr{E}, \mathscr{H})$ is the space of all bounded linear operators from a Banach space $\mathscr{E}$ into a Banach space $\mathscr{H}$ and if $\mathscr{E} = \mathscr{H}$, we write $\mathscr{L}(\mathscr{E})$, which transforms every mapping $J \in \mathscr{L}(\mathscr{E}, \mathscr{H})$ to $(s_y(J))_{y=0}^{\infty}$ is said to be s-number, if it is satisfying the following conditions:

(a) $\|J\| = s_0(J) \geq s_1(J) \geq s_2(J) \geq \cdots \geq 0$, for every $J \in \mathscr{L}(\mathscr{E}, \mathscr{H})$

(b) $s_{x+y-1}(J_1 + J_2) \leq s_x(J_1) + s_y(J_2)$, for every $J_1, J_2 \in \mathscr{L}(\mathscr{E}, \mathscr{H})$ and $x, y \in \mathbb{N}_0$

(c) Ideal property: $s_y(WHJ) \leq \|W\| s_y(H) \|J\|$, for every $J \in \mathscr{L}(\mathscr{E}_0, \mathscr{E})$, $H \in \mathscr{L}(\mathscr{E}, \mathscr{H})$, and $W \in \mathscr{L}(\mathscr{H}, \mathscr{H}_0)$, where $\mathscr{E}_0$ and $\mathscr{H}_0$ are any two Banach spaces

(d) If $J \in \mathscr{L}(\mathscr{E}, \mathscr{H})$ and $\delta \in \mathbb{R}$, we have $s_y(\delta J) = |\delta| s_y(J)$

(e) Rank property: if rank $(J) \leq y$, then $s_y(J) = 0$, for all $J \in \mathscr{L}(\mathscr{E}, \mathscr{H})$

(f) Norming property: $s_{k\geq y}(I_y) = 0$ or $s_{k<y}(I_y) = 1$

The yth approximation number, $\alpha_y(J)$, is defined as

$$\alpha_y(J) = \inf \{\|J - \mathscr{K}\| : \mathscr{K} \in \mathscr{L}(\mathscr{E}, \mathscr{H}) \text{ and rank } (\mathscr{K}) \leq y\}. \tag{1}$$

Notations 2 (see [2]). $S_C := \{S_C(\mathscr{E}, \mathscr{H})\}$, where $S_C(\mathscr{E}, \mathscr{H}) := \{J \in \mathscr{L}(\mathscr{E}, \mathscr{H}) : ((s_y(J))_{y=0}^{\infty} \in C\}$. Also, $\{S_C^{\text{app}}(\mathscr{E}, \mathscr{H})\}$, where $S_C^{\text{app}}(\mathscr{E}, \mathscr{H}) := \{J \in \mathscr{L}(\mathscr{E}, \mathscr{H}) : ((\alpha_y(J))_{y=0}^{\infty} \in C\}$.

Fixed point theory, Banach space geometry, normal series theory, ideal transformations, and approximation theory are all examples of ideal operator theorems and summability. The concept of a pre-quasi-operator ideal is introduced and studied by Faried and Bakery [2]. Bakery and Abou Elmatty investigated some topological and geometric structures of $\ell(\gamma, \lambda)$ in [3]. They proved that the space $S^{\mathrm{app}}_{\ell(\gamma,\lambda)}$ is a small pre-quasi-operator ideal and gave a strictly inclusion relation for different weights and powers. Several mathematicians were able to investigate many extensions for contraction mappings defined on the space or on the space itself thanks to the Banach fixed point theorem [4]. Kannan [5] investigated an example of a class of operators that perform the same fixed point actions as contractions but are not continuous. Ghoncheh [6] demonstrated the existence of a Kannan mapping fixed point in complete modular spaces with Fatou property (also see [7–10]). Bakery and Mohamed [11] examined the sufficient requirements on $\ell^{((\lambda_z))}$, variable exponent in $(0, 1]$ under definite pre-quasi-norm so that there is a fixed point of Kannan pre-quasi-norm contraction mappings on this space. For the construction of pre-quasi-Banach and closed spaces, we use a weighted Nakano sequence space, $(\ell(\gamma, \lambda))_P$, with various pre-quasi-norms in this study. Weighted Nakano sequence space's pre-quasi-normal structural features, including the fixed point idea of Kannan pre-quasi-norm contraction and the Kannan pre-quasi-norm nonexpansive mapping in weighted Nakano sequence space, are improved. The existence of a fixed point for the Kannan pre-quasi-norm contraction mapping has been demonstrated using weighted Nakano sequence space and s-numbers. Our talk concluded with various instances of how the information gathered could be put to good use in resolving a problem.

2. Preliminaries and Definitions

We indicate the space of all mappings $P : \mathfrak{A} \longrightarrow [0,\infty)$, by $[0,\infty)^{\mathfrak{A}}$.

Definition 3 (see [12]). If $\mathfrak{A}$ is a vector space and $\theta = (0, 0, 0,)$, a mapping $P \in [0,\infty]^{\mathfrak{A}}$ is said to be modular:

(a) If $g \in \mathfrak{A}$, $g = \theta \Longleftrightarrow P(g) = 0$ with $P(g) \geq 0$

(b) $P(\delta g) = P(g)$ holds, for each $g \in \mathfrak{A}$ and $|\delta| = 1$

(c) The inequality $P(\delta g + (1-\delta) f) \leq P(g) + P(f)$ verifies, for every $g, f \in \mathfrak{A}$ and $\delta \in [0, 1]$.

Definition 4 (see [2]). If the following conditions hold:

$$\{e_y\}_{y\in\mathbb{N}_0} \subseteq \mathfrak{A}. \tag{2}$$

(1) $\mathfrak{A}$ is solid. This means if $g = (g_y) \in \mathbb{R}^{\mathbb{N}_0}$, $f = (f_y) \in \mathfrak{A}$, and $|g_y| \leq |f_y|$, for every $y \in \mathbb{N}_0$, then $g \in \mathfrak{A}$

(2) $(g_{[y/2]})_{y=0}^{\infty} \in \mathfrak{A}$, where $[y/2]$ denotes the integral part of $y/2$, when $(g_y)_{y=0}^{\infty} \in \mathfrak{A}$

Then, $\mathfrak{A}$ is said to be a special space of sequences (sss).

Definition 5 (see [2]). If we have $P \in [0,\infty)^{\mathfrak{A}}$ with the following:

(i) if $g \in \mathfrak{A}$, $g = \theta \Longleftrightarrow P(g) = 0$

(ii) suppose $g \in \mathfrak{A}$ and $\delta \in \mathbb{R}$, then there is $B \geq 1$ for which $P(\delta g) \leq B |\delta| P(g)$

(iii) the inequality, $P(g + f) \leq J(P(g) + P(f))$, for each $g, f \in \mathfrak{A}$, verifies for some $J \geq 1$

(iv) if $z \in \mathbb{N}_0$ and $|g_z| \leq |f_z|$, then $P((g_z)) \leq P((f_z))$

(v) the inequality, $P((g_z)) \leq P((g_{[z/2]})) \leq J_0 P((g_z))$, satisfies, for some $J_0 \geq 1$

(vi) assume F is the space of finite sequences, one has $\bar{F} = \mathfrak{A}_P$

(vii) we have $\sigma > 0$ so that $P(\eta, 0, 0, 0, \cdots) \geq \sigma |\eta| P(1, 0, 0, 0, \cdots)$, for every $\eta \in \mathbb{R}$

Then, $\mathfrak{A}_P \subseteq \mathfrak{A}$ is said to be a premodular sss.

Example 1. Since for all $v, t \in \ell(((a+1)/(2a+5))_{a=0}^{\infty})$, we have

$$P\left(\frac{v+t}{2}\right) = \left(\sum_{a\in\mathbb{N}_0} \left|\frac{v_a + t_a}{2}\right|^{(a+1)/(2a+5)}\right)^5 \leq 8(P(v) + P(t)). \tag{3}$$

Hence, $P(v) = \left(\sum_{a\in\mathbb{N}_0} |v_a|^{(a+1)/(2a+5)}\right)^5$ is a premodular (not a modular) on $\ell(((a+1)/(2a+5))_{a=0}^{\infty})$.

Definition 6 (see [11]). Assume $\mathfrak{A}$ is a sss. The function $P \in [0,\infty)^{\mathfrak{A}}$ is called a pre-quasi-norm on $\mathfrak{A}$, if it satisfies the conditions (i), (ii), and (iii) of Definition 5.

Theorem 7 (see [11]. *Suppose $\mathfrak{A}$ is a premodular sss; then, $\mathfrak{A}$ is a pre-quasi-normed sss.*

Theorem 8 (see [11]). *Quasinormed sss is contained in pre-quasi-normed sss.*

Definition 9 (see [13]).

(a) The pre-quasi-norm P on $\ell(\gamma, \Delta)$ is called P-convex, if $P(\varepsilon g + (1-\varepsilon) f) \leq \varepsilon P(g) + (1-\varepsilon) P(f)$, for each $\varepsilon \in [0, 1]$ and $g, f \in \ell(\gamma, \Delta)$

(b) $\{g_z\}_{z\in\mathbb{N}_0} \subseteq (\ell(\gamma, \Delta))_P$ is P-convergent to $g \in (\ell(\gamma, \Delta))_P$, if and only if $\lim_{z\longrightarrow\infty} P(g_z - g) = 0$. If the P-limit exists, then it is unique

(c) $\{g_z\}_{z\in\mathbb{N}_0} \subseteq (\ell(\gamma,\Delta))_P$ is P-Cauchy, if $\lim_{z,y\longrightarrow\infty} P(g_z - g_y) = 0$

(d) $\Delta \subset (\ell(\gamma,\Delta))_P$ is P-closed, if for every P-converging $\{g_z\}_{z\in\mathbb{N}_0} \subset \Delta$ to g, then $g \in \Delta$

(e) $\Delta \subset (\ell(\gamma,\lambda))_P$ is P-bounded, if $\delta_P(\Delta) = \sup\{P(g-f): g, f \in \Delta\} < \infty$

(f) The P-ball of radius $l \geq 0$ and center g, for every $g \in (\ell(\gamma,\lambda))_P$, is defined as

$$\mathscr{B}_P(g,l) = \{f \in (\ell(\gamma,\lambda))_P : P(g-f) \leq l\} \tag{4}$$

(g) A pre-quasi-norm P on $\ell(\gamma,\lambda)$ satisfies the Fatou property, when for every sequence $\{f^z\} \subseteq (\ell(\gamma,\lambda))_P$ with $\lim_{z\longrightarrow\infty} P(f^z - f) = 0$ and every $g \in (\ell(\gamma,\lambda))_P$ then $P(g-f) \leq \sup_y \inf_{z\geq y} P(g - f^z)$

Recall that the Fatou property implies the P-closed of the P-balls.

Definition 10 (see [14]). Let $\mathscr{L}$ be the class of each bounded linear operators between any two Banach spaces. A subclass $\mathscr{U}$ of $\mathscr{L}$ is known as an operator ideal, if all element $\mathscr{U}(\mathscr{E}, \mathscr{H}) = \mathscr{U} \cap \mathscr{L}(\mathscr{E}, \mathscr{H})$ fulfills the following conditions:

(i) $I_\Gamma \in \mathscr{U}$, where Γ indicates Banach space of one dimension

(ii) The space $\mathscr{U}(\mathscr{E}, \mathscr{H})$ is linear over $\mathbb{R}$

(iii) If $G_1 \in \mathscr{L}(\mathscr{E}_0, \mathscr{E})$, $G_2 \in \mathscr{U}(\mathscr{E}, \mathscr{H})$, and $G_3 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_0)$, then $G_3G_2G_1 \in \mathscr{U}(\mathscr{E}_0, \mathscr{H}_0)$, where $\mathscr{E}_0$ and $\mathscr{H}_0$ are normed spaces

Pre-quasi-operator ideals are more general than quasioperator ideals.

Definition 11 (see [2]). A mapping $\mathbb{P} \in [0,\infty)^{\mathscr{U}}$ is called a pre-quasi-norm when

(a) let $J \in \mathscr{U}(\mathscr{E}, \mathscr{H})$, $\mathbb{P}(J) \geq 0$, and $\mathbb{P}(J) = 0 \Leftrightarrow J = 0$

(b) we have $D \geq 1$ with $\mathbb{P}(\gamma J) \leq D|\gamma|\mathbb{P}(J)$, when $J \in \mathscr{U}(\mathscr{E}, \mathscr{H})$ and $\gamma \in \mathbb{R}$

(c) we have $J \geq 1$ so that $\mathbb{P}(J_1 + J_2) \leq J[\mathbb{P}(J_1) + \mathbb{P}(J_2)]$, for all $J_1, J_2 \in \mathscr{U}(\mathscr{E}, \mathscr{H})$

(d) we have $\tau \geq 1$ such that $J \in \mathscr{L}(\mathscr{E}_0, \mathscr{E})$, $J_1 \in \mathscr{U}(\mathscr{E}, \mathscr{H})$ and $J_2 \in \mathscr{L}(\mathscr{H}, \mathscr{H}_0)$ then $\mathbb{P}(J_2J_1J) \leq \tau\|J_2\|\mathbb{P}(J_1)\|J\|$

Theorem 12 (see [11]). *Suppose $\mathfrak{A}_P$ is a premodular sss, then $\mathbb{P}(J) = P(s_z(J))_{z=0}^{\infty}$ is a pre-quasi-norm on $S_{\mathfrak{A}_P}$.*

Theorem 13 (see [2]). *Quasi-normed ideal is contained in pre-quasi-normed ideal.*

Lemma 14 (see [15]). *If $\lambda \geq 2$ and for all $g, f \in \mathbb{R}$, then*

$$\left|\frac{g+f}{2}\right|^\lambda + \left|\frac{g-f}{2}\right|^\lambda \leq \frac{1}{2}\left(|g|^\lambda + |f|^\lambda\right). \tag{5}$$

Lemma 15 (see [16]). *Let $1 < \lambda \leq 2$ and $g, f \in \mathbb{R}$ with $|g| + |f| \neq 0$; then,*

$$\left|\frac{g+f}{2}\right|^\lambda + \frac{\lambda(\lambda-1)}{2}\left|\frac{g-f}{|g|+|f|}\right|^{2-\lambda}\left|\frac{g-f}{2}\right|^\lambda \leq \frac{1}{2}\left(|g|^\lambda + |f|^\lambda\right). \tag{6}$$

Lemma 16 (see [17]). *Suppose $\lambda_z > 0$ and $g_z, f_z \in \mathbb{R}$, for all $z \in \mathbb{N}_0$; then,*

$$|g_z + f_z|^{\lambda_z} \leq 2^{K-1}\left(|g_z|^{\lambda_z} + |f_z|^{\lambda_z}\right), \tag{7}$$

where $K = \max\{1, \sup_z \lambda_z\}$.

3. Main Results

3.1. The Sequence Space $\ell(\gamma,\lambda)$. Assume $\lambda = (\lambda_z) \in \mathbb{R}^{+\mathbb{N}_0}$ and $\gamma = (\gamma_z) \in \mathbb{R}^{+\mathbb{N}_0}$, where $\mathbb{R}^+$ denotes the set of positive reals. In [3], the weighted Nakano sequence space was defined as

$$\ell(\gamma,\lambda) = \{\nu = (\nu_a) \in \mathbb{R}^{\mathbb{N}_0} : P(\mu\nu) < \infty, \text{ for some } \mu > 0\}, \tag{8}$$

while $P(\nu) = \sum_{a=0}^{\infty} \gamma_a |\nu_a|^{\lambda_a}$.

Theorem 17. *If $(\lambda_a) \in \ell_\infty$, then*

$$\ell(\gamma,\lambda) = \{\nu = (\nu_a) \in \mathbb{R}^{\mathbb{N}_0} : P(\mu\nu) < \infty, \textit{for any } \mu > 0\}. \tag{9}$$

Proof.

$$\begin{aligned}
\ell(\gamma,\lambda) &= \{\nu = (\nu_a) \in \mathbb{R}^{\mathbb{N}_0} : P(\mu\nu) < \infty, \text{ for some } \mu > 0\} \\
&= \left\{\nu = (\nu_a) \in \mathbb{R}^{\mathbb{N}_0} : \inf_a |\mu|^{\lambda_a} \sum_{a=0}^{\infty} \gamma_a |\nu_a|^{\lambda_a}\right. \\
&\quad \left.\leq \sum_{a=0}^{\infty} \gamma_a |\mu\nu_a|^{\lambda_a} < \infty, \text{ for some } \mu > 0\right\} \\
&= \left\{\nu = (\nu_a) \in \mathbb{R}^{\mathbb{N}_0} : \sum_{a=0}^{\infty} \gamma_a |\nu_a|^{\lambda_a} < \infty\right\}
\end{aligned}$$

$$= \left\{ v = (v_a) \in \mathbb{R}^{\mathbb{N}_0} : P(\mu v) < \infty, \text{for any } \mu > 0 \right\}. \quad (10)$$

(1) If $\gamma_a = 1$, with $a \in \mathbb{N}_0$, then $\ell(\gamma, \lambda) = \ell^{((\lambda_a))}$ defined and considered in [18, 19]

(2) If $\gamma_a = 1/\lambda_a$, with $a \in \mathbb{N}_0$, then $\ell(\gamma, \lambda) = \ell_{\lambda(.)}$ examined by many authors [16, 20, 21]

□

Theorem 18. *The space $(\ell(\gamma, \lambda), \|.\|)$ is a Banach space, where $\|v\| = inf\{\kappa > 0 : P(v/\kappa) \leq 1\}$.*

Proof. Since we have the following:

(i) $\|v\| \geq 0$, for each $v \in \ell(\gamma, \lambda)$ and $\|v\| = 0$, if and only if, $v = \theta$

(ii) suppose $v \in \mathbb{R}$, $v \in \ell(\gamma, \lambda)$ without loss of generality, let $v \neq 0$ then

$$\begin{aligned} \|vv\| &= \inf \left\{ \zeta > 0 : P\left(\frac{vv}{\zeta}\right) \leq 1 \right\} \\ &= \inf \left\{ |v|\mu > 0 : P\left(\frac{v}{\mu}\right) \leq 1 \right\} = |v| \|v\| \end{aligned} \quad (11)$$

(iii) assume $v, t \in \ell(\gamma, \lambda)$, then there are $\zeta_1 > 0$ and $\zeta_2 > 0$ be such that $P(v/\zeta_1) \leq 1$ and $P(t/\zeta_2) \leq 1$. Let $\zeta = \zeta_1 + \zeta_2$; since P is nondecreasing and convex, one has

$$P\left(\frac{v+t}{\zeta}\right) = P\left(\frac{v+t}{\zeta_1+\zeta_2}\right) \leq \frac{\zeta_1}{\zeta_1+\zeta_2} P\left(\frac{v}{\zeta_1}\right) + \frac{\zeta_2}{\zeta_1+\zeta_2} P\left(\frac{t}{\zeta_2}\right) \leq 1 \quad (12)$$

As the ζ's are nonnegative, one can see

$$\begin{aligned} \|v+t\| &= \inf \left\{ \zeta > 0 : P\left(\frac{v+t}{\zeta}\right) \leq 1 \right\} \\ &\leq \inf \left\{ \zeta_1 > 0 : P\left(\frac{v}{\zeta_1}\right) \leq 1 \right\} \\ &\quad + \inf \left\{ \zeta_2 > 0 : P\left(\frac{t}{\zeta_2}\right) \leq 1 \right\} \\ &= \|v\| + \|t\|. \end{aligned} \quad (13)$$

Then, the space $(\ell(\gamma, \lambda), \|.\|)$ is a normed space. Next, let $g^x = (g_z^x)_{z=0}^{\infty}$ be a Cauchy sequence in $\ell(\gamma, \lambda)$. Therefore, for every $\varepsilon \in (0, 1)$, we have $x_0 \in \mathbb{N}_0$ such that for all $x, y \geq x_0$, we obtain

$$\|g^x - g^y\| = \inf \left\{ \zeta > 0 : P\left(\frac{g^x - g^y}{\zeta}\right) \leq 1 \right\} < \varepsilon. \quad (14)$$

So, for $x, y \geq x_0$ and $z \in \mathbb{N}_0$, one can see $|g_z^x - g_z^y| < \varepsilon$. Hence, (g_z^y) is a Cauchy sequence in $\mathbb{R}$, for fixed $z \in \mathbb{N}_0$. This implies $\lim_{y \to \infty} g_z^y = g_z^0$, for fixed $z \in \mathbb{N}_0$. Hence, $\|g^x - g^0\| < \varepsilon$, for every $x \geq x_0$. Since $\|g^0\| \leq \|g^x - g^0\| + \|g^x\| < \infty$, therefore, $g^0 \in \ell(\gamma, \lambda)$. This implies that $(\ell(\gamma, \lambda), \|.\|)$ is a Banach space. □

4. Pre-Quasi-Normed Sequence Space

To create pre-quasi-Banach and closed sequence space, we study the conditions on $(\ell(\gamma, \lambda))_P$, where $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$, for each $g \in \ell(\gamma, \lambda)$. The Fatou property of $(\ell(\gamma, \lambda))_P$ has been investigated for various P.

