and Abstract Applied Analysis

Special Issue
Recent Progress in Differential and Difference Equations

Guest Editors
J. Diblík, E. Braverman, Yu. Rogovchenko,
and M. Růžičková

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Editorial Board

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Editorial

Recent Progress in Differential and Difference Equations

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This special issue on *Recent Progress in Differential and Difference Equations* contains forty-four research articles. Most papers originate from the talks at the Conference on Differential and Difference Equations and Applications (where all Guest Editors served as organizers) held in Rajecké Teplice, Slovak Republic, during June 21–25, 2010 (http://fpv.uniza.sk/cddea2010/page.php?id=19&action=show). At the conference, more than 50 contributed papers and posters were presented along with eighteen invited lectures delivered by leading researchers such as Professors A. A. Boichuk (Slovak Republic), T. A. Burton (USA), V. Covachev (Oman), O. Došlý (Czech Republic), J. Džurina (Slovak Republic), J. Jaroš (Slovak Republic), D. Khusainov (Ukraine), W. Kratz (Germany), N. Partsvania (Georgia), I. Rachůnková (Czech Republic), V. Răsvan (Romania), M. Rontó (Hungary), S. Staněk (Czech Republic), M. Tvrdý (Czech Republic), and F. Sadyrbaev (Latvia). In addition to the papers discussed at the conference, many articles have been written for publication in this issue. As a result, seventy-seven authors from sixteen countries contributed to the success of this thematic collection of papers.

The issue covers a wide variety of problems for different classes of ordinary, functional, impulsive, stochastic, fractional, partial differential equations, as well as difference and integrodifferential equations, inclusions, and dynamic equations on time scales. The topics discussed in the contributed papers are traditional for qualitative theory of differential, functional differential, difference, and other classes of equations. The issue contains papers on asymptotic behavior of positive solutions of functional differential equations of delayed type, solutions to third-order trinomial delay differential equations, half-linear q-difference equations, solutions to discrete equations with two delays in the critical case, solutions to delay difference and integrodifferential equations, as well as research on weighted asymptotically periodic solutions to linear Volterra difference equations. Furthermore, asymptotic properties of third-order nonlinear functional differential equations with mixed arguments and variational equations are investigated.

Traditionally, stability problems receive a great deal of attention at various conferences. Papers included in this issue address stability of linear differential equations with several delays, stability of linear delay differential equations under Perron's condition, exponential stability of solutions to stochastic control systems with delay, and instability of the trivial solution of autonomous differential systems with quadratic right-hand sides in a cone.

As usual, many papers deal with oscillation and nonoscillation of various classes of equations. In particular, a number of papers are concerned with oscillation of second-order neutral delay dynamic equations of Emden-Fowler type, second-order neutral functional differential equations with mixed nonlinearities and of mixed type, second-order superlinear neutral differential equations, singular nonlinear differential equations, second-order sublinear impulsive differential equations, and half-linear differential equations. In addition, nonoscillation of advanced differential equations with several terms, second-order dynamic equations with several delays, and first-order neutral differential equations are studied.

Several authors deal with different aspects of the theory of boundary value problems for nonlinear fractional differential equations, q-difference inclusions, and weakly nonlinear delay differential systems. Interesting results are obtained for a class of fourth-order boundary value problems, singular boundary value problems for nonlinear fractional differential equations, nonseparated three-point boundary value problems for linear functional differential equations, and periodic problems for difference equations.

Papers collected in this special issue are also concerned with a maximal number of period annuli, Lie groups in infinite dimension, Weyl-Titchmarsh theory for time-scale symplectic systems on a half line, compatible and incompatible nonuniqueness conditions for the classical Cauchy problem, optimization of solutions to dynamic systems with random structure, application of discrete Mittag-Leffler functions in linear fractional difference equations, conjugacy of self-adjoint even order difference equations, H_∞ estimates for Lipschitz nonlinear discrete-time systems with delay, and reducibility of quasiperiodic Hamiltonian systems with a small perturbation. Existence of invariant sets for impulsive differential equations with particularities in ω-limit set and existence of pseudosymmetric solutions to p-Laplacian differential equations involving derivative are explored.

Finally, some applied problems are also considered—a two-species cooperative Lotka-Volterra system of degenerate parabolic equations, equations of Emden-Fowler type, and oscillatory periodic solutions for two differential-difference equations that model phase-locked loop control of high-frequency generators and nonlinear growth of a fluctuating population.

Although it is not possible to adequately represent in this special issue all directions of current research in ordinary, functional, partial, impulsive, dynamic, stochastic differential equations, difference, and integrodifferential equations, we believe that it reflects many important recent trends in research, indicates current challenging problems, and outlines new ideas for future studies in the field.

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Research Article

A Final Result on the Oscillation of Solutions of the Linear Discrete Delayed Equation $\Delta x(n) = -p(n)x(n-k)$ with a Positive Coefficient

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A linear $(k+1)$th-order discrete delayed equation $\Delta x(n) = -p(n)x(n-k)$ where $p(n)$ a positive sequence is considered for $n \to \infty$. This equation is known to have a positive solution if the sequence $p(n)$ satisfies an inequality. Our aim is to show that, in the case of the opposite inequality for $p(n)$, all solutions of the equation considered are oscillating for $n \to \infty$.

1. Introduction

The existence of positive solutions of difference equations is often encountered when analysing mathematical models describing various processes. This is a motivation for an intensive study of the conditions for the existence of positive solutions of discrete or continuous equations. Such analysis is related to investigating the case of all solutions being oscillating (for investigation in both directions, we refer, e.g., to [1–30] and to the references therein). The existence of monotonous and nontrivial solutions of nonlinear difference equations (the first one implies the existence of solutions of the same sign) also has attracted some attention recently (see, e.g., several, mostly asymptotic methods in [31–42] and the related references therein). In this paper, sharp conditions are derived for all the solutions being oscillating for a class of linear $(k+1)$-order delayed discrete equations.

We consider the delayed $(k+1)$-order linear discrete equation

$$\Delta x(n) = -p(n)x(n-k), \tag{1.1}$$

where $n \in \mathbb{Z}_a^\infty := \{a, a+1, \dots\}$, $a \in \mathbb{N} := \{1, 2, \dots\}$ is fixed, $\Delta x(n) = x(n+1) - x(n)$, $p : \mathbb{Z}_a^\infty \to \mathbb{R}$, $k \in \mathbb{N}$. In what follows, we will also use the sets $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_a^b := \{a, a+1, \dots, b\}$ for $a, b \in \mathbb{N}$, $a < b$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is positive (negative) on $\mathbb{Z}_a^\infty$ if $x(n) > 0$ ($x(n) < 0$) for every $n \in \mathbb{Z}_a^\infty$. A solution $x = x(n) : \mathbb{Z}_a^\infty \to \mathbb{R}$ of (1.1) is oscillating on $\mathbb{Z}_a^\infty$ if it is not positive or negative on $\mathbb{Z}_{a_1}^\infty$ for an arbitrary $a_1 \in \mathbb{Z}_a^\infty$.

Definition 1.1. Let us define the expression $\ln_q t$, $q \geq 1$, by $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t \equiv t$, where $t > \exp_{q-2} 1$ and $\exp_s t = \exp(\exp_{s-1} t)$, $s \geq 1$, $\exp_0 t \equiv t$, and $\exp_{-1} t \equiv 0$ (instead of $\ln_0 t$, $\ln_1 t$, we will only write t and $\ln t$).

In [4] difference equation (1.1) is considered and the following result on the existence of a positive solution is proved.

Theorem 1.2 (see [4]). *Let $q \in \mathbb{N}_0$ be a fixed integer, let $a \in \mathbb{N}$ be sufficiently large and*

$$0 < p(n) \leq \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k}{8(n \ln n \cdots \ln_q n)^2}\right] \tag{1.2}$$

for every $n \in \mathbb{Z}_a^\infty$. Then there exists a positive integer $a_1 \geq a$ and a solution $x = x(n)$, $n \in \mathbb{Z}_{a_1}^\infty$ of equation (1.1) *such that*

$$0 < x(n) \leq \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n \ln n \ln_2 n \cdots \ln_q n} \tag{1.3}$$

holds for every $n \in \mathbb{Z}_{a_1}^\infty$.

Our goal is to answer the open question whether all solutions of (1.1) are oscillating if inequality (1.2) is replaced with the opposite inequality

$$p(n) \geq \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n \ln n)^2} + \cdots + \frac{k\theta}{8(n \ln n \cdots \ln_q n)^2}\right] \tag{1.4}$$

assuming $\theta > 1$, and n is sufficiently large. Below we prove that if (1.4) holds and $\theta > 1$, then all solutions of (1.1) are oscillatory. This means that the result given by Theorem 1.2 is a final in a sense. This is discussed in Section 4. Moreover, in Section 3, we show that all solutions of (1.1) will be oscillating if (1.4) holds only on an infinite sequence of subintervals of $\mathbb{Z}_a^\infty$.

Because of its simple form, equation (1.1) (as well as its continuous analogue) attracts permanent attention of investigators. Therefore, in Section 4 we also discuss some of the known results.

The proof of our main result will use the next consequence of one of Domshlak's results [13, Theorem 4, page 66].

Lemma 1.3. *Let s and r be fixed natural numbers such that $r - s > k$. Let $\{\varphi(n)\}_1^\infty$ be a bounded sequence of real numbers and ν_0 be a positive number such that there exists a number $\nu \in (0, \nu_0)$*

satisfying

$$0 \le \sum_{n=s+1}^{i} \varphi(n) \le \frac{\pi}{\nu}, \quad i \in \mathbb{Z}_{s+1}^{r}, \quad \frac{\pi}{\nu} \le \sum_{n=s+1}^{i} \varphi(n) \le \frac{2\pi}{\nu}, \quad i \in \mathbb{Z}_{r+1}^{r+k}, \tag{1.5}$$

$$\varphi(i) \ge 0, \quad i \in \mathbb{Z}_{r+1-k}^{r}, \quad \sum_{n=i+1}^{i+k} \varphi(n) > 0, \quad i \in \mathbb{Z}_a^{\infty}, \quad \sum_{n=i}^{i+k} \varphi(n) > 0, \quad i \in \mathbb{Z}_a^{\infty}. \tag{1.6}$$

Then, if $p(n) \ge 0$ *for* $n \in \mathbb{Z}_{s+1}^{s+k}$ *and*

$$p(n) \ge \mathcal{R} := \left(\prod_{\ell=n-k}^{n} \frac{\sin\left(\nu \sum_{i=\ell+1}^{\ell+k} \varphi(i)\right)}{\sin\left(\nu \sum_{i=\ell}^{\ell+k} \varphi(i)\right)} \right) \cdot \frac{\sin(\nu\varphi(n-k))}{\sin\left(\nu \sum_{i=n+1-k}^{n} \varphi(i)\right)} \tag{1.7}$$

for $n \in \mathbb{Z}_{s+k+1}^{r}$, *any solution of* (1.1) *has at least one change of sign on* $\mathbb{Z}_{s-k}^{r}$.

Throughout the paper, symbols "o" and "O" (for $n \to \infty$) will denote the well-known Landau order symbols.

2. Main Result

In this section, we give sufficient conditions for all solutions of (1.1) to be oscillatory as $n \to \infty$. It will be necessary to develop asymptotic decompositions of some auxiliary expressions. As the computations needed are rather cumbersome, some auxiliary computations are collected in the appendix to be utilized in the proof of the main result (Theorem 2.1) below.

Theorem 2.1. *Let* $a \in \mathbb{N}$ *be sufficiently large,* $q \in \mathbb{N}_0$ *and* $\theta > 1$. *Assuming that the function* $p : \mathbb{Z}_a^{\infty} \to (0, \infty)$ *satisfies inequality* (1.4) *for every* $n \in \mathbb{Z}_a^{\infty}$, *all solutions of* (1.1) *are oscillating as* $n \to \infty$.

Proof. As emphasized above, in the proof, we will use Lemma 1.3. We define

$$\varphi(n) := \frac{1}{n \ln n \ln_2 n \ln_3 n \cdots \ln_q n}, \tag{2.1}$$

where n is sufficiently large, and $q \ge 0$ is a fixed integer. Although the idea of the proof is simple, it is very technical and we will refer to notations and auxiliary computations contained in the appendix. We will develop an asymptotic decomposition of the right-hand side $\mathcal{R}$ of inequality (1.7) with the function $\varphi(n)$ defined by (2.1). We show that this will lead to the desired inequality (1.4). We set

$$\mathcal{R}_1 := \frac{\prod_{i=1}^{k} V(n+i)}{\prod_{i=0}^{k} V^{+}(n+i)} \cdot \varphi(n-k), \tag{2.2}$$

where V and V^+ are defined by (A.4) and (A.5). Moreover, $\mathcal{R}$ can be expressed as

$$\begin{aligned}
\mathcal{R} &= \frac{\sin\left(\nu \sum_{i=n+1-k}^{n} \varphi(i)\right) \prod_{\ell=n-k+1}^{n} \sin\left(\nu \sum_{i=\ell+1}^{\ell+k} \varphi(i)\right)}{\prod_{\ell=n-k}^{n} \sin\left(\nu \sum_{i=\ell}^{\ell+k} \varphi(i)\right)} \cdot \frac{\sin(\nu\varphi(n-k))}{\sin\left(\nu \sum_{i=n+1-k}^{n} \varphi(i)\right)} \\
&= \frac{\prod_{\ell=n-k+1}^{n} \sin\left(\nu \sum_{i=\ell+1}^{\ell+k} \varphi(i)\right)}{\prod_{\ell=n-k}^{n} \sin\left(\nu \sum_{i=\ell}^{\ell+k} \varphi(i)\right)} \cdot \sin(\nu\varphi(n-k)) \\
&= \frac{\prod_{p=1}^{k} \sin(\nu V(n+p))}{\prod_{p=0}^{k} \sin(\nu V^+(n+p))} \cdot \sin(\nu\varphi(n-k)).
\end{aligned} \tag{2.3}$$

Recalling the asymptotic decomposition of $\sin x$ when $x \to 0$: $\sin x = x + O(x^3)$, we get (since $\lim_{n\to\infty}\varphi(n-k) = 0$, $\lim_{n\to\infty} V(n+p) = 0$, $p = 1,\ldots,k$, and $\lim_{n\to\infty} V^+(n+p) = 0$, $p = 0,\ldots,k$)

$$\begin{aligned}
\sin \nu\varphi(n-k) &= \nu\varphi(n-k) + O\left(\nu^3\varphi^3(n-k)\right), \\
\sin \nu V(n+p) &= \nu V(n+p) + O\left(\nu^3 V^3(n+p)\right), \quad p = 1,\ldots,k, \\
\sin \nu V^+(n+p) &= \nu V^+(n+p) + O\left(\nu^3 (V^+)^3(n+p)\right), \quad p = 0,\ldots,k
\end{aligned} \tag{2.4}$$

as $n \to \infty$. Then, it is easy to see that, by (A.13), we have $\varphi(n-\ell) = O(\varphi(n))$, $n \to \infty$ for every $\ell \in \mathbb{R}$ and

$$\mathcal{R} = \mathcal{R}_1 \cdot \left(1 + O\left(\nu^2\varphi^2(n)\right)\right), \quad n \longrightarrow \infty. \tag{2.5}$$

Moreover, for $\mathcal{R}_1$, we will get an asymptotic decomposition as $n \to \infty$. Using formulas (A.13), (A.57), and (A.60), we get

$$\begin{aligned}
\mathcal{R}_1 = {} & \frac{k^k}{(k+1)^{k+1}} \cdot \frac{1 - k\alpha(n) - (k/24)\left(k^2 - 12k + 11\right)\alpha^2(n) + (k/6)\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n) + O(1/n^3)}{1 - (k/24)(k^2+3k+2)\alpha^2(n) + (k/6)(k^2+3k+2)\sum_{i=0}^{q}\omega_i(n) + O(1/n^3)} \\
& \times \left(1 + k\alpha(n) + k^2\sum_{i=0}^{q}\omega_i(n) + O\left(\frac{1}{n^3}\right)\right).
\end{aligned} \tag{2.6}$$

Since $\lim_{n\to\infty}\alpha(n) = 0$, $\lim_{n\to\infty}\omega_i(n) = 0$, $i = 1,\ldots,q$, we can decompose the denominator of the second fraction as the sum of the terms of a geometric sequence. Keeping only terms with the order of accuracy necessary for further analysis (i.e. with order $O(1/n^3)$), we get

$$\begin{aligned}&\left(1-\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)+\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)^{-1}\\&\quad=1+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right).\end{aligned}\tag{2.7}$$

We perform an auxiliary computation in $\mathcal{R}_1$,

$$\begin{aligned}&\left(1-k\alpha(n)-\frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)+\frac{k}{6}\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)\\&\quad\times\left(1+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)\\&\quad\times\left(1+k\alpha(n)+k^2\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)\\&\quad=\left(1-k\alpha(n)-\frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)+\frac{k}{6}\left(k^2+5\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)\\&\qquad\times\left(1+k\alpha(n)+\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)-\frac{k}{6}\left(k^2-3k+2\right)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)\right)\\&\quad=1-\frac{3}{8}k(k+1)\alpha^2(n)+\frac{1}{2}k(k+1)\sum_{i=0}^{q}\omega_i(n)+O\left(\frac{1}{n^3}\right)=(\text{we use formula (A.15)})\\&\quad=1+\frac{1}{8}k(k+1)\Omega(n)+O\left(\frac{1}{n^3}\right)\\&\quad=1+\frac{1}{8}k(k+1)\left(\frac{1}{n^2}+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n\ln_2 n)^2}+\cdots+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right)\\&\qquad+O\left(\frac{1}{n^3}\right).\end{aligned}\tag{2.8}$$

Thus, we have

$$\begin{aligned}\mathcal{R}_1&=\frac{k^k}{(k+1)^{k+1}}\times\left[1+\frac{1}{8}k(k+1)\left(\frac{1}{n^2}+\frac{1}{(n\ln n)^2}+\frac{1}{(n\ln n\ln_2 n)^2}+\cdots\right.\right.\\&\qquad\left.\left.+\frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right)\right]+O\left(\frac{1}{n^3}\right)\\&=\left(\frac{k}{k+1}\right)^k\times\left[\frac{1}{k+1}+\frac{k}{8n^2}+\frac{k}{8(n\ln n)^2}+\cdots+\frac{k}{8(n\ln n\cdots\ln_q n)^2}\right]+O\left(\frac{1}{n^3}\right).\end{aligned}\tag{2.9}$$

Finalizing our decompositions, we see that

$$\begin{aligned}
\mathcal{R} &= \mathcal{R}_1 \cdot \left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \left(\left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] + O\left(\frac{1}{n^3}\right)\right) \\
&\quad \times \left(1 + O\left(\nu^2\varphi^2(n)\right)\right) \\
&= \left(\frac{k}{k+1}\right)^k \times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] \\
&\quad + O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right).
\end{aligned} \tag{2.10}$$

It is easy to see that inequality (1.7) becomes

$$\begin{aligned}
p(n) \geq \left(\frac{k}{k+1}\right)^k &\times \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2}\right] \\
&+ O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right)
\end{aligned} \tag{2.11}$$

and will be valid if (see (1.4))

$$\begin{aligned}
&\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k\theta}{8(n\ln n\cdots\ln_q n)^2} \\
&\quad \geq \frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n\cdots\ln_q n)^2} + O\left(\frac{\nu^2}{(n\ln n\cdots\ln_q n)^2}\right)
\end{aligned} \tag{2.12}$$

or

$$\theta \geq 1 + O\left(\nu^2\right) \tag{2.13}$$

for $n \to \infty$. If $n \geq n_0$, where n_0 is sufficiently large, (2.13) holds for $\nu \in (0, \nu_0)$ with ν_0 sufficiently small because $\theta > 1$. Consequently, (2.11) is satisfied and the assumption (1.7) of Lemma 1.3 holds for $n \in \mathbb{Z}_{n_0}^{\infty}$. Let $s \geq n_0$ in Lemma 1.3 be fixed, $r > s + k + 1$ be so large (and ν_0 so small if necessary) that inequalities (1.5) hold. Such choice is always possible since the series $\sum_{n=s+1}^{\infty} \varphi(n)$ is divergent. Then Lemma 1.3 holds and any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s-k}^{r}$. Obviously, inequalities (1.5) can be satisfied for another pair of (s, r), say (s_1, r_1) with $s_1 > r$ and $r_1 > s_1 + k$ sufficiently large and, by Lemma 1.3, any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s_1-k}^{r_1}$. Continuing this process, we will get a sequence of pairs (s_j, r_j) with $\lim_{j\to\infty} s_j = \infty$ such that any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s_j-k}^{r_j}$. This concludes the proof. □

3. A Generalization

The coefficient p in Theorem 2.1 is supposed to be positive on $\mathbb{Z}_a^\infty$. For all sufficiently large n, the expression $\mathcal{R}$, as can easily be seen from (2.10), is positive. Then, as follows from Lemma 1.3, any solution of (1.1) has at least one change of sign on $\mathbb{Z}_{s-k}^r$ if p is nonnegative on $\mathbb{Z}_{s+1}^{s+k}$ and satisfies inequality (1.4) on $\mathbb{Z}_{s+k+1}^r$.

Owing to this remark, Theorem 2.1 can be generalized because (and the following argumentation was used at the end of the proof of Theorem 2.1) all solutions of (1.1) will be oscillating as $n \to \infty$ if a sequence of numbers $\{s_i, r_i\}$, $r_i > s_i + k + 1$, $s_1 \geq a$, $i = 1, 2, \ldots$ exists such that $s_{i+1} > r_i$ (i.e., the sets $\mathbb{Z}_{s_i}^{r_i}$, $\mathbb{Z}_{s_{i+1}}^{r_{i+1}}$ are disjoint and $\lim_{i\to\infty} s_i = \infty$), and, for every pair (s_i, r_i), all assumptions of Lemma 1.3 are satisfied (because of the specification of function φ by (2.1), inequalities (1.6) are obviously satisfied). This means that, on the set

$$\mathcal{M} := \mathbb{Z}_a^\infty \setminus \bigcup_{i=1}^{\infty} \mathbb{Z}_{s_i}^{r_i}, \tag{3.1}$$

function p can assume even negative values, and, moreover, there is no restriction on the behavior of $p(n)$ for $n \in \mathcal{M}$. This leads to the following generalization of Theorem 2.1 with a proof similar to that of Theorem 2.1 and, therefore, omitted.

Theorem 3.1. *Let $a \in \mathbb{N}$ be sufficiently large, $q \in \mathbb{N}_0$, ν_0 be a positive number, $\theta > 1$ and $p : \mathbb{Z}_a^\infty \to \mathbb{R}$. Let there exists a sequence on integers $\{s_j, r_j\}$, $r_j > s_j + k + 1$, $j = 1, 2, \ldots$, $s_1 \geq a$, s_1 sufficiently large and $s_{j+1} > r_j$ such that, for function φ (defined by (2.1)) and for each pair (s_j, r_j), $j = 1, 2, \ldots$, there exists a number $\nu_j \in (0, \nu_0)$ such that*

$$0 \leq \sum_{n=s_j+1}^{i} \varphi(n) \leq \frac{\pi}{\nu_j}, \quad i \in \mathbb{Z}_{s_j+1}^{r_j}, \quad \frac{\pi}{\nu_j} \leq \sum_{n=s_j+1}^{i} \varphi(n) \leq \frac{2\pi}{\nu_j}, \quad i \in \mathbb{Z}_{r_j+1}^{r_j+k}, \tag{3.2}$$

$p(n) \geq 0$ for $n \in \mathbb{Z}_{s_j+1}^{s_j+k}$, and (1.4) holds for $n \in \mathbb{Z}_{s_j+k+1}^{r_j}$, then all solutions of (1.1) are oscillating as $n \to \infty$.

4. Comparisons, Concluding Remarks, and Open Problems

Equation (1.1) with $k = 1$ was considered in [5], where a particular case of Theorem 2.1 is proved. In [4], a hypothesis is formulated together with the proof of Theorem 1.2 (Conjecture 1) about the oscillation of all solutions of (1.1) almost coinciding with the formulation of Theorem 2.1. For its simple form, (1.1) is often used for testing new results and is very frequently investigated.

Theorems 1.2 and 2.1 obviously generalize several classical results. We mention at least some of the simplest ones (see, e.g., [16, Theorem 7.7] or [19, Theorem 7.5.1]),

Theorem 4.1. *Let $p(n) \equiv p = \text{const}$. Then every solution of (1.1) oscillates if and only if*

$$p > \frac{k^k}{(k+1)^{k+1}}. \tag{4.1}$$

Or the following result holds as well (see, e.g., [16, Theorem 7.6]) [18, 19]).

Theorem 4.2. *Let* $p(n) \geq 0$ *and*

$$\sup_n p(n) < \frac{k^k}{(k+1)^{k+1}}. \tag{4.2}$$

Then (1.1) *has a nonoscillatory solution.*

In [9] a problem on oscillation of all solutions of equation

$$\Delta u(n) + p(n)u(\tau(n)) = 0, \quad n \in \mathbb{N} \tag{4.3}$$

is considered where $p : \mathbb{N} \to \mathbb{R}_+$, $\tau : \mathbb{N} \to \mathbb{N}$, and $\lim_{n\to\infty}\tau(n) = +\infty$. Since in (4.3) delay τ is variable, we can formulate

Open Problem 1. It is an interesting open question whether Theorems 1.2 and 2.1 can be extended to linear difference equations with a variable delay argument of the form, for example,

$$\Delta u(n) = -p(n)u(h(n)), \quad n \in \mathbb{Z}_a^\infty, \tag{4.4}$$

where $0 \leq n - h(n) \leq k$. For some of the related results for the differential equation

$$\dot{x}(t) = -p(t)x(h(t)), \tag{4.5}$$

see the results in [3, 12] that are described below.

Open Problem 2. It is well known [19, 22] that the following condition is also sufficient for the oscillation of all solutions of (4.5) with $h(n) = n - k$*:*

$$\liminf_{n\to\infty} \frac{1}{k} \sum_{i=n-k}^{n-1} p_i > \frac{k^k}{(k+1)^{k+1}}. \tag{4.6}$$

The right-hand side of (4.6) is a critical value for this criterion since this number cannot be replaced with a smaller one.

In [30] equation (1.1) is considered as well. The authors prove that all solutions oscillate if $p(n) \geq 0$, $\varepsilon > 0$ and

$$\limsup_{n\to\infty} p(n) > \frac{k^k}{(k+1)^{k+1}} - \frac{\varepsilon}{k} + 4k\varepsilon^{1/4}, \tag{4.7}$$

where

$$\varepsilon = \left(\frac{k}{k+1}\right)^{k+1} - \liminf_{n\to\infty} \sum_{i=n-k}^{n-1} p_i. \tag{4.8}$$

An open problem is to obtain conditions similar to Theorem 2.1 for this kind of oscillation criteria. Some results on this problem for delay differential equations were also obtained in paper [3].

In [26] the authors establish an equivalence between the oscillation of (1.1) and the equation

$$\Delta^2 y(n-1) + \frac{2(k+1)^k}{k^{k+1}}\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)y(n) = 0 \tag{4.9}$$

under the critical state

$$\liminf_{n\to\infty} p(n) = \frac{k^k}{(k+1)^{k+1}}, \tag{4.10}$$

$$p(n) \geq \frac{k^k}{(k+1)^{k+1}}. \tag{4.11}$$

Then they obtain some sharp oscillation and nonoscillation criteria for (1.1). One of the results obtained there is the following.

Theorem 4.3. *Assume that, for sufficiently large n, inequality* (4.11) *holds. Then the following statements are valid.*

(i) *If*

$$\liminf_{n\to\infty}\left[\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)n^2\right] > \frac{k^{k+1}}{8(k+1)^k}, \tag{4.12}$$

then every solution of (1.1) *is oscillatory.*

(ii) *If, on the other hand,*

$$\left(p(n) - \frac{k^k}{(k+1)^{k+1}}\right)n^2 \leq \frac{k^{k+1}}{8(k+1)^k}, \tag{4.13}$$

then (1.1) *has a nonoscillatory solution.*

Regarding our results, it is easy to see that statement (i) is a particular case of Theorem 2.1 while statement (ii) is a particular case of Theorem 1.2.

In [27], the authors investigate (1.1) for $n \geq n_0$ and prove that (1.1) is oscillatory if

$$\sum_{i=n_0}^{\infty} p(i)\left\{\frac{k+1}{k}\cdot\sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1\right\} = \infty. \tag{4.14}$$

Comparing (4.14) with Theorem 2.1, we can see that (4.14) gives not sharp sufficient condition. Set, for example, $k = 1$, $\theta > 1$ and

$$p(n) = \frac{1}{2}\left[\frac{1}{2} + \frac{\theta}{8n^2}\right]. \tag{4.15}$$

Then,

$$\frac{k+1}{k} \cdot \sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1 = 2 \cdot \sqrt{\frac{1}{4}\left(1 + \frac{\theta}{4(i+1)^2}\right)} - 1 = \frac{\theta/4(i+1)^2}{1 + \sqrt{1 + \theta/4(i+1)^2}} \tag{4.16}$$

and the series in the left-hand side of (4.14) converges since

$$\begin{aligned}
&\sum_{i=n_0}^{\infty} p(i)\left\{\frac{k+1}{k} \sqrt[k+1]{\sum_{j=i+1}^{i+k} p(j)} - 1\right\} \\
&\quad = \sum_{i=n_0}^{\infty} \frac{1}{2}\left[\frac{1}{2} + \frac{\theta}{8i^2}\right] \frac{\theta/4(i+1)^2}{1 + \sqrt{1 + \theta/4(i+1)^2}} \le \theta \sum_{i=n_0}^{\infty} \frac{1}{i^2}\left[1 + \frac{\theta}{i^2}\right] < \infty.
\end{aligned} \tag{4.17}$$

But, by Theorem 2.1 all solutions of (1.1) are oscillating as $n \to \infty$. Nevertheless (4.14) is not a consequence of Theorem 2.1.

Let us consider a continuous variant of (1.1): a delayed differential linear equation of the form

$$\dot{x}(t) = -a(t)x(t - \tau), \tag{4.18}$$

where $\tau > 0$ is a constant delay and $a : [t_0, \infty) \to (0, \infty)$ (or $a : [t_0, \infty) \to \mathbb{R}$), $t_0 \in \mathbb{R}$. This equation, too, for its simple form, is often used for testing new results and is very frequently investigated. It is, for example, well known that a scalar linear equation with delay

$$\dot{x}(t) + \frac{1}{e}x(t - 1) = 0 \tag{4.19}$$

has a nonoscillatory solution as $t \to \infty$. This means that there exists an eventually positive solution. The coefficient $1/e$ is called critical with the following meaning: for any $\alpha > (1/e)$, all solutions of the equation

$$\dot{x}(t) + \alpha x(t - 1) = 0 \tag{4.20}$$

are oscillatory while, for $\alpha \le (1/e)$, there exists an eventually positive solution. In [10], the third author considered (4.18), where $a : [t_0, \infty) \to (0, \infty)$ is a continuous function, and t_0 is sufficiently large. For the critical case, he obtained the following result (being a continuous variant of Theorems 1.2 and 2.1).

Theorem 4.4. *(a) Let an integer $k \geq 0$ exist such that $a(t) \leq a_k(t)$ if $t \to \infty$ where*

$$a_k(t) := \frac{1}{e\tau} + \frac{\tau}{8et^2} + \frac{\tau}{8e(t\ln t)^2} + \cdots + \frac{\tau}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{4.21}$$

Then there exists an eventually positive solution x of (4.18).

(b) Let an integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$ exist such that

$$a(t) > a_{k-2}(t) + \frac{\theta\tau}{8e(t\ln t\ln_2 t\cdots\ln_{k-1} t)^2}, \tag{4.22}$$

if $t \to \infty$. Then all solutions of (4.18) *oscillate.*

Further results on the critical case for (4.18) can be found in [1, 11, 14, 17, 24].

In [12], Theorem 7 was generalized for equations with a variable delay

$$\dot{x}(t) + a(t)x(t - \tau(t)) = 0, \tag{4.23}$$

where $a : [t_0, \infty) \to (0, \infty)$ and $\tau : [t_0, \infty) \to (0, \infty)$ are continuous functions. The main results of this paper include the following.

Theorem 4.5 (see [12]). *Let $t - \tau(t) \geq t_0 - \tau(t_0)$ if $t \geq t_0$. Let an integer $k \geq 0$ exist such that $a(t) \leq a_{k\tau}(t)$ for $t \to \infty$, where*

$$a_{k\tau}(t) := \frac{1}{e\tau(t)} + \frac{\tau(t)}{8et^2} + \frac{\tau(t)}{8e(t\ln t)^2} + \cdots + \frac{\tau(t)}{8e(t\ln t\ln_2 t\cdots\ln_k t)^2}. \tag{4.24}$$

If moreover

$$\begin{gathered}\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq 1 \quad \text{when } t \longrightarrow \infty, \\ \lim_{t\to\infty} \tau(t) \cdot \left(\frac{1}{t}\ln t\ln_2 t\cdots\ln_k t\right) = 0,\end{gathered} \tag{4.25}$$

then there exists an eventually positive solution x of (4.23) *for $t \to \infty$.*

Finally, the last results were generalized in [3]. We reproduce some of the results given there.

Theorem 4.6. *(A) Let $\tau > 0$, $0 \leq \tau(t) \leq \tau$ for $t \to \infty$, and let condition (a) of Theorem 4.4 holds. Then* (4.23) *has a nonoscillatory solution.*

(B) Let $\tau(t) \geq \tau > 0$ for $t \to \infty$, and let condition (b) of Theorem 4.4 holds. Then all solutions of (4.23) *oscillate.*

For every integer $k \geq 0$, $\delta > 0$ and $t \to \infty$, we define

$$A_k(t) := \frac{1}{e\delta\tau(t)} + \frac{\delta}{8e\tau(t)s^2} + \frac{\delta}{8e\tau(t)(s\ln s)^2} + \cdots + \frac{\delta}{8e\tau(t)(s\ln s\ln_2 s\cdots\ln_k s)^2}, \tag{4.26}$$

where

$$s = p(t) := \int_{t_0}^{t} \frac{1}{\tau(\xi)} d\xi. \tag{4.27}$$

Theorem 4.7. *Let for t_0 sufficiently large and $t \geq t_0$: $\tau(t) > 0$ a.e., $1/\tau(t)$ be a locally integrable function,*

$$\lim_{t\to\infty} (t - \tau(t)) = \infty, \quad \int_{t_0}^{\infty} \frac{1}{\tau(\xi)} d\xi = \infty, \tag{4.28}$$

and let there exists $t_1 > t_0$ such that $t - \tau(t) \geq t_0$, $t \geq t_1$.

(a) *If there exists a $\delta \in (0, \infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \leq \delta, \quad t \geq t_1, \tag{4.29}$$

and, for a fixed integer $k \geq 0$,

$$a(t) \leq A_k(t), \quad t \geq t_1, \tag{4.30}$$

then there exists an eventually positive solution of (4.23).

(b) *If there exists a $\delta \in (0, \infty)$ such that*

$$\int_{t-\tau(t)}^{t} \frac{1}{\tau(\xi)} d\xi \geq \delta, \quad t \geq t_1, \tag{4.31}$$

and, for a fixed integer $k \geq 2$ and $\theta > 1$, $\theta \in \mathbb{R}$,

$$a(t) > A_{k-2}(t) + \frac{\theta\delta}{8e\tau(t)(s \ln s \ln_2 s \cdots \ln_{k-1} s)^2}, \tag{4.32}$$

if $t \geq t_1$, then all solutions of (4.23) *oscillate.*

Appendix

A. Auxiliary Computations

This part includes auxiliary results with several technical lemmas proved. Part of them is related to the asymptotic decomposition of certain functions and the rest deals with computing the sums of some algebraic expressions. The computations are referred to in the proof of the main result (Theorem 2.1) in Section 2.

First we define auxiliary functions (recalling also the definition of function φ given by (2.1)):

$$\begin{aligned}
\varphi(n) &:= \frac{1}{n \ln n \ln_2 n \ln_3 n \cdots \ln_q n}, \\
\alpha(n) &:= \frac{1}{n} + \frac{1}{n \ln n} + \frac{1}{n \ln n \ln_2 n} + \cdots + \frac{1}{n \ln n \ln_2 n \cdots \ln_q n}, \\
\omega_0(n) &:= \frac{1}{n^2} + \frac{3}{2n^2 \ln n} + \frac{3}{2n^2 \ln n \ln_2 n} + \cdots + \frac{3}{2n^2 \ln n \ln_2 n \cdots \ln_q n}, \\
\omega_1(n) &:= \frac{1}{(n \ln n)^2} + \frac{3}{2(n \ln n)^2 \ln_2 n} + \cdots + \frac{3}{2(n \ln n)^2 \ln_2 n \cdots \ln_q n}, \\
&\ \,\vdots \\
\omega_{q-1}(n) &:= \frac{1}{(n \ln n \cdots \ln_{q-1} n)^2} + \frac{3}{2(n \ln n \cdots \ln_{q-1} n)^2 \ln_q n}, \\
\omega_q(n) &:= \frac{1}{(n \ln n \cdots \ln_q n)^2}, \\
\Omega(n) &:= \frac{1}{n^2} + \frac{1}{(n \ln n)^2} + \frac{1}{(n \ln n \ln_2 n)^2} + \cdots + \frac{1}{(n \ln n \ln_2 n \cdots \ln_q n)^2},
\end{aligned} \tag{A.1}$$

where n is sufficiently large and $q \in \mathbb{N}_0$. Moreover, we set (for admissible values of arguments)

$$\Sigma(p) := \sum_{\ell=1}^{k} (k - p - \ell), \tag{A.2}$$

$$\Sigma^+(p) := \Sigma(p) + (k - p), \tag{A.3}$$

$$V(n+p) := \sum_{\ell=1}^{k} \varphi(n + p - k + \ell), \tag{A.4}$$

$$V^+(n+p) := V(n+p) + \varphi(n + p - k), \tag{A.5}$$

$$S(p) := \sum_{\ell=1}^{k} (k - p - \ell)^2, \tag{A.6}$$

$$S^+(p) := S(p) + (k - p)^2. \tag{A.7}$$

A.1. Asymptotic Decomposition of Iterative Logarithms

In the proof of the main result, we use auxiliary results giving asymptotic decompositions of iterative logarithms. The following lemma is proved in [11].

Lemma A.1. *For fixed $r, \sigma \in \mathbb{R} \setminus \{0\}$ and a fixed integer $s \geq 1$, the asymptotic representation*

$$\begin{aligned}\frac{\ln_s^{\sigma}(n-r)}{\ln_s^{\sigma} n} = 1 &- \frac{r\sigma}{n \ln n \cdots \ln_s n} - \frac{r^2\sigma}{2n^2 \ln n \cdots \ln_s n} \\ &- \frac{r^2\sigma}{2(n\ln n)^2 \ln_2 n \cdots \ln_s n} - \cdots - \frac{r^2\sigma}{2(n \ln n \cdots \ln_{s-1} n)^2 \ln_s n} \\ &+ \frac{r^2\sigma(\sigma-1)}{2(n \ln n \cdots \ln_s n)^2} - \frac{r^3\sigma(1+o(1))}{3n^3 \ln n \cdots \ln_s n}\end{aligned} \tag{A.8}$$

holds for $n \to \infty$.

A.2. Formulas for $\Sigma(p)$ and for $\Sigma^+(p)$

Lemma A.2. *The following formulas hold:*

$$\Sigma(p) = \frac{k}{2} \cdot (k - 2p - 1), \tag{A.9}$$

$$\Sigma^+(p) = \frac{k+1}{2} \cdot (k - 2p). \tag{A.10}$$

Proof. It is easy to see that

$$\begin{aligned}\Sigma(p) &= \sum_{\ell=-p}^{k-p-1} \ell = (k-p-1) + (k-p-2) + \cdots + (-p) \\ &= (k-(p+1)) + (k-(p+2)) + \cdots + (k-(p+k)) = \frac{k}{2} \cdot (k-2p-1), \\ \Sigma^+(p) &= \Sigma(p) + (k-p) = \frac{k+1}{2} \cdot (k-2p).\end{aligned} \tag{A.11}$$

□

A.3. Formula for the Sum of the Terms of an Arithmetical Sequence

Denote by $u_1, u_2, \ldots, u_r$ the terms of an arithmetical sequence of kth order (kth differences are constant), $d'_1, d'_2, d'_3, \ldots$, the first differences ($d'_1 = u_2 - u_1$, $d'_2 = u_3 - u_2, \ldots$), $d''_1, d''_2, d''_3, \ldots$, the second differences ($d''_1 = d'_2 - d'_1, \ldots$), and so forth. Then the following result holds (see, e.g., [43]).

Lemma A.3. *For the sum of r terms of an arithmetical sequence of kth order, the following formula holds*

$$\sum_{i=1}^{r} u_i = \frac{r!}{(r-1)! \cdot 1!} \cdot u_1 + \frac{r!}{(r-2)! \cdot 2!} \cdot d'_1 + \frac{r!}{(r-3)! \cdot 3!} \cdot d''_1 + \cdots. \tag{A.12}$$

A.4. Asymptotic Decomposition of $\varphi(n-l)$

Lemma A.4. *For fixed* $\ell \in \mathbb{R}$ *and* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$\varphi(n-\ell) = \varphi(n)\left(1 + \ell\alpha(n) + \ell^2 \sum_{i=0}^{q} \omega_q(n)\right) + O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.13}$$

holds for $n \to \infty$.

Proof. The function $\varphi(n)$ is defined by (2.1). We develop the asymptotic decomposition of $\varphi(n-\ell)$ when n is sufficiently large and $\ell \in \mathbb{R}$. Applying Lemma A.1 (for $\sigma = -1$, $r = \ell$ and $s = 1, 2, \ldots, q$), we get

$$\begin{aligned}
\varphi(n-\ell) &= \frac{1}{(n-\ell)\ln(n-\ell)\ln_2(n-\ell)\ln_3(n-\ell)\cdots\ln_q(n-\ell)} \\
&= \frac{1}{n(1-\ell/n)\ln(n-\ell)\ln_2(n-\ell)\ln_3(n-\ell)\cdots\ln_q(n-\ell)} \\
&= \varphi(n)\cdot\frac{1}{1-\ell/n}\cdot\frac{\ln n}{\ln(n-\ell)}\cdot\frac{\ln_2 n}{\ln_2(n-\ell)}\cdot\frac{\ln_3 n}{\ln_3(n-\ell)}\cdots\frac{\ln_q n}{\ln_q(n-\ell)} \\
&= \varphi(n)\left(1 + \frac{\ell}{n} + \frac{\ell^2}{n^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n} + \frac{\ell^2}{2n^2\ln n} + \frac{\ell^2}{(n\ln n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n\ln_2 n} + \frac{\ell^2}{2n^2\ln n\ln_2 n} + \frac{\ell^2}{2(n\ln n)^2\ln_2 n} + \frac{\ell^2}{(n\ln n\ln_2 n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\left(1 + \frac{\ell}{n\ln n\ln_2 n\ln_3 n} + \frac{\ell^2}{2n^2\ln n\ln_2 n\ln_3 n} + \frac{\ell^2}{2(n\ln n)^2\ln_2 n\ln_3 n}\right. \\
&\qquad\left. + \frac{\ell^2}{2(n\ln n\ln_2 n)^2\ln_3 n} + \frac{\ell^2}{(n\ln n\ln_2 n\ln_3 n)^2} + O\left(\frac{1}{n^3}\right)\right) \\
&\quad\times\cdots\times\left(1 + \frac{\ell}{n\ln n\ln_2 n\ln_3 n\cdots\ln_q n} + \frac{\ell^2}{2n^2\ln n\cdots\ln_q n} + \frac{\ell^2}{2(n\ln n)^2\ln_2\cdots n\ln_q n}\right. \\
&\qquad\left. + \cdots + \frac{\ell^2}{2(n\ln n\cdots\ln_{q-1}n)^2\ln_q n} + \frac{\ell^2}{(n\ln n\cdots\ln_q n)^2} + O\left(\frac{1}{n^3}\right)\right).
\end{aligned} \tag{A.14}$$

Finally, gathering the same functional terms and omitting the terms having a higher order of accuracy than is necessary, we obtain the asymptotic decomposition (A.13). □

A.5. Formula for $\alpha^2(n)$

Lemma A.5. *For fixed* $q \in \mathbb{N}_0$*, the formula*

$$\alpha^2(n) = \frac{4}{3}\sum_{i=0}^{q}\omega_i(n) - \frac{1}{3}\Omega(n) \tag{A.15}$$

holds for all sufficiently large n*.*

Proof. It is easy to see that

$$\begin{aligned}
\alpha^2(n) &= \frac{1}{n^2} + \frac{2}{n^2\ln n} + \frac{2}{n^2\ln n\ln_2 n} + \cdots + \frac{2}{n^2\ln n\ln_2 n\cdots\ln_q n} \\
&\quad + \frac{1}{(n\ln n)^2} + \frac{2}{(n\ln n)^2\ln_2 n} + \cdots + \frac{2}{(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\quad + \frac{1}{(n\ln n\ln_2 n)^2} + \frac{2}{(n\ln n\ln_2 n)^2\ln_3 n} + \cdots + \frac{2}{(n\ln n\ln_2 n)^2\cdots\ln_q n} \\
&\quad + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2} \\
&= \frac{4}{3}\left(\frac{1}{n^2} + \frac{3}{2n^2\ln n} + \frac{3}{2n^2\ln n\ln_2 n} + \cdots + \frac{3}{2n^2\ln n\ln_2 n\cdots\ln_q n}\right. \\
&\quad + \frac{1}{(n\ln n)^2} + \frac{3}{2(n\ln n)^2\ln_2 n} + \cdots + \frac{3}{2(n\ln n)^2\ln_2 n\cdots\ln_q n} \\
&\quad + \frac{1}{(n\ln n\ln_2 n)^2} + \frac{2}{(n\ln n\ln_2 n)^2\ln_3 n} + \cdots + \frac{2}{(n\ln n\ln_2 n)^2\cdots\ln_q n} \\
&\quad \left. + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right) \\
&\quad - \frac{1}{3}\left(\frac{1}{n^2} + \frac{1}{(n\ \ln n)^2} + \frac{1}{(n\ln n\ln_2 n)^2} + \cdots + \frac{1}{(n\ln n\ln_2 n\cdots\ln_q n)^2}\right) \\
&= \frac{4}{3}\sum_{i=0}^{q}\omega_i(n) - \frac{1}{3}\Omega(n).
\end{aligned} \tag{A.16}$$

□

A.6. Asymptotic Decomposition of $V(n+p)$

Lemma A.6. *For fixed* $p \in \mathbb{N}$ *and* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$V(n+p) = \varphi(n)\left[k + \Sigma(p)\alpha(n) + S(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.17}$$

holds for $n \to \infty$*.*

Proof. It is easy to deduce from formula (A.13) with $\ell = k-p-1, k-p-2, \dots, -p$ that

$$
\begin{aligned}
V(n+p) &:= \varphi(n+p-k+1) + \varphi(n+p-k+2) + \cdots + \varphi(n+p) = \sum_{\ell=-p}^{k-p-1} \varphi(n-\ell) = \varphi(n) \\
&\times \sum_{\ell=-p}^{k-p-1} \left(1 + \frac{\ell}{n} + \frac{\ell}{n \ln n} + \frac{\ell}{n \ln n \ln_2 n} + \cdots + \frac{\ell}{n \ln n \ln_2 n \cdots \ln_q n} \right. \\
&+ \frac{\ell^2}{n^2} + \frac{3\ell^2}{2n^2 \ln n} + \cdots + \frac{3\ell^2}{2n^2 \ln n \ln_2 n \cdots \ln_q n} + \frac{\ell^2}{(n \ln n)^2} \\
&+ \frac{3\ell^2}{2(n \ln n)^2 \ln_2 n} + \frac{3\ell^2}{2(n \ln n)^2 \ln_3 n} + \cdots + \frac{3\ell^2}{2(n \ln n)^2 \ln_3 n \cdots \ln_q n} \\
&+ \frac{\ell^2}{(n \ln n \ln_2 n)^2} + \frac{3\ell^2}{2(n \ln n \ln_2 n)^2 \ln_3 n} + \cdots + \frac{3\ell^2}{2(n \ln n \ln_2 n)^2 \ln_3 n \cdots \ln_q n} \\
&+ \frac{\ell^2}{(n \ln n \ln_2 n \ln_3 n)^2} + \cdots + \frac{3\ell^2}{2(n \ln n \ln_2 n \ln_3 n)^2 \ln_4 n \cdots \ln_q n} \\
&+ \cdots + \frac{\ell^2}{(n \ln n \ln_2 n \cdots \ln_{q-1} n)^2} + \frac{3\ell^2}{2(n \ln n \ln_2 n \cdots \ln_{q-1} n)^2 \ln_q n} \\
&\left. + \frac{\ell^2}{(n \ln n \ln_2 n \cdots \ln_q n)^2} + O\left(\frac{1}{n^3}\right) \right).
\end{aligned}
\tag{A.18}
$$

Then

$$
\begin{aligned}
V(n+p) &:= \varphi(n) \sum_{\ell=-p}^{k-p-1} \left[1 + \ell \alpha(n) + \ell^2 \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right) \right] \\
&= \varphi(n) \left[\sum_{\ell=-p}^{k-p-1} 1 + \alpha(n) \cdot \sum_{\ell=-p}^{k-p-1} \ell + \sum_{\ell=-p}^{k-p-1} \ell^2 \cdot \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right) \right] \\
&= \varphi(n) \left[k + \Sigma(p) \alpha(n) + S(p) \cdot \sum_{i=0}^{q} \omega_i(n) + O\left(\frac{1}{n^3}\right) \right] \\
&= \varphi(n) \left[k + \Sigma(p) \alpha(n) + S(p) \sum_{i=0}^{q} \omega_i(n) \right] + O\left(\frac{\varphi(n)}{n^3}\right).
\end{aligned}
\tag{A.19}
$$

□

A.7. Asymptotic Decomposition of $V^+(n+p)$

Lemma A.7. *For fixed* $p \in \mathbb{N}_0$ *and* $q \in \mathbb{N}_0$*, the asymptotic representation*

$$V^+(n+p) = \varphi(n)\left[k + 1 + \Sigma^+(p)\alpha(n) + S^+(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) \tag{A.20}$$

holds for $n \to \infty$.

Proof. By (A.5), (A.13), (A.17), (A.10), and (A.7), we get

$$\begin{aligned}
V^+(n+p) &:= V(n+p) + \varphi(n+p-k) \\
&= \varphi(n)\left[k + \Sigma(p)\alpha(n) + S(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) + \varphi(n+p-k) \\
&= \varphi(n)\left[k + \Sigma(p)\alpha(n) + S(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) \\
&\quad + \varphi(n)\Big(1 + (k-p)\alpha(n) + (k-p)^2\omega_0(n) + (k-p)^2\omega_1(n) \\
&\qquad + \cdots + (k-p)^2\omega_{q-1}(n) + (k-p)^2\omega_q(n) + O\left(\frac{1}{n^3}\right)\Big) \\
&= \varphi(n)\left[k + 1 + (\Sigma(p) + (k-p))\alpha(n) + \left(S(p) + (k-p)^2\right)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right) \\
&= \varphi(n)\left[k + 1 + \Sigma^+(p)\alpha(n) + S^+(p)\sum_{i=0}^{q}\omega_i(n)\right] + O\left(\frac{\varphi(n)}{n^3}\right).
\end{aligned} \tag{A.21}$$

□

A.8. Formula for $\sum_{p=1}^{k}\Sigma(p)$

Lemma A.8. *For the above sum, the following formula holds:*

$$\sum_{p=1}^{k}\Sigma(p) = -k^2. \tag{A.22}$$

Proof. Using formula (A.9), we get

$$\begin{aligned}
\sum_{p=1}^{k}\Sigma(p) &= \Sigma(1) + \Sigma(2) + \Sigma(3) + \cdots + \Sigma(k) \\
&= \frac{k}{2}\cdot[(k-3) + (k-5) + (k-7) + \cdots + (k-(2k+1))] \\
&= \frac{k}{2}\cdot(-2k) = -k^2.
\end{aligned} \tag{A.23}$$

□

A.9. Formula for $\sum_{p=1}^{k} \Sigma^2(p)$

Lemma A.9. *For the above sum, the following formula holds:*

$$\sum_{p=1}^{k} \Sigma^2(p) = \frac{k^3}{12}\left(k^2 + 11\right). \tag{A.24}$$

Proof. Using formula (A.9), we get

$$\begin{aligned}\sum_{p=1}^{k} \Sigma^2(p) &= \frac{k^2}{4}\sum_{p=1}^{k}(k-2p-1)^2 \\ &= \frac{k^2}{4}\cdot\left[(k-3)^2 + (k-5)^2 + (k-7)^2 + \cdots + (k-(2k+1))^2\right].\end{aligned} \tag{A.25}$$

We compute the sum in the square brackets. We use formula (A.12). In our case,

$$\begin{gathered} r = k, \quad u_1 = (k-3)^2, \quad u_2 = (k-5)^2, \quad u_3 = (k-7)^2, \ldots, \quad u_k = (k-2k-1)^2 = (k+1)^2, \\ d_1' = u_2 - u_1 = (k-5)^2 - (k-3)^2 = -4k + 16, \\ d_2' = u_3 - u_2 = (k-7)^2 - (k-5)^2 = -4k + 24, \end{gathered} \tag{A.26}$$

the second differences are constant, and

$$d_1'' = d_2' - d_1' = (-4k + 24) - (-4k + 16) = 8. \tag{A.27}$$

Then the sum in the square brackets equals

$$\frac{k!}{(k-1)!\cdot 1!}\cdot(k-3)^2 + \frac{k!(-4)}{(k-2)!\cdot 2!}\cdot(k-4) + \frac{k!}{(k-3)!\cdot 3!}\cdot 8 = \frac{k}{3}\left(k^2 + 11\right), \tag{A.28}$$

and formula (A.24) is proved. □

A.10. Formula for $2\prod_{i,j=0,i>j}^{k}\Sigma(i)\Sigma(j)$

Lemma A.10. *For the above product, the following formula holds:*

$$2\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma(i)\Sigma(j) = k^4 - \frac{k^3}{12}\left(k^2 + 11\right). \tag{A.29}$$

Proof. We have

$$2\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma(i)\ \Sigma(j) = \left(\sum_{p=1}^{k}\Sigma(p)\right)^2 - \sum_{p=1}^{k}(\Sigma(p))^2. \tag{A.30}$$

Then, using formulas (A.22), and (A.24), we get

$$\left(\sum_{p=1}^{k}\Sigma(p)\right)^2 - \sum_{p=1}^{k}(\Sigma(p))^2 = \left(-k^2\right)^2 - \frac{k^3}{12}\left(k^2+11\right) = k^4 - \frac{k^3}{12}\left(k^2+11\right). \tag{A.31}$$

□

A.11. Formula for $\sum_{p=0}^{k}\Sigma^{+}(p)$

Lemma A.11. *For the above sum, the following formula holds:*

$$\sum_{p=0}^{k}\Sigma^{+}(p) = 0. \tag{A.32}$$

Proof. Using formulas (A.9), (A.10), and (A.22), we get

$$\sum_{p=0}^{k}\Sigma^{+}(p) = \Sigma(0) + \sum_{p=1}^{k}\Sigma(p) + \sum_{p=0}^{k}(k-p) = \frac{k}{2}(k-1) - k^2 + \frac{k}{2}(k+1) = 0. \tag{A.33}$$

□

A.12. Formula for $\sum_{p=0}^{k}(\Sigma^{+}(p))^2$

Lemma A.12. *For the above sum, the following formula holds:*

$$\sum_{p=0}^{k}\left(\Sigma^{+}(p)\right)^2 = \frac{(k+1)^2 k}{12}\cdot\left(k^2+3k+2\right). \tag{A.34}$$

Proof. Using formula (A.10), we get

$$\sum_{p=0}^{k}\left(\Sigma^{+}(p)\right)^2 = \frac{(k+1)^2}{4}\left[(k-0)^2 + (k-2)^2 + (k-4)^2 + \cdots + (k-2k)^2\right]. \tag{A.35}$$

We compute the sum in the square brackets. We use formula (A.12). In our case,

$$\begin{gathered}
r = k+1, \quad u_1 = k^2, \quad u_2 = (k-2)^2, \quad u_3 = (k-4)^2, \ldots, \quad u_{k+1} = (k-2k)^2 = k^2, \\
d_1' = u_2 - u_1 = (k-2)^2 - k^2 = -4k+4, \\
d_2' = u_3 - u_2 = (k-4)^2 - (k-2)^2 = -4k+12,
\end{gathered} \tag{A.36}$$

the second differences are constant, and

$$d_1'' = d_2' - d_1' = (-4k + 12) - (-4k + 4) = 8. \tag{A.37}$$

Then, the sum in the square brackets equals

$$\frac{(k+1)!}{k! \cdot 1!} \cdot k^2 + \frac{4(k+1)!}{(k-1)! \cdot 2!} \cdot (-k+1) + \frac{(k+1)!}{(k-2)! \cdot 3!} \cdot 8 = \frac{k}{3}\left(k^2 + 3k + 2\right), \tag{A.38}$$

and formula (A.34) is proved. □

A.13. Formula for $2\prod_{i,j=0,\ i>j}^{k} \Sigma^+(i)\Sigma^+(j)$

Lemma A.13. *For the above product, the following formula holds:*

$$2\prod_{\substack{i,j=0 \\ i>j}}^{k} \Sigma^+(i)\Sigma^+(j) = -\frac{(k+1)^2 k}{12}\left(k^2 + 3k + 2\right). \tag{A.39}$$

Proof. We have

$$2\prod_{\substack{i,j=0 \\ i>j}}^{k} \Sigma^+(i)\Sigma^+(j) = \left(\sum_{p=0}^{k} \Sigma^+(p)\right)^2 - \sum_{p=0}^{k} \left(\Sigma^+(p)\right)^2. \tag{A.40}$$

Then, using formulas (A.32), and (A.34), we get

$$\left(\sum_{p=1}^{k} \Sigma^+(p)\right)^2 - \sum_{p=1}^{k} \left(\Sigma^+(p)\right)^2 = -\sum_{p=1}^{k} \left(\Sigma^+(p)\right)^2 = -\frac{(k+1)^2 k}{12} \cdot \left(k^2 + 3k + 2\right). \tag{A.41}$$

□

A.14. Formula for $S(p)$

Lemma A.14. *For a fixed integer p, the formula*

$$S(p) = \frac{k}{6}\left[2k^2 - 3(1 + 2p)k + \left(6p^2 + 6p + 1\right)\right] \tag{A.42}$$

holds.

Proof. We use formula (A.12). In our case

$$
\begin{aligned}
r = k, \quad u_1 &= (k-p-1)^2, \ldots, \quad u_k = (k-p-k)^2 = p^2, \\
d_1' = u_2 - u_1 &= (k-p-2)^2 - (k-p-1)^2 = (2k-2p-3)(-1), \\
d_2' = u_3 - u_2 &= (k-p-3)^2 - (k-p-2)^2 = (2k-2p-5)(-1),
\end{aligned}
\tag{A.43}
$$

the second differences are constant, and

$$
d_1'' = d_2' - d_1' = (2k-2p-5)(-1) - (2k-2p-3)(-1) = 2. \tag{A.44}
$$

Then the formula

$$
S(p) = \frac{k!}{(k-1)! \cdot 1!} \cdot (k-p-1)^2 + \frac{k!(-1)}{(k-2)! \cdot 2!} \cdot (2k-2p-3) + \frac{k!}{(k-3)! \cdot 3!} \cdot 2 \tag{A.45}
$$

directly follows from (A.12). After some simplification, we get

$$
\begin{aligned}
S(p) &= k \cdot (k-p-1)^2 - \frac{k(k-1)}{2} \cdot (2k-2p-3) + \frac{k(k-1)(k-2)}{3} \\
&= \frac{k}{6} \cdot \left[6\left(k^2 - 2k(p+1) + (p+1)^2\right) - 3\left(2k^2 - k(2p+5) + (2p+3)\right) + 2\left(k^2 - 3k + 2\right)\right] \\
&= \frac{k}{6}\left[2k^2 - 3(1+2p)k + \left(6p^2 + 6p + 1\right)\right].
\end{aligned}
\tag{A.46}
$$

Formula (A.42) is proved. □

A.15. Formula for $\sum_{p=1}^{k} S(p)$

Lemma A.15. *For a fixed integer p, the formula*

$$
\sum_{p=1}^{k} S(p) = \frac{k}{6}\left(k^3 + 5k\right) \tag{A.47}
$$

holds.

Proof. Since, by (A.42),

$$
\frac{6}{k} S(p) = 2k^2 - 3(1+2p)k + \left(6p^2 + 6p + 1\right), \tag{A.48}
$$

we get

$$\begin{aligned}
\frac{6}{k}\sum_{p=1}^{k}S(p) &= 2\sum_{p=1}^{k}k^2 - 3k\sum_{p=1}^{k}(1+2p) + 6\sum_{p=1}^{k}p^2 + 6\sum_{p=1}^{k}p + \sum_{p=1}^{k}1 \\
&= 2k^3 - 3k\left(k^2+2k\right) + k\left(2k^2+3k+1\right) + 3\left(k^2+k\right) + k \\
&= k^3 + 5k.
\end{aligned} \tag{A.49}$$

This yields (A.47). □

A.16. Formula for $S^+(p)$

Lemma A.16. *The above expression equals*

$$S^+(p) = \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]. \tag{A.50}$$

Proof. We have the following:

$$\begin{aligned}
S^+(p) &= (k-p)^2 + S(p) \\
&= \left[(k-p)^2 + \frac{k}{6}\left[2k^2 - 3(1+2p)k + \left(6p^2+6p+1\right)\right]\right. \\
&\quad \left. -\frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right]\right] + \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right] \\
&= \frac{1}{6}\left[6k^2 - 12kp + 6p^2 + k[-4k+6p+1] - 2k^2 + (6p-1)k - 6p^2\right] \\
&\quad + \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right] \\
&= \frac{k+1}{6}\left[2k^2 + (-6p+1)k + 6p^2\right].
\end{aligned} \tag{A.51}$$

This yields (A.50). □

A.17. Formula for $\sum_{p=0}^{k} S^+(p)$

Lemma A.17. *The above expression equals*

$$\sum_{p=0}^{k}S^+(p) = \frac{(k+1)k}{6}\left(k^2+3k+2\right). \tag{A.52}$$

Proof. Since, by (A.50),

$$\frac{6}{k+1}S^+(p) = 2k^2 + (-6p+1)k + 6p^2, \tag{A.53}$$

we get

$$\begin{aligned}\frac{6}{k+1}\sum_{p=0}^{k}S^{+}(p) &= 2\sum_{p=0}^{k}k^{2}+k\sum_{p=0}^{k}(-6p+1)+6\sum_{p=0}^{k}p^{2}\\ &= 2k^{2}(k+1)+k(-3k(k+1)+(k+1))+k\left(2k^{2}+3k+1\right)\\ &= k^{3}+3k^{2}+2k.\end{aligned} \tag{A.54}$$

This yields (A.52). □

A.18. Formula for $(1/k)\sum_{p=1}^{k}S(p)-(1/(k+1))\sum_{p=0}^{k}S^{+}(p)$

Lemma A.18. *The above expression equals*

$$\frac{1}{k}\sum_{p=1}^{k}S(p)-\frac{1}{k+1}\sum_{p=0}^{k}S^{+}(p)=\frac{1}{2}\cdot\left(-k^{2}+k\right). \tag{A.55}$$

Proof. By (A.47) and (A.50), we obtain

$$\begin{aligned}\frac{1}{k}\sum_{p=1}^{k}S(p)-\frac{1}{k+1}\sum_{p=0}^{k}S^{+}(p) &= \frac{1}{6}\cdot\left(k^{3}+5k\right)-\frac{1}{6}\cdot\left(k^{3}+3k^{2}+2k\right)\\ &= \frac{1}{6}\cdot\left(-3k^{2}+3k\right)=\frac{1}{2}\cdot\left(-k^{2}+k\right).\end{aligned} \tag{A.56}$$

This yields (A.55). □

A.19. Asymptotic Decomposition of $\prod_{p=1}^{k}V(n+p)$

Lemma A.19. *For a fixed* $q\in\mathbb{N}_0$, *the asymptotic representation*

$$\begin{aligned}\prod_{p=1}^{k}V(n+p) &= k^{k}\varphi^{k}(n)\left[1-k\alpha(n)-\frac{k}{24}\left(k^{2}-12k+11\right)\alpha^{2}(n)+\frac{k}{6}\left(k^{2}+5\right)\sum_{i=0}^{q}\omega_{i}(n)\right]\\ &\quad+O\left(\frac{\varphi^{k}(n)}{n^{3}}\right)\end{aligned} \tag{A.57}$$

holds for $n\to\infty$.

Proof. Using formula (A.17), we get

$$\prod_{p=1}^{k} V(n+p) = \prod_{p=1}^{k}\left[\varphi(n)\left[k+\Sigma(p)\alpha(n)+S(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\right]$$

$$= \varphi^k(n)\left[k^k + k^{k-1}\alpha(n)\sum_{i=1}^{k}\Sigma(i) + k^{k-2}\alpha^2(n)\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma(i)\ \Sigma(j) + k^{k-1}\sum_{i=1}^{k}S(i)\sum_{j=0}^{q}\omega_j(n)\right]$$

$$+ O\left(\frac{\varphi^k(n)}{n^3}\right) = (*). \tag{A.58}$$

Now, by (A.22), (A.29), and (A.47)

$$(*) = \varphi^k(n)\left[k^k + k^{k-1}(-k)^2\alpha(n) + \frac{1}{2}k^{k-2}\left(k^4 - \frac{k^3}{12}\left(k^2+11\right)\right)\alpha^2(n)\right.$$

$$\left. + \frac{1}{6}k^{k-1}k\left(k^3+5k\right)\sum_{j=0}^{q}\omega_j(n)\right] + O\left(\frac{\varphi^k(n)}{n^3}\right) \tag{A.59}$$

$$= k^k\varphi^k(n)\left[1 - k\alpha(n) - \frac{k}{24}\left(k^2-12k+11\right)\alpha^2(n)\right.$$

$$\left. + \frac{k}{6}\left(k^2+5\right)\sum_{j=0}^{q}\omega_j(n)\right] + O\left(\frac{\varphi^k(n)}{n^3}\right).$$

□

A.20. Asymptotic Decomposition of $\prod_{p=0}^{k} V^{+}(n+p)$

Lemma A.20. *For a fixed* $q \in \mathbb{N}_0$, *the asymptotic representation*

$$\prod_{p=0}^{k} V^{+}(n+p) = (k+1)^{k+1}\varphi^{k+1}(n)\left[1 - \frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n) + \frac{k}{6}\left(k^2+3k+2\right)\sum_{i=0}^{q}\omega_i(n)\right]$$

$$+ O\left(\frac{\varphi^{k+1}(n)}{n^3}\right) \tag{A.60}$$

holds for $n \to \infty$.

Proof. Using formula (A.20), we get

$$\begin{aligned}
\prod_{p=0}^{k} V^{+}(n+p) &= \prod_{p=0}^{k}\left[\varphi(n)\left[k+1+\Sigma^{+}(p)\alpha(n)+S^{+}(p)\sum_{i=0}^{q}\omega_i(n)\right]+O\left(\frac{\varphi(n)}{n^3}\right)\right] \\
&= \varphi^{k+1}(n)\Bigg[(k+1)^{k+1}+(k+1)^{k}\alpha(n)\sum_{i=0}^{k}\Sigma^{+}(i) \\
&\quad + (k+1)^{k-1}\alpha^2(n)\prod_{\substack{i,j=0\\ i>j}}^{k}\Sigma^{+}(i)\Sigma^{+}(j) \\
&\quad + (k+1)^{k}\sum_{i=0}^{k}S^{+}(i)\sum_{j=0}^{q}\omega_j(n)\Bigg]+O\left(\frac{\varphi^{k+1}(n)}{n^3}\right)=(*).
\end{aligned} \tag{A.61}$$

Now, by (A.32), (A.39), and (A.52), we derive

$$\begin{aligned}
(*) &= \varphi^{k+1}(n)\Bigg[(k+1)^{k+1}-(k+1)^{k-1}\frac{(k+1)^2k}{24}\left(k^2+3k+2\right)\alpha^2(n) \\
&\quad +(k+1)^{k}\frac{(k+1)k}{6}\left(k^2+3k+2\right)\sum_{j=0}^{q}\omega_j(n)\Bigg]+O\left(\frac{\varphi^{k+1}(n)}{n^3}\right) \\
&= (k+1)^{k+1}\varphi^{k+1}(n)\left[1-\frac{k}{24}\left(k^2+3k+2\right)\alpha^2(n)+\frac{k}{6}\left(k^2+3k+2\right)\sum_{j=0}^{q}\omega_j(n)\right] \\
&\quad +O\left(\frac{\varphi^{k+1}(n)}{n^3}\right).
\end{aligned} \tag{A.62}$$

□

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Research Article

A Two-Species Cooperative Lotka-Volterra System of Degenerate Parabolic Equations

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We consider a cooperating two-species Lotka-Volterra model of degenerate parabolic equations. We are interested in the coexistence of the species in a bounded domain. We establish the existence of global generalized solutions of the initial boundary value problem by means of parabolic regularization and also consider the existence of the nontrivial time-periodic solution for this system.

1. Introduction

In this paper, we consider the following two-species cooperative system:

$$u_t = \Delta u^{m_1} + u^{\alpha}(a - bu + cv), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$v_t = \Delta v^{m_2} + v^{\beta}(d + eu - fv), \quad (x,t) \in \Omega \times \mathbb{R}_+, \tag{1.2}$$

$$u(x,t) = 0, \quad v(x,t) = 0, \quad (x,t) \in \partial\Omega \times \mathbb{R}_+, \tag{1.3}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{1.4}$$

where $m_1, m_2 > 1$, $0 < \alpha < m_1$, $0 < \beta < m_2$, $1 \le (m_1 - \alpha)(m_2 - \beta)$, $a = a(x,t)$, $b = b(x,t)$, $c = c(x,t)$, $d = d(x,t)$, $e = e(x,t)$, $f = f(x,t)$ are strictly positive smooth functions and periodic in time with period $T > 0$ and $u_0(x)$ and $v_0(x)$ are nonnegative functions and satisfy $u_0^{m_1}, v_0^{m_2} \in W_0^{1,2}(\Omega)$.

In dynamics of biological groups, the system (1.1)-(1.2) can be used to describe the interaction of two biological groups. The diffusion terms Δu^{m_1} and Δv^{m_2} represent the effect of dispersion in the habitat, which models a tendency to avoid crowding and the speed of the diffusion is rather slow. The boundary conditions (1.3) indicate that the habitat is surrounded by a totally hostile environment. The functions u and v represent the spatial densities of the species at time t and a, d are their respective net birth rate. The functions b and f are intra-specific competitions, whereas c and e are those of interspecific competitions.

As famous models for dynamics of population, two-species cooperative systems like (1.1)-(1.2) have been studied extensively, and there have been many excellent results, for detail one can see [1–6] and references therein. As a special case, men studied the following two-species Lotka-Volterra cooperative system of ODEs:

$$\begin{aligned} u'(t) &= u(t)\big(a(t) - b(t)u(t) + c(t)v(t)\big), \\ v'(t) &= v(t)\big(d(t) + e(t)u(t) - f(t)v(t)\big). \end{aligned} \tag{1.5}$$

For this system, Lu and Takeuchi [7] studied the stability of positive periodic solution and Cui [1] discussed the persistence and global stability of it.

When $m_1 = m_2 = \alpha = \beta = 1$, from (1.1)-(1.2) we get the following classical cooperative system:

$$\begin{aligned} u_t &= \Delta u + u(a - bu + cv), \\ v_t &= \Delta v + v(d + eu - fv). \end{aligned} \tag{1.6}$$

For this system, Lin et al. [5] showed the existence and asymptotic behavior of T-periodic solutions when a, b, c, e, d, f are all smooth positive and periodic in time with period $T > 0$. When a, b, c, e, d, f are all positive constants, Pao [6] proved that the Dirichlet boundary value problem of this system admits a unique solution which is uniformly bounded when $ce < bf$, while the blowup solutions are possible when the two species are strongly mutualistic ($ce > bf$). For the homogeneous Neumann boundary value problem of this system, Lou et al. [4] proved that the solution will blow up in finite time under a sufficient condition on the initial data. When $c = e = 0$ and $\alpha = \beta = 1$, from (1.1) we get the single degenerate equation

$$u_t = \Delta u^m + u(a - bu). \tag{1.7}$$

For this equation, Sun et al. [8] established the existence of nontrivial nonnegative periodic solutions by monotonicity method and showed the attraction of nontrivial nonnegative periodic solutions.

In the recent years, much attention has been paid to the study of periodic boundary value problems for parabolic systems; for detail one can see [9–15] and the references therein. Furthermore, many researchers studied the periodic boundary value problem for degenerate parabolic systems, such as [16–19]. Taking into account the impact of periodic factors on the species dynamics, we are also interested in the existence of the nontrivial periodic solutions of the cooperative system (1.1)-(1.2). In this paper, we first show the existence of the global generalized solution of the initial boundary value problem (1.1)–(1.4). Then under the condition that

$$b_l f_l > c_M e_M, \tag{1.8}$$

where $f_M = \sup\{f(x,t) \mid (x,t) \in \Omega \times \mathbb{R}\}$, $f_l = \inf\{f(x,t) \mid (x,t) \in \Omega \times \mathbb{R}\}$, we show that the generalized solution is uniformly bounded. At last, by the method of monotone iteration, we establish the existence of the nontrivial periodic solutions of the system (1.1)-(1.2), which follows from the existence of a pair of large periodic supersolution and small periodic subsolution. At last, we show the existence and the attractivity of the maximal periodic solution.

Our main efforts center on the discussion of generalized solutions, since the regularity follows from a quite standard approach. Hence we give the following definition of generalized solutions of the problem (1.1)–(1.4).

Definition 1.1. A nonnegative and continuous vector-valued function (u,v) is said to be a generalized solution of the problem (1.1)–(1.4) if, for any $0 \le \tau < T$ and any functions $\varphi_i \in C^1(\overline{Q}_\tau)$ with $\varphi_i|_{\partial\Omega\times[0,\tau)} = 0$ $(i = 1,2)$, $\nabla u^{m_1}, \nabla v^{m_2} \in L^2(Q_\tau)$, $\partial u^{m_1}/\partial t$, $\partial v^{m_2}/\partial t \in L^2(Q_\tau)$ and

$$\iint_{Q_\tau} u\frac{\partial \varphi_1}{\partial t} - \nabla u^{m_1}\nabla\varphi_1 + u^\alpha(a - bu + cv)\varphi_1 dx\, dt = \int_\Omega u(x,\tau)\varphi_1(x,\tau)dx - \int_\Omega u_0(x)\varphi_1(x,0)dx,$$
$$\iint_{Q_\tau} v\frac{\partial \varphi_2}{\partial t} - \nabla v^{m_2}\nabla\varphi_2 + v^\beta(d + eu - fv)\varphi_2 dx\, dt = \int_\Omega v(x,\tau)\varphi_2(x,\tau)dx - \int_\Omega v_0(x)\varphi_2(x,0)dx, \tag{1.9}$$

where $Q_\tau = \Omega \times (0,\tau)$.

Similarly, we can define a weak supersolution $(\overline{u},\overline{v})$ (subsolution $(\underline{u},\underline{v})$) if they satisfy the inequalities obtained by replacing "=" with "≤" ("≥") in (1.3), (1.4), and (1.9) and with an additional assumption $\varphi_i \ge 0$ $(i = 1,\ 2)$.

Definition 1.2. A vector-valued function (u,v) is said to be a T-periodic solution of the problem (1.1)–(1.3) if it is a solution in $[0,T]$ such that $u(\cdot,0) = u(\cdot,T)$, $v(\cdot,0) = v(\cdot,T)$ in Ω. A vector-valued function $(\overline{u},\overline{v})$ is said to be a T-periodic supersolution of the problem (1.1)–(1.3) if it is a supersolution in $[0,T]$ such that $\overline{u}(\cdot,0) \ge \overline{u}(\cdot,T)$, $\overline{v}(\cdot,0) \ge \overline{v}(\cdot,T)$ in Ω. A vector-valued function $(\underline{u},\underline{v})$ is said to be a T-periodic subsolution of the problem (1.1)–(1.3), if it is a subsolution in $[0,T]$ such that $\underline{u}(\cdot,0) \le \underline{u}(\cdot,T)$, $\underline{v}(\cdot,0) \le \underline{v}(\cdot,T)$ in Ω.

This paper is organized as follows. In Section 2, we show the existence of generalized solutions to the initial boundary value problem and also establish the comparison principle. Section 3 is devoted to the proof of the existence of the nonnegative nontrivial periodic solutions by using the monotone iteration technique.

2. The Initial Boundary Value Problem

To solve the problem (1.1)–(1.4), we consider the following regularized problem:

$$\frac{\partial u_\varepsilon}{\partial t} = \operatorname{div}\left(\left(mu_\varepsilon^{m_1-1} + \varepsilon\right)\nabla u_\varepsilon\right) + u_\varepsilon^\alpha(a - bu_\varepsilon + cv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.1}$$

$$\frac{\partial v_\varepsilon}{\partial t} = \operatorname{div}\left(\left(mv_\varepsilon^{m_2-1} + \varepsilon\right)\nabla v_\varepsilon\right) + v_\varepsilon^\beta(d + eu_\varepsilon - fv_\varepsilon), \quad (x,t) \in Q_T, \tag{2.2}$$

$$u_\varepsilon(x,t) = 0, \quad v_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{2.3}$$

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x), \quad v_\varepsilon(x,0) = v_{0\varepsilon}(x), \quad x \in \Omega, \tag{2.4}$$

where $Q_T = \Omega\times(0,T)$, $0<\varepsilon<1$, $u_{0\varepsilon}, v_{0\varepsilon} \in C_0^\infty(\Omega)$ are nonnegative bounded smooth functions and satisfy

$$\begin{gathered}
0 \le u_{0\varepsilon} \le \|u_0\|_{L^\infty(\Omega)}, \qquad 0 \le v_{0\varepsilon} \le \|v_0\|_{L^\infty(\Omega)}, \\
u_{0\varepsilon}^{m_1} \longrightarrow u_0^{m_1}, \quad v_{0\varepsilon}^{m_2} \longrightarrow v_0^{m_2}, \quad \text{in } W_0^{1,2}(\Omega) \text{ as } \varepsilon \longrightarrow 0.
\end{gathered} \tag{2.5}$$

The standard parabolic theory (cf. [20, 21]) shows that (2.1)–(2.4) admits a nonnegative classical solution $(u_\varepsilon, v_\varepsilon)$. So, the desired solution of the problem (1.1)–(1.4) will be obtained as a limit point of the solutions $(u_\varepsilon, v_\varepsilon)$ of the problem (2.1)–(2.4). In the following, we show some important uniform estimates for $(u_\varepsilon, v_\varepsilon)$.

Lemma 2.1. *Let $(u_\varepsilon, v_\varepsilon)$ be a solution of the problem* (2.1)–(2.4).

(1) *If $1 < (m_1-\alpha)(m_2-\beta)$, then there exist positive constants r and s large enough such that*

$$\frac{1}{m_2-\beta} < \frac{m_1+r-1}{m_2+s-1} < m_1-\alpha, \tag{2.6}$$

$$\|u_\varepsilon\|_{L^r(Q_T)} \le C, \qquad \|v_\varepsilon\|_{L^s(Q_T)} \le C, \tag{2.7}$$

where C is a positive constant only depending on $m_1, m_2, \alpha, \beta, r, s$, $|\Omega|$, and T.

(2) *If $1 = (m_1-\alpha)(m_2-\beta)$, then* (2.7) *also holds when $|\Omega|$ is small enough.*

Proof. Multiplying (2.1) by u_ε^{r-1} $(r>1)$ and integrating over Ω, we have that

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t}dx = -\frac{4r(r-1)m_1}{(m_1+r-1)^2}\int_\Omega \left|\nabla u_\varepsilon^{(m_1+r-1)/2}\right|^2 dx + r\int_\Omega u_\varepsilon^{\alpha+r-1}(a-bu_\varepsilon+cv_\varepsilon)dx. \tag{2.8}$$

By Poincaré's inequality, we have that

$$K\int_\Omega u_\varepsilon^{m_1+r-1}dx \le \int_\Omega \left|\nabla u_\varepsilon^{(m_1+r-1)/2}\right|^2 dx, \tag{2.9}$$

where K is a constant depending only on $|\Omega|$ and N and becomes very large when the measure of the domain Ω becomes small. Since $\alpha < m_1$, Young's inequality shows that

$$\begin{gathered}
au_\varepsilon^{\alpha+r-1} \le \frac{Kr(r-1)m_1}{(m_1+r-1)^2}u_\varepsilon^{m_1+r-1} + CK^{-(\alpha+r-1)/(m_1-\alpha)}, \\
cu_\varepsilon^{\alpha+r-1}v_\varepsilon \le \frac{Kr(r-1)m_1}{(m_1+r-1)^2}u_\varepsilon^{m_1+r-1} + CK^{-(\alpha+r-1)/(m_1-\alpha)}v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)}.
\end{gathered} \tag{2.10}$$

For convenience, here and below, C denotes a positive constant which is independent of ε and may take different values on different occasions. Complying (2.8) with (2.9) and (2.10), we obtain

$$\int_\Omega \frac{\partial u_\varepsilon^r}{\partial t}dx \le -\frac{2Kr(r-1)m_1}{(m_1+r-1)^2}\int_\Omega u_\varepsilon^{m_1+r-1}dx + CK^{-(\alpha+r-1)/(m_1-\alpha)}\int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)}dx$$
$$+ CK^{-(\alpha+r-1)/(m_1-\alpha)}. \tag{2.11}$$

As a similar argument as above, for v_ε and positive constant $s>1$, we have that

$$\int_\Omega \frac{\partial v_\varepsilon^s}{\partial t}dx \le -\frac{2Ks(s-1)m_2}{(m_2+s-1)^2}\int_\Omega v_\varepsilon^{m_2+s-1}dx + CK^{-(\beta+s-1)/(m_2-\beta)}\int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)}dx$$
$$+ CK^{-(\beta+s-1)/(m_2-\beta)}. \tag{2.12}$$

Thus we have that

$$\begin{aligned}\int_\Omega \left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le{}& -\frac{2Kr(r-1)m_1}{(m_1+r-1)^2}\int_\Omega u_\varepsilon^{m_1+r-1}dx + CK^{-(\beta+s-1)/(m_2-\beta)}\int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)}dx\\ &-\frac{2Ks(s-1)m_2}{(m_2+s-1)^2}\int_\Omega v_\varepsilon^{m_2+s-1}dx + CK^{-(\alpha+r-1)/(m_1-\alpha)}\int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)}dx\\ &+ CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}.\end{aligned} \tag{2.13}$$

For the case of $1<(m_1-\alpha)(m_2-\beta)$, there exist r, s large enough such that

$$\frac{1}{m_1-\alpha} < \frac{m_2+s-1}{m_1+r-1} < m_2-\beta. \tag{2.14}$$

By Young's inequality, we have that

$$\begin{aligned}\int_\Omega u_\varepsilon^{(m_2+s-1)/(m_2-\beta)}dx &\le \frac{r(r-1)m_1K^{(m_2+s-1)/(m_2-\beta)}}{C(m_1+r-1)^2}\int_\Omega u_\varepsilon^{m_1+r-1}dx + CK^{-\gamma_1},\\ \int_\Omega v_\varepsilon^{(m_1+r-1)/(m_1-\alpha)}dx &\le \frac{s(s-1)m_2K^{(m_1+r-1)/(m_1-\alpha)}}{C(m_2+s-1)^{p_2}}\int_\Omega v_\varepsilon^{m_2+s-1}dx + CK^{-\gamma_2},\end{aligned} \tag{2.15}$$

where

$$\begin{aligned}\gamma_1 &= \frac{(m_2+s-1)^2}{[m_2-\beta]\left[(m_2-\beta)(m_1+r-1)-(m_2+s-1)\right]},\\ \gamma_2 &= \frac{(m_1+r-1)^2}{[m_1-\alpha]\left[(m_1-\alpha)(m_2+s-1)-(m_1+r-1)\right]}.\end{aligned} \tag{2.16}$$

Together with (2.13), we have that

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega\left(u_\varepsilon^{m_1+r-1}+v_\varepsilon^{m_2+s-1}\right)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right) + CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}, \tag{2.17}$$

where

$$\theta_1 = \frac{(m_2+s-1)+(m_1+r-1)(\beta+s-1)}{(m_2-\beta)(m_1+r-1)-(m_2+s-1)}, \qquad \theta_2 = \frac{(m_1+r-1)+(m_2+s-1)(\alpha+r-1)}{(m_1-\alpha)(m_2+s-1)-(m_1+r-1)}. \tag{2.18}$$

Furthermore, by Hölder's and Young's inequalities, from (2.17) we obtain

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega (u_\varepsilon^r+v_\varepsilon^s)dx + C\left(K^{-\theta_1}+K^{-\theta_2}\right) + 2K|\Omega| + CK^{-(\alpha+r-1)/(m_1-\alpha)} + CK^{-(\beta+s-1)/(m_2-\beta)}. \tag{2.19}$$

Then by Gronwall's inequality, we obtain

$$\int_\Omega (u_\varepsilon^r+v_\varepsilon^s)dx \le C. \tag{2.20}$$

Now we consider the case of $1=(m_1-\alpha)(m_2-\beta)$. It is easy to see that there exist positive constants r, s large enough such that

$$\frac{1}{m_1-\alpha} = \frac{m_2+s-1}{m_1+r-1} = m_2-\beta. \tag{2.21}$$

Due to the continuous dependence of K upon $|\Omega|$ in (2.9), from (2.13) we have that

$$\int_\Omega\left(\frac{\partial u_\varepsilon^r}{\partial t}+\frac{\partial v_\varepsilon^s}{\partial t}\right)dx \le -K\int_\Omega\left(u_\varepsilon^{m_1+r-1}+v_\varepsilon^{m_2(p_2-1)+s-1}\right)dx + C \tag{2.22}$$

when $|\Omega|$ is small enough. Then by Young's and Gronwall's inequalities we can also obtain (2.20), and thus we complete the proof of this lemma. □

Taking $u_\varepsilon^{m_1}, v_\varepsilon^{m_2}$ as the test functions, we can easily obtain the following lemma.

Lemma 2.2. *Let* $(u_\varepsilon, v_\varepsilon)$ *be a solution of* (2.1)–(2.4)*; then*

$$\iint_{Q_T} |\nabla u_\varepsilon^{m_1}|^2 dx\,dt \le C, \qquad \iint_{Q_T} |\nabla v_\varepsilon^{m_2}|^2 dx\,dt \le C, \tag{2.23}$$

where C *is a positive constant independent of* ε.

Lemma 2.3. *Let $(u_\varepsilon, v_\varepsilon)$ be a solution of* (2.1)–(2.4)*, then*

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le C, \qquad \|v_\varepsilon\|_{L^\infty(Q_T)} \le C, \tag{2.24}$$

where C is a positive constant independent of ε.

Proof. For a positive constant $k > \|u_{0\varepsilon}\|_{L^\infty(\Omega)}$, multiplying (2.1) by $(u_\varepsilon - k)_+^{m_1}\chi_{[t_1,t_2]}$ and integrating the results over Q_T, we have that

$$\begin{aligned} &\frac{1}{m_1+1}\iint_{Q_T}\frac{\partial(u_\varepsilon-k)_+^{m_1+1}\chi_{[t_1,t_2]}}{\partial t}dx\,dt+\iint_{Q_T}\left|\nabla(u_\varepsilon-k)_+^{m_1}\chi_{[t_1,t_2]}\right|^2dx\,dt \\ &\quad\le\iint_{Q_T}au_\varepsilon^{\alpha+m_1}(a+cv_\varepsilon)dx\,dt, \end{aligned} \tag{2.25}$$

where $s_+ = \max\{0, s\}$ and $\chi_{[t_1,t_2]}$ is the characteristic function of $[t_1, t_2]$ $(0 \le t_1 < t_2 \le T)$. Let

$$I_k(t) = \int_\Omega (u_\varepsilon - k)_+^{m_1+1}dx; \tag{2.26}$$

then $I_k(t)$ is absolutely continuous on $[0, T]$. Denote by σ the point where $I_k(t)$ takes its maximum. Assume that $\sigma > 0$, for a sufficient small positive constant ϵ. Taking $t_1 = \sigma - \epsilon$, $t_2 = \sigma$ in (2.25), we obtain

$$\begin{aligned} &\frac{1}{(m_1+1)\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\frac{\partial(u_\varepsilon-k)_+^{m_1+1}}{\partial t}dx\,dt+\frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\left|\nabla(u_\varepsilon-k)_+^{m_1}\right|^2dx\,dt \\ &\quad\le\frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega u_\varepsilon^{\alpha+m_1}(a+cv_\varepsilon)dx\,dt. \end{aligned} \tag{2.27}$$

From

$$\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\frac{\partial(u_\varepsilon-k)_+^{m_1+1}}{\partial t}dx\,dt = I_k(\sigma) - I_k(\sigma-\epsilon) \ge 0, \tag{2.28}$$

we have that

$$\frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega\left|\nabla(u_\varepsilon-k)_+^{m_1}\right|^2dx\,dt \le \frac{1}{\epsilon}\int_{\sigma-\epsilon}^{\sigma}\int_\Omega u_\varepsilon^{\alpha+m_1}(a+cv_\varepsilon)dx\,dt. \tag{2.29}$$

Letting $\epsilon \to 0^+$, we have that

$$\int_\Omega \left|\nabla (u_\varepsilon(x,\sigma)-k)_+^{m_1}\right|^2 dx \le \int_\Omega u_\varepsilon^{\alpha+m_1}(x,\sigma)(a+cv_\varepsilon(x,\sigma))dx. \tag{2.30}$$

Denote $A_k(t)=\{x : u_\varepsilon(x,t)>k\}$ and $\mu_k=\sup_{t\in(0,T)}|A_k(t)|$; then

$$\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx \le \int_{A_k(\sigma)} u_\varepsilon^{\alpha+m_1}(a+cv_\varepsilon)dx. \tag{2.31}$$

By Sobolev's theorem,

$$\left(\int_{A_k(\sigma)} ((u_\varepsilon-k)_+^{m_1})^p dx\right)^{1/p} \le C\left(\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx\right)^{1/2}, \tag{2.32}$$

with

$$2<p<\begin{cases} +\infty, & N\le 2,\\ \dfrac{2N}{N-2}, & N>2, \end{cases} \tag{2.33}$$

we obtain

$$\begin{aligned}
\left(\int_{A_k(\sigma)} ((u_\varepsilon-k)_+^{m_1})^p dx\right)^{2/p} &\le C\int_{A_k(\sigma)} \left|\nabla (u_\varepsilon-k)_+^{m_1}\right|^2 dx\\
&\le C\int_{A_k(\sigma)} u_\varepsilon^{\alpha+m_1}(a+v_\varepsilon)dx\\
&\le C\left(\int_{A_k(\sigma)} u_\varepsilon^r dx\right)^{(m_1+\alpha)/r}\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^{r/(r-m_1-\alpha)}dx\right)^{(r-m_1-\alpha)/r}\\
&\le C\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^{r/(r-m_1-\alpha)}dx\right)^{(r-m_1-\alpha)/r}\\
&\le C\left(\int_{A_k(\sigma)} (a+v_\varepsilon)^s dx\right)^{1/s} |A_k(\sigma)|^{(s(r-m_1-\alpha)-r)/sr}\\
&\le C\mu_k^{(s(r-m_1-\alpha)-r)/sr},
\end{aligned} \tag{2.34}$$

where $r > p(m_1 + \alpha)/(p - 2)$, $s > pr/(p(r - m_1 - \alpha) - 2r)$ and C denotes various positive constants independent of ε. By Hölder's inequality, it yields

$$\begin{aligned} I_k(\sigma) &= \int_\Omega (u_\varepsilon - k)_+^{m_1+1} dx = \int_{A_k(\sigma)} (u_\varepsilon - k)_+^{m_1+1} dx \\ &\le \left(\int_{A_k(\sigma)} (u_\varepsilon - k)_+^{m_1 p} dx \right)^{(m_1+1)/m_1 p} \mu_k^{1-(m_1+1)/m_1 p} \\ &\le C \mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}. \end{aligned} \tag{2.35}$$

Then

$$I_k(t) \le I_k(\sigma) \le C \mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}, \quad t \in [0, T]. \tag{2.36}$$

On the other hand, for any $h > k$ and $t \in [0, T]$, we have that

$$I_k(t) \ge \int_{A_k(t)} (u_\varepsilon - k)_+^{m_1+1} dx \ge (h - k)^{m_1+1} |A_h(t)|. \tag{2.37}$$

Combined with (2.35), it yields

$$(h - k)^{m_1+1} \mu_h \le C \mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}, \tag{2.38}$$

that is,

$$\mu_h \le \frac{C}{(h - k)^{m_1+1}} \mu_k^{1+[sp(r-m_1-\alpha)-pr-2sr](m_1+1)/2psrm_1}. \tag{2.39}$$

It is easy to see that

$$\gamma = 1 + \frac{\left[sp(r - m_1 - \alpha) - pr - 2sr \right] (m_1 + 1)}{2psrm_1} > 1. \tag{2.40}$$

Then by the De Giorgi iteration lemma [22], we have that

$$\mu_{l+d} = \sup |A_{l+d}(t)| = 0, \tag{2.41}$$

where $d = C^{1/(m_1+1)} \mu_l^{(\gamma-1)/(m_1+1)} 2^{\gamma/(\gamma-1)}$. That is,

$$u_\varepsilon \le l + d \quad \text{a.e. in } Q_T. \tag{2.42}$$

It is the same for the second inequality of (2.24). The proof is completed. □

Lemma 2.4. *The solution* $(u_\varepsilon, v_\varepsilon)$ *of* (2.1)–(2.4) *satisfies the following:*

$$\iint_{Q_T} \left| \frac{\partial u_\varepsilon^{m_1}}{\partial t} \right|^2 dx\, dt \le C, \qquad \iint_{Q_T} \left| \frac{\partial v_\varepsilon^{m_2}}{\partial t} \right|^2 dx\, dt \le C, \tag{2.43}$$

where C *is a positive constant independent of* ε.

Proof. Multiplying (2.1) by $(\partial/\partial t)u_\varepsilon^{m_1}$ and integrating over Ω, by (2.3), (2.4) and Young's inequality we have that

$$\begin{aligned}
&\frac{4m_1}{(m_1+1)^2} \iint_{Q_T} \left| \frac{\partial}{\partial t} u_\varepsilon^{(m_1+1)/2} \right|^2 dx\, dt \\
&\quad = \iint_{Q_T} \frac{\partial u_\varepsilon}{\partial t} \frac{\partial u_\varepsilon^{m_1}}{\partial t} dx\, dt \\
&\quad = \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx - \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^{m_1}(x,T)|^2 dx \\
&\qquad + \iint_{Q_T} m_1 u_\varepsilon^{\alpha+m_1-1} (a - b u_\varepsilon + c v_\varepsilon) \frac{\partial u_\varepsilon}{\partial t} dx\, dt \\
&\quad = \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx - \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^{m_1}(x,T)|^2 dx \\
&\qquad + \iint_{Q_T} \frac{2m_1}{m_1+1} u_\varepsilon^{(2\alpha+m_1-1)/2} (a - b u_\varepsilon + c v_\varepsilon) \frac{\partial u_\varepsilon^{(m_1+1)/2}}{\partial t} dx\, dt \\
&\quad \le \frac{1}{2} \int_\Omega |\nabla u_\varepsilon^{m_1}(x,0)|^2 dx + 2m_1 \iint_{Q_T} u_\varepsilon^{2\alpha+m_1-1} (a - b u_\varepsilon + c v_\varepsilon)^2 dx\, dt \\
&\qquad + \frac{2m_1}{(m_1+1)^2} \iint_{Q_T} \left| \frac{\partial}{\partial t} u_\varepsilon^{(m_1+1)/2} \right|^2 dx\, dt,
\end{aligned} \tag{2.44}$$

which together with the bound of $a, b, c, u_\varepsilon, v_\varepsilon$ shows that

$$\iint_{Q_T} \left| \frac{\partial u_\varepsilon^{(m_1+1)/2}}{\partial t} \right|^2 dx\, dt \le C, \tag{2.45}$$

where C is a positive constant independent of ε. Noticing the bound of u_ε, we have that

$$\iint_{Q_T} \left| \frac{\partial u_\varepsilon^{m_1}}{\partial t} \right|^2 dx\, dt = \frac{4m_1^2}{(m_1+1)^2} \iint_{Q_T} u_\varepsilon^{m_1-1} \left| \frac{\partial}{\partial t} u_\varepsilon^{(m_1+1)/2} \right|^2 dx\, dt \le C. \tag{2.46}$$

It is the same for the second inequality. The proof is completed. □

From the above estimates of $u_\varepsilon, v_\varepsilon$, we have the following results.

Theorem 2.5. *The problem* (1.1)–(1.4) *admits a generalized solution.*

Proof. By Lemmas 2.2, 2.3, and 2.4, we can see that there exist subsequences of $\{u_\varepsilon\}, \{v_\varepsilon\}$ (denoted by themselves for simplicity) and functions u, v such that

$$\begin{gathered} u_\varepsilon \longrightarrow u, \quad v_\varepsilon \longrightarrow v, \quad \text{a.e in } Q_T, \\ \frac{\partial u_\varepsilon^{m_1}}{\partial t} \longrightarrow \frac{\partial u^{m_1}}{\partial t}, \quad \frac{\partial v_\varepsilon^{m_2}}{\partial t} \longrightarrow \frac{\partial v^{m_2}}{\partial t}, \quad \text{weakly in } L^2(Q_T), \\ \nabla u_\varepsilon^{m_1} \longrightarrow \nabla u^{m_1}, \quad \nabla v_\varepsilon^{m_2} \longrightarrow \nabla v^{m_2}, \quad \text{weakly in } L^2(Q_T), \end{gathered} \tag{2.47}$$

as $\varepsilon \to 0$. Then a rather standard argument as [23] shows that (u, v) is a generalized solution of (1.1)–(1.4) in the sense of Definition 1.1. □

In order to prove that the generalized solution of (1.1)–(1.4) is uniformly bounded, we need the following comparison principle.

Lemma 2.6. *Let* $(\underline{u}, \underline{v})$ *be a subsolution of the problem* (1.1)–(1.4) *with the initial value* $(\underline{u}_0, \underline{v}_0)$ *and* $(\overline{u}, \overline{v})$ *a supersolution with a positive lower bound of the problem* (1.1)–(1.4) *with the initial value* $(\overline{u}_0, \overline{v}_0)$. *If* $\underline{u}_0 \le \overline{v}_0$, $\underline{u}_0 \le \overline{v}_0$, *then* $\underline{u}(x,t) \le \overline{u}(x,t)$, $\underline{v}(x,t) \le \overline{v}(x,t)$ *on* Q_T.

Proof. Without loss of generality, we might assume that $\|\underline{u}(x,t)\|_{L^\infty(Q_T)}$, $\|\overline{u}(x,t)\|_{L^\infty(Q_T)}$, $\|\underline{v}(x,t)\|_{L^\infty(Q_T)}$, $\|\overline{v}(x,t)\|_{L^\infty(Q_T)} \le M$, where M is a positive constant. By the definitions of subsolution and supersolution, we have that

$$\begin{aligned} &\int_0^t \int_\Omega -\underline{u}\frac{\partial \varphi}{\partial t} + \nabla \underline{u}^{m_1} \nabla \varphi dx\, d\tau + \int_\Omega \underline{u}(x,t)\varphi(x,t)dx - \int_\Omega \underline{u}_0(x)\varphi(x,0)dx \\ &\quad \le \int_0^t \int_\Omega \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi dx\, d\tau, \\ &\int_0^t \int_\Omega -\overline{u}\frac{\partial \varphi}{\partial t} + \nabla \overline{u}^{m_1} \nabla \varphi dx\, d\tau + \int_\Omega \overline{u}(x,t)\varphi(x,t)dx - \int_\Omega \overline{v}_0(x)\varphi(x,0)dx \\ &\quad \ge \int_0^t \int_\Omega \overline{u}^\alpha (a - b\overline{u} + c\overline{v})\varphi dx\, d\tau. \end{aligned} \tag{2.48}$$

Take the test function as

$$\varphi(x,t) = H_\varepsilon\big(\underline{u}^{m_1}(x,t) - \overline{u}^{m_1}(x,t)\big), \tag{2.49}$$

where $H_\varepsilon(s)$ is a monotone increasing smooth approximation of the function $H(s)$ defined as follows:

$$H(s) = \begin{cases} 1, & s > 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.50}$$

It is easy to see that $H'_\varepsilon(s) \to \delta(s)$ as $\varepsilon \to 0$. Since $\partial \underline{u}^{m_1}/\partial t$, $\partial \overline{u}^{m_1}/\partial t \in L^2(Q_T)$, the test function $\varphi(x,t)$ is suitable. By the positivity of a, b, c we have that

$$\begin{aligned}
&\int_\Omega (\underline{u}-\overline{u}) H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1})dx - \int_0^t\int_\Omega (\underline{u}-\overline{u}) \frac{\partial H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1})}{\partial t} dx\, d\tau \\
&\quad + \int_0^t\int_\Omega H'_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1}) \left|\nabla(\underline{u}^{m_1}-\overline{u}^{m_1})\right|^2 dx\, d\tau \qquad (2.51)\\
&\quad \le \int_0^t\int_\Omega a(\underline{u}^\alpha-\overline{u}^\alpha)H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1}) + c(\underline{u}^\alpha\underline{v}-\overline{u}^\alpha\overline{v})H_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1}) dx\, d\tau,
\end{aligned}$$

where C is a positive constant depending on $\|a(x,t)\|_{C(Q_t)}$, $\|c(x,t)\|_{C(Q_t)}$. Letting $\varepsilon \to 0$ and noticing that

$$\int_0^t\int_\Omega H'_\varepsilon(\underline{u}^{m_1}-\overline{u}^{m_1}) \left|\nabla(\underline{u}^{m}-\overline{u}^{m})\right|^2 dx\, d\tau \ge 0, \qquad (2.52)$$

we arrive at

$$\int_\Omega [\underline{u}(x,t)-\overline{u}(x,t)]_+ dx \le C\int_0^t\int_\Omega (\underline{u}^\alpha-\overline{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha-\overline{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v}-\overline{v})_+ dx\, d\tau. \qquad (2.53)$$

Let $(\overline{u},\overline{v})$ be a supsolution with a positive lower bound σ. Noticing that

$$\begin{aligned}
&(x^\alpha-y^\alpha)_+ \le C(\alpha)(x-y)_+, \quad \text{for } \alpha \ge 1,\\
&(x^\alpha-y^\alpha)_+ \le x^{\alpha-1}(x-y)_+ \le y^{\alpha-1}(x-y)_+, \quad \text{for } \alpha < 1,
\end{aligned} \qquad (2.54)$$

with $x, y > 0$, we have that

$$\int_0^t\int_\Omega (\underline{u}^\alpha-\overline{u}^\alpha)_+ + \underline{v}(\underline{u}^\alpha-\overline{u}^\alpha)_+ + \overline{u}^\alpha(\underline{v}-\overline{v})_+ dx\, d\tau \le C\int_0^t\int_\Omega (\underline{u}-\overline{u})_+ + (\underline{v}-\overline{v})_+ dx\, d\tau, \qquad (2.55)$$

where C is a positive constant depending upon α, σ, M.

Similarly, we also have that

$$\int_\Omega [\underline{v}(x,t)-\overline{v}(x,t)]_+ dx \le C\int_0^t\int_\Omega (\underline{u}-\overline{u})_+ + (\underline{v}-\overline{v})_+ dx\, d\tau. \qquad (2.56)$$

Combining the above two inequalities, we obtain

$$\int_{\Omega} \left[\underline{u}(x,t)-\overline{u}(x,t)\right]_{+}+\left[\underline{v}(x,t)-\overline{v}(x,t)\right]_{+} dx \leq C\int_{0}^{t}\int_{\Omega}\left(\underline{u}-\overline{u}\right)_{+}+\left(\underline{v}-\overline{v}\right)_{+} dx\, d\tau. \tag{2.57}$$

By Gronwall's lemma, we see that $\underline{u} \leq \overline{u}$, $\underline{v} \leq \overline{v}$. The proof is completed. □

Corollary 2.7. *If $b_l f_l > c_M e_M$, then the problem* (1.1)–(1.4) *admits at most one global solution which is uniformly bounded in $\overline{\Omega} \times [0,\infty)$.*

Proof. The uniqueness comes from the comparison principle immediately. In order to prove that the solution is global, we just need to construct a bounded positive supersolution of (1.1)–(1.4).

Let $\rho_1 = (a_M f_l + d_M c_M)/(b_l f_l - c_M e_M)$ and $\rho_2 = (a_M e_M + d_M b_l)/(b_l f_l - c_M e_M)$, since $b_l f_l > c_M e_M$; then $\rho_1, \rho_2 > 0$ and satisfy

$$a_M - b_l\rho_1 + c_M\rho_2 = 0, \qquad d_M + e_M\rho_1 - f_l\rho_2 = 0. \tag{2.58}$$

Let $(\overline{u},\overline{v}) = (\eta\rho_1,\eta\rho_2)$, where $\eta > 1$ is a constant such that $(u_0, v_0) \leq (\eta\rho_1,\eta\rho_2)$; then we have that

$$\overline{u}_t - \Delta\overline{u}^{m_1} = 0 \geq \overline{u}^{\alpha}\left(a - b\overline{u} + c\overline{v}\right), \qquad \overline{v}_t - \Delta\overline{v}^{m_2} = 0 \geq \overline{v}^{\beta}\left(d + e\overline{u} - f\overline{v}\right). \tag{2.59}$$

That is, $(\overline{u},\overline{v}) = (\eta\rho_1,\eta\rho_2)$ is a positive supersolution of (1.1)–(1.4). Since $\overline{u},\overline{v}$ are global and uniformly bounded, so are u and v. □

3. Periodic Solutions

In order to establish the existence of the nontrivial nonnegative periodic solutions of the problem (1.1)–(1.3), we need the following lemmas. Firstly, we construct a pair of T-periodic supersolution and T-periodic subsolution as follows.

Lemma 3.1. *In case of $b_l f_l > c_M e_M$, there exists a pair of T-periodic supersolution and T-periodic subsolution of the problem* (1.1)–(1.3).

Proof. We first construct a T-periodic subsolution of (1.1)–(1.3). Let λ be the first eigenvalue and ϕ be the uniqueness solution of the following elliptic problem:

$$-\Delta\phi = \lambda\phi, \quad x \in \Omega, \quad \phi = 0, \quad x \in \partial\Omega; \tag{3.1}$$

then we have that

$$\lambda > 0, \quad \phi(x) > 0 \quad \text{in } \Omega, \quad \left|\nabla\phi\right| > 0 \quad \text{on } \partial\Omega, \quad M = \max_{x\in\overline{\Omega}}\phi(x) < \infty. \tag{3.2}$$

Let

$$(\underline{u}, \underline{v}) = \left(\varepsilon\phi^{2/m_1}(x), \varepsilon\phi^{2/m_2}(x)\right), \tag{3.3}$$

where $\varepsilon > 0$ is a small constant to be determined. We will show that $(\underline{u}, \underline{v})$ is a (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Taking the nonnegative function $\varphi_1(x,t) \in C^1(\overline{Q}_T)$ as the test function, we have that

$$\begin{aligned}
&\iint_{Q_T} \left(\underline{u}\frac{\partial \varphi_1}{\partial t} + \Delta \underline{u}^{m_1}\varphi_1 + \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1\right) dx\,dt \\
&\qquad + \int_{\Omega} \underline{u}(x,0)\varphi_1(x,0) - \underline{u}(x,T)\varphi_1(x,T)\,dx \\
&\quad = \iint_{Q_T} \left(\underline{u}^{\alpha}(a - b\underline{u} + c\underline{v}) + \Delta \underline{u}^{m_1}\right)\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - \iint_{Q_T} \nabla \underline{u}^{m_1}\nabla\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T} \phi\nabla\phi\cdot\nabla\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T} \nabla\phi\nabla(\phi\varphi_1) - \left|\nabla\phi\right|^2\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T} -\operatorname{div}(\nabla\phi)\phi\varphi_1 - \left|\nabla\phi\right|^2\varphi_1 dx\,dt \\
&\quad = \iint_{Q_T} \underline{u}^{\alpha}(a - b\underline{u} + c\underline{v})\varphi_1 dx\,dt - 2\varepsilon^{m_1}\iint_{Q_T} \left(\lambda\phi^2 - \left|\nabla\phi\right|^2\right)\varphi_1 dx\,dt.
\end{aligned} \tag{3.4}$$

Similarly, for any nonnegative test function $\varphi_2(x,t) \in C^1(\overline{Q}_T)$, we have that

$$\begin{aligned}
&\iint_{Q_T} \left(\underline{v}\frac{\partial \varphi_2}{\partial t} + \Delta \underline{v}^{m_2}\varphi_2 + \underline{v}^{\beta}(d + e\underline{u} - f\underline{v})\varphi_2\right) dx\,dt + \int_{\Omega} \underline{v}(x,0)\varphi_2(x,0) - \underline{v}(x,T)\varphi_2(x,T)\,dx \\
&\quad = \iint_{Q_T} \underline{v}^{\beta}(d + e\underline{u} - f\underline{v})\varphi_2 dx\,dt - 2\varepsilon^{m_2}\iint_{Q_T} \left(\lambda\phi^2 - \left|\nabla\phi\right|^2\right)\varphi_2 dx\,dt.
\end{aligned} \tag{3.5}$$

We just need to prove the nonnegativity of the right-hand side of (3.4) and (3.5).

Since $\phi_1 = \phi_2 = 0$, $|\nabla\phi_1|, |\nabla\phi_2| > 0$ on $\partial\Omega$, then there exists $\delta > 0$ such that

$$\lambda\phi^2 - \left|\nabla\phi\right|^2 \le 0, \qquad x \in \overline{\Omega}_{\delta}, \tag{3.6}$$

where $\overline{\Omega}_\delta = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \le \delta\}$. Choosing

$$\varepsilon \le \min\left\{ \frac{a_l}{b_M M^{2/m_1}}, \frac{d_l}{f_M M^{2/m_2}} \right\}, \tag{3.7}$$

then we have that

$$\begin{aligned} 2\varepsilon^{m_1} \int_0^T \int_{\Omega_\delta} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_1 dx\, dt \le 0 \le \int_0^T \int_{\Omega_\delta} \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi_1 dx\, dt, \\ 2\varepsilon^{m_2} \int_0^T \int_{\Omega_\delta} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_2 dx\, dt \le 0 \le \int_0^T \int_{\Omega_\delta} \underline{v}^\beta (d + e\underline{u} - f\underline{v})\varphi_2 dx\, dt, \end{aligned} \tag{3.8}$$

which shows that $(\underline{u}, \underline{v})$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3) on $\overline{\Omega}_\delta \times (0, T)$.

Moreover, we can see that, for some $\sigma > 0$,

$$\phi(x) \ge \sigma > 0, \quad x \in \Omega \setminus \overline{\Omega}_\delta. \tag{3.9}$$

Choosing

$$\varepsilon \le \min\left\{ \frac{a_l}{2b_M M^{2/m_1}}, \left(\frac{a_l}{4\lambda M^{2(m_1-\alpha)/m_1}}\right)^{1/(m_1-\alpha)}, \frac{d_l}{2f_M M^{2/m_2}}, \left(\frac{d_l}{4\lambda M^{2(m_2-\beta)/m_2}}\right)^{1/(m_2-\beta)} \right\}, \tag{3.10}$$

then

$$\begin{aligned} \varepsilon^\alpha \phi^{2\alpha/m_1} a - b\varepsilon^{\alpha+1}\phi^{2(\alpha+1)/m_1} + c\varepsilon^\alpha \phi^{2\alpha/m_1} \varepsilon\phi^{2/m_2} - 2\varepsilon^{m_1}\lambda\phi^2 \ge 0, \\ \varepsilon^\beta \phi^{2\beta/m_2} d + e\varepsilon\phi^{2/m_1}\varepsilon^\beta\phi^{2\beta/m_2} - f\varepsilon^{\beta+1}\phi^{2(\beta+1)/m_2} - 2\varepsilon^{m_2}\lambda\phi^2 \ge 0 \end{aligned} \tag{3.11}$$

on Q_T, that is

$$\begin{aligned} \iint_{Q_T} \underline{u}^\alpha (a - b\underline{u} + c\underline{v})\varphi_1 dx\, dt - 2\varepsilon^{m_1} \iint_{Q_T} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_1 dx\, dt \ge 0, \\ \iint_{Q_T} \underline{v}^\beta (d + e\underline{u} - f\underline{v})\varphi_2 dx\, dt - 2\varepsilon^{m_2} \iint_{Q_T} \left(\lambda\phi^2 - |\nabla\phi|^2\right)\varphi_2 dx\, dt \ge 0. \end{aligned} \tag{3.12}$$

These relations show that $(\underline{u}, \underline{v}) = (\varepsilon\phi_1^{2/m_1}(x), \varepsilon\phi_2^{2/m_2}(x))$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Letting $(\overline{u}, \overline{v}) = (\eta\rho_1, \eta\rho_2)$, where η, ρ_1, ρ_2 are taken as those in Corollary 2.7, it is easy to see that $(\overline{u}, \overline{v})$ is a positive (time independent, hence T-periodic) subsolution of (1.1)–(1.3).

Obviously, we may assume that $\underline{u}(x,t) \le \overline{u}(x,t)$, $\underline{v}(x,t) \le \overline{v}(x,t)$ by changing η, ε appropriately. □

Lemma 3.2 (see [24, 25]). *Let u be the solution of the following Dirichlet boundary value problem*

$$\frac{\partial u}{\partial t} = \Delta u^m + f(x,t), \quad (x,t) \in \Omega \times (0,T),$$
$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.13}$$

where $f \in L^\infty(\Omega \times (0,T))$; then there exist positive constants K and $\alpha \in (0,1)$ depending only upon $\tau \in (0,T)$ and $\|f\|_{L^\infty(\Omega\times(0,T))}$, such that, for any $(x_i,t_i) \in \Omega \times [\tau,T]$ $(i=1,2)$,

$$|u(x_1,t_1) - u(x_2,t_2)| \le K\left(|x_1-x_2|^\alpha + |t_1-t_2|^{\alpha/2}\right). \tag{3.14}$$

Lemma 3.3 (see [26]). *Define a Poincaré mapping*

$$P_t : L^\infty(\Omega) \times L^\infty(\Omega) \longrightarrow L^\infty(\Omega) \times L^\infty(\Omega),$$
$$P_t(u_0(x), v_0(x)) := (u(x,t), v(x,t)) \quad (t>0), \tag{3.15}$$

where $(u(x,t),v(x,t))$ is the solution of (1.1)–(1.4) *with initial value $(u_0(x),v_0(x))$. According to Lemmas 2.6 and 3.2 and Theorem 2.5, the map P_t has the following properties:*

(i) *P_t is defined for any $t>0$ and order preserving;*

(ii) *P_t is order preserving;*

(iii) *P_t is compact.*

Observe that the operator P_T is the classical Poincaré map and thus a fixed point of the Poincaré map gives a T-periodic solution setting. This will be made by the following iteration procedure.

Theorem 3.4. *Assume that $b_l f_l > c_M e_M$ and there exists a pair of nontrivial nonnegative T-periodic subsolution $(\underline{u}(x,t),\underline{v}(x,t))$ and T-periodic supersolution $(\overline{u}(x,t),\overline{v}(x,t))$ of the problem* (1.1)–(1.3) *with $\underline{u}(x,0) \le \overline{u}(x,0)$; then the problem* (1.1)–(1.3) *admits a pair of nontrivial nonnegative periodic solutions $(u_*(x,t),v_*(x,t))$, $(u^*(x,t),v^*(x,t))$ such that*

$$\underline{u}(x,t) \le u_*(x,t) \le u^*(x,t) \le \overline{u}(x,t), \quad \underline{v}(x,t) \le v_*(x,t) \le v^*(x,t) \le \overline{v}(x,t), \quad \text{in } Q_T. \tag{3.16}$$

Proof. Taking $\overline{u}(x,t)$, $\underline{u}(x,t)$ as those in Lemma 3.1 and choosing suitable $B(x_0,\delta)$, $B(x_0,\delta')$, Ω', k_1, k_2, and K, we can obtain $\underline{u}(x,0) \le \overline{u}(x,0)$. By Lemma 2.6, we have that $P_T(\underline{u}(\cdot,0)) \ge \underline{u}(\cdot,T)$. Hence by Definition 1.2 we get $P_T(\underline{u}(\cdot,0)) \ge \underline{u}(\cdot,0)$, which implies $P_{(k+1)T}(\underline{u}(\cdot,0)) \ge P_{kT}(\underline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. Similarly we have that $P_T(\overline{u}(\cdot,0)) \le \overline{u}(\cdot,T) \le \overline{u}(\cdot,0)$, and hence $P_{(k+1)T}(\overline{u}(\cdot,0)) \le P_{kT}(\overline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. By Lemma 2.6, we have that $P_{kT}(\underline{u}(\cdot,0)) \le P_{kT}(\overline{u}(\cdot,0))$ for any $k \in \mathbb{N}$. Then

$$u_*(x,0) = \lim_{k\to\infty} P_{kT}(\underline{u}(x,0)), \qquad u^*(x,0) = \lim_{k\to\infty} P_{kT}(\overline{u}(x,0)) \tag{3.17}$$

exist for almost every $x \in \Omega$. Since the operator P_T is compact (see Lemma 3.3), the above limits exist in $L^\infty(\Omega)$, too. Moreover, both $u_*(x,0)$ and $u^*(x,0)$ are fixed points of P_T. With the similar method as [26], it is easy to show that the even extension of the function $u_*(x,t)$, which is the solution of the problem (1.1)–(1.4) with the initial value $u_*(x,0)$, is indeed a nontrivial nonnegative periodic solution of the problem (1.1)–(1.3). It is the same for the existence of $u^*(x,t)$. Furthermore, by Lemma 2.6, we obtain (3.16) immediately, and thus we complete the proof. □

Furthermore, by De Giorgi iteration technique, we can also establish a prior upper bound of all nonnegative periodic solutions of (1.1)–(1.3). Then with a similar method as [18], we have the following remark which shows the existence and attractivity of the maximal periodic solution.

Remark 3.5. If $b_l f_l > c_M e_M$, the problem (1.1)–(1.3) admits a maximal periodic solution (U, V). Moreover, if (u, v) is the solution of the initial boundary value problem (1.1)–(1.4) with nonnegative initial value (u_0, v_0), then, for any $\varepsilon > 0$, there exists t depending on u_0, v_0, and ε, such that

$$0 \le u \le U + \varepsilon, \quad 0 \le v \le V + \varepsilon, \quad \text{for } x \in \Omega,\ t \ge t. \tag{3.18}$$

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Research Article

Asymptotic Behavior of Solutions of Delayed Difference Equations

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This contribution is devoted to the investigation of the asymptotic behavior of delayed difference equations with an integer delay. We prove that under appropriate conditions there exists at least one solution with its graph staying in a prescribed domain. This is achieved by the application of a more general theorem which deals with systems of first-order difference equations. In the proof of this theorem we show that a good way is to connect two techniques—the so-called retract-type technique and Liapunov-type approach. In the end, we study a special class of delayed discrete equations and we show that there exists a positive and vanishing solution of such equations.

1. Introduction

Throughout this paper, we use the following notation: for an integer q, we define

$$\mathbb{Z}_q^{\infty} := \{q, q+1, \dots\}. \tag{1.1}$$

We investigate the asymptotic behavior for $n \to \infty$ of the solutions of the discrete delayed equation of the $(k+1)$-th order

$$\Delta v(n) = f(n, v(n), v(n-1), \dots, v(n-k)), \tag{1.2}$$

where n is the independent variable assuming values from the set $\mathbb{Z}_a^{\infty}$ with a fixed $a \in \mathbb{N}$. The number $k \in \mathbb{N}$, $k \geq 1$ is the fixed delay, $\Delta v(n) = v(n+1) - v(n)$, and $f : \mathbb{Z}_a^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$.

A function $v : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ is a solution of (1.2) if it satisfies (1.2) for every $n \in \mathbb{Z}_a^{\infty}$.

We will study (1.2) together with $k+1$ initial conditions

$$v(a+s-k) = v^{a+s-k} \in \mathbb{R}, \quad s = 0, 1, \ldots, k. \tag{1.3}$$

Initial problem (1.2), (1.3) obviously has a unique solution, defined for every $n \in \mathbb{Z}_{a-k}^{\infty}$. If the function f is continuous with respect to its last $k+1$ arguments, then the solution of (1.2) continuously depends on initial conditions (1.3).

Now we give a general description of the problem solved in this paper.

Problem 1. Let $b, c : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ be functions such that $b(n) < c(n)$ for every $n \in \mathbb{Z}_{a-k}^{\infty}$. The problem under consideration is to find sufficient conditions for the right-hand side of (1.2) that will guarantee the existence of a solution $v = v^*(n)$ of initial problem (1.2), (1.3) such that

$$b(n) < v^*(n) < c(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}. \tag{1.4}$$

This problem can be solved with help of a result which is valid for systems of first-order difference equations and which will be presented in the next section. This is possible because the considered equation (1.2) can be rewritten as a system of $k+1$ first-order difference equations, similarly as a differential equation of a higher order can be transformed to a special system of first-order differential equations. Although the process of transforming a $(k+1)$-st order difference equation to a system of first order equations is simple and well-known (it is described in Section 3), the determination of the asymptotic properties of the solutions of the resulting system using either Liapunov approach or retract-type method is not trivial. These analogies of classical approaches, known from the qualitative theory of differential equations, were developed for difference systems in [1] (where an approach based on Liapunov method was formulated) and in [2–5] (where retract-type analysis was modified for discrete equations). It occurs that for the mentioned analysis of asymptotic problems of system (1.2), neither the ideas of Liapunov, nor the retract-type technique can be applied directly. However, in spite of the fact that each of the two mentioned methods fails when used independently, it appears that the combination of both these techniques works for this type of systems. Therefore, in Section 2 we prove the relevant result suitable for the asymptotic analysis of systems arising by transformation of (1.2) to a system of first-order differential equations (Theorem 2.1), where the assumptions put to the right-hand side of the system are of both types: those caused by the application of the Liapunov approach and those which are typical for the retract-type technique. Such an idea was applied in a particular case of investigation of asymptotic properties of solutions of the discrete analogue of the Emden-Fowler equation in [6, 7]. The approach is demonstrated in Section 3 where, moreover, its usefulness is illustrated on the problem of detecting the existence of positive solutions of linear equations with a single delay (in Section 3.4) and asymptotic estimation of solutions (in Section 3.3).

Advantages of our approach can be summarized as follows. We give a general method of analysis which is different from the well-known comparison method (see, e.g., [8, 9]). Comparing our approach with the scheme of investigation in [10, 11] which is based on a result from [12], we can see that the presented method is more general because it unifies the investigation of systems of discrete equations and delayed discrete equations thanks to the Liapunov-retract-type technique.

For related results concerning positive solutions and the asymptotics of solutions of discrete equations, the reader is referred also to [13–25].

2. The Result for Systems of First-Order Equations

Consider the system of m difference equations

$$\Delta u(n) = F(n, u(n)), \tag{2.1}$$

where $n \in \mathbb{Z}_a^\infty$, $u = (u_1, \ldots, u_m)$, and $F : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{R}^m$, $F = (F_1, \ldots, F_m)$. The solution of system (2.1) is defined as a vector function $u : \mathbb{Z}_a^\infty \to \mathbb{R}^m$ such that for every $n \in \mathbb{Z}_a^\infty$, (2.1) is fulfilled. Again, if we prescribe initial conditions

$$u_i(a) = u_i^a \in \mathbb{R}, \quad i = 1, \ldots, m \tag{2.2}$$

the initial problem (2.1), (2.2) has a unique solution. Let us define a set $\Omega \subset \mathbb{Z}_a^\infty \times \mathbb{R}^m$ as

$$\Omega := \bigcup_{n=a}^{\infty} \Omega(n), \tag{2.3}$$

where

$$\Omega(n) := \{(n, u) : n \in \mathbb{Z}_a^\infty,\ u_i \in \mathbb{R},\ b_i(n) < u_i < c_i(n),\ i = 1, \ldots, m\} \tag{2.4}$$

with $b_i, c_i : \mathbb{Z}_a^\infty \to \mathbb{R}$, $i = 1, \ldots, m$, being auxiliary functions such that $b_i(n) < c_i(n)$ for each $n \in \mathbb{Z}_a^\infty$. Such set Ω is called a *polyfacial set*.

Our aim (in this part) is to solve, in correspondence with formulated Problem 1, the following similar problem for systems of difference equations.

Problem 2. Derive sufficient conditions with respect to the right-hand sides of system (2.1) which guarantee the existence of at least one solution $u(n) = (u_1^*(n), \ldots, u_m^*(n))$, $n \in \mathbb{Z}_a^\infty$, satisfying

$$\left(n, u_1^*(n), \ldots, u_m^*(n)\right) \in \Omega(n) \tag{2.5}$$

for every $n \in \mathbb{Z}_a^\infty$.

As we mentioned above, in [1] the above described problem is solved via a Liapunov-type technique. Here we will combine this technique with the retract-type technique which was used in [2–5] so as the result can be applied easily to the system arising after transformation of (1.2). This brings a significant increase in the range of systems we are able to investigate. Before we start, we recall some basic notions that will be used.

2.1. Consequent Point

Define the mapping $\mathcal{C} : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{Z}_a^\infty \times \mathbb{R}^m$ as

$$\mathcal{C} : (n, u) \longmapsto (n+1, u + F(n, u)). \tag{2.6}$$

For any point $M = (n, u) \in \mathbb{Z}_a^\infty \times \mathbb{R}^m$, the point $\mathcal{C}(M)$ is called the *first consequent point* of the point M. The geometrical meaning is that if a point M lies on the graph of some solution of system (2.1), then its first consequent point $\mathcal{C}(M)$ is the next point on this graph.

2.2. Liapunov-Type Polyfacial Set

We say that a polyfacial set Ω is *Liapunov-type* with respect to discrete system (2.1) if

$$b_i(n+1) < u_i + F_i(n, u) < c_i(n+1) \tag{2.7}$$

for every $i = 1, \ldots, m$ and every $(n, u) \in \Omega$. The geometrical meaning of this property is this: if a point $M = (n, u)$ lies inside the set $\Omega(n)$, then its first consequent point $\mathcal{C}(M)$ stays inside $\Omega(n+1)$.

In this contribution we will deal with sets that need not be of Liapunov-type, but they will have, in a certain sense, a similar property. We say that a polyfacial set Ω is *Liapunov-type with respect to the jth variable* ($j \in \{1, \ldots, m\}$) and to discrete system (2.1) if

$$(n, u) \in \Omega \Longrightarrow b_j(n+1) < u_j + F_j(n, u) < c_j(n+1). \tag{2.8}$$

The geometrical meaning is that if $M = (n, u) \in \Omega(n)$, then the u_j-coordinate of its first consequent point stays between $b_j(n+1)$ and $c_j(n+1)$, meanwhile the other coordinates of $\mathcal{C}(M)$ may be arbitrary.

2.3. Points of Strict Egress and Their Geometrical Sense

An important role in the application of the retract-type technique is played by the so called strict egress points. Before we define these points, let us describe the boundaries of the sets $\Omega(n)$, $n \in \mathbb{Z}_a^\infty$, in detail. As one can easily see,

$$\bigcup_{n \in \mathbb{Z}_a^\infty} \partial\Omega(n) = \left(\bigcup_{j=1}^m \Omega_B^j \right) \cup \left(\bigcup_{j=1}^m \Omega_C^j \right) \tag{2.9}$$

with

$$\begin{aligned} \Omega_B^j &:= \left\{ (n, u) : n \in \mathbb{Z}_a^\infty,\ u_j = b_j(n),\ b_i(n) \le u_i \le c_i(n),\ i = 1, \ldots, m,\ i \ne j \right\}, \\ \Omega_C^j &:= \left\{ (n, u) : n \in \mathbb{Z}_a^\infty,\ u_j = c_j(n),\ b_i(n) \le u_i \le c_i(n),\ i = 1, \ldots, m,\ i \ne j \right\}. \end{aligned} \tag{2.10}$$

In accordance with [3, Lemmas 1 and 2], a point $(n,u) \in \partial\Omega(n)$ is a *point of the type of strict egress* for the polyfacial set Ω with respect to discrete system (2.1) if and only if for some $j \in \{1,\dots,m\}$

$$u_j = b_j(n), \quad F_j(n,u) < b_j(n+1) - b_j(n), \tag{2.11}$$

or

$$u_j = c_j(n), \quad F_j(n,u) > c_j(n+1) - c_j(n). \tag{2.12}$$

Geometrically these inequalities mean the following: if a point $M = (n,u) \in \partial\Omega(n)$ is a point of the type of strict egress, then the first consequent point $\mathcal{C}(M) \notin \overline{\Omega(n+1)}$.

2.4. Retract and Retraction

If $A \subset B$ are any two sets in a topological space and $\pi : B \to A$ is a continuous mapping from B onto A such that $\pi(p) = p$ for every $p \in A$, then π is said to be a *retraction* of B onto A. If there exists a retraction of B onto A, then A is called a *retract* of B.

2.5. The Existence Theorem for the System of First-Order Equations (Solution of Problem 2)

The following result, solving Problem 2, gives sufficient conditions with respect to the right-hand sides of (2.1) which guarantee the existence of at least one solution satisfying (2.5) for every $n \in \mathbb{Z}_a^\infty$.

Theorem 2.1. *Let $b_i(n), c_i(n), b_i(n) < c_i(n)$, $i = 1,\dots,m$, be real functions defined on $\mathbb{Z}_a^\infty$ and let $F_i : \mathbb{Z}_a^\infty \times \mathbb{R}^m \to \mathbb{R}$, $i = 1,\dots,m$, be continuous functions. Suppose that for one fixed $j \in \{1,\dots,m\}$ all the points of the sets Ω_B^j, Ω_C^j are points of strict egress, that is, if $(n,u) \in \Omega_B^j$, then*

$$F_j(n,u) < b_j(n+1) - b_j(n), \tag{2.13}$$

and if $(n,u) \in \Omega_C^j$, then

$$F_j(n,u) > c_j(n+1) - c_j(n). \tag{2.14}$$

Further suppose that the set Ω is of Liapunov-type with respect to the ith variable for every $i \in \{1,\dots,m\}$, $i \neq j$, that is, that for every $(n,u) \in \Omega$

$$b_i(n+1) < u_i + F_i(n,u) < c_i(n+1). \tag{2.15}$$

Then there exists a solution $u = (u_1^*(n),\dots,u_m^*(n))$ of system (2.1) satisfying the inequalities

$$b_i(n) < u_i^*(n) < c_i(n), \quad i = 1,\dots,m, \tag{2.16}$$

for every $n \in \mathbb{Z}_a^\infty$.

Proof. The proof will be by contradiction. We will suppose that there exists no solution satisfying inequalities (2.16) for every $n \in \mathbb{Z}_a^\infty$. Under this supposition we prove that there exists a continuous mapping (a retraction) of a closed interval onto both its endpoints which is, by the intermediate value theorem of calculus, impossible.

Without the loss of generality we may suppose that the index j in Theorem 2.1 is equal to 1, that is, all the points of the sets Ω_B^1 and Ω_C^1 are strict egress points. Each solution of system (2.1) is uniquely determined by the chosen initial condition

$$u(a) = (u_1(a), \dots, u_m(a)) = (u_1^a, \dots, u_m^a) = u^a. \tag{2.17}$$

For the following considerations, let u_i^a with $u_i^a \in (b_i(a), c_i(a))$, $i = 2, \dots, m$, be chosen arbitrarily but fixed. Now the solution of (2.1) is given just by the choice of u_1^a, we can write

$$u = u(n, u_1^a) = (u_1(n, u_1^a), \dots, u_m(n, u_1^a)). \tag{2.18}$$

Define the closed interval $I := [b_1(a), c_1(a)]$. Hereafter we show that, under the supposition that there exists no solution satisfying inequalities (2.16), there exists a retraction $\mathcal{R}$ (which will be a composition of two auxiliary mappings $\mathcal{R}_1$ and $\mathcal{R}_2$ defined below) of the set $B := I$ onto the set $A := \partial I = \{b_1(a), c_1(a)\}$. This contradiction will prove our result. To arrive at such a contradiction, we divide the remaining part of the proof into several steps.

Construction of the Leaving Value n^*

Let a point $\tilde{u}_1 \in I$ be fixed. The initial condition $u_1(a) = \tilde{u}_1$ defines a solution $u = u(n, \tilde{u}_1) = (u_1(n, \tilde{u}_1), \dots, u_m(n, \tilde{u}_1))$. According to our supposition, this solution does not satisfy inequalities (2.16) for every $n \in \mathbb{Z}_a^\infty$. We will study the moment the solution leaves the domain Ω for the first time. The first value of n for which inequalities (2.16) are not valid will be denoted as s.

(I) First consider the case $\tilde{u}_1 \in \operatorname{int} I$. Then there exists a value $s > 1$ in $\mathbb{Z}_{a+1}^\infty$ such that

$$(s, u(s, \tilde{u}_1)) \notin \Omega(s) \tag{2.19}$$

while

$$(r, u(r, \tilde{u}_1)) \in \Omega(r) \quad \text{for } a \le r \le s-1. \tag{2.20}$$

As the set Ω is of the Liapunov-type with respect to all variables except the first one and $(s-1, u(s-1, \tilde{u}_1)) \in \Omega(s-1)$, then

$$b_i(s) < u_i(s, \tilde{u}_1) < c_i(s), \quad i = 2, \dots, m. \tag{2.21}$$

Because $j = 1$ was assumed, and Ω is of Liapunov-type for each variable u_i, $i \neq j$, then the validity of inequalities (2.16) has to be violated in the u_1-coordinate. The geometrical meaning was explained in Section 2.2.

Now, two cases are possible: either $(s, u(s, \tilde{u}_1)) \notin \overline{\Omega(s)}$ or $(s, u(s, \tilde{u}_1)) \in \partial\Omega(s)$. In the first case $u_1(s, \tilde{u}_1) < b_1(s)$ or $u_1(s, \tilde{u}_1) > c_1(s)$. In the second case $u_1(s, \tilde{u}_1) = b_1(s)$ or

$u_1(s, \tilde{u}_1) = c_1(s)$ and, due to (2.13) and (2.14), $u_1(s+1, \tilde{u}_1) < b_1(s+1)$ or $u_1(s+1, \tilde{u}_1) > c_1(s+1)$, respectively.

(II) If $\tilde{u}_1 \in \partial I$, then $(a, u(a, \tilde{u}_1)) \notin \Omega(a)$. Thus, for this case, we could put $s = a$. Further, because of the strict egress property of Ω_B^1 and Ω_C^1, either $u_1(a+1, \tilde{u}_1) < b_1(a+1)$ (if $\tilde{u}_1 = b_1(a)$) or $u_1(a+1, \tilde{u}_1) > c_1(a+1)$ (if $\tilde{u}_1 = c_1(a)$) and thus $(a+1, u(a+1, \tilde{u}_1)) \notin \overline{\Omega(a+1)}$.

Unfortunately, for the next consideration the value s (the first value of the independent variable for which the graph of the solution is out of Ω) would be of little use. What we will need is the last value for which the graph of the solution stays in $\overline{\Omega}$. We will denote this value as n^* and will call it the *leaving value*. We can define n^* as

$$\begin{gathered} n^* = s - 1 \quad \text{if } (s, u(s, \tilde{u}_1)) \notin \overline{\Omega(s)}, \\ n^* = s \quad \text{if } (s, u(s, \tilde{u}_1)) \in \partial\Omega(s). \end{gathered} \tag{2.22}$$

As the value of n^* depends on the chosen initial point $\tilde{u}_1$, we could write $n^* = n^*(\tilde{u}_1)$ but we will mostly omit the argument $\tilde{u}_1$, unless it is necessary. From the above considerations it follows that

$$\begin{gathered} b_1(n^*) \le u_1(n^*, \tilde{u}_1) \le c_1(n^*), \\ u_1(n^*+1, \tilde{u}_1) < b_1(n^*+1) \quad \text{or} \quad u_1(n^*+1, \tilde{u}_1) > c_1(n^*+1). \end{gathered} \tag{2.23}$$

Auxiliary Mapping $\mathcal{R}_1$

Now we construct the auxiliary mapping $\mathcal{R}_1 : I \to \mathbb{R} \times \mathbb{R}$. First extend the discrete functions b_1, c_1 onto the whole interval $[a, \infty)$:

$$\begin{aligned} b_1(t) &:= b_1(\lfloor t \rfloor) + (b_1(\lfloor t \rfloor + 1) - b_1(\lfloor t \rfloor))(t - \lfloor t \rfloor), \\ c_1(t) &:= c_1(\lfloor t \rfloor) + (c_1(\lfloor t \rfloor + 1) - c_1(\lfloor t \rfloor))(t - \lfloor t \rfloor), \end{aligned} \tag{2.24}$$

$\lfloor t \rfloor$ being the integer part of t (the floor function). Note that b_1, c_1 are now piecewise linear continuous functions of a real variable t such that $b_1(t) < c_1(t)$ for every t and that the original values of $b_1(n), c_1(n)$ for $n \in \mathbb{Z}_a^\infty$ are preserved. This means that the graphs of these functions connect the points $(n, b_1(n))$ or $(n, c_1(n))$ for $n \in \mathbb{Z}_a^\infty$, respectively. Denote V the set

$$V := \{(t, u_1) : t \in [a, \infty),\ b_1(t) \le u_1 \le c_1(t)\}. \tag{2.25}$$

The boundary of V consists of three mutually disjoint parts V_a, V_b, and V_c:

$$\partial V = V_a \cup V_b \cup V_c, \tag{2.26}$$

where

$$\begin{aligned} V_a &:= \{(a, u_1) : b_1(a) < u_1 < c_1(a)\}, \\ V_b &:= \{(t, u_1) : t \in [a, \infty), u_1 = b_1(t)\}, \\ V_c &:= \{(t, u_1) : t \in [a, \infty), u_1 = c_1(t)\}. \end{aligned} \tag{2.27}$$

Define the mapping $\mathcal{R}_1 : I \to V_b \cup V_c$ as follows: let $\mathcal{R}_1(\tilde{u}_1)$ be the point of intersection of the line segment defined by its end points $(n^*, u_1(n^*, \tilde{u}_1))$, $(n^*+1, u_1(n^*+1, \tilde{u}_1))$ with $V_b \cup V_c$ (see Figure 1). The mapping $\mathcal{R}_1$ is obviously well defined on I and $\mathcal{R}_1(b_1(a)) = (a, b_1(a))$, $\mathcal{R}_1(c_1(a)) = (a, c_1(a))$.