Theorem 19.

(a1) Let $(\lambda_z) \in \mathbb{R}^{+\mathbb{N}_0} \cap \ell_\infty$ be an increase.

(a2) Either (γ_z) is a monotonic decrease or monotonic increase so that there is $E \geq 1$, where $\gamma_{2a+1} \leq E\gamma_z$.

Then, $(\ell(\gamma, \lambda))_P$ is a premodular sss.

Proof.

(i) Evidently, $P(g) \geq 0$ and $P(g) = 0 \Longleftrightarrow g = \theta$.

(1-i) and (iii). Let $g, f \in \ell(\gamma, \lambda)$. As $(\lambda_z) \in \mathbb{R}^{+\mathbb{N}_0} \cap \ell_\infty$, one gets

$$\begin{aligned} P(g+f) &= \left[\sum_{z=0}^{\infty} \gamma_z |g_z + f_z|^{\lambda_z} \right]^{1/K} \\ &\leq \left[\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z} \right]^{1/K} + \left[\sum_{z=0}^{\infty} \gamma_z |f_z|^{\lambda_z} \right]^{1/K} \\ &= P(g) + P(f) < \infty. \end{aligned} \quad (15)$$

Hence, $g + f \in \ell(\gamma, \lambda)$.

(1-ii) and (ii). Let $\eta \in \mathbb{R}$ and $g \in \ell(\gamma, \lambda)$. Since $(\lambda_z) \in \mathbb{R}^{+\mathbb{N}_0} \cap \ell_\infty$, one has

$$\begin{aligned} P(\eta g) &= \left[\sum_{z=0}^{\infty} \gamma_z |\eta g_z|^{\lambda_z} \right]^{1/K} \\ &\leq \sup_z |\eta|^{\lambda_z/K} \left[\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z} \right]^{1/K} \\ &\leq D|\eta| P(g) < \infty, \end{aligned} \quad (16)$$

where $D = \max\{1, \sup_z |\eta|^{(\lambda_z/K)-1}\} \geq 1$. Therefore, $\eta g \in \ell(\gamma, \lambda)$. From conditions (1-i) and (1-ii), one has$\ell(\gamma, \lambda)$ which is linear. And $e_z \in \ell(\gamma, \lambda)$, for all $z \in \mathbb{N}_0$, as

$$P(e_z) = \left[\sum_{r=0}^{\infty} \gamma_r |e_z(r)|^{\lambda_r}\right]^{1/K} = (\gamma_z)^{\lambda_z/K}. \quad (17)$$

(2) and (iv). Let $|g_z| \leq |f_z|$, for all $z \in \mathbb{N}_0$ and $f \in \ell(\gamma, \lambda)$. Since $\gamma_z > 0$, for all $z \in \mathbb{N}_0$, then

$$P(g) = \left[\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}\right]^{1/K} \leq \left[\sum_{z=0}^{\infty} \gamma_z |f_z|^{\lambda_z}\right]^{1/K} = P(f) < \infty, \quad (18)$$

we get $g \in \ell(\gamma, \lambda)$.

(3) and (v). Suppose $(g_z) \in \ell(\gamma, \lambda)$ and (γ_z) is increasing. There is $E \geq 1$ so that $\gamma_{2z+1} \leq E\gamma_z$ and $(\lambda_z) \in \mathbb{R}^{+\mathbb{N}_0} \cap \ell_\infty$ is increasing; one can see

$$\begin{aligned} P\left(\left(g_{\left[\frac{z}{2}\right]}\right)\right) &= \left[\sum_{z=0}^{\infty} \gamma_z \left|g_{[z/2]}\right|^{\lambda_z}\right] \\ &= \left[\sum_{z=0}^{\infty} \gamma_{2z} |g_z|^{\lambda_{2z}} + \sum_{z=0}^{\infty} \gamma_{2z+1} |g_z|^{\lambda_{2z+1}}\right]^{1/K} \\ &\leq \left[\sum_{z=0}^{\infty} \gamma_{2z} |g_z|^{\lambda_z} + \sum_{z=0}^{\infty} \gamma_{2z+1} |g_z|^{\lambda_z}\right]^{1/K} \\ &\leq (2E)^{1/K} \left[\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}\right]^{1/K} \\ &= (2E)^{1/K} P((g_z)), \end{aligned} \quad (19)$$

Then, $(g_{[z/2]}) \in \ell(\gamma, \lambda)$.

(vi) Obviously, $\bar{F} = \ell(\gamma, \lambda)$

(vii) We have $0 < \kappa \leq |\xi|^{(\lambda_0/K)-1}$, for $\xi \neq 0$ or $\kappa > 0$, for $\xi = 0$ such that

$$P(\xi, 0, 0, 0, \cdots) \geq \kappa |\xi| P(1, 0, 0, 0, \cdots) \quad (20)$$

□

Theorem 20. *Let the conditions (a1) and (a2) of Theorem 19 be satisfied, then $(\ell(\gamma, \lambda))_P$ be a pre-quasi-Banach sss.*

Proof. From Theorems 19 and 7, we have $(\ell(\gamma, \lambda))_P$ which is a pre-quasi-normed sss. Suppose $g^x = (g_z^x)_{z=0}^{\infty}$ is a Cauchy sequence in $(\ell(\gamma, \lambda))_P$. Therefore, for all $\varepsilon \in (0, 1)$, we have $x_0 \in \mathbb{N}_0$ such that for all $x, y \geq x_0$, we get

$$P(g^x - g^y) = \left[\sum_{z=0}^{\infty} \gamma_z |g_z^x - g_z^y|^{\lambda_z}\right]^{1/K} < \varepsilon. \quad (21)$$

Hence, for $x, y \geq x_0$ and $z \in \mathbb{N}_0$, one obtains $|g_z^x - g_z^y| < \varepsilon$. This implies (g_z^y) is a Cauchy sequence in $\mathbb{R}$, for fixed $z \in \mathbb{N}_0$. This explains $\lim_{y\longrightarrow\infty} g_z^y = g_z^0$, with fixed $z \in \mathbb{N}_0$. Therefore, $P(g^x - g^0) < \varepsilon$, for all $x \geq x_0$. Also, one has $P(g^0) = P(g^0 - g^x + g^x) \leq P(g^x - g^0) + P(g^x) < \infty$; hence, $g^0 \in \ell(\gamma, \lambda)$. □

Theorem 21. *The space $(\ell(\gamma, \lambda))_P$ is a pre-quasi-closed sss, whenever the conditions (a1) and (a2) of Theorem 19 are satisfied.*

Proof. Let $g^x = (g_z^x)_{z=0}^{\infty} \in (\ell(\gamma, \lambda))_P$ and $\lim_{x\longrightarrow\infty} P(g^x - g^0) = 0$; hence, for all $\varepsilon \in (0, 1)$, one has $x_0 \in \mathbb{N}_0$ such that for every $x \geq x_0$, we obtain

$$\varepsilon > P(g^x - g^0) = \left[\sum_{z=0}^{\infty} \gamma_z |g_z^x - g_z^0|^{\lambda_z}\right]^{1/K}. \quad (22)$$

This gives $|g_z^x - g_z^0| < \varepsilon$. Therefore, (g_z^x) is a convergent sequence in $\mathbb{R}$, for constant $z \in \mathbb{N}_0$. Hence, $\lim_{x\longrightarrow\infty} g_z^x = g_z^0$, with constant $z \in \mathbb{N}_0$. Also, one gets

$$P(g^0) = P(g^0 - g^x + g^x) \leq P(g^x - g^0) + P(g^x) < \infty. \quad (23)$$

Hence, $g^0 \in \ell(\gamma, \lambda)$. □

Theorem 22. *The function $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$, for every $g \in \ell(\gamma, \lambda)$, has the Fatou property, if the conditions (a1) and (a2) of Theorem 19conditions (a1) and (a2) of Theorem 19 are satisfied.*

Proof. Assume $\{f^y\} \subseteq (\ell(\gamma, \lambda))_P$ and $\lim_{y\longrightarrow\infty} P(f^y - f) = 0$. As $(\ell(\gamma, \lambda))_P$ is a pre-quasi-closed space, this implies $f \in (\ell(\gamma, \lambda))_P$. Hence, for all $g \in (\ell(\gamma, \lambda))_P$, we have

$$\begin{aligned} P(g - f) &= \left[\sum_{z=0}^{\infty} \gamma_z |g_z - f_z|^{\lambda_z}\right]^{1/K} \\ &\leq \left[\sum_{z=0}^{\infty} \gamma_z |g_z - f_z^y|^{\lambda_z}\right]^{1/K} + \left[\sum_{z=0}^{\infty} \gamma_z |f_z^y - f_z|^{\lambda_z}\right]^{1/K} \\ &\leq \sup_l \inf_{y \geq l} P(g - f^y). \end{aligned} \quad (24)$$

□

Theorem 23. *The function $P(g) = \sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}$ does not hold the Fatou property, if the setups (a1) and (a2) of Theorem 19 are satisfied with $\lambda_0 > 1$.*

Proof. Assume $\{f^y\} \subseteq (\ell(\gamma, \lambda))_P$ and $\lim_{y\longrightarrow\infty} P(f^y - f) = 0$. As $(\ell(\gamma, \lambda))_P$ is a pre-quasi-closed space, this implies $f \in (\ell(\gamma, \lambda))_P$. Hence, for all $g \in (\ell(\gamma, \lambda))_P$, we have

$$
\begin{aligned}
P(g-f) &= \sum_{z=0}^{\infty} \gamma_z |g_z - f_z|^{\lambda_z} \\
&\le 2^{\sup_z \lambda_z - 1} \left[\sum_{z=0}^{\infty} \gamma_z |g_z - f_z^y|^{\lambda_z} + \sum_{z=0}^{\infty} \gamma_z |f_z^y - f_z|^{\lambda_z} \right] \\
&\le 2^{\sup_z \lambda_z - 1} \sup_l \inf_{y \ge l} P(g - f^y).
\end{aligned}
\tag{25}
$$

□

Example 2. The function $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$ is a pre-quasi-norm and not quasi-norm, for all $g \in \ell(\gamma, \lambda)$.

Example 3. The function $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda}]^{1/\lambda}$ is a pre-quasi-norm, quasi-norm, and not a norm on $\ell(\gamma, (\lambda))$, for $0 < \lambda < 1$.

Example 4. The function $P(g) = \inf\{\kappa > 0 : \sum_{z=0}^{\infty} \gamma_z (|g_z|/\kappa)^{\lambda_z} \le 1\}$ is a norm on $\ell(\gamma, \lambda)$.

5. Kannan Contraction's Fixed Points

Here, P-Lipschitzian mapping acting on $(\ell(\gamma, \lambda))_P$ as Kannan P-Lipschitzian mapping has been defined. We investigate the adequate requirements for a fixed point of Kannan contraction mapping on $(\ell(\gamma, \lambda))_P$ equipped with various pre-quasi-norms.

Definition 24. A mapping $J : \mathfrak{A}_P \longrightarrow \mathfrak{A}_P$ is said to be a Kannan P-Lipschitzian, if there exists $\iota \ge 0$, such that

$$
P(Jg - Jf) \le \iota\{P(Jg - g) + P(Jf - f)\}, \tag{26}
$$

for every $g, f \in \mathfrak{A}_P$.

(1) Let $\iota \in [0, 1/2)$; then, the operator J is called Kannan P-contraction

(2) For $\iota = 1/2$, then the operator J is said to be Kannan P-non-expansive

A vector $v \in \mathfrak{A}_P$ is said to be a fixed point of J, if $J(g) = g$.

Theorem 25. *Assume the conditions (a1) and (a2) of Theorem 19 are satisfied, and $J : (\ell(\gamma, \lambda))_P \longrightarrow (\ell(\gamma, \lambda))_P$ is Kannan P-contraction mapping, where $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$, for all $g \in \ell(\gamma, \lambda)$; hence, J has a unique fixed point.*

Proof. Let the setups be satisfied. Assume $g \in \ell(\gamma, \lambda)$; hence, $J^x g \in \ell(\gamma, \lambda)$. Since J is a Kannan P-contraction mapping, we obtain

$$
P(J^{x+1}g - J^x g) \le \iota\left(P(J^{x+1}g - J^x g) + P(J^x g - J^{x-1}g)\right) \Longrightarrow
$$

$$
\begin{aligned}
P(J^{x+1}g - J^x g) &\le \frac{\iota}{1-\iota} P(J^x g - J^{x-1}g) \\
&\le \left(\frac{\iota}{1-\iota}\right)^2 P(J^{x-1}g - J^{x-2}g) \le \cdots \le \left(\frac{\iota}{1-\iota}\right)^x P(Jg - g).
\end{aligned}
\tag{27}
$$

Therefore, for $y > x$ with $x, y \in \mathbb{N}_0$, one has

$$
\begin{aligned}
P(J^x g - J^y g) &\le \iota\left(P(J^x g - J^{x-1}g) + P(J^y g - J^{y-1}g)\right) \\
&\le \iota\left(\left(\frac{\iota}{1-\iota}\right)^{x-1} + \left(\frac{\iota}{1-\iota}\right)^{y-1}\right) P(Jg - g).
\end{aligned}
\tag{28}
$$

Hence,$\{J^x g\}$ is a Cauchy sequence in$(\ell(\gamma, \lambda))_P$, since $(\ell(\gamma, \lambda))_P$ is pre-quasi-Banach space. We have $f \in (\ell(\gamma, \lambda))_P$ with$\lim_{x \longrightarrow \infty} J^x g = f$, to show that $Jf = f$. As P verifies the Fatou property, we get

$$
P(Jf - f) \le \sup_l \inf_{x \ge l} P(J^{x+1}g - J^x g) \le \sup_l \inf_{x \ge l} \left(\frac{\iota}{1-\iota}\right)^p P(Jg - g) = 0. \tag{29}
$$

Hence, $Jf = f$. So f is a fixed point of J. To show the uniqueness of f, let us have two different fixed points f, $g \in (\ell(\gamma, \lambda))_P$ of J. So, one has

$$
P(f - g) \le P(Jf - Jg) \le \iota(P(Jf - f) + P(Jg - g)) = 0. \tag{30}
$$

This implies $f = g$. □

Example 5. Assume $J : (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P \longrightarrow (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$, where $P(g) = \sqrt{\sum_{a \in \mathbb{N}_0} (a+2)/(2a+3) |g_a|^{(2a+3)/(a+2)}}$, for every $g \in \ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty})$ and

$$
J(g) = \begin{cases} \dfrac{g}{4}, & P(g) \in [0, 1), \\[2mm] \dfrac{g}{5}, & P(g) \in [1, \infty). \end{cases} \tag{31}
$$

As for each $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1), P(g_2) \in [0, 1)$, one has

$$
\begin{aligned}
P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{4} - \frac{g_2}{4}\right) \le \frac{1}{\sqrt[4]{27}}\left(P\left(\frac{3g_1}{4}\right) + P\left(\frac{3g_2}{4}\right)\right) \\
&= \frac{1}{\sqrt[4]{27}}\left(P(Jg_1 - g_1) + P(Jg_2 - g_2)\right).
\end{aligned}
\tag{32}
$$

For all $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1), P(g_2) \in [1, \infty)$, one has

$$\begin{aligned}P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{5} - \frac{g_2}{5}\right) \le \frac{1}{\sqrt[4]{64}}\left(P\left(\frac{4g_1}{5}\right) + P\left(\frac{4g_2}{5}\right)\right)\\ &= \frac{1}{\sqrt[4]{64}}(P(Jg_1 - g_1) + P(Jg_2 - g_2)).\end{aligned} \tag{33}$$

For all $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1) \in [0,1)$ and $P(g_2) \in [1,\infty)$, we get

$$\begin{aligned}P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{4} - \frac{g_2}{5}\right) \le \frac{1}{\sqrt[4]{27}}P\left(\frac{3g_1}{4}\right) + \frac{1}{\sqrt[4]{64}}P\left(\frac{4g_2}{5}\right)\\ &\le \frac{1}{\sqrt[4]{27}}\left(P\left(\frac{3g_1}{4}\right) + P\left(\frac{4g_2}{5}\right)\right)\\ &= \frac{1}{\sqrt[4]{27}}(P(Jg_1 - g_1) + P(Jg_2 - g_2)).\end{aligned} \tag{34}$$

Hence, J is Kannan P-contraction and holds one element $\theta \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$, so that $J(\theta) = \theta$, by Theorem 25.

Corollary 26. *Let conditions (a1) and (a2) of Theorem 19 be satisfied, and $J : (\ell(\gamma,\lambda))_P \longrightarrow (\ell(\gamma,\lambda))_P$ is Kannan P -contraction mapping, where $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$, for all $g \in \ell(\gamma,\lambda)$; then, J has a unique fixed point f with $P(J^x g - f) \le \iota(\iota/(1-\iota))^{x-1} P(Jg - g)$.*

Proof. In view of Theorem 25, we have a unique fixed point f of J. Therefore, one gets

$$\begin{aligned}P(J^x g - f) &= P(J^x g - Jf)\\ &\le \iota\left(P\left(J^x g - J^{x-1} g\right) + P(Jf - f)\right)\\ &= \iota\left(\frac{\iota}{1-\iota}\right)^{x-1} P(Jg - g).\end{aligned} \tag{35}$$

□

Definition 27. Assume $\mathfrak{A}_P$ is a pre-quasi-normed sss, $J : \mathfrak{A}_P \longrightarrow \mathfrak{A}_P$ and $f \in \mathfrak{A}_P$. The operator J is said to be P-sequentially continuous at f, if and only if, when $\lim_{x\longrightarrow\infty} P(g_x - f) = 0$, then $\lim_{x\longrightarrow\infty} P(Jg_x - Jf) = 0$.

Example 6. Suppose $J : (\ell(((2z+4)/(z+1))_{z=0}^{\infty}, ((z+1)/(2z+4))_{z=0}^{\infty}))_P \longrightarrow (\ell(((2z+4)/(z+1))_{z=0}^{\infty}, ((z+1)/(2z+4))_{z=0}^{\infty}))_P$, where $P(g) = [\sum_{z\in\mathcal{N}} (2z+4)/(z+1) |g_z|^{(z+1)/(2z+4)}]^4$, for every $g \in \ell(((2z+4)/(z+1))_{z=0}^{\infty}, ((z+1)/(2z+4))_{z=0}^{\infty})$ and

$$J(g) = \begin{cases}\dfrac{1}{18}(e_0 + g), & g_0 \in \left(-\infty, \dfrac{1}{17}\right),\\[2mm] \dfrac{1}{17}e_0, & g_0 = \dfrac{1}{17},\\[2mm] \dfrac{1}{18}e_0, & g_0 \in \left(\dfrac{1}{17}, \infty\right).\end{cases} \tag{36}$$

J is clearly both P-sequentially continuous and discontinuous at $(1/17)e_0 \in (\ell(((2z+4)/(z+1))_{z=0}^{\infty}, ((z+1)/(2z+4))_{z=0}^{\infty}))_P$.

Example 7. Assume J is defined as in Example 5. Suppose $\{g^{(n)}\} \subseteq (\ell(((z+2)/(2z+3))_{z=0}^{\infty}, ((2z+3)/(z+2))_{z=0}^{\infty}))_P$ is such that $\lim_{n\longrightarrow\infty} P(g^{(n)} - g^{(0)}) = 0$, where $g^{(0)} \in (\ell(((z+2)/(2z+3))_{z=0}^{\infty}, ((2z+3)/(z+2))_{z=0}^{\infty}))_P$ with $P(g^{(0)}) = 1$.

As the pre-quasi-norm P is continuous, we have

$$\lim_{n\longrightarrow\infty} P\left(Jg^{(n)} - Jg^{(0)}\right) = \lim_{n\longrightarrow\infty} P\left(\frac{g^{(n)}}{4} - \frac{g^{(0)}}{5}\right) = P\left(\frac{g^{(0)}}{20}\right) > 0. \tag{37}$$

Therefore, J is not P-sequentially continuous at $g^{(0)}$.