Prove that the mapping $\mathcal{R}_1$ is continuous. The point $\mathcal{R}_1(\tilde{u}_1) = (t(\tilde{u}_1), u_1(\tilde{u}_1))$ lies either on V_b or on V_c. Without the loss of generality, consider the second case (the first one is analogical). The relevant boundary line segment for $t \in [n^*, n^*+1]$, which is a part of V_c, is described by (see (2.24))

$$u_1 = c(n^*) + (c(n^*+1) - c(n^*))(t - n^*), \tag{2.28}$$

and the line segment joining the points $(n^*, u_1(n^*, \tilde{u}_1))$, $(n^*+1, u_1(n^*+1, \tilde{u}_1))$ by the equation

$$u_1 = u_1(n^*, \tilde{u}_1) + (u_1(n^*+1, \tilde{u}_1) - u_1(n^*, \tilde{u}_1))(t - n^*), \quad t \in [n^*, n^*+1]. \tag{2.29}$$

The coordinates of the point $\mathcal{R}_1(\tilde{u}_1) = (t(\tilde{u}_1), u_1(\tilde{u}_1))$, which is the intersection of both these line segments, can be obtained as the solution of the system consisting of (2.28) and (2.29). Solving this system with respect to t and u_1, we get

$$t(\tilde{u}_1) = n^* + \frac{u_1(n^*, \tilde{u}_1) - c_1(n^*)}{c_1(n^*+1) - u_1(n^*+1, \tilde{u}_1) + u_1(n^*, \tilde{u}_1) - c_1(n^*)}, \tag{2.30}$$

$$u_1(\tilde{u}_1) = c_1(n^*) + \frac{(u_1(n^*, \tilde{u}_1) - c_1(n^*))(c_1(n^*+1) - c_1(n^*))}{c_1(n^*+1) - u_1(n^*+1, \tilde{u}_1) + u_1(n^*, \tilde{u}_1) - c_1(n^*)}. \tag{2.31}$$

Let $\{v_k\}_{k=1}^{\infty}$ be any sequence with $v_k \in I$ such that $v_k \to \tilde{u}_1$. We will show that $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$. Because of the continuity of the functions F_i, $i = 1, \ldots, m$,

$$u_1(n, v_k) \to u_1(n, \tilde{u}_1) \quad \text{for every fixed } n \in \mathbb{Z}_a^{\infty}. \tag{2.32}$$

We have to consider two cases:

(I) $(n^*, u(n^*, \tilde{u}_1)) \in \Omega(n^*)$, that is, $b_1(n^*) < u_1(n^*, \tilde{u}_1) < c_1(n^*)$,

(II) $(n^*, u(n^*, \tilde{u}_1)) \in \partial\Omega(n^*)$, that is, $u_1(n^*, \tilde{u}_1) = c_1(n^*)$.

Recall that (due to our agreement) in both cases $u_1(n^*+1, \tilde{u}_1) > c_1(n^*+1)$.

(I) In this case also $u_1(n^*, v_k) < c_1(n^*)$ and $u_1(n^*+1, v_k) > c_1(n^*+1)$ for k sufficiently large. That means that the leaving value $n^*(v_k)$ is the same as n^* given by $\tilde{u}_1$ and thus the point $\mathcal{R}_1(v_k) = (t(v_k), u_1(v_k))$ is given by

$$t(v_k) = n^* + \frac{u_1(n^*, v_k) - c_1(n^*)}{c_1(n^*+1) - u_1(n^*+1, v_k) + u_1(n^*, v_k) - c_1(n^*)}, \tag{2.33}$$

$$u_1(v_k) = c_1(n^*) + \frac{(u_1(n^*, v_k) - c_1(n^*))(c_1(n^*+1) - c_1(n^*))}{c_1(n^*+1) - u_1(n^*+1, v_k) + u_1(n^*, v_k) - c_1(n^*)}. \tag{2.34}$$

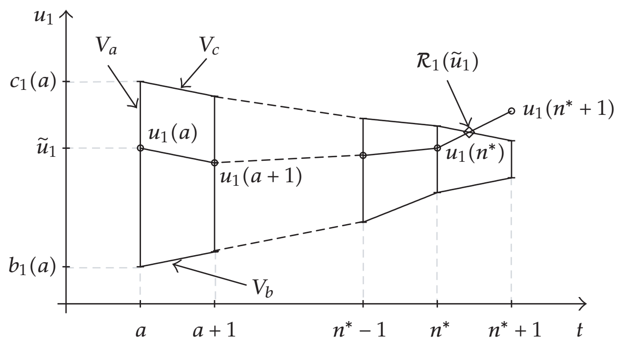

Figure 1: Construction of the mapping $\mathcal{R}_1$.

The desired convergence $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$ is implied by equations (2.30) to (2.34).

(II) Suppose $n^* = a$. Then $\tilde{u}_1 = c_1(a)$, $v_k = u_1(a, v_k) < c_1(a)$ for all k and as $k \to \infty$, $u_1(a+1, v_k) > c_1(a+1)$. A minor edit of the text in the case (I) proof provides the continuity proof.

Suppose $n^* > a$. In this case there can be $u_1(n^*, v_k) \le c_1(n^*)$ for some members of the sequence $\{v_k\}$ and $u_1(n^*, v_k) > c_1(n^*)$ for the others. Without the loss of generality, we can suppose that $\{v_k\}$ splits into two infinite subsequences $\{v_{q_k}\}$ and $\{v_{r_k}\}$ such that

$$u_1(n^*, v_{q_k}) \le c_1(n^*), \qquad u_1(n^*+1, v_{q_k}) > c_1(n^*+1)$$
$$u_1(n^*, v_{r_k}) > c_1(n^*). \tag{2.35}$$

For the subsequence $\{v_{q_k}\}$, the text of the proof of (I) can be subjected to a minor edit to provide the proof of continuity. As for the subsequence $\{v_{r_k}\}$, the leaving value $n^*(v_{r_k})$ is different from n^* given by $\tilde{u}_1$ because $(n^*, u_1(n^*, v_{r_k}))$ is already out of $\overline{\Omega}$. For k sufficiently large,

$$n^*(v_{r_k}) = n^* - 1 \tag{2.36}$$

because $u_1(n^*-1, \tilde{u}_1) < c_1(n^*-1)$ and thus, as $k \to \infty$, $u_1(n^*-1, v_{r_k}) < c_1(n^*-1)$.

Hence, the value of the mapping $\mathcal{R}_1$ for v_{r_k} is (in (2.33), (2.34) we replace n^* by n^*-1)

$$t(v_{r_k}) = n^* - 1 + \frac{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}{c_1(n^*) - u_1(n^*, v_{r_k}) + u_1(n^*-1, v_{r_k}) - c_1(n^*-1)},$$
$$u_1(v_{r_k}) = c(n^*-1) + \frac{(u_1(n^*-1, v_{r_k}) - c_1(n^*-1))(c_1(n^*) - c_1(n^*-1))}{c_1(n^*) - u_1(n^*, v_{r_k}) + u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}. \tag{2.37}$$

Due to (2.32), $u_1(n^*, v_{r_k}) \to u_1(n^*, \tilde{u}_1) = c_1(n^*)$ and thus

$$
\begin{gathered}
t(v_{r_k}) \longrightarrow n^* - 1 + \frac{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)}{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)} = n^*, \\
u_1(v_{r_k}) \longrightarrow c(n^*-1) + \frac{(u_1(n^*-1, v_{r_k}) - c_1(n^*-1))(c_1(n^*) - c_1(n^*-1))}{u_1(n^*-1, v_{r_k}) - c_1(n^*-1)} = c_1(n^*), \\
\mathcal{R}_1(v_{r_k}) = (t(v_{r_k}), u_1(v_{r_k})) \longrightarrow (n^*, c_1(n^*)) = \mathcal{R}_1(\tilde{u}_1).
\end{gathered}
\tag{2.38}
$$

We have shown that $\mathcal{R}_1(v_{q_k}) \to \mathcal{R}_1(\tilde{u}_1)$ and $\mathcal{R}_1(v_{r_k}) \to \mathcal{R}_1(\tilde{u}_1)$ and thus $\mathcal{R}_1(v_k) \to \mathcal{R}_1(\tilde{u}_1)$.

Auxiliary Mapping $\mathcal{R}_2$

Define $\mathcal{R}_2 : V_b \cup V_c \to \{b_1(a), c_1(a)\}$ as

$$
\mathcal{R}_2(P) = \begin{cases} b_1(a) & \text{if } P \in V_b, \\ c_1(a) & \text{if } P \in V_c. \end{cases}
\tag{2.39}
$$

The mapping $\mathcal{R}_2$ is obviously continuous.

Resulting Mapping $\mathcal{R}$ and Its Properties

Define $\mathcal{R} := \mathcal{R}_2 \circ \mathcal{R}_1$. Due to construction we have

$$
\mathcal{R}(b_1(a)) = b_1(a), \; \mathcal{R}(c_1(a)) = c_1(a),
\tag{2.40}
$$

and $\mathcal{R}(I) = \partial I$. The mapping $\mathcal{R}$ is continuous because of the continuity of the two mappings $\mathcal{R}_1$ and $\mathcal{R}_2$. Hence, it is the sought retraction of I onto ∂I. But such a retraction cannot exist and thus we get a contradiction and the proof is complete. □

3. Application of Theorem 2.1 to the Delayed Discrete Equation

Now, let us return to the original delayed discrete equation (1.2), that is,

$$
\Delta v(n) = f(n, v(n), v(n-1), \ldots, v(n-k)).
\tag{3.1}
$$

As it was said in Section 1, this equation will be transformed to a system of $k+1$ first-order discrete equations. Then we will apply Theorem 2.1 to this system and prove that under certain conditions there exists a solution of delayed equation (1.2) that stays in the prescribed domain. In the end, we will study a special case of (1.2).

3.1. *Transformation of* (1.2) *to the System of First-Order Equations*

We will proceed in accordance with the well-known scheme similarly as when constructing the system of first-order differential equations from a differential equation of a higher order. Put

$$
\begin{aligned}
u_1(n) &:= v(n),\\
u_2(n) &:= v(n-1),\\
&\cdots\\
u_{k+1}(n) &:= v(n-k),
\end{aligned}
\tag{3.2}
$$

where $u_1, u_2, \ldots, u_{k+1}$ are new unknown functions. From (1.2) we get $\Delta u_1(n) = f(n, u_1(n), u_2(n), \ldots, u_{k+1}(n))$. Obviously $u_2(n+1) = u_1(n), \ldots, u_{k+1}(n+1) = u_k(n)$. Rewriting these equalities in terms of differences, we have $\Delta u_2(n) = u_1(n) - u_2(n), \ldots, \Delta u_{k+1}(n) = u_k(n) - u_{k+1}(n)$. Altogether, we get the system

$$
\begin{aligned}
\Delta u_1(n) &= f(n, u_1(n), \ldots, u_{k+1}(n)),\\
\Delta u_2(n) &= u_1(n) - u_2(n),\\
&\cdots\\
\Delta u_{k+1}(n) &= u_k(n) - u_{k+1}(n)
\end{aligned}
\tag{3.3}
$$

which is equivalent to (1.2).

3.2. *The Existence Theorem for the Delayed Equation* (1.2) *(Solution of Problem 1)*

The following theorem is a consequence of Theorem 2.1. In fact, this theorem has been already proved in [12]. There, the proof is based upon a modification of the retract method for delayed equations. Our method (rearranging a delayed equation to a system of first-order equations) is, by its principle, more general than that used in [12].

Theorem 3.1. *Let $b(n)$, $c(n)$, $b(n) < c(n)$, be real functions defined on $\mathbb{Z}_{a-k}^{\infty}$. Further, let $f : \mathbb{Z}_a^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function and let the inequalities*

$$b(n) + f(n, b(n), v_2, \ldots, v_{k+1}) < b(n+1), \tag{3.4}$$

$$c(n) + f(n, c(n), v_2, \ldots, v_{k+1}) > c(n+1) \tag{3.5}$$

hold for every $n \in \mathbb{Z}_a^{\infty}$ and every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1. \tag{3.6}$$

Then there exists a solution $v = v^*(n)$ *of* (1.2) *satisfying the inequalities*

$$b(n) < v^*(n) < c(n) \tag{3.7}$$

for every $n \in \mathbb{Z}_{a-k}^{\infty}$.

Proof. We have shown that (1.2) is equivalent to system (3.3) which can be seen as a special case of system (2.1) with $m = k+1$ and $F = (F_1, \dots F_{k+1})$ where

$$\begin{gathered} F_1(n, u_1, \dots, u_{k+1}) := f(n, u_1, \dots, u_{k+1}), \\ F_2(n, u_1, \dots, u_{k+1}) := u_1 - u_2, \\ \cdots \\ F_k(n, u_1, \dots, u_{k+1}) := u_{k-1} - u_k, \\ F_{k+1}(n, u_1, \dots, u_{k+1}) := u_k - u_{k+1}. \end{gathered} \tag{3.8}$$

Define the polyfacial set Ω as

$$\Omega := \{(n, u) : n \in \mathbb{Z}_a^{\infty},\ b_i(n) < u_i < c_i(n),\ i = 1, \dots, k+1\} \tag{3.9}$$

with

$$b_i(n) := b(n - i + 1), \qquad c_i(n) := c(n - i + 1), \quad i = 1, \dots, k+1. \tag{3.10}$$

We will show that for system (3.3) and the set Ω, all the assumptions of Theorem 2.1 are satisfied.

As the function f is supposed to be continuous, the mapping F is continuous, too. Put the index j from Theorem 2.1, characterizing the points of egress, equal to 1. We will verify that the set Ω is of Liapunov-type with respect to the ith variable for any $i = 2, \dots, k+1$, that is, (see (2.8)) that for every $(n, u) \in \Omega$

$$b_i(n+1) < u_i + F_i(n, u) < c_i(n+1) \quad \text{for } i = 2, \dots, k+1. \tag{3.11}$$

First, we compute

$$u_i + F_i(n, u) = u_i + u_{i-1} - u_i = u_{i-1} \quad \text{for } i = 2, \dots, k+1. \tag{3.12}$$

Thus we have to show that for $i = 2, \dots, k+1$

$$b_i(n+1) < u_{i-1} < c_i(n+1). \tag{3.13}$$

Because $(n, u) \in \Omega$, then $b_p(n) < u_p < c_p(n)$ for any $p \in \{1, \dots, k+1\}$, and therefore

$$b_{i-1}(n) < u_{i-1} < c_{i-1}(n) \quad \text{for } i = 2, \dots, k+1. \tag{3.14}$$

But, by (3.10), we have

$$b_{i-1}(n) = b(n-i+1+1) = b(n-i+2), \tag{3.15}$$

meanwhile

$$b_i(n+1) = b(n+1-i+1) = b(n-i+2), \tag{3.16}$$

and thus $b_{i-1}(n) = b_i(n+1)$. Analogously we get that $c_{i-1}(n) = c_i(n+1)$. Thus inequalities (3.11) are fulfilled.

Further we will show that all the boundary points $M \in \Omega_B^1 \cup \Omega_C^1$ are points of strict egress for the set Ω with respect to system (3.3). According to (2.11), we have to show that if $u_1 = b_1(n)$ and $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$, then

$$b_1(n) + F_1(n,u) < b_1(n+1), \tag{3.17}$$

that is,

$$b_1(n) + f(n, b_1(n), u_2, \dots, u_{k+1}) < b_1(n+1). \tag{3.18}$$

Notice that the condition $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$ is equivalent with condition $b(n-i+1) < u_i < c(n-i+1)$ (see (3.10)). Looking at the supposed inequality (3.4) and realizing that $b_1(n) = b(n)$ and $b_1(n+1) = b(n+1)$, we can see that inequality (3.18) is fulfilled.

Analogously, according to (2.12), we have to prove that for $u_1 = c_1(n)$ and $b_i(n) < u_i < c_i(n)$ for $i = 2, \dots, k+1$ the inequality

$$c_1(n) + F_1(n,u) > c_1(n+1), \tag{3.19}$$

that is,

$$c_1(n) + f(n, c_1(n), u_2, \dots, u_{k+1}) > c_1(n+1) \tag{3.20}$$

holds.

Again, considering (3.5) and the fact that $c_1(n) = c(n)$ and $c_1(n+1) = c(n+1)$, we can see that this inequality really holds.

Thus, by the assertion of Theorem 2.1, there exists a solution $u = u^*(n)$ of system (3.3) such that for every $n \in \mathbb{Z}_a^\infty$

$$b_i(n) < u_i^*(n) < c_i(n) \quad \text{for } i = 1, \dots, k+1. \tag{3.21}$$

In our case, $v = v^*(n) = u_1^*(n)$ is the solution of the original equation (1.2). Further, $b_1(n) = b(n)$ and $c_1(n) = c(n)$, and thus the existence of a solution of the delayed equation (1.2) such that inequalities (3.7) are satisfied is guaranteed. □

3.3. Asymptotic Solution Estimates for Delayed Difference Equations

Let us suppose that two functions $u, w : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ are given such that

$$u(n) < w(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}, \tag{3.22}$$

$$\Delta u(n) \geq f(n, u(n), u(n-1), \ldots, u(n-k)), \quad n \in \mathbb{Z}_{a}^{\infty}, \tag{3.23}$$

$$\Delta w(n) \leq f(n, w(n), w(n-1), \ldots, w(n-k)), \quad n \in \mathbb{Z}_{a}^{\infty}. \tag{3.24}$$

Consider the problem of whether there exists a solution $v = v^*(n)$, $n \in \mathbb{Z}_{a-k}^{\infty}$ of (1.2) such that

$$u(n) < v^*(n) < w(n), \quad n \in \mathbb{Z}_{a-k}^{\infty}. \tag{3.25}$$

The following corollary of Theorem 3.1 presents sufficient conditions for the existence of a solution of this problem.

Corollary 3.2. *Let functions $u, w : \mathbb{Z}_{a-k}^{\infty} \to \mathbb{R}$ satisfy inequalities* (3.22)–(3.24). *Let $f : \mathbb{Z}_{a}^{\infty} \times \mathbb{R}^{k+1} \to \mathbb{R}$ be a continuous function such that*

$$f(n, u(n), y_2, \ldots, y_{k+1}) > f(n, u(n), z_2, \ldots, z_{k+1}), \tag{3.26}$$

$$f(n, w(n), y_2, \ldots, y_{k+1}) > f(n, w(n), z_2, \ldots, z_{k+1}) \tag{3.27}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $y_2, \ldots, y_{k+1}, z_2, \ldots, z_{k+1} \in \mathbb{R}$ such that

$$y_i < z_i, \quad i = 2, \ldots, k+1. \tag{3.28}$$

Then there exists a solution $v = v^*(n)$ of (1.2) *satisfying inequalities* (3.25) *for every $n \in \mathbb{Z}_{a-k}^{\infty}$.*

Proof. This assertion is an easy consequence of Theorem 3.1.

Put $b(n) := u(n)$, $c(n) := w(n)$. Considering inequalities (3.23) and (3.26), we can see that

$$\Delta u(n) > f(n, u(n), v_2, \ldots, v_{k+1}) \tag{3.29}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $v_2, \ldots, v_{k+1}$ such that

$$b(n-i+1) < v_i < c(n-i+1), \quad i = 2, \ldots, k+1. \tag{3.30}$$

Similarly,

$$\Delta w(n) < f(n, w(n), v_2, \ldots, v_{k+1}) \tag{3.31}$$

for every $n \in \mathbb{Z}_{a}^{\infty}$ and every $b(n-i+1) < v_i < c(n-i+1)$, $i = 2, \ldots, k+1$.

Obviously, inequalities (3.29) and (3.31) are equivalent with inequalities (3.4) and (3.5), respectively. Thus, all the assumptions of Theorem 3.1 are satisfied and there exists a solution $v = v^*(n)$ of (1.2) satisfying inequalities (3.25) for every $n \in \mathbb{Z}_{a-k}^{\infty}$. $\square$

Example 3.3. Consider the equation

$$\Delta v(n) = v^2(n) - v(n-1) \tag{3.32}$$

for $n \in \mathbb{Z}_3^{\infty}$ which is a second-order delayed discrete equation with delay $k = 1$. We will show that there exists a solution $v = v^*(n)$ of (3.32) that satisfies the inequalities

$$1 < v^*(n) < n \tag{3.33}$$

for $n \in \mathbb{Z}_2^{\infty}$.

We will prove that for the functions

$$u(n) := 1, \quad w(n) := n, \quad f(n, v_1, v_2) := v_1^2 - v_2 \tag{3.34}$$

all the assumptions of Corollary 3.2 are satisfied. Inequality (3.22) is obviously fulfilled for $n \in \mathbb{Z}_2^{\infty}$. Inequality (3.23) can be also proved very easily:

$$\Delta u(n) = 0, \qquad f(n, u(n), u(n-1)) = 1^2 - 1 = 0, \tag{3.35}$$

and thus for every $n \in \mathbb{Z}_3^{\infty}$, $\Delta u(n) \geq f(n, u(n), u(n-1))$.

As for inequality (3.24), we get

$$\Delta w(n) = 1, \qquad f(n, w(n), w(n-1)) = n^2 - n + 1 \tag{3.36}$$

and thus $\Delta w(n) \leq f(n, w(n), w(n-1))$ for $n \in \mathbb{Z}_3^{\infty}$.

Finally, the functions

$$f(n, u(n), v_2) = 1 - v_2, \qquad f(n, w(n), v_2) = n^2 - v_2 \tag{3.37}$$

are decreasing with respect to v_2. Therefore, conditions (3.26) and (3.27) are satisfied, too. Hence, due to Corollary 3.2, there exists a solution of (3.32) satisfying (3.33).

3.4. Positive Solutions of a Linear Equation with a Single Delay

We will apply the result of Theorem 3.1 to the investigation of a simple linear difference equation of the $(k+1)$-st order with only one delay, namely, the equation

$$\Delta v(n) = -p(n)v(n-k), \tag{3.38}$$

where, again, $n \in \mathbb{Z}_a^{\infty}$ is the independent variable and $k \in \mathbb{N}$, $k \geq 1$, is the fixed delay. The function $p : \mathbb{Z}_a^{\infty} \to \mathbb{R}$ is assumed to be positive. Our goal is to give sharp sufficient conditions for the existence of positive solutions. The existence of such solutions is very often substantial for a concrete model considered. For example, in biology, when a model of population dynamics is described by an equation, the positivity of a solution may mean that the studied biological species can survive in the supposed environment.

For its simple form, (3.38) often serves for testing new results and is very frequently investigated. It was analyzed, for example, in papers [10, 11, 26]. A sharp result on existence of positive solutions given in [26] is proved by a comparison method [8, 9]. Here we will use Theorem 3.1 to generalize this result.

For the purposes of this section, define the expression $\ln_q t$, where $q \in \mathbb{N}$, as

$$\begin{gathered} \ln_q t := \ln\left(\ln_{q-1} t\right) \\ \ln_0 t := t. \end{gathered} \tag{3.39}$$

We will write only $\ln t$ instead of $\ln_1 t$. Further, for a fixed integer $\ell \geq 0$ define auxiliary functions

$$\begin{gathered} \mu_\ell(n) := \frac{1}{8n^2} + \frac{1}{8(n\ln n)^2} + \cdots + \frac{1}{8(n\ln n \cdots \ln_\ell n)^2}, \\ p_\ell(n) := \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right), \end{gathered} \tag{3.40}$$

$$\nu_\ell(n) := \left(\frac{k}{k+1}\right)^n \cdot \sqrt{n\ln n \ln_2 n \cdots \ln_\ell n}. \tag{3.41}$$

In [26], it was proved that if $p(n)$ in (3.38) is a positive function bounded by $p_\ell(n)$ for some $\ell \geq 0$, then there exists a positive solution of (3.38) bounded by the function $\nu_\ell(n)$ for n sufficiently large. Since $\lim_{n\to\infty}\nu_\ell(n) = 0$, such solution will vanish for $n \to \infty$. Here we show that (3.38) has a positive solution bounded by $\nu_\ell(n)$ even if the coefficient $p(n)$ satisfies a less restrictive inequality (see inequality (3.58) below). The proof of this statement will be based on the following four lemmas. The symbols "o" and "O" stand for the Landau order symbols and are used for $n \to \infty$.

Lemma 3.4. *The formula*

$$\ln(y - z) = \ln y - \sum_{i=1}^{\infty} \frac{z^i}{iy^i} \tag{3.42}$$

holds for any numbers $y, z \in \mathbb{R}$ such that $y > 0$ and $|z| < y$.

Proof. The assertion is a simple consequence of the well-known Maclaurin expansion

$$\ln(1 - x) = -\sum_{i=1}^{\infty} \frac{1}{i} x^i \quad \text{for } -1 \leq x < 1. \tag{3.43}$$

As $\ln(y-z)-\ln y=\ln(1-z/y)$, substituting $x=z/y$ we get

$$\ln(y-z)-\ln y=-\sum_{i=1}^{\infty}\frac{z^i}{iy^i}\quad \text{for } -y\le z<y \tag{3.44}$$

and adding $\ln y$ to both sides of this equality, we get (3.42). □

Lemma 3.5. *For fixed $r\in\mathbb{R}\setminus\{0\}$ and fixed $q\in\mathbb{N}$, the asymptotic representation*

$$\begin{aligned}\ln_q(n-r)=\ln_q n&-\frac{r}{n\ln n\cdots\ln_{q-1}n}-\frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n}\\&-\frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_{q-1}n}-\cdots-\frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2}\\&-\frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}\end{aligned} \tag{3.45}$$

holds for $n\to\infty$.

Proof. We will prove relation (3.45) by induction with respect to q. For $q=1$, (3.45) reduces to

$$\ln(n-r)=\ln n-\frac{r}{n}-\frac{r^2}{2n^2}-\frac{r^3(1+o(1))}{3n^3} \tag{3.46}$$

which holds due to Lemma 3.4. Suppose that relation (3.45) holds for some q. We can write $\ln_q(n-r)=y-z$ with $y=\ln_q n$ and

$$\begin{aligned}z=&\frac{r}{n\ln n\cdots\ln_{q-1}n}+\frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n}+\frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_{q-1}n}\\&+\cdots+\frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2}+\frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}.\end{aligned} \tag{3.47}$$

Now we will show that (3.45) holds for $q+1$. Notice that in our case, the condition $|z|<y$ from Lemma 3.4 is fulfilled for n sufficiently large because $z\to 0$ for $n\to\infty$, meanwhile $y\to\infty$ for $n\to\infty$. Thus we are justified to use Lemma 3.4 and doing so, we get

$$
\begin{aligned}
\ln_{q+1}(n-r) &= \ln\left(\ln_q(n-r)\right)\\
&= \ln(y-z) = \ln y - \frac{1}{y}\ z - \frac{1}{2y^2}\ z^2 - \cdots\\
&= \ln(\ln_q n) - \frac{1}{\ln_q n}\cdot\left(\frac{r}{n\ln n\cdots\ln_{q-1}n} + \frac{r^2}{2n^2\ln n\cdots\ln_{q-1}n} + \cdots\right.\\
&\qquad\left. + \frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2} + \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_{q-1}n}\right)\\
&\quad - \frac{1}{2\left(\ln_q n\right)^2}\cdot\left(\frac{r^2}{\left(n\ln n\cdots\ln_{q-1}n\right)^2} + O\left(\frac{1}{n^3\left(\ln n\cdots\ln_{q-1}n\right)^2}\right)\right)\\
&\quad + O\left(\frac{1}{\left(n\ln n\cdots\ln_q n\right)^3}\right)\\
&= \ln_{q+1}n - \frac{r}{n\ln n\cdots\ln_q n} - \frac{r^2}{2n^2\ln n\cdots\ln_q n} - \frac{r^2}{2(n\ln n)^2\ln_2 n\cdots\ln_q n}\\
&\quad - \cdots - \frac{r^2}{2\left(n\ln n\cdots\ln_q n\right)^2} - \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_q n}.
\end{aligned}
\tag{3.48}
$$

Thus, formula (3.45) holds for $q+1$, too, which ends the proof. □

Lemma 3.6. *For fixed $r \in \mathbb{R}\setminus\{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representations*

$$
\begin{aligned}
\sqrt{\frac{\ln_q(n-r)}{\ln_q n}} &= 1 - \frac{r}{2n\ln n\cdots\ln_q n} - \frac{r^2}{4n^2\ln n\cdots\ln_q n} - \frac{r^2}{4(n\ln n)^2\ln_2 n\cdots\ln_q n} - \cdots\\
&\quad - \frac{r^2}{4\left(n\ln n\cdots\ln_{q-1}n\right)^2\ln_q n} - \frac{r^2}{8\left(n\ln n\cdots\ln_q n\right)^2} - \frac{r^3(1+o(1))}{6n^3\ln n\cdots\ln_q n},
\end{aligned}
\tag{3.49}
$$

$$
\sqrt{\frac{n-r}{n}} = 1 - \frac{r}{2n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right)
\tag{3.50}
$$

hold for $n \to \infty$.

Proof. Both these relations are simple consequences of the asymptotic formula

$$
\sqrt{1-x} = 1 - \frac{1}{2}\ x - \frac{1}{8}\ x^2 - \frac{1}{16}\ x^3 + o\left(x^3\right) \quad \text{for } x \longrightarrow 0
\tag{3.51}
$$

and of Lemma 3.5 (for formula (3.49)). In the case of relation (3.49), we put

$$
x = \frac{r}{n\ln n\cdots\ln_q n} + \frac{r^2}{2n^2\ln n\cdots\ln_q n} + \cdots + \frac{r^2}{2\left(n\ln n\cdots\ln_{q-1}n\right)^2\ln_q n} + \frac{r^3(1+o(1))}{3n^3\ln n\cdots\ln_q n}
\tag{3.52}
$$

and in the case of relation (3.50), we put $x = r/n$. □

Lemma 3.7. *For fixed $r \in \mathbb{R} \setminus \{0\}$ and fixed $q \in \mathbb{N}$, the asymptotic representation*

$$\begin{aligned}
&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_q(n-r)}{\ln_q n}} \\
&\quad = 1 - r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_q n}\right) - r^2\mu_q(n) - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right)
\end{aligned} \tag{3.53}$$

holds for $n \to \infty$.

Proof. We will prove relation (3.53) by induction with respect to q. For $q = 1$, (3.53) reduces to

$$\begin{aligned}
\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}} &= 1 - r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right) - r^2\mu_1(n) - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right) \\
&= 1 - r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right) - r^2\left(\frac{1}{8n^2}+\frac{1}{8(n\ln n)^2}\right) - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3.54}$$

On the other hand, using Lemma 3.6, we get

$$\begin{aligned}
&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}} \\
&\quad = \left(1 - \frac{r}{2n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right)\right) \\
&\qquad \times \left(1 - \frac{r}{2n\ln n} - \frac{r^2}{4n^2\ln n} - \frac{r^2}{8(n\ln n)^2} - \frac{r^3(1+o(1))}{6n^3\ln n}\right) \\
&\quad = 1 - \frac{r}{2n\ln n} - \frac{r^2}{4n^2\ln n} - \frac{r^2}{8(n\ln n)^2} - \frac{r}{2n} + \frac{r^2}{4n^2\ln n} - \frac{r^2}{8n^2} - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right) \\
&\quad = 1 - r\left(\frac{1}{2n}+\frac{1}{2n\ln n}\right) - r^2\left(\frac{1}{8n^2}+\frac{1}{8(n\ln n)^2}\right) - \frac{r^3}{16n^3} + o\left(\frac{1}{n^3}\right).
\end{aligned} \tag{3.55}$$

Thus, for $q = 1$, relation (3.53) holds. Now suppose that (3.53) holds for some q and prove that it holds for $q + 1$. In the following calculations, we use Lemma 3.6 and we skip some tedious expressions handling.

$$\begin{aligned}
&\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_{q+1}(n-r)}{\ln_{q+1}n}}\\
&\quad=\sqrt{\frac{(n-r)}{n}\frac{\ln(n-r)}{\ln n}\cdots\frac{\ln_{q}(n-r)}{\ln_{q}n}}\cdot\sqrt{\frac{\ln_{q+1}(n-r)}{\ln_{q+1}n}}\\
&\quad=\left(1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_q n}\right)-r^2\mu_q(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right)\right)\\
&\qquad\times\left(1-\frac{r}{2n\ln n\cdots\ln_{q+1}n}-\frac{r^2}{4n^2\ln n\cdots\ln_{q+1}n}-\cdots\right.\\
&\qquad\qquad\left.-\frac{r^2}{4(n\ln n\cdots\ln_q n)^2\ln_{q+1}n}-\frac{r^2}{8(n\ln n\cdots\ln_{q+1}n)^2}+o\left(\frac{1}{n^3}\right)\right)\\
&\quad=1-r\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{q+1}n}\right)-r^2\mu_{q+1}(n)-\frac{r^3}{16n^3}+o\left(\frac{1}{n^3}\right).
\end{aligned}\tag{3.56}$$

We can see that formula (3.53) holds for $q+1$, too, which ends the proof. □

Now we are ready to prove that there exists a bounded positive solution of (3.38). Remind that functions p_ℓ and v_ℓ were defined by (3.40) and (3.41), respectively.

Theorem 3.8. *Let $\omega:\mathbb{Z}_a^\infty\to\mathbb{R}$ satisfy the inequality*

$$|\omega(n)|\le\varepsilon\left(\frac{k}{k+1}\right)^k\cdot\frac{k(2k^2+k-1)}{16n^3(k+1)},\quad n\in\mathbb{Z}_a^\infty,\tag{3.57}$$

for a fixed $\varepsilon\in(0,1)$. Suppose that there exists an integer $\ell\ge 0$ such that the function p satisfies the inequalities

$$0<p(n)\le p_\ell(n)+\omega(n)\tag{3.58}$$

for every $n\in\mathbb{Z}_a^\infty$. Then there exists a solution $v=v^*(n)$, $n\in\mathbb{Z}_{a-k}^\infty$ of (3.38) such that for n sufficiently large the inequalities

$$0<v^*(n)<v_\ell(n)\tag{3.59}$$

hold.

Proof. Show that all the assumptions of Theorem 3.1 are fulfilled. For (3.38), $f(n,v_1,\ldots,v_{k+1})=-p(n)v_{k+1}$. This is a continuous function. Put

$$b(n):=0,\qquad c(n):=v_\ell(n).\tag{3.60}$$

We have to prove that for every $v_2, \ldots, v_{k+1}$ such that $b(n-i+1) < v_i < c(n-i+1)$, $i = 2, \ldots, k+1$, the inequalities (3.4) and (3.5) hold for n sufficiently large. Start with (3.4). That gives that for $b(n-k) < v_{k+1} < c(n-k)$, it has to be

$$0 - p(n) \cdot v_{k+1} < 0. \tag{3.61}$$

This certainly holds, because the function p is positive and so is v_{k+1}.

Next, according to (3.5), we have to prove that

$$\nu_\ell(n) - p(n)v_{k+1} > \nu_\ell(n+1) \tag{3.62}$$

which is equivalent to the inequality

$$-p(n)v_{k+1} > \nu_\ell(n+1) - \nu_\ell(n). \tag{3.63}$$

Denote the left-hand side of (3.63) as $L_{(3.63)}$. As $v_{k+1} < c(n-k) = \nu_\ell(n-k)$ and as by (3.40), (3.58), and (3.57)

$$p(n) \le \left(\frac{k}{k+1}\right)^k \cdot \left(\frac{1}{k+1} + k\mu_\ell(n)\right) + \varepsilon\left(\frac{k}{k+1}\right)^k \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}, \tag{3.64}$$

we have

$$\begin{aligned} L_{(3.63)} &> -\left(\frac{k}{k+1}\right)^k \left(\frac{1}{k+1} + k\mu_\ell(n) + \varepsilon \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}\right) \\ &\quad \times \left(\frac{k}{k+1}\right)^{n-k} \sqrt{(n-k)\ln(n-k)\cdots\ln_\ell(n-k)} \\ &= -\left(\frac{k}{k+1}\right)^n \left(\frac{1}{k+1} + k\mu_\ell(n) + \varepsilon \cdot \frac{k(2k^2+k-1)}{16n^3(k+1)}\right) \cdot \sqrt{(n-k)\ln(n-k)\cdots\ln_\ell(n-k)}. \end{aligned} \tag{3.65}$$

Further, we can easily see that

$$\nu_\ell(n+1) - \nu_\ell(n) = \left(\frac{k}{k+1}\right)^n \sqrt{n\ln n\cdots\ln_\ell n}\left(\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_\ell(n+1)}{\ln_\ell n}} - 1\right). \tag{3.66}$$

Thus, to prove (3.63), it suffices to show that for n sufficiently large,

$$
\begin{aligned}
&-\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k\left(2k^{2}+k-1\right)}{16n^{3}(k+1)}\right)\sqrt{\frac{(n-k)}{n}\frac{\ln(n-k)}{\ln n}\cdots\frac{\ln_{\ell}(n-k)}{\ln_{\ell}n}}\\
&\quad>\frac{k}{k+1}\sqrt{\frac{(n+1)}{n}\frac{\ln(n+1)}{\ln n}\cdots\frac{\ln_{\ell}(n+1)}{\ln_{\ell}n}}-1.
\end{aligned}
\tag{3.67}
$$

Denote the left-hand side of inequality (3.67) as $L_{(3.67)}$ and the right-hand side as $R_{(3.67)}$. Using Lemma 3.7 with $r=k$ and $q=\ell$, we can write

$$
\begin{aligned}
L_{(3.67)}&=-\left(\frac{1}{k+1}+k\mu_{\ell}(n)+\varepsilon\cdot\frac{k\left(2k^{2}+k-1\right)}{16n^{3}(k+1)}\right)\\
&\quad\times\left(1-k\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right)-k^{2}\mu_{\ell}(n)-\frac{k^{3}}{16n^{3}}+o\left(\frac{1}{n^{3}}\right)\right)\\
&=-\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right)\\
&\quad+\frac{k^{2}}{k+1}\mu_{\ell}(n)+\frac{k^{3}}{16n^{3}(k+1)}-k\mu_{\ell}(n)+\frac{k^{2}}{16n^{3}}-\varepsilon\cdot\frac{k\left(2k^{2}+k-1\right)}{16n^{3}(k+1)}+o\left(\frac{1}{n^{3}}\right)\\
&=-\frac{1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right)\\
&\quad-\frac{k}{k+1}\mu_{\ell}(n)+\frac{2k^{3}(1-\varepsilon)+k^{2}(1-\varepsilon)+k\varepsilon}{16n^{3}(k+1)}+o\left(\frac{1}{n^{3}}\right).
\end{aligned}
\tag{3.68}
$$

Using Lemma 3.7 with $r=-1$ and $q=\ell$, we get for $R_{(3.67)}$

$$
\begin{aligned}
R_{(3.67)}&=\frac{k}{k+1}\left(1+\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}-\mu_{\ell}(n)+\frac{1}{16n^{3}}+o\left(\frac{1}{n^{3}}\right)\right)-1\\
&=\frac{-1}{k+1}+\frac{k}{k+1}\left(\frac{1}{2n}+\frac{1}{2n\ln n}+\cdots+\frac{1}{2n\ln n\cdots\ln_{\ell}n}\right)\\
&\quad-\frac{k}{k+1}\cdot\mu_{\ell}(n)+\frac{k}{16n^{3}(k+1)}+o\left(\frac{1}{n^{3}}\right).
\end{aligned}
\tag{3.69}
$$

It is easy to see that the inequality (3.67) reduces to

$$
\frac{2k^{3}(1-\varepsilon)+k^{2}(1-\varepsilon)+k\varepsilon}{16n^{3}(k+1)}+o\left(\frac{1}{n^{3}}\right)>\frac{k}{16n^{3}(k+1)}+o\left(\frac{1}{n^{3}}\right).
\tag{3.70}
$$

This inequality is equivalent to

$$\frac{k\left(2k^2(1-\varepsilon)+k(1-\varepsilon)-(1-\varepsilon)\right)}{16n^3(k+1)}+o\left(\frac{1}{n^3}\right)>0. \tag{3.71}$$

The last inequality holds for n sufficiently large because $k \geq 1$ and $1-\varepsilon \in (0,1)$. We have proved that all the assumptions of Theorem 3.1 are fulfilled and hence there exists a solution of (3.38) satisfying conditions (3.59). □

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Research Article

Asymptotic Behavior of Solutions to Half-Linear q-Difference Equations

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We derive necessary and sufficient conditions for (some or all) positive solutions of the half-linear q-difference equation $D_q(\Phi(D_q y(t))) + p(t)\Phi(y(qt)) = 0$, $t \in \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$, to behave like q-regularly varying or q-rapidly varying or q-regularly bounded functions (that is, the functions y, for which a special limit behavior of $y(qt)/y(t)$ as $t \to \infty$ is prescribed). A thorough discussion on such an asymptotic behavior of solutions is provided. Related Kneser type criteria are presented.

1. Introduction

In this paper we recall and survey the theory of q-Karamata functions, that is, of the functions $y : q^{\mathbb{N}_0} \to (0, \infty)$, where $q^{\mathbb{N}_0} := \{q^k : k \in \mathbb{N}_0\}$ with $q > 1$, and for which some special limit behavior of $y(qt)/y(t)$ as $t \to \infty$ is prescribed, see [1–3]. This theory corresponds with the classical "continuous" theory of regular variation, see, for example, [4], but shows some special features (see Section 2), not known in the continuous case, which are due to the special structure of $q^{\mathbb{N}_0}$. The theory of q-Karamata functions provides a powerful tool, which we use in this paper to establish sufficient and necessary conditions for some or all positive solutions of the half-linear q-difference equation

$$D_q\big(\Phi\big(D_q y(t)\big)\big) + p(t)\Phi\big(y(qt)\big) = 0, \tag{1.1}$$

where $\Phi(u) = |u|^{\alpha-1} \operatorname{sgn} u$ with $\alpha > 1$, to behave like q-regularly varying or q-rapidly varying or q-regularly bounded functions. We stress that there is no sign condition on p. We also present Kneser type (non)oscillation criteria for (1.1), existing as well as new ones, which are somehow related to our asymptotic results.

The main results of this paper can be understood as a q-version of the continuous results for

$$\left(\Phi\left(y'(t)\right)\right)' + p(t)\Phi\left(y(t)\right) = 0 \tag{1.2}$$

from [5] (with noting that some substantial differences between the parallel results are revealed), or as a half-linear extension of the results for $D_q^2 y(t) + p(t)y(qt) = 0$ from [1]. In addition, we provide a thorough description of asymptotic behavior of solutions to (1.1) with respect to the limit behavior of $t^\alpha p(t)$ in the framework of q-Karamata theory. For an explanation why the q-Karamata theory and its applications are not included in a general theory of regular variation on measure chains see [6]. For more information on (1.2) see, for example, [7]. Many applications of the theory of regular variation in differential equations can be found, for example, in [8]. Linear q-difference equations were studied, for example, in [1, 9–11]; for related topics see, for example, [12, 13]. Finally note that the theory of q-calculus is very extensive with many aspects; some people speak bout different tongues of q-calculus. In our paper we follow essentially its "time-scale dialect".

2. Preliminaries

We start with recalling some basic facts about q-calculus. For material on this topic see [9, 12, 13]. See also [14] for the calculus on time-scales which somehow contains q-calculus. First note that some of the below concepts may appear to be described in a "nonclassical q-way", see, for example, our definition of q-integral versus original Jackson's definition [9, 12, 13], or the q-exponential function. But, working on the lattice $q^{\mathbb{N}_0}$ (which is a time-scale), we can introduce these concepts in an alternative and "easier" way (and, basically, we avoid some classical q-symbols). Our definitions, of course, naturally correspond with the original definitions. The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $D_q f(t) = [f(qt) - f(t)]/[(q-1)t]$. The q-integral $\int_a^b f(t)\mathrm{d}_q t$, $a, b \in q^{\mathbb{N}_0}$, of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ is defined by $\int_a^b f(t)\mathrm{d}_q t = (q-1)\sum_{t\in[a,b)\cap q^{\mathbb{N}_0}} tf(t)$ if $a < b$; $\int_a^b f(t)\mathrm{d}_q t = 0$ if $a = b$; $\int_a^b f(t)\mathrm{d}_q t = (1-q)\sum_{t\in[b,a)\cap q^{\mathbb{N}_0}} tf(t)$ if $a > b$. The improper q-integral is defined by $\int_a^\infty f(t)\mathrm{d}_q t = \lim_{b\to\infty}\int_a^b f(t)\mathrm{d}_q t$. We use the notation $[a]_q = (q^a - 1)/(q-1)$ for $a \in \mathbb{R}$. Note that $\lim_{q\to 1^+}[a]_q = a$. It holds that $D_q t^\vartheta = [\vartheta]_q t^{\vartheta-1}$. In view of the definition of $[a]_q$, it is natural to introduce the notation $[\infty]_q = \infty$, $[-\infty]_q = 1/(1-q)$. For $p \in \mathcal{R}$ (i.e., for $p : q^{\mathbb{N}_0} \to \mathbb{R}$ satisfying $1 + (q-1)tp(t) \neq 0$ for all $t \in q^{\mathbb{N}_0}$) we denote $e_p(t,s) = \prod_{u\in[s,t)\cap q^{\mathbb{N}_0}}[(q-1)up(u) + 1]$ for $s < t$, $e_p(t,s) = 1/e_p(s,t)$ for $s > t$, and $e_p(t,t) = 1$, where $s, t \in q^{\mathbb{N}_0}$. For $p \in \mathcal{R}$, $e(\cdot, a)$ is a solution of the IVP $D_q y = p(t)y$, $y(a) = 1$, $t \in q^{\mathbb{N}_0}$. If $s \in q^{\mathbb{N}_0}$ and $p \in \mathcal{R}^+$, where $\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + (q-1)tp(t) > 0 \text{ for all } t \in q^{\mathbb{N}_0}\}$. then $e_p(t,s) > 0$ for all $t \in q^{\mathbb{N}_0}$. If $p, r \in \mathcal{R}$, then $e_p(t,s)e_p(s,u) = e_p(t,u)$ and $e_p(t,s)e_r(t,s) = e_{p+r+t(q-1)pr}(t,s)$. Intervals having the subscript q denote the intervals in $q^{\mathbb{N}_0}$, for example, $[a,\infty)_q = \{a, aq, aq^2, \ldots\}$ with $a \in q^{\mathbb{N}_0}$.

Next we present auxiliary statements which play important roles in proving the main results. Define $F : (0,\infty) \to \mathbb{R}$ by $F(x) = \Phi(x/q - 1/q) - \Phi(1 - 1/x)$ and $h : (\Phi([-\infty]_q), \infty) \to \mathbb{R}$ by

$$h(x) = \frac{x}{1-q^{1-\alpha}}\left[1 - \left(1 + (q-1)\Phi^{-1}(x)\right)^{1-\alpha}\right]. \tag{2.1}$$

For $y : q^{\mathbb{N}_0} \to \mathbb{R} \setminus \{0\}$ define the operator $\mathcal{L}$ by

$$\mathcal{L}[y](t) = \Phi\left(\frac{y(q^2t)}{qy(qt)} - \frac{1}{q}\right) - \Phi\left(1 - \frac{y(t)}{y(qt)}\right). \tag{2.2}$$

We denote $\omega_q = ([(\alpha-1)/\alpha]_q)^{\alpha}$. Let β mean the conjugate number of α, that is, $1/\alpha + 1/\beta = 1$.

The following lemma lists some important properties of F, h, $\mathcal{L}$ and relations among them.

Lemma 2.1. (i) *The function F has the global minimum on $(0,\infty)$ at $q^{(\alpha-1)/\alpha}$ with*

$$F\left(q^{(\alpha-1)/\alpha}\right) = -\frac{\omega_q(q-1)^{\alpha}}{q^{\alpha-1}} \tag{2.3}$$

and $F(1) = 0 = F(q)$. Further, F is strictly decreasing on $(0, q^{(\alpha-1)/\alpha})$ and strictly increasing on $(q^{(\alpha-1)/\alpha}, \infty)$ with $\lim_{x\to 0+}F(x) = \infty$, $\lim_{t\to\infty}F(x) = \infty$.

(ii) *The graph of $x \mapsto h(x)$ is a parabola-like curve with the minimum at the origin. The graph of $x \mapsto h(x) + \gamma_\alpha$ touches the line $x \mapsto x$ at $x = \lambda_0 := ([(\alpha-1)/\alpha]_q)^{\alpha-1}$. The equation $h(\lambda) + \gamma = \lambda$ has*

(a) *no real roots if $\gamma > \omega_q/[\alpha-1]_q$,*

(b) *the only root λ_0 if $\gamma = \omega_q/[\alpha-1]_q$,*

(c) *two real roots λ_1, λ_2 with $0 < \lambda_1 < \lambda_0 < \lambda_2 < 1$ if $\gamma \in (0, \omega_q/[\alpha-1]_q)$,*

(d) *two real roots 0 and 1 if $\gamma = 0$,*

(e) *two real roots λ_1, λ_2 with $\lambda_1 < 0 < 1 < \lambda_2$ if $\gamma < 0$.*

(iii) *It holds that $F(q^{\vartheta_1}) = F(q^{\vartheta_2})$, where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i) + 1]$, $i = 1, 2$, with $\lambda_1 < \lambda_2$ being the real roots of the equation $\lambda = h(\lambda) + A$ with $A \in (-\infty, \omega_q/[\alpha-1]_q)$.*

(iv) *If $q \to 1+$, then $h(x) \to |x|^{\beta}$.*

(v) *For $\vartheta \in \mathbb{R}$ it hold that $\Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}} = \Phi([\vartheta]_q) - h(\Phi([\vartheta]_q))$.*

(vi) *For $\vartheta \in \mathbb{R}$ it hold that $F(q^{\vartheta}) = (q-1)^{\alpha}[1-\alpha]_q\Phi([\vartheta]_q)[1-\vartheta]_{q^{\alpha-1}}$.*

(vii) *For $y \neq 0$, (1.1) can be written as $\mathcal{L}[y](t) = -(q-1)^{\alpha}t^{\alpha}p(t)$.*

(viii) *If the $\lim_{t\to\infty}y(qt)/y(t)$ exists as a positive real number, then $\lim_{t\to\infty}\mathcal{L}[y](t) = \lim_{t\to\infty}F(y(qt)/y(t))$.*

Proof. We prove only (iii). The proofs of other statements are either easy or can be found in [3].

(iii) Let λ_1, λ_2 be the real roots of $\lambda = h(\lambda) + A$. We have $\lambda_i = \Phi([\vartheta_i]_q)$, $i = 1, 2$, and so, by virtue of identities (v) and (vi), we get $F(q^{\vartheta_1}) = (q-1)^{\alpha}[1-\alpha]_q(\lambda_1 - h(\lambda_1)) = (q-1)^{\alpha}[1-\alpha]_q A = (q-1)^{\alpha}[1-\alpha]_q(\lambda_2 - h(\lambda_2)) = F(q^{\vartheta_2})$. □

Next we define the basic concepts of q-Karamata theory. Note that the original definitions (see [1–3]) was more complicated; they were motivated by the classical continuous and the discrete (on the uniform lattices) theories. But soon it has turned out that simpler (and equivalent)definitions can be established. Also, there is no need to introduce the concept of normality, since every q-regularly varying or q-rapidly varying or q-regularly bounded function is automatically normalized. Such (and some other) simplifications are not possible in the original continuous theory or in the classical discrete theory; in q-calculus, they are practicable thanks to the special structure of $q^{\mathbb{N}_0}$, which is somehow natural for examining regularly varying behavior.

For $f : q^{\mathbb{N}_0} \to (0, \infty)$ denote

$$K_* = \liminf_{t \to \infty} \frac{f(qt)}{f(t)}, \qquad K^* = \limsup_{t \to \infty} \frac{f(qt)}{f(t)}, \qquad K = \lim_{t \to \infty} \frac{f(qt)}{f(t)}. \tag{2.4}$$

Definition 2.2. A function $f : q^{\mathbb{N}_0} \to (0, \infty)$ is said to be

(i) *q-regularly varying of index ϑ*, $\vartheta \in \mathbb{R}$, if $K = q^{\vartheta}$; we write $f \in \mathcal{RV}_q(\vartheta)$,

(ii) *q-slowly varying* if $K = 1$; we write $f \in \mathcal{SV}_q$,

(iii) *q-rapidly varying of index ∞* if $K = \infty$; we write $f \in \mathcal{RPV}_q(\infty)$,

(iv) *q-rapidly varying of index $-\infty$* if $K = 0$; we write $f \in \mathcal{RPV}_q(-\infty)$,

(v) *q-regularly bounded* if $0 < K_* \le K^* < \infty$; we write $f \in \mathcal{RB}_q$.

Clearly, $\mathcal{SV}_q = \mathcal{RV}_q(0)$. We have defined q-regular variation, q-rapid variation, and q-regular boundedness at infinity. If we consider a function $f : q^{\mathbb{Z}} \to (0, \infty)$, $q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$, then $f(t)$ is said to be q-regularly varying/q-rapidly varying/q-regularly bounded at zero if $f(1/t)$ is q-regularly varying/q-rapidly varying/q-regularly bounded at infinity. But it is apparent that it is sufficient to examine just the behavior at ∞.

Next we list some selected important properties of the above-defined functions. We define $\tau : [1, \infty) \to q^{\mathbb{N}_0}$ as $\tau(x) = \max\{s \in q^{\mathbb{N}_0} : s \le x\}$.

Proposition 2.3. (i) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow \lim_{t \to \infty} t D_q f(t)/f(t) = [\vartheta]_q$.

(ii) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t) = \varphi(t) e_{\psi}(t, 1)$, *where a positive* φ *satisfies* $\lim_{t \to \infty} \varphi(t) = C \in (0, \infty)$, $\lim_{t \to \infty} t\psi(t) = [\vartheta]_q$, $\psi \in \mathcal{R}^+$ *(w.l.o.g., φ can be replaced by C).*

(iii) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t) = t^{\vartheta} L(t)$, *where* $L \in \mathcal{SV}_q$.

(iv) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow f(t)/t^{\gamma}$ *is eventually increasing for each* $\gamma < \vartheta$ *and* $f(t)/t^{\eta}$ *is eventually decreasing for each* $\eta > \vartheta$.

(v) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow \lim_{t \to \infty} f(\tau(\lambda t))/f(t) = (\tau(\lambda))^{\vartheta}$ *for every* $\lambda \ge 1$.

(vi) $f \in \mathcal{RV}_q(\vartheta) \Leftrightarrow R : [1, \infty) \to (0, \infty)$ *defined by* $R(x) = f(\tau(x))(x/\tau(x))^{\vartheta}$ *for* $x \in [1, \infty)$ *is regularly varying of index* ϑ.

(vii) $f \in \mathcal{RV}_q(\vartheta) \Rightarrow \lim_{t \to \infty} \log f(t)/\log t = \vartheta$.

Proof. See [2]. □

Proposition 2.4. (i) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow \lim_{t\to\infty} tD_q f(t)/f(t) = [\pm\infty]_q$.

(ii) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow f(t) = \varphi(t)e_\psi(t,1)$, *where a positive* φ *satisfies* $\liminf_{t\to\infty}\varphi(qt)/\varphi(t) > 0$ *for index* ∞, $\limsup_{t\to\infty}\varphi(qt)/\varphi(t) < \infty$ *for index* $-\infty$, *and* $\lim_{t\to\infty} t\psi(t) = [\pm\infty]_q$, $\psi \in \mathcal{R}^+$ *(w.l.o.g.,* φ *can be replaced by* $C \in (0,\infty)$*).*

(iii) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow$ *for each* $\vartheta \in [0,\infty)$, $f(t)/t^\vartheta$ *is eventually increasing (towards* ∞*) for index* ∞ *and* $f(t)t^\vartheta$ *is eventually decreasing (towards 0) for index* $-\infty$.

(iv) $f \in \mathcal{RPV}_q(\pm\infty) \Leftrightarrow$ *for every* $\lambda \in [q,\infty)$ *it holds,* $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = \infty$ *for index* ∞ *and* $\lim_{t\to\infty} f(\tau(\lambda t))/f(t) = 0$ *for index* $-\infty$.

(v) *Let* $R : [1,\infty) \to (0,\infty)$ *be defined by* $R(x) = f(\tau(x))$ *for* $x \in [1,\infty)$. *If* R *is rapidly varying of index* $\pm\infty$, *then* $f \in \mathcal{RPV}_q(\pm\infty)$. *Conversely, if* $f \in \mathcal{RPV}_q(\pm\infty)$, *then* $\lim_{x\to\infty} R(\lambda x)/R(x) = \infty$, *resp.,* $\lim_{x\to\infty} R(\lambda x)/R(x) = 0$ *for* $\lambda \in [q,\infty)$.

(vi) $f \in \mathcal{RPV}_q(\pm\infty) \Rightarrow \lim_{t\to\infty} \log f(t)/\log t = \pm\infty$.

Proof. We prove only the "if" part of (iii). The proofs of (iv), (v), and (vi) can be found in [1]. The proofs of other statements can be found in [3].

Assume that $f(t)/t^\vartheta$ is eventually increasing (towards ∞) for each $\vartheta \in [0,\infty)$. Because of monotonicity, we have $f(t)/t^\vartheta \le f(qt)/(q^\vartheta t^\vartheta)$, and so $f(qt)/f(t) \ge q^\vartheta$ for large t. Since ϑ is arbitrary, we have $f(qt)/f(t) \to \infty$ as $t \to \infty$, thus $f \in \mathcal{RPV}_q(\infty)$. The case of the index $-\infty$ can be treated in a similar way. □

Proposition 2.5. (i) $f \in \mathcal{RB}_q \Leftrightarrow [-\infty]_q < \liminf_{t\to\infty} tD_q f(t)/f(t) \le \limsup_{t\to\infty} tD_q f(t)/f(t) < [\infty]_q$.

(ii) $f \in \mathcal{RB}_q \Leftrightarrow f(t) = t^\vartheta\varphi(t)e_\psi(t,1)$, *where* $0 < C_1 \le \varphi(t) \le C_2 < \infty$, $[-\infty]_q < D_1 \le t\psi(t) \le D_2 < [\infty]_q$ *(w.l.o.g.,* φ *can be replaced by* $C \in (0,\infty)$*).*

(iii) $f \in \mathcal{RB}_q \Leftrightarrow f(t)/t^{\gamma_1}$ *is eventually increasing and* $f(t)/t^{\gamma_2}$ *is eventually decreasing for some* $\gamma_1 < \gamma_2$ *(w.l.o.g., monotonicity can be replaced by almost monotonicity; a function* $f : q^{\mathbb{N}_0} \to (0,\infty)$ *is said to be almost increasing (almost decreasing) if there exists an increasing (decreasing) function* $g : q^{\mathbb{N}_0} \to (0,\infty)$ *and* $C, D \in (0,\infty)$ *such that* $Cg(t) \le f(t) \le Dg(t)$*).*

(iv) $f \in \mathcal{RB}_q \Leftrightarrow 0 < \liminf_{t\to\infty} f(\tau(\lambda t))/f(t) \le \limsup_{t\to\infty} f(\tau(\lambda t))/f(t) < \infty$ *for every* $\lambda \in [q,\infty)$ *or for every* $\lambda \in (0,1)$.

(v) $f \in \mathcal{RB}_q \Leftrightarrow R : [1,\infty) \to (0,\infty)$ *defined by* $R(x) = f(\tau(x))$ *for* $x \in [1,\infty)$ *is regularly bounded.*

(vi) $f \in \mathcal{RB}_q \Rightarrow -\infty < \liminf_{t\to\infty} \log f(t)/\log t \le \limsup_{t\to\infty} \log f(t)/\log t < \infty$.

Proof. See [1]. □

For more information on q-Karamata theory see [1–3].

3. Asymptotic Behavior of Solutions to (1.1) in the Framework of q-Karamata Theory

First we establish necessary and sufficient conditions for positive solutions of (1.1) to be q-regularly varying or q-rapidly varying or q-regularly bounded. Then we use this result to provide a thorough discussion on Karamata-like behavior of solutions to (1.1).

Theorem 3.1. (i) *Equation* (1.1) *has eventually positive solutions* u, v *such that* $u \in \mathcal{RV}_q(\vartheta_1)$ *and* $v \in \mathcal{RV}_q(\vartheta_2)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = P \in \left(-\infty, \frac{\omega_q}{q^{\alpha-1}}\right), \tag{3.1}$$

where $\vartheta_i = \log_q[(q-1)\Phi^{-1}(\lambda_i)+1]$, $i = 1,2$, *with* $\lambda_1 < \lambda_2$ *being the real roots of the equation* $\lambda = h(\lambda) - P/[1-\alpha]_q$. *For the indices* ϑ_i, $i = 1,2$, *it holds that* $\vartheta_1 < 0 < 1 < \vartheta_2$ *provided* $P < 0$; $\vartheta_1 = 0$, $\vartheta_2 = 1$ *provided* $P = 0$; $0 < \vartheta_1 < (\alpha-1)/\alpha < \vartheta_2 < 1$ *provided* $P > 0$. *Any of two conditions* $u \in \mathcal{RV}_q(\vartheta_1)$ *and* $v \in \mathcal{RV}_q(\vartheta_2)$ *implies* (3.1).

(ii) *Let* (1.1) *be nonoscillatory (which can be guaranteed, for example, by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*; with the note that it allows* (3.2)*). Equation* (1.1) *has an eventually positive solution* u *such that* $u \in \mathcal{RV}_q((\alpha-1)/\alpha)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = \frac{\omega_q}{q^{\alpha-1}}. \tag{3.2}$$

All eventually positive solutions of (1.1) *are* q*-regularly varying of index* $(\alpha-1)/\alpha$ *provided* (3.2) *holds.*

(iii) *Equation* (1.1) *has eventually positive solutions* u, v *such that* $u \in \mathcal{RPV}_q(-\infty)$ *and* $u \in \mathcal{RPV}_q(\infty)$ *if and only if*

$$\lim_{t\to\infty} t^\alpha p(t) = -\infty. \tag{3.3}$$

All eventually positive solutions of (1.1) *are* q*-rapidly varying provided* (3.3) *holds.*

(iv) *If* (1.1) *is nonoscillatory (which can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*) and*

$$\liminf_{t\to\infty} t^\alpha p(t) > -\infty, \tag{3.4}$$

then all eventually positive solutions of (1.1) *are* q*-regularly bounded.*

Conversely, if there exists an eventually positive solution u *of* (1.1) *such that* $u \in \mathcal{RB}_q$, *then*

$$-\infty < \liminf_{t\to\infty} t^\alpha p(t) \le \limsup_{t\to\infty} t^\alpha p(t) < \frac{1+q^{1-\alpha}}{(q-1)^\alpha}. \tag{3.5}$$

If, in addition, p *is eventually positive or* u *is eventually increasing, then the constant on the right-hand side of* (3.5) *can be improved to* $1/(q-1)^\alpha$.

Proof. (i) *Necessity*. Assume that u is a solution of (1.1) such that $u \in \mathcal{RV}_q(\vartheta_1)$. Then, by Lemma 2.1,

$$\begin{aligned}
\lim_{t\to\infty} t^\alpha p(t) &= -(q-1)^{-\alpha} \lim_{t\to\infty} \mathcal{L}[u](t) = -(q-1)^{-\alpha} \lim_{t\to\infty} F\left(\frac{u(qt)}{u(t)}\right) \\
&= -(q-1)^{-\alpha} F\left(q^{\vartheta_1}\right) = -[1-\alpha]_q \left[\Phi\left([\vartheta_1]_q\right) - h\left(\Phi[\vartheta_1]_q\right)\right] \\
&= \frac{[1-\alpha]_q P}{[1-\alpha]_q} = P.
\end{aligned} \tag{3.6}$$

The same arguments work when dealing with $v \in \mathcal{RV}_q(\vartheta_2)$ instead of u.

Sufficiency. Assume that (3.1) holds. Then there exist $N \in [0, \infty)$, $t_0 \in q^{\mathbb{N}_0}$, and $P_\eta \in (0, \omega_q/q^{\alpha-1})$ such that $-N \leq t^\alpha p(t) \leq P_\eta$ for $t \in [t_0, \infty)_q$. Let $\mathcal{X}$ be the Banach space of all bounded functions $[t_0, \infty)_q \to \mathbb{R}$ endowed with the supremum norm. Denote $\Omega = \{w \in \mathcal{X} : \Phi(q^{-\eta}-1) \leq w(t) \leq \widetilde{N}$ for $t \in [t_0, \infty)_q\}$, where $\widetilde{N} = N(q-1)^\alpha + q^{1-\alpha}$, $\eta = \log_q[(q-1)\Phi^{-1}(\lambda_\eta)+1]$, λ_η being the smaller root of $\lambda = h(\lambda) - P_\eta/[1-\alpha]_q$. In view of Lemma 2.1, it holds that $\eta < (\alpha-1)/\alpha$. Moreover, if $P_\eta \geq P$ (which must be valid in our case), then $\vartheta_1 \leq \eta$. Further, by Lemma 2.1, $-(q-1)^P_\eta = \Phi(q^{-\eta}-1)(1-q^{(\alpha-1)(\eta-1)})$. Let $\mathcal{T} : \Omega \to \mathcal{X}$ be the operator defined by

$$(\mathcal{T}w)(t) = -(q-1)^\alpha t^\alpha p(t) - \Phi\left(\frac{1}{q\Phi^{-1}(w(qt)) + q} - \frac{1}{q}\right). \tag{3.7}$$

By means of the contraction mapping theorem we will prove that $\mathcal{T}$ has a fixed-point in Ω. First we show that $\mathcal{T}\Omega \subseteq \Omega$. Let $w \in \Omega$. Then, using identities (v) and (vi) from Lemma 2.1,

$$\begin{aligned}(\mathcal{T}w)(t) &\geq -(q-1)^\alpha P_\eta - \Phi\left(\frac{1}{qq^{-\eta}} - \frac{1}{q}\right)\\ &= (\lambda_\eta - h(\lambda_\eta))(q-1)^\alpha[1-\alpha]_q - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\ &= F(q^\eta) - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\ &= \Phi(q^{-\eta}-1)\left(1-q^{(\alpha-1)(\eta-1)}\right) - q^{(\eta-1)(\alpha-1)}\Phi(1-q^{-\eta})\\ &= \Phi(q^{-\eta}-1)\end{aligned} \tag{3.8}$$

and $(\mathcal{T}w)(t) \leq -(q-1)^\alpha t^\alpha p(t) + q^{1-\alpha} \leq \widetilde{N}$ for $t \in [t_0, \infty)_q$. Now we prove that $\mathcal{T}$ is a contraction mapping on Ω. Consider the function $g : (-1, \infty) \to \mathbb{R}$ defined by $g(x) = -\Phi(1/(q\Phi^{-1}(x) + q) - 1/q)$. It is easy to see that $|g'(x)| = q^{1-\alpha}(\Phi^{-1}(x)+1)^{-\alpha}$. Let $w, z \in \Omega$. The Lagrange mean value theorem yields $|g(w(t)) - g(z(t))| = |w(t) - z(t)||g'(\xi(t))|$, where $\xi : q^{\mathbb{N}_0} \to \mathbb{R}$ is such that $\min\{w(t), z(t)\} \leq \xi(t) \leq \max\{w(t), z(t)\}$ for $t \in [t_0, \infty)_q$. Hence,

$$\begin{aligned}|(\mathcal{T}w)(t) - (\mathcal{T}z)(t)| &= |g(w(qt)) - g(z(qt))|\\ &= |w(qt) - z(qt)||g'(\xi(t))|\\ &\leq |w(qt) - z(qt)||g'(\Phi(q^{-\eta}-1))|\\ &= q^{\eta\alpha+1-\alpha}|w(qt) - z(qt)|\\ &\leq q^{\eta\alpha+1-\alpha}\|w - z\|\end{aligned} \tag{3.9}$$

for $t \in [t_0, \infty)_q$. Thus $\|\mathcal{T}w - \mathcal{T}z\| \leq q^{\eta\alpha+1-\alpha}\|w-z\|$, where $q^{\eta\alpha+1-\alpha} \in (0,1)$ by virtue of $q > 1$ and $\eta < (\alpha-1)/\alpha$. The Banach fixed-point theorem now guarantees the existence of $w \in \Omega$ such that $w = \mathcal{T}w$. Define u by $u(t) = \prod_{s\in[t_0,t)_q}(\Phi^{-1}(w(s))+1)^{-1}$. Then u is a positive solution of $\mathcal{L}[u](t) = -(q-1)^\alpha t^\alpha p(t)$ on $[t_0, \infty)_q$, and, consequently, of (1.1) (this implies nonoscillation of (1.1)). Moreover, $q^{-\eta} \leq \Phi^{-1}(w(t)) + 1 \leq 1/\overline{N}$, where $\overline{N} = 1/(\Phi^{-1}(\widetilde{N})+1)$, and thus

$\overline{N} \le u(qt)/u(t) \le q^{\eta}$. Denote $M_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $M^* = \limsup_{t\to\infty} u(qt)/u(t)$. Rewrite $\mathcal{L}[u](t) = -(q-1)^{\alpha} t^{\alpha} p(t)$ as

$$\Phi\left(\frac{u(q^2t)}{qu(qt)} - \frac{1}{q}\right) = \Phi\left(1 - \frac{u(t)}{u(qt)}\right) - (q-1)^{\alpha} t^{\alpha} p(t). \tag{3.10}$$

Taking lim inf and lim sup as $t \to \infty$ in (3.10), we get $\Phi(M_*/q - 1/q) = \Phi(1-1/M_*) - (q-1)^{\alpha} P$ and $\Phi(M^*/q - 1/q) = \Phi(1 - 1/M^*) - (q-1)^{\alpha} P$, respectively. Hence, $F(M_*) = F(M^*)$. Since $M_*, M^* \in [\overline{N}, q^{\eta}]$ and F is strictly decreasing on $(0, q^{(\alpha-1)/\alpha})$ (by Lemma 2.1), we have $M := M_* = M^*$. Moreover,

$$F(M) = -(q-1)^{\alpha} P = (q-1)^{\alpha} [1-\alpha]_q \Big(\Phi\big([\vartheta_i]_q\big) - h\big(\Phi[\vartheta_i]_q\big)\Big) = F\big(q^{\vartheta_i}\big), \tag{3.11}$$

$i = 1, 2$, which implies $M = q^{\vartheta_1}$, in view of the facts that $M, q^{\vartheta_1} \in (0, q^{(\alpha-1)/\alpha})$, $q^{\vartheta_2} > q^{(\alpha-1)/\alpha}$, and F is monotone on $(0, q^{(\alpha-1)/\alpha})$. Thus $u \in \mathcal{RV}_q(\vartheta_1)$. Now we show that there exists a solution v of (1.1) with $v \in \mathcal{RV}_q(\vartheta_2)$. We can assume that N, t_0, and P_{η} are the same as in the previous part. Consider the set $\Gamma = \{w \in \mathcal{X} : \Phi(q^{\zeta-1} - 1/q) \le w(t) \le \widetilde{M}$ for $t \in [t_0, \infty)_q\}$, where $\widetilde{M} = 1 + (q-1)^{\alpha} N$, $\zeta = \log_q[(q-1)\Phi^{-1}(\lambda_{\zeta}) + 1]$, λ_{ζ} being the larger root of $\lambda = h(\lambda) - P_{\eta}/[1-\alpha]_q$. It is clear that N can be chosen in such a way that $\Phi(q^{\vartheta_2 - 1} - 1/q) < \widetilde{M}$. It holds $(\alpha-1)/\alpha < \zeta \le \vartheta_2$ and $-(q-1)^{\alpha} P_{\eta} = \Phi(q^{-\zeta} - 1)(1 - q^{(\alpha-1)(\zeta-1)})$. Define $\mathcal{S} : \Gamma \to \mathcal{X}$ by $(\mathcal{S}w)(t) = \Phi(1 - 1/(q\Phi^{-1}(w(t/q)) + 1)) - (q-1)^{\alpha} t^{\alpha} p(t)$ for $t \in [qt_0, \infty)_q$, and $(\mathcal{S}w)(t_0) = \Phi(q^{\vartheta_2 - 1} - 1/q)$. Using similar arguments as above it is not difficult to see that $\mathcal{S}\Gamma \subseteq \Gamma$ and $\|\mathcal{S}w - \mathcal{S}z\| < q^{\alpha - 1 - \alpha\zeta} \|w - z\|$ for $w, z \in \Gamma$. So there exists $w \in \Gamma$ such that $w = \mathcal{S}w$. If we define $v(t) = \prod_{s \in [qt_0, t)_q} (q\Phi^{-1}(w(s/q)) + 1)$, then v is a positive solution of (1.1) on $[qt_0, \infty)_q$, which satisfies $q^{\zeta} \le v(qt)/v(t) \le q\Phi^{-1}(\widetilde{M}) + 1$. Arguing as above we show that $v \in \mathcal{RV}_q(\vartheta_2)$.

(ii) *Necessity*. The proof is similar to that of (i).

Sufficiency. The condition $t^{\alpha} p(t) \le \omega_q / q^{\alpha-1}$ for large t implies nonoscillation of (1.1). Indeed, it is easy to see that $y(t) = t^{(\alpha-1)/\alpha}$ is a nonoscillatory solution of the Euler type equation $D_q(\Phi(D_q y(t))) + \omega_q q^{1-\alpha} t^{-\alpha} \Phi(y(qt)) = 0$. Nonoscillation of (1.1) then follows by using the Sturm type comparison theorem, see also Section 4(i). Let us write P as $P = [1 - \alpha]_q (h(\Phi([(\alpha-1)/\alpha]_q)) - \Phi([(\alpha-1)/\alpha]_q))$, with noting that $\lambda = \Phi([(\alpha-1)/\alpha]_q)$ is the double root of $\lambda = h(\lambda) - \omega_q q^{1-\alpha}/[1-\alpha]_q$, see Lemma 2.1. Then, in view of Lemma 2.1, we obtain

$$\begin{aligned} F\big(q^{\vartheta}\big) &= (q-1)^{\alpha} [1-\alpha]_q \Big[\Phi\big([\vartheta]_q\big) - h\big(\Phi[\vartheta]_q\big)\Big] = -\frac{(q-1)^{\alpha} \omega_q}{q^{\alpha-1}} \\ &= -(q-1)^{\alpha} \lim_{t\to\infty} t^{\alpha} p(t) = \lim_{t\to\infty} \mathcal{L}[u](t). \end{aligned} \tag{3.12}$$

Let us denote $U_* = \liminf_{t\to\infty} u(qt)/u(t)$ and $U^* = \limsup_{t\to\infty} u(qt)/u(t)$. It is impossible to have $U_* = 0$ or $U^* = \infty$, otherwise $\lim_{t\to\infty} \mathcal{L}[u](t) = \infty$, which contradicts to (3.12). Thus $0 < U_* \le U^* < \infty$. Consider (1.1) in the form (3.10). Taking lim sup, respectively, lim inf as $t \to \infty$ in (3.10), into which our u is plugged, we obtain $F(U_*) = F(q^{(\alpha-1)/\alpha}) = F(U^*)$. Thanks to the properties of F, see Lemma 2.1, we get $U_* = U^* = q^{(\alpha-1)/\alpha}$. Hence, $u \in \mathcal{RV}_q((\alpha-1)/\alpha)$.

Since we worked with an arbitrary positive solution, it implies that all positive solutions must be q-regularly varying of index $(\alpha-1)/\alpha$.

(iii) The proof repeats the same arguments as that of [3, Theorem 1] (in spite of no sign condition on p). Note just that condition (3.3) compels p to be eventually negative and the proof of necessity does not depend on the sign of p.

(iv) *Sufficiency*. Let u be an eventually positive solution of (1.1). Assume by a contradiction that $\limsup_{t\to\infty} y(qt)/y(t)=\infty$. Then, in view of Lemma 2.1(vii),

$$\infty=\limsup_{t\to\infty}\left(\Phi\left(\frac{y(q^2t)}{qy(qt)}-\frac{1}{q}\right)-1\right)\le\limsup_{t\to\infty}\mathcal{L}[y](t)=-(q-1)^{\alpha}\liminf_{t\to\infty}t^{\alpha}p(t)<\infty \quad (3.13)$$

by (3.4), a contradiction. If $\liminf_{t\to\infty} y(qt)/y(t)=0$, then $\limsup_{t\to\infty} y(t)/y(qt)=\infty$ and we proceed similarly as in the previous case. Since we worked with an arbitrary positive solution, it implies that all positive solutions must be q-regularly bounded.

Necessity. Let $y\in\mathcal{RB}_q$ be a solution of (1.1). Taking lim sup as $t\to\infty$ in $-(q-1)^{\alpha}t^{\alpha}p(t)=\mathcal{L}[y](t)$, we get

$$\begin{aligned}&-(q-1)^{\alpha}\liminf_{t\to\infty}t^{\alpha}p(t)\\&\quad=\limsup_{t\to\infty}\mathcal{L}[y](t)\le\limsup_{t\to\infty}\Phi\left(\frac{y(q^2t)}{qy(qt)}-\frac{1}{q}\right)+\limsup_{t\to\infty}\Phi\left(\frac{y(t)}{y(qt)}-1\right)<\infty,\end{aligned} \quad (3.14)$$

which implies the first inequality in (3.5). Similarly, the lim inf as $t\to\infty$ yields $-(q-1)^{\alpha}\limsup_{t\to\infty}t^{\alpha}p(t)>-1/q^{\alpha-1}-1$, which implies the last inequality in (3.5). If p is eventually positive, then every eventually positive solution of (1.1) is eventually increasing, which can be easily seen from its concavity. Hence, $y(qt)/y(t)\ge 1$ for large t. Thus the last inequality becomes $-(q-1)^{\alpha}\limsup_{t\to\infty}t^{\alpha}p(t)>-1$. □

We are ready to provide a summarizing thorough discussion on asymptotic behavior of solutions to (1.1) with respect to the limit behavior of $t^{\alpha}p(t)$ in the framework of q-Karamata theory. Denote

$$P=\lim_{t\to\infty}t^{\alpha}p(t),\qquad P_*=\liminf_{t\to\infty}t^{\alpha}p(t),\qquad P^*=\limsup_{t\to\infty}t^{\alpha}p(t). \quad (3.15)$$

The set of all q-regularly varying and q-rapidly varying functions is said to be *q-Karamata functions*. With the use of the previous results we obtain the following statement.

Corollary 3.2. (i) *Assume that there exists $P\in\mathbb{R}\cup\{-\infty,\infty\}$. In this case,* (1.1) *possesses solutions that are q-Karamata functions provided* (1.1) *is nonoscillatory. Moreover, we distinguish the following subcases:*

(a) $P=-\infty$*:* (1.1) *is nonoscillatory and all its positive solutions are q-rapidly varying (of index $-\infty$ or ∞).*

(b) $P\in(-\infty,\omega_q/q^{\alpha-1})$*:* (1.1) *is nonoscillatory and there exist a positive solution which is q-regularly varying of index ϑ_1 and a positive solution which is q-regularly varying of index ϑ_2.*

(c) $P = \gamma_q$: (1.1) *either oscillatory or nonoscillatory (the latter one can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*). In case of nonoscillation of* (1.1) *all its positive solutions are* q*-regularly varying of index* $(\alpha-1)/\alpha$.

(d) $P \in (\omega_q/q^{\alpha-1}, \infty) \cup \{\infty\}$: (1.1) *is oscillatory.*

(ii) *Assume that* $\mathbb{R} \cup \{-\infty\} \ni P_* < P^* \in \mathbb{R} \cup \{\infty\}$*. In this case, there are no* q*-Karamata functions among positive solutions of* (1.1)*. Moreover, we distinguish the following subcases:*

(a) $P_* \in (\omega_q/q^{\alpha-1}, \infty) \cup \{\infty\}$: (1.1) *is oscillatory.*

(b) $P_* \in \{-\infty\} \cup (-\infty, \omega_q/q^{\alpha-1}]$: (1.1) *is either oscillatory (this can be guaranteed, e.g., by* $P^* > (1+q^{1-\alpha})/(q-1)^\alpha$ *or by* $p > 0$ *and* $P^* \ge 1/(q-1)^\alpha$*) or nonoscillatory (this can be guaranteed, e.g., by* $t^\alpha p(t) \le \omega_q/q^{\alpha-1}$ *for large* t*). If, in addition to nonoscillation of* (1.1)*, it holds* $P_* > -\infty$*, then all its positive solutions are* q*-regularly bounded, but there is no* q*-regularly varying solution. If* $P_* = -\infty$*, then there is no* q*-regularly bounded or* q*-rapidly varying solution.*

4. Concluding Remarks

(i) We start with some remarks to Kneser type criteria. As a by product of Theorem 3.1(i) we get the following nonoscillation Kneser type criterion: if $\lim_{t\to\infty} t^\alpha p(t) < \omega_q/q^{\alpha-1}$, then (1.1) is nonoscillatory. However, its better variant is known (it follows from a more general time-scale case involving Hille-Nehari type criterion [15]), where the sufficient condition is relaxed to $\limsup_{t\to\infty} t^\alpha p(t) < \omega_q/q^{\alpha-1}$. The constant $\omega_q/q^{\alpha-1}$ is sharp, since $\liminf_{t\to\infty} t^\alpha p(t) > \omega_q/q^{\alpha-1}$ implies oscillation of (1.1), see [15]. But no conclusion can be generally drawn if the equality occurs in these conditions. The above lim sup nonoscillation criterion can be alternatively obtained also from the observation presented at the beginning of the proof of Theorem 3.1(ii) involving the Euler type q-difference equation. And it is worthy of note that the conclusion of that observation can be reached also when modifying the proof of Hille-Nehari type criterion in [15]. A closer examination of the proof of Theorem 3.1(iv) shows that a necessary condition for nonoscillation of (1.1) is $-(q-1)^\alpha \limsup_{t\to\infty} t^\alpha p(t) \ge -q^{1-\alpha} - 1$. Thus we have obtained quite new Kneser type oscillation criterion: if $\limsup_{t\to\infty} t^\alpha p(t) > (1+q^{1-\alpha})/(q-1)^\alpha$, then (1.1) is oscillatory. If p is eventually positive, then the constant on the right-hand side can be improved to $1/(q-1)^\alpha$ and the strict inequality can be replaced by the nonstrict one (this is because of q-regular boundedness of possible positive solutions). A continuous analog of this criterion is not known, which is quite natural since $1/(q-1)^\alpha \to \infty$ as $q \to 1$. Compare these results with the Hille-Nehari type criterion, which was proved in general setting for dynamic equations and time-scales, and is valid no matter what the graininess is (see [15]); in q-calculus it reads as follows: if $p \ge 0$ and $\limsup_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\mathrm{d}_q s > 1$, then (1.1) is oscillatory. This criterion holds literally also in the continuous case. Finally note that, in general, $\limsup_{t\to\infty} t^{\alpha-1} \int_t^\infty p(s)\mathrm{d}_q s \le \limsup_{t\to\infty} -[1-\alpha]_q t^\alpha p(t)$.

(ii) The results contained in Theorem 3.1 can understood at least in the three following ways:

(a) As a q-version of the continuous results for (1.2) from [5]. However, there are several substantial differences: The conditions in the continuous case are (and somehow must be) in the integral form (see also the item (iii) of this section); there is a different approach in the proof (see also the item (iv) of this section); the rapid variation has not been treated in such detail in the continuous case; in the case of the existence of the double root, we show that all (and not just some) positive solutions are q-regularly varying under quite mild assumptions; for positive solutions to be q-regularly bounded we obtain quite simple and natural sufficient and also necessary conditions.

(b) As a half-linear extension of the results for $D_q^2 y(t) + p(t)y(qt) = 0$ from [1]. In contrast to the linear case, in the half-linear case a reduction of order formula is not at disposal. Thus to prove that there are two q-regularly varying solutions of two different indices we need immediately to construct both of them. Lack of a fundamental like system for half-linear equations causes that, for the time being, we are not able to show that all positive solutions are q-regularly varying. This is however much easier task when $p(t) < 0$, see [3].

(c) As a generalization of the results from [3] in the sense of no sign condition on the coefficient p.

(iii) From the continuous theory we know that the sufficient and necessary conditions for regularly or rapidly varying behavior of solutions to (1.2) are in terms of limit behavior of integral expressions, typically $t^{\alpha-1}\int_t^\infty p(s)\mathrm{d}s$ or $t^{\alpha-1}\int_t^{\lambda t} p(s)\mathrm{d}s$. In contrast to that, in q-calculus case the conditions have nonintegral form. This is the consequence of specific properties of q-calculus: one thing is that we use a different approach which does not apply in the continuous case. Another thing is that the limit $\lim_{t\to\infty} t^{\alpha-1}\int_t^\infty p(s)\mathrm{d}_q s$ can be expressed in terms of $\lim_{t\to\infty} t^\alpha p(t)$ (and vice versa), provided it exists. Such a relation does not work in the continuous case.

(iv) As already said, our approach in the proof of Theorem 3.1 is different from what is known in the continuous theory. Our method is designed just for q-difference equations and roughly speaking, it is based on rewriting a q-difference equation in terms of the fractions which appear in Definition 2.2. Such a technique cannot work in the continuous case. Since this method uses quite natural and simple relations (which are possible thanks to the special structure of $q^{\mathbb{N}_0}$), we believe that it will enable us to prove also another results which are q-versions of existing or nonexisting continuous results; in the latter case, such results may serve to predict a possible form of the continuous counterpart, which may be difficult to handle directly. We just take, formally, the limit as $q \to 1+$.

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Research Article

Asymptotic Formula for Oscillatory Solutions of Some Singular Nonlinear Differential Equation

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Singular differential equation $(p(t)u')' = p(t)f(u)$ is investigated. Here f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0,\infty)$ and has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. An asymptotic formula for oscillatory solutions is derived.

1. Introduction

In this paper, we investigate the equation

$$\left(p(t)u'\right)' = p(t)f(u), \quad t \in (0,\infty), \tag{1.1}$$

where f satisfies

$$f \in Lip_{\text{loc}}(\mathbb{R}), \quad f(0) = f(L) = 0, \; f(x) < 0, \; x \in (0,L), \tag{1.2}$$

$$\exists \overline{B} \in (-\infty,0)\colon f(x) > 0, \quad x \in \left[\overline{B},0\right), \tag{1.3}$$

$$F\left(\overline{B}\right) = F(L), \quad \text{where } F(x) = -\int_0^x f(z)\mathrm{d}z, \; x \in \mathbb{R}, \tag{1.4}$$

and p fulfils

$$p \in C[0,\infty) \cap C^1(0,\infty), \quad p(0) = 0, \tag{1.5}$$

$$p'(t) > 0, \quad t \in (0,\infty), \quad \lim_{t\to\infty}\frac{p'(t)}{p(t)} = 0. \tag{1.6}$$

Equation (1.1) is a generalization of the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0,\infty), \tag{1.7}$$

which arises for $k > 1$ and special forms of f in many areas, for example: in the study of phase transitions of Van der Waals fluids [1–3], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [4, 5], in the homogeneous nucleation theory [6], in the relativistic cosmology for the description of particles which can be treated as domains in the universe [7], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1.1), where f is a polynomial with three zeros have been presented in [9–11]. Close problems about the existence of positive solutions can be found in [12–14].

Due to $p(0) = 0$, (1.1) has a singularity at $t = 0$.

Definition 1.1. A function $u \in C^1[0,\infty) \cap C^2(0,\infty)$ which satisfies (1.1) for all $t \in (0,\infty)$ is called a *solution* of (1.1).

Definition 1.2. Let u be a solution of (1.1) and let L be of (1.2). Denote $u_{\sup} = \sup\{u(t) : t \in [0,\infty)\}$. If $u_{\sup} < L$ ($u_{\sup} = L$ or $u_{\sup} > L$), then u is called a *damped* solution (a *bounding homoclinic* solution or an *escape* solution).

These three types of solutions have been investigated in [15–19]. In particular, the existence of damped oscillatory solutions which converge to 0 has been proved in [19].

The main result of this paper is contained in Section 3 in Theorem 3.1, where we provide an asymptotic formula for damped oscillatory solutions of (1.1).

2. Existence of Oscillatory Solutions

Here, we will study solutions of (1.1) satisfying the initial conditions

$$u(0) = B, \qquad u'(0) = 0, \tag{2.1}$$

with a parameter $B \le L$. Reason is that we focus our attention on damped solutions of (1.1) and that each solution u of (1.1) must fulfil $u'(0) = 0$ (see [19]).

First, we bring two theorems about the existence of damped and oscillatory solutions.

Theorem 2.1 (see [19]). *Assume that* (1.2)–(1.6) *hold. Then for each* $B \in [\overline{B}, L)$ *problem* (1.1), (2.1) *has a unique solution. This solution is damped.*

Theorem 2.2. *Assume that* (1.2)–(1.6) *hold. Further, let there exists* $k_0 \in (0,\infty)$ *such that*

$$p \in C^2(0,\infty), \qquad \limsup_{t\to\infty}\left|\frac{p''(t)}{p'(t)}\right| < \infty, \qquad \liminf_{t\to\infty}\frac{p(t)}{t^{k_0}} \in (0,\infty], \tag{2.2}$$

$$\lim_{x\to 0+}\frac{f(x)}{x} < 0, \qquad \lim_{x\to 0-}\frac{f(x)}{x} < 0. \tag{2.3}$$

Then for each $B \in [\overline{B}, L)$ *problem* (1.1), (2.1) *has a unique solution* u. *If* $B \neq 0$, *then the solution* u *is damped and oscillatory with decreasing amplitudes and*

$$\lim_{t\to\infty} u(t) = 0. \tag{2.4}$$

Proof. The assertion follows from Theorems 2.3, 2.10 and 3.1 in [19]. □

Example 2.3. The functions

(i) $p(t) = t^k$, $p(t) = t^k \ln(t^\ell + 1)$, $k, \ell \in (0, \infty)$,

(ii) $p(t) = t + \alpha \sin t$, $\alpha \in (-1, 1)$,

(iii) $p(t) = t^k/(1 + t^\ell)$, $k, \ell \in (0, \infty)$, $\ell < k$

satisfy (1.5), (1.6), and (2.2).

The functions

(i) $p(t) = \ln(t + 1)$, $p(t) = \arctan t$, $p(t) = t^k/(1 + t^k)$, $k \in (0, \infty)$

satisfy (1.5), (1.6), but not (2.2) (the third condition).

The function

(i) $p(t) = t^k + \alpha \sin t^k$, $\alpha \in (-1, 1)$, $k \in (1, \infty)$,

satisfy (1.5), (1.6) but not (2.2) (the second and third conditions).

Example 2.4. Let $k \in (0, \infty)$.

(i) The function

$$f(x) = \begin{cases} -kx, & \text{for } x \le 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \tag{2.5}$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\overline{B} = -(3k)^{-1/2}$ and (2.3).

(ii) The function

$$f(x) = \begin{cases} kx^2, & \text{for } x \le 0, \\ x(x-1), & \text{for } x > 0, \end{cases} \tag{2.6}$$

satisfies (1.2) with $L = 1$, (1.3), (1.4) with $\overline{B} = -(2k)^{-1/3}$ but not (2.3) (the second condition).

In the next section, the generalized Matell's theorem which can be found as Theorem 6.5 in the monograph by Kiguradze will be useful. For our purpose, we provide its following special case.

Consider an interval $J \subset \mathbb{R}$. We write $AC(J)$ for the set of functions absolutely continuous on J and $AC_{\mathrm{loc}}(J)$ for the set of functions belonging to $AC(I)$ for each compact interval $I \subset J$. Choose $t_0 > 0$ and a function matrix $A(t) = (a_{i,j}(t))_{i,j\le 2}$ which is defined on (t_0, ∞). Denote by $\lambda(t)$ and $\mu(t)$ eigenvalues of $A(t)$, $t \in (t_0, \infty)$. Further, suppose

$$\lambda = \lim_{t\to\infty} \lambda(t), \qquad \mu = \lim_{t\to\infty} \mu(t) \tag{2.7}$$

be different eigenvalues of the matrix $A = \lim_{t\to\infty} A(t)$, and let $\mathbf{l}$ and $\mathbf{m}$ be eigenvectors of A corresponding to λ and μ, respectively.

Theorem 2.5 (see [20]). *Assume that*

$$a_{i,j} \in AC_{\mathrm{loc}}(t_0, \infty), \qquad \left|\int_{t_0}^{\infty} a'_{i,j}(t)dt\right| < \infty, \; i,j = 1,2, \tag{2.8}$$

and that there exists $c_0 > 0$ such that

$$\int_s^t \mathrm{Re}\big(\lambda(\tau) - \mu(\tau)\big)\mathrm{d}\tau \le c_0, \quad t_0 \le s < t, \tag{2.9}$$

or

$$\int_{t_0}^{\infty} \mathrm{Re}\big(\lambda(\tau) - \mu(\tau)\big)\mathrm{d}\tau = \infty, \qquad \int_s^t \mathrm{Re}\big(\lambda(\tau) - \mu(\tau)\big)\mathrm{d}\tau \ge -c_0, \quad t_0 \le s < t. \tag{2.10}$$

Then the differential system

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) \tag{2.11}$$

has a fundamental system of solutions $x(t)$, $y(t)$ such that

$$\lim_{t\to\infty} \mathbf{x}(t)\mathrm{e}^{-\int_{t_0}^t \lambda(\tau)\mathrm{d}\tau} = \mathbf{l}, \qquad \lim_{t\to\infty} \mathbf{y}(t)\mathrm{e}^{-\int_{t_0}^t \mu(\tau)\mathrm{d}\tau} = \mathbf{m}. \tag{2.12}$$

3. Asymptotic Formula

In order to derive an asymptotic formula for a damped oscillatory solution u of problem (1.1), (2.1), we need a little stronger assumption than (2.3). In particular, the function $f(x)/x$ should have a negative derivative at $x = 0$.

Theorem 3.1. *Assume that* (1.2)–(1.6), *and* (2.2) *hold. Assume, moreover, that there exist $\eta > 0$ and $c > 0$ such that*

$$\frac{f(x)}{x} \in AC[-\eta, \eta], \qquad \lim_{x\to 0} \frac{f(x)}{x} = -c. \tag{3.1}$$

Then for each $B \in [\overline{B}, L)$ *problem* (1.1), (2.1) *has a unique solution* u*. If* $B \neq 0$*, then the solution* u *is damped and oscillatory with decreasing amplitudes such that*

$$\limsup_{t\to\infty} \sqrt{p(t)}|u(t)| < \infty. \tag{3.2}$$

Proof. We have the following steps:

Step 1 (construction of an auxiliary linear differential system). Choose $B \in [\overline{B}, L)$, $B \neq 0$. By Theorem 2.2, problem (1.1), (2.1) has a unique oscillatory solution u with decreasing amplitudes and satisfying (2.4). Having this solution u, define a linear differential equation

$$v'' + \frac{p'(t)}{p(t)}v' = \frac{f(u(t))}{u(t)}v, \tag{3.3}$$

and the corresponding linear differential system

$$x_1' = x_2, \qquad x_2' = \frac{f(u(t))}{u(t)}x_1 - \frac{p'(t)}{p(t)}x_2. \tag{3.4}$$

Denote

$$A(t) = \left(a_{i,j}(t)\right)_{i,j\le 2} = \begin{pmatrix} 0 & 1 \\ \dfrac{f(u(t))}{u(t)} & -\dfrac{p'(t)}{p(t)} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ -c & 0 \end{pmatrix}. \tag{3.5}$$

By (1.6), (2.4), and (3.1),

$$A = \lim_{t\to\infty} A(t). \tag{3.6}$$

Eigenvalues of A are numbers $\lambda = i\sqrt{c}$ and $\mu = -i\sqrt{c}$, and eigenvectors of A are $\mathbf{l} = (1, i\sqrt{c})$ and $\mathbf{m} = (1, -i\sqrt{c})$, respectively. Denote

$$D(t) = \left(\frac{p'(t)}{2p(t)}\right)^2 + \frac{f(u(t))}{u(t)}, \quad t \in (0,\infty). \tag{3.7}$$

Then eigenvalues of $A(t)$ have the form

$$\lambda(t) = -\frac{p'(t)}{2p(t)} + \sqrt{D(t)}, \qquad \mu(t) = -\frac{p'(t)}{2p(t)} - \sqrt{D(t)}, \quad t \in (0,\infty). \tag{3.8}$$

We see that

$$\lim_{t\to\infty} \lambda(t) = \lambda, \qquad \lim_{t\to\infty} \mu(t) = \mu. \tag{3.9}$$

Step 2 (verification of the assumptions of Theorem 2.5). Due to (1.6), (2.4), and (3.1), we can find $t_0 > 0$ such that

$$u(t_0) \neq 0, \qquad |u(t)| \leq \eta, \qquad D(t) < 0, \quad t \in (t_0, \infty). \tag{3.10}$$

Therefore, by (3.1),

$$a_{21}(t) = \frac{f(u(t))}{u(t)} \in AC_{\mathrm{loc}}(t_0, \infty), \tag{3.11}$$

and so

$$\left| \int_{t_0}^{\infty} \left(\frac{f(u(t))}{u(t)} \right)' \mathrm{d}t \right| = \left| \lim_{t \to \infty} \frac{f(u(t))}{u(t)} - \frac{f(u(t_0))}{u(t_0)} \right| = \left| -c - \frac{f(u(t_0))}{u(t_0)} \right| < \infty. \tag{3.12}$$

Further, by (2.2), $a_{22}(t) = -p'(t)/p(t) \in C^1(t_0, \infty)$. Hence, due to (1.6),

$$\left| \int_{t_0}^{\infty} \left(\frac{p'(t)}{p(t)} \right) \mathrm{d}t \right| = \left| \lim_{t \to \infty} \frac{p'(t)}{p(t)} - \frac{p'(t_0)}{p(t_0)} \right| = \frac{p'(t_0)}{p(t_0)} < \infty. \tag{3.13}$$

Since $a_{11}(t) \equiv 0$ and $a_{12}(t) \equiv 1$, we see that (2.8) is satisfied. Using (3.8) we get $\mathrm{Re}(\lambda(t) - \mu(t)) \equiv 0$. This yields

$$\int_s^t \mathrm{Re}\big(\lambda(\tau) - \mu(\tau)\big) \mathrm{d}\tau = 0 < c_0, \quad t_0 \leq s < t, \tag{3.14}$$

for any positive constant c_0. Consequently (2.9) is valid.

Step 3 (application of Theorem 2.5). By Theorem 2.5 there exists a fundamental system $\mathbf{x}(t) = (x_1(t), x_2(t))$, $\mathbf{y}(t) = (y_1(t), y_2(t))$ of solutions of (3.4) such that (2.12) is valid. Hence

$$\lim_{t \to \infty} x_1(t) \mathrm{e}^{-\int_{t_0}^t \lambda(\tau) \mathrm{d}\tau} = 1, \qquad \lim_{t \to \infty} y_1(t) \mathrm{e}^{-\int_{t_0}^t \mu(\tau) \mathrm{d}\tau} = 1. \tag{3.15}$$

Using (3.8) and (3.10), we get

$$\begin{aligned} \exp\left(-\int_{t_0}^t \lambda(\tau) \mathrm{d}\tau \right) &= \exp\left(\int_{t_0}^t \left(\frac{p'(\tau)}{2p(\tau)} - \sqrt{D(\tau)} \right) \mathrm{d}\tau \right) \\ &= \exp\left(\frac{1}{2} \ln \frac{p(t)}{p(t_0)} \right) \exp\left(-i \int_{t_0}^t \sqrt{|D(\tau)|} \mathrm{d}\tau \right), \end{aligned} \tag{3.16}$$

and, hence,

$$\left|e^{-\int_{t_0}^{t}\lambda(\tau)d\tau}\right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \tag{3.17}$$

Similarly

$$\left|e^{-\int_{t_0}^{t}\mu(\tau)d\tau}\right| = \sqrt{\frac{p(t)}{p(t_0)}}, \quad t \in (t_0, \infty). \tag{3.18}$$

Therefore, (3.15) implies

$$\begin{aligned} 1 &= \lim_{t\to\infty}\left|x_1(t)e^{-\int_{t_0}^{t}\lambda(\tau)d\tau}\right| = \lim_{t\to\infty}|x_1(t)|\sqrt{\frac{p(t)}{p(t_0)}}, \\ 1 &= \lim_{t\to\infty}\left|y_1(t)e^{-\int_{t_0}^{t}\mu(\tau)d\tau}\right| = \lim_{t\to\infty}|y_1(t)|\sqrt{\frac{p(t)}{p(t_0)}}. \end{aligned} \tag{3.19}$$

Step 4 (asymptotic formula). In Step 1, we have assumed that u is a solution of (1.1), which means that

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) = f(u(t)), \quad \text{for } t \in (0, \infty). \tag{3.20}$$

Consequently

$$u''(t) + \frac{p'(t)}{p(t)}u'(t) = \frac{f(u(t))}{u(t)}u(t), \quad \text{for } t \in (0, \infty), \tag{3.21}$$

and, hence, u is also a solution of (3.3). This yields that there are $c_1, c_2 \in \mathbb{R}$ such that $u(t) = c_1x_1(t) + c_2y_1(t),\ t \in (0, \infty)$. Therefore,

$$\limsup_{t\to\infty}\sqrt{p(t)}|u(t)| \leq (|c_1| + |c_2|)\sqrt{p(t_0)} < \infty. \tag{3.22}$$

□

Remark 3.2. Due to (2.2) and (3.2), we have for a solution u of Theorem 3.1

$$u(t) = O\left(t^{-k_0/2}\right), \quad \text{for } t \longrightarrow \infty. \tag{3.23}$$

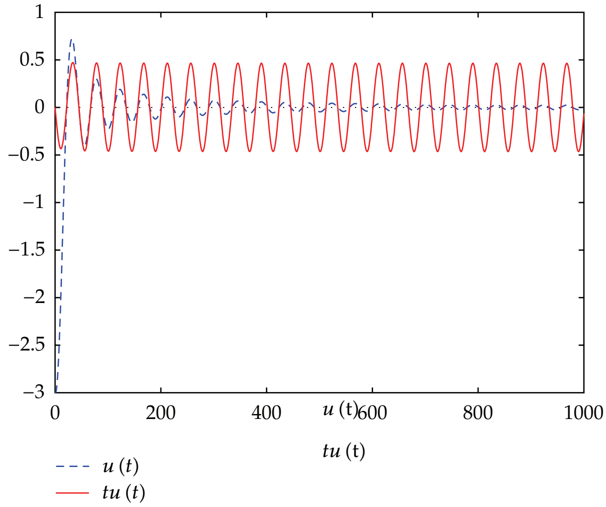

Figure 1

Example 3.3. Let $k \in (1, \infty)$.

(i) The functions $f(x) = x(x-1)$ and $f(x) = x(x-1)(x+2)$ satisfy all assumptions of Theorem 3.1.

(ii) The functions $f(x) = x^{2k-1}(x-1)$ and $f(x) = x^{2k-1}(x-1)(x+2)$

satisfy (1.2)–(1.4) but not (3.1) (the second condition).

Example 3.4. Consider the initial problem

$$\left(t^2 u'\right)' = t^2 u(u-5)(u+10), \qquad u(0) = -3, \;\; u'(0) = 0. \tag{3.24}$$

Here $L_0 = -10$, $L = 5$ and we can check that $\overline{B} < -3$. Further, all assumptions of Theorems 2.2 and 3.1 are fulfilled. Therefore, by Theorem 2.2, there exists a unique solution u of problem (3.24) which is damped and oscillatory and converges to 0. By Theorem 3.1, we have

$$\limsup_{t \to \infty} t|u(t)| < \infty, \quad \text{that is,} \;\; u(t) = O\left(\frac{1}{t}\right), \quad \text{for } t \longrightarrow \infty. \tag{3.25}$$

The behaviour of the solution $u(t)$ and of the function $tu(t)$ is presented on Figure 1.

Remark 3.5. Our further research of this topic will be focused on a deeper investigation of all types of solutions defined in Definition 1.2. For example, we have proved in [15, 19] that damped solutions of (1.1) can be either oscillatory or they have a finite number of zeros or no zero and converge to 0. A more precise characterization of behaviour of nonoscillatory solutions are including their asymptotic formulas in as open problem. The same can be said about homoclinic solutions. In [17] we have found some conditions which guarantee their existence, and we have shown that if u is a homoclinic solution of (1.1), then $\lim_{t\to\infty} u(t) = L$.

In order to discover other existence conditions for homoclinic solutions, we would like to estimate their convergence by proper asymptotic formulas.

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Research Article

Asymptotic Properties of Third-Order Delay Trinomial Differential Equations

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The aim of this paper is to study properties of the third-order delay trinomial differential equation $((1/r(t))y''(t))' + p(t)y'(t) + q(t)y(\sigma(t)) = 0$, by transforming this equation onto the second-/third-order binomial differential equation. Using suitable comparison theorems, we establish new results on asymptotic behavior of solutions of the studied equations. Obtained criteria improve and generalize earlier ones.

1. Introduction

In this paper, we will study oscillation and asymptotic behavior of solutions of third-order delay trinomial differential equations of the form

$$\left(\frac{1}{r(t)}\, y''(t)\right)' + p(t)y'(t) + q(t)y(\sigma(t)) = 0. \tag{E}$$

Throughout the paper, we assume that $r(t), p(t), q(t), \sigma(t) \in C([t_0, \infty))$ and

(i) $r(t) > 0, p(t) \geq 0, q(t) > 0, \sigma(t) > 0$,

(ii) $\sigma(t) \leq t$, $\lim_{t\to\infty}\sigma(t) = \infty$,

(iii) $R(t) = \int_{t_0}^{t} r(s)\, \mathrm{d}s \to \infty$ as $t \to \infty$.

By a solution of (E), we mean a function $y(t) \in C^2([T_x, \infty))$, $T_x \geq t_0$, that satisfies (E) on $[T_x, \infty)$. We consider only those solutions $y(t)$ of (E) which satisfy $\sup\{|y(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and, otherwise, it is nonoscillatory. Equation (E) itself is said to be oscillatory if all its solutions are oscillatory.

Recently, increased attention has been devoted to the oscillatory and asymptotic properties of second- and third-order differential equations (see [1–22]). Various techniques appeared for the investigation of such differential equations. Our method is based on establishing new comparison theorems, so that we reduce the examination of the third-order trinomial differential equations to the problem of the observation of binomial equations.

In earlier papers [11, 13, 16, 20], a particular case of (E), namely, the ordinary differential equation (without delay)

$$y'''(t) + p(t)y'(t) + g(t)y(t) = 0, \tag{E_1}$$

has been investigated, and sufficient conditions for all its nonoscillatory solutions $y(t)$ to satisfy

$$y(t)y'(t) < 0 \tag{1.1}$$

or the stronger condition

$$\lim_{t\to\infty} y(t) = 0 \tag{1.2}$$

are presented. It is known that (E_1) has always a solution satisfying (1.1). Recently, various kinds of sufficient conditions for all nonoscillatory solutions to satisfy (1.1) or (1.2) appeared. We mention here [9, 11, 13, 16, 21]. But there are only few results for differential equations with deviating argument. Some attempts have been made in [8, 10, 18, 19]. In this paper we generalize these, results and we will study conditions under which all nonoscillatory solutions of (E) satisfy (1.1) and (1.2). For our further references we define as following.

Definition 1.1. We say that (E) has property (P_0) if its every nonoscillatory solution $y(t)$ satisfies (1.1).

In this paper, we have two purposes. In the first place, we establish comparison theorems for immediately obtaining results for third-order delay equation from that of third order equation without delay. This part extends and complements earlier papers [7, 8, 10, 18].

Secondly, we present a comparison principle for deducing the desired property of (E) from the oscillation of a second-order differential equation without delay. Here, we generalize results presented in [8, 9, 14, 15, 21].

Remark 1.2. All functional inequalities considered in this paper are assumed to hold eventually;0 that is, they are satisfied for all t large enough.

2. Main Results

It will be derived that properties of (E) are closely connected with the corresponding second-order differential equation

$$\left(\frac{1}{r(t)}\, v'(t)\right)' + p(t)v(t) = 0 \tag{E_v}$$

as the following theorem says.

Theorem 2.1. *Let $v(t)$ be a positive solution of (E_v). Then (E) can be written as*

$$\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' + q(t)v(t)y(\sigma(t)) = 0. \tag{E^c}$$

Proof. The proof follows from the fact that

$$\frac{1}{v(t)}\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' = \left(\frac{1}{r(t)}\, y''(t)\right)' + p(t)y'(t). \tag{2.1}$$

$\square$

Now, in the sequel, instead of studying properties of the trinomial equation (E), we will study the behavior of the binomial equation (E^c). For our next considerations, it is desirable for (E^c) to be in a canonical form; that is,

$$\int^{\infty} v(t)\mathrm{d}t = \infty, \tag{2.2}$$

$$\int^{\infty} \frac{r(t)}{v^2(t)}\mathrm{d}t = \infty, \tag{2.3}$$

because properties of the canonical equations are nicely explored.

Now, we will study the properties of the positive solutions of (E_v) to recognize when (2.2)-(2.3) are satisfied. The following result (see, e.g., [7, 9] or [14]) is a consequence of Sturm's comparison theorem.

Lemma 2.2. *If*

$$\frac{R^2(t)}{r(t)}p(t) \le \frac{1}{4}, \tag{2.4}$$

then (E_v) possesses a positive solution $v(t)$.

To be sure that (E_v) possesses a positive solution, we will assume throughout the paper that (2.4) holds. The following result is obvious.

Lemma 2.3. *If $v(t)$ is a positive solution of (E_v), then $v'(t) > 0$, $((1/r(t))v'(t))' < 0$, and, what is more,* (2.2) *holds and there exists $c > 0$ such that $v(t) \le cR(t)$.*

Now, we will show that if (E_v) is nonoscillatory, then we always can choose a positive solution $v(t)$ of (E_v) for which (2.3) holds.

Lemma 2.4. *If $v_1(t)$ is a positive solution of (E_v) for which* (2.3) *is violated, then*

$$v_2(t) = v_1(t)\int_{t_0}^{\infty} \frac{r(s)}{v_1^2(s)}\mathrm{d}s \tag{2.5}$$

is another positive solution of (E_v) and, for $v_2(t)$, (2.3) *holds.*

Proof. First note that

$$v_2''(t) = v_1''(t)\int_{t_0}^{t} \frac{r(s)}{v_1^2(s)}\mathrm{d}s = -p(t)v_1(t)\int_{t_0}^{t} v_1^{-2}(s)\mathrm{d}s = -p(t)v_2(t). \tag{2.6}$$

Thus, $v_2(t)$ is a positive solution of (E_v). On the other hand, to insure that (2.3) holds for $v_2(t)$, let us denote $w(t) = \int_t^{\infty} r(s)/v_1^2(s)\,\mathrm{d}s$. Then $\lim_{t\to\infty} w(t) = 0$ and

$$\int_{t_1}^{\infty} \frac{r(s)}{v_2^2(s)}\mathrm{d}s = \int_{t_1}^{\infty} \frac{-w'(s)}{w(s)}\mathrm{d}s = \lim_{t\to\infty}\left(\frac{1}{w(t)} - \frac{1}{w(t_1)}\right) = \infty. \tag{2.7}$$

□

Combining Lemmas 2.2, 2.3, and 2.4, we obtain the following result.

Lemma 2.5. *Let* (2.4) *hold. Then trinomial (E) can be represented in its binomial canonical form (E^c).*

Now we can study properties of (E) with help of its canonical representation (E^c). For our reference, let us denote for (E^c)

$$L_0y = y, \quad L_1y = \frac{1}{v}\,(L_0y)', \quad L_2y = \frac{v^2}{r}\,(L_1y)', \quad L_3y = (L_2y)'. \tag{2.8}$$

Now, (E^c) can be written as $L_3y(t) + v(t)q(t)y(\sigma(t)) = 0$.

We present a structure of the nonoscillatory solutions of (E^c). Since (E^c) is in a canonical form, it follows from the well-known lemma of Kiguradze (see, e.g., [7, 9, 14]) that every nonoscillatory solution $y(t)$ of (E^c) is either of *degree 0*, that is,

$$yL_0y(t) > 0, \quad yL_1y(t) < 0, \quad yL_2y(t) > 0, \quad yL_3y(t) < 0, \tag{2.9}$$

or of *degree 2*, that is,

$$yL_0y(t) > 0, \quad yL_1y(t) > 0, \quad yL_2y(t) > 0, \quad yL_3y(t) < 0. \tag{2.10}$$

Definition 2.6. We say that (E^c) has property (A) if its every nonoscillatory solution $y(t)$ is of degree 0; that is, it satisfies (2.9).

Now we verify that property (P_0) of (E) and property (A) of (E^c) are equivalent in the sense that $y(t)$ satisfies (1.1) if and only if it obeys (2.9).

Theorem 2.7. *Let* (2.4) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *Then (E^c) has property (A) if and only if (E) has property (P_0).*

Proof. $\rightarrow$ We suppose that $y(t)$ is a positive solution of (E). We need to verify that $y'(t) < 0$. Since $y(t)$ is also a solution of (E^c), then it satisfies (2.9). Therefore, $0 > L_1 y(t) = y'(t)/v(t)$.

$\leftarrow$ Assume that $y(t)$ is a positive solution of (E^c). We will verify that (2.9) holds. Since $y(t)$ is also a solution of (E), we see that $y'(t) < 0$; that is, $L_1 y(t) < 0$. It follows from (E^c) that $L_3 y(t) = -v(t)q(t)y(\sigma(t)) < 0$. Thus, $L_2 y(t)$ is decreasing. If we admit $L_2 y(t) < 0$ eventually, then $L_1 y(t)$ is decreasing, and integrating the inequality $L_1 y(t) < L_1 y(t_1)$, we get $y(t) < y(t_1) + L_1 y(t_1) \int_{t_1}^{t} v(s)\, ds \rightarrow -\infty$ as $t \rightarrow \infty$. Therefore, $L_2 y(t) > 0$ and (2.9) holds. □

The following result which can be found in [9, 14] presents the relationship between property (A) of delay equation and that of equation without delay.

Theorem 2.8. *Let* (2.4) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *Let*

$$\sigma(t) \in C^1([t_0, \infty)), \quad \sigma'(t) > 0. \tag{2.11}$$

If

$$\left(\frac{v^2(t)}{r(t)}\left(\frac{1}{v(t)}\, y'(t)\right)'\right)' + \frac{v(\sigma^{-1}(t))q(\sigma^{-1}(t))}{\sigma'(\sigma^{-1}(t))}\, y(t) = 0 \tag{E_2}$$

has property (A), then so does (E^c).

Combining Theorems 2.7 and 2.8, we get a criterion that reduces property (P_0) of (E) to the property (A) of (E_2).

Corollary 2.9. *Let* (2.4) *and* (2.11) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *If (E_2) has property (A) then (E) has property (P_0).*

Employing any known or future result for property (A) of (E_2), then in view of Corollary 2.9, we immediately obtain that property (P_0) holds for (E).

Example 2.10. We consider the third-order delay trinomial differential equation

$$\left(\frac{1}{t}y''(t)\right)' + \frac{\alpha(2-\alpha)}{t^3}y'(t) + q(t)y(\sigma(t)) = 0, \tag{2.12}$$

where $0 < \alpha < 1$ and $\sigma(t)$ satisfies (2.11). The corresponding equation (E_v) takes the form

$$\left(\frac{1}{t}v'(t)\right)' + \frac{\alpha(2-\alpha)}{t^3}\, v(t) = 0, \tag{2.13}$$

and it has the pair of the solutions $v(t) = t^\alpha$ and $\widehat{v}(t) = t^{2-\alpha}$. Thus, $v(t) = t^\alpha$ is our desirable solution, which permits to rewrite (2.12) in its canonical form. Then, by Corollary 2.9, (2.12) has property (P_0) if the equation

$$\left(t^{2\alpha-1}\left(t^{-\alpha}\, y'(t)\right)'\right)' + \frac{\left(\sigma^{-1}(t)\right)^\alpha q\left(\sigma^{-1}(t)\right)}{\sigma'\left(\sigma^{-1}(t)\right)} y(t) = 0 \tag{2.14}$$

has property (A).

Now, we enhance our results to guarantee stronger asymptotic behavior of the nonoscillatory solutions of (E). We impose an additional condition on the coefficients of (E) to achieve that every nonoscillatory solution of (E) tends to zero as $t \to \infty$.

Corollary 2.11. *Let* (2.4) *and* (2.11) *hold. Assume that $v(t)$ is a positive solution of (E_v) satisfying* (2.2)-(2.3). *If (E_2) has property (A) and*

$$\int_{t_0}^\infty v(s_3) \int_{s_3}^\infty \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^\infty v(s_1)q(s_1)ds_1\, ds_2\, ds_3 = \infty, \tag{2.15}$$

then every nonoscillatory solution $y(t)$ of (E) satisfies (1.2).

Proof. Assume that $y(t)$ is a positive solution of (E). Then, it follows from Corollary 2.9 that $y'(t) < 0$. Therefore, $\lim_{t\to\infty} y(t) = \ell \geq 0$. Assume $\ell > 0$. On the other hand, $y(t)$ is also a solution of (E^c), and, in view of Theorem 2.7, it has to be of *degree 0*; that is, (2.9) is fulfilled. Then, integrating (E^c) from t to ∞, we get

$$L_2 y(t) \geq \int_t^\infty v(s)q(s)y(\sigma(s))\mathrm{d}s \geq \ell \int_t^\infty v(s)q(s)\mathrm{d}s. \tag{2.16}$$

Multiplying this inequality by $r(t)/v^2(t)$ and then integrating from t to ∞, we have

$$-L_1 y(t) \geq \ell \int_t^\infty \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^\infty v(s_1)q(s_1)\mathrm{d}s_1\, \mathrm{d}s_2. \tag{2.17}$$

Multiplying this by $v(t)$ and then integrating from t_1 to t, we obtain

$$y(t_1) \geq \ell \int_{t_1}^t v(s_3) \int_{s_3}^\infty \frac{r(s_2)}{v^2(s_2)} \int_{s_2}^\infty v(s_1)q(s_1)\mathrm{d}s_1\, \mathrm{d}s_2\, \mathrm{d}s_3 \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{2.18}$$

This is a contradiction, and we deduce that $\ell = 0$. The proof is complete. □

Example 2.12. We consider once more the third-order equation (2.12). It is easy to see that (2.15) takes the form

$$\int_{t_0}^{\infty} s_3^{\alpha} \int_{s_3}^{\infty} s_2^{1-2\alpha} \int_{s_2}^{\infty} s_1^{\alpha} q(s_1) \mathrm{d}s_1 \, \mathrm{d}s_2 \, \mathrm{d}s_3 = \infty. \tag{2.19}$$

Then, by Corollary 2.11, every nonoscillatory solution of (2.12) tends to zero as $t \to \infty$ provided that (2.19) holds and (2.14) has property (A).

In the second part of this paper, we derive criteria that enable us to deduce property (P_0) of (E) from the oscillation of a suitable second-order differential equation. The following theorem is a modification of Tanaka's result [21].

Theorem 2.13. *Let* (2.4) *and* (2.11) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *Let*

$$\int^{\infty} v(s) q(s) ds < \infty. \tag{2.20}$$

If the second-order equation

$$\left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) q(s) ds\right) z(\sigma(t)) = 0 \tag{E_3}$$

is oscillatory, then (E^c) *has property* (A).

Proof. Assume that $y(t)$ is a positive solution of (E^c), then $y(t)$ is either of *degree 0* or of *degree 2*. Assume that $y(t)$ is of *degree 2*; that is, (2.10) holds. An integration of (E^c) yields

$$L_2 y(t) \geq \int_t^{\infty} v(s) q(s) y(\sigma(s)) \mathrm{d}s. \tag{2.21}$$

On the other hand,

$$y(t) \geq \int_{t_1}^{t} v(x) L_1 y(x) \mathrm{d}x. \tag{2.22}$$

Combining the last two inequalities, we get

$$\begin{aligned} L_2 y(t) &\geq \int_t^{\infty} v(s) q(s) \int_{t_1}^{\sigma(s)} v(x) L_1 y(x) \mathrm{d}x \, \mathrm{d}s \\ &\geq \int_t^{\infty} v(s) q(s) \int_{\sigma(t)}^{\sigma(s)} v(x) L_1 y(x) \mathrm{d}x \, \mathrm{d}s \\ &= \int_{\sigma(t)}^{\infty} L_1 y(x) v(x) \int_{\sigma^{-1}(x)}^{\infty} v(s) q(s) \mathrm{d}s \, \mathrm{d}x. \end{aligned} \tag{2.23}$$

Integrating the previous inequality from t_1 to t, we see that $w(t) \equiv L_1 y(t)$ satisfies

$$w(t) \geq w(t_1) + \int_{t_1}^{t} \frac{r(s)}{v^2(s)} \int_{\sigma(s)}^{\infty} L_1 y(x) v(x) \int_{\sigma^{-1}(x)}^{\infty} v(\delta) q(\delta) \mathrm{d}\delta \, \mathrm{d}x \, \mathrm{d}s. \tag{2.24}$$

Denoting the right-hand side of (2.24) by $z(t)$, it is easy to see that $z(t) > 0$ and

$$\begin{aligned} 0 &= \left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) g(s) ds\right) w(\sigma(t)) = 0 \\ &\geq \left(\frac{v^2(t)}{r(t)} z'(t)\right)' + \left(v(\sigma(t))\sigma'(t) \int_t^{\infty} v(s) g(s) ds\right) z(\sigma(t)) = 0. \end{aligned} \tag{2.25}$$

By Theorem 2 in [14], the corresponding equation (E_3) also has a positive solution. This is a contradiction. We conclude that $y(t)$ is of *degree 0*; that is, (E^c) has property (A). □

If (2.20) does not hold, then we can use the following result.

Theorem 2.14. *Let* (2.4) *and* (2.11) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If*

$$\int^{\infty} v(s) q(s) ds = \infty, \tag{2.26}$$

then (E^c) *has property (A).*

Proof. Assume that $y(t)$ is a positive solution of (E^c) and $y(t)$ is of *degree 2*. An integration of (E^c) yields

$$\begin{aligned} L_2 y(t_1) &\geq \int_{t_1}^{t} v(s) q(s) y(\sigma(s)) \mathrm{d}s \\ &\geq y(\sigma(t_1)) \int_{t_1}^{t} v(s) q(s) \mathrm{d}s \longrightarrow \infty \quad \text{as } t \longrightarrow \infty, \end{aligned} \tag{2.27}$$

which is a contradiction. Thus, $y(t)$ is of *degree 0*. The proof is complete now. □

Taking Theorem 2.13 and Corollary 2.9 into account, we get the following criterion for property (P_0) of (E).

Corollary 2.15. *Let* (2.4), (2.11), *and* (2.20) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If* (E_3) *is oscillatory, then* (E) *has property* (P_0).

Applying any criterion for oscillation of (E_3), Corollary 2.15 yields a sufficient condition property (P_0) of (E).

Corollary 2.16. *Let* (2.4), (2.11), *and* (2.20) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If*

$$\liminf_{t\to\infty}\left(\int_{t_0}^{\sigma(t)}\frac{r(s)}{v^2(s)}ds\right)\left(\int_t^{\infty}v(\sigma(x))\sigma'(x)\int_x^{\infty}v(s)g(s)ds\,dx\right)>\frac{1}{4}, \tag{2.28}$$

then (E) *has property* (P_0).

Proof. It follows from Theorem 11 in [9] that condition (2.28) guarantees the oscillation of (E_3). The proof arises from Corollary 2.16. □

Imposing an additional condition on the coefficients of (E), we can obtain that every nonoscillatory solution of (E) tends to zero as $t\to\infty$.

Corollary 2.17. *Let* (2.4) *and* (2.11) *hold. Assume that* $v(t)$ *is a positive solution of* (E_v) *satisfying* (2.2)-(2.3). *If* (2.28) *and* (2.15) *hold, then every nonoscillatory solution* $y(t)$ *of* (E) *satisfies* (1.2).

Example 2.18. We consider again (2.12). By Corollary 2.17, every nonoscillatory solution of (2.12) tends to zero as $t\to\infty$ provided that (2.19) holds and

$$\liminf_{t\to\infty}\sigma^{2-2\alpha}(t)\left(\int_t^{\infty}\sigma^{\alpha}(x)\sigma'(x)\int_x^{\infty}s^{\alpha}q(s)\mathrm{d}s\,\mathrm{d}x\right)>\frac{2-2\alpha}{4}. \tag{2.29}$$

For a special case of (2.12), namely, for

$$\left(\frac{1}{t}\,y''(t)\right)'+\frac{\alpha(2-\alpha)}{t^3}y'(t)+\frac{a}{t^4}\,y(\lambda t))=0, \tag{2.30}$$

with $0<\alpha<1$, $0<\lambda<1$, and $a>0$, we get that every nonoscillatory solution of (2.30) tends to zero as $t\to\infty$ provided that

$$\frac{a\lambda^{3-\alpha}}{(3-\alpha)(1-\alpha)^2}>1. \tag{2.31}$$

If we set $a=\beta[(\beta+1)(\beta+3)+\alpha(2-\alpha)]\lambda^{\beta}$, where $\beta>0$, then one such solution of (2.12) is $y(t)=t^{-\beta}$.

On the other hand, if for some $\gamma\in(1+\alpha,3-\alpha)$ we have $a=\gamma[(\gamma-1)(3-\gamma)+\alpha(\alpha-2)]\lambda^{-\gamma}>0$, then (2.31) is violated and (2.12) has a nonoscillatory solution $y(t)=t^{\gamma}$ which is of *degree 2*.

3. Summary

In this paper, we have introduced new comparison theorems for the investigation of properties of third-order delay trinomial equations. The comparison principle established in Corollaries 2.9 and 2.11 enables us to deduce properties of the trinomial third-order equations from that of binomial third-order equations. Moreover, the comparison theorems presented in Corollaries 2.15–2.17 permit to derive properties of the trinomial third-order equations from the oscillation of suitable second-order equations. The results obtained are of high generality, are easily applicable, and are illustrated on suitable examples.

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Research Article

Asymptotic Convergence of the Solutions of a Discrete Equation with Two Delays in the Critical Case

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A discrete equation $\Delta y(n) = \beta(n)[y(n-j) - y(n-k)]$ with two integer delays k and j, $k > j \ge 0$ is considered for $n \to \infty$. We assume $\beta : \mathbb{Z}_{n_0-k}^{\infty} \to (0,\infty)$, where $\mathbb{Z}_{n_0}^{\infty} = \{n_0, n_0+1, \ldots\}$, $n_0 \in \mathbb{N}$ and $n \in \mathbb{Z}_{n_0}^{\infty}$. Criteria for the existence of strictly monotone and asymptotically convergent solutions for $n \to \infty$ are presented in terms of inequalities for the function β. Results are sharp in the sense that the criteria are valid even for some functions β with a behavior near the so-called critical value, defined by the constant $(k-j)^{-1}$. Among others, it is proved that, for the asymptotic convergence of all solutions, the existence of a strictly monotone and asymptotically convergent solution is sufficient.

1. Introduction

We use the following notation: for integers $s, q, s \le q$, we define $\mathbb{Z}_s^q := \{s, s+1, \ldots, q\}$, where the cases $s = -\infty$ and $q = \infty$ are admitted too. Throughout this paper, using the notation $\mathbb{Z}_s^q$ or another one with a pair of integers s, q, we assume $s \le q$.

In this paper we study a discrete equation with two delays

$$\Delta y(n) = \beta(n)\left[y(n-j) - y(n-k)\right] \tag{1.1}$$

as $n \to \infty$. Integers k and j in (1.1) satisfy the inequality $k > j \ge 0$ and $\beta : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+ := (0,\infty)$, where $n_0 \in \mathbb{N}$ and $n \in \mathbb{Z}_{n_0}^{\infty}$. Without loss of generality, we assume $n_0 - k > 0$ throughout the paper (this is a technical detail, necessary for some expressions to be well defined).

The results concern the asymptotic convergence of all solutions of (1.1). We focus on what is called the *critical case* (with respect to the function β) which separates the case when all solutions are convergent from the case when there exist divergent solutions.

Such a critical case is characterized by the constant value

$$\beta(n) \equiv \beta_{\mathrm{cr}} := (k-j)^{-1}, \quad n \in \mathbb{Z}_{n_0-k}^{\infty}, \tag{1.2}$$

and below we explain its meaning and importance by an analysis of the asymptotic behavior of solutions of (1.1).

Consider (1.1) with $\beta(n) = \beta_0$, where β_0 is a positive constant; that is, we consider the following equation:

$$\Delta y(n) = \beta_0 \cdot [y(n-j) - y(n-k)]. \tag{1.3}$$

Looking for a solution of (1.3) in the form $y(n) = \lambda^n$, $\lambda \in \mathbb{C} \setminus \{0\}$ using the usual procedure, we get the characteristic equation

$$\lambda^{k+1} - \lambda^k = \beta_0 \cdot \left[\lambda^{k-j} - 1\right]. \tag{1.4}$$

Denote its roots by λ_i, $i = 1, \ldots, k+1$. Then characteristic equation (1.4) has a root $\lambda_{k+1} = 1$. Related solution of (1.3) is $y_{k+1}(n) = 1$. Then there exists a one-parametric family of constant solutions of (1.3) $y(n) = c_{k+1} y_{k+1}(n) = c_{k+1}$, where c_{k+1} is an arbitrary constant. Equation (1.4) can be rewritten as

$$\lambda^k(\lambda - 1) = \beta_0 \cdot (\lambda - 1)\left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right), \tag{1.5}$$

and, instead of (1.4), we can consider the following equation:

$$f(\lambda) := \lambda^k - \beta_0 \cdot \left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right) = 0. \tag{1.6}$$

Let $\beta_0 = \beta_{\mathrm{cr}}$. Then (1.6) has a root $\lambda_k = 1$ which is a double root of (1.4). By the theory of linear difference equations, (1.3) has a solution $y_k(n) = n$, linearly independent with $y_{k+1}(n)$. There exists a two-parametric family of solutions of (1.3)

$$y(n) = c_k y_k(n) + c_{k+1} y_{k+1}(n) = c_k n + c_{k+1}, \tag{1.7}$$

where c_k, c_{k+1} are arbitrary constants. Then $\lim_{n \to \infty} y(n) = \infty$ if $c_k \neq 0$. This means that solutions with $c_k \neq 0$ are divergent.

Let $\beta_0 < \beta_{\mathrm{cr}}$ and $k - j > 1$. We define two functions of a complex variable λ

$$F(\lambda) := \lambda^k, \qquad \Psi(\lambda) := \beta_0 \cdot \left(\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right), \tag{1.8}$$

and (1.6) can be written as

$$F(\lambda) - \Psi(\lambda) = 0. \tag{1.9}$$

By Rouche's theorem, all roots λ_i, $i = 1, 2, \dots, k$ of (1.6) satisfy $|\lambda_i| < 1$ because, on the boundary C of a unit circle $|\lambda| < 1$, we have

$$|\Psi(\lambda)|_C = \beta_0 \cdot \left|\lambda^{k-j-1} + \lambda^{k-j-2} + \cdots + 1\right| < \frac{1}{k-j}(k-j) = 1 = |F(\lambda)|_C, \tag{1.10}$$

and the functions $F(\lambda)$, $F(\lambda) - \Psi(\lambda)$ have the same number of zeros in the domain $|\lambda| < 1$.

The case $\beta_0 < \beta_{\mathrm{cr}}$ and $k - j = 1$ is trivial because (1.6) turns into

$$\lambda^k - \beta_0 = 0 \tag{1.11}$$

and, due to inequality $|\lambda|^k = \beta_0 < \beta_{\mathrm{cr}} = 1$, has all its roots in the domain $|\lambda| < 1$.

Then the relevant solutions $y_i(n)$, $i = 1, 2, \dots, k$ satisfy $\lim_{n\to\infty} y_i(n) = 0$, and the limit of the general solution of (1.3), $y(n) = \lim_{n\to\infty} \sum_{i=1}^{k+1} c_i y_i(n)$ where c_i are arbitrary constants, is finite because

$$\lim_{n\to\infty} y(n) = \lim_{n\to\infty} \sum_{i=1}^{k+1} c_i y_i(n) = c_{k+1}. \tag{1.12}$$

Let $\beta_0 > \beta_{\mathrm{cr}}$. Since $f(1) = 1 - \beta_0 \cdot (k - j) < 0$ and $f(+\infty) = +\infty$, there exists a root $\lambda = \lambda_* > 1$ of (1.6) and a solution $y_*(n) = (\lambda_*)^n$ of (1.3) satisfying $\lim_{n\to\infty} y_*(n) = \infty$. This means that solution $y_*(n)$ is divergent.

Gathering all the cases considered, we have the following:

(i) if $0 < \beta_0 < \beta_{\mathrm{cr}}$, then all solutions of (1.3) have a finite limit as $n \to \infty$,

(ii) if $\beta_0 \geq \beta_{\mathrm{cr}}$, then there exists a divergent solution of (1.3) when $n \to \infty$.

The above analysis is not applicable in the case of a nonconstant function $\beta(n)$ in (1.1). To overcome some difficulties, the method of auxiliary inequalities is applied to investigate (1.1). From our results it follows that, for example, all solutions of (1.1) have a finite limit for $n \to \infty$ (or, in accordance with the below definition, are asymptotically convergent) if there exists a $p > 1$ such that the inequality

$$\beta(n) \leq \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \tag{1.13}$$

holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$, where n_0 is a sufficiently large natural number. The limit of the right-hand side of (1.13) as $n \to \infty$ equals the critical value β_{cr}:

$$\lim_{n\to\infty} \left(\frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \right) = \frac{1}{k-j} = \beta_{\mathrm{cr}}. \tag{1.14}$$

It means that the function $\beta(n)$ in (1.1) can be sufficiently close to the critical value β_{cr} but such that all solutions of (1.1) are convergent as $n \to \infty$.

The proofs of the results are based on comparing the solutions of (1.1) with those of an auxiliary inequality that formally copies (1.1). First, we prove that, under certain conditions, (1.1) has an increasing and convergent solution $y = y(n)$ (i.e., there exists a finite limit $\lim_{n\to\infty} y(n)$). Then we extend this statement to all the solutions of (1.1). It is an interesting fact that, in the general case, the asymptotic convergence of all solutions is characterized by the existence of a strictly increasing and bounded solution.

The problem concerning the asymptotic convergence of solutions in the continuous case, that is, in the case of delayed differential equations or other classes of equations, is a classical one and has attracted much attention recently. The problem of the asymptotic convergence of solutions of discrete and difference equations with delay has not yet received much attention. We mention some papers from both of these fields (in most of them, equations and systems with a structure similar to the discrete equation (1.1) are considered).

Arino and Pituk [1], for example, investigate linear and nonlinear perturbations of a linear autonomous functional-differential equation which has infinitely many equilibria. Bereketoğlu and Karakoç [2] derive sufficient conditions for the asymptotic constancy and estimates of the limits of solutions for an impulsive system, and Györi et al. give sufficient conditions for the convergence of solutions of a nonhomogeneous linear system of impulsive delay differential equations and a limit formula in [3]. Bereketoğlu and Pituk [4] give sufficient conditions for the asymptotic constancy of solutions of nonhomogeneous linear delay differential equations with unbounded delay. The limits of the solutions can be computed in terms of the initial conditions and a special matrix solution of the corresponding adjoint equation. In [5] Diblík studies the scalar equation under the assumption that every constant is its solution. Criteria and sufficient conditions for the convergence of solutions are found. The paper by Diblík and Růžičková [6] deals with the asymptotic behavior of a first-order linear homogeneous differential equation with double delay. The convergence of solutions of the delay Volterra equation in the critical case is studied by Messina et al. in [7]. Berezansky and Braverman study a behavior of solutions of a food-limited population model with time delay in [8].

Bereketoğlu and Huseynov [9] give sufficient conditions for the asymptotic constancy of the solutions of a system of linear difference equations with delays. The limits of the solutions, as $t \to \infty$, can be computed in terms of the initial function and a special matrix solution of the corresponding adjoint equation. Dehghan and Douraki [10] study the global behavior of a certain difference equation and show, for example, that zero is always an equilibrium point which satisfies a necessary and suffient condition for its local asymptotic stability. Györi and Horváth [11] study a system of linear delay difference equations such that every solution has a finite limit at infinity. The stability of difference equations is studied intensively in papers by Stević [12, 13]. In [12], for example, he proves the global asymptotic stability of a class of difference equations. Baštinec and Diblík [14] study a class of positive and vanishing at infinity solutions of a linear difference equation with delay. Nonoscillatory solutions of second-order difference equations of the Poincaré type are investigated by Medina and Pituk in [15].

Comparing the known investigations with the results presented, we can see that our results can be applied to the critical case giving strong sufficient conditions of asymptotic convergence of solutions for this case. Nevertheless, we are not concerned with computing the limits of the solutions as $n \to \infty$.

The paper is organized as follows. In Section 2 auxiliary results are collected, an auxiliary inequality is studied, and the relationship of its solutions with the solutions of (1.1) is derived. The existence of a strictly increasing and convergent solution of (1.1) is established in Section 3. Section 4 contains results concerning the convergence of all solutions of (1.1). An example illustrating the sharpness of the results derived is given as well.

Throughout the paper we adopt the customary notation $\sum_{i=k+s}^{k}\mathcal{B}(i)=0$, where k is an integer, s is a positive integer, and $\mathcal{B}$ denotes the function under consideration regardless of whether it is defined for the arguments indicated or not.

2. Auxiliary Results

Let $\mathcal{C} := \mathcal{C}(\mathbb{Z}_{-k}^{0},\mathbb{R})$ be the space of discrete functions mapping the discrete interval $\mathbb{Z}_{-k}^{0}$ into $\mathbb{R}$. Let $v\in\mathbb{Z}_{n_0}^{\infty}$ be given. The function $y:\mathbb{Z}_{v-k}^{\infty}\to\mathbb{R}$ is said to be a *solution of* (1.1) *on* $\mathbb{Z}_{v-k}^{\infty}$ if it satisfies (1.1) for every $n\in\mathbb{Z}_{v}^{\infty}$. A solution y of (1.1) on $\mathbb{Z}_{v-k}^{\infty}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty}y(n)$ exists and is finite. For a given $v\in\mathbb{Z}_{n_0}^{\infty}$ and $\varphi\in\mathcal{C}$, we say that $y=y_{(v,\varphi)}$ is a *solution of* (1.1) defined by the initial conditions (v,φ) if $y_{(v,\varphi)}$ is a solution of (1.1) on $\mathbb{Z}_{v-k}^{\infty}$ and $y_{(v,\varphi)}(v+m)=\varphi(m)$ for $m\in\mathbb{Z}_{-k}^{0}$.

2.1. Auxiliary Inequality

The auxiliary inequality

$$\Delta\omega(n)\geq\beta(n)\left[\omega(n-j)-\omega(n-k)\right] \tag{2.1}$$

will serve as a helpful tool in the analysis of (1.1). Let $v\in\mathbb{Z}_{n_0}^{\infty}$. The function $\omega:\mathbb{Z}_{v-k}^{\infty}\to\mathbb{R}$ is said to be a *solution* of (2.1) on $\mathbb{Z}_{v-k}^{\infty}$ if ω satisfies inequality (2.1) for $n\in\mathbb{Z}_{v}^{\infty}$. A solution ω of (2.1) on $\mathbb{Z}_{v-k}^{\infty}$ is *asymptotically convergent* if the limit $\lim_{n\to\infty}\omega(n)$ exists and is finite.

We give some properties of solutions of inequalities of the type (2.1), which will be utilized later on. We will also compare the solutions of (1.1) with the solutions of inequality (2.1).

Lemma 2.1. *Let $\varphi\in\mathcal{C}$ be strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $\mathbb{Z}_{-k}^{0}$. Then the corresponding solution $y_{(n^*,\varphi)}(n)$ of* (1.1) *with $n^*\in\mathbb{Z}_{n_0}^{\infty}$ is strictly increasing (nondecreasing, strictly decreasing, nonincreasing) on $\mathbb{Z}_{n^*-k}^{\infty}$ too.*

If φ is strictly increasing (nondecreasing) and $\omega:\mathbb{Z}_{n_0-k}^{\infty}\to\mathbb{R}$ is a solution of inequality (2.1) *with $\omega(n_0+m)=\varphi(m)$, $m\in\mathbb{Z}_{n_0-k}^{n_0}$, then ω is strictly increasing (nondecreasing) on $\mathbb{Z}_{n_0-k}^{\infty}$.*

Proof. This follows directly from (1.1), inequality (2.1), and from the properties $\beta(n)>0$, $n\in\mathbb{Z}_{n_0-k}^{\infty}$, $k>j\geq0$. □

Theorem 2.2. *Let $\omega(n)$ be a solution of inequality* (2.1) *on $\mathbb{Z}_{n_0-k}^{\infty}$. Then there exists a solution $y(n)$ of* (1.1) *on $\mathbb{Z}_{n_0-k}^{\infty}$ such that the inequality*

$$y(n)\leq\omega(n) \tag{2.2}$$

holds on $\mathbb{Z}_{n_0-k}^{\infty}$. *In particular, a solution* $y(n_0,\phi)$ *of* (1.1) *with* $\phi \in C$ *defined by the equation*

$$\phi(m) := \omega(n_0+m), \quad m \in \mathbb{Z}_{-k}^{0} \tag{2.3}$$

is such a solution.

Proof. Let $\omega(n)$ be a solution of inequality (2.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. We will show that the solution $y(n) := y_{(n_0,\phi)}(n)$ of (1.1) satisfies inequality (2.2), that is,

$$y_{(n_0,\phi)}(n) \leq \omega(n) \tag{2.4}$$

on $\mathbb{Z}_{n_0-k}^{\infty}$. Let $W : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}$ be defined by $W(n) = \omega(n) - y(n)$. Then $W = 0$ on $\mathbb{Z}_{n_0-k}^{n_0}$, and, in addition, W is a solution of (2.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. Lemma 2.1 implies that W is nondecreasing. Consequently, $\omega(n) - y(n) \geq \omega(n_0) - y(n_0) = 0$ for all $n \geq n_0$. □

2.2. Comparison Lemma

Now we consider an inequality of the type (2.1)

$$\Delta\omega^*(n) \geq \beta_1(n)\left[\omega^*(n-j) - \omega^*(n-k)\right], \tag{2.5}$$

where $\beta_1 : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ is a discrete function satisfying $\beta_1(n) \geq \beta(n)$ on $\mathbb{Z}_{n_0-k}^{\infty}$. The following comparison lemma holds.

Lemma 2.3. *Let* $\omega^* : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ *be a nondecreasing positive solution of inequality* (2.5) *on* $\mathbb{Z}_{n_0-k}^{\infty}$. *Then* ω^* *is a solution of inequality* (2.1) *on* $\mathbb{Z}_{n_0-k}^{\infty}$ *too.*

Proof. Let ω^* be a nondecreasing solution of (2.5) on $\mathbb{Z}_{n_0-k}^{\infty}$. We have

$$\omega^*(n-j) - \omega^*(n-k) \geq 0 \tag{2.6}$$

because $n-k < n-j$. Then

$$\Delta\omega^*(n) \geq \beta_1(n)\left[\omega^*(n-j) - \omega^*(n-k)\right] \geq \beta(n)\left[\omega^*(n-j) - \omega^*(n-k)\right] \tag{2.7}$$

on $\mathbb{Z}_{n_0}^{\infty}$. Consequently, the function $\omega := \omega^*$ solves inequality (2.1) on $\mathbb{Z}_{n_0}^{\infty}$, too. □

2.3. A Solution of Inequality (2.1)

We will construct a solution of inequality (2.1). In the following lemma, we obtain a solution of inequality (2.1) in the form of a sum. This auxiliary result will help us derive sufficient conditions for the existence of a strictly increasing and asymptotically convergent solution of (1.1) (see Theorem 3.2 below).

Lemma 2.4. *Let there exist a discrete function* $\varepsilon : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^+$ *such that*

$$\varepsilon(n+1) \geq \sum_{i=n-k+1}^{n-j} \beta(i-1)\varepsilon(i) \tag{2.8}$$

on $\mathbb{Z}_{n_0}^{\infty}$*. Then there exists a solution* $\omega(n) = \omega_\varepsilon(n)$ *of inequality* (2.1) *defined on* $\mathbb{Z}_{n_0-k}^{\infty}$ *having the form*

$$\omega_\varepsilon(n) := \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i). \tag{2.9}$$

Proof. For $n \in \mathbb{Z}_{n_0}^{\infty}$, we get

$$\begin{aligned}
\Delta\omega_\varepsilon(n) &= \omega_\varepsilon(n+1) - \omega_\varepsilon(n) \\
&= \sum_{i=n_0-k+1}^{n+1} \beta(i-1)\varepsilon(i) - \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i) \\
&= \beta(n)\varepsilon(n+1), \\
\omega_\varepsilon(n-j) - \omega_\varepsilon(n-k) &= \sum_{i=n_0-k+1}^{n-j} \beta(i-1)\varepsilon(i) - \sum_{i=n_0-k+1}^{n-k} \beta(i-1)\varepsilon(i) \\
&= \sum_{i=n-k+1}^{n-j} \beta(i-1)\varepsilon(i).
\end{aligned} \tag{2.10}$$

We substitute ω_ε for ω in (2.1). Using (2.10), we get

$$\beta(n)\varepsilon(n+1) \geq \beta(n) \sum_{n-k+1}^{n-j} \beta(i-1)\varepsilon(i). \tag{2.11}$$

This inequality will be satisfied if inequality (2.8) holds. Indeed, reducing the last inequality by $\beta(n)$, we obtain

$$\varepsilon(n+1) \geq \sum_{n-k+1}^{n-j} \beta(i-1)\varepsilon(i), \tag{2.12}$$

which is inequality (2.8). □

2.4. Decomposition of a Function into the Difference of Two Strictly Increasing Functions

It is well known that every absolutely continuous function is representable as the difference of two increasing absolutely continuous functions [16, page 318]. We will need a simple discrete analogue of this result.

Lemma 2.5. *Every function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two strictly increasing functions $\varphi_j \in \mathcal{C}$, $j = 1, 2$, that is,*

$$\varphi(n) = \varphi_1(n) - \varphi_2(n), \qquad n \in \mathbb{Z}_{-k}^{0}. \tag{2.13}$$

Proof. Let constants $M_n > 0$, $n \in \mathbb{Z}_{-k}^{0}$ be such that inequalities

$$M_{n+1} > M_n + \max\{0, \varphi(n) - \varphi(n+1)\} \tag{2.14}$$

are valid for $n \in \mathbb{Z}_{-k}^{-1}$. We set

$$\begin{aligned} &\varphi_1(n) := \varphi(n) + M_n, \quad n \in \mathbb{Z}_{-k}^{0}, \\ &\varphi_2(n) := M_n, \quad n \in \mathbb{Z}_{-k}^{0}. \end{aligned} \tag{2.15}$$

It is obvious that (2.13) holds. Now we verify that both functions φ_j, $j = 1, 2$ are strictly increasing. The first one should satisfy $\varphi_1(n+1) > \varphi_1(n)$ for $n \in \mathbb{Z}_{-k}^{-1}$, which means that

$$\varphi(n+1) + M_{n+1} > \varphi(n) + M_n \tag{2.16}$$

or

$$M_{n+1} > M_n + \varphi(n) - \varphi(n+1). \tag{2.17}$$

We conclude that the last inequality holds because, due to (2.14), we have

$$M_{n+1} > M_n + \max\{0, \varphi(n) - \varphi(n+1)\} \ge M_n + \varphi(n) - \varphi(n+1). \tag{2.18}$$

The inequality $\varphi_2(n+1) > \varphi_2(n)$ obviously holds for $n \in \mathbb{Z}_{-k}^{-1}$ due to (2.14) as well. □

2.5. Auxiliary Asymptotic Decomposition

The following lemma can be proved easily by induction. The symbol $\mathcal{O}$ stands for the Landau order symbol.

Lemma 2.6. *For fixed $r, \sigma \in \mathbb{R} \setminus \{0\}$, the asymptotic representation*

$$(n-r)^{\sigma} = n^{\sigma}\left[1 - \frac{\sigma r}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \tag{2.19}$$

holds for $n \to \infty$.

3. Convergent Solutions of (1.1)

This part deals with the problem of detecting the existence of asymptotically convergent solutions. The results shown below provide sufficient conditions for the existence of an asymptotically convergent solution of (1.1). First we present two obvious statements concerning asymptotic convergence. From Lemma 2.1 and Theorem 2.2, we immediately get the following.

Theorem 3.1. *Let $\omega(n)$ be a strictly increasing and bounded solution of (2.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. Then there exists a strictly increasing and asymptotically convergent solution $y(n)$ of (1.1) on $\mathbb{Z}_{n_0-k}^{\infty}$.*

From Lemma 2.1, Theorem 2.2, and Lemma 2.4, we get the following.

Theorem 3.2. *Let there exist a function $\varepsilon : \mathbb{Z}_{n_0-k}^{\infty} \to \mathbb{R}^{+}$ satisfying*

$$\sum_{i=n_0-k+1}^{\infty} \beta(i-1)\varepsilon(i) < \infty \tag{3.1}$$

and inequality (2.8) on $\mathbb{Z}_{n_0}^{\infty}$. Then the initial function

$$\varphi(n) = \sum_{i=n_0-k+1}^{n_0+n} \beta(i-1)\varepsilon(i), \quad n \in \mathbb{Z}_{-k}^{0} \tag{3.2}$$

defines a strictly increasing and asymptotically convergent solution $y_{(n_0,\varphi)}(n)$ of (1.1) on $\mathbb{Z}_{n_0-k}^{\infty}$ satisfying the inequality

$$y(n) \le \sum_{i=n_0-k+1}^{n} \beta(i-1)\varepsilon(i) \tag{3.3}$$

on $\mathbb{Z}_{n_0}^{\infty}$.

Assuming that the coefficient $\beta(n)$ in (1.1) can be estimated by a suitable function, we can prove that (1.1) has a convergent solution.

Theorem 3.3. *Let there exist a $p > 1$ such that the inequality*

$$\beta(n) \le \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} \tag{3.4}$$

holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$*. Then there exists a strictly increasing and asymptotically convergent solution* $y(n)$ *of* (1.1) *as* $n \to \infty$.

Proof. In the proof, we assume (without loss of generality) that n_0 is sufficiently large for asymptotic computations to be valid. Let us verify that inequality (2.8) has a solution ε such that

$$\sum_{i=n_0-k+1}^{\infty} \beta(i-1)\varepsilon(i) < \infty. \tag{3.5}$$

We put

$$\beta(n) = \beta^*(n) := \frac{1}{k-j} - \frac{p^*}{2n}, \qquad \varepsilon(n) := \frac{1}{n^{\alpha}} \tag{3.6}$$

in inequality (2.8), where $p^* > 0$ and $\alpha > 1$ are constants. Then, for the right-hand side $\mathcal{R}(n)$ of (2.8), we have

$$\begin{aligned}
\mathcal{R}(n) &= \sum_{i=n-k+1}^{n-j} \left[\frac{1}{k-j} - \frac{p^*}{2(i-1)}\right] \frac{1}{i^{\alpha}} \\
&= \frac{1}{k-j} \sum_{i=n-k+1}^{n-j} \frac{1}{i^{\alpha}} - \frac{p^*}{2} \sum_{i=n-k+1}^{n-j} \frac{1}{(i-1)i^{\alpha}} \\
&= \frac{1}{k-j}\left[\frac{1}{(n-k+1)^{\alpha}} + \frac{1}{(n-k+2)^{\alpha}} + \cdots + \frac{1}{(n-j)^{\alpha}}\right] \\
&\quad - \frac{p^*}{2}\left[\frac{1}{(n-k)(n-k+1)^{\alpha}} + \frac{1}{(n-k+1)(n-k+2)^{\alpha}} + \cdots + \frac{1}{(n-j-1)(n-j)^{\alpha}}\right].
\end{aligned} \tag{3.7}$$

We asymptotically decompose $\mathcal{R}(n)$ as $n \to \infty$ using decomposition formula (2.19) in Lemma 2.6. We apply this formula to each term in the first square bracket with $\sigma = -\alpha$ and with $r = k-1$ for the first term, $r = k-2$ for the second term, and so forth, and, finally, $r = j$ for the last term. To estimate the terms in the second square bracket, we need only the first terms of the decomposition and the order of accuracy, which can be computed easily without using Lemma 2.6. We get

$$\begin{aligned}
\mathcal{R}(n) &= \frac{1}{(k-j)n^{\alpha}}\left[1 + \frac{\alpha(k-1)}{n} + 1 + \frac{\alpha(k-2)}{n} + \cdots + 1 + \frac{\alpha j}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] \\
&\quad - \frac{p^*}{2n^{\alpha+1}}\left[1 + 1 + \cdots + 1 + \mathcal{O}\left(\frac{1}{n}\right)\right]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(k-j)n^{\alpha+1}}\left[(k-j)n + \alpha(k-1) + \alpha(k-2) + \cdots + \alpha j + \mathcal{O}\left(\frac{1}{n}\right)\right] \\
&\quad - \frac{p^*}{2n^{\alpha+1}}\left[(k-j) + \mathcal{O}\left(\frac{1}{n}\right)\right] \\
&= \frac{1}{n^\alpha} + \frac{\alpha}{(k-j)n^{\alpha+1}}\frac{(k+j-1)(k-j)}{2} - \frac{p^*}{2n^{\alpha+1}}(k-j) + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right),
\end{aligned} \tag{3.8}$$

and, finally,

$$\mathcal{R}(n) = \frac{1}{n^\alpha} + \frac{\alpha}{2n^{\alpha+1}}(k+j-1) - \frac{p^*}{2n^{\alpha+1}}(k-j) + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right). \tag{3.9}$$

A similar decomposition of the left-hand side $\mathcal{L}(n) = \varepsilon(n+1) = (n+1)^{-\alpha}$ in inequality (2.8) leads to

$$\mathcal{L}(n) \equiv \frac{1}{(n+1)^\alpha} = \frac{1}{n^\alpha}\left[1 - \frac{\alpha}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right] = \frac{1}{n^\alpha} - \frac{\alpha}{n^{\alpha+1}} + \mathcal{O}\left(\frac{1}{n^{\alpha+2}}\right) \tag{3.10}$$

(we use decomposition formula (2.19) in Lemma 2.6 with $\sigma = -\alpha$ and $r = -1$).

Comparing $\mathcal{L}(n)$ and $\mathcal{R}(n)$, we see that, for $\mathcal{L}(n) \geq \mathcal{R}(n)$, it is necessary to match the coefficients of the terms $n^{-\alpha-1}$ because the coefficients of the terms $n^{-\alpha}$ are equal. It means that we need the inequality

$$-\alpha > \frac{\alpha(k+j-1)}{2} - \frac{p^*}{2}(k-j). \tag{3.11}$$

Simplifying this inequality, we get

$$\begin{aligned}
&\frac{p^*}{2}(k-j) > \alpha + \frac{\alpha(k+j-1)}{2}, \\
&p^*(k-j) > \alpha(k+j+1),
\end{aligned} \tag{3.12}$$

and, finally,

$$p^* > \frac{\alpha(k+j+1)}{k-j}. \tag{3.13}$$

We set

$$p^* := p\,\frac{k+j+1}{k-j}, \tag{3.14}$$

where $p = \text{const}$. Then the previous inequality holds for $p > \alpha$, that is, for $p > 1$. Consequently, the function β^* defined by (3.6) has the form

$$\beta^*(n) = \frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)n} \tag{3.15}$$

with $p > 1$, and, for the function ω_ε defined by formula (2.9), we have

$$\omega_\varepsilon(n) = \sum_{i=n_0-k+1}^{n} \left(\frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)(i-1)} \right) \frac{1}{i^\alpha}. \tag{3.16}$$

Function $\omega_\varepsilon(n)$ is a positive solution of inequality (2.1), and, moreover, it is easy to verify that $\omega_\varepsilon(\infty) < \infty$ since $\alpha > 1$. This is a solution to every inequality of the type (2.1) if the function β^* fixed by formula (3.15) is changed by an arbitrary function β satisfying inequality (3.4). This is a straightforward consequence of Lemma 2.3 if, in its formulation, we set

$$\beta_1(n) := \beta^*(n) = \frac{1}{k-j} - \frac{p(k+j+1)}{2(k-j)n} \tag{3.17}$$

with $p > 1$ since $\omega^* \equiv \omega_\varepsilon$ is the desired solution of inequality (2.5). Finally, by Theorem 3.1 with $\omega := \omega_\varepsilon$ as defined by (3.16), we conclude that there exists a strictly increasing and convergent solution $y(n)$ of (1.1) as $n \to \infty$ satisfying the inequality

$$y(n) < \omega_\varepsilon(n) \tag{3.18}$$

on $\mathbb{Z}_{n_0-k}^{\infty}$. □

4. Convergence of All Solutions

In this part we present results concerning the convergence of all solutions of (1.1). First we use inequality (3.4) to state the convergence of all the solutions.

Theorem 4.1. *Let there exist a $p > 1$ such that inequality* (3.4) *holds for all $n \in \mathbb{Z}_{n_0-k}^{\infty}$. Then all solutions of* (1.1) *are convergent as $n \to \infty$.*

Proof. First we prove that every solution defined by a monotone initial function is convergent. We will assume that a strictly monotone initial function $\varphi \in \mathcal{C}$ is given. For definiteness, let φ be strictly increasing or nondecreasing (the case when it is strictly decreasing or nonincreasing can be considered in much the same way). By Lemma 2.1, the solution $y_{(n_0,\varphi)}$ is monotone; that is, it is either strictly increasing or nondecreasing. We prove that $y_{(n_0,\varphi)}$ is convergent.

By Theorem 3.3 there exists a strictly increasing and asymptotically convergent solution $y = Y(n)$ of (1.1) on $\mathbb{Z}_{n_0-k}^{\infty}$. Without loss of generality we assume $y_{(n_0,\varphi)} \not\equiv Y(n)$ on

$\mathbb{Z}_{n_0-k}^{\infty}$ since, in the opposite case, we can choose another initial function. Similarly, without loss of generality, we can assume

$$\Delta Y(n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}. \tag{4.1}$$

Hence, there is a constant $\gamma > 0$ such that

$$\Delta Y(n) - \gamma \Delta y(n) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1} \tag{4.2}$$

or

$$\Delta \big(Y(n) - \gamma y(n)\big) > 0, \quad n \in \mathbb{Z}_{n_0-k}^{n_0-1}, \tag{4.3}$$

and the function $Y(n) - \gamma y(n)$ is strictly increasing on $\mathbb{Z}_{n_0-k}^{n_0-1}$. Then Lemma 2.1 implies that $Y(n) - \gamma y(n)$ is strictly increasing on $\mathbb{Z}_{n_0-k}^{\infty}$. Thus

$$Y(n) - \gamma y(n) > Y(n_0) - \gamma y(n_0), \quad n \in \mathbb{Z}_{n_0}^{\infty} \tag{4.4}$$

or

$$y(n) < \frac{1}{\gamma}(Y(n) - Y(n_0)) + y(n_0), \quad n \in \mathbb{Z}_{n_0}^{\infty}, \tag{4.5}$$

and, consequently, $y(n)$ is a bounded function on $\mathbb{Z}_{n_0-k}^{\infty}$ because of the boundedness of $Y(n)$. Obviously, in such a case, $y(n)$ is asymptotically convergent and has a finite limit.

Summarizing the previous section, we state that every monotone solution is convergent. It remains to consider a class of all nonmonotone initial functions. For the behavior of a solution $y_{(n_0,\varphi)}$ generated by a nonmonotone initial function $\varphi \in \mathcal{C}$, there are two possibilities: $y_{(n_0,\varphi)}$ is either eventually monotone and, consequently, convergent, or $y_{(n_0,\varphi)}$ is eventually nonmonotone.

Now we use the statement of Lemma 2.5 that every discrete function $\varphi \in \mathcal{C}$ can be decomposed into the difference of two strictly increasing discrete functions $\varphi_j \in \mathcal{C}$, $j = 1,2$. In accordance with the previous part of the proof, every function $\varphi_j \in \mathcal{C}$, $j = 1,2$ defines a strictly increasing and asymptotically convergent solution $y_{(n_0,\varphi_j)}$. Now it is clear that the solution $y_{(n_0,\varphi)}$ is asymptotically convergent. □

We will finish the paper with two obvious results. Inequality (3.4) in Theorem 3.3 was necessary only for the proof of the existence of an asymptotically convergent solution. If we assume the existence of an asymptotically convergent solution rather than inequality (3.4), we can formulate the following result, the proof of which is an elementary modification of the proof of Theorem 4.1.

Theorem 4.2. *If* (1.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}_{n_0-k}^{\infty}$, *then all the solutions of* (1.1) *defined on* $\mathbb{Z}_{n_0-k}^{\infty}$ *are asymptotically convergent.*

Combining the statements of Theorems 2.2, 3.1, and 4.2, we get a series of equivalent statements below.

Theorem 4.3. *The following three statements are equivalent.*

(a) *Equation* (1.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}_{n_0-k}^{\infty}$.

(b) *All solutions of* (1.1) *defined on* $\mathbb{Z}_{n_0-k}^{\infty}$ *are asymptotically convergent.*

(c) *Inequality* (2.1) *has a strictly monotone and asymptotically convergent solution on* $\mathbb{Z}_{n_0-k}^{\infty}$.

Example 4.4. We will demonstrate the sharpness of the criterion (3.4) by the following example. Let $k = 1$, $j = 0$, $\beta(n) = 1 - 1/n$, $n \in \mathbb{Z}_{n_0-1}^{\infty}$, $n_0 = 2$ in (1.1); that is, we consider the equation

$$\Delta y(n) = \left(1 - \frac{1}{n}\right)\left[y(n) - y(n-1)\right]. \tag{4.6}$$

By Theorems 3.3 and 4.3, all solutions are asymptotically convergent if

$$\beta(n) \le \frac{1}{k-j} - \frac{p(k+j+1)}{2n(k-j)} = 1 - \frac{p}{n}, \tag{4.7}$$

where a constant $p > 1$. In our case the inequality (4.7) does not hold since inequality

$$\beta(n) = 1 - \frac{1}{n} \le 1 - \frac{p}{n} \tag{4.8}$$

is valid only for $p \le 1$. Inequality (4.7) is sharp because there exists a solution $y = y^*(n)$ of (4.6) having the form of an nth partial sum of harmonic series, that is,

$$y^*(n) = \sum_{i=1}^{n} \frac{1}{i} \tag{4.9}$$

with the obvious property $\lim_{n\to\infty} y^*(n) = +\infty$. Then (by Theorem 4.3), all strictly monotone (increasing or decreasing) solutions of (4.6) tend to infinity.

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Research Article

Boundary Value Problems for q-Difference Inclusions

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We investigate the existence of solutions for a class of second-order q-difference inclusions with nonseparated boundary conditions. By using suitable fixed-point theorems, we study the cases when the right-hand side of the inclusions has convex as well as nonconvex values.

1. Introduction

The discretization of the ordinary differential equations is an important and necessary step towards finding their numerical solutions. Instead of the standard discretization based on the arithmetic progression, one can use an equally efficient q-discretization related to geometric progression. This alternative method leads to q-difference equations, which in the limit $q \to 1$ correspond to the classical differential equations. q-difference equations are found to be quite useful in the theory of quantum groups [1]. For historical notes and development of the subject, we refer the reader to [2, 3] while some recent results on q-difference equations can be found in [4–6]. However, the theory of boundary value problems for nonlinear q-difference equations is still in the initial stages, and many aspects of this theory need to be explored.

Differential inclusions arise in the mathematical modelling of certain problems in economics, optimal control, stochastic analysis, and so forth and are widely studied by many authors; see [7–13] and the references therein. For some works concerning difference inclusions and dynamic inclusions on time scales, we refer the reader to the papers [14–17].

In this paper, we study the existence of solutions for second-order q-difference inclusions with nonseparated boundary conditions given by

$$D_q^2 u(t) \in F(t, u(t)), \quad 0 \leq t \leq T, \tag{1.1}$$

$$u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T), \tag{1.2}$$

where $F : [0, T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact valued multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all subsets of $\mathbb{R}$, T is a fixed constant, and $\eta \neq 1$ is a fixed real number.

The aim of our paper is to establish some existence results for the Problems (1.1)-(1.2), when the right-hand side is convex as well as nonconvex valued. First of all, an integral operator is found by applying the tools of q-difference calculus, which plays a pivotal role to convert the given boundary value problem to a fixed-point problem. Our approach is simpler as it does not involve the typical series solution form of q-difference equations. The first result relies on the nonlinear alternative of Leray-Schauder type. In the second result, we will combine the nonlinear alternative of Leray-Schauder type for single-valued maps with a selection theorem due to Bressan and Colombo for lower semicontinuous multivalued maps with nonempty closed and decomposable values, while in the third result, we will use the fixed-point theorem for generalized contraction multivalued maps due to Wegrzyk. The methods used are standard; however, their exposition in the framework of Problems (1.1)-(1.2) is new.

The paper is organized as follows: in Section 2, we recall some preliminary facts that we need in the sequel, and we prove our main results in Section 3.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts which we need for the forthcoming analysis.

2.1. q-Calculus

Let us recall some basic concepts of q-calculus [1–3].

For $0 < q < 1$, we define the q-derivative of a real-valued function f as

$$D_q f(t) = \frac{f(t) - f(qt)}{(1-q)t}, \qquad D_q f(0) = \lim_{t \to 0} D_q f(t). \tag{2.1}$$

The higher-order q-derivatives are given by

$$D_q^0 f(t) = f(t), \qquad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{2.2}$$

The q-integral of a function f defined in the interval $[a, b]$ is given by

$$\int_a^x f(t) d_q t := \sum_{n=0}^{\infty} x(1-q) q^n f(xq^n) - a f(q^n a), \quad x \in [a, b], \tag{2.3}$$

and for $a = 0$, we denote

$$I_q f(x) = \int_0^x f(t) d_q t = \sum_{n=0}^{\infty} x(1-q) q^n f(xq^n), \tag{2.4}$$

provided the series converges. If $a \in [0, b]$ and f is defined in the interval $[0, b]$, then

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t. \tag{2.5}$$

Similarly, we have

$$I_q^0 f(t) = f(t), \qquad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}. \tag{2.6}$$

Observe that

$$D_q I_q f(x) = f(x), \tag{2.7}$$

and if f is continuous at $x = 0$, then

$$I_q D_q f(x) = f(x) - f(0). \tag{2.8}$$

In q-calculus, the integration by parts formula is

$$\int_0^x f(t) D_q g(t) d_q t = \left[f(t) g(t)\right]_0^x - \int_0^x D_q f(t) g(qt) d_q t. \tag{2.9}$$

2.2. Multivalued Analysis

Let us recall some basic definitions on multivalued maps [18, 19].

Let X denote a normed space with the norm $|\cdot|$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for all bounded sets B in X (i.e., $\sup_{x \in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on X if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every bounded set B in X. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e., $x_n \to x_*$, $y_n \to y_*, y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed-point if there is $x \in X$ such that $x \in G(x)$. The fixed-point set of the multivalued operator G will be denoted by $\operatorname{Fix} G$.

For more details on multivalued maps, see the books of Aubin and Cellina [20], Aubin and Frankowska [21], Deimling [18], and Hu and Papageorgiou [19].

Let $C([0,T],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$\|u\|_{\infty} = \sup\{|u(t)| : t \in [0,T]\}. \tag{2.10}$$

Let $L^1([0,T],\mathbb{R})$ be the Banach space of measurable functions $u : [0,T] \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|u\|_{L^1} = \int_0^T |u(t)|dt, \quad \forall u \in L^1([0,T],\mathbb{R}). \tag{2.11}$$

Definition 2.1. A multivalued map $G : [0,T] \to \mathcal{P}(\mathbb{R})$ with nonempty compact convex values is said to be measurable if for any $x \in \mathbb{R}$, the function

$$\mathrm{t} \longmapsto d(x,F(t)) = \inf\{|x-z| : z \in F(t)\} \tag{2.12}$$

is measurable.

Definition 2.2. A multivalued map $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be Carathéodory if

(i) $t \mapsto F(t,x)$ is measurable for each $x \in \mathbb{R}$,

(ii) $x \mapsto F(t,x)$ is upper semicontinuous for almost all $t \in [0,T]$.

Further a Carathéodory function F is called L^1-Carathéodory if

(iii) for each $\alpha > 0$, there exists $\varphi_\alpha \in L^1([0,T],\mathbb{R}^+)$ such that

$$\|F(t,x)\| = \sup\{|v| : v \in F(t,x)\} \leq \varphi_\alpha(t) \tag{2.13}$$

for all $\|x\|_\infty \leq \alpha$ and for a.e. $t \in [0,T]$.

Let E be a Banach space, let X be a nonempty closed subset of E, and let $G : X \to \mathcal{P}(E)$ be a multivalued operator with nonempty closed values. G is lower semicontinuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set B in E. Let A be a subset of $[0,T]\times\mathbb{R}$. A is $\mathcal{L}\otimes\mathcal{B}$ measurable if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J} \times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,T]$ and D is Borel measurable in $\mathbb{R}$. A subset A of $L^1([0,T],\mathbb{R})$ is decomposable if for all $u,v \in A$ and $\mathcal{J} \subset [0,T]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{J-\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

Definition 2.3. If $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values and $u(\cdot) \in C([0,\mathrm{T}],\mathbb{R})$, then $F(\cdot,\cdot)$ is of lower semicontinuous type if

$$S_F(u) = \left\{w \in L^1([0,T],\mathbb{R}) : w(t) \in F(t,u(t)) \quad \text{for a.e. } t \in [0,T]\right\} \tag{2.14}$$

is lower semicontinuous with closed and decomposable values.

Let (X, d) be a metric space associated with the norm $|\cdot|$. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, d^*(A, B) = \sup\{d(a, B) : a \in A\}, \tag{2.15}$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

Definition 2.4. A function $l : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function (see [25]) if it is continuous strictly increasing and $\sum_{n=1}^{\infty} l^n(t) < \infty$, for each $t > 0$.

Definition 2.5. A multivalued operator N on X with nonempty values in X is called

(a) γ-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(N(x), N(y)) \leq \gamma d(x, y), \quad \text{for each } x, y \in X, \tag{2.16}$$

(b) a contraction if and only if it is γ-Lipschitz with $\gamma < 1$,

(c) a generalized contraction if and only if there is a strict comparison function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_H(N(x), N(y)) \leq l(d(x, y)), \quad \text{for each } x, y \in X. \tag{2.17}$$

The following lemmas will be used in the sequel.

Lemma 2.6 (see [22]). *Let X be a Banach space. Let $F : [0, T] \times X \to \mathcal{P}(X)$ be an L^1-Carathéodory multivalued map with $S_F \neq \emptyset$, and let Γ be a linear continuous mapping from $L^1([0, T], X)$ to $C([0, T], X)$, then the operator*

$$\Gamma \circ S_F : C([0, T], X) \longrightarrow \mathcal{P}(C([0, T], X)) \tag{2.18}$$

defined by $(\Gamma \circ S_F)(x) = \Gamma(S_F(x))$ has compact convex values and has a closed graph operator in $C([0, T], X) \times C([0, T], X)$.

In passing, we remark that if $\dim X < \infty$, then $S_F(x) \neq \emptyset$ for any $x(\cdot) \in C([0, T], X)$ with $F(\cdot, \cdot)$ as in Lemma 2.6.

Lemma 2.7 (nonlinear alternative for Kakutani maps [23]). *Let E be a Banach space, C, a closed convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to \mathcal{P}_{c,cv}(C)$ is an upper semicontinuous compact map; here, $\mathcal{P}_{c,cv}(C)$ denotes the family of nonempty, compact convex subsets of C, then either*

(i) *F has a fixed-point in $\overline{U}$,*

(ii) *or there is a $u \in \partial U$ and $\lambda \in (0, 1)$ with $u \in \lambda F(u)$.*

Lemma 2.8 (see [24]). *Let Y be a separable metric space, and let $N : Y \to \mathcal{P}(L^1([0, T], \mathbb{R}))$ be a lower semicontinuous multivalued map with closed decomposable values, then $N(\cdot)$ has a continuous* *selection; that is, there exists a continuous mapping (single-valued)* $g : Y \to L^1([0,T],\mathbb{R})$ *such that* $g(y) \in N(y)$ *for every* $y \in Y$.

Lemma 2.9 (Wegrzyk's fixed-point theorem [25, 26]). *Let* (X,d) *be a complete metric space. If* $N : X \to \mathcal{P}(X)$ *is a generalized contraction with nonempty closed values, then* $\mathrm{Fix}N \neq \emptyset$.

Lemma 2.10 (Covitz and Nadler's fixed-point theorem [27]). *Let* (X,d) *be a complete metric space. If* $N : X \to \mathcal{P}(X)$ *is a multivalued contraction with nonempty closed values, then* N *has a fixed-point* $z \in X$ *such that* $z \in N(z)$, *that is,* $\mathrm{Fix}N \neq \emptyset$.

3. Main Results

In this section, we are concerned with the existence of solutions for the Problems (1.1)-(1.2) when the right-hand side has convex as well as nonconvex values. Initially, we assume that F is a compact and convex valued multivalued map.

To define the solution for the Problems (1.1)-(1.2), we need the following result.

Lemma 3.1. *Suppose that* $\sigma : [0,T] \to \mathbb{R}$ *is continuous, then the following problem*

$$\begin{gathered} D_q^2 u(t) = \sigma(t), \quad \text{a.e. } t \in [0,T], \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T) \end{gathered} \tag{3.1}$$

has a unique solution

$$u(t) = \int_0^T G(t,qs)\sigma(s)\,d_q s, \tag{3.2}$$

where $G(t,qs)$ *is the Green's function given by*

$$G(t,qs) = \frac{1}{(\eta-1)^2}\begin{cases} \eta(\eta-1)(qs-t) + \eta T, & \text{if } 0 \le t < s \le T, \\ (\eta-1)(qs-t) + \eta T, & \text{if } 0 \le s \le t \le T. \end{cases} \tag{3.3}$$

Proof. In view of (2.7) and (2.9), the solution of $D_q^2 u = \sigma(t)$ can be written as

$$u(t) = \int_0^t (t-qs)\sigma(s)\,d_q s + a_1 t + a_2, \tag{3.4}$$

where a_1, a_2 are arbitrary constants. Using the boundary conditions (1.2) and (3.4), we find that

$$a_1 = \frac{-\eta}{(\eta-1)} \int_0^T \sigma(s) d_q s,$$
$$a_2 = \frac{\eta^2 T}{(\eta-1)^2} \int_0^T \sigma(s) d_q s - \frac{\eta}{(\eta-1)} \int_0^T (T-qs)\sigma(s) d_q s. \tag{3.5}$$

Substituting the values of a_1 and a_2 in (3.4), we obtain (3.2). □

Let us denote

$$G_1 = \max_{t,s\in[0,T]} |G(t,qs)|. \tag{3.6}$$

Definition 3.2. A function $u \in C([0,T],\mathbb{R})$ is said to be a solution of (1.1)-(1.2) if there exists a function $v \in L^1([0,T],\mathbb{R})$ with $v(t) \in F(t,x(t))$ a.e. $t \in [0,T]$ and

$$u(t) = \int_0^T G(t,qs) v(s) d_q s, \tag{3.7}$$

where $G(t,qs)$ is given by (3.3).

Theorem 3.3. *Suppose that*

(H1) *the map $F : [0,T]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ has nonempty compact convex values and is Carathéodory,*

(H2) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in L^1([0,T],\mathbb{R}_+)$ such that*

$$\|F(t,u)\|_{\mathcal{P}} := \sup\{|v| : v \in F(t,u)\} \le p(t)\psi(\|u\|_\infty) \tag{3.8}$$

for each $(t,u) \in [0,T]\times\mathbb{R}$,

(H3) *there exists a number $M > 0$ such that*

$$\frac{M}{G_1\psi(M)\|p\|_{L^1}} > 1, \tag{3.9}$$

then the BVP (1.1)-(1.2) has at least one solution.

Proof. In view of Definition 3.2, the existence of solutions to (1.1)-(1.2) is equivalent to the existence of solutions to the integral inclusion

$$u(t) \in \int_0^T G(t,qs) F(s,u(s)) d_q s, \quad t \in [0,T]. \tag{3.10}$$

Let us introduce the operator

$$N(u) := \left\{ h \in C([0,T],\mathbb{R}) : h(t) = \int_0^T G(t,qs)v(s)d_qs,\ v \in S_{F,u} \right\}. \tag{3.11}$$

We will show that N satisfies the assumptions of the nonlinear alternative of Leray-Schauder type. The proof will be given in several steps.

Step 1 ($N(u)$ is convex for each $u \in C([0,T],\mathbb{R})$). Indeed, if h_1, h_2 belong to $N(u)$, then there exist $v_1, v_2 \in S_{F,u}$ such that for each $t \in [0,T]$, we have

$$h_i(t) = \int_0^T G(t,qs)v_i(s)d_qs, \quad (i = 1,2). \tag{3.12}$$

Let $0 \leq d \leq 1$, then, for each $t \in [0,T]$, we have

$$(dh_1 + (1-d)h_2)(t) = \int_0^T G(t,qs)[dv_1(s) + (1-d)v_2(s)]d_qs. \tag{3.13}$$

Since $S_{F,u}$ is convex (because F has convex values); therefore,

$$dh_1 + (1-d)h_2 \in N(u). \tag{3.14}$$

Step 2 (N maps bounded sets into bounded sets in $C([0,T],\mathbb{R})$). Let $B_m = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty \leq m, m > 0\}$ be a bounded set in $C([0,T],\mathbb{R})$ and $u \in B_m$, then for each $h \in N(u)$, there exists $v \in S_{F,u}$ such that

$$h(t) = \int_0^T G(t,qs)v(s)d_qs. \tag{3.15}$$

Then, in view of (H2), we have

$$\begin{aligned} |h(t)| &\leq \int_0^T \left|G(t,qs)\right| |v(s)| d_qs \\ &\leq G_1 \int_0^T p(s)\psi(\|u\|_\infty)d_qs \\ &\leq G_1\psi(m)\int_0^T p(s)d_qs. \end{aligned} \tag{3.16}$$

Thus,

$$\|h\|_\infty \leq G_1\psi(m)\left\|p\right\|_{L^1}. \tag{3.17}$$

Step 3 (N maps bounded sets into equicontinuous sets of $C([0,T],\mathbb{R})$). Let $r_1, r_2 \in [0, T]$, $r_1 < r_2$ and B_m be a bounded set of $C([0,T],\mathbb{R})$ as in Step 2 and $x \in B_m$. For each $h \in N(u)$

$$
\begin{aligned}
|h(r_2) - h(r_1)| &\le \int_0^T |G(r_2, s) - G(r_1, s)||v(s)|d_q s \\
&\le \varphi(\|u\|_\infty) \int_0^T |G(r_2, s) - G(r_1, s)|p(s)d_q s \\
&\le \varphi(m) \int_0^T |G(r_2, s) - G(r_1, s)|p(s)d_q s.
\end{aligned} \tag{3.18}
$$

The right-hand side tends to zero as $r_2 - r_1 \to 0$. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli Theorem, we can conclude that $N : C([0,T],\mathbb{R}) \to \mathcal{P}(C([0,T],\mathbb{R}))$ is completely continuous.

Step 4 (N has a closed graph). Let $u_n \to u_*$, $h_n \in N(u_n)$, and $h_n \to h_*$. We need to show that $h_* \in N(u_*)$. $h_n \in N(u_n)$ means that there exists $v_n \in S_{F,u_n}$ such that, for each $t \in [0,T]$,

$$
h_n(t) = \int_0^T G(t, qs) v_n(s) d_q s. \tag{3.19}
$$

We must show that there exists $h_* \in S_{F,u_*}$ such that, for each $t \in [0,T]$,

$$
h_*(t) = \int_0^T G(t, qs) v_*(s) d_q s. \tag{3.20}
$$

Clearly, we have

$$
\|h_n - h_*\|_\infty \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{3.21}
$$

Consider the continuous linear operator

$$
\Gamma : L^1([0,T],\mathbb{R}) \longrightarrow C([0,T],\mathbb{R}), \tag{3.22}
$$

defined by

$$
v \longmapsto (\Gamma v)(t) = \int_0^T G(t, qs) v(s) d_q s. \tag{3.23}
$$

From Lemma 2.6, it follows that $\Gamma \circ S_F$ is a closed graph operator. Moreover, we have

$$
h_n(t) \in \Gamma(S_{F,u_n}). \tag{3.24}
$$

Since $u_n \to u_*$, it follows from Lemma 2.6 that

$$h_*(t) = \int_0^T G(t, qs) v_*(s) d_q s \tag{3.25}$$

for some $v_* \in S_{F,u_*}$.

Step 5 (a priori bounds on solutions). Let u be a possible solution of the Problems (1.1)-(1.2), then there exists $v \in L^1([0,T],\mathbb{R})$ with $v \in S_{F,u}$ such that, for each $t \in [0,T]$,

$$u(t) = \int_0^T G(t, qs) v(s) d_q s. \tag{3.26}$$

For each $t \in [0,T]$, it follows by (H2) and (H3) that

$$\begin{aligned} |u(t)| &\le G_1 \int_0^T p(s)\psi(\|u\|_\infty) d_q s \\ &\le G_1 \psi(\|u\|_\infty) \int_0^T p(s) d_q s. \end{aligned} \tag{3.27}$$

Consequently,

$$\frac{\|u\|_\infty}{G_1 \psi(\|u\|_\infty) \|p\|_{L^1}} \le 1. \tag{3.28}$$

Then by (H3), there exists M such that $\|u\|_\infty \neq M$.

Let

$$U = \{u \in C([0,T],\mathbb{R}) : \|u\|_\infty < M + 1\}. \tag{3.29}$$

The operator $N : \overline{U} \to \mathcal{P}(C([0,T],\mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $u \in \lambda N(u)$ for some $\lambda \in (0,1)$. Consequently, by Lemma 2.7, it follows that N has a fixed-point u in $\overline{U}$ which is a solution of the Problems (1.1)-(1.2). This completes the proof. □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the nonlinear alternative of Leray-Schauder type combined with the selection theorem of Bressan and Colombo for lower semicontinuous maps with decomposable values.

Theorem 3.4. *Suppose that the conditions (H2) and (H3) hold. Furthermore, it is assumed that*

(H4) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact values and*

(a) $(t,u) \mapsto F(t,u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable,*

(b) $u \mapsto F(t,u)$ *is lower semicontinuous for a.e.* $t \in [0,T]$, (H5) *for each* $\rho > 0$, *there exists* $\varphi_\rho \in L^1([0,T],\mathbb{R}_+)$ *such that*

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le \varphi_\rho(t) \quad \forall \|u\|_\infty \le \rho \text{ and for a.e. } t \in [0,T]. \tag{3.30}$$

then, the BVP (1.1)-(1.2) *has at least one solution.*

Proof. Note that (H4) and (H5) imply that F is of lower semicontinuous type. Thus, by Lemma 2.8, there exists a continuous function $f : C([0,T],\mathbb{R}) \to L^1([0,T],\mathbb{R})$ such that $f(u) \in \mathcal{F}(u)$ for all $u \in C([0,T],\mathbb{R})$. So we consider the problem

$$\begin{gathered} D_q^2 u(t) = f(u(t)), \quad 0 \le t \le T, \\ u(0) = \eta u(T), \qquad D_q u(0) = \eta D_q u(T). \end{gathered} \tag{3.31}$$

Clearly, if $u \in C([0,T],\mathbb{R})$ is a solution of (3.31), then u is a solution to the Problems (1.1)-(1.2). Transform the Problem (3.31) into a fixed-point theorem

$$u(t) = \left(\overline{N}u\right)(t), \quad t \in [0,T], \tag{3.32}$$

where

$$\left(\overline{N}u\right)(t) = \int_0^T G(t,qs) f(u(s)) d_q s, \quad t \in [0,T]. \tag{3.33}$$

We can easily show that $\overline{N}$ is continuous and completely continuous. The remainder of the proof is similar to that of Theorem 3.3. □

Now, we prove the existence of solutions for the Problems (1.1)-(1.2) with a nonconvex valued right-hand side by applying Lemma 2.9 due to Wegrzyk.

Theorem 3.5. *Suppose that*

(H6) $F : [0,T] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ *has nonempty compact values and* $F(\cdot,u)$ *is measurable for each* $u \in \mathbb{R}$,

(H7) $d_H(F(t,u),F(t,\overline{u})) \le k(t) l(|u-\overline{u}|)$ *for almost all* $t \in [0,1]$ *and* $u,\overline{u} \in \mathbb{R}$ *with* $k \in L^1([0,1],\mathbb{R}_+)$ *and* $d(0,F(t,0)) \le k(t)$ *for almost all* $t \in [0,1]$, *where* $l : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing,*

then the BVP (1.1)-(1.2) *has at least one solution on* $[0,T]$ *if* $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strict comparison function, where* $\gamma = G_1 \|k\|_{L^1}$.

Proof. Suppose that $\gamma l : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions (H6) and (H7), $F(\cdot, u(\cdot))$ is measurable and has a measurable selection $v(\cdot)$ (see Theorem 3.6 [28]). Also $k \in L^1([0,1],\mathbb{R})$ and

$$\begin{aligned} |v(t)| &\le d(0, F(t,0)) + H_d(F(t,0), F(t,u(t))) \\ &\le k(t) + k(t) l(|u(t)|) \\ &\le (1 + l(\|u\|_\infty)) k(t). \end{aligned} \tag{3.34}$$

Thus, the set $S_{F,u}$ is nonempty for each $u \in C([0,T],\mathbb{R})$.

As before, we transform the Problems (1.1)-(1.2) into a fixed-point problem by using the multivalued map N given by (3.11) and show that the map N satisfies the assumptions of Lemma 2.9. To show that the map $N(u)$ is closed for each $u \in C([0,T],\mathbb{R})$, let $(u_n)_{n\ge 0} \in N(u)$ such that $u_n \to \tilde{u}$ in $C([0,T],\mathbb{R})$, then $\tilde{u} \in C([0,T],\mathbb{R})$ and there exists $v_n \in S_{F,u}$ such that, for each $t \in [0,T]$,

$$u_n(t) = \int_0^T G(\mathrm{t}, qs) v_n(s) d_q s. \tag{3.35}$$

As F has compact values, we pass onto a subsequence to obtain that v_n converges to v in $L^1([0,T],\mathbb{R})$. Thus, $v \in S_{F,u}$ and for each $t \in [0,T]$,

$$u_n(t) \longrightarrow \tilde{u}(t) = \int_0^T G(t, qs) v(s) d_q s. \tag{3.36}$$

So, $\tilde{u} \in N(u)$ and hence $N(u)$ is closed.

Next, we show that

$$d_H(N(u), N(\overline{u})) \le \gamma l(\|u - \overline{u}\|_\infty) \quad \text{for each } u, \overline{u} \in C([0,T],\mathbb{R}). \tag{3.37}$$

Let $u, \overline{u} \in C([0,T],\mathbb{R})$ and $h_1 \in N(u)$. Then, there exists $v_1(t) \in S_{F,u}$ such that for each $t \in [0,T]$,

$$h_1(t) = \int_0^T G(t, qs) v_1(s) d_q s. \tag{3.38}$$

From (H7), it follows that

$$d_H(F(t, u(t)), F(t, \overline{u}(t))) \le k(t) l(|u(t) - \overline{u}(t)|). \tag{3.39}$$

So, there exists $w \in F(t, \overline{u}(t))$ such that

$$|v_1(t) - w| \le k(t) l(|u(t) - \overline{u}(t)|), \quad t \in [0,T]. \tag{3.40}$$

Define $U : [0,T] \to \mathcal{P}(\mathbb{R})$ as

$$U(t) = \{w \in \mathbb{R} : |v_1(t) - w| \le k(t)l(|u(t) - \overline{u}(t)|)\}. \tag{3.41}$$

Since the multivalued operator $U(t) \cap F(t, \overline{u}(t))$ is measurable (see Proposition 3.4 in [28]), there exists a function $v_2(t)$ which is a measurable selection for $U(t) \cap F(t, \overline{u}(t))$. So, $v_2(t) \in F(t, \overline{u}(t))$, and for each $t \in [0,T]$,

$$|v_1(t) - v_2(t)| \le k(t)l(|u(t) - \overline{u}(t)|). \tag{3.42}$$

For each $t \in [0,T]$, let us define

$$h_2(t) = \int_0^T G(t, qs)v_2(s)d_qs, \tag{3.43}$$

then

$$\begin{aligned} |h_1(t) - h_2(t)| &\le \int_0^T |G(t, qs)||v_1(s) - v_2(s)|d_qs \\ &\le G_1 \int_0^T k(s)l(\|u - \overline{u}\|)d_qs. \end{aligned} \tag{3.44}$$

Thus,

$$\|h_1 - h_2\|_\infty \le G_1\|k\|_{L^1} l(\|u - \overline{u}\|_\infty) = \gamma l(\|u - \overline{u}\|_\infty). \tag{3.45}$$

By an analogous argument, interchanging the roles of u and $\overline{u}$, we obtain

$$d_H(N(u), N(\overline{u})) \le G_1\|k\|_{L^1} l(\|u - \overline{u}\|_\infty) = \gamma l(\|u - \overline{u}\|_\infty) \tag{3.46}$$

for each $u, \overline{u} \in C([0,T], \mathbb{R})$. So, N is a generalized contraction, and thus, by Lemma 2.9, N has a fixed-point u which is a solution to (1.1)-(1.2). This completes the proof. □

Remark 3.6. We notice that Theorem 3.5 holds for several values of the function l, for example, $l(t) = \ln(1+t)/\chi$, where $\chi \in (0,1)$, $l(t) = t$, and so forth. Here, we emphasize that the condition (H7) reduces to $d_H(F(t,u), F(t,\overline{u})) \le k(t)|u - \overline{u}|$ for $l(t) = t$, where a contraction principle for multivalued map due to Covitz and Nadler [27] (Lemma 2.10) is applicable under the condition $G_1\|k\|_{L^1} < 1$. Thus, our result dealing with a nonconvex valued right-hand side of (1.1) is more general, and the previous results for nonconvex valued right-hand side of the inclusions based on Covitz and Nadler's fixed-point result (e.g., see [8]) can be extended to generalized contraction case.

Remark 3.7. Our results correspond to the ones for second-order q-difference inclusions with antiperiodic boundary conditions $(u(0) = -u(T), D_q u(0) = -D_q u(T))$ for $\eta = -1$. The results for an initial value problem of second-order q-difference inclusions follow for $\eta = 0$. These results are new in the present configuration.

Remark 3.8. In the limit $q \to 1$, the obtained results take the form of their "continuous" (i.e., differential) counterparts presented in Sections 4 (ii) for $\lambda_1 = \lambda_2 = \eta, \mu_1 = 0 = \mu_2$ of [29].

Example 3.9. Consider a boundary value problem of second-order q-difference inclusions given by

$$\begin{gathered} D_q^2 u(t) \in F(t, u(t)), \quad 0 \leq t \leq 1 \\ u(0) = -u(1), \qquad D_q u(0) = -D_q u(1), \end{gathered} \tag{3.47}$$

where $\eta = -1$ and $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map given by

$$(t, u) \longrightarrow F(t, u) = \left[\frac{u^3}{u^3 + 3} + t^3 + 3, \frac{u}{u+1} + t + 1 \right]. \tag{3.48}$$

For $f \in F$, we have

$$|f| \leq \max\left(\frac{u^3}{u^3 + 3} + t^3 + 3, \frac{u}{u+1} + t + 1 \right) \leq 5, \quad u \in \mathbb{R}. \tag{3.49}$$

Thus,

$$\|F(t, u)\|_{\mathcal{P}} := \sup\{|y| : y \in F(t, u)\} \leq 5 = p(t)\psi(\|u\|_\infty), \quad u \in \mathbb{R}, \tag{3.50}$$

with $p(t) = 1, \psi(\|u\|_\infty) = 5$. Further, using the condition

$$\frac{M}{G_1 \psi(M) \|p\|_{L^1}} > 1, \tag{3.51}$$

we find that $M > 5G_2$, where $G_2 = G_1|_{\eta=-1, T=1}$. Clearly, all the conditions of Theorem 3.3 are satisfied. So, the conclusion of Theorem 3.3 applies to the Problem (3.47).

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Research Article

Boundary-Value Problems for Weakly Nonlinear Delay Differential Systems

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Conditions are derived of the existence of solutions of nonlinear boundary-value problems for systems of n ordinary differential equations with constant coefficients and single delay (in the linear part) and with a finite number of measurable delays of argument in nonlinearity: $\dot{z}(t) = Az(t-\tau) + g(t) + \varepsilon Z(z(h_i(t), t, \varepsilon),\ t \in [a,b]$, assuming that these solutions satisfy the initial and boundary conditions $z(s) := \psi(s)$ if $s \notin [a,b]$, $\ell z(\cdot) = \alpha \in \mathbb{R}^m$. The use of a delayed matrix exponential and a method of pseudoinverse by Moore-Penrose matrices led to an *explicit* and *analytical* form of sufficient conditions for the existence of solutions in a given space and, moreover, to the construction of an iterative process for finding the solutions of such problems in a general case when the number of boundary conditions (defined by a linear vector functional ℓ) does not coincide with the number of unknowns in the differential system with a single delay.

1. Introduction

First, we derive some auxiliary results concerning the theory of differential equations with delay. Consider a system of linear differential equations with concentrated delay

$$\dot{z}(t) - A(t)z(h_0(t)) = g(t) \quad \text{if } t \in [a,b], \tag{1.1}$$

assuming that

$$z(s) := \psi(s) \quad \text{if } s \notin [a,b], \tag{1.2}$$

where A is an $n\times n$ real matrix and g is an n-dimensional real column-vector with components in the space $L_p[a,b]$ (where $p \in [1,\infty)$) of functions summable on $[a,b]$; the delay $h_0(t) \leq t$ is a function $h_0 : [a,b] \to \mathbb{R}$ measurable on $[a,b]$; $\psi : \mathbb{R} \setminus [a,b] \to \mathbb{R}^n$ is a given function. Using the denotations

$$(S_{h_0}z)(t) := \begin{cases} z(h_0(t)) & \text{if } h_0(t) \in [a,b], \\ \theta & \text{if } h_0(t) \notin [a,b], \end{cases} \tag{1.3}$$

$$\psi^{h_0}(t) := \begin{cases} \theta & \text{if } h_0(t) \in [a,b], \\ \psi(h_0(t)) & \text{if } h_0(t) \notin [a,b], \end{cases} \tag{1.4}$$

where θ is an n-dimensional zero column-vector and assuming $t \in [a,b]$, it is possible to rewrite (1.1), (1.2) as

$$(Lz)(t) := \dot{z}(t) - A(t)(S_{h_0}z)(t) = \varphi(t), \quad t \in [a,b], \tag{1.5}$$

where φ is an n-dimensional column-vector defined by the formula

$$\varphi(t) := g(t) + A(t)\psi^{h_0}(t) \in L_p[a,b]. \tag{1.6}$$

We will investigate (1.5) assuming that the operator L maps a Banach space $D_p[a,b]$ of absolutely continuous functions $z : [a,b] \to \mathbb{R}^n$ into a Banach space $L_p[a,b]$ $(1 \leq p < \infty)$ of functions $\varphi : [a,b] \to \mathbb{R}^n$ summable on $[a,b]$; the operator S_{h_0} maps the space $D_p[a,b]$ into the space $L_p[a,b]$. Transformations (1.3), (1.4) make it possible to add the initial function $\psi(s)$, $s < a$ to nonhomogeneity generating an additive and homogeneous operation not depending on ψ and without the classical assumption regarding the continuous connection of solution $z(t)$ with the initial function $\psi(t)$ at the point $t = a$.

A solution of differential system (1.5) is defined as an n-dimensional column vector-function $z \in D_p[a,b]$, absolutely continuous on $[a,b]$, with a derivative $\dot{z} \in L_p[a,b]$ satisfying (1.5) almost everywhere on $[a,b]$.

Such approach makes it possible to apply well-developed methods of linear functional analysis to (1.5) with a linear and bounded operator L. It is well-known (see: [1, 2]) that a nonhomogeneous operator equation (1.5) with delayed argument is solvable in the space $D_p[a,b]$ for an arbitrary right-hand side $\varphi \in L_p[a,b]$ and has an n-dimensional family of solutions ($\dim \ker L = n$) in the form

$$z(t) = X(t)c + \int_a^b K(t,s)\varphi(s)ds \quad \forall c \in \mathbb{R}^n, \tag{1.7}$$

where the kernel $K(t,s)$ is an $n \times n$ Cauchy matrix defined in the square $[a,b] \times [a,b]$ being, for every fixed $s \le t$, a solution of the matrix Cauchy problem

$$(LK(\cdot,s))(t) := \frac{\partial K(t,s)}{\partial t} - A(t)(S_{h_0}K(\cdot,s))(t) = \Theta, \quad K(s,s) = I, \tag{1.8}$$

where $K(t,s) \equiv \Theta$ if $a \le t < s \le b$, Θ is $n \times n$ null matrix and I is $n \times n$ identity matrix. A fundamental $n \times n$ matrix $X(t)$ for the homogeneous ($\varphi \equiv \theta$) equation (1.5) has the form $X(t) = K(t,a)$, $X(a) = I$ [2]. Throughout the paper, we denote by Θ_s an $s \times s$ null matrix if $s \neq n$, by $\Theta_{s,p}$ an $s \times p$ null matrix, by I_s an $s \times s$ identity matrix if $s \neq n$, and by θ_s an s-dimensional zero column-vector if $s \neq n$.

A serious disadvantage of this approach, when investigating the above-formulated problem, is the necessity to find the Cauchy matrix $K(t,s)$ [3, 4]. It exists but, as a rule, can only be found numerically. Therefore, it is important to find systems of differential equations with delay such that this problem can be solved directly. Below we consider the case of a system with so-called single delay [5]. In this case, the problem of how to construct the Cauchy matrix is successfully solved *analytically* due to a delayed matrix exponential defined below.

1.1. A Delayed Matrix Exponential

Consider a Cauchy problem for a linear nonhomogeneous differential system with constant coefficients and with a single delay τ

$$\dot{z}(t) = Az(t-\tau) + g(t), \tag{1.9}$$

$$z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0], \tag{1.10}$$

with an $n \times n$ constant matrix A, $g : [0,\infty) \to \mathbb{R}^n$, $\psi : [-\tau, 0] \to \mathbb{R}^n$, $\tau > 0$ and an unknown vector-solution $z : [-\tau,\infty) \to \mathbb{R}^n$. Together with a nonhomogeneous problem (1.9), (1.10), we consider a related homogeneous problem

$$\begin{gathered} \dot{z}(t) = Az(t-\tau), \\ z(s) = \psi(s), \quad \text{if } s \in [-\tau, 0]. \end{gathered} \tag{1.11}$$

Denote by e_τ^{At} a matrix function called a delayed matrix exponential (see [5]) and defined as

$$e_\tau^{At} := \begin{cases} \Theta & \text{if } -\infty < t < -\tau, \\ I & \text{if } -\tau \le t < 0, \\ I + A\dfrac{t}{1!} & \text{if } 0 \le t < \tau, \\ I + A\dfrac{t}{1!} + A^2\dfrac{(t-\tau)^2}{2!} & \text{if } \tau \le t < 2\tau, \\ \cdots & \\ I + A\dfrac{t}{1!} + \cdots + A^k\dfrac{(t-(k-1)\tau)^k}{k!} & \text{if } (k-1)\tau \le t < k\tau, \\ \cdots . & \end{cases} \tag{1.12}$$

This definition can be reduced to the following expression:

$$e_\tau^{At} = \sum_{n=0}^{[t/\tau]+1} A^n \, \frac{(t-(n-1)\tau)^n}{n!}, \tag{1.13}$$

where $[t/\tau]$ is the greatest integer function. The delayed matrix exponential equals the unit matrix I on $[-\tau, 0]$ and represents a fundamental matrix of a homogeneous system with single delay. Thus, the delayed matrix exponential solves the Cauchy problem for a homogeneous system (1.11), satisfying the unit initial conditions

$$z(s) = \varphi(s) \equiv e_\tau^{As} = I \quad \text{if } -\tau \le s \le 0, \tag{1.14}$$

and the following statement holds (see, e.g., [5], [6, Remark 1], [7, Theorem 2.1]).

Lemma 1.1. *A solution of a Cauchy problem for a nonhomogeneous system with single delay* (1.9)*, satisfying a constant initial condition*

$$z(s) = \varphi(s) = c \in \mathbb{R}^n \quad \textit{if } s \in [-\tau, 0] \tag{1.15}$$

has the form

$$z(t) = e_\tau^{A(t-\tau)}c + \int_0^t e_\tau^{A(t-\tau-s)} g(s)\,ds. \tag{1.16}$$

The delayed matrix exponential was applied, for example, in [6, 7] to investigation of boundary value problems of diffferential systems with a single delay and in [8] to investigation of the stability of linear perturbed systems with a single delay.

1.2. Fredholm Boundary-Value Problem

Without loss of generality, let $a = 0$ and, with a view of the above, the problem (1.9), (1.10) can be transformed $(h_0(t) := t - \tau)$ to an equation of the type (1.1) (see (1.5))

$$\dot{z}(t) - A(S_{h_0}z)(t) = \varphi(t), \quad t \in [0, b], \tag{1.17}$$

where, in accordance with (1.3),(1.4)

$$\begin{gathered}
(S_{h_0}z)(t) = \begin{cases} z(t-\tau) & \text{if } t-\tau \in [0,b], \\ \theta & \text{if } t-\tau \notin [0,b], \end{cases} \\
\varphi(t) = \ g(t) + A\ \psi^{h_0}(t) \ \in \ L_p[0,b], \\
\psi^{h_0}(t) = \begin{cases} \theta & \text{if } t-\tau \in [0,b], \\ \psi(t-\tau) & \text{if } t-\tau \notin [0,b]. \end{cases}
\end{gathered} \tag{1.18}$$

A general solution of problem (1.17) for a nonhomogeneous system with single delay and zero initial data has the form (1.7)

$$z(t) = X(t)c + \int_0^b K(t,s)\varphi(s)ds \quad \forall c \in \mathbb{R}^n, \tag{1.19}$$

where, as can easily be verified (in view of the above-defined delayed matrix exponential) by substituting into (1.17),

$$X(t) = e_\tau^{A(t-\tau)}, \qquad X(0) = e_\tau^{-A\tau} = I \tag{1.20}$$

is a normal fundamental matrix of the homogeneous system related to (1.17) (or (1.9)) with initial data $X(0) = I$, and the Cauchy matrix $K(t,s)$ has the form

$$\begin{gathered}
K(t,s) = e_\tau^{A(t-\tau-s)} \quad \text{if } 0 \le s < t \le b, \\
K(t,s) \equiv \Theta \quad \text{if } 0 \le t < s \le b.
\end{gathered} \tag{1.21}$$

Obviously

$$K(t,0) = e_\tau^{A(t-\tau)} = X(t), \qquad K(0,0) = e_\tau^{A(-\tau)} = X(0) = I, \tag{1.22}$$

and, therefore, the initial problem (1.17) for systems of ordinary differential equations with constant coefficients and single delay has an n-parametric family of linearly independent solutions (1.16).

Now, we will deal with a general boundary-value problem for system (1.17). Using the results [2, 9], it is easy to derive statements for a general boundary-value problem if the number m of boundary conditions does not coincide with the number n of unknowns in a differential system with single delay.

We consider a boundary-value problem

$$\begin{gathered} \dot{z}(t) - Az(t-\tau) = g(t), \quad t \in [0,b], \\ z(s) := \psi(s), \quad s \notin [0,b], \end{gathered} \tag{1.23}$$

assuming that

$$\ell z(\cdot) = \alpha \in \mathbb{R}^m, \tag{1.24}$$

or, using (1.18), its equivalent form

$$\begin{gathered} \dot{z}(t) - A(S_{h_0}z)(t) = \varphi(t), \quad t \in [0,b], \\ \ell z(\cdot) = \alpha \in \mathbb{R}^m, \end{gathered} \tag{1.25}$$

where α is an m-dimensional constant vector-column ℓ is an m-dimensional linear vector-functional defined on the space $D_p[0,b]$ of an n-dimensional vector-functions

$$\ell = \operatorname{col}\,(\ell_1, \dots, \ell_m) : D_p[0,b] \longrightarrow \mathbb{R}^m, \qquad \ell_i : D_p[0,b] \longrightarrow \mathbb{R}, \quad i = 1, \dots, m, \tag{1.26}$$

absolutely continuous on $[0,b]$. Such problems for functional-differential equations are of Fredholm's type (see, e.g., [1, 2]). In order to formulate the following result, we need several auxiliary abbreviations. We set

$$Q := \ell X(\cdot) = \ell e_\tau^{A(\cdot-\tau)}. \tag{1.27}$$

We define an $n \times n$-dimensional matrix (orthogonal projection)

$$P_Q := I - Q^+ Q, \tag{1.28}$$

projecting space $\mathbb{R}^n$ to $\ker Q$ of the matrix Q.

Moreover, we define an $m \times m$-dimensional matrix (orthogonal projection)

$$P_{Q^*} := I_m - QQ^+, \tag{1.29}$$

projecting space $\mathbb{R}^m$ to $\ker Q^*$ of the transposed matrix $Q^* = Q^T$, where I_m is an $m \times m$ identity matrix and Q^+ is an $n \times m$-dimensional matrix pseudoinverse to the $m \times n$-dimensional matrix Q. Denote $d := \operatorname{rank} P_{Q^*}$ and $n_1 := \operatorname{rank} Q = \operatorname{rank} Q^*$. Since

$$\operatorname{rank} P_{Q^*} = m - \operatorname{rank} Q^*, \tag{1.30}$$

we have $d = m - n_1$.

We will denote by $P_{Q_d^*}$ an $d \times m$-dimensional matrix constructed from d linearly independent rows of the matrix P_{Q^*}. Denote $r := \operatorname{rank} P_Q$. Since

$$\operatorname{rank} P_Q = n - \operatorname{rank} Q, \tag{1.31}$$

we have $r = n - n_1$. By P_{Q_r} we will denote an $n \times r$-dimensional matrix constructed from r linearly independent columns of the matrix P_Q. Finally, we define

$$X_r(t) := X(t) P_{Q_r}, \tag{1.32}$$

and a generalized Green operator

$$(G\varphi)(t) := \int_0^b G(t,s)\varphi(s)ds, \tag{1.33}$$

where

$$G(t,s) := K(t,s) - e_\tau^{A(t-\tau)} Q^+ \ell K(\cdot, s) \tag{1.34}$$

is a generalized Green matrix corresponding to the boundary-value problem (1.25) (the Cauchy matrix $K(t,s)$ has the form (1.21)).

In [6, Theorem 4], the following result (formulating the necessary and sufficient conditions of solvability and giving representations of the solutions $z \in D_p[0,b]$, $\dot{z} \in L_p[0,b]$ of the boundary-value problem (1.25) in an *explicit analytical* form) is proved.

Theorem 1.2. *If* $n_1 \le \min(m,n)$*, then:*

(i) *the homogeneous problem*

$$\begin{gathered} \dot{z}(t) - A(S_{h_0} z)(t) = \theta, \quad t \in [0,b], \\ \ell z(\cdot) = \theta_m \in \mathbb{R}^m \end{gathered} \tag{1.35}$$

corresponding to problem (1.25) *has exactly* r *linearly independent solutions*

$$z(t, c_r) = X_r(t) c_r = e_\tau^{A(t-\tau)} P_{Q_r} c_r \in D_p[0,b], \tag{1.36}$$

(ii) *nonhomogeneous problem* (1.25) *is solvable in the space* $D_p[0,b]$ *if and only if* $\varphi \in L_p[0,b]$ *and* $\alpha \in \mathbb{R}^m$ *satisfy* d *linearly independent conditions*

$$P_{Q_d^*} \cdot \left(\alpha - \ell \int_0^b K(\cdot, s)\varphi(s)ds \right) = \theta_d, \tag{1.37}$$

(iii) *in that case the nonhomogeneous problem* (1.25) *has an r-dimensional family of linearly independent solutions represented in an analytical form*

$$z(t) = z_0(t, c_r) := X_r(t)c_r + (G\varphi)(t) + X(t)Q^+\alpha \quad \forall c_r \in \mathbb{R}^r. \tag{1.38}$$

2. Perturbed Weakly Nonlinear Boundary Value Problems

As an example of applying Theorem 1.2, we consider a problem of the branching of solutions $z : [0, b] \to \mathbb{R}^n$, $b > 0$ of systems of nonlinear ordinary differential equations with a small parameter ε and with a finite number of measurable delays $h_i(t)$, $i = 1, 2, \ldots, k$ of argument of the form

$$\dot{z}(t) = Az(t-\tau) + g(t) + \varepsilon Z(z(h_i(t)), t, \varepsilon), \quad t \in [0, b], \; h_i(t) \leq t, \tag{2.1}$$

satisfying the initial and boundary conditions

$$z(s) = \psi(s), \quad \text{if } s < 0, \qquad \ell z(\cdot) = \alpha, \quad \alpha \in \mathbb{R}^m, \tag{2.2}$$

and such that its solution $z = z(t, \varepsilon)$, satisfying

$$\begin{gathered} z(\cdot, \varepsilon) \in D_p[0, b], \\ \dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ z(t, \cdot) \in C[0, \varepsilon_0], \end{gathered} \tag{2.3}$$

for a sufficiently small $\varepsilon_0 > 0$, for $\varepsilon = 0$, turns into one of the generating solutions (1.38); that is, $z(t, 0) = z_0(t, c_r)$ for a $c_r \in \mathbb{R}^r$. We assume that the $n \times 1$ vector-operator Z satisfies

$$\begin{gathered} Z(\cdot, t, \varepsilon) \in C^1\left[\|z - z_0\| \leq q\right], \\ Z(z(h_i(t)), \cdot, \varepsilon) \in L_p[0, b], \\ Z(z(h_i(t)), t, \cdot) \in C[0, \varepsilon_0], \end{gathered} \tag{2.4}$$

where $q > 0$ is sufficiently small. Using denotations (1.3), (1.4), and (1.6), it is easy to show that the perturbed nonlinear boundary value problem (2.1), (2.2) can be rewritten in the form

$$\dot{z}(t) = A(S_{h_0}z)(t) + \varepsilon Z((S_h z)(t), t, \varepsilon) + \varphi(t), \quad \ell z(\cdot) = \alpha, \; t \in [0, b]. \tag{2.5}$$

In (2.5), A is an $n \times n$ constant matrix, $h_0 : [0, b] \to \mathbb{R}$ is a single delay defined by $h_0(t) := t - \tau$, $\tau > 0$,

$$(S_h z)(t) = \operatorname{col}[(S_{h_1}z)(t), \ldots, (S_{h_k}z)(t)] \tag{2.6}$$

is an N-dimensional column vector, where $N = nk$, and φ is an n-dimensional column vector given by

$$\varphi(t) = g(t) + A\ \psi^{h_0}(t). \tag{2.7}$$

The operator S_h maps the space D_p into the space

$$L_p^N = \underbrace{L_p \times \cdots \times L_p}_{k\text{-times}}, \tag{2.8}$$

that is, $S_h : D_p \to L_p^N$. Using denotation (1.3) for the operator $S_{h_i} : D_p \to L_p$, $i = 1, \dots, k$, we have the following representation:

$$(S_{h_i} z)(t) = \int_0^b \chi_{h_i}(t,s)\dot{z}(s)ds + \chi_{h_i}(t,0)z(0), \tag{2.9}$$

where

$$\chi_{h_i}(t,s) = \begin{cases} 1, & \text{if } (t,s) \in \Omega_i, \\ 0, & \text{if } (t,s) \notin \Omega_i \end{cases} \tag{2.10}$$

is the characteristic function of the set

$$\Omega_i := \{(t,s) \in [0,b] \times [0,b] : 0 \le s \le h_i(t) \le b\}. \tag{2.11}$$

Assume that the generating boundary value problem

$$\dot{z}(t) = A(S_{h_0} z)(t) + \varphi(t), \quad lz = \alpha, \tag{2.12}$$

being a particular case of (2.5) for $\varepsilon = 0$, has solutions for nonhomogeneities $\varphi \in L_p[0,b]$ and $\alpha \in \mathbb{R}^m$ that satisfy conditions (1.37). In such a case, by Theorem 1.2, the problem (2.12) possesses an r-dimensional family of solutions of the form (1.38).

Problem 1. Below, we consider the following problem: derive the necessary and sufficient conditions indicating when solutions of (2.5) turn into solutions (1.38) of the boundary value problem (2.12) for $\varepsilon = 0$.

Using the theory of generalized inverse operators [2], it is possible to find conditions for the solutions of the boundary value problem (2.5) to be branching from the solutions of (2.5) with $\varepsilon = 0$. Below, we formulate statements, solving the above problem. As compared with an earlier result [10, page 150], the present result is derived in an *explicit analytical* form. The progress was possible by using the delayed matrix exponential since, in such a case, all the necessary calculations can be performed to the full.

Theorem 2.1 (necessary condition). *Consider the system* (2.1)*; that is,*

$$\dot{z}(t) = Az(t-\tau) + g(t) + \varepsilon Z(z(h_i(t)), t, \varepsilon), \quad t \in [0, b], \tag{2.13}$$

where $h_i(t) \le t$, $i = 1, \ldots, k$, with the initial and boundary conditions (2.2)*; that is,*

$$z(s) = \psi(s), \quad \text{if } s < 0 < b, \qquad \ell z(\cdot) = \alpha \ \in \mathbb{R}^m, \tag{2.14}$$

and assume that, for nonhomogeneities

$$\varphi(t) = \ g(t) + A\ \psi^{h_0}(t) \ \in L_p[0, b], \tag{2.15}$$

and for $\alpha \in \mathbb{R}^m$, the generating boundary value problem

$$\dot{z}(t) = A(S_{h_0} z)(t) + \varphi(t), \quad \ell z(\cdot) = \alpha, \tag{2.16}$$

corresponding to the problem (1.25)*, has exactly an r-dimensional family of linearly independent solutions of the form* (1.38)*. Moreover, assume that the boundary value problem* (2.13), (2.14) *has a solution $z(t, \varepsilon)$ which, for $\varepsilon = 0$, turns into one of solutions $z_0(t, c_r)$ in* (1.38) *with a vector-constant $c_r := c_r^0 \in \mathbb{R}^r$.*

Then, the vector c_r^0 satisfies the equation

$$F\left(c_r^0\right) := \int_0^b H(s) Z\Big((S_h z_0)\left(s, c_r^0\right), s, 0\Big) ds = \theta_d, \tag{2.17}$$

where

$$H(s) := P_{Q_d^*} \ell K(\cdot, s) = P_{Q_d^*} \ell e_\tau^{A(\cdot-\tau-s)}. \tag{2.18}$$

Proof. We consider the nonlinearity in system (2.13), that is, the term $\varepsilon Z(z(h_i(t)), t, \varepsilon)$ as an inhomogeneity, and use Theorem 1.2 assuming that condition (1.37) is satisfied. This gives

$$\int_0^b H(s) Z((S_h z)(s, \varepsilon), s, \varepsilon) ds = \theta_d. \tag{2.19}$$

In this integral, letting $\varepsilon \to 0$, we arrive at the required condition (2.17). □

Corollary 2.2. *For periodic boundary-value problems, the vector-constant $c_r \in \mathbb{R}^r$ has a physical meaning-it is the amplitude of the oscillations generated. For this reason,* (2.17) *is called an equation generating the amplitude [11]. By analogy with the investigation of periodic problems, it is natural to say* (2.17) *is an equation for generating the constants of the boundary value problem* (2.13), (2.14).

If (2.17) *is solvable, then the vector constant* $c_r^0 \in \mathbb{R}^r$ *specifies the generating solution* $z_0(t, c_r^0)$ *corresponding to the solution* $z = z(t, \varepsilon)$ *of the original problem such that*

$$\begin{gathered} z(\cdot, \varepsilon) : [0, b] \longrightarrow \mathbb{R}^n, \\ z(\cdot, \varepsilon\ ) \in D_p[0, b], \\ \dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ z(t, \cdot) \in C[0, \varepsilon_0], \\ z(t, 0) = z_0\left(t, c_r^0\right). \end{gathered} \tag{2.20}$$

Also, if (2.17) *is unsolvable, the problem* (2.13), (2.14) *has no solution in the analyzed space. Note that, here and in what follows, all expressions are obtained in the real form and hence, we are interested in real solutions of* (2.17), *which can be algebraic or transcendental.*

Sufficient conditions for the existence of solutions of the boundary-value problem (2.13), (2.14) can be derived using results in [10, page 155] and [2]. By changing the variables in system (2.13), (2.14)

$$z(t, \varepsilon) = z_0\left(t, c_r^0\right) + y(t, \varepsilon), \tag{2.21}$$

we arrive at a problem of finding sufficient conditions for the existence of solutions of the problem

$$\dot{y}(t) = A\left(S_{h_0} y\right)(t) + \varepsilon Z\left(S_h\left(z_0 + y\right)(t), t, \varepsilon\right), \quad \ell y = \theta_m, \ t \in [0, b], \tag{2.22}$$

and such that

$$\begin{gathered} y(\cdot, \varepsilon) : [0, b] \longrightarrow \mathbb{R}^n, \\ y(\cdot, \varepsilon) \in D_p[0, b], \\ \dot{y}(\cdot, \varepsilon) \in L_p[0, b], \\ y(t, \cdot) \in C[0, \varepsilon_0], \\ y(t, 0) = \theta. \end{gathered} \tag{2.23}$$

Since the vector function $Z((S_h z)(t), t, \varepsilon)$ is continuously differentiable with respect to z and continuous in ε in the neighborhood of the point

$$(z, \varepsilon) = \left(z_0\left(t, c_r^0\right), 0\right), \tag{2.24}$$

we can separate its linear term as a function depending on y and terms of order zero with respect to ε

$$Z\left(S_h\left(z_0\left(t,c_r^0\right)+y\right),t,\varepsilon\right)=f_0\left(t,c_r^0\right)+A_1(t)(S_hy)(t)+R((S_hy)(t),t,\varepsilon), \tag{2.25}$$

where

$$\begin{gathered}
f_0\left(t,c_r^0\right):=Z\left((S_hz_0)\left(t,c_r^0\right),t,0\right),\quad f_0\left(\cdot,c_r^0\right)\in L_p[0,b],\\
A_1(t)=A_1\left(t,c_r^0\right)=\left.\frac{\partial Z(S_hx,t,0)}{\partial S_hx}\right|_{x=z_0(t,c_r^0)},\quad A_1(\cdot)\in L_p[0,b],\\
R(\theta,t,0)=\theta,\quad \frac{\partial R(\theta,t,0)}{\partial y}=\Theta,\quad R(y,\cdot,\varepsilon)\in L_p[0,b].
\end{gathered} \tag{2.26}$$

We now consider the vector function $Z((S_h(z_0+y))(t),t,\varepsilon)$ in (2.22) as an inhomogeneity and we apply Theorem 1.2 to this system. As the result, we obtain the following representation for the solution of (2.22):

$$y(t,\varepsilon)=X_r(t)c+y^{(1)}(t,\varepsilon). \tag{2.27}$$

In this expression, the unknown vector of constants $c=c(\varepsilon)\in C[0,\varepsilon_0]$ is determined from a condition similar to condition (1.37) for the existence of solution of problem (2.22):

$$B_0c=\int_0^b H(s)\left[A_1(s)\left(S_hy^{(1)}\right)(s,\varepsilon)+R((S_hy)(s,\varepsilon),s,\varepsilon)\right]ds, \tag{2.28}$$

where

$$B_0=\int_0^b H(s)A_1(s)(S_hX_r)(s)ds \tag{2.29}$$

is a $d\times r$ matrix, and

$$H(s):=P_{Q_d^*}\ell K(\cdot,s)=P_{Q_d^*}\ell e_\tau^{A(\cdot-\tau-s)}. \tag{2.30}$$

The unknown vector function $y^{(1)}(t,\varepsilon)$ is determined by using the generalized Green operator as follows:

$$y^{(1)}(t,\varepsilon)=\varepsilon\left(G\left[Z\left(S_h\left(z_0\left(s,c_r^0\right)+y\right),s,\varepsilon\right)\right]\right)(t). \tag{2.31}$$

Let $P_{N(B_0)}$ be an $r \times r$ matrix orthoprojector $\mathbb{R}^r \rightarrow N(B_0)$, and let $P_{N(B_0^*)}$ be a $d \times d$ matrix-orthoprojector $\mathbb{R}^d \rightarrow N(B_0^*)$. Equation (2.28) is solvable with respect to $c \in \mathbb{R}^r$ if and only if

$$P_{N(B_0^*)} \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds = \theta_d. \tag{2.32}$$

For

$$P_{N(B_0^*)} = \Theta_d, \tag{2.33}$$

the last condition is always satisfied and (2.28) is solvable with respect to $c \in \mathbb{R}^r$ up to an arbitrary vector constant $P_{N(B_0)}c \in \mathbb{R}^r$ from the null space of the matrix B_0

$$c = B_0^+ \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds + P_{N(B_0)}c. \tag{2.34}$$

To find a solution $y = y(t,\varepsilon)$ of (2.28) such that

$$\begin{gathered} y(\cdot,\varepsilon) : [0,b] \longrightarrow R^n, \\ y(\cdot,\varepsilon) \in D_p[0,b], \\ \dot{y}(\cdot,\varepsilon) \in L_p[0,b], \\ y(t,\cdot) \in C[0,\varepsilon_0], \\ y(t,0) = \theta, \end{gathered} \tag{2.35}$$

it is necessary to solve the following operator system:

$$\begin{gathered} y(t,\varepsilon) = X_r(t)c + y^{(1)}(t,\varepsilon), \\ c = B_0^+ \int_0^b H(s)\Big[A_1(s)\Big(S_h y^{(1)}\Big)(s,\varepsilon) + R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)\Big] ds, \\ y^{(1)}(t,\varepsilon) = \varepsilon G\Big[Z\Big(S_h\Big(z_0\Big(s, c_r^0\Big) + y\Big), s, \varepsilon\Big)\Big](t). \end{gathered} \tag{2.36}$$

The operator system (2.36) belongs to the class of systems solvable by the method of simple iterations, convergent for sufficiently small $\varepsilon \in [0,\varepsilon_0]$ (see [10, page 188]). Indeed, system (2.36) can be rewritten in the form

$$u = L^{(1)}u + Fu, \tag{2.37}$$

where $u = \operatorname{col}\,(y(t,\varepsilon), c(\varepsilon), y^{(1)}(t,\varepsilon))$ is a $(2n+r)$-dimensional column vector, $L^{(1)}$ is a linear operator

$$L^{(1)} := \begin{pmatrix} \Theta & X_r(t) & I \\ \Theta_{r,n} & \Theta_{r,r} & L_1 \\ \Theta & \Theta_{n,r} & \Theta \end{pmatrix}, \tag{2.38}$$

where

$$L_1(*) = B_0^+ \int_0^b H(s)A_1(s)(*)ds, \tag{2.39}$$

and F is a nonlinear operator

$$Fu := \begin{pmatrix} \theta \\ B_0^+ \int_0^b H(s)R\big((S_h y)(s,\varepsilon), s, \varepsilon\big)ds \\ \varepsilon\big(G\big[Z\big((S_h z_0)(s, c_r^0), s, 0\big) + A_1(s)\big(S_h y\big)(s,\varepsilon) + R\big(\big(S_h y\big)(s,\varepsilon), s, \varepsilon\big)\big]\big)(t) \end{pmatrix}. \tag{2.40}$$

In view of the structure of the operator $L^{(1)}$ containing zero blocks on and below the main diagonal, the inverse operator

$$\left(I_{2n+r} - L^{(1)}\right)^{-1} \tag{2.41}$$

exists. System (2.37) can be transformed into

$$u = Su, \tag{2.42}$$

where

$$S := \left(I_{2n+r} - L^{(1)}\right)^{-1} F \tag{2.43}$$

is a contraction operator in a sufficiently small neighborhood of the point

$$(z,\varepsilon) = \left(z_0\left(t, c_r^0\right), 0\right). \tag{2.44}$$

Thus, the solvability of the last operator system can be established by using one of the existing versions of the fixed-point principles [12] applicable to the system for sufficiently small $\varepsilon \in [0,\varepsilon_0]$. It is easy to prove that the sufficient condition $P_{N(B_0^*)} = \Theta_d$ for the existence of solutions of the boundary value problem (2.13), (2.14) means that the constant $c_r^0 \in \mathbb{R}^r$ of the equation for generating constant (2.17) is a simple root of equation (2.17) [2]. By using the method of simple iterations, we can find the solution of the operator system and hence the solution of the original boundary value problem (2.13), (2.14). Now, we arrive at the following theorem.

Theorem 2.3 (sufficient condition). *Assume that the boundary value problem* (2.13), (2.14) *satisfies the conditions listed above and the corresponding linear boundary value problem* (1.25) *has an r-dimensional family of linearly independent solutions of the form* (1.38)*. Then, for any simple root $c_r = c_r^0 \in \mathbb{R}^r$ of the equation for generating the constants* (2.17)*, there exist at least one solution of the boundary value problem* (2.13), (2.14)*. The indicated solution $z(t, \varepsilon)$ is such that*

$$\begin{aligned} &z(\cdot, \varepsilon) \in D_p[0, b], \\ &\dot{z}(\cdot, \varepsilon) \in L_p[0, b], \\ &z(t, \cdot) \in C[0, \varepsilon_0], \end{aligned} \tag{2.45}$$

and, for $\varepsilon = 0$, turns into one of the generating solutions (1.38) *with a constant $c_r^0 \in \mathbb{R}^r$; that is, $z(t, 0) = z_0(t, c_r^0)$. This solution can be found by the method of simple iterations, which is convergent for a sufficiently small $\varepsilon \in [0, \varepsilon_0]$.*

Corollary 2.4. *If the number n of unknown variables is equal to the number m of boundary conditions (and hence $r = d$), the boundary value problem* (2.13), (2.14) *has a unique solution. In such a case, the problems considered for functional-differential equations are of Fredholm's type with a zero index. By using the procedure proposed in [2] with some simplifying assumptions, we can generalize the proposed method to the case of multiple roots of equation* (2.17) *to determine sufficient conditions for the existence of solutions of the boundary-value problem* (2.13), (2.14).

3. Example

We will illustrate the above proved theorems on the example of a weakly perturbed linear boundary value problem. Consider the following simplest boundary value problem-a periodic problem for the delayed differential equation:

$$\begin{gathered} \dot{z}(t) = z(t - \tau) + \varepsilon \sum_{i=1}^{k} B_i(t) z(h_i(t)) + g(t), \quad t \in (0, T], \\ z(s) = \psi(s), \quad \text{if } s < 0, \\ z(0) = z(T), \end{gathered} \tag{3.1}$$

where $0 < \tau, T = \text{const}$, B_i are $n \times n$ matrices, $B_i, g \in L_p[0, T]$, $\psi : \mathbb{R}^1 \setminus (0, T] \to \mathbb{R}^n$, $h_i(t) \leq t$ are measurable functions. Using the symbols S_{h_i} and ψ^{h_i} (see (1.3), (1.4), (2.9)), we arrive at the following operator system:

$$\begin{gathered} \dot{z}(t) = z(t - \tau) + \varepsilon B(t)(S_h z)(t) + \varphi(t, \varepsilon), \\ \ell z := z(0) - z(T) = \theta_n, \end{gathered} \tag{3.2}$$

where $B(t) := (B_1(t), \dots, B_k(t))$ is an $n \times N$ matrix ($N = nk$), and

$$\varphi(t, \varepsilon) := g(t) + \psi^{h_0}(t) + \varepsilon \sum_{i=1}^{k} B_i(t) \psi^{h_i}(t) \in L_p[0, T]. \tag{3.3}$$

We will consider the simplest case with $T \leq \tau$. Utilizing the delayed matrix exponential, it can be easily verified that in this case, the matrix

$$X(t) = e_\tau^{I(t-\tau)} = I \tag{3.4}$$

is a normal fundamental matrix for the homogeneous generating system

$$\dot{z}(t) = z(t - \tau). \tag{3.5}$$

Then,

$$\begin{gathered}
Q := \ell X(\cdot) = e_\tau^{-I\tau} - e_\tau^{I(T-\tau)} = \theta_n, \\
P_Q = P_{Q^*} = I, \quad (r = n, d = m = n), \\
K(t, s) = \begin{cases} e_\tau^{I(t-\tau-s)} = I, & 0 \leq s \leq t \leq T, \\ \Theta, & s > t, \end{cases} \\
\ell K(\cdot, s) = K(0, s) - K(T, s) = -I, \\
H(\tau) = P_{Q^*} \ell K(\cdot, s) = -I, \\
(S_{h_i} I)(t) = \chi_{h_i}(t, 0) \cdot I = I \cdot \begin{cases} 1, & \text{if } 0 \leq h_i(t) \leq T, \\ 0, & \text{if } h_i(t) \ < 0. \end{cases}
\end{gathered} \tag{3.6}$$

To illustrate the theorems proved above, we will find the conditions for which the boundary value problem (3.1) has a solution $z(t, \varepsilon)$ that, for $\varepsilon = 0$, turns into one of solutions (1.38) $z_0(t, c_r)$ of the generating problem. In contrast to the previous works [7, 9], we consider the case when the unperturbed boundary-value problem

$$\begin{gathered}
\dot{z}(t) = z(t - \tau) + \varphi(t, 0), \\
z(0) = z(T)
\end{gathered} \tag{3.7}$$

has an n-parametric family of linear-independent solutions of the form(1.38)

$$z := z_0(t, c_n) = c_n + \big(G\varphi\big)(t), \quad \forall c_n \in \mathbb{R}^n. \tag{3.8}$$

For this purpose, it is necessary and sufficient for the vector function

$$\varphi(t) = g(t) + \psi^{h_0}(t) \tag{3.9}$$

to satisfy the condition of type (1.37)

$$\int_0^T H(s)\varphi(s)\ ds = -\int_0^T \varphi(s)\ ds = \theta_n. \tag{3.10}$$

Then, according to the Theorem 2.1, the constant $c_n = c_n^0 \in \mathbb{R}^n$ must satisfy (2.17), that is, the equation

$$F\left(c_n^0\right) := \int_0^T H(s)Z\Big((S_h z_0)\left(s, c_n^0\right), s, 0\Big)ds = \theta_n, \tag{3.11}$$

which in our case is a linear algebraic system

$$B_0 c_n^0 = -\int_0^T B(s)(S_h(G\varphi))(s)ds, \tag{3.12}$$

with the $n \times n$ matrix B_0 in the form

$$\begin{aligned} B_0 &= \int_0^T H(s)B(s)(S_h I)(s)ds \\ &= -\int_0^T \sum_{i=1}^k B_i(s)(S_{h_i} I)(s)ds = -\sum_{i=1}^k \int_0^T B_i(s)\chi_{h_i}(s,0)ds. \end{aligned} \tag{3.13}$$

According to Corollary 2.4, if $\det B_0 \neq 0$, the problem (3.1) for the case $T \leq \tau$ has a unique solution $z(t,\varepsilon)$ with the properties

$$\begin{gathered} z(\cdot,\varepsilon) \in D_p^n[0,T], \\ \dot{z}(\cdot,\varepsilon) \in L_p^n[0,T], \\ z(t,\cdot) \in C[0,\varepsilon_0], \\ z(t,0) = z_0\left(t, c_n^0\right), \end{gathered} \tag{3.14}$$

for $g \in L_p[0,T]$, $\psi(t) \in L_p[0,T]$, and for measurable delays h_i that which satisfy the criterion (3.10) of the existence of a generating solution where

$$c_n^0 = -B_0^+ \int_0^T B(s)(S_h(G\varphi))(s)ds. \tag{3.15}$$

A solution $z(t,\varepsilon)$ of the boundary value problem (3.1) can be found by the convergent method of simple iterations (see Theorem 2.3).

If, for example, $h_i(t) = t - \Delta_i$, where $0 < \Delta_i = \text{const} < T$, $i = 1, \ldots, k$, then

$$\chi_{h_i}(t,0) = \begin{cases} 1 & \text{if } 0 \leq h_i(t) = t - \Delta_i \leq T, \\ 0 & \text{if } h_i(t) = t - \Delta_i < 0, \end{cases} = \begin{cases} 1 & \text{if } \Delta_i \leq t \leq T + \Delta_i, \\ 0, & \text{if } t < \Delta_i. \end{cases} \tag{3.16}$$

The $n \times n$ matrix B_0 can be rewritten in the form

$$\begin{aligned} B_0 &= \int_0^T H(s) \sum_{i=1}^{k} B_i(s) \chi_{h_i}(s,0) d\tau \\ &= -\sum_{i=1}^{k} \int_0^T B_i(s) \chi_{h_i}(s,0) ds = -\sum_{i=1}^{k} \int_{\Delta_i}^T B_i(s) ds, \end{aligned} \tag{3.17}$$

and the unique solvability condition of the boundary value problem (3.1) takes the form

$$\det\left[\sum_{i=1}^{k} \int_{\Delta_i}^T B_i(s) ds\right] \neq 0. \tag{3.18}$$

It is easy to see that if the vector function $Z(z(h_i(t)), t, \varepsilon)$ is nonlinear in z, for example as a square, then (3.11) generating the constants will be a square-algebraic system and, in this case, the boundary value problem (3.1) can have two solutions branching from the point $\varepsilon = 0$.

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Research Article

Bounds of Solutions of Integrodifferential Equations

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Some new integral inequalities are given, and bounds of solutions of the following integro-differential equation are determined: $x'(t)-\digamma(t,x(t),\int_0^t k(t,s,x(t),x(s))ds) = h(t)$, $x(0) = x_0$, where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\digamma : R_+ \times R^2 \to R$ are continuous functions, $R_+ = [0,\infty)$.

1. Introduction

Ou Yang [1] established and applied the following useful nonlinear integral inequality.

Theorem 1.1. *Let u and h be nonnegative and continuous functions defined on R_+ and let $c \geq 0$ be a constant. Then, the nonlinear integral inequality*

$$u^2(t) \leq c^2 + 2\int_0^t h(s)u(s)ds, \quad t \in R_+ \tag{1.1}$$

implies

$$u(t) \leq c + \int_0^t h(s)ds, \quad t \in R_+. \tag{1.2}$$

This result has been frequently used by authors to obtain global existence, uniqueness, boundedness, and stability of solutions of various nonlinear integral, differential, and integrodifferential equations. On the other hand, Theorem 1.1 has also been extended and generalized by many authors; see, for example, [2–19]. Like Gronwall-type inequalities, Theorem 1.1 is also used to obtain *a priori* bounds to unknown functions. Therefore, integral inequalities of this type are usually known as Gronwall-Ou Yang type inequalities.

In the last few years there have been a number of papers written on the discrete inequalities of Gronwall inequality and its nonlinear version to the Bihari type, see [13, 16, 20]. Some applications discrete versions of integral inequalities are given in papers [21–23].

Pachpatte [11, 12, 14–16] and Salem [24] have given some new integral inequalities of the Gronwall-Ou Yang type involving functions and their derivatives. Lipovan [7] used the modified Gronwall-Ou Yang inequality with logarithmic factor in the integrand to the study of wave equation with logarithmic nonlinearity. Engler [5] used a slight variant of the Haraux's inequality for determination of global regular solutions of the dynamic antiplane shear problem in nonlinear viscoelasticity. Dragomir [3] applied his inequality to the stability, boundedness, and asymptotic behaviour of solutions of nonlinear Volterra integral equations.

In this paper, we present new integral inequalities which come out from above-mentioned inequalities and extend Pachpatte's results (see [11, 16]) especially. Obtained results are applied to certain classes of integrodifferential equations.

2. Integral Inequalities

Lemma 2.1. *Let u, f, and g be nonnegative continuous functions defined on R_+. If the inequality*

$$u(t) \le u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds \tag{2.1}$$

holds where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \le u_0\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right] \tag{2.2}$$

for $t \in R_+$.

Proof. Define a function $v(t)$ by the right-hand side of (2.1)

$$v(t) = u_0 + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds. \tag{2.3}$$

Then, $v(0) = u_0$, $u(t) \le v(t)$ and

$$\begin{aligned} v'(t) &= f(t)u(t) + f(t)\int_0^t g(s)(u(t) + u(s))ds \\ &\le f(t)v(t) + f(t)\int_0^t g(s)(v(t) + v(s))ds. \end{aligned} \tag{2.4}$$

Define a function $m(t)$ by

$$m(t) = v(t) + \int_0^t g(s)v(s)ds + v(t)\int_0^t g(s)ds, \tag{2.5}$$

then $m(0) = v(0) = u_0$, $v(t) \le m(t)$,

$$v'(t) \le f(t)m(t), \tag{2.6}$$

$$\begin{aligned} m'(t) &= 2g(t)v(t) + v'(t)\left(1 + \int_0^t g(s)ds\right) \\ &\le m(t)\left[2g(t) + f(t)\left(1 + \int_0^t g(s)ds\right)\right]. \end{aligned} \tag{2.7}$$

Integrating (2.7) from 0 to t, we have

$$m(t) \le u_0 \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.8}$$

Using (2.8) in (2.6), we obtain

$$v'(t) \le u_0 f(t) \exp\left(\int_0^t \left(2g(s) + f(s)\left(1 + \int_0^s g(\sigma)d\sigma\right)\right)ds\right). \tag{2.9}$$

Integrating from 0 to t and using $u(t) \le v(t)$, we get inequality (2.2). The proof is complete. □

Lemma 2.2. *Let u, f, and g be nonnegative continuous functions defined on R_+, $w(t)$ be a positive nondecreasing continuous function defined on R_+. If the inequality*

$$u(t) \le w(t) + \int_0^t f(s)\left(u(s) + \int_0^s g(\tau)(u(s) + u(\tau))d\tau\right)ds, \tag{2.10}$$

holds, where u_0 is a nonnegative constant, $t \in R_+$, then

$$u(t) \le w(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.11}$$

where $t \in R_+$.

Proof. Since the function $w(t)$ is positive and nondecreasing, we obtain from (2.10)

$$\frac{u(t)}{w(t)} \leq 1 + \int_0^t f(s)\left(\frac{u(s)}{w(s)} + \int_0^s g(\tau)\left(\frac{u(s)}{w(s)} + \frac{u(\tau)}{w(\tau)}\right)d\tau\right)ds. \tag{2.12}$$

Applying Lemma 2.1 to inequality (2.12), we obtain desired inequality (2.11). □

Lemma 2.3. *Let u, f, g, and h be nonnegative continuous functions defined on R_+, and let c be a nonnegative constant.*

If the inequality

$$u^2(t) \leq c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds \tag{2.13}$$

holds for $t \in R_+$, then

$$u(t) \leq p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \tag{2.14}$$

where

$$p(t) = c + \int_0^t h(s)ds. \tag{2.15}$$

Proof. Define a function $z(t)$ by the right-hand side of (2.13)

$$z(t) = c^2 + 2\left[\int_0^t f(s)u(s)\left(u(s) + \int_0^s g(\tau)(u(\tau) + u(s))d\tau\right) + h(s)u(s)\right]ds. \tag{2.16}$$

Then $z(0) = c^2$, $u(t) \leq \sqrt{z(t)}$ and

$$\begin{aligned} z'(t) &= 2\left[f(t)u(t)\left(u(t) + \int_0^t g(s)(u(t) + u(s))ds\right) + h(t)u(t)\right] \\ &\leq 2\sqrt{z(t)}\left[f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t)\right]. \end{aligned} \tag{2.17}$$

Differentiating $\sqrt{z(t)}$ and using (2.17), we get

$$\begin{aligned} \frac{d}{dt}\left(\sqrt{z(t)}\right) &= \frac{z'(t)}{2\sqrt{z(t)}} \\ &\leq f(t)\left(\sqrt{z(t)} + \int_0^t g(s)\left(\sqrt{z(t)} + \sqrt{z(s)}\right)ds\right) + h(t). \end{aligned} \tag{2.18}$$

Integrating inequality (2.18) from 0 to t, we have

$$\sqrt{z(t)} \le p(t) + \int_0^t f(s)\left(\sqrt{z(s)} + \int_0^s g(\tau)\left(\sqrt{z(s)} + \sqrt{z(\tau)}\right)d\tau\right)ds, \tag{2.19}$$

where $p(t)$ is defined by (2.15), $p(t)$ is positive and nondecreasing for $t \in R_+$. Now, applying Lemma 2.2 to inequality (2.19), we get

$$\sqrt{z(t)} \le p(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right]. \tag{2.20}$$

Using (2.20) and the fact that $u(t) \le \sqrt{z(t)}$, we obtain desired inequality (2.14). □

3. Application of Integral Inequalities

Consider the following initial value problem

$$x'(t) - \mathcal{F}\left(t, x(t), \int_0^t k(t,s,x(t),x(s))ds\right) = h(t), \quad x(0) = x_0, \tag{3.1}$$

where $h : R_+ \to R$, $k : R_+^2 \times R^2 \to R$, $\mathcal{F} : R_+ \times R^2 \to R$ are continuous functions. We assume that a solution $x(t)$ of (3.1) exists on R_+.

Theorem 3.1. *Suppose that*

$$\begin{gathered} |k(t,s,u_1,u_2)| \le f(t)g(s)(|u_1| + |u_2|) \quad \text{for } (t,s,u_1,u_2) \in R_+^2 \times R^2, \\ |\mathcal{F}(t,u_1,v_1)| \le f(t)|u_1| + |v_1| \quad \text{for } (t,u_1,v_1) \in R_+ \times R^2, \end{gathered} \tag{3.2}$$

where f, g are nonnegative continuous functions defined on R_+. Then, for the solution $x(t)$ of (3.1) the inequality

$$\begin{gathered} |x(t)| \le r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma\right)\right)d\tau\right)ds\right], \\ r(t) = |x_0| + \int_0^t |h(t)|dt \end{gathered} \tag{3.3}$$

holds on R_+.

Proof. Multiplying both sides of (3.1) by $x(t)$ and integrating from 0 to t we obtain

$$x^2(t) = x_0^2 + 2\int_0^t \left[x(s)\mathcal{F}\left(s, x(s), \int_0^s k(s,\tau,x(s),x(\tau))d\tau\right) + x(s)h(s)\right]ds. \tag{3.4}$$

From (3.2) and (3.4), we get

$$|x(t)|^2 \le |x_0|^2 + 2\int_0^t \left[f(s)|x(s)| \times \left(|x(s)| + \int_0^s g(\tau)(|x(s)| + |x(\tau)|)d\tau \right) + |h(s)||x(s)| \right] ds. \tag{3.5}$$

Using inequality (2.14) in Lemma 2.3, we have

$$|x(t)| \le r(t)\left[1 + \int_0^t f(s)\exp\left(\int_0^s \left(2g(\tau) + f(\tau)\left(1 + \int_0^\tau g(\sigma)d\sigma \right) \right) d\tau \right) ds \right], \tag{3.6}$$

where

$$r(t) = |x_0| + \int_0^t |h(t)|dt, \tag{3.7}$$

which is the desired inequality (3.3). □

Remark 3.2. It is obvious that inequality (3.3) gives the bound of the solution $x(t)$ of (3.1) in terms of the known functions.

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Research Article

Compatible and Incompatible Nonuniqueness Conditions for the Classical Cauchy Problem

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In the first part of this paper sufficient conditions for nonuniqueness of the classical Cauchy problem $\dot{x} = f(t, x)$, $x(t_0) = x_0$ are given. As the essential tool serves a method which estimates the "distance" between two solutions with an appropriate Lyapunov function and permits to show that under certain conditions the "distance" between two different solutions vanishes at the initial point. In the second part attention is paid to conditions that are obtained by a formal inversion of uniqueness theorems of Kamke-type but cannot guarantee nonuniqueness because they are incompatible.

1. Introduction

Consider the initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x_0, \tag{1.1}$$

where $t_0 \in \mathbb{R}$, $t \in J := [t_0, t_0 + a]$ with $a > 0$, $x, x_0 \in \mathbb{R}^n$ and $f : J \times \mathbb{R}^n \to \mathbb{R}^n$.

In the first part (Section 2) we give sufficient conditions for nonuniqueness of the classical n-dimensional Cauchy problem (1.1). As the essential tool serves a method which estimates the "distance" between two solutions with an appropriate Lyapunov function and permits to show that under certain conditions the "distance" between two different solutions vanishes at the initial point. In the second part (Section 3) we analyze for the one-dimensional case a set of conditions that takes its origin in an inversion of the uniqueness theorem by Kamke (see, e.g., [1, page 56]) but cannot guarantee nonuniqueness since it contains an inner contradiction. Several attempts were made to get nonuniqueness criteria by using conditions that are (in a certain sense) reverse uniqueness conditions of Kamke type. But this inversion process has to be handled very carefully. It can yield incompatible conditions. This is illustrated by a general set of conditions (in Theorems 3.2, 3.5 and 3.6) that would ensure nonuniqueness, but unfortunately they are inconsistent.

In this paper we study Cauchy problems where f is continuous at the initial point. Related results can be found in [1–5]. In literature there are several investigations for the discontinuous case [1, 6–13] with different qualitative behaviour.

2. Main Result

In the following let $\mathbb{R}_+ := [0, \infty)$, $b > 0$, $\rho > 0$ and

$$S_\rho^n(x_0) := \{x \in \mathbb{R}^n : \|x - x_0\| < \rho\}, \tag{2.1}$$

where $\|\cdot\|$ means the Euclidean norm.

Definition 2.1. We say that the initial value problem (1.1) has at least two different solutions on the interval J if there exist solutions $\varphi(t)$, $\psi(t)$ defined on J and $\varphi \not\equiv \psi$.

The following notions are used in our paper (see, e.g., [14, pages 136 and 137]).

Definition 2.2. A function $\varphi : [0, \rho) \to \mathbb{R}_+$ is said to belong to the class $\mathcal{K}_\rho$ if it is continuous, strictly increasing on $[0, \rho)$ and $\varphi(0) = 0$.

Definition 2.3. A function $V : J \times S_\rho^n(0) \to \mathbb{R}_+$ with $V(t, 0) \equiv 0$ is said to be positive definite if there exists a function $\varphi \in \mathcal{K}_\rho$ such that the relation

$$V(t, x) \geq \varphi(\|x\|) \tag{2.2}$$

is satisfied for $(t, x) \in J \times S_\rho^n(0)$.

For the convenience of the reader we recall the definition of a uniformly Lipschitzian function with respect to a given variable.

Definition 2.4. A function $V(t, \cdot) : S_\rho^n(0) \to \mathbb{R}_+$ is said to be Lipschitzian uniformly with respect to $t \in J$ if for arbitrarily given $x^* \in S_\rho^n(0)$ there exists a constant $k = k(x^*)$ such that

$$\left\|V(t, x_1^*) - V(t, x_2^*)\right\| \leq k\left\|x_1^* - x_2^*\right\| \tag{2.3}$$

holds for every $t \in J$ and for every x_1^*, x_2^* within a small neighbourhood of x^* in $S_\rho^n(0)$.

In [1, 15, 16] generalized derivatives of a Lipschitzian function along solutions of an associated differential system are analyzed. A slight modification of Theorem 4.3 [15, Appendix I] is the following lemma.

Lemma 2.5. *Let $V : J \times S_\rho^n(0) \to \mathbb{R}_+$ be continuous and let $V(t, \cdot) : S_\rho^n(0) \to \mathbb{R}_+$ be Lipschitzian uniformly with respect to $t \in J$. Let $x_1, x_2 : J \to S_\rho^n(0)$ be any two solutions of*

$$\dot{x} = f(t,x), \tag{2.4}$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function. Then for the upper right Dini derivative the equality

$$\begin{aligned} &D^+V(t, x_2(t) - x_1(t)) \\ &\quad := \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, x_2(t+h) - x_1(t+h)) - V(t, x_2(t) - x_1(t))] \\ &\quad = \limsup_{h \to 0^+} \frac{1}{h}\left[V\left(t+h, x_2(t) - x_1(t) + h\left(f(t, x_2(t)) - f(t, x_1(t))\right)\right) - V(t, x_2(t) - x_1(t))\right] \end{aligned} \tag{2.5}$$

holds.

In the proof of Theorem 2.8 we require the following lemmas which are slight adaptations of Theorem 1.4.1 [14, page 15] and Theorem 1.3.1 [1, page 10] for the left side of the initial point.

Lemma 2.6. *Let E be an open (t,u)-set in $\mathbb{R}^2$, let $g : E \to \mathbb{R}$ be a continuous function, and let u be the unique solution of*

$$\dot{u} = g(t,u), \quad u(t_2) = u_2, \tag{2.6}$$

to the left with $t_2 > t_0$, $(t_2, u_2) \in E$. Further, we assume that the scalar continuous function $m : (t_0, t_2] \to \mathbb{R}$ with $(t, m(t)) \in E$ satisfies $m(t_2) \le u(t_2)$ and

$$D^+m(t) \ge g(t, m(t)), \quad t_0 < t \le t_2. \tag{2.7}$$

Then

$$m(t) \le u(t) \tag{2.8}$$

holds as far as the solution u exists left of t_2 in $(t_0, t_2]$.

Lemma 2.7. *Let $S := \{(t,x) : t_0 - a \le t \le t_0,\ |x - x_0| \le b\}$ and $f : S \to \mathbb{R}$ be continuous and nondecreasing in x for each fixed t in $[t_0 - a, t_0]$. Then, the initial value problem* (1.1) *has at most one solution in $[t_0 - a, t_0]$.*

Theorem 2.8 (main result). *Suppose that*

(i) *$f : J \times S_b^n(x_0) \to \mathbb{R}^n$ is a continuous function such that*

$$M := \sup\left\{ \|f(t,x)\| : t \in J,\ x \in S_b^n(x_0) \right\} < \frac{b}{a}. \tag{2.9}$$

Let x_1 *be a solution of problem* (1.1) *on* J. *Let, moreover, there exist numbers* $t_1 \in (t_0, t_0+a]$, $r \in (0, 2b)$ *and continuous functions* $g : (t_0, t_1] \times \mathbb{R}_+ \to \mathbb{R}$, $V : [t_0, t_1] \times S_r^n(0) \to \mathbb{R}_+$ *such that*

(ii) g *is nondecreasing in the second variable, and the problem*

$$\dot{u} = g(t, u), \qquad \lim_{t \to t_0+} u(t) = 0 \tag{2.10}$$

has a positive solution u^* *on* $(t_0, t_1]$;

(iii) V *is positive definite and* $V(t, \cdot) : S_r^n(0) \to \mathbb{R}_+$ *is Lipschitzian uniformly with respect to* $t \in J$;

(iv) *for* $t_0 < t \le t_1$, $\|y - x_1(t)\| < r$, *the inequality*

$$\dot{V}(t, y - x_1(t)) \ge g(t, V(t, y - x_1(t))) \tag{2.11}$$

holds where

$$\begin{aligned} &\dot{V}(t, y - x_1(t)) \\ &\quad := \limsup_{h \to 0^+} \frac{1}{h}\left[V(t+h, y - x_1(t) + h[f(t,y) - f(t, x_1(t))]) - V(t, y - x_1(t))\right]. \end{aligned} \tag{2.12}$$

Then the set of different solutions of problem (1.1) *on interval* J *has the cardinality of the continuum.*

Remark 2.9. If condition (i) is fulfilled then, as it is well known, problem (1.1) is globally solvable and every global solution admits the estimate

$$\|x(t) - x_0\| \le M(t - t_0), \quad t \in J. \tag{2.13}$$

Moreover, for any local solution x_* of problem (1.1), defined on some interval $[t_0, t_1] \subset J$, there exists a global solution x of that problem such that $x(t) = x_*(t)$ for $t \in [t_0, t_1]$.

Remark 2.10. For the case $M = 0$ the initial value problem is unique and the assumptions of Theorem 2.8 cannot be satisfied. Therefore, without loss of generality, we assume $M > 0$ in the proof below.

Proof. At first we show that (1.1) has at least two different solutions on $[t_0, t_1^*]$, where $t_1^* \le t_1$, $t_1^* \le t_0 + \min\{a, b/(3M)\}$ is sufficiently close to t_0. We construct a further solution of (1.1) by finding a point (t_2, x_2) not lying on the solution $x_1(t)$ and starting from this point backwards to the initial point (t_0, x_0).

First we show that there exist values t_2 and x_2, $t_0 < t_2 \le t_1^*$, $\|x_2 - x_0\| \le 2b/3$ such that

$$u^*(t_2) = V(t_2, x_2 - x_1(t_2)) \tag{2.14}$$

holds for the nontrivial solution $u^*(t)$ of $\dot{u} = g(t,u)$. From Lemma 2.7 it follows that $u^*(t)$ is determined uniquely to the left by the initial data $(t_2, u^*(t_2))$. We consider the ε-tubes

$$S(\varepsilon) := \left\{(t,x) : t_0 \leq t \leq t_1^*,\ \|x - x_1(t)\| = \varepsilon\right\} \tag{2.15}$$

for $\varepsilon > 0$ around the solution $x_1(t)$. There exists $\varepsilon_1 > 0$ such that $S(\varepsilon)$ with $0 < \varepsilon \leq \varepsilon_1 < r$ is contained in the set

$$\left\{(t,x) : t_0 \leq t \leq t_1^*,\ \|x - x_0\| \leq \frac{2b}{3}\right\}. \tag{2.16}$$

For $0 \leq \delta \leq \varepsilon_1$, $t \in [t_0, t_1^*]$ we define

$$\begin{gathered}\Psi(\delta,t) := \max_{\|x - x_1(t)\| = \delta} V(t, x - x_1(t)),\\ \Psi(\delta) := \max_{t \in [t_0, t_1^*]} \Psi(\delta,t) \equiv \max_{(t,x) \in S(\delta)} V(t, x - x_1(t)).\end{gathered} \tag{2.17}$$

The function $\Psi(\delta,t)$ is continuous in t for $t_0 \leq t \leq t_1^*$. Since $\lim_{\delta \to 0}\Psi(\delta) = 0$, there exists a δ_2, $0 < \delta_2 \leq \min\{\varepsilon_1, b/3\}$, such that $\Psi(\delta_2) \leq u^*(t_1^*)$. It is clear that inequalities

$$\Psi\left(\delta_2, t_1^*\right) \leq \Psi(\delta_2) \leq u^*\left(t_1^*\right) \tag{2.18}$$

and (due to positive definiteness of V)

$$\Psi(\delta_2, t_0) > 0 = \lim_{t \to t_0+} u^*(t) \tag{2.19}$$

hold. We define a function

$$\omega(t) := \Psi(\delta_2, t) - u^*(t), \tag{2.20}$$

continuous on $[t_0, t_1^*]$. Taking into account inequalities $\omega(t_0) > 0$ and $\omega(t_1^*) \leq 0$ we conclude that there exists t_2, $t_0 < t_2 \leq t_1^*$, with

$$\Psi(\delta_2, t_2) = u^*(t_2). \tag{2.21}$$

The value $\Psi(\delta_2, t_2)$ is taken by $V(t_2, x - x_1(t_2))$ at a point $x = x_2$ such that $\|x_2 - x_1(t_2)\| = \delta_2$ and clearly (in view of the construction) $x_2 \neq x_1(t_2)$. The above statement is proved and (2.14) is valid for (t_2, x_2) determined above.

Now consider the initial value problem

$$\dot{x} = f(t,x), \qquad x(t_2) = x_2. \tag{2.22}$$

Obviously $t_2 - t_0 \le b/(3M)$ since

$$0 < t_2 - t_0 \le t_1^* - t_0 \le \min\left\{a, \frac{b}{3M}\right\} \le \frac{b}{3M} \tag{2.23}$$

and $\|x_2 - x_0\| \le 2b/3$ because

$$\begin{aligned}
\|x_2 - x_0\| &= \|x_2 - x_1(t_2) + x_1(t_2) - x_0\| \\
&\le \|x_2 - x_1(t_2)\| + \|x_1(t_2) - x_0\| = \delta_2 + \left\| \int_{t_0}^{t_2} f(s, x_1(s))ds \right\| \\
&\le \delta_2 + M(t_2 - t_0) \le \delta_2 + M\frac{b}{3M} = \delta_2 + \frac{b}{3} \le \frac{2b}{3}.
\end{aligned} \tag{2.24}$$

Peano's theorem implies that there exists a solution $x_2(t)$ of problem (2.22) on $t_0 \le t \le t_2$. We will show that $x_2(t_0) = x_0$. Set

$$m(t) := V(t, x_2(t) - x_1(t)). \tag{2.25}$$

Note that $m(t_2) = u^*(t_2)$. Lemma 2.5 and condition (iv) imply

$$\begin{aligned}
D^+ m(t) &:= \limsup_{h \to 0^+} \frac{m(t+h) - m(t)}{h} \\
&= D^+ V(t, x_2(t) - x_1(t)) \\
&= \dot{V}(t, x_2(t) - x_1(t)) \ge g(t, V(t, x_2(t) - x_1(t))) = g(t, m(t))
\end{aligned} \tag{2.26}$$

for $t_0 < t \le t_2$.

Applying Lemma 2.6 we get $m(t) \le u^*(t)$ for $t_0 < t \le t_2$. As $m(t) \ge 0$ for $t_0 < t \le t_2$ and m is continuous at t_0, we find $m(t_0) = 0$. Therefore we have $x_2(t_0) = x_1(t_0) = x_0$ and, as noted above, $x_2(t_2) = x_2 \ne x_1(t_2)$. Thus problem (1.1) has two different solutions.

According to the well-known Kneser theorem [17, Theorem 4.1, page 15] the set of solutions of problem (1.1) either consists of one element or has the cardinality of the continuum. Consequently, if problem (1.1) has two different solutions on interval $[t_0, t_1^*]$ and condition (i) is satisfied, then the set of different solutions of problem (1.1) on interval J has the cardinality of the continuum. The proof is completed. □

Remark 2.11. Note that in the scalar case with $V(t, x) := |x|$ condition (2.11) has the form

$$\big(f(t, y) - f(t, x_1(t))\big) \cdot \operatorname{sign}\big(y - x_1(t)\big) \ge g\big(t, |y - x_1(t)|\big). \tag{2.27}$$

Example 2.12. Consider for $a = 0.1$, $b = 1$, $t_0 = 0$ and $x_0 = 0$ the scalar differential equation

$$\dot{x} = f(t,x) := \begin{cases} 2x^{1/3} - \dfrac{1}{2} \cdot t^{1/2} \cdot \sin \dfrac{|x|}{t} & \text{if } t \neq 0, \\ 2x^{1/3} & \text{if } t = 0, \end{cases} \tag{2.28}$$

with the initial condition $x(0) = 0$. Let us show that the set of different solutions of this problem on interval J has the cardinality of $\mathbb{R}$. Obviously we can set $x_1(t) \equiv 0$. Put

$$g(t,u) := 2u^{1/3} - \frac{1}{2} \cdot t^{1/2}, \qquad u^*(t) := t^{3/2}, \qquad V(t,x) := |x|. \tag{2.29}$$

Conditions (i), (ii), and (iii) are satisfied. Let us verify that the last condition (iv) is valid, too. We get

$$\begin{aligned} \dot{V}(t, y - x_1(t)) = \dot{V}(t,y) &= (\operatorname{sign} y) \cdot \left[2y^{1/3} - \frac{1}{2} \cdot t^{1/2} \cdot \sin \frac{|y|}{t} \right] \\ &\geq 2|y|^{1/3} - \frac{1}{2} \cdot t^{1/2} = 2V(t,y)^{1/3} - \frac{1}{2} \cdot t^{1/2} = g(t, V(t,y)) \\ &= g(t, V(t, y - x_1(t))). \end{aligned} \tag{2.30}$$

Thus, all conditions of Theorem 2.8 hold and, consequently, the set of different solutions on J of given problem has the cardinality of $\mathbb{R}$.

3. Incompatible Conditions

In this section we show that the formulation of condition (iv) in Theorem 2.8 without knowledge of a solution of the Cauchy problem can lead to an incompatible set of conditions. In the proof of Theorem 3.2 for the one-dimensional case we use the following result given by Nekvinda [18, page 1].

Lemma 3.1. *Let $D \subset \mathbb{R}^2$ and let $f : D \to \mathbb{R}$ be a continuous function in D. Let equation*

$$\dot{x} = f(t,x) \tag{3.1}$$

has the property of left uniqueness. For any $t_0 \in \mathbb{R}$ let A be the set of all $x_0 \in \mathbb{R}$ such that $(t_0, x_0) \in D$ and, for some $\varepsilon > 0$, the initial-value problem (1.1) *has more than one solution in the interval* $[t_0, t_0+\varepsilon)$. *Then A is at most countable.*

Theorem 3.2. *The set of conditions (i)–(iv):*

(i) *$f : R_0 \to \mathbb{R}$ with $R_0 := \{(t,x) \in J \times \mathbb{R},\ |x - x_0| \leq b\}$ is continuous;*

(ii) $g : (t_0, t_0 + a] \times (0, \infty) \to \mathbb{R}_+$ *is continuous, nondecreasing in the second variable, and has the following property: there exists a continuous function* $u^*(t)$ *on* J, *which satisfies the differential equation*

$$\dot{u}(t) = g(t, u) \tag{3.2}$$

for $t_0 < t \leq t_0 + a$ *with* $u^*(t_0) = 0$ *and does not vanish for* $t \neq t_0$;

(iii) $V : J \times S^1_{2b}(0) \to \mathbb{R}_+$ *is continuous, positive definite, and Lipschitzian uniformly with respect to* $t \in J$;

(iv) *for* $t_0 < t \leq t_0 + a$, $|x - x_0| \leq b$, $|y - x_0| \leq b$, $x \neq y$,

$$\dot{V}(t, x - y) \geq g(t, V(t, x - y)), \tag{3.3}$$

where we define

$$\dot{V}(t, x - y) := \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x - y + h[f(t, x) - f(t, y)]) - V(t, x - y)] \tag{3.4}$$

contains a contradiction.

Proof. Any initial value problem

$$\dot{x} = f(t, x), \qquad x(t_0) = x^* \tag{3.5}$$

with $|x^* - x_0| \leq b$ has at least two different solutions due to Theorem 2.8. Thus we have an uncountable set of nonuniqueness points. We show that solutions passing through different initial points are left unique. Suppose that it does not hold. Let $x_1(t)$ be a solution starting from (t_0, x_1), and let $x_2(t)$ be a solution starting from (t_0, x_2) with $x_2 \neq x_1$. If we assume that these solutions cross at a point $t_1 > t_0$ and if we set

$$m(t) := V(t, x_1(t) - x_2(t)) \tag{3.6}$$

then $m(t_0) > 0$, $m(t_1) = 0$. Therefore there exists a point $t \in (t_0, t_1)$ such that (we apply Lemma 2.5)

$$D^+ m(t) = D^+ V(t, x_1(t) - x_2(t)) = \dot{V}(t, x_1(t) - x_2(t)) < 0, \tag{3.7}$$

in contradiction to (3.3). Thus we obtain left uniqueness. From Lemma 3.1 we conclude in contrast to the above conclusion that the set of nonuniqueness points (t_0, x^*) can be at most countable. □

In [1, Theorem 1.24.1, page 99] the following nonuniqueness result (see [14, Theorem 2.2.7, page 55], too) is given which uses an inverse Kamke's condition (condition (3.9) below).

Theorem 3.3. *Let $g(t,u)$ be continuous on $0 < t \leq a$, $0 \leq u \leq 2b$, $g(t,0) \equiv 0$, and $g(t,u) > 0$ for $u > 0$. Suppose that, for each t_1, $0 < t_1 < a$, $u(t) \not\equiv 0$ is a differentiable function on $0 < t < t_1$, and continuous on $0 \leq t < t_1$ for which $\dot{u}_+(0)$ exists,*

$$\begin{aligned} \dot{u} &= g(t,u), \qquad 0 < t < t_1, \\ u(0) &= \dot{u}_+(0) = 0. \end{aligned} \tag{3.8}$$

Let $f \in C[R_0, \mathbb{R}]$, where $R_0 : 0 \leq t \leq a$, $|x| \leq b$, and, for $(t,x), (t,y) \in R_0$, $t \neq 0$,

$$\left| f(t,x) - f(t,y) \right| \geq g(t, |x-y|). \tag{3.9}$$

Then, the scalar problem $\dot{x} = f(t,x)$, $x(0) = 0$ has at least two solutions on $0 \leq t \leq a$.

Remark 3.4. In the proof of Theorem 3.3 at first $f(t,0) = 0$ is assumed. Putting $y = 0$ in (3.9) leads to the inequality

$$\left| f(t,x) \right| \geq g(t, |x|). \tag{3.10}$$

As $f(t,x)$ is continuous and $g(t,u) > 0$ for $u > 0$ it follows that $f(t,x)$ must have constant sign for each of the half planes $x > 0$ and $x < 0$. For the upper half plane this implies that

$$\begin{aligned} f(t,x) &\geq g(t,x), \\ f(t,x) &\leq -g(t,x). \end{aligned} \tag{3.11}$$

For the first inequality nonuniqueness is shown in [1]. But a similar argumentation cannot be used for the second inequality as the following example in [5] shows. We consider the initial value problem $\dot{x} = f(t,x)$, $x(0) = 0$, with

$$f(t,x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0 \\ \sqrt{-x} & \text{if } x < 0 \end{cases} \tag{3.12}$$

and $g(t,u) := \sqrt{u}$. Thus inequality $|f(t,x)| = \sqrt{|x|} \geq g(t,|x|)$ holds. In the upper half-plane we have $f(t,x) \leq -g(t,x)$. The function $u(t) = t^2/4$ is a nontrivial solution of the comparison equation. Therefore all assumptions are fulfilled, but the initial value problem has at most one solution because of Theorem 1.3.1 [1, page 10].

The next theorem analyzes in the scalar case (for $(t_0, x_0) = (0,0)$) that even fulfilling a rather general condition (see condition (3.14) in the following theorem) cannot ensure nonuniqueness since the set of all conditions contains an inner contradiction. The proof was motivated by the paper [5].

Theorem 3.5. *There exists no system of three functions f, g, and V satisfying the following suppositions:*

(i) $f : R_0 \to \mathbb{R}$ *with* $R_0 := \{(t,x) \in \mathbb{R} \times \mathbb{R},\ 0 \le t \le a, 0 \le x \le b\}$ *is a continuous function;*

(ii) *the continuous function* $g : (0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t,0) := 0$ *if* $t \in (0,a]$, *has the following property: there exists a continuously differentiable function* $u^*(t)$ *on* $0 \le t \le a$, *satisfying the differential equation*

$$\dot{u} = g(t,u) \tag{3.13}$$

for $0 < t \le a$ *such that* $u^*(0) = 0$ *and* $u^*(t) > 0$ *for* $t \ne 0$;

(iii) *the continuous function* $V : [0,a] \times S_b^1(0) \to \mathbb{R}_+$ *is positive definite, and for all* $0 < t \le a$, $0 < x < b$ *continuously differentiable;*

(iv) *for* $0 < t \le a$, $0 < y < x \le b$,

$$\dot{V}(t, x-y) \ge g(t, V(t, x-y)) \ge 0, \tag{3.14}$$

where we define

$$\dot{V}(t, x-y) := V_1'(t, x-y) + V_2'(t, x-y) \cdot [f(t,x) - f(t,y)] \tag{3.15}$$

and subscript indices denote the derivative with respect to the first and second argument, respectively;

(v) *there exist a positive constant* ϑ *and a function* $\xi : (0,b] \to (0,\infty)$ *such that for* $0 < t \le a$ *and* $0 < x \le b$

$$\begin{gathered} 0 \le V'_1(t,x) \le \vartheta \cdot \xi(x), \quad 0 < V_2'(t,x) \le \vartheta \cdot \frac{\xi(x)}{x}, \\ V(t,x) \ge \xi(x); \end{gathered} \tag{3.16}$$

(vi) *for* $t \in [0,a]$ *and* x, y *with* $0 < y < x \le b$ *the inequality*

$$f(t,x) - f(t,y) \ge 0 \tag{3.17}$$

holds.

Proof. Let us show that the above properties are not compatible. For fixed numbers x, y with $0 < y < x \le b$ consider the auxiliary function

$$F(t) := \frac{f(t,x) - f(t,y)}{x - y} + 1, \quad t \in [0,a]\ . \tag{3.18}$$

Clearly, F is continuous and assumes a (positive) maximum. Set

$$K = \max_{[0,a]} F(t) \geq 1. \tag{3.19}$$

If the function g fulfills the inequality

$$g(t,u) \leq \Lambda \cdot u \tag{3.20}$$

with a positive constant Λ in a domain $0 < t \leq A \leq a$, $0 \leq u \leq B$, $B > 0$, then the initial value problem

$$\dot{u} = g(t,u), \qquad u(0) = 0 \tag{3.21}$$

has the unique trivial solution $u = 0$. Really, since $u^*(t) > 0$ for $t \in (0,a]$, by integrating inequality

$$\frac{\dot{u}^*(t)}{u^*(t)} \leq \Lambda \tag{3.22}$$

with limits t, $A^* \in (0,A)$ we get

$$u^*(A^*) \leq u^*(t) \exp[\Lambda(A^* - t)] \tag{3.23}$$

and for $t \rightarrow 0^+$

$$u^*(A^*) \leq 0 \tag{3.24}$$

which contradicts positivity of u^*. Therefore problem (3.21) has only the trivial solution. Hence, there exist a sequence $\{(t_n, u_n)\}$ with $t_n \in (0,a]$, $u_n > 0$, $\lim_{n\to\infty}(t_n, u_n) = (0,0)$ and a sequence $\{\lambda_n\}$, $\lambda_n > 0$, with $\lim_{n\to\infty}\lambda_n = \infty$ such that the inequality

$$g(t_n, u_n) > \lambda_n u_n \tag{3.25}$$

holds for every n. Consider now the relation

$$V(t,x) = 0. \tag{3.26}$$

Due to the properties of V we conclude that for all sufficiently small positive numbers t_n, u_n (i.e., for all sufficiently large n) there exists a (sufficiently small and positive) number $\widetilde{u}_n$ such that the equation

$$V(t_n, x) = u_n \tag{3.27}$$

has the solution $x = \tilde{u}_n$. Thus a sequence $\{\tilde{u}_n\}$ with $\tilde{u}_n > 0$ and $\lim_{n\to\infty}\tilde{u}_n = 0$ corresponds to the sequence $\{(t_n, u_n)\}$. For every n define a number j_n as

$$j_n = \left\lceil \frac{x-y}{\tilde{u}_n} - 1 \right\rceil, \tag{3.28}$$

where $\lceil\cdot\rceil$ is the ceiling function. Without loss of generality we can suppose that

$$\frac{x-y}{\tilde{u}_n} > 4. \tag{3.29}$$

Obviously,

$$\frac{x-y}{\tilde{u}_n} - 1 \leq j_n < \frac{x-y}{\tilde{u}_n}. \tag{3.30}$$

Moreover, without loss of generality we can suppose that for every sufficiently large n the inequality

$$\lambda_n > 2\vartheta K \tag{3.31}$$

holds. Set

$$\begin{aligned} x_0 &:= y, \\ x_1 &:= y + \tilde{u}_n, \\ x_2 &:= y + 2\tilde{u}_n, \\ &\vdots \\ x_{j_n} &:= y + j_n \cdot \tilde{u}_n, \\ x_{j_n+1} &:= x. \end{aligned} \tag{3.32}$$

Consider for all sufficiently large n the expression

$$\mathcal{E}_n := j_n V_1'(t_n, \tilde{u}_n) + V_2'(t_n,\ \tilde{u}_n) \cdot \left[f(t_n, x) - f(t_n, y) \right]. \tag{3.33}$$

Then

$$\begin{aligned}
\mathcal{E}_n &= j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot\sum_{i=1}^{j_n+1}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] \\
&= j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot\sum_{i=1}^{j_n}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] + \left[f(t_n,x)-f\left(t_n,x_{j_n}\right)\right] \\
&\geq [\text{due to (vi)}] \geq j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot\sum_{i=1}^{j_n}\left[f(t_n,x_i)-f(t_n,x_{i-1})\right] \\
&= [\text{due to (iv) and (v)}] = j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot j_n\left[\frac{-V_1'(t_n,\tilde{u}_n)+\dot{V}(t_n,\tilde{u}_n)}{V_2'(t_n,\tilde{u}_n)}\right] \\
&\geq [\text{due to (iv)}] \\
&\geq j_n V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot j_n\left[\frac{-V_1'(t_n,\tilde{u}_n)+g(t_n,V(t_n,\tilde{u}_n))}{V'_2(t_n,\tilde{u}_n)}\right] \\
&= j_n\cdot g(t_n,V(t_n,\tilde{u}_n)) = [\text{due to (3.27)}] = j_n\cdot g(t_n,u_n) \geq [\text{due to (3.25)}] \\
&\geq j_n\lambda_n u_n \geq [\text{due to (3.31)}] \geq j_n u_n\cdot 2\vartheta K \geq [\text{due to (3.30)}] \\
&\geq \left(\frac{x-y}{\tilde{u}_n}-1\right) u_n\cdot 2\vartheta K \\
&= \left(\frac{x-y}{\tilde{u}_n}-1\right) V(t_n,\tilde{u}_n)\cdot 2\vartheta K \\
&\geq [\text{due to (3.16) }] \geq \left(\frac{x-y}{\tilde{u}_n}-1\right)\xi(\tilde{u}_n)\cdot 2\vartheta K \\
&= (x-y-\tilde{u}_n)\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot 2\vartheta K \\
&\geq [\text{due to (3.29)}] \geq \frac{3}{4}\cdot(x-y)\cdot 2\vartheta K\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n} \\
&= \frac{3}{2}\cdot(x-y)\cdot\vartheta K\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n} > 0.
\end{aligned} \tag{3.34}$$

Estimating the expression $\mathcal{E}_n$ from above we get (see (3.32))

$$\begin{aligned}
\mathcal{E}_n &\leq \frac{x-y}{\tilde{u}_n} V_1'(t_n,\tilde{u}_n) + V_2'(t_n,\tilde{u}_n)\cdot\left[f(t_n,x)-f(t_n,y)\right] \\
&\leq [\text{due to (v)}] \\
&\leq \frac{x-y}{\tilde{u}_n}\cdot\vartheta\cdot\xi(\tilde{u}_n) + \vartheta\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot(K-1)(x-y) = \vartheta\cdot\frac{\xi(\tilde{u}_n)}{\tilde{u}_n}\cdot K(x-y).
\end{aligned} \tag{3.35}$$

These two above estimations yield

$$0 < \frac{3}{2} \cdot (x-y) \cdot \vartheta K \cdot \frac{\xi(\widetilde{u}_n)}{\widetilde{u}_n} \leq \mathcal{E}_n \leq (\mathrm{x}-y) \cdot \vartheta K \cdot \frac{\xi(\widetilde{u}_n)}{\widetilde{u}_n}\ , \tag{3.36}$$

in contrast to $(3/2) \nleq 1$. Since the initially taken points x and y, $0 < y < x$, can be chosen arbitrarily close to zero, the theorem is proved. □

The following result is a consequence of Theorem 3.5 if $V(t,x) := |x|$, $\xi(x) := x$ and $\vartheta = 1$. Condition (3.38) below was discussed previously in [5].