Theorem 28. *If the conditions (a1) and (a2) of Theorem 19 are satisfied with $\lambda_0 > 1$ and $J : (\ell(\gamma,\lambda))_P \longrightarrow (\ell(\gamma,\lambda))_P$, where $P(g) = \sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}$, for all $g \in \ell(\gamma,\lambda)$,*

(1) suppose J is Kannan P-contraction mapping

(2) assume J is P-sequentially continuous at a point $g \in (\ell(\gamma,\lambda))_P$

(3) we have $t \in (\ell(\gamma,\lambda))_P$ such that $\{J^x t\}$ has a subsequence $\{J^{x_j} t\}$ converging to f; then, $f \in (\ell(\gamma,\lambda))_P$ is the only fixed point of J

Proof. Suppose the settings are verified. Assume f is not a fixed point of J, then $Jf \ne f$. From parts (54) and (55), one gets

$$\lim_{x_j\longrightarrow\infty} P(J^{x_j} t - f) = 0 \text{ and } \lim_{x_j\longrightarrow\infty} P\left(J^{x_j+1} t - Jf\right) = 0. \tag{38}$$

Since J is Kannan P-contraction, we obtain

$$\begin{aligned}0 < P(Jf - f) &= P\left(\left(Jf - J^{x_j+1} t\right) + \left(J^{x_j} t - f\right) + \left(J^{x_j+1} t - J^{x_j} t\right)\right)\\ &\le 2^{2\sup_j \lambda_j - 2} P\left(J^{x_j+1} t - Jf\right) + 2^{2\sup_j \lambda_j - 2} P\left(J^{x_j} t - f\right)\\ &\quad + 2^{\sup_j \lambda_j - 1} \iota\left(\frac{\iota}{1-\iota}\right)^{x_j - 1} P(Jt - t).\end{aligned} \tag{39}$$

We get a contradiction when $x_j \longrightarrow \infty$. To show the uniqueness of f, suppose we have two different fixed points $f, g \in (\ell(\gamma,\lambda))_P$ of J. Therefore, one obtains

$$P(f-g) \le P(Jf - Jg) \le \iota(P(Jf-f) + P(Jg-g)) = 0. \tag{40}$$

Hence, $f = g$. □

Example 8. Assume J is defined as in Example 5. Let $P(g) = \sum_{a\in\mathbb{N}_0} (a+2)/(2a+3)|g_a|^{(2a+3)/(a+2)}$, for every $g \in \ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty})$.

As for each $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1), P(g_2) \in [0,1)$, one has

$$\begin{aligned} P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{4} - \frac{g_2}{4}\right) \le \frac{2}{\sqrt{27}}\left(P\left(\frac{3g_1}{4}\right) + P\left(\frac{3g_2}{4}\right)\right) \\ &= \frac{2}{\sqrt{27}}(P(Jg_1 - g_1) + P(Jg_2 - g_2)). \end{aligned} \tag{41}$$

For all $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1), P(g_2) \in [1,\infty)$, one has

$$\begin{aligned} P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{5} - \frac{g_2}{5}\right) \le \frac{1}{4}\left(P\left(\frac{4g_1}{5}\right) + P\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{1}{4}(P(Jg_1 - g_1) + P(Jg_2 - g_2)). \end{aligned} \tag{42}$$

For all $g_1, g_2 \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ with $P(g_1) \in [0,1)$ and $P(g_2) \in [1,\infty)$, we obtain

$$\begin{aligned} P(Jg_1 - Jg_2) &= P\left(\frac{g_1}{4} - \frac{g_2}{5}\right) \le \frac{2}{\sqrt{27}}P\left(\frac{3g_1}{4}\right) + \frac{1}{4}P\left(\frac{4g_2}{5}\right) \\ &\le \frac{2}{\sqrt{27}}\left(P\left(\frac{3g_1}{4}\right) + P\left(\frac{4g_2}{5}\right)\right) \\ &= \frac{2}{\sqrt{27}}(P(Jg_1 - g_1) + P(Jg_2 - g_2)). \end{aligned} \tag{43}$$

So, the mapping J is Kannan P-contraction and

$$J^p(g) = \begin{cases} \dfrac{g}{4^p}, & P(g) \in [0,1), \\ \dfrac{g}{5^p}, & P(g) \in [1,\infty). \end{cases} \tag{44}$$

Obviously, J is P-sequentially continuous at $\theta \in (\ell(((a+2(/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ and $\{J^p g\}$ contains a subsequence $\{J^{p_i} g\}$ converging to θ. From Theorem 28, the vector $\theta \in (\ell(((a+2)/(2a+3))_{a=0}^{\infty}, ((2a+3)/(a+2))_{a=0}^{\infty}))_P$ is the unique fixed point of J.

6. Kannan Nonexpansive Fixed Points

The uniform convexity (UUC 2) defined in [22] of the space $(\ell(\gamma,\lambda))_P$ has been investigated, where $P(g) = [\sum_{d=0}^{\infty} \gamma_d |g_d|^{\lambda_d}]^{1/K}$, for all $g \in \ell(\gamma,\lambda)$. The property (R) and the P-normal structure property of this space have been discussed. Finally, we present the sufficient conditions on this space such that the Kannan pre-quasi-norm nonexpansive mapping on it has a fixed point.

Definition 29 (see [23, 24]).

(1) [25] Suppose $r > 0$ and $h > 0$. Let

$$\mathbb{H}_1(r,h) = \{(g,f) : g, h \in \mathfrak{A}_P, P(g) \le r, P(f) \le h, P(g-h) \ge rh\} \tag{45}$$

If $\mathbb{H}_1(r,h) \ne \varnothing$, we put

$$\mathrm{H}_1(r,h) = \inf\left\{1 - \frac{1}{r}P\left(\frac{g+f}{2}\right) : (g,f) \in \mathbb{H}_1(r,h)\right\}. \tag{46}$$

If $\mathbb{H}_1(r,h) = \varnothing$, we put $H_1(r,h) = 1$. The function P holds the uniform convexity (UC), if for all $r > 0$ and $h > 0$, one has $H_1(r,h) > 0$. Note that for every $r > 0$, then $\mathbb{H}_1(r, h) \ne \varnothing$, for very small $h > 0$.

(2) [22] The function P holds (UUC1), if for every $x \ge 0$ and $h > 0$, we have $\beta_1(x,h)$ with

$$H_1(r,h) > \beta_1(x,h) > 0, \text{and } r > x \tag{47}$$

(3) [22] Let $r > 0$ and $h > 0$. Suppose

$$\mathbb{H}_2(r,h) = \left\{(g,f) : g, f \in \mathfrak{A}_P, P(g) \le r, P(f) \le r, P\left(\frac{g-f}{2}\right) \ge rh\right\} \tag{48}$$

When $\mathbb{H}_2(r,h) \ne \varnothing$, we put

$$H_2(r,h) = \inf\left\{1 - \frac{1}{r}P\left(\frac{g+f}{2}\right) : (g,f) \in \mathbb{H}_2(r,h)\right\}. \tag{49}$$

If $\mathbb{H}_2(r,h) = \varnothing$, we place $H_2(r,h) = 1$. The function P supports (UC 2), if for every $r > 0$ and $h > 0$, one has $H_2(r,h) > 0$. Note that for all $r > 0$, $\mathbb{H}_2(r,h) \ne \varnothing$, with very small $h > 0$.

(4) [22] P verifies (UUC 2); when $x \ge 0$ and $h > 0$, one has $\beta_2(x,h)$ with

$$H_2(r,h) > \beta_2(x,h) > 0, \quad \text{for } r > x \tag{50}$$

(5) [25] The function P is strictly convex, (SC); when $g, f \in \mathfrak{A}_P$ with $P(g) = P(f)$ and $P((g+f)/2) = (P(g) + P(f))/2$, one obtains $g = f$

We will need the following comment here and later: $P_V(g) = [\sum_{m\in V} \gamma_m |g_m|^{\lambda_m}]^{1/K}$, for all $V \subset \mathbb{N}_0$ and $g \in (\ell(\gamma,\lambda))_P$. If $V = \varnothing$, we set $P_V(g) = 0$.

Theorem 30. *If the conditions (a1) and (a2) of Theorem 19 are satisfied with $\lambda_0 > 1$, then the pre-quasi-norm P on $\ell(\gamma, \lambda)$ is (UUC2).*

Proof. Assume the settings are satisfied, $r > x \geq 0$ and $h > 0$. Let $g, t \in \ell(\gamma, \lambda)$ so that

$$\begin{gathered} P(g) \leq r, \\ P(f) \leq r, \\ P\left(\frac{g-f}{2}\right) \geq rh. \end{gathered} \tag{51}$$

From the definition of P, we have

$$\begin{aligned} rh \leq P\left(\frac{g-f}{2}\right) &= \left[\sum_{y=0}^{\infty} \gamma_y \left|\frac{g_y - f_y}{2}\right|^{\lambda_y}\right]^{1/K} \\ &\leq \left[2^{-\lambda_0} \sum_{y=0}^{\infty} \gamma_y \left|g_y - f_y\right|^{\lambda_y}\right]^{1/K} \\ &\leq 2^{-\lambda_0/K} \left(\left[\sum_{y=0}^{\infty} \gamma_y \left|g_y\right|^{\lambda_y}\right]^{1/K} + \left[\sum_{y=0}^{\infty} \gamma_y \left|f_y\right|^{\lambda_y}\right]^{1/K}\right) \\ &= 2^{-\lambda_0/K}(P(g) + P(f)) \leq 2r, \end{aligned} \tag{52}$$

which implies $h \leq 2$. Consequent, put $A = \{y \in \mathbb{N}_0 : \lambda_y \geq 2\}$ and $B = \{y \in \mathbb{N}_0 : 1 < \lambda_y < 2\} = \mathbb{N}_0 \setminus A$. Let $\kappa \in \ell(\gamma, \lambda)$; we get $P^K(\kappa) = P_A^K(\kappa) + P_B^K(\kappa)$. By using the conditions, we get $P_A((g-f)/2) \geq (rh/2)$ or $P_B((g-f)/2) \geq (rh/2)$. Assume first $P_A((g-f)/2) \geq (rh/2)$. Using Lemma 14, one gets

$$P_A^K\left(\frac{g+f}{2}\right) + P_A^K\left(\frac{g-f}{2}\right) \leq \frac{P_A^K(g) + P_A^K(f)}{2}. \tag{53}$$

This explains

$$P_A^K\left(\frac{g+f}{2}\right) \leq \frac{P_A^K(g) + P_A^K(f)}{2} - \left(\frac{rh}{2}\right)^K. \tag{54}$$

Since

$$P_B^K\left(\frac{g+f}{2}\right) \leq \frac{P_B^K(g) + P_B^K(f)}{2}, \tag{55}$$

by adding inequalities 2 and 3, and from inequality 1, we have

$$P^K\left(\frac{g+f}{2}\right) \leq \frac{P^K(g) + P^K(f)}{2} - \left(\frac{rh}{2}\right)^K \leq r^K\left(1 - \left(\frac{h}{2}\right)^K\right). \tag{56}$$

This gives

$$P\left(\frac{g+f}{2}\right) \leq r\left(1 - \left(\frac{h}{2}\right)^K\right)^{1/K}. \tag{57}$$

Next, suppose $P_B((g-f)/2) \geq (rh/2)$. Set $D = (h/4)^K$,

$$\begin{gathered} B_1 = \left\{y \in Q : \left|g_y - f_y\right| \leq D\left(\left|g_y\right| + \left|f_y\right|\right)\right\}, \\ B_2 = B \setminus B_1. \end{gathered} \tag{58}$$

As $D \leq 1$ and the power function is convex, so

$$\begin{aligned} P_{B_1}^K\left(\frac{g-f}{2}\right) &\leq \sum_{y\in B_1} D^{\lambda_y} \gamma_y \left|\frac{|g_y| + |f_y|}{2}\right|^{\lambda_y} \\ &\leq \left(\frac{D}{2}\right)^{\lambda_0} \left(P_{B_1}^K(g) + P_{B_1}^K(f)\right) \\ &\leq \frac{D}{2}\left(P_B^K(g) + P_B^K(f)\right) \leq Dr^K. \end{aligned} \tag{59}$$

Since $P_B((g-f)/2) \geq (rh/2)$, we get

$$P_{B_2}^K\left(\frac{g-f}{2}\right) = P_B^K\left(\frac{g-f}{2}\right) - P_{B_1}^K\left(\frac{g-f}{2}\right) \geq r^K\left(\left(\frac{h}{2}\right)^K - \left(\frac{h}{4}\right)^K\right). \tag{60}$$

For any $d \in B_2$, we have

$$\begin{gathered} \lambda_0 - 1 < \lambda_0(\lambda_0 - 1) \leq \leq \lambda_{y-1}\left(\lambda_{y-1} - 1\right) \leq \lambda_y\left(\lambda_y - 1\right), \\ D \leq D^{2-\lambda_y} \leq \left|\frac{g_y - f_y}{\left|g_y\right| + \left|f_y\right|}\right|^{2-\lambda_y}. \end{gathered} \tag{61}$$

By Lemma 15, one gets

$$\gamma_y\left|\frac{g_y + f_y}{2}\right|^{\lambda_y} + \frac{(\lambda_0 - 1)D}{2}\gamma_y\left|\frac{g_y - f_y}{2}\right|^{\lambda_y} \leq \frac{1}{2}\left(\gamma_y\left|g_y\right|^{\lambda_y} + \gamma_y\left|f_y\right|^{\lambda_y}\right). \tag{62}$$

Hence,

$$P_{B_2}^K\left(\frac{g+f}{2}\right)+\frac{(\lambda_0-1)D}{2}P_{B_2}^K\left(\frac{g-f}{2}\right)\le\frac{P_{B_2}^K(g)+P_{B_2}^K(f)}{2}. \tag{63}$$

This investigates

$$P_{B_2}^K\left(\frac{g+f}{2}\right)\le\frac{P_{B_2}^K(g)+P_{B_2}^K(f)}{2}-\frac{(\lambda_0-1)D}{2}r^K\left(\left(\frac{h}{2}\right)^K-\left(\frac{h}{4}\right)^K\right). \tag{64}$$

Since

$$P_{B_1}^K\left(\frac{g+f}{2}\right)\le\frac{P_{B_1}^K(g)+P_{B_1}^K(f)}{2}, \tag{65}$$

by adding inequalities 5 and 6, one has

$$\begin{aligned}P_B^K\left(\frac{g+f}{2}\right)&\le\frac{P_B^K(g)+P_B^K(f)}{2}-\frac{(\lambda_0-1)D}{2}r^K\left(\left(\frac{h}{2}\right)^K-\left(\frac{h}{4}\right)^K\right)\\&\le\frac{P_B^K(g)+P_B^K(f)}{2}-\frac{(\lambda_0-1)}{2}\left(\frac{h}{4}\right)^{2K}r^K\left(2^K-1\right)\\&\le\frac{P_B^K(g)+P_B^K(f)}{2}-\frac{(\lambda_0-1)}{2^K-1}\left(\frac{h}{4}\right)^{2K}r^K.\end{aligned} \tag{66}$$

Since

$$P_A^K\left(\frac{g+f}{2}\right)\le\frac{P_A^K(g)+P_A^K(f)}{2}, \tag{67}$$

by adding inequalities 7 and 8, and from inequality 1, we obtain

$$\begin{aligned}P^K\left(\frac{g+f}{2}\right)&\le\frac{P^K(g)+P^K(f)}{2}-\frac{(\lambda_0-1)}{2^K-1}\left(\frac{h}{4}\right)^{2K}r^K\\&\le r^K\left[1-\frac{(\lambda_0-1)}{2^K-1}\left(\frac{h}{4}\right)^{2K}\right].\end{aligned} \tag{68}$$

This implies

$$P\left(\frac{g+f}{2}\right)\le r\left[1-\frac{(\lambda_0-1)}{2^K-1}\left(\frac{h}{4}\right)^{2K}\right]^{1/K}. \tag{69}$$

It is clear that

$$1<\lambda_0\le K<2^K\Rightarrow 0<\frac{\lambda_0-1}{2^K-1}<1. \tag{70}$$

By using inequalities 4 and 9 and Definition 29, if we put

$$\beta_2(x,h)=\min\left(1-\left(1-\left(\frac{h}{2}\right)^K\right)^{1/K},1-\left[1-\frac{(\lambda_0-1)}{2^K-1}\left(\frac{h}{4}\right)^{2K}\right]^{1/K}\right), \tag{71}$$

therefore, we have $H_2(r,h)>\beta_2(x,h)>0$; this implies P is (UUC2). ☐

Definition 31. Space $\mathfrak{A}_P$ is said to satisfy the property (R), if for all decreasing sequence $\{\Delta_j\}_{j\in\mathbb{N}_0}$ of P -closed and P -convex nonempty subsets of $\mathfrak{A}_P$ with $\sup_{j\in\mathbb{N}_0}d_P(g,\Delta_j)<\infty$, for some $g\in\mathfrak{A}_P$, then one has $\bigcap_{j\in\mathbb{N}_0}\Delta_j\neq\emptyset$.

Theorem 32. *If conditions (a1) and (a2) of Theorem 19 are satisfied with $\lambda_0>1$, then*

(1) Assume Δ is a nonempty P-closed and P-convex subset of $(\ell(\gamma,\lambda))_P$. Let $g\in(\ell(\gamma,\lambda))_P$ be with

$$d_P(g,\Delta)=\inf\{P(g-f):f\in\Delta\}<\infty \tag{72}$$

Hence, one has a unique $\eta\in\Delta$ so that $d_P(g,\Delta)=P(g-\eta)$.

(2) The property (R) holds on $(\ell(\gamma,\lambda))_P$

Proof. Suppose the conditions are satisfied. To show (51), assume $g\notin\Delta$ as Δ is P-closed. So, one has $D\coloneqq d_P(g,\Delta)>0$. Hence, for every $x\in\mathbb{N}_0$, one has $f_x\in\Delta$ with $P(g-f_x)<D(1+1/x)$. Assume $\{f_x/2\}$ is not P-Cauchy. Therefore, we get a subsequence $\{f_{m(x)}/2\}$ and $y_0>0$ with $P((f_{m(x)}-f_{m(a)})/2)\ge y_0$, for every $x>a\ge 0$. Furthermore, we have $\mathrm{H}_2(D(1+(1/x)),y_0/2D)>\iota\coloneqq\beta_2(D(1+(1/x)),y_0/2D)>0$, for all $x\in\mathbb{N}_0$. As

$$\max\left(P\left(g-f_{m(x)}\right),P\left(g-f_{m(a)}\right)\right)\le D\left(1+\frac{1}{m(a)}\right) \tag{73}$$

and

$$P\left(\frac{f_{m(x)}-f_{m(a)}}{2}\right)\ge y_0\ge D\left(1+\frac{1}{m(a)}\right)\frac{y_0}{2D}, \tag{74}$$

for all $x>a\ge 0$, we obtain

$$P\left(g-\frac{f_{m(x)}+f_{m(a)}}{2}\right)\le D\left(1+\frac{1}{m(a)}\right)(1-\iota). \tag{75}$$

So

$$D=d_P(g,\Delta)\le D\left(1+\frac{1}{m(a)}\right)(1-\iota), \tag{76}$$

for any $a\in\mathbb{N}_0$. If we let $a\longrightarrow\infty$, we get

$$0 < D \le D\left(1 + \frac{1}{m(a)}\right)(1 - \iota) < D, \tag{77}$$

which is a contradiction. So, $\{f_x/2\}$ is P-Cauchy. Since $(\ell(\gamma,\lambda))_P$ is P-complete, then $\{f_x/2\}P$-converges to some f. For every $a \in \mathbb{N}_0$, one has the sequence $\{(f_x + f_a)/2\}P$ converges to $f + (f_a/2)$. As Δ is P-closed and P-convex, one gets $f + (f_a/2) \in \Delta$. Surely $f + (f_a/2)P$ converges to 2 f, which implies $2f \in \Delta$. By setting $\eta = 2f$ and using Theorem 22, as P satisfies the Fatou property, we obtain

$$\begin{aligned} d_P(g,\Delta) &\le P(g-\eta) \le \sup_i \inf_{a\ge i} P\left(g - \left(f + \frac{f_a}{2}\right)\right) \\ &\le \sup_i \inf_{a\ge i} \sup_i \inf_{x\ge i} P\left(g - \frac{f_x + f_a}{2}\right) \\ &\le \frac{1}{2} \sup_i \inf_{a\ge i} \sup_i \inf_{x\ge i} [P(g-f_x) + P(g-f_a)] = d_P(g,\Delta). \end{aligned} \tag{78}$$