Theorem 3.6. *There exists no system of two functions f and g satisfying the following suppositions:*

(i) *$f : R_0 \to \mathbb{R}$ with $R_0 := \{(t,x) \in \mathbb{R} \times \mathbb{R},\ 0 \leq t \leq a, 0 \leq x \leq b\}$ is a continuous function;*

(ii) *the continuous function $g : (0,a] \times \mathbb{R}_+ \to \mathbb{R}_+$, $g(t,0) := 0$ if $t \in (0,a]$, has the following property: there exists a continuously differentiable function $u^*(t)$ on $0 \leq t \leq a$, satisfying the differential equation*

$$\dot{u}(t) = g(t,u) \tag{3.37}$$

for $0 < t \leq a$ such that $u^*(0) = 0$ and $u^*(t) > 0$ for $t \neq 0$;

(iii) *for $0 < t \leq a$, $0 < y < x \leq b$*

$$f(t,x) - f(t,y) \geq g(t,x-y) \geq 0; \tag{3.38}$$

(iv) *for $0 < y < x \leq b$ the inequality $f(0,x) - f(0,y) \geq 0$ holds.*

Remark 3.7. Let us note that in the singular case, that is, when we permit that the function $f(t,x)$ is not continuous at $t = 0$, the given sets of conditions in Theorems 3.5 and 3.6 can be compatible. This can be seen from the proof where the continuity of f is substantial. Such singular case was considered in [13].

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Research Article

Conjugacy of Self-Adjoint Difference Equations of Even Order

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We study oscillation properties of $2n$-order Sturm-Liouville difference equations. For these equations, we show a conjugacy criterion using the p-criticality (the existence of linear dependent recessive solutions at ∞ and $-\infty$). We also show the equivalent condition of p-criticality for one term $2n$-order equations.

1. Introduction

In this paper, we deal with $2n$-order Sturm-Liouville difference equations and operators

$$L(y)_k = \sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(r_k^{[\nu]} \Delta^{\nu} y_{k+n-\nu} \right) = 0, \quad r_k^{[n]} > 0,\ k \in \mathbb{Z}, \tag{1.1}$$

where Δ is the forward difference operator, that is, $\Delta y_k = y_{k+1} - y_k$, and $r^{[\nu]}, \nu = 0, \ldots, n$, are real-valued sequences. The main result is the conjugacy criterion which we formulate for the equation $L(y)_k + q_k y_{k+n} = 0$, that is viewed as a perturbation of (1.1), and we suppose that (1.1) is at least p-critical for some $p \in \{1, \ldots, n\}$. The concept of p-criticality (a disconjugate equation is said to be p-critical if and only if it possesses p solutions that are recessive both at ∞ and $-\infty$, see Section 3) was introduced for second-order difference equations in [1], and later in [2] for (1.1). For the continuous counterpart of the used techniques, see [3–5] from where we get an inspiration for our research.

The paper is organized as follows. In Section 2, we recall necessary preliminaries. In Section 3, we recall the concept of p-criticality, as introduced in [2], and show the first result—the equivalent condition of p-criticality for the one term difference equation

$$\Delta^n \left(r_k \Delta^n y_k \right) = 0 \tag{1.2}$$

(Theorem 3.4). In Section 4 we show the conjugacy criterion for equation

$$(-\Delta)^n \left(r_k \Delta^n y_k \right) + q_k y_{k+n} = 0, \tag{1.3}$$

and Section 5 is devoted to the generalization of this criterion to the equation with the middle terms

$$\sum_{\nu=0}^{n} (-\Delta)^\nu \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) + q_k y_{k+n} = 0. \tag{1.4}$$

2. Preliminaries

The proof of our main result is based on equivalency of (1.1) and the linear Hamiltonian difference systems

$$\Delta x_k = A x_{k+1} + B_k u_k, \qquad \Delta u_k = C_k x_{k+1} - A^T u_k, \tag{2.1}$$

where A, B_k, and C_k are $n \times n$ matrices of which B_k and C_k are symmetric. Therefore, we start this section recalling the properties of (2.1), which we will need later. For more details, see the papers [6–11] and the books [12, 13].

The substitution

$$x_k^{[y]} = \begin{pmatrix} y_{k+n-1} \\ \Delta y_{k+n-2} \\ \vdots \\ \Delta^{n-1} y_k \end{pmatrix}, \qquad u_k^{[y]} = \begin{pmatrix} \sum_{\nu=1}^{n} (-\Delta)^{\nu-1} \left(r_k^{[\nu]} \Delta^\nu y_{k+n-\nu} \right) \\ \vdots \\ -\Delta \left(r_k^{[n]} \Delta^n y_k \right) + r_k^{[n-1]} \Delta^{n-1} y_{k+1} \\ r_k^{[n]} \Delta^n y_k \end{pmatrix} \tag{2.2}$$

transforms (1.1) to linear Hamiltonian system (2.1) with the $n \times n$ matrices A, B_k, and C_k given by

$$\begin{gathered} A = \left(a_{ij} \right)_{i,j=1}^{n}, \qquad a_{ij} = \begin{cases} 1, & \text{if } j = i+1,\ i = 1, \dots, n-1, \\ 0, & \text{elsewhere,} \end{cases} \\ B_k = \operatorname{diag} \left\{ 0, \dots, 0, \frac{1}{r_k^{[n]}} \right\}, \qquad C_k = \operatorname{diag} \left\{ r_k^{[0]}, \dots, r_k^{[n-1]} \right\}. \end{gathered} \tag{2.3}$$

Then, we say that the solution (x, u) of (2.1) is generated by the solution y of (1.1).

Let us consider, together with system (2.1), the matrix linear Hamiltonian system

$$\Delta X_k = AX_{k+1} + B_k U_k, \qquad \Delta U_k = C_k X_{k+1} - A^T U_k, \tag{2.4}$$

where the matrices A, B_k, and C_k are also given by (2.3). We say that the matrix solution (X, U) of (2.4) is generated by the solutions $y^{[1]}, \ldots, y^{[n]}$ of (1.1) if and only if its columns are generated by $y^{[1]}, \ldots, y^{[n]}$, respectively, that is, $(X, U) = (x^{[y_1]}, \ldots, x^{[y_n]}, u^{[y_1]}, \ldots, u^{[y_n]})$. Reversely, if we have the solution (X, U) of (2.4), the elements from the first line of the matrix X are exactly the solutions $y^{[1]}, \ldots, y^{[n]}$ of (1.1). Now, we can define the oscillatory properties of (1.1) via the corresponding properties of the associated Hamiltonian system (2.1) with matrices A, B_k, and C_k given by (2.3), for example, (1.1) is disconjugate if and only if the associated system (2.1) is disconjugate, the system of solutions $y^{[1]}, \ldots, y^{[n]}$ is said to be recessive if and only if it generates the recessive solution X of (2.4), and so forth. Therefore, we define the following properties just for linear Hamiltonian systems.

For system (2.4), we have an analog of the continuous *Wronskian identity*. Let (X, U) and $(\tilde{X}, \tilde{U})$ be two solutions of (2.4). Then,

$$X_k^T \tilde{U}_k - U_k^T \tilde{X}_k \equiv W \tag{2.5}$$

holds with a constant matrix W. We say that the solution (X, U) of (2.4) is a *conjoined basis*, if

$$X_k^T U_k \equiv U_k^T X_k, \qquad \operatorname{rank}\begin{pmatrix} X \\ U \end{pmatrix} = n. \tag{2.6}$$

Two conjoined bases $(X, U), (\tilde{X}, \tilde{U})$ of (2.4) are called *normalized* conjoined bases of (2.4) if $W = I$ in (2.5) (where I denotes the identity operator).

System (2.1) is said to be *right disconjugate* in a discrete interval $[l, m]$, $l, m \in \mathbb{Z}$, if the solution $\binom{X}{U}$ of (2.4) given by the initial condition $X_l = 0, U_l = I$ satisfies

$$\ker X_{k+1} \subseteq \ker X_k, \qquad X_k X_{k+1}^{\dagger} (I - A)^{-1} B_k \geq 0, \tag{2.7}$$

for $k = l, \ldots, m-1$, see [6]. Here ker, †, and ≥ stand for kernel, Moore-Penrose generalized inverse, and nonnegative definiteness of the matrix indicated, respectively. Similarly, (2.1) is said to be *left disconjugate* on $[l, m]$, if the solution given by the initial condition $X_m = 0$, $U_m = -I$ satisfies

$$\ker X_k \subseteq \ker X_{k+1}, \quad X_{k+1} X_k^{\dagger} B_k (I - A)^{T-1} \geq 0, \quad k = l, \ldots, m-1. \tag{2.8}$$

System (2.1) is disconjugate on $\mathbb{Z}$, if it is right disconjugate, which is the same as left disconjugate, see [14, Theorem 1], on $[l, m]$ for every $l, m \in \mathbb{Z}$, $l < m$. System (2.1) is said to be *nonoscillatory* at ∞ (*nonoscillatory* at $-\infty$), if there exists $l \in \mathbb{Z}$ such that it is right disconjugate on $[l, m]$ for every $m > l$ (there exists $m \in \mathbb{Z}$ such that (2.1) is left disconjugate on $[l, m]$ for every $l < m$).

We call a conjoined basis $\binom{\tilde{X}}{\tilde{U}}$ of (2.4) the *recessive solution* at ∞, if the matrices $\tilde{X}_k$ are nonsingular, $\tilde{X}_k\tilde{X}_{k+1}^{-1}(I-A_k)^{-1}B_k \geq 0$ (both for large k), and for any other conjoined basis $\binom{X}{U}$, for which the (constant) matrix $X^T\tilde{U} - U^T\tilde{X}$ is nonsingular, we have

$$\lim_{k\to\infty} X_k^{-1}\tilde{X}_k = 0. \tag{2.9}$$

The solution (X,U) is called the *dominant solution* at ∞. The recessive solution at ∞ is determined uniquely up to a right multiple by a nonsingular constant matrix and exists whenever (2.4) is nonoscillatory and eventually controllable. (System is said to be *eventually controllable* if there exist $N,\kappa \in \mathbb{N}$ such that for any $m \geq N$ the trivial solution $\binom{x}{u} = \binom{0}{0}$ of (2.1) is the only solution for which $x_m = x_{m+1} = \cdots = x_{m+\kappa} = 0$.) The equivalent characterization of the recessive solution $\binom{\tilde{X}}{\tilde{U}}$ of eventually controllable Hamiltonian difference systems (2.1) is

$$\lim_{k\to\infty}\left(\sum k\tilde{X}_{j+1}^{-1}(I-A)^{-1}B_j\tilde{X}_j^{T-1}\right)^{-1} = 0, \tag{2.10}$$

see [12]. Similarly, we can introduce the recessive and the dominant solutions at $-\infty$. For related notions and results for second-order dynamic equations, see, for example, [15, 16].

We say that a pair (x,u) is *admissible* for system (2.1) if and only if the first equation in (2.1) holds.

The energy functional of (1.1) is given by

$$\mathcal{F}(y) := \sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{n} r_k^{[\nu]}\left(\Delta^\nu y_{k+n-\nu}\right)^2. \tag{2.11}$$

Then, for admissible (x,u), we have

$$\begin{aligned}\mathcal{F}(y) &= \sum_{k=-\infty}^{\infty}\sum_{\nu=0}^{n} r_k^{[\nu]}\left(\Delta^\nu y_{k+n-\nu}\right)^2 \\ &= \sum_{k=-\infty}^{\infty}\left[\sum_{\nu=0}^{n-1} r_k^{[\nu]}\left(\Delta^\nu y_{k+n-\nu}\right)^2 + \frac{1}{r_k^{[n]}}\left(r_k^{[n]}\Delta^n y_k\right)^2\right] \\ &= \sum_{k=-\infty}^{\infty}\left[x_{k+1}^T C_k x_{k+1} + u_k^T B_k u_k\right] =: \mathcal{F}(x,u).\end{aligned} \tag{2.12}$$

To prove our main result, we use a variational approach, that is, the equivalency of disconjugacy of (1.1) and positivity of $\mathcal{F}(x,u)$; see [6].

Now, we formulate some auxiliary results, which are used in the proofs of Theorems 3.4 and 4.1. The following Lemma describes the structure of the solution space of

$$\Delta^n\left(r_k\Delta^n y_k\right) = 0, \quad r_k > 0. \tag{2.13}$$

Lemma 2.1 (see [17, Section 2]). *Equation* (2.13) *is disconjugate on* $\mathbb{Z}$ *and possesses a system of solutions* $y^{[j]}, \tilde{y}^{[j]}$, $j = 1, \ldots, n$, *such that*

$$y^{[1]} \prec \cdots \prec y^{[n]} \prec \tilde{y}^{[1]} \prec \cdots \prec \tilde{y}^{[n]} \tag{2.14}$$

as $k \to \infty$, *where* $f \prec g$ *as* $k \to \infty$ *for a pair of sequences* f, g *means that* $\lim_{k\to\infty}(f_k/g_k) = 0$. *If* (2.14) *holds, the solutions* $y^{[j]}$ *form the recessive system of solutions at* ∞, *while* $\tilde{y}^{[j]}$ *form the dominant system,* $j = 1, \ldots, n$. *The analogous statement holds for the ordered system of solutions as* $k \to -\infty$.

Now, we recall the transformation lemma.

Lemma 2.2 (see [14, Theorem 4]). *Let* $h_k > 0$, $L(y) = \sum_{\nu=0}^{n}(-\Delta)^{\nu}(r_k^{[\nu]}\Delta^{\nu}y_{k+n-\nu})$ *and consider the transformation* $y_k = h_k z_k$. *Then, one has*

$$h_{k+n}L(y) = \sum_{\nu=0}^{n}(-\Delta)^{\nu}\left(R_k^{[\nu]}\Delta^{\nu}z_{k+n-\nu}\right), \tag{2.15}$$

where

$$R_k^{[n]} = h_{k+n}h_k r_k^{[n]}, \qquad R_k^{[0]} = h_{k+n}L(h), \tag{2.16}$$

that is, y *solves* $L(y) = 0$ *if and only if* z *solves the equation*

$$\sum_{\nu=0}^{n}(-\Delta)^{\nu}\left(R_k^{[\nu]}\Delta^{\nu}z_{k+n-\nu}\right) = 0. \tag{2.17}$$

The next lemma is usually called the second mean value theorem of summation calculus.

Lemma 2.3 (see [17, Lemma 3.2]). *Let* $n \in \mathbb{N}$ *and the sequence* a_k *be monotonic for* $k \in [K+n-1, L+n-1]$ *(i.e.,* Δa_k *does not change its sign for* $k \in [K+n-1, L+n-2]$*). Then, for any sequence* b_k *there exist* $n_1, n_2 \in [K, L-1]$ *such that*

$$\begin{gathered}
\sum_{j=K}^{L-1}a_{n+j}b_j \le a_{K+n-1}\sum_{i=K}^{n_1-1}b_i + a_{L+n-1}\sum_{i=n_1}^{L-1}b_i, \\
\sum_{j=K}^{L-1}a_{n+j}b_j \ge a_{K+n-1}\sum_{i=K}^{n_2-1}b_i + a_{L+n-1}\sum_{i=n_2}^{L-1}b_i.
\end{gathered} \tag{2.18}$$

Now, let us consider the linear difference equation

$$y_{k+n} + a_k^{[n-1]}y_{k+n-1} + \cdots + a_k^{[0]}y_k = 0, \tag{2.19}$$

where $k \ge n_0$ for some $n_0 \in \mathbb{N}$ and $a_k^{[0]} \neq 0$, and let us recall the main ideas of [18] and [19, Chapter IX].

An integer $m > n_0$ is said to be a *generalized zero* of multiplicity k of a nontrivial solution y of (2.19) if $y_{m-1} \neq 0$, $y_m = y_{m+1} = \cdots = y_{m+k-2} = 0$, and $(-1)^k y_{m-1} y_{m+k-1} \geq 0$. Equation (2.19) is said to be eventually disconjugate if there exists $N \in \mathbb{N}$ such that no non-trivial solution of this equation has n or more generalized zeros (counting multiplicity) on $[N, \infty)$.

A system of sequences $u_k^{[1]}, \ldots, u_k^{[n]}$ is said to form the *D-Markov system* of sequences for $k \in [N, \infty)$ if Casoratians

$$C\left(u^{[1]}, \ldots, u^{[j]}\right)_k = \begin{vmatrix} u_k^{[1]} & \cdots & u_k^{[j]} \\ u_{k+1}^{[1]} & \cdots & u_{k+1}^{[j]} \\ \vdots & & \vdots \\ u_{k+j-1}^{[1]} & \cdots & u_{k+j-1}^{[j]} \end{vmatrix}, \quad j = 1, \ldots, n \tag{2.20}$$

are positive on $(N + j, \infty)$.

Lemma 2.4 (see [19, Theorem 9.4.1]). *Equation* (2.19) *is eventually disconjugate if and only if there exist* $N \in \mathbb{N}$ *and solutions* $y^{[1]}, \ldots, y^{[n]}$ *of* (2.19) *which form a D-Markov system of solutions on* (N, ∞). *Moreover, this system can be chosen in such a way that it satisfies the additional condition*

$$\lim_{k \to \infty} \frac{y_k^{[i]}}{y_k^{[i+1]}} = 0, \quad i = 1, \ldots, n-1. \tag{2.21}$$

3. Criticality of One-Term Equation

Suppose that (1.1) is disconjugate on $\mathbb{Z}$ and let $\hat{y}^{[i]}$ and $\tilde{y}^{[i]}$, $i = 1, \ldots, n$, be the recessive systems of solutions of $L(y) = 0$ at $-\infty$ and ∞, respectively. We introduce the linear space

$$\mathcal{H} = \operatorname{Lin}\left\{\hat{y}^{[1]}, \ldots, \hat{y}^{[n]}\right\} \cap \operatorname{Lin}\left\{\tilde{y}^{[1]}, \ldots, \tilde{y}^{[n]}\right\}. \tag{3.1}$$

Definition 3.1 (see [2]). Let (1.1) be disconjugate on $\mathbb{Z}$ and let $\dim \mathcal{H} = p \in \{1, \ldots, n\}$. Then, we say that the operator L (or (1.1)) is *p-critical* on $\mathbb{Z}$. If $\dim \mathcal{H} = 0$, we say that L is *subcritical* on $\mathbb{Z}$. If (1.1) is not disconjugate on $\mathbb{Z}$, that is, $L \ngeq 0$, we say that L is *supercritical* on $\mathbb{Z}$.

To prove the result in this section, we need the following statements, where we use the generalized power function

$$k^{(0)} = 1, \qquad k^{(i)} = k(k-1)\cdots(k-i+1), \quad i \in \mathbb{N}. \tag{3.2}$$

For reader's convenience, the first statement in the following lemma is slightly more general than the corresponding one used in [2] (it can be verified directly or by induction).

Lemma 3.2 (see [2]). *The following statements hold.*

(i) *Let z_k be any sequence, $m \in \{0, \ldots, n\}$, and*

$$y_k := \sum_{j=0}^{k-1} (k-j-1)^{(n-1)} z_j, \tag{3.3}$$

then

$$\Delta^m y_k = \begin{cases} (n-1)^{(m)} \displaystyle\sum_{j=0}^{k-1} (k-j-1)^{(n-1-m)} z_j, & m \leq n-1, \\ (n-1)! z_k, & m = n. \end{cases} \tag{3.4}$$

(ii) *The generalized power function has the binomial expansion*

$$(k-j)^{(n)} = \sum_{i=0}^{n} (-1)^i \binom{n}{i} k^{(n-i)} (j+i-1)^{(i)}. \tag{3.5}$$

We distinguish two types of solutions of (2.13). The *polynomial* solutions $k^{(i)}$, $i = 0, \ldots, n-1$, for which $\Delta^n y_k = 0$, and *nonpolynomial* solutions

$$\sum_{j=0}^{k-1} (k-j-1)^{(n-1)} j^{(i)} r_j^{-1}, \quad i = 0, \ldots, n-1, \tag{3.6}$$

for which $\Delta^n y_k \neq 0$. (Using Lemma 3.2(i) we obtain $\Delta^n y_k = (n-1)! k^{(i)} r_k^{-1}$.)

Now, we formulate one of the results of [20].

Proposition 3.3 (see [20, Theorem 4]). *If for some $m \in \{0, \ldots, n-1\}$*

$$\sum_{k=-\infty}^{0} \left[k^{(n-m-1)} \right]^2 r_k^{-1} = \infty = \sum_{k=0}^{\infty} \left[k^{(n-m-1)} \right]^2 r_k^{-1}, \tag{3.7}$$

then

$$\operatorname{Lin}\left\{1, \ldots, k^{(m)}\right\} \subseteq \mathscr{U}, \tag{3.8}$$

that is, (2.13) *is at least $(m+1)$-critical on $\mathbb{Z}$.*

Now, we show that (3.7) is also sufficient for (2.13) to be at least $(m+1)$-critical.

Theorem 3.4. *Let $m \in \{0, \ldots, n-1\}$. Equation* (2.13) *is at least $(m+1)$-critical if and only if* (3.7) *holds.*

Proof. Let $\mathcal{U}^+$ and $\mathcal{U}^-$ denote the subspaces of the solution space of (2.13) generated by the recessive system of solutions at ∞ and $-\infty$, respectively. Necessity of (3.7) follows directly from Proposition 3.3. To prove sufficiency, it suffices to show that if one of the sums in (3.7) is convergent, then $\{1,\dots,k^{(m)}\} \not\subseteq \mathcal{U}^+ \cap \mathcal{U}^-$. We show this statement for the sum $\sum^{\infty}$. The other case is proved similarly, so it will be omitted. Particularly, we show

$$\sum_{k=0}^{\infty}\left[k^{(n-m-1)}\right]^2 r_k^{-1} < \infty \Longrightarrow k^{(m)} \notin \mathcal{U}^+. \tag{3.9}$$

Let us denote $p := n-m-1$, and let us consider the following nonpolynomial solutions of (2.13):

$$y_k^{[\ell]} = \sum_{j=0}^{k-1}(k-j-1)^{(n-1)} j^{(p+\ell-1)} r_j^{-1} - \sum_{i=0}^{p}\left[(-1)^i\binom{n-1}{i}(k-1)^{(n-1-i)}\sum_{j=0}^{\infty} j^{(p+\ell-1)}(j+i-1)^{(i)} r_j^{-1}\right], \tag{3.10}$$

where $\ell = 1-p,\dots,m+1$. By Stolz-Cesàro theorem, since (using Lemma 3.2(i)) $\Delta^n y_k^{[\ell]} = (n-1)!k^{(p+\ell-1)}r_k^{-1}$, these solutions are ordered, that is, $y^{[i]} \prec y^{[i+1]},\ i=1-p,\dots,m$, as well as the polynomial solutions, that is, $k^{(i)} \prec k^{(i+1)},\ i=0,\dots,n-2$.

By some simple calculation and by Lemma 3.2 (at first, we use (i), and at the end, we use (ii)), we have

$$\begin{aligned}
\Delta^m y_k^{[1]}
&= \frac{(n-1)!}{(n-m-1)!}\sum_{j=0}^{k-1}(k-j-1)^{(n-m-1)} j^{(p)} r_j^{-1} \\
&\quad - \sum_{i=0}^{p}\left[(-1)^i\binom{n-1}{i}\frac{(n-1-i)!}{(n-m-1-i)!}(k-1)^{(n-m-1-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right] \\
&= \frac{(n-1)!}{p!}\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} \\
&\quad - \sum_{i=0}^{p}\left[(-1)^i\frac{(n-1)!(n-1-i)!}{(n-1-i)!i!(p-i)!}(k-1)^{(p-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right] \\
&= \frac{(n-1)!}{p!}\left\{\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} - \sum_{i=0}^{p}\left[(-1)^i\binom{p}{i}(k-1)^{(p-i)}\sum_{j=0}^{\infty} j^{(p)}(j+i-1)^{(i)} r_j^{-1}\right]\right\} \\
&= \frac{(n-1)!}{p!}\left[\sum_{j=0}^{k-1}(k-j-1)^{(p)} j^{(p)} r_j^{-1} - \sum_{j=0}^{\infty}(k-j-1)^{(p)} j^{(p)} r_j^{-1}\right]
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{(n-1)!}{p!}\sum_{j=k}^{\infty}(k-j-1)^{(p)}j^{(p)}r_j^{-1} \\
&= (-1)^{p+1}\frac{(n-1)!}{p!}\sum_{j=k}^{\infty}(j+1-k)^{(p)}j^{(p)}r_j^{-1}, \\
&\qquad\qquad \sum_{j=k}^{\infty}(j+1-k)^{(p)}j^{(p)}r_j^{-1} \le \sum_{j=k}^{\infty}\left[j^{(p)}\right]^2 r_j^{-1}.
\end{aligned}
\tag{3.11}
$$

Hence, from this and by Stolz-Cesàro theorem, we get

$$
\lim_{k\to\infty}\frac{y_k^{[1]}}{k^{(m)}} = \frac{1}{m!}\lim_{k\to\infty}\Delta^m y_k^{[1]} = 0, \tag{3.12}
$$

thus $y_k^{[1]} \prec k^{(m)}$. We obtained that $\{1, k, \dots, k^{(m-1)}, y^{[1-p]}, \dots, y^{[1]}\} \prec k^{(m)}$, which means that we have n solutions less than $k^{(m)}$, therefore $k^{(m)} \notin \mathcal{U}^+$ and (2.13) is at most m-critical. □

4. Conjugacy of Two-Term Equation

In this section, we show the conjugacy criterion for two-term equation.

Theorem 4.1. *Let $n > 1$, q_k be a real-valued sequence, and let there exist an integer $m \in \{0, \dots, n-1\}$ and real constants $c_0, \dots, c_m$ such that* (2.13) *is at least* $(m+1)$*-critical and the sequence $h_k := c_0 + c_1 k + \cdots + c_m k^{(m)}$ satisfies*

$$
\limsup_{K\downarrow -\infty,\, L\uparrow\infty}\sum_{k=K}^{L} q_k h_{k+n}^2 \le 0. \tag{4.1}
$$

If $q \not\equiv 0$, then

$$
(-\Delta)^n(r_k\Delta^n y_k) + q_k y_{k+n} = 0 \tag{4.2}
$$

is conjugate on $\mathbb{Z}$.

Proof. We prove this theorem using the variational principle; that is, we find a sequence $y \in \ell_0^2(\mathbb{Z})$ such that the energy functional $F(y) = \sum_{k=-\infty}^{\infty}[r_k(\Delta^n y_k)^2 + q_k y_{k+n}^2] < 0$.

At first, we estimate the first term of $F(y)$. To do this, we use the fact that this term is an energy functional of (2.13). Let us denote it by $\widetilde{F}$ that is,

$$
\widetilde{F}(y) = \sum_{k=-\infty}^{\infty} r_k(\Delta^n y_k)^2. \tag{4.3}
$$

Using the substitution (2.2), we find out that (2.13) is equivalent to the linear Hamiltonian system (2.1) with the matrix $C_k \equiv 0$; that is,

$$\Delta x_k = A_k x_{k+1} + B_k u_k, \qquad \Delta u_k = -A^T u_k, \tag{4.4}$$

and to the matrix system

$$\Delta X_k = A_k X_{k+1} + B_k U_k, \qquad \Delta U_k = -A^T U_k. \tag{4.5}$$

Now, let us denote the recessive solutions of (4.5) at $-\infty$ and ∞ by (X^-, U^-) and (X^+, U^+), respectively, such that the first $m+1$ columns of X^+ and X^- are generated by the sequences $1, k, \ldots, k^{(m)}$. Let K, L, M, and N be arbitrary integers such that $N - M > 2n$, $M - L > 2n$, and $L - K > 2n$ (some additional assumptions on the choice of K, L, M, N will be specified later), and let $(x^{[f]}, u^{[f]})$ and $(x^{[g]}, u^{[g]})$ be the solutions of (4.4) given by the formulas

$$\begin{aligned}
x_k^{[f]} &= X_k^-\left(\sum_{j=K}^{k-1}\mathcal{B}_j^-\right)\left(\sum_{j=K}^{L-1}\mathcal{B}_j^-\right)^{-1}\left(X_L^-\right)^{-1}x_L^{[h]},\\
u_k^{[f]} &= U_k^-\left(\sum_{j=K}^{k-1}\mathcal{B}_j^-\right)\left(\sum_{j=K}^{L-1}\mathcal{B}_j^-\right)^{-1}\left(X_L^-\right)^{-1}x_L^{[h]} + \left(X_k^-\right)^{T-1}\left(\sum_{j=K}^{L-1}\mathcal{B}_j^-\right)^{-1}\left(X_L^-\right)^{-1}x_L^{[h]},\\
x_k^{[g]} &= X_k^+\left(\sum_{j=k}^{N-1}\mathcal{B}_j^+\right)\left(\sum_{j=M}^{N-1}\mathcal{B}_j^+\right)^{-1}\left(X_M^+\right)^{-1}x_M^{[h]},\\
u_k^{[g]} &= U_k^+\left(\sum_{j=k}^{N-1}\mathcal{B}_j^+\right)\left(\sum_{j=M}^{N-1}\mathcal{B}_j^+\right)^{-1}\left(X_M^+\right)^{-1}x_M^{[h]} - \left(X_k^+\right)^{T-1}\left(\sum_{j=M}^{N-1}\mathcal{B}_j^+\right)^{-1}\left(X_M^+\right)^{-1}x_M^{[h]},
\end{aligned} \tag{4.6}$$

where

$$\begin{aligned}
\mathcal{B}_k^- &= \left(X_{k+1}^-\right)^{-1}(I-A)^{-1}B_k\left(X_k^-\right)^{T-1},\\
\mathcal{B}_k^+ &= \left(X_{k+1}^+\right)^{-1}(I-A)^{-1}B_k\left(X_k^+\right)^{T-1},
\end{aligned} \tag{4.7}$$

and $(x^{[h]}, u^{[h]})$ is the solution of (4.4) generated by h. By a direct substitution, and using the convention that $\sum_k^{k-1} = 0$, we obtain

$$x_K^{[f]} = 0, \qquad x_L^{[f]} = x_L^{[h]}, \qquad x_M^{[g]} = x_M^{[h]}, \qquad x_N^{[g]} = 0. \tag{4.8}$$

Now, from (4.1), together with the assumption $q \not\equiv 0$, we have that there exist $\tilde{k} \in \mathbb{Z}$ and $\varepsilon > 0$ such that $q_{\tilde{k}} \le -\varepsilon$. Because the numbers K, L, M, and N have been "almost free" so far, we may choose them such that $L < \tilde{k} < M - n - 1$.

Let us introduce the test sequence

$$y_k := \begin{cases} 0, & k \in (-\infty, K-1], \\ f_k, & k \in [K, L-1], \\ h_k(1+D_k), & k \in [L, M-1], \\ g_k, & k \in [M, N-1], \\ 0, & k \in [N, \infty), \end{cases} \tag{4.9}$$

where

$$D_k = \begin{cases} \delta > 0, & k = \tilde{k} + n, \\ 0, & \text{otherwise}. \end{cases} \tag{4.10}$$

To finish the first part of the proof, we use (4.4) to estimate the contribution of the term

$$\widetilde{F}(y) = \sum_{k=-\infty}^{\infty} r_k (\Delta^n y_k)^2 = \sum_{k=-\infty}^{\infty} u_k^{[y]T} B_k u_k^{[y]} = \sum_{k=K}^{N-1} u_k^{[y]T} B_k u_k^{[y]}. \tag{4.11}$$

Using the definition of the test sequence y, we can split $\widetilde{F}$ into three terms. Now, we estimate two of them as follows. Using (4.4), we obtain

$$\begin{aligned} \sum_{k=K}^{L-1} u_k^{[f]T} B_k u_k^{[f]} &= \sum_{k=K}^{L-1} \left[u_k^{[f]T} \left(\Delta x_k^{[f]} - A x_{k+1}^{[f]} \right) \right] = \sum_{k=K}^{L-1} \left[u_k^{[f]T} \Delta x_k^{[f]} - u_k^{[f]T} A x_{k+1}^{[f]} \right] \\ &= \sum_{k=K}^{L-1} \left[\Delta \left(u_k^{[f]T} x_k^{[f]} \right) - \Delta u_k^{[f]T} x_{k+1}^{[f]} - u_k^{[f]T} A x_{k+1}^{[f]} \right] \\ &= \sum_{k=K}^{L-1} \left[\Delta \left(u_k^{[f]T} x_k^{[f]} \right) - x_{k+1}^{[f]T} \left(\Delta u_k^{[f]} + A^T u_k^{[f]} \right) \right] = u_k^{[f]T} x_k^{[f]} \Big|_K^L = x_L^{[f]T} u_L^{[f]} \\ &= x_L^{[h]T} \left[U_L^- (X_L^-)^{-1} x_L^{[h]} + (X_L^-)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} (X_L^-)^{-1} x_L^{[h]} \right] \\ &= x_L^{[h]T} (X_L^-)^{T-1} \left(\sum_{j=K}^{L-1} \mathcal{B}_j^- \right)^{-1} (X_L^-)^{-1} x_L^{[h]} =: \mathcal{G}, \end{aligned} \tag{4.12}$$

where we used the fact that $x_L^{[h]T} U_L^- (X_L^-)^{-1} x_L^{[h]} \equiv 0$ (recall that the last $n-m-1$ entries of $x_L^{[h]}$ are zeros and that the first $m+1$ columns of X^- and U^- are generated by the solutions $1, \dots, k^{(m)}$). Similarly,

$$\sum_{k=M}^{N-1} u_k^{[g]T} B_k u_k^{[g]} = -x_M^{[g]T} u_M^{[g]} = x_M^{[h]T} \left(X_M^+\right)^{T-1} \left(\sum_{j=M}^{N-1} \mathcal{B}_j^+\right)^{-1} \left(X_M^+\right)^{-1} x_M^{[h]} =: \mathcal{H}. \tag{4.13}$$

Using property (2.10) of recessive solutions of the linear Hamiltonian difference systems, we can see that $\mathcal{G} \to 0$ as $K \to -\infty$ and $\mathcal{H} \to 0$ as $N \to \infty$. We postpone the estimation of the middle term of $\widetilde{F}$ to the end of the proof.

To estimate the second term of $F(y)$, we estimate at first its terms

$$\sum_{k=K}^{L-1} q_k f_{k+n}^2, \qquad \sum_{k=M}^{N-1} q_k g_{k+n}^2. \tag{4.14}$$

For this estimation, we use Lemma 2.3. To do this, we have to show the monotonicity of the sequences

$$\begin{gathered} \frac{f_k}{h_k} \quad \text{for } k \in [K+n-1, L+n-1], \\ \frac{g_k}{h_k} \quad \text{for } k \in [M+n-1, N+n-1]. \end{gathered} \tag{4.15}$$

Let $x^{[1]}, \dots, x^{[2n]}$ be the ordered system of solutions of (2.13) in the sense of Lemma 2.1. Then, again by Lemma 2.1, there exist real numbers $d_1, \dots, d_n$ such that $h = d_1 x^{[1]} + \cdots + d_n x^{[n]}$. Because $h \not\equiv 0$, at least one coefficient d_i is nonzero. Therefore, we can denote $p := \max\{i \in [1,n] : d_i \neq 0\}$, and we replace the solution $x^{[p]}$ by h. Let us denote this new system again $x^{[1]}, \dots, x^{[2n]}$ and note that this new system has the same properties as the original one.

Following Lemma 2.2, we transform (2.13) via the transformation $y_k = h_k z_k$, into

$$\sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(R_k^{[\nu]} \Delta^{\nu} z_{k+n-\nu}\right) = 0, \tag{4.16}$$

that is,

$$(-\Delta)^n \left(r_k h_k h_{k+n} \Delta^{n-1} w_k\right) + \cdots - \Delta\left(R_k^{[1]} w_{k+n-1}\right) = 0 \tag{4.17}$$

possesses the fundamental system of solutions

$$w^{[1]} = -\Delta\left(\frac{x^{[1]}}{h}\right), \dots, w^{[p-1]} = -\Delta\left(\frac{x^{[p-1]}}{h}\right),$$
$$w^{[p]} = \Delta\left(\frac{x^{[p+1]}}{h}\right), \dots, w^{[2n-1]} = \Delta\left(\frac{x^{[2n]}}{h}\right). \tag{4.18}$$

Now, let us compute the Casoratians

$$\begin{aligned} C\left(w^{[1]}\right) &= w^{[1]} = -\Delta\left(\frac{x^{[1]}}{h}\right) = \frac{C(x^{[1]}, h)}{h_k h_{k+1}} > 0, \\ C\left(w^{[1]}, w^{[2]}\right) &= \frac{C(x^{[1]}, x^{[2]}, h)}{h_k h_{k+1} h_{k+2}} > 0, \\ &\vdots \\ C\left(w^{[1]}, \dots, w^{[2n-1]}\right) &= \frac{C(x^{[1]}, \dots, x^{[p-1]}, x^{[p+1]}, \dots, x^{[2n]}, h)}{h_k \cdots h_{k+2n-1}} > 0. \end{aligned} \tag{4.19}$$

Hence, $w^{[1]}, \dots, w^{[2n-1]}$ form the D-Markov system of sequences on $[M, \infty)$, for M sufficiently large. Therefore, by Lemma 2.4, (4.17) is eventually disconjugate; that is, it has at most $2n-2$ generalized zeros (counting multiplicity) on $[M, \infty)$. The sequence $\Delta(g/h)$ is a solution of (4.17), and we have that this sequence has generalized zeros of multiplicity $n-1$ both at M and at N; that is,

$$\Delta\left(\frac{g_{M+i}}{h_{M+i}}\right) = 0 = \Delta\left(\frac{g_{N+i}}{h_{N+i}}\right), \quad i = 0, \dots, n-2. \tag{4.20}$$

Moreover, $g_M/h_M = 1$ and $g_N/h_N = 0$. Hence, $\Delta(g_k/h_k) \le 0, k \in [M, N+n-1]$. We can proceed similarly for the sequence f/h.

Using Lemma 2.3, we have that there exist integers $\xi_1 \in [K, L-1]$ and $\xi_2 \in [M, N-1]$ such that

$$\begin{aligned} \sum_{k=K}^{L-1} q_k f_{k+n}^2 &= \sum_{k=K}^{L-1}\left[q_k h_{k+n}^2 \left(\frac{f_{k+n}}{h_{k+n}}\right)^2\right] \le \sum_{k=\xi_1}^{L-1} q_k h_{k+n}^2, \\ \sum_{k=M}^{N-1} q_k g_{k+n}^2 &= \sum_{k=M}^{N-1}\left[q_k h_{k+n}^2 \left(\frac{g_{k+n}}{h_{k+n}}\right)^2\right] \le \sum_{k=M}^{\xi_2-1} q_k h_{k+n}^2. \end{aligned} \tag{4.21}$$

Finally, we estimate the remaining term of $F(y)$. By (4.9), we have

$$\begin{aligned}
&\sum_{k=L}^{M-1}\left[r_k(\Delta^n y_k)^2+q_k y_{k+n}^2\right]\\
&\quad=\sum_{k=L}^{M-1}\left\{r_k[\Delta^n h_k+\Delta^n(h_kD_k)]^2+q_k(h_{k+n}+h_{k+n}D_{k+n})^2\right\}\\
&\quad=\sum_{k=L}^{M-1}\left\{r_k[\Delta^n(h_kD_k)]^2+q_kh_{k+n}^2+2q_kh_{k+n}^2D_{k+n}+q_kh_{k+n}^2D_{k+n}^2\right\}\\
&\quad=\sum_{k=\tilde{k}}^{\tilde{k}+n}\left\{r_k[\Delta^n(h_kD_k)]^2\right\}+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]+2q_{\tilde{k}}h_{\tilde{k}+n}^2D_{\tilde{k}+n}+q_{\tilde{k}}h_{\tilde{k}+n}^2D_{\tilde{k}+n}^2\\
&\quad=\sum_{k=\tilde{k}}^{\tilde{k}+n}\left\{r_k\left[(-1)^{k-\tilde{k}}\binom{n}{k-\tilde{k}}h_{\tilde{k}+n}\delta\right]^2\right\}+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]+2\delta q_{\tilde{k}}h_{\tilde{k}+n}^2+\delta^2q_{\tilde{k}}h_{\tilde{k}+n}^2\\
&\quad\le\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2-\delta^2\varepsilon h_{\tilde{k}+n}^2\\
&\quad<\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2.
\end{aligned}\tag{4.22}$$

Altogether, we have

$$\begin{aligned}
F(y)&<\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]+\sum_{k=L}^{M-1}\left[q_kh_{k+n}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2+\mathcal{G}+\mathcal{H}+\sum_{k=\xi_1}^{L-1}q_kh_{k+n}^2+\sum_{k=M}^{\xi_2-1}q_kh_{k+n}^2\\
&=\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2+\mathcal{G}+\mathcal{H}+\sum_{k=\xi_1}^{\xi_2-1}q_kh_{k+n}^2,
\end{aligned}\tag{4.23}$$

where for K sufficiently small is $\mathcal{G}<\delta^2/3$, for N sufficiently large is $\mathcal{H}<\delta^2/3$, and, from (4.1), $\sum_{k=\xi_1}^{\xi_2-1}q_kh_{k+n}^2<\delta^2/3$ for $\xi_1<L$ and $\xi_2>M$. Therefore,

$$\begin{aligned}
F(y)&<\delta^2+\delta^2h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]-2\delta\varepsilon h_{\tilde{k}+n}^2\\
&=\delta\left\{\delta\left[1+h_{\tilde{k}+n}^2\sum_{k=\tilde{k}}^{\tilde{k}+n}\left[r_k\binom{n}{k-\tilde{k}}^2\right]\right]-\varepsilon h_{\tilde{k}+n}^2\right\},
\end{aligned}\tag{4.24}$$

which means that $F(y)<0$ for δ sufficiently small, and (4.2) is conjugate on $\mathbb{Z}$. □

5. Equation with the Middle Terms

Under the additional condition $q_k \leq 0$ for large $|k|$, and by combining of the proof of Theorem 4.1 with the proof of [2, Lemma 1], we can establish the following criterion for the full $2n$-order equation.

Theorem 5.1. *Let $n > 1$, q_k be a real-valued sequence, and let there exist an integer $m \in \{0, \ldots, n-1\}$ and real constants $c_0, \ldots, c_m$ such that* (1.1) *is at least* $(m+1)$*-critical and the sequence $h_k := c_0 + c_1 k + \cdots + c_m k^{(m)}$ satisfies*

$$\limsup_{K \downarrow -\infty, L \uparrow \infty} \sum_{k=K}^{L} q_k h_{k+n}^2 \leq 0. \tag{5.1}$$

If $q_k \leq 0$ for large $|k|$ and $q \not\equiv 0$, then

$$L(y)_k + q_k y_{k+n} = \sum_{\nu=0}^{n} (-\Delta)^{\nu} \left(r_k^{[\nu]} \Delta^{\nu} y_{k+n-\nu} \right) + q_k y_{k+n} = 0 \tag{5.2}$$

is conjugate on $\mathbb{Z}$.

Remark 5.2. Using Theorem 3.4, we can see that the statement of Theorem 4.1 holds if and only if (3.7) holds. Finding a criterion similar to Theorem 3.4 for (1.1) is still an open question.

Remark 5.3. In the view of the matrix operator associated to (1.1) in the sense of [21], we can see that the perturbations in Theorem 4.1 affect the diagonal elements of the associated matrix operator. A description of behavior of (1.1), with regard to perturbations of limited part of the associated matrix operator (but not only of the diagonal elements), is given in [2].

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Research Article

Discrete Mittag-Leffler Functions in Linear Fractional Difference Equations

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This paper investigates some initial value problems in discrete fractional calculus. We introduce a linear difference equation of fractional order along with suitable initial conditions of fractional type and prove the existence and uniqueness of the solution. Then the structure of the solutions space is discussed, and, in a particular case, an explicit form of the general solution involving discrete analogues of Mittag-Leffler functions is presented. All our observations are performed on a special time scale which unifies and generalizes ordinary difference calculus and q-difference calculus. Some of our results are new also in these particular discrete settings.

1. Introduction

The fractional calculus is a research field of mathematical analysis which may be taken for an old as well as a modern topic. It is an old topic because of its long history starting from some notes and ideas of G. W. Leibniz and L. Euler. On the other hand, it is a modern topic due to its enormous development during the last two decades. The present interest of many scientists and engineers in the theory of fractional calculus has been initiated by applications of this theory as well as by new mathematical challenges.

The theory of discrete fractional calculus belongs among these challenges. Foundations of this theory were formulated in pioneering works by Agarwal [1] and Diaz and Osler [2], where basic approaches, definitions, and properties of the theory of fractional sums and differences were reported (see also [3, 4]). The cited papers discussed these notions on discrete sets formed by arithmetic or geometric sequences (giving rise to fractional difference calculus or q-difference calculus). Recently, a series of papers continuing this research has appeared (see, e.g., [5, 6]).

The extension of basic notions of fractional calculus to other discrete settings was performed in [7], where fractional sums and differences have been introduced and studied in the framework of (q, h)-calculus, which can be reduced to ordinary difference calculus and q-difference calculus via the choice $q = h = 1$ and $h = 0$, respectively. This extension follows recent trends in continuous and discrete analysis, characterized by a unification and generalization, and resulting into the origin and progressive development of the time scales theory (see [8, 9]). Discussing problems of fractional calculus, a question concerning the introduction of (Hilger) fractional derivative or integral on arbitrary time scale turns out to be a difficult matter. Although first attempts have been already performed (see, e.g., [10]), results obtained in this direction seem to be unsatisfactory.

The aim of this paper is to introduce some linear nabla (q, h)-fractional difference equations (i.e., equations involving difference operators of noninteger orders) and investigate their basic properties. Some particular results concerning this topic are already known, either for ordinary difference equations or q-difference equations of fractional order (some relevant references will be mentioned in Section 4). We wish to unify them and also present results which are new even also in these particular discrete settings.

The structure of the paper is the following: Section 2 presents a necessary mathematical background related to discrete fractional calculus. In particular, we are going to make some general remarks concerning fractional calculus on arbitrary time scales. In Section 3, we consider a linear nabla (q, h)-difference equation of noninteger order and discuss the question of the existence and uniqueness of the solution for the corresponding initial value problem, as well as the question of a general solution of this equation. In Section 4, we consider a particular case of the studied equation and describe the base of its solutions space by the use of eigenfunctions of the corresponding difference operator. We show that these eigenfunctions can be taken for discrete analogues of the Mittag-Leffler functions.

2. Preliminaries

The basic definitions of fractional calculus on continuous or discrete settings usually originate from the Cauchy formula for repeated integration or summation, respectively. We state here its general form valid for arbitrary time scale $\mathbb{T}$. Before doing this, we recall the notion of Taylor monomials introduced in [9]. These monomials $\widehat{h}_n : \mathbb{T}^2 \to \mathbb{R}$, $n \in \mathbb{N}_0$ are defined recursively as follows:

$$\widehat{h}_0(t, s) = 1 \quad \forall s, t \in \mathbb{T} \tag{2.1}$$

and, given $\widehat{h}_n$ for $n \in \mathbb{N}_0$, we have

$$\widehat{h}_{n+1}(t, s) = \int_s^t \widehat{h}_n(\tau, s)\nabla\tau \quad \forall s, t \in \mathbb{T} \ . \tag{2.2}$$

Now let $f : \mathbb{T} \to \mathbb{R}$ be ∇-integrable on $[a, b] \cap \mathbb{T}$, $a,\ b \in \mathbb{T}$. We put

$${}_a\nabla^{-1} f(t) = \int_a^t f(\tau)\nabla\tau \quad \forall t \in \mathbb{T},\ a \le t \le b \tag{2.3}$$

and define recursively

$$_a\nabla^{-n} f(t) = \int_a^t {_a\nabla^{-n+1}} f(\tau)\nabla\tau \tag{2.4}$$

for $n = 2, 3, \ldots$. Then we have the following.

Proposition 2.1 (Nabla Cauchy formula). *Let $n \in \mathbb{Z}^+$, a, $b \in \mathbb{T}$ and let $f : \mathbb{T} \to \mathbb{R}$ be ∇-integrable on $[a, b] \cap \mathbb{T}$. If $t \in \mathbb{T}$, $a \leq t \leq b$, then*

$$_a\nabla^{-n} f(t) = \int_a^t \widehat{h}_{n-1}(t, \rho(\tau)) f(\tau)\nabla\tau\ . \tag{2.5}$$

Proof. This assertion can be proved by induction. If $n = 1$, then (2.5) obviously holds. Let $n \geq 2$ and assume that (2.5) holds with n replaced with $n - 1$, that is,

$$_a\nabla^{-n+1} f(t) = \int_a^t \widehat{h}_{n-2}(t, \rho(\tau)) f(\tau)\nabla\tau. \tag{2.6}$$

By the definition, the left-hand side of (2.5) is an antiderivative of $_a\nabla^{-n+1} f(t)$. We show that the right-hand side of (2.5) is an antiderivative of $\int_a^t \widehat{h}_{n-2}(t, \rho(\tau)) f(\tau)\nabla\tau$. Indeed, it holds

$$\nabla \int_a^t \widehat{h}_{n-1}(t, \rho(\tau)) f(\tau)\nabla\tau = \int_a^t \nabla\widehat{h}_{n-1}(t, \rho(\tau)) f(\tau)\nabla\tau = \int_a^t \widehat{h}_{n-2}(t, \rho(\tau)) f(\tau)\nabla\tau, \tag{2.7}$$

where we have employed the property

$$\nabla \int_a^t g(t, \tau)\nabla\tau = \int_a^t \nabla g(t, \tau)\nabla\tau + g(\rho(t), t) \tag{2.8}$$

(see [9, page 139]). Consequently, the relation (2.5) holds up to a possible additive constant. Substituting $t = a$, we can find this additive constant zero. □

The formula (2.5) is a corner stone in the introduction of the nabla fractional integral $_a\nabla^{-\alpha} f(t)$ for positive reals α. However, it requires a reasonable and natural extension of a discrete system of monomials $(\widehat{h}_n, n \in \mathbb{N}_0)$ to a continuous system $(\widehat{h}_\alpha, \alpha \in \mathbb{R}^+)$. This matter is closely related to a problem of an explicit form of $\widehat{h}_n$. Of course, it holds $\widehat{h}_1(t, s) = t - s$ for all $t, s \in \mathbb{T}$. However, the calculation of $\widehat{h}_n$ for $n > 1$ is a difficult task which seems to be answerable only in some particular cases. It is well known that for $\mathbb{T} = \mathbb{R}$, it holds

$$\widehat{h}_n(t, s) = \frac{(t - s)^n}{n!}, \tag{2.9}$$

while for discrete time scales $\mathbb{T} = \mathbb{Z}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k, k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$, we have

$$\widehat{h}_n(t,s) = \frac{\prod_{j=0}^{n-1}(t-s+j)}{n!}, \qquad \widehat{h}_n(t,s) = \prod_{j=0}^{n-1} \frac{q^j t - s}{\sum_{r=0}^{j} q^r}, \tag{2.10}$$

respectively. In this connection, we recall a conventional notation used in ordinary difference calculus and q-calculus, namely,

$$(t-s)^{(n)} = \prod_{j=0}^{n-1}(t-s+j), \qquad (t-s)_{\widetilde{q}}^{(n)} = t^n \prod_{j=0}^{n-1}\left(1 - \frac{\widetilde{q}^j s}{t}\right) \quad (0 < \widetilde{q} < 1) \tag{2.11}$$

and $[j]_q = \sum_{r=0}^{j-1} q^r \, (q > 0)$, $[n]_q! = \prod_{j=1}^{n}[j]_q$. To extend the meaning of these symbols also for noninteger values (as it is required in the discrete fractional calculus), we recall some other necessary background of q-calculus. For any $x \in \mathbb{R}$ and $0 < q \neq 1$, we set $[x]_q = (q^x - 1)/(q - 1)$. By the continuity, we put $[x]_1 = x$. Further, the q-Gamma function is defined for $0 < \widetilde{q} < 1$ as

$$\Gamma_{\widetilde{q}}(x) = \frac{(\widetilde{q}, \widetilde{q})_\infty (1 - \widetilde{q})^{1-x}}{(\widetilde{q}^x, \widetilde{q})_\infty}, \tag{2.12}$$

where $(p, \widetilde{q})_\infty = \prod_{j=0}^{\infty}(1 - p\widetilde{q}^j)$, $x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Note that this function satisfies the functional relation $\Gamma_{\widetilde{q}}(x+1) = [x]_{\widetilde{q}} \Gamma_{\widetilde{q}}(x)$ and the condition $\Gamma_{\widetilde{q}}(1) = 1$. Using this, the q-binomial coefficient can be introduced as

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\widetilde{q}} = \frac{\Gamma_{\widetilde{q}}(x+1)}{\Gamma_{\widetilde{q}}(k+1)\Gamma_{\widetilde{q}}(x-k+1)}, \quad x \in \mathbb{R}, \quad k \in \mathbb{Z}. \tag{2.13}$$

Note that although the q-Gamma function is not defined at nonpositive integers, the formula

$$\frac{\Gamma_{\widetilde{q}}(x+m)}{\Gamma_{\widetilde{q}}(x)} = (-1)^m \widetilde{q}^{xm + \binom{m}{2}} \frac{\Gamma_{\widetilde{q}}(1-x)}{\Gamma_{\widetilde{q}}(1-x-m)}, \quad x \in \mathbb{R}, \ m \in \mathbb{Z}^+ \tag{2.14}$$

permits to calculate this ratio also at such the points. It is well known that if $\widetilde{q} \to 1^-$ then $\Gamma_{\widetilde{q}}(x)$ becomes the Euler Gamma function $\Gamma(x)$ (and analogously for the q-binomial coefficient). Among many interesting properties of the q-Gamma function and q-binomial coefficients, we mention q-Pascal rules

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\widetilde{q}} = \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_{\widetilde{q}} + \widetilde{q}^k \begin{bmatrix} x-1 \\ k \end{bmatrix}_{\widetilde{q}}, \quad x \in \mathbb{R}, \ k \in \mathbb{Z}, \tag{2.15}$$

$$\begin{bmatrix} x \\ k \end{bmatrix}_{\widetilde{q}} = \widetilde{q}^{x-k} \begin{bmatrix} x-1 \\ k-1 \end{bmatrix}_{\widetilde{q}} + \begin{bmatrix} x-1 \\ k \end{bmatrix}_{\widetilde{q}}, \quad x \in \mathbb{R}, \ k \in \mathbb{Z} \tag{2.16}$$

and the q-Vandermonde identity

$$\sum_{j=0}^{m}\begin{bmatrix} x \\ m-j \end{bmatrix}_{\tilde{q}}\begin{bmatrix} y \\ j \end{bmatrix}_{\tilde{q}}\tilde{q}^{j^2-mj+xj}=\begin{bmatrix} x+y \\ m \end{bmatrix}_{\tilde{q}}, \quad x,y\in\mathbb{R},\ m\in\mathbb{N}_0 \tag{2.17}$$

(see [11]) that turn out to be very useful in our further investigations.

The computation of an explicit form of $\widehat{h}_n(t,s)$ can be performed also in a more general case. We consider here the time scale

$$\mathbb{T}^{t_0}_{(q,h)}=\left\{t_0q^k+[k]_qh, k\in\mathbb{Z}\right\}\cup\left\{\frac{h}{1-q}\right\}, \quad t_0>0,\ q\ge 1,\ h\ge 0,\ q+h>1 \tag{2.18}$$

(see also [7]). Note that if $q=1$ then the cluster point $h/(1-q)=-\infty$ is not involved in $\mathbb{T}^{t_0}_{(q,h)}$. The forward and backward jump operator is the linear function $\sigma(t)=qt+h$ and $\rho(t)=q^{-1}(t-h)$, respectively. Similarly, the forward and backward graininess is given by $\mu(t)=(q-1)t+h$ and $\nu(t)=q^{-1}\mu(t)$, respectively. In particular, if $t_0=q=h=1$, then $\mathbb{T}^{t_0}_{(q,h)}$ becomes $\mathbb{Z}$, and if $t_0=1$, $q>1$, $h=0$, then $\mathbb{T}^{t_0}_{(q,h)}$ is reduced to $\overline{q^{\mathbb{Z}}}$.

Let $a\in\mathbb{T}^{t_0}_{(q,h)}$, $a>h/(1-q)$ be fixed. Then we introduce restrictions of the time scale $\mathbb{T}^{t_0}_{(q,h)}$ by the relation

$$\widetilde{\mathbb{T}}^{\sigma^i(a)}_{(q,h)}=\left\{t\in\mathbb{T}^{t_0}_{(q,h)}, t\ge\sigma^i(a)\right\}, \quad i=0,1,\ldots, \tag{2.19}$$

where the symbol σ^i stands for the ith iterate of σ (analogously, we use the symbol ρ^i). To simplify the notation, we put $\tilde{q}=1/q$ whenever considering the time scale $\mathbb{T}^{t_0}_{(q,h)}$ or $\widetilde{\mathbb{T}}^{\sigma^i(a)}_{(q,h)}$.

Using the induction principle, we can verify that Taylor monomials on $\mathbb{T}^{t_0}_{(q,h)}$ have the form

$$\widehat{h}_n(t,s)=\frac{\prod_{j=0}^{n-1}\left(\sigma^j(t)-s\right)}{[n]_q!}=\frac{\prod_{j=0}^{n-1}\left(t-\rho^j(s)\right)}{[n]_{\tilde{q}}!}. \tag{2.20}$$

Note that this result generalizes previous forms (2.10) and, moreover, enables its unified notation. In particular, if we introduce the symbolic (q,h)-power

$$(t-s)^{(n)}_{(\tilde{q},h)}=\prod_{j=0}^{n-1}\left(t-\rho^j(s)\right) \tag{2.21}$$

unifying (2.11), then the Cauchy formula (2.5) can be rewritten for $\mathbb{T}=\mathbb{T}^{t_0}_{(q,h)}$ as

$${}_a\nabla^{-n}f(t)=\int_a^t\frac{\left(t-\rho(\tau)\right)^{(n-1)}_{(\tilde{q},h)}}{[n-1]_{\tilde{q}}!}f(\tau)\nabla\tau. \tag{2.22}$$

Discussing a reasonable generalization of (q,h)-power (2.21) to real values α instead of integers n, we recall broadly accepted extensions of its particular cases (2.11) in the form

$$(t-s)^{(\alpha)}=\frac{\Gamma(t-s+\alpha)}{\Gamma(t-s)},\qquad (t-s)_{\widetilde{q}}^{(\alpha)}=t^{\alpha}\frac{(s/t,\widetilde{q})_{\infty}}{(\widetilde{q}^{\alpha}s/t,\widetilde{q})_{\infty}},\qquad t\neq 0. \tag{2.23}$$

Now, we assume $s,t\in\mathbb{T}_{(q,h)}^{t_0}$, $t\geq s>h/(1-q)$. First, consider (q,h)-power (2.21) corresponding to the time scale $\mathbb{T}_{(q,h)}^{t_0}$, where $q>1$. Then we can rewrite (2.21) as

$$(t-s)_{(\widetilde{q},h)}^{(n)}=\left(t+\frac{h\widetilde{q}}{1-\widetilde{q}}\right)^{n}\prod_{j=0}^{n-1}\left(1-\widetilde{q}^{j}\frac{s+h\widetilde{q}/(1-\widetilde{q})}{t+h\widetilde{q}/(1-\widetilde{q})}\right)=([t]-[s])_{\widetilde{q}}^{(n)}, \tag{2.24}$$

where $[t]=t+h\widetilde{q}/(1-\widetilde{q})$ and $[s]=s+h\widetilde{q}/(1-\widetilde{q})$. A required extension of (q,h)-power (2.21) is then provided by the formula

$$(t-s)_{(\widetilde{q},h)}^{(\alpha)}=([t]-[s])_{\widetilde{q}}^{(\alpha)}. \tag{2.25}$$

Now consider (q,h)-power (2.21) corresponding to the time scale $\mathbb{T}_{(q,h)}^{t_0}$, where $q=1$. Then

$$(t-s)_{(1,h)}^{(n)}=\prod_{j=0}^{n-1}(t-s+jh)=h^{n}\prod_{j=0}^{n-1}\left(\frac{t-s}{h}+j\right)=h^{n}\frac{((t-s)/h+n-1)!}{((t-s)/h-1)!} \tag{2.26}$$

and the formula (2.21) can be extended by

$$(t-s)_{(1,h)}^{(\alpha)}=\frac{h^{\alpha}\Gamma((t-s)/h+\alpha)}{\Gamma((t-s)/h)}. \tag{2.27}$$

These definitions are consistent, since it can be shown that

$$\lim_{\widetilde{q}\to 1^{-}}([t]-[s])_{\widetilde{q}}^{(\alpha)}=(t-s)_{(1,h)}^{(\alpha)}. \tag{2.28}$$

Now the required extension of the monomial $\widehat{h}_n(t,s)$ corresponding to $\mathbb{T}_{(q,h)}^{t_0}$ takes the form

$$\widehat{h}_{\alpha}(t,s)=\frac{(t-s)_{(\widetilde{q},h)}^{(\alpha)}}{\Gamma_{\widetilde{q}}(\alpha+1)}. \tag{2.29}$$

Another (equivalent) expression of $\widehat{h}_{\alpha}(t,s)$ is provided by the following assertion.

Proposition 2.2. *Let $\alpha \in \mathbb{R}$, $s,t \in \mathbb{T}_{(q,h)}^{t_0}$ and $n \in \mathbb{N}_0$ be such that $t = \sigma^n(s)$. Then*

$$\widehat{h}_\alpha(t,s) = (\nu(t))^\alpha \begin{bmatrix} \alpha + n - 1 \\ n-1 \end{bmatrix}_{\widetilde{q}} = (\nu(t))^\alpha \begin{bmatrix} -\alpha - 1 \\ n-1 \end{bmatrix}_{\widetilde{q}} (-1)^{n-1} \widetilde{q}^{\alpha(n-1)+\binom{n}{2}}. \tag{2.30}$$

Proof. Let $q > 1$. Using the relations

$$[t] = \frac{\nu(t)}{(1-\widetilde{q})}, \qquad \frac{[s]}{[t]} = \widetilde{q}^n, \tag{2.31}$$

we can derive that

$$\begin{aligned} \widehat{h}_\alpha(t,s) &= \frac{[t]^\alpha ([s]/[t],\widetilde{q})_\infty}{\Gamma_{\widetilde{q}}(\alpha+1)(\widetilde{q}^\alpha [s]/[t],\widetilde{q})_\infty} = \frac{(1-\widetilde{q})^{-\alpha}\nu(t)^\alpha (\widetilde{q}^n,\widetilde{q})_\infty}{\Gamma_{\widetilde{q}}(\alpha+1)(\widetilde{q}^{\alpha+n},\widetilde{q})_\infty} \\ &= (\nu(t))^\alpha \frac{\Gamma_{\widetilde{q}}(\alpha+n)}{\Gamma_{\widetilde{q}}(\alpha+1)\Gamma_{\widetilde{q}}(n)} = (\nu(t))^\alpha \begin{bmatrix} \alpha+n-1 \\ n-1 \end{bmatrix}_{\widetilde{q}}. \end{aligned} \tag{2.32}$$

The second equality in (2.30) follows from the identity (2.14). The case $q = 1$ results from (2.27). □

The key property of $\widehat{h}_\alpha(t,s)$ follows from its differentiation. The symbol $\nabla_{(q,h)}^m$ used in the following assertion (and also undermentioned) is the mth order nabla (q,h)-derivative on the time scale $\mathbb{T}_{(q,h)}^{t_0}$, defined for $m = 1$ as

$$\nabla_{(q,h)} f(t) = \frac{f(t) - f(\rho(t))}{\nu(t)} = \frac{f(t) - f(\widetilde{q}(t-h))}{(1-\widetilde{q})t + \widetilde{q}h} \tag{2.33}$$

and iteratively for higher orders.

Lemma 2.3. *Let $m \in \mathbb{Z}^+$, $\alpha \in \mathbb{R}$, $s,t \in \mathbb{T}_{(q,h)}^{t_0}$ and $n \in \mathbb{Z}^+$, $n \geq m$ be such that $t = \sigma^n(s)$. Then*

$$\nabla_{(q,h)}^m \widehat{h}_\alpha(t,s) = \begin{cases} \widehat{h}_{\alpha-m}(t,s), & \alpha \notin \{0,1,\ldots,m-1\}, \\ 0, & \alpha \in \{0,1,\ldots,m-1\}. \end{cases} \tag{2.34}$$

Proof. First let $m = 1$. For $\alpha = 0$ we get $\widehat{h}_0(t,s) = 1$ and the first nabla (q,h)-derivative is zero. If $\alpha \neq 0$, then by (2.30) and (2.16), we have

$$\begin{aligned}
\nabla_{(q,h)}\widehat{h}_\alpha(t,s) &= \frac{\widehat{h}_\alpha(t,s) - \widehat{h}_\alpha(\rho(t),s)}{\nu(t)} \\
&= \frac{1}{\nu(t)}\left((\nu(t))^\alpha \begin{bmatrix} \alpha+n-1 \\ n-1 \end{bmatrix}_{\widetilde{q}} - (\nu(\rho(t)))^\alpha \begin{bmatrix} \alpha+n-2 \\ n-2 \end{bmatrix}_{\widetilde{q}} \right) \\
&= (\nu(t))^{\alpha-1}\left(\begin{bmatrix} \alpha+n-1 \\ n-1 \end{bmatrix}_{\widetilde{q}} - \widetilde{q}^\alpha \begin{bmatrix} \alpha+n-2 \\ n-2 \end{bmatrix}_{\widetilde{q}} \right) = \widehat{h}_{\alpha-1}(t,s).
\end{aligned} \tag{2.35}$$

The case $m \geq 2$ can be verified by the induction principle. □

We note that an extension of this property for derivatives of noninteger orders will be performed in Section 4.

Now we can continue with the introduction of (q,h)-fractional integral and derivative of a function $f : \widetilde{\mathbb{T}}^a_{(q,h)} \to \mathbb{R}$. Let $t \in \widetilde{\mathbb{T}}^a_{(q,h)}$. Our previous considerations (in particular, the Cauchy formula (2.5) along with the relations (2.22) and (2.29)) warrant us to introduce the nabla (q,h)-fractional integral of order $\alpha \in \mathbb{R}^+$ over the time scale interval $[a,t] \cap \widetilde{\mathbb{T}}^a_{(q,h)}$ as

$$_a\nabla^{-\alpha}_{(q,h)} f(t) = \int_a^t \widehat{h}_{\alpha-1}(t,\rho(\tau)) f(\tau) \nabla\tau \tag{2.36}$$

(see also [7]). The nabla (q,h)-fractional derivative of order $\alpha \in \mathbb{R}^+$ is then defined by

$$_a\nabla^{\alpha}_{(q,h)} f(t) = \nabla^m_{(q,h)}\, {}_a\nabla^{-(m-\alpha)}_{(q,h)} f(t), \tag{2.37}$$

where $m \in \mathbb{Z}^+$ is given by $m-1 < \alpha \leq m$. For the sake of completeness, we put

$$_a\nabla^{0}_{(q,h)} f(t) = f(t). \tag{2.38}$$

As we noted earlier, a reasonable introduction of fractional integrals and fractional derivatives on arbitrary time scales remains an open problem. In the previous part, we have consistently used (and in the sequel, we shall consistently use) the time scale notation of main procedures and operations to outline a possible way out to further generalizations.

3. A Linear Initial Value Problem

In this section, we are going to discuss the linear initial value problem

$$\sum_{j=1}^m p_{m-j+1}(t)\, {}_a\nabla^{\alpha-j+1}_{(q,h)} y(t) + p_0(t) y(t) = 0\,, \qquad t \in \widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}, \tag{3.1}$$

$$
{}_a\nabla^{\alpha-j}_{(q,h)} y(t)\Big|_{t=\sigma^m(a)} = y_{\alpha-j}, \quad j = 1, 2, \dots, m, \tag{3.2}
$$

where $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ are such that $m - 1 < \alpha \leq m$. Further, we assume that $p_j(t)$ are arbitrary real-valued functions on $\widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}$ $(j = 1, \dots, m-1)$, $p_m(t) = 1$ on $\widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}$ and $y_{\alpha-j}$ $(j = 1, \dots, m)$ are arbitrary real scalars.

If α is a positive integer, then (3.1)-(3.2) becomes the standard discrete initial value problem. If α is not an integer, then applying the definition of nabla (q, h)-fractional derivatives, we can observe that (3.1) is of the general form

$$
\sum_{i=0}^{n-1} a_i(t) y\left(\rho^i(t)\right) = 0 , \quad t \in \widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}, \ n \text{ being such that } t = \sigma^n(a), \tag{3.3}
$$

which is usually referred to as the equation of Volterra type. If such an equation has two different solutions, then their values differ at least at one of the points $\sigma(a), \sigma^2(a), \dots, \sigma^m(a)$. In particular, if $a_0(t) \neq 0$ for all $t \in \widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}$, then arbitrary values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a))$ determine uniquely the solution $y(t)$ for all $t \in \widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}$. We show that the values $y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-m}$, introduced by (3.2), keep the same properties.

Proposition 3.1. *Let $y : \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)} \to \mathbb{R}$ be a function. Then* (3.2) *represents a one-to-one mapping between the vectors $(y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a)))$ and $(y_{\alpha-1}, y_{\alpha-2}, \dots, y_{\alpha-m})$.*

Proof. The case $\alpha \in \mathbb{Z}^+$ is well known from the literature. Let $\alpha \notin \mathbb{Z}^+$. We wish to show that the values of $y(\sigma(a)), y(\sigma^2(a)), \dots, y(\sigma^m(a))$ determine uniquely the values of

$$
{}_a\nabla^{\alpha-1}_{(q,h)} y(t)\Big|_{t=\sigma^m(a)}, \quad {}_a\nabla^{\alpha-2}_{(q,h)} y(t)\Big|_{t=\sigma^m(a)}, \dots, \ {}_a\nabla^{\alpha-m}_{(q,h)} y(t)\Big|_{t=\sigma^m(a)} \tag{3.4}
$$

and vice versa. Utilizing the relation

$$
{}_a\nabla^{\alpha-j}_{(q,h)} y(t)\Big|_{t=\sigma^m(a)} = \sum_{k=1}^{m} \nu\left(\sigma^{m-k+1}(a)\right) \widehat{h}_{j-1-\alpha}\left(\sigma^m(a), \sigma^{m-k}(a)\right) y\left(\sigma^{m-k+1}(a)\right) \tag{3.5}
$$

(see [7, Propositions 1 and 3] with respect to (2.30)), we can rewrite (3.2) as the linear mapping

$$
\sum_{k=1}^{m} r_{jk} y\left(\sigma^{m-k+1}(a)\right) = y_{\alpha-j}, \quad j = 1, \dots, m, \tag{3.6}
$$

where

$$
r_{jk} = \nu\left(\sigma^{m-k+1}(a)\right) \widehat{h}_{j-1-\alpha}\left(\sigma^m(a), \sigma^{m-k}(a)\right), \quad j, k = 1, \dots, m \tag{3.7}
$$

are elements of the transformation matrix R_m. We show that R_m is regular. Obviously,

$$\det R_m = \left(\prod_{k=1}^{m} \nu\left(\sigma^k(a)\right)\right) \det H_m, \tag{3.8}$$

where

$$H_m = \begin{pmatrix} \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{-\alpha}(\sigma^m(a), a) \\ \widehat{h}_{1-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \widehat{h}_{1-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{1-\alpha}(\sigma^m(a), a) \\ \vdots & \vdots & \ddots & \vdots \\ \widehat{h}_{m-1-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \widehat{h}_{m-1-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{m-1-\alpha}(\sigma^m(a), a) \end{pmatrix}. \tag{3.9}$$

To calculate $\det H_m$, we employ some elementary operations preserving the value of $\det H_m$. Using the properties

$$\begin{gathered} \widehat{h}_{i-\alpha}\left(\sigma^m(a), \sigma^{\ell}(a)\right) - \nu(\sigma^m(a))\widehat{h}_{i-\alpha-1}\left(\sigma^m(a), \sigma^{\ell}(a)\right) = \widehat{h}_{i-\alpha}\left(\sigma^{m-1}(a), \sigma^{\ell}(a)\right) \\ (i = 1, 2, \ldots, m-1,\ l = 0, 1, \ldots m-2), \\ \widehat{h}_{i-\alpha}\left(\sigma^m(a), \sigma^{m-1}(a)\right) - \nu(\sigma^m(a))\widehat{h}_{i-\alpha-1}\left(\sigma^m(a), \sigma^{m-1}(a)\right) = 0, \end{gathered} \tag{3.10}$$

which follow from Lemma 2.3, we multiply the ith row $(i = 1, 2, \ldots, m-1)$ of H_m by $-\nu(\sigma^m(a))$ and add it to the successive one. We arrive at the form

$$\left(\begin{array}{c|ccc} \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{-\alpha}(\sigma^m(a), a) \\ \hline 0 & & & \\ \vdots & & H_{m-1} & \\ 0 & & & \end{array}\right). \tag{3.11}$$

Then we apply repeatedly this procedure to obtain the triangular matrix

$$\begin{pmatrix} \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-1}(a)) & \widehat{h}_{-\alpha}(\sigma^m(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{-\alpha}(\sigma^m(a), a) \\ 0 & \widehat{h}_{1-\alpha}(\sigma^{m-1}(a), \sigma^{m-2}(a)) & \cdots & \widehat{h}_{1-\alpha}(\sigma^{m-1}(a), a) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \widehat{h}_{m-1-\alpha}(\sigma(a), a) \end{pmatrix}. \tag{3.12}$$

Since $\widehat{h}_{i-\alpha}(\sigma^k(a), \sigma^{k-1}(a)) = (\nu(\sigma^k(a))^{i-\alpha}(i = 0, 1, \ldots, m-1)$, we get

$$\det H_m = \prod_{k=1}^{m} \left(\nu\left(\sigma^k(a)\right)\right)^{m-k-\alpha}, \text{ that is,} \quad \det R_m = \prod_{k=1}^{m} \left(\nu\left(\sigma^k(a)\right)\right)^{m-k-\alpha+1} \neq 0. \tag{3.13}$$

Thus the matrix R_m is regular, hence the corresponding mapping (3.6) is one to one. □

Now we approach a problem of the existence and uniqueness of (3.1)-(3.2). First we recall the general notion of ν-regressivity of a matrix function and a corresponding linear nabla dynamic system (see [9]).

Definition 3.2. An $n \times n$-matrix-valued function $A(t)$ on a time scale $\mathbb{T}$ is called ν-regressive provided

$$\det(I - \nu(t)A(t)) \neq 0 \quad \forall t \in \mathbb{T}_\kappa, \tag{3.14}$$

where I is the identity matrix. Further, we say that the linear dynamic system

$$\nabla z(t) = A(t)z(t) \tag{3.15}$$

is ν-regressive provided that $A(t)$ is ν-regressive.

Considering a higher order linear difference equation, the notion of ν-regressivity for such an equation can be introduced by means of its transformation to the corresponding first order linear dynamic system. We are going to follow this approach and generalize the notion of ν-regressivity for the linear fractional difference equation (3.1).

Definition 3.3. Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m - 1 < \alpha \leq m$. Then (3.1) is called ν-regressive provided the matrix

$$A(t) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -\dfrac{p_0(t)}{\nu^{m-\alpha}(t)} & -p_1(t) & \cdots & -p_{m-2}(t) & -p_{m-1}(t) \end{pmatrix} \tag{3.16}$$

is ν-regressive.

Remark 3.4. The explicit expression of the ν-regressivity property for (3.1) can be read as

$$1 + \sum_{j=1}^{m-1} p_{m-j}(t)(\nu(t))^j + p_0(t)(\nu(t))^\alpha \neq 0 \quad \forall t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}. \tag{3.17}$$

If α is a positive integer, then both these introductions agree with the definition of ν-regressivity of a higher order linear difference equation presented in [9].

Theorem 3.5. *Let* (3.1) *be* ν*-regressive. Then the problem* (3.1)-(3.2) *has a unique solution defined for all* $t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$.

Proof. The conditions (3.2) enable us to determine the values of $y(\sigma(a)), y(\sigma^2(a)), \ldots, y(\sigma^m(a))$ by the use of (3.6). To calculate the values of $y(\sigma^{m+1}(a)), y(\sigma^{m+2}(a)), \ldots$, we perform the transformation

$$z_j(t) = {}_a\nabla_{(q,h)}^{\alpha-m+j-1} y(t), \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^j(a)}, \ j = 1, 2, \ldots, m \tag{3.18}$$

which allows us to rewrite (3.1) into a matrix form. Before doing this, we need to express $y(t)$ in terms of $z_1(t),\ z_1(\rho(t)), \ldots, z_1(\sigma(a))$. Applying the relation ${}_a\nabla_{(q,h)}^{m-\alpha}\, {}_a\nabla_{(q,h)}^{-(m-\alpha)} y(t) = y(t)$ (see [7]) and expanding the fractional derivative, we arrive at

$$y(t) = {}_a\nabla_{(q,h)}^{m-\alpha} z_1(t) = \frac{z_1(t)}{\nu^{m-\alpha}(t)} + \int_a^{\rho(t)} \widehat{h}_{\alpha-m-1}(t, \rho(\tau)) z_1(\tau) \nabla\tau. \tag{3.19}$$

Therefore, the problem (3.1)-(3.2) can be rewritten to the vector form

$$\begin{gathered} {}_a\nabla_{(q,h)} z(t) = A(t) z(t) + b(t), \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \\ z(\sigma^m(a)) = \left(y_{\alpha-m}, \ldots, y_{\alpha-1}\right)^T, \end{gathered} \tag{3.20}$$

where

$$z(t) = (z_1(t), \ldots, z_m(t))^T, \qquad b(t) = \left(0, \ldots, 0, -p_0(t) \int_a^{\rho(t)} \widehat{h}_{\alpha-m-1}(t, \rho(\tau)) z_1(\tau) \nabla\tau\right)^T \tag{3.21}$$

and $A(t)$ is given by (3.16). The ν-regressivity of the matrix $A(t)$ enables us to write

$$z(t) = (I - \nu(t) A(t))^{-1} \left(z(\rho(t)) + \nu(t) b(t)\right), \quad t \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}, \tag{3.22}$$

hence, using the value of $z(\sigma^m(a))$, we can solve this system by the step method starting from $t = \sigma^{m+1}(a)$. The solution $y(t)$ of the original initial value problem (3.1)-(3.2) is then given by the formula (3.19). □

Remark 3.6. The previous assertion on the existence and uniqueness of the solution can be easily extended to the initial value problem involving nonhomogeneous linear equations as well as some nonlinear equations.

The final goal of this section is to investigate the structure of the solutions of (3.1). We start with the following notion.

Definition 3.7. Let $\gamma \in \mathbb{R}$, $0 \le \gamma < 1$. For m functions $y_j : \widetilde{\mathbb{T}}^{a}_{(q,h)} \to \mathbb{R} (j = 1, 2, \ldots, m)$, we define the γ-Wronskian $W_\gamma(y_1, \ldots, y_m)(t)$ as determinant of the matrix

$$V_\gamma(y_1, \ldots, y_m)(t) = \begin{pmatrix} {}_a\nabla^{-\gamma}_{(q,h)} y_1(t) & {}_a\nabla^{-\gamma}_{(q,h)} y_2(t) & \cdots & {}_a\nabla^{-\gamma}_{(q,h)} y_m(t) \\ {}_a\nabla^{1-\gamma}_{(q,h)} y_1(t) & {}_a\nabla^{1-\gamma}_{(q,h)} y_2(t) & \cdots & {}_a\nabla^{1-\gamma}_{(q,h)} y_m(t) \\ \vdots & \vdots & \ddots & \vdots \\ {}_a\nabla^{m-1-\gamma}_{(q,h)} y_1(t) & {}_a\nabla^{m-1-\gamma}_{(q,h)} y_2(t) & \cdots & {}_a\nabla^{m-1-\gamma}_{(q,h)} y_m(t) \end{pmatrix}, \quad t \in \widetilde{\mathbb{T}}^{\sigma^m(a)}_{(q,h)}. \tag{3.23}$$

Remark 3.8. Note that the first row of this matrix involves fractional order integrals. It is a consequence of the form of initial conditions utilized in our investigations. Of course, this introduction of $W_\gamma(y_1, \ldots, y_m)(t)$ coincides for $\gamma = 0$ with the classical definition of the Wronskian (see [8]). Moreover, it holds $W_\gamma(y_1, \ldots, y_m)(t) = W_0({}_a\nabla^{-\gamma}_{(q,h)} y_1, \ldots, {}_a\nabla^{-\gamma}_{(q,h)} y_m)(t)$.

Theorem 3.9. *Let functions $y_1(t), \ldots, y_m(t)$ be solutions of the ν-regressive equation* (3.1) *and let $W_{m-\alpha}(y_1, \ldots, y_m)(\sigma^m(a)) \neq 0$. Then any solution $y(t)$ of* (3.1) *can be written in the form*

$$y(t) = \sum_{k=1}^{m} c_k y_k(t), \quad t \in \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \tag{3.24}$$

where $c_1, \ldots, c_m$ are real constants.

Proof. Let $y(t)$ be a solution of (3.1). By Proposition 3.1, there exist real scalars $y_{\alpha-1}, \ldots, y_{\alpha-m}$ such that $y(t)$ is satisfying (3.2). Now we consider the function $u(t) = \sum_{k=1}^{m} c_k y_k(t)$, where the m-tuple $(c_1, \ldots, c_m)$ is the unique solution of

$$V_{m-\alpha}(y_1, \ldots, y_m)(\sigma^m(a)) \cdot \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} = \begin{pmatrix} y_{\alpha-m} \\ y_{\alpha-m+1} \\ \vdots \\ y_{\alpha-1} \end{pmatrix}. \tag{3.25}$$

The linearity of (3.1) implies that $u(t)$ has to be its solution. Moreover, it holds

$${}_a\nabla^{\alpha-j}_{(q,h)} u(t) \Big|_{t=\sigma^m(a)} = y_{\alpha-j}, \quad j = 1, 2, \ldots, m, \tag{3.26}$$

hence $u(t)$ is a solution of the initial value problem (3.1)-(3.2). By Theorem 3.5, it must be $y(t) = u(t)$ for all $t \in \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ and (3.24) holds. □

Remark 3.10. The formula (3.24) is essentially an expression of the general solution of (3.1).

4. Two-Term Equation and (q,h)-Mittag-Leffler Function

Our main interest in this section is to find eigenfunctions of the fractional operator ${}_a\nabla^{\alpha}_{(q,h)}$, $\alpha \in \mathbb{R}^+$. In other words, we wish to solve (3.1) in a special form

$$ {}_a\nabla^{\alpha}_{(q,h)} y(t) = \lambda y(t), \quad \lambda \in \mathbb{R}, \quad t \in \widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}. \tag{4.1} $$

Throughout this section, we assume that ν-regressivity condition for (4.1) is ensured, that is,

$$ \lambda(\nu(t))^{\alpha} \neq 1. \tag{4.2} $$

Discussions on methods of solving fractional difference equations are just at the beginning. Some techniques how to explicitly solve these equations (at least in particular cases) are exhibited, for example, in [12–14], where a discrete analogue of the Laplace transform turns out to be the most developed method. In this section, we describe the technique not utilizing the transform method, but directly originating from the role which is played by the Mittag-Leffler function in the continuous fractional calculus (see, e.g., [15]). In particular, we introduce the notion of a discrete Mittag-Leffler function in a setting formed by the time scale $\widetilde{\mathbb{T}}^{a}_{(q,h)}$ and demonstrate its significance with respect to eigenfunctions of the operator ${}_a\nabla^{\alpha}_{(q,h)}$. These results generalize and extend those derived in [16, 17].

We start with the power rule stated in Lemma 2.3 and perform its extension to fractional integrals and derivatives.

Proposition 4.1. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and $t \in \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Then it holds*

$$ {}_a\nabla^{-\alpha}_{(q,h)} \widehat{h}_{\beta}(t,a) = \widehat{h}_{\alpha+\beta}(t,a). \tag{4.3} $$

Proof. Let $t \in \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ be such that $t = \sigma^n(a)$ for some $n \in \mathbb{Z}^+$. We have

$$ \begin{aligned} {}_a\nabla^{-\alpha}_{(q,h)} \widehat{h}_{\beta}(t,a) &= \sum_{k=0}^{n-1} \widehat{h}_{\alpha-1}\left(t, \rho^{k+1}(t)\right) \nu\left(\rho^k(t)\right) \widehat{h}_{\beta}\left(\rho^k(t), a\right) \\ &= \sum_{k=0}^{n-1} (\nu(t))^{\alpha-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_{\widetilde{q}} (-1)^k \widetilde{q}^{(\alpha-1)k+\binom{k+1}{2}} \widetilde{q}^k \nu(t) \\ &\quad \times \left(\nu\left(\rho^k(t)\right)\right)^{\beta} \begin{bmatrix} -\beta-1 \\ n-k-1 \end{bmatrix}_{\widetilde{q}} (-1)^{n-k-1} \widetilde{q}^{\beta(n-k-1)+\binom{n-k}{2}} \\ &= (\nu(t))^{\alpha+\beta} \sum_{k=0}^{n-1} \begin{bmatrix} -\alpha \\ k \end{bmatrix}_{\widetilde{q}} \begin{bmatrix} -\beta-1 \\ n-k-1 \end{bmatrix}_{\widetilde{q}} (-1)^{n-1} \widetilde{q}^{k^2-k(n-1)+k\alpha+\binom{n}{2}+\beta(n-1)} \end{aligned} $$

$$
\begin{aligned}
&= (\nu(t))^{\alpha+\beta}\sum_{k=0}^{n-1}\begin{bmatrix}-\alpha\\ n-k-1\end{bmatrix}_{\widetilde{q}}\begin{bmatrix}-\beta-1\\ k\end{bmatrix}_{\widetilde{q}}\\
&\qquad\times(-1)^{n-1}\widetilde{q}^{(n-k-1)^2-(n-k-1)(n-1)+(n-k-1)\alpha+\binom{n}{2}+\beta(n-1)}\\
&= (\nu(t))^{\alpha+\beta}\sum_{k=0}^{n-1}\begin{bmatrix}-\alpha\\ n-k-1\end{bmatrix}_{\widetilde{q}}\begin{bmatrix}-\beta-1\\ k\end{bmatrix}_{\widetilde{q}}(-1)^{n-1}\widetilde{q}^{k^2-k(n-1)-k\alpha+(\alpha+\beta)(n-1)+\binom{n}{2}}\\
&= (\nu(t))^{\alpha+\beta}\begin{bmatrix}-\alpha-\beta-1\\ n-1\end{bmatrix}_{\widetilde{q}}(-1)^{n-1}\widetilde{q}^{(\alpha+\beta)(n-1)+\binom{n}{2}} = \widehat{h}_{\alpha+\beta}(t,a),
\end{aligned}
\tag{4.4}
$$

where we have used (2.30) on the second line and (2.17) on the last line. □

Corollary 4.2. *Let $\alpha\in\mathbb{R}^+$, $\beta\in\mathbb{R}$, $t\in\widetilde{\mathbb{T}}_{(q,h)}^{\sigma^{m+1}(a)}$, where $m\in\mathbb{Z}^+$ is satisfying $m-1<\alpha\le m$. Then*

$$
{}_a\nabla^{\alpha}_{(q,h)}\widehat{h}_{\beta}(t,a)=\begin{cases}\widehat{h}_{\beta-\alpha}(t,a), & \beta-\alpha\notin\{-1,\ldots,-m\},\\ 0, & \beta-\alpha\in\{-1,\ldots,-m\}.\end{cases}
\tag{4.5}
$$

Proof. Proposition 4.1 implies that

$$
{}_a\nabla^{\alpha}_{(q,h)}\widehat{h}_{\beta}(t,a)=\nabla^{m}_{(q,h)}\left({}_a\nabla^{-(m-\alpha)}_{(q,h)}\widehat{h}_{\beta}(t,a)\right)=\nabla^{m}_{(q,h)}\widehat{h}_{m+\beta-\alpha}(t,a).
\tag{4.6}
$$

Then the statement is an immediate consequence of Lemma 2.3. □

Now we are in a position to introduce a (q,h)-discrete analogue of the Mittag-Leffler function. We recall that this function is essentially a generalized exponential function, and its two-parameter form (more convenient in the fractional calculus) can be introduced for $\mathbb{T}=\mathbb{R}$ by the series expansion

$$
E_{\alpha,\beta}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(\alpha k+\beta)},\quad \alpha,\ \beta\in\mathbb{R}^+,\ t\in\mathbb{R}.
\tag{4.7}
$$

The fractional calculus frequently employs (4.7), because the function

$$
t^{\beta-1}E_{\alpha,\beta}(\lambda t^{\alpha})=\sum_{k=0}^{\infty}\lambda^k\frac{t^{\alpha k+\beta-1}}{\Gamma(\alpha k+\beta)}
\tag{4.8}
$$

(a modified Mittag-Leffler function, see [15]) satisfies under special choices of β a continuous (differential) analogy of (4.1). Some extensions of the definition formula (4.7) and their utilization in special fractional calculus operators can be found in [18, 19].

Considering the discrete calculus, the form (4.8) seems to be much more convenient for discrete extensions than the form (4.7), which requires, among others, the validity of the law of exponents. The following introduction extends the discrete Mittag-Leffler function defined and studied in [20] for the case $q = h = 1$.

Definition 4.3. Let $\alpha,\ \beta,\ \lambda \in \mathbb{R}$. We introduce the (q,h)-Mittag-Leffler function $E^{s,\lambda}_{\alpha,\beta}(t)$ by the series expansion

$$E^{s,\lambda}_{\alpha,\beta}(t) = \sum_{k=0}^{\infty}\lambda^k \widehat{h}_{\alpha k+\beta-1}(t,s)\left(= \sum_{k=0}^{\infty}\lambda^k \frac{(t-s)^{(\alpha k+\beta-1)}_{(\widetilde{q},h)}}{\Gamma_{\widetilde{q}}(\alpha k+\beta)}\right), \quad s,t\in \widetilde{\mathbb{T}}^{a}_{(q,h)},\ t\geq s\ . \tag{4.9}$$

It is easy to check that the series on the right-hand side converges (absolutely) if $|\lambda|(\nu(t))^{\alpha} < 1$. As it might be expected, the particular (q,h)-Mittag-Leffler function

$$E^{a,\lambda}_{1,1}(t) = \prod_{k=0}^{n-1}\frac{1}{1-\lambda\nu(\rho^k(t))}, \tag{4.10}$$

where $n\in\mathbb{Z}^+$ satisfies $t=\sigma^n(a)$, is a solution of the equation

$$\nabla_{(q,h)}y(t) = \lambda y(t), \qquad t\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}, \tag{4.11}$$

that is, it is a discrete (q,h)-analogue of the exponential function.

The main properties of the (q,h)-Mittag-Leffler function are described by the following assertion.

Theorem 4.4. (i) *Let $\eta\in\mathbb{R}^+$ and $t\in\widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$. Then*

$${}_a\nabla^{-\eta}_{(q,h)}E^{a,\lambda}_{\alpha,\beta}(t) = E^{a,\lambda}_{\alpha,\beta+\eta}(t). \tag{4.12}$$

(ii) *Let $\eta\in\mathbb{R}^+$, $m\in\mathbb{Z}^+$ be such that $m-1<\eta\leq m$ and let $\alpha k+\beta-1\notin\{0,-1,\ldots,-m+1\}$ for all $k\in\mathbb{Z}^+$. If $t\in\widetilde{\mathbb{T}}^{\sigma^{m+1}(a)}_{(q,h)}$, then*

$${}_a\nabla^{\eta}_{(q,h)}E^{a,\lambda}_{\alpha,\beta}(t) = \begin{cases} E^{a,\lambda}_{\alpha,\beta-\eta}(t), & \beta-\eta\notin\{0,-1,\ldots,-m+1\},\\ \lambda E^{a,\lambda}_{\alpha,\beta-\eta+\alpha}(t), & \beta-\eta\in\{0,-1,\ldots,-m+1\}.\end{cases} \tag{4.13}$$

Proof. The part (i) follows immediately from Proposition 4.1. Considering the part (ii), we can write

$${}_a\nabla^{\eta}_{(q,h)}E^{a,\lambda}_{\alpha,\beta}(t) = {}_a\nabla^{\eta}_{(q,h)}\sum_{k=0}^{\infty}\lambda^k\widehat{h}_{\alpha k+\beta-1}(t,a) = \sum_{k=0}^{\infty}\lambda^k\,{}_a\nabla^{\eta}_{(q,h)}\widehat{h}_{\alpha k+\beta-1}(t,a) \tag{4.14}$$

due to the absolute convergence property.

If $k \in \mathbb{Z}^+$, then Corollary 4.2 implies the relation

$$ {}_a\nabla^{\eta}_{(q,h)} \widehat{h}_{\alpha k+\beta-1}(t,a) = \widehat{h}_{\alpha k+\beta-\eta-1}(t,a) \tag{4.15}$$

due to the assumption $\alpha k + \beta - 1 \notin \{0, -1, \ldots, -m+1\}$. If $k = 0$, then two possibilities may occur. If $\beta - \eta \notin \{0, -1, \ldots, -m+1\}$, we get (4.15) with $k = 0$ which implies the validity of $(4.13)_1$. If $\beta - \eta \in \{0, -1, \ldots, -m+1\}$, the nabla (q,h)-fractional derivative of this term is zero and by shifting the index k, we obtain $(4.13)_2$. □

Corollary 4.5. *Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m-1 < \alpha \le m$. Then the functions*

$$ E^{a,\lambda}_{\alpha,\beta}(t), \qquad \beta = \alpha - m + 1, \ldots, \alpha - 1, \alpha \tag{4.16}$$

define eigenfunctions of the operator ${}_a\nabla^{\alpha}_{(q,h)}$ on each set $[\sigma(a), b] \cap \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$, where $b \in \widetilde{\mathbb{T}}^{\sigma(a)}_{(q,h)}$ is satisfying $|\lambda|(\nu(b))^{\alpha} < 1$.

Proof. The assertion follows from Theorem 4.4 by the use of $\eta = \alpha$. □

Our final aim is to show that any solution of (4.1) can be written as a linear combination of (q,h)-Mittag-Leffler functions (4.16).

Lemma 4.6. *Let $\alpha \in \mathbb{R}^+$ and $m \in \mathbb{Z}^+$ be such that $m-1 < \alpha \le m$. Then*

$$ W_{m-\alpha}\left(E^{a,\lambda}_{\alpha,\alpha-m+1}, E^{a,\lambda}_{\alpha,\alpha-m+2}, \ldots, E^{a,\lambda}_{\alpha,\alpha}\right)(\sigma^m(a)) = \prod_{k=1}^{m} \frac{1}{1-\lambda\left(\nu\left(\sigma^k(a)\right)\right)^{\alpha}} \neq 0. \tag{4.17}$$

Proof. The case $m = 1$ is trivial. For $m \ge 2$, we can formally write $\lambda E^{a,\lambda}_{\alpha,\alpha-\ell}(t) = E^{a,\lambda}_{\alpha,-\ell}(t)$ for all $t \in \widetilde{\mathbb{T}}^{\sigma^m(a)}_{(q,h)}$ $(\ell = 0, \ldots, m-2)$. Consequently, applying Theorem 4.4, the Wronskian can be expressed as

$$ W_{m-\alpha}\left(E^{a,\lambda}_{\alpha,\alpha-m+1}, E^{a,\lambda}_{\alpha,\alpha-m+2}, \ldots, E^{a,\lambda}_{\alpha,\alpha}\right)(\sigma^m(a)) = \det M_m(\sigma^m(a)), \tag{4.18}$$

where

$$ M_m(\sigma^m(a)) = \begin{pmatrix} E^{a,\lambda}_{\alpha,1}(\sigma^m(a)) & E^{a,\lambda}_{\alpha,2}(\sigma^m(a)) & \ldots & E^{a,\lambda}_{\alpha,m}(\sigma^m(a)) \\ E^{a,\lambda}_{\alpha,0}(\sigma^m(a)) & E^{a,\lambda}_{\alpha,1}(\sigma^m(a)) & \ldots & E^{a,\lambda}_{\alpha,m-1}(\sigma^m(a)) \\ \ldots & \ldots & \ddots & \ldots \\ E^{a,\lambda}_{\alpha,2-m}(\sigma^m(a)) & E^{a,\lambda}_{\alpha,3-m}(\sigma^m(a)) & \ldots & E^{a,\lambda}_{\alpha,1}(\sigma^m(a)) \end{pmatrix}. \tag{4.19}$$

Using the q-Pascal rule (2.15), we obtain the equality

$$ E^{a,\lambda}_{\alpha,i}(\sigma^m(a)) - \nu(\sigma(a)) E^{a,\lambda}_{\alpha,i-1}(\sigma^m(a)) = E^{\sigma(a),\lambda}_{\alpha,i}(\sigma^m(a)), \quad i \in \mathbb{Z},\ i \ge 3-m. \tag{4.20}$$

Starting with the first row, $\binom{m}{2}$ elementary row operations of the type (4.20) transform the matrix $M_m(\sigma^m(a))$ into the matrix

$$\widehat{M}_m(\sigma^m(a)) = \begin{pmatrix} E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & E_{\alpha,2}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) \\ E_{\alpha,0}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & E_{\alpha,1}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & \dots & E_{\alpha,m-1}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) \\ \dots & \dots & \ddots & \dots \\ E_{\alpha,2-m}^{a,\lambda}(\sigma^m(a)) & E_{\alpha,3-m}^{a,\lambda}(\sigma^m(a)) & \dots & E_{\alpha,1}^{a,\lambda}(\sigma^m(a)) \end{pmatrix} \tag{4.21}$$

with the property $\det \widehat{M}_m(\sigma^m(a)) = \det M_m(\sigma^m(a))$. By Lemma 2.3, we have

$$\begin{gathered} E_{\alpha,p}^{\sigma^i(a),\lambda}(\sigma^m(a)) - \nu(\sigma^m(a))E_{\alpha,p-1}^{\sigma^i(a),\lambda}(\sigma^m(a)) = E_{\alpha,p}^{\sigma^i(a),\lambda}\left(\sigma^{m-1}(a)\right), \quad i = 0,\dots,m-2, \\ E_{\alpha,p}^{\sigma^i(a),\lambda}(\sigma^m(a)) - \nu(\sigma^m(a))E_{\alpha,p-1}^{\sigma^i(a),\lambda}(\sigma^m(a)) = 0, \quad i = m-1, \end{gathered} \tag{4.22}$$

where $p \in \mathbb{Z},\ p \geq 3 - m + i$. Starting with the last column, using $m-1$ elementary column operations of the type (4.22), we obtain the matrix

$$\left(\begin{array}{c|ccc} E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) & 0 & \cdots & 0 \\ \cline{2-4} E_{\alpha,0}^{\sigma^{m-2}(a),\lambda}(\sigma^m(a)) & & & \\ \vdots & & \widehat{M}_{m-1}(\sigma^{m-1}(a)) & \\ E_{\alpha,2-m}^{a,\lambda}(\sigma^m(a)) & & & \end{array}\right) \tag{4.23}$$

preserving the value of $\det \widehat{M}_m(\sigma^m(a))$. Since

$$E_{\alpha,1}^{\sigma^{m-1}(a),\lambda}(\sigma^m(a)) = \sum_{k=0}^{\infty}\lambda^k(\nu(\sigma^m(a)))^{\alpha k} = \frac{1}{1-\lambda(\nu(\sigma^m(a)))^{\alpha}}, \tag{4.24}$$

we can observe the recurrence

$$\det \widehat{M}_m(\sigma^m(a)) = \frac{1}{1-\lambda(\sigma^m(a))^{\alpha}} \det \widehat{M}_{m-1}\left(\sigma^{m-1}(a)\right), \tag{4.25}$$

which implies the assertion. □

Now we summarize the results of Theorem 3.9, Corollary 4.5, and Lemma 4.6 to obtain

Theorem 4.7. *Let $y(t)$ be any solution of (4.1) defined on $[\sigma(a), b] \cap \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$, where $b \in \widetilde{\mathbb{T}}_{(q,h)}^{\sigma(a)}$ is satisfying $|\lambda|(\nu(b))^{\alpha} < 1$. Then*

$$y(t) = \sum_{j=1}^{m} c_j E_{\alpha,\alpha-m+j}^{a,\lambda}(t), \tag{4.26}$$

where $c_1, \ldots, c_m$ *are real constants.*

We conclude this paper by the illustrating example.

Example 4.8. Consider the initial value problem

$$\begin{gathered} {}_a\nabla_{(q,h)}^{\alpha} y(t) = \lambda\, y(t), \quad \sigma^3(a) \le t \le \sigma^n(a),\ 1 < \alpha \le 2, \\ {}_a\nabla_{(q,h)}^{\alpha-1} y(t)\Big|_{t=\sigma^2(a)} = y_{\alpha-1}, \\ {}_a\nabla_{(q,h)}^{\alpha-2} y(t)\Big|_{t=\sigma^2(a)} = y_{\alpha-2}, \end{gathered} \tag{4.27}$$

where n is a positive integer given by the condition $|\lambda|\nu(\sigma^n(a))^{\alpha} < 1$. By Theorem 4.7, its solution can be expressed as a linear combination

$$y(t) = c_1 E_{\alpha,\alpha-1}^{a,\lambda}(t) + c_2 E_{\alpha,\alpha}^{a,\lambda}(t). \tag{4.28}$$

The constants c_1, c_2 can be determined from the system

$$V_{2-\alpha}\left(E_{\alpha,\alpha-1}^{a,\lambda}, E_{\alpha,\alpha}^{a,\lambda}\right)\left(\sigma^2(a)\right) \cdot \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} y_{\alpha-2} \\ y_{\alpha-1} \end{pmatrix} \tag{4.29}$$

with the matrix elements

$$\begin{aligned} v_{11} = v_{22} &= \frac{[1]_q + \left([\alpha]_q - [1]_q\right)\lambda\nu(\sigma(a))^{\alpha}}{\left(1-\lambda\nu(\sigma(a))^{\alpha}\right)\left(1-\lambda\nu(\sigma^2(a))^{\alpha}\right)}, \\ v_{12} &= \frac{[2]_q\nu(\sigma(a)) + \left([\alpha]_q - [2]_q\right)\lambda\nu(\sigma(a))^{\alpha+1}}{\left(1-\lambda\nu(\sigma(a))^{\alpha}\right)\left(1-\lambda\nu(\sigma^2(a))^{\alpha}\right)}, \\ v_{21} &= \frac{[\alpha]_q\lambda\nu(\sigma(a))^{\alpha-1}}{\left(1-\lambda\nu(\sigma(a))^{\alpha}\right)\left(1-\lambda\nu(\sigma^2(a))^{\alpha}\right)}. \end{aligned} \tag{4.30}$$

By Lemma 4.6, the matrix $V_{2-\alpha}(E_{\alpha,\alpha-1}^{a,\lambda}, E_{\alpha,\alpha}^{a,\lambda})(\sigma^2(a))$ has a nonzero determinant, hence applying the Cramer rule, we get

$$\begin{aligned} c_1 &= \frac{y_{\alpha-2}v_{22} - y_{\alpha-1}v_{12}}{W_{2-\alpha}\left(E_{\alpha,\alpha-1}^{a,\lambda}, E_{\alpha,\alpha}^{a,\lambda}\right)(\sigma^2(a))}, \\ c_2 &= \frac{y_{\alpha-1}v_{11} - y_{\alpha-2}v_{21}}{W_{2-\alpha}\left(E_{\alpha,\alpha-1}^{a,\lambda}, E_{\alpha,\alpha}^{a,\lambda}\right)(\sigma^2(a))}. \end{aligned} \tag{4.31}$$

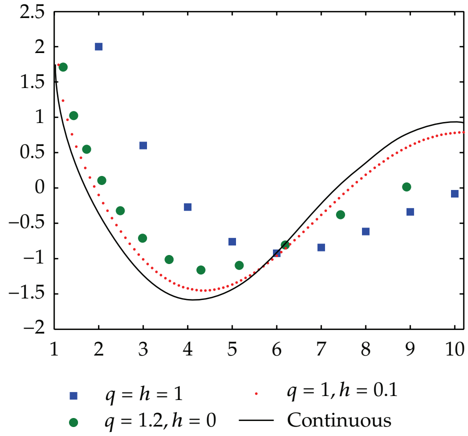

Figure 1: $\alpha = 1.8, a = 1, \lambda = -1/3, y_{\alpha-1} = -1, y_{\alpha-2} = 1$.

Now we make a particular choice of the parameters α, a, λ, $y_{\alpha-1}$ and $y_{\alpha-2}$ and consider the initial value problem in the form

$$\begin{gathered}
{}_1\nabla^{1.8}_{(q,h)} y(t) = -\frac{1}{3}\, y(t), \quad \sigma^3(1) \le t \le \sigma^n(1), \\
{}_1\nabla^{0.8}_{(q,h)} y(t)\Big|_{t=\sigma^2(1)} = -1, \\
{}_1\nabla^{-0.2}_{(q,h)} y(t)\Big|_{t=\sigma^2(1)} = 1,
\end{gathered} \tag{4.32}$$

where n is a positive integer satisfying $\nu(\sigma^n(1)) < 3^{5/9}$. If we take the time scale of integers (the case $q = h = 1$), then the solution $y(t)$ of the corresponding initial value problem takes the form

$$y(t) = \frac{14}{5}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2}(j + 1.8k - 0.2)}{(t-2)!} - \frac{2}{15}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k \frac{\prod_{j=1}^{t-2}(j + 1.8k + 0.8)}{(t-2)!}, \quad t = 2, 3, \ldots. \tag{4.33}$$

Similarly we can determine $y(t)$ for other choices of q and h. For comparative reasons, Figure 1 depicts (in addition to the above case $q = h = 1$) the solution $y(t)$ under particular choices $q = 1.2$, $h = 0$ (the pure q-calculus), $q = 1$, $h = 0.1$ (the pure h-calculus) and also the solution of the corresponding continuous (differential) initial value problem.

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Research Article

Estimates of Exponential Stability for Solutions of Stochastic Control Systems with Delay

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A nonlinear stochastic differential-difference control system with delay of neutral type is considered. Sufficient conditions for the exponential stability are derived by using Lyapunov-Krasovskii functionals of quadratic form with exponential factors. Upper bound estimates for the exponential rate of decay are derived.

1. Introduction

The theory and applications of functional differential equations form an important part of modern nonlinear dynamics. Such equations are natural mathematical models for various real life phenomena where the aftereffects are intrinsic features of their functioning. In recent years, functional differential equations have been used to model processes in different areas such as population dynamics and ecology, physiology and medicine, economics, and other natural sciences [1–3]. In many of the models the initial data and parameters are subjected to random perturbations, or the dynamical systems themselves represent stochastic processes. For this reason, stochastic functional differential equations are widely studied [4, 5].

One of the principal problems of the corresponding mathematical analysis of equations is a comprehensive study of their global dynamics and the related prediction of long-term behaviors in applied models. Of course, the problem of stability of a particular solution plays a significant role. Therefore, the study of stability of linear equations is the first natural and important step in the analysis of more complex nonlinear systems.

When applying the mathematical theory to real-world problems a mere statement of the stability in the system is hardly sufficient. In addition to stability as such, it is of significant importance to obtain constructive and verifiable estimates of the rate of convergence of solutions in time. One of the principal tools used in the related studies is the second Lyapunov method [6–8]. For functional differential equations, this method has been developing in two main directions in recent years. The first one is the method of finite Lyapunov functions with the additional assumption of Razumikhin type [9, 10]. The second one is the method of Lyapunov-Krasovskii functionals [11, 12]. For stochastic functional differential equations, some aspects of these two lines of research have been developed, for example, in [11, 13–19] and [11, 18, 20–25], respectively. In the present paper, by using the method of Lyapunov-Krasovskii functionals, we derive sufficient conditions for stability together with the rate of convergence to zero of solutions for a class of linear stochastic functional differential equation of a neutral type.

2. Preliminaries

In solving control problems for linear systems, very often, a scalar function $u = u(x)$ needs to be found such that the system

$$\dot{x}(t) = Ax(t) + bu(x(t)) \tag{2.1}$$

is asymptotically stable. Frequently, such a function depends on a scalar argument which is a linear combination of phase coordinates and its graph lies in the first and the third quadrants of the plane. An investigation of the asymptotic stability of systems with a control function

$$u(x(t)) = f(\sigma(t)), \quad \sigma(t) = c^T x(t), \tag{2.2}$$

that is, an investigation of systems

$$\dot{x}(t) = Ax(t) + bf(\sigma(t)), \quad \sigma(t) = c^T x(t), \tag{2.3}$$

with a function f satisfying $f(0) = 0$, $f(\sigma)(k\sigma - f(\sigma)) > 0$ for $\sigma \neq 0$ and a $k > 0$ is called an analysis of the absolute stability of control systems [26]. One of the fundamental methods (called a frequency method) was developed by Gelig et al. (see, e.g., the book [27]). Another basic method is the method of Lyapunov's functions and Lyapunov-Krasovskii functionals. Very often, the appropriate Lyapunov functions and Lyapunov-Krasovskii functionals are constructed as quadratic forms with integral terms containing a given nonlinearity [28, 29]. An overview of the present state can be found, for example, in [30, 31]. Problems of absolute stability of stochastic equations are treated, for example, in [11, 14, 15, 24].

3. Main Results

Consider the following control system of stochastic differential-difference equations of a neutral type

$$
\begin{aligned}
d[x(t) - Dx(t-\tau)] = {} & \left[A_0 x(t) + A_1 x(t-\tau) + a_2 f(\sigma(t))\right] dt \\
& + \left[B_0 x(t) + B_1 x(t-\tau) + b_2 f(\sigma(t))\right] dw(t),
\end{aligned}
\tag{3.1}
$$

where

$$
\sigma(t) := c^T [x(t) - Dx(t-\tau)],
\tag{3.2}
$$

$x : [0,\infty) \to \mathbb{R}^n$ is an n-dimensional column vector, A_0, A_1, B_0, B_1, and D are real $n \times n$ constant matrices, a_2, b_2, and c are $n \times 1$ constant vectors, $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\tau > 0$ is a constant delay, and $w(t)$ is a standard scalar Wiener process with

$$
M\{dw(t)\} = 0, \qquad M\left\{dw^2(t)\right\} = dt, \qquad M\{dw(t_1)\, dw(t_2), t_1 \neq t_2\} = 0.
\tag{3.3}
$$

An F_t-measurable random process $\{x(t) \equiv x(t,\omega)\}$ is called a solution of (3.1) if it satisfies, with a probability one, the following integral equation

$$
\begin{aligned}
x(t) = {} & Dx(t-\tau) + [x(0) - Dx(-\tau)] \\
& + \int_0^t \left[A_0 x(s) + A_1 x(s-\tau) + a_2 f(\sigma(s))\right] ds \\
& + \int_0^t \left[B_0 x(s) + B_1 x(s-\tau) + b_2 f(\sigma(s))\right] dw(s), \quad t \geq 0
\end{aligned}
\tag{3.4}
$$

and the initial conditions

$$
x(t) = \varphi(t), \quad x'(t) = \psi(t), \quad t \in [-\tau, 0],
\tag{3.5}
$$

where $\varphi, \psi : [-\tau, 0] \to \mathbb{R}^n$ are continuous functions. Here and in the remaining part of the paper, we will assume that the initial functions φ and ψ are continuous random processes. Under those assumptions, a solution to the initial value problem (3.1), (3.5) exists and is unique for all $t \geq 0$ up to its stochastic equivalent solution on the space (Ω, F, P) [4].

We will use the following norms of matrices and vectors

$$\begin{aligned}
\|A\| &:= \sqrt{\lambda_{\max}\left(A^T A\right)}\,, \\
\|x(t)\| &:= \sqrt{\sum_{i=1}^{n} x_i^2(t)}, \\
\|x(t)\|_\tau &:= \max_{-\tau \le s \le 0} \{\|x(t+s)\|\}, \\
\|x(t)\|_{\tau,\gamma}^2 &:= \int_{t-\tau}^{t} e^{-\gamma(t-s)} \|x(s)\|^2 ds,
\end{aligned} \tag{3.6}$$

where $\lambda_{\max}(*)$ is the largest eigenvalue of the given symmetric matrix (similarly, the symbol $\lambda_{\min}(*)$ denotes the smallest eigenvalue of the given symmetric matrix), and γ is a positive parameter.

Throughout this paper, we assume that the function f satisfies the inequality

$$0 \le f(\sigma)\sigma \le k\sigma^2 \tag{3.7}$$

if $\sigma \in \mathbb{R}$ where k is a positive constant.

For the reader's convenience, we recall that the zero solution of (3.1) is called stable in the square mean if, for every $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon) > 0$ such that every solution $x = x(t)$ of (3.1) satisfies $M\{\|x(t)\|^2\} < \varepsilon$ provided that the initial conditions (3.5) are such that $\|\varphi(0)\|_\tau < \delta$ and $\|\psi(0)\|_\tau < \delta$. If the zero solution is stable in the square mean and, moreover,

$$\lim_{t \to +\infty} M\left\{\|x(t)\|^2\right\} = 0, \tag{3.8}$$

then it is called asymptotically stable in the square mean.

Definition 3.1. If there exist positive constants N, γ, and θ such that the inequality

$$M\left\{\|x(t)\|_{\tau,\gamma}^2\right\} \le N\ \|x(0)\|_\tau^2\ e^{-\theta t} \tag{3.9}$$

holds on $[0, \infty)$, then the zero solution of (3.1) is called exponentially γ-integrally stable in the square mean.

In this paper, we prove the exponential γ-integral stability in the square mean of the differential-difference equation with constant delay (3.1). We employ the method of stochastic Lyapunov-Krasovskii functionals. In [11, 18, 22, 24] the Lyapunov-Krasovskii functional is chosen to be of the form

$$V[x(t), t] = h[x(t) - cx(t-\tau)]^2 + g\int_{-\tau}^{0} x^2(t+s)ds, \tag{3.10}$$

where constants $h > 0$ and $g > 0$ are such that the total stochastic differential of the functional along solutions is negative definite.

In the present paper, we consider the Lyapunov-Krasovskii functional in the following form:

$$V[x(t),t] = [x(t) - Dx(t-\tau)]^T H[x(t) - Dx(t-\tau)] \\ + \int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s)Gx(s)ds + \beta \int_0^{\sigma(t)} f(\xi)d\xi, \tag{3.11}$$

where constants $\gamma > 0$, $\beta > 0$ and $n \times n$ positive definite symmetric matrices G, H are to be restricted later on. This allows us not only to derive sufficient conditions for the stability of the zero solution but also to obtain coefficient estimates of the rate of the exponential decay of solutions.

We set

$$P := \begin{pmatrix} H & -HD \\ -D^T H & D^T HD \end{pmatrix}. \tag{3.12}$$

Then, by using introduced norms, the functional (3.11) yields two-sided estimates

$$\lambda_{\min}(G)\|x(t)\|_{\tau,\gamma}^2 \le V[x(t),t] \le \left[\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right]\|x(t)\|^2 \\ + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right]\|x(t-\tau)\|^2 + \lambda_{\max}(G)\|x(t)\|_{\tau,\gamma}^2, \tag{3.13}$$

where $t \in [0,\infty)$.

We will use an auxiliary $(2n+1) \times (2n+1)$-dimensional matrix:

$$S = S(\beta,\gamma,\nu,G,H) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix}, \tag{3.14}$$

where

$$
\begin{aligned}
s_{11} &:= -A_0 H - H A_0 - B_0^T H B_0 - G, \\
s_{12} &:= A_0^T H D - H A_1 - B_0^T H B_1, \\
s_{13} &:= -H a_2 - B_0^T H b_2 - \frac{1}{2}(\beta A_0 + \nu I)^T c, \\
s_{21} &:= s_{12}^T, \\
s_{22} &:= D^T H A_1 + A_1^T H D - B_1^T H B_1 + e^{-\gamma\tau} G, \\
s_{23} &:= D^T H a_2 - B_1^T H b_2 - \frac{1}{2}\beta A_1 c, \\
s_{31} &:= s_{13}^T, \\
s_{32} &:= s_{23}^T, \\
s_{33} &:= \frac{\nu}{k} - b_2^T H b_2 - \beta c^T a_2,
\end{aligned}
\tag{3.15}
$$

where ν is a parameter.

Now we establish our main result on the exponential γ-integral stability of a trivial solution in the square mean of system (3.1) when $t \to \infty$.

Theorem 3.2. *Let $\|D\| < 1$. Let there exist positive constants β, γ, ν and positive definite symmetric matrices G, H such that the matrix S is positively definite as well. Then the zero solution of the system* (3.1) *is exponentially γ-integrally stable in the square mean on $[0, \infty)$. Moreover, every solution $x(t)$ of* (3.1) *satisfies the inequality*

$$
M\left\{ \|x(t)\|_{\tau,\gamma}^2 \right\} \leq N \|x(0)\|_\tau^2 \, e^{-\theta t}
\tag{3.16}
$$

for all $t \geq 0$ where

$$
\begin{aligned}
N &:= \frac{1}{\lambda_{\min}(G)} \cdot \left(2\lambda_{\max}(P) + 0.5\beta k \|c\|^2 + 0.5\beta k \left\| c^T D \right\|^2 + \frac{1}{\gamma}\lambda_{\max}(G) \right), \\
\theta &:= \min\left\{ \frac{\gamma \lambda_{\min}(G)}{\lambda_{\max}(G)}, \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k \|c\|^2} \right\}.
\end{aligned}
\tag{3.17}
$$

Proof. We will apply the method of Lyapunov-Krasovskii functionals using functional (3.11). Using the Itô formula, we compute the stochastic differential of (3.11) as follows

$$
\begin{aligned}
dV[x(t),t] = &\Big(\left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right]^T dt \\
&+ \left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right]^T dw(t)\Big) \\
&\times H[x(t) - Dx(t-\tau)] + [x(t) - Dx(t-\tau)]^T \\
&\times H\Big(\left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right] dt \\
&\qquad + \left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right]^T dw(t)\Big) \\
&+ \left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right]^T \\
&\times H\left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right] d\Big(w^2(t)\Big) \\
&+ x^T(t)Gx(t)dt - e^{-\gamma\tau} x^T(t-\tau)Gx(t-\tau)dt + \beta f(\sigma(t))c^T \\
&\times \Big(\left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right] dt \\
&\quad + \left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right]^T dw(t)\Big) \\
&- \gamma \int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s)Gx(s)ds\,dt.
\end{aligned}
\tag{3.18}
$$

Taking the mathematical expectation we obtain (we use properties (3.3))

$$
\begin{aligned}
M\{dV[x(t),t]\} = &M\Big\{ \left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right]^T \\
&\quad \times H[x(t) - Dx(t-\tau)]dt \Big\} \\
&+ M\Big\{ [x(t) - Dx(t-\tau)]^T \\
&\qquad \times H\left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right] dt \Big\} \\
&+ M\Big\{ \left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right]^T \\
&\qquad \times H\left[B_0x(t) + B_1x(t-\tau) + b_2 f(\sigma(t))\right] d\Big(w^2(t)\Big) \Big\} \\
&+ M\Big\{ \Big[x^T(t)Gx(t)dt - e^{-\gamma\tau} x^T(t-\tau)Gx(t-\tau)dt\Big] \Big\} \\
&+ \beta M\Big\{ f(\sigma(t))c^T \left[A_0x(t) + A_1x(t-\tau) + a_2 f(\sigma(t))\right] dt \Big\} \\
&- \gamma M\left\{ \int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s)Gx(s)ds\,dt \right\}.
\end{aligned}
\tag{3.19}
$$

Utilizing the matrix S defined by (3.14), the last expression can be rewritten in the following vector matrix form

$$\frac{d}{dt}M\{V[x(t),t]\} = -M\left\{\left(x^T(t), x^T(t-\tau), f(\sigma(t))\right) \times S \times \left(x^T(t), x^T(t-\tau), f(\sigma(t))\right)^T\right\} - \nu\left[\sigma(t) - \frac{f(\sigma(t))}{k}\right] f(\sigma(t)) - \gamma M\left\{\int_{t-\tau}^{t} e^{-\gamma(t-s)} x^T(s) G x(s) ds\right\}. \tag{3.20}$$

We will show next that solutions of (3.1) decay exponentially by calculating the corresponding exponential rate.

The full derivative of the mathematical expectation for the Lyapunov-Krasovskii functional (3.11) satisfies

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \le &-\lambda_{\min}(S) M\left\{\|x(t)\|^2\right\} \\ &- \lambda_{\min}(S) M\left\{\|x(t-\tau)\|^2\right\} \\ &- \gamma\lambda_{\min}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}.\end{aligned} \tag{3.21}$$

In the following we will use inequalities being a consequence of (3.13).

$$\begin{aligned}\lambda_{\min}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} &\le M\{V[x(t)]\} \\ &\le \left[\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right] \times M\left\{\|x(t)\|^2\right\} \\ &\quad + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\} \\ &\quad + \lambda_{\max}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}.\end{aligned} \tag{3.22}$$

Let us derive conditions for the coefficients of (3.1) and parameters of the Lyapunov-Krasovskii functional (3.11) such that the following inequality:

$$\frac{d}{dt}M\{V[x(t),t]\} \le -\theta M\{V[x(t),t]\} \tag{3.23}$$

holds. We use a sequence of the following calculations supposing that either inequality

$$\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k|c|^2}\lambda_{\max}(G) \ge 0 \tag{3.24}$$

holds, or the opposite inequality

$$\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k|c|^2}\lambda_{\max}(G) \le 0 \tag{3.25}$$

is valid.

(1) Let inequality (3.24) holds. Rewrite the right-hand part of inequality (3.22) in the form

$$\begin{aligned} -M\left\{\|x(t)\|^2\right\} \le{}& \frac{1}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \\ &\times \Big[- M\{V[x(t),t]\} + \lambda_{\max}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \\ &+ \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\}\Big] \end{aligned} \tag{3.26}$$

and substitute the latter into inequality (3.21). This results in

$$\begin{aligned} \frac{d}{dt}M\{V[x(t),t]\} \le{}& -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \\ &\times \Big[- M\{V[x(t),t]\} + \lambda_{\max}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \\ &+ \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right] M\left\{\|x(t-\tau)\|^2\right\}\Big] \\ &- \gamma\lambda_{\min}(G) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} - \lambda_{\min}(S) M\left\{\|x(t-\tau)\|^2\right\}, \end{aligned} \tag{3.27}$$

or, equivalently,

$$\begin{aligned} \frac{d}{dt}M\{V[x(t),t]\} \le{}& -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} M\{V[x(t),t]\} \\ &- \lambda_{\min}(S)\left(1 - \frac{\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2}\right) M\left\{\|x(t-\tau)\|^2\right\} \\ &- \left(\gamma\lambda_{\min}(G) - \frac{\lambda_{\min}(S)}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2}\lambda_{\max}(G)\right) M\left\{\|x(t)\|^2_{\tau,\gamma}\right\}. \end{aligned} \tag{3.28}$$

The inequality

$$\frac{\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2}{\lambda_{\max}(P) + 0.5\beta k\|c\|^2} \le 1 \tag{3.29}$$

always holds. Because inequality (3.24) is valid, a differential inequality

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} &\leq -\frac{\lambda_{\min}(S)}{\lambda_{\max}(P)+0.5\beta k\|c\|^2}M\{V[x(t),t]\}\\ &\leq -\theta M\{V[x(t),t]\}\end{aligned} \tag{3.30}$$

will be true as well.

(2) Let inequality (3.25) hold. We rewrite the right-hand side of inequality (3.22) in the form

$$\begin{aligned}-M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} \leq \frac{1}{\lambda_{\max}(G)} \times \Big(&- M\{V[x(t),t]\} + \left(\lambda_{\max}(P)+0.5\beta k\|c\|^2\right)M\left\{\|x(t)\|^2\right\}\\ &+\left[\lambda_{\max}(P)+0.5\beta k\left\|c^T D\right\|^2\right]M\left\{\|x(t-\tau)\|^2\right\}\Big)\end{aligned} \tag{3.31}$$

and substitute the latter again into inequality (3.21). This results in

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \leq -\lambda_{\min}(S)M\left\{\|x(t)\|^2\right\} - \lambda_{\min}(S)M\left\{\|x(t-\tau)\|^2\right\} + \gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}&\\ \times\Big\{ - M\{V[x(t),t]\} + \left(\lambda_{\max}(P)+0.5\beta k\|c\|^2\right)M\left\{\|x(t)\|^2\right\}&\\ +\left[\lambda_{\max}(P)+0.5\beta k\left\|c^T D\right\|^2\right]M\left\{\|x(t-\tau)\|^2\right\}\Big\}&\end{aligned} \tag{3.32}$$

or in

$$\begin{aligned}\frac{d}{dt}M\{V[x(t),t]\} \leq{}& -\gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}M\{V[x(t),t]\}\\ &-\left(\lambda_{\min}(S) - \frac{\lambda_{\max}(P)+0.5\beta k\|c\|^2}{\lambda_{\max}(G)}\gamma\lambda_{\min}(G)\right)M\left\{\|x(t)\|^2\right\}\\ &-\left(\lambda_{\min}(S) - \frac{\gamma\lambda_{\min}(G)\left[\lambda_{\max}(P)+0.5\beta k\left\|c^T D\right\|^2\right]}{\lambda_{\max}(G)}\right)M\left\{\|x(t-\tau)\|^2\right\}.\end{aligned} \tag{3.33}$$

Because inequality (3.25) is valid, a differential inequality

$$\frac{d}{dt}M\{V[x(t),t]\} \leq -\gamma\frac{\lambda_{\min}(G)}{\lambda_{\max}(G)}M\{V[x(t),t]\} \leq -\theta M\{V[x(t),t]\} \tag{3.34}$$

will be valid as well.

Analysing inequalities (3.30) and (3.34) we conclude that (3.23) always holds. Solving inequality (3.23) we obtain

$$M\{V[x(t),t]\} \le M\{V[x(0),0]\}e^{-\theta t}. \tag{3.35}$$

Now we derive estimates of the rate of the exponential decay of solutions. We use inequalities (3.22), (3.35). It is easy to see that

$$\begin{aligned}
\lambda_{\min}(G)M\left\{\|x(t)\|^2_{\tau,\gamma}\right\} &\le M\{V[x(t),t]\} \le M\{V[x(0),0]\}e^{-\theta t} \\
&\le \left(\left(\lambda_{\max}(P) + 0.5\beta k\|c\|^2\right)\|x(0)\|^2 \right.\\
&\quad \left. + \left[\lambda_{\max}(P) + 0.5\beta k\left\|c^T D\right\|^2\right]\|x(-\tau)\|^2 + \lambda_{\max}(G)\|x(0)\|^2_{\tau,\gamma}\right)e^{-\theta t} \\
&\le \left(2\lambda_{\max}(P) + 0.5\beta k\|c\|^2 + 0.5\beta k\left\|c^T D\right\|^2 + \frac{1}{\gamma}\lambda_{\max}(G)\right)\|x(0)\|^2_{\tau}e^{-\theta t}.
\end{aligned} \tag{3.36}$$

Now, inequality (3.16) is a simple consequence of the latter chain of inequalities. □

4. A Scalar Case

As an example, we will apply Theorem 3.2 to a scalar control stochastic differential-difference equation of a neutral type

$$\begin{aligned}
d[x(t) - d_0x(t-\tau)] = {}& \left[a_0x(t) + a_1x(t-\tau) + a_2f(\sigma(t))\right]dt \\
& + \left[b_0x(t) + b_1x(t-\tau) + b_2f(\sigma(t))\right]dw(t),
\end{aligned} \tag{4.1}$$

where $\sigma(t) = c[x(t) - d_0x(t-\tau)]$, $x \in \mathbb{R}$, a_0, a_1, a_2, b_0, b_1, d_2, d_0, and c are real constants, $\tau > 0$ is a constant delay, and $w(t)$ is a standard scalar Wiener process satisfying (3.3). An F_t-measurable random process $\{x(t) \equiv x(t,\omega)\}$ is called a solution of (4.1) if it satisfies, with a probability one, the following integral equation:

$$\begin{aligned}
x(t) = {}& d_0x(t-\tau) + [x(0) - d_0x(-\tau)] \\
& + \int_0^t \left[a_0x(s) + a_1x(s-\tau) + a_2f(\sigma(t))\right]ds \\
& + \int_0^t \left[b_0x(s) + b_1x(s-\tau) + b_2f(\sigma(t))\right]dw(s), \quad t \ge 0.
\end{aligned} \tag{4.2}$$

The Lyapunov-Krasovskii functional V reduces to

$$V[x(t),t] = [x(t) - d_0x(t-\tau)]^2 + g\int_{t-\tau}^{t} e^{-\gamma(t-s)}x^2(s)ds + \beta\int_0^{\sigma(t)} f(\xi)d\xi, \tag{4.3}$$

where we assume $g > 0$ and $\beta > 0$. The matrix S reduces to (for simplicity we set $H = (1)$)

$$S = S(g,\beta,\gamma,\nu) := \begin{pmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{pmatrix} \tag{4.4}$$

and has entries

$$\begin{aligned} s_{11} &:= -2a_0 - b_0^2 - g, \\ s_{12} &:= a_0d_0 - a_1 - b_0b_1, \\ s_{13} &:= -a_2 - b_0b_2 - \frac{1}{2}(\beta a_0 + \nu)c, \\ s_{21} &:= s_{12}, \\ s_{22} &:= 2a_1d_0 - b_1^2 + e^{-\gamma\tau}g, \\ s_{23} &:= a_2d_0 - b_1b_2 - 0.5\beta a_1c, \\ s_{31} &:= s_{13}, \\ s_{32} &:= s_{23}, \\ s_{33} &:= \frac{\nu}{k} - b_2^2 - \beta c a_2, \end{aligned} \tag{4.5}$$

where ν is a parameter. Therefore, the above calculation yields the following result.

Theorem 4.1. *Let $|d_0| < 1$. Assume that positive constants β, γ, g, and ν are such that the matrix S is positive definite. Then the zero solution of (4.1) is exponentially γ-integrally stable in the square mean on $[0,\infty)$. Moreover, every solution $x(t)$ satisfies the following convergence estimate:*

$$M\left\{\|x(t)\|_{\tau,\gamma}^2\right\} \le N\|x(0)\|_\tau^2 e^{-\theta t} \tag{4.6}$$

for all $t \ge 0$ where

$$\begin{aligned} N &:= \frac{1}{g}\left(2 + 2d_0^2 + 0.5\beta kc^2 + 0.5\beta k(cd_0)^2\right) + \frac{1}{\gamma}, \\ \theta &:= \min\left\{\gamma, \frac{\lambda_{\min}(S)}{1 + d_0^2 + 0.5\beta kc^2}\right\}. \end{aligned} \tag{4.7}$$

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Research Article

Existence and Asymptotic Behavior of Positive Solutions of Functional Differential Equations of Delayed Type

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Solutions of the equation $\dot{y}(t) = -f(t, y_t)$ are considered for $t \to \infty$. The existence of two classes of positive solutions which are asymptotically different is proved using the retract method combined with Razumikhin's technique. With the aid of two auxiliary linear equations, which are constructed using upper and lower linear functional estimates of the right-hand side of the equation considered, inequalities for both types of positive solutions are given as well.

1. Introduction

Let $C([a,b],\mathbb{R}^n)$, where $a, b \in \mathbb{R}$, $a < b$, be the Banach space of the continuous mappings from the interval $[a,b]$ into $\mathbb{R}^n$ equipped with the supremum norm

$$\|\psi\|_C = \sup_{\theta\in[a,b]} \|\psi(\theta)\|, \quad \psi \in C([a,b],\mathbb{R}^n), \tag{1.1}$$

where $\|\cdot\|$ is the maximum norm in $\mathbb{R}^n$. In the case of $a = -r < 0$ and $b = 0$, we will denote this space as C_r^n, that is,

$$\mathrm{C}_r^n := C([-r,0],\mathbb{R}^n). \tag{1.2}$$

If $\sigma \in \mathbb{R}^n$, $A \geq 0$, and $y \in C([\sigma - r, \sigma + A],\mathbb{R}^n)$, then, for each $t \in [\sigma, \sigma + A]$, we define $y_t \in \mathrm{C}_r^n$ by $y_t(\theta) = y(t+\theta)$, $\theta \in [-r, 0]$.

The present article is devoted to the problem of the existence of two classes of asymptotically different positive solutions of the delayed equation

$$\dot{y}(t) = -f(t, y_t), \tag{1.3}$$

for $t \to +\infty$, where $f : \Omega \to \mathbb{R}$ is a continuous quasibounded functional that satisfies a local Lipschitz condition with respect to the second argument and Ω is an open subset in $\mathbb{R} \times C_r^1$ such that conditions which use f are well defined.

The main supposition of our investigation is that the right-hand side of (1.3) can be estimated as follows:

$$C_A(t) y_t(-r) \le f(t, y_t) \le C_B(t) y_t(-r), \tag{1.4}$$

where $(t, y_t) \in \Omega$, and $C_A, C_B : [t_0 - r, \infty) \to \mathbb{R}^+ := (0, \infty)$, $t_0 \in \mathbb{R}$ are continuous functions satisfying

$$0 < C_A(t) \le C_B(t) \le \frac{1}{(r\mathrm{e})}, \quad t \in [t_0 - r, \infty), \tag{1.5}$$

$$\int_{t_0 - r}^{\infty} C_B(t) \mathrm{d}t < 1. \tag{1.6}$$

Quite lots of investigations are devoted to existence of positive solutions of different classes of equations (we mention at least monographs [1–6] and papers [7–12]). The investigation of two classes of asymptotically different solutions of (1.3) has been started in the paper [13] using a monotone iterative technique and a retract principle. Assumptions of results obtained are too cumbersome and are applied to narrow classes of equations. In the presented paper we derive more general statements under weaker conditions. This progress is related to more general inequalities (1.4) for the right-hand side of (1.3) which permit to omit utilization of properties of solutions of transcendental equations used in [13].

1.1. Ważewski's Principle

In this section we introduce Ważewski's principle for a system of retarded functional differential equations

$$\dot{y}(t) = F(t, y_t), \tag{1.7}$$

where $F : \Omega^* \mapsto \mathbb{R}^n$ is a continuous quasibounded map which satisfies a local Lipschitz condition with respect to the second argument and Ω^* is an open subset in $\mathbb{R} \times C_r^n$. We recall that the functional F is quasibounded if F is bounded on every set of the form $[t_1, t_2] \times C_{rL}^n \subset \Omega^*$, where $t_1 < t_2$, $C_{rL}^n := C([-r, 0], L)$ and L is a closed bounded subset of $\mathbb{R}^n$ (compare [2, page 305]).

In accordance with [14], a function $y(t)$ is said to be a *solution of system* (1.7) *on* $[\sigma - r, \sigma + A)$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that $y \in C([\sigma - r, \sigma + A), \mathbb{R}^n)$, $(t, y_t) \in \Omega^*$, and $y(t)$ satisfies the system (1.7) for $t \in [\sigma, \sigma + A)$. For a given $\sigma \in \mathbb{R}$, $\varphi \in C$, we say $y(\sigma, \varphi)$ is a solution of the system (1.7) through $(\sigma,\varphi) \in \Omega^*$ if there is an $A > 0$ such that $y(\sigma,\varphi)$ is a solution of the system (1.7) on $[\sigma - r, \sigma + A)$ and $y_\sigma(\sigma,\varphi) = \varphi$. In view of the above conditions, each element $(\sigma,\varphi) \in \Omega^*$ determines a unique solution $y(\sigma,\varphi)$ of the system (1.7) through $(\sigma,\varphi) \in \Omega^*$ on its maximal interval of existence $I_{\sigma,\varphi} = [\sigma, a)$, $\sigma < a \le \infty$ which depends continuously on initial data [14]. A solution $y(\sigma,\varphi)$ of the system (1.7) is said to be *positive* if

$$y_i(\sigma,\varphi) > 0 \tag{1.8}$$

on $[\sigma - r, \sigma] \cup I_{\sigma,\varphi}$ for each $i = 1, 2, \ldots, n$. A nontrivial solution $y(\sigma,\varphi)$ of the system (1.7) is said to be *oscillatory* on $I_{\sigma,\varphi}$ (under condition $I_{\sigma,\varphi} = [\sigma,\infty)$) if (1.8) does not hold on any subinterval $[\sigma_1,\infty) \subset [\sigma,\infty)$, $\sigma_1 \ge \sigma$.

As a method of proving the existence of positive solutions of (1.3), we use Ważewski's retract principle which was first introduced by Ważewski [15] for ordinary differential equations and later extended to retarded functional differential equations by Rybakowski [16] and which is widely applicable to concrete examples. A summary of this principle is given below.

As usual, if a set $\omega \subset \mathbb{R} \times \mathbb{R}^n$, then $\operatorname{int}\omega$ and $\partial\omega$ denote the interior and the boundary of ω, respectively.

Definition 1.1 (see [16]). Let the continuously differentiable functions $l_i(t,y)$, $i = 1, 2, \ldots, p$ and $m_j(t,y)$, $j = 1, 2, \ldots, q$, $p^2 + q^2 > 0$ be defined on some open set $\omega_0 \subset \mathbb{R} \times \mathbb{R}^n$. The set

$$\omega^* = \{(t,y) \in \omega_0 : l_i(t,y) < 0, m_j(t,y) < 0,\ i = 1,\ldots,p,\ j = 1,\ldots,q\} \tag{1.9}$$

is called a *regular polyfacial set* with respect to the system (1.7), provided that it is nonempty, if (α) to (γ) below hold.

(α) For $(t,\pi) \in \mathbb{R} \times C_r^n$ such that $(t+\theta, \pi(\theta)) \in \omega^*$ for $\theta \in [-r, 0)$, we have $(t,\pi) \in \Omega^*$.

(β) For all $i = 1, 2, \ldots, p$, all $(t,y) \in \partial\omega^*$ for which $l_i(t,y) = 0$, and all $\pi \in C_r^n$ for which $\pi(0) = y$ and $(t+\theta, \pi(\theta)) \in \omega^*$, $\theta \in [-r, 0)$. It follows that $Dl_i(t,y) > 0$, where

$$Dl_i(t,y) \equiv \sum_{k=1}^{n} \frac{\partial l_i(t,y)}{\partial y_k} f_k(t,\pi) + \frac{\partial l_i(t,y)}{\partial t}. \tag{1.10}$$

(γ) For all $j = 1, 2, \ldots, q$, all $(t,y) \in \partial\omega^*$ for which $m_j(t,y) = 0$, and all $\pi \in C_r^n$ for which $\pi(0) = y$ and $(t+\theta, \pi(\theta)) \in \omega^*$, $\theta \in [-r, 0)$. It follows that $Dm_j(t,y) < 0$, where

$$Dm_j(t,y) \equiv \sum_{k=1}^{n} \frac{\partial m_j(t,y)}{\partial y_k} f_k(t,\pi) + \frac{\partial m_j(t,y)}{\partial t}. \tag{1.11}$$

The elements $(t,\pi) \in \mathbb{R} \times C_r^n$ in the sequel are assumed to be such that $(t,\pi) \in \Omega^*$.

In the following definition, a set ω^* is an arbitrary set without any connection with a regular polyfacial set ω^* defined by (1.9) in Definition 1.1.

Definition 1.2. A *system of initial functions* p_{A,ω^*} with respect to the nonempty sets A and ω^*, where $A \subset \overline{\omega^*} \subset \mathbb{R} \times \mathbb{R}^n$ is defined as a continuous mapping $p : A \to C_r^n$ such that (α) and (β) below hold.

(α) If $z = (t, y) \in A \cap \operatorname{int} \omega^*$, then $(t + \theta, p(z)(\theta)) \in \omega^*$ for $\theta \in [-r, 0]$.

(β) If $z = (t, y) \in A \cap \partial\omega^*$, then $(t + \theta, p(z)(\theta)) \in \omega^*$ for $\theta \in [-r, 0)$ and $(t, p(z)(0)) = z$.

Definition 1.3 (see [17]). If $\mathcal{A} \subset \mathcal{B}$ are subsets of a topological space and $\pi : \mathcal{B} \to \mathcal{A}$ is a continuous mapping from $\mathcal{B}$ onto $\mathcal{A}$ such that $\pi(p) = p$ for every $p \in \mathcal{A}$, then π is said to be a *retraction* of $\mathcal{B}$ onto $\mathcal{A}$. When a retraction of $\mathcal{B}$ onto $\mathcal{A}$ exists, $\mathcal{A}$ is called a *retract* of $\mathcal{B}$.

The following lemma describes the main result of the paper [16].

Lemma 1.4. *Let $\omega^* \subset \omega_0$ be a regular polyfacial set with respect to the system* (1.7), *and let W be defined as follows:*

$$W = \{(t, y) \in \partial\omega^* : m_j(t, y) < 0,\ j = 1, 2, \ldots, q\}. \tag{1.12}$$

Let $Z \subset W \cup \omega^*$ be a given set such that $Z \cap W$ is a retract of W but not a retract of Z. Then for each fixed system of initial functions p_{Z,ω^*}, there is a point $z_0 = (\sigma_0, y_0) \in Z \cap \omega^*$ such that for the corresponding solution $y(\sigma_0, p(z_0))(t)$ of (1.7), *one has*

$$(t, y(\sigma_0, p(z_0))(t)) \in \omega^* \tag{1.13}$$

for each $t \in D_{\sigma_0, p(z_0)}$.

Remark 1.5. When Lemma 1.4 is applied, a lot of technical details should be fulfilled. In order to simplify necessary verifications, it is useful, without loss of generality, to vary the first coordinate t in definition of the set ω^* in (1.9) within a half-open interval open at the right. Then the set ω^* is not open, but tracing the proof of Lemma 1.4, it is easy to see that for such sets it remains valid. Such possibility is used below. We will apply similar remark and explanation to sets of the type Ω, Ω^* which serve as domains of definitions of functionals on the right-hand sides of equations considered.

For continuous vector functions

$$\rho^* = (\rho_1^*, \rho_2^*, \ldots, \rho_n^*), \qquad \delta^* = (\delta_1^*, \delta_2^*, \ldots, \delta_n^*) : [t_0 - r, \infty) \longrightarrow \mathbb{R}^n, \tag{1.14}$$

with $\rho^*(t) \ll \delta^*(t)$ for $t \in [t_0 - r, \infty)$ (the symbol $\ll$ here and below means that $\rho_i^*(t) < \delta_i^*(t)$ for all $i = 1, 2, \ldots, n$), continuously differentiable on $[t_0, \infty)$, we define the set

$$\omega^* := \{(t, y) : t \in [t_0, \infty), \rho^*(t) \ll y \ll \delta^*(t)\}. \tag{1.15}$$

In the sequel, we employ the following result from [18, Theorem 1], which is proved with the aid of the retract technique combined with Razumikhin's approach.

Theorem 1.6. *Let there be a* $p \in \{0, \ldots, n\}$ *such that*

(i) *if* $t \geq t_0$, $\phi \in C_r^n$ *and* $(t + \theta, \phi(\theta)) \in \omega^*$ *for any* $\theta \in [-r, 0)$, *then*

$$\begin{aligned} \left(\delta^{*i}\right)'(t) &< F^i(t, \phi), \quad \textit{when } \phi^i(0) = \delta^{*i}(t), \\ \left(\rho^{*i}\right)'(t) &> F^i(t, \phi), \quad \textit{when } \phi^i(0) = \rho^{*i}(t) \end{aligned} \tag{1.16}$$

for any $i = 1, 2, \ldots, p$, *(If* $p = 0$, *this condition is omitted.)*

(ii) *if* $t \geq t_0$, $\phi \in C_r^n$ *and* $(t + \theta, \phi(\theta)) \in \omega^*$ *for any* $\theta \in [-r, 0)$ *then*

$$\begin{aligned} \left(\rho^{*i}\right)'(t) &< F^i(t, \phi), \quad \textit{when } \phi^i(0) = \rho^{*i}(t), \\ \left(\delta^{*i}\right)'(t) &> F^i(t, \phi), \quad \textit{when } \phi^i(0) = \delta^{*i}(t) \end{aligned} \tag{1.17}$$

for any $i = p + 1, p + 2, \ldots, n$. *(If* $p = n$, *this condition is omitted.)*

Then, there exists an uncountable set $\mathcal{Y}$ *of solutions of* (1.7) *on* $[t_0 - r, \infty)$ *such that each* $y \in \mathcal{Y}$ *satisfies*

$$\rho^*(t) \ll y(t) \ll \delta^*(t), \quad t \in [t_0 - r, \infty). \tag{1.18}$$

1.2. Structure of Solutions of a Linear Equation

In this section we focus our attention to structure of solutions of scalar linear differential equation of the type (1.3) with variable bounded delay of the form

$$\dot{x}(t) = -c(t)x(t - \tau(t)) \tag{1.19}$$

with continuous functions $c : [t_0 - r, \infty) \to \mathbb{R}^+$ and $\tau : [t_0, \infty) \to (0, r]$.

In accordance with above definitions of positive or oscillatory solutions, we call a solution of (1.19) oscillatory if it has arbitrary large zeros, otherwise it is called nonoscillatory (positive or negative).

Let us mention properties of (1.19) which will be used later. Theorem 13 from [19] describes sufficient conditions for existence of positive solutions of (1.19) with nonzero limit.

Theorem 1.7 (see [19, Theorem 13]). *Linear equation* (1.19) *has a positive solution with nonzero limit if and only if*

$$\int^{\infty} c(t)\mathrm{d}t < \infty. \tag{1.20}$$

Remark 1.8. Tracing the proof of Theorem 1.7, we conclude that a positive solution $x = x(t)$ of (1.19) with nonzero limit exists on $[t_0 - r, \infty)$ if

$$\int_{t_0-r}^{\infty} c(t)\mathrm{d}t < 1. \tag{1.21}$$

The following theorem is a union of parts of results from [20] related to the structure formulas for solutions of (1.19).

Theorem 1.9. *Suppose the existence of a positive solution of* (1.19) *on* $[t_0 - r, \infty)$. *Then there exist two positive solutions* x_d *and* x_s *of* (1.19) *on* $[t_0 - r, \infty)$ *satisfying the relation*

$$\lim_{t\to\infty} \frac{x_s(t)}{x_d(t)} = 0 \tag{1.22}$$

such that every solution $x = x(t)$ *of* (1.19) *on* $[t_0 - r, \infty)$ *can be represented by the formula*

$$x(t) = Kx_d(t) + O(x_s(t)), \tag{1.23}$$

where the constant K *depends on* x.

The symbol O, applied in (1.23) and below, is the Landau order symbol frequently used in asymptotic analysis.

Moreover, Theorem 9 in [20] gives a possibility to replace the pair of solutions $x_d(t)$ and $x_s(t)$ in (1.23) by another pairs of solutions $\widetilde{x}_d(t)$ and $\widetilde{x}_s(t)$ if

$$\lim_{t\to\infty} \frac{\widetilde{x}_s(t)}{\widetilde{x}_d(t)} = 0 \tag{1.24}$$

as given in the following theorem.

Theorem 1.10. *Let* $\widetilde{x}_d(t)$ *and* $\widetilde{x}_s(t)$ *be positive solutions of* (1.19) *on* $[t_0 - r, \infty)$ *such that* (1.24) *holds. Then every solution* $x = x(t)$ *of* (1.19) *on* $[t_0 - r, \infty)$ *can be represented by the formula*

$$x(t) = K^*\widetilde{x}_d(t) + O(\widetilde{x}_s(t)), \tag{1.25}$$

where the constant K^* *depends on* x.

The next definition is based on the properties of solutions x_d, $\widetilde{x}_d$, x_s, and $\widetilde{x}_s$ described in Theorems 1.9 and 1.10.

Definition 1.11 (see [20, Definition 2]). Suppose that the positive solutions x_d and x_s of (1.19) on $[t_0 - r, \infty)$ satisfy the relation (1.22). Then, we call the solution x_d a *dominant* solution and the solution x_s a *subdominant* solution.

Due to linearity of (1.19), there are infinitely many dominant and subdominant solutions. Obviously, another pair of a dominant and a subdominant solutions is the pair $\widetilde{x}_d(t)$, $\widetilde{x}_s(t)$ in Theorem 1.10.

2. Main Results

Let us consider two auxiliary linear equations:

$$\dot{x}(t) = -C_A(t)x(t-r), \tag{2.1}$$

$$\dot{z}(t) = -C_B(t)z(t-r), \tag{2.2}$$

where $r \in \mathbb{R}^+$ and C_A, C_B are positive continuous functions on $[t_0 - r, \infty)$, $t_0 \in \mathbb{R}$. According to the Theorems 1.7 and 1.9, both (2.1) and (2.2) have two types of positive solutions (subdominant and dominant). Let us denote them $x_d(t)$, $x_s(t)$ for (2.1) and $z_d(t)$, $z_s(t)$ for (2.2), respectively, such that

$$\lim_{t\to\infty} \frac{x_s(t)}{x_d(t)} = 0, \qquad \lim_{t\to\infty} \frac{z_s(t)}{z_d(t)} = 0. \tag{2.3}$$

Without loss of generality, we can suppose that $x_s(t) < x_d(t)$ and $z_s(t) < z_d(t)$ on $[t_0 - r, \infty)$.

2.1. Auxiliary Linear Result

The next lemma states that if $z_d(t)$, $z_s(t)$ are dominant and subdominant solutions for (2.2), then there are dominant and subdominant solutions $x_d^*(t)$, $x_s^*(t)$ for (2.1) satisfying certain inequalities.

Lemma 2.1. *Let* (1.5) *be valid. Let $z_d(t)$, $z_s(t)$ be dominant and subdominant solutions for* (2.2)*. Then there are positive solutions $x_s^*(t)$, $x_d^*(t)$ of* (2.1) *on $[t_0 - r, \infty)$ such that:*

(a) $x_s^*(t) < z_s(t)$, $t \in [t_0 - r, \infty)$,

(b) $z_d(t) < x_d^*(t)$, $t \in [t_0 - r, \infty)$,

(c) *$x_d^*(t)$ and $x_s^*(t)$ are dominant and subdominant solutions for* (2.1).

Proof. (a) To prove the part (a), we employ Theorem 1.6 with $p = n = 1$; that is, we apply the case (i). Consider (2.1), set $F(t, \phi) := -C_A(t)\phi(-r)$, $\rho^*(t) := 0$, $\delta^*(t) := z_s(t)$, and assume (see the case (i)):

$$0 < \phi(\theta) < z_s(t+\theta), \quad \theta \in [-r, 0), \;\; \phi(0) = z_s(t), \;\; t \geq t_0. \tag{2.4}$$

Now we have to verify the inequalities (1.16), that is, in our case:

$$\begin{aligned} F(t,\phi) - (\delta^*)'(t) &= -C_A(t)\phi(-r) - (\delta^*)'(t) \\ &= -C_A(t)\phi(-r) - z_s'(t) \\ &= -C_A(t)\phi(-r) + C_B(t)z_s(t-r) \\ &\geq (\text{we use (1.5)}) \\ &\geq -C_B(t)\phi(-r) + C_B(t)z_s(t-r) \\ &> C_B(t)[z_s(t-r) - z_s(t-r)] = 0 \end{aligned} \tag{2.5}$$

and $F(t,\phi) > (\delta^*)'(t)$ if $t \in [t_0,\infty)$. Further, we have

$$-F(t,\phi) + (\rho^*)'(t) = C_A(t)\phi(-r) + 0 = C_A(t)\phi(-r) > 0 \tag{2.6}$$

and $F(t,\phi) < (\rho^*)'(t)$ if $t \in [t_0,\infty)$. Since both inequalities are fulfilled and all assumptions of Theorem 1.6 are satisfied for the case in question, there exists a solution $x_s^*(t)$ of (2.1) on $[t_0 - r,\infty)$ such that $x_s^*(t) < z_s(t)$ for $t \in [t_0 - r,\infty)$.

(b) To prove the part (b), we consider a solution $x = x_d^*(t)$ of the following initial problem:

$$\dot{x}(t) = -C_A(t)x(t-r), \quad t \in [t_0 - r,\infty), \tag{2.7}$$

$$x(t) = z_d(t), \quad t \in [t_0 - r, t_0]. \tag{2.8}$$

Now, let us define a function

$$W(t,x) = z_d(t) - x(t), \quad t \in [t_0 - r,\infty). \tag{2.9}$$

We find the sign of the full derivative of W along the trajectories of (2.7) if $t \in [t_0, t_0 + r]$:

$$\begin{aligned} \left.\frac{\mathrm{d}W(t,x)}{\mathrm{d}t}\right|_{t\in[t_0,t_0+r]} &= -C_B(t)z_d(t-r) + C_A(t)x(t-r) \\ &= (\text{due to (2.8)}) \\ &= -C_B(t)z_d(t-r) + C_A(t)z_d(t-r) \\ &\leq [C_A(t) - C_B(t)]z_d(t-r) \leq (\text{due to (1.5)}) \leq 0. \end{aligned} \tag{2.10}$$

It means that function W is nonincreasing and it holds

$$\begin{aligned} W(t_0, x(t_0)) &= z_d(t_0) - x(t_0) = z_d(t_0) - z_d(t_0) \\ &= 0 \geq W(t_0+\varepsilon, x(t_0+\varepsilon)) = z_d(t_0+\varepsilon) - x(t_0+\varepsilon), \quad \varepsilon \in [0,r], \end{aligned} \tag{2.11}$$

and hence $z_d(t_0+\varepsilon) \le x(t_0+\varepsilon)$. It will be showed that this inequality holds also for every $t > t_0 + r$.

On the contrary, let us suppose that the inequality is not true, that is, there exists a point $t = t^{**}$ such that $z_d(t^{**}) > x(t^{**})$. Then there exists a point $t^* \in [t_0, t^{**})$ such that $z_d(t^*) < x(t^*)$, otherwise $z_d(t) \equiv x(t)$ on $[t_0, t^{**}]$. Without loss of generality, we can suppose that $x(t) \equiv z_d(t)$ on $[t_0, t^{***}]$ with a $t^{***} \in [t_0, t^*)$ and $x(t) > z_d(t)$ on (t^{***}, t^*). Then, there exists a point $t^\diamond \in (t^{***}, t^*)$ such that $x(t^\diamond) = Kz_d(t^\diamond)$ for a constant $K > 1$ and

$$Kz_d(t) > x(t), \quad \text{for } t \in [t_0, t^\diamond). \tag{2.12}$$

Hence, for a function $W^*(t,x)$ defined as $W^*(t,x) := Kz_d(t) - x(t)$, $t \in [t_0, t^\diamond]$, we get

$$\begin{aligned} \left.\frac{\mathrm{d}W^*(t,x)}{\mathrm{d}t}\right|_{t=t^\diamond} &= K(-C_B(t)z_d(t-r)) + C_A(t)x(t-r) \\ &< (\text{due to } (2.12)) \\ &< K(-C_B(t)z_d(t-r)) + C_A(t)Kz_d(t-r) \\ &= Kz_d(t-r)[C_A(t) - C_B(t)] \le \big(\text{by } (1.5)\big) \le 0. \end{aligned} \tag{2.13}$$

It means that $Kz_d(t) < x(t)$ on a right-hand neighborhood of $t^\diamond$. This is a contradiction with inequality

$$z_d(t) < Kz_d(t) < x(t), \tag{2.14}$$

hence it is proved that the existence of a solution $x_d^*(t)$ satisfies $z_d(t) < x_d^*(t)$ on $[t_0 - r, \infty)$.

(c) To prove the part (c), we consider $\lim_{t\to\infty} x_s^*(t)/x_d^*(t)$. Due to (a) and (b), we get

$$0 \le \lim_{t\to\infty} \frac{x_s^*(t)}{x_d^*(t)} \le \lim_{t\to\infty} \frac{z_s(t)}{z_d(t)} = 0, \tag{2.15}$$

and $x_d^*(t)$ and $x_s^*(t)$ are (by Definition 1.11) dominant and subdominant solutions for (2.1). □

2.2. Existence of Positive Solutions of (1.3)

The next theorems state that there exist two classes of positive solutions of (1.3) such that graphs of each solution of the first class are between graphs of dominant solutions of (2.1) and (2.2), and graphs of each solution of the second class are between graphs of subdominant solutions of (2.1) and (2.2), respectively. It means that we prove there are two classes of asymptotically different positive solutions of (1.3). Without loss of generality (see Remark 1.5), we put $\Omega := [t_0, \infty) \times C_r^1$. In the following, we will use our main supposition (1.4); that is, we assume that for $(t, \phi) \in \Omega$ inequalities,

$$C_A(t)\phi(-r) \le f(t,\phi) \le C_B(t)\phi(-r) \tag{2.16}$$

hold, where ϕ is supposed to be positive.

Theorem 2.2. *Let $f : \Omega \to \mathbb{R}$ be a continuous quasibounded functional. Let inequality* (1.5) *be valid, and* (2.16) *holds for any $(t,\phi) \in \Omega$ with $\phi(\theta) > 0$, $\theta \in [-r,0]$. Let $x(t)$ be a positive solution of* (2.1) *on $[t_0 - r, \infty)$, and let $z(t)$ be a positive solution of* (2.2) *on $[t_0 - r, \infty)$ such that $x(t) < z(t)$ on $[t_0 - r, \infty)$. Then there exists an uncountable set $\mathcal{Y}$ of positive solutions of* (1.3) *on $[t_0 - r, \infty)$ such that each solution $y \in \mathcal{Y}$ satisfies*

$$x(t) < y(t) < z(t) \tag{2.17}$$

for $t \in [t_0 - r, \infty)$.

Proof. To prove this theorem, we employ Theorem 1.6 with $p = n = 1$; that is, we apply the case (i). Set $F(t, y_t) := -f(t, y_t)$, $\rho^*(t) := x(t)$, $\delta^*(t) := z(t)$; hence, the set ω^* will be defined as

$$\omega^* := \{(t,y) : t \in [t_0 - r, \infty),\ x(t) < y(t) < z(t)\}. \tag{2.18}$$

Now, we have to verify the inequalities (1.16). In our case

$$\begin{aligned}
F(t,\phi) - (\delta^*)'(t) &= -f(t,\phi) - (\delta^*)'(t) \\
&= -f(t,\phi) - z'(t) \\
&= -f(t,\phi) + C_B(t)z(t-r) \\
&\geq \text{(we use (2.16))} \\
&\geq -C_B(t)\phi(-r) + C_B(t)z(t-r) \\
&> \left(\text{we use (2.18)} : \phi(-r) < z(t-r)\right) \\
&> C_B(t)[z(t-r) - z(t-r)] = 0, \\
-F(t,\phi) + (\rho^*)'(t) &= f(t,\phi) + (\rho^*)'(t) \\
&= f(t,\phi) + x'(t) \\
&= f(t,\phi) - C_A(t)x(t-r) \\
&\geq \text{(we use (2.16))} \\
&\geq C_A(t)\phi(-r) - C_A(t)x(t-r) \\
&> \left(\text{we use (2.18)} : \phi(-r) > x(t-r)\right) \\
&> C_A(t)[x(t-r) - x(t-r)] = 0.
\end{aligned} \tag{2.19}$$

Therefore,

$$\begin{aligned}
F(t,\phi) - (\delta^*)'(t) &> 0, \\
-F(t,\phi) + (\rho^*)'(t) &> 0.
\end{aligned} \tag{2.20}$$

Both inequalities (1.16) are fulfilled, and all assumptions of Theorem 1.6 are satisfied for the case in question. There exists class of positive solutions $\mathcal{Y}$ of (1.3) on $[t_0 - r, \infty)$ that for each solution $y \in \mathcal{Y}$ from this class it is satisfied that $x(t) < y(t) < z(t)$ for $t \in [t_0 - r, \infty)$. □

Corollary 2.3. *Let, in accordance with Lemma 2.1, $x_s(t)$ be the subdominant solution of* (2.1)*, and let $z_s(t)$ be the subdominant solution of* (2.2)*, that is, $x_s(t) < z_s(t)$ on $[t_0 - r, \infty)$. Then, there exists an uncountable set $\mathcal{Y}_s$ of positive solutions of* (1.3) *on $[t_0 - r, \infty)$ such that each solution $y_s \in \mathcal{Y}_s$ satisfies*

$$x_s(t) < y_s(t) < z_s(t). \tag{2.21}$$

If inequality (1.6) holds, then dominant solutions $x_d(t)$ of (2.1) and $z_d(t)$ of (2.2) have finite positive limits

$$\begin{aligned} C_x &:= \lim_{t\to\infty} x_d(t), \quad C_x > 0, \\ C_z &:= \lim_{t\to\infty} z_d(t), \quad C_z > 0. \end{aligned} \tag{2.22}$$

This is a simple consequence of positivity of solutions $x_d(t)$, $z_d(t)$ and properties of dominant and subdominant solutions (see Theorem 1.7, Remark 1.8, Theorem 1.9, formulas (1.22)–(1.25) and (2.3)). Then, due to linearity of (2.1) and (2.2), it is clear that there are dominant solutions $x_d(t)$, $z_d(t)$ of both equations such that $z_d(t) < x_d(t)$ on $[t_0 - r, \infty)$. In the following lemma, we without loss of generality suppose that $x_d(t)$ and $z_d(t)$ are such solutions and their initial functions are nonincreasing on initial interval $[t_0 - r, t_0]$. We will need constants M and L satisfying

$$\begin{aligned} M > M^* &:= \frac{x_d(t_0 - r)}{C_z}, \\ L > L^* &:= \frac{M z_d(t_0 - r)}{C_x}. \end{aligned} \tag{2.23}$$

Lemma 2.4. *Let $f : \Omega \to \mathbb{R}$ be a continuous quasibounded functional. Let inequalities* (1.5) *and* (1.6) *be valid, and* (2.16) *holds for any $(t, \phi) \in \Omega$ with $\phi(\theta) > 0$, $\theta \in [-r, 0]$. Let $x_d(t)$, $t \in [t_0 - r, \infty)$ be a dominant solution of* (2.1)*, nonincreasing on $[t_0 - r, t_0]$, and let $z_d(t)$, $t \in [t_0 - r, \infty)$ be a dominant solution of* (2.2)*, nonincreasing on $[t_0 - r, t_0]$, such that $z_d(t) < x_d(t)$, $t \in [t_0 - r, \infty)$. Then there exists another dominant solution $z_d^*(t)$ of* (2.2) *and a positive solution $y = y_d(t)$ of* (1.3) *on $[t_0 - r, \infty)$ such that it holds that*

$$x_d(t) < y_d(t) < z_d^*(t) \tag{2.24}$$

for $t \in [t_0 - r, \infty)$ and $z_d^*(t) = M z_d(t)$.

Proof. Both dominant solutions $x_d(t)$ and $z_d(t)$, of (2.1) and (2.2), respectively, have nonzero positive limits C_x and C_z. From linearity of (2.1) and (2.2), it follows that solutions multiplied by an arbitrary constant are also solutions of (2.1) and (2.2), respectively. It holds that

$$z_d^*(t_0 - r) = Mz_d(t_0 - r) \geq Mz_d(t) = z_d^*(t) > MC_z > x_d(t_0 - r) \geq x_d(t), \tag{2.25}$$

where $t \in [t_0 - r, \infty)$.

Now, we define the set ω^* in the same way as (2.18) in the proof of Theorem 2.2, but with $x_d(t)$ instead of $x(t)$ and with $z_d^*(t)$ instead of $z(t)$, that is,

$$\omega^* := \{(t, y) : t \in [t_0 - r, \infty), x_d(t) < y(t) < z_d^*(t)\}. \tag{2.26}$$

According to the Theorem 2.2 (with $x_d(t)$ instead of $x(t)$ and with $z_d^*(t)$ instead of $z(t)$), it is visible that there exists a positive solution $y = y_d(t)$ of (1.3) satisfying

$$x_d(t) < y_d(t) < z_d^*(t), \tag{2.27}$$

where $t \in [t_0, \infty)$; that is, inequalities (2.24) hold. □

Theorem 2.5. *Let all suppositions of Lemma 2.4 be valid, and let $y_d(t)$ be a solution of* (1.3) *satisfying inequalities* (2.24)*. Then, there exists a positive solution $x_d^{**}(t)$ of* (2.1) *on* $[t_0 - r, \infty)$ *satisfying*

$$z_d(t) < y_d(t) < x_d^{**}(t), \tag{2.28}$$

where $x_d^{**}(t) = Lx_d(t)$ and $t \in [t_0 - r, \infty)$.

Proof. Multiplying solution $x_d(t)$ by the constant L, we have

$$Lx_d(t) > LC_x > Mz_d(t_0 - r). \tag{2.29}$$

Using (2.29) and (2.24), we get

$$x_d^{**}(t) = Lx_d(t) > Mz_d(t_0 - r) = z_d^*(t_0 - r) > z_d^*(t) > y_d(t) > x_d(t) > z_d(t), \tag{2.30}$$

where $t \in [t_0 - r, \infty)$. Hence, there exists a solution $y_d(t)$ of (1.3) such that inequalities (2.28) hold. □

2.3. Asymptotically Different Behavior of Positive Solutions of (1.3)

Somewhat reformulating the statement of Theorem 2.5, we can define a class of positive solutions $\mathcal{Y}_d$ of (1.3) such that every solution $y_d \in \mathcal{Y}_d$ is defined on $[t_0 - r, \infty)$ and satisfies

$$Cz_d(t) < y_d(t) < Cx_d^{**}(t), \tag{2.31}$$

where $t \in [t_0 - r, \infty)$ for a positive constant C and, for every positive constant C, there exists a solution $y_d \in \mathcal{Y}_d$ satisfying (2.31) on $[t_0 - r, \infty)$.

The following theorem states that positive solutions $y_s(t)$ and $y_d(t)$ of (1.3) have a different order of vanishing.

Theorem 2.6. *Let all the assumptions of Corollary 2.3 and Theorem 2.5 be met. Then there exist two classes $\mathcal{Y}_s$ and $\mathcal{Y}_d$ of positive solutions of* (1.3) *described by inequalities* (2.21) *and* (2.31). *Every two solutions y_s, y_d, such that $y_s \in \mathcal{Y}_s$ and $y_d \in \mathcal{Y}_d$, have asymptotically different behavior, that is,*

$$\lim_{t \to +\infty} \frac{y_s(t)}{y_d(t)} = 0. \tag{2.32}$$

Proof. Let the solution $y_s(t)$ be the one specified in Corollary 2.3 and the solution $y_d(t)$ specified by (2.31) with a positive constant C. Now let us verify that (2.32) holds. With the aid of inequalities (2.21) and (2.31), we get

$$0 \le \lim_{t \to +\infty} \frac{y_s(t)}{y_d(t)} \le \lim_{t \to +\infty} \frac{z_s(t)}{C z_d(t)} = 0 \tag{2.33}$$

in accordance with (1.22), since $z_s(t)$ and $z_d(t)$ are positive (subdominant and dominant) solutions of linear equation (2.2). □

Another final statement, being a consequence of Lemma 2.1 and Theorems 2.2 and 2.5, is the following.

Theorem 2.7. *Let $f : \Omega \to \mathbb{R}$ be a continuous quasibounded functional. Let inequalities* (1.5) *and* (1.6) *be valid, and* (2.16) *holds for any $(t, \phi) \in \Omega$ with $\phi(\theta) > 0$, $\theta \in [-r, 0]$. Then on $[t_0 - r, \infty)$ there exist*

(a) *dominant and subdominant solutions $x_d(t)$, $x_s(t)$ of* (2.1),

(b) *dominant and subdominant solutions $z_d(t)$, $z_s(t)$ of* (2.2),

(c) *solutions $y_d(t)$, $y_s(t)$ of* (1.3)

such that

$$0 < x_s(t) < y_s(t) < z_s(t) < z_d(t) < y_d(t) < x_d(t), \tag{2.34}$$

$$\lim_{t \to \infty} \frac{x_s(t)}{x_d(t)} = \lim_{t \to \infty} \frac{z_s(t)}{z_d(t)} = \lim_{t \to \infty} \frac{y_s(t)}{y_d(t)} = 0. \tag{2.35}$$

Example 2.8. Let (1.3) be reduced to

$$\dot{y}(t) = -f(t, y_t) := -3t \exp\left(-3t + \frac{1}{2}\cos\left(ty(t-1)\right)\right) \cdot y(t-1), \tag{2.36}$$

and let auxiliary linear equations (2.1) and (2.2) be reduced to

$$\dot{x}(t) = -4t \exp(2-4t) \cdot x(t-1), \tag{2.37}$$

$$\dot{z}(t) = -2t \exp(1-2t) \cdot z(t-1), \tag{2.38}$$

that is,

$$C_A(t) := 4t \exp(2-4t), \qquad C_B(t) := 2t \exp(1-2t), \qquad r = 1. \tag{2.39}$$

Let t_0 be sufficiently large. Inequalities (1.5), (1.6), and (2.16) hold. In view of linearity and by Remark 1.8, we conclude that there exist dominant solutions $x_d(t)$ of (2.37) and $z_d(t)$ of (2.38) such that

$$\lim_{t\to\infty} x_d(t) = 11, \quad \lim_{t\to\infty} z_d(t) = 2, \quad z_d(t) < x_d(t), \ t \in [t_0-1, \infty). \tag{2.40}$$

Moreover, there exist subdominant solutions $x_s(t)$ of (2.37) and $z_s(t)$ of (2.38) such that $x_s(t) < z_s(t)$, $t \in [t_0-1, \infty)$ which are defined as

$$x_s(t) := \exp\left(-2t^2\right), \qquad z_s(t) := \exp\left(-t^2\right). \tag{2.41}$$

By Theorem 2.7, we conclude that there exist solutions $y_s(t)$ and $y_d(t)$ of (2.36) satisfying inequalities (2.34), and (without loss of generality) inequalities

$$0 < x_s(t) = \exp\left(-2t^2\right) < y_s(t) < z_s(t) = \exp\left(-t^2\right) < 1 \le z_d(t) < y_d(t) < 10 \le x_d(t) \tag{2.42}$$

hold on $[t_0-1, \infty)$.

3. Conclusions and Open Problems

The following problems were not answered in the paper and present interesting topics for investigation.

Open Problem 3.1. In Lemma 2.4 and Theorems 2.5–2.7 we used the convergence assumption (1.6) being, without loss of generality, equivalent to

$$\int^{\infty} C_B(t) \mathrm{d}t < \infty. \tag{3.1}$$

It is an open question whether similar results could be proved if the integral is divergent, that is, if

$$\int^{\infty} C_B(t) \mathrm{d}t = \infty. \tag{3.2}$$

Open Problem 3.2. Dominant and subdominant solutions are used for representation of family of all solutions of scalar linear differential delayed equation, for example, by formula (1.25). Investigation in this line of the role of solutions $y_d(t)$ and $y_s(t)$ of (1.3) (see Theorems 2.6 and 2.7) is an important question. Namely, it seems to be an interesting question to establish sufficient conditions for the right-hand side of (1.3) such that its every solution $y = y(t)$ can be represented on $[t_0 - r, \infty)$ by the formula

$$y(t) = Ky_d(t) + O\big(y_s(t)\big), \tag{3.3}$$

where the constant K depends only on $y(t)$.

Open Problem 3.3. The notions dominant and subdominant solutions are in the cited papers defined for scalar differential delayed equations only. It is a rather interesting question if the results presented can be enlarged to systems of differential delayed equations.

Remark 3.4. Except for papers and books mentioned in this paper we refer, for example, to sources [21–23], treating related problems as well. Note that the topic is connected with similar questions for discrete equations (e.g., [24–27]).

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Research Article

Existence Conditions for Bounded Solutions of Weakly Perturbed Linear Impulsive Systems

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The weakly perturbed linear nonhomogeneous impulsive systems in the form $\dot{x} = A(t)x + \varepsilon A_1(t)x + f(t)$, $t \in \mathbb{R}$, $t \notin \mathcal{T} := \{\tau_i\}_{\mathbb{Z}}$, $\Delta x|_{t=\tau_i} = \gamma_i + \varepsilon A_{1i}x(\tau_i-)$, $\tau_i \in \mathcal{T} \subset \mathbb{R}$, $\gamma_i \in \mathbb{R}^n$, and $i \in \mathbb{Z}$ are considered. Under the assumption that the generating system (for $\varepsilon = 0$) does not have solutions bounded on the entire real axis for some nonhomogeneities and using the Vishik-Lyusternik method, we establish conditions for the existence of solutions of these systems bounded on the entire real axis in the form of a Laurent series in powers of small parameter ε with finitely many terms with negative powers of ε, and we suggest an algorithm of construction of these solutions.

1. Introduction

In this contribution we study the problem of existence and construction of solutions of weakly perturbed linear differential systems with impulsive action bounded on the entire real axis. The application of the theory of differential systems with impulsive action (developed in [1–3]), the well-known results on the splitting index by Sacker [4] and by Palmer [5] on the Fredholm property of bounded solutions of linear systems of ordinary differential equations [6–9], the theory of pseudoinverse matrices [10] and results obtained in analyzing boundary-value problems for ordinary differential equations (see [10–12]), enables us to obtain existence conditions and to propose an algorithm for the construction of solutions bounded on the entire real axis of weakly perturbed linear impulsive differential systems.

2. Initial Problem

We consider the problem of existence and construction of solutions bounded on the entire real axis of linear systems of ordinary differential equations with impulsive action at fixed points of time

$$\begin{gathered}\dot{x} = A(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathfrak{T}, \\ \Delta x|_{t=\tau_i} = \gamma_i, \quad \tau_i \in \mathfrak{T},\ i \in \mathbb{Z},\end{gathered} \tag{2.1}$$

where $A \in BC_{\mathfrak{T}}(\mathbb{R})$ is an $n \times n$ matrix of functions, $f \in BC_{\mathfrak{T}}(\mathbb{R})$ is an $n \times 1$ vector function, $BC_{\mathfrak{T}}(\mathbb{R})$ is the Banach space of real vector functions bounded on $\mathbb{R}$ and left-continuous for $t \in \mathbb{R}$ with discontinuities of the first kind at $t \in \mathfrak{T} := \{\tau_i\}_{\mathbb{Z}}$ with the norm: $\|x\|_{BC_{\mathfrak{T}}(\mathbb{R})} := \sup_{t\in\mathbb{R}}\|x(t)\|$, γ_i are n-dimensional column constant vectors: $\gamma_i \in \mathbb{R}^n$; $\cdots < \tau_{-2} < \tau_{-1} < \tau_0 = 0 < \tau_1 < \tau_2 < \cdots$, and $\Delta x|_{t=\tau_i} := x(\tau_i+) - x(\tau_i-)$.

The solution $x(t)$ of the system (2.1) is sought in the Banach space of n-dimensional bounded on $\mathbb{R}$ and piecewise continuously differentiable vector functions with discontinuities of the first kind at $t \in \mathfrak{T}:\ x \in BC^1_{\mathfrak{T}}(\mathbb{R})$.

Parallel with the nonhomogeneous impulsive system (2.1), we consider the corresponding homogeneous system

$$\dot{x} = A(t)x, \qquad \Delta x|_{t=\tau_i} = 0, \tag{2.2}$$

which is the homogeneous system without impulses, and let $X(t)$ be the fundamental matrix of (2.2) such that $X(0) = I$.

Assume that the homogeneous system (2.2) is exponentially dichotomous (e-dichotomous) [5, 10] on semiaxes $\mathbb{R}_- = (-\infty, 0]$ and $\mathbb{R}_+ = [0, \infty)$, that is, there exist projectors P and Q ($P^2 = P,\ Q^2 = Q$) and constants $K_i \geq 1$, $\alpha_i > 0$ ($i = 1, 2$) such that the following inequalities are satisfied:

$$\begin{gathered}\left\|X(t)PX^{-1}(s)\right\| \leq K_1 e^{-\alpha_1(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-P)X^{-1}(s)\right\| \leq K_1 e^{-\alpha_1(s-t)}, \quad s \geq t,\ t, s \in \mathbb{R}_+, \\ \left\|X(t)QX^{-1}(s)\right\| \leq K_2 e^{-\alpha_2(t-s)}, \quad t \geq s, \\ \left\|X(t)(I-Q)X^{-1}(s)\right\| \leq K_2 e^{-\alpha_2(s-t)}, \quad s \geq t,\ t, s \in \mathbb{R}_-.\end{gathered} \tag{2.3}$$

For getting the solution $x \in BC^1_{\mathcal{T}}(\mathbb{R})$ bounded on the entire axis, we assume that $t = 0 \notin \mathcal{T}$, that is, $x(0+) - x(0-) = \gamma_0 = 0$.

We use the following notation: $D = P - (I - Q)$; D^+ is a Moore-Penrose pseudoinverse matrix to D; P_D and P_{D^*} are $n \times n$ matrices (orthoprojectors) projecting $\mathbb{R}^n$ onto $N(D) = \ker D$ and onto $N(D^*) = \ker D^*$, respectively, that is, $P_D : \mathbb{R}^n \to N(D)$, $P_D^2 = P_D = P_D^*$, and $P_{D^*} : \mathbb{R}^n \to N(D^*)$, $P_{D^*}^2 = P_{D^*} = P_{D^*}^*$; $H(t) = [P_{D^*}Q]X^{-1}(t)$; $d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I - P)]$ and $r = \operatorname{rank}[PP_D] = \operatorname{rank}[(I - Q)P_D]$.

The existence conditions and the structure of solutions of system (2.1) bounded on the entire real axis was analyzed in [13]. Here the following theorem was formulated and proved.

Theorem 2.1. *Assume that the linear nonhomogeneous impulsive differential system* (2.1) *has the corresponding homogeneous system* (2.2) *e-dichotomous on the semiaxes* $\mathbb{R}_- = (-\infty, 0]$ *and* $\mathbb{R}_+ = [0, \infty)$ *with projectors* P *and* Q*, respectively. Then the homogeneous system* (2.2) *has exactly* r *linearly independent solutions bounded on the entire real axis. If nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *satisfy* d *linearly independent conditions*

$$\int_{-\infty}^{\infty} H_d(t) f(t) dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)\gamma_i = 0, \tag{2.4}$$

then the nonhomogeneous system (2.1) *possesses an* r*-parameter family of linearly independent solutions bounded on* $\mathbb{R}$ *in the form*

$$x(t, c_r) = X_r(t)c_r + \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix} \right)(t), \quad \forall c_r \in \mathbb{R}^r. \tag{2.5}$$

Here, $H_d(t) = [P_{D^*}Q]_d X^{-1}(t)$ is a $d \times n$ matrix formed by a complete system of d linearly independent rows of matrix $H(t)$,

$$X_r(t) := X(t)[PP_D]_r = X(t)[(I - Q)P_D]_r \tag{2.6}$$

is an $n \times r$ matrix formed by a complete system of r linearly independent solutions bounded on $\mathbb{R}$ of homogeneous system (2.2), and $\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix} \right)(t)$ is the generalized Green operator of the problem of finding bounded solutions of the nonhomogeneous impulsive system (2.1), acting upon $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, defined by the formula

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(t) = X(t)\begin{cases} \displaystyle\int_0^t PX^{-1}(s)f(s)ds - \int_t^{\infty}(I-P)X^{-1}(s)f(s)ds \\ \displaystyle\quad +\sum_{i=1}^{j} PX^{-1}(\tau_i)\gamma i - \sum_{i=j+1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad +PD^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \displaystyle\qquad \left. +\sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \geq 0; \\ \displaystyle\int_{-\infty}^{t} QX^{-1}(s)f(s)ds - \int_t^{0}(I-Q)X^{-1}(s)f(s)ds \\ \displaystyle\quad +\sum_{i=-\infty}^{-(j+1)} QX^{-1}(\tau_i)\gamma_i - \sum_{i=-j}^{-1}(I-Q)X^{-1}(\tau_i)\gamma_i \\ \displaystyle\quad +(I-Q)D^{+}\left\{\int_{-\infty}^{0} QX^{-1}(s)f(s)ds + \int_0^{\infty}(I-P)X^{-1}(s)f(s)ds \right. \\ \displaystyle\qquad \left. +\sum_{i=-\infty}^{-1} QX^{-1}(\tau_i)\gamma_i + \sum_{i=1}^{\infty}(I-P)X^{-1}(\tau_i)\gamma_i \right\}, & t \leq 0, \end{cases} \tag{2.7}$$

with the following property

$$\left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0-) - \left(G\begin{bmatrix} f \\ \gamma_i \end{bmatrix}\right)(0+) = \int_{-\infty}^{\infty} H(t)f(t)dt + \sum_{i=-\infty}^{\infty} H(\tau_i)\gamma_i. \tag{2.8}$$

These results are required to establish new conditions for the existence of solutions of weakly perturbed linear impulsive systems bounded on the entire real axis.

3. Perturbed Problems

Consider a weakly perturbed nonhomogeneous linear impulsive system in the form

$$\begin{aligned} \dot{x} &= A(t)x + \varepsilon A_1(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T}, \\ \Delta x|_{t=\tau_i} &= \gamma_i + \varepsilon A_{1i}x(\tau_i -), \quad \tau_i \in \mathcal{T},\ \gamma_i \in \mathbb{R}^n,\ i \in \mathbb{Z}, \end{aligned} \tag{3.1}$$

where $A_1 \in BC_{\mathcal{T}}(\mathbb{R})$ is an $n \times n$ matrix of functions, A_{1i} are $n \times n$ constant matrices.

Assume that the condition of solvability (2.4) of the generating system (2.1) (obtained from system (3.1) for $\varepsilon = 0$) is not satisfied for all nonhomogeneities $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, that is, system (2.1) does not have solutions bounded on the entire real axis. Therefore, we analyze whether the system (2.1) can be made solvable by introducing linear perturbations to the differential system and to the pulsed conditions. Also it is important to determine perturbations $A_1(t)$ and A_{1i} required to make the problem (3.1) solvable in the space of functions bounded on the entire real axis, that is, it is necessary to specify pertubations for which the corresponding homogeneous system

$$\dot{x} = A(t)x + \varepsilon A_1(t)x, \quad t \in \mathbb{R} \setminus \mathcal{T},$$
$$\Delta x|_{t=\tau_i} = \varepsilon A_{1i}x(\tau_i -), \quad \tau_i \in \mathcal{T}, \; i \in \mathbb{Z}, \tag{3.2}$$

turns into a system e-trichotomous or e-dichotomous on the entire real axis [10].

We show that this problem can be solved using the $d \times r$ matrix

$$B_0 = \int_{-\infty}^{\infty} H_d(t)A_1(t)X_r(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}X_r(\tau_i -), \tag{3.3}$$

constructed with the coefficients of the system (3.1). The Vishik-Lyusternik method developed in [14] enables us to establish conditions under which a solution of impulsive system (3.1) can be represented by a function bounded on the entire real axis in the form of a Laurent series in powers of the small fixed parameter ε with finitely many terms with negative powers of ε.

We use the following notation: B_0^+ is the unique matrix pseudoinverse to B_0 in the Moore-Penrose sense, P_{B_0} is the $r \times r$ matrix (orthoprojector) projecting the space R^r to the null space $N(B_0)$ of the $d \times r$ matrix B_0, that is, $P_{B_0}: R^r \to N(B_0)$, and $P_{B_0^*}$ is the $d \times d$ matrix (orthoprojector) projecting the space $\mathbb{R}^d$ to the null space $N(B_0^*)$ of the $r \times d$ matrix B_0^* ($B_0^* = B^T$), that is, $P_{B_0^*} : \mathbb{R}^d \to N(B_0^*)$.

Now we formulate and prove a theorem that enables us to solve indicated problem.

Theorem 3.1. *Suppose that the system* (3.1) *satisfies the conditions imposed above, and the homogeneous system* (2.2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively. Let nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *be given such that the condition* (2.4) *is not satisfied and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If*

$$P_{B_0^*} = 0, \tag{3.4}$$

then the system (3.2) *is e-trichotomous on* $\mathbb{R}$ *and, for all nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$*, the system* (3.1) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of a series*

$$x(t, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k x_k(t), \tag{3.5}$$

uniformly convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*]$.

Here, ε_* is a proper constant characterizing the range of convergence of the series (3.5) and the coefficients $x_k(t)$ of the series (3.5) are determined from the corresponding impulsive systems as

$$\begin{gathered}
x_k(t) = x_k(t,c_k) = X_r(t)c_k + \left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x(\tau_i-,c_{k-1}) \end{bmatrix}\right)(t) \quad \text{for } k = 1,2,\dots, \\
c_k = -B_0^+\left[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x_{k-1}(\tau_i-,c_{k-1}) \end{bmatrix}\right)(t)dt\right. \\
\left. + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot,c_{k-1}) \\ A_{1i}x_{k-1}(\cdot,c_{k-1}) \end{bmatrix}\right)(\tau_i-)\right], \\
x_{-1}(t) = x_{-1}(t,c_{-1}) = X_r(t)c_{-1}, \quad c_{-1} = B_0^+\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\}, \\
x_0(t) = x_0(t,c_0) = X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)X_r(t)c_{-1} + f(\cdot) \\ \gamma_i + A_{1i}X_r(\tau_i-)c_{-1} \end{bmatrix}\right)(t), \\
c_0 = -B_0^+\left[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)dt\right. \\
\left. + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\right].
\end{gathered} \tag{3.6}$$

Proof. We suppose that the problem (3.1) has a solution in the form of a Laurent series (3.5). We substitute this solution into the system (3.1) and equate the coefficients at the same powers of ε. The problem of determination of the coefficient $x_{-1}(t)$ of the term with ε^{-1} in series (3.5) is reduced to the problem of finding solutions of homogeneous system without impulses

$$\begin{gathered}
\dot{x}_{-1} = A(t)x_{-1}, \quad t \notin \mathcal{T}, \\
\Delta x_{-1}|_{t=\tau_i} = 0, \quad i \in \mathbb{Z},
\end{gathered} \tag{3.7}$$

bounded on the entire real axis. According to the Theorem 2.1, the homogeneous system (3.7) possesses r-parameter family of solutions

$$x_{-1}(t,c_{-1}) = X_r(t)c_{-1} \tag{3.8}$$

bounded on the entire real axis, where c_{-1} is an r-dimensional vector column $c_{-1} \in \mathbb{R}^r$ and is determined from the condition of solvability of the problem used for determining the coefficient x_0 of the series (3.5).

For ε^0, the problem of determination of the coefficient $x_0(t)$ of series (3.5) reduces to the problem of finding solutions of the following nonhomogeneous system:

$$\begin{aligned} \dot{x}_0 &= A(t)x_0 + A_1(t)x_{-1} + f(t), \quad t \notin \mathcal{T}, \\ \Delta x_0|_{t=\tau_i} &= A_{1i}x_{-1}(\tau_i-) + \gamma_i, \quad i \in \mathbb{Z}, \end{aligned} \tag{3.9}$$

bounded on the entire real axis. According to the Theorem 2.1, the condition of solvability of this problem takes the form

$$\int_{-\infty}^{\infty} H_d(t)\left[A_1(t)X_r(t)c_{-1} + f(t)\right]dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i)\left[A_{1i}X_r(\tau_i-)c_{-1} + \gamma_i\right] = 0. \tag{3.10}$$

Using the matrix B_0, we get the following algebraic system for $c_{-1} \in \mathbb{R}^r$:

$$B_0 c_{-1} = -\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i, \tag{3.11}$$

which is solvable if and only if the condition

$$P_{B_0^*}\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\} = 0 \tag{3.12}$$

is satisfied, that is, if

$$P_{B_0^*} = 0. \tag{3.13}$$

In this case, this algebraic system is solvable with respect to $c_{-1} \in \mathbb{R}^r$ within an arbitrary vector constant $P_{B_0}c(\forall c \in \mathbb{R}^r)$ from the null space of the matrix B_0, and one of its solutions has the form

$$c_{-1} = B_0^+\left\{\int_{-\infty}^{\infty} H_d(t)f(t)dt + \sum_{i=-\infty}^{\infty} H_d(\tau_i-)\gamma_i\right\}. \tag{3.14}$$

Therefore, under condition (3.4), the nonhomogeneous system (3.9) possesses an r-parameter set of solution bounded on $\mathbb{R}$ in the form

$$x_0(t, c_0) = X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot, c_{-1}) + f(\cdot) \\ \gamma_i + A_{1i}x_{-1}(\tau_i-, c_{-1}) \end{bmatrix}\right)(t), \tag{3.15}$$

where $(G[\begin{smallmatrix} * \\ * \end{smallmatrix}])(t)$ is the generalized Green operator (2.7) of the problem of finding bounded solutions of system (3.9), and c_0 is an r-dimensional constant vector determined in the next step of the process from the condition of solvability of the impulsive problem for coefficient $x_1(t)$.

We continue this process by problem of determination of the coefficient $x_1(t)$ of the term with ε^1 in the series (3.5). It reduces to the problem of finding solutions of the system

$$\begin{aligned} \dot{x}_1 &= A(t)x_1 + A_1(t)x_0, \quad t \notin \mathcal{T}, \\ \Delta x_1|_{t=\tau_i} &= A_{1i}x_0(\tau_i-), \quad i \in \mathbb{Z}, \end{aligned} \tag{3.16}$$

bounded on the entire real axis. If the condition (3.4) is satisfied and by using the condition of solvability of this problem, that is,

$$\begin{aligned} &\int_{-\infty}^{\infty} H_d(t)A_1(t)\left[X_r(t)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)\right]dt \\ &\quad + \sum_{i=-\infty}^{\infty} H_d(\tau_{i-})A_{1i}\left[X_r(\tau_i-)c_0 + \left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\right] = 0, \end{aligned} \tag{3.17}$$

we determine the vector $c_0 \in \mathbb{R}^r$ (within an arbitrary vector constant $P_{B_0}c, \forall c \in \mathbb{R}^r$) as

$$\begin{aligned} c_0 = -B_0^+ &\left[\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\tau_i-,c_{-1}) + \gamma_i \end{bmatrix}\right)(t)dt \right. \\ &\quad \left. + \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{-1}(\cdot,c_{-1}) + f(\cdot) \\ A_{1i}x_{-1}(\cdot,c_{-1}) + \gamma_i \end{bmatrix}\right)(\tau_i-)\right]. \end{aligned} \tag{3.18}$$

Thus, under the condition (3.4), system (3.16) possesses an r-parameter set of solutions bounded on $\mathbb{R}$ in the form

$$x_1(t,c_1) = X_r(t)c_1 + \left(G\begin{bmatrix} A_1(\cdot)x_0(\cdot,c_0) \\ A_{1i}x(\tau_i-,c_0) \end{bmatrix}\right)(t), \tag{3.19}$$

where $(G[\begin{smallmatrix} * \\ * \end{smallmatrix}])(t)$ is the generalized Green operator (2.7) of the problem of finding bounded solutions of system (3.16), and c_1 is an r-dimensional constant vector determined in the next stage of the process from the condition of solvability of the problem for $x_2(t)$.

If we continue this process, we prove (by induction) that the problem of determination of the coefficient $x_k(t)$ in the series (3.5) is reduced to the problem of finding solutions of the system

$$\begin{aligned} \dot{x}_k &= A(t)x_k + A_1(t)x_{k-1}, \quad t \notin \mathcal{T}, \\ \Delta x_k|_{t=\tau_i} &= A_{1i}x_{k-1}(\tau_i-), \quad i \in \mathbb{Z}, \ k = 1,2,\ldots, \end{aligned} \tag{3.20}$$

bounded on the entire real axis. If the condition (3.4) is satisfied, then a solution of this problem bounded on $\mathbb{R}$ has the form

$$x_k(t) = x_k(t, c_k) = X_r(t)c_k + \left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1k}x_{k-1}(\tau_i-, c_{k-1}) \end{bmatrix} \right)(t), \tag{3.21}$$

where $(G[{*\atop *}])(t)$ is the generalized Green operator of the problem of finding bounded solutions of impulsive system (3.20) and the constant vector $c_k \in R^r$ is given by the formula

$$\begin{aligned} c_k = -B_0^+ \Bigg[&\int_{-\infty}^{\infty} H_d(t)A_1(t)\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1i}x_{k-1}(\tau_i-, c_{k-1}) \end{bmatrix} \right)(t)dt \\ &+ \sum_{i=-\infty}^{\infty} H_d(\tau_i)A_{1i}\left(G\begin{bmatrix} A_1(\cdot)x_{k-1}(\cdot, c_{k-1}) \\ A_{1i}x_{k-1}(\cdot, c_{k-1}) \end{bmatrix} \right)(\tau_i-) \Bigg] \end{aligned} \tag{3.22}$$

(within an arbitrary vector constant $P_{B_0}c$, $c \in R^r$).

The fact that the series (3.5) is convergent can be proved by using the procedure of majorization. □

In the case where the number $r = \operatorname{rank} PP_D = \operatorname{rank}(I - Q)P_D$ of linear independent solutions of system (2.2) bounded on $\mathbb{R}$ is equal to the number $d = \operatorname{rank}[P_{D^*}Q] = \operatorname{rank}[P_{D^*}(I-P)]$, Theorem 3.1 yields the following assertion.

Corollary 3.2. *Suppose that the system* (3.1) *satisfies the conditions imposed above, and the homogeneous system* (2.2) *is e-dichotomous on $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors P and Q, respectively. Let nonhomogeneities $f \in BC_{\tau}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$ be given such that the condition* (2.4) *is not satisfied, and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If condition*

$$\det B_0 \neq 0 \quad (r = d), \tag{3.23}$$

is satisfied, then the system (3.1) *possesses a unique solution bounded on $\mathbb{R}$ in the form of series* (3.5) *uniformly convergent for sufficiently small fixed $\varepsilon \in (0, \varepsilon_*]$.*

Proof. If $r = d$, then B_0 is a square matrix. Therefore, it follows from condition (3.4) that $P_{B_0} = P_{B_0^*} = 0$, which is equivalent to the condition (3.23). In this case, the constant vectors $c_k \in \mathbb{R}^r$ are uniquely determined from (3.22). The coefficients of the series (3.5) are also uniquely determined by (3.21), and, for all $f \in BC_{\tau}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$, the system (3.1) possesses a unique solution bounded on $\mathbb{R}$, which means that system (3.2) is e-dichotomous. □

We now illustrate the assertions proved above.

Example 3.3. Consider the impulsive system

$$\dot{x} = A(t)x + \varepsilon A_1(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T},$$
$$\Delta x|_{t=\tau_i} = \gamma_i + \varepsilon A_{1i} x(\tau_i -), \quad \gamma_i = \begin{Bmatrix} \gamma_i^{(1)} \\ \gamma_i^{(2)} \\ \gamma_i^{(3)} \end{Bmatrix} \in \mathbb{R}^3, \; i \in \mathbb{Z}, \tag{3.24}$$

where

$$A(t) = \operatorname{diag}\{-\tanh t, \; -\tanh t, \tanh t\},$$
$$f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R}), \tag{3.25}$$
$$A_1(t) = \{a_{ij}(t)\}_{i,j=1}^3 \in BC_{\mathcal{T}}(\mathbb{R}), \qquad A_{1i} = \{\tilde{a}_{ij}\}_{i,j=1}^3.$$

The generating homogenous system (for $\varepsilon = 0$) has the form

$$\dot{x} = A(t)x, \qquad \Delta x|_{t=\tau_i} = 0 \tag{3.26}$$

and is e-dichotomous (as shown in [6]) on the semiaxes $\mathbb{R}_+$ and $\mathbb{R}_-$ with projectors $P = \operatorname{diag}\{1,1,0\}$ and $Q = \operatorname{diag}\{0,0,1\}$. The normal fundamental matrix of this system is

$$X(t) = \operatorname{diag}\left\{ \frac{2}{e^t + e^{-t}}, \frac{2}{e^t + e^{-t}}, \frac{e^t + e^{-t}}{2} \right\}. \tag{3.27}$$

Thus, we have

$$D = 0, \qquad D^+ = 0, \qquad P_D = P_{D^*} = I_3,$$
$$r = \operatorname{rank} P P_D = 2, \qquad d = \operatorname{rank} P_{D^*} Q = 1,$$
$$X_r(t) = \begin{pmatrix} \dfrac{2}{e^t + e^{-t}} & 0 \\ 0 & \dfrac{2}{e^t + e^{-t}} \\ 0 & 0 \end{pmatrix}, \tag{3.28}$$
$$H_d(t) = \left(0, 0, \frac{2}{e^t + e^{-t}}\right). \tag{3.29}$$

In order that the generating impulsive system (2.1) with the matrix $A(t)$ specified above has solutions bounded on the entire real axis, the nonhomogeneities $f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i = \operatorname{col}\{\gamma_i^{(1)}, \gamma_i^{(2)}, \gamma_i^{(3)}\} \in \mathbb{R}^3$ must satisfy condition (2.4). In this analyzed impulsive problem, this condition takes the form

$$\int_{-\infty}^{\infty} \frac{2 f_3(t)}{e^t + e^{-t}} dt + \sum_{i=-\infty}^{\infty} \frac{2}{e^{\tau_i} + e^{-\tau_i}} \gamma_i^{(3)} = 0, \quad \forall f_1(t), f_2(t) \in BC_{\mathcal{T}}(\mathbb{R}), \; \forall \gamma_i^{(1)}, \gamma_i^{(2)} \in \mathbb{R}. \tag{3.30}$$

Let f_3 and $\gamma_i^{(3)}$ be given such that the condition (3.30) is not satisfied and the corresponding generating system (2.1) does not have solutions bounded on the entire real axis. The system (3.24) will be an e-trichotomous on $\mathbb{R}$ if the coefficients $a_{31}(t), a_{32}(t) \in BC_{\mathcal{T}}(\mathbb{R})$ of the perturbing matrix $A_1(t)$ and the coefficients $\tilde{a}_{31}, \tilde{a}_{32} \in \mathbb{R}$ of the perturbing matrix A_{1i} satisfy condition (3.4), that is, $P_{B_0^*} = 0$, where the matrix B_0 has the form

$$B_0 = \int_{-\infty}^{\infty} \left[\frac{a_{31}(t)}{(e^t + e^{-t})^2}, \frac{a_{32}(t)}{(e^t + e^{-t})^2} \right] dt + \sum_{i=-\infty}^{\infty} \left[\frac{\tilde{a}_{31}}{(e^{\tau_i -} + e^{-\tau_i -})^2}, \frac{\tilde{a}_{32}}{(e^{\tau_i -} + e^{-\tau_i -})^2} \right]. \tag{3.31}$$

Therefore, if $a_{31}(t), a_{32}(t) \in BC_{\mathcal{T}}(\mathbb{R})$ and $\tilde{a}_{31}, \tilde{a}_{32} \in \mathbb{R}$ are such that at least one of the following inequalities

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{a_{31}(t)}{(e^t + e^{-t})^2} dt + \sum_{i=-\infty}^{\infty} \frac{\tilde{a}_{31}}{(e^{\tau_i -} + e^{-\tau_i -})^2} \neq 0, \\ &\int_{-\infty}^{\infty} \frac{a_{32}(t)}{(e^t + e^{-t})^2} dt + \sum_{i=-\infty}^{\infty} \frac{\tilde{a}_{32}}{(e^{\tau_i -} + e^{-\tau_i -})^2} \neq 0 \end{aligned} \tag{3.32}$$

is satisfied, then either the condition (3.4) or the equivalent condition rank $B_0 = d = 1$ from Theorem 3.1 is satisfied and the system (3.2) is e-trichotomous on $\mathbb{R}$. In this case, the coefficients $a_{11}(t), a_{12}(t), a_{13}(t), a_{21}(t), a_{22}(t), a_{23}(t), a_{33}(t)$ are arbitrary functions from the space $BC_{\mathcal{T}}(\mathbb{R})$, and $\tilde{a}_{11}, \tilde{a}_{12}, \tilde{a}_{13}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{a}_{23}, \tilde{a}_{33}$ are arbitrary constants from $\mathbb{R}$. Moreover, for any

$$f(t) = \operatorname{col}\{f_1(t), f_2(t), f_3(t)\} \in BC_{\mathcal{T}}(\mathbb{R}) \tag{3.33}$$

a solution of the system (3.24) bounded on $\mathbb{R}$ is given by the series (3.5) (within a constant from the null space $N(B_0)$, $\dim N(B_0) = r - \operatorname{rank} B_0 = 1$).

Another Perturbed Problem

In this part, we show that the problem of finding bounded solutions of nonhomogeneous system (2.1), in the case if the condition (2.4) is not satisfied, can be made solvable by introducing linear perturbations only to the pulsed conditions.

Therefore, we consider the weakly perturbed nonhomogeneous linear impulsive system in the form

$$\begin{aligned} \dot{x} &= A(t)x + f(t), \quad t \in \mathbb{R} \setminus \mathcal{T}, \; A, f \in BC_{\mathcal{T}}(\mathbb{R}), \\ \Delta x|_{t=\tau_i} &= \gamma_i + \varepsilon A_{1i} x(\tau_i -), \quad \gamma_i \in \mathbb{R}^n, \; i \in \mathbb{Z}, \end{aligned} \tag{3.34}$$

where A_{1i} are $n \times n$ constant matrices. For $\varepsilon = 0$, we obtain the generating system (2.1). We assume that this generating system does not have solutions bounded on the entire real axis, which means that the condition of solvability (2.4) is not satisfied (for some nonhomogeneities $f \in BC_{\mathcal{T}}(\mathbb{R})$ and $\gamma_i \in \mathbb{R}^n$). Let us show that it is possible to make this problem solvable by adding linear perturbation only to the pulsed conditions. In the case, if this is possible, it is necessary to determine perturbations A_{1i} for which the corresponding homogeneous system

$$\dot{x} = A(t)x, \quad t \in \mathbb{R} \setminus \mathcal{T},$$
$$\Delta x|_{t=\tau_i} = \varepsilon A_{1i} x(\tau_i-), \quad i \in \mathbb{Z}, \tag{3.35}$$

turns into the system e-trichotomous or e-dichotomous on the entire real axis.

This problem can be solved with help of the $d \times r$ matrix

$$B_0 = \sum_{i=-\infty}^{\infty} H_d(\tau_i) A_{1i} X_r(\tau_i-) \tag{3.36}$$

constructed with the coefficients from the impulsive system (3.34).

By using Theorem 3.1, we seek a solution in the form of the series (3.5). Thus, we have the following corollary.

Corollary 3.4. *Suppose that the system* (3.34) *satisfies the conditions imposed above and the generating homogeneous system* (2.2) *is e-dichotomous on* $\mathbb{R}_+$ *and* $\mathbb{R}_-$ *with projectors* P *and* Q*, respectively. Let nonhomogeneities* $f \in BC_{\mathcal{T}}(\mathbb{R})$ *and* $\gamma_i \in \mathbb{R}^n$ *be given such that the condition* (2.4) *is not satisfied, and the generating system* (2.1) *does not have solutions bounded on the entire real axis. If the condition* (3.4) *is satisfied, then the system* (3.35) *is e-trichotomous on* $\mathbb{R}$*, and the system* (3.34) *possesses at least one solution bounded on* $\mathbb{R}$ *in the form of series* (3.5) *uniformly convergent for sufficiently small fixed* $\varepsilon \in (0, \varepsilon_*]$.

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Research Article

Existence of Nonoscillatory Solutions of First-Order Neutral Differential Equations

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This paper contains some sufficient conditions for the existence of positive solutions which are bounded below and above by positive functions for the first-order nonlinear neutral differential equations. These equations can also support the existence of positive solutions approaching zero at infinity

1. Introduction

This paper is concerned with the existence of a positive solution of the neutral differential equations of the form

$$\frac{d}{dt}[x(t) - a(t)x(t-\tau)] = p(t)f(x(t-\sigma)), \quad t \geq t_0, \tag{1.1}$$

where $\tau > 0$, $\sigma \geq 0$, $a \in C([t_0,\infty),(0,\infty))$, $p \in C(R,(0,\infty))$, $f \in C(R,R)$, f is nondecreasing function, and $xf(x) > 0$, $x \neq 0$.

By a solution of (1.1) we mean a function $x \in C([t_1 - m,\infty),R)$, $m = \max\{\tau,\sigma\}$, for some $t_1 \geq t_0$, such that $x(t) - a(t)x(t-\tau)$ is continuously differentiable on $[t_1,\infty)$ and such that (1.1) is satisfied for $t \geq t_1$.

The problem of the existence of solutions of neutral differential equations has been studied by several authors in the recent years. For related results we refer the reader to [1–11] and the references cited therein. However there is no conception which guarantees the existence of positive solutions which are bounded below and above by positive functions. In this paper we have presented some conception. The method also supports the existence of positive solutions approaching zero at infinity.

As much as we know, for (1.1) in the literature, there is no result for the existence of solutions which are bounded by positive functions. Only the existence of solutions which are bounded by constants is treated, for example, in [6, 10, 11]. It seems that conditions of theorems are rather complicated, but cannot be simpler due to Corollaries 2.3, 2.6, and 3.2.

The following fixed point theorem will be used to prove the main results in the next section.

Lemma 1.1 ([see [6, 10] Krasnoselskii's fixed point theorem]). *Let X be a Banach space, let Ω be a bounded closed convex subset of X, and let S_1, S_2 be maps of Ω into X such that $S_1x + S_2y \in \Omega$ for every pair $x, y \in \Omega$. If S_1 is contractive and S_2 is completely continuous, then the equation*

$$S_1x + S_2x = x \tag{1.2}$$

has a solution in Ω.

2. The Existence of Positive Solution

In this section we will consider the existence of a positive solution for (1.1). The next theorem gives us the sufficient conditions for the existence of a positive solution which is bounded by two positive functions.

Theorem 2.1. *Suppose that there exist bounded functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that*

$$u(t) \leq v(t), \quad t \geq t_0, \tag{2.1}$$

$$v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1, \tag{2.2}$$

$$\begin{aligned} \frac{1}{u(t-\tau)}\left(u(t) + \int_t^\infty p(s)f(v(s-\sigma))ds\right) &\leq a(t) \\ \leq \frac{1}{v(t-\tau)}\left(v(t) + \int_t^\infty p(s)f(u(s-\sigma))ds\right) &\leq c < 1, \quad t \geq t_1. \end{aligned} \tag{2.3}$$

Then (1.1) *has a positive solution which is bounded by functions u, v.*

Proof. Let $C([t_0, \infty), R)$ be the set of all continuous bounded functions with the norm $\|x\| = \sup_{t \geq t_0}|x(t)|$. Then $C([t_0, \infty), R)$ is a Banach space. We define a closed, bounded, and convex subset Ω of $C([t_0, \infty), R)$ as follows:

$$\Omega = \{x = x(t) \in C([t_0, \infty), R) : u(t) \leq x(t) \leq v(t), \ t \geq t_0\}. \tag{2.4}$$

We now define two maps S_1 and $S_2 : \Omega \to C([t_0, \infty), R)$ as follows:

$$(S_1x)(t) = \begin{cases} a(t)x(t-\tau), & t \geq t_1, \\ (S_1x)(t_1), & t_0 \leq t \leq t_1, \end{cases}$$
$$(S_2x)(t) = \begin{cases} -\int_t^\infty p(s)f(x(s-\sigma))ds, & t \geq t_1, \\ (S_2x)(t_1) + v(t) - v(t_1), & t_0 \leq t \leq t_1. \end{cases} \tag{2.5}$$

We will show that for any $x, y \in \Omega$ we have $S_1x + S_2y \in \Omega$. For every $x, y \in \Omega$ and $t \geq t_1$, we obtain

$$(S_1x)(t) + (S_2y)(t) \leq a(t)v(t-\tau) - \int_t^\infty p(s)f(u(s-\sigma))ds \leq v(t). \tag{2.6}$$

For $t \in [t_0, t_1]$, we have

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\leq v(t_1) + v(t) - v(t_1) = v(t).\end{aligned} \tag{2.7}$$

Furthermore, for $t \geq t_1$, we get

$$(S_1x)(t) + (S_2y)(t) \geq a(t)u(t-\tau) - \int_t^\infty p(s)f(v(s-\sigma))ds \geq u(t). \tag{2.8}$$

Let $t \in [t_0, t_1]$. With regard to (2.2), we get

$$v(t) - v(t_1) + u(t_1) \geq u(t), \quad t_0 \leq t \leq t_1. \tag{2.9}$$

Then for $t \in [t_0, t_1]$ and any $x, y \in \Omega$, we obtain

$$\begin{aligned}(S_1x)(t) + (S_2y)(t) &= (S_1x)(t_1) + (S_2y)(t_1) + v(t) - v(t_1) \\ &\geq u(t_1) + v(t) - v(t_1) \geq u(t).\end{aligned} \tag{2.10}$$

Thus we have proved that $S_1x + S_2y \in \Omega$ for any $x, y \in \Omega$.

We will show that S_1 is a contraction mapping on Ω. For $x, y \in \Omega$ and $t \geq t_1$ we have

$$\left|(S_1x)(t) - (S_1y)(t)\right| = |a(t)|\left|x(t-\tau) - y(t-\tau)\right| \leq c\|x - y\|. \tag{2.11}$$

This implies that

$$\|S_1x - S_1y\| \leq c\|x - y\|. \tag{2.12}$$

Also for $t \in [t_0, t_1]$, the previous inequality is valid. We conclude that S_1 is a contraction mapping on Ω.

We now show that S_2 is completely continuous. First we will show that S_2 is continuous. Let $x_k = x_k(t) \in \Omega$ be such that $x_k(t) \to x(t)$ as $k \to \infty$. Because Ω is closed, $x = x(t) \in \Omega$. For $t \geq t_1$ we have

$$\begin{aligned} |(S_2 x_k)(t) - (S_2 x)(t)| &\leq \left| \int_t^\infty p(s)\left[f(x_k(s-\sigma)) - f(x(s-\sigma))\right] ds \right| \\ &\leq \int_{t_1}^\infty p(s)\left|f(x_k(s-\sigma)) - f(x(s-\sigma))\right| ds. \end{aligned} \tag{2.13}$$

According to (2.8), we get

$$\int_{t_1}^\infty p(s) f(v(s-\sigma)) ds < \infty. \tag{2.14}$$

Since $|f(x_k(s-\sigma)) - f(x(s-\sigma))| \to 0$ as $k \to \infty$, by applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \|(S_2 x_k)(t) - (S_2 x)(t)\| = 0. \tag{2.15}$$

This means that S_2 is continuous.

We now show that $S_2\Omega$ is relatively compact. It is sufficient to show by the Arzela-Ascoli theorem that the family of functions $\{S_2 x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$. The uniform boundedness follows from the definition of Ω. For the equicontinuity we only need to show, according to Levitans result [7], that for any given $\varepsilon > 0$ the interval $[t_0, \infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have a change of amplitude less than ε. Then with regard to condition (2.14), for $x \in \Omega$ and any $\varepsilon > 0$, we take $t^* \geq t_1$ large enough so that

$$\int_{t^*}^\infty p(s) f(x(s-\sigma)) ds < \frac{\varepsilon}{2}. \tag{2.16}$$

Then, for $x \in \Omega$, $T_2 > T_1 \geq t^*$, we have

$$\begin{aligned} |(S_2 x)(T_2) - (S_2 x)(T_1)| &\leq \int_{T_2}^\infty p(s) f(x(s-\sigma)) ds \\ &\quad + \int_{T_1}^\infty p(s) f(x(s-\sigma)) ds < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \tag{2.17}$$

For $x \in \Omega$ and $t_1 \leq T_1 < T_2 \leq t^*$, we get

$$\begin{aligned}|(S_2x)(T_2) - (S_2x)(T_1)| &\leq \int_{T_1}^{T_2} p(s)f(x(s-\sigma))ds \\ &\leq \max_{t_1 \leq s \leq t^*} \{p(s)f(x(s-\sigma))\}(T_2 - T_1).\end{aligned} \tag{2.18}$$

Thus there exists $\delta_1 = \varepsilon/M$, where $M = \max_{t_1 \leq s \leq t^*}\{p(s)f(x(s-\sigma))\}$, such that

$$|(S_2x)(T_2) - (S_2x)(T_1)| < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_1. \tag{2.19}$$

Finally for any $x \in \Omega$, $t_0 \leq T_1 < T_2 \leq t_1$, there exists a $\delta_2 > 0$ such that

$$\begin{aligned}|(S_2x)(T_2) - (S_2x)(T_1)| = |v(T_1) - v(T_2)| &= \left|\int_{T_1}^{T_2} v'(s)ds\right| \\ &\leq \max_{t_0 \leq s \leq t_1}\{|v'(s)|\}(T_2 - T_1) < \varepsilon \quad \text{if } 0 < T_2 - T_1 < \delta_2.\end{aligned} \tag{2.20}$$

Then $\{S_2x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, \infty)$, and hence $S_2\Omega$ is relatively compact subset of $C([t_0, \infty), R)$. By Lemma 1.1 there is an $x_0 \in \Omega$ such that $S_1x_0 + S_2x_0 = x_0$. We conclude that $x_0(t)$ is a positive solution of (1.1). The proof is complete. □

Corollary 2.2. *Suppose that there exist functions $u, v \in C^1([t_0, \infty), (0, \infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.3) *hold and*

$$v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1. \tag{2.21}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v.*

Proof. We only need to prove that condition (2.21) implies (2.2). Let $t \in [t_0, t_1]$ and set

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1). \tag{2.22}$$

Then with regard to (2.21), it follows that

$$H'(t) = v'(t) - u'(t) \leq 0, \quad t_0 \leq t \leq t_1. \tag{2.23}$$

Since $H(t_1) = 0$ and $H'(t) \leq 0$ for $t \in [t_0, t_1]$, this implies that

$$H(t) = v(t) - v(t_1) - u(t) + u(t_1) \geq 0, \quad t_0 \leq t \leq t_1. \tag{2.24}$$

Thus all conditions of Theorem 2.1 are satisfied. □

Corollary 2.3. *Suppose that there exists a function $v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that*

$$a(t) = \frac{1}{v(t-\tau)}\left(v(t) + \int_t^{\infty} p(s)f(v(s-\sigma))ds\right) \leq c < 1, \quad t \geq t_1. \tag{2.25}$$

Then (1.1) *has a solution $x(t) = v(t)$, $t \geq t_1$.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.1. □

Theorem 2.4. *Suppose that there exist functions $u, v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.2), *and* (2.3) *hold and*

$$\lim_{t\to\infty} v(t) = 0. \tag{2.26}$$

Then (1.1) *has a positive solution which is bounded by the functions u, v and tends to zero.*

Proof. The proof is similar to that of Theorem 2.1 and we omit it. □

Corollary 2.5. *Suppose that there exist functions $u, v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.1), (2.3), (2.21), *and* (2.26) *hold. Then* (1.1) *has a positive solution which is bounded by the functions u, v and tends to zero.*

Proof. The proof is similar to that of Corollary 2.2, and we omitted it. □

Corollary 2.6. *Suppose that there exists a function $v \in C^1([t_0,\infty),(0,\infty))$, constant $c > 0$ and $t_1 \geq t_0 + m$ such that* (2.25), (2.26) *hold. Then* (1.1) *has a solution $x(t) = v(t)$, $t \geq t_1$ which tends to zero.*

Proof. We put $u(t) = v(t)$ and apply Theorem 2.4. □

3. Applications and Examples

In this section we give some applications of the theorems above.

Theorem 3.1. *Suppose that*

$$\int_{t_0}^{\infty} p(t)dt = \infty, \tag{3.1}$$

$0 < k_1 \le k_2$ and there exist constants $c > 0,\ \gamma \ge 0,\ t_1 \ge t_0 + m$ such that

$$\frac{k_1}{k_2}\exp\left((k_2 - k_1)\int_{t_0-\gamma}^{t_0} p(t)dt\right) \ge 1, \tag{3.2}$$

$$\begin{aligned}
&\exp\left(-k_2\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k_2\int_{t_0-\gamma}^{t-\tau} p(s)ds\right)\\
&\quad\times\int_t^{\infty} p(s)f\left(\exp\left(-k_1\int_{t_0-\gamma}^{s-\sigma} p(\xi)d\xi\right)\right)ds \le a(t)\\
&\le \exp\left(-k_1\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k_1\int_{t_0-\gamma}^{t-\tau} p(s)ds\right)\\
&\quad\times\int_t^{\infty} p(s)f\left(\exp\left(-k_2\int_{t_0-\gamma}^{s-\sigma} p(\xi)d\xi\right)\right)ds \le c < 1, \quad t \ge t_1.
\end{aligned} \tag{3.3}$$

Then (1.1) *has a positive solution which tends to zero.*

Proof. We set

$$u(t) = \exp\left(-k_2\int_{t_0-\gamma}^{t} p(s)ds\right), \quad v(t) = \exp\left(-k_1\int_{t_0-\gamma}^{t} p(s)ds\right), \quad t \ge t_0. \tag{3.4}$$

We will show that the conditions of Corollary 2.5 are satisfied. With regard to (2.21), for $t \in [t_0, t_1]$, we get

$$\begin{aligned}
v'(t) - u'(t) &= -k_1p(t)v(t) + k_2p(t)u(t)\\
&= p(t)v(t)\left[-k_1 + k_2u(t)\exp\left(k_1\int_{t_0-\gamma}^{t} p(s)ds\right)\right]\\
&= p(t)v(t)\left[-k_1 + k_2\exp\left((k_1 - k_2)\int_{t_0-\gamma}^{t} p(s)ds\right)\right]\\
&\le p(t)v(t)\left[-k_1 + k_2\exp\left((k_1 - k_2)\int_{t_0-\gamma}^{t_0} p(s)ds\right)\right] \le 0.
\end{aligned} \tag{3.5}$$

Other conditions of Corollary 2.5 are also satisfied. The proof is complete. □

Corollary 3.2. *Suppose that* $k > 0,\ c > 0,\ t_1 \geq t_0 + m$, (3.1) *holds, and*

$$a(t) = \exp\left(-k\int_{t-\tau}^{t} p(s)ds\right) + \exp\left(k\int_{t_0}^{t-\tau} p(s)ds\right) \times \int_{t}^{\infty} p(s) f\left(\exp\left(-k\int_{t_0}^{s-\sigma} p(\xi)d\xi\right)\right)ds \leq c < 1, \quad t \geq t_1. \tag{3.6}$$

Then (1.1) *has a solution*

$$x(t) = \exp\left(-k\int_{t_0}^{t} p(s)ds\right), \quad t \geq t_1, \tag{3.7}$$

which tends to zero.

Proof. We put $k_1 = k_2 = k,\ \gamma = 0$ and apply Theorem 3.1. □

Example 3.3. Consider the nonlinear neutral differential equation

$$[x(t) - a(t)x(t-2)]' = px^3(t-1), \quad t \geq t_0, \tag{3.8}$$

where $p \in (0, \infty)$. We will show that the conditions of Theorem 3.1 are satisfied. Condition (3.1) obviously holds and (3.2) has a form

$$\frac{k_1}{k_2}\exp((k_2 - k_1)p\gamma) \geq 1, \tag{3.9}$$

$0 < k_1 \leq k_2,\ \gamma \geq 0$. For function $a(t)$, we obtain

$$\begin{aligned}
&\exp(-2pk_2) + \frac{1}{3k_1}\exp(p[k_2(\gamma - t_0 - 2) - 3k_1(\gamma - t_0 - 1) + (k_2 - 3k_1)t]) \\
&\quad\leq a(t) \leq \exp(-2pk_1) \\
&\qquad + \frac{1}{3k_2}\exp(p[k_1(\gamma - t_0 - 2) - 3k_2(\gamma - t_0 - 1) + (k_1 - 3k_2)t]), \quad t \geq t_0.
\end{aligned} \tag{3.10}$$

For $p = 1,\ k_1 = 1,\ k_2 = 2,\ \gamma = 1,\ t_0 = 1$, condition (3.9) is satisfied and

$$e^{-4} + \frac{1}{3e}e^{-t} \leq a(t) \leq e^{-2} + \frac{e^4}{6}e^{-5t}, \quad t \geq t_1 \geq 3. \tag{3.11}$$

If the function $a(t)$ satisfies (3.11), then (3.8) has a solution which is bounded by the functions $u(t) = \exp(-2t),\ v(t) = \exp(-t),\ t \geq 3$.

For example if $p = 1, k_1 = k_2 = 1.5, \gamma = 1, t_0 = 1$, from (3.11) we obtain

$$a(t) = e^{-3} + \frac{e^{1.5}}{4.5} e^{-3t}, \tag{3.12}$$

and the equation

$$\left[x(t) - \left(e^{-3} + \frac{e^{1.5}}{4.5} e^{-3t}\right) x(t-2)\right]' = x^3(t-1), \quad t \geq 3, \tag{3.13}$$

has the solution $x(t) = \exp(-1.5t)$ which is bounded by the function $u(t)$ and $v(t)$.

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Research Article

Existence of Oscillatory Solutions of Singular Nonlinear Differential Equations

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Asymptotic properties of solutions of the singular differential equation $(p(t)u'(t))' = p(t)f(u(t))$ are described. Here, f is Lipschitz continuous on $\mathbb{R}$ and has at least two zeros 0 and $L > 0$. The function p is continuous on $[0,\infty)$ and has a positive continuous derivative on $(0,\infty)$ and $p(0) = 0$. Further conditions for f and p under which the equation has oscillatory solutions converging to 0 are given.

1. Introduction

For $k \in \mathbb{N}$, $k > 1$, and $L \in (0,\infty)$, consider the equation

$$u'' + \frac{k-1}{t}u' = f(u), \quad t \in (0,\infty), \tag{1.1}$$

where

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R}), \qquad f(0) = f(L) = 0, \qquad f(x) < 0, \quad x \in (0,L), \tag{1.2}$$

$$\exists \overline{B} \in (-\infty,0) : f(x) > 0, \quad x \in \left[\overline{B},0\right). \tag{1.3}$$

Let us put

$$F(x) = -\int_0^x f(z)\mathrm{d}z \quad \text{for } x \in \mathbb{R}. \tag{1.4}$$

Moreover, we assume that f fulfils

$$F\left(\overline{B}\right) = F(L), \tag{1.5}$$

and denote

$$L_0 = \inf\left\{x < \overline{B} : f(x) > 0\right\} \geq -\infty. \tag{1.6}$$

Due to (1.2)–(1.4), we see that $F \in C^1(\mathbb{R})$ is decreasing and positive on $(L_0, 0)$ and increasing and positive on $(0, L]$.

Equation (1.1) arises in many areas. For example, in the study of phase transitions of Van der Waals fluids [1–3], in population genetics, where it serves as a model for the spatial distribution of the genetic composition of a population [4, 5], in the homogenous nucleation theory [6], and in relativistic cosmology for description of particles which can be treated as domains in the universe [7], in the nonlinear field theory, in particular, when describing bubbles generated by scalar fields of the Higgs type in the Minkowski spaces [8]. Numerical simulations of solutions of (1.1), where f is a polynomial with three zeros, have been presented in [9–11]. Close problems about the existence of positive solutions can be found in [12–14].

In this paper, we investigate a generalization of (1.1) of the form

$$\left(p(t)u'\right)' = p(t)f(u), \quad t \in (0, \infty), \tag{1.7}$$

where f satisfies (1.2)–(1.5) and p fulfils

$$p \in C[0, \infty) \cap C^1(0, \infty), \quad p(0) = 0, \tag{1.8}$$

$$p'(t) > 0,\ t \in (0, \infty), \quad \lim_{t \to \infty} \frac{p'(t)}{p(t)} = 0. \tag{1.9}$$

Equation (1.7) is singular in the sense that $p(0) = 0$. If $p(t) = t^{k-1}$, with $k > 1$, then p satisfies (1.8), (1.9), and (1.7) is equal to (1.1).

Definition 1.1. A function $u \in C^1[0, \infty) \cap C^2(0, \infty)$ which satisfies (1.7) for all $t \in (0, \infty)$ is called a *solution* of (1.7).

Consider a solution u of (1.7). Since $u \in C^1[0, \infty)$, we have $u(0), u'(0) \in \mathbb{R}$ and the assumption, $p(0) = 0$ yields $p(0)u'(0) = 0$. We can find $M > 0$ and $\delta > 0$ such that $|f(u(t))| \leq M$ for $t \in (0, \delta)$. Integrating (1.7), we get

$$\left|u'(t)\right| = \left|\frac{1}{p(t)} \int_0^t p(s)f(u(s))\mathrm{d}s\right| \leq \frac{M}{p(t)} \int_0^t p(s)\mathrm{d}s \leq Mt, \quad t \in (0, \delta). \tag{1.10}$$

Consequently, the condition

$$u'(0) = 0 \tag{1.11}$$

is necessary for each solution of (1.7). Denote

$$u_{\sup} = \sup\{u(t) : t \in [0, \infty)\}. \tag{1.12}$$

Definition 1.2. Let u be a solution of (1.7). If $u_{\sup} < L$, then u is called a *damped* solution.

If a solution u of (1.7) satisfies $u_{\sup} = L$ or $u_{\sup} > L$, then we call u a bounding homoclinic solution or an escape solution. These three types of solutions have been investigated in [15–18]. Here, we continue the investigation of the existence and asymptotic properties of damped solutions. Due to (1.11) and Definition 1.2, it is reasonable to study solutions of (1.7) satisfying the initial conditions

$$u(0) = u_0 \in (L_0, L], \qquad u'(0) = 0. \tag{1.13}$$

Note that if $u_0 > L$, then a solution u of the problem (1.7), (1.13) satisfies $u_{\sup} > L$, and consequently u is not a damped solution. Assume that $L_0 > -\infty$, then $f(L_0) = 0$, and if we put $u_0 = L_0$, a solution u of (1.7), (1.13) is a constant function equal to L_0 on $[0, \infty)$. Since we impose no sign assumption on $f(x)$ for $x < L_0$, we do not consider the case $u_0 < L_0$. In fact, the choice of u_0 between two zeros L_0 and 0 of f has been motivated by some hydrodynamical model in [11].

A lot of papers are devoted to oscillatory solutions of nonlinear differential equations. Wong [19] published an account on a nonlinear oscillation problem originated from earlier works of Atkinson and Nehari. Wong's paper is concerned with the study of oscillatory behaviour of second-order Emden-Fowler equations

$$y''(x) + a(x)|y(x)|^{\gamma-1}y(x) = 0, \quad \gamma > 0, \tag{1.14}$$

where a is nonnegative and absolutely continuous on $(0, \infty)$. Both superlinear case ($\gamma > 1$) and sublinear case ($\gamma \in (0, 1)$) are discussed, and conditions for the function a giving oscillatory or nonoscillatory solutions of (1.14) are presented; see also [20]. Further extensions of these results have been proved for more general differential equations. For example, Wong and Agarwal [21] or Li [22] worked with the equation

$$\left(a(t)\left(y'(t)\right)^{\sigma}\right)' + q(t)f\left(y(t)\right) = 0, \tag{1.15}$$

where $\sigma > 0$ is a positive quotient of odd integers, $a \in C^1(\mathbb{R})$ is positive, $q \in C(\mathbb{R})$, $f \in C^1(\mathbb{R})$, $xf(x) > 0$, $f'(x) \geq 0$ for all $x \neq 0$. Kulenović and Ljubović [23] investigated an equation

$$\left(r(t)g\left(y'(t)\right)\right)' + p(t)f\left(y(t)\right) = 0, \tag{1.16}$$

where $g(u)/u \le m$, $f(u)/u \ge k > 0$, or $f'(u) \ge k$ for all $u \ne 0$. The investigation of oscillatory and nonoscillatory solutions has been also realized in the class of quasilinear equations. We refer to the paper [24] by Ho, dealing with the equation

$$\left(t^{n-1}\Phi_p(u')\right)' + t^{n-1}\sum_{i=1}^{N}\alpha_i t^{\beta_i}\Phi_{q_i}(u) = 0, \tag{1.17}$$

where $1 < p < n$, $\alpha_i > 0$, $\beta_i \ge -p$, $q_i > p-1$, $i = 1,\dots,N$, $\Phi_p(y) = |y|^{p-2}y$.

Oscillation results for the equation

$$\left(a(t)\Phi_p(x')\right)' + b(t)\Phi_q(x) = 0, \tag{1.18}$$

where $a, b \in C([0,\infty))$ are positive, can be found in [25]. We can see that the nonlinearity $f(y) = |y|^{\gamma-1}y$ in (1.14) is an increasing function on $\mathbb{R}$ having a unique zero at $y = 0$.

Nonlinearities in all the other (1.15)–(1.18) have similar globally monotonous behaviour. We want to emphasize that, in contrast to the above papers, the nonlinearity f in our (1.7) needs not be globally monotonous. Moreover, we deal with solutions of (1.7) starting at a singular point $t = 0$, and we provide an interval for starting values u_0 giving oscillatory solutions (see Theorems 2.3, 2.10, and 2.16). We specify a behaviour of oscillatory solutions in more details (decreasing amplitudes—see Theorems 2.10 and 2.16), and we show conditions which guarantee that oscillatory solutions converge to 0 (Theorem 3.1).

The paper is organized in this manner: Section 2 contains results about existence, uniqueness, and other basic properties of solutions of the problem (1.7), (1.13). These results which mainly concern damped solutions are taken from [18] and extended or modified a little. We also provide here new conditions for the existence of oscillatory solutions in Theorem 2.16. Section 3 is devoted to asymptotic properties of oscillatory solutions, and the main result is contained in Theorem 3.1.

2. Solutions of the Initial Problem (1.7), (1.13)

Let us give an account of this section in more details. The main objective of this paper is to characterize asymptotic properties of oscillatory solutions of the problem (1.7), (1.13). In order to present more complete results about the solutions, we start this section with the unique solvability of the problem (1.7), (1.13) on $[0,\infty)$ (Theorem 2.1). Having such global solutions, we have proved (see papers [15–18]) that oscillatory solutions of the problem (1.7), (1.13) can be found just in the class of damped solutions of this problem. Therefore, we give here one result about the existence of damped solutions (Theorem 2.3). Example 2.5 shows that there are damped solutions which are not oscillatory. Consequently, we bring results about the existence of oscillatory solutions in the class of damped solutions. This can be found in Theorem 2.10, which is an extension of Theorem 3.4 of [18] and in Theorem 2.16, which are new. Theorems 2.10 and 2.16 cover different classes of equations which is illustrated by examples.

Theorem 2.1 (existence and uniqueness). *Assume that* (1.2)–(1.5), (1.8), (1.9) *hold and that there exists* $C_L \in (0,\infty)$ *such that*

$$0 \le f(x) \le C_L \quad \textit{for } x \ge L \tag{2.1}$$

then the initial problem (1.7), (1.13) *has a unique solution* u. *The solution* u *satisfies*

$$\begin{aligned} u(t) &\ge u_0 \quad \textit{if} \;\; u_0 < 0, \\ u(t) &> \overline{B} \quad \textit{if} \;\; u_0 \ge 0, \end{aligned} \quad \textit{for } t \in [0,\infty). \tag{2.2}$$

Proof. Let $u_0 < 0$, then the assertion is contained in Theorem 2.1 of [18]. Now, assume that $u_0 \in [0, L]$, then the proof of Theorem 2.1 in [18] can be slightly modified. □

For close existence results, see also Chapters 13 and 14 of [26], where this kind of equations is studied.

Remark 2.2. Clearly, for $u_0 = 0$ and $u_0 = L$, the problem (1.7), (1.13) has a unique solution $u \equiv 0$ and $u \equiv L$, respectively. Since $f \in \mathrm{Lip}_{\mathrm{loc}}(\mathbb{R})$, no solution of the problem (1.7), (1.13) with $u_0 < 0$ or $u_0 \in (0, L)$ can touch the constant solutions $u \equiv 0$ and $u \equiv L$.

In particular, assume that $C \in \{0, L\}$, $a > 0$, u is a solution of the problem (1.7), (1.13) with $u_0 < L$, $u_0 \ne 0$, and (1.2), (1.8), and (1.9) hold. If $u(a) = C$, then $u'(a) \ne 0$, and if $u'(a) = 0$, then $u(a) \ne C$.

The next theorem provides an extension of Theorem 2.4 in [18].

Theorem 2.3 (existence of damped solutions). *Assume that* (1.2)–(1.5), (1.8), *and* (1.9) *hold, then for each* $u_0 \in [\overline{B}, L)$, *the problem* (1.7), (1.13) *has a unique solution. This solution is damped.*

Proof. First, assume that there exists $C_L > 0$ such that f satisfies (2.1), then, by Theorem 2.1, the problem (1.7), (1.13) has a unique solution u satisfying (2.2). Assume that u is not damped, that is,

$$\sup\{u(t) : t \in [0,\infty)\} \ge L. \tag{2.3}$$

By (1.3)–(1.5), the inequality $F(u_0) \le F(L)$ holds. Since u fulfils (1.7), we have

$$u''(t) + \frac{p'(t)}{p(t)} u'(t) = f(u(t)) \quad \text{for } t \in (0,\infty). \tag{2.4}$$

Multiplying (2.4) by u' and integrating between 0 and $t > 0$, we get

$$0 < \frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (0,\infty), \tag{2.5}$$

and consequently

$$0 < \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \mathrm{d}s \le F(u_0) - F(u(t)), \quad t \in (0, \infty). \tag{2.6}$$

By (2.3), we can find that $b \in (0, \infty]$ such that $u(b) \ge L$, $(u(\infty) = \limsup_{t\to\infty} u(t))$, and hence, according to (1.5),

$$0 < \int_0^b \frac{p'(s)}{p(s)} u'^2(s) \mathrm{d}s \le F(u_0) - F(u(b)) \le F(B) - F(L) \le 0, \tag{2.7}$$

which is a contradiction. We have proved that $\sup\{u(t) : t \in [0, \infty)\} < L$, that is, u is damped. Consequently, assumption (2.1) can be omitted. □

Example 2.4. Consider the equation

$$u'' + \frac{2}{t} u' = u(u-1)(u+2), \tag{2.8}$$

which is relevant to applications in [9–11]. Here, $p(t) = t^2$, $f(x) = x(x-1)(x+2)$, $L_0 = -2$, and $L = 1$. Hence $f(\mathrm{x}) < 0$ for $x \in (0,1)$, $f(x) > 0$ for $x \in (-2, 0)$, and

$$F(x) = -\int_0^x f(z) \mathrm{d}z = -\frac{x^4}{4} - \frac{x^3}{3} + x^2. \tag{2.9}$$

Consequently, F is decreasing and positive on $[-2, 0)$ and increasing and positive on $(0, 1]$. Since $F(1) = 5/12$ and $F(-1) = 13/12$, there exists a unique $\overline{B} \in (-1, 0)$ such that $F(\overline{B}) = 5/12 = F(1)$. We can see that all assumptions of Theorem 2.3 are fulfilled and so, for each $u_0 \in [\overline{B}, 1)$, the problem (2.8), (1.13) has a unique solution which is damped. We will show later (see Example 2.11), that each damped solution of the problem (2.8), (1.13) is oscillatory.

In the next example, we will show that damped solutions can be nonzero and monotonous on $[0, \infty)$ with a limit equal to zero at ∞. Clearly, such solutions are not oscillatory.

Example 2.5. Consider the equation

$$u'' + \frac{3}{t} u' = f(u), \tag{2.10}$$

where

$$f(x) = \begin{cases} -x^3 & \text{for } x \le 1, \\ x - 2 & \text{for } x \in (1, 3), \\ 1 & \text{for } x \ge 3. \end{cases} \tag{2.11}$$

We see that $p(t) = t^3$ in (2.10) and the functions f and p satisfy conditions (1.2)–(1.5), (1.8), and (1.9) with $L = 2$. Clearly, $L_0 = -\infty$. Further,

$$F(x) = -\int_0^x f(z)\mathrm{d}z = \begin{cases} \dfrac{x^4}{4} & \text{for } x \le 1, \\ -\dfrac{x^2}{2} + 2x - \dfrac{5}{4} & \text{for } x \in (1,3), \\ -x + \dfrac{13}{4} & \text{for } x \ge 3. \end{cases} \tag{2.12}$$

Since $F(L) = F(2) = 3/4$, assumption (1.5) yields $F(\overline{B}) = \overline{B}^4/4 = 3/4$ and $\overline{B} = -3^{1/4}$. By Theorem 2.3, for each $u_0 \in [-3^{1/4}, 2)$, the problem (2.10), (1.13) has a unique solution u which is damped. On the other hand, we can check by a direct computation that for each $u_0 \le 1$ the function

$$u(t) = \frac{8u_0}{8 + u_0^2 t^2}, \quad t \in [0, \infty) \tag{2.13}$$

is a solution of equation (2.10) and satifies conditions (1.13). If $u_0 < 0$, then $u < 0$, $u' > 0$ on $(0, \infty)$, and if $u_0 \in (0, 1]$, then $u > 0$, $u' < 0$ on $(0, \infty)$. In both cases, $\lim_{t\to\infty} u(t) = 0$.

In Example 2.5, we also demonstrate that there are equations fulfilling Theorem 2.3 for which all solutions with $u_0 < L$, not only those with $u_0 \in [\overline{B}, L)$, are damped. Some additional conditions giving, moreover, bounding homoclinic solutions and escape solutions are presented in [15–17].

In our further investigation of asymptotic properties of damped solutions the following lemmas are useful.

Lemma 2.6. *Assume* (1.2), (1.8), *and* (1.9). *Let u be a damped solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, L)$ which is eventually positive or eventually negative, then*

$$\lim_{t\to\infty} u(t) = 0, \qquad \lim_{t\to\infty} u'(t) = 0. \tag{2.14}$$

Proof. Let u be eventually positive, that is, there exists $t_0 \ge 0$ such that

$$u(t) > 0 \quad \text{for } t \in [t_0, \infty). \tag{2.15}$$

Denote $\theta = \inf\{t_0 \ge 0 : u(t) > 0,\ t \in [t_0, \infty)\}$.

Let $\theta > 0$, then $u(\theta) = 0$ and, by Remark 2.2, $u'(\theta) > 0$. Assume that $u' > 0$ on (θ, ∞), then u is increasing on (θ, ∞), and there exists $\lim_{t\to\infty} u(t) = \ell \in (0, L)$. Multiplying (2.4) by u', integrating between θ and t, and using notation (1.4), we obtain

$$\frac{u'^2(t)}{2} + \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s = F(u_0) - F(u(t)), \quad t \in (\theta, \infty). \tag{2.16}$$

Letting $t \to \infty$, we get

$$\lim_{t\to\infty} \frac{u'^2(t)}{2} = -\lim_{t\to\infty} \int_\theta^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s + F(u_0) - F(\ell). \tag{2.17}$$

Since the function $\int_\theta^t (p'(s)/p(s))u'^2(s)\mathrm{d}s$ is positive and increasing, it follows that it has a limit at ∞, and hence there exists also $\lim_{t\to\infty} u'(t) \geq 0$. If $\lim_{t\to\infty} u'(t) > 0$, then $L > l = \lim_{t\to\infty} u(t) = \infty$, which is a contradiction. Consequently

$$\lim_{t\to\infty} u'(t) = 0. \tag{2.18}$$

Letting $t \to \infty$ in (2.4) and using (1.2), (1.9) and $\ell \in (0, L)$, we get $\lim_{t\to\infty} u''(t) = f(\ell) < 0$, and so $\lim_{t\to\infty} u'(t) = -\infty$, which is contrary to (2.18). This contradiction implies that the inequality $u' > 0$ on (θ, ∞) cannot be satisfied and that there exists $a > \theta$ such that $u'(a) = 0$. Since $u > 0$ on (a, ∞), we get by (1.2), (1.7), and (1.13) that $(pu')' < 0$ on (a, ∞). Due to $p(a)u'(a) = 0$, we see that $u' < 0$ on (a, ∞). Therefore, u is decreasing on (a, ∞) and $\lim_{t\to\infty} u(t) = \ell_0 \in [0, L)$. Using (2.16) with a in place of θ, we deduce as above that (2.18) holds and that $\lim_{t\to\infty} u''(t) = f(\ell_0) = 0$. Consequently, $\ell_0 = 0$. We have proved that (2.14) holds provided $\theta > 0$.

If $\theta = 0$, then we take $a = 0$ and use the above arguments. If u is eventually negative, we argue similarly. □

Lemma 2.7. *Assume* (1.2)–(1.5), (1.8), (1.9), *and*

$$p \in C^2(0, \infty), \quad \limsup_{t\to\infty} \left| \frac{p''(t)}{p'(t)} \right| < \infty, \tag{2.19}$$

$$\lim_{x\to 0+} \frac{f(x)}{x} < 0. \tag{2.20}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (0, L)$, then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \qquad u'(t) < 0 \quad \text{for } t \in (0, \delta_1]. \tag{2.21}$$

Proof. Assume that such δ_1 does not exist, then u is positive on $[0, \infty)$ and, by Lemma 2.6, u satisfies (2.14). We define a function

$$v(t) = \sqrt{p(t)}\, u(t), \quad t \in [0, \infty). \tag{2.22}$$

By (2.19), we have $v \in C^2(0, \infty)$ and

$$v'(t) = \frac{p'(t)u(t)}{2\sqrt{p(t)}} + \sqrt{p(t)}\, u'(t), \tag{2.23}$$

$$v''(t) = v(t) \left[\frac{1}{2} \frac{p''(t)}{p(t)} - \frac{1}{4} \left(\frac{p'(t)}{p(t)} \right)^2 + \frac{f(u(t))}{u(t)} \right], \quad t \in (0, \infty). \tag{2.24}$$

By (1.9) and (2.19), we get

$$\lim_{t\to\infty}\left[\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2\right]=\frac{1}{2}\lim_{t\to\infty}\frac{p''(t)}{p'(t)}\cdot\frac{p'(t)}{p(t)}=0. \tag{2.25}$$

Since u is positive on $(0,\infty)$, conditions (2.14) and (2.20) yield

$$\lim_{t\to\infty}\frac{f(u(t))}{u(t)}=\lim_{x\to 0+}\frac{f(x)}{x}<0. \tag{2.26}$$

Consequently, there exist $\omega>0$ and $R>0$ such that

$$\frac{1}{2}\frac{p''(t)}{p(t)}-\frac{1}{4}\left(\frac{p'(t)}{p(t)}\right)^2+\frac{f(u(t))}{u(t)}<-\omega \quad \text{for } t\ge R. \tag{2.27}$$

By (2.22), v is positive on $(0,\infty)$ and, due to (2.24) and (2.27), we get

$$v''(t)<-\omega v(t)<0 \quad \text{for } t\ge R. \tag{2.28}$$

Thus, v' is decreasing on $[R,\infty)$ and $\lim_{t\to\infty}v'(t)=V$. If $V<0$, then $\lim_{t\to\infty}v(t)=-\infty$, contrary to the positivity of v. If $V\ge 0$, then $v'>0$ on $[R,\infty)$ and $v(t)\ge v(R)>0$ for $t\in[R,\infty)$. Then (2.28) yields $0>-\omega v(R)\ge -\omega v(t)>v''(t)$ for $t\in[R,\infty)$. We get $\lim_{t\to\infty}v'(t)=-\infty$ which contradicts $V\ge 0$. The obtained contradictions imply that u has at least one zero in $(0,\infty)$. Let $\delta_1>0$ be the first zero of u. Then $u>0$ on $[0,\delta_1)$ and, by (1.2) and (1.7), $u'<0$ on $(0,\delta_1)$. Due to Remark 2.2, we have also $u'(\delta_1)<0$. □

For negative starting value, we can prove a dual lemma by similar arguments.

Lemma 2.8. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.19) *and*

$$\lim_{x\to 0-}\frac{f(x)}{x}<0. \tag{2.29}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0\in(L_0,0)$, then there exists $\theta_1>0$ such that*

$$u(\theta_1)=0, \qquad u'(t)>0 \quad \textit{for } t\in(0,\theta_1]. \tag{2.30}$$

The arguments of the proof of Lemma 2.8 can be also found in the proof of Lemma 3.1 in [18], where both (2.20) and (2.29) were assumed. If one argues as in the proofs of Lemmas 2.7 and 2.8 working with a_1, A_1 and b_1, B_1 in place of 0, and u_0, one gets the next corollary.

Corollary 2.9. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.19), (2.20), *and* (2.29). *Let u be a solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$.*

(I) Assume that there exist $b_1 > 0$ and $B_1 \in (L_0, 0)$ such that

$$u(b_1) = B_1, \quad u'(b_1) = 0, \tag{2.31}$$

then there exists $\theta > b_1$ such that

$$u(\theta) = 0, \qquad u'(t) > 0 \quad \textit{for } t \in (b_1, \theta]. \tag{2.32}$$

(II) Assume that there exist $a_1 > 0$ and $A_1 \in (0, L)$ such that

$$u(a_1) = A_1, \quad u'(a_1) = 0, \tag{2.33}$$

then there exists $\delta > a_1$ such that

$$u(\delta) = 0, \qquad u'(t) < 0 \quad \textit{for } t \in (a_1, \delta]. \tag{2.34}$$

Note that if all conditions of Lemmas 2.7 and 2.8 are satisfied, then each solution of the problem (1.7), (1.13) with $u_0 \in (L_0, 0) \cup (0, L)$ has at least one simple zero in $(0, \infty)$. Corollary 2.9 makes possible to construct an unbounded sequence of all zeros of any damped solution u. In addition, these zeros are simple (see the proof of Theorem 2.10). In such a case, u has either a positive maximum or a negative minimum between each two neighbouring zeros. If we denote sequences of these maxima and minima by $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$, respectively, then we call the numbers $|A_n - B_n|$, $n \in \mathbb{N}$ amplitudes of u.

In [18], we give conditions implying that each damped solution of the problem (1.7), (1.13) with $u_0 < 0$ has an unbounded set of zeros and decreasing sequence of amplitudes. Here, there is an extension of this result for $u_0 \in (0, L)$.

Theorem 2.10 (existence of oscillatory solutions I). *Assume that* (1.2)–(1.5), (1.8), (1.9), (2.19), (2.20), *and* (2.29) *hold, Then each damped solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$ is oscillatory and its amplitudes are decreasing.*

Proof. For $u_0 < 0$, the assertion is contained in Theorem 3.4 of [18]. Let u be a damped solution of the problem (1.7), (1.13) with $u_0 \in (0, L)$. By (2.2) and Definition 1.2, we can find $L_1 \in (0, L)$ such that

$$\overline{B} < u(t) \le L_1 \quad \text{for } t \in [0, \infty). \tag{2.35}$$

Step 1. Lemma 2.7 yields $\delta_1 > 0$ satisfying (2.21). Hence, there exists a maximal interval (δ_1, b_1) such that $u' < 0$ on (δ_1, b_1). If $b_1 = \infty$, then u is eventually negative and decreasing. On the other hand, by Lemma 2.6, u satisfies (2.14). But this is not possible. Therefore, $b_1 < \infty$ and there exists $B_1 \in (\overline{B}, 0)$ such that (2.31) holds. Corollary 2.9 yields $\theta_1 > b_1$ satisfying (2.32) with $\theta = \theta_1$. Therefore, u has just one negative local minimum $B_1 = u(b_1)$ between its first zero δ_1 and second zero θ_1.

Step 2. By (2.32) there exists a maximal interval (θ_1, a_1), where $u' > 0$. If $a_1 = \infty$, then u is eventually positive and increasing. On the other hand, by Lemma 2.6, u satisfies (2.14). We get a contradiction. Therefore $a_1 < \infty$ and there exists $A_1 \in (0, L)$ such that (2.33) holds. Corollary 2.9 yields $\delta_2 > a_1$ satisfying (2.34) with $\delta = \delta_2$. Therefore u has just one positive maximum $A_1 = u(a_1)$ between its second zero θ_1 and third zero δ_2.

Step 3. We can continue as in Steps 1 and 2 and get the sequences $\{A_n\}_{n=1}^{\infty} \subset (0, L)$ and $\{B_n\}_{n=1}^{\infty} \subset [u_0, 0)$ of positive local maxima and negative local minima of u, respectively. Therefore u is oscillatory. Using arguments of the proof of Theorem 3.4 of [18], we get that the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. In particular, we use (2.5) and define a Lyapunov function V_u by

$$V_u(t) = \frac{u'^2(t)}{2} + F(u(t)) = F(u_0) - \int_0^t \frac{p'(s)}{p(s)} u'^2(s) \mathrm{d}s, \quad t \in (0, \infty), \tag{2.36}$$

then

$$V_u(t) > 0, \quad V_u'(t) = -\frac{p'(t)}{p(t)} u'^2(t) \le 0 \quad \text{for } t \in (0, \infty), \tag{2.37}$$

$$V_u'(t) < 0 \quad \text{for } t \in (0, \infty),\ t \ne a_n, b_n,\ n \in \mathbb{N}. \tag{2.38}$$

Consequently,

$$c_u := \lim_{t \to \infty} V_u(t) \ge 0. \tag{2.39}$$

So, sequences $\{V_u(a_n)\}_{n=1}^{\infty} = \{F(A_n)\}_{n=1}^{\infty}$ and $\{V_u(b_n)\}_{n=1}^{\infty} = \{F(B_n)\}_{n=1}^{\infty}$ are decreasing and

$$\lim_{n \to \infty} F(A_n) = \lim_{n \to \infty} F(B_n) = c_u. \tag{2.40}$$

Finally, due to (1.4), the sequence $\{A_n\}_{n=1}^{\infty}$ is decreasing and the sequence $\{B_n\}_{n=1}^{\infty}$ is increasing. Hence, the sequence of amplitudes $\{A_n - B_n\}_{n=1}^{\infty}$ is decreasing, as well. $\square$

Example 2.11. Consider the problem (1.7), (1.13), where $p(t) = t^2$ and $f(x) = x(x-1)(x+2)$. In Example 2.4, we have shown that (1.2)–(1.5), (1.8), and (1.9) with $L_0 = -2$, $L = 1$ are valid. Since

$$\begin{gathered} \lim_{t \to \infty} \frac{p''(t)}{p'(t)} = \lim_{t \to \infty} \frac{1}{t} = 0, \\ \lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} (x-1)(x+2) = -2 < 0, \end{gathered} \tag{2.41}$$

we see that (2.19), (2.20), and (2.29) are satisfied. Therefore, by Theorem 2.10, each damped solution of (2.8), (1.13) with $u_0 \in (-2,0) \cup (0,1)$ is oscillatory and its amplitudes are decreasing.

Example 2.12. Consider the problem (1.7), (1.13), where

$$p(t) = \frac{t^k}{1+t^\ell}, \quad k > \ell \geq 0,$$
$$f(x) = \begin{cases} x(x-1)(x+3), & \text{for } x \leq 0, \\ x(x-1)(x+4), & \text{for } x > 0, \end{cases} \tag{2.42}$$

then $L_0 = -3$, $L = 1$,

$$\lim_{t\to\infty} \frac{p''(t)}{p'(t)} = 0, \qquad \lim_{x\to 0-} \frac{f(x)}{x} = -3, \qquad \lim_{x\to 0+} \frac{f(x)}{x} = -4. \tag{2.43}$$

We can check that also all remaining assumptions of Theorem 2.10 are satisfied, and this theorem is applicable here.

Assume that f does not fulfil (2.20) and (2.29). It occurs, for example, if $f(x) = -|x|^\alpha \operatorname{sign} x$ with $\alpha > 1$ for x in some neighbourhood of 0, then Theorem 2.10 cannot be applied. Now, we will give another sufficient conditions for the existence of oscillatory solutions. For this purpose, we introduce the following lemmas.

Lemma 2.13. *Assume* (1.2)–(1.5), (1.8), (1.9), *and*

$$\int_1^\infty \frac{1}{p(s)} \mathrm{d}s = \infty, \tag{2.44}$$

$$\exists \epsilon > 0 : f \in C^1(0,\epsilon), \qquad f' \leq 0 \quad \text{on } (0,\epsilon). \tag{2.45}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (0,L)$, then there exists $\delta_1 > 0$ such that*

$$u(\delta_1) = 0, \qquad u'(t) < 0 \quad \text{for } t \in (0,\delta_1]. \tag{2.46}$$

Proof. Assume that such δ_1 does not exist, then u is positive on $[0,\infty)$ and, by Lemma 2.6, u satisfies (2.14). In view of (1.7) and (1.2), we have $u' < 0$ on $(0,\infty)$. From (2.45), it follows that there exists $t_0 > 0$ such that

$$0 < u(t) < \epsilon, \quad \text{for } t \in [t_0,\infty). \tag{2.47}$$

Motivated by arguments of [27], we divide (1.7) by $f(u)$ and integrate it over interval $[t_0, t]$. We get

$$\int_{t_0}^{t} \frac{(p(s)u'(s))'}{f(u(s))} \mathrm{d}s = \int_{t_0}^{t} p(s)\mathrm{d}s \quad \text{for } t \in [t_0, \infty). \tag{2.48}$$

Using the per partes integration, we obtain

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s = \frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s)\mathrm{d}s, \quad t \in [t_0, \infty). \tag{2.49}$$

From (1.8) and (1.9), it follows that there exists $t_1 \in (t_0, \infty)$ such that

$$\frac{p(t_0)u'(t_0)}{f(u(t_0))} + \int_{t_0}^{t} p(s)\mathrm{d}s \geq 1, \quad t \in [t_1, \infty), \tag{2.50}$$

and therefore

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_0}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq 1, \quad t \in [t_1, \infty). \tag{2.51}$$

From the fact that $f'(u(s)) \leq 0$ for $s > t_0$ (see (2.45)), we have

$$\frac{p(t)u'(t)}{f(u(t))} + \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq 1, \quad t \in [t_1, \infty), \tag{2.52}$$

then

$$\frac{p(t)u'(t)}{f(u(t))} \geq 1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s > 0, \quad t \in [t_1, \infty), \tag{2.53}$$

$$\frac{p(t)u'(t)}{f(u(t))\left(1 - \int_{t_1}^{t} p(s)f'(u(s))u'^2(s)f^{-2}(u(s))\mathrm{d}s\right)} \geq 1, \quad t \in [t_1, \infty). \tag{2.54}$$

Multiplying this inequality by $-f'(u(t))u'(t)/f(u(t)) \geq 0$, we get

$$\left(\ln\left(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s\right)\right)' \geq -\left(\ln\left|f(u(t))\right|\right)', \quad t \in [t_1, \infty), \tag{2.55}$$

and integrating it over $[t_1, t]$, we obtain

$$\ln\left(1 - \int_{t_1}^{t} \frac{p(s)f'(u(s))u'^2(s)}{f^2(u(s))} \mathrm{d}s\right) \geq \ln\left(\frac{f(u(t_1))}{f(u(t))}\right), \tag{2.56}$$

and therefore,

$$1 - \int_{t_1}^{t} \frac{p(s) f'(u(s)) u'^2(s)}{f^2(u(s))} \mathrm{d}s \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty). \tag{2.57}$$

According to (2.53), we have

$$\frac{p(t)u'(t)}{f(u(t))} \geq \frac{f(u(t_1))}{f(u(t))}, \quad t \in [t_1, \infty), \tag{2.58}$$

and consequently,

$$u'(t) \leq f(u(t_1)) \frac{1}{p(t)}, \quad t \in [t_1, \infty). \tag{2.59}$$

Integrating it over $[t_1, t]$, we get

$$u(t) \leq u(t_1) + f(u(t_1)) \int_{t_1}^{t} \frac{1}{p(s)} \mathrm{d}s, \quad t \in [t_1, \infty). \tag{2.60}$$

From (2.44), it follows that

$$\lim_{t \to \infty} u(t) = -\infty, \tag{2.61}$$

which is a contradiction. □

By similar arguments, we can prove a dual lemma.

Lemma 2.14. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.44), *and*

$$\exists \epsilon > 0 : f \in C^1(-\epsilon, 0), \quad f' \leq 0 \text{ on } (-\epsilon, 0). \tag{2.62}$$

Let u be a solution of the problem (1.7), (1.13) *with $u_0 \in (L_0, 0)$, then, there exists $\theta_1 > 0$ such that*

$$u(\theta_1) = 0, \qquad u'(t) > 0 \quad \textit{for } t \in (0, \theta_1]. \tag{2.63}$$

Following ideas before Corollary 2.9, we get the next corollary.

Corollary 2.15. *Assume* (1.2)–(1.5), (1.8), (1.9), (2.44), (2.45), *and* (2.62). *Let u be a solution of the problem* (1.7), (1.13) *with $u_0 \in (L_0, 0) \cup (0, L)$, then the assertions I and II of Corollary 2.9 are valid.*

Now, we are able to formulate another existence result for oscillatory solutions. Its proof is almost the same as the proof of Theorem 2.10 for $u_0 \in (L_0, 0)$ and the proof of Theorem 3.4 in [18] for $u_0 \in (0, L)$. The only difference is that we use Lemmas 2.13, 2.14, and Corollary 2.15, in place of Lemmas 2.7, 2.8, and Corollary 2.9, respectively.

Theorem 2.16 (existence of oscillatory solutions II). *Assume that* (1.2)–(1.5), (1.8), (1.9), (2.44), (2.45), *and* (2.62) *hold, then each damped solution of the problem* (1.7), (1.13) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *is oscillatory and its amplitudes are decreasing.*

Example 2.17. Let us consider (1.7) with

$$p(t) = t^{\alpha}, \quad t \in [0, \infty),$$
$$f(x) = \begin{cases} -|x|^{\lambda}\operatorname{sgn} x, & x \le 1, \\ x - 2, & x \in (1, 3), \\ 1, & x \ge 3, \end{cases} \tag{2.64}$$

where λ and α are real parameters.

Case 1. Let $\lambda \in (1, \infty)$ and $\alpha \in (0, 1]$, then all assumptions of Theorem 2.16 are satisfied. Note that f satisfies neither (2.20) nor (2.29) and hence Theorem 2.10 cannot be applied.

Case 2. Let $\lambda = 1$ and $\alpha \in (0, \infty)$, then all assumptions of Theorem 2.10 are satisfied. If $\alpha \in (0, 1]$, then also all assumptions of Theorem 2.16 are fulfilled, but for $\alpha \in (1, \infty)$, the function p does not satisfy (2.44), and hence Theorem 2.16 cannot be applied.

3. Asymptotic Properties of Oscillatory Solutions

In Lemma 2.6 we show that if u is a damped solution of the problem (1.7), (1.13) which is not oscillatory then u converges to 0 for $t \to \infty$. In this section, we give conditions under which also oscillatory solutions converge to 0.

Theorem 3.1. *Assume that* (1.2)–(1.5), (1.8), *and* (1.9) *hold and that there exists* $k_0 > 0$ *such that*

$$\liminf_{t \to \infty} \frac{p(t)}{t^{k_0}} > 0, \tag{3.1}$$

then each damped oscillatory solution u of the problem (1.7), (1.13) *with* $u_0 \in (L_0, 0) \cup (0, L)$ *satisfies*

$$\lim_{t \to \infty} u(t) = 0, \qquad \lim_{t \to \infty} u'(t) = 0. \tag{3.2}$$

Proof. Consider an oscillatory solution u of the problem (1.7), (1.13) with $u_0 \in (0, L)$.

Step 1. Using the notation and some arguments of the proof of Theorem 2.10, we have the unbounded sequences $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\theta_n\}_{n=1}^{\infty}$, and $\{\delta_n\}_{n=1}^{\infty}$, such that

$$0 < \delta_1 < b_1 < \theta_1 < a_1 < \delta_2 < \cdots < \delta_n < b_n < \theta_n < a_n < \delta_{n+1} < \cdots, \tag{3.3}$$

where $u(\theta_n) = u(\delta_n) = 0$, $u(a_n) = A_n > 0$ is a unique local maximum of u in (θ_n, δ_{n+1}), $u(b_n) = B_n < 0$ is a unique local minimum of u in (δ_n, θ_n), $n \in \mathbb{N}$. Let V_u be given by (2.36) and then (2.39) and (2.40) hold and, by (1.2)–(1.4), we see that

$$\lim_{t\to\infty} u(t) = 0 \Longleftrightarrow c_u = 0. \tag{3.4}$$

Assume that (3.2) does not hold. Then $c_u > 0$. Motivated by arguments of [28], we derive a contradiction in the following steps.

Step 2 (estimates of u). By (2.36) and (2.39), we have

$$\lim_{n\to\infty} \frac{u'^2(\delta_n)}{2} = \lim_{n\to\infty} \frac{u'^2(\theta_n)}{2} = c_u > 0, \tag{3.5}$$

and the sequences $\{u'^2(\delta_n)\}_{n=1}^{\infty}$ and $\{u'^2(\theta_n)\}_{n=1}^{\infty}$ are decreasing. Consider $n \in \mathbb{N}$. Then $u'^2(\delta_n)/2 > c_u$ and there are α_n, β_n satisfying $a_n < \alpha_n < \delta_n < \beta_n < b_n$ and such that

$$u'^2(\alpha_n) = u'^2(\beta_n) = c_u, \quad u'^2(t) > c_u, \quad t \in (\alpha_n, \beta_n). \tag{3.6}$$

Since $V_u(t) > c_u$ for $t > 0$ (see (2.39)), we get by (2.36) and (3.6) the inequalities $c_u/2 + F(u(\alpha_n)) > c_u$ and $c_u/2 + F(u(\beta_n)) > c_u$, and consequently $F(u(\alpha_n)) > c_u/2$ and $F(u(\beta_n)) > c_u/2$. Therefore, due to (1.4), there exists $\tilde{c} > 0$ such that

$$u(\alpha_n) > \tilde{c}, \quad u(\beta_n) < -\tilde{c}, \quad n \in \mathbb{N}. \tag{3.7}$$

Similarly, we deduce that there are $\tilde{\alpha}_n, \tilde{\beta}_n$ satisfying $b_n < \tilde{\alpha}_n < \theta_n < \tilde{\beta}_n < a_{n+1}$ and such that

$$u(\tilde{\alpha}_n) < -\tilde{c}, \quad u\left(\tilde{\beta}_n\right) > \tilde{c}, \quad n \in \mathbb{N}. \tag{3.8}$$

The behaviour of u and inequalities (3.7) and (3.8) yield

$$|u(t)| > \tilde{c}, \qquad t \in [\beta_n, \tilde{\alpha}_n] \cup \left[\tilde{\beta}_n, \alpha_{n+1}\right], \ n \in \mathbb{N}. \tag{3.9}$$

Step 3 (estimates of $\beta_n - \alpha_n$). We prove that there exist $c_0, c_1 \in (0, \infty)$ such that

$$c_0 < \beta_n - \alpha_n < c_1, \quad n \in \mathbb{N}. \tag{3.10}$$

Assume on the contrary that there exists a subsequence satisfying $\lim_{\ell\to\infty}(\beta_\ell - \alpha_\ell) = 0$. By the mean value theorem and (3.7), there is $\xi_\ell \in (\alpha_\ell, \beta_\ell)$ such that $0 < 2\tilde{c} < u(\alpha_\ell) - u(\beta_\ell) = |u'(\xi_\ell)|(\beta_\ell - \alpha_\ell)$. Since $F(u(t)) \ge 0$ for $t \in [0, \infty)$, we get by (2.16) the inequality

$$\left|u'(t)\right| < \sqrt{2F(u_0)}, \quad t \in [0, \infty), \tag{3.11}$$

and consequently

$$0 < 2\tilde{c} \leq \sqrt{2F(u_0)} \lim_{\ell \to \infty} (\beta_\ell - \alpha_\ell) = 0, \tag{3.12}$$

which is a contradiction. So, c_0 satisfying (3.10) exists. Using the mean value theorem again, we can find $\tau_n \in (\alpha_n, \delta_n)$ such that $u(\delta_n) - u(\alpha_n) = u'(\tau_n)(\delta_n - \alpha_n)$ and, by (3.6),

$$\delta_n - \alpha_n = \frac{-u(\alpha_n)}{u'(\tau_n)} = \frac{u(\alpha_n)}{|u'(\tau_n)|} < \frac{A_1}{\sqrt{c_u}}. \tag{3.13}$$

Similarly, we can find $\eta_n \in (\delta_n, \beta_n)$ such that

$$\beta_n - \delta_n = \frac{u(\beta_n)}{u'(\eta_n)} = \frac{|u(\beta_n)|}{|u'(\eta_n)|} < \frac{|B_1|}{\sqrt{c_u}}. \tag{3.14}$$

If we put $c_1 = (A_1 + |B_1|)/\sqrt{c_u}$, then (3.10) is fulfilled. Similarly, we can prove

$$c_0 < \tilde{\beta}_n - \tilde{\alpha}_n < c_1, \quad n \in \mathbb{N}. \tag{3.15}$$

Step 4 (estimates of $\alpha_{n+1} - \alpha_n$). We prove that there exist $c_2 \in (0, \infty)$ such that

$$\alpha_{n+1} - \alpha_n < c_2, \quad n \in \mathbb{N}. \tag{3.16}$$

Put $m_1 = \min\{f(x) : B_1 \leq x \leq -\tilde{c}\} > 0$. By (3.9), $B_1 \leq u(t) < -\tilde{c}$ for $t \in [\beta_n, \tilde{\alpha}_n]$, $n \in \mathbb{N}$. Therefore,

$$f(u(t)) \geq m_1, \quad t \in [\beta_n, \tilde{\alpha}_n], \ n \in \mathbb{N}. \tag{3.17}$$

Due to (1.9), we can find $t_1 > 0$ such that

$$\frac{p'(t)}{p(t)} \sqrt{2F(u_0)} < \frac{m_1}{2}, \quad t \in [t_1, \infty). \tag{3.18}$$

Let $n_1 \in \mathbb{N}$ fulfil $\alpha_{n_1} \geq t_1$, then, according to (2.4), (3.11), (3.17), and (3.18), we have

$$u''(t) > -\frac{m_1}{2} + m_1 = \frac{m_1}{2}, \quad t \in [\beta_n, \tilde{\alpha}_n], \ n \geq n_1. \tag{3.19}$$

Integrating (3.19) from b_n to β_n and using (3.6), we get $2\sqrt{c_u} > m_1(b_n - \beta_n)$ for $n \geq n_1$. Similarly we get $2\sqrt{c_u} > m_1(\tilde{\alpha}_n - b_n)$ for $n \geq n_1$. Therefore

$$\frac{4}{m_1}\sqrt{c_u} > \tilde{\alpha}_n - \beta_n, \quad n \geq n_1. \tag{3.20}$$

By analogy, we put $m_2 = \min\{-f(x) : \widetilde{c} \le x \le A_1\} > 0$ and prove that there exists $n_2 \in \mathbb{N}$ such that

$$\frac{4}{m_2}\sqrt{c_u} > \alpha_{n+1} - \widetilde{\beta}_n, \quad n \ge n_2. \tag{3.21}$$

Inequalities (3.10), (3.15), (3.20), and (3.21) imply the existence of c_2 fulfilling (3.16).

Step 5 (construction of a contradiction). Choose $t_0 > c_1$ and integrate the equality in (2.37) from t_0 to $t > t_0$. We have

$$V_u(t) = V_u(t_0) - \int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau, \qquad t \ge t_0. \tag{3.22}$$

Choose $n_0 \in \mathbb{N}$ such that $\alpha_{n_0} > t_0$. Further, choose $n \in \mathbb{N}$, $n > n_0$ and assume that $t > \beta_n$, then, by (3.6),

$$\begin{aligned}\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau &> \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau \\ &> c_u \sum_{j=n_0}^{n} \int_{\alpha_j}^{\beta_j} \frac{p'(\tau)}{p(\tau)}\mathrm{d}\tau = c_u \sum_{j=n_0}^{n} [\ln p(\tau)]_{\alpha_j}^{\beta_j}.\end{aligned} \tag{3.23}$$

By virtue of (3.1) there exists $c_3 > 0$ such that $p(t)/t^{k_0} > c_3$ for $t \in [t_0, \infty)$. Thus, $\ln p(t) > \ln c_3 + k_0 \ln t$ and

$$\int_{t_0}^{t} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau > c_u \sum_{j=n_0}^{n} [\ln c_3 + k_0 \ln t]_{\alpha_j}^{\beta_j} = c_u k_0 \sum_{j=n_0}^{n} \ln \frac{\beta_j}{\alpha_j}. \tag{3.24}$$

Due to (3.10) and $c_1 < \alpha_{n_0}$, we have

$$1 < \frac{\beta_j}{\alpha_j} < 1 + \frac{c_1}{\alpha_j} < 2, \quad j = n_0, \dots, n, \tag{3.25}$$

and the mean value theorem yields $\xi_j \in (1, 2)$ such that

$$\ln \frac{\beta_j}{\alpha_j} = \left(\frac{\beta_j}{\alpha_j} - 1\right)\frac{1}{\xi_j} > \frac{\beta_j - \alpha_j}{2\alpha_j}, \quad j = n_0, \dots, n. \tag{3.26}$$

By (3.10) and (3.16), we deduce

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{\alpha_j}, \quad \alpha_j < jc_2 + \alpha_1, \ j = n_0, \dots, n. \tag{3.27}$$

Thus,

$$\frac{\beta_j - \alpha_j}{\alpha_j} > \frac{c_0}{jc_2 + \alpha_1}, \quad j = n_0, \ldots, n. \tag{3.28}$$

Using (3.24)–(3.28) and letting t to ∞, we obtain

$$\begin{aligned}\int_{t_0}^{\infty} \frac{p'(\tau)}{p(\tau)} u'^2(\tau)\mathrm{d}\tau &\geq c_u k_0 \sum_{n=n_0}^{\infty} \ln \frac{\beta_n}{\alpha_n} \geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{\beta_n - \alpha_n}{\alpha_n} \\ &\geq \frac{1}{2} c_u k_0 \sum_{n=n_0}^{\infty} \frac{c_0}{nc_2 + \alpha_1} = \infty.\end{aligned} \tag{3.29}$$

Using it in (3.22), we get $\lim_{t\to\infty} V_u(t) = -\infty$, which is a contradiction. So, we have proved that $c_u = 0$.

Using (2.4) and (3.4), we have

$$\lim_{t\to\infty} \left(\frac{u'^2(t)}{2} + \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \right) = F(u_0) - F(0) = F(u_0). \tag{3.30}$$

Since the function $\int_0^t (p'(s)/p(s)) u'^2(s)\mathrm{d}s$ is increasing, there exists

$$\lim_{t\to\infty} \int_0^t \frac{p'(s)}{p(s)} u'^2(s)\mathrm{d}s \leq F(u_0). \tag{3.31}$$

Therefore, there exists

$$\lim_{t\to\infty} u'^2(t) = \ell^2. \tag{3.32}$$

If $\ell > 0$, then $\lim_{t\to\infty} |u'(t)| = \ell$, which contradicts (3.4). Therefore, $\ell = 0$ and (3.2) is proved.

If $u_0 \in (L_0, 0)$, we argue analogously. □

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Research Article

Existence Results for Singular Boundary Value Problem of Nonlinear Fractional Differential Equation

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By applying a fixed point theorem for mappings that are decreasing with respect to a cone, this paper investigates the existence of positive solutions for the nonlinear fractional boundary value problem: $D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0$, $0 < t < 1$, $u(0) = u'(0) = u'(1) = 0$, where $2 < \alpha < 3$, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative.

1. Introduction

Many papers and books on fractional calculus differential equation have appeared recently. Most of them are devoted to the solvability of the linear initial fractional equation in terms of a special function [1–4]. Recently, there has been significant development in the existence of solutions and positive solutions to boundary value problems for fractional differential equations by the use of techniques of nonlinear analysis (fixed point theorems, Leray-Schauder theory, etc.), see [5, 6] and the references therein.

In this paper, we consider the following boundary value problems of the nonlinear fractional differential equation

$$\begin{gathered} D_{0^+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1,\ 2 < \alpha < 3, \\ u(0) = u'(0) = u'(1) = 0, \end{gathered} \tag{1.1}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative and $f(t,x)$ is singular at

$x = 0$. Our assumptions throughout are

(H_1) $f(t,x) : (0,1) \times (0,\infty) \to [0,\infty)$ is continuous,

(H_2) $f(t,x)$ is decreasing in x, for each fixed t,

(H_3) $\lim_{x\to 0^+} f(t,x) = \infty$ and $\lim_{x\to\infty} f(t,x) = 0$, uniformly on compact subsets of (0,1), and

(H_4) $0 < \int_0^1 f(t, q_\theta(t))dt < \infty$ for all $\theta > 0$ and q_θ as defined in (3.1).

The seminal paper by Gatica et al. [7] in 1989 has had a profound impact on the study of singular boundary value problems for ordinary differential equations (ODEs). They studied singularities of the type in (H_1)–(H_4) for second order Sturm-Louiville problems, and their key result hinged on an application of a particular fixed point theorem for operators which are decreasing with respect to a cone. Various authors have used these techniques to study singular problems of various types. For example, Henderson and Yin [8] as well as Eloe and Henderson [9, 10] have studied right focal, focal, conjugate, and multipoint singular boundary value problems for ODEs. However, as far as we know, no paper is concerned with boundary value problem for fractional differential equation by using this theorem. As a result, the goal of this paper is to fill the gap in this area.

Motivated by the above-mentioned papers and [11], the purpose of this paper is to establish the existence of solutions for the boundary value problem (1.1) by the use of a fixed point theorem used in [7, 11]. The paper has been organized as follows. In Section 2, we give basic definitions and provide some properties of the corresponding Green's function which are needed later. We also state the fixed point theorem from [7] for mappings that are decreasing with respect to a cone. In Section 3, we formulate two lemmas which establish a priori upper and lower bounds on solutions of (1.1). We then state and prove our main existence theorem.

For fractional differential equation and applications, we refer the reader to [1–3]. Concerning boundary value problems (1.1) with ordinary derivative (not fractional one), we refer the reader to [12, 13].

2. Some Preliminaries and a Fixed Point Theorem

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions and properties can be found in the literature.

Definition 2.1 (see [3]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, \tag{2.1}$$

provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.2 (see [3]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,\infty) \to R$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n-1 \le \alpha < n$, provided that the right-hand side is pointwise defined on $(0,\infty)$.

Definition 2.3. By a solution of the boundary value problem (1.1) we understand a function $u \in C[0,1]$ such that $D_{0^+}^{\alpha} u$ is continuous on $(0,1)$ and u satisfies (1.1).

Lemma 2.4 (see [3]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N} \tag{2.3}$$

for some $c_i \in R, i = 1,\ldots,N$, $N = [\alpha]$.