Hence, $P(g-\lambda) = d_P(g,\Delta)$. As P is (UUC2), then P is (SC); this implies η is only one. To show (2), let $g \notin \Delta_{x_0}$, for some $x_0 \in \mathbb{N}_0$. Since $(d_P(g,\Delta_x))_{x\in\mathbb{N}_0} \in \ell_\infty$ is increasing. Set $\lim_{x\longrightarrow\infty} d_P(g,\Delta_x) = D$. If $D > 0$. Else $g \in \Delta_x$, for all $x \in \mathbb{N}_0$. From (51), we get one point $f_x \in \Delta_x$ with $d_P(g,\Delta_x) = P(g - f_x)$, for every $x \in \mathbb{N}_0$. A consistent proof will show that $\{f_x/2\}P$ converges to some $f \in (\ell(\gamma,\lambda))_P$. Since $\{\Delta_x\}$ are P-convex, decreasing, and P-closed, we get $2f \in \cap_{x\in\mathbb{N}_0}\Delta_x$. □

Definition 33. $(\ell(\gamma,\lambda))_P$ satisfies the P -normal structure property, if for all nonempty P -bounded, P -convex, and P -closed subset Δ of $(\ell(\gamma,\lambda))_P$ not decreased to one point, we have $g \in \Delta$ with

$$\sup_{f\in\Delta} P(g-f) < \delta_P(\Delta) \coloneqq \sup\{P(g-f) \colon g,f \in \Delta\} < \infty. \tag{79}$$

Theorem 34. *If the conditions (a1) and (a2) of Theorem 19 are satisfied with $\lambda_0 > 1$, then $(\ell(\gamma,\lambda))_P$ has the P-normal structure property.*

Proof. Suppose the setups are satisfied. Theorem 30 implies that P is (UUC2). Let Δ be a P-bounded, P-convex, and P -closed subset of $(\ell(\gamma,\lambda))_P$ not decreased to one point. Hence, $\delta_P(\Delta) > 0$. Put $D = \delta_P(\Delta)$. Suppose $g, f \in \Delta$ with $g \ne f$. Hence, $P((g-f)/2) = y > 0$. For every $\eta \in \Delta$, we have $P(g-\eta) \le D$ and $P(f-\eta) \le D$. Since Δ is P-convex, one obtains $((g+f)/2) \in \Delta$. Hence,

$$P\left(\frac{g+f}{2} - \eta\right) = P\left(\frac{(g-\eta)+(f-\eta)}{2}\right) \le D\left(1 - \mathrm{H}_2\left(D, \frac{y}{D}\right)\right), \tag{80}$$

for every $\eta \in \Delta$. Hence,

$$\sup_{\eta\in\Delta} P\left(\frac{g+f}{2} - \eta\right) \le D\left(1 - \mathrm{H}_2\left(D, \frac{y}{D}\right)\right) < D = \delta_P(\Delta). \tag{81}$$

□

Lemma 35. *Let Δ be a nonempty P-bounded, P-convex, and P-closed subset of $(\ell(\gamma,\lambda))_P$, where $(\ell(\gamma,\lambda))_P$ verifies the (R) property and the P -quasi-normal property, and $J : \Delta \longrightarrow \Delta$ be a Kannan P -nonexpansive mapping. For $z > 0$, suppose $G_z = \{g \in \Delta : P(g - J(g)) \le z\} \ne \varnothing$. Take*

$$\Delta_z = \bigcap \{\mathscr{B}_P(x,y) \colon J(K_z) \subset \mathscr{B}_P(x,y)\} \cap \Delta. \tag{82}$$

Then, Δ_z is a nonempty, P-convex, and P-closed subset of Δ and

$$J(\Delta_z) \subset \Delta_z \subset K_z \text{ and } \delta_P(\Delta_z) \le z. \tag{83}$$

Proof. As $J(K_z) \subset \Delta_z$, which implies $\Delta_z \ne \varnothing$, Δ_z is a P-closed and P-convex subset of Δ, as the P-balls are P-convex and P -closed. Assume $g \in \Delta_z$. When $P(g - J(g)) = 0$, we get $g \in K_z$. Else, suppose $P(g - J(g)) > 0$. Take

$$x = \sup\{P(J(\kappa) - J(g)) \colon \kappa \in K_z\}. \tag{84}$$

From the definition of x, so $J(K_z) \subset \mathscr{B}_P(J(g), x)$. Hence, $\Delta_z \subset \mathscr{B}_P(J(g), x)$, which implies $P(g - J(g)) \le x$. Assume $r > 0$. Hence, there is $\kappa \in K_z$ with $x - r \le P(J(\kappa) - J(g))$. Then,

$$\begin{aligned} P(g - J(g)) - r &\le x - r \le P(J(\kappa) - J(g)) \\ &\le \frac{1}{2}(P(g - J(g)) + P(\kappa - J(\kappa))) \\ &\le \frac{1}{2}(P(g - J(g)) + z). \end{aligned} \tag{85}$$

Since r is randomly positive, one has $P(g - J(g)) \le z$; then, we have $g \in K_z$. This implies $\Delta_z \subset K_z$. As $J(K_z) \subset \Delta_z$, we get $J(\Delta_z) \subset J(K_z) \subset \Delta_z$; this indicates Δ_z is J-invariant, consequent to prove that $\delta_P(\Delta_z) \le z$. As

$$P(J(g) - J(f)) \le \frac{1}{2}(P(g - J(g)) + P(f - J(f))), \tag{86}$$

for every $g, f \in K_z$, let $g \in K_z$. Then, $J(K_z) \subset \mathscr{B}_P(J(g), z)$. The definition of Δ_z implies $\Delta_z \subset \mathscr{B}_P(J(g), z)$. So, $J(g) \in \bigcap_{f\in\Delta_z} \mathscr{B}_P(f, z)$. Therefore, we obtain $P(f - \kappa) \le z$, for every $f, \kappa \in \Delta_z$; this gives $\delta_P(\Delta_z) \le z$. □

Theorem 36. *Let Δ be a nonempty, P -convex, P -closed, and P -bounded subset of $(\ell(\gamma,\lambda))_P$, where $(\ell(\gamma,\lambda))_P$ holds the P -quasi-normal property and the (R) property, and $J : \Delta \longrightarrow \Delta$ be a Kannan P -non-expansive mapping. Then, J has a fixed point.*

Proof. Take $z_0 = \inf\{P(g - J(g)) \colon g \in \Delta\}$ and $z_x = z_0 + (1/x)$, for every $x \ge 1$. In view of the definition of z_0, we get

$K_{z_x} = \{g \in \Delta : P(g - J(g)) \leq z_x\} \neq \emptyset$, for all $x \geq 1$. Let Δ_{z_x} as defined in Lemma 35. Clearly, $\{\Delta_{z_x}\}$ is a decreasing sequence of nonempty P-bounded, P-closed, and P-convex subsets of Δ. One has $\Delta_\infty = \bigcap_{x\geq 1} \Delta_{z_x} \neq \emptyset$, from the property (R). Let $g \in \Delta_\infty$; we have $P(g - J(g)) \leq z_x$, for every $x \geq 1$. Suppose $x \longrightarrow \infty$; one has $P(g - J(g)) \leq z_0$, which implies $P(g - J(g)) = z_0$. Then, $K_{z_0} \neq \emptyset$. One gets $z_0 = 0$. Otherwise, $z_0 > 0$; this implies that J has no a fixed point. Assume Δ_{z_0} is as defined in Lemma 35. Since J has not a fixed point and Δ_{z_0} is J-invariant, then $\delta_P(\Delta_{z_0}) > 0$. From the P-quasi-normal property, we have $g \in \Delta_{z_0}$ with

$$P(g - f) < \delta_P\left(\Delta_{z_0}\right) \leq z_0, \tag{87}$$

for every $f \in \Delta_{z_0}$. According to Lemma 35, one gets $\Delta_{z_0} \subset K_{z_0}$. From definition of Δ_{z_0}, hence, $J(g) \in K_{z_0} \subset \Delta_{z_0}$. Obviously, this explains

$$P(g - J(g)) < \delta_P\left(\Delta_{z_0}\right) \leq z_0, \tag{88}$$

which contradicts the definition of z_0. Therefore, $z_0 = 0$, which means that J has a fixed point in Δ. □

We have the next corollary according to Theorems 32, 34, and 36.

Corollary 37. *Pick up the conditions (a1) and (a2) of Theorem 19 to be satisfied with $\lambda_0 > 1$. Assume Δ is a nonempty, P -convex, P -closed, and P -bounded subset of $(\ell(\gamma, \lambda))_P$, and $J : \Delta \longrightarrow \Delta$ is a Kannan P -nonexpansive operator. Hence, J has a fixed point.*

Example 9. Suppose $J : \Delta \longrightarrow \Delta$, where

$$J(g) = \begin{cases} \dfrac{g}{4}, & P(g) \in [0, 1), \\ \dfrac{g}{5}, & P(g) \in [1,\infty), \end{cases} \tag{89}$$

where $\Delta = \{g \in (\ell(((z+2)/(2z+3))_{z=0}^\infty, ((2z+3)/(z+2))_{z=0}^\infty))_P : g_0 = g_1 = 0\}$ and $P(g) = \sqrt{\sum_{z\in\mathbb{N}_0} (z+2)/(2z+3)|g_z|^{(2z+3)/(z+2)}}$, for every $g \in (\ell(((z+2)/(2z+3))_{z=0}^\infty, ((2z+3)/(z+2))_{z=0}^\infty))_P$. According to Example 5, the operator J is Kannan P-contraction. Hence, it is a Kannan P-nonexpansive mapping. Evidently, Δ is a nonempty, P-convex, P-closed, and P-bounded subset of $(\ell(((z+2)/(2z+3))_{z=0}^\infty, ((2z+3)/(z+2))_{z=0}^\infty))_P$. In view of Corollary 37, the operator J has a fixed point $\theta \in \Delta$.

7. Kannan Contraction's Fixed Points on Pre-Quasi-Ideal

In this part, we suppose $\mathscr{E}$ and $\mathscr{H}$ are Banach spaces. The Kannan contraction's fixed points on $(S_{(\ell(\gamma,\lambda))_P}, \mathbb{P})$, where $\mathbb{P}(C) = P((s_a(C))_{a=0}^\infty)$, has been examined.

Theorem 38 (see [3]). *If the conditions (a1) and (a2) of Theorem 19 are fulfilled, then $(S_{(\ell(\gamma,\lambda))_P}, \mathbb{P})$ is a pre-quasi-Banach operator ideal.*

Theorem 39. *If the conditions (a1) and (a2) of Theorem 19 are satisfied, then $(S_{(\ell(\gamma,\lambda))_P}, \mathbb{P})$ is a pre-quasi-closed operator ideal.*

Proof. The class $S_{(\ell(\gamma,\lambda))_P}$ is a pre-quasi-operator ideal and follows from Theorems 19 and 12. Let $C_y \in S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$, for every $y \in \mathbb{N}_0$ and $\lim_{y\longrightarrow\infty}\mathbb{P}(C_y - C) = 0$. Hence, there is $\varsigma > 0$ and since $\mathscr{L}(\mathscr{E}, \mathscr{H}) \supseteq S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$, one obtains

$$\begin{aligned}\mathbb{P}\left(C_y - C\right) &= P\left(\left(s_x\left(C_y - C\right)\right)_{x=0}^\infty\right) \geq P\left(s_0\left(C_y - C\right), 0, 0, 0, \cdots\right) \\ &= P\left(\|C_y - C\|, 0, 0, 0, \cdots\right) \geq \varsigma\gamma_0^{1/K} \|C_y - C\|.\end{aligned} \tag{90}$$

Hence, $(C_y)_{y\in\mathbb{N}_0}$ is convergent in $\mathscr{L}(\mathscr{E}, \mathscr{H})$. This implies $\lim_{y\longrightarrow\infty}\|C_y - C\| = 0$ and while $(s_x(C_y))_{x=0}^\infty \in (\ell(\gamma, \lambda))_P$, for every $y \in \mathbb{N}_0$ and $(\ell(\gamma, \lambda))_P$ is a premodular sss. Hence, there exists $E \geq 1$ with

$$\begin{aligned}\mathbb{P}(C) &= P\left((s_x(C))_{x=0}^\infty\right) = P\left(\left(s_x\left(C - C_y + C_y\right)\right)_{x=0}^\infty\right) \\ &\leq P\left(\left(s_{\left[\frac{x}{2}\right]}\left(C - C_y\right)\right)_{x=0}^\infty\right) + P\left(\left(s_{\left[\frac{x}{2}\right]}\left(C_y\right)_{x=0}^\infty\right)\right) \\ &\leq P\left(\left(\|C_y - C\|\right)_{x=0}^\infty\right) + (2E)^{1/K} P\left(\left(s_x\left(C_y\right)_{x=0}^\infty\right)\right) < \varepsilon.\end{aligned} \tag{91}$$

We have $(s_x(C))_{x=0}^\infty \in (\ell(\gamma, \lambda))_P$, then $C \in S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$. □

Definition 40. The function $\mathbb{P}$ on $S_{\mathfrak{A}_P}$ satisfies the Fatou property, if for any sequence $\{C_x\}_{x\in\mathbb{N}_0} \subseteq S_{\mathfrak{A}_P}(\mathscr{E}, \mathscr{H})$ with $\lim_{x\longrightarrow\infty}\mathbb{P}(C_x - C) = 0$ and any $\mathscr{V} \in S_{\mathfrak{A}_P}(\mathscr{E}, \mathscr{H})$, then

$$\mathbb{P}(\mathscr{V} - C) \leq \sup_x \inf_{j\geq x} P\left(\mathscr{V} - C_j\right). \tag{92}$$

Theorem 41. *Suppose the conditions (a1) and (a2) of Theorem 19 are satisfied, then the function $\mathbb{P}(C) = [\sum_{x=0}^\infty \gamma_x |s_x(C)|^{\lambda_x}]^{1/K}$, for all $C \in S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$, does not hold the Fatou property.*

Proof. Assume $\{C_p\}_{p\in\mathbb{N}_0} \subseteq S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$ with $\lim_{p\longrightarrow\infty}\mathbb{P}(C_p - C) = 0$. Since the space $S_{(\ell(\gamma,\lambda))_P}$ is a pre-quasi-closed ideal, then $C \in S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$. Hence, for any $\mathscr{V} \in S_{(\ell(\gamma,\lambda))_P}(\mathscr{E}, \mathscr{H})$, we have

$$\begin{aligned}\mathbb{P}(\mathscr{V}-C)&=\left[\sum_{x=0}^{\infty}\gamma_x|s_x(\mathscr{V}-C)|^{\lambda_x}\right]^{1/K}\\&\le\left[\sum_{x=0}^{\infty}\gamma_x\left|s_{[x/2]}(\mathscr{V}-C_i)\right|^{\lambda_x}\right]^{1/K}\\&\quad+\left[\sum_{x=0}^{\infty}\gamma_x\left|s_{[x/2]}(C_i-C)\right|^{\lambda_x}\right]^{1/K}\\&\le(2E)^{1/K}\sup_p\inf_{i\ge p}\left[\sum_{x=0}^{\infty}\gamma_x|s_x(\mathscr{V}-C_i)|^{\lambda_x}\right]^{1/K}.\end{aligned}\tag{93}$$

□

Definition 42. A mapping $\mathscr{G}:S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})\longrightarrow S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})$ is said to be a Kannan $\mathbb{P}$ -Lipschitzian, if we have $\iota\ge 0$, such that

$$\mathbb{P}(\mathscr{G}C-\mathscr{G}A)\le\iota\{\mathbb{P}(\mathscr{G}C-C)+\mathbb{P}(\mathscr{G}A-A)\},\tag{94}$$

for every $C,A\in S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})$.

(1) The mapping $\mathscr{G}$ is said to be Kannan $\mathbb{P}$-contraction, whenever $\iota\in[0,1/2)$

(2) The mapping $\mathscr{G}$ is said to be Kannan $\mathbb{P}$-non-expansive, whenever $\iota=1/2$

Definition 43. Suppose $\mathscr{G}:S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})\longrightarrow S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})$ and $B\in S_{\mathfrak{A}_p}(\mathscr{E},\mathscr{H})$.$\mathscr{G}$ is said to be $\mathbb{P}$ -sequentially continuous at B, if and only if, if $\lim_{p\longrightarrow\infty}\mathbb{P}(C_p-B)=0$, then $\lim_{p\longrightarrow\infty}\mathbb{P}(\mathscr{G}C_p-\mathscr{G}B)=0$.

Example 10. Assume $\mathscr{G}:S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})\longrightarrow S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})$, where $\mathbb{P}(C)=\sqrt{\sum_{a=0}^{\infty}(a+3)/(2a+1)|s_a(C)|^{(2a+1)/(a+3)}}$, for every $C\in S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})$ and

$$\mathscr{G}(C)=\begin{cases}\dfrac{C}{263170}, & \mathbb{P}(C)\in[0,1),\\[2mm]\dfrac{C}{263171}, & \mathbb{P}(C)\in[1,\infty).\end{cases}\tag{95}$$

Evidently, $\mathscr{G}$ is $\mathbb{P}$-sequentially continuous at the zero operator $\Theta\in S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}$.

Let $\{C^{(n)}\}\subseteq S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}$ with $\lim_{m\longrightarrow\infty}\mathbb{P}(C^{(m)}-C^{(0)})=0$, where $C^{(0)}\in S_{(\ell(((a+3)/(2a+1))_{a=0}^{\infty},((2a+1)/(a+3))_{a=0}^{\infty}))_p}$ with $\mathbb{P}(C^{(0)})=1$. As $\mathbb{P}$ is continuous, one has

$$\begin{aligned}\lim_{n\longrightarrow\infty}\mathbb{P}\left(\mathscr{G}C^{(n)}-\mathscr{G}C^{(0)}\right)&=\lim_{n\longrightarrow\infty}\mathbb{P}\left(\frac{C^{(n)}}{263170}-\frac{C^{(0)}}{263171}\right)\\&=\mathbb{P}\left(\frac{C^{(0)}}{69258712070}\right)>0.\end{aligned}\tag{96}$$

So $\mathscr{G}$ is not $\mathbb{P}$-sequentially continuous at $C^{(0)}$.

Theorem 44. *Assume the conditions (a1) and (a2) of Theorem 19 are satisfied and* $\mathscr{G}:S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})\longrightarrow S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$, *where* $\mathbb{P}(C)=[\sum_{y=0}^{\infty}\gamma_y|s_y(C)|^{\lambda_y}]^{1/K}$, *for every* $C\in S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$.

(i) Let $\mathscr{G}$ is Kannan $\mathbb{P}$-contraction mapping

(ii) $\mathscr{G}$ is $\mathbb{P}$-sequentially continuous at a point $A\in S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$

(iii) We have $Y\in S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$ such that the sequence $\{\mathscr{G}^pY\}$ has a subsequence $\{\mathscr{G}^{p_i}Y\}$ converging to X

Then, $X\in S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$ is the only fixed point of $\mathscr{G}$.

Proof. Assume X is not a fixed point of $\mathscr{G}$; we have $\mathscr{G}X\ne X$. By parts (ii) and (iii), one can see

$$\begin{aligned}&\lim_{p_i\longrightarrow\infty}\mathbb{P}(\mathscr{G}^{p_i}Y-X)=0,\\&\lim_{p_i\longrightarrow\infty}\mathbb{P}\left(\mathscr{G}^{p_i+1}Y-\mathscr{G}X\right)=0.\end{aligned}\tag{97}$$

As $\mathscr{G}$ is Kannan $\mathbb{P}$-contraction mapping, we get

$$\begin{aligned}0<\mathbb{P}(\mathscr{G}X-X)&=\mathbb{P}\left(\left(\mathscr{G}X-\mathscr{G}^{p_i+1}Y\right)+\left(\mathscr{G}^{p_i}Y-X\right)\right.\\&\quad\left.+\left(\mathscr{G}^{p_i+1}Y-\mathscr{G}^{p_i}Y\right)\right)\\&\le(2E)^{1/K}\mathbb{P}\left(\mathscr{G}^{p_i+1}Y-\mathscr{G}X\right)\\&\quad+(2E)^{2/K}\mathbb{P}\left(\mathscr{G}^{p_i}Y-X\right)\\&\quad+(2E)^{2/K}\iota\left(\frac{\iota}{1-\iota}\right)^{p_i-1}\mathbb{P}(\mathscr{G}Y-Y).\end{aligned}\tag{98}$$

□

This gives a contradiction as $p_i\longrightarrow\infty$. Hence, X is a fixed point of $\mathscr{G}$. For the uniqueness of the fixed point X, assume we have two different fixed points $X,Y\in S_{(\ell(\gamma,\lambda))_p}(\mathscr{E},\mathscr{H})$ of $\mathscr{G}$. Therefore, one gets $\mathbb{P}(X-Y)\le\mathbb{P}(\mathscr{G}X-\mathscr{G}Y)\le\iota(\mathbb{P}(\mathscr{G}X-X)+\mathbb{P}(\mathscr{G}Y-Y))=0$. This implies $X=Y$.