Lemma 2.5. *Given $f \in C[0,1]$, and $2 < \alpha < 3$, the unique solution of*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + f(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0 \end{gathered} \tag{2.4}$$

is

$$u(t) = \int_0^1 G(t,s) f(s) ds, \tag{2.5}$$

where

$$G(t,s) = \begin{cases} \dfrac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.6}$$

Proof. We may apply Lemma 2.4 to reduce (2.4) to an equivalent integral equation

$$u(t) = -I_{0+}^{\alpha} f(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} \tag{2.7}$$

for some $c_i \in R$, $i = 1,2,3$. From $u(0) = u'(0) = u'(1) = 0$, one has

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s) ds, \quad c_2 = c_3 = 0. \tag{2.8}$$

Therefore, the unique solution of problem (2.4) is

$$\begin{aligned} u(t) &= \int_0^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds \\ &= \int_0^t \left[\frac{t^{\alpha-1}(1-s)^{\alpha-2} - (t-s)^{\alpha-1}}{\Gamma(\alpha)} \right] f(s) ds + \int_t^1 \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} f(s) ds \\ &= \int_0^1 G(t,s) f(s) ds. \end{aligned} \tag{2.9}$$

□

Lemma 2.6. *The function $G(t,s)$ defined by* (2.6) *satisfies the following conditions:*

(i) $G(t,s) > 0$, $0 < t, s < 1$,

(ii) $q(t)G(1,s) \le G(t,s) \le G(1,s) = s(1-s)^{\alpha-2}/(\Gamma(\alpha))$ *for* $0 \le t, s \le 1$, *where* $q(t) = t^{\alpha-1}$.

Proof. Observing the expression of $G(t,s)$, it is clear that $G(t,s) > 0$ for $0 < t, s < 1$. For given $s \in (0,1)$, $G(t,s)$ is increasing with respect to t. Consequently, $G(t,s) \le G(1,s)$ for $0 \le t, s \le 1$. If $s \le t$, we have

$$\begin{aligned} G(t,s) &= \frac{t(t-ts)^{\alpha-2} - (t-s)(t-s)^{\alpha-2}}{\Gamma(\alpha)} \\ &\ge \frac{t(t-ts)^{\alpha-2} - (t-s)(t-ts)^{\alpha-2}}{\Gamma(\alpha)} \\ &= \frac{st^{\alpha-2}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \ge \frac{st^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} = q(t)G(1,s). \end{aligned} \tag{2.10}$$

If $t \le s$, we have

$$G(t,s) = \frac{t^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} \ge \frac{st^{\alpha-1}(1-s)^{\alpha-2}}{\Gamma(\alpha)} = q(t)G(1,s). \tag{2.11}$$

□

Let E be a Banach space, $P \subset E$ be a cone in E. Every cone P in E defines a partial ordering in E given by $x \le y$ if and only if $y - x \in P$. If $x \le y$ and $x \ne y$, we write $x < y$. A cone P is said to be normal if there exists a constant $N > 0$ such that $\theta \le x \le y$ implies $\|x\| \le N\|y\|$. If P is normal, then every order interval $[x,y] = \{z \in E \mid x \le z \le y\}$ is bounded. For the concepts and properties about the cone theory we refer to [14, 15].

Next we state the fixed point theorem due to Gatica et al. [7] which is instrumental in proving our existence results.

Theorem 2.7 (Gatica-Oliker-Waltman fixed point theorem). *Let E be a Banach space, $P \subset E$ be a normal cone, and $D \subset P$ be such that if $x, y \in D$ with $x \le y$, then $[x,y] \subset D$. Let $T : D \to P$ be a continuous, decreasing mapping which is compact on any closed order interval contained in D, and suppose there exists an $x_0 \subset D$ such that T^2x_0 is defined (where $T^2x_0 = T(Tx_0)$) and Tx_0, T^2x_0 are order comparable to x_0. Then T has a fixed point in D provided that either:*

(i) $Tx_0 \le x_0$ *and* $T^2x_0 \le x_0$;

(ii) $x_0 \le Tx_0$ *and* $x_0 \le T^2x_0$; *or*

(iii) *The complete sequence of iterates* $\{T^n x_0\}_{n=0}^{\infty}$ *is defined and there exists* $y_0 \in D$ *such that* $Ty_0 \in D$ *with* $y_0 \le T^n x_0$ *for all* $n \in \mathbb{N}$.

3. Main Results

In this section, we apply Theorem 2.7 to a sequence of operators that are decreasing with respect to a cone. These obtained fixed points provide a sequence of iterates which converges to a solution of (1.1).

Let the Banach space $E = C[0,1]$ with the maximum norm $\|u\| = \max_{t\in[0,1]}|u(t)|$, and let $P = \{u \in E \mid u(t) \geq 0, t \in [0,1]\}$. P is a norm cone in E. For $\theta > 0$, let

$$q_\theta(t) = \theta \cdot q(t), \tag{3.1}$$

where $q(t)$ is given in Lemma 2.6. Define $D \subset P$ by

$$D = \{u \in P \mid \exists\theta(u) > 0 \text{ such that } u(t) \geq q_\theta(t), t \in [0,1]\}, \tag{3.2}$$

and the integral operator $T : D \to P$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds, \tag{3.3}$$

where $G(t,s)$ is given in (2.6). It suffices to define D as above, since the singularity in f precludes us from defining T on all of P. Furthermore, it can easily be verified that T is well defined. In fact, note that for $u \in D$ there exists $\theta(u) > 0$ such that $u(t) \geq q_\theta(t)$ for all $t \in [0,1]$. Since $f(t,x)$ decreases with respect to x, we see $f(t,u(t)) \leq f(t,q_\theta(t))$ for $t \in [0,1]$. Thus,

$$0 \leq \int_0^1 G(t,s)f(s,u(s))ds \leq \int_0^1 f(s,q_\theta(s))ds < \infty. \tag{3.4}$$

Similarly, T is decreasing with respect to D.

Lemma 3.1. *$u \in D$ is a solution of* (1.1) *if and only if $Tu = u$.*

Proof. One direction of the lemma is obviously true. To see the other direction, let $u \in D$. Then $(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))ds$, and Tu satisfies (1.1). Moreover, by Lemma 2.6, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq q(t)\int_0^1 G(1,s)f(s,u(s))ds = q(t)\|Tu\|, \quad \forall t \in [0,1].\end{aligned} \tag{3.5}$$

Thus, there exists some $\theta(Tu)$ such that $(Tu)(t) \geq q_\theta(t)$, which implies that $Tu \in D$. That is, $T : D \to D$.

We now present two lemmas that are required in order to apply Theorem 2.7. The first establishes a priori upper bound on solutions, while the second establishes a priori lower bound on solutions. □

Lemma 3.2. *If f satisfies (H_1)–(H_4), then there exists an $S > 0$ such that $\|u\| \leq S$ for any solution $u \in D$ of* (1.1).

Proof. For the sake of contradiction, suppose that the conclusion is false. Then there exists a sequence $\{u_n\}_{n=1}^{\infty}$ of solutions to (1.1) such that $\|u_n\| \leq \|u_{n+1}\|$ with $\lim_{n\to\infty}\|u_n\| = \infty$. Note that for any solution $u_n \in D$ of (1.1), by (3.5), we have

$$u_n(t) = (Tu_n)(t) \geq q(t)\|u_n\|, \quad t \in [0,1], \ n \geq 1. \tag{3.6}$$

Then, assumptions (H_2) and (H_4) yield, for $0 \leq t \leq 1$ and all $n \geq 1$,

$$\begin{aligned} u_n(t) = (Tu_n)(t) &= \int_0^1 G(t,s) f(s, u_n(s)) ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^1 s(1-s)^{\alpha-2} f\left(s, q_{\|u_1\|}(s)\right) ds = N, \end{aligned} \tag{3.7}$$

for some $0 < N < +\infty$. In particular, $\|u_n\| \leq N$, for all $n \geq 1$, which contradicts $\lim_{n\to\infty}\|u_n\| = \infty$. □

Lemma 3.3. *If f satisfies (H_1)–(H_4), then there exists an $R > 0$ such that $\|u\| \geq R$ for any solution $u \in D$ of* (1.1)*.*

Proof. For the sake of contradiction, suppose $u_n(t) \to 0$ uniformly on $[0,1]$ as $n \to \infty$. Let $M = \inf\{G(t,s) : (t,s) \in [1/4, 3/4] \times [1/4, 3/4]\} > 0$. From ($H_3$), we see that $\lim_{x\to 0^+} f(t,x) = \infty$ uniformly on compact subsets of $(0,1)$. Hence, there exists some $\delta > 0$ such that for $t \in [1/4, 3/4]$ and $0 < x < \delta$, we have $f(t,x) \geq 2/M$. On the other hand, there exists an $n_0 \in N$ such that $n \geq n_0$ implies $0 < u_n(t) < \delta/2$, for $t \in (0,1)$. So, for $t \in [1/4, 3/4]$ and $n \geq n_0$,

$$\begin{aligned} u_n(t) = (Tu_n)(t) &= \int_0^1 G(t,s) f(s, u_n(s)) ds \geq \int_{1/4}^{3/4} G(t,s) f(s, u_n(s)) ds \\ &\geq M \int_{1/4}^{3/4} f\left(s, \frac{\delta}{2}\right) ds \geq M \int_{1/4}^{3/4} \frac{2}{M} d\text{s} = 1. \end{aligned} \tag{3.8}$$

But this contradicts the assumption that $\|u_n\| \to 0$ uniformly on $[0,1]$ as $n \to \infty$. Hence, there exists an $R > 0$ such that $R \leq \|u\|$. □

We now present the main result of the paper.

Theorem 3.4. *If f satisfies (H_1)–(H_4), then* (1.1) *has at least one positive solution.*

Proof. For each $n \geq 1$, defined $v_n : [0,1] \to [0, +\infty)$ by

$$v_n(t) = \int_0^1 G(t,s) f(s,n) ds. \tag{3.9}$$

By conditions (H_1)–(H_4), for $n \geq 1$,

$$0 < v_{n+1}(t) \leq v_n(t), \quad \text{on } (0,1], \tag{3.10}$$

$$\lim_{n\to\infty} v_n(t) = 0 \quad \text{uniformly on } [0,1]. \tag{3.11}$$

Now define a sequence of functions $f_n : (0,1) \times [0,+\infty)$, $n \geq 1$, by

$$f_n(t,x) = f(t, \max\{x, v_n(t)\}). \tag{3.12}$$

Then, for each $n \geq 1$, f_n is continuous and satisfies (H_2). Furthermore, for $n \geq 1$,

$$\begin{aligned} f_n(t,x) &\leq f(t,x) \quad \text{on } (0,1) \times (0,+\infty), \\ f_n(t,x) &\leq f(t,v_n(t)) \quad \text{on } (0,1) \times (0,+\infty). \end{aligned} \tag{3.13}$$

Note that f_n has effectively "removed the singularity" in $f(t,x)$ at $x = 0$, then we define a sequence of operators $T_n : P \to P$, $n \geq 1$, by

$$(T_n u)(t) = \int_0^1 G(t,s) f_n(s,u(s))ds, \ u \in P. \tag{3.14}$$

From standard arguments involving the Arzela-Ascoli Theorem, we know that each T_n is in fact a compact mapping on P. Furthermore, $T_n(0) \geq 0$ and $T_n^2(0) \geq 0$. By Theorem 2.7, for each $n \geq 1$, there exists $u_n \in P$ such that $T_n u_n(x) = u_n(t)$ for $t \in [0,1]$. Hence, for each $n \geq 1$, u_n satisfies the boundary conditions of the problem. In addition, for each u_n,

$$\begin{aligned} (T_n u_n)(t) &= \int_0^1 G(t,s) f_n(s,u_n(s))ds = \int_0^1 G(t,s) f_n(s, \max\{u_n(s), v_n(s)\})ds \\ &\leq \int_0^1 G(t,s) f_n(s, v_n(s))ds \leq T v_n(t), \end{aligned} \tag{3.15}$$

which implies

$$u_n(t) = (T_n u_n)(t) \leq T v_n(t), \quad t \in [0,1], \ n \in \mathrm{N}. \tag{3.16}$$

Arguing as in Lemma 3.2 and using (3.11), it is fairly straightforward to show that there exists an $S > 0$ such that $\|u_n\| \leq S$ for all $n \in \mathbb{N}$. Similarly, we can follow the argument of Lemma 3.3 and (3.5) to show that there exists an $R > 0$ such that

$$u_n(t) \geq q(t)R, \quad \text{on } [0,1], \ \text{for } n \geq 1. \tag{3.17}$$

Since $T : D \to D$ is a compact mapping, there is a subsequence of $\{Tu_n\}$ which converges to some $u^* \in D$. We relabel the subsequence as the original sequence so that $Tu_n \to u^*$ as $n \to \infty$.

To conclude the proof of this theorem, we need to show that

$$\lim_{n\to\infty} \|Tu_n - u_n\| = 0. \tag{3.18}$$

To that end, fixed $\theta = R$, and let $\varepsilon > 0$ be give. By the integrability condition (H_4), there exists $0 < \delta < 1$ such that

$$\int_0^\delta s(1-s)^{\alpha-2} f(s, q_\theta(s))\,ds < \frac{\Gamma(\alpha)}{2}\varepsilon. \tag{3.19}$$

Further, by (3.11), there exists an n_0 such that, for $n \geq n_0$,

$$v_n(t) \leq q_\theta(t) \quad \text{on } [\delta, 1], \tag{3.20}$$

so that

$$v_n(t) \leq q_\theta(t) \leq u_n(t) \quad \text{on } [\delta, 1]. \tag{3.21}$$

Thus, for $s \in [\delta, 1]$ and $n \geq n_0$,

$$f_n(s, u_n(s)) = f(s, \max\{u_n(s), v_n(s)\}) = f(s, u_n(s)), \tag{3.22}$$

and for $t \in [0, 1]$,

$$\begin{aligned} Tu_n(t) - u_n(t) &= Tu_n(t) - T_n u_n(t) \\ &= \int_0^1 G(t,s)\left[f(s, u_n(s)) - f_n(s, u_n(s))\right] ds. \end{aligned} \tag{3.23}$$

Thus, for $t \in [0, 1]$,

$$\begin{aligned} |Tu_n(t) - u_n(t)| &\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^\delta s(1-s)^{\alpha-2} f(s, u_n(s))\,ds + \int_0^\delta s(1-s)^{\alpha-2} f(s, \max\{u_n(s), v_n(s)\})\,ds\right] \\ &\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^\delta s(1-s)^{\alpha-2} f(s, u_n(s))\,ds + \int_0^\delta s(1-s)^{\alpha-2} f(s, u_n(s))\,ds\right] \\ &\leq \frac{2}{\Gamma(\alpha)} \int_0^\delta s(1-s)^{\alpha-2} f(s, q_\theta(s))\,ds < \varepsilon. \end{aligned} \tag{3.24}$$

Since $t \in [0, 1]$ was arbitrary, we conclude that $\|Tu_n - u_n\| \leq \varepsilon$ for all $n \geq n_0$. Hence, $u^* \in [q_R, S]$ and for $t \in [0, 1]$

$$Tu^*(t) = T\left(\lim_{n\to\infty} Tu_n(t)\right) = T\left(\lim_{n\to\infty} u_n(t)\right) = \lim_{n\to\infty} Tu_n = u^*(t). \tag{3.25}$$

□

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Research Article

Existence Theory for Pseudo-Symmetric Solution to p-Laplacian Differential Equations Involving Derivative

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We all-sidedly consider a three-point boundary value problem for p-Laplacian differential equation with nonlinear term involving derivative. Some new sufficient conditions are obtained for the existence of at least one, triple, or arbitrary odd positive pseudosymmetric solutions by using pseudosymmetric technique and fixed-point theory in cone. As an application, two examples are given to illustrate the main results.

1. Introduction

Recent research results indicate that considerable achievement was made in the existence of positive solutions to dynamic equations; for details, please see [1–6] and the references therein. In particular, the existence of positive pseudosymmetric solutions to p-Laplacian difference and differential equations attract many researchers' attention, such as [7–11]. The reason is that the pseudosymmetry problem not only has theoretical value, such as in the study of metric manifolds [12], but also has practical value itself; for example, we can apply this characteristic into studying the chemistry structure [13]. On another hand, there are much attention paid to the positive solutions of boundary value problems (BVPs) for differential equation with the nonlinear term involved with the derivative explicitly [14–18]. Hence, it is natural to continue study pseudosymmetric solutions to p-Laplacian differential equations with the nonlinear term involved with the first-order derivative explicitly.

First, let us recall some relevant results about BVPs with p-Laplacian, We would like to mention the results of Avery and Henderson [7, 8], Ma and Ge [11] and Sun and Ge [16].

Throughout this paper, we denote the p-Laplacian operator by $\varphi_p(u)$; that is, $\varphi_p(u) = |u|^{p-2}u$ for $p > 1$ with $(\varphi_p)^{-1} = \varphi_q$ and $1/p + 1/q = 1$.

For the three-point BVPs with p-Laplacian

$$\begin{gathered}(\varphi_p(u'(t)))' + h(t)f(t,u(t)) = 0 \quad \text{for } t \in [0,1], \\ u(0) = 0, \quad u(\eta) = u(1),\end{gathered} \tag{1.1}$$

here, $\eta \in (0,1)$ is constant, by using the five functionals fixed point theorem in a cone [19], Avery and Henderson [8] established the existence of at least *three* positive pseudosymmetric solutions to BVPs (1.1). The authors also obtained the similar results in their paper [7] for the discrete case. In addition, Ma and Ge [11] developed the existence of at least *two* positive pseudosymmetric solutions to BVPs (1.1) by using the monotone iterative technique.

For the three-point p-Laplacian BVPs with dependence on the first-order derivative

$$\begin{gathered}(\varphi_p(u'(t)))' + h(t)f(t,u(t),u'(t)) = 0 \quad \text{for } t \in [0,1], \\ u(0) = 0, \quad u(\eta) = u(1),\end{gathered} \tag{1.2}$$

Sun and Ge [16] obtained the existence of at least *two* positive pseudosymmetric solutions to BVPs (1.2) via the monotone iterative technique again. However, it is worth mentioning that the above-mentioned papers [7, 8, 10, 11, 16], the authors only considered results on the existence of positive pseudosymmetric solutions partly, they failed to further provide comprehensive results on the existence of positive pseudosymmetric solutions to p-Laplacian. Naturally, in this paper, we consider the existence of positive pseudosymmetric solutions for p-Laplacian differential equations in all respects.

Motivated by the references [7, 8, 10, 11, 16, 18], in present paper, we consider all-sidedly p-Laplacian BVPs (1.2), using the compression and expansion fixed point theorem [20] and Avery-Peterson fixed point theorem [21]. We obtain that there exist at least *one, triple or arbitrary odd* positive pseudosymmetric solutions to problem (1.2). In particular, we not only get some local properties of pseudosymmetric solutions, but also obtain that the position of pseudosymmetric solutions is determined under some conditions, which is much better than the results in papers [8, 11, 16]. Correspondingly, we generalize and improve the results in papers Avery and Henderson [8]. From the view of applications, two examples are given to illustrate the main results.

Throughout this paper, we assume that

(S1) $f(t,u,u') : [0,1] \times [0,\infty) \times (-\infty,+\infty) \to [0,\infty)$ is continuous, does not vanish identically on interval $[0,1]$, and $f(t,u,u')$ is pseudosymmetric about η on $[0,1]$,

(S2) $h(t) \in L([0,1],[0,\infty))$ is pseudosymmetric about η on $[0,1]$, and does not vanish identically on any closed subinterval of $[0,1]$. Furthermore, $0 < \int_0^1 h(t)dt < \infty$.

2. Preliminaries

In the preceding of this section, we state the definition of cone and several fixed point theorems needed later [20, 22]. In the rest of this section, we will prove that solving BVPs (1.2) is equivalent to finding the fixed points of a completely continuous operator.

We first list the definition of cone and the compression and expansion fixed point theorem [20, 22].

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(i) if $x \in P$ and $\lambda \geq 0$, then $\lambda x \in P$,

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Lemma 2.2 (see [20, 22]). *Let P be a cone in a Banach space E. Assume that Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. If $A : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|, \forall x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|, \forall x \in P \cap \partial\Omega_2$.

Then, A has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Given a nonnegative continuous functional γ on a cone P of a real Banach space E, we define, for each $d > 0$, the set $P(\gamma, d) = \{x \in P : \gamma(x) < d\}$.

Let γ and θ be nonnegative continuous convex functionals on P, α a nonnegative continuous concave functional on P, and ψ a nonnegative continuous functional on P respectively. We define the following convex sets:

$$\begin{gathered}
P(\gamma, \alpha, b, d) = \{x \in P : b \leq \alpha(x),\ \gamma(x) \leq d\}, \\
P(\gamma, \theta, \alpha, b, c, d) = \{x \in P : b \leq \alpha(x),\ \theta(x) \leq c, \gamma(x) \leq d\},
\end{gathered} \tag{2.1}$$

and a closed set $R(\gamma, \psi, a, d) = \{x \in P : a \leq \psi(x), \gamma(x) \leq d\}$.

Next, we list the fixed point theorem due to Avery-Peterson [21].

Lemma 2.3 (see [21]). *Let P be a cone in a real Banach space E and $\gamma,\ \theta,\ \alpha,\ \psi$ defined as above; moreover, ψ satisfies $\psi(\lambda' x) \leq \lambda' \psi(x)$ for $0 \leq \lambda' \leq 1$ such that for some positive numbers h and d,*

$$\alpha(x) \leq \psi(x),\ \|x\| \leq h\gamma(x), \tag{2.2}$$

for all $x \in \overline{P(\gamma, d)}$. Suppose that $A : \overline{P(\gamma, d)} \to \overline{P(\gamma, d)}$ is completely continuous and there exist positive real numbers a, b, c with $a < b$ such that

(i) $\{x \in P(\gamma, \theta, \alpha, b, c, d) : \alpha(x) > b\} \neq \emptyset$ *and* $\alpha(A(x)) > b$ *for* $x \in P(\gamma,\ \theta,\ \alpha,\ b,\ c,\ d)$,

(ii) $\alpha(A(x)) > b$ *for* $x \in P(\gamma,\ \alpha,\ b,\ d)$ *with* $\theta(A(x)) > c$,

(iii) $0 \notin R(\gamma, \psi, a, d)$ *and* $\psi(A(x)) < a$ *for all* $x \in R(\gamma,\ \psi,\ a,\ d)$ *with* $\psi(x) = a$.

Then, A has at least three fixed points $x_1,\ x_2,\ x_3 \in \overline{P(\gamma, d)}$ such that

$$\gamma(x_i) \leq d \quad \text{for } i = 1, 2, 3, b < \alpha(x_1),\ a < \psi(x_2),\ \alpha(x_2) < b \text{ with } \psi(x_3) < a. \tag{2.3}$$

Now, let $E = C^1([0,1],\mathbb{R})$. Then, E is a Banach space with norm

$$\|u\| = \max\left\{\max_{t\in[0,1]}|u(t)|, \max_{t\in[0,1]}\left|u'(t)\right|\right\}. \tag{2.4}$$

Define a cone $P \subset E$ by

$$P = \left\{u \in E \mid u(0) = 0, u \text{ is concave, nonnegative on } [0,1]\text{and } u \text{ is symmetricon}\left[\eta, 1\right]\right\}. \tag{2.5}$$

The following lemma can be founded in [11], which is necessary to prove our result.

Lemma 2.4 (see [11]). *If $u \in P$, then the following statements are true:*

(i) $u(t) \geq (u(\omega_1)/\omega_1)\min\{t, 1+\eta-t\}$ *for* $t \in [0,1]$, *here* $\omega_1 = (\eta+1)/2$,

(ii) $u(t) \geq (\eta/\omega_1)u(\omega_1)$ *for* $t \in [\eta, \omega_1]$,

(iii) $\max_{t\in[0,1]}u(t) = u(\omega_1)$.

Lemma 2.5. *If $u \in P$, then the following statements are true:*

(i) $u(t) \leq \max_{t\in[0,1]}|u'(t)|$,

(ii) $\|u(t)\| = \max_{t\in[0,1]}|u'(t)| = \max\{|u'(0)|, |u'(1)|\}$,

(iii) $\min_{t\in[0,\omega_1]}u(t) = u(0)$ *and* $\min_{t\in[\omega_1,1]}u(t) = u(1)$.

Proof. (i) Since

$$u(t) = u(0) + \int_0^t u'(t)dt \quad \text{for } t \in [0,1], \tag{2.6}$$

which reduces to

$$u(t) \leq \int_0^t \left|u'(t)\right|dt \leq \max_{t\in[0,1]}\left|u'(t)\right|. \tag{2.7}$$

(ii) By using $u''(t) \leq 0$ for $t \in [0,1]$, we have $u'(t)$ is monotone decreasing function on $[0,1]$. Moreover,

$$\max_{t\in[0,1]}u(t) = u\left(\frac{\eta+1}{2}\right) = u(\omega_1), \tag{2.8}$$

which implies that $u'(\omega_1) = 0$, so, $u'(t) \geq 0$ for $t \in [0,\omega_1]$ and $u'(t) \leq 0$ for $t \in [\omega_1, 1]$. □

Now, we define the operator $A : P \to E$ by

$$(Au)(t) = \begin{cases} \int_0^t \varphi_q\left(\int_s^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds & \text{for } t \in [0, \omega_1], \\ w(\eta) + \int_t^1 \varphi_q\left(\int_{\omega_1}^s h(r) f(r, u(r), u'(r)) dr\right) ds & \text{for } t \in [\omega_1, 1], \end{cases} \tag{2.9}$$

here, $w(\eta) = (Au)(\eta)$.

Lemma 2.6. *$A : P \to P$ is a completely continuous operator.*

Proof. In fact, $(Au)(t) \geq 0$ for $t \in [0,1]$, $(Au)(\eta) = (Au)(1)$ and $(Au)(0) = 0$.

It is easy to see that the operator A is pseudosymmetric about ω_1 on $[0,1]$.

In fact, for $t \in [\eta, \omega_1]$, we have $1-t+\eta \in [\omega_1, 1]$, and according to the integral transform, one has

$$\begin{aligned} &\int_{1-t+\eta}^1 \varphi_q\left(\int_{\omega_1}^s h(r) f(r, u(r), u'(r)) dr\right) ds \\ &\quad = \int_\eta^t \varphi_q\left(\int_{s_1}^{\omega_1} h(r_1) f(r_1, u(r_1), u'(r_1)) dr_1\right) ds_1, \end{aligned} \tag{2.10}$$

here, $s = 1 - s_1 + \eta$, $r = 1 - r_1 + \eta$. Hence,

$$\begin{aligned} (Au)(1-t+\eta) &= w(\eta) + \int_{1-t+\eta}^1 \varphi_q\left(\int_{\omega_1}^s h(r) f(r, u(r), u'(r)) dr\right) ds \\ &= w(\eta) + \int_\eta^t \varphi_q\left(\int_{s_1}^{\omega_1} h(r_1) f(r_1, u(r_1), u'(r_1)) dr_1\right) ds_1 \\ &= \int_0^\eta \varphi_q\left(\int_s^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds \\ &\quad + \int_\eta^t \varphi_q\left(\int_s^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds \\ &= \int_0^t \varphi_q\left(\int_s^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds = (Au)(t). \end{aligned} \tag{2.11}$$

For $t \in [\omega_1, 1]$, we note that $1 - t + \eta \in [\eta, \omega_1]$, by using the integral transform, one has

$$\begin{aligned} &\int_\eta^{1-t+\eta} \varphi_q\left(\int_s^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds \\ &\quad = \int_t^1 \varphi_q\left(\int_{\omega_1}^{s_1} h(r_1) f(r_1, u(r_1), u'(r_1)) dr_1\right) ds_1, \end{aligned} \tag{2.12}$$

where $s = 1 - s_1 + \eta, r = 1 - r_1 + \eta$. Thus,

$$
\begin{aligned}
(Au)(1-t+\eta) &= \int_0^{1-t+\eta} \varphi_q\left(\int_s^{\omega_1} h(r)f(r,u(r),u'(r))dr\right)ds \\
&= w(\eta) + \int_\eta^{1-t+\eta} \varphi_q\left(\int_s^{\omega_1} h(r)f(r,u(r),u'(r))dr\right)ds \\
&= w(\eta) + \int_t^1 \varphi_q\left(\int_{\omega_1}^{s_1} h(r_1)f(r_1,u(r_1),u'(r_1))dr_1\right)ds_1 \\
&= w(\eta) + \int_t^1 \varphi_q\left(\int_{\omega_1}^{s} h(r)f(r,u(r),u'(r))dr\right)ds = (Au)(t).
\end{aligned}
\tag{2.13}
$$

Hence, A is pseudosymmetric about η on $[0,1]$.
In addition,

$$
(Au)'(t) = \varphi_q\left(\int_t^{\omega_1} h(r)f(r,u(r),u'(r))ds\right) \geq 0, t \in [0,\omega_1]
\tag{2.14}
$$

is continuous and nonincreasing in $[0,\omega_1]$; moreover, $\varphi_q(x)$ is a monotone increasing continuously differentiable function

$$
\left(\int_t^{\omega_1} h(s)f(s,u(s),u'(s))ds\right)' = -h(t)f(t,u(t),u'(t)) \leq 0, t \in [0,\omega_1],
\tag{2.15}
$$

it is easy to obtain $(Au)''(t) \leq 0$ for $t \in [0,\omega_1]$. By using the similar way, we can deduce $(Au)''(t) \leq 0$ for $t \in [\omega_1,1]$. So, $A : P \to P$. It is easy to obtain that $A : P \to P$ is completely continuous. □

Hence, the solutions of BVPs (1.2) are fixed points of the completely continuous operator A.

3. One Solutions

In this section, we will study the existence of one positive pseudosymmetric solution to problem (1.2) by Krasnosel'skii's fixed point theorem in a cone.

Motivated by the notations in reference [23], for $u \in P$, let

$$\begin{aligned} f^0 &= \sup_{t\in[0,1]} \lim_{(u,u')\to(0,0)} \frac{f(t,u,u')}{\varphi_p(|u'|)}, \\ f_0 &= \inf_{t\in[0,1]} \lim_{(u,u')\to(0,0)} \frac{f(t,u,u')}{\varphi_p(|u'|)}, \\ f^\infty &= \sup_{t\in[0,1]} \lim_{(u,u')\to(\infty,\infty)} \frac{f(t,u,u')}{\varphi_p(|u'|)}, \\ f_\infty &= \inf_{t\in[0,1]} \lim_{(u,u')\to(\infty,\infty)} \frac{f(t,u,u')}{\varphi_p(|u'|)}. \end{aligned} \tag{3.1}$$

In the following, we discuss the problem (1.2) under the following four possible cases.

Theorem 3.1. *If $f^0 = 0$ and $f_\infty = \infty$, problem* (1.2) *has at least one positive pseudosymmetric solution u.*

Proof. In view of $f^0 = 0$, there exists an $H_1 > 0$ such that

$$f(t,u,u') \le \varphi_p(\varepsilon)\varphi_p(|u'|) = \varphi_p(\varepsilon|u'|) \quad \text{for } (t,u,u') \in [0,1] \times (0,H_1] \times [-H_1,H_1], \tag{3.2}$$

here, $\varepsilon > 0$ and satisfies

$$\varepsilon\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le 1. \tag{3.3}$$

If $u \in P$ with $\|u\| = H_1$, by Lemma 2.5, we have

$$u(t) \le \max_{t\in[0,1]} |u'(t)| \le \|u\| = H_1 \quad \text{for } t \in [0,1], \tag{3.4}$$

hence,

$$\begin{aligned} \|Au\| &= \max\{|(Au)'(0)|, |(Au)'(1)|\} \\ &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right), \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\} \\ &\le \varepsilon \max_{t\in[0,1]} |u'(t)|\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le \|u\|. \end{aligned} \tag{3.5}$$

If set $\Omega_{H_1} = \{u \in E : \|u\| < H_1\}$, one has $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_{H_1}$.

According to $f_\infty = \infty$, there exists an $H_2' > 0$ such that

$$f(t,u,u') \ge \max_{t\in[0,1]} \varphi_p(k)\varphi_p(|u'|) = \max_{t\in[0,1]} \varphi_p(k|u'|), \tag{3.6}$$

where $(t,u,u') \in [0,1] \times [H_2',\infty) \times (-\infty, H_2'] \cup [H_2',\infty)$, $k > 0$ and satisfies

$$k\varphi_q\left(\int_{\omega_1}^{1} h(r)dr\right) \geq 1. \tag{3.7}$$

Set

$$\begin{gathered} H_2 = \max\left\{2H_1, \frac{\omega_1}{\eta}H_2'\right\},\ \Omega_{H_2^*} = \{u \in E : \|u\| < 5H_2\}, \\ \Omega_{H_2} = \left\{u \in \Omega_{H_2^*} : u(\omega_1) < H_2\right\}. \end{gathered} \tag{3.8}$$

For $u \in P \cap \partial\Omega_{H_2}$, we have $u(\omega_1) = H_2$ since $u(t) \leq |u'(t)|$ for $u \in P$.
If $u \in P$ with $u(\omega_1) = H_2$, Lemmas 2.4 and 2.5 reduce to

$$\min_{t\in[\omega_1,1]}\left|u'(t)\right| \geq \min_{t\in[\omega_1,1]} u(t) = u(1) \geq \frac{\eta u(\omega_1)}{\omega_1} \geq H_2'. \tag{3.9}$$

For $u \in P \cap \partial\Omega_{H_2}$, according to (3.6), (3.7) and (3.9), we get

$$\begin{aligned} \|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right), \varphi_q\left(\int_{\omega_1}^{1} h(r)f(r,u(r),u'(r))dr\right)\right\} \\ &\geq \varphi_q\left(\int_{\omega_1}^{1} h(r)f(r,u(r),u'(r))dr\right) \\ &\geq k\max_{t\in[0,1]}\left|u'(t)\right|\varphi_q\left(\int_1^{\omega_1} h(r)dr\right) = \|u\|. \end{aligned} \tag{3.10}$$

Thus, by (i) of Lemma 2.2, the problem (1.2) has at least one positive pseudosymmetric solution u in $P \cap (\overline{\Omega}_{H_2} \setminus \Omega_{H_1})$. □

Theorem 3.2. *If $f_0 = \infty$ and $f^\infty = 0$, problem* (1.2) *has at least one positive pseudosymmetric solution u.*

Proof. Since $f_0 = \infty$, there exists an $H_3 > 0$ such that

$$f(t,u,u') \geq \max_{t\in[0,1]}\varphi_p(m)\varphi_p\left(\left|u'\right|\right) = \max_{t\in[0,1]}\varphi_p\left(m\left|u'\right|\right), \tag{3.11}$$

here, $(t,u,u') \in [0,1] \times (0,H_3) \times [-H_3, H_3]$ and m is such that

$$m\varphi_q\left(\int_{\omega_1}^{1} h(r)dr\right) \geq 1. \tag{3.12}$$

If $u \in P$ with $\|u\| = H_3$, Lemma 2.5 implies that

$$u(t) \le \max_{t\in[0,1]}\left|u'(t)\right| \le \|u\| = H_3 \quad \text{for } t \in [0,1], \tag{3.13}$$

now, by (3.11), (3.12), and (3.13), we have

$$\begin{aligned}\|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right), \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\}\\ &\ge \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right) \ge m\max_{t\in[0,1]}\left|u'(t)\right|\varphi_q\left(\int_{\omega_1}^1 h(r)dr\right) = \|u\|.\end{aligned} \tag{3.14}$$

If let $\Omega_{H_3} = \{u \in E : \|u\| < H_3\}$, one has $\|Au\| \ge \|u\|$ for $u \in P \cap \partial\Omega_{H_3}$.

Now, we consider $f^\infty = 0$.

Suppose that f is bounded, for some constant $K > 0$, then

$$f(t,u,u') \le \varphi_p(K) \quad \forall (t,u,u') \in [0,1] \times [0,\infty) \times (-\infty,\infty). \tag{3.15}$$

Pick

$$H_4 \ge \max\left\{H_4', 2H_3,\ K\varphi_q\left(\int_0^{\omega_1} h(s)ds\right), \frac{C}{\delta}\right\}, \tag{3.16}$$

here, C is an arbitrary positive constant and satisfy the (3.21). Let

$$\Omega_{H_4} = \{u \in E : \|u\| < H_4\}. \tag{3.17}$$

If $u \in P \cap \partial\Omega_{H_4}$, one has $\|u\| = H_4$, then (3.15) and (3.16) imply that

$$\begin{aligned}\|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right), \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\}\\ &\le K\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le H_4 = \|u\|.\end{aligned} \tag{3.18}$$

Suppose that f is unbounded.

By definition of $f^\infty = 0$, there exists $H_4' > 0$ such that

$$f(t,u,u') \le \varphi_p(\delta)\varphi_p\left(\left|u'\right|\right) = \varphi_p\left(\delta\left|u'\right|\right), \tag{3.19}$$

where $(t,u,u') \in [0,\omega_1] \times [H_4',\infty) \times (-\infty,H_4'] \cup [H_4',\infty)$ and $\delta > 0$ satisfies

$$\delta\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le 1. \tag{3.20}$$

From $f \in C([0,1] \times [0,+\infty) \times (-\infty,\infty), [0,+\infty))$, we have

$$f(t,u,u') \le \varphi_p(C) \quad \text{for } (t,u,u') \in [0,1] \times [0,H_4'] \times [-H_4', H_4'], \tag{3.21}$$

here, C is an arbitrary positive constant.

Then, for $(t,u,u') \in [0,1] \times [0,\infty) \times (-\infty,\infty)$, we have

$$f(t,u,u') \le \max\{\varphi_p(C), \varphi_p(\delta)\varphi_p(|u'|)\}. \tag{3.22}$$

If $u \in P \cap \partial\Omega_{H_4}$, one has $\|u\| = H_4$, which reduces to

$$\begin{aligned}
\|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right), \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\} \\
&\le \max\{C, \delta\|u'\|\}\varphi_q\left(\int_0^{\omega_1} h(r)dr\right) \\
&\le H_4 = \|u\|.
\end{aligned} \tag{3.23}$$

Consequently, for any cases, if we take $\Omega_{H_4} = \{u \in E : \|u\| < H_4\}$, we have $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_{H_4}$. Thus, the condition (ii) of Lemma 2.2 is satisfied.

Consequently, the problem (1.2) has at least one positive pseudosymmetric solution

$$u \in P \cap \left(\overline{\Omega}_{H_4} \setminus \Omega_{H_3}\right) \text{ with } H_3 \le \|u\| \le H_4. \tag{3.24}$$

□

Theorem 3.3. *Suppose that the following conditions hold:*

(i) *there exist nonzero finite constants c_1 and c_2 such that $f^0 = c_1$ and $f_\infty = c_2$,*

(ii) *there exist nonzero finite constants c_3 and c_4 such that $f_0 = c_3$ and $f^\infty = c_4$.*

Then, problem (1.2) *has at least one positive pseudosymmetric solution u.*

Proof. (i) In view of $f^0 = c_1$, there exists an $H_5 > 0$ such that

$$\begin{aligned}
f(t,u,u') &\le \varphi_p(\varepsilon + c_{11})\varphi_p(|u'|) \\
&= \varphi_p((\varepsilon + c_{11})|u'|) \quad \text{for } (t,u,u') \in [0,1] \times (0,H_5] \times [-H_5, H_5],
\end{aligned} \tag{3.25}$$

here, $c_1 = \varphi_p(c_{11} + \varepsilon)$, $\varepsilon > 0$ and satisfies

$$(\varepsilon + c_{11})\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le 1. \tag{3.26}$$

If $u \in P$ with $\|u\| = H_5$, by Lemma 2.5, we have

$$u(t) \le |u'(t)| \le \|u\| = H_5 \quad \text{for } t \in [0,1], \tag{3.27}$$

hence,

$$\begin{aligned}\|Au\| &= \max\{|(Au)'(0)|,|(Au)'(1)|\}\\ &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right),\varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\}\\ &\le (\varepsilon + c_{11})\max_{t\in[0,1]}|u'(t)|\varphi_q\left(\int_0^{\omega_1} h(s)ds\right) \le \|u\|.\end{aligned} \tag{3.28}$$

If set $\Omega_{H_5} = \{u \in E : \|u\| < H_5\}$, one has $\|Au\| \le \|u\|$ for $u \in P \cap \partial\Omega_{H_5}$.

According to $f_\infty = c_2$, there exists an $H_6' > 0$ such that

$$f(t,u,u') \ge \max_{t\in[0,1]}\varphi_p(c_{22}-\varepsilon)\varphi_p(|u'|) = \max_{t\in[0,1]}\varphi_p((c_{22}-\varepsilon)|u'|), \tag{3.29}$$

where $(t,u,u') \in [0,1] \times [H_6',\infty) \times (-\infty,H_6'] \cup [H_6',\infty)$, $c_2 = \varphi_p(c_{22}-\varepsilon)$, $\varepsilon > 0$ and satisfies

$$(c_{22}-\varepsilon)\varphi_q\left(\int_{\omega_1}^1 h(r)dr\right) \ge 1. \tag{3.30}$$

Set

$$\begin{gathered}H_6 = \max\left\{2H_5,\frac{\omega_1}{\eta}H_6'\right\}, \qquad \Omega_{H_6^*} = \{u \in E : \|u\| < 5H_6\},\\ \Omega_{H_6} = \left\{u \in \Omega_{H_6^*} : u(\omega_1) < H_6\right\}.\end{gathered} \tag{3.31}$$

If $u \in P$ with $u(\omega_1) = H_6$, Lemmas 2.4 and 2.5 reduce to

$$\min_{t\in[\omega_1,1]}|u'(t)| \ge \min_{t\in[\omega_1,1]} u(t) = u(1) \ge \frac{\eta u(\omega_1)}{\omega_1} \ge H_6'. \tag{3.32}$$

For $u \in P \cap \partial\Omega_{H_6}$, according to (3.29), (3.30) and (3.32), we get

$$\begin{aligned}\|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right),\varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\}\\ &\ge \varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\\ &\ge (c_{22}-\varepsilon)\max_{t\in[0,1]}|u'(t)|\varphi_q\left(\int_1^{\omega_1} h(r)dr\right) = \|u\|.\end{aligned} \tag{3.33}$$

Thus, by (i) of Lemma 2.2, the problem (1.2) has at least one positive pseudosymmetric solution u in $P \cap (\overline{\Omega}_{H_6} \setminus \Omega_{H_5})$.

(ii) By using the similar way as to Theorem 3.2, we can prove to it. □

4. Triple Solutions

In the previous section, some results on the existence of at least one positive pseudosymmetric solutions to problem (1.2) are obtained. In this section, we will further discuss the existence criteria for at least *three* and arbitrary odd positive pseudosymmetric solutions of problems (1.2) by using the Avery-Peterson fixed point theorem [21].

Choose a $r \in (\eta, \omega_1)$, for the notational convenience, we denote

$$M = \omega_1 \varphi_q\left(\int_0^{\omega_1} h(r)dr\right), \qquad N = \eta\varphi_q\left(\int_\eta^{\omega_1} h(r)dr\right), \qquad W = \varphi_q\left(\int_0^{\omega_1} h(r)dr\right). \tag{4.1}$$

Define the nonnegative continuous convex functionals θ and γ, nonnegative continuous concave functional α, and nonnegative continuous functional ψ, respectively, on P by

$$\begin{gathered}\gamma(u) = \max_{t\in[0,1]}\left|u'(t)\right| = \max\{u'(0), u'(1)\} = \|u\|,\\ \psi(u) = \theta(u) = \max_{t\in[0,\omega_1]} u(t) = u(\omega_1) \le \|u\|,\\ \alpha(u) = \min_{t\in[\eta,\omega_1]} u(t) = u(\eta).\end{gathered} \tag{4.2}$$

Now, we state and prove the results in this section.

Theorem 4.1. *Suppose that there exist constants a^*, b^*, and d^* such that $0 < a^* < b^* < (N/W)d^*$. In addition, f satisfies the following conditions:*

(i) $f(t,u,u') \le \varphi_p(d^*/W)$ *for* $(t,u,u') \in [0,1] \times [0,d^*] \times [-d^*, d^*]$,

(ii) $f(t,u,u') > \varphi_p(b^*/N)$ *for* $(t,u,u') \in [\eta,\omega_1] \times [b^*,d^*] \times [-d^*, d^*]$,

(iii) $f(t,u,u') < \varphi_p(a^*/M)$ *for* $(t,u,u') \in [0,\omega_1] \times [0,a^*] \times [-d^*, d^*]$.

Then, problem (1.2) *has at least three positive pseudosymmetric solutions u_1, u_2, and u_3 such that*

$$\begin{gathered}\|x_i\| \le d^* \quad \textit{for } i = 1,2,3,\ b^* < \min_{t\in[\eta,\omega_1]} u_1(t),\ a^* < \max_{t\in[0,1]} u_2(t),\\ \min_{t\in[\eta,\omega_1]} u_2(t) < b^* \quad \textit{with} \max_{t\in[0,1]} u_3(t) < a^*.\end{gathered} \tag{4.3}$$

Proof. According to the definition of completely continuous operator A and its properties, we need to show that all the conditions of Lemma 2.3 hold with respect to A.

It is obvious that

$$\begin{gathered}\psi(\lambda' u) = \lambda' u(\omega_1) = \lambda'\psi(u) \quad \text{for } 0 < \lambda' < 1,\\ \alpha(u) \le \psi(u) \quad \forall u \in P,\\ \|u\| = \gamma(u) \quad \forall u \in P.\end{gathered} \tag{4.4}$$

Firstly, we show that $A:\overline{P(\gamma,d^*)} \longrightarrow \overline{P(\gamma,d^*)}$.

For any $u \in \overline{P(\gamma,d^*)}$, we have

$$u(t) \leq \max_{t\in[0,1]}\left|u'(t)\right| \leq \|u\| = \gamma(u) \leq d^* \quad \text{for } t \in [0,1], \tag{4.5}$$

hence, the assumption (i) implies that

$$\begin{aligned}\|Au\| &= \max\left\{\varphi_q\left(\int_0^{\omega_1} h(r)f(r,u(r),u'(r))dr\right),\varphi_q\left(\int_{\omega_1}^1 h(r)f(r,u(r),u'(r))dr\right)\right\}\\ &\leq \frac{d^*}{W}\varphi_q\left(\int_0^{\omega_1} h(r)dr\right) = d^*.\end{aligned} \tag{4.6}$$

From the above analysis, it remains to show that (i)–(iii) of Lemma 2.3 hold.

Secondly, we verify that condition (i) of Lemma 2.3 holds; let $u(t) \equiv (tb^*/\eta) + b^*, t \in [0,1]$, and it is easy to see that

$$\begin{aligned}\alpha(u) &= u(\eta) \ = 2b^* > b^*,\\ \theta(u) &= u(\omega_1) = \frac{\omega_1 b^*}{\eta} + b^* \leq \frac{\omega_1 b^*}{\eta} + b^*,\end{aligned} \tag{4.7}$$

in addition, we have $\gamma(u) = (b^*/\eta) < d^*$, since $b^* < (N/W)d^*$. Thus

$$\left\{u \in P\left(\gamma,\theta,\alpha,b^*,\frac{\omega_1 b^*}{\eta}+b^*,d^*\right) : \alpha(x) > b^*\right\} \neq \emptyset. \tag{4.8}$$

For any

$$u \in P\left(\gamma,\theta,\alpha,b^*,\frac{\omega_1 b^*}{\eta}+b^*,d^*\right), \tag{4.9}$$

one has

$$b^* \leq u(t) \leq \|u\| \leq d^* \quad \forall t \in [\eta,\omega_1], \tag{4.10}$$

it follows from the assumption (ii) that

$$\begin{aligned}\alpha(Au) = (Au)(\eta) &= \int_0^\eta \varphi_q\left(\int_s^{\omega_1} h(r)f(r,u(r),u'(r))dr\right)ds\\ &\geq \int_0^\eta \varphi_q\left(\int_\eta^{\omega_1} h(r)f(r,u(r),u'(r))dr\right)ds\\ &> \frac{b^*}{N}\eta\varphi_q\left(\int_\eta^{\omega_1} h(r)dr\right) = b^*.\end{aligned} \tag{4.11}$$

Thirdly, we prove that the condition (ii) of Lemma 2.3 holds. In fact,

$$\alpha(Au) = Au(\eta),$$
$$\theta(Au) = \max_{t\in[0,\omega_1]} A(u) = Au(\omega_1). \tag{4.12}$$

For any $u \in P(\gamma, \alpha, b^*, d^*)$ with $\theta(Au) > (\omega_1 b^*/\eta) + b^*$, we have

$$\alpha(Au) = Au(\eta) \geq \frac{\eta}{\omega_1} Au(\omega_1) \geq \frac{\eta}{\omega_1}\theta(Au) = b^* + \frac{\omega_1 b^*}{\eta} > b^*. \tag{4.13}$$

Finally, we check condition (iii) of Lemma 2.3.

Clearly, since $\psi(0) = 0 < a^*$, we have $0 \notin R(\gamma, \psi, a^*, d^*)$. If

$$u \in R(\gamma, \psi, a^*, d^*) \text{ with } \psi\ (u) = \max_{t\in[0,\omega_1]} u(t) = u(\omega_1) = a^*, \tag{4.14}$$

then

$$0 \leq u(t) \leq a^* \quad \forall t \in [0, \omega_1],$$
$$\max_{t\in[0,1]} |u'(t)| = \|u\| = \gamma(u) \leq d^*. \tag{4.15}$$

Hence, by assumption (iii), we have

$$\begin{aligned}\psi(Au) &= (Au)(\omega_1)\\ &\leq \int_0^{\omega_1} \varphi_q\left(\int_0^{\omega_1} h(r) f(r, u(r), u'(r)) dr\right) ds\\ &< \frac{a^*}{M}\omega_1 \varphi_q\left(\int_0^{\omega_1} h(r) dr\right) = a^*.\end{aligned} \tag{4.16}$$

Consequently, from above, all the conditions of Lemma 2.3 are satisfied. The proof is completed. □

Corollary 4.2. *If the condition* (i) *in Theorem 4.1 is replaced by the following condition (i′):*

(i′) $\lim_{(u,u')\to(\infty,\infty)} (f(t,u,u')/(\varphi_p(|u'|))) \leq \varphi_p(1/W)$,

then the conclusion of Theorem 4.1 also holds.

Proof. From Theorem 4.1, we only need to prove that (i′) implies that (i) holds. That is, assume that (i′) holds, then there exists a number $d^* \geq (W/N)b^*$ such that

$$f(t, u, u') \leq \varphi_p\left(\frac{d^*}{W}\right) \quad \text{for } (t, u, u') \in [0,1] \times [0, d^*] \times [-d^*, d^*]. \tag{4.17}$$

Suppose on the contrary that for any $d^* \geq (W/N)b^*$, there exists $(u_c, u'_c) \in [0, d^*] \times [-d^*, d^*]$ such that

$$f(t, u_c, u'_c) > \varphi_p\left(\frac{d^*}{W}\right) \quad \text{for } t \in [0,1]. \tag{4.18}$$

Hence, if we choose $c_n^* > (W/N)b^*$ $(n = 1, 2, \ldots)$ with $c_n^* \to \infty$, then there exist $(u_n, u'_n) \in [0, c_n^*] \times [-c_n^*, c_n^*]$ such that

$$f(t, u_n, u'_n) > \varphi_p\left(\frac{c_n^*}{W}\right) \quad \text{for } t \in [0,1], \tag{4.19}$$

and so

$$\lim_{n \to \infty} f(t, u_n, u'_n) = \infty \quad \text{for } t \in [0,1]. \tag{4.20}$$

Since the condition (i$'$) holds, there exists $\tau > 0$ satisfying

$$f(t, u, u') \leq \varphi_p\left(\frac{|u'|}{W}\right) \quad \text{for } (t, u, u') \in [0,1] \times [\tau, \infty) \times (-\infty, \tau] \cup [\tau, \infty). \tag{4.21}$$

Hence, we have

$$u_n < |u'_n| \leq \tau. \tag{4.22}$$

Otherwise, if

$$|u'_n| > u_n > \tau \quad \text{for } t \in [0,1], \tag{4.23}$$

it follows from (4.21) that

$$f(t, u_n, u'_n) \leq \varphi_p\left(\frac{u_n}{W}\right) \leq \varphi_p\left(\frac{c_n^*}{W}\right) \quad \text{for } t \in [0,1], \tag{4.24}$$

which contradicts (4.19).

Let

$$W = \max_{(t,u,u') \in [0,1] \times [0,\tau] \times [-\tau,\tau]} f(t, u, u'), \tag{4.25}$$

then

$$f(t, u_n, u'_n) \leq W (n = 1, 2, \ldots), \tag{4.26}$$

which also contradicts (4.20). $\square$

Theorem 4.3. *Suppose that there exist constants a_i^*, b_i^*, and d_i^* such that*

$$0 < a_1^* < b_1^* < \frac{N}{W} d_2^* < a_2^* < b_2^* < \frac{N}{W} d_3^* < \cdots < a_n^* < b_n^* < \frac{N}{W} d_{n+1}^*, \tag{4.27}$$

here, $n \in \mathbb{N}$ and $i = 1, 2, \ldots, n$. In addition, suppose that f satisfies the following conditions:

(i) $f(t, u, u') \leq \varphi_p(d_i^*/W)$ *for* $(t, u, u') \in [0,1] \times [0, d_i^*] \times [-d_i^*, d_i^*]$,

(ii) $f(t, u, u') > \varphi_p(b_i^*/N)$ *for* $(t, u, u') \in [\eta, \omega_1] \times [b_i^*, d_i^*] \times [-d_i^*, d_i^*]$,

(iii) $f(t, u, u') < \varphi_p(a_i^*/M)$ *for* $(t, u, u') \in [0, \omega_1] \times [0, a_i^*] \times [-d_i^*, d_i^*]$.

Then, problem (1.2) *has at least* $2n - 1$ *positive pseudosymmetric solutions.*

Proof. When $n = 1$, it is immediate from condition (i) that

$$A : \overline{P}_{a_1^*} \longrightarrow P_{a_1^*} \subset \overline{P}_{a_1^*}. \tag{4.28}$$

It follows from the Schauder fixed point theorem that A has at least one fixed point

$$u_1 \in \overline{P}_{a_1^*}, \tag{4.29}$$

which means that

$$\|u_1\| \leq a_1^*. \tag{4.30}$$

When $n = 2$, it is clear that Theorem 4.1 holds (with $a^* = a_1^*$, $b^* = b_1^*$, and $d^* = d_2^*$). Then, there exists at least three positive pseudosymmetric solutions u_1, u_2, and u_3 such that

$$\begin{gathered} \|x_1\| \leq d_2^*, \quad \|x_2\| \leq d_2^*, \quad \|x_3\| \leq d_2^*, \quad b^* < \min_{t \in [\eta, \omega_1]} u_1(t), \quad a_1^* < \max_{t \in [0,1]} u_2(t), \\ \min_{t \in [\eta, \omega_1]} u_2(t) < b_1^* \text{ with } \max_{t \in [0,1]} u_3(t) < a_1^*. \end{gathered} \tag{4.31}$$

Following this way, we finish the proof by induction. The proof is complete. □

5. Examples

In this section, we present two simple examples to illustrate our results.

Example 5.1. Consider the following BVPs:

$$\begin{gathered} (\varphi_p(u'(t)))' + t\left(t + 1 + |u'(t)|^{p-2}\right) = 0, \quad t \in [0,1], \\ u(0) = 0, \; u(0.2) = u(1). \end{gathered} \tag{5.1}$$

Note that

$$f_0 = \inf_{t\in[0,1]} \lim_{(u,u')\to(0,0)} \frac{t+1+|u'(t)|^{p-2}}{|u'(t)|^{p-2}u'(t)} = \infty,$$
$$f^\infty = \sup_{t\in[0,1]} \lim_{(u,u')\to(\infty,\infty)} \frac{t+1+|u'(t)|^{p-2}}{|u'(t)|^{p-2}u'(t)} = 0. \tag{5.2}$$

Hence, Theorem 3.2 implies that the BVPs in (5.1) have at least one pseudosymmetric solution u.

Example 5.2. Consider the following BVPs with $p = 3$:

$$(\varphi_p(u'(t)))' + h(t)f(t,u(t),u'(t)) = 0, t \in [0,1],$$
$$u(0) = 0,\ u\ (0.2) = u(1), \tag{5.3}$$

where $h(t) = 2t$ and

$$f(t,u,u') = \begin{cases} t + 4 + \left(\dfrac{u'}{5.5}\right)^2, & u \in [0, 0.9], \\ t + 750u - 671 + \left(\dfrac{u'}{5.5}\right)^2, & u \in [0.9, 1], \\ t + 79 + \left(\dfrac{u'}{5.5}\right)^2, & u \in [1, 5.5], \\ t + 14.364u + \left(\dfrac{u'}{5.5}\right)^2, & u \in [5.5, +\infty). \end{cases} \tag{5.4}$$

Note that $\eta = 0.2,\ \omega_1 = 0.6$, then a direct calculation shows that

$$M = \omega_1\varphi_q\left(\int_0^{\omega_1} h(r)dr\right) = 0.6 \times 0.6 = 0.36,\ N \approx 0.1131,\ W = 0.6. \tag{5.5}$$

If we take $a' = 0.9,\ b' = 1,\ d' = 5.5$, then $a' < b' < (N/W)d'$ holds; furthermore,

$$f(t,u,u') < 82 < 84.028 \approx \varphi_p\left(\frac{d'}{W}\right) \quad \text{for } (t,u,u') \in [0,0.6] \times [0,5.5] \times [-5.5,5.5],$$
$$f(t,u,u') > 79 > 78.176 \approx \varphi_p\left(\frac{b'}{N}\right) \quad \text{for } (t,u,u') \in [0.6,1] \times [1,5.5] \times [-5.5,5.5], \tag{5.6}$$
$$f(t,u,u') < 6.2 < 6.25 = \varphi_p\left(\frac{a'}{M}\right) \quad \text{for } (t,u,u') \in [0,0.6] \times [0,0.9] \times [-5.5,5.5].$$

By Theorem 4.1, we see that the BVPs in (5.3) have at least *three* positive pseudosymmetric solutions u_1,u_2 and u_3 such that

$$\begin{gathered}\|x_i\| \leq 5.5 \quad \text{for } i = 1,\ 2,\ 3,\ 1 < \min_{t\in[0.2,0.6]} u_1(t), 0.9 < \max_{t\in[0,1]} u_2(t),\\ \min_{t\in[0.2,0.6]} u_2(t) < 1 \quad \text{with } \max_{t\in[0,1]} u_3(t) < 0.9.\end{gathered} \tag{5.7}$$

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Research Article

Global Nonexistence of Positive Initial-Energy Solutions for Coupled Nonlinear Wave Equations with Damping and Source Terms

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This work is concerned with a system of nonlinear wave equations with nonlinear damping and source terms acting on both equations. We prove a global nonexistence theorem for certain solutions with positive initial energy.

1. Introduction

In this paper we study the initial-boundary-value problem

$$\begin{gathered}
u_{tt} - \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) + |u_t|^{m-1}u_t = f_1(u,v), \quad (x,t) \in \Omega \times (0,T), \\
v_{tt} - \operatorname{div}\left(g\left(|\nabla v|^2\right)\nabla v\right) + |v_t|^{r-1}v_t = f_2(u,v), \quad (x,t) \in \Omega \times (0,T), \\
u(x,t) = v(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \\
u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega, \\
v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), \quad x \in \Omega,
\end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$, $m, r \geq 1$, and $f_i(\cdot,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$ are given functions to be specified later. We assume that g is a function which satisfies

$$g \in C^1, \quad g(s) > 0, \quad g(s) + 2sg'(s) > 0 \tag{1.2}$$

for $s > 0$.

To motivate our work, let us recall some results regarding $g \equiv 1$. The single-wave equation of the form

$$u_{tt} - \Delta u + h(u_t) = f(u), \quad x \in \Omega, \ t > 0 \tag{1.3}$$

in $\Omega \times (0, \infty)$ with initial and boundary conditions has been extensively studied, and many results concerning global existence, blow-up, energy decay have been obtained. In the absence of the source term, that is, ($f = 0$), it is well known that the damping term $h(u_t)$ assures global existence and decay of the solution energy for arbitrary initial data (see [1]). In the absence of the damping term, the source term causes finite time blow-up of solutions with a large initial data (negative initial energy) (see [2, 3]). The interaction between the damping term and the source term makes the problem more interesting. This situation was first considered by Levine [4, 5] in the linear damping case $h(u_t) = au_t$ and a polynomial source term of the form $f(u) = b|u|^{p-2}u$. He showed that solutions with negative initial energy blow up in finite time. The main tool used in [4, 5] is the "concavity method." Georgiev and Todorova in [6] extended Levine's result to the nonlinear damping case $h(u_t) = a|u_t|^{m-2}u_t$. In their work, the authors considered problem (1.3) with $f(u) = b|u|^{p-2}u$ and introduced a method different from the one known as the concavity method and showed that solutions with negative energy continue to exist globally in time if $m \geq p \geq 2$ and blow up in finite time if $p > m \geq 2$ and the initial energy is sufficiently negative. This latter result has been pushed by Messaoudi [7] to the situation where the initial energy $E(0) < 0$ and has been improved by the same author in [8] to accommodate certain solutions with positive initial energy.

In the case of g being a given nonlinear function, the following equation:

$$u_{tt} - g(u_x)_x - u_{xxt} + \delta |u_t|^{m-1} \, u_t = \mu |u|^{p-1} u, \quad x \in (0,1), \ t > 0, \tag{1.4}$$

with initial and boundary conditions has been extensively studied. Equation of type of (1.4) is a class of nonlinear evolution governing the motion of a viscoelastic solid composed of the material of the rate type, see [9–12]. It can also be seen as field equation governing the longitudinal motion of a viscoelastic bar obeying the nonlinear Voigt model, see [13]. In two- and three-dimensional cases, they describe antiplane shear motions of viscoelastic solids. We refer to [14–16] for physical origins and derivation of mathematical models of motions of viscoelastic media and only recall here that, in applications, the unknown u naturally represents the displacement of the body relative to a fixed reference configuration. When $\delta = \mu = 0$, there have been many impressive works on the global existence and other properties of solutions of (1.4), see [9, 10, 17, 18]. Especially, in [19] the authors have proved the global existence and uniqueness of the generalized and classical solution for the initial boundary value problem (1.4) when we replace $\delta |u_t|^{m-1} u_t$ and $\mu |u|^{p-1} u$ by $g(u_t)$ and $f(u)$, respectively. But about the blow-up of the solution for problem, in this paper there has not been any discussion. Chen et al. [20] considered problem (1.4) and first established an ordinary differential inequality, next given the sufficient conditions of blow-up of the solution of (1.4) by the inequality. In [21], Hao et al. considered the single-wave equation of the form

$$u_{tt} - \operatorname{div}\left(g\left(|\nabla u|^2\right)\nabla u\right) + h(u_t) = f(u), \quad x \in \Omega,\ t > 0 \tag{1.5}$$

with initial and Dirichlet boundary condition, where g satisfies condition (1.2) and

$$g(s) \geq b_1 + b_2 s^q, \quad q \geq 0. \tag{1.6}$$

The damping term has the form

$$h(u_t) = d_1 u_t + d_2 |u_t|^{r-1}\, u_t, \quad r > 1. \tag{1.7}$$

The source term is

$$f(u) = a_1 u + a_2 |u|^{p-1} u \tag{1.8}$$

with $p \geq 1$ for $n = 1, 2$ and $1 \leq n \leq 2n/(n-2)$ for $n \geq 3$, $a_1, a_2, b_1, b_2, d_1, d_2$ are nonnegative constants, and $b_1 + b_2 > 0$. By using the energy compensation method [7, 8, 22], they proved that under some conditions on the initial value and the growth orders of the nonlinear strain term, the damping term, and the source term, the solution to problem (1.5) exists globally and blows up in finite time with negative initial energy, respectively.

Some special cases of system (1.1) arise in quantum field theory which describe the motion of charged mesons in an electromagnetic field, see [23, 24]. Recently, some of the ideas in [6, 22] have been extended to study certain systems of wave equations. Agre and Rammaha [25] studied the system of (1.1) with $g \equiv 1$ and proved several results concerning local and global existence of a weak solution and showed that any weak solution with negative initial energy blows up in finite time, using the same techniques as in [6]. This latter blow-up result has been improved by Said-Houari [26] by considering a larger class of initial data for which the initial energy can take positive values. Recently, Wu et al. [27] considered problem (1.1) with the nonlinear functions $f_1(u, v)$ and $f_2(u, v)$ satisfying appropriate conditions. They proved under some restrictions on the parameters and the initial data several results on global existence of a weak solution. They also showed that any weak solution with initial energy $E(0) < 0$ blows up in finite time.

In this paper, we also consider problem (1.1) and improve the global nonexistence result obtained in [27], for a large class of initial data in which our initial energy can take positive values. The main tool of the proof is a technique introduced by Payne and Sattinger [28] and some estimates used firstly by Vitillaro [29], in order to study a class of a single-wave equation.

2. Preliminaries and Main Result

First, let us introduce some notation used throughout this paper. We denote by $\|\cdot\|_q$ the $L^q(\Omega)$ norm for $1 \le q \le \infty$ and by $\|\nabla \cdot\|_2$ the Dirichlet norm in $H_0^1(\Omega)$ which is equivalent to the $H^1(\Omega)$ norm. Moreover, we set

$$(\varphi, \psi) = \int_\Omega \varphi(x)\psi(x)dx \tag{2.1}$$

as the usual $L^2(\Omega)$ inner product.

Concerning the functions $f_1(u,v)$ and $f_2(u,v)$, we take

$$\begin{aligned} f_1(u,v) &= \left[a|u+v|^{2(p+1)}(u+v) + b|u|^p u|v|^{(p+2)}\right], \\ f_2(u,v) &= \left[a|u+v|^{2(p+1)}(u+v) + b|u|^{(p+2)}|v|^p v\right], \end{aligned} \tag{2.2}$$

where $a, b > 0$ are constants and p satisfies

$$\begin{cases} p > -1, & \text{if } n = 1, 2, \\ -1 < p \le \dfrac{4-n}{n-2}, & \text{if } n \ge 3. \end{cases} \tag{2.3}$$

One can easily verify that

$$uf_1(u,v) + vf_2(u,v) = 2(p+2)F(u,v), \quad \forall (u,v) \in \mathbb{R}^2, \tag{2.4}$$

where

$$F(u,v) = \frac{1}{2(p+2)}\left[a|u+v|^{2(p+2)} + 2b|uv|^{p+2}\right]. \tag{2.5}$$

We have the following result.

Lemma 2.1 (see [30, Lemma 2.1]). *There exist two positive constants c_0 and c_1 such that*

$$\frac{c_0}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right) \le F(u,v) \le \frac{c_1}{2(p+2)}\left(|u|^{2(p+2)} + |v|^{2(p+2)}\right). \tag{2.6}$$

Throughout this paper, we define g by

$$g(s) = b_1 + b_2 s^q, \quad q \ge 0, \ b_1 + b_2 > 0, \tag{2.7}$$

where b_1, b_2 are nonnegative constants. Obviously, g satisfies conditions (1.2) and (1.6). Set

$$G(s) = \int_0^s g(s)ds, \quad s \ge 0. \tag{2.8}$$

In order to state and prove our result, we introduce the following function space:

$$
\begin{aligned}
Z = \Big\{ (u,v) \mid u,v \in L^{\infty}\Big([0,T]; W_0^{1,2(q+1)}(\Omega) \cap L^{2(p+2)}(\Omega)\Big), \\
u_t \in L^{\infty}\Big([0,T]; L^2(\Omega)\Big) \cap L^{m+1}(\Omega \times (0,T)), \\
v_t \in L^{\infty}\Big([0,T]; L^2(\Omega)\Big) \cap L^{r+1}(\Omega \times (0,T)), u_{tt}, v_{tt} \in L^{\infty}\Big([0,T), L^2(\Omega)\Big) \Big\}.
\end{aligned}
\tag{2.9}
$$

Define the energy functional $E(t)$ associated with our system

$$
E(t) = \frac{1}{2}\Big(\|u_t(t)\|_2^2 + \|v_t(t)\|_2^2\Big) + \frac{1}{2}\int_{\Omega}\Big(G\Big(|\nabla u|^2\Big) + G\Big(|\nabla v|^2\Big)\Big)dx - \int_{\Omega} F(u,v)dx. \tag{2.10}
$$

A simple computation gives

$$
\frac{dE(t)}{dt} = -\|u\|_{m+1}^{m+1} - \|v\|_{r+1}^{r+1} \le 0. \tag{2.11}
$$

Our main result reads as follows.

Theorem 2.2. *Assume that* (2.3) *holds. Assume further that* $2(p+2) > \max\{2q+2, m+1, r+1\}$. *Then any solution of* (1.1) *with initial data satisfying*

$$
\left(\int_{\Omega}\Big(G\Big(|\nabla u_0|^2\Big) + G\Big(|\nabla v_0|^2\Big)\Big)dx\right)^{1/2} > \alpha_1, \qquad E(0) < E_2, \tag{2.12}
$$

cannot exist for all time, where the constant α_1 *and* E_2 *are defined in* (3.7).

3. Proof of Theorem 2.2

In this section, we deal with the blow-up of solutions of the system (1.1). Before we prove our main result, we need the following lemmas.

Lemma 3.1. *Let* $\Theta(t)$ *be a solution of the ordinary differential inequality*

$$
\frac{d\Theta(t)}{dt} \ge C\Theta^{1+\varepsilon}(t), \quad t > 0, \tag{3.1}
$$

where $\varepsilon > 0$. *If* $\Theta(0) > 0$, *then the solution ceases to exist for* $t \ge \Theta^{-\varepsilon}(0)C^{-1}\varepsilon^{-1}$.

Lemma 3.2. *Assume that* (2.3) *holds. Then there exists* $\eta > 0$ *such that for any* $(u,v) \in Z$, *one has*

$$
\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \le \eta\left(\int_{\Omega}\Big(G\Big(|\nabla u|^2\Big) + G\Big(|\nabla v|^2\Big)\Big)dx\right)^{p+2}. \tag{3.2}
$$

Proof. By using Minkowski's inequality, we get

$$\|u+v\|_{2(p+2)}^2 \leq 2\left(\|u\|_{2(p+2)}^2 + \|v\|_{2(p+2)}^2\right). \tag{3.3}$$

Also, Hölder's and Young's inequalities give us

$$\|uv\|_{p+2} \leq \|u\|_{2(p+2)}\|v\|_{2(p+2)} \leq \frac{1}{2}\left(\|u\|_{2(p+2)}^2 + \|v\|_{2(p+2)}^2\right). \tag{3.4}$$

If $b_1 > 0$, then we have

$$\int_\Omega \left(G\left(|\nabla u|^2\right) + G\left(|\nabla v|^2\right)\right)dx \geq c\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right). \tag{3.5}$$

If $b_1 = 0$, from $b_1 + b_2 > 0$, we have $b_2 > 0$. Since $W_0^{1,2(q+1)}(\Omega) \hookrightarrow H_0^1(\Omega)$, we have

$$\|\nabla u\|_2^2 + \|\nabla v\|_2^2 \leq c_1\left(\|\nabla u\|_{2(q+1)}^2 + \|\nabla v\|_{2(q+1)}^2\right), \tag{3.6}$$

which implies that (3.5) still holds for $b_1 = 0$. Combining (3.3), (3.4) with (3.5) and the embedding $H_0^1(\Omega) \hookrightarrow L^{2(p+2)}(\Omega)$, we have (3.2). □

In order to prove our result and for the sake of simplicity, we take $a = b = 1$ and introduce the following:

$$\begin{gathered} B = \eta^{1/(2(p+2))}, \qquad \alpha_1 = B^{-(p+2/(p+1))}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{2(p+2)}\right)\alpha_1^2, \\ E_2 = \left(\frac{1}{2(q+1)} - \frac{1}{2(p+2)}\right)\alpha_1^2, \end{gathered} \tag{3.7}$$

where η is the optimal constant in (3.2). The following lemma will play an essential role in the proof of our main result, and it is similar to a lemma used first by Vitillaro [29].

Lemma 3.3. *Assume that* (2.3) *holds. Let* $(u, v) \in Z$ *be the solution of the system* (1.1). *Assume further that* $E(0) < E_1$ *and*

$$\left(\int_\Omega \left(G\left(|\nabla u_0|^2\right) + G\left(|\nabla v_0|^2\right)\right)dx\right)^{1/2} > \alpha_1. \tag{3.8}$$

Then there exists a constant $\alpha_2 > \alpha_1$ *such that*

$$\left(\int_\Omega \left(G\left(|\nabla u|^2\right) + G\left(|\nabla v|^2\right)\right)dx\right)^{1/2} \geq \alpha_2, \quad \text{for } t > 0, \tag{3.9}$$

$$\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right)^{1/(2(p+2))} \geq B\alpha_2, \quad \text{for } t > 0. \tag{3.10}$$

Proof. We first note that, by (2.10), (3.2), and the definition of B, we have

$$\begin{aligned}E(t) &\geq \frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx-\frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right)\\ &\geq \frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx-\frac{B^{2(p+2)}}{2(p+2)}\left(\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx\right)^{p+2}\\ &= \frac{1}{2}\alpha^2-\frac{B^{2(p+2)}}{2(p+2)}\alpha^{2(p+2)},\end{aligned} \tag{3.11}$$

where $\alpha = (\int_{\Omega}(G(|\nabla u|^2)+G(|\nabla v|^2))dx)^{1/2}$. It is not hard to verify that g is increasing for $0<\alpha<\alpha_1$, decreasing for $\alpha>\alpha_1$, $g(\alpha)\to-\infty$ as $\alpha\to+\infty$, and

$$g(\alpha_1)=\frac{1}{2}\alpha_1^2-\frac{B^{2(p+2)}}{2(p+2)}\alpha_1^{2(p+2)}=E_1, \tag{3.12}$$

where α_1 is given in (3.7). Since $E(0)<E_1$, there exists $\alpha_2>\alpha_1$ such that $g(\alpha_2)=E(0)$.

Set $\alpha_0=(\int_{\Omega}(G(|\nabla u_0|^2)+G(|\nabla v_0|^2))dx)^{1/2}$. Then by (3.11) we get $g(\alpha_0)\leq E(0)=g(\alpha_2)$, which implies that $\alpha_0\geq\alpha_2$. Now, to establish (3.9), we suppose by contradiction that

$$\left(\int_{\Omega}\left(G\left(|\nabla u(t_0)|^2\right)+G\left(|\nabla v(t_0)|^2\right)\right)dx\right)^{1/2}<\alpha_2, \tag{3.13}$$

for some $t_0>0$. By the continuity of $\int_{\Omega}(G(|\nabla u|^2)+G(|\nabla v|^2))dx$, we can choose t_0 such that

$$\left(\int_{\Omega}\left(G\left(|\nabla u(t_0)|^2\right)+G\left(|\nabla v(t_0)|^2\right)\right)dx\right)^{1/2}>\alpha_1. \tag{3.14}$$

Again, the use of (3.11) leads to

$$E(t_0)\geq g\left(\left(\int_{\Omega}\left(G\left(|\nabla u(t_0)|^2\right)+G\left(|\nabla v(t_0)|^2\right)\right)dx\right)^{1/2}\right)>g(\alpha_2)=E(0). \tag{3.15}$$

This is impossible since $E(t)\leq E(0)$ for all $t\in[0,T)$. Hence (3.9) is established.

To prove (3.10), we make use of (2.10) to get

$$\frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx\leq E(0)+\frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right). \tag{3.16}$$

Consequently, (3.9) yields

$$\begin{aligned}\frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right) &\geq \frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx-E(0)\\ &\geq \frac{1}{2}\alpha_2^2-E(0)\geq \frac{1}{2}\alpha_2^2-g(\alpha_2)=\frac{B^{2(p+2)}}{2(p+2)}\alpha_2^{2(p+2)}.\end{aligned} \tag{3.17}$$

Therefore, (3.17) and (3.7) yield the desired result. □

Proof of Theorem 2.2. We suppose that the solution exists for all time and we reach to a contradiction. Set

$$H(t)=E_2-E(t). \tag{3.18}$$

By using (2.10) and (3.18), we have

$$\begin{aligned}0<H(0)\leq H(t)=E_2-\frac{1}{2}\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right)-\frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx\\ +\frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right).\end{aligned} \tag{3.19}$$

From (3.9), we have

$$\begin{aligned}E_2-\frac{1}{2}\left(\|u_t(t)\|_2^2+\|v_t(t)\|_2^2\right)-\frac{1}{2}\int_{\Omega}\left(G\left(|\nabla u|^2\right)+G\left(|\nabla v|^2\right)\right)dx\\ \leq E_2-\frac{1}{2}\alpha_1^2\leq E_1-\frac{1}{2}\alpha_1^2=-\frac{1}{2(p+2)}\alpha_1^2<0,\quad \forall t\geq 0.\end{aligned} \tag{3.20}$$

Hence, by the above inequality and (2.6), we have

$$0<H(0)\leq H(t)\leq \frac{1}{2(p+2)}\left(\|u+v\|_{2(p+2)}^{2(p+2)}+2\|uv\|_{p+2}^{p+2}\right), \tag{3.21}$$

$$\leq \frac{c_1}{2(p+2)}\left(\|u\|_{2(p+2)}^{2(p+2)}+\|v\|_{2(p+2)}^{2(p+2)}\right). \tag{3.22}$$

We then define

$$\Theta(t)=H^{1-\delta}(t)+\epsilon\int_{\Omega}(uu_t+vv_t)dx, \tag{3.23}$$

where ϵ small enough is to be chosen later and

$$0<\delta\leq \min\left\{\frac{p+1}{2(p+2)},\frac{2(p+2)-(m+1)}{2m(p+2)},\frac{2(p+2)-(r+1)}{2r(p+2)}\right\}. \tag{3.24}$$

Our goal is to show that $\Theta(t)$ satisfies the differential inequality (3.1) which leads to a blow-up in finite time. By taking a derivative of (3.23), we get

$$\begin{aligned}\Theta'(t) &= (1-\delta)H^{-\delta}(t)H'(t) + \epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - \epsilon\int_\Omega\left(g\left(|\nabla u|^2\right)|\nabla u|^2 + g\left(|\nabla v|^2\right)|\nabla v|^2\right)dx \\ &\quad - \epsilon\int_\Omega\left(|u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v\right)dx + \epsilon\int_\Omega\left(uf_1(u,v) + vf_2(u,v)\right)dx \\ &= (1-\delta)H^{-\delta}(t)H'(t) + \epsilon\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) - b_1\epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - \epsilon b_2\|\nabla u\|_{2(q+2)}^{2(q+2)} \\ &\quad - \epsilon b_2\|\nabla v\|_{2(q+2)}^{2(q+2)} - \epsilon\int_\Omega\left(|u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v\right)dx + \epsilon\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right).\end{aligned} \tag{3.25}$$

From the definition of $H(t)$, it follows that

$$\begin{aligned}-b_2\|\nabla u\|_{2(q+2)}^{2(q+2)} - b_2\|\nabla v\|_{2(q+2)}^{2(q+2)} &= 2(q+1)H(t) - 2(q+1)E_2 + (q+1)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ &\quad + (q+1)b_1\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) - 2(q+1)\int_\Omega F(u,v)dx,\end{aligned} \tag{3.26}$$

which together with (3.25) gives

$$\begin{aligned}\Theta'(t) &= (1-\delta)H^{-\delta}(t)H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + b_1 q\epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &\quad - \epsilon\int_\Omega\left(|u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v\right)dx + \epsilon\left(1 - \frac{q+1}{p+2}\right)\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right) \\ &\quad + 2(q+1)H(t) - 2(q+1)E_2.\end{aligned} \tag{3.27}$$

Then, using (3.10), we obtain

$$\begin{aligned}\Theta'(t) &\geq (1-\delta)H^{-\delta}(t)H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + b_1 q\epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + 2(q+1)H(t) \\ &\quad + \epsilon\overline{c}\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right) - \epsilon\int_\Omega\left(|u_t|^{m-1}u_t u + |v_t|^{r-1}v_t v\right)dx,\end{aligned} \tag{3.28}$$

where $\overline{c} = 1-(q+1)/(p+2)-2(q+1)E_2(B\alpha_2)^{-2(p+2)}$. It is clear that $\overline{c} > 0$, since $\alpha_2 > B^{-(p+2)/(p+1)}$. We now exploit Young's inequality to estimate the last two terms on the right side of (3.28)

$$\begin{aligned}\left|\int_\Omega |u_t|^{m-1}u_t u dx\right| &\leq \frac{\eta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1} + \frac{m\eta_1^{-((m+1)/m)}}{m+1}\|u_t\|_{m+1}^{m+1}, \\ \left|\int_\Omega |v_t|^{r-1}v_t v dx\right| &\leq \frac{\eta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1} + \frac{r\eta_2^{-((r+1)/r)}}{r+1}\|v_t\|_{r+1}^{r+1},\end{aligned} \tag{3.29}$$

where η_1, η_2 are parameters depending on the time t and specified later. Inserting the last two estimates into (3.28), we have

$$\begin{aligned}\Theta'(t) \geq{}& (1-\delta)H^{-\delta}(t)H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + b_1 q\epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) + 2(q+1)H(t) \\ &+ \epsilon\overline{c}\left(\|u+v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}\right) - \epsilon\frac{\eta_1^{m+1}}{m+1}\|u\|_{m+1}^{m+1} - \epsilon\frac{m\eta_1^{-((m+1)/m)}}{m+1}\|u_t\|_{m+1}^{m+1} \\ &- \epsilon\frac{\eta_2^{r+1}}{r+1}\|v\|_{r+1}^{r+1} - \epsilon\frac{r\eta_2^{-((r+1)/r)}}{r+1}\|v_t\|_{r+1}^{r+1}.\end{aligned} \tag{3.30}$$

By choosing η_1 and η_2 such that

$$\eta_1^{-(m+1)/m} = M_1 H^{-\delta}(t), \qquad \eta_2^{-(r+1)/r} = M_2 H^{-\delta}(t), \tag{3.31}$$

where M_1 and M_2 are constants to be fixed later. Thus, by using (2.6) and (3.31), inequality (3.31) then takes the form

$$\begin{aligned}\Theta'(t) \geq{}& ((1-\delta) - M\epsilon)H^{-\delta}(t)H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) + b_1 q\epsilon\left(\|\nabla u\|_2^2 + \|\nabla v\|_2^2\right) \\ &+ 2(q+1)H(t) + \epsilon c_2\left(\|u\|_{2(p+2)}^{2(p+2)} + 2\|v\|_{2(p+2)}^{2(p+2)}\right) - \epsilon M_1^{-m} H^{\delta m}(t)\|u\|_{m+1}^{m+1} \\ &- \epsilon M_2^{-r} H^{\delta r}(t)\|v\|_{r+1}^{r+1},\end{aligned} \tag{3.32}$$

where $M = m/(m+1)M_1 + r/(r+1)M_2$ and c_2 is a positive constant.

Since $2(p+2) > \max\{m+1, r+1\}$, taking into account (2.6) and (3.21), then we have

$$\begin{aligned}H^{\delta m}(t)\|u\|_{m+1}^{m+1} &\leq c_3\left(\|u\|_{2(p+2)}^{2\delta m(p+2)+(m+1)} + \|v\|_{2(p+2)}^{2\delta m(p+2)}\|u\|_{m+1}^{m+1}\right), \\ H^{\delta r}(t)\|v\|_{r+1}^{r+1} &\leq c_4\left(\|v\|_{2(p+2)}^{2\delta r(p+2)+(r+1)} + \|u\|_{2(p+2)}^{2\delta r(p+2)}\|v\|_{r+1}^{r+1}\right),\end{aligned} \tag{3.33}$$

for some positive constants c_3 and c_4. By using (3.24) and the algebraic inequality

$$z^{\nu} \leq z+1 \leq \left(1+\frac{1}{a}\right)(z+a), \quad \forall z \geq 0,\ 0 < \nu \leq 1,\ a \geq 0, \tag{3.34}$$

we have

$$\|u\|_{2(p+2)}^{2\delta m(p+2)+(m+1)} \leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + H(0)\right) \leq d\left(\|u\|_{2(p+2)}^{2(p+2)} + H(t)\right), \quad \forall t \geq 0, \tag{3.35}$$

where $d = 1 + 1/H(0)$. Similarly,

$$\|v\|_{2(p+2)}^{2\delta r(p+2)+(r+1)} \le d\left(\|v\|_{2(p+2)}^{2(p+2)} + H(t)\right), \quad \forall t \ge 0. \tag{3.36}$$

Also, since

$$(X+Y)^s \le C(X^s + Y^s), \quad X, Y \ge 0, \ s > 0, \tag{3.37}$$

by using (3.24) and (3.34), we conclude that

$$\begin{aligned} \|v\|_{2(p+2)}^{2\delta m(p+2)} \|u\|_{m+1}^{m+1} &\le C\left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{(m+1)}^{2(p+2)}\right) \le C\left(\|v\|_{2(p+2)}^{2(p+2)} + \|u\|_{2(p+2)}^{2(p+2)}\right), \\ \|u\|_{2(p+2)}^{2\delta r(p+2)} \|v\|_{r+1}^{r+1} &\le C\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{(r+1)}^{2(p+2)}\right) \le C\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right), \end{aligned} \tag{3.38}$$

where C is a generic positive constant. Taking into account (3.33)–(3.38), estimate (3.32) takes the form

$$\begin{aligned} \Theta'(t) \ge{} & ((1-\delta) - M\epsilon) H^{-\delta}(t) H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \epsilon\left(2(q+1) - C_1 M_1^{-m} - C_1 M_2^{-r}\right) H(t) \\ & + \epsilon\left(c_2 - C_2 M_1^{-m} - C_2 M_2^{-r}\right)\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right), \end{aligned} \tag{3.39}$$

where $C_1 = \max\{c_3 d + C, c_4 d + C\}$, $C_2 = \max\{c_3 d, c_4 d\}$. At this point, and for large values of M_1 and M_2, we can find positive constants κ_1 and κ_2 such that (3.39) becomes

$$\begin{aligned} \Theta'(t) \ge{} & ((1-\delta) - M\epsilon) H^{-\delta}(t) H'(t) + \epsilon(q+2)\left(\|u_t\|_2^2 + \|v_t\|_2^2\right) \\ & + \epsilon\kappa_1 H(t) + \epsilon\kappa_2\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right). \end{aligned} \tag{3.40}$$

Once M_1 and M_2 are fixed, we pick ϵ small enough so that $(1-\delta) - M\epsilon \ge 0$ and

$$\Theta(0) = H^{1-\delta}(0) + \epsilon \int_\Omega (u_0 u_1 + v_0 v_1)\, dx > 0. \tag{3.41}$$

Since $H'(t) \ge 0$, there exists $\Lambda > 0$ such that (3.40) becomes

$$\Theta'(t) \ge \epsilon\Lambda\left(H(t) + \|u_t\|_2^2 + \|v_t\|_2^2 + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}\right). \tag{3.42}$$

Then, we have

$$\Theta(t) \ge \Theta(0), \quad \forall t \ge 0. \tag{3.43}$$

Next, we have by Hölder's and Young's inequalities

$$\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right)^{1/(1-\delta)} \le C\left(\|u\|_{2(p+2)}^{\tau/(1-\delta)} + \|u_t\|_2^{s/(1-\delta)} + \|v\|_{2(p+2)}^{\tau/(1-\delta)} + \|v_t\|_2^{s/(1-\delta)}\right), \tag{3.44}$$

for $1/\tau + 1/s = 1$. We take $s = 2(1-\delta)$, to get $\tau/(1-\delta) = 2/(1-2\delta)$. Here and in the sequel, C denotes a positive constant which may change from line to line. By using (3.24) and (3.34), we have

$$\|u\|_{2(p+2)}^{2/(1-2\delta)} \le d\left(\|u\|_{2(p+2)}^{2(p+2)} + H(t)\right), \qquad \|v\|_{2(p+2)}^{2/(1-2\delta)} \le d\left(\|v\|_{2(p+2)}^{2(p+2)} + H(t)\right), \quad \forall t \ge 0. \tag{3.45}$$

Therefore, (3.44) becomes

$$\left(\int_\Omega uu_t dx + \int_\Omega vv_t dx\right)^{1/(1-\delta)} \le C\left(\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right). \tag{3.46}$$

Note that

$$\begin{aligned}\Theta^{1/(1-\delta)}(t) &= \left(H^{1-\delta}(t) + \epsilon\int_\Omega (uu_t + vv_t)dx\right)^{1/(1-\delta)}\\ &\le C\left(H(t) + \left|\int_\Omega uu_t dx + \int_\Omega vv_t dx\right|^{1/(1-\delta)}\right)\\ &\le C\left(H(t) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_2^2 + \|v_t\|_2^2\right).\end{aligned} \tag{3.47}$$

Combining (3.42) with (3.47), we have

$$\Theta(t) \ge C\Theta^{1/(1-\delta)}(t), \quad \forall t \ge 0. \tag{3.48}$$

A simple application of Lemma 3.1 gives the desired result. □

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Research Article

H_∞ Estimation for a Class of Lipschitz Nonlinear Discrete-Time Systems with Time Delay

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The issue of H_∞ estimation for a class of Lipschitz nonlinear discrete-time systems with time delay and disturbance input is addressed. First, through integrating the H_∞ filtering performance index with the Lipschitz conditions of the nonlinearity, the design of robust estimator is formulated as a positive minimum problem of indefinite quadratic form. Then, by introducing the Krein space model and applying innovation analysis approach, the minimum of the indefinite quadratic form is obtained in terms of innovation sequence. Finally, through guaranteeing the positivity of the minimum, a sufficient condition for the existence of the H_∞ estimator is proposed and the estimator is derived in terms of Riccati-like difference equations. The proposed algorithm is proved to be effective by a numerical example.

1. Introduction

In control field, nonlinear estimation is considered to be an important task which is also of great challenge, and it has been a very active area of research for decades [1–7]. Many kinds of methods on estimator design have been proposed for different types of nonlinear dynamical systems. Generally speaking, there are three approaches widely adopted for nonlinear estimation. In the first one, by using an extended (nonexact) linearization of the nonlinear systems, the estimator is designed by employing classical linear observer techniques [1]. The second approach, based on a nonlinear state coordinate transformation which renders the dynamics driven by nonlinear output injection and the output linear on the new coordinates, uses the quasilinear approaches to design the nonlinear estimator [2–4]. In the last one, methods are developed to design nonlinear estimators for systems which consist of an observable linear part and a locally or globally Lipschitz nonlinear part [5–7]. In this paper, the problem of H_∞ estimator design is investigated for a class of Lipschitz nonlinear discrete-time systems with time delay and disturbance input.

In practice, most nonlinearities can be regarded as Lipschitz, at least locally when they are studied in a given neighborhood [6]. For example, trigonometric nonlinearities occurring in many robotic problems, non-linear softening spring models frequently used in mechanical systems, nonlinearities which are square or cubic in nature, and so forth. Thus, in recent years, increasing attention has been paid to estimator design for Lipschitz nonlinear systems [8–19]. For the purpose of designing this class of nonlinear estimator, a number of approaches have been developed, such as sliding mode observers [8, 9], H_∞ optimization techniques [10–13], adaptive observers [14, 15], high-gain observers [16], loop transfer recovery observers [17], proportional integral observers [18], and integral quadratic constraints approach [19]. All of the above results are obtained in the assumption that the Lipschitz nonlinear systems are delay free. However, time delay is an inherent characteristic of many physical systems, and it can result in instability and poor performances if it is ignored. The estimator design for time-delay Lipschitz nonlinear systems has become a substantial need. Unfortunately, compared with estimator design for delay-free Lipschitz nonlinear systems, less research has been carried out on the time-delay case. In [20], the linear matrix inequality-(LMI-) based full-order and reduced-order robust H_∞ observers are proposed for a class of Lipschitz nonlinear discrete-time systems with time delay. In [21], by using Lyapunov stability theory and LMI techniques, a delay-dependent approach to the H_∞ and L_2-L_∞ filtering is proposed for a class of uncertain Lipschitz nonlinear time-delay systems. In [22], by guaranteeing the asymptotic stability of the error dynamics, the robust observer is presented for a class of uncertain discrete-time Lipschitz nonlinear state delayed systems; In [23], based on the sliding mode techniques, a discontinuous observer is designed for a class of Lipschitz nonlinear systems with uncertainty. In [24], an LMI-based convex optimization approach to observer design is developed for both constant-delay and time-varying delay Lipschitz nonlinear systems.

In this paper, the H_∞ estimation problem is studied for a class of Lipschitz nonlinear discrete time-delay systems with disturbance input. Inspired by the recent study on H_∞ fault detection for linear discrete time-delay systems in [25], a recursive Kalman-like algorithm in an indefinite metric space, named the Krein space [26], will be developed to the design of H_∞ estimator for time-delay Lipschitz nonlinear systems. Unlike [20], the delay-free nonlinearities and the delayed nonlinearities in the presented systems are decoupling. For the case presented in [20], the H_∞ observer design problem, utilizing the technical line of this paper, can be solved by transforming it into a delay-free system through state augmentation. Indeed, the state augmentation results in a higher system dimension and, thus, a much more expensive computational cost. Therefore, this paper based on the presented time-delay Lipschitz nonlinear systems, focuses on the robust estimator design without state augmentation by employing innovation analysis approach in the Krein space. The major contribution of this paper can be summarized as follows: (i) it extends the Krein space linear estimation methodology [26] to the state estimation of the time-delay Lipschitz nonlinear systems and (ii) it develops a recursive Kalman-like robust estimator for time-delay Lipschitz nonlinear systems without state augmentation.

The remainder of this paper is arranged as follows. In Section 2, the interest system, the Lipschitz conditions, and the H_∞ estimation problem are introduced. In Section 3, a partially equivalent Krein space problem is constructed, the H_∞ estimator is obtained by computed Riccati-like difference equations, and sufficient existence condition is derived in terms of matrix inequalities. An example is given to show the effect of the proposed algorithm in Section 4. Finally, some concluding remarks are made in Section 5.

In the sequel, the following notation will be used: elements in the Krein space will be denoted by **boldface** letters, and elements in the Euclidean space of complex numbers will be denoted by normal letters; $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space; $\|\cdot\|$ denotes the Euclidean norm; $\theta(k) \in l_2[0,N]$ means $\sum_{k=0}^{N}(\theta^T(k)\theta(k)) < \infty$; the superscripts "$-1$" and "$T$" stand for the inverse and transpose of a matrix, resp.; I is the identity matrix with appropriate dimensions; For a real matrix, $P > 0$ ($P < 0$, resp.) means that P is symmetric and positive (negative, resp.) definite; $\langle *, * \rangle$ denotes the inner product in the Krein space; diag$\{\cdots\}$ denotes a block-diagonal matrix; $\mathcal{L}\{\cdots\}$ denotes the linear space spanned by sequence $\{\cdots\}$.

2. System Model and Problem Formulation

Consider a class of nonlinear systems described by the following equations:

$$\begin{aligned} x(k+1) &= Ax(k) + A_d x(k_d) + f(k, Fx(k), u(k)) \\ &\quad + h(k, Hx(k_d), u(k)) + Bw(k), \\ y(k) &= Cx(k) + v(k), \\ z(k) &= Lx(k), \end{aligned} \tag{2.1}$$

where $k_d = k - d$, and the positive integer d denotes the known state delay; $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^p$ is the measurable information, $w(k) \in \mathbb{R}^q$ and $v(k) \in \mathbb{R}^m$ are the disturbance input belonging to $l_2[0,N]$, $y(k) \in \mathbb{R}^m$ is the measurement output, and $z(k) \in \mathbb{R}^r$ is the signal to be estimated; the initial condition $x_0(s)$ $(s = -d, -d+1, \ldots, 0)$ is unknown; the matrices $A \in \mathbb{R}^{n\times n}$, $A_d \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times q}$, $C \in \mathbb{R}^{m\times n}$ and $L \in \mathbb{R}^{r\times n}$, are real and known constant matrices.

In addition, $f(k, Fx(k), u(k))$ and $h(k, Hx(k_d), u(k))$ are assumed to satisfy the following Lipschitz conditions:

$$\begin{aligned} \|f(k, Fx(k), u(k)) - f(k, F\check{x}(k), u(k))\| &\le \alpha \|F(x(k) - \check{x}(k))\|, \\ \|h(k, Hx(k_d), u(k)) - h(k, H\check{x}(k_d), u(k))\| &\le \beta \|H(x(k_d) - \check{x}(k_d))\|, \end{aligned} \tag{2.2}$$

for all $k \in \{0, 1, \ldots, N\}$, $u(k) \in \mathbb{R}^p$ and $x(k), \check{x}(k), x(k_d), \check{x}(k_d) \in \mathbb{R}^n$. where $\alpha > 0$ and $\beta > 0$ are known Lipschitz constants, and F, H are real matrix with appropriate dimension.

The H_∞ estimation problem under investigation is stated as follows. Given the desired noise attenuation level $\gamma > 0$ and the observation $\{y(j)\}_{j=0}^{k}$, find an estimate $\check{z}(k \mid k)$ of the signal $z(k)$, if it exists, such that the following inequality is satisfied:

$$\sup_{(x_0, w, v) \neq 0} \frac{\sum_{k=0}^{N} \|\check{z}(k \mid k) - z(k)\|^2}{\sum_{k=-d}^{0} \|x_0(k)\|^2_{\Pi^{-1}(k)} + \sum_{k=0}^{N} \|w(k)\|^2 + \sum_{k=0}^{N} \|v(k)\|^2} < \gamma^2, \tag{2.3}$$

where $\Pi(k)$ $(k = -d, -d+1, \ldots, 0)$ is a given positive definite matrix function which reflects the relative uncertainty of the initial state $x_0(k)$ $(k = -d, -d+1, \ldots, 0)$ to the input and measurement noises.

Remark 2.1. For the sake of simplicity, the initial state estimate $\hat{x}_0(k)$ $(k = -d, -d+1, \ldots, 0)$ is assumed to be zero in inequality (2.3).

Remark 2.2. Although the system given in [20] is different from the one given in this paper, the problem mentioned in [20] can also be solved by using the presented approach. The resolvent first converts the system given in [20] into a delay-free one by using the classical system augmentation approach, and then designs estimator by employing the similar but easier technical line with our paper.

3. Main Results

In this section, the Krein space-based approach is proposed to design the H_∞ estimator for Lipschitz nonlinear systems. To begin with, the H_∞ estimation problem (2.3) and the Lipschitz conditions (2.2) are combined in an indefinite quadratic form, and the nonlinearities are assumed to be obtained by $\{y(i)\}_{i=0}^{k}$ at the time step k. Then, an equivalent Krein space problem is constructed by introducing an imaginary Krein space stochastic system. Finally, based on projection formula and innovation analysis approach in the Krein space, the recursive estimator is derived.

3.1. *Construct a Partially Equivalent Krein Space Problem*

It is proved in this subsection that the H_∞ estimation problem can be reduced to a positive minimum problem of indefinite quadratic form, and the minimum can be obtained by using the Krein space-based approach.

Since the denominator of the left side of (2.3) is positive, the inequality (2.3) is equivalent to

$$\underbrace{\sum_{k=-d}^{0}\|x_0(k)\|^2_{\Pi^{-1}(k)}+\sum_{k=0}^{N}\|w(k)\|^2+\sum_{k=0}^{N}\|v(k)\|^2-\gamma^{-2}\sum_{k=0}^{N}\|v_z(k)\|^2}_{\triangleq J_N^*}>0,\quad \forall(x_0,w,v)\neq 0, \tag{3.1}$$

where $v_z(k)=\check{z}(k\mid k)-z(k)$.

Moreover, we denote

$$\begin{aligned} z_f(k)&=Fx(k), & \check{z}_f(k\mid k)&=F\check{x}(k\mid k),\\ z_h(k_d)&=Hx(k_d), & \check{z}_h(k_d\mid k)&=H\check{x}(k_d\mid k), \end{aligned} \tag{3.2}$$

where $\check{z}_f(k\mid k)$ and $\check{z}_h(k_d\mid k)$ denote the optimal estimation of $z_f(k)$ and $z_h(k_d)$ based on the observation $\{y(j)\}_{j=0}^{k}$, respectively. And, let

$$\begin{aligned} w_f(k)&=f\big(k,z_f(k),u(k)\big)-f\big(k,\check{z}_f(k\mid k),u(k)\big),\\ w_h(k_d)&=h(k,z_h(k_d),u(k))-h(k,\check{z}_h(k_d\mid k),u(k)),\\ v_{z_f}(k)&=\check{z}_f(k\mid k)-z_f(k),\\ v_{z_h}(k_d)&=\check{z}_h(k_d\mid k)-z_h(k_d). \end{aligned} \tag{3.3}$$

From the Lipschitz conditions (2.2), we derive that

$$\underbrace{J_N^* + \sum_{k=0}^{N} \left\| w_f(k) \right\|^2 + \sum_{k=0}^{N} \| w_h(k_d) \|^2 - \alpha^2 \sum_{k=0}^{N} \left\| v_{z_f}(k) \right\|^2 - \beta^2 \sum_{k=0}^{N} \| v_{z_h}(k_d) \|^2}_{\triangleq J_N} \le J_N^*. \tag{3.4}$$

Note that the left side of (3.1) and (3.4), J_N, can be recast into the form

$$\begin{aligned} J_N = & \sum_{k=-d}^{0} \| x_0(k) \|^2_{\overline{\Pi}^{-1}(k)} + \sum_{k=0}^{N} \| \overline{w}(k) \|^2 + \sum_{k=0}^{N} \| v(k) \|^2 \\ & - \gamma^{-2} \sum_{k=0}^{N} \| v_z(k) \|^2 - \alpha^2 \sum_{k=0}^{N} \left\| v_{z_f}(k) \right\|^2 - \beta^2 \sum_{k=d}^{N} \| v_{z_h}(k_d) \|^2, \end{aligned} \tag{3.5}$$

where

$$\begin{gathered} \overline{\Pi}(k) = \begin{cases} \left(\Pi^{-1}(k) - \beta^2 H^T H \right)^{-1}, & k = -d, \ldots, -1, \\ \Pi(k), & k = 0, \end{cases} \\ \overline{w}(k) = \left[w^T(k) \;\; w_f^T(k) \;\; w_h^T(k_d) \right]^T. \end{gathered} \tag{3.6}$$

Since $J_N \le J_N^*$, it is natural to see that if $J_N > 0$ then the H_∞ estimation problem (2.3) is satisfied, that is, $J_N^* > 0$. Hence, the H_∞ estimation problem (2.3) can be converted into finding the estimate sequence $\{\{\check{z}(k \mid k)\}_{k=0}^N; \{\check{z}_f(k \mid k)\}_{k=0}^N; \{\check{z}_h(k_d \mid k)\}_{k=d}^N\}$ such that J_N has a minimum with respect to $\{x_0, \overline{w}\}$ and the minimum of J_N is positive. As mentioned in [25, 26], the formulated H_∞ estimation problem can be solved by employing the Krein space approach.

Introduce the following Krein space stochastic system

$$\begin{aligned} \mathbf{x}(k+1) &= A\mathbf{x}(k) + A_d \mathbf{x}(k_d) + f\left(k, \check{\mathbf{z}}_f(k \mid k), \mathbf{u}(k)\right) \\ &\quad + h(k, \check{\mathbf{z}}_h(k_d \mid k), \mathbf{u}(k)) + \overline{B}\overline{\mathbf{w}}(k), \\ \mathbf{y}(k) &= C\mathbf{x}(k) + \mathbf{v}(k), \\ \check{\mathbf{z}}_f(k \mid k) &= F\mathbf{x}(k) + \mathbf{v}_{z_f}(k), \\ \check{\mathbf{z}}(k \mid k) &= L\mathbf{x}(k) + \mathbf{v}_z(k), \\ \check{\mathbf{z}}_h(k_d \mid k) &= H\mathbf{x}(k_d) + \mathbf{v}_{z_h}(k_d), \quad k \ge d, \end{aligned} \tag{3.7}$$

where $\overline{B} = [B \;\; I \;\; I]$; the initial state $\mathbf{x}_0(s)$ $(s = -d, -d+1, \ldots, 0)$ and $\overline{\mathbf{w}}(k)$, $\mathbf{v}(k)$, $\mathbf{v}_{z_f}(k)$, $\mathbf{v}_z(k)$ and $\mathbf{v}_{z_h}(k)$ are mutually uncorrelated white noises with zero means and known covariance matrices $\overline{\Pi}(s)$, $Q_{\overline{w}}(k) = I$, $Q_v(k) = I$, $Q_{v_{z_f}}(k) = -\alpha^{-2} I$, $Q_{v_z}(k) = -\gamma^2 I$, and $Q_{v_{zh}}(k) = -\beta^{-2} I$; $\check{\mathbf{z}}_f(k \mid k)$, $\check{\mathbf{z}}(k \mid k)$ and $\check{\mathbf{z}}_h(k_d \mid k)$ are regarded as the imaginary measurement at time k for the linear combination $F\mathbf{x}(k)$, $L\mathbf{x}(k)$, and $H\mathbf{x}(k_d)$, respectively.

Let

$$\mathbf{y}_z(k) = \begin{cases} \left[\mathbf{y}^T(k) \;\; \check{\mathbf{z}}_m^T(k \mid k)\right]^T, & 0 \le k < d, \\ \left[\mathbf{y}^T(k) \;\; \check{\mathbf{z}}_m^T(k \mid k) \;\; \check{\mathbf{z}}_h^T(k_d \mid k)\right]^T, & k \ge d, \end{cases}$$

$$\mathbf{v}_{z,a}(k) = \begin{cases} \left[\mathbf{v}^T(k) \;\; \mathbf{v}_{z_f}^T(k) \;\; \mathbf{v}_z^T(k)\right]^T, & 0 \le k < d, \\ \left[\mathbf{v}^T(k) \;\; \mathbf{v}_{z_f}^T(k) \;\; \mathbf{v}_z^T(k) \;\; \mathbf{v}_{z_h}^T(k_d)\right]^T, & k \ge d, \end{cases} \tag{3.8}$$

$$\check{\mathbf{z}}_m(k \mid k) = \left[\check{\mathbf{z}}_f^T(k \mid k) \;\; \check{\mathbf{z}}^T(k \mid k)\right]^T.$$

Definition 3.1. The estimator $\hat{\mathbf{y}}(i \mid i-1)$ denotes the optimal estimation of $\mathbf{y}(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}\}$; the estimator $\hat{\mathbf{z}}_m(i \mid i)$ denotes the optimal estimation of $\check{\mathbf{z}}_m(i \mid i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i)\}$; the estimator $\hat{\mathbf{z}}_h(i_d \mid i)$ denotes the optimal estimation of $\check{\mathbf{z}}_h(i_d \mid i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i), \check{\mathbf{z}}_m(i \mid i)\}$.

Furthermore, introduce the following stochastic vectors and the corresponding covariance matrices

$$\begin{aligned} \tilde{\mathbf{y}}(i \mid i-1) &= \mathbf{y}(i) - \hat{\mathbf{y}}(ii-1), & R_{\tilde{y}}(ii-1) &= \langle \tilde{\mathbf{y}}(ii-1), \tilde{\mathbf{y}}(ii-1) \rangle, \\ \tilde{\mathbf{z}}_m(i \mid i) &= \check{\mathbf{z}}_m(ii) - \hat{\mathbf{z}}_m(ii), & R_{\tilde{z}_m}(ii) &= \langle \tilde{\mathbf{z}}_m(ii), \tilde{\mathbf{z}}_m(ii) \rangle, \\ \tilde{\mathbf{z}}_h(i_d \mid i) &= \check{\mathbf{z}}_h(i_d i) - \hat{\mathbf{z}}_h(i_d i), & R_{\tilde{z}_h}(i_d i) &= \langle \tilde{\mathbf{z}}_h(i_d i), \tilde{\mathbf{z}}_h(i_d i) \rangle. \end{aligned} \tag{3.9}$$

And, denote

$$\tilde{\mathbf{y}}_z(i) = \begin{cases} \left[\tilde{\mathbf{y}}^T(i \mid i-1) \;\; \tilde{\mathbf{z}}_m^T(i \mid i)\right]^T, & 0 \le i < d, \\ \left[\tilde{\mathbf{y}}^T(i \mid i-1) \;\; \tilde{\mathbf{z}}_m^T(i \mid i) \;\; \tilde{\mathbf{z}}_h^T(i_d \mid i)\right]^T, & i \ge d, \end{cases} \tag{3.10}$$

$$R_{\tilde{y}_z}(i) = \langle \tilde{\mathbf{y}}_z(i), \tilde{\mathbf{y}}_z(i) \rangle.$$

For calculating the minimum of J_N, we present the following Lemma 3.2.

Lemma 3.2. $\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^{k}\}$ *is the innovation sequence which spans the same linear space as that of* $\mathcal{L}\{\{\mathbf{y}_z(i)\}_{i=0}^{k}\}$.

Proof. From Definition 3.1 and (3.9), $\tilde{\mathbf{y}}(i \mid i-1)$, $\tilde{\mathbf{z}}_m(i \mid i)$ and $\tilde{\mathbf{z}}_h(i_d \mid i)$ are the linear combination of the observation sequence $\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i)\}$, $\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i), \check{\mathbf{z}}_m(i \mid i)\}$, and $\{\{\mathbf{y}_z(j)\}_{j=0}^{i}\}$, respectively. Conversely, $\mathbf{y}(i)$, $\check{\mathbf{z}}_m(i \mid i)$ and $\check{\mathbf{z}}_h(i_d \mid i)$ can be given by the linear combination of $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1)\}$, $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1), \tilde{\mathbf{z}}_m(i \mid i)\}$ and $\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i}\}$, respectively. Hence,

$$\mathcal{L}\left\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^{k}\right\} = \mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{k}\right\}. \tag{3.11}$$

It is also shown by (3.9) that $\tilde{\mathbf{y}}(i \mid i-1)$, $\tilde{\mathbf{z}}_m(i \mid i)$ and $\tilde{\mathbf{z}}_h(i_d \mid i)$ satisfy

$$\begin{aligned} \tilde{\mathbf{y}}(i \mid i-1) &\perp \mathcal{L}\left\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}\right\}, \\ \tilde{\mathbf{z}}_m(i \mid i) &\perp \mathcal{L}\left\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i)\right\}, \\ \tilde{\mathbf{z}}_h(i_d \mid i) &\perp \mathcal{L}\left\{\{\mathbf{y}_z(j)\}_{j=0}^{i-1}; \mathbf{y}(i), \tilde{\mathbf{z}}_m(i \mid i)\right\}. \end{aligned} \tag{3.12}$$

Consequently,

$$\begin{aligned} \tilde{\mathbf{y}}(i \mid i-1) &\perp \mathcal{L}\left\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}\right\}, \\ \tilde{\mathbf{z}}_m(i \mid i) &\perp \mathcal{L}\left\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1)\right\}, \\ \tilde{\mathbf{z}}_h(i_d \mid i) &\perp \mathcal{L}\left\{\{\tilde{\mathbf{y}}_z(j)\}_{j=0}^{i-1}; \tilde{\mathbf{y}}(i \mid i-1), \tilde{\mathbf{z}}_m(i \mid i)\right\}. \end{aligned} \tag{3.13}$$

This completes the proof. □

Now, an existence condition and a solution to the minimum of J_N are derived as follows.

Theorem 3.3. *Consider system* (2.1)*, given a scalar* $\gamma > 0$ *and the positive definite matrix* $\Pi(k)$ $(k = -d, -d+1, \ldots, 0)$*, then* J_N *has the minimum if only if*

$$\begin{aligned} R_{\tilde{y}}(k \mid k-1) &> 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_m}(k \mid k) &< 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_h}(k_d \mid k) &< 0, \quad d \le k \le N. \end{aligned} \tag{3.14}$$

In this case the minimum value of J_N *is given by*

$$\begin{aligned} \min J_N = &\sum_{k=0}^{N} \tilde{y}^T(k \mid k-1) R_{\tilde{y}}^{-1}(k \mid k-1) \tilde{y}(k \mid k-1) + \sum_{k=0}^{N} \tilde{z}_m^T(k \mid k) R_{\tilde{z}_m}^{-1}(k \mid k) \tilde{z}_m(k \mid k) \\ &+ \sum_{k=d}^{N} \tilde{z}_h^T(k_d \mid k) R_{\tilde{z}_h}^{-1}(k_d \mid k) \tilde{z}_h(k_d \mid k), \end{aligned} \tag{3.15}$$

where

$$\begin{aligned} \tilde{y}(k \mid k-1) &= y(k) - \hat{y}(k \mid k-1), \\ \tilde{z}_m(k \mid k) &= \check{z}_m(k \mid k) - \hat{z}_m(k \mid k), \\ \tilde{z}_h(k_d \mid k) &= \check{z}_h(k_d \mid k) - \hat{z}_h(k_d \mid k), \\ \check{z}_m(k \mid k) &= \left[\check{z}_f^T(k \mid k) \quad \check{z}^T(k \mid k)\right]^T, \end{aligned} \tag{3.16}$$

$\hat{y}(k \mid k-1)$ is obtained from the Krein space projection of $\mathbf{y}(k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}\}$, $\hat{z}_m(k \mid k)$ is obtained from the Krein space projection of $\check{\mathbf{z}}_m(k \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1};\ \mathbf{y}(k)\}$, and $\hat{z}_h(k_d \mid k)$ is obtained from the Krein space projection of $\check{\mathbf{z}}_h(k_d \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1};\ \mathbf{y}(k), \check{\mathbf{z}}_m(k \mid k)\}$.

Proof. Based on the definition (3.2) and (3.3), the state equation in system (2.1) can be rewritten as

$$\begin{aligned} x(k+1) = {} & Ax(k) + A_d x(k_d) + f\left(k, \check{z}_f(k \mid k), u(k)\right) \\ & + h(k, \check{z}_h(k_d \mid k), u(k)) + \overline{B}\overline{w}(k). \end{aligned} \tag{3.17}$$

In this case, it is assumed that $f(k, \check{z}_f(k \mid k), u(k))$ and $h(k, \check{z}_h(k_d \mid k), u(k))$ are known at time k. Then, we define

$$y_z(k) = \begin{cases} \left[y^T(k) \ \ \check{z}_f^T(k \mid k) \ \ \check{z}^T(k \mid k)\right]^T, & 0 \le k < d, \\ \left[y^T(k) \ \ \check{z}_f^T(k \mid k) \ \ \check{z}^T(k \mid k) \ \ \check{z}_h^T(k_d \mid k)\right]^T, & k \ge d. \end{cases} \tag{3.18}$$

By introducing an augmented state

$$x_a(k) = \left[x^T(k) \ \ x^T(k-1) \ \ \cdots \ \ x^T(k-d)\right]^T, \tag{3.19}$$

we obtain an augmented state-space model

$$\begin{aligned} x_a(k+1) &= A_a x_a(k) + B_{u,a}\overline{u}(k) + \overline{B}_a\overline{w}(k), \\ y_z(k) &= C_{z,a}(k)x_a(k) + v_{z,a}(k), \end{aligned} \tag{3.20}$$

where

$$A_a = \begin{bmatrix} A & 0 & \cdots & 0 & A_d \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{bmatrix}, \qquad B_{u,a} = \begin{bmatrix} I & I \\ 0 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}, \qquad \overline{B}_a = \begin{bmatrix} \overline{B} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$
\begin{aligned}
\mathrm{C}_{z,a}(k) &= \begin{cases} \begin{bmatrix} C & 0 & \cdots & 0 \\ F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \end{bmatrix}, & 0 \le k < d, \\ \begin{bmatrix} C & 0 & \cdots & 0 \\ F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \\ 0 & \cdots & 0 & H \end{bmatrix}, & k \ge d, \end{cases} \\
v_{z,a}(k) &= \begin{cases} \begin{bmatrix} v^T(k) & v_{z_f}^T(k) & v_z^T(k) \end{bmatrix}^T, & 0 \le k < d, \\ \begin{bmatrix} v^T(k) & v_{z_f}^T(k) & v_z^T(k) & v_{z_h}^T(k_d) \end{bmatrix}^T, & k \ge d, \end{cases} \\
\overline{u}(k) &= \left[f^T\left(k, \breve{z}_f(k \mid k), u(k)\right) \;\; h^T\left(k, \breve{z}_h(k_d \mid k), u(k)\right) \right]^T.
\end{aligned} \tag{3.21}
$$

Additionally, we can rewrite J_N as

$$
J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ v_{z,aN} \end{bmatrix}^T \begin{bmatrix} P_a(0) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{v_{z,aN}} \end{bmatrix}^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ v_{z,aN} \end{bmatrix}, \tag{3.22}
$$

where

$$
\begin{aligned}
P_a(0) &= \operatorname{diag}\left\{ \overline{\Pi}(0), \overline{\Pi}(-1), \ldots, \overline{\Pi}(-d) \right\}, \\
\overline{w}_N &= \begin{bmatrix} \overline{w}^T(0) & \overline{w}^T(1) & \cdots & \overline{w}^T(N) \end{bmatrix}^T, \\
v_{z,aN} &= \left[v_{z,a}^T(0) \;\; v_{z,a}^T(1) \;\; \cdots \;\; v_{z,a}^T(N) \right]^T, \\
Q_{v_{z,aN}} &= \operatorname{diag}\left\{ Q_{v_{z,a}}(0), Q_{v_{z,a}}(1), \ldots, Q_{v_{z,a}}(N) \right\}, \\
Q_{v_{z,a}}(k) &= \begin{cases} \operatorname{diag}\left\{ I, -\gamma^2, -\alpha^{-2} \right\}, & 0 \le k < d, \\ \operatorname{diag}\left\{ I, -\gamma^2, -\alpha^{-2}, -\beta^{-2} \right\}, & k \ge d. \end{cases}
\end{aligned} \tag{3.23}
$$

Define the following state transition matrix

$$
\begin{aligned}
\Phi(k+1, m) &= A_a \Phi(k, m), \\
\Phi(m, m) &= I,
\end{aligned} \tag{3.24}
$$

and let

$$y_{zN} = \left[y_z^T(0) \;\; y_z^T(1) \;\; \cdots \;\; y_z^T(N)\right]^T,$$
$$\overline{u}_N = \left[\overline{u}^T(0) \;\; \overline{u}^T(1) \;\; \cdots \;\; \overline{u}^T(N)\right]^T. \tag{3.25}$$

Using (3.20) and (3.24), we have

$$y_{zN} = \Psi_{0N} x_a(0) + \Psi_{\overline{u}N}\overline{u}_N + \Psi_{\overline{w}N}\overline{w}_N + v_{z,aN}, \tag{3.26}$$

where

$$\Psi_{0N} = \begin{bmatrix} C_{z,a}(0)\Phi(0,0) \\ C_{z,a}(1)\Phi(1,0) \\ \vdots \\ C_{z,a}(N)\Phi(N,0) \end{bmatrix}, \quad \Psi_{\overline{u}N} = \begin{bmatrix} \varphi_{00} & \varphi_{01} & \cdots & \varphi_{0N} \\ \varphi_{10} & \varphi_{11} & \cdots & \varphi_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{N0} & \varphi_{N1} & \cdots & \varphi_{NN} \end{bmatrix},$$
$$\varphi_{ij} = \begin{cases} C_{z,a}(i)\Phi(i,j+1)B_{u,a}, & i > j, \\ 0, & i \le j. \end{cases} \tag{3.27}$$

The matrix $\Psi_{\overline{w}N}$ is derived by replacing $B_{u,a}$ in $\Psi_{\overline{u}N}$ with $\overline{B}_a$.

Thus, J_N can be reexpressed as

$$J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}^T \left\{ \Gamma_N \begin{bmatrix} P_a(0) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & Q_{v_{z,aN}} \end{bmatrix} \Gamma_N^T \right\}^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}, \tag{3.28}$$

where

$$\overline{y}_{zN} = y_{zN} - \Psi_{\overline{u}N}\overline{u}_N,$$
$$\Gamma_N = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \Psi_{0N} & \Psi_{\overline{w}N} & I \end{bmatrix}. \tag{3.29}$$

Considering the Krein space stochastic system defined by (3.7) and state transition matrix (3.24), we have

$$\mathbf{y}_{zN} = \Psi_{0N}\mathbf{x}_a(0) + \Psi_{\overline{u}N}\overline{\mathbf{u}}_N + \Psi_{\overline{w}N}\overline{\mathbf{w}}_N + \mathbf{v}_{z,aN}, \tag{3.30}$$

where matrices Ψ_{0N}, $\Psi_{\overline{u}N}$, and $\Psi_{\overline{w}N}$ are the same as given in (3.26), vectors $\mathbf{y}_{zN}$ and $\overline{\mathbf{u}}_N$ are, respectively, defined by replacing Euclidean space element y_z and $\overline{u}$ in y_{zN} and $\overline{u}_N$ given by (3.25) with the Krein space element $\mathbf{y}_z$ and $\overline{\mathbf{u}}$, vectors $\overline{\mathbf{w}}_N$ and $\mathbf{v}_{z,aN}$ are also defined by replacing Euclidean space element $\overline{w}$ and $v_{z,a}$ in $\overline{w}_N$ and $v_{z,aN}$ given by (3.23) with the Krein space element $\overline{\mathbf{w}}$ and $\mathbf{v}_{z,a}$, and vector $\mathbf{x}_a(0)$ is given by replacing Euclidean space element x in $x_a(k)$ given by (3.19) with the Krein space element $\mathbf{x}$ when $k = 0$.

Using the stochastic characteristic of $\mathbf{x}_a(0)$, $\overline{\mathbf{w}}_N$ and $\mathbf{v}_{z,a}$, we have

$$J_N = \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}^T \left\langle \begin{bmatrix} \mathbf{x}_a(0) \\ \overline{\mathbf{w}}_N \\ \overline{\mathbf{y}}_{zN} \end{bmatrix}, \begin{bmatrix} \mathbf{x}_a(0) \\ \overline{\mathbf{w}}_N \\ \overline{\mathbf{y}}_{zN} \end{bmatrix} \right\rangle^{-1} \begin{bmatrix} x_a(0) \\ \overline{w}_N \\ \overline{y}_{zN} \end{bmatrix}, \tag{3.31}$$

where $\overline{\mathbf{y}}_{zN} = \mathbf{y}_{zN} - \Psi_{\overline{u}N}\overline{\mathbf{u}}_N$.

In the light of Theorem 2.4.2 and Lemma 2.4.3 in [26], J_N has a minimum over $\{x_a(0), \overline{w}_N\}$ if and only if $R_{\overline{y}_{zN}} = \langle \overline{\mathbf{y}}_{zN}, \overline{\mathbf{y}}_{zN} \rangle$ and $Q_{v_{z,aN}} = \langle \mathbf{v}_{z,aN}, \mathbf{v}_{z,aN} \rangle$ have the same inertia. Moreover, the minimum of J_N is given by

$$\min J_N = \overline{y}_{zN}^T R_{\overline{y}_{zN}}^{-1} \overline{y}_{zN}. \tag{3.32}$$

On the other hand, applying the Krein space projection formula, we have

$$\overline{\mathbf{y}}_{zN} = \Theta_N \tilde{\mathbf{y}}_{zN}, \tag{3.33}$$

where

$$\tilde{\mathbf{y}}_{zN} = \begin{bmatrix} \tilde{\mathbf{y}}_z^T(0) & \tilde{\mathbf{y}}_z^T(1) & \cdots & \tilde{\mathbf{y}}_z^T(N) \end{bmatrix}^T,$$

$$\Theta_N = \begin{bmatrix} \theta_{00} & \theta_{01} & \cdots & \theta_{0N} \\ \theta_{10} & \theta_{11} & \cdots & \theta_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N0} & \theta_{N1} & \cdots & \theta_{NN} \end{bmatrix},$$

$$\theta_{ij} = \begin{cases} \langle \overline{\mathbf{y}}_z(i), \widetilde{\mathbf{y}}_z(j) \rangle R^{-1}_{\widetilde{y}_z}(j), & i > j \geq 0, \\ \begin{bmatrix} I & 0 \\ m_1 & I \end{bmatrix}, & d > i = j \geq 0, \\ \begin{bmatrix} I & 0 & 0 \\ m_1 & I & 0 \\ m_2 & m_3 & I \end{bmatrix}, & i = j \geq d, \\ 0, & 0 \leq i < j, \end{cases}$$

$$m_1 = \left\langle \overline{\check{\mathbf{z}}}_m(i \mid i), \widetilde{\mathbf{y}}(j \mid j-1) \right\rangle R^{-1}_{\widetilde{y}}(j \mid j-1),$$

$$m_2 = \left\langle \overline{\check{\mathbf{z}}}_h(i_d \mid i), \widetilde{\mathbf{y}}(j \mid j-1) \right\rangle R^{-1}_{\widetilde{y}}(j \mid j-1),$$

$$m_3 = \left\langle \overline{\check{\mathbf{z}}}_h(i_d \mid i), \widetilde{\mathbf{z}}_m(j \mid j) \right\rangle R^{-1}_{\widetilde{z}_m}(j \mid j),$$

$$\overline{\mathbf{y}}_z(i) = \mathbf{y}_z(i) - \sum_{j=0}^{N} \varphi_{ij} \overline{\mathbf{u}}(j),$$

$$\overline{\check{\mathbf{z}}}_m(i \mid i) = \check{\mathbf{z}}_m(i \mid i) - \sum_{j=0}^{N} \varphi_{m,ij} \overline{\mathbf{u}}(j),$$

$$\overline{\check{\mathbf{z}}}_h(i_d \mid i) = \check{\mathbf{z}}_h(i_d \mid i) - \sum_{j=0}^{N} \varphi_{h,ij} \overline{\mathbf{u}}(j),$$

$$(3.34)$$

where $\varphi_{m,ij}$ is derived by replacing $C_{z,a}$ in φ_{ij} with $\left[\begin{smallmatrix} F & 0 & \cdots & 0 \\ L & 0 & \cdots & 0 \end{smallmatrix}\right]$, $\varphi_{h,ij}$ is derived by replacing $C_{z,a}$ in φ_{ij} with $[0 \ \ 0 \ \ \cdots \ \ H]$ Furthermore, it follows from (3.33) that

$$R_{\overline{y}_{zN}} = \Theta_N R_{\widetilde{y}_{zN}} \Theta_N^T, \qquad \overline{y}_{zN} = \Theta_N \widetilde{y}_{zN}, \tag{3.35}$$

where

$$R_{\widetilde{y}_{zN}} = \langle \widetilde{\mathbf{y}}_{zN}, \widetilde{\mathbf{y}}_{zN} \rangle,$$

$$\widetilde{y}_{zN} = \left[\widetilde{y}_z^T(0) \ \ \widetilde{y}_z^T(1) \ \ \cdots \ \ \widetilde{y}_z^T(N) \right]^T,$$

$$\widetilde{y}_z(i) = \begin{cases} \left[\widetilde{y}^T(i \mid i-1) \ \ \widetilde{z}_m^T(i \mid i) \right]^T, & 0 \leq i < d, \\ \left[\widetilde{y}^T(i \mid i-1) \ \ \widetilde{z}_m^T(i \mid i) \ \ \widetilde{z}_h^T(i_d \mid i) \right]^T, & i \geq d. \end{cases} \tag{3.36}$$

Since matrix Θ_N is nonsingular, it follows from (3.35) that $R_{\overline{y}_{zN}}$ and $R_{\tilde{y}_{zN}}$ are congruent, which also means that $R_{\overline{y}_{zN}}$ and $R_{\tilde{y}_{zN}}$ have the same inertia. Note that both $R_{\tilde{y}_{zN}}$ and $Q_{v_{z,aN}}$ are block-diagonal matrices, and

$$R_{\tilde{y}_z}(k) = \begin{cases} \operatorname{diag}\{R_{\tilde{y}}(k \mid k-1), R_{\tilde{z}_m}(k \mid k)\}, & 0 \le k < d, \\ \operatorname{diag}\{R_{\tilde{y}}(k \mid k-1), R_{\tilde{z}_m}(k \mid k), R_{\tilde{z}_h}(k_d \mid k)\}, & k \le d, \end{cases} \tag{3.37}$$

$Q_{v_{z,a}}(k)$ is given by (3.23). It follows that $R_{\tilde{y}_{zN}}$ and $Q_{v_{z,aN}}$ have the same inertia if and only if $R_{\tilde{y}}(k \mid k-1) > 0$ $(0 \le k \le N)$, $R_{\tilde{z}_m}(k \mid k) < 0$ $(0 \le k \le N)$ and $R_{\tilde{z}_h}(k_d \mid k) < 0$ $(d \le k \le N)$.

Therefore, J_N subject to system (2.1) with Lipschitz conditions (2.2) has the minimum if and only if $R_{\tilde{y}}(k \mid k-1) > 0$ $(0 \le k \le N)$, $R_{\tilde{z}_m}(k \mid k) < 0$ $(0 \le k \le N)$ and $R_{\tilde{z}_h}(k_d \mid k) < 0$ $(d \le k \le N)$. Moreover, the minimum value of J_N can be rewritten as

$$\begin{aligned} \min J_N &= \overline{y}_{zN}^T R_{\overline{y}_{zN}}^{-1} \overline{y}_{zN} = \tilde{y}_{zN}^T R_{\tilde{y}_{zN}}^{-1} \tilde{y}_{zN} \\ &= \sum_{k=0}^{N} \tilde{y}^T(k \mid k-1) R_{\tilde{y}}^{-1}(k \mid k-1) \tilde{y}(k \mid k-1) + \sum_{k=0}^{N} \tilde{z}_m^T(k \mid k) R_{\tilde{z}_m}^{-1}(k \mid k) \tilde{z}_m(k \mid k) \\ &\quad + \sum_{k=d}^{N} \tilde{z}_h^T(k_d \mid k) R_{\tilde{z}_h}^{-1}(k_d \mid k) \tilde{z}_h(k_d \mid k). \end{aligned} \tag{3.38}$$

The proof is completed. □

Remark 3.4. Due to the built innovation sequence $\{\{\tilde{\mathbf{y}}_z(i)\}_{i=0}^k\}$ in Lemma 3.2, the form of the minimum on indefinite quadratic form J_N is different from the one given in [26–28]. It is shown from (3.15) that the estimation errors $\tilde{y}(k \mid k-1)$, $\tilde{z}_m(k \mid k)$ and $\tilde{z}_h(k_d \mid k)$ are mutually uncorrelated, which will make the design of H_∞ estimator much easier than the one given in [26–28].

3.2. Solution of the H_∞ Estimation Problem

In this subsection, the Kalman-like recursive H_∞ estimator is presented by using orthogonal projection in the Krein space.

Denote

$$\begin{aligned} \mathbf{y}_0(i) &= \mathbf{y}(i), \\ \mathbf{y}_1(i) &= \left[\mathbf{y}^T(i) \;\; \check{\mathbf{z}}_m^T(i \mid i)\right]^T, \\ \mathbf{y}_2(i) &= \left[\mathbf{y}^T(i) \;\; \check{\mathbf{z}}_m^T(i \mid i) \;\; \check{\mathbf{z}}_h^T(i \mid i+d)\right]^T. \end{aligned} \tag{3.39}$$

Observe from (3.8), we have

$$\begin{aligned} \mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{j}\right\} &= \mathcal{L}\left\{\{\mathbf{y}_1(i)\}_{i=0}^{j}\right\}, \quad 0 \le j < d, \\ \mathcal{L}\left\{\{\mathbf{y}_z(i)\}_{i=0}^{j}\right\} &= \mathcal{L}\left\{\{\mathbf{y}_2(i)\}_{i=0}^{j_d}\right\}; \left\{\{\mathbf{y}_1(i)\}_{i=j_d+1}^{j}\right\}, \quad j \ge d. \end{aligned} \tag{3.40}$$

Definition 3.5. Given $k \ge d$, the estimator $\hat{\xi}(i \mid j, 2)$ for $0 \le j < k_d$ denotes the optimal estimate of $\xi(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_2(s)\}_{s=0}^{j}\}$, and the estimator $\hat{\xi}(i \mid j, 1)$ for $k_d \le j \le k$ denotes the optimal estimate of $\xi(i)$ given the observation $\mathcal{L}\{\{\mathbf{y}_2(s)\}_{s=0}^{k_d-1}; \{\mathbf{y}_1(\tau)\}_{\tau=k_d}^{j}\}$. For simplicity, we use $\hat{\xi}(i,2)$ to denote $\hat{\xi}(i \mid i-1, 2)$, and use $\hat{\xi}(i,1)$ to denote $\hat{\xi}(i \mid i-1,1)$ throughout the paper.

Based on the above definition, we introduce the following stochastic sequence and the corresponding covariance matrices

$$\begin{aligned} \tilde{\mathbf{y}}_2(i,2) &= \mathbf{y}_2(i) - \hat{\mathbf{y}}_2(i,2), \qquad R_{\tilde{y}_2}(i,2) = \langle \tilde{\mathbf{y}}_2(i,2), \tilde{\mathbf{y}}_2(i,2) \rangle, \\ \tilde{\mathbf{y}}_1(i,1) &= \mathbf{y}_1(i) - \hat{\mathbf{y}}_1(i,1), \qquad R_{\tilde{y}_1}(i,1) = \langle \tilde{\mathbf{y}}_1(i,1), \tilde{\mathbf{y}}_1(i,1) \rangle, \\ \tilde{\mathbf{y}}_0(i,0) &= \mathbf{y}_0(i) - \hat{\mathbf{y}}_0(i,1), \qquad R_{\tilde{y}_0}(i,0) = \langle \tilde{\mathbf{y}}_0(\mathrm{i},0), \tilde{\mathbf{y}}_0(i,0) \rangle. \end{aligned} \tag{3.41}$$

Similar to the proof of Lemma 2.2.1 in [27], we can obtain that $\{\tilde{\mathbf{y}}_2(0,2), \ldots, \tilde{\mathbf{y}}_2(k_d - 1,2); \tilde{\mathbf{y}}_1(k_d,1), \ldots, \tilde{\mathbf{y}}_1(k-1,1)\}$ is the innovation sequence which is a mutually uncorrelated white noise sequence and spans the same linear space as $\mathcal{L}\{\mathbf{y}_2(0), \ldots, \mathbf{y}_2(k_d - 1); \mathbf{y}_1(k_l), \ldots, \mathbf{y}_1(k-1)\}$ or equivalently $\mathcal{L}\{\mathbf{y}_z(0), \ldots, \mathbf{y}_z(k-1)\}$.

Applying projection formula in the Krein space, $\hat{\mathbf{x}}(i,2)$ $(i = 0,1,\ldots,k_d)$ is computed recursively as

$$\tag{3.42}$$

$$\begin{aligned} \hat{\mathbf{x}}(i+1,2) &= \sum_{j=0}^{i} \langle \mathbf{x}(i+1), \tilde{\mathbf{y}}_2(j,2) \rangle R_{\tilde{y}_2}^{-1}(j,2) \tilde{\mathbf{y}}_2(j,2) \\ &= A\hat{\mathbf{x}}(i \mid i,2) + A_d \hat{\mathbf{x}}(i_d \mid i,2) + f\left(i, \check{\mathbf{z}}_f(i \mid i), \mathbf{u}(i)\right) \\ &\quad + h(i, \check{\mathbf{z}}_h(i_d \mid i), \mathbf{u}(i)), \quad i = 0,1,\ldots,k_d - 1, \\ \hat{\mathbf{x}}(\tau,2) &= 0, \quad (\tau = -d, -d+1, \ldots, 0). \end{aligned} \tag{3.43}$$

Note that

$$\begin{aligned} \hat{\mathbf{x}}(i \mid i,2) &= \hat{\mathbf{x}}(i,2) + P_2(i,i) C_2^T R_{\tilde{y}_2}^{-1}(i,2) \tilde{\mathbf{y}}_2(i,2), \\ \hat{\mathbf{x}}(i_d \mid i,2) &= \hat{\mathbf{x}}(i_d,2) + \sum_{j=i_d}^{i} P_2(i_d,j) C_2^T R_{\tilde{y}_2}^{-1}(j,2) \tilde{\mathbf{y}}_2(j,2), \end{aligned} \tag{3.44}$$

where

$$\begin{aligned}
C_2 &= \begin{bmatrix} C^T & F^T & L^T & H^T \end{bmatrix}^T, \\
P_2(i,j) &= \langle \mathbf{e}(i,2), \mathbf{e}(j,2) \rangle, \\
\mathbf{e}(i,2) &= \mathbf{x}(i) - \widehat{\mathbf{x}}(i,2), \\
R_{\widetilde{y}_2}(i,2) &= C_2 P_2(i,i) C_2^T + Q_{v_2}(i), \\
Q_{v_2}(i) &= \operatorname{diag}\left\{ I, -\alpha^{-2} I, -\gamma^2 I, -\beta^{-2} I \right\}.
\end{aligned} \tag{3.45}$$

Substituting (3.44) into (3.43), we have

$$\begin{aligned}
\widehat{\mathbf{x}}(i+1,2) &= A\widehat{\mathbf{x}}(i,2) + A_d \widehat{\mathbf{x}}(i_d,2) + f\left(i, \breve{\mathbf{z}}_f(i \mid i), \mathbf{u}(i)\right) + h(i, \breve{\mathbf{z}}_h(i_d \mid i), \mathbf{u}(i)) \\
&\quad + A_d \sum_{j=i_d}^{i-1} P_2(i_d, j) C_2^T R_{\widetilde{y}_2}^{-1}(j,2) \widetilde{\mathbf{y}}_2(j,2) + K_2(i) \widetilde{\mathbf{y}}_2(i,2), \\
K_2(i) &= A_d P_2(i_d, i) C_2^T R_{\widetilde{y}_2}^{-1}(i,2) + A P_2(i,i) C_2^T R_{\widetilde{y}_2}^{-1}(i,2).
\end{aligned} \tag{3.46}$$

Moreover, taking into account (3.7) and (3.46), we obtain

$$\begin{aligned}
\mathbf{e}(i+1,2) &= A\mathbf{e}(i,2) + A_d \mathbf{e}(i_d,2) + \overline{B}\overline{\mathbf{w}}(i) - K_2(i)\widetilde{\mathbf{y}}_2(i,2) \\
&\quad - A_d \sum_{j=i_d}^{i-1} P_2(i_d, j) C_2^T R_{\widetilde{y}_2}^{-1}(j,2) \widetilde{\mathbf{y}}_2(j,2), \quad i = 0, 1, \ldots, k_d - 1.
\end{aligned} \tag{3.47}$$

Consequently,

$$\begin{aligned}
P_2(i-j, i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,2) \rangle \\
&= P_2(i-j,i) A^T + P_2^T(i_d, i-j) A_d^T - P_2(i-j,i) C_2^T K_2^T(i) \\
&\quad - \sum_{t=i-j}^{i-1} P_2(i-j,t) C_2^T R_{\widetilde{y}_2}^{-1}(t,2) C_2 P_2^T(i_d,t) A_d^T, \quad j = 0, 1, \ldots, d, \\
P_2(i+1, i+1) &= \langle \mathbf{e}(i+1,2), \mathbf{e}(i+1,2) \rangle \\
&= A P_2(i, i+1) + A_d P_2(i_d, i+1) + \overline{B} Q_{\overline{w}}(i) \overline{B}^T,
\end{aligned} \tag{3.48}$$

where $Q_{\overline{w}}(i) = I$. Thus, $P_2(i,i)$ $(i = 0,1,\ldots,k_d)$ can be computed recursively as

$$\begin{aligned}
P_2(i-j,i+1) &= P_2(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_2(i-j,i)C_2^T K_2^T(i) \\
&\quad - \sum_{t=i-j}^{i-1} P_2(i-j,t)C_2^T R_{\widetilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T, \\
P_2(i+1,i+1) &= AP_2(i,i+1) + A_d P_2(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T, \quad j = 0,1,\ldots,d.
\end{aligned} \tag{3.49}$$

Similarly, employing the projection formula in the Krein space, the optimal estimator $\widehat{\mathbf{x}}(i,1)$ $(i = k_d+1,\ldots,k)$ can be computed by

$$\begin{aligned}
\widehat{\mathbf{x}}(i+1,1) &= A\widehat{\mathbf{x}}(i,1) + A_d\widehat{\mathbf{x}}(i_d,2) + f\left(i,\check{\mathbf{z}}_f(i \mid i),\mathbf{u}(i)\right) + h(i,\check{\mathbf{z}}_h(i_d \mid i),\mathbf{u}(i)) \\
&\quad + K_1(i)\widetilde{\mathbf{y}}_1(i,1) + A_d\sum_{j=i_d}^{k_d-1} P_2(i_d,j)C_2^T R_{\widetilde{y}_2}^{-1}(j,2)\widetilde{\mathbf{y}}_2(j,2) \\
&\quad + A_d\sum_{j=k_d}^{i-1} P_1(i_d,j)C_1^T R_{\widetilde{y}_1}^{-1}(j,1)\widetilde{\mathbf{y}}_1(j,1), \\
\widehat{\mathbf{x}}(k_d,1) &= \widehat{\mathbf{x}}(k_d,2),
\end{aligned} \tag{3.50}$$

where

$$\begin{gathered}
C_1 = \left[C^T \quad F^T \quad L^T\right]^T, \\
P_1(i,j) = \begin{cases} \langle \mathbf{e}(i,2),\mathbf{e}(j,1)\rangle, & i < k_d, \\ \langle \mathbf{e}(i,1),\mathbf{e}(j,1)\rangle, & i \geq k_d, \end{cases} \\
\mathbf{e}(i,1) = \mathbf{x}(i) - \widehat{\mathbf{x}}(i,1), \\
R_{\widetilde{y}_1}(i,1) = C_1 P_1(i,i)C_1^T + Q_{v_1}(i), \\
Q_{v_1}(i) = \operatorname{diag}\left\{I, -\alpha^{-2}I, -\gamma^2 I\right\}, \\
K_1(i) = AP_1(i,i)C_1^T R_{\widetilde{y}_1}^{-1}(i,1) + A_d P_1(i_d,i)C_1^T R_{\widetilde{y}_1}^{-1}(i,1).
\end{gathered} \tag{3.51}$$

Then, from (3.7) and (3.50), we can yield

$$\begin{aligned}
\mathbf{e}(i+1,1) &= A\mathbf{e}(i,1) + A_d\mathbf{e}(i_d,2) + \overline{B}\overline{\mathbf{w}}(i) - K_1(i)\widetilde{\mathbf{y}}_1(i,1) \\
&\quad - A_d\sum_{j=i_d}^{k_d-1} P_2(i_d,j)C_2^T R_{\widetilde{y}_2}^{-1}(j,2)\widetilde{\mathbf{y}}_2(j,2) \\
&\quad - A_d\sum_{j=k_d}^{i-1} P_1(i_d,j)C_1^T R_{\widetilde{y}_1}^{-1}(j,1)\widetilde{\mathbf{y}}_1(j,1).
\end{aligned} \tag{3.52}$$

Thus, we obtain that

(1) if $i-j \geq k_d$, we have

$$\begin{aligned} P_1(i-j,i+1) &= \langle \mathbf{e}(i-j,1), \mathbf{e}(i+1,1)\rangle \\ &= P_1(i-j,i)A^T + P_1^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \end{aligned} \tag{3.53}$$

(2) if $i-j < k_d$, we have

$$\begin{aligned} P_1(i-j,i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,1)\rangle \\ &= P_1(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{k_d-1} P_2(i-j,t)C_2^T R_{\tilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T \\ &\quad - \sum_{t=k_d}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \end{aligned} \tag{3.54}$$

$$\begin{aligned} P_1(i+1,i+1) &= \langle \mathbf{e}(i-j,2), \mathbf{e}(i+1,1)\rangle \\ &= AP_1(i,i+1) + A_d P_1(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T. \end{aligned} \tag{3.55}$$

It follows from (3.53), (3.54), and (3.55) that $P_1(i,i)$ $(i = k_d+1,\ldots,k)$ can be computed by

$$\begin{aligned} P_1(i-j,i+1) &= P_1(i-j,i)A^T + P_2^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{k_d-1} P_2(i-j,t)C_2^T R_{\tilde{y}_2}^{-1}(t,2)C_2 P_2^T(i_d,t)A_d^T \\ &\quad - \sum_{t=k_d}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \quad i-j < k_d, \\ P_1(i-j,i+1) &= P_1(i-j,i)A^T + P_1^T(i_d,i-j)A_d^T - P_1(i-j,i)C_1^T K_1^T(i) \\ &\quad - \sum_{t=i-j}^{i-1} P_1(i-j,t)C_1^T R_{\tilde{y}_1}^{-1}(t,1)C_1 P_1^T(i_d,t)A_d^T, \quad i-j \geq k_d, \\ P_1(i+1,i+1) &= AP_1(i,i+1) + A_d P_1(i_d,i+1) + \overline{B}Q_{\overline{w}}(i)\overline{B}^T, \quad j = 0,1,\ldots,d. \end{aligned} \tag{3.56}$$

Next, according to the above analysis, $\hat{\mathbf{z}}_m(k \mid k)$ as the Krein space projections of $\check{\mathbf{z}}_m(k \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}; \mathbf{y}_0(k)\}$ can be computed by the following formula

$$\hat{\mathbf{z}}_m(k \mid k) = C_m\hat{\mathbf{x}}(k,1) + C_m P_1(k,k)C^T R_{\tilde{y}_0}^{-1}(k,0)\tilde{\mathbf{y}}_0(k,0), \tag{3.57}$$

where

$$\begin{aligned} C_m &= \left[F^T \quad L^T\right]^T, \\ R_{\tilde{y}_0}(k,0) &= CP_1(k,k)C^T + Q_v(k). \end{aligned} \tag{3.58}$$

And, $\hat{\mathbf{z}}_h(k_d \mid k)$ as the Krein space projections of $\check{\mathbf{z}}_h(k_d \mid k)$ onto $\mathcal{L}\{\{\mathbf{y}_z(j)\}_{j=0}^{k-1}; \mathbf{y}_1(k)\}$ can be computed by the following formula

$$\hat{\mathbf{z}}(k_d \mid k) = H\hat{\mathbf{x}}(k_d,1) + \sum_{j=k_d}^{k} HP_1(k_d,j)C_1^T R_{\tilde{y}_1}^{-1}(j,1)\tilde{\mathbf{y}}_1(j,1). \tag{3.59}$$

Based on Theorem 3.3 and the above discussion, we propose the following results.

Theorem 3.6. *Consider system* (2.1) *with Lipschitz conditions* (2.2)*, given a scalar* $\gamma > 0$ *and matrix* $\Pi(k)$ $(k = -d, \ldots, 0)$*, then the* H_∞ *estimator that achieves* (2.3) *if*

$$\begin{aligned} R_{\tilde{y}}(k \mid k-1) &> 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_m}(k \mid k) &< 0, \quad 0 \le k \le N, \\ R_{\tilde{z}_h}(k_d \mid k) &< 0, \quad d \le k \le N, \end{aligned} \tag{3.60}$$

where

$$\begin{aligned} R_{\tilde{y}}(k \mid k-1) &= R_{\tilde{y}_0}(k,0), \\ R_{\tilde{z}_m}(k \mid k) &= C_m P_1(k,k)C_m^T - C_m P_1(k,k)C^T R_{\tilde{y}_0}^{-1}(k,0)CP_1(k,k)C_m^T + Q_{v_m}(k), \\ R_{\tilde{z}_h}(k_d \mid k) &= HP_1(k_d,k_d)H^T - \sum_{j=k_d}^{k} HP_1(k_d,j)C_1^T R_{\tilde{y}_1}^{-1}(j,1)C_1 P_1^T(k_d,j)H^T - \beta^{-2}I, \\ Q_{v_m}(k) &= \operatorname{diag}\left\{-\alpha^{-2}I, -\gamma^2 I\right\}, \end{aligned} \tag{3.61}$$

$R_{\tilde{y}_0}(k,0)$*,* $P_1(i,j)$*, and* $R_{\tilde{y}_1}(j,1)$ *are calculated by* (3.58)*,* (3.56)*, and* (3.51)*, respectively.*

Moreover, one possible level-γ H_∞ *estimator is given by*

$$\check{z}(k \mid k) = E\hat{z}_m(k \mid k), \tag{3.62}$$

where $E = [0 \quad I]$*, and* $\hat{z}_m(k \mid k)$ *is computed by* (3.57).

Proof. In view of Definitions 3.1 and 3.5, it follows from (3.9) and (3.41) that $R_{\tilde{y}}(k \mid k-1) = R_{\tilde{y}_0}(k, 0)$. In addition, according to (3.7), (3.9), and (3.57), the covariance matrix $R_{\tilde{z}_m}(k \mid k)$ can be given by the second equality in (3.61). Similarly, based on (3.7), (3.9), and (3.59), the covariance matrix $R_{\tilde{z}_h}(k_d \mid k)$ can be obtained by the third equality in (3.61). Thus, from Theorem 3.3, it follows that J_N has a minimum if (3.60) holds.

On the other hand, note that the minimum value of J_N is given by (3.15) in Theorem 3.3 and any choice of estimator satisfying min $J_N > 0$ is an acceptable one. Therefore, Taking into account (3.60), one possible estimator can be obtained by setting $\check{z}_m(k \mid k) = \hat{z}_m(k \mid k)$ and $\check{z}_h(k_d \mid k) = \hat{z}_h(k_d \mid k)$. This completes the proof. □

Remark 3.7. It is shown from (3.57) and (3.59) that $\hat{\mathbf{z}}_m(k \mid k)$ and $\hat{\mathbf{z}}(k_d \mid k)$ are, respectively, the filtering estimate and fixed-lag smoothing of $\check{\mathbf{z}}_m(k \mid k)$ and $\check{\mathbf{z}}(k_d \mid k)$ in the Krein space. Additionally, it follows from Theorem 3.6 that $\check{z}_m(k \mid k)$ and $\check{z}_h(k_d \mid k)$ achieving the H_∞ estimation problem (2.3) can be, respectively, computed by the right side of (3.57) and (3.59). Thus, it can be concluded that the proposed results in this paper are related with both the H_2 filtering and H_2 fixed-lag smoothing in the Krein space.

Remark 3.8. Recently, the robust H_∞ observers for Lipschitz nonlinear delay-free systems with Lipschitz nonlinear additive uncertainties and time-varying parametric uncertainties have been studied in [10, 11], where the optimization of the admissible Lipschitz constant and the disturbance attenuation level are discussed simultaneously by using the multiobjective optimization technique. In addition, the sliding mode observers with H_∞ performance have been designed for Lipschitz nonlinear delay-free systems with faults (matched uncertainties) and disturbances in [8]. Although the Krein space-based robust H_∞ filter has been proposed for discrete-time uncertain linear systems in [28], it cannot be applied to solving the H_∞ estimation problem given in [10] since the considered system contains Lipschitz nonlinearity and Lipschitz nonlinear additive uncertainty. However, it is meaningful and promising in the future, by combining the algorithm given in [28] with our proposed method in this paper, to construct a Krein space-based robust H_∞ filter for discrete-time Lipschitz nonlinear systems with nonlinear additive uncertainties and time-varying parametric uncertainties.

4. A Numerical Example

Consider the system (2.1) with time delay $d = 3$ and the parameters

$$\begin{gathered} A = \begin{bmatrix} 0.7 & 0 \\ 0 & -0.4 \end{bmatrix}, \qquad A_d = \begin{bmatrix} -0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, \qquad F = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.01 \end{bmatrix}, \\ H = \begin{bmatrix} 0.03 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad B = \begin{bmatrix} 1.2 \\ 0.7 \end{bmatrix}, \qquad C = \begin{bmatrix} 1.7 & 0.9 \end{bmatrix}, \qquad L = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, \\ f(k, Fx(k), u(k)) = \sin(Fx(k)), \qquad h(k, Hx(k_d), u(k)) = \cos(Hx(k_d)). \end{gathered} \tag{4.1}$$

Then we have $\alpha = \beta = 1$. Set $x(k) = [-0.2k \;\; 0.1k]^T$ $(k = -3, -2, -1, 0)$, and $\Pi(k) = I$ $(k = -3, -2, -1, 0)$. Both the system noise $w(k)$ and the measurement noise $v(k)$ are supposed to be band-limited white noise with power 0.01. By applying Theorem 3.1 in [20], we obtain the minimum disturbance attenuation level $\gamma_{\min} = 1.6164$ and the observer

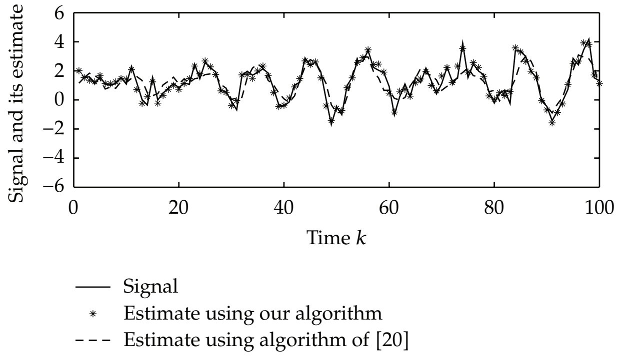

Figure 1: Signal $z(k)$ (solid), its estimate using our algorithm (star), and its estimate using algorithm in [20] (dashed).

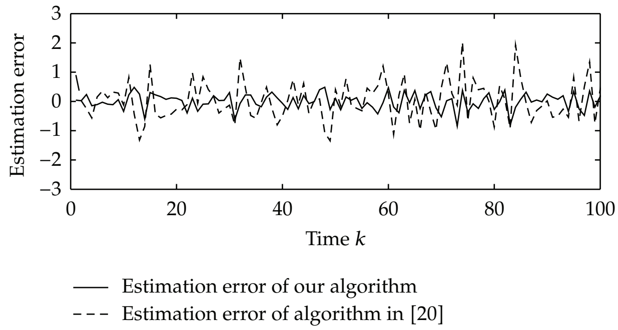

Figure 2: Estimation error of our algorithm (solid) and estimation error of algorithm in [20] (dashed).

parameter $L = [-0.3243 \;\; 0.0945]^T$ of (5) in [20]. In this numerical example, we compare our algorithm with the one given in [20] in case of $\gamma = 1.6164$. Figure 1 shows the true value of signal $z(k)$, the estimate using our algorithm, and the estimate using the algorithm given in [20]. Figure 2 shows the estimation error of our approach and the estimation error of the approach in [20]. It is shown in Figures 1 and 2 that the proposed algorithm is better than the one given in [20]. Figure 3 shows the ratios between the energy of the estimation error and input noises for the proposed H_∞ estimation algorithm. It is shown that the maximum energy ratio from the input noises to the estimation error is less than γ^2 by using our approach. Figure 4 shows the value of indefinite quadratic form J_N for the given estimation algorithm. It is shown that the value of indefinite quadratic form J_N is positive by employing the proposed algorithm in Theorem 3.6.

5. Conclusions

A recursive H_∞ filtering estimate algorithm for discrete-time Lipschitz nonlinear systems with time-delay and disturbance input is proposed. By combining the H_∞-norm estimation condition with the Lipschitz conditions on nonlinearity, the H_∞ estimation problem is converted to the positive minimum problem of indefinite quadratic form. Motivated by the observation that the minimum problem of indefinite quadratic form coincides with Kalman filtering in the Krein space, a novel Krein space-based H_∞ filtering estimate algorithm is

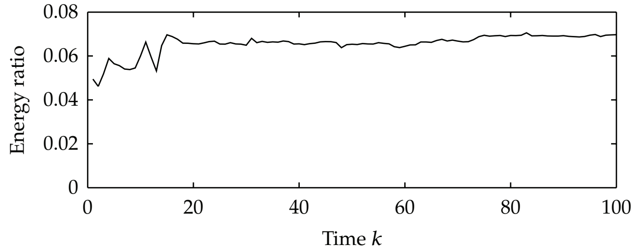

Figure 3: The energy ratio between estimation error and all input noises for the proposed H_∞ estimation algorithm.

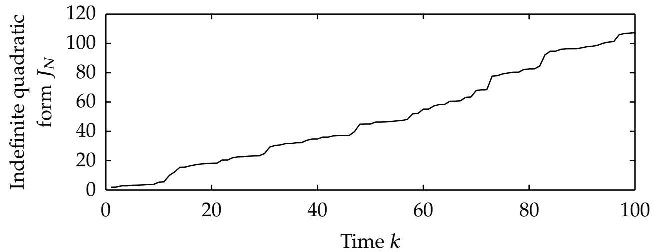

Figure 4: The value of indefinite quadratic form J_N for the given estimation algorithm.

developed. Employing projection formula and innovation analysis technology in the Krein space, the H_∞ estimator and its sufficient existence condition are presented based on Riccati-like difference equations. A numerical example is provided in order to demonstrate the performances of the proposed approach.

Future research work will extend the proposed method to investigate more general nonlinear system models with nonlinearity in observation equations. Another interesting research topic is the H_∞ multistep prediction and fixed-lag smoothing problem for time-delay Lipschitz nonlinear systems.

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Research Article

Instable Trivial Solution of Autonomous Differential Systems with Quadratic Right-Hand Sides in a Cone

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The present investigation deals with global instability of a general n-dimensional system of ordinary differential equations with quadratic right-hand sides. The global instability of the zero solution in a given cone is proved by Chetaev's method, assuming that the matrix of linear terms has a simple positive eigenvalue and the remaining eigenvalues have negative real parts. The sufficient conditions for global instability obtained are formulated by inequalities involving norms and eigenvalues of auxiliary matrices. In the proof, a result is used on the positivity of a general third-degree polynomial in two variables to estimate the sign of the full derivative of an appropriate function in a cone.

1. Introduction

Recently, there has been a rapidly growing interest in investigating the instability conditions of differential systems. The number of papers dealing with instability problems is rather low compared with the huge quantity of papers in which the stability of the motion of differential systems is investigated. The first results on the instability of zero solution of differential systems were obtained in a general form by Lyapunov [1] and Chetaev [2].

Further investigation on the instability of solutions of systems was carried out to weaken the conditions of the Lyapunov and Chetaev theorems for special-form systems. Some results are presented, for example, in [3–10], but instability problems are analysed only locally. For example, in [7], a linear system of ordinary differential equations in the matrix form is considered, and conditions such that the corresponding forms (of the second and the third power) have fixed sign in some cone of the space $\mathbb{R}^n$ are derived. To investigate this property another problem inverse to the known Lyapunov problem for the construction of Lyapunov functions is solved.

In the present paper, instability solutions of systems with quadratic right-hand sides is investigated in a cone dealing with a general n-dimensional system with quadratic right-hand sides. We assume that the matrix of linear terms has a simple positive eigenvalue and the remaining eigenvalues have negative real parts.

Unlike the previous investigations, we prove the global instability of the zero solution in a given cone and the conditions for global instability are formulated by inequalities involving norms and eigenvalues of auxiliary matrices. The main tool is the method of Chetaev and application of a suitable Chetaev-type function. A novelty in the proof of the main result (Theorem 3.1) is the utilization of a general third-order polynomial inequality of two variables to estimate the sign of the full derivative of an appropriate function along the trajectories of a given system in a cone.

In the sequel, the norms used for vectors and matrices are defined as

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}, \tag{1.1}$$

for a vector $x = (x_1, \ldots, x_n)^T$ and

$$\|\mathcal{F}\| = \left(\lambda_{\max}\left(\mathcal{F}^T\mathcal{F}\right)\right)^{1/2}, \tag{1.2}$$

for any $m \times n$ matrix $\mathcal{F}$. Here and throughout the paper, $\lambda_{\max}(\cdot)$ (or $\lambda_{\min}(\cdot)$) is the maximal (or minimal) eigenvalue of the corresponding symmetric and positive-semidefinite matrix $\mathcal{F}^T\mathcal{F}$ (see, e.g., [11]).

In this paper, we consider the instability of the trivial solution of a nonlinear autonomous differential system with quadratic right-hand sides

$$\dot{x}_i = \sum_{s=1}^{n} a_{is}x_s + \sum_{s,q=1}^{n} b_{sq}^i x_s x_q, \quad i = 1, \ldots, n, \tag{1.3}$$

where coefficients a_{is} and b_{sq}^i are constants. Without loss of generality, throughout this paper we assume

$$b_{sq}^i = b_{qs}^i. \tag{1.4}$$

As emphasized, for example, in [2, 10–12], system (1.3) can be written in a general vector-matrix form

$$\dot{x} = Ax + X^T Bx, \tag{1.5}$$

where A is an $n \times n$ constant square matrix, matrix X^T is an $n \times n^2$ rectangular matrix

$$X^T = \left\{X_1^T, X_2^T, \ldots, X_n^T\right\}, \tag{1.6}$$

where the entries of the $n \times n$ square matrices X_i, $i = 1, \ldots, n$ are equal to zero except the ith row with entries $x^T = (x_1, x_2, \ldots, x_n)$, that is,

$$X_i^T = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \tag{1.7}$$

and B is a rectangular $n^2 \times n$ matrix such that

$$B^T = \{B_1, B_2, \ldots, B_n\}, \tag{1.8}$$

where matrices $B_i = \{b_{sq}^i\}$, $i, s, q = 1, \ldots, n$, that is, matrices

$$B_i = \begin{pmatrix} b_{11}^i & b_{12}^i & \cdots & b_{1n}^i \\ b_{21}^i & b_{22}^i & \cdots & b_{2n}^i \\ \cdots & \cdots & \cdots & \cdots \\ b_{n1}^i & b_{n2}^i & \cdots & b_{nn}^i \end{pmatrix} \tag{1.9}$$

are $n \times n$ constant and symmetric. Representation (1.5) permits an investigation of differential systems with quadratic right-hand sides by methods of matrix analysis. Such approach was previously used, for example, in [13].

If matrix A admits one simple positive eigenvalue, the system (1.5) can be transformed, using a suitable linear transformation of the dependent variables, to the same form (1.5) but with the matrix A having the form

$$A = \begin{pmatrix} A_0 & \theta \\ \theta^T & \lambda \end{pmatrix}, \tag{1.10}$$

where A_0 is an $(n-1) \times (n-1)$ constant matrix, $\theta = (0, 0, \ldots, 0)^T$ is the $(n-1)$-dimensional zero vector and $\lambda > 0$. With regard to this fact, we do not introduce new notations for the coefficients b_{sq}^i, $i, s, q = 1, 2, \ldots, n$ in (1.5), assuming throughout the paper that A in (1.5) has the form (1.10), preserving the old notations a_{ij} for entries of matrix A_0. This means that we assume that $A = \{a_{is}\}$, $i, s = 1, 2, \ldots, n$ with $a_{ns} = a_{sn} = 0$ for $s = 1, 2, \ldots, n-1$ and $a_{nn} = \lambda$, and $A_0 = \{a_{is}\}$, $i, s = 1, 2, \ldots, n-1$.

We will give criteria of the instability of a trivial solution of the system (1.5) if the matrix A of linear terms is defined by (1.10).

2. Preliminaries

In this part we collect the necessary material-the definition of a cone, auxiliary Chetaev-type results on instability in a cone and, finally, a third degree polynomial inequality, which will be used to estimate the sign of the full derivative of a Chetaev-type function along the trajectories of system (1.5).

2.1. Instability of the Zero Solution of Systems of Differential Equations in a Cone

We consider an autonomous system of differential equations

$$\dot{x} = f(x), \tag{2.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies a local Lipschitz condition and $f(0) = 0$, that is, (2.1) admits the trivial solution. We will consider solutions of (2.1) determined by points $(x, t) = (x_0, 0)$ where $x_0 \in \mathbb{R}^n$. The symbol $x(x_0, t)$ denotes the solution $x = x(t)$ of (2.1), satisfying initial condition $x(0) = x_0$.

Definition 2.1. The zero solution $x \equiv 0$ of (2.1) is called unstable if there exists $\varepsilon > 0$ such that, for arbitrary $\delta > 0$, there exists an $x_0 \in \mathbb{R}^n$ with $\|x_0\| < \delta$ and $T \geq 0$ such that $\|x(x_0, T)\| \geq \varepsilon$.

Definition 2.2. A set $K \subset R^n$ is called a cone if $\alpha x \in K$ for arbitrary $x \in K$ and $\alpha > 0$.

Definition 2.3. A cone K is said to be a global cone of instability for (2.1) if $x(x_0, t) \in K$ for arbitrary $x_0 \in K$ and $t \geq 0$ and $\lim_{t\to\infty}\|x(x_0, t)\| = \infty$.

Definition 2.4. The zero solution $x \equiv 0$ of (2.1) is said to be globally unstable in a cone K if K is a global cone of instability for (2.1).

Now, we prove results analogous to the classical Chetaev theorem (see, e.g., [2]) on instability in a form suitable for our analysis. As usual, if $\mathcal{S}$ is a set, then $\partial\mathcal{S}$ denotes its boundary and $\overline{\mathcal{S}}$ its closure, that is, $\overline{\mathcal{S}} := \mathcal{S} \cup \partial\mathcal{S}$.

Theorem 2.5. *Let $V : \mathbb{R}^n \to \mathbb{R}$, $V(0, \ldots, 0) = 0$ be a continuously differentiable function. Assume that the set*

$$K = \{x \in R^n : V(x) > 0\} \tag{2.2}$$

is a cone. If the full derivative of V along the trajectories of (2.1) *is positive for every $x \in K$, that is, if*

$$\dot{V}(x) := grad^T V(x) f(x) > 0, \quad x \in K, \tag{2.3}$$

then K is a global cone of instability for the system (2.1).

Proof. Let ε be a positive number. We define a neighborhood of the origin

$$U_\varepsilon := \{x \in R^n : \|x\| < \varepsilon\}, \tag{2.4}$$

and a constant

$$M_\varepsilon := \max_{x \in \overline{U}_\varepsilon \cap \overline{K}} V(x). \tag{2.5}$$

Moreover, define a set

$$W_\delta := \left\{x \in \overline{U}_\varepsilon \cap \overline{K}, V(x) \geq \delta\right\}, \tag{2.6}$$

where δ is a positive number such that $\delta < M_\varepsilon$. Then, $W_\delta \neq \emptyset$.

Let $x_0 \in W_\delta \cap K$, then $V(x_0) = \delta_1 \in [\delta, M_\varepsilon]$. We show that there exists a $t = t_T = t_T(\varepsilon, x_0)$ such that $x(x_0, t_T) \notin \overline{U}_\varepsilon$ and $x(x_0, t_T) \in K$.

Suppose to the contrary that this is not true and $x(x_0, t) \in \overline{U}_\varepsilon$ for all $t \geq 0$. Since $\dot{V}(x) > 0$, the function V is increasing along the solutions of (2.1). Thus $x(x_0, t)$ remains in K. Due to the compactness of W_δ, there exists a positive value β such that for $x(x_0, t) \in W_\delta$

$$\frac{d}{dt} V(x(x_0, t)) = \operatorname{grad}^T V(x(x_0, t)) f(x(x_0, t)) > \beta. \tag{2.7}$$

Integrating this inequality over the interval $[0, t]$, we get

$$V(x(x_0, t)) - V(x_0) = V(x(x_0, t)) - \delta_1 > \beta t. \tag{2.8}$$

Then there exists a $t = t_T = t_T(\varepsilon, x_0)$ satisfying

$$t_T > \frac{(M_\varepsilon - \delta_1)}{\beta}, \tag{2.9}$$

such that $V(x(x_0, t_T)) > M_\varepsilon$ and, consequently, $x(x_0, t_T) \notin \overline{U}_\varepsilon$. This is contrary to our supposition. Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t \to \infty} \|x(x_0, t)\| = \infty, \tag{2.10}$$

that is, the zero solution is globally unstable, and K is a global cone of instability. □

Theorem 2.6. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function and let $S, Z : \mathbb{R}^n \to \mathbb{R}$, $Z(0, \ldots, 0) = 0$ be continuous functions such that $V = S \cdot Z$. Assume that the set*

$$K_1 = \{x \in R^n : Z(x) > 0\} \tag{2.11}$$

is a cone, and $S(x) > 0$ *for any* $x \in K_1$*. If the full derivative* (2.3) *of* V *along the trajectories of* (2.1) *is positive for every* $x \in K_1$*, that is, if* $\dot{V}(x) > 0$ *for every* $x \in K_1$*, then* K_1 *is a global cone of instability for the system* (2.1)*.*

Proof. The proof is a modification of the proof of Theorem 2.5. Let ε be a positive number. We define a neighborhood U_ε of the origin by formula (2.4) and a constant

$$M_\varepsilon := \max_{x \in \overline{U}_\varepsilon \cap \overline{K_1}} V(x). \tag{2.12}$$

Moreover, define a set

$$W_\delta := \left\{ x \in \overline{U}_\varepsilon \cap \overline{K_1}, V(x) \geq \delta \right\}, \tag{2.13}$$

where δ is a positive number such that $\delta < M_\varepsilon$. Then $W_\delta \neq \emptyset$.

Let $x_0 \in W_\delta \cap K_1$. Then $V(x_0) = \delta_1 \in [\delta, M_\varepsilon]$. We show that there exists a $t = t_T = t_T(\varepsilon, x_0)$ such that $x(x_0, t_T) \notin \overline{U}_\varepsilon$ and $x(x_0, t_T) \in K_1$.

Suppose to the contrary that this is not true and $x(x_0, t) \in \overline{U}_\varepsilon$ for all $t \geq 0$. Since $\dot{V}(x) > 0$, the function V is increasing along the solutions of (2.1). Due to the compactness of W_δ, there exists a positive value β such that for $x(x_0, t) \in W_\delta$

$$\frac{d}{dt} V(x(x_0, t)) = \operatorname{grad}^T V(x(x_0, t)) f(x(x_0, t)) > \beta. \tag{2.14}$$

Integrating this inequality over interval $[0, t]$, we get

$$V(x(x_0, t)) - V(x_0) = V(x(x_0, t)) - \delta_1 = S(x(x_0, t)) Z(x(x_0, t)) - \delta_1 > \beta t. \tag{2.15}$$

Since $S(x(x_0, t)) > 0$, the inequality

$$Z(x(x_0, t)) > \frac{\delta_1 + \beta t}{S(x(x_0, t))} > 0 \tag{2.16}$$

is an easy consequence of (2.15). Thus $x(x_0, t)$ remains in K_1. Apart from this, (2.15) also implies the existence of a $t = t_T = t_T(\varepsilon, x_0)$ satisfying

$$t_T > \frac{(M_\varepsilon - \delta_1)}{\beta}, \tag{2.17}$$

such that $V(x(x_0, t_T)) > M_\varepsilon$. Consequently, $x(x_0, t_T) \notin \overline{U}_\varepsilon$. This is contrary to our supposition. Since $\varepsilon > 0$ is arbitrary, we have

$$\lim_{t \to \infty} \| x(x_0, t) \| = \infty, \tag{2.18}$$

that is, the zero solution is globally unstable and K_1 is a global cone of instability. □

Definition 2.7. A function V satisfying all the properties indicated in Theorem 2.5 is called a Chetaev function for the system (2.1). A function V satisfying all the properties indicated in Theorem 2.6 is called a Chetaev-type function for the system (2.1).

2.2. Auxiliary Inequality

Our results will be formulated in terms of global cones of instability. These will be derived using an auxiliary inequality valid in a given cone. Let $(x, y) \in \mathbb{R}^2$ and let k be a positive number. We define a cone

$$\mathcal{K} := \left\{ (x, y) \in \mathbb{R}^2 : y > k|x| \right\}. \tag{2.19}$$

Lemma 2.8. *Let a, b, c, d, and k be given constants such that $b > 0$, $d > 0$, $k > 0$, and $|c| \leq kd$. Assume, moreover, either*

$$|a| \leq kb, \tag{2.20}$$

or

$$|a| > kb, \tag{2.21}$$

$$|c| \neq kd, \quad k \geq \max\left\{ \sqrt{\frac{|a+kb|}{c+kd}}, \sqrt{\frac{|a-kb|}{|c-kd|}} \right\}, \tag{2.22}$$

then

$$ax^3 + bx^2y + cxy^2 + dy^3 > 0, \tag{2.23}$$

for every $(x, y) \in \mathcal{K}$.

Proof. We partition $\mathcal{K}$ into two disjoint cones

$$\begin{aligned} \mathcal{K}_1 &:= \left\{ (x, y) \in \mathbb{R}^2 : y > k|x|, \ x > 0 \right\}, \\ \mathcal{K}_2 &:= \left\{ (x, y) \in \mathbb{R}^2 : y > k|x|, \ x \leq 0 \right\}, \end{aligned} \tag{2.24}$$

and rewrite (2.23) as

$$x\left(ax^2 + cy^2\right) + y\left(bx^2 + dy^2\right) > 0. \tag{2.25}$$

We prove the validity of (2.23) in each of the two cones separately.

The case of the cone $\mathcal{K}_1$. Suppose that (2.20) holds. Estimating the left-hand side of (2.25), we get

$$\begin{aligned} x\left(ax^2+cy^2\right)+y\left(bx^2+dy^2\right) &> x\left(ax^2+cy^2\right)+kx\left(bx^2+dy^2\right) \\ &= x\left[x^2(a+kb)+y^2(c+kd)\right] > 0, \end{aligned} \tag{2.26}$$

and (2.23) holds.

If inequalities (2.21) and (2.22) are valid, then, estimating the left-hand side of (2.25), we get

$$\begin{aligned} x\left(ax^2+cy^2\right)+y\left(bx^2+dy^2\right) &> x\left(ax^2+cy^2\right)+kx\left(bx^2+dy^2\right) \\ &= x\left[x^2(a+kb)+y^2(c+kd)\right] \\ &\geq x\left[-|a+kb|x^2+(c+kd)y^2\right] \\ &= (c+kd)x\left[y^2-\frac{|a+kb|}{c+kd}x^2\right] \\ &= (c+kd)x\left[y-\sqrt{\frac{|a+kb|}{c+kd}}x\right]\left[y+\sqrt{\frac{|a+kb|}{c+kd}}x\right] \\ &= (c+kd)x^2\left[k-\sqrt{\frac{|a+kb|}{c+kd}}\right]\left[k+\sqrt{\frac{|a+kb|}{c+kd}}\right] \\ &\geq 0, \end{aligned} \tag{2.27}$$

and (2.23) holds again.

The case of the cone $\mathcal{K}_2$. Suppose that (2.20) hold, then, estimating the left-hand side of (2.25), we get

$$\begin{aligned} x\left(ax^2+cy^2\right)+y\left(bx^2+dy^2\right) &= -|x|\left(ax^2+cy^2\right)+y\left(bx^2+dy^2\right) \\ &> -|x|\left(ax^2+cy^2\right)+k|x|\left(bx^2+dy^2\right) \\ &= -|x|\left[(a-kb)x^2+(c-kd)y^2\right] \\ &\geq 0, \end{aligned} \tag{2.28}$$

and (2.23) holds.

If inequalities (2.21) and (2.22) are valid, then the estimation of (2.25) implies (we use (2.28))

$$\begin{aligned}
&x\left(ax^2+cy^2\right)+y\left(bx^2+dy^2\right)\\
&\quad > -|x|\left[(a-kb)x^2+(c-kd)y^2\right]\\
&\quad = |c-kd||x|\left[y^2-\frac{a-kb}{|c-kd|}x^2\right]\\
&\quad = \begin{cases} \geq 0 & \text{if } a-kb<0,\\[2mm] |c-kd||x|\left[y-\sqrt{\dfrac{a-kb}{|c-kd|}}\,x\right]\left[y+\sqrt{\dfrac{a-kb}{|c-kd|}}\,x\right] & \\[2mm] \geq |c-kd|x^2\left[k+\sqrt{\dfrac{a-kb}{|c-kd|}}\right]\left[k-\sqrt{\dfrac{a-kb}{|c-kd|}}\right]\geq 0 & \text{if } a-kb>0. \end{cases}
\end{aligned} \tag{2.29}$$

Hence, (2.23) holds again. □

3. Global Cone of Instability

In this part we derive a result on the instability of system (1.5) in a cone. In order to properly formulate the results, we have to define some auxiliary vectors and matrices (some definitions copy the previous ones used in Introduction, but with a dimension of $n-1$ rather than n). We denote

$$\begin{gathered}
x_{(n-1)}=\left(x_1,x_2,\ldots,x_{n-1}\right)^T,\\
b_i=\left(b^i_{1n},b^i_{2n},\ldots,b^i_{n-1,n}\right)^T,\quad i=1,2,\ldots,n,\\
\widetilde{b}=\left(b^1_{nn},b^2_{nn},\ldots,b^{n-1}_{nn}\right)^T.
\end{gathered} \tag{3.1}$$

Apart from this, we define symmetric $(n-1)\times(n-1)$ matrices

$$B^0_i=\left\{b^i_{sq}\right\},\quad i=1,2,\ldots,n,\quad s,q=1,2,\ldots,n-1, \tag{3.2}$$

that is,

$$B_i^0 = \begin{pmatrix} b_{11}^i & b_{12}^i & \cdots & b_{1,n-1}^i \\ b_{21}^i & b_{22}^i & \cdots & b_{2,n-1}^i \\ \cdots & \cdots & \cdots & \cdots \\ b_{n-1,1}^i & b_{n-1,2}^i & \cdots & b_{n-1,n-1}^i \end{pmatrix},$$
$$\widetilde{B} = \begin{pmatrix} b_{1n}^1 & b_{2n}^1 & \cdots & b_{n-1,n}^1 \\ b_{1n}^2 & b_{2n}^2 & \cdots & b_{n-1,n}^2 \\ \cdots & \cdots & \cdots & \cdots \\ b_{1n}^{n-1} & b_{2n}^{n-1} & \cdots & b_{n-1,n}^{n-1} \end{pmatrix}. \tag{3.3}$$

Finally, we define an $(n-1)\times(n-1)^2$ matrix

$$\overline{B}^T = \left\{ \overline{B}_1^T, \overline{B}_2^T, \ldots, \overline{B}_{n-1}^T \right\}, \tag{3.4}$$

where $(n-1)\times(n-1)$ matrices $\overline{B}_i^T$, $i = 1, 2, \ldots, n-1$ are defined as

$$\overline{B}_i^T = \begin{pmatrix} b_{i1}^1 & b_{i2}^1 & \cdots & b_{i,n-1}^1 \\ b_{i1}^2 & b_{i2}^2 & \cdots & b_{i,n-1}^2 \\ \cdots & \cdots & \cdots & \cdots \\ b_{i1}^{n-1} & b_{i2}^{n-1} & \cdots & b_{i,n-1}^{n-1} \end{pmatrix}. \tag{3.5}$$

We consider a matrix equation

$$A_0^T H + H A_0 = -C, \tag{3.6}$$

where H and C are $(n-1)\times(n-1)$ matrices. It is well-known (see, e.g., [14]) that, for a given positive definite symmetric matrix C, (3.6) can be solved for a positive definite symmetric matrix H if and only if the matrix A_0 is asymptotically stable.

Theorem 3.1 (Main result). *Assume that the matrix A_0 is asymptotically stable, $b_{nn}^n > 0$ and h is a positive number. Let C be an $(n-1)\times(n-1)$ positive definite symmetric matrix and H be a related $(n-1)\times(n-1)$ positive definite symmetric matrix solving equation* (3.6). *Assume that the matrix $(-H\widetilde{B}^T - \widetilde{B}H + h(B_n^0)^T)$ is positive definite,*

$$\left\| 2hb_n - H\widetilde{b} \right\| \le \sqrt{\lambda_{\min}(H)h} \cdot b_{nn}^n, \tag{3.7}$$

and, in addition, one of the following conditions is valid:
either

$$\left\|H\overline{B}^{T}\right\| \le \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_n^0\right)^{T}\right) \tag{3.8}$$

or

$$\left\|H\overline{B}^{T}\right\| > \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_n^0\right)^{T}\right), \tag{3.9}$$

a strong inequality holds in (3.7)*, and*

$$\sqrt{\frac{\lambda_{\min}(H)}{h}} \ge \max\left\{\sqrt{\tau_1}, \sqrt{\tau_2}\right\}, \tag{3.10}$$

where

$$\begin{aligned} \tau_1 &= \frac{\left\|H\overline{B}^{T}\right\| - \sqrt{\lambda_{\min}(H)/h} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h(B_n^0)^{T}\right)}{-\left\|2hb_n - H\widetilde{b}\right\| + \sqrt{\lambda_{\min}(H)h} \cdot b_{nn}^{n}}, \\ \tau_2 &= \frac{\left\|H\overline{B}^{T}\right\| + \sqrt{\lambda_{\min}(H)/h} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h(B_n^0)^{T}\right)}{\left\|2hb_n - H\widetilde{b}\right\| + \sqrt{\lambda_{\min}(H)h} \cdot b_{nn}^{n}}. \end{aligned} \tag{3.11}$$

Then the set

$$K := \left\{\left(x_{(n-1)}^{T}, x_n\right) : \sqrt{h}x_n > \sqrt{x_{(n-1)}^{T} H x_{(n-1)}}\right\} \tag{3.12}$$

is a global cone of instability for the system (1.5)*.*

Proof. First we make auxiliary computations. For the reader's convenience, we recall that, for two $(n-1)\times(n-1)$ matrices $\mathcal{A}$, $\mathcal{A}_1$, two $1\times(n-1)$ vectors ℓ, ℓ_1, two $(n-1)\times 1$ vectors $\mathcal{C}$, $\mathcal{C}_1$ and two 1×1 "matrices" m, m_1, the multiplicative rule

$$\begin{pmatrix} \mathcal{A} & \mathcal{C} \\ \ell & m \end{pmatrix}\begin{pmatrix} \mathcal{A}_1 & \mathcal{C}_1 \\ \ell_1 & m_1 \end{pmatrix} = \begin{pmatrix} \mathcal{A}\mathcal{A}_1 + \mathcal{C}\ell_1 & A\mathcal{C}_1 + \mathcal{C}m_1 \\ \ell\mathcal{A}_1 + m\ell_1 & \ell\mathcal{C}_1 + mm_1 \end{pmatrix} \tag{3.13}$$

holds. This rule can be modified easily for the case of arbitrary rectangular matrices under the condition that all the products are well defined.

We will rewrite system (1.5) in an equivalent form, suitable for further investigation. With this in mind, we define an $(n-1)^2 \times (n-1)$ matrix $X_{(n-1)}$ as

$$X^T_{(n-1)} = \left(X^T_{1(n-1)}, X^T_{2(n-1)}, \dots, X^T_{n-1(n-1)}\right), \tag{3.14}$$

where all the elements of the $(n-1) \times (n-1)$ matrices $X^T_{i(n-1)}$, $i = 1, 2, \dots, n-1$ are equal to zero except the ith row, which equals $x^T_{(n-1)}$, that is,

$$X^T_{i(n-1)} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ x_1 & x_2 & \cdots & x_{n-1} \\ 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}. \tag{3.15}$$

Moreover, we define $1 \times (n-1)$ vectors Y_i, $i = 1, 2, \dots, n-1$ with components equal to zero except the ith element, which equals x_n, that is,

$$Y_i = (0, \dots, 0, x_n, 0, \dots, 0), \tag{3.16}$$

and $(n-1) \times (n-1)$ zero matrix Θ.

It is easy to see that matrices X^T and B in (1.5) can be expressed as

$$X^T = \begin{pmatrix} X^T_{1(n-1)} & Y^T_1 & \cdots & X^T_{n-1(n-1)} & Y^T_{n-1} & \Theta & \theta \\ \theta^T & 0 & \cdots & \theta^T & 0 & x^T_{(n-1)} & x_n \end{pmatrix},$$
$$B = \begin{pmatrix} B^0_1 & b_1 \\ b^T_1 & b^1_{nn} \\ \cdots & \cdots \\ B^0_n & b_n \\ b^T_n & b^n_{nn} \end{pmatrix}. \tag{3.17}$$

Now we are able to rewrite the system (1.5) under the above assumption regarding the representation of the matrix A in the form (1.10) in an equivalent form

$$\begin{aligned}
\begin{pmatrix} \dot{x}_{(n-1)} \\ \dot{x}_n \end{pmatrix} &= \begin{pmatrix} A_0 & \theta \\ \theta^T & \lambda \end{pmatrix} \begin{pmatrix} x_{(n-1)} \\ x_n \end{pmatrix} \\
&+ \begin{pmatrix} X^T_{1(n-1)} & Y^T_1 & \cdots & X^T_{n-1(n-1)} & Y^T_{n-1} & \Theta & \theta \\ \theta^T & 0 & \cdots & \theta^T & 0 & x^T_{(n-1)} & x_n \end{pmatrix} \\
&\times \begin{pmatrix} B^0_1 & b_1 \\ b^{\mathrm{T}}_1 & b^1_{nn} \\ \cdots & \cdots \\ B^0_n & b_n \\ b^T_n & b^n_{nn} \end{pmatrix} \begin{pmatrix} x_{(n-1)} \\ x_n \end{pmatrix}.
\end{aligned} \tag{3.18}$$

Finally, since the equalities

$$\begin{aligned}
\sum_{j=1}^{n-1} X^T_{j(n-1)} B^0_j &= \overline{B}^T X_{(n-1)}, \\
\sum_{j=1}^{n-1} Y^T_j b^T_j &= \widetilde{B} x_n, \\
\sum_{j=1}^{n-1} X^T_{j(n-1)} b_j &= \widetilde{B} x_{(n-1)}, \\
\sum_{j=1}^{n-1} Y^T_j b^j_{nn} &= \widetilde{b} x_n
\end{aligned} \tag{3.19}$$

can be verified easily using (3.13), we have

$$\begin{pmatrix} \dot{x}_{(n-1)} \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} A_0 + r_{11}\left(x^T_{(n-1)}, x_n\right) & r_{12}\left(x^T_{(n-1)}, x_n\right) \\ r_{21}\left(x^T_{(n-1)}, x_n\right) & \lambda + r_{22}\left(x^T_{(n-1)}, x_n\right) \end{pmatrix} \begin{pmatrix} x_{(n-1)} \\ x_n \end{pmatrix}, \tag{3.20}$$

where

$$\begin{aligned}
r_{11}\left(x_{(n-1)}^T, x_n\right) &= \sum_{j=1}^{n-1}\left[X_{j(n-1)}^T B_j^0 + Y_j^T b_j^T\right] = \overline{B}^T X_{(n-1)} + \widetilde{B} x_n, \\
r_{12}\left(x_{(n-1)}^T, x_n\right) &= \sum_{j=1}^{n-1}\left[X_{j(n-1)}^T b_j + Y_j^T b_{nn}^j\right] = \widetilde{B} x_{(n-1)} + \widetilde{b} x_n, \\
r_{21}\left(x_{(n-1)}^T, x_n\right) &= x_{(n-1)}^T B_n^0 + x_n b_n^T, \\
r_{22}\left(x_{(n-1)}^T, x_n\right) &= x_{(n-1)}^T b_n + x_n b_{nn}^n.
\end{aligned} \tag{3.21}$$

The remaining part of the proof is based on Theorem 2.6 with a Chetaev-type function $V = S \cdot Z$ and with suitable functions S and Z. Such functions we define as

$$V\left(x_{(n-1)}^T, x_n\right) = \begin{pmatrix} x_{(n-1)}^T & x_n \end{pmatrix} \begin{pmatrix} -H & \theta \\ \theta^T & h \end{pmatrix} \begin{pmatrix} x_{(n-1)} \\ x_n \end{pmatrix}, \tag{3.22}$$

that is,

$$\begin{aligned}
V\left(x_{(n-1)}^T, x_n\right) &= -x_{(n-1)}^T H x_{(n-1)} + h x_n^2, \\
S\left(x_{(n-1)}^T, x_n\right) &= \sqrt{x_{(n-1)}^T H x_{(n-1)}} + \sqrt{h} x_n, \\
Z\left(x_{(n-1)}^T, x_n\right) &= -\sqrt{x_{(n-1)}^T H x_{(n-1)}} + \sqrt{h} x_n.
\end{aligned} \tag{3.23}$$

We will verify the necessary properties. Obviously, $V = S \cdot Z$, the set

$$\begin{aligned}
K_1 : &= \left\{\left(x_{(n-1)}^T, x_n\right) \in \mathbb{R}^n : Z\left(x_{(n-1)}, x_n\right) > 0\right\} \\
&= \left\{\left(x_{(n-1)}^T, x_n\right) \in \mathbb{R}^n : \sqrt{h} x_n > \sqrt{x_{(n-1)}^T H x_{(n-1)}}\right\}
\end{aligned} \tag{3.24}$$

is a cone and $S(x_{(n-1)}^T, x_n) > 0$ for every $(x_{(n-1)}^T, x_n) \in K_1$.

The full derivative of V (in the form (3.22)) along the trajectories of the system (1.5) (we use its transformed form (3.20)) equals

$$\begin{aligned}\dot V\left(x^T_{(n-1)}, x_n\right) &= \begin{pmatrix}\dot x^T_{(n-1)} & \dot x_n\end{pmatrix}\begin{pmatrix}-H & \theta\\ \theta^T & h\end{pmatrix}\begin{pmatrix}x_{(n-1)}\\ x_n\end{pmatrix} + \begin{pmatrix}x^T_{(n-1)} & x_n\end{pmatrix}\begin{pmatrix}-H & \theta\\ \theta^T & h\end{pmatrix}\begin{pmatrix}\dot x_{(n-1)}\\ \dot x_n\end{pmatrix}\\
&= \begin{pmatrix}x^T_{(n-1)} & x_n\end{pmatrix}\begin{pmatrix}A_0^T + r^T_{11}\left(x^T_{(n-1)}, x_n\right) & r^T_{21}\left(x^T_{(n-1)}, x_n\right)\\ r^T_{12}\left(x^T_{(n-1)}, x_n\right) & \lambda + r_{22}\left(x^T_{(n-1)}, x_n\right)\end{pmatrix}\begin{pmatrix}-H & \theta\\ \theta^T & h\end{pmatrix}\begin{pmatrix}x_{(n-1)}\\ x_n\end{pmatrix}\\
&\quad + \begin{pmatrix}x^T_{(n-1)} & x_n\end{pmatrix}\begin{pmatrix}-H & \theta\\ \theta^T & h\end{pmatrix}\begin{pmatrix}A_0 + r_{11}\left(x^T_{(n-1)}, x_n\right) & r_{12}\left(x^T_{(n-1)}, x_n\right)\\ r_{21}\left(x^T_{(n-1)}, x_n\right) & \lambda + r_{22}\left(x^T_{(n-1)}, x_n\right)\end{pmatrix}\\
&\quad \times \begin{pmatrix}x_{(n-1)}\\ x_n\end{pmatrix}.\end{aligned} \tag{3.25}$$

Using formula (3.13), we get

$$\dot V\left(x^T_{(n-1)}, x_n\right) = \begin{pmatrix}x^T_{(n-1)} & x_n\end{pmatrix}\begin{pmatrix}c_{11}\left(x^T_{(n-1)}, x_n\right) & c_{12}\left(x^T_{(n-1)}, x_n\right)\\ c_{21}\left(x^T_{(n-1)}, x_n\right) & c_{22}\left(x^T_{(n-1)}, x_n\right)\end{pmatrix}\begin{pmatrix}x_{(n-1)}\\ x_n\end{pmatrix}, \tag{3.26}$$

where

$$\begin{aligned}c_{11}\left(x^T_{(n-1)}, x_n\right) &= -\left[A_0 + r_{11}(x^T_{(n-1)}, x_n)\right]^T H - H\left[A_0 + r_{11}\left(x^T_{(n-1)}, x_n\right)\right],\\
c_{12}\left(x^T_{(n-1)}, x_n\right) &= hr^T_{21}\left(x^T_{(n-1)}, x_n\right) - Hr_{12}\left(x^T_{(n-1)}, x_n\right),\\
c_{21}\left(x^T_{(n-1)}, x_n\right) &= hr_{21}\left(x^T_{(n-1)}, x_n\right) - r^T_{12}\left(x^T_{(n-1)}, x_n\right)H = c^T_{12}\left(x^T_{(n-1)}, x_n\right),\\
c_{22}\left(x^T_{(n-1)}, x_n\right) &= 2h\left[\lambda + r_{22}\left(x^T_{(n-1)}, x_n\right)\right].\end{aligned} \tag{3.27}$$

We reduce these formulas using (3.21). Then,

$$\begin{aligned}c_{11}\left(x^T_{(n-1)}, x_n\right) &= -\left(A_0^T H + HA_0\right) - \left(\overline{B}^T X_{(n-1)} + \widetilde{B}x_n\right)^T H - H\left(\overline{B}^T X_{(n-1)} + \widetilde{B}x_n\right),\\
c_{12}\left(x^T_{(n-1)}, x_n\right) &= h\left(x^T_{(n-1)}B^0_n + x_n b^T_n\right)^T - H\left(\widetilde{B}x_{(n-1)} + \widetilde{b}x_n\right),\\
c_{21}\left(x^T_{(n-1)}, x_n\right) &= h\left(x^T_{(n-1)}B^0_n + x_n b^T_n\right) - \left(\widetilde{B}x_{(n-1)} + \widetilde{b}x_n\right)^T H,\\
c_{22}\left(x^T_{(n-1)}, x_n\right) &= 2h\left(\lambda + x^T_{(n-1)}b_n + x_n b^n_{nn}\right).\end{aligned} \tag{3.28}$$

The derivative (3.26) turns into

$$\begin{aligned}
\dot{V}\left(x_{(n-1)}^{T}, x_n\right) &= x_{(n-1)}^{T} c_{11}\left(x_{(n-1)}^{T}, x_n\right) x_{(n-1)} + x_{(n-1)}^{T} c_{12}\left(x_{(n-1)}^{T}, x_n\right) x_n \\
&\quad + x_n c_{21}\left(x_{(n-1)}^{T}, x_n\right) x_{(n-1)} + x_n c_{22}\left(x_{(n-1)}^{T}, x_n\right) x_n \\
&= x_{(n-1)}^{T}\left[-\left(A_0^T H + H A_0\right) - \left(\overline{B}^T X_{(n-1)} + \widetilde{B} x_n\right)^T H - H\left(\overline{B}^T X_{(n-1)} + \widetilde{B} x_n\right)\right] x_{(n-1)} \\
&\quad + x_{(n-1)}^{T}\left[h\left(x_{(n-1)}^{T} B_n^0 + x_n b_n^T\right)^T - H\left(\widetilde{B} x_{(n-1)} + \widetilde{b} x_n\right)\right] x_n \\
&\quad + x_n\left[h\left(x_{(n-1)}^{T} B_n^0 + x_n b_n^T\right) - \left(\widetilde{B} x_{(n-1)} + \widetilde{b} x_n\right)^T H\right] x_{(n-1)} \\
&\quad + x_n\left[2h\left(\lambda + x_{(n-1)}^{T} b_n + x_n b_{nn}^n\right)\right] x_n \\
&= -x_{(n-1)}^{T}\left(A_0^T H + H A_0\right) x_{(n-1)} + 2h\lambda x_n^2 \\
&\quad - x_{(n-1)}^{T}\left(\left(\overline{B}^T X_{(n-1)}\right)^T H + H\overline{B}^T X_{(n-1)}\right) x_{(n-1)} \\
&\quad - x_{(n-1)}^{T}\left(\left(\widetilde{B} x_n\right)^T H + H\widetilde{B} x_n\right) x_{(n-1)} \\
&\quad + x_{(n-1)}^{T}\left(2h\left(B_n^0\right)^T - H\widetilde{B} - \widetilde{B} H\right) x_{(n-1)} x_n \\
&\quad + 2x_{(n-1)}^{T}\left(h b_n - H\widetilde{b}\right) x_n^2 \\
&\quad + 2h\left(x_{(n-1)}^{T} b_n + x_n b_{nn}^n\right) x_n^2.
\end{aligned} \tag{3.29}$$

Finally, using (3.6), we get

$$\begin{aligned}
\dot{V}\left(x_{(n-1)}^{T}, x_n\right) &= x_{(n-1)}^{T} C x_{(n-1)} + 2h\lambda x_n^2 - 2x_{(n-1)}^{T} H\overline{B}^T X_{(n-1)} x_{(n-1)} \\
&\quad + 2x_{(n-1)}^{T}\left[-H\widetilde{B}^T - \widetilde{B} H + h\left(B_n^0\right)^T\right] x_{(n-1)} x_n + 2x_{(n-1)}^{T}\left(2h b_n - H\widetilde{b}\right) x_n^2 + 2h b_{nn}^n x_n^3.
\end{aligned} \tag{3.30}$$

Let us find the conditions for the positivity of $\dot{V}(x^T_{(n-1)}, x_n)$ in the cone K_1. We use (3.30). If $(x^T_{(n-1)}, x_n) \in K_1$, then $x_n \geq 0$ and

$$\begin{aligned}\dot{V}\left(x^T_{(n-1)}, x_n\right) \geq{}& x^T_{(n-1)} C x_{(n-1)} + 2h\lambda x_n^2 - 2\left\| H\overline{B}^T \right\| \cdot \left\| x_{(n-1)} \right\|^3 \\ &+ 2\lambda_{\min}\left(-H\widetilde{B}^T - \widetilde{B}H + h\left(B_n^0\right)^T \right) \cdot \left\| x_{(n-1)} \right\|^2 \cdot x_n \\ &- 2\left\| 2hb_n - H\widetilde{b} \right\| \cdot \left\| x_{(n-1)} \right\| \cdot x_n^2 + 2hb^n_{nn} x_n^3.\end{aligned} \tag{3.31}$$

We set

$$\begin{aligned} a &= -2\left\| H\overline{B}^T \right\|, \\ b &= 2\lambda_{\min}\left(-H\widetilde{B}^T - \widetilde{B}H + h\left(B_n^0\right)^T \right), \\ c &= -2\left\| 2hb_n - H\widetilde{b} \right\|, \\ d &= 2hb^n_{nn}. \end{aligned} \tag{3.32}$$

If

$$a\left\| x_{(n-1)} \right\|^3 + b\left\| x_{(n-1)} \right\|^2 \cdot x_n + c\left\| x_{(n-1)} \right\| \cdot x_n^2 + dx_n^3 > 0 \tag{3.33}$$

in K_1, then $\dot{V}(x^T_{(n-1)}, x_n) > 0$ since C is a positive definite matrix and

$$x^T_{(n-1)} C x_{(n-1)} + 2h\lambda x_n^2 \geq \lambda_{\min}(C)\left\| x_{(n-1)} \right\|^2 + 2h\lambda x_n^2 > 0. \tag{3.34}$$

If $(x^T_{(n-1)}, x_n) \in K_1$, then

$$x_n > \sqrt{\frac{x^T_{(n-1)} H x_{(n-1)}}{h}} \geq \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \left\| x_{(n-1)} \right\|, \tag{3.35}$$

$$K_1 \subset \mathcal{K}^* := \left\{ \left(x^T_{(n-1)}, x_n\right) \in \mathbb{R}^n : x_n > \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \left\| x_{(n-1)} \right\| \right\}. \tag{3.36}$$

Now, we use Lemma 2.8 with $\mathcal{K} = \mathcal{K}^*$, $y = x_n$, $x = \|x_{(n-1)}\|$, with coefficients a, b, c, and d defined by formula (3.32) and with $k := \sqrt{\lambda_{\min}(H)/h}$.

Obviously $|c| \leq kd$ because, due to (3.7), inequality

$$\left\| 2hb_n - H\widetilde{b} \right\| \leq \sqrt{\lambda_{\min}(H)h} \cdot b^n_{nn} \tag{3.37}$$

holds. Moreover, $|a| \leq kb$ if (3.8) holds, that is, if

$$\left\| H\overline{B}^{T} \right\| \leq \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_{n}^{0}\right)^{T} \right). \tag{3.38}$$

Further, $|a| > kb$ if (3.9) holds, that is, if

$$\left\| H\overline{B}^{T} \right\| > \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h_{n}\left(B_{n}^{0}\right)^{T} \right), \tag{3.39}$$

and (2.22) holds due to (4.10) and the condition $|c| \neq kd$. Thus the assumptions of Lemma 2.8 are true, the inequality (3.33) holds in the cone $\mathcal{K}^*$ and, due to embedding (3.36), in the cone K_1 as well.

All the assumptions of Theorem 2.6 are fulfilled with regard to system (1.5) and the theorem is proved, because $K_1 = K$. □

Remark 3.2. We will focus our attention to Lemma 2.8 about the positivity of a third-degree polynomial in two variables in the cone $\mathcal{K}$. We used it to estimate the derivative $\dot{V}$ expressed by formula (3.30). Obviously, there are other possibilities of estimating its sign. Let us demonstrate one of them. Let us, for example, estimate the right-hand side of (3.31) in the cone K_1 using inequality (3.35), then

$$\begin{aligned}
\dot{V}\left(x_{(n-1)}^{T}, x_{n}\right) &\geq x_{(n-1)}^{T} C x_{(n-1)} + 2h\lambda x_{n}^{2} - 2\left\| H\overline{B}^{T} \right\| \cdot \left\| x_{(n-1)} \right\|^{3} \\
&\quad + 2\lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_{n}^{0}\right)^{T} \right) \cdot \left\| x_{(n-1)} \right\|^{2} \cdot x_{n} \\
&\quad - 2\left\| 2hb_{n} - H\widetilde{b} \right\| \cdot \left\| x_{(n-1)} \right\| \cdot x_{n}^{2} + 2hb_{nn}^{n} x_{n}^{3} \\
&\geq \lambda_{\min}(C) \left\| x_{(n-1)} \right\|^{2} + 2h\lambda x_{n}^{2} - 2\left\| H\overline{B}^{T} \right\| \cdot \left\| x_{(n-1)} \right\|^{3} \\
&\quad + 2\sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_{n}^{0}\right)^{T} \right) \cdot \left\| x_{(n-1)} \right\|^{3} \\
&\quad - 2\left\| 2hb_{n} - H\widetilde{b} \right\| \cdot \left\| x_{(n-1)} \right\| \cdot x_{n}^{2} + 2\sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \left\| x_{(n-1)} \right\| \cdot hb_{nn}^{n} \cdot x_{n}^{2},
\end{aligned} \tag{3.40}$$

and the positivity of $\dot{V}(x_{(n-1)}^{T}, x_{n})$ will be guaranteed if

$$\begin{gathered}
\left\| H\overline{B}^{T} \right\| \leq \sqrt{\frac{\lambda_{\min}(H)}{h}} \cdot \lambda_{\min}\left(-H\widetilde{B}^{T} - \widetilde{B}H + h\left(B_{n}^{0}\right)^{T} \right), \\
\left\| 2hb_{n} - H\widetilde{b} \right\| \leq \sqrt{\lambda_{\min}(H)h} \cdot b_{nn}^{n}.
\end{gathered} \tag{3.41}$$

We see that this approach produces only one set of inequalities for the positivity of $\dot{V}(x^T_{(n-1)}, x_n)$, namely the case when (3.7) and (3.8) holds. Unfortunately, using such approach, we are not able to detect the second case (3.7) and (3.9) when $\dot{V}(x^T_{(n-1)}, x_n)$ is positive. This demonstrates the advantage of detailed estimates using the above third-degree polynomial in two variables.

4. Planar Case

Now we consider a particular case of the system (1.5) for $n = 2$. This means that, in accordance with (1.5) and (1.10), we consider a system

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t) + b^1_{11}x^2_1(t) + 2b^1_{12}x_1(t)x_2(t) + b^1_{22}x^2_2(t), \\ \dot{x}_2(t) &= \lambda x_2(t) + b^2_{11}x^2_1(t) + 2b^2_{12}x_1(t)x_2(t) + b^2_{22}x^2_2(t), \end{aligned} \tag{4.1}$$

where $a < 0$ and $\lambda > 0$. The solution of matrix equation (3.6) for $A_0 = (a)$, $H = (h_{11})$, and $C = (c)$ with $c > 0$, that is,

$$(ah_{11}) + (h_{11}a) = -(c) \tag{4.2}$$

gives

$$H = (h_{11}) = \left(-\frac{c}{2a}\right), \tag{4.3}$$

with $h_{11} = -c/2a > 0$. The set K defined by (3.12) where $h > 0$ and $x_{(n-1)} = x_1$ reduces to

$$K = \left\{(x_1, x_2) : x_2 > \sqrt{\frac{c}{2|a|h}} \cdot |x_1|\right\}. \tag{4.4}$$

Now, from Theorem 3.1, we will deduce sufficient conditions indicating K being a global cone of instability for system (4.1). In our particular case, we have

$$\begin{gathered} b_i = \left(b^i_{12}\right), \quad i = 1, 2, \quad \widetilde{b} = \left(b^1_{22}\right), \\ B^0_i = \left(b^i_{11}\right), \quad i = 1, 2, \quad \widetilde{B} = \left(b^1_{12}\right), \quad \overline{B}^T = \left(b^1_{11}\right) = B^0_1. \end{gathered} \tag{4.5}$$

Now, we compute all necessary expressions used in Theorem 3.1. We have

$$-H\widetilde{B}^T - \widetilde{B}H + h\left(B_n^0\right)^T = -\left(-\frac{c}{2a}\right)\left(b_{12}^1\right) - \left(b_{12}^1\right)\left(-\frac{c}{2a}\right) + h\left(b_{11}^2\right) = \left(hb_{11}^2 - \frac{c}{|a|}b_{12}^1\right),$$

$$\left\|2hb_n - H\widetilde{b}\right\| = \left|2hb_{12}^2 - \frac{c}{2|a|}b_{22}^1\right|,$$

$$\sqrt{\lambda_{\min}(H)h} = \sqrt{\frac{ch}{2|a|}},$$

$$\sqrt{\frac{\lambda_{\min}(H)}{h}} = \sqrt{\frac{c}{2|a|h}},$$

$$\left\|H\overline{B}^T\right\| = \left|\frac{c}{2|a|}b_{11}^1\right| = \frac{c}{2|a|}\left|b_{11}^1\right|,$$

$$\lambda_{\min}\left(-H\widetilde{B}^T - \widetilde{B}H + h\left(B_n^0\right)^T\right) = hb_{11}^2 - \frac{c}{|a|}b_{12}^1, \tag{4.6}$$

$$\begin{aligned}\tau_1 &= \frac{\left\|H\overline{B}^T\right\| - \sqrt{\lambda_{\min}(H)/h}\cdot\lambda_{\min}\left(-H\widetilde{B}^T - \widetilde{B}H + h(B_n^0)^T\right)}{-\left\|2hb_n - H\widetilde{b}\right\| + \sqrt{\lambda_{\min}(H)h}\cdot b_{nn}^n}\\ &= \frac{(c/2|a|)\left|b_{11}^1\right| - \sqrt{c/2|a|h}\cdot\left(hb_{11}^2 - (c/|a|)b_{12}^1\right)}{-\left|2hb_{12}^2 - (c/2|a|)b_{22}^1\right| + \sqrt{ch/2|a|}\cdot b_{22}^2},\\ \tau_2 &= \frac{\left\|H\overline{B}^T\right\| + \sqrt{\lambda_{\min}(H)/h}\cdot\lambda_{\min}\left(-H\widetilde{B}^T - \widetilde{B}H + h(B_n^0)^T\right)}{\left\|2hb_n - H\widetilde{b}\right\| + \sqrt{\lambda_{\min}(H)h}\cdot b_{nn}^n}\\ &= \frac{(c/2|a|)\left|b_{11}^1\right| + \sqrt{(c/2|a|h)}\cdot\left(hb_{11}^2 - (c/|a|)b_{12}^1\right)}{\left|2hb_{12}^2 - (c/2|a|)b_{22}^1\right| + \sqrt{(ch/2|a|)}\cdot b_{22}^2}.\end{aligned}$$

Theorem 4.1 (Planar Case). *Assume that $a < 0$, $b_{22}^2 > 0$, $h > 0$, $c > 0$ and $hb_{11}^2|a| > cb_{12}^1$. Let*

$$\left|2hb_{12}^2 - \frac{c}{2|a|}b_{22}^1\right| \le \sqrt{\frac{ch}{2|a|}}\cdot b_{22}^2, \tag{4.7}$$

and, in addition, one of the following conditions is valid:

either

$$\frac{c}{2|a|}\left|b_{11}^1\right| \le \sqrt{\frac{c}{2|a|h}}\cdot\left(hb_{11}^2 - \frac{c}{|a|}b_{12}^1\right) \tag{4.8}$$

or

$$\frac{c}{2|a|}\left|b_{11}^{1}\right| > \sqrt{\frac{c}{2|a|h}} \cdot \left(hb_{11}^{2} - \frac{c}{|a|}b_{12}^{1}\right), \tag{4.9}$$

strong inequality holds in (4.7)*, and*

$$\sqrt{\frac{c}{2|a|h}} \geq \max\left\{\sqrt{\tau_1}, \sqrt{\tau_2}\right\}, \tag{4.10}$$

where τ_1 *and* τ_1 *are defined by* (4.6)*. Then the set* K *defined by* (4.4) *is a global cone of instability for the system* (4.1)*.*

It is easy to see that the choice $h = 1$, $c = |a|$ significantly simplifies all assumptions. Therefore we give such a particular case of Theorem 4.1.

Corollary 4.2 (Planar Case). *Assume that* $a < 0$, $b_{22}^{2} > 0$ *and* $b_{11}^{2} > b_{12}^{1}$*. Let*

$$\left|2b_{12}^{2} - \frac{1}{2}b_{22}^{1}\right| \leq \frac{1}{\sqrt{2}} \cdot b_{22}^{2}, \tag{4.11}$$

and, in addition, one of the following conditions is valid:
either

$$\frac{1}{2}\left|b_{11}^{1}\right| \leq \frac{1}{\sqrt{2}} \cdot \left(b_{11}^{2} - b_{12}^{1}\right) \tag{4.12}$$

or

$$\frac{1}{2}\left|b_{11}^{1}\right| > \frac{1}{\sqrt{2}} \cdot \left(b_{11}^{2} - b_{12}^{1}\right), \tag{4.13}$$

strong inequality holds in (4.11)*, and*

$$\frac{1}{\sqrt{2}} \geq \max\left\{\sqrt{\tau_1}, \sqrt{\tau_2}\right\}, \tag{4.14}$$

where

$$\tau_1 = \frac{(1/2)\left|b_{11}^{1}\right| - \left(1/\sqrt{2}\right) \cdot \left(b_{11}^{2} - b_{12}^{1}\right)}{-\left|2b_{12}^{2} - (1/2)b_{22}^{1}\right| + \left(1/\sqrt{2}\right) \cdot b_{22}^{2}}, \qquad \tau_2 = \frac{(1/2)\left|b_{11}^{1}\right| + \left(1/\sqrt{2}\right) \cdot \left(b_{11}^{2} - b_{12}^{1}\right)}{\left|2b_{12}^{2} - (1/2)b_{22}^{1}\right| + \left(1/\sqrt{2}\right) \cdot b_{22}^{2}}, \tag{4.15}$$

Then the set

$$K = \left\{ (x_1, x_2) : x_2 > \frac{1}{\sqrt{2}} \cdot |x_1| \right\} \tag{4.16}$$

is a global cone of instability for the system (4.1).

Example 4.3. The set K defined by (4.16) is a global cone of instability for the system

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t) + x_1^2(t) + 2\sqrt{2}x_1(t)x_2(t) + x_2^2(t), \\ \dot{x}_2(t) &= \lambda x_2(t) + 2\sqrt{2}x_1^2(t) + 2x_1(t)x_2(t) + 2\sqrt{2}x_2^2(t), \end{aligned} \tag{4.17}$$

where $a < 0$ and $\lambda > 0$ since inequalities (4.11) and (4.12) in Corollary 4.2 hold.

Example 4.4. The set K defined by (4.16) is a global cone of instability for the system

$$\begin{aligned} \dot{x}_1(t) &= ax_1(t) + 4x_1^2(t) + 2\sqrt{2}x_1(t)x_2(t) + x_2^2(t), \\ \dot{x}_2(t) &= \lambda x_2(t) + 2\sqrt{2}x_1^2(t) + 2x_1(t)x_2(t) + 20\sqrt{2}x_2^2(t), \end{aligned} \tag{4.18}$$

where $a < 0$ and $\lambda > 0$ since inequalities (4.11), (4.13), (4.14) in Corollary 4.2 hold.

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Research Article

Invariant Sets of Impulsive Differential Equations with Particularities in ω-Limit Set

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Sufficient conditions for the existence and asymptotic stability of the invariant sets of an impulsive system of differential equations defined in the direct product of a torus and an Euclidean space are obtained.

1. Introduction

The evolution of variety of processes in physics, chemistry, biology, and so forth, frequently undergoes short-term perturbations. It is sometimes convenient to neglect the duration of the perturbations and to consider these perturbations to be "instantaneous." This leads to the necessity of studying the differential equations with discontinuous trajectories, the so-called impulsive differential equations. The fundamentals of the mathematical theory of impulsive differential equations are stated in [1–4]. The theory is developing intensively due to its applied value in simulations of the real world phenomena.

At the same time, this paper is closely related to the oscillation theory. In the middle of the 20th century, a sharp turn towards the investigations of the oscillating processes that were characterized as "almost exact" iterations within "almost the same" periods of time took place. Quasiperiodic oscillations were brought to the primary focus of investigations of the oscillation theory [5].

Quasiperiodic oscillations are a sufficiently complicated and sensitive object for investigating. The practical value of indicating such oscillations is unessential. Due to the instability of frequency basis, quasiperiodic oscillation collapses easily and may be transformed into periodic oscillation via small shift of the right-hand side of the system. This fact has led to search for more rough object than the quasiperiodic solution. Thus the minimal set that is covered by the trajectories of the quasiperiodic motions becomes the main object of investigations. As it is known, such set is a torus. The first profound assertions regarding the invariant toroidal manifolds were obtained by Bogoliubov et al. [6, 7]. Further results in this area were widely extended by many authors.

Consider the system of differential equations

$$\frac{dz}{dt} = F(z), \tag{1.1}$$

where the function $F(z)$ is defined in some subset D of the $(m+n)$-dimensional Euclidean space E^{m+n}, continuous and satisfies a Lipschitz condition. Let M be an invariant toroidal manifold of the system. While investigating the trajectories that begin in the neighborhood of the manifold M, it is convenient to make the change of variables from Euclidean coordinates $(z_1, \ldots, z_{m+n})$ to so-called local coordinates $\varphi = (\varphi_1, \ldots, \varphi_m)$, $x = (x_1, \ldots, x_n)$, where φ is a point on the surface of an m-dimensional torus $\mathcal{T}^m$ and x is a point in an n-dimensional Euclidean space E^n. The change of variables is performed in a way such that the equation, which defines the invariant manifold M, transforms into $x = 0$, $\varphi \in \mathcal{T}^m$ in the new coordinates. In essence, the manifold $x = 0$, $\varphi \in \mathcal{T}^m$ is the m-dimensional torus in the space $\mathcal{T}^m \times E^n$. The character of stability of the invariant torus M is closely linked with stability of the set $x = 0$, $\varphi \in \mathcal{T}^m$: from stability, asymptotic stability, and instability of the manifold M, there follow the stability, asymptotic stability, and instability of the torus $x = 0$, $\varphi \in \mathcal{T}^m$ correspondingly and vice versa. This is what determines the relevance and value of the investigation of conditions for the existence and stability of invariant sets of the systems of differential equations defined in $\mathcal{T}^m \times E^n$. Theory of the existence and perturbation, properties of smoothness, and stability of invariant sets of systems defined in $\mathcal{T}^m \times E^n$ are considered in [8].

2. Preliminaries

The main object of investigation of this paper is the system of differential equations, defined in the direct product of an m-dimensional torus $\mathcal{T}^m$ and an n-dimensional Euclidean space E^n that undergo impulsive perturbations at the moments when the phase point φ meets a given set in the phase space. Consider the system

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= A(\varphi)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} &= B(\varphi)x + g(\varphi), \end{aligned} \tag{2.1}$$

where $\varphi = (\varphi_1, \ldots, \varphi_m)^T \in \mathcal{T}^m$, $x = (x_1, \ldots, x_n)^T \in E^n$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector function that satisfies a Lipschitz condition

$$\|a(\varphi'') - a(\varphi')\| \le L\|\varphi'' - \varphi'\| \tag{2.2}$$

for every $\varphi', \varphi'' \in \mathcal{T}^m$. $A(\varphi), B(\varphi)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ square matrices; $f(\varphi)$, $g(\varphi)$ are continuous (piecewise continuous with first kind discontinuities in the set Γ) 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector functions.

Some aspects regarding existence and stability of invariant sets of systems similar to (2.1) were considered by different authors in [9–12].

We regard the point $\varphi = (\varphi_1, \ldots, \varphi_m)^T$ as a point of the m-dimensional torus $\mathcal{T}^m$ so that the domain of the functions $A(\varphi), B(\varphi), f(\varphi), g(\varphi)$, and $a(\varphi)$ is the torus $\mathcal{T}^m$. We assume that the set Γ is a subset of the torus $\mathcal{T}^m$, which is a manifold of dimension $m-1$ defined by the equation $\Phi(\varphi) = 0$ for some continuous scalar 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ function.

The system of differential equations

$$\frac{d\varphi}{dt} = a(\varphi) \tag{2.3}$$

defines a dynamical system on the torus $\mathcal{T}^m$. Denote by $\varphi_t(\varphi)$ the solution of (2.3) that satisfies the initial condition $\varphi_0(\varphi) = \varphi$. The Lipschitz condition (2.2) guarantees the existence and uniqueness of such solution. Moreover, the solutions $\varphi_t(\varphi)$ satisfies a group property [8]

$$\varphi_t(\varphi_\tau(\varphi)) = \varphi_{t+\tau}(\varphi) \tag{2.4}$$

for all $t, \tau \in \mathbb{R}$ and $\varphi \in \mathcal{T}^m$.

Denote by $t_i(\varphi)$, $i \in \mathbb{Z}$ the solutions of the equation $\Phi(\varphi_t(\varphi)) = 0$ that are the moments of impulsive action in system (2.1). Let the function $\Phi(\varphi)$ be such that the solutions $t = t_i(\varphi)$ exist, since otherwise, system (2.1) would not be an impulsive system. Assume that

$$\begin{gathered} \lim_{i \to \pm\infty} t_i(\varphi) = \pm\infty, \\ \lim_{T \to \pm\infty} \frac{i(t, t+T)}{T} = p < \infty \end{gathered} \tag{2.5}$$

uniformly with respect to $t \in \mathbb{R}$, where $i(a, b)$ is the number of the points $t_i(\varphi)$ in the interval (a, b). Hence, the moments of impulsive perturbations $t_i(\varphi)$ satisfy the equality [10, 11]

$$t_i(\varphi_{-t}(\varphi)) - t_i(\varphi) = t. \tag{2.6}$$

Together with system (2.1), we consider the linear system

$$\begin{gathered} \frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} = B(\varphi_{t_i(\varphi)}(\varphi))x + g(\varphi_{t_i(\varphi)}(\varphi)) \end{gathered} \tag{2.7}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. We obtain system (2.7) by substituting $\varphi_t(\varphi)$ for φ in the second and third equations of system (2.1). By invariant set of system (2.1), we understand a set that is defined by a function $u(\varphi)$, which has a period 2π with respect to each of the components φ_v, $v = 1, \ldots, m$, such that the function $x(t, \varphi) = u(\varphi_t(\varphi))$ is a solution of system (2.7) for every $\varphi \in \mathcal{T}^m$.

We call a point φ^* an ω-limit point of the trajectory $\varphi_t(\varphi)$ if there exists a sequence $\{t_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that

$$\lim_{n\to+\infty} t_n = +\infty, \\ \lim_{n\to+\infty} \varphi_{t_n}(\varphi) = \varphi^*. \tag{2.8}$$

The set of all ω-limit points for a given trajectory $\varphi_t(\varphi)$ is called ω-limit set of the trajectory $\varphi_t(\varphi)$ and denoted by Ω_φ.

Referring to system (2.7), the matrices $A(\varphi_t(\varphi))$ and $B(\varphi_t(\varphi))$, that influence the behavior of the solution $x(t,\varphi)$ of the system (2.7), depend not only on the functions $A(\varphi)$ and $B(\varphi)$ but also on the behavior of the trajectories $\varphi_t(\varphi)$. Moreover, in [9], sufficient conditions for the existence and stability of invariant sets of a system similar to (2.1) were obtained in terms of a Lyapunov function $V(\varphi,x)$ that satisfies some conditions in the domain $Z = \{\varphi \in \Omega,\ x \in \overline{J}_h\}$, where $\overline{J}_h = \{x \in E^n,\ \|x\| \le h,\ h > 0\}$,

$$\Omega = \bigcup_{\varphi\in\mathcal{T}^m} \Omega_\varphi. \tag{2.9}$$

Since the Lyapunov function has to satisfy some conditions not on the whole surface of the torus $\mathcal{T}^m$ but only in the ω-limit set Ω, it is interesting to consider system (2.1) with specific properties in the domain Ω.

3. Main Result

Consider system (2.1) assuming that the matrices $A(\varphi)$ and $B(\varphi)$ are constant in the domain Ω:

$$A(\varphi)\big|_{\varphi\in\Omega} = \widetilde{A}, \\ B(\varphi)\big|_{\varphi\in\Omega} = \widetilde{B}. \tag{3.1}$$

Therefore, for every $\varphi \in \mathcal{T}^m$

$$\lim_{t\to+\infty} A(\varphi_t(\varphi)) = \widetilde{A}, \\ \lim_{t\to+\infty} B(\varphi_t(\varphi)) = \widetilde{B}. \tag{3.2}$$

We will obtain sufficient conditions for the existence and asymptotic stability of an invariant set of the system (2.1) in terms of the eigenvalues of the matrices $\widetilde{A}$ and $\widetilde{B}$. Denote by

$$\gamma = \max_{j=1,\ldots,n} \operatorname{Re}\lambda_j\left(\widetilde{A}\right), \\ \alpha^2 = \max_{j=1,\ldots,n} \lambda_j\left(\left(E+\widetilde{B}\right)^T\left(E+\widetilde{B}\right)\right). \tag{3.3}$$

Similar systems without impulsive perturbations have been considered in [13].

Theorem 3.1. *Let the moments of impulsive perturbations* $\{t_i(\varphi)\}$ *be such that uniformly with respect to* $t \in \mathbb{R}$ *there exists a finite limit*

$$\lim_{\widetilde{T}\to\infty} \frac{i\left(t, t+\widetilde{T}\right)}{\widetilde{T}} = p. \tag{3.4}$$

If the following inequality holds

$$\gamma + p \ln \alpha < 0, \tag{3.5}$$

then system (2.1) *has an asymptotically stable invariant set.*

Proof. Consider a homogeneous system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= A(\varphi_t(\varphi))x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= B(\varphi_{t_i(\varphi)}(\varphi))x \end{aligned} \tag{3.6}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. By $\Omega_\tau^t(\varphi)$, we denote the fundamental matrix of system (3.6), which turns into an identity matrix at the point $t = \tau$, that is, $\Omega_\tau^\tau(\varphi) \equiv E$. It can be readily verified [4] that $\Omega_\tau^t(\varphi)$ satisfies the equalities

$$\begin{aligned} \frac{\partial}{\partial t}\Omega_\tau^t(\varphi) &= A(\varphi_t(\varphi))\Omega_\tau^t(\varphi), \\ \Omega_\tau^t(\varphi) &= \Omega_\tau^t(\varphi + 2\pi e_k), \\ \Omega_{t+\tau}^t(\varphi_{-t}(\varphi)) &= \Omega_\tau^0(\varphi) \end{aligned} \tag{3.7}$$

for all $t, \tau \in \mathbb{R}$ and $\varphi \in \mathcal{T}^m$. Rewrite system (3.6) in the form

$$\begin{aligned} \frac{dx}{dt} &= \widetilde{A}x + \left(A(\varphi_t(\varphi)) - \widetilde{A}\right)x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= \widetilde{B}x + \left(B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B}\right)x. \end{aligned} \tag{3.8}$$

The fundamental matrix $\Omega_\tau^t(\varphi)$ of the system (3.6) may be represented in the following way [4]:

$$\begin{aligned} \Omega_\tau^t(\varphi) = X_\tau^t(\varphi) &+ \int_\tau^t X_s^t(\varphi)\left(A(\varphi_s(\varphi)) - \widetilde{A}\right)\Omega_s^t(\varphi)ds \\ &+ \sum_{\tau \le t_i(\varphi) < t} X_{t_i(\varphi)}^t(\varphi)\left(B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B}\right)\Omega_\tau^{t_i(\varphi)}(\varphi), \end{aligned} \tag{3.9}$$

where $X_\tau^t(\varphi)$ is the fundamental matrix of the homogeneous impulsive system with constant coefficients

$$\begin{aligned} \frac{dx}{dt} &= \widetilde{A}x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= \widetilde{B}x \end{aligned} \tag{3.10}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. Taking into account that the matrix $X_\tau^t(\varphi)$ satisfies the estimate [14]

$$\left\| X_\tau^t(\varphi) \right\| \le K e^{-\mu(t-\tau)}, \quad t \ge \tau \tag{3.11}$$

for every $\varphi \in \mathcal{T}^m$ and some $K \ge 1$, where $\gamma + p\ln\alpha < -\mu < 0$, we obtain

$$\begin{aligned} \left\| \Omega_\tau^t(\varphi) \right\| \le{}& K e^{-\mu(t-\tau)} + \int_\tau^t K e^{-\mu(t-s)} \left\| A(\varphi_s(\varphi)) - \widetilde{A} \right\| \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \sum_{\tau \le t_i(\varphi) < t} K e^{-\mu(t-t_i(\varphi))} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B} \right\| \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\|. \end{aligned} \tag{3.12}$$

It follows from (3.2) that for arbitrary small ε_A and ε_B, there exists a moment T such that

$$\begin{aligned} \left\| A(\varphi_t(\varphi)) - \widetilde{A} \right\| &\le \varepsilon_A, \\ \left\| B(\varphi_t(\varphi)) - \widetilde{B} \right\| &\le \varepsilon_B \end{aligned} \tag{3.13}$$

for all $t \ge T$. Hence, multiplying (3.12) by $e^{\mu(t-\tau)}$, utilizing (3.13), and weakening the inequality, we obtain

$$\begin{aligned} e^{\mu(t-\tau)} \left\| \Omega_\tau^t(\varphi) \right\| \le{}& K + \int_\tau^T K e^{\mu(s-\tau)} \left\| A(\varphi_s(\varphi)) - \widetilde{A} \right\| \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \int_\tau^t K\varepsilon_A e^{\mu(s-\tau)} \left\| \Omega_\tau^s(\varphi) \right\| ds \\ &+ \sum_{\tau \le t_i(\varphi) < T} K e^{\mu(t_i(\varphi)-\tau)} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B} \right\| \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\| \\ &+ \sum_{\tau \le t_i(\varphi) < t} K\varepsilon_B e^{\mu(t_i(\varphi)-\tau)} \left\| \Omega_\tau^{t_i(\varphi)}(\varphi) \right\|. \end{aligned} \tag{3.14}$$

Using the Gronwall-Bellman inequality for piecewise continuous functions [4], we obtain the estimate for the fundamental matrix $\Omega_\tau^t(\varphi)$ of the system (3.6)

$$\left\| \Omega_\tau^t(\varphi) \right\| \le K_1 e^{-(\mu - K\varepsilon_A - p\ln(1+K\varepsilon_B))(t-\tau)}, \tag{3.15}$$

where

$$K_1 = K + \int_{\tau}^{T} K e^{\mu(s-\tau)} \left\| A(\varphi_s(\varphi)) - \widetilde{A} \right\| \left\| \Omega_{\tau}^{s}(\varphi) \right\| ds + \sum_{\tau \le t_i(\varphi) < T} K e^{\mu(t_i(\varphi)-\tau)} \left\| B(\varphi_{t_i(\varphi)}(\varphi)) - \widetilde{B} \right\| \left\| \Omega_{\tau}^{t_i(\varphi)}(\varphi) \right\|. \tag{3.16}$$

Choosing ε_A and ε_B so that $\mu > K\varepsilon_A + p\ln(1 + K\varepsilon_B)$, the following estimate holds

$$\left\| \Omega_{\tau}^{t}(\varphi) \right\| \le K_1 e^{-\gamma_1(t-\tau)} \tag{3.17}$$

for all $t \ge \tau$ and some $K_1 \ge 1$, $\gamma_1 > 0$.

Estimate (3.17) is a sufficient condition for the existence and asymptotic stability of an invariant set of system (2.1). Indeed, it is easy to verify that invariant set $x = u(\varphi)$ of the system (2.1) may be represented as

$$u(\varphi) = \int_{-\infty}^{0} \Omega_{\tau}^{0}(\varphi) f(\varphi_{\tau}(\varphi)) d\tau + \sum_{t_i(\varphi) < 0} \Omega_{t_i(\varphi)}^{0}(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)). \tag{3.18}$$

The integral and the sum from (3.18) converge since inequality (3.17) holds and limit (3.4) exists. Utilizing the properties (3.7) of the matrix $\Omega_{\tau}^{t}(\varphi)$ (2.4), and (2.6), one can show that the function $u(\varphi_t(\varphi))$ satisfies the equation

$$\frac{dx}{dt} = A(\varphi_t(\varphi))x + f(\varphi_t(\varphi)) \tag{3.19}$$

for $t \ne t_i(\varphi)$ and has discontinuities $B(\varphi_{t_i(\varphi)}(\varphi))u(\varphi_{t_i(\varphi)}(\varphi)) + g(\varphi_{t_i(\varphi)}(\varphi))$ at the points $t = t_i(\varphi)$. It means that the function $x(t,\varphi) = u(\varphi_t(\varphi))$ is a solution of the system (2.7). Hence, $u(\varphi)$ defines the invariant set of system (2.1).

Let us prove the asymptotic stability of the invariant set. Let $x = x(t,\varphi)$ be an arbitrary solutions of the system (2.7), and $x^* = u(\varphi_t(\varphi))$ is the solution that belongs to the invariant set. The difference of these solutions admits the representation

$$x(t,\varphi) - u(\varphi_t(\varphi)) = \Omega_0^{t}(\varphi)(x(0,\varphi) - u(\varphi)). \tag{3.20}$$

Taking into account estimate (3.17), the following limit exists

$$\lim_{t\to\infty} \left\| x(t,\varphi) - u(\varphi_t(\varphi)) \right\| = 0. \tag{3.21}$$

It proves the asymptotic stability of the invariant set $x = u(\varphi)$. □

4. Perturbation Theory

Let us show that small perturbations of the right-hand side of the system (2.1) do not ruin the invariant set. Let $A_1(\varphi)$ and $B_1(\varphi)$ be continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ square matrices. Consider the perturbed system

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= (A(\varphi) + A_1(\varphi))x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi\in\Gamma} &= (B(\varphi) + B_1(\varphi))x + g(\varphi). \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let the moments of impulsive perturbations $\{t_i(\varphi)\}$ be such that uniformly with respect to $t \in \mathbb{R}$, there exists a finite limit*

$$\lim_{\tilde{T}\to\infty} \frac{i\left(t, t + \tilde{T}\right)}{\tilde{T}} = p \tag{4.2}$$

and the following inequality holds

$$\gamma + p \ln \alpha < 0. \tag{4.3}$$

Then there exist sufficiently small constants $a_1 > 0$ and $b_1 > 0$ such that for any continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ functions $A_1(\varphi)$ and $B_1(\varphi)$ such that

$$\begin{aligned} \max_{\varphi\in\mathcal{T}^m} \|A_1(\varphi)\| &\le a_1, \\ \max_{\varphi\in\mathcal{T}^m} \|B_1(\varphi)\| &\le b_1, \end{aligned} \tag{4.4}$$

system (4.1) *has an asymptotically stable invariant set.*

Proof. The constants a_1 and b_1 exist since the matrices $A_1(\varphi)$ and $B_1(\varphi)$ are continuous functions defined in the torus $\mathcal{T}^m$, which is a compact manifold.

Consider the impulsive system that corresponds to system(4.1)

$$\begin{aligned} \frac{dx}{dt} &= A(\varphi_t(\varphi))x + A_1(\varphi_t(\varphi))x, \quad t \ne t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= B(\varphi_{t_i(\varphi)}(\varphi))x + B_1(\varphi_{t_i(\varphi)}(\varphi))x \end{aligned} \tag{4.5}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter. The fundamental matrix $\Psi_\tau^t(\varphi)$ of the system (4.5) may be represented in the following way

$$\Psi_\tau^t(\varphi) = \Omega_\tau^t(\varphi) + \int_\tau^t \Omega_s^t(\varphi) A_1(\varphi_s(\varphi)) \Psi_s^t(\varphi) ds + \sum_{\tau \le t_i(\varphi) < t} \Omega_{t_i(\varphi)}^t(\varphi) B_1(\varphi_{t_i(\varphi)}(\varphi)) \Psi_\tau^{t_i(\varphi)}(\varphi), \tag{4.6}$$

where $\Omega_\tau^t(\varphi)$ is the fundamental matrix of the system (3.6). Then taking estimate (3.17) into account,

$$e^{\gamma_1(t-\tau)} \|\Psi_\tau^t(\varphi)\| \le K_1 + \int_\tau^t K_1 a_1 e^{\gamma_1(s-\tau)} \|\Psi_\tau^s(\varphi)\| ds + \sum_{\tau \le t_i(\varphi) < t} K_1 b_1 e^{\gamma_1(t_i(\varphi)-\tau)} \left\|\Psi_\tau^{t_i(\varphi)}(\varphi)\right\|. \tag{4.7}$$

Using the Gronwall-Bellman inequality for piecewise continuous functions, we obtain the estimate for the fundamental matrix $\Psi_\tau^t(\varphi)$ of the system (4.5)

$$\|\Psi_\tau^t(\varphi)\| \le K_1 e^{-(\gamma_1 - K_1 a_1 - p \ln(1+K_1 b_1))(t-\tau)}. \tag{4.8}$$

Let the constants a_1 and b_1 be such that $\gamma_1 > K_1 a_1 + p \ln(1 + K_1 b_1)$. Hence, the matrix $\Psi_\tau^t(\varphi)$ satisfies the estimate

$$\|\Psi_\tau^t(\varphi)\| \le K_2 e^{-\gamma_2(t-\tau)} \tag{4.9}$$

for all $t \ge \tau$ and some $K_2 \ge 1$, $\gamma_2 > 0$. As in Theorem 3.1, from estimate (4.9), we conclude that the system (4.1) has an asymptotically stable invariant set $x = u(\varphi)$, which admits the representation

$$u(\varphi) = \int_{-\infty}^0 \Psi_\tau^0(\varphi) f(\varphi_\tau(\varphi)) d\tau + \sum_{t_i(\varphi) < 0} \Psi_{t_i(\varphi)}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)). \tag{4.10}$$

□

Consider the nonlinear system of differential equations with impulsive perturbations of the form

$$\begin{aligned} \frac{d\varphi}{dt} &= a(\varphi), \\ \frac{dx}{dt} &= F(\varphi, x), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} &= I(\varphi, x), \end{aligned} \tag{4.11}$$

where $\varphi \in \mathcal{T}^m$, $x \in \overline{J}_h$, $a(\varphi)$ is a continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ vector function and satisfies Lipschitz conditions (2.2); $F(\varphi, x)$ and $I(\varphi, x)$ are continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \ldots, m$ functions that have continuous partial derivatives with respect to x up to the second order inclusively. Taking these assumptions into account, system (4.11) may be rewritten in the following form:

$$\begin{gathered}\frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A_0(\varphi)x + A_1(\varphi, x)x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} = B_0(\varphi)x + B_1(\varphi, x)x + g(\varphi),\end{gathered} \tag{4.12}$$

where

$$A(\varphi, x) = \int_0^1 \frac{\partial F(\varphi, \tau x)}{\partial(\tau x)} d\tau, \qquad B(\varphi, x) = \int_0^1 \frac{\partial I(\varphi, \tau x)}{\partial(\tau x)} d\tau, \tag{4.13}$$

$A_0(\varphi) = A(\varphi, 0)$, $A_1(\varphi, x) = A(\varphi, x) - A(\varphi, 0)$, $B_0(\varphi) = B(\varphi, 0)$, $B_1(\varphi, x) = B(\varphi, x) - B(\varphi, 0)$, $f(\varphi) = F(\varphi, 0)$, and $g(\varphi) = I(\varphi, 0)$. We assume that the matrices $A_0(\varphi)$ and $B_0(\varphi)$ are constant in the domain Ω:

$$\begin{gathered}A_0(\varphi)\big|_{\varphi \in \Omega} = \widetilde{A}, \\ B_0(\varphi)\big|_{\varphi \in \Omega} = \widetilde{B}\end{gathered} \tag{4.14}$$

and the inequality $\gamma + p \ln \alpha < 0$ holds.

We will construct the invariant set of system (4.12) using an iteration method proposed in [8]. As initial invariant set M_0, we consider the set $x = 0$, as M_k—the invariant set of the system

$$\begin{gathered}\frac{d\varphi}{dt} = a(\varphi), \\ \frac{dx}{dt} = A_0(\varphi)x + A_1(\varphi, u_{k-1}(\varphi))x + f(\varphi), \quad \varphi \notin \Gamma, \\ \Delta x|_{\varphi \in \Gamma} = B_0(\varphi)x + B_1(\varphi, u_{k-1}(\varphi))x + g(\varphi),\end{gathered} \tag{4.15}$$

where $x = u_{k-1}(\varphi)$ is the invariant set on $(k-1)$-step.

Using Theorem 4.1, the invariant set $x = u_k(\varphi)$, $k = 1, 2, \ldots$ may be represented as

$$u_k(\varphi) = \int_{-\infty}^0 \Psi_\tau^0(\varphi, k) f(\varphi_\tau(\varphi)) d\tau + \sum_{t_i(\varphi) < 0} \Psi_{t_i(\varphi)+0}^0(\varphi, k) g(\varphi_{t_i(\varphi)}(\varphi)), \tag{4.16}$$

where $\Psi_\tau^t(\varphi, k)$ is the fundamental matrix of the homogeneous system

$$\begin{aligned}\frac{dx}{dt} &= \left(A_0(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right)x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= \left(B_0(\varphi_t(\varphi)) + B_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right)x\end{aligned} \tag{4.17}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter and satisfies the estimate

$$\left\|\Psi_\tau^t(\varphi, k)\right\| \leq K_2 e^{-\gamma_2(t-\tau)} \tag{4.18}$$

for all $t \geq \tau$ and some $K_2 \geq 1$, $\gamma_2 > 0$ only if

$$\begin{aligned}\max_{\varphi \in \mathcal{T}^m}\left\|A_1(\varphi, u_{k-1}(\varphi))\right\| &\leq a_1, \\ \max_{\varphi \in \mathcal{T}^m}\left\|B_1(\varphi, u_{k-1}(\varphi))\right\| &\leq b_1.\end{aligned} \tag{4.19}$$

Let us prove that the invariant sets $x = u_k(\varphi)$ belong to the domain $\overline{J}_h$. Denote by

$$\begin{aligned}\max_{\varphi \in \mathcal{T}^m}\|f(\varphi)\| &\leq M_f, \\ \max_{\varphi \in \mathcal{T}^m}\|g(\varphi)\| &\leq M_g.\end{aligned} \tag{4.20}$$

Since the torus $\mathcal{T}^m$ is a compact manifold, such constants M_f and M_g exist. Analogously to [4], using the representation (4.16) and estimate (4.18), we obtain that

$$\|u_k(\varphi)\| \leq \frac{K_2}{\gamma_2} M_f + \frac{K_2}{1 - e^{-\gamma_2 \theta_1}} M_g, \tag{4.21}$$

where θ_1 is a minimum gap between moments of impulsive actions. Condition (3.4) guarantees that such constant θ_1 exists. Assume that the constants K_2 and γ_2 are such that $\|u(\varphi)\| \leq h$.

Let us obtain the conditions for the convergence of the sequence $\{u_k(\varphi)\}$. For this purpose, we estimate the difference $w_{k+1}(\varphi) = u_{k+1}(\varphi) - u_k(\varphi)$ and take into account that the functions $u_k(\varphi_t(\varphi))$ satisfy the relations

$$\begin{aligned}\frac{d}{dt}u_k(\varphi_t(\varphi)) &= \left(A_0(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right) \\ &\quad \times u_k(\varphi_t(\varphi)) + f(\varphi_t(\varphi)), \quad t \neq t_i(\varphi), \\ \Delta u_k(\varphi_t(\varphi))\big|_{t=t_i(\varphi)} &= \left(B_0(\varphi_t(\varphi)) + B_1(\varphi_t(\varphi), u_{k-1}(\varphi_t(\varphi)))\right) \\ &\quad \times u_k(\varphi_t(\varphi)) + g(\varphi_t(\varphi))\end{aligned} \tag{4.22}$$

for all $\varphi \in \mathcal{T}^m$, $k = 1, 2, \ldots$. Hence, the difference $w_{k+1}(\varphi) = u_{k+1}(\varphi) - u_k(\varphi)$ is the invariant set of the linear impulsive system

$$\begin{gathered}
\frac{d\varphi}{dt} = a(\varphi), \\
\frac{dx}{dt} = \left(A_0(\varphi) + A_1(\varphi, u_k(\varphi))\right)x + \left(A_1(\varphi, u_k(\varphi)) - A_1(\varphi, u_{k-1}(\varphi))\right)u_k(\varphi), \quad \varphi \notin \Gamma, \\
\Delta x|_{\varphi\in\Gamma} = \left(B_0(\varphi) + B_1(\varphi, u_k(\varphi))\right)x + \left(B_1(\varphi, u_k(\varphi)) - B_1(\varphi, u_{k-1}(\varphi))\right)u_k(\varphi).
\end{gathered} \tag{4.23}$$

Then, taking (4.21) into account,

$$\begin{aligned}
\max_{\varphi\in\mathcal{T}^m}\|u_{k+1}(\varphi) - u_k(\varphi)\| \le{}& \frac{K_2}{\gamma_2}\|A_1(\varphi, u_k(\varphi)) - A_1(\varphi, u_{k-1}(\varphi))\|\|u_k(\varphi)\| \\
&+ \frac{K_2}{1 - e^{-\gamma_2\theta}}\|B_1(\varphi, u_k(\varphi)) - B_1(\varphi, u_{k-1}(\varphi))\|\|u_k(\varphi)\|.
\end{aligned} \tag{4.24}$$

Let the functions $A_1(\varphi, x)$ and $B_1(\varphi, x)$ satisfy the Lipschitz condition with constants L_A and L_B correspondingly. Then

$$\begin{aligned}
\max_{\varphi\in\mathcal{T}^m}\|u_{k+1}(\varphi) - u_k(\varphi)\| &\le \frac{K_2}{\gamma_2}L_A h\|u_k(\varphi) - u_{k-1}(\varphi)\| + \frac{K_2}{1 - e^{-\gamma_2\theta}}L_B h\|u_k(\varphi) - u_{k-1}(\varphi)\| \\
&= \left(\frac{K_2 h}{\gamma_2}L_A + \frac{K_2 h}{1 - e^{-\gamma_2\theta}}L_B\right)\|u_k(\varphi) - u_{k-1}(\varphi)\|.
\end{aligned} \tag{4.25}$$

Assuming that the constants L_A and L_B are so small that

$$\frac{K_2 h}{\gamma_2}L_A + \frac{K_2 h}{1 - e^{-\gamma_2\theta}}L_B < 1, \tag{4.26}$$

we conclude that the sequence $\{u_k(\varphi)\}$ converges uniformly with respect to $\varphi \in \mathcal{T}^m$ and

$$\lim_{k\to\infty} u_k(\varphi) = u(\varphi). \tag{4.27}$$

Thus, the invariant set $x = u(\varphi)$ admits the representation

$$u(\varphi) = \int_{-\infty}^{0} \Psi_\tau^0(\varphi) f(\varphi_\tau(\varphi))d\tau + \sum_{t_i(\varphi)<0} \Psi_{t_i(\varphi)}^0(\varphi) g(\varphi_{t_i(\varphi)}(\varphi)), \tag{4.28}$$

where $\Psi_\tau^t(\varphi)$ is the fundamental matrix of the homogeneous system

$$\begin{aligned}\frac{dx}{dt} &= (A(\varphi_t(\varphi)) + A_1(\varphi_t(\varphi), u(\varphi_t(\varphi))))x, \quad t \neq t_i(\varphi), \\ \Delta x|_{t=t_i(\varphi)} &= (B(\varphi_t(\varphi)) + B(\varphi_t(\varphi), u(\varphi_t(\varphi))))x\end{aligned} \tag{4.29}$$

that depends on $\varphi \in \mathcal{T}^m$ as a parameter and satisfies the estimation

$$\left\|\Psi_\tau^t(\varphi)\right\| \leq K_2 e^{-\gamma_2(t-\tau)} \tag{4.30}$$

for all $t \geq \tau$ and some $K_2 \geq 1$, $\gamma_2 > 0$. The following assertion has been proved.

Theorem 4.2. *Let the matrices $A_0(\varphi)$ and $B_0(\varphi)$ be constant in the domain Ω:*

$$\begin{aligned}A_0(\varphi)\big|_{\varphi\in\Omega} &= \widetilde{A}, \\ B_0(\varphi)\big|_{\varphi\in\Omega} &= \widetilde{B},\end{aligned} \tag{4.31}$$

uniformly with respect to $t \in \mathbb{R}$, there exists a finite limit

$$\lim_{\widetilde{T}\to\infty} \frac{i\left(t, t+\widetilde{T}\right)}{\widetilde{T}} = p \tag{4.32}$$

and the following inequality holds

$$\gamma + p \ln \alpha < 0, \tag{4.33}$$

where

$$\begin{aligned}\gamma &= \max_{j=1,\dots,n} \operatorname{Re} \lambda_j\left(\widetilde{A}\right), \\ \alpha^2 &= \max_{j=1,\dots,n} \lambda_j\left(\left(E+\widetilde{B}\right)^T\left(E+\widetilde{B}\right)\right).\end{aligned} \tag{4.34}$$

Then there exist sufficiently small constants a_1 and b_1 and sufficiently small Lipschitz constants L_A and L_B such that for any continuous 2π-periodic with respect to each of the components φ_v, $v = 1, \dots, m$ matrices $F(\varphi, x)$ and $I(\varphi, x)$, which have continuous partial derivatives with respect to x up to the second order inclusively, such that

$$\begin{aligned}\max_{\varphi\in\mathcal{T}^m, x\in\bar{J}_h} \left\|A_1(\varphi, x)\right\| &\leq a_1, \\ \max_{\varphi\in\mathcal{T}^m, x\in\bar{J}_h} \left\|B_1(\varphi, x)\right\| &\leq b_1\end{aligned} \tag{4.35}$$

and for any $x', x'' \in \overline{J}_h$

$$\begin{aligned} \|A_1(\varphi, x') - A_1(\varphi, x'')\| &\le L_A \|x' - x''\|, \\ \|B_1(\varphi, x') - B_1(\varphi, x'')\| &\le L_B \|x' - x''\|, \end{aligned} \tag{4.36}$$

system (4.11) *has an asymptotically stable invariant set.*

5. Conclusion

In summary, we have obtained sufficient conditions for the existence and asymptotic stability of invariant sets of a linear impulsive system of differential equations defined in $\mathcal{T}^m \times E^n$ that has specific properties in the ω-limit set Ω of the trajectories $\varphi_t(\varphi)$. We have proved that it is sufficient to impose some restrictions on system (2.1) only in the domain Ω to guarantee the existence and asymptotic stability of the invariant set.

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Research Article

New Stability Conditions for Linear Differential Equations with Several Delays

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New explicit conditions of asymptotic and exponential stability are obtained for the scalar nonautonomous linear delay differential equation $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0$ with measurable delays and coefficients. These results are compared to known stability tests.

1. Introduction

In this paper we continue the study of stability properties for the scalar linear differential equation with several delays and an arbitrary number of positive and negative coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

which was begun in [1–3]. Equation (1.1) and its special cases were intensively studied, for example, in [4–21]. In [2] we gave a review of stability tests obtained in these papers.

In almost all papers on stability of delay-differential equations coefficients and delays are assumed to be continuous, which is essentially used in the proofs of main results. In real-world problems, for example, in biological and ecological models with seasonal fluctuations of parameters and in economical models with investments, parameters of differential equations are not necessarily continuous.

There are also some mathematical reasons to consider differential equations without the assumption that parameters are continuous functions. One of the main methods to investigate impulsive differential equations is their reduction to a nonimpulsive differential equation with discontinuous coefficients. Similarly, difference equations can sometimes be reduced to the similar problems for delay-differential equations with discontinuous piecewise constant delays.

In paper [1] some problems for differential equations with several delays were reduced to similar problems for equations with one delay which generally is not continuous.

One of the purposes of this paper is to extend and partially improve most popular stability results for linear delay equations with continuous coefficients and delays to equations with measurable parameters.

Another purpose is to generalize some results of [1–3]. In these papers, the sum of coefficients was supposed to be separated from zero and delays were assumed to be bounded. So the results of these papers are not applicable, for example, to the following equations:

$$\begin{gathered}\dot{x}(t)+|\sin t|x(t-\tau)=0,\\ \dot{x}(t)+\alpha(|\sin t|-\sin t)x(t-\tau)=0,\\ \dot{x}(t)+\frac{1}{t}x(t)+\frac{\alpha}{t}x\left(\frac{t}{2}\right)=0.\end{gathered}\tag{1.2}$$

In most results of the present paper these restrictions are omitted, so we can consider all the equations mentioned above. Besides, necessary stability conditions (probably for the first time) are obtained for (1.1) with nonnegative coefficients and bounded delays. In particular, if this equation is exponentially stable then the ordinary differential equation

$$\dot{x}(t)+\sum_{k=1}^{m}a_k(t)x(t)=0\tag{1.3}$$

is also exponentially stable.

2. Preliminaries

We consider the scalar linear equation with several delays (1.1) for $t \ge t_0$ with the initial conditions (for any $t_0 \ge 0$)

$$x(t)=\varphi(t),\quad t<t_0,\qquad x(t_0)=x_0,\tag{2.1}$$

and under the following assumptions:

(a1) $a_k(t)$ are Lebesgue measurable essentially bounded on $[0,\infty)$ functions;

(a2) $h_k(t)$ are Lebesgue measurable functions,

$$h_k(t)\le t,\qquad \limsup_{t\to\infty} h_k(t)=\infty;\tag{2.2}$$

(a3) $\varphi:(-\infty,t_0)\to R$ is a Borel measurable bounded function.

We assume conditions (a1)–(a3) hold for all equations throughout the paper.

Definition 2.1. A locally absolutely continuous for $t \geq t_0$ function $x : R \to R$ is called *a solution* of problem (1.1), (2.1) if it satisfies (1.1) for almost all $t \in [t_0, \infty)$ and the equalities (2.1) for $t \leq t_0$.

Below we present a solution representation formula for the nonhomogeneous equation with locally Lebesgue integrable right-hand side $f(t)$:

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = f(t), \quad t \geq t_0. \tag{2.3}$$

Definition 2.2. A solution $X(t,s)$ of the problem

$$\begin{gathered}\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \quad t \geq s \geq 0, \\ x(t) = 0, \quad t < s, \qquad x(s) = 1,\end{gathered} \tag{2.4}$$

is called *the fundamental function* of (1.1).

Lemma 2.3 (see [22, 23]). *Suppose conditions (a1)–(a3) hold. Then the solution of* (2.3), (2.1) *has the following form*

$$x(t) = X(t,t_0)x_0 - \int_{t_0}^{t} X(t,s)\sum_{k=1}^{m} a_k(s)\varphi(h_k(s))ds + \int_{t_0}^{t} X(t,s)f(s)ds, \tag{2.5}$$

where $\varphi(t) = 0,\ t \geq t_0$.

Definition 2.4 (see [22]). Equation (1.1) is *stable* if for any initial point t_0 and number $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality $\sup_{t<t_0}|\varphi(t)| + |x(t_0)| < \delta$ implies $|x(t)| < \varepsilon$, $t \geq t_0$, for the solution of problem (1.1), (2.1).

Equation (1.1) is *asymptotically stable* if it is stable and all solutions of (1.1)-(2.1) for any initial point t_0 tend to zero as $t \to \infty$.

In particular, (1.1) is asymptotically stable if the fundamental function is uniformly bounded: $|X(t,s)| \leq K$, $t \geq s \geq 0$ and all solutions tend to zero as $t \to \infty$.

We apply in this paper only these two conditions of asymptotic stability.

Definition 2.5. Equation (1.1) is *(uniformly) exponentially stable*, if there exist $M > 0$, $\mu > 0$ such that the solution of problem (1.1), (2.1) has the estimate

$$|x(t)| \leq Me^{-\mu(t-t_0)}\left(|x(t_0)| + \sup_{t<t_0}\left|\varphi(t)\right|\right), \quad t \geq t_0, \tag{2.6}$$

where M and μ do not depend on t_0.

Definition 2.6. The fundamental function $X(t,s)$ of (1.1) *has an exponential estimation* if there exist $K > 0$, $\lambda > 0$ such that

$$|X(t,s)| \le K e^{-\lambda(t-s)}, \quad t \ge s \ge 0. \tag{2.7}$$

For the linear (1.1) with bounded delays the last two definitions are equivalent. For unbounded delays estimation (2.7) implies asymptotic stability of (1.1).

Under our assumptions the exponential stability does not depend on values of equation parameters on any finite interval.

Lemma 2.7 (see [24, 25]). *Suppose $a_k(t) \ge 0$. If*

$$\int_{\max\{h(t),t_0\}}^{t} \sum_{i=1}^{m} a_i(s)ds \le \frac{1}{e}, \quad h(t) = \min_k\{h_k(t)\}, \quad t \ge t_0, \tag{2.8}$$

or there exists $\lambda > 0$, such that

$$\lambda \ge \sum_{k=1}^{m} A_k e^{\lambda\sigma_k}, \tag{2.9}$$

where

$$0 \le a_k(t) \le A_k, \quad t - h_k(t) \le \sigma_k, \quad t \ge t_0, \tag{2.10}$$

then $X(t,s) > 0$, $t \ge s \ge t_0$, where $X(t,s)$ is the fundamental function of (1.1).

Lemma 2.8 (see [3]). *Suppose $a_k(t) \ge 0$,*

$$\liminf_{t\to\infty} \sum_{k=1}^{m} a_k(t) > 0, \tag{2.11}$$

$$\limsup_{t\to\infty} (t - h_k(t)) < \infty, \quad k = 1, \ldots, m, \tag{2.12}$$

and there exists $r(t) \le t$ such that for sufficiently large t

$$\int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds \le \frac{1}{e}. \tag{2.13}$$

If

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{r(t)} \sum_{i=1}^{m} a_i(s)ds \right| < 1, \tag{2.14}$$

then (1.1) *is exponentially stable.*

Lemma 2.9 (see [3]). *Suppose* (2.12) *holds and there exists a set of indices* $I \subset \{1,\dots,m\}$*, such that* $a_k(t) \geq 0$, $k \in I$,

$$\liminf_{t\to\infty} \sum_{k\in I} a_k(t) > 0, \tag{2.15}$$

and the fundamental function of the equation

$$\dot{x}(t) + \sum_{k\in I} a_k(t)x(h_k(t)) = 0 \tag{2.16}$$

is eventually positive. If

$$\limsup_{t\to\infty} \frac{\sum_{k\notin I}|a_k(t)|}{\sum_{k\in I} a_k(t)} < 1, \tag{2.17}$$

then (1.1) *is exponentially stable.*

The following lemma was obtained in [26, Corollary 2], see also [27].

Lemma 2.10. *Suppose for* (1.1) *condition* (2.12) *holds and this equation is exponentially stable. If*

$$\int_0^\infty \sum_{k=1}^n |b_k(s)|ds < \infty, \qquad \limsup_{t\to\infty}\left(t - g_k(t)\right) < \infty, \quad g_k(t) \leq t, \tag{2.18}$$

then the equation

$$\dot{x}(t) + \sum_{k=1}^m a_k(t)x(h_k(t)) + \sum_{k=1}^n b_k(t)x\left(g_k(t)\right) = 0 \tag{2.19}$$

is exponentially stable.

The following elementary result will be used in the paper.

Lemma 2.11. *The ordinary differential equation*

$$\dot{x}(t) + a(t)x(t) = 0 \tag{2.20}$$

is exponentially stable if and only if there exists $R > 0$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+R} a(s)ds > 0. \tag{2.21}$$

The following example illustrates that a stronger than (2.21) sufficient condition

$$\liminf_{t,s\to\infty} \frac{1}{t-s}\int_s^t a(\tau)d\tau > 0 \tag{2.22}$$

is not necessary for the exponential stability of the ordinary differential equation (2.20).

Example 2.12. Consider the equation

$$\dot{x}(t) + a(t)x(t) = 0, \quad \text{where } a(t) = \begin{cases} 1, & t \in [2n, 2n+1), \\ 0, & t \in [2n+1, 2n+2), \end{cases} \quad n = 0, 1, 2, \ldots \tag{2.23}$$

Then lim inf in (2.22) equals zero, but $|X(t,s)| < ee^{-0.5(t-s)}$, so the equation is exponentially stable. Moreover, if we consider lim inf in (2.22) under the condition $t - s \geq R$, then it is still zero for any $R \leq 1$.

3. Main Results

Lemma 3.1. *Suppose* $a_k(t) \geq 0$, (2.11), (2.12) *hold and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < 1 + \frac{1}{e}. \tag{3.1}$$

Then (1.1) *is exponentially stable.*

Proof. By (2.11) there exists function $r(t) \leq t$ such that for sufficiently large t

$$\int_{r(t)}^{t} \sum_{k=1}^{m} a_k(s)ds = \frac{1}{e}. \tag{3.2}$$

For this function condition (2.14) has the form

$$\begin{aligned} &\limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds - \int_{r(t)}^{t} \sum_{i=1}^{m} a_i(s)ds \right| \\ &\quad = \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \left| \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds - \frac{1}{e} \right| < 1. \end{aligned} \tag{3.3}$$

The latter inequality follows from (3.1). The reference to Lemma 2.8 completes the proof. □

Corollary 3.2. *Suppose* $a_k(t) \geq 0$, (2.11), (2.12) *hold and*

$$\limsup_{t\to\infty} \int_{\min_k\{h_k(t)\}}^{t} \sum_{i=1}^{m} a_i(s)ds < 1 + \frac{1}{e}. \tag{3.4}$$

Then (1.1) *is exponentially stable.*

The following theorem contains stability conditions for equations with unbounded delays. We also omit condition (2.11) in Lemma 3.1.

We recall that $b(t) > 0$ in the space of Lebesgue measurable essentially bounded functions means $b(t) \geq 0$ and $b(t) \not\equiv 0$ almost everywhere.

Theorem 3.3. *Suppose* $a_k(t) \geq 0$, *condition* (3.1) *holds,* $\sum_{k=1}^{m} a_k(t) > 0$ *and*

$$\int_0^\infty \sum_{k=1}^{m} a_k(t)dt = \infty, \qquad \limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < \infty. \tag{3.5}$$

Then (1.1) *is asymptotically stable.*

If in addition there exists $R > 0$ *such that*

$$\liminf_{t\to\infty} \int_t^{t+R} \sum_{k=1}^{m} a_k(\tau)d\tau > 0 \tag{3.6}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If condition (2.12) *also holds then* (1.1) *is exponentially stable.*

Proof. Let $s = p(t) := \int_0^t \sum_{k=1}^{m} a_k(\tau)d\tau$, $y(s) = x(t)$, where $p(t)$ is a strictly increasing function. Then $x(h_k(t)) = y(l_k(s))$, $l_k(s) \leq s$, $l_k(s) = \int_0^{h_k(t)} \sum_{i=1}^{m} a_i(\tau)d\tau$ and (1.1) can be rewritten in the form

$$\dot{y}(s) + \sum_{k=1}^{m} b_k(s)y(l_k(s)) = 0, \tag{3.7}$$

where $b_k(s) = a_k(t)/\sum_{i=1}^{m} a_i(t)$, $s - l_k(s) = \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(\tau)d\tau$. Since $\sum_{k=1}^{m} b_k(s) = 1$ and $\limsup_{s\to\infty}(s - l_k(s)) < \infty$, then Lemma 3.1 can be applied to (3.7). We have

$$\begin{aligned}
&\limsup_{s\to\infty} \sum_{k=1}^{m} \frac{b_k(s)}{\sum_{i=1}^{m} b_i(s)} \int_{l_k(s)}^{s} \sum_{i=1}^{m} b_i(\tau)d\tau \\
&\quad= \limsup_{s\to\infty} \sum_{k=1}^{m} b_k(s)(s - l_k(s)) \\
&\quad= \limsup_{t\to\infty} \sum_{k=1}^{m} \frac{a_k(t)}{\sum_{i=1}^{m} a_i(t)} \int_{h_k(t)}^{t} \sum_{i=1}^{m} a_i(s)ds < 1 + \frac{1}{e}.
\end{aligned} \tag{3.8}$$

By Lemma 3.1, (3.7) is exponentially stable. Due to the first equality in (3.5) $t \to \infty$ implies $s \to \infty$. Hence $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$.

Equation (3.7) is exponentially stable, thus the fundamental function $Y(u,v)$ of (3.7) has an exponential estimation

$$|Y(u,v)| \le Ke^{-\lambda(u-v)}, \quad u \ge v \ge 0, \tag{3.9}$$

with $K > 0$, $\lambda > 0$. Since $X(t,s) = Y(\int_0^t \sum_{k=1}^m a_k(\tau)d\tau, \int_0^s \sum_{k=1}^m a_k(\tau)d\tau)$, where $X(t,s)$ is the fundamental function of (1.1), then (3.9) yields

$$|X(t,s)| \le K \exp\left\{-\lambda \int_s^t \sum_{k=1}^m a_k(\tau)d\tau\right\}. \tag{3.10}$$

Hence $|X(t,s)| \le K$, $t \ge s \ge 0$, which together with $\lim_{t\to\infty} x(t) = 0$ yields that (1.1) is asymptotically stable.

Suppose now that (3.6) holds. Without loss of generality we can assume that for some $R > 0$, $\alpha > 0$ we have

$$\int_t^{t+R} \sum_{k=1}^m a_k(\tau)d\tau \ge \alpha > 0, \quad t \ge s \ge 0. \tag{3.11}$$

Hence

$$\exp\left\{-\lambda \int_s^t \sum_{k=1}^m a_k(\tau)d\tau\right\} \le \exp\left\{\lambda R \sup_{t\ge 0} \sum_{k=1}^m a_k(t)\right\} e^{-\lambda\alpha(t-s)/R}. \tag{3.12}$$

Thus, condition (3.6) implies the exponential estimate for $X(t,s)$.

The last statement of the theorem is evident. □

Remark 3.4. The substitution $s = p(t) := \int_0^t \sum_{k=1}^m a_k(\tau)d\tau$, $y(s) = x(t)$ was first used in [28].

Note that in [10, Lemma 2] this idea was extended to a more general equation

$$\dot{x}(t) + \int_{t_0}^t x(s)d_s r(t,s) = 0. \tag{3.13}$$

The ideas of [10] allow to generalize the results of the present paper to equations with a distributed delay.

Corollary 3.5. *Suppose* $a_k(t) \ge 0$, $\sum_{k=1}^m a_k(t) \equiv \alpha > 0$, *condition* (2.12) *holds and*

$$\limsup_{t\to\infty} \sum_{k=1}^m a_k(t)(t - h_k(t)) < 1 + \frac{1}{e}. \tag{3.14}$$

Then (1.1) *is exponentially stable.*

Corollary 3.6. *Suppose* $a_k(t) = \alpha_k p(t)$, $\alpha_k > 0$, $p(t) > 0$, $\int_0^\infty p(t)dt = \infty$ *and*

$$\limsup_{t\to\infty} \sum_{k=1}^{m} \alpha_k \int_{h_k(t)}^{t} p(s)ds < 1 + \frac{1}{e}. \tag{3.15}$$

Then (1.1) *is asymptotically stable.*

If in addition there exists $R > 0$ *such that*

$$\liminf_{t\to\infty} \int_{t}^{t+R} p(\tau)d\tau > 0, \tag{3.16}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If also (2.12) *holds then* (1.1) *is exponentially stable.*

Remark 3.7. Let us note that similar results for (3.13) were obtained in [10], see Corollary 3.4 and remark after it, Theorem 4 and Corollaries 4.1 and 4.2 in [10], where an analogue of condition (3.16) was applied. This allows to extend the results of the present paper to equations with a distributed delay.

Corollary 3.8. *Suppose* $a(t) \geq 0$, $b(t) \geq 0$, $a(t) + b(t) > 0$,

$$\begin{gathered} \int_0^\infty (a(t) + b(t))dt = \infty, \qquad \limsup_{t\to\infty} \int_{h(t)}^{t} (a(s) + b(s))ds < \infty, \\ \limsup_{t\to\infty} \frac{b(t)}{a(t) + b(t)} \int_{h(t)}^{t} (a(s) + b(s))ds < 1 + \frac{1}{e}. \end{gathered} \tag{3.17}$$

Then the following equation is asymptotically stable

$$\dot{x}(t) + a(t)x(t) + b(t)x(h(t)) = 0. \tag{3.18}$$

If in addition there exists $R > 0$ *such that* $\liminf_{t\to\infty} \int_t^{t+R} (a(\tau) + b(\tau))d\tau > 0$ *then the fundamental function of* (3.18) *has an exponential estimation.*

If also $\limsup_{t\to\infty}(t - h(t)) < \infty$ *then* (3.18) *is exponentially stable.*

In the following theorem we will omit the condition $\sum_{k=1}^{m} a_k(t) > 0$ of Theorem 3.3.

Theorem 3.9. *Suppose* $a_k(t) \geq 0$, *condition* (3.4) *and the first inequality in* (3.5) *hold. Then* (1.1) *is asymptotically stable.*

If in addition (3.6) *holds then the fundamental function of* (1.1) *has an exponential estimation.*

If also (2.12) *holds then* (1.1) *is exponentially stable.*

Proof. For simplicity suppose that $m = 2$ and consider the equation

$$\dot{x}(t) + a(t)x(h(t)) + b(t)x\big(g(t)\big) = 0, \tag{3.19}$$

where $a(t) \geq 0$, $b(t) \geq 0$, $\int_0^\infty (a(s)+b(s))ds = \infty$ and there exist $t_0 \geq 0$, $\varepsilon > 0$ such that

$$\int_{\min\{h(t),g(t)\}}^t (a(s)+b(s))ds < 1 + \frac{1}{e} - \varepsilon, \quad t \geq t_0. \tag{3.20}$$

Let us find $t_1 \geq t_0$ such that $e^{-h(t)} < \varepsilon/4$, $e^{-g(t)} < \varepsilon/4$, $t \geq t_1$, such t_1 exists due to (a2). Then $\int_{\min\{h(t),g(t)\}}^t e^{-s}ds < \varepsilon/2$, $t \geq t_1$. Rewrite (3.19) in the form

$$\dot{x}(t) + \left(a(t)+e^{-t}\right)x(h(t)) + b(t)x\left(g(t)\right) - e^{-t}x(h(t)) = 0, \tag{3.21}$$

where $a(t)+b(t)+e^{-t} > 0$. After the substitution $s = \int_{t_1}^t (a(\tau)+b(\tau)+e^{-\tau})d\tau$, $y(s) = x(t)$, (3.21) has the form

$$\dot{y}(s) + \frac{a(t)+e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) + \frac{b(t)}{a(t)+b(t)+e^{-t}}y\left(p(s)\right) - \frac{e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) = 0, \tag{3.22}$$

where similar to the proof of Theorem 3.3

$$s - l(s) = \int_{h(t)}^t \left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau, \qquad s - p(s) = \int_{g(t)}^t \left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau. \tag{3.23}$$

First we will show that by Corollary 3.2 the equation

$$\dot{y}(s) + \frac{a(t)+e^{-t}}{a(t)+b(t)+e^{-t}}y(l(s)) + \frac{b(t)}{a(t)+b(t)+e^{-t}}y\left(p(s)\right) = 0 \tag{3.24}$$

is exponentially stable. Since $(a(t)+e^{-t})/(a(t)+b(t)+e^{-t}) + b(t)/(a(t)+b(t)+e^{-t}) = 1$, then (2.11) holds. Condition (3.20) implies (2.12). So we have to check only condition (3.4) where the sum under the integral is equal to 1. By (3.20), (3.23) we have

$$\begin{aligned} \int_{\min\{l(s),p(s)\}}^s 1ds = s - \min\{l(s),p(s)\}, \qquad s - l(s) &= \int_{h(t)}^t \left(a(\tau)+b(\tau)+e^{-\tau}\right)d\tau \\ &= \int_{h(t)}^t (a(\tau)+b(\tau))d\tau + \int_{h(t)}^t e^{-\tau}d\tau < 1 + \frac{1}{e} - \varepsilon + \frac{\varepsilon}{2} = 1 + \frac{1}{e} - \frac{\varepsilon}{2}, \quad t \geq t_1. \end{aligned} \tag{3.25}$$

The same calculations give $s - p(s) < 1 + (1/e) - \varepsilon/2$, thus condition (3.4) holds.

Hence (3.24) is exponentially stable.

We return now to (3.22), $t \geq t_1$. We have $ds = (a(t)+b(t)+e^{-t})dt$, then

$$\int_{t_1}^\infty \frac{e^{-t}}{a(t)+b(t)+e^{-t}}ds = \int_{t_1}^\infty \frac{e^{-t}}{a(t)+b(t)+e^{-t}}\left(a(t)+b(t)+e^{-t}\right)dt < \infty. \tag{3.26}$$

By Lemma 2.10, (3.22) is exponentially stable. Since $t \to \infty$ implies $s \to \infty$ then $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$, which completes the proof of the first part of the theorem. The rest of the proof is similar to the proof of Theorem 3.3. □

Corollary 3.10. *Suppose $a(t) \geq 0$, $\int_0^\infty a(t)dt = \infty$ and*

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < 1 + \frac{1}{e}. \tag{3.27}$$

Then the equation

$$\dot{x}(t) + a(t)x(h(t)) = 0 \tag{3.28}$$

is asymptotically stable. If in addition condition (2.21) *holds then the fundamental function of* (3.28) *has an exponential estimation. If also* $\limsup_{t\to\infty}(t - h(t)) < \infty$ *then* (3.28) *is exponentially stable.*

Now consider (1.1), where only some of coefficients are nonnegative.

Theorem 3.11. *Suppose there exists a set of indices $I \subset \{1, \ldots, m\}$ such that $a_k(t) \geq 0$, $k \in I$,*

$$\int_0^\infty \sum_{k\in I} a_k(t)dt = \infty, \quad \limsup_{t\to\infty} \int_{h_k(t)}^{t} \sum_{i\in I} a_i(s)ds < \infty, \quad k = 1, \ldots, m, \tag{3.29}$$

$$\sum_{k\notin I} |a_k(t)| = 0, \quad t \in E, \qquad \limsup_{t\to\infty, t\notin E} \frac{\sum_{k\notin I} |a_k(t)|}{\sum_{k\in I} a_k(t)} < 1, \quad \text{where } E = \left\{ t \geq 0, \sum_{k\in I} a_k(t) = 0 \right\}. \tag{3.30}$$

If the fundamental function $X_0(t,s)$ of (2.16) *is eventually positive then all solutions of* (1.1) *tend to zero as $t \to \infty$.*

If in addition there exists $R > 0$ such that

$$\liminf_{t\to\infty} \int_{t}^{t+R} \sum_{k\in I} a_k(\tau)d\tau > 0 \tag{3.31}$$

then the fundamental function of (1.1) *has an exponential estimation.*

If condition (2.12) *also holds then* (1.1) *is exponentially stable.*

Proof. Without loss of generality we can assume $X_0(t,s) > 0$, $t \geq s \geq 0$. Rewrite (1.1) in the form

$$\dot{x}(t) + \sum_{k\in I} a_k(t)x(h_k(t)) + \sum_{k\notin I} a_k(t)x(h_k(t)) = 0. \tag{3.32}$$

Suppose first that $\sum_{k\in I} a_k(t) \neq 0$. After the substitution $s = p(t) := \int_0^t \sum_{k\in I} a_k(\tau)d\tau$, $y(s) = x(t)$ we have $x(h_k(t)) = y(l_k(s))$, $l_k(s) \leq s$, $l_k(s) = \int_0^{h_k(t)} \sum_{i\in I} a_i(\tau)d\tau$, $k = 1,\dots,m$, and (1.1) can be rewritten in the form

$$\dot{y}(s) + \sum_{k=1}^{m} b_k(s)y(l_k(s)) = 0, \tag{3.33}$$

where $b_k(s) = a_k(t)/\sum_{i\in I} a_i(t)$. Denote by $Y_0(u,v)$ the fundamental function of the equation

$$\dot{y}(s) + \sum_{k\in I} b_k(s)y(l_k(s)) = 0. \tag{3.34}$$

We have

$$\begin{gathered} X_0(t,s) = Y_0\left(\int_0^t \sum_{k\in I} a_k(\tau)d\tau, \int_0^s \sum_{k\in I} a_k(\tau)d\tau\right), \\ Y_0(u,v) = X_0\left(p^{-1}(u), p^{-1}(v)\right) > 0, \quad u \geq v \geq 0. \end{gathered} \tag{3.35}$$

Let us check that other conditions of Lemma 2.9 hold for (3.33). Since $\sum_{k\in I} b_k(s) = 1$ then condition (2.15) is satisfied. In addition,

$$\limsup_{s\to\infty,\, p^{-1}(s)\notin E} \frac{\sum_{k\notin I}|b_k(s)|}{\sum_{k\in I} b_k(s)} = \limsup_{t\to\infty,\, t\notin E} \frac{\sum_{k\notin I}|a_k(t)|}{\sum_{k\in I} a_k(t)} < 1. \tag{3.36}$$

By Lemma 2.9, (3.33) is exponentially stable. Hence for any solution $x(t)$ of (1.1) we have $\lim_{t\to\infty} x(t) = \lim_{s\to\infty} y(s) = 0$. The end of the proof is similar to the proof of Theorem 3.9. In particular, to remove the condition $\sum_{k\in I} a_k(t) \neq 0$ we rewrite the equation by adding the term e^{-t} to one of $a_k(t)$, $k \in I$. □

Remark 3.12. Explicit positiveness conditions for the fundamental function were presented in Lemma 2.7.

Corollary 3.13. *Suppose*

$$\begin{gathered} a(t) \geq 0, \quad \int_0^\infty a(t)dt = \infty, \quad \limsup_{t\to\infty} \int_{g_k(t)}^t a(s)ds < \infty, \\ \sum_{k=1}^{n} |b_k(t)| = 0, \quad t \in E, \qquad \limsup_{t\to\infty,\, t\notin E} \frac{\sum_{k=1}^{n}|b_k(t)|}{a(t)} < 1, \end{gathered} \tag{3.37}$$

where $E = \{t \geq 0, a(t) = 0\}$. Then the equation

$$\dot{x}(t) + a(t)x(t) + \sum_{k=1}^{n} b_k(t)x(g_k(t)) = 0 \tag{3.38}$$

is asymptotically stable. If in addition (2.21) *holds then the fundamental function of* (3.38) *has an exponential estimation. If also* $\limsup_{t\to\infty}(t - g_k(t)) < \infty$ *then* (3.38) *is exponentially stable.*

Theorem 3.14. *Suppose* $\int_0^\infty \sum_{k=1}^m |a_k(s)|ds < \infty$*. Then all solutions of* (1.1) *are bounded and* (1.1) *is not asymptotically stable.*

Proof. For the fundamental function of (1.1) we have the following estimation

$$|X(t,s)| \le \exp\left\{\int_s^t \sum_{k=1}^m |a_k(\tau)|d\tau\right\}. \tag{3.39}$$

Then by solution representation formula (2.5) for any solution $x(t)$ of (1.1) we have

$$\begin{aligned}|x(t)| &\le \exp\left\{\int_{t_0}^t \sum_{k=1}^m |a_k(s)|ds\right\}|x(t_0)| + \int_{t_0}^t \exp\left\{\int_s^t \sum_{k=1}^m |a_k(\tau)|d\tau\right\}\sum_{k=1}^m |a_k(s)|\big|\varphi(h_k(s))\big|ds \\ &\le \exp\left\{\int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds\right\}\left(|x(t_0)| + \int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds\|\varphi\|\right),\end{aligned} \tag{3.40}$$

where $\|\varphi\| = \max_{t<0}|\varphi(t)|$. Then $x(t)$ is a bounded function.