Example 11. Assume $\mathscr{G}:S_{(\ell(((y+2)/(y+1))_{y=0}^{\infty},((y+1)/(y+2))_{y=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})\longrightarrow S_{(\ell(((y+2)/(y+1))_{y=0}^{\infty},((y+1)/(y+2))_{y=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})$, where $\mathbb{P}(C)=\sum_{y=0}^{\infty}(y+2)/(y+1)|s_y(C)|^{(y+1)/(y+2)}$, for every $C\in S_{(\ell(((y+2)/(y+1))_{y=0}^{\infty},((y+1)/(y+2))_{y=0}^{\infty}))_p}(\mathscr{E},\mathscr{H})$ and

$$\mathcal{G}(C) = \begin{cases} \dfrac{C}{26}, & \mathbb{P}(C) \in [0,1), \\ \dfrac{C}{37}, & \mathbb{P}(C) \in [1,\infty). \end{cases} \quad (99)$$

If $C_1, C_2 \in S_{(\ell(((a+2)/(a+1))_{a=0}^{\infty},((a+1)/(a+2))_{a=0}^{\infty}))_P}$ with $\mathbb{P}(C_1), \mathbb{P}(C_2) \in [0,1)$, one has

$$\begin{aligned}\mathbb{P}(\mathcal{G}C_1 - \mathcal{G}C_2) &= \mathbb{P}\left(\frac{C_1}{26} - \frac{C_2}{26}\right) \le \frac{2}{5}\left(\mathbb{P}\left(\frac{25C_1}{26}\right) + \mathbb{P}\left(\frac{25C_2}{26}\right)\right) \\ &= \frac{2}{5}(\mathbb{P}(\mathcal{G}C_1 - C_1) + \mathbb{P}(\mathcal{G}C_2 - C_2)).\end{aligned} \quad (100)$$

For each $C_1, C_2 \in S_{(\ell(((a+2)/(a+1))_{a=0}^{\infty},((a+1)/(a+2))_{a=0}^{\infty}))_P}$ with $\mathbb{P}(C_1), \mathbb{P}(C_2) \in [1,\infty)$, we get

$$\begin{aligned}\mathbb{P}(\mathcal{G}C_1 - \mathcal{G}C_2) &= \mathbb{P}\left(\frac{C_1}{37} - \frac{C_2}{37}\right) \le \frac{1}{3}\left(\mathbb{P}\left(\frac{36C_1}{37}\right) + \mathbb{P}\left(\frac{36C_2}{37}\right)\right) \\ &= \frac{1}{3}(\mathbb{P}(\mathcal{G}C_1 - C_1) + \mathbb{P}(\mathcal{G}C_2 - C_2)).\end{aligned} \quad (101)$$

For each $C_1, C_2 \in S_{(\ell(((a+2)/(a+1))_{a=0}^{\infty},((a+1)/(a+2))_{a=0}^{\infty}))_P}$ with $\mathbb{P}(C_1) \in [0,1)$ and $\mathbb{P}(C_2) \in [1,\infty)$, one can see

$$\begin{aligned}\mathbb{P}(\mathcal{G}C_1 - \mathcal{G}C_2) &= \mathbb{P}\left(\frac{C_1}{26} - \frac{C_2}{37}\right) \le \frac{2}{5}\mathbb{P}\left(\frac{25C_1}{26}\right) + \frac{1}{3}\mathbb{P}\left(\frac{36C_2}{37}\right) \\ &\le \frac{2}{5}\left(\mathbb{P}\left(\frac{25C_1}{26}\right) + \mathbb{P}\left(\frac{36C_2}{37}\right)\right) \\ &= \frac{2}{5}(\mathbb{P}(\mathcal{G}C_1 - C_1) + \mathbb{P}(\mathcal{G}C_2 - C_2)).\end{aligned} \quad (102)$$

So, the mapping C is Kannan $\mathbb{P}$-contraction and

$$\mathcal{G}^p(C) = \begin{cases} \dfrac{C}{26^p}, & \mathbb{P}(C) \in [0,1), \\ \dfrac{C}{37^p}, & \mathbb{P}(C) \in [1,\infty). \end{cases} \quad (103)$$

From Theorem 44, $\Theta \in S_{(\ell(((a+2)/(a+1))_{a=0}^{\infty},((a+1)/(a+2))_{a=0}^{\infty}))_P}$ is the unique fixed point of $\mathcal{G}$, since $\mathcal{G}$ is $\mathbb{P}$-sequentially continuous and $\{\mathcal{G}^{p_i}C\}$ converging to Θ.

8. The Presence of Solutions to Summable Equations

The solution to (104), which is studied by some authors (see [26–28]), in $(\ell(\gamma,\lambda))_P$, where $(\lambda_l) \in \mathbb{R}^{+\mathbb{N}_0}$ is an increase and $P(g) = [\sum_{z=0}^{\infty} \gamma_z |g_z|^{\lambda_z}]^{1/K}$, for all $g \in \ell(\gamma,\lambda)$, has been examined.

Consider the summable equations

$$g_z = x_z + \sum_{r=0}^{\infty} G(z,r) W(r, g_r), \quad (104)$$

and suppose $J : (\ell(\gamma,\lambda))_P \longrightarrow (\ell(\gamma,\lambda))_P$ is defined by

$$J(g_z)_{z\in\mathbb{N}_0} = \left(x_z + \sum_{r=0}^{\infty} G(z,r) W(r, g_r)\right)_{z\in\mathbb{N}_0}. \quad (105)$$

Theorem 45. *Assume $G : \mathbb{N}_0^2 \longrightarrow \mathbb{R}, W : \mathbb{N}_0 \times \mathbb{R} \longrightarrow \mathbb{R}$, $x : \mathbb{N}_0 \longrightarrow \mathbb{R}$, and for all $z \in \mathbb{N}_0$, there exists $\iota \in [0, 1/2)$ with*

$$\begin{aligned}&\left|\sum_{r\in\mathbb{N}_0} G(z,r)(W(r,g_r) - W(r,f_r))\right|^{\lambda_z} \\ &\le \iota^K \left[\left|x_z - g_z + \sum_{r=0}^{\infty} G(z,r)W(r,g_r)\right|^{\lambda_z} \right. \\ &\left. + \left|x_z - f_z + \sum_{r=0}^{\infty} G(z,r)W(r,f_r)\right|^{\lambda_z}\right].\end{aligned} \quad (106)$$

Then, the summable equation (104) has a solution in $(\ell(\gamma,\lambda))_P$.

Proof. Suppose $J : (\ell(\gamma,\lambda))_P \longrightarrow (\ell(\gamma,\lambda))_P$ is defined by (105). We have

$$\begin{aligned}P(Jg - Jf) &= \left[\sum_{z\in\mathbb{N}_0} \gamma_z |Jg_z - Jf_z|^{\lambda_z}\right]^{1/K} \\ &= \left[\sum_{z\in\mathbb{N}_0} \gamma_z \left|\sum_{r\in\mathbb{N}_0} G(z,r)[W(r,g_r) - W(r,f_r)]\right|^{\lambda_z}\right]^{1/K} \\ &\le \iota\left(\left[\sum_{z\in\mathbb{N}_0} \gamma_z \left|x_z - g_z + \sum_{r=0}^{\infty} G(z,r)W(r,g_r)\right|^{\lambda_z}\right]^{1/K}\right. \\ &\left. + \left[\sum_{z\in\mathbb{N}_0} \gamma_z \left|x_z - f_z + \sum_{r=0}^{\infty} G(z,r)W(r,f_r)\right|^{\lambda_z}\right]^{1/K}\right) \\ &= \iota(P(Jg - g) + P(Jf - f)).\end{aligned} \quad (107)$$

□

Hence, there is a unique solution of equation (104) in $(\ell(\gamma,\lambda))_P$ according to Theorem 25.

Example 12. Suppose we have $(\ell(((z+2)/(z+1))_{z=0}^{\infty}, ((z+1)/(z+2))_{z=0}^{\infty}))_P$, where $P(g) = \sum_{z\in\mathbb{N}_0} (z+2)/(z+1) |g_z|^{(z+1)/(z+2)}$, for all $g \in \ell(((z+2)/(z+1))_{z=0}^{\infty}, ((z+1)/(z+2))_{z=0}^{\infty})$. Consider the summable equations

$$g_z = e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{e^{|g_z|}}{z^2 + r^2 + 1} \right)^l, \quad (108)$$

where $l > 2$, and let $J : (\ell(((z+2)/(z+1))_{z=0}^{\infty}, ((z+1)/(z+2))_{z=0}^{\infty}))_P \longrightarrow (\ell(((z+2)/(z+1))_{z=0}^{\infty}, ((z+1)/(z+2))_{z=0}^{\infty}))_P$ be defined by

$$J(g_z)_{z\in\mathbb{N}_0} = \left(e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{e^{|g_z|}}{z^2 + r^2 + 1} \right)^l \right)_{z\in\mathbb{N}_0}. \quad (109)$$

Obviously,

$$\begin{aligned} &\left| \sum_{r=0}^{\infty} (-1)^z \left(\frac{e^{|g_z|}}{z^2 + r^2 + 1} \right)^l ((-1)^r - (-1)^r) \right|^{(z+1)/(z+2)} \\ &\le \frac{1}{3} \left[\left| e^{-(3z+6)} - g_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{e^{|g_z|}}{z^2 + r^2 + 1} \right)^l \right|^{(z+1)/(z+2)} \right. \\ &\left. + \left| e^{-(3z+6)} - f_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{e^{|f_z|}}{z^2 + r^2 + 1} \right)^l \right|^{(z+1)/(z+2)} \right]. \end{aligned} \quad (110)$$

By Theorem 45, the summable equation (108) has a unique solution in $(\ell(((z+2)/(z+1))_{z=0}^{\infty}, ((z+1)/(z+2))_{z=0}^{\infty}))_P$.

Example 13. Assume we have $(\ell(((z+3)/(2z+1))_{z=0}^{\infty}, ((2z+1)/(z+3))_{z=0}^{\infty}))_P$, where $P(g) = \sqrt{\sum_{z\in\mathbb{N}_0} (z+3)/(2z+1)|g_z|^{(2z+1)/(z+3)}}$, for all $g \in \ell(((z+3)/(2z+1))_{z=0}^{\infty}, ((2z+1)/(z+3))_{z=0}^{\infty})$. Consider the summable equations

$$g_z = e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l, \quad (111)$$

where $l > 2$, and let $J : (\ell(((z+3)/(2z+1))_{z=0}^{\infty}, ((2z+1)/(z+3))_{z=0}^{\infty}))_P \longrightarrow (\ell(((z+3)/(2z+1))_{z=0}^{\infty}, ((2z+1)/(z+3))_{z=0}^{\infty}))_P$ be defined by

$$J(g_z)_{z\in\mathbb{N}_0} = \left(e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l \right)_{z\in\mathbb{N}_0}. \quad (112)$$

It is easy to see that

$$\begin{aligned} &\left| \sum_{r=0}^{\infty} (-1)^z \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l ((-1)^r - (-1)^r) \right|^{(2z+1)/(z+3)} \\ &\le \frac{1}{9} \left[\left| e^{-(3z+6)} - g_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l \right|^{(2z+1)/(z+3)} \right. \\ &\left. + \left| e^{-(3z+6)} - f_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{f_z}{z^2 + r^2 + 1} \right)^l \right|^{(2z+1)/(z+3)} \right]. \end{aligned} \quad (113)$$

By Theorem 45, the summable equation (111) has one solution in $(\ell(((z+3)/(2z+1))_{z=0}^{\infty}, ((2z+1)/(z+3))_{z=0}^{\infty}))_P$.

Example 14. If we have $(\ell(((z+2)/(2z+3))_{z=0}^{\infty}, ((2z+3)/(z+2))_{z=0}^{\infty}))_P$, where $P(g) = \sqrt{\sum_{z\in\mathbb{N}_0} (z+2)/(2z+3)|g_z|^{(2z+3)/(z+2)}}$, for all $g \in \ell(((z+2)/(2z+3))_{z=0}^{\infty}, ((2z+3)/(z+2))_{z=0}^{\infty})$. Consider the summable equations

$$g_z = e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l, \quad (114)$$

such that $z \ge 2$ and $l > 2$, and assume $J : \Delta \longrightarrow \Delta$, where $\Delta = \{v \in (\ell(((z+2)/(2z+3))_{z=0}^{\infty}, ((2z+3)/(z+2))_{z=0}^{\infty}))_P : g_0 = g_1 = 0\}$ is defined by

$$J(g_z)_{z\ge2} = \left(e^{-(3z+6)} + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l \right)_{z\ge2},$$

$$\begin{aligned} &\left| \sum_{r=0}^{\infty} (-1)^z \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l ((-1)^r - (-1)^r) \right|^{(2z+3)/(z+2)} \\ &\le \frac{1}{9} \left[\left| e^{-(3z+6)} - g_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{g_z}{z^2 + r^2 + 1} \right)^l \right|^{(2z+3)/(z+2)} \right. \\ &\left. + \left| e^{-(3z+6)} - f_z + \sum_{r=0}^{\infty} (-1)^{z+r} \left(\frac{f_z}{z^2 + r^2 + 1} \right)^l \right|^{(2z+3)/(z+2)} \right]. \end{aligned} \quad (115)$$

According to Theorem 45, the summable equation (114) has a solution in Δ.

9. Conclusion

There is a pre-quasi-normed space theorem that is more general than quasi-normed space. In weighted Nakano sequence space with the well-known pre-quasi-norm, we investigate the appropriate conditions for the generation of pre-quasi-Banach and closed spaces. Pre-quasi-normal structural properties of weighted Nakano sequence space, including the fixed point idea of Kannan pre-quasi-norm contraction and Kannan pre-quasi-norm nonexpansive mapping in weighted Nakano sequence space, are improved. Using weighted Nakano sequence space and s-numbers, the presence of a fixed point for Kannan pre-quasi-norm contraction mapping has been proved. Toward the end of our discussion, we provided several examples of how the collected data could be used to solve an issue. The weight and power of the weighted Nakano sequence space can be used to define a wide range of circumstances under which existence findings can be found. Banach lattices are introduced in this article, and a new space of solutions for many difference equations is introduced, the spectrum of any bounded linear operator between any two Banach spaces with s-numbers in this sequence space.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was funded by the University of Jeddah, Saudi Arabia, under grant No. UJ-20-110-DR. The authors, therefore, acknowledge with thanks the University's technical and financial support.

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Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9129992, 7 pages
https://doi.org/10.1155/2021/9129992

Research Article

Solving an Integral Equation by Using Fixed Point Approach in Fuzzy Bipolar Metric Spaces

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Received 14 September 2021; Revised 26 October 2021; Accepted 22 November 2021; Published 3 December 2021

Academic Editor: Anita Tomar

The purpose of this manuscript is to obtain some fixed point results under mild contractive conditions in fuzzy bipolar metric spaces. Our results generalize and extend many of the previous findings in the same approach. Moreover, two examples to support our theorems are obtained. Finally, to examine and strengthen the theoretical results, the existence and uniqueness of the solution to a nonlinear integral equation was studied as a kind of applications.

1. Introduction

The notion of the continuous triangular norm was introduced in 1960 by Schweizer and Sklar in their paper [1]. The concept of fuzzy set theory was initiated by Zadeh [2] in 1965. Some references to a fuzzy logic-based education system can be found in [3–6]. The other direction of the fuzzy set is the fuzzy metric theory. The idea of fuzzy metric space (FM-space) was presented by Kramosil and Michalek [7]. With the help of continuous t-norm property, they obtained some pivotal fixed point results under the mild contractive conditions in the mentioned space. Many authors worked in this direction; they either modified the definition of FM-spaces [8] or extended the well-known fixed point theorem of Banach to fuzzy metric spaces [9]. Moreover, Gregori and Sapena [5, 10] obtained some contractive-type fixed point theorems in FM-spaces. Recently, in 2020, Li et al. [11] showed some strongly coupled fixed point theorems by using cyclic contractive-type mappings in complete FM-spaces. In 2019, Beloul and Tomar [12] proved integral-type common fixed point theorems in modified intuitionistic fuzzy metric spaces. Prasad et al. [13] presented coincidence theorems via contractive mappings in ordered non-Archimedean fuzzy metric spaces. Again Prasad [14] analyzed coincidence points of relational ψ-contractions in 2021. The bipolar metric space has been studied by many authors, and important results have been obtained [15–18].

Recently, FM-space was extended and generalized to fuzzy bipolar metric space (FBM-space) by Mutlu and Gurdal [19]. They gave new concepts for measurement of the distance between the elements of two different sets. Bartwal et al. [20] introduced the notion of fuzzy bipolar metric space and obtained some fixed point results under mild conditions.

A continuation of this approach, in this manuscript, we shall obtain some fixed point theorems via contractive-type mappings in FBM-spaces. Our results generalize, unify, and extend the results of Bartwal et al. [20] and many other papers in this direction. Also, two examples are given to support our theorems. Ultimately, the existence and uniqueness solution to an integral equation in the sense of Lebesgue measurable functions are obtained as an application.

2. Basic Facts

This part is devoted to present some basic definitions, lemmas, and propositions of FBM-spaces as follows.

Definition 1. (see [8]). Let Π be a nonvoid set. A 3-triple $(\Omega, \Gamma, *)$ is called an FM-space if Γ is a fuzzy set on $\Omega^2 \times (0, \infty)$ and $*$ is a continuous η-norm justifying the hypotheses below:

(1) $\Gamma(\mu, \sigma, \eta) > 0$

(2) $\Gamma(\mu, \sigma, \eta) = 1$ iff $\mu = \sigma$

(3) $\Gamma(\mu, \sigma, \eta) = \Gamma(\sigma, \mu, \eta)$

(4) $\Gamma(\mu, \vartheta, \eta + \zeta) \geq \Gamma(\mu, \sigma, \eta) * \Gamma(\sigma, \vartheta, \zeta)$

(5) $\Gamma(\mu, \sigma, .)$: $(0,\infty) \longrightarrow (0, 1]$ is continuous

for all $\mu, \sigma, \vartheta \in \Omega$ and $\eta, \zeta > 0$.

Lemma 2. (see [21]). *Let $(\Pi, \Gamma, *)$ be an FM-space. If for all $\mu, \sigma \in \Pi$ and $\eta > 0$.*

$$\Gamma(\mu, \sigma, k\eta) \geq \Gamma(\mu, \sigma, \eta), \tag{1}$$

where $k \in (0, 1)$, then $\mu = \sigma$.

Definition 3. (see [20]). Let Π and Ω be two nonvoid sets. A 4-tuple $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is said to be an FBM-space, where $*$ is continuous η-norm and $\Gamma_{\mathfrak{b}}$ is a fuzzy set on $\Pi \times \Omega \times (0,\infty)$, fulfilling the subsequent assumptions:

(1) $\Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) > 0$ for all $(\mu, \sigma) \in \Pi \times \Omega$

(2) $\Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = 1$ iff $\mu = \sigma$ for $\mu \in \Pi$ and $\sigma \in \Omega$

(3) $\Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = \Gamma_{\mathfrak{b}}(\sigma, \mu, \eta)$ for all $\mu, \sigma \in \Pi \cap \Omega$

(4) $\Gamma_{\mathfrak{b}}(\mu_1, \sigma_2, \eta + \zeta + r) \geq \Gamma_{\mathfrak{b}}(\mu_1, \sigma_1, \eta) * \Gamma_{\mathfrak{b}}(\mu_2, \sigma_1, \zeta) * \Gamma_{\mathfrak{b}}(\mu_2, \sigma_2, r)$ for all $\mu_1, \mu_2 \in \Pi$ and $\sigma_1, \sigma_2 \in \Omega$

(5) $\Gamma_{\mathfrak{b}}(\mu, \sigma, .)$: $[0,\infty) \longrightarrow [0, 1]$ is left continuous

(6) $\Gamma_{\mathfrak{b}}(\mu, \sigma, .)$ is nondecreasing for all $\mu \in \Pi$ and $\sigma \in \Omega$, for all $\eta, \zeta, r > 0$

Remark 4. (see [20]). In an FBM-space $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$, if $\Pi = \Omega$, then $(\Pi, \Gamma_{\mathfrak{b}}, *)$ is an FM-space.

Lemma 5. (see [20]). *Let $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ be an FBM-space so that*

$$\Gamma_{\mathfrak{b}}(\mu, \sigma, k\eta) \geq \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta), \tag{2}$$

for $\mu \in \Pi, \sigma \in \Omega$ and $k \in (0, 1)$. Then, $\mu = \sigma$.

Definition 6. (see [20]). Let $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ be an FBM-space. A point $\sigma \in \Pi \cup \Omega$ is called a left point if $\sigma \in \Pi$, a right point if $\sigma \in \Omega$, and a central point if it is both a left and a right point. Similarly, a sequence $\{\sigma_\alpha\}$ on the set Π is called a left sequence, and a sequence $\{\sigma_\alpha\}$ on Ω is called a right sequence. In an FBM-space, a left or a right sequence is called simply a sequence. A sequence $\{\sigma_\alpha\}$ is said to be convergent to a point σ, iff $\{\sigma_\alpha\}$ is a left sequence, σ is a right point, and $\lim_{\alpha \longrightarrow \infty} \Gamma_{\mathfrak{b}}(\sigma_\alpha, \sigma, \eta) = 1$. A bisequence $(\{\sigma_\alpha\}, \{\mu_\alpha\})$ on $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is a sequence on the set $\Pi \times \Omega$. If the sequence $\{\sigma_\alpha\}$ and $\{\mu_\alpha\}$ are convergent, then the bisequence $(\{\sigma_\alpha\}, \{\mu_\alpha\})$ is said to be convergent, and if $\{\sigma_\alpha\}$ and $\{\mu_\alpha\}$ converge to a common point, then $(\{\sigma_\alpha\}, \{\mu_\alpha\})$ is called biconvergent. A bisequence $(\{\sigma_\alpha\}, \{\mu_\alpha\})$ is a Cauchy bisequence, if $\lim_{\alpha, \beta \longrightarrow \infty} \Gamma_{\mathfrak{b}}(\sigma_\alpha, \mu_\beta, \eta) = 1$. An FBM-space is called complete, if every Cauchy bisequence is convergent, hence biconvergent.

Lemma 7. (see [20]). *In an FBM-space, every convergent Cauchy bisequence is biconvergent.*

Lemma 8. (see [20]). *Let$(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$be an FBM-space, and if$\mu \in \Pi \cap \Omega$is a limit of a sequence, then it is a unique limit of the sequence.*

Definition 9. A point $\mu \in \Pi \cap \Omega$ is said to be common fixed point for the mappings (Λ, Θ) on $\mu \in \Pi \cap \Omega$ such that $\mu = \Lambda\mu = \Theta\mu$.

3. Main Results

Now, we present the first main theorem.

Theorem 10. *Let $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ be a complete FBM-space such that*

$$\lim_{\eta \longrightarrow \infty} \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = 1 \quad \text{for all } \mu \in \Pi, \sigma \in \Omega. \tag{3}$$

Let $\Lambda, \Theta : \Pi \cup \Omega \longrightarrow \Pi \cup \Omega$ be two mappings satisfying

(1) $\Lambda(\Pi) \subseteq \Pi$, $\Theta(\Pi) \subseteq \Pi$, and $\Lambda(\Omega) \subseteq \Omega$, $\Theta(\Omega) \subseteq \Omega$

(2) $\Gamma_{\mathfrak{b}}(\Lambda(\mu), \Theta(\sigma), k\eta) \geq \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta)$ for all $\mu \in \Pi, \sigma \in \Omega$ and $\eta > 0$, where $0 < k < 1$

Then, Λ and Θ have a unique common fixed point.

Proof. Fix $\mu_0 \in \Pi$ and $\sigma_0 \in \Omega$ and assume that$\Lambda(\mu_{2\alpha}) = \mu_{2\alpha+1}$, $\Theta(\mu_{2\alpha+1}) = \mu_{2\alpha+2}$, $\Lambda(\sigma_{2\alpha}) = \sigma_{2\alpha+1}$, and$\Theta(\sigma_{2\alpha+1}) = \sigma_{2\alpha+2}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Then, we get $(\mu_\alpha, \sigma_\alpha)$ as a bisequence on the

FBM-space $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$. Now, we have

$$\Gamma_{\mathfrak{b}}(\mu_1, \sigma_1, \eta) = \Gamma_{\mathfrak{b}}(\Lambda(\mu_0), \Theta(\sigma_0), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{k}\right), \quad (4)$$

$\forall \eta > 0$ and $\alpha \in \mathbb{N}$. By induction, we obtain

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_{2\alpha+1}, \sigma_{2\alpha+1}, \eta) &= \Gamma_{\mathfrak{b}}(\Lambda(\mu_{2\alpha}), \Theta(\sigma_{2\alpha}), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{k^{2\alpha+1}}\right),\\ \Gamma_{\mathfrak{b}}(\mu_{2\alpha+1}, \sigma_{2\alpha+2}, \eta) &= \Gamma_{\mathfrak{b}}(\Lambda(\mu_{2\alpha}), \Theta(\sigma_{2\alpha+1}), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_1, \frac{\eta}{k^{2\alpha+1}}\right),\end{aligned} \quad (5)$$

for all $\eta > 0$ and $\alpha \in \mathbb{N}$. □

Letting $\alpha < \beta$, for $\alpha, \beta \in \mathbb{N}$. Then, from the definition of the FBM-space, we get

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\mu_\alpha, \sigma_\alpha, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\alpha, \frac{\eta}{3}\right)\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\beta, \frac{\eta}{3}\right),\\ &\vdots\\ &\geq \Gamma_{\mathfrak{b}}\left(\mu_\alpha, \sigma_\alpha, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\alpha, \frac{\eta}{3}\right) * \cdots\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\beta-1}, \sigma_{\beta-1}, \frac{\eta}{3^{\beta-1}}\right) * \Gamma_{\mathfrak{b}}\left(\mu_\beta, \sigma_{\beta-1}, \frac{\eta}{3^{\beta-1}}\right)\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_\beta, \sigma_\beta, \frac{\eta}{3^{\beta-1}}\right).\end{aligned} \quad (6)$$

Therefore,

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{3k^\alpha}\right) * \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_1, \frac{\eta}{3k^{\alpha+1}}\right) * \cdots,\\ &\quad \cdots * \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{3^{\beta-1}k^{\beta+1}}\right).\end{aligned} \quad (7)$$

From (3), as $\alpha, \beta \longrightarrow \infty$, we get

$$\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) \geq 1 \quad \forall \eta > 0. \quad (8)$$

Thus, bisequence $(\{\mu_\alpha\}, \{\sigma_\alpha\})$ is a Cauchy bisequence. Since $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is a complete FBM-space. By Lemma 7, bisequence $(\{\mu_\alpha\}, \{\sigma_\alpha\})$ is a biconvergent sequence. Therefore, $\{\mu_\alpha\} \longrightarrow u$ and $\{\sigma_\alpha\} \longrightarrow u$, where $u \in \Pi \cap \Omega$. By Lemma 8, both sequences $\{\mu_\alpha\}$ and $\{\sigma_\alpha\}$ have a unique limit. From the triangular property of fuzzy bipolar metric spaces, we have

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\Lambda(u), u, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\Lambda(u), \sigma_{\alpha+1}, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_{\alpha+1}, \frac{\eta}{3}\right)\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right) = \Gamma_{\mathfrak{b}}\left(\Lambda(u), \Theta(\sigma_\alpha), \frac{\eta}{3}\right)\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_{\alpha+1}, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right) \geq \Gamma_{\mathfrak{b}}\left(u, \sigma_\alpha, \frac{\eta}{3}\right)\\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_{\alpha+1}, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right),\end{aligned} \quad (9)$$

for all $\alpha \in \mathbb{N}$ and $\eta > 0$ and as $\alpha \longrightarrow \infty$,

$$\Gamma_{\mathfrak{b}}(\Lambda(u), u, \eta) \longrightarrow 1 * 1 * 1 = 1. \quad (10)$$

From Definition 3 condition (2), $\Lambda(u) = u$. Again,

$$\Gamma_{\mathfrak{b}}(u, \Theta(u), \eta) = \Gamma_{\mathfrak{b}}(\Lambda(u), \Theta(u), \eta) \geq \Gamma_{\mathfrak{b}}\left(u, u, \frac{\eta}{k}\right) = 1. \quad (11)$$

Therefore, $\Theta(u) = u$. Hence, u is a common fixed point of Λ and Θ.

Let $v \in \Pi \cap \Omega$ be another fixed point of Λ and Θ. Then,

$$\Gamma_{\mathfrak{b}}(u, v, \eta) = \Gamma_{\mathfrak{b}}(\Lambda(u), \Theta(v), \eta) \geq \Gamma_{\mathfrak{b}}\left(u, v, \frac{\eta}{k}\right), \quad (12)$$

for $0 < k < 1$ and $\forall \eta > 0$. By Lemma 5, we have $u = v$.

The following example supports the above theorem.

Example 11. Let $\Pi = [0, 2]$ and $\Omega = \{0\} \cup \mathbb{N} - \{1, 2\}$. Define $\Gamma_{\mathfrak{b}} = \eta/(\eta + |\mu - \sigma|)$ for all $\eta > 0$, $\mu \in \Pi$, and $\sigma \in \Omega$. Clearly, $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is a complete FBM-space, where $*$ is a continuous η-norm defined as $\mathfrak{p} * \mathfrak{q} = \mathfrak{p}\mathfrak{q}$.

Let $\Lambda, \Theta : \Pi \cup \Omega \longrightarrow \Pi \cup \Omega$ be mappings defined by

$$\begin{aligned}\Lambda(\mu) &= \begin{cases} 2 - \mu, & \text{if } \mu \in [0, 2],\\ 2, & \text{if } \mu \in \mathbb{N} - \{1, 2\},\end{cases}\\ \Theta(\mu) &= \begin{cases} \mu, & \text{if } \mu \in [0, 2],\\ 2, & \text{if } \mu \in \mathbb{N} - \{1, 2\},\end{cases}\end{aligned} \quad (13)$$

for all $\mu \in \Pi \cup \Omega$. Now, suppose that $k = 1/2$, then for all $\eta > 0$, we discuss the following cases:

Case 1. If $\mu \in [0, 2]$ and $\sigma \in \mathbb{N} - \{1, 2\}$, then

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\Lambda(\mu), \Theta(\sigma), k\eta) &= \Gamma_{\mathfrak{b}}(2 - \mu, 2, k\eta) = \frac{k\eta}{k\eta + |2 - \mu - 2|}\\ &\geq \frac{\eta}{\eta + |\mu - \sigma|} = \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta).\end{aligned} \quad (14)$$

Case 2. If $\mu \in \mathbb{N} - \{1, 2\}$ and $\sigma \in [0, 2]$, then

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\Lambda(\mu), \Theta(\sigma), k\eta) = \Gamma_{\mathfrak{b}}(2, \sigma, k\eta) &= \frac{k\eta}{k\eta + |2-\sigma|} \\ &\geq \frac{\eta}{\eta + |\mu - \sigma|} = \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta).\end{aligned} \tag{15}$$

Therefore, the conditions 1 and 2 of Theorem 10 are fulfilled byΛandΘ. By Theorem 10, Λ and Θ have a unique common fixed point, i.e., $\mu = 1$.

The second result of this part is as follows.

Theorem 12. *Let $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ be a complete FBM-space such that*

$$\lim_{\eta \longrightarrow \infty} \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = 1 \quad \text{for all } \mu \in \Pi, \sigma \in \Omega. \tag{16}$$

Let $\Lambda : \Pi \cup \Omega \longrightarrow \Pi \cup \Omega$ be two mappings satisfying

(1) $\Lambda(\Pi) \subseteq \Omega$, $\Lambda(\Omega) \subseteq \Pi$, and $\Theta(\Pi) \subseteq \Omega$, $\Theta(\Omega) \subseteq \Pi$

(2) $\Gamma_{\mathfrak{b}}(\Lambda(\sigma), \Theta(\mu), k\eta) \geq \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta)$ for all $\mu \in \Pi, \sigma \in \Omega$, and $\eta > 0$, where $0 < k < 1$

Then, Λ and Θ have a unique common fixed point.

Proof. Fix $\mu_0 \in \Pi$ and $\sigma_0 \in \Omega$ and assume that $\Lambda(\mu_{2\alpha}) = \sigma_{2\alpha}$, $\Theta(\mu_{2\alpha+1}) = \sigma_{2\alpha+1}$, $\Lambda(\sigma_{2\alpha}) = \mu_{2\alpha+1}$, and $\Theta(\sigma_{2\alpha+1}) = \mu_{2\alpha+2}$ for all $\alpha \in \mathbb{N} \cup \{0\}$. Then, we get $(\mu_\alpha, \sigma_\alpha)$ as a bisequence on the FBM-space $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$. Now, we have

$$\Gamma_{\mathfrak{b}}(\mu_1, \sigma_0, \eta) = \Gamma_{\mathfrak{b}}(\Lambda(\sigma_0), \Theta(\mu_0), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{k}\right), \tag{17}$$

$\forall \eta > 0$ and $\alpha \in \mathbb{N}$. By induction, we get

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_{2\alpha+1}, \sigma_{2\alpha+1}, \eta) &= \Gamma_{\mathfrak{b}}(\Lambda(\sigma_{2\alpha}), \Theta(\mu_{2\alpha+1}), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{k^{4\alpha+1}}\right), \\ \Gamma_{\mathfrak{b}}(\mu_{2\alpha+1}, \sigma_{2\alpha}, \eta) &= \Gamma_{\mathfrak{b}}(\Lambda(\sigma_{2\alpha}), \Theta(\mu_{2\alpha}), \eta) \geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{k^{4\alpha}}\right),\end{aligned} \tag{18}$$

for all $\eta > 0$ and $\alpha \in \mathbb{N}$. Letting $\alpha < \beta$, for $\alpha, \beta \in \mathbb{N}$. Then, from the definition of the fuzzy bipolar metric space, we get

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\mu_\alpha, \sigma_\alpha, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\alpha, \frac{\eta}{3}\right) \\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\beta, \frac{\eta}{3}\right), \\ &\vdots \\ &\geq \Gamma_{\mathfrak{b}}\left(\mu_\alpha, \sigma_\alpha, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_\alpha, \frac{\eta}{3}\right) * \cdots * \Gamma_{\mathfrak{b}}\left(\mu_{\beta-1}, \sigma_{\beta-1}, \frac{\eta}{3^{\beta-1}}\right) \\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_\beta, \sigma_{\beta-1}, \frac{\eta}{3^{\beta-1}}\right) * \Gamma_{\mathfrak{b}}\left(\mu_\beta, \sigma_\beta, \frac{\eta}{3^{\beta-1}}\right).\end{aligned} \tag{19}$$

Therefore,

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{3k^{2\alpha+1}}\right) * \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{3k^{2\alpha}}\right) * \cdots, \\ &\quad \cdots * \Gamma_{\mathfrak{b}}\left(\mu_0, \sigma_0, \frac{\eta}{3^{\beta-1}k^{2\beta+1}}\right).\end{aligned} \tag{20}$$

From (16), as $\alpha, \beta \longrightarrow \infty$, we get

$$\Gamma_{\mathfrak{b}}(\mu_\alpha, \sigma_\beta, \eta) \geq 1 \quad \forall \eta > 0. \tag{21}$$

Thus, bisequence $(\{\mu_\alpha\}, \{\sigma_\alpha\})$ is a Cauchy bisequence, since$(\Pi, \Omega, \Gamma_{\mathfrak{b}}, \star)$is a complete FBM-space. By Lemma 7, bisequence $(\{\mu_\alpha\}, \{\sigma_\alpha\})$ is a biconvergent sequence. Therefore, $\{\mu_\alpha\} \longrightarrow u$ and $\{\sigma_\alpha\} \longrightarrow u$, where $u \in \Pi \cap \Omega$. By Lemma 8, both sequences $\{\mu_\alpha\}$ and $\{\sigma_\alpha\}$ have a unique limit. From the triangular property of fuzzy bipolar metric spaces, we have

$$\begin{aligned}\Gamma_{\mathfrak{b}}(\Lambda(u), u, \eta) &\geq \Gamma_{\mathfrak{b}}\left(\Lambda(u), \sigma_{\alpha+1}, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, \sigma_{\alpha+1}, \frac{\eta}{3}\right) \\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right) = \Gamma_{\mathfrak{b}}\left(\Lambda(u), \Theta(\mu_{\alpha+1}), \frac{\eta}{3}\right) \\ &\quad * \Gamma_{\mathfrak{b}}\left(\Lambda\sigma_\alpha, \Theta\mu_{\alpha+1}, \frac{\eta}{3}\right) * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right) \\ &\geq \Gamma_{\mathfrak{b}}\left(u, \mu_{\alpha+1}, \frac{\eta}{3k}\right) * \Gamma_{\mathfrak{b}}\left(\sigma_\alpha, \mu_{\alpha+1}, \frac{\eta}{3k}\right) \\ &\quad * \Gamma_{\mathfrak{b}}\left(\mu_{\alpha+1}, u, \frac{\eta}{3}\right),\end{aligned} \tag{22}$$

for all $\alpha \in \mathbb{N}$ and $\eta > 0$ and as $\alpha \longrightarrow \infty$,

$$\Gamma_{\mathfrak{b}}(\Lambda(u), u, \eta) \longrightarrow 1 * 1 * 1 = 1. \tag{23}$$

From Definition 3 condition (2), $\Lambda(u) = u$. Again,

$$\Gamma_{\mathfrak{b}}(u, \Theta(u), \eta) = \Gamma_{\mathfrak{b}}(\Lambda(u), \Theta(u), \eta) \geq \Gamma_{\mathfrak{b}}\left(u, u, \frac{\eta}{k}\right) = 1. \tag{24}$$

Therefore, $\Theta(u) = u$. Hence, u is common fixed point of Λ and Θ. Let $v \in \Pi \cap \Omega$ be a another fixed point of Λ and Θ. Then,

$$\Gamma_{\mathfrak{b}}(u, v, \eta) = \Gamma_{\mathfrak{b}}(\Lambda(u), \Theta(v), \eta) \geq \Gamma_{\mathfrak{b}}\left(u, v, \frac{\eta}{k}\right), \tag{25}$$

for $0 < k < 1$ and $\forall \eta > 0$. By Lemma 5, we have $u = v$. □

To support the above theorem, we present the following example.

Example 13. Let $\Pi = \{0, 1, 2, 7\}$ and $\Omega = \{0, 1/3, 1/2, 3\}$ and define a continuous η-norm as $r * \zeta = \min\{r, \zeta\}$. Define $\Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = \exp^{-}(|\mu - \sigma|/\eta)$ for all $\eta > 0, \mu \in \Pi$, and $\sigma \in \Omega$. Then, $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is a complete FBM-space. Suppose we define a mapping $\Lambda, \Theta : \Pi \cup \Omega \longrightarrow \Pi \cup \Omega$ by

$$\Lambda(\mu) = \begin{cases} \frac{1}{3}, & \text{if } \mu \in \{7, 2\}, \\ 0, & \text{if } \mu \in \left\{0, \frac{1}{3}, \frac{1}{2}, 1, 3\right\}, \end{cases}$$
$$\Theta(\mu) = \begin{cases} \frac{1}{2}, & \text{if } \mu \in \{7, 2\}, \\ 0, & \text{if } \mu \in \left\{0, \frac{1}{3}, \frac{1}{2}, 1, 3\right\}. \end{cases} \tag{26}$$

Now, suppose that $k = 1/2$, then for all $\eta > 0$, we obtain the following cases.

Case 1. Let $\mu \in \{7, 2\}$ and $\sigma \in \{0, 1/3, 1/2, 1, 3\}$, then

$$\begin{aligned} \Gamma_{\mathfrak{b}}(\Lambda(\mu), \Theta(\sigma), k\eta) &= \Gamma_{\mathfrak{b}}\left(\frac{1}{3}, 0, k\eta\right) = \frac{k\eta}{k\eta + |1/3|} \\ &\geq \frac{\eta}{\eta + |\mu - \sigma|} = \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta). \end{aligned} \tag{27}$$

Case 2. Let $\mu \in \{0, 1/3, 1/2, 1, 3\}$ and $\sigma \in \{7, 2\}$, then

$$\begin{aligned} \Gamma_{\mathfrak{b}}(\Lambda(\mu), \Theta(\sigma), k\eta) &= \Gamma_{\mathfrak{b}}\left(0, \frac{1}{2}, k\eta\right) = \frac{k\eta}{k\eta + |1/2|} \\ &\geq \frac{\eta}{\eta + |\mu - \sigma|} = \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta). \end{aligned} \tag{28}$$

Therefore, the conditions 1 and 2 of Theorem 12 were also satisfied by Λ and Θ. Based on Theorem 12, we get Λ and Θ that have a unique common fixed point, i.e., $\mu = 0$.

4. Supportive Application

In this section, we apply Theorem 10 to discuss the existence and uniqueness solution to the following nonlinear integral equations:

$$\begin{cases} \mu(\gamma) = \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_1(\gamma, \zeta, \mu(\zeta)) d\zeta, \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2, \\ \mu(\gamma) = \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_2(\gamma, \zeta, \mu(\zeta)) d\zeta, \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2, \end{cases} \tag{29}$$

where $\mathscr{E}_1 \cup \mathscr{E}_2$ is a Lebesgue measurable set with $m(\mathscr{E}_1 \cup \mathscr{E}_2) < \infty$. Let $\Pi = L^\infty(\mathscr{E}_1)$ and $\Omega = L^\infty(\mathscr{E}_2)$ be two normed linear spaces. Define $\Gamma_{\mathfrak{b}} : \Pi \times \Omega \times (0,\infty) \longrightarrow 0, 1]$ by

$$\Gamma_{\mathfrak{b}}(\mu, \sigma, \eta) = e^{-\left(\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} |\mu(\gamma) - \sigma(\gamma)|/\eta\right)}, \tag{30}$$

for all $\mu \in \Pi, \sigma \in \Omega$. Clearly, $(\Pi, \Omega, \Gamma_{\mathfrak{b}}, *)$ is a complete FBM-space.

System (29) will be considered under the following hypotheses:

(i) $\mathscr{G}_1, \mathscr{G}_2 : (\mathscr{E}_1^2 \cup \mathscr{E}_2^2) \times 0,\infty) \longrightarrow 0,\infty)$ and $b \in L^\infty(\mathscr{E}_1) \cup L^\infty(\mathscr{E}_2)$

(ii) There is a continuous function $\theta : \mathscr{E}_1^2 \cup \mathscr{E}_2^2 \longrightarrow 0, \infty$ and $k \in (0, 1)$ such that $|\mathscr{G}_1(\gamma, \zeta, \mu(\zeta)) - \mathscr{G}_2(\gamma, \zeta, \sigma(\zeta))| \leq k\theta(\gamma, \zeta)(|\mu(\gamma) - \sigma(\gamma)|)$, for $\gamma, \zeta \in \mathscr{E}_1^2 \cup \mathscr{E}_2^2$

(iii) $\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \theta(\gamma, \zeta) d\zeta \leq 1$

Theorem 14. *Under hypotheses (i)–(iii), System (29) has a unique common solution in $L^\infty(\mathscr{E}_1) \cup L^\infty(\mathscr{E}_2)$.*

Proof. Define the mappings $\Lambda, \Theta : L^\infty(\mathscr{E}_1) \cup L^\infty(\mathscr{E}_2) \longrightarrow L^\infty(\mathscr{E}_1) \cup L^\infty(\mathscr{E}_2)$ by

$$\begin{aligned} \Lambda(\mu(\gamma)) &= \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_1(\gamma, \zeta, \mu(\zeta)) d\zeta, \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2, \\ \Theta(\mu(\gamma)) &= \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_2(\gamma, \zeta, \mu(\zeta)) d\zeta, \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2. \end{aligned} \tag{31}$$

Now, we have

$$\begin{aligned} &\Gamma_{\mathfrak{b}}(\Lambda\mu(\gamma), \Theta\sigma(\gamma), k\eta) \\ &= e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} (|\Lambda\mu(\gamma) - \Theta\sigma(\gamma)|/k\eta)} \\ &= e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \left(\left|\mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_1(\gamma,\zeta,\mu(\zeta)) d\zeta - \mathfrak{b}(\gamma) - \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_2(\gamma,\zeta,\sigma(\zeta)) d\zeta\right| / k\eta\right)} \\ &= e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \left(\left|\mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_1(\gamma,\zeta,\mu(\zeta)) d\zeta - \left(\mathfrak{b}(\gamma) + \int_{\mathscr{E}_1 \cup \mathscr{E}_2} \mathscr{G}_2(\gamma,\zeta,\sigma(\zeta)) d\zeta\right)\right| / k\eta\right)} \\ &\geq e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \left(\int_{\mathscr{E}_1 \cup \mathscr{E}_2} |\mathscr{G}_1(\gamma,\zeta,\mu(\zeta)) - \mathscr{G}_2(\gamma,\zeta,\sigma(\zeta))| d\zeta / k\eta\right)} \\ &\geq e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \left(\int_{\mathscr{E}_1 \cup \mathscr{E}_2} k\theta(\gamma,\zeta)(|\mu(\gamma) - \sigma(\gamma)|) d\zeta / k\eta\right)} \\ &\geq e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} \left(\int_{\mathscr{E}_1 \cup \mathscr{E}_2} k\theta(\gamma,\zeta)(|\mu(\gamma) - \sigma(\gamma)|) d\zeta / k\eta\right)} \\ &\geq e^{-\sup_{\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2} (|\mu(\gamma) - \sigma(\gamma)|/\eta)} = \Gamma_{\mathfrak{b}}(\mu, \sigma, \eta). \end{aligned} \tag{32}$$

Hence, all hypotheses of Theorem 10 are fulfilled, and consequently, the system (29) has a unique common solution. □

Example 15. Let $\mathscr{E}_1 = [0, 1]$, $\mathscr{E}_2 = [1, 2]$, $\Pi = L^\infty(\mathscr{E}_1)$, and $\Omega = L^\infty(\mathscr{E}_2)$. Now, consider the following nonlinear integral equations as

$$\Lambda(\mu(\gamma)) = \gamma + \int_{\mathscr{E}_1\cup\mathscr{E}_2}\left(\gamma+\zeta+\frac{\mu(\zeta)}{8(1+\mu(\zeta))}\right)d\zeta,$$
$$\Theta(\mu(\gamma)) = \gamma + \int_{\mathscr{E}_1\cup\mathscr{E}_2}\left(\gamma+\zeta+\frac{\mu(\zeta)}{8(1+\mu(\zeta))}\right)d\zeta, \tag{33}$$

for all $\gamma \in \mathscr{E}_1 \cup \mathscr{E}_2$. Then clearly, the above equation is in the form of the following equation:

$$\Lambda(\mu(\gamma)) = \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}_1(\gamma,\zeta,\mu(\zeta))d\zeta, \quad \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2,$$
$$\Theta(\mu(\gamma)) = \mathfrak{b}(\gamma) + \int_{\mathscr{E}_1\cup\mathscr{E}_2}\mathscr{G}_2(\gamma,\zeta,\mu(\zeta))d\zeta, \quad \gamma \in \mathscr{E}_1 \cup \mathscr{E}_2, \tag{34}$$

where $\mathfrak{b}(\gamma) = \gamma$ and

$$\mathscr{G}_1(\gamma,\zeta,\mu(\zeta)) = \gamma+\zeta+\frac{\mu(\zeta)}{8(1+\mu(\zeta))},$$
$$\mathscr{G}_2(\gamma,\zeta,\mu(\zeta)) = \gamma+\zeta+\frac{\mu(\zeta)}{8(1+\mu(\zeta))}. \tag{35}$$

That is, (33) is a particular case of system (29). Now, we have

$$\begin{aligned}
&|\mathscr{G}_1(\gamma,\zeta,\mu(\zeta)) - \mathscr{G}_2(\gamma,\zeta,\sigma(\zeta))| \\
&= \left|\frac{\mu(\zeta)}{8(1+\mu(\zeta))} - \frac{\sigma(\zeta)}{8(1+\sigma(\zeta))}\right| \\
&= \left|\frac{1}{8}\frac{\mu(\zeta)-\sigma(\zeta)}{(1+\mu(\zeta))(1+\sigma(\zeta))}\right| \\
&\le \frac{1}{8}(|\mu(\zeta)-\sigma(\zeta)|).
\end{aligned} \tag{36}$$

Consider a continuous function $\theta : \mathscr{E}_1^2 \cup \mathscr{E}_2^2 \longrightarrow [0,\infty)$ defined by $\theta(\gamma,\zeta) = 1/4 \forall \gamma, \zeta \in \mathscr{E}_1^2 \cup \mathscr{E}_2^2$. Then, we obtain

$$\sup_{\gamma\in\mathscr{E}_1\cup\mathscr{E}_2}\int_{\mathscr{E}_1\cup\mathscr{E}_2}\theta(\gamma,\zeta)d\zeta \le 1. \tag{37}$$

Therefore, all the conditions of Theorem 14 are satisfied. Hence, system (33) has a unique common solution in $L^\infty(\mathscr{E}_1) \cup L^\infty(\mathscr{E}_2)$.

5. Conclusion

First of all, we proved common fixed point theorems on fuzzy bipolar metric space with an application. On the basis of the ideas of this paper along with the literature present on FBM-spaces, we encourage the interested researcher to explore more interesting results for these spaces.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally in this research article. All authors read and approved the final manuscript.

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Hindawi
Journal of Function Spaces
Volume 2021, Article ID 9587604, 7 pages
https://doi.org/10.1155/2021/9587604

Research Article

New Results on Perov-Interpolative Contractions of Suzuki Type Mappings

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Received 16 August 2021; Revised 19 September 2021; Accepted 8 October 2021; Published 20 October 2021

Academic Editor: Santosh Kumar

In this paper, we introduce some common fixed point theorems for interpolative contraction operators using Perov operator which satisfy Suzuki type mappings. Further, some results are given. These results generalize several new results present in the literature.

1. Introduction

Banach [1] introduced the Banach contraction principle that generalized in various wide directions by many researchers. One of the generalizations was supposed by Kannan [2] in 1968 and later with other researchers such as C′iric′ Reich Rus. In 2018, Karapınar [3] adopted the interpolative approach to define the generalized Kannan type contraction on a complete metric space and proved the following.

A mapping $F : \mathscr{P} \longrightarrow \mathscr{P}$ on $(\mathscr{P}, d)$ a complete metric space such that

$$d(Fx, Fz) \leq k[d(x, Fx)]^{\alpha} \cdot [d(z, Fz)]^{1-\alpha}, \tag{1}$$

where $k \in [0, 1)$ and $\alpha \in (0, 1)$, for each $x, z \in \mathscr{P} \setminus Fix(F)$. Then, F has a unique fixed point in $\mathscr{P}$. Afterward, this concept has been extended in different aspects for example [4–14] and also see e.g. [15–19].

On the other hand, Perov [20, 21] gave a characterization of Banach contraction principle in the framework vector-valued metric space.

For a nonempty set $\mathscr{P}$, a function $d : \mathscr{P} \times \mathscr{P} \longrightarrow \mathbb{R}^k$ is called a vector-valued metric on $\mathscr{P}$ if the followings are fulfilled:

(1) $d(x, z) \geq 0$ for all $x, z \in \mathscr{P}$; if $d(x, z) = 0$, then $x = z$

(2) $d(x, z) = d(z, x)$ for all $x, z \in \mathscr{P}$

(3) $d(x, z) \leq d(x, t) + d(t, z)$ for all $x, z \in \mathscr{P}$

(4) where $0 := \underbrace{(0, 0, \cdots, 0)}_{k\text{-times}}$. We mention that, for $x, y \in \mathbb{R}^k$, that is, $x = (x_i)_{i=1}^{k}$ and $y = (y_i)_{i=1}^{k}$

$$x \leq y \Leftrightarrow x_i \leq y_i \quad \text{for each} \quad i \in \{1, 2, 3, \cdots, k\}. \tag{2}$$

The notations $M_{mm}(\mathbb{R})$ (respectively, $M_{mm}(\mathbb{R}_0^+)$) denote the collection of all square matrices of real numbers (respectively, nonnegative real numbers) where $\mathbb{R}$ (respectively, $\mathbb{R}_0^+ := [0,\infty)$) is the set of real numbers (respectively, nonnegative real numbers). Furthermore, $\mathbb{C}$ denotes the complex numbers, as usual.

A matrix $A \in M_{mm}(\mathbb{R})$ converges to zero if and only if its spectral radius is strictly less than 1, that is, $\rho(A) < 1$, see, e.g., [22]. It is equivalent to saying that all the eigenvalues of A are in the open unit disc, that is, $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with det $(A - \lambda I) = 0$, where I denotes the unit matrix of $M_{mm}(\mathbb{R})$.

Theorem 1 (see, e.g., [22, 23]). *Let $A \in M_{mm}(\mathbb{R}_0^+)$. Then, the following assertions are equivalent:*

(i) A converges to zero

$$\rho(A) < 1; \tag{3}$$

(ii) $A^n \longrightarrow 0$ as $n \longrightarrow \infty$

(iii) The matrix $(I - A)$ is nonsingular and

$$(I - A)^{-1} = I + A + \cdots + A^n + \cdots; \tag{4}$$

(iv) The matrix $(I - A)$ is nonsingular and $(I - A)^{-1}$ has nonnegative elements

(v) $A^n q$ and qA^n are convergent towards zero as $n \longrightarrow \infty$, for each $q \in \mathbb{R}^m$

Note also that if $A, B \in M_{mm}(\mathbb{R}_0^+)$ with $A \leq B$ (in the component-wise meaning), then, $\rho(B) < 1$ implies $\rho(A) < 1$.

Theorem 2 (Perov Cauchy, perov1966certain). *Let $(\mathscr{P}, d)$ be a complete vector-valued metric space and the operator $F : \mathscr{P} \longrightarrow \mathscr{P}$ with the property that there exists a matrix $A \in M_{mm}(\mathbb{R}_0^+)$ convergent towards zero such that*

$$d(F(x), F(z)) \leq A d(x, z), \quad \textit{for all} \quad x, z \in \mathscr{P}. \tag{5}$$

Then, F possesses a unique fixed point.

Ali et al. [24] defined Λ admissible that a generalized of α-admissible given by Samet et al. [25].

Definition 3 (see [24]). Let $\mathscr{P} \neq \emptyset$, $\Lambda : \mathscr{P} \times \mathscr{P} \longrightarrow M_{mm}(\mathbb{R}_+)$. A mapping $F : \mathscr{P} \longrightarrow \mathscr{P}$
is said to be Λ-admissible if

$$\Lambda(x, z) \geq I \quad \text{implies} \quad \Lambda(Fx, Fz) \geq I, \quad \text{for all} \quad x, z \in \mathscr{P}, \tag{6}$$

where I is the $m \times m$ identity matrix and the inequality between matrices mans entrywise inequality.

We define some related to Λ-admissible the following concept of admissibility using the above definition and given by some authors [25–30].

Definition 4. Let $\mathscr{P} \neq \emptyset$, $\Lambda : \mathscr{P} \times \mathscr{P} \longrightarrow M_{mm}(\mathbb{R}_+)$, and $F : \mathscr{P} \longrightarrow \mathscr{P}$ be a mapping. We say that F is an Λ-orbital admissible mapping if

$$\Lambda(x, Fx) \geq I \quad \text{implies} \quad \Lambda\left(Fx, F^2 x\right) \geq I, \quad \text{for all} \quad x, z \in \mathscr{P}. \tag{7}$$

Moreover, an Λ-orbital admissible mapping F is said to be triangular Λ-orbital admissible if for all $x, z \in \mathscr{P}$, we have

$$\Lambda(x, z) \geq I \quad \text{and} \quad \Lambda(z, Fz) \geq I, \quad \text{implies} \quad \Lambda(x, Fz) \geq I. \tag{8}$$

Definition 5. For a nonempty set $\mathscr{P}$, let $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ and $\Lambda : \mathscr{P} \times \mathscr{P} \longrightarrow M_{mm}(\mathbb{R}_+)$ be mappings. We say that $(F, \mathscr{G})$ is a generalized Λ -admissible pair if for all $x, z \in \mathscr{P}$, we have

$$\Lambda(x, z) \geq I \Rightarrow \Lambda(Fx, \mathscr{G}z) \geq I \quad \text{and} \quad \Lambda(\mathscr{G}z, Fx) \geq I. \tag{9}$$

Lemma 6. *Let $\mathscr{P} \neq \emptyset$ and $F : \mathscr{P} \longrightarrow \mathscr{P}$ be a triangular Λ -admissible map. Suppose that there exists $x_0 \in \mathscr{P}$ such that $\Lambda(x_0, Fx_0) \geq I$. Identify a sequence $\{x_n\}$ using $x_{n+1} = Fx_n$. Thus, we have $\Lambda(x_n, x_m) \geq I$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Lemma 7. *Let $\mathscr{P} \neq \emptyset$ and $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ be a triangular Λ -admissible mapping. Assume that there exists $x_0 \in \mathscr{P}$ such that $\Lambda(x_0, Fx_0) \geq I$. Identify sequence $x_{2r+1} = Fx_{2r}$, and $x_{2r+2} = \mathscr{G}x_{2r+1}$, where $r = 0, 1, 2, \cdots$. So, we have $\Lambda(x_n, x_m) \geq I$ for all $m, n \in \mathbb{N} \cup \{0\}$ with $n < m$.*

Recently, one of the interesting generalizations was given by Suzuki [31, 32] which characterizes the completeness of underlying metric spaces. Suzuki introduced and generalized versions of Banach's and Edelstein's basic results. In addition, Popescu [33] has modified the nonexpansiveness situation with the weaker C-condition presented by Suzuki. As stated, the existence of fixed points of maps satisfying the C-condition has been extensively studied; see [34–38]. Karapnar [39] investigated the definition of a nonexpansive mapping satisfying the C-condition.

Definition 8. A mapping F on a metric space $(\mathscr{P}, d)$ satisfies the C-condition if

$$\frac{1}{2} d(x, Fx) \leq d(x, z) \Rightarrow d(Fx, Fz) \leq d(x, z), \tag{10}$$

for each $x, z \in \mathscr{P}$.

Theorem 9. *Let $(\mathscr{P}, d)$ be a compact metric space and $F : \mathscr{P} \longrightarrow \mathscr{P}$ be a mapping satisfying condition (C) for all $x, z \in \mathscr{P}$. Then, F has a unique fixed point.*

2. Main Results

For the rest of the paper, we use the following notation: $\mathscr{P}_{\mathrm{F}} = \mathscr{P} \setminus Fix(\mathrm{F})$, where $Fix(\mathrm{F}) = \{x \in \mathscr{P} \mid \mathrm{F}x = x\}$.

Definition 10. Let $(\mathscr{P}, d)$ be a complete vector-valued metric space and $\Lambda : \mathscr{P} \times \mathscr{P} \longrightarrow M_{mm}(\mathbb{R}_+)$, $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ be mappings. We say that $(F, \mathscr{G})$ forms a pair of Perov-interpolative $C'iric'$-Reich-Rus contractions of Suzuki type, if there exist $A, B \in M_{mm}(\mathbb{R}_+)$ converges towards zero, (where $B = A^q$, $q > 1$) such that

$$\begin{aligned}\frac{1}{2}\min\{d(x, Fx), d(z, \mathscr{G}z)\} \le d(x, z) \Rightarrow \Lambda(x, z)d(Fx, \mathscr{G}z) \\ \le A\left([d(x, z)]^{\beta} \cdot [d(x, Fx)]^{\alpha} \cdot [d(z, \mathscr{G}z)]^{1-\alpha-\beta}\right),\end{aligned} \tag{11}$$

for each $(x, z) \in \mathscr{P}_{\mathrm{F}} \times \mathscr{P}_{\mathscr{G}}$, where $\beta \ge 0$, $\alpha > 0$ are such that $\beta + \alpha < 1$.

Theorem 11. *Let $(\mathscr{P}, d)$ be a complete vector-valued metric space and $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ be two mappings such that $(F, \mathscr{G})$ is a pair of Perov-interpolative $C'iric'$-Reich-Rus contractions of Suzuki type. Assume that*

(i) $(F, \mathscr{G})$ is a generalized Λ-admissible pair

(ii) There exists $x_0 \in \mathscr{P}$ such that $\Lambda(x_0, Fx_0) \ge I$ and $\Lambda(Fx_0, x_0) \ge I$

(iii) F and $\mathscr{G}$ are continuous mappings

Then, F and $\mathscr{G}$ have a common fixed point.

Proof. Let $x_0 \in \mathscr{P}$ such that $\Lambda(x_0, \mathrm{F}x_0) \ge I$. We define the sequence $\{x_r\}$ in $\mathscr{P}$ as following

$$x_{2r+1} = Fx_{2r} \text{ and } x_{2r+2} = \mathscr{G}x_{2r+1}, \tag{12}$$

for every $r \in \mathbb{N}$. From (i) and (ii), $\Lambda(x_0, Fx_0) \ge I$ and $\Lambda(Fx_0, x_0) \ge I$, and then $\Lambda(x_1, x_2) = \Lambda(Fx_0, \mathscr{G}x_1) \ge I$ and $\Lambda(x_2, x_1) = \Lambda(\mathscr{G}x_1, Fx_0) \ge I$. Similarly, we get $\Lambda(x_2, x_3) = \Lambda(Fx_1, \mathscr{G}x_2) \ge I$ and $\Lambda(x_3, x_2) = \Lambda(\mathscr{G}x_2, Fx_1) \ge I$. Repeating this process, we write

$$\Lambda(x_m, x_{m+1}) \ge I \quad \text{and} \quad \Lambda(x_{m+1}, x_m) \ge I, \tag{13}$$

for every $m \in \mathbb{N}$.

On the other hand, we have

$$\begin{aligned}&\frac{1}{2}\min\{d(x_{2r}, Fx_{2r}), d(x_{2r+1}, \mathscr{G}x_{2r+1})\} \\ &= \frac{1}{2}\min\{d(x_{2r}, x_{2r+1}), d(x_{2r+1}, x_{2r+2})\} \le d(x_{2r}, x_{2r+1}).\end{aligned} \tag{14}$$

So, since the mappings $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ forms a pair of Perov-interpolative contractions of Suzuki type, we get

$$\begin{aligned}d(x_{2r+1}, x_{2r+2}) &= Id(Fx_{2r}, \mathscr{G}x_{2r+1}) \le \Lambda(x_{2r}, x_{2r+1})d(Fx_{2r}, \mathscr{G}x_{2r+1}) \\ &\le A\Big([d(x_{2r}, x_{2r+1})]^{\beta} \cdot [d(x_{2r}, Fx_{2r})]^{\alpha} \\ &\quad \cdot [d(x_{2r+1}, \mathscr{G}x_{2r+1})]^{1-\alpha-\beta}\Big) \\ &= A\left([d(x_{2r}, x_{2r+1})]^{\beta+\alpha} \cdot [d(x_{2r+1}, x_{2r+2})]^{1-\alpha-\beta}\right).\end{aligned} \tag{15}$$

Therefore, it follows that

$$[d(x_{2r+1}, x_{2r+2})]^{\alpha+\beta} \le A[d(x_{2r}, x_{2r+1})]^{\beta+\alpha}, \tag{16}$$

or equivalent

$$d(x_{2r+1}, x_{2r+2}) \le A^q d(x_{2r}, x_{2r+1}), \tag{17}$$

where $q = 1/\beta + \alpha > 1$. Then,

$$d(x_{2r+1}, x_{2r+2}) \le Bd(x_{2r}, x_{2r+1}) \quad \text{where} \quad A^q = B. \tag{18}$$

Letting $x = x_{2\mathrm{r}}$ and $z = x_{2\mathrm{r}-1}$, since

$$\begin{aligned}&\frac{1}{2}\min\{d(x_{2r}, Fx_{2r}), d(x_{2r-1}, \mathscr{G}x_{2r-1})\} \\ &= \frac{1}{2}\min\{d(x_{2r}, x_{2r+1}), d(x_{2r-1}, x_{2r})\} \le d(x_{2r}, x_{2r-1}),\end{aligned} \tag{19}$$

similarly, we get

$$d(x_{2r}, x_{2r+1}) \le Bd(x_{2r-1}, x_{2r}). \tag{20}$$

Thus, combining (18) and (20), we have that

$$d(x_{2r+1}, x_{2r+2}) \le B^{2r}d(x_1, x_2), \tag{21}$$

$$d(x_{2r}, x_{2r+1}) \le B^{2r}d(x_0, x_1). \tag{22}$$

Take into account (21) and (22), we obtain that for each $m \in \mathbb{N}$

$$d(x_m, x_{m+1}) \le B^m w(x), \tag{23}$$

where $w(x) = \max\{d(x_0, x_1), d(x_1, x_2)\}$. By this way, using triangular inequality and (23), for $p \ge 0$, we get

$$\begin{aligned}d(x_m, x_{m+p}) &\le d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \cdots + d(x_{m+p-1}, x_{m+p}) \\ &= \sum_{m=l}^{m+p-1} d(x_i, x_{i+1}) \le B^m\left(\sum_{i=0}^{\infty} B^i\right)w(x) \\ &= B^m(I + B + \cdots + B^m + \cdots)w(x).\end{aligned} \tag{24}$$

Because B is convergent to zero, we attain that $(I - B)$ is nonsingular and

$$(I-B)^{-1} = I + B + \cdots + B^m + \cdots. \tag{25}$$

Therefore,

$$d(x_m, x_{m+p}) B^m (I-B)^{-1} w(x), \tag{26}$$

$$d(x_m, x_{m+p}) \longrightarrow 0 asm \longrightarrow \infty. \tag{27}$$

So, the sequence (x_m) is a fundamental (Cauchy), and using the completeness of the space $(\mathscr{P}, d)$, there exists $t \in \mathscr{P}$ such that

$$\lim_{m\longrightarrow\infty} d(x_m, t) = 0. \tag{28}$$

We claim that the point t is a common fixed point of F and $\mathscr{G}$. If (iii.) is provide, that is, the mapping F and $\mathscr{G}$ are continuous, we have

$$\lim_{r\longrightarrow\infty} d(x_{2r+1}, \mathrm{Ft}) = \lim_{r\longrightarrow\infty} d(Fx_{2r}, \mathrm{Ft}) = 0, \tag{29}$$

then, $\mathrm{F}t = t$. Also, similarly, we get

$$\lim_{r\longrightarrow\infty} d(x_{2r+2}, \mathscr{G}t) = \lim_{r\longrightarrow\infty} d(\mathscr{G}x_{2r+1}, \mathscr{G}t) = 0. \tag{30}$$

$\mathscr{G}t = t$. Therefore, t is a common fixed point of F and $\mathscr{G}$. The proof is complete.□ □

In the following theorem, we remove the assumption of the continuity of the mappings F and $\mathscr{G}$.

Theorem 12. *Besides the hypothesis (i) and (ii) of Theorem 11, if we assume that the condition:*

(i) If $\{x_r\}$ is a sequence in $\mathscr{P}$ such that $x_r \longrightarrow x \in \mathscr{P}$ as $r \longrightarrow \infty$ and, there exists a subsequence $\{x_{r_k}\}$ of $\{x_r\}$ such that $\Lambda(x_{r_k}, x) \geq I$ and $\Lambda(x, x_{r_k}) \geq I$, for all k

holds, then, the mappings F and $\mathscr{G}$ have a common fixed point.

Proof. From Theorem 11, the sequence $\{x_r\}$ defined by (12) is a Cauchy sequence and converges to some $t \in \mathscr{P}$. Similarly, using (13) and the condition (H_Λ), there exists a subsequence $\{x_{r_k}\}$ of $\{x_r\}$ such that $\Lambda(x_{2r_k}, t) \geq I$ and $\Lambda(t, x_{2r_{k-1}}) \geq I$ for all k. We claim that for all $k \geq 0$

$$\frac{1}{2} \min \left\{ d\left(x_{2r_k}, Fx_{2r_k}\right), d(t, \mathscr{G}t) \right\} \leq d\left(x_{2r_k}, t\right). \tag{31}$$

Supposing on the contrary,

$$\frac{1}{2} \min \left\{ d\left(x_{2r_k}, Fx_{2r_k}\right), d(t, \mathscr{G}t) \right\} > d\left(x_{2r_k}, t\right), \tag{32}$$

we get

$$\frac{1}{2} \min \left\{ d\left(x_{2r_k}, x_{2r_{k+1}}\right), d(t, \mathscr{G}t) \right\} > d\left(x_{2r_k}, t\right), \tag{33}$$

and letting $k \longrightarrow \infty$, we acquire that a contradiction. Therefore, the condition (31) holds, and from (11), we obtain

$$\begin{aligned} d\left(x_{2r_k+1}, \mathscr{G}t\right) &= Id\left(Fx_{2r_k}, \mathscr{G}t\right) \leq \Lambda\left(x_{2r_k}, t\right) d\left(Fx_{2r_k}, \mathscr{G}t\right) \\ &\leq A\left(\left[d\left(x_{2r_k}, t\right)\right]^\beta \cdot \left[d\left(x_{2r_k}, Fx_{2r_k}\right)\right]^\alpha \cdot [d(t, \mathscr{G}t)]^{1-\alpha-\beta}\right) \\ &= A\left(\left[d\left(x_{2r_k}, t\right)\right]^\beta \cdot \left[d\left(x_{2r_k}, x_{2r_{k+1}}\right)\right]^\alpha \cdot [d(t, \mathscr{G}t)]^{1-\alpha-\beta}\right). \end{aligned} \tag{34}$$

On the taking k tend to infinity, it follows that we get $\mathscr{G}t = t$. Similarly, we assert that, for all $k \geq 0$

$$\frac{1}{2} \min \left\{ d(t, \mathrm{Ft}), d\left(x_{2r_{k-1}}, \mathscr{G}x_{2r_{k-1}}\right) \right\} \leq d\left(t, x_{2r_{k-1}}\right), \tag{35}$$

Supposing on the contrary,

$$\frac{1}{2} \min \left\{ d(t, \mathrm{Ft}), d\left(x_{2r_{k-1}}, \mathscr{G}x_{2r_{k-1}}\right) \right\} > d\left(t, x_{2r_{k-1}}\right), \tag{36}$$

and, so

$$\frac{1}{2} \min \left\{ d(t, \mathrm{Ft}), d\left(x_{2r_{k-1}}, x_{2r_k}\right) \right\} > d\left(t, x_{2r_{k-1}}\right), \tag{37}$$

taking $k \longrightarrow \infty$, we obtain that a contradiction. Hence, condition (35) is true and from (11), we obtain

$$\begin{aligned} d\left(\mathrm{Ft}, x_{2r_k}\right) &= Id\left(\mathrm{Ft}, \mathscr{G}x_{2r_{k-1}}\right) \leq \Lambda\left(t, x_{2r_{k-1}}\right) d\left(\mathrm{Ft}, \mathscr{G}x_{2r_{k-1}}\right) \\ &\leq A\left(\left[d\left(t, x_{2r_{k-1}}\right)\right]^\beta \cdot [d(t, \mathrm{Ft})]^\alpha \cdot \left[d\left(x_{2r_{k-1}}, \mathscr{G}x_{2r_{k-1}}\right)\right]^{1-\alpha-\beta}\right) \\ &= A\left(\left[d\left(t, x_{2r_{k-1}}\right)\right]^\beta \cdot [d(t, \mathrm{Ft})]^\alpha \cdot \left[d\left(x_{2r_{k-1}}, x_{2r_k}\right)\right]^{1-\alpha-\beta}\right). \end{aligned} \tag{38}$$

Letting k tend to infinity, it follows that we acquire $\mathrm{Ft} = t$. Thus, t is a common fixed point of F and $\mathscr{G}$.□ □

Corollary 13. *Let $(\mathscr{P}, d)$ be a complete vector-valued metric space and $F, \mathscr{G} : \mathscr{P} \longrightarrow \mathscr{P}$ be two continuous mappings such that*

$$\begin{aligned} &\frac{1}{2} \min \{ d(x, Fx), d(z, \mathscr{G}z) \} \leq d(x, z) \Rightarrow d(Fx, \mathscr{G}z) \\ &\quad \leq A\left([d(x, z)]^\beta \cdot [d(x, Fx)]^\alpha \cdot [d(z, \mathscr{G}z)]^{1-\alpha-\beta}\right), \end{aligned} \tag{39}$$

for each $(x, z) \in \mathscr{P}_F \times \mathscr{P}_\mathscr{G}$, where $A, A^q \in M_{mm}(\mathbb{R}_+)$, $q > 1$, converges towards zero and $\beta \geq 0, \alpha > 0$, are such that $\beta + \alpha < 1$. Then, F and $\mathscr{G}$ have a common fixed point.

Corollary 14. *Let $(\mathscr{P}, d)$ be a complete vector-valued metric space and the mappings $\Lambda : \mathscr{P} \times \mathscr{P} \longrightarrow M_{mm}(\mathbb{R}_+)$ and $F, \mathscr{G}$*

: $\mathscr{P} \longrightarrow \mathscr{P}$ *such that*

$$\Lambda(x,z)d(Fx,\mathscr{G}z) \le A\left([d(x,z)]^{\beta}\cdot[d(x,Fx)]^{\alpha}\cdot[d(z,\mathscr{G}z)]^{1-\alpha-\beta}\right), \tag{40}$$

for each $(x,z)\in\mathscr{P}_F\times\mathscr{P}_{\mathscr{G}}$, *where* $A, A^q \in M_{mm}(\mathbb{R}_+)$, $q>1$, *converges towards zero and the constants* $\beta\ge 0$, $\alpha>0$, *are such that* $\beta+\alpha<1$. *Assume that*

(i) $(F,\mathscr{G})$ *is a generalized* Λ*-admissible pair*

(ii) *There exists* $x_0\in\mathscr{P}$ *such that* $\Lambda(x_0,Fx_0)\ge I$ *and* $\Lambda(Fx_0,x_0)\ge I$

(iii) *The condition* (H_Λ) *holds or* F *and* $\mathscr{G}$ *are continuous mappings*

Then, F *and* $\mathscr{G}$ *have a common fixed point.*

Letting $F=\mathscr{G}$ in Theorem 11, we obtain the next results.

Corollary 15. *Let* $(\mathscr{P},d)$ *be a generalized metric spaces and* $\Lambda:\mathscr{P}\times\mathscr{P}\longrightarrow M_{mm}(\mathbb{R}_+)$. *Let* $F:\mathscr{P}\longrightarrow\mathscr{P}$ *be a* Λ*-orbital admissible mapping such that*

$$\begin{aligned}\frac{1}{2}d(x,Fx)\le d(x,z) &\Rightarrow \Lambda(x,z)d(Fx,Fz)\\ &\le A[d(x,z)]^{\beta}\cdot[d(x,Fx)]^{\alpha}\cdot[d(z,Fz)]^{1-\alpha-\beta},\end{aligned} \tag{41}$$

for each $x,z\in\mathscr{P}_F$, *where* $A, A^q\in M_{mm}(\mathbb{R}_+)$, $q>1$, *converges towards zero, and* β,α *are constants, such that* $\beta\ge 0$, $\alpha>0$, *and* $\beta+\alpha<1$. *Assume that*

(i) F *is a triangular* Λ*-orbital admissible*

(ii) *There exists* $x_0\in\mathscr{P}$ *such that* $\Lambda(x_0,Fx_0)\ge I$ *and* $\Lambda(Fx_0,x_0)\ge I$

(iii) *Either* F *is continuous, or the condition* (H_Λ) *holds*

Then, F *has a fixed point.*

Definition 16. Let $(\mathscr{P},d)$ be a vector-valued metric space and $\Lambda:\mathscr{P}\times\mathscr{P}\longrightarrow M_{mm}(\mathbb{R}_+)$, $F,\mathscr{G}:\mathscr{P}\longrightarrow\mathscr{P}$, be mappings. We say that $F,\mathscr{G}:\mathscr{P}\longrightarrow\mathscr{P}$ are Perov-interpolative Kannan contractions of Suzuki type, if there exist a real number $\alpha\in(0,1)$ and $A,C\in M_{mm}(\mathbb{R}_+)$ converges towards zero, where $A^{1/a}=C$, such that

$$\begin{aligned}\frac{1}{2}\min\{d(x,Fx),d(z,\mathscr{G}z)\}\le d(x,z) &\Rightarrow \Lambda(x,z)d(Fx,\mathscr{G}z)\\ &\le A\left([d(x,Fx)]^{\alpha}\cdot[d(z,\mathscr{G}z)]^{1-\alpha}\right),\end{aligned} \tag{42}$$

for each $(x,z)\in\mathscr{P}_F\times\mathscr{P}_{\mathscr{G}}$.

Theorem 17. *Let* $(\mathscr{P},d)$ *be a complete vector-valued metric space and* $F,\mathscr{G}:\mathscr{P}\longrightarrow\mathscr{P}$ *be Perov-interpolative Kannan contractions of Suzuki type. Assume that*

(i) $(F,\mathscr{G})$ *is a generalized* Λ*-admissible pair*

(ii) *There exists* $x_0\in\mathscr{P}$ *such that* $\Lambda(x_0,Fx_0)\ge I$ *and* $\Lambda(Fx_0,x_0)\ge I$

(iii) *Either,* F *and* $\mathscr{G}$ *are continuous mappings or, the condition* (H_Λ) *holds*

Then, F *and* $\mathscr{G}$ *have a common fixed point.*

Proof. Taking $\beta=0$ in Theorem 11.□ □

Remark 18. If $m=1$ and $A=\kappa\in(0,1)$ in the above Theorems, then, we find the concept of the usual metric spaces and interpolative Kannan contraction of Suzuki type and interpolative Ćirić–Reich–Rus contraction of Suzuki type.

Example 19. Let $\mathscr{P}=[0,2]$, $d:\mathscr{P}\times\mathscr{P}\longrightarrow[0,+\infty)$, where $d(x,z)=\begin{pmatrix}|x-z|\\|x-z|\end{pmatrix}$, and two mappings $F,\mathscr{G}:\mathscr{P}\longrightarrow\mathscr{P}$, defined as

$$\mathrm{F}x=\begin{cases}\frac{1}{3}, & if\, x\in[0,1],\\ \frac{x}{4}, & if\, x\in(1,2],\end{cases} \tag{43}$$

respectively,

$$\mathscr{G}x=\begin{cases}\frac{1}{3}, & if\, x\in[0,1],\\ \frac{x}{8}, & if\, x\in(1,2].\end{cases} \tag{44}$$

We choose $\beta=1/3$, $\alpha=1/3$ and $A=\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}$. Let also $\Lambda:\mathscr{P}\times\mathscr{P}\longrightarrow M_{22}(\mathbb{R}_+)$, where

$$\Lambda(x,z)=\begin{cases}\begin{pmatrix}2&0\\0&2\end{pmatrix} & \text{if}\quad x,z\in[0,1,]\\ \begin{pmatrix}1&0\\0&1\end{pmatrix} & \text{if}\quad x=0, z=2,\\ \begin{pmatrix}0&0\\0&0\end{pmatrix} & \text{otherwise.}\end{cases} \tag{45}$$

Then, we have to check that (11) holds. We have to examine the following cases:

(1) $x,z\in[0,1]$. Let $U_1=\{1/2\}\times\{1/2n : n\in\{2,3,4,\cdots\}\}$ and $U_2=\{1/2n : n\in\{2,3,4,\cdots\}\}\times\{1/2\}$. For $(x,z)\in A_1\cup A_2$

$$\frac{1}{2}\min\{d(x,Fx),d(z,\mathscr{G}z)\} \leq \begin{pmatrix} \frac{1}{12} \\ \frac{1}{12} \end{pmatrix} < \begin{pmatrix} \frac{n-1}{2n} \\ \frac{n-1}{2n} \end{pmatrix} = d(x,z)\Lambda(x,z)d(Fx,\mathscr{G}z) \leq A[d(x,z)]^{1/3}[d(x,Fx]^{1/3}[d(z,\mathscr{G}z)]^{1/3}, \tag{46}$$

and since $d(Fx,\mathscr{G}z)=0$, the inequality (11) holds.

(2) $x,z \in (1,2]$. Similarly, since $\Lambda(x,z)=0$, the relation (11) holds

(3) For $x=0$ and $z=2$

$$\begin{aligned} \frac{1}{2}\min\left\{\left(d\left(0,\frac{1}{3}\right),d\left(2,\frac{1}{4}\right)\right\}\right. &= \frac{1}{2}\min\left\{\begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix},\begin{pmatrix} \frac{7}{4} \\ \frac{7}{4} \end{pmatrix}\right\} \\ &= \begin{pmatrix} \frac{1}{6} \\ \frac{1}{6} \end{pmatrix} < \begin{pmatrix} 2 \\ 2 \end{pmatrix} = d(0,2) \Rightarrow \Lambda(0,2)d(F0,\mathscr{G}2) \\ &= \begin{pmatrix} \frac{1}{12} \\ \frac{1}{12} \end{pmatrix} < \begin{pmatrix} \left(\frac{7}{48}\right)^{1/3} \\ \left(\frac{7}{48}\right)^{1/3} \end{pmatrix} \\ &= A[d(0,2)]^{1/3}\cdot[d(0,\mathrm{F}0)]^{1/3}\cdot[d(2,\mathscr{G}2)]^{1/3}. \end{aligned} \tag{47}$$

Then, (11) holds. Consequently, the assumptions of Theorem 11 being satisfied, it follows that the mappings F and $\mathscr{G}$ have a fixed point, which is $x=1/3$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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