Moreover, $|X(t,s)| \le A := \exp\{\int_0^\infty \sum_{k=1}^m |a_k(s)|ds\}$, $t \ge s \ge 0$. Let us choose $t_0 \ge 0$ such that $\int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|ds < 1/(2A)$, then $X_t'(t,t_0) + \sum_{k=1}^m a_k(t)X(h_k(t),t_0) = 0$, $X(t_0,t_0) = 1$ implies $X(t,t_0) \ge 1 - \int_{t_0}^\infty \sum_{k=1}^m |a_k(s)|Ads > 1 - A(1/(2A)) = 1/2$, thus $X(t,t_0)$ does not tend to zero, so (1.1) is not asymptotically stable. □

Theorems 3.11 and 3.14 imply the following results.

Corollary 3.15. *Suppose* $a_k(t) \ge 0$*, there exists a set of indices* $I \subset \{1,\dots,m\}$ *such that condition* (3.30) *and the second condition in* (3.29) *hold. Then all solutions of* (1.1) *are bounded.*

Proof. If $\int_0^\infty \sum_{k\in I}|a_k(t)|dt = \infty$, then all solutions of (1.1) are bounded by Theorem 3.11. Let $\int_0^\infty \sum_{k\in I}|a_k(t)|dt < \infty$. By (3.30) we have $\int_0^\infty \sum_{k\notin I}|a_k(t)|dt \le \int_0^\infty \sum_{k\in I}|a_k(t)|dt < \infty$. Then $\int_0^\infty \sum_{k=1}^m |a_k(t)|dt < \infty$. By Theorem 3.14 all solutions of (1.1) are bounded. □

Theorem 3.16. *Suppose* $a_k(t) \ge 0$*. If* (1.1) *is asymptotically stable, then the ordinary differential equation*

$$\dot{x}(t) + \left(\sum_{k=1}^m a_k(t)\right)x(t) = 0 \tag{3.41}$$

is also asymptotically stable. If in addition (2.12) *holds and* (1.1) *is exponentially stable, then* (3.41) *is also exponentially stable.*

Proof. The solution of (3.41), with the initial condition $x(t_0) = x_0$, can be presented as $x(t) = x_0 \exp\{-\int_{t_0}^{t} \sum_{k=1}^{m} a_k(s)ds\}$, so (3.41) is asymptotically stable, as far as

$$\int_0^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty \tag{3.42}$$

and is exponentially stable if (3.6) holds (see Lemma 2.11).

If (3.42) does not hold, then by Theorem 3.14, (1.1) is not asymptotically stable.

Further, let us demonstrate that exponential stability of (1.1) really implies (3.6).

Suppose for the fundamental function of (1.1) inequality (2.7) holds and condition (3.6) is not satisfied. Then there exists a sequence $\{t_n\}$, $t_n \to \infty$, such that

$$\int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau < \frac{1}{n} < \frac{1}{e}, \quad n \geq 3. \tag{3.43}$$

By (2.12) there exists $n_0 > 3$ such that $t - h_k(t) \leq n_0$, $k = 1, \dots, m$. Lemma 2.7 implies $X(t,s) > 0$, $t_n \leq s \leq t \leq t_n + n$, $n \geq n_0$. Similar to the proof of Theorem 3.14 and using the inequality $1 - x \geq e^{-x}$, $x > 0$, we obtain

$$X(t_n, t_n + n) \geq 1 - \int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau \geq \exp\left\{-\int_{t_n}^{t_n+n} \sum_{k=1}^{m} a_k(\tau)d\tau\right\} > e^{-1/n}. \tag{3.44}$$

Inequality (2.7) implies $|X(t_n + n, t_n)| \leq Ke^{-\lambda n}$. Hence $Ke^{-\lambda n} \geq e^{-1/n}$, $n \geq n_0$, or $K > e^{\lambda n - 1/3}$ for any $n \geq n_0$. The contradiction proves the theorem. □

Theorems 3.11 and 3.16 imply the following statement.

Corollary 3.17. *Suppose $a_k(t) \geq 0$ and the fundamental function of* (1.1) *is positive. Then* (1.1) *is asymptotically stable if and only if the ordinary differential equation* (3.41) *is asymptotically stable.*

If in addition (2.12) *holds then* (1.1) *is exponentially stable if and only if* (3.41) *is exponentially stable.*

4. Discussion and Examples

In paper [2] we gave a review of known stability tests for the linear equation (1.1). In this part we will compare the new results obtained in this paper with known stability conditions.

First let us compare the results of the present paper with our papers [1–3]. In all these three papers we apply the same method based on Bohl-Perron-type theorems and comparison with known exponentially stable equations.

In [1–3] we considered exponential stability only. Here we also give explicit conditions for asymptotic stability. For this type of stability, we omit the requirement that the delays are bounded and the sum of the coefficients is separated from zero. We also present some new stability tests, based on the results obtained in [3].

Compare now the results of the paper with some other known results [5–7, 9, 10, 22]. First of all we replace the constant $3/2$ in most of these tests by the constant $1 + 1/e$. Evidently

$1 + 1/e = 1.3678\cdots < 3/2$, so we have a worse constant, but it is an open problem to obtain $(3/2)$-stability results for equations with measurable coefficients and delays.

Consider now (3.28) with a single delay. This equation is well studied beginning with the classical stability result by Myshkis [29]. We present here several statements which cover most of known stability tests for this equation.

Statement 1 (see [5]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are continuous functions and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds \leq \frac{3}{2}. \tag{4.1}$$

Then all solutions of (3.28) are bounded.

If in addition

$$\liminf_{t\to\infty} \int_{h(t)}^{t} a(s)ds > 0, \tag{4.2}$$

and the strict inequality in (4.1) holds then (3.28) is exponentially stable.

Statement 2 (see [7]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are continuous functions, the strict inequality (4.1) holds and $\int_0^\infty a(s)ds = \infty$. Then all solutions of (3.28) tend to zero as $t \to \infty$.

Statement 3 (see [9, 10]). Suppose $a(t) \geq 0$, $h(t) \leq t$ are measurable functions, $\int_0^\infty a(s)ds = \infty$, $A(t) = \int_0^t a(s)ds$ is a strictly monotone increasing function and

$$\limsup_{t\to\infty} \int_{h(t)}^{t} a(s)ds < \sup_{0<\omega<\pi/2} \left(\omega + \frac{1}{\Phi(\omega)}\right) \approx 1.45\ldots, \tag{4.3}$$

$\Phi(\omega) = \int_0^\infty u(t,\omega)dt$, where $u(t,\omega)$ is a solution of the initial value problem

$$\dot{y}(t) + y(t-\omega) = 0, \quad y(t) = 0, \quad t < 0, \quad y(0) = 1. \tag{4.4}$$

Then (3.28) is asymptotically stable.

Note that instead of the equation $\dot{y}(t) + y(t-\omega) = 0$ with a constant delay, the equation

$$\dot{y}(t) + y(t - \tau(t)) = 0 \tag{4.5}$$

can be used as the model equation. For example, the following results are valid.

Statement 4 (see [10]). Equation (4.5) is exponentially stable if $|\tau(t) - \omega| \leq k/\chi(\omega)$, where $k \in [0,\omega)$, $0 \leq \omega < \pi/2$ and $\chi(\omega) = \int_0^\infty |u(t,\omega)|dt$.

Obviously in this statement the delay can exceed 2.

Statement 5 (see [10]). Let $\tau(t) \leq k + \omega\{t/\omega\}$, where $k \in (0,1)$, $0 < \omega < 1$, $\{q\}$ is the fractional part of q. Then (4.5) is exponentially stable.

Here the delay $\tau(t)$ can be in the neighbourhood of ω which is close to 1.

Example 4.1. Consider the equation

$$\dot{x}(t) + \alpha(|\sin t| - \sin t)x(h(t)) = 0, \quad h(t) \leq t, \tag{4.6}$$

where $h(t)$ is an arbitrary measurable function such that $t - h(t) \leq \pi$ and $\alpha > 0$.

This equation has the form (3.28) where $a(t) = \alpha(|\sin t| - \sin t)$. Let us check that the conditions of Corollary 3.10 hold. It is evident that $\int_0^\infty a(s)ds = \infty$. We have

$$\limsup_{t\to\infty} \int_{h(t)}^t a(s)ds \leq \limsup_{t\to\infty} \int_{t-\pi}^t a(s)ds \leq -\alpha \int_\pi^{2\pi} 2\sin s\, ds = 4\alpha. \tag{4.7}$$

If $\alpha < 0.25(1 + 1/e)$, then condition (3.27) holds, hence all solutions of (4.6) tend to zero as $t \to \infty$.

Statements 1–3 fail for this equation. In Statements 1 and 2 the delay should be continuous. In Statement 3 function $A(t) = \int_0^t a(s)ds$ should be strictly increasing.

Consider now the general equation (1.1) with several delays. The following two statements are well known for this equation.

Statement 6 (see [6]). Suppose $a_k(t) \geq 0$, $h_k(t) \leq t$ are continuous functions and

$$\limsup_{t\to\infty} a_k(t) \limsup_{t\to\infty}(t - h_k(t)) \leq 1. \tag{4.8}$$

Then all solutions of (1.1) are bounded and 1 in the right-hand side of (4.8) is the best possible constant.

If $\sum_{k=1}^m a_k(t) > 0$ and the strict inequality in (4.8) is valid then all solutions of (1.1) tend to zero as $t \to \infty$.

If $a_k(t)$ are constants then in (4.8) the number 1 can be replaced by $3/2$.

Statement 7 (see [7]). Suppose $a_k(t) \geq 0$, $h_k(t) \leq t$ are continuous, $h_1(t) \leq h_2(t) \leq \cdots \leq h_m(t)$ and

$$\limsup_{t\to\infty} \int_{h_1(t)}^t \sum_{k=1}^m a_k(s)ds \leq \frac{3}{2}. \tag{4.9}$$

Then any solution of (1.1) tends to a constant as $t \to \infty$.

If in addition $\int_0^\infty \sum_{k=1}^m a_k(s)ds = \infty$, then all solutions of (1.1) tend to zero as $t \to \infty$.

Example 4.2. Consider the equation

$$\dot{x}(t) + \frac{\alpha}{t} x\left(\frac{t}{2} - \sin t\right) + \frac{\beta}{t} x\left(\frac{t}{2}\right) = 0, \quad t \geq t_0 > 0, \tag{4.10}$$

where $\alpha > 0$, $\beta > 0$. Denote $p(t) = 1/t$, $h(t) = t/2 - \sin t$, $g(t) = t/2$.

We apply Corollary 3.6. Since $\lim_{t\to\infty}[\ln(t/2) - \ln(t/2 - \sin t)] = 0$, then

$$\limsup_{t\to\infty}\left(\alpha\int_{h(t)}^{t} p(s)ds + \beta\int_{g(t)}^{t} p(s)ds\right) \le (\alpha+\beta)\ln 2. \tag{4.11}$$

Hence if $\alpha + \beta < (1/\ln 2)(1 + 1/e)$ then (4.10) is asymptotically stable. Statement 4 fails for this equation since the delays are unbounded. Statement 5 fails for this equation since neither $h(t) \le g(t)$ nor $g(t) \le h(t)$ holds.

Stability results where the nondelay term dominates over the delayed terms are well known beginning with the book of Krasovskii [30]. The following result is cited from the monograph [22].

Statement 8 (see [22]). Suppose $a(t)$, $b_k(t)$, $t - h_k(t)$ are bounded continuous functions, there exist δ, k, $\delta > 0$, $0 < k < 1$, such that $a(t) \ge \delta$ and $\sum_{k=1}^{m} |b_k(t)| < k\delta$. Then the equation

$$\dot{x}(t) + a(t)x(t) + \sum_{k=1}^{m} b_k(t)x(h_k(t)) = 0 \tag{4.12}$$

is exponentially stable.

In Corollary 3.13 we obtained a similar result without the assumption that the parameters of the equation are continuous functions and the delays are bounded.

Example 4.3. Consider the equation

$$\dot{x}(t) + \frac{1}{t}x(t) + \frac{\alpha}{t}x\left(\frac{t}{2}\right) = 0, \quad t \ge t_0 > 0. \tag{4.13}$$

If $\alpha < 1$ then by Corollary 3.13 all solutions of (4.13) tend to zero. The delay is unbounded, thus Statement 8 fails for this equation.

In [31] the authors considered a delay autonomous equation with linear and nonlinear parts, where the differential equation with the linear part only has a positive fundamental function and the linear part dominates over the nonlinear one. They generalized the early result of Győri [32] and some results of [33].

In Theorem 3.11 we obtained a similar result for a linear nonautonomous equation without the assumption that coefficients and delays are continuous.

In all the results of the paper we assumed that all or several coefficients of equations considered here are nonnegative. Stability results for (3.28) with oscillating coefficient $a(t)$ were obtained in [34, 35].

We conclude this paper with some open problems.

(1) Is the constant $1 + 1/e$ sharp? Prove or disprove that in Corollary 3.10 the constant $1 + 1/e$ can be replaced by the constant $3/2$.
Note that all known proofs with the constant $3/2$ apply methods which are not applicable for equations with measurable parameters.

(2) Suppose (2.11), (2.12) hold and

$$\limsup_{t\to\infty}\sum_{k=1}^{m}\frac{|a_k(t)|}{\sum_{i=1}^{m}a_i(t)}\int_{h_k(t)}^{t}\sum_{i=1}^{m}a_i(s)ds < 1+\frac{1}{e}. \tag{4.14}$$

Prove or disprove that (1.1) is exponentially stable.
The solution of this problem will improve Theorem 3.3.

(3) Suppose (1.1) is exponentially stable. Prove or disprove that the ordinary differential equation (3.41) is also exponentially (asymptotically) stable, without restrictions on the signs of coefficients $a_k(t) \geq 0$, as in Theorem 3.16. The solution of this problem would improve Theorem 3.16.

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Research Article

Nonoscillation of Second-Order Dynamic Equations with Several Delays

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Existence of nonoscillatory solutions for the second-order dynamic equation $(A_0x^{\Delta})^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$ is investigated in this paper. The results involve nonoscillation criteria in terms of relevant dynamic and generalized characteristic inequalities, comparison theorems, and explicit nonoscillation and oscillation conditions. This allows to obtain most known nonoscillation results for second-order delay differential equations in the case $A_0(t) \equiv 1$ for $t \in [t_0,\infty)_{\mathbb{R}}$ and for second-order nondelay difference equations ($\alpha_i(t) = t+1$ for $t \in [t_0,\infty)_{\mathbb{N}}$). Moreover, the general results imply new nonoscillation tests for delay differential equations with arbitrary A_0 and for second-order delay difference equations. Known nonoscillation results for quantum scales can also be deduced.

1. Introduction

This paper deals with second-order linear delay dynamic equations on time scales. Differential equations of the second order have important applications and were extensively studied; see, for example, the monographs of Agarwal et al. [1], Erbe et al. [2], Győri and Ladas [3], Ladde et al. [4], Myškis [5], Norkin [6], Swanson [7], and references therein. Difference equations of the second order describe finite difference approximations of second-order differential equations, and they also have numerous applications.

We study nonoscillation properties of these two types of equations and some of their generalizations. The main result of the paper is that under some natural assumptions for a delay dynamic equation the following four assertions are equivalent: nonoscillation of solutions of the equation on time scales and of the corresponding dynamic inequality, positivity of the fundamental function, and the existence of a nonnegative solution for a generalized Riccati inequality. The equivalence of oscillation properties of the differential equation and the corresponding differential inequality can be applied to obtain new explicit nonoscillation and oscillation conditions and also to prove some well-known results in a different way. A generalized Riccati inequality is used to compare oscillation properties of two equations without comparing their solutions. These results can be regarded as a natural generalization of the well-known Sturm-Picone comparison theorem for a second-order ordinary differential equation; see [7, Section 1.1]. Applying positivity of the fundamental function, positive solutions of two equations can be compared. There are many results of this kind for delay differential equations of the first-order and only a few for second-order equations. Myškis [5] obtained one of the first comparison theorems for second-order differential equations. The results presented here are generalizations of known nonoscillation tests even for delay differential equations (when the time scale is the real line).

The paper also contains conditions on the initial function and initial values which imply that the corresponding solution is positive. Such conditions are well known for first-order delay differential equations; however, to the best of our knowledge, the only paper concerning second-order equations is [8].

From now on, we will without furthermore mentioning suppose that the time scale $\mathbb{T}$ is unbounded from above. The purpose of the present paper is to study nonoscillation of the delay dynamic equation

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \tag{1.1}$$

where $n \in \mathbb{N}$, $t_0 \in \mathbb{T}$, $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the forcing term, $A_0 \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, and for all $i \in [1,n]_{\mathbb{N}}$, $A_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the coefficient corresponding to the function α_i, where $\alpha_i \le \sigma$ on $[t_0,\infty)_{\mathbb{T}}$.

In this paper, we follow the method employed in [8] for second-order delay differential equations. The method can also be regarded as an application of that used in [9] for first-order dynamic equations.

As a special case, the results of the present paper allow to deduce nonoscillation criteria for the delay differential equation

$$\left(A_0x'\right)'(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.2}$$

in the case $A_0(t) \equiv 1$ for $t \in [t_0,\infty)_{\mathbb{R}}$, they coincide with theorems in [8]. The case of a "quickly growing" function A_0 when the integral of its reciprocal can converge is treated separately.

Let us recall some known nonoscillation and oscillation results for the ordinary differential equations

$$\left(A_0x'\right)'(t) + A_1(t)x(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.3}$$

$$x''(t) + A_1(t)x(t) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{R}}, \tag{1.4}$$

where A_1 is nonnegative, which are particular cases of (1.2) with $n = 1$, $\alpha_1(t) = t$, and $A_0(t) \equiv 1$ for all $t \in [t_0,\infty)_{\mathbb{R}}$.

In [10], Leighton proved the following well-known oscillation test for (1.4); see [10, 11].

Theorem A (see [10]). *Assume that*

$$\int_{t_0}^{\infty} \frac{1}{A_0(\eta)} d\eta = \infty, \qquad \int_{t_0}^{\infty} A_1(\eta) d\eta = \infty, \tag{1.5}$$

then (1.3) *is oscillatory.*

This result for (1.4) was obtained by Wintner in [12] without imposing any sign condition on the coefficient A_1.

In [13], Kneser proved the following result.

Theorem B (see [13]). *Equation* (1.4) *is nonoscillatory if* $t^2 A_1(t) \leq 1/4$ *for all* $t \in [t_0, \infty)_{\mathbb{R}}$, *while oscillatory if* $t^2 A_1(t) > \lambda_0/4$ *for all* $t \in [t_0, \infty)_{\mathbb{R}}$ *and some* $\lambda_0 \in (1, \infty)_{\mathbb{T}}$.

In [14], Hille proved the following result, which improves the one due to Kneser; see also [14–16].

Theorem C (see [14]). *Equation* (1.4) *is nonoscillatory if*

$$t \int_{t}^{\infty} A_1(\eta) d\eta \leq \frac{1}{4} \quad \forall t \in [t_0, \infty)_{\mathbb{R}}, \tag{1.6}$$

while it is oscillatory if

$$t \int_{t}^{\infty} A_1(\eta) d\eta > \frac{\lambda_0}{4} \quad \forall t \in [t_0, \infty)_{\mathbb{R}} \text{ and some } \lambda_0 \in (1, \infty)_{\mathbb{R}}. \tag{1.7}$$

Another particular case of (1.1) is the second-order delay difference equation

$$\Delta(A_0 \Delta x)(k) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(k) x(\alpha_i(k)) = 0 \quad \text{for} \in [k_0, \infty)_{\mathbb{N}}, \tag{1.8}$$

to the best of our knowledge, there are very few nonoscillation results for this equation; see, for example, [17]. However, nonoscillation properties of the nondelay equations

$$\Delta(A_0 \Delta x)(k) + A_1(k) x(k+1) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.9}$$

$$\Delta^2 x(k) + A_1(k) x(k+1) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}} \tag{1.10}$$

have been extensively studied in [1, 18–22]; see also [23]. In particular, the following result is valid.

Theorem D. *Assume that*

$$\sum_{j=k_0}^{\infty} A_1(j) = \infty, \tag{1.11}$$

then (1.10) *is oscillatory.*

The following theorem can be regarded as the discrete analogue of the nonoscillation result due to Kneser.

Theorem E. *Assume that* $k(k+1)A_1(k) \le 1/4$ *for all* $k \in [k_0, \infty)_{\mathbb{N}}$, *then* (1.10) *is nonoscillatory.*

Hille's result in [14] also has a counterpart in the discrete case. In [22], Zhou and Zhang proved the nonoscillation part, and in [24], Zhang and Cheng justified the oscillation part which generalizes Theorem E.

Theorem F (see [22, 24]). *Equation* (1.10) *is nonoscillatory if*

$$k\sum_{j=k}^{\infty} A_1(j) \le \frac{1}{4} \quad \forall k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.12}$$

while is oscillatory if

$$k\sum_{j=k}^{\infty} A_1(j) > \frac{\lambda_0}{4} \quad \forall k \in [k_0, \infty)_{\mathbb{N}} \text{ and some } \lambda_0 \in (1, \infty)_{\mathbb{R}}. \tag{1.13}$$

In [23], Tang et al. studied nonoscillation and oscillation of the equation

$$\Delta^2 x(k) + A_1(k)x(k) = 0 \quad \text{for } k \in [k_0, \infty)_{\mathbb{N}}, \tag{1.14}$$

where $\{A_1(k)\}$ is a sequence of nonnegative reals and obtained the following theorem.

Theorem G (see [23]). *Equation* (1.14) *is nonoscillatory if* (1.12) *holds, while is it oscillatory if* (1.13) *holds.*

These results together with some remarks on the q-difference equations will be discussed in Section 7. The readers can find some nonoscillation results for second-order nondelay dynamic equations in the papers [20, 25–29], some of which generalize some of those mentioned above.

The paper is organized as follows. In Section 2, some auxiliary results are presented. In Section 3, the equivalence of the four above-mentioned properties is established. Section 4 is dedicated to comparison results. Section 5 includes some explicit nonoscillation and oscillation conditions. A sufficient condition for existence of a positive solution is given in Section 6. Section 7 involves some discussion and states open problems. Section 7 as an appendix contains a short account on the fundamentals of the time scales theory.

2. Preliminary Results

Consider the following delay dynamic equation:

$$\begin{gathered}\left(A_0x^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}}A_i(t)x(\alpha_i(t))=f(t)\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\\ x(t_0)=x_1,\quad x^{\Delta}(t_0)=x_2,\quad x(t)=\varphi(t)\quad\text{for } t\in[t_{-1},t_0)_{\mathbb{T}},\end{gathered}\tag{2.1}$$

where $n\in\mathbb{N}$, $\mathbb{T}$ is a time scale unbounded above, $t_0\in\mathbb{T}$, $x_1,x_2\in\mathbb{R}$ are the initial values, $\varphi\in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ is the initial function, such that φ has a finite left-sided limit at the initial point t_0 provided that it is left dense, $f\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the forcing term, $A_0\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^{+})$, and for all $i\in[1,n]_{\mathbb{N}}$, $A_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is the coefficient corresponding to the function $\alpha_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$, which satisfies $\alpha_i(t)\le\sigma(t)$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ and $\lim_{t\to\infty}\alpha_i(t)=\infty$. Here, we denoted

$$\alpha_{\min}(t):=\min_{i\in[1,n]_{\mathbb{N}}}\{\alpha_i(t)\}\quad\text{for } t\in[t_0,\infty)_{\mathbb{T}},\quad t_{-1}:=\inf_{t\in[t_0,\infty)_{\mathbb{T}}}\{\alpha_{\min}(t)\},\tag{2.2}$$

then t_{-1} is finite, since $\alpha_{\min}$ asymptotically tends to infinity.

Definition 2.1. A function $x:[t_{-1},\infty)_{\mathbb{T}}\to\mathbb{R}$ with $x\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ and a derivative satisfying $A_0x^{\Delta}\in C^1_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is called a *solution* of (2.1) if it satisfies the equation in the first line of (2.1) identically on $[t_0,\infty)_{\mathbb{T}}$ and also the initial conditions in the second line of (2.1).

For a given function $\varphi\in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}},\mathbb{R})$ with a finite left-sided limit at the initial point t_0 provided that it is left-dense and $x_1,x_2\in\mathbb{R}$, problem (2.1) admits a unique solution satisfying $x=\varphi$ on $[t_{-1},t_0)_{\mathbb{T}}$ with $x(t_0)=x_1$ and $x^{\Delta}(t_0)=x_2$ (see [30] and [31, Theorem 3.1]).

Definition 2.2. A solution of (2.1) is called *eventually positive* if there exists $s\in[t_0,\infty)_{\mathbb{T}}$ such that $x>0$ on $[s,\infty)_{\mathbb{T}}$, and if $(-x)$ is eventually positive, then x is called *eventually negative*. If (2.1) has a solution which is either eventually positive or eventually negative, then it is called *nonoscillatory*. A solution, which is neither eventually positive nor eventually negative, is called *oscillatory*, and (2.1) is said to be *oscillatory* provided that every solution of (2.1) is oscillatory.

For convenience in the notation and simplicity in the proofs, we suppose that functions vanish out of their specified domains, that is, let $f:D\to\mathbb{R}$ be defined for some $D\subset\mathbb{R}$, then it is always understood that $f(t)=\chi_D(t)f(t)$ for $t\in\mathbb{R}$, where χ_D is the characteristic function of the set $D\subset\mathbb{R}$ defined by $\chi_D(t)\equiv 1$ for $t\in D$ and $\chi_D(t)\equiv 0$ for $t\notin D$.

Definition 2.3. Let $s \in \mathbb{T}$ and $s_{-1} := \inf_{t\in[s,\infty)_{\mathbb{T}}}\{\alpha_{\min}(t)\}$. The solutions $\mathcal{X}_1 = \mathcal{X}_1(\cdot, s)$ and $\mathcal{X}_2 = \mathcal{X}_2(\cdot, s)$ of the problems

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [s,\infty)_{\mathbb{T}}, \\ x^{\Delta}(s) = \frac{1}{A_0(s)}, \quad x(t) \equiv 0 \quad \text{for } t \in [s_{-1}, s]_{\mathbb{T}},\end{gathered} \tag{2.3}$$

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [s,\infty)_{\mathbb{T}}, \\ x^{\Delta}(s) = 0, \quad x(t) = \chi_{\{s\}}(t) \quad \text{for } t \in [s_{-1}, s]_{\mathbb{T}},\end{gathered} \tag{2.4}$$

which satisfy $\mathcal{X}_1(\cdot, s), \mathcal{X}_2(\cdot, s) \in C^1_{\mathrm{rd}}([s,\infty)_{\mathbb{T}}, \mathbb{R})$, are called the *first fundamental solution* and the *second fundamental solution* of (2.1), respectively.

The following lemma plays the major role in this paper; it presents a representation formula to solutions of (2.1) by the means of the fundamental solutions $\mathcal{X}_1$ and $\mathcal{X}_2$.

Lemma 2.4. *Let x be a solution of* (2.1), *then x can be written in the following form:*

$$x(t) = x_2\mathcal{X}_1(t,t_0) + x_1\mathcal{X}_2(t,t_0) + \int_{t_0}^{t} \mathcal{X}_1(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta \tag{2.5}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. For $t \in [t_{-1},\infty)_{\mathbb{T}}$, let

$$y(t) := \begin{cases} \displaystyle\int_{t_0}^{t} \mathcal{X}_1(t,\sigma(\eta))\left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta))\right]\Delta\eta & \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ \varphi(t) & \text{for } t \in [t_1, t_0)_{\mathbb{T}}. \end{cases} \tag{2.6}$$

We recall that $\mathcal{X}_1(\cdot, t_0)$ and $\mathcal{X}_2(\cdot, t_0)$ solve (2.3) and (2.4), respectively. To complete the proof, let us demonstrate that y solves

$$\begin{gathered}\left(A_0 y^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ y(t_0) = 0, \quad y^{\Delta}(t_0) = 0, \quad y(t) = \varphi(t) \quad \text{for } t \in [t_{-1}, t_0)_{\mathbb{T}}.\end{gathered} \tag{2.7}$$

This will imply that the function z defined by $z := x_2\mathcal{X}_1(\cdot, t_0) + x_1\mathcal{X}_2(\cdot, t_0) + y$ on $[t_0, \infty)_{\mathbb{T}}$ is a solution of (2.1). Combining this with the uniqueness result in [31, Theorem 3.1] will complete the proof. For all $t \in [t_0, \infty)_{\mathbb{T}}$, we can compute that

$$\begin{aligned} y^{\Delta}(t) &= \int_{t_0}^{t} \mathcal{X}_1^{\Delta}(t, \sigma(\eta)) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\quad + \mathcal{X}_1(\sigma(t), \sigma(t)) \left[f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) \right] \\ &= \int_{t_0}^{t} \mathcal{X}_1^{\Delta}(t, \sigma(\eta)) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta. \end{aligned} \tag{2.8}$$

Therefore, $y(t_0) = 0$, $y^{\Delta}(t_0) = 0$, and $y = \varphi$ on $[t_{-1}, t_0)_{\mathbb{T}}$, that is, y satisfies the initial conditions in (2.7). Differentiating y^{Δ} after multiplying by A_0 and using the properties of the first fundamental solution $\mathcal{X}_1$, we get

$$\begin{aligned} \left(A_0 y^{\Delta}\right)^{\Delta}(t) &= \int_{t_0}^{t} \left(A_0\mathcal{X}_1^{\Delta}(\cdot, \sigma(\eta))\right)^{\Delta}(t) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\quad + A_0^{\sigma}(t)\mathcal{X}_1^{\Delta}(\sigma(t), \sigma(t)) \left[f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) \right] \\ &= -\sum_{j\in[1,n]_{\mathbb{N}}} A_j(t) \int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t), \sigma(\eta)) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\quad - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) + f(t) \end{aligned} \tag{2.9}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. For $t \in [t_0, \infty)_{\mathbb{T}}$, set $I(t) = \{i \in [1, n]_{\mathbb{N}} : \chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_i(t)) = 1\}$ and $J(t) := \{i \in [1, n]_{\mathbb{N}} : \chi_{[t_{-1},t_0)_{\mathbb{T}}}(\alpha_i(t)) = 1\}$. Making some arrangements, for all $t \in [t_0, \infty)_{\mathbb{T}}$, we find

$$\begin{aligned} \left(A_0 y^{\Delta}\right)^{\Delta}(t) &= -\sum_{j\in I(t)} A_j(t) \int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t), \sigma(\eta)) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\quad - \sum_{j\in J(t)} A_j(t) \int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t), \sigma(\eta)) \left[f(\eta) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\quad - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) + f(t), \end{aligned} \tag{2.10}$$

and thus

$$\begin{aligned}\left(A_0 y^{\Delta}\right)^{\Delta}(t) &= -\sum_{j\in I(t)} A_j(t)\int_{t_0}^{\alpha_j(t)} \mathcal{X}_1(\alpha_j(t),\sigma(\eta))f(\eta)\Delta\eta - \sum_{j\in J(t)} A_i(t)\varphi(\alpha_i(t)) + f(t) \\ &= -\sum_{j\in I(t)} A_j(t)y(\alpha_j(t)) - \sum_{j\in J(t)} A_j(t)y(\alpha_j(t)) + f(t),\end{aligned} \tag{2.11}$$

which proves that y satisfies (2.7) on $[t_0,\infty)_{\mathbb{T}}$ since $I(t)\cap J(t)=\emptyset$ and $I(t)\cup J(t)=[1,n]_{\mathbb{N}}$ for each $t\in[t_0,\infty)_{\mathbb{T}}$. The proof is therefore completed. □

Next, we present a result from [9] which will be used in the proof of the main result.

Lemma 2.5 (see [9, Lemma 2.5]). *Let $t_0\in\mathbb{T}$ and assume that K is a nonnegative Δ-integrable function defined on $\{(t,s)\in\mathbb{T}\times\mathbb{T}: t\in[t_0,\infty)_{\mathbb{T}}, s\in[t_0,t]_{\mathbb{T}}\}$. If $f,g\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ satisfy*

$$f(t)=\int_{t_0}^{t} K(t,\eta)f(\eta)\Delta\eta + g(t) \quad \forall t\in[t_0,\infty)_{\mathbb{T}}, \tag{2.12}$$

then $g(t)\geq 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$ implies $f(t)\geq 0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$.

3. Nonoscillation Criteria

Consider the delay dynamic equation

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}} \tag{3.1}$$

and its corresponding inequalities

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) \leq 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}, \tag{3.2}$$

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) \geq 0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}}. \tag{3.3}$$

We now prove the following result, which plays a major role throughout the paper.

Theorem 3.1. *Suppose that the following conditions hold:*

($A1$) $A_0\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$,

($A2$) *for* $i\in[1,n]_{\mathbb{N}}$, $A_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$,

($A3$) *for* $i\in[1,n]_{\mathbb{N}}$, $\alpha_i\in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{T})$ *satisfies* $\alpha_i(t)\leq\sigma(t)$ *for all* $t\in[t_0,\infty)_{\mathbb{T}}$ *and* $\lim_{t\to\infty}\alpha_i(t)=\infty$, *then the following conditions are equivalent:*

(i) *the second-order dynamic equation* (3.1) *has a nonoscillatory solution,*

(ii) *the second-order dynamic inequality* (3.2) *has an eventually positive solution and/or* (3.3) *has an eventually negative solution,*

(iii) *there exist a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and a function* $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ *with* $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ *satisfying the first-order dynamic Riccati inequality*

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(t, \alpha_i(t)) \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.4}$$

(iv) *the first fundamental solution* $\mathcal{X}_1$ *of* (3.1) *is eventually positive, that is, there exists a sufficiently large* $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *such that* $\mathcal{X}_1(t, s) > 0$ *for all* $t \in (s, \infty)_{\mathbb{T}}$ *and all* $s \in [t_1, \infty)_{\mathbb{T}}$.

Proof. The proof follows the scheme: (i)⇒(ii)⇒(iii)⇒(iv)⇒(i).

(i)⇒(ii) This part is trivial, since any eventually positive solution of (3.1) satisfies (3.2) too, which indicates that its negative satisfies (3.3).

(ii)⇒(iii) Let x be an eventually positive solution of (3.2), then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. We may assume without loss of generality that $x(t_1) = 1$ (otherwise, we may proceed with the function $x/x(t_1)$, which is also a solution since (3.2) is linear). Let

$$\Lambda(t) := A_0(t)\frac{x^{\Delta}(t)}{x(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.5}$$

then evidently $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ and

$$1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} = 1 + \mu(t)\frac{x^{\Delta}(t)}{x(t)} = \frac{x^{\sigma}(t)}{x(t)} > 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.6}$$

which proves that $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$. This implies that the exponential function $\mathrm{e}_{\Lambda/A_0}(\cdot, t_1)$ is well defined and is positive on the entire time scale $[t_1, \infty)_{\mathbb{T}}$; see [32, Theorem 2.48]. From (3.5), we see that Λ satisfies the ordinary dynamic equation

$$\begin{gathered} x^{\Delta}(t) = \frac{\Lambda(t)}{A_0(t)}x(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x(t_1) = 1, \end{gathered} \tag{3.7}$$

whose unique solution is

$$x(t) = \mathrm{e}_{\Lambda/A_0}(t, t_1) \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.8}$$

see [32, Theorem 2.77]. Hence, using (3.8), for all $t \in [t_1, \infty)_{\mathbb{T}}$, we get

$$\begin{aligned} x^{\Delta}(t) &= \frac{\Lambda(t)}{A_0(t)} \mathrm{e}_{\Lambda/A_0}(t, t_1), \\ \left(A_0 x^{\Delta}\right)^{\Delta}(t) &= \left(\Lambda \mathrm{e}_{\Lambda/A_0}(\cdot, t_1)\right)^{\Delta}(t) = \Lambda^{\Delta}(t)\mathrm{e}_{\Lambda/A_0}(t, t_1) + \Lambda^{\sigma}(t)\mathrm{e}^{\Delta}_{\Lambda/A_0}(t, t_1) \\ &= \Lambda^{\Delta}(t)\mathrm{e}_{\Lambda/A_0}(t, t_1) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t)\mathrm{e}_{\Lambda/A_0}(t, t_1), \end{aligned} \tag{3.9}$$

which gives by substituting into (3.2) and using [32, Theorem 2.36] that

$$\begin{aligned} 0 &\geq \Lambda^{\Delta}(t)\mathrm{e}_{\Lambda/A_0}(t, t_1) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t)\mathrm{e}_{\Lambda/A_0}(t, t_1) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\Lambda/A_0}(\alpha_i(t), t_1) \\ &= \mathrm{e}_{\Lambda/A_0}(t, t_1)\left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)\frac{\mathrm{e}_{\Lambda/A_0}(\alpha_i(t), t_1)}{\mathrm{e}_{\Lambda/A_0}(t, t_1)}\right] \\ &= \mathrm{e}_{\Lambda/A_0}(t, t_1)\left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(t, \alpha_i(t))\right] \end{aligned} \tag{3.10}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. Since the expression in the brackets is the same as the left-hand side of (3.4) and $\mathrm{e}_{\Lambda/A_0}(\cdot, t_1) > 0$ on $[t_1, \infty)_{\mathbb{T}}$, the function Λ is a solution of (3.4).

(iii)⇒(iv) Consider the initial value problem

$$\begin{gathered} \left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x^{\Delta}(t_1) = 0, \quad x(t) \equiv 0 \quad \text{for } t \in [t_{-1}, t_1]_{\mathbb{T}}. \end{gathered} \tag{3.11}$$

Denote

$$g(t) := A_0(t)x^{\Delta}(t) - \Lambda(t)x(t) \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.12}$$

where x is any solution of (3.11) and Λ is a solution of (3.4). From (3.12), we have

$$\begin{gathered} x^{\Delta}(t) = \frac{\Lambda(t)}{A_0(t)}x(t) + \frac{g(t)}{A_0(t)} \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \\ x(t_1) = 0, \end{gathered} \tag{3.13}$$

whose unique solution is

$$x(t) = \int_{t_1}^{t} \mathrm{e}_{\Lambda/A_0}(t, \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.14}$$

see [32, Theorem 2.77]. Now, for all $t \in [t_1, \infty)_{\mathbb{T}}$, we compute that

$$\begin{aligned} x(t) &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), t)\left[\int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A(\eta)}\Delta\eta - \mu(t)\mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(t))\frac{g(t)}{A_0(t)}\right] \\ &= \frac{A_0(t)}{A_0(t) + \mu(t)\Lambda(t)}\left[x^{\sigma}(t) - \mu(t)\frac{g(t)}{A_0(t)}\right] \\ &= \frac{1}{A_0(t) + \mu(t)\Lambda(t)}\left[A_0(t)x^{\sigma}(t) - \mu(t)g(t)\right], \end{aligned} \tag{3.15}$$

and similarly

$$\begin{aligned} x(\alpha_i(t)) &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \\ &\quad \times \left[\int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A(\eta)}\Delta\eta - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta\right] \\ &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))\left[x^{\sigma}(t) - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\sigma(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta\right] \\ &= \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))x^{\sigma}(t) - \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t), \sigma(\eta))\frac{g(\eta)}{A_0(\eta)}\Delta\eta \end{aligned} \tag{3.16}$$

for $i \in [1, n]_{\mathbb{N}}$. From (3.12) and (3.15), we have

$$\begin{aligned} \left(A_0 x^{\Delta}\right)^{\Delta}(t) &= (\Lambda x + g)^{\Delta}(t) = \Lambda^{\Delta}(t)x^{\sigma}(t) + \Lambda(t)x^{\Delta}(t) + g^{\Delta}(t) \\ &= \Lambda^{\Delta}(t)x^{\sigma}(t) + \frac{\Lambda^2(t)}{A_0(t)}x(t) + \frac{\Lambda(t)}{A_0(t)}g(t) + g^{\Delta}(t) \end{aligned} \tag{3.17}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. We substitute (3.14), (3.15), (3.16), and (3.17) into (3.11) and find that

$$\begin{aligned} f(t) &= \left[\Lambda^{\Delta}(t)x^{\sigma}(t) + \frac{\Lambda^2(t)}{A_0(t)}x(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t))\right] + \frac{\Lambda(t)}{A_0(t)}g(t) + g^{\Delta}(t) \\ &= \left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t))\right]x^{\sigma}(t) \end{aligned}$$

$$\begin{aligned}
&- \left[\frac{\mu(t)\Lambda^2(t)}{A_0(t)(A_0(t)+\mu(t)\Lambda(t))} g(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \right] \\
&+ \frac{\Lambda(t)}{A_0(t)} g(t) + g^{\Delta}(t) \\
&= \left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t)+\mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t),\alpha_i(t)) \right] \\
&\times \left[1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right] \int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(t,\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
&- \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
&+ \frac{\Lambda(t)}{A_0(t)+\mu(t)\Lambda(t)} g(t) + g^{\Delta}(t)
\end{aligned} \tag{3.18}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$. Then, (3.18) can be rewritten as

$$\begin{aligned}
g^{\Delta}(t) = {} & -\frac{\Lambda(t)}{A_0(t)+\mu(t)\Lambda(t)} g(t) + \Upsilon(t) \int_{t_1}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(t,\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta \\
& + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \int_{\alpha_i(t)}^{\sigma(t)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(t),\sigma(\eta)) \frac{g(\eta)}{A_0(\eta)} \Delta\eta + f(t)
\end{aligned} \tag{3.19}$$

for all $t \in [t_1,\infty)_{\mathbb{T}}$, where

$$\Upsilon(t) := -\left[1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right] \left[\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t)+\mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t),\alpha_i(t)) \right] \tag{3.20}$$

for $t \in [t_1,\infty)_{\mathbb{T}}$. We now show that $\Upsilon \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$. Indeed, by using (3.4) and the simple useful formula (A.2), we get

$$\begin{aligned}
\Upsilon(t) &= -\left[\left(1 + \mu(t)\frac{\Lambda(t)}{A_0(t)} \right) \Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^2(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \right] \\
&= -\left[\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \right] \geq 0
\end{aligned} \tag{3.21}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. On the other hand, from (3.11) and (3.12), we see that $g(t_1) = 0$. Then, by [32, Theorem 2.77], we can write (3.19) in the equivalent form

$$g = \mathscr{A} g + h \quad on \ [t_1, \infty)_{\mathbb{T}}, \tag{3.22}$$

where, for $t \in [t_1, \infty)_{\mathbb{T}}$, we have defined

$$\begin{aligned}(\mathscr{A} g)(t) := \int_{t_1}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Bigg[&\Upsilon(\eta) \int_{t_1}^{\sigma(\eta)} \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta \\ &+ \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta) \int_{\alpha_i(\eta)}^{\sigma(\eta)} \mathrm{e}_{\Lambda/A_0}(\alpha_i(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta \Bigg] \Delta\eta,\end{aligned} \tag{3.23}$$

$$h(t) := \int_{t_1}^{t} \mathrm{e}_{\Lambda/A_0}(t, \sigma(\zeta)) f(\eta) \Delta\eta. \tag{3.24}$$

Note that $\Lambda/A_0 \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ implies $-\Lambda/(A_0 + \mu\Lambda) \in \mathcal{R}^+([t_1, \infty)_{\mathbb{T}}, \mathbb{R})$ (indeed, we have $1 - \mu\Lambda/(A_0 + \mu\Lambda) = A_0/(A_0 + \mu\Lambda) > 0$ on $[t_1, \infty)_{\mathbb{T}}$), and thus the exponential function $\mathrm{e}_{\ominus(\Lambda/A_0)}(\cdot, t_1)$ is also well defined and positive on the entire time scale $[t_1, \infty)_{\mathbb{T}}$, see [32, Exercise 2.28]. Thus, $f \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$ implies $h \geq 0$ on $[t_1, \infty)_{\mathbb{T}}$. For simplicity of notation, for $s, t \in [t_1, \infty)_{\mathbb{T}}$, we let

$$\begin{aligned}K_1(t, s) &:= \frac{1}{A_0(s)} \int_{s}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(s)) \Delta\eta, \\ K_2(t, s) &:= \frac{1}{A_0(s)} \int_{s}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta) \chi_{[\alpha_i(\eta),\infty)_{\mathbb{T}}}(s) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(s)) \Delta\eta.\end{aligned} \tag{3.25}$$

Using the change of integration order formula in [33, Lemma 1], for all $t \in [t_1, \infty)_{\mathbb{T}}$, we obtain

$$\begin{aligned}&\int_{t_1}^{t} \int_{t_1}^{\sigma(\eta)} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\zeta\Delta\eta \\ &\quad = \int_{t_1}^{t} \int_{\zeta}^{t} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t, \sigma(\eta)) \Upsilon(\eta) \mathrm{e}_{\Lambda/A_0}(\sigma(\eta), \sigma(\zeta)) \frac{g(\zeta)}{A_0(\zeta)} \Delta\eta\Delta\zeta \\ &\quad = \int_{t_1}^{t} K_1(t, \zeta) g(\zeta) \Delta\zeta,\end{aligned} \tag{3.26}$$

and similarly

$$\begin{aligned}
&\int_{t_1}^{t}\int_{t_1}^{\sigma(\eta)} \mathrm{e}_{-\Lambda/(A_0+\mu\Lambda)}(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\chi_{[\alpha_i(\eta),\infty)_{\mathbb{T}}}(\zeta)\mathrm{e}_{\Lambda/A_0}(\sigma(\eta),\sigma(\zeta))\frac{g(\zeta)}{A_0(\zeta)}\Delta\zeta\Delta\eta \\
&\quad = \int_{t_1}^{t} K_2(t,\zeta)g(\zeta)\Delta\zeta.
\end{aligned} \tag{3.27}$$

Therefore, we can rewrite (3.23) in the equivalent form of the integral operator

$$(\mathscr{H}g)(t) = \int_{t_1}^{t} [K_1(t,\eta) + K_2(t,\eta)]g(\eta)\Delta\eta \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}, \tag{3.28}$$

whose kernel is nonnegative. Consequently, using (3.22), (3.24), and (3.28), we obtain that $f \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ implies $h \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$; this and Lemma 2.5 yield that $g \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$. Therefore, from (3.14), we infer that if $f \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$, then $x \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ too. On the other hand, by Lemma 2.4, x has the following representation:

$$x(t) = \int_{t_1}^{t} \mathcal{X}_1(t,\sigma(\eta))f(\eta)\Delta\eta \quad \text{for } t \in [t_1,\infty)_{\mathbb{T}}. \tag{3.29}$$

Since x is eventually nonnegative for any eventually nonnegative function f, we infer that the kernel $\mathcal{X}_1$ of the integral on the right-hand side of (3.29) is eventually nonnegative. Indeed, assume to the contrary that $x \geq 0$ on $[t_1,\infty)_{\mathbb{T}}$ but $\mathcal{X}_1$ is not nonnegative, then we may pick $t_2 \in [t_1,\infty)_{\mathbb{T}}$ and find $s \in [t_1,t_2)_{\mathbb{T}}$ such that $\mathcal{X}_1(t_2,\sigma(s)) < 0$. Then, letting $f(t) := -\min\{\mathcal{X}_1(t_2,\sigma(t)),0\} \geq 0$ for $t \in [t_1,\infty)_{\mathbb{T}}$, we are led to the contradiction $x(t_2) < 0$, where x is defined by (3.29). To prove that $\mathcal{X}_1$ is eventually positive, set $x(t) := \mathcal{X}_1(t,s)$ for $t \in [t_0,\infty)_{\mathbb{T}}$, where $s \in [t_1,\infty)_{\mathbb{T}}$, to see that $x \geq 0$ and $(A_0x^{\Delta})^{\Delta} \leq 0$ on $[s,\infty)_{\mathbb{T}}$, which implies A_0x^{Δ} is nonincreasing on $[s,\infty)_{\mathbb{T}}$. So that, we may let $t_1 \in [t_0,\infty)_{\mathbb{T}}$ so large that x^{Δ} (i.e., A_0x^{Δ}) is of fixed sign on $[s,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$. The initial condition and (A1) together with $x^{\Delta}(s) = 1/A_0(s) > 0$ imply that $x^{\Delta} > 0$ on $[s,\infty)_{\mathbb{T}}$. Consequently, we have $x(t) = \mathcal{X}_1(t,s) > \mathcal{X}_1(s,s) = 0$ for all $t \in (s,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$.

(iv)⇒(i) Clearly, $\mathcal{X}_1(\cdot,t_0)$ is an eventually positive solution of (3.1).

The proof is completed. □

Let us introduce the following condition:

$(A4)$ $A_0 \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ with

$$\int_{t_0}^{\infty} \frac{1}{A_0(\eta)}\Delta\eta = \infty. \tag{3.30}$$

Remark 3.2. It is well known that (A4) ensures existence of $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t)x^{\Delta}(t) \geq 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$, for any nonoscillatory solution x of (3.1). This fact follows from the formula

$$x(t) = x(s) + A_0(s)x^{\Delta}(s)\int_s^t \frac{1}{A_0(\eta)}\Delta\eta - \int_s^t \frac{1}{A_0(\eta)}\left[\int_s^{\eta} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\zeta)x(\alpha_i(\zeta))\Delta\zeta\right]\Delta\eta \tag{3.31}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$, obtained by integrating (3.1) twice, where $s \in [t_0, \infty)_{\mathbb{T}}$. In the case when (A4) holds, (iii) of Theorem 3.1 can be assumed to hold with $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$, which means that any positive (negative) solution is nondecreasing (nonincreasing).

Remark 3.3. Let (A4) hold and exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and the function $\Lambda \in C^1_{\mathrm{rd}}([t_1, \infty)_{\mathbb{T}}, \mathbb{R}^+_0)$ satisfying inequality (3.4), then the assertions (i), (iii), and (iv) of Theorem 3.1 are also valid on $[t_1, \infty)_{\mathbb{T}}$.

Remark 3.4. It should be noted that (3.4) is also equivalent to the inequality

$$\Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{3.32}$$

see (3.20) and compare with [26, 28, 29, 34].

Example 3.5. For $\mathbb{T} = \mathbb{R}$, (3.4) has the form

$$\Lambda'(t) + \frac{1}{A_0(t)}\Lambda^2(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\exp\left\{-\int_{\alpha_i(t)}^t \frac{\Lambda(\eta)}{A_0(\eta)}\mathrm{d}\eta\right\} \leq 0 \quad \forall t \in [t_1, \infty)_{\mathbb{R}}, \tag{3.33}$$

see [8] for the case $A_0(t) \equiv 1,\ t \in [t_0, \infty)_{\mathbb{R}}$, and [35] for $n = 1,\ \alpha_1(t) = t,\ t \in [t_0, \infty)_{\mathbb{R}}$.

Example 3.6. For $\mathbb{T} = \mathbb{N}$, (3.4) becomes

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{A_0(k) + \Lambda(k)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(k)\prod_{j=\alpha_i(k)}^{k} \frac{A_0(j)}{A_0(j) + \Lambda(j)} \leq 0 \quad \forall k \in [k_1, \infty)_{\mathbb{N}}, \tag{3.34}$$

where the product over the empty set is assumed to be equal to one; see [1, 18] (or (1.10)) for $n = 1$, $\alpha_1(k) = k + 1$, $k \in [k_0, \infty)_{\mathbb{N}}$, and [20] for $n = 1$, $A_0(k) \equiv 1$, $\alpha_1(k) = k + 1$, $k \in [k_0, \infty)_{\mathbb{N}}$. It should be mentioned that in the literature all the results relating difference equations with discrete Riccati equations consider only the nondelay case. This result in the discrete case is therefore new.

Example 3.7. For $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ with $q \in (1,\infty)_{\mathbb{R}}$, under the same assumption on the product as in the previous example, condition (3.4) reduces to the inequality

$$D_q\Lambda(t) + \frac{\Lambda^2(t)}{A_0(t) + (q-1)t\Lambda(t)} + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t) \prod_{\eta=\log_q(\alpha_i(t))}^{\log_q(t)} \frac{A_0(q^\eta)}{A_0(q^\eta) + (q-1)q^\eta\Lambda(q^\eta)} \leq 0 \quad (3.35)$$

for all $t \in [t_1,\infty)_{\overline{q^{\mathbb{Z}}}}$.

4. Comparison Theorems

Theorem 3.1 can be employed to obtain comparison nonoscillation results. To this end, together with (3.1), we consider the second-order dynamic equation

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \quad (4.1)$$

where $B_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ for $i \in [1,n]_{\mathbb{N}}$.

The following theorem establishes the relation between the first fundamental solution of the model equation with positive coefficients and comparison (4.1) with coefficients of arbitrary signs.

Theorem 4.1. *Suppose that (A2), (A3), (A4), and the following condition hold:*

(A5) *for $i \in [1,n]_{\mathbb{N}}$, $B_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ with $A_i(t) \geq B_i(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.*

Assume further that (3.4) *admits a solution $\Lambda \in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ for some $t_1 \in [t_0,\infty)_{\mathbb{T}}$, then the first fundamental solution $\mathcal{Y}_1$ of* (4.1) *satisfies $\mathcal{Y}_1(t,s) \geq \mathcal{X}_1(t,s) > 0$ for all $t \in (s,\infty)_{\mathbb{T}}$ and all $s \in [t_1,\infty)_{\mathbb{T}}$, where $\mathcal{X}_1$ denotes the first fundamental solution of* (3.1).

Proof. We consider the initial value problem

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\alpha_i(t)) = f(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ x^{\Delta}(t_0) = 0, \quad x(t) \equiv 0 \quad \text{for } t \in [t_{-1},t_0]_{\mathbb{T}},\end{gathered} \quad (4.2)$$

where $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$. Let $g \in C_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R})$, and define the function x as

$$x(t) = \int_{t_1}^{t} \mathcal{X}_1(t,\sigma(\eta))g(\eta)\Delta\eta \quad \forall t \in [t_1,\infty)_{\mathbb{T}}. \quad (4.3)$$

By the Leibnitz rule (see [32, Theorem 1.117]), for all $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$x^{\Delta}(t) = \int_{t_1}^{t} \mathcal{X}_1^{\Delta}(t, \sigma(\eta)) g(\eta) \Delta\eta, \tag{4.4}$$

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) = \int_{t_1}^{t} \left(A_0 \mathcal{X}_1^{\Delta}(\cdot, \sigma(\eta))\right)^{\Delta}(t) g(\eta) \Delta\eta + g(t). \tag{4.5}$$

Substituting (4.3) and (4.5) into (4.2), we get

$$\begin{aligned} f(t) &= \int_{t_1}^{t} \left(A_0 \mathcal{X}_1^{\Delta}(\cdot, \sigma(\eta))\right)^{\Delta}(t) g(\eta) \Delta\eta + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) \int_{t_1}^{\alpha_i(t)} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t) \\ &= \sum_{i \in [1,n]_{\mathbb{N}}} [B_i(t) - A_i(t)] \int_{t_1}^{\alpha_i(t)} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t) \\ &= \sum_{i \in [1,n]_{\mathbb{N}}} [B_i(t) - A_i(t)] \int_{t_1}^{t} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) g(\eta) \Delta\eta + g(t), \end{aligned} \tag{4.6}$$

where in the last step, we have used the fact that $\mathcal{X}_1(t, \sigma(s)) \equiv 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$ and all $s \in [t, \infty)_{\mathbb{T}}$. Therefore, we obtain the operator equation

$$g = \mathscr{H} g + f \quad \text{on } [t_1, \infty)_{\mathbb{T}}, \tag{4.7}$$

where

$$(\mathscr{H} g)(t) := \int_{t_1}^{t} \sum_{i \in [1,n]_{\mathbb{N}}} \mathcal{X}_1(\alpha_i(t), \sigma(\eta)) [A_i(t) - B_i(t)] g(\eta) \Delta\eta \quad \text{for } t \in [t_1, \infty)_{\mathbb{T}}, \tag{4.8}$$

whose kernel is nonnegative. An application of Lemma 2.5 shows that nonnegativity of f implies the same for g, and thus x is nonnegative by (4.3). On the other hand, by Lemma 2.4, x has the representation

$$x(t) = \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) f(\eta) \Delta\eta \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.9}$$

Proceeding as in the proof of the part (iii)⇒(iv) of Theorem 3.1, we conclude that the first fundamental solution $\mathcal{Y}_1$ of (4.1) satisfies $\mathcal{Y}_1(t, s) \geq 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_1, \infty)_{\mathbb{T}}$. To complete the proof, we have to show that $\mathcal{Y}_1(t, s) \geq \mathcal{X}_1(t, s) > 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_1, \infty)_{\mathbb{T}}$. Clearly, for any fixed $s \in [t_1, \infty)_{\mathbb{T}}$ and all $t \in [s, \infty)_{\mathbb{T}}$, we have

$$\left(A_0 \mathcal{Y}_1^{\Delta}(\cdot, s)\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathcal{Y}_1(\alpha_i(t), s) = \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(t) - B_i(t)] \mathcal{Y}_1(\alpha_i(t), s), \tag{4.10}$$

which by the solution representation formula yields that

$$\mathcal{Y}_1(t,s) = \mathcal{X}_1(t,s) + \int_s^t \mathcal{X}_1(t,\sigma(\eta)) \sum_{i\in[1,n]_{\mathbb{N}}} [A_i(\eta) - B_i(\eta)] \mathcal{Y}_1(\alpha_i(\eta),s)\Delta\eta \geq \mathcal{X}_1(t,s) \quad (4.11)$$

for all $t \in [s,\infty)_{\mathbb{T}}$. This completes the proof since the first fundamental solution $\mathcal{X}_1$ satisfies $\mathcal{X}_1(t,s) > 0$ for all $t \in (s,\infty)_{\mathbb{T}}$ and all $s \in [t_1,\infty)_{\mathbb{T}}$ by Remark 3.3. □

Corollary 4.2. *Suppose that (A1), (A2), (A3), and (A5) hold, and* (3.1) *has a nonoscillatory solution on* $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$, *then* (4.1) *admits a nonoscillatory solution on* $[t_2,\infty)_{\mathbb{T}} \subset [t_1,\infty)_{\mathbb{T}}$.

Corollary 4.3. *Assume that (A2) and (A3) hold.*

(i) *If (A1) holds and the dynamic inequality*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t) x(\alpha_i(t)) \leq 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \quad (4.12)$$

where $A_i^{+}(t) := \max\{A_i(t),0\}$ *for* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $i \in [1,n]_{\mathbb{N}}$, *has a positive solution on* $[t_0,\infty)_{\mathbb{T}}$, *then* (3.1) *also admits a positive solution on* $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$.

(ii) *If (A4) holds and there exist a sufficiently large* $t_1 \in [t_0,\infty)_{\mathbb{T}}$ *and a function* $\Lambda \in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}},\mathbb{R}^+_0)$ *satisfying the inequality*

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t) e_{\ominus(\Lambda/A_0)}(t,\alpha_i(t)) \leq 0 \quad \forall t \in [t_1,\infty)_{\mathbb{T}}, \quad (4.13)$$

then the first fundamental solution $\mathcal{X}_1$ *of* (3.1) *satisfies* $\mathcal{X}_1(t,s) > 0$ *for all* $t \in (s,\infty)_{\mathbb{T}}$ *and all* $s \in [t_1,\infty)_{\mathbb{T}}$.

Proof. Consider the dynamic equation

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i^{+}(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}. \quad (4.14)$$

Theorem 3.1 implies that for this equation the assertions (i) and (ii) hold. Since for all $i \in [1,n]_{\mathbb{N}}$, we have $A_i(t) \leq A_i^{+}(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, the application of Corollary 4.2 and Theorem 4.1 completes the proof. □

Now, let us compare the solutions of problem (2.1) and the following initial value problem:

$$\begin{aligned} &\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t) x(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ &x(t_0) = y_1, \quad x^{\Delta}(t_0) = y_2, \quad x(t) = \varphi(t) \quad \text{for } t \in [t_{-1},t_0)_{\mathbb{T}}, \end{aligned} \quad (4.15)$$

where $y_1, y_2 \in \mathbb{R}$ are the initial values, $\psi \in C_{\rm rd}([t_{-1}, t_0)_{\mathbb{T}}, \mathbb{R})$ is the initial function such that ψ has a finite left-sided limit at the initial point t_0 provided that it is left dense, $g \in C_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ is the forcing term.

Theorem 4.4. *Suppose that (A2), (A3), (A4), (A5), and the following condition hold:*

($A6$) *$f, g \in C_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ and $\varphi, \psi \in C_{\rm rd}([t_{-1}, t_0)_{\mathbb{T}}, \mathbb{R})$ satisfy*

$$f(t) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t)\varphi(\alpha_i(t)) \leq g(t) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t)\psi(\alpha_i(t)) \quad \forall t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.16}$$

Moreover, let (2.1) *have a positive solution x on $[t_0, \infty)_{\mathbb{T}}$, $y_1 = x_1$, and $y_2 \geq x_2$, then the solution y of* (4.15) *satisfies $y(t) \geq x(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$.*

Proof. By Theorem 3.1 and Remark 3.3, we can assume that $\Lambda \in C_{\rm rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ is a solution of the dynamic Riccati inequality (3.4), then by ($A5$), the function Λ is also a solution of the dynamic Riccati inequality

$$\Lambda^{\Delta}(t) + \frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t) + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) e_{\ominus(\Lambda/A_0)}(t, \alpha_i(t)) \leq 0 \quad \forall t \in [t_0, \infty)_{\mathbb{T}}, \tag{4.17}$$

which is associated with (4.15). Hence, by Theorem 3.1 and Remark 3.3, the first fundamental solution $\mathcal{Y}_1$ of (4.15) satisfies $\mathcal{Y}_1(t, s) > 0$ for all $t \in (s, \infty)_{\mathbb{T}}$ and all $s \in [t_0, \infty)_{\mathbb{T}}$. Rewriting (2.1) in the form

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i \in [1,n]_{\mathbb{N}}} B_i(t) x(\alpha_i(t)) = f(t) - \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(t) - B_i(t)] x(\alpha_i(t)), \quad t \in [t_0, \infty)_{\mathbb{T}} \\ x(t_0) = x_1, \quad x^{\Delta}(t_0) = x_2, \quad x(t) = \varphi(t), \quad t \in [t_{-1}, t_0)_{\mathbb{T}},\end{gathered} \tag{4.18}$$

applying Lemma 2.4, and using ($A6$), we have

$$\begin{aligned} x(t) &= x_2 \mathcal{Y}_1(t, t_0) + x_1 \mathcal{Y}_2(t, t_0) + \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) \\ &\quad \times \left[f(\eta) - \sum_{i \in [1,n]_{\mathbb{N}}} [A_i(\eta) - B_i(\eta)] \chi_{[t_0,\infty)_{\mathbb{T}}}(\alpha_i(\eta)) x(\alpha_i(\eta)) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(\eta)\varphi(\alpha_i(\eta)) \right] \Delta\eta \\ &\leq y_2 \mathcal{Y}_1(t, t_0) + y_1 \mathcal{Y}_2(t, t_0) + \int_{t_0}^{t} \mathcal{Y}_1(t, \sigma(\eta)) \left[g(\eta) - \sum_{i \in [1,n]_{\mathbb{N}}} B_i(\eta)\psi(\alpha_i(\eta)) \right] \Delta\eta \\ &= y(t) \end{aligned} \tag{4.19}$$

for all $t \in [t_0, \infty)_{\mathbb{T}}$. This completes the proof. □

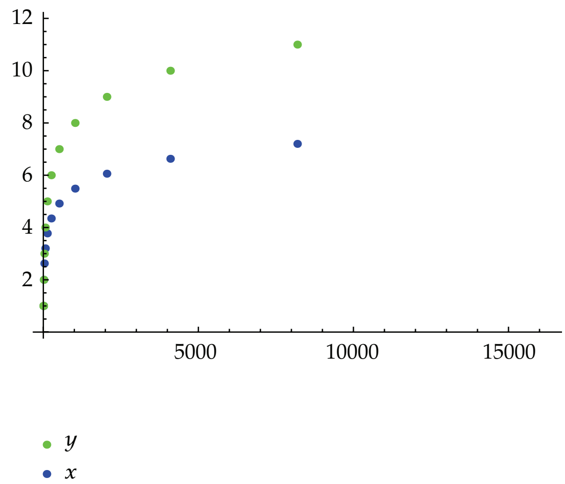

Figure 1: The graph of 10 iterates for the solutions of (4.20) and (4.22) illustrates the result of Theorem 4.4, here $y(t) > x(t)$ for all $t \in (1, \infty)_{\overline{2^{\mathbb{Z}}}}$.

Remark 4.5. If $B_i \in C_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ for $i \in [1, n]_{\mathbb{N}}$, $f(t) \leq g(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $\varphi(t) \geq \psi(t)$ for all $t \in [t_{-1}, t_0)_{\mathbb{T}}$, then $(A6)$ holds.

The following example illustrates Theorem 4.4 for the quantum time scale $\mathbb{T} = \overline{2^{\mathbb{Z}}}$.

Example 4.6. Let $\overline{2^{\mathbb{Z}}} := \{2^k : k \in \mathbb{Z}\} \cup \{0\}$, and consider the following initial value problems:

$$\begin{gathered} D_2\left(\mathrm{Id}_{\overline{2^{\mathbb{Z}}}} D_2 x\right)(t) + \frac{2}{t^4} x\left(\frac{t}{4}\right) = -\frac{1}{t^4} \quad \text{for } t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}, \\ D_2 x(1) = 1, \quad x(t) \equiv 1 \quad \text{for } t \in \left[\frac{1}{4}, 1\right]_{\overline{2^{\mathbb{Z}}}}, \end{gathered} \tag{4.20}$$

where $\mathrm{Id}_{\overline{2^{\mathbb{Z}}}}$ is the identity function on $\overline{2^{\mathbb{Z}}}$, that is, $\mathrm{Id}_{\overline{2^{\mathbb{Z}}}}(t) = t$ for $t \in \overline{2^{\mathbb{Z}}}$, and

$$D_2 x(t) = \frac{1}{t}(x(2t) - x(t)) \quad \text{for } t \in \overline{2^{\mathbb{Z}}}, \tag{4.21}$$

$$\begin{gathered} D_2\left(\mathrm{Id}_{\overline{2^{\mathbb{Z}}}} D_2 x\right)(t) + \frac{1}{t^4} x\left(\frac{t}{4}\right) = \frac{1}{t^4} \quad \text{for } t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}, \\ D_2 x(1) = 1, \quad x(t) \equiv 1 \quad \text{for } t \in \left[\frac{1}{4}, 1\right]_{\overline{2^{\mathbb{Z}}}}. \end{gathered} \tag{4.22}$$

Denoting by x and y the solutions of (4.20) and (4.22), respectively, we obtain $y(t) \geq x(t)$ for all $t \in [1, \infty)_{\overline{2^{\mathbb{Z}}}}$ by Theorem 4.4. For the graph of the first 10 iterates, see Figure 1.

As an immediate consequence of Theorem 4.4, we obtain the following corollary.

Corollary 4.7. *Suppose that (A1), (A2), and (A3) hold and that* (3.1) *is nonoscillatory, then, for* $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$*, the dynamic equation*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = f(t) \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}} \tag{4.23}$$

is also nonoscillatory.

We now consider the following dynamic equation:

$$\begin{gathered}\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)x(\alpha_i(t)) = g(t) \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \\ x(t_0) = y_1, \quad x^{\Delta}(t_0) = y_2, \quad x(t) = \psi(t) \quad \text{for } t \in [t_{-1},t_0)_{\mathbb{T}},\end{gathered} \tag{4.24}$$

where the parameters are the same as in (4.15).

We obtain the most complete result if we compare solutions of (2.1) and (4.24) by omitting the condition ($A2$) and assuming that the solution of (2.1) is positive.

Corollary 4.8. *Suppose that (A3), (A4), and the following condition hold:*

($A7$) $f, g \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$ *and* $\varphi, \psi \in C_{\mathrm{rd}}([t_{-1},t_0)_{\mathbb{T}}, \mathbb{R})$ *satisfy*

$$f(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\varphi(\alpha_i(t)) \le g(t) - \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\psi(\alpha_i(t)) \quad \forall t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.25}$$

If x is a positive solution of (2.1) *on* $[t_0,\infty)_{\mathbb{T}}$ *with* $x_1 = y_1$ *and* $y_2 \ge x_2$*, then for the solution y of* (4.24)*, one has* $y(t) \ge x(t)$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. Corollary 4.3 and Remark 3.3 imply that the first fundamental solution $\mathcal{X}_1$ associated with (2.1) (and (4.24)) satisfies $\mathcal{X}_1(t,s) > 0$ for all $t \in (s,\infty)_{\mathbb{T}}$ and all $s \in [t_0,\infty)_{\mathbb{T}}$. Hence, the claim follows from the solution representation formula. □

Remark 4.9. If at least one of the inequalities in the statements of Theorem 4.4 and Corollary 4.8 is strict, then the conclusions hold with the strict inequality too.

Let us compare equations with different coefficients and delays. Now, we consider

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} B_i(t)x(\beta_i(t)) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{4.26}$$

Theorem 4.10. *Suppose that (A2), (A4), (A5), and the following condition hold:*

($A8$) *for* $i \in [1,n]_{\mathbb{N}}$, $\beta_i \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{T})$ *satisfies* $\beta_i(t) \le \alpha_i(t)$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$ *and* $\lim_{t\to\infty}\beta_i(t) = \infty$.

Assume further that the first-order dynamic Riccati inequality (3.4) *has a solution* $\Lambda \in C^1_{\mathrm{rd}}([t_1,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$ *for some* $t_1 \in [t_0,\infty)_{\mathbb{T}}$*, then the first fundamental solution* $\mathcal{Y}_1$ *of* (4.26) *satisfies* $\mathcal{Y}_1(t,s) > 0$ *for all* $t \in (s,\infty)_{\mathbb{T}}$ *and all* $s \in [t_1,\infty)_{\mathbb{T}}$.

Proof. Note that $(A5)$ implies $A_i(t) \geq B_i^+(t)$ for all $t \in [t_0, \infty)_{\mathbb{T}}$ and $i \in [1, n]_{\mathbb{N}}$, then we have

$$\begin{aligned} 0 &\geq \Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \alpha_i(t)) \\ &\geq \Lambda^{\Delta}(t) + \frac{\Lambda^2(t)}{A_0(t) + \mu(t)\Lambda(t)} + \sum_{i \in [1,n]_{\mathbb{N}}} B_i^+(t) \mathrm{e}_{\ominus(\Lambda/A_0)}(\sigma(t), \beta_i(t)) \end{aligned} \tag{4.27}$$

for all $t \in [t_1, \infty)_{\mathbb{T}}$. The reference to Corollary 4.3 (ii) concludes the proof. □

Remark 4.11. If the condition $(A4)$ in Theorem 4.1, Theorem 4.4, Corollary 4.8, and Theorem 4.10 is replaced with $(A1)$, then the claims of the theorems are valid eventually.

Let us introduce the function

$$\alpha_{\max}(t) := \max_{i \in [1,n]_{\mathbb{N}}} \{\alpha_i(t)\} \quad \text{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{4.28}$$

Corollary 4.12. *Suppose that (A1), (A2), (A3), and (A5) hold. If*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + \left(\sum_{i \in [1,n]_{\mathbb{N}}} A_i(t)\right) x(\alpha_{\max}(t)) = 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{4.29}$$

is nonoscillatory, then (4.1) *is also nonoscillatory.*

Remark 4.13. The claim of Corollary 4.12 is also true when $\alpha_{\max}$ is replaced by σ.

5. Explicit Nonoscillation and Oscillation Results

Theorem 5.1. *Suppose that (A1), (A2), and (A3) hold and that*

$$\frac{\sigma(t)}{2tA_0(t) + \mu(t)} + 2t\sigma(t) \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \mathrm{e}_{\ominus(1/(2\mathrm{Id}_{\mathbb{T}} A_0))}(\sigma(t), \alpha_i(t)) \leq 1 \quad \forall t \in [t_1, \infty)_{\mathbb{T}}, \tag{5.1}$$

where $t_1 \in [t_0, \infty)_{\mathbb{T}}$ *and* $\mathrm{Id}_{\mathbb{T}}$ *is the identity function on* $\mathbb{T}$, *then* (3.1) *is nonoscillatory.*

Proof. The statement of the theorem yields that $\Lambda(t) = 1/(2t)$ for $t \in [t_0, \infty)_{\mathbb{T}^+}$ is a positive solution of the Riccati inequality (3.32). □

Next, let us apply Theorem 5.1 to delay differential equations.

Corollary 5.2. *Let* $A_0 \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}^+)$*, for* $i \in [1,n]_{\mathbb{N}}$*,* $A_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R}_0^+)$*, and* $\alpha_i \in C([t_0,\infty)_{\mathbb{R}},\mathbb{R})$ *such that* $\alpha_i(t) \le t$ *for all* $t \in [t_0,\infty)_{\mathbb{R}}$ *and* $\lim_{t\to\infty}\alpha_i(t) = \infty$*. If*

$$\frac{1}{2A_0(t)} + 2t^2 \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\exp\left\{-\int_{\alpha_i(t)}^{t} \frac{1}{2\eta A_0(\eta)}d\eta\right\} \le 1 \quad \forall t \in [t_1,\infty)_{\mathbb{R}} \tag{5.2}$$

for some $t_1 \in [t_0,\infty)_{\mathbb{R}}$*, then* (1.2) *is nonoscillatory.*

Now, let us proceed with the discrete case.

Corollary 5.3. *Let* $\{A_0(k)\}$ *be a positive sequence, for* $i \in [1,n]_{\mathbb{N}}$*, let* $\{A_i(k)\}$ *be a nonnegative sequence, and let* $\{\alpha_i(k)\}$ *be a divergent sequence such that* $\alpha_i(k) \le k+1$ *for all* $k \in [k_0,\infty)_{\mathbb{N}}$*. If*

$$\frac{k+1}{2kA_0(k)+1} + 2k(k+1)\sum_{i\in[1,n]_{\mathbb{N}}} A_i(k)\prod_{j=\alpha_i(k)}^{k}\frac{2jA_0(j)}{2jA_0(j)+1} \le 1 \quad \forall k \in [k_1,\infty)_{\mathbb{N}} \tag{5.3}$$

for some $k_1 \in [k_0,\infty)_{\mathbb{N}}$*, then* (1.8) *is nonoscillatory.*

Let us introduce the function

$$A(t,s) := \int_s^t \frac{1}{A_0(\eta)}\Delta\eta \quad \text{for } s,t \in [t_0,\infty)_{\mathbb{T}}. \tag{5.4}$$

Theorem 5.4. *Suppose that (A1), (A2), and (A3) hold, and for every* $t_1 \in [t_0,\infty)_{\mathbb{T}}$*, the dynamic equation*

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \frac{1}{A(\alpha_{\max}(t),t_1)}\left(\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)A(\alpha_i(t),t_1)\right)x(\alpha_{\max}(t)) = 0, \quad t \in [t_2,\infty)_{\mathbb{T}} \tag{5.5}$$

is oscillatory, where $t_2 \in [t_1,\infty)_{\mathbb{T}}$ *satisfies* $\alpha_{\min}(t) > t_1$ *for all* $t \in [t_2,\infty)_{\mathbb{T}}$*, then* (3.1) *is also oscillatory.*

Proof. Assume to the contrary that (3.1) is nonoscillatory, then there exists a solution x of (3.1) such that $x > 0$, $(A_0x^{\Delta})^{\Delta} \le 0$ on $[t_1,\infty)_{\mathbb{T}} \subset [t_0,\infty)_{\mathbb{T}}$. This implies that A_0x^{Δ} is nonincreasing on $[t_1,\infty)_{\mathbb{T}}$, then it follows that

$$x(t) \ge x(t) - x(t_1) = \int_{t_1}^{t}\frac{1}{A_0(\eta)}A_0(\eta)x^{\Delta}(\eta)\Delta\eta \ge A(t,t_1)A_0(t)x^{\Delta}(t) \quad \forall t \in [t_1,\infty)_{\mathbb{T}}, \tag{5.6}$$

or simply by using (5.4),

$$x(t) - A(t,t_1)A_0(t)x^{\Delta}(t) \ge 0 \quad \forall t \in [t_1,\infty)_{\mathbb{T}}. \tag{5.7}$$

Now, let

$$\varphi(t) := \frac{x(t)}{A(t,t_1)} \quad \text{for } t \in (t_1,\infty)_{\mathbb{T}}. \tag{5.8}$$

By the quotient rule, (5.4) and (5.7), we have

$$\varphi^{\Delta}(t) = \frac{A(t,t_1)A_0(t)x^{\Delta}(t) - x(t)}{A(\sigma(t),t_1)A(t,t_1)A_0(t)} \le 0 \quad \forall t \in (t_1,\infty)_{\mathbb{T}}, \tag{5.9}$$

proving that φ is nonincreasing on $(t_1,\infty)_{\mathbb{T}}$. Therefore, for all $i \in [1,n]_{\mathbb{N}}$, we obtain

$$\frac{x(\alpha_{\max}(t))}{A(\alpha_{\max}(t),t_1)} = \varphi(\alpha_{\max}(t)) \le \varphi(\alpha_i(t)) = \frac{x(\alpha_i(t))}{A(\alpha_i(t),t_1)} \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.10}$$

where $t_2 \in [t_1,\infty)_{\mathbb{T}}$ satisfies $\alpha_{\min}(t) > t_1$ for all $t \in [t_2,\infty)_{\mathbb{T}}$. Using (5.10) in (3.1), we see that x solves

$$\left(A_0x^{\Delta}\right)^{\Delta}(t) + \frac{1}{A(\alpha_{\max}(t),t_1)}\left(\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)A(\alpha_i(t),t_1)\right)x(\alpha_{\max}(t)) \le 0 \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.11}$$

which shows that (5.5) is also nonoscillatory by Theorem 3.1. This is a contradiction, and the proof is completed. □

The following theorem can be regarded as the dynamic generalization of Leighton's result (Theorem A).

Theorem 5.5. *Suppose that (A2), (A3), and (A4) hold and that*

$$\int_{t_2}^{\infty} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta)\mathrm{e}_{\ominus(1/(A_0A(\cdot,t_1)))}(\sigma(\eta),\alpha_i(\eta))\Delta\eta = \infty, \tag{5.12}$$

where $t_2 \in (t_1,\infty) \subset [t_0,\infty)_{\mathbb{T}}$*, then every solution of* (3.1) *is oscillatory.*

Proof. Assume to the contrary that (3.1) is nonoscillatory. It follows from Theorem 3.1 and Remark 3.2 that (3.4) has a solution $\Lambda \in \mathrm{C}_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$. Using (3.5) and (5.7), we see that

$$\Lambda(t) \le \frac{1}{A(t,t_1)} \quad \forall t \in [t_2,\infty)_{\mathbb{T}}, \tag{5.13}$$

which together with (3.4) implies that

$$\Lambda^{\Delta}(t) + \sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(1/(A_0A(\cdot,t_1)))}(\sigma(t),\alpha_i(t)) \le 0 \quad \forall t \in [t_2,\infty)_{\mathbb{T}}. \tag{5.14}$$

Integrating the last inequality, we get

$$\Lambda(t) - \Lambda(t_2) + \int_{t_2}^{t} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta) e_{\ominus(1/(A_0 A(\cdot,t_1)))}(\sigma(\eta), \alpha_i(\eta)) \Delta\eta \le 0 \quad \forall t \in [t_2, \infty)_{\mathbb{T}}, \tag{5.15}$$

which is in a contradiction with (5.12). This completes the proof. □

We conclude this section with applications of Theorem 5.5 to delay differential equations and difference equations.

Corollary 5.6. *Let* $A_0 \in C([t_0,\infty)_{\mathbb{R}}, \mathbb{R}^+)$, *for* $i \in [1,n]_{\mathbb{N}}$, $A_i \in C([t_0,\infty)_{\mathbb{R}}, \mathbb{R}_0^+)$, *and* $\alpha_i \in C([t_0,\infty)_{\mathbb{R}}, \mathbb{R})$ *such that* $\alpha_i(t) \le t$ *for all* $t \in [t_0,\infty)_{\mathbb{R}}$ *and* $\lim_{t\to\infty}\alpha_i(t) = \infty$. *If*

$$\lim_{t\to\infty} A(t,t_0) = \infty, \qquad \int_{t_0}^{\infty} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(\eta) \frac{A(\alpha_i(\eta), t_0)}{A(\eta, t_0)} d\eta = \infty, \tag{5.16}$$

where

$$A(t,s) := \int_s^t \frac{1}{A_0(\eta)} d\eta \quad \text{for } s,t \in [t_0,\infty)_{\mathbb{R}}, \tag{5.17}$$

then (1.2) *is oscillatory.*

Corollary 5.7. *Let* $\{A_0(k)\}$ *be a positive sequence, for* $i \in [1,n]_{\mathbb{N}}$, *let* $\{A_i(k)\}$ *be a nonnegative sequence and let* $\{\alpha_i(k)\}$ *be a divergent sequence such that* $\alpha_i(k) \le k+1$ *for all* $k \in [k_0,\infty)_{\mathbb{N}}$. *If*

$$\lim_{k\to\infty} A(k,k_0) = \infty, \qquad \sum_{j=k_0}^{\infty} \sum_{i\in[1,n]_{\mathbb{N}}} A_i(j) \prod_{\ell=\alpha_i(j)}^{j} \frac{A_0(\ell)A(\ell,k_0)}{A_0(\ell)A(\ell,k_0)+1} = \infty, \tag{5.18}$$

where

$$A(k,l) := \sum_{j=l}^{k-1} \frac{1}{A_0(j)} \quad \text{for } l,k \in [k_0,\infty)_{\mathbb{N}}, \tag{5.19}$$

then (1.8) *is oscillatory.*

6. Existence of a Positive Solution

Theorem 6.1. *Suppose that (A2), (A3), and (A4) hold,* $f \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$, *and the first-order dynamic Riccati inequality* (3.4) *has a solution* $\Lambda \in C_{\mathrm{rd}}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}_0^+)$. *Moreover, suppose that there exist* $x_1, x_2 \in \mathbb{R}^+$ *such that* $\varphi(t) \le x_1$ *for all* $t \in [t_{-1}, t_0)_{\mathbb{T}}$ *and* $x_2 \ge \Lambda(t_0)x_1/A_0(t_0)$, *then* (2.1) *admits a positive solution* x *such that* $x(t) \ge x_1$ *for all* $t \in [t_0,\infty)_{\mathbb{T}}$.

Proof. First assume that y is the solution of the following initial value problem:

$$\begin{gathered}\left(A_0 y^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)y(\alpha_i(t))=0 \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},\\ y^{\Delta}(t_0)=\frac{\Lambda(t_0)}{A_0(t_0)}x_1,\quad y(t)\equiv x_1 \quad \text{for } t\in[t_{-1},t_0]_{\mathbb{T}}.\end{gathered}\tag{6.1}$$

Denote

$$z(t):=\begin{cases} x_1 \mathrm{e}_{\Lambda/A_0}(t,t_0) & \text{for } t\in[t_0,\infty)_{\mathbb{T}},\\ x_1 & \text{for } t\in[t_{-1},t_0)_{\mathbb{T}},\end{cases}\tag{6.2}$$

then, by following similar arguments to those in the proof of the part (ii)⇒(iii) of Theorem 3.1, we obtain

$$\begin{aligned} g(t)&:=\left(A_0 z^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)z(\alpha_i(t))\\ &=x_1\mathrm{e}_{\Lambda/A_0}(t,t_0)\left[\Lambda^{\Delta}(t)+\frac{1}{A_0(t)}\Lambda^{\sigma}(t)\Lambda(t)+\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)\mathrm{e}_{\ominus(\Lambda/A_0)}(t,\alpha_i(t))\right]\leq 0\end{aligned}\tag{6.3}$$

for all $t\in[t_0,\infty)_{\mathbb{T}}$. So z is a solution to

$$\begin{gathered}\left(A_0 z^{\Delta}\right)^{\Delta}(t)+\sum_{i\in[1,n]_{\mathbb{N}}} A_i(t)z(\alpha_i(t))=g(t) \quad \text{for } t\in[t_0,\infty)_{\mathbb{T}},\\ z^{\Delta}(t_0)=\frac{\Lambda(t_0)}{A_0(t_0)}x_1,\quad z(t)\equiv x_1 \quad \text{for } t\in[t_{-1},t_0]_{\mathbb{T}}.\end{gathered}\tag{6.4}$$

Theorem 4.4 implies that $y(t)\geq z(t)\geq x_1>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$. By the hypothesis of the theorem, Theorem 4.4, and Corollary 4.8, we have $x(t)\geq y(t)\geq x_1>0$ for all $t\in[t_0,\infty)_{\mathbb{T}}$. This completes the proof for the case $f\equiv 0$ and $g\equiv 0$ on $[t_0,\infty)_{\mathbb{T}}$.

The general case where $f\not\equiv 0$ on $[t_0,\infty)_{\mathbb{T}}$ is also a consequence of Theorem 4.4. □

Let us illustrate the result of Theorem 6.1 with the following example.

Example 6.2. Let $\sqrt{\mathbb{N}_0}:=\{\sqrt{k}:k\in\mathbb{N}_0\}$, and consider the following delay dynamic equation:

$$\begin{gathered}\left(\mathrm{Id}_{\sqrt{\mathbb{N}_0}}x^{\Delta}\right)^{\Delta}(t)+\frac{1}{8t\sqrt{t^2+1}}\left(x(t)+\frac{1}{2}x\left(\sqrt{t^2-1}\right)\right)=\frac{1}{t\sqrt{t^2+1}},\quad t\in[1,\infty)_{\sqrt{\mathbb{N}_0}},\\ x^{\Delta}(1)=2,\quad x(t)\equiv 2 \quad \text{for } t\in[0,1]_{\sqrt{\mathbb{N}_0}},\end{gathered}\tag{6.5}$$

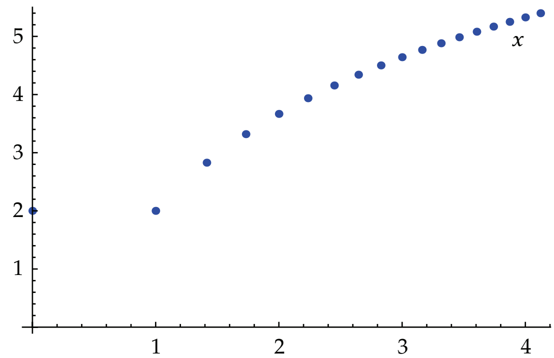

Figure 2: The graph of 15 iterates for the solution of (6.5) illustrates the result of Theorem 6.1.

then (5.1) takes the form $\Phi(t) \le 1$ for all $t \in [1,\infty)_{\sqrt{\mathbb{N}_0}}$, where the function Φ is defined by

$$\Phi(t) := \frac{1}{2t^2 + \left(\sqrt{t^2+1} - t\right)} \left(\sqrt{t^2+1} + \frac{t^2}{2} \left(1 + \frac{t^2-1}{2(t^2-1) + \left(t - \sqrt{t^2-1}\right)} \right) \right) \quad \text{for } t \in [1,\infty)_{\mathbb{R}} \tag{6.6}$$

and is decreasing on $[1,\infty)_{\mathbb{R}}$ and thus is not greater than $\Phi(1) \approx 0.79$, that is, Theorem 5.1 holds. Theorem 6.1 therefore ensures that the solution is positive on $[1,\infty)_{\sqrt{\mathbb{N}_0}}$. For the graph of 15 iterates, see Figure 2.

7. Discussion and Open Problems

We start this section with discussion of explicit nonoscillation conditions for delay differential and difference equations. Let us first consider the continuous case. Corollary 5.6 with $n = 1$ and $\alpha_1(t) = t$ for $t \in [t_0,\infty)_{\mathbb{R}}$ reduces to Theorem A. Nonoscillation part of Kneser's result for (1.4) follows from Corollary 5.2 by letting $n = 1$, $A_0(t) \equiv 1$, and $\alpha_1(t) = t$ for $t \in [t_0,\infty)_{\mathbb{R}}$. Theorem E is obtained by applying Corollary 5.3 to (1.10).

Known nonoscillation tests for difference equations can also be deduced from the results of the present paper. In [18, Lemma 1.2], Chen and Erbe proved that (1.9) is nonoscillatory if and only if there exists a sequence $\{\Lambda(k)\}$ with $A_0(k) + \Lambda(k) > 0$ for all $k \in [k_1,\infty)_{\mathbb{N}}$ and some $k_1 \in [k_0,\infty)_{\mathbb{N}}$ satisfying

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{A_0(k) + \Lambda(k)} + A_1(k) \le 0 \quad \forall k \in [k_1,\infty)_{\mathbb{N}}. \tag{7.1}$$

Since this result is a necessary and sufficient condition, the conclusion of Theorem F could be deduced from

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{1 + \Lambda(k)} + A_1(k) \le 0 \quad \forall k \in [k_1,\infty)_{\mathbb{N}}, \tag{7.2}$$

which is a particular case of (7.1) with $A_0(k) \equiv 1$ for $k \in [k_0, \infty)_{\mathbb{N}}$. We present below a short proof for the nonoscillation part only. Assuming (1.12) and letting

$$\Lambda(k) := \frac{1}{4(k-1)} + \sum_{j=k}^{\infty} A_1(j) \quad \text{for } k \in [k_1, \infty)_{\mathbb{N}} \subset [2, \infty)_{\mathbb{N}}, \tag{7.3}$$

we get

$$\frac{1}{4(k-1)} + \frac{1}{4k} \geq \Lambda(k) \geq \frac{1}{4(k-1)} \quad \forall k \in [k_1, \infty)_{\mathbb{N}}, \tag{7.4}$$

and this yields

$$\Delta\Lambda(k) + \frac{\Lambda^2(k)}{1+\Lambda(k)} + A_1(k) \leq -\frac{1}{4k^2(4k-3)} < 0 \quad \forall\, k \in [k_1, \infty)_{\mathbb{N}}. \tag{7.5}$$

That is, the discrete Riccati inequality (7.2) has a positive solution implying that (1.10) is nonoscillatory. It is not hard to prove that (1.13) implies nonexistence of a sequence $\{\Lambda(k)\}$ satisfying the discrete Riccati inequality (7.2) (see the proof of [23, Lemma 3]). Thus, oscillation/nonoscillation results for (1.10) in [21] can be deduced from nonexistence/existence of a solution for the discrete Riccati inequality (7.2); see also [20].

An application of Theorem 3.1 with $\Lambda(t) := \lambda/t$ for $t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}^+}$ and $\lambda \in \mathbb{R}^+$ implies the following result for quantum scales.

Example 7.1. Let $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$ with $q \in (1, \infty)_{\mathbb{R}}$. If there exist $\lambda \in \mathbb{R}_0^+$ and $t_1 \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}^+}$ such that

$$\frac{\lambda^2}{A_0(t) + (q-1)\lambda} + t^2 \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) \prod_{\eta=\log_q(\alpha_i(t))}^{\log_q(t)} \frac{A_0(q^{\eta})}{A_0(q^{\eta}) + (q-1)\lambda} \leq \frac{\lambda}{q}, \quad t \in [t_1, \infty)_{\overline{q^{\mathbb{Z}}}}, \tag{7.6}$$

then the delay q-difference equation

$$D_q\big(A_0 D_q x\big)(t) + \sum_{i \in [1,n]_{\mathbb{N}}} A_i(t) x(\alpha_i(t)) = 0 \quad \text{for } t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}} \tag{7.7}$$

is nonoscillatory.

In [36], Bohner and Ünal studied nonoscillation and oscillation of the q-difference equation

$$D_q^2 x(t) + \frac{a}{qt^2} x(qt) = 0 \quad \text{for } t \in [t_0, \infty)_{\overline{q^{\mathbb{Z}}}}, \tag{7.8}$$

where $a \in \mathbb{R}_0^+$, and proved that (7.7) is nonoscillatory if and only if

$$a \leq \frac{1}{(\sqrt{q}+1)^2}. \tag{7.9}$$

For the above q-difference equation, (7.6) reduces to the algebraic inequality

$$\frac{\lambda^2}{1+(q-1)\lambda} + \frac{a}{q} \leq \frac{\lambda}{q} \quad \text{or} \quad \lambda^2 - (1-(q-1)a)\lambda + a \leq 0, \tag{7.10}$$

whose discriminant is $(1-(q-1)a)^2 - 4a = (q-1)^2a^2 - (q+1)a + 1$. The discriminant is nonnegative if and only if

$$a \geq \frac{q+2\sqrt{q}+1}{q^2-2q+1} = \frac{1}{(\sqrt{q}-1)^2} \quad \text{or} \quad a \leq \frac{q-2\sqrt{q}+1}{q^2-2q+1} = \frac{1}{(\sqrt{q}+1)^2}. \tag{7.11}$$

If the latter one holds, then the inequality (7.6) holds with an equality for the value

$$\lambda := \frac{1}{2}\left(1-(q-1)a + \sqrt{(1-(q-1)a)^2 - 4a}\right). \tag{7.12}$$

It is easy to check that this value is not less than $2/(\sqrt{q}+1)^2$, that is, the solution is nonnegative. This gives us the nonoscillation part of [36, Theorem 3].

Let us also outline connections to some known results in the theory of second-order ordinary differential equations. For example, the Sturm-Picone comparison theorem is an immediate corollary of Theorem 4.10 if we remark that a solution $\Lambda \in C_{\mathrm{rd}}^1([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ of the inequality (3.32) satisfying $\Lambda/A_0 \in \mathcal{R}^+([t_1,\infty)_{\mathbb{T}},\mathbb{R})$ is also a solution of (3.32) with B_i instead of A_i for $i = 0, 1$.

Proposition 7.2 (see [28, 32, 36]). *Suppose that $B_0(t) \geq A_0(t) > 0$, $A_1(t) \geq 0$, and $A_1(t) \geq B_1(t)$ for all $t \in [t_0,\infty)_{\mathbb{T}}$, then nonoscillation of*

$$\left(A_0 x^\Delta\right)^\Delta(t) + A_1(t)x^\sigma(t) = 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}} \tag{7.13}$$

implies nonoscillation of

$$\left(B_0 x^\Delta\right)^\Delta(t) + B_1(t)x^\sigma(t) = 0 \quad \textit{for } t \in [t_0,\infty)_{\mathbb{T}}. \tag{7.14}$$

The following result can also be regarded as another generalization of the Sturm-Picone comparison theorem. It is easily deduced that there is a solution $\Lambda \in C_{\mathrm{rd}}^1([t_1,\infty)_{\mathbb{T}},\mathbb{R}_0^+)$ of the inequality (3.4).

Proposition 7.3. *Suppose that (A4) and the conditions of Proposition 7.2 are fulfilled, then nonoscillation of*

$$\left(A_0 x^{\Delta}\right)^{\Delta}(t) + A_1(t)x(t) = 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{T}} \tag{7.15}$$

implies the same for

$$\left(B_0 x^{\Delta}\right)^{\Delta}(t) + B_1(t)x(t) = 0 \quad \textit{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{7.16}$$

Finally, let us present some open problems. To this end, we will need the following definition.

Definition 7.4. A solution x of (3.1) is said to be *slowly oscillating* if for every $t_1 \in [t_0, \infty)_{\mathbb{T}}$ there exist $t_2 \in (t_1, \infty)_{\mathbb{T}}$ with $\alpha_{\min}(t) \geq t_1$ for all $t \in [t_2, \infty)_{\mathbb{T}}$ and $t_3 \in [t_2, \infty)_{\mathbb{T}}$ such that $x(t_1)x^{\sigma}(t_1) \leq 0$, $x(t_2)x^{\sigma}(t_2) \leq 0$, $x(t) > 0$ for all $t \in (t_1, t_2)_{\mathbb{T}}$.

Following the method of [8, Theorem 10], we can demonstrate that if (A1), (A2) with positive coefficients and (A3) hold, then the existence of a slowly oscillating solution of (3.1) which has infinitely many zeros implies oscillation of all solutions.

(P1) Generally, will existence of a slowly oscillating solution imply oscillation of all solutions? To the best of our knowledge, slowly oscillating solutions have not been studied for difference equations yet, the only known result is [9, Proposition 5.2].

All the results of the present paper are obtained under the assumptions that all coefficients of (3.1) are nonnegative, and if some of them are negative, it is supposed that the equation with the negative terms omitted has a positive solution.

(P2) Obtain sufficient nonoscillation conditions for (3.1) with coefficients of an arbitrary sign, not assuming that all solutions of the equation with negative terms omitted are nonoscillatory. In particular, consider the equation with one oscillatory coefficient.

(P3) Describe the asymptotic and the global properties of nonoscillatory solutions.

(P4) Deduce nonoscillation conditions for linear second-order impulsive equations on time scales, where both the solution and its derivative are subject to the change at impulse points (and these changes can be matched or not). The results of this type for second-order delay differential equations were obtained in [37].

(P5) Consider the same equation on different time scales. In particular, under which conditions will nonoscillation of (1.8) imply nonoscillation of (1.2)?

(P6) Obtain nonoscillation conditions for neutral delay second-order equations. In particular, for difference equations some results of this type (a necessary oscillation conditions) can be found in [17].

(P7) In the present paper, all parameters of the equation are rd-continuous which corresponds to continuous delays and coefficients for differential equations. However, in [8], nonoscillation of second-order equations is studied under a more general assumption that delays and coefficients are Lebesgue measurable functions. Can the restrictions of rd-continuity of the parameters be relaxed to involve, for example, discontinuous coefficients which arise in the theory of impulsive equations?

Appendix

Time Scales Essentials

A *time scale*, which inherits the standard topology on $\mathbb{R}$, is a nonempty closed subset of reals. Here, and later throughout this paper, a time scale will be denoted by the symbol $\mathbb{T}$, and the intervals with a subscript $\mathbb{T}$ are used to denote the intersection of the usual interval with $\mathbb{T}$. For $t \in \mathbb{T}$, we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf(t,\infty)_{\mathbb{T}}$ while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$, and the *graininess function* $\mu : \mathbb{T} \to \mathbb{R}_0^+$ is defined to be $\mu(t) := \sigma(t) - t$. A point $t \in \mathbb{T}$ is called *right dense* if $\sigma(t) = t$ and/or equivalently $\mu(t) = 0$ holds; otherwise, it is called *right scattered*, and similarly *left dense* and *left scattered* points are defined with respect to the backward jump operator. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}$, the Δ-derivative $f^{\Delta}(t)$ of f at the point t is defined to be the number, provided it exists, with the property that, for any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\left|\left[f^{\sigma}(t) - f(s)\right] - f^{\Delta}(t)\left[\sigma(t) - s\right]\right| \leq \varepsilon|\sigma(t) - s| \quad \forall s \in U, \tag{A.1}$$

where $f^{\sigma} := f \circ \sigma$ on $\mathbb{T}$. We mean the Δ-derivative of a function when we only say derivative unless otherwise is specified. A function f is called *rd-continuous* provided that it is continuous at right-dense points in $\mathbb{T}$ and has a finite limit at left-dense points, and the *set of rd-continuous functions* is denoted by $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$. The set of functions $C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ includes the functions whose derivative is in $C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ too. For a function $f \in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the so-called *simple useful formula* holds

$$f^{\sigma}(t) = f(t) + \mu(t) f^{\Delta}(t) \quad \forall t \in \mathbb{T}^{\kappa}, \tag{A.2}$$

where $\mathbb{T}^{\kappa} := \mathbb{T} \setminus \{\sup \mathbb{T}\}$ if $\sup \mathbb{T} = \max \mathbb{T}$ and satisfies $\rho(\max \mathbb{T}) \neq \max \mathbb{T}$; otherwise, $\mathbb{T}^{\kappa} := \mathbb{T}$. For $s, t \in \mathbb{T}$ and a function $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$, the Δ-integral of f is defined by

$$\int_s^t f(\eta)\Delta\eta = F(t) - F(s) \quad \text{for } s,t \in \mathbb{T}, \tag{A.3}$$

where $F \in C^1_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is an antiderivative of f, that is, $F^{\Delta} = f$ on $\mathbb{T}^{\kappa}$. Table 1 gives the explicit forms of the forward jump, graininess, Δ-derivative, and Δ-integral on the well-known time scales of reals, integers, and the quantum set, respectively.

A function $f \in C_{\mathrm{rd}}(\mathbb{T},\mathbb{R})$ is called *regressive* if $1 + \mu f \neq 0$ on $\mathbb{T}^{\kappa}$, and *positively regressive* if $1 + \mu f > 0$ on $\mathbb{T}^{\kappa}$. The *set of regressive functions* and the *set of positively regressive functions* are denoted by $\mathcal{R}(\mathbb{T},\mathbb{R})$ and $\mathcal{R}^{+}(\mathbb{T},\mathbb{R})$, respectively, and $\mathcal{R}^{-}(\mathbb{T},\mathbb{R})$ is defined similarly.

Table 1: Forward jump, Δ-derivative, and Δ-integral.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\sigma(t)$	t	$t+1$	qt
$f^{\Delta}(t)$	$f'(t)$	$\Delta f(t)$	$D_q f(t) := (f(qt)-f(t))/((q-1)t)$
$\int_s^t f(\eta)\Delta\eta$	$\int_s^t f(\eta)d\eta$	$\sum_{\eta=s}^{t-1} f(\eta)$	$\int_s^t f(\eta)d_q\eta := (q-1)\sum_{\eta=\log_q(s)}^{\log_q(t/q)} f(q^\eta)q^\eta$

Table 2: The exponential function.

$\mathbb{T}$	$\mathbb{R}$	$\mathbb{Z}$	$\overline{q^{\mathbb{Z}}}$, $(q>1)$
$\mathrm{e}_f(t,s)$	$\exp\{\int_s^t f(\eta)d\eta\}$	$\prod_{\eta=s}^{t-1}(1+f(\eta))$	$\prod_{\eta=\log_q(s)}^{\log_q(t/q)}(1+(q-1)q^\eta f(q^\eta))$

Let $f \in \mathcal{R}(\mathbb{T},\mathbb{R})$, then the *exponential function* $\mathrm{e}_f(\cdot,s)$ on a time scale $\mathbb{T}$ is defined to be the unique solution of the initial value problem

$$\begin{gathered} x^{\Delta}(t) = f(t)x(t) \quad \text{for } t\in\mathbb{T}^{\kappa}, \\ x(s) = 1 \end{gathered} \tag{A.4}$$

for some fixed $s\in\mathbb{T}$. For $h\in\mathbb{R}^+$, set $\mathbb{C}_h := \{z\in\mathbb{C} : z\neq -1/h\}$, $\mathbb{Z}_h := \{z\in\mathbb{C} : -\pi/h < \mathrm{Im}(z) \leq \pi/h\}$, and $\mathbb{C}_0 := \mathbb{Z}_0 := \mathbb{C}$. For $h\in\mathbb{R}_0^+$, we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) := \begin{cases} z, & h=0, \\ \dfrac{1}{h}\mathrm{Log}(1+hz), & h>0 \end{cases} \tag{A.5}$$

for $z\in\mathbb{C}_h$, then the exponential function can also be written in the form

$$\mathrm{e}_f(t,s) := \exp\left\{\int_s^t \xi_{\mu(\eta)}(f(\eta))\Delta\eta\right\} \quad \text{for } s,t\in\mathbb{T}. \tag{A.6}$$

Table 2 illustrates the explicit forms of the exponential function on some well-known time scales.

The exponential function $\mathrm{e}_f(\cdot,s)$ is strictly positive on $[s,\infty)_{\mathbb{T}}$ if $f \in \mathcal{R}^+([s,\infty)_{\mathbb{T}},\mathbb{R})$, while $\mathrm{e}_f(\cdot,s)$ alternates in sign at right-scattered points of the interval $[s,\infty)_{\mathbb{T}}$ provided that $f \in \mathcal{R}^-([s,\infty)_{\mathbb{T}},\mathbb{R})$. For $h\in\mathbb{R}_0^+$, let $z,w\in\mathbb{C}_h$, the *circle plus* $\oplus_h$ and the *circle minus* $\ominus_h$ are defined by $z\oplus_h w := z+w+hzw$ and $z\ominus_h w := (z-w)/(1+hw)$, respectively. Further throughout the paper, we will abbreviate the operations $\oplus_\mu$ and $\ominus_\mu$ simply by $\oplus$ and $\ominus$, respectively. It is also known that $\mathcal{R}^+(\mathbb{T},\mathbb{R})$ is a subgroup of $\mathcal{R}(\mathbb{T},\mathbb{R})$, that is, $0\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$, $f,g\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$ implies $f\oplus_\mu g\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$ and $\ominus_\mu f\in\mathcal{R}^+(\mathbb{T},\mathbb{R})$, where $\ominus_\mu f := 0\ominus_\mu f$ on $\mathbb{T}$.

The readers are referred to [32] for further interesting details in the time scale theory.

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Research Article

On a Maximal Number of Period Annuli

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We consider equation $x'' + g(x) = 0$, where $g(x)$ is a polynomial, allowing the equation to have multiple period annuli. We detect the maximal number of possible period annuli for polynomials of odd degree and show how the respective optimal polynomials can be constructed.

1. Introduction

Consider equation

$$x'' + g(x) = 0, \tag{1.1}$$

where $g(x)$ is an odd degree polynomial with simple zeros.

The equivalent differential system

$$x' = y, \qquad y' = -g(x) \tag{1.2}$$

has critical points at $(p_i, 0)$, where p_i are zeros of $g(x)$. Recall that a critical point O of (1.2) is *a center* if it has a punctured neighborhood covered with nontrivial cycles.

We will use the following definitions.

Definition 1.1 (see [1]). A *central region* is the largest connected region covered with cycles surrounding O.

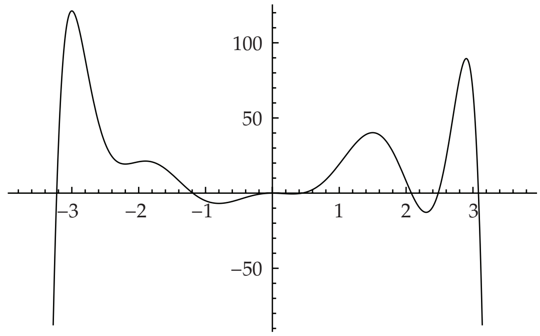

Figure 1

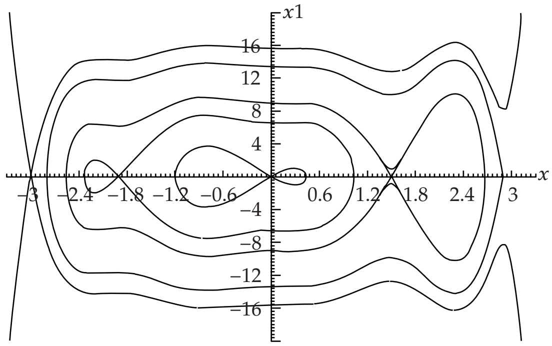

Figure 2: The phase portrait for (1.1), where $G(x)$ is as in Figure 1.

Definition 1.2 (see [1]). A *period annulus* is every connected region covered with nontrivial concentric cycles.

Definition 1.3. We will call a period annulus associated with a central region *a trivial period annulus.* Periodic trajectories of a trivial period annulus encircle exactly one critical point of the type center.

Definition 1.4. Respectively, a period annulus enclosing several (more than one) critical points will be called a *nontrivial period annulus.*

For example, there are four central regions and three nontrivial period annuli in the phase portrait depicted in Figure 2.

Period annuli are the continua of periodic solutions. They can be used for constructing examples of nonlinear equations which have a prescribed number of solutions to the Dirichlet problem

$$x'' + g(x) = 0, \qquad x(0) = 0, \qquad x(1) = 0, \tag{1.3}$$

or a given number of positive solutions [2] to the same problem.

Under certain conditions, period annuli of (1.1) give rise to limit cycles in a dissipative equation

$$x'' + f(x)x' + g(x) = 0. \tag{1.4}$$

The Liénard equation with a quadratical term

$$x'' + f(x)x'^2 + g(x) = 0 \tag{1.5}$$

can be reduced to the form (1.1) by Sabatini's transformation [3]

$$u := \Phi(x) = \int_0^x e^{F(s)} ds, \tag{1.6}$$

where $F(x) = \int_0^x f(s)ds$. Since $du/dx > 0$, this is one-to-one correspondence and the inverse function $x = x(u)$ is well defined.

Lemma 1.5 (see [3, Lemma 1]). *The function $x(t)$ is a solution of* (1.5) *if and only if $u(t) = \Phi(x(t))$ is a solution to*

$$u'' + g(x(u))e^{F(x(u))} = 0. \tag{1.7}$$

Our task in this article is to define the maximal number of nontrivial period annuli for (1.1).

(A) We suppose that $g(x)$ is an odd degree polynomial with simple zeros and with a negative coefficient at the principal term (so $g(-\infty) = +\infty$ and $g(+\infty) = -\infty$). A zero z is called simple if $g(z) = 0$ and $g'(z) \neq 0$.

The graph of a primitive function $G(x) = \int_0^x g(s)ds$ is an even degree polynomial with possible multiple local maxima.

The function $g(x) = -x(x^2 - p^2)(x^2 - q^2)$ is a sample.

We discuss nontrivial period annuli in Section 2. In Section 3, a maximal number of *regular pairs* is detected. Section 4 is devoted to construction of polynomials $g(x)$ which provide the maximal number of *regular pairs* or, equivalently, nontrivial period annuli in (1.1).

2. Nontrivial Period Annuli

The result below provides the criterium for the existence of nontrivial period annuli.

Theorem 2.1 (see [4]). *Suppose that $g(x)$ in* (1.1) *is a polynomial with simple zeros. Assume that M_1 and M_2 ($M_1 < M_2$) are nonneighboring points of maximum of the primitive function $G(x)$. Suppose that any other local maximum of $G(x)$ in the interval (M_1, M_2) is (strictly) less than* $\min\{G(M_1); G(M_2)\}$.

Then, there exists a nontrivial period annulus associated with a pair (M_1, M_2).

It is evident that if $G(x)$ has m pairs of non-neighboring points of maxima then m nontrivial period annuli exist.

Consider, for example, (1.1), where

$$g(x) = -x(x+3)(x+2.2)(x+1.9)(x+0.8)(x-0.3)(x-1.5)(x-2.3)(x-2.9). \tag{2.1}$$

The equivalent system has alternating "saddles" and "centers", and the graph of $G(x)$ is depicted in Figure 1.

There are three pairs of non-neighboring points of maxima and three nontrivial period annuli exist, which are depicted in Figure 2.

3. Polynomials

Consider a polynomial $G(x)$. Points of local maxima x_i and x_j of $G(x)$ are non-neighboring if the interval (x_i, x_j) contains at least one point of local maximum of $G(x)$.

Definition 3.1. Two non-neighboring points of maxima $x_i < x_j$ of $G(x)$ will be called a *regular pair* if $G(x) < \min\{G(x_i), G(x_j)\}$ at any other point of maximum lying in the interval (x_i, x_j).

Theorem 3.2. *Suppose $g(x)$ is a polynomial which satisfies the condition A. Let $G(x)$ be a primitive function for $g(x)$ and n a number of local maxima of $G(x)$.*

Then, the maximal possible number of regular pairs is $n-2$.

Proof. By induction, let $x_1, x_2, \dots, x_n$ be successive points of maxima of $G(x)$, $x_1 < x_2 < \cdots < x_n$.

(1) Let $n = 3$. The following combinations are possible at three points of maxima:

(a) $G(x_1) \geq G(x_2) \geq G(x_3)$,

(b) $G(x_2) < G(x_1), G(x_2) < G(x_3)$,

(c) $G(x_1) \leq G(x_2) \leq G(x_3)$,

(d) $G(x_2) \geq G(x_1), G(x_2) \geq G(x_3)$.

Only the case (b) provides *a regular pair*. In this case, therefore, the maximal number of *regular pairs* is 1.

(2) Suppose that for any sequence of $n > 3$ ordered points of maxima of $G(x)$ the maximal number of *regular pairs* is $n-2$. Without loss of generality, add to the right one more point of maximum of the function $G(x)$. We get a sequence of $n+1$ consecutive points of maximum $x_1, x_2, \dots, x_n, x_{n+1}$, $x_1 < x_2 < \dots < x_n < x_{n+1}$. Let us prove that the maximal number of *regular pairs* is $n-1$. For this, consider the following possible variants.

(a) The couple x_1, x_n is *a regular pair*. If $G(x_1) > G(x_n)$ and $G(x_{n+1}) > G(x_n)$, then, beside the *regular pairs* in the interval $[x_1, x_n]$, only one new *regular pair* can appear, namely, x_1, x_{n+1}. Then, the maximal number of *regular pairs* which can be composed of the points $x_1, x_2, \dots, x_n, x_{n+1}$, is not greater than $(n-2)+1 = n-1$. If $G(x_1) \leq G(x_n)$ or $G(x_{n+1}) \leq G(x_n)$, then the additional *regular pair* does not appear. In a particular case $G(x_2) < G(x_3) < \cdots < G(x_n) < G(x_{n+1})$ and $G(x_1) > G(x_n)$ the following *regular pairs* exist, namely, x_1 and x_3, x_1 and x_4,…, x_1 and x_n, and the new pair x_1 and x_{n+1} appears, totally $n-1$ pairs.

(b) Suppose that x_1, x_n is not *a regular pair*. Let x_i and x_j be a *regular pair*, $1 \le i < j \le n$, and there is no other *regular pair* x_p, x_q such that $1 \le p \le i < j \le q \le n$. Let us mention that if such a pair x_i, x_j does not exist, then the function $G(x)$ does not have *regular pairs* at all and the sequence $\{G(x_k)\}$, $k = 1, \cdots, n$, is monotone. Then, if $G(x_{n+1})$ is greater than any other maximum, there are exactly $(n+1) - 2 = n - 1$ *regular pairs*.

Otherwise, we have two possibilities:

$$\text{either } G(x_i) \ge G(x_p), \quad p = 1, \ldots, i-1,$$
$$\text{or } G(x_j) \ge G(x_q), \quad q = j+1, \ldots, n.$$

In the first case, the interval $[x_1, x_i]$ contains i points of maximum of $G(x)$, $i < n$, and hence the number of *regular pairs* in this interval does not exceed $i - 2$. There are no *regular pairs* x_p, x_k for $1 \le p < i$, $i < k \le n+1$. The interval $[x_i, x_{n+1}]$ contains $(n+1) - (i-1)$ points of maximum of $G(x)$, and hence the number of *regular pairs* in this interval does not exceed $(n+1) - (i-1) - 2 = n - i$. Totally, there are no more *regular pairs* than $(i-2) + (n-i) = n-2$.

In the second case, the number of *regular pairs* in $[x_i, x_j]$ does not exceed $j - (i-1) - 2 = j - i - 1$. In $[x_j, x_{n+1}]$,there are no more than $(n+1) - (j-1) - 2 = n - j$ *regular pairs*. The points x_p, $p = 1, \ldots, i-1$, x_q, $j < q \le n$ do not form *regular pairs*, by the choice of x_p and x_q. The points $x_p, p = 1, \ldots, i$, together with x_{n+1} (it serves as the $i+1$th point in a collection of points) form not more than $(i+1) - 2 = i - 1$ *regular pairs.* Totally, the number of *regular pairs* is not greater than $(j-i-1) + (n-j) + (i-1) = n-2$. □

4. Existence of Polynomials with Optimal Distribution

Theorem 4.1. *Given number n, a polynomial $g(x)$ can be constructed such that*

(a) *the condition (A) is satisfied,*

(b) *the primitive function $G(x)$ has exactly n points of maximum and the number of regular pairs is exactly $n-2$.*

Proof of the Theorem. Consider the polynomial

$$G(x) = -\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{5}{2}\right)\left(x - \frac{5}{2}\right)\left(x + \frac{7}{2}\right)\left(x - \frac{7}{2}\right). \tag{4.1}$$

It is an even function with the graph depicted in Figure 3.

Consider now the polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2} + \varepsilon\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\left(x + \frac{5}{2}\right)\left(x - \frac{5}{2}\right)\left(x + \frac{7}{2}\right)\left(x - \frac{7}{2}\right), \tag{4.2}$$

where $\varepsilon > 0$ is small enough. The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Figure 4.

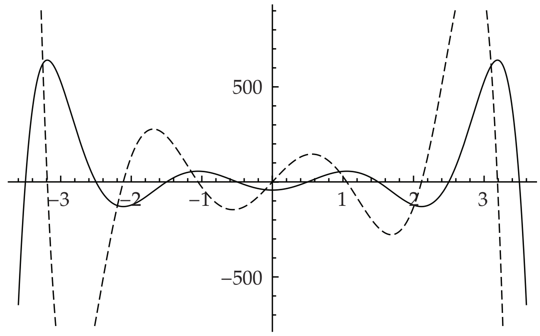

Figure 3: $G(x)$ (solid) and $G'(x) = g(x)$ (dashed).

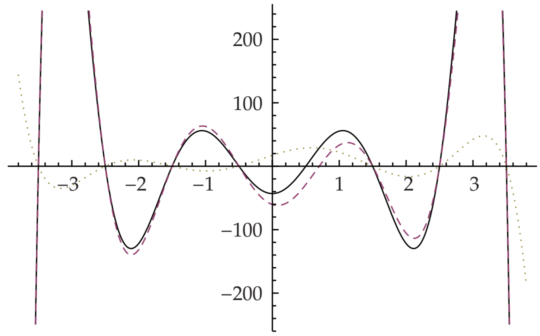

Figure 4: $G(x)$ (solid line), $G_\varepsilon(x)$ (dashed line), and $G(x) - G_\varepsilon(x)$ (dotted line).

Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the right of $x = 0$ m_1^+, m_2^+. Denote the maximal values of $G(x)$ and $G_\varepsilon(x)$ to the left of $x = 0$ m_1^-, m_2^-. One has for $G(x)$ that $m_1^+ = m_1^- < m_2^- = m_2^+$. One has for $G_\varepsilon(x)$ that $m_1^+ < m_1^- < m_2^+ < m_2^-$. Then, there are two regular pairs (resp., m_1^- and m_2^+, m_2^+ and m_2^-).

For arbitrary even n the polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2}\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\cdots\left(x + \frac{2n-1}{2}\right)\left(x - \frac{2n-1}{2}\right), \tag{4.3}$$

is to be considered where the maximal values $m_1^+, m_2^+, \ldots, m_{n/2}^+$ to the right of $x = 0$ form ascending sequence, and, respectively, the maximal values $m_1^-, m_2^-, \ldots, m_{n/2}^-$ to the left of $x = 0$ also form ascending sequence. One has that $m_i^+ = m_i^-$ for all i. For a slightly modified polynomial

$$G_\varepsilon(x) = -\left(x + \frac{1}{2} + \varepsilon\right)\left(x - \frac{1}{2}\right)\left(x + \frac{3}{2}\right)\left(x - \frac{3}{2}\right)\ldots\left(x + \frac{2n-1}{2}\right)\left(x - \frac{2n-1}{2}\right), \tag{4.4}$$

the maximal values are arranged as

$$m_1^+ < m_1^- < m_2^+ < m_2^- < \cdots < m_{n/2}^+ < m_{n/2}^-. \tag{4.5}$$

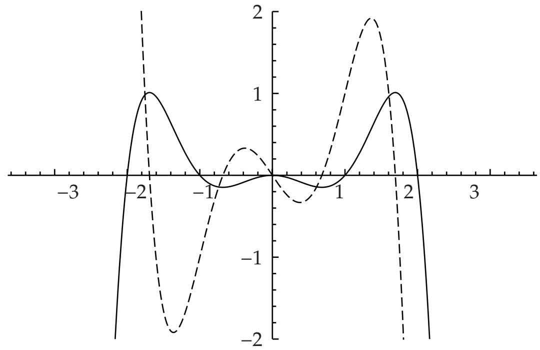

Figure 5: $G(x)$ (solid) and $G'(x) = g(x)$ (dashed).

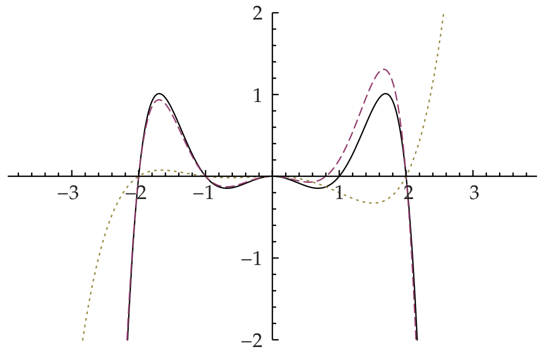

Figure 6: $G(x)$ (solid), $G_\varepsilon(x)$ (dashed), and $G(x) - G_\varepsilon(x)$ (dotted).

Therefore, there exist exactly $n - 2$ *regular pairs* and, consequently, $n - 2$ nontrivial period annuli in the differential equation (1.1).

If n is odd, then the polynomial

$$G(x) = -x^2(x-1)(x+1)(x-2)(x+2)\cdots(x-(n-1))(x+(n-1)) \tag{4.6}$$

with n local maxima is to be considered. The maxima are descending for $x < 0$ and ascending if $x > 0$. The polynomial with three local maxima is depicted in Figure 5.

The slightly modified polynomial

$$G(x) = -x^2(x-1-\varepsilon)(x+1)(x-2)(x+2)\cdots(x-(n-1))(x+(n-1)) \tag{4.7}$$

has maxima which are not equal and are arranged in an optimal way in order to produce the maximal $(n-2)$ *regular pairs*.

The graph of $G_\varepsilon(x)$ with $\varepsilon = 0.2$ is depicted in Figure 6. □

Acknowledgments

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Research Article

On Nonoscillation of Advanced Differential Equations with Several Terms

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Existence of positive solutions for advanced equations with several terms $\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0$, $h_k(t) \geq t$ is investigated in the following three cases: (a) all coefficients a_k are positive; (b) all coefficients a_k are negative; (c) there is an equal number of positive and negative coefficients. Results on asymptotics of nonoscillatory solutions are also presented.

1. Introduction

This paper deals with nonoscillation properties of scalar advanced differential equations. Advanced differential equations appear in several applications, especially as mathematical models in economics; an advanced term may, for example, reflect the dependency on anticipated capital stock [1, 2].

It is not quite clear how to formulate an initial value problem for such equations, and existence and uniqueness of solutions becomes a complicated issue. To study oscillation, we need to assume that there exists a solution of such equation on the halfline. In the beginning of 1980s, sufficient oscillation conditions for first-order linear advanced equations with constant coefficients and deviations of arguments were obtained in [3] and for nonlinear equations in [4]. Later oscillation properties were studied for other advanced and mixed differential equations (see the monograph [5], the papers [6–12] and references therein). Overall, these publications mostly deal with sufficient oscillation conditions; there are only few results [7, 9, 12] on existence of positive solutions for equations with several advanced terms and variable coefficients, and the general nonoscillation theory is not complete even for first-order linear equations with variable advanced arguments and variable coefficients of the same sign.

The present paper partially fills up this gap. We obtain several nonoscillation results for advanced equations using the generalized characteristic inequality [13]. The main method of this paper is based on fixed point theory; thus, we also state the existence of a solution in certain cases.

In the linear case, the best studied models with advanced arguments were the equations of the types

$$\begin{gathered}\dot{x}(t)-a(t)x(h(t))+b(t)x(t)=0,\\ \dot{x}(t)-a(t)x(t)+b(t)x(g(t))=0,\end{gathered}\tag{1.1}$$

where $a(t)\geq 0$, $b(t)\geq 0$, $h(t)\geq t$, and $g(t)\geq t$.

Let us note that oscillation of higher order linear and nonlinear equations with advanced and mixed arguments was also extensively investigated, starting with [14]; see also the recent papers [15–19] and references therein.

For equations with an advanced argument, the results obtained in [20, 21] can be reformulated as Theorems A–C below.

Theorem A (see [20]). *If a, b, and h are equicontinuous on $[0,\infty)$, $a(t)\geq 0$, $b(t)\geq 0$, $h(t)\geq t$, and $\limsup_{t\to\infty}[h(t)-t]<\infty$, then the advanced equation*

$$\dot{x}(t)+a(t)x(h(t))+b(t)x(t)=0\tag{1.2}$$

has a nonoscillatory solution.

In the present paper, we extend Theorem A to the case of several deviating arguments and coefficients (Theorem 2.10).

Theorem B (see [20]). *If a, b, and h are equicontinuous on $[0,\infty)$, $a(t)\geq 0$, $b(t)\geq 0$, $h(t)\geq t$, $\limsup_{t\to\infty}[h(t)-t]<\infty$, and*

$$\limsup_{t\to\infty}\int_t^{h(t)}a(s)\exp\left\{\int_s^{h(s)}b(\tau)d\tau\right\}ds<\frac{1}{e},\tag{1.3}$$

then the advanced equation

$$\dot{x}(t)-a(t)x(h(t))-b(t)x(t)=0\tag{1.4}$$

has a nonoscillatory solution.

Corollary 2.3 of the present paper extends Theorem B to the case of several coefficients $a_k\geq 0$ and advanced arguments h_k (generally, $b(t)\equiv 0$); if

$$\int_t^{\max_k h_k(t)}\sum_{i=1}^{m}a_i(s)ds\leq\frac{1}{e},\tag{1.5}$$

then the equation

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0 \tag{1.6}$$

has an eventually positive solution. To the best of our knowledge, only the opposite inequality (with $\min_k h_k(t)$ rather than $\max_k h_k(t)$ in the upper bound) was known as a sufficient oscillation condition. Coefficients and advanced arguments are also assumed to be of a more general type than in [20]. Comparison to equations with constant arguments deviations, and coefficients (Corollary 2.8) is also outlined.

For advanced equations with coefficients of different sign, the following result is known.

Theorem C (see [21]). *If $0 \le a(t) \le b(t)$ and $h(t) \ge t$, then there exists a nonoscillatory solution of the equation*

$$\dot{x}(t) - a(t)x(h(t)) + b(t)x(t) = 0. \tag{1.7}$$

This result is generalized in Theorem 2.13 to the case of several positive and negative terms and several advanced arguments; moreover, positive terms can also be advanced as far as the advance is not greater than in the corresponding negative terms.

We also study advanced equations with positive and negative coefficients in the case when positive terms dominate rather than negative ones; some sufficient nonoscillation conditions are presented in Theorem 2.15; these results are later applied to the equation with constant advances and coefficients. Let us note that analysis of nonoscillation properties of the mixed equation with a positive advanced term

$$\dot{x}(t) + a(t)x(h(t)) - b(t)x\big(g(t)\big) = 0, \quad h(t) \ge t,\ g(t) \le t,\ a(t) \ge 0,\ b(t) \ge 0 \tag{1.8}$$

was also more complicated compared to other cases of mixed equations with positive and negative coefficients [21].

In nonoscillation theory, results on asymptotic properties of nonoscillatory solutions are rather important; for example, for equations with several delays and positive coefficients, all nonoscillatory solutions tend to zero if the integral of the sum of coefficients diverges; under the same condition for negative coefficients, all solutions tend to infinity. In Theorems 2.6 and 2.11, the asymptotic properties of nonoscillatory solutions for advanced equations with coefficients of the same sign are studied.

The paper is organized as follows. Section 2 contains main results on the existence of nonoscillatory solutions to advanced equations and on asymptotics of these solutions: first for equations with coefficients of the same sign, then for equations with both positive and negative coefficients. Section 3 involves some comments and open problems.

2. Main Results

Consider first the equation

$$\dot{x}(t) - \sum_{k=1}^{m} a_k(t)x(h_k(t)) = 0, \tag{2.1}$$

under the following conditions:

(a1) $a_k(t) \geq 0$, $k = 1, \ldots, m$, are Lebesgue measurable functions locally essentially bounded for $t \geq 0$,

(a2) $h_k : [0, \infty) \to \mathbb{R}$ are Lebesgue measurable functions, $h_k(t) \geq t$, $k = 1, \ldots, m$.

Definition 2.1. A locally absolutely continuous function $x : [t_0, \infty) \to R$ is called *a solution of problem* (2.1) if it satisfies (2.1) for almost all $t \in [t_0, \infty)$.

The same definition will be used for all further advanced equations.

Theorem 2.2. *Suppose that the inequality*

$$u(t) \geq \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u(s)ds\right\}, \quad t \geq t_0 \tag{2.2}$$

has a nonnegative solution which is integrable on each interval $[t_0, b]$, *then* (2.1) *has a positive solution for* $t \geq t_0$.

Proof. Let $u_0(t)$ be a nonnegative solution of inequality (2.2). Denote

$$u_{n+1}(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\}, \quad n = 0, 1, \ldots, \tag{2.3}$$

then

$$u_1(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} \leq u_0(t). \tag{2.4}$$

By induction we have $0 \leq u_{n+1}(t) \leq u_n(t) \leq u_0(t)$. Hence, there exists a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$. By the Lebesgue convergence theorem, we have

$$u(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{\int_t^{h_k(t)} u(s)ds\right\}. \tag{2.5}$$

Then, the function

$$x(t) = x(t_0) \exp\left\{\int_{t_0}^{t} u(s)ds\right\} \quad \text{for any } x(t_0) > 0 \tag{2.6}$$

is a positive solution of (2.1). □

Corollary 2.3. *If*

$$\int_t^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds \le \frac{1}{e}, \quad t \ge t_0, \tag{2.7}$$

then (2.1) *has a positive solution for* $t \ge t_0$.

Proof. Let $u_0(t) = e\sum_{k=1}^{m} a_k(t)$, then u_0 satisfies (2.2) at any point t where $\sum_{k=1}^{m} a_k(t) = 0$. In the case when $\sum_{k=1}^{m} a_k(t) \neq 0$, inequality (2.7) implies

$$\begin{aligned}
&\frac{u_0(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\}} \\
&\quad\ge \frac{u_0(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{\max_k h_k(t)} u_0(s)ds\right\}} \\
&\quad= \frac{e\sum_{k=1}^{m} a_k(t)}{\sum_{k=1}^{m} a_k(t)\exp\left\{e\int_t^{\max_k h_k(t)} \sum_{i=1}^{m} a_i(s)ds\right\}} \\
&\quad\ge \frac{e\sum_{k=1}^{m} a_k(t)}{\sum_{k=1}^{m} a_k(t)e} = 1.
\end{aligned} \tag{2.8}$$

Hence, $u_0(t)$ is a positive solution of inequality (2.2). By Theorem 2.2, (2.1) has a positive solution for $t \ge t_0$. □

Corollary 2.4. *If there exists* $\sigma > 0$ *such that* $h_k(t) - t \le \sigma$ *and* $\int_0^\infty \sum_{k=1}^{m} a_k(s)ds < \infty$, *then* (2.1) *has an eventually positive solution.*

Corollary 2.5. *If there exists* $\sigma > 0$ *such that* $h_k(t) - t \le \sigma$ *and* $\lim_{t\to\infty} a_k(t) = 0$, *then* (2.1) *has an eventually positive solution.*

Proof. Under the conditions of either Corollary 2.4 or Corollary 2.5, obviously there exists $t_0 \ge 0$ such that (2.7) is satisfied. □

Theorem 2.6. *Let* $\int^\infty \sum_{k=1}^{m} a_k(s)ds = \infty$ *and* x *be an eventually positive solution of* (2.1), *then* $\lim_{t\to\infty} x(t) = \infty$.

Proof. Suppose that $x(t) > 0$ for $t \ge t_1$, then $\dot{x}(t) \ge 0$ for $t \ge t_1$ and

$$\dot{x}(t) \ge \sum_{k=1}^{m} a_k(t)x(t_1), \quad t \ge t_1, \tag{2.9}$$

which implies

$$x(t) \ge x(t_1)\int_{t_1}^{t} \sum_{k=1}^{m} a_k(s)ds. \tag{2.10}$$

Thus, $\lim_{t\to\infty} x(t) = \infty$. □

Consider together with (2.1) the following equation:

$$\dot{x}(t) - \sum_{k=1}^{m} b_k(t) x\left(g_k(t)\right) = 0, \tag{2.11}$$

for $t \geq t_0$. We assume that for (2.11) conditions (a1)-(a2) also hold.

Theorem 2.7. *Suppose that $t \leq g_k(t) \leq h_k(t)$, $0 \leq b_k(t) \leq a_k(t)$, $t \geq t_0$, and the conditions of Theorem 2.2 hold, then* (2.11) *has a positive solution for $t \geq t_0$.*

Proof. Let $u_0(t) \geq 0$ be a solution of inequality (2.2) for $t \geq t_0$, then this function is also a solution of this inequality if $a_k(t)$ and $h_k(t)$ are replaced by $b_k(t)$ and $g_k(t)$. The reference to Theorem 2.2 completes the proof. □

Corollary 2.8. *Suppose that there exist $a_k > 0$ and $\sigma_k > 0$ such that $0 \leq a_k(t) \leq a_k$, $t \leq h_k(t) \leq t + \sigma_k$, $t \geq t_0$, and the inequality*

$$\lambda \geq \sum_{k=1}^{m} a_k e^{\lambda \sigma_k} \tag{2.12}$$

has a solution $\lambda \geq 0$, then (2.1) *has a positive solution for $t \geq t_0$.*

Proof. Consider the equation with constant parameters

$$\dot{x}(t) - \sum_{k=1}^{m} a_k x(t + \sigma_k) = 0. \tag{2.13}$$

Since the function $u(t) \equiv \lambda$ is a solution of inequality (2.2) corresponding to (2.13), by Theorem 2.2, (2.13) has a positive solution. Theorem 2.7 implies this corollary. □

Corollary 2.9. *Suppose that $0 \leq a_k(t) \leq a_k$, $t \leq h_k(t) \leq t + \sigma$ for $t \geq t_0$, and*

$$\sum_{k=1}^{m} a_k \leq \frac{1}{e\sigma}, \tag{2.14}$$

then (2.1) *has a positive solution for $t \geq t_0$.*

Proof. Since $\sum_{k=1}^{m} a_k \leq 1/e\sigma$, the number $\lambda = 1/\sigma$ is a positive solution of the inequality

$$\lambda \geq \left(\sum_{k=1}^{m} a_k\right) e^{\lambda \sigma}, \tag{2.15}$$

which completes the proof. □

Consider now the equation with positive coefficients

$$\dot{x}(t) + \sum_{k=1}^{m} a_k(t) x(h_k(t)) = 0. \tag{2.16}$$

Theorem 2.10. *Suppose that $a_k(t) \geq 0$ are continuous functions bounded on $[t_0, \infty)$ and h_k are equicontinuous functions on $[t_0, \infty)$ satisfying $0 \leq h_k(t) - t \leq \delta$, then* (2.16) *has a nonoscillatory solution.*

Proof. In the space $C[t_0, \infty)$ of continuous functions on $[t_0, \infty)$, consider the set

$$M = \left\{ u \mid 0 \leq u \leq \sum_{k=1}^{m} a_k(t) \right\}, \tag{2.17}$$

and the operator

$$(Hu)(t) = \sum_{k=1}^{m} a_k(t) \exp\left\{ -\int_t^{h_k(t)} u(s) ds \right\}. \tag{2.18}$$

If $u \in M$, then $Hu \in M$.

For the integral operator

$$(Tu)(t) := \int_t^{h_k(t)} u(s) ds, \tag{2.19}$$

we will demonstrate that TM is a compact set in the space $C[t_0, \infty)$. If $u \in M$, then

$$\|(Tu)(t)\|_{C[t_0,\infty)} \leq \sup_{t \geq t_0} \int_t^{t+\delta} |u(s)| ds \leq \sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t) \delta < \infty. \tag{2.20}$$

Hence, the functions in the set TM are uniformly bounded in the space $C[t_0, \infty)$.

Functions h_k are equicontinuous on $[t_0, \infty)$, so for any $\varepsilon > 0$, there exists a $\sigma_0 > 0$ such that for $|t - s| < \sigma_0$, the inequality

$$|h_k(t) - h_k(s)| < \frac{\varepsilon}{2} \left(\sup_{t \geq t_0} \sum_{k=1}^{m} a_k(t) \right)^{-1}, \quad k = 1, \ldots, m \tag{2.21}$$

holds. From the relation

$$\int_{t_0}^{h_k(t_0)} - \int_t^{h_k(t)} = \int_{t_0}^{t} + \int_t^{h_k(t_0)} - \int_t^{h_k(t_0)} - \int_{h_k(t_0)}^{h_k(t)} = \int_{t_0}^{t} - \int_{h_k(t_0)}^{h_k(t)}, \tag{2.22}$$

we have for $|t-t_0| < \min\{\sigma_0, \varepsilon/2\sup_{t\geq t_0}\sum_{k=1}^m a_k(t)\}$ and $u \in M$ the estimate

$$\begin{aligned}
|(Tu)(t)-(Tu)(t_0)| &= \left|\int_{t}^{h_k(t)} u(s) - \int_{t_0}^{h_k(t_0)} u(s)ds\right| \\
&\leq \int_{t_0}^{t} |u(s)|ds + \int_{h_k(t_0)}^{h_k(t)} |u(s)|ds \\
&\leq |t-t_0| \sup_{t\geq t_0} \sum_{k=1}^{m} a_k(t) + |h_k(t)-h_k(t_0)| \sup_{t\geq t_0} \sum_{k=1}^{m} a_k(t) \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned} \tag{2.23}$$

Hence, the set TM contains functions which are uniformly bounded and equicontinuous on $[t_0, \infty)$, so it is compact in the space $C[t_0, \infty)$; thus, the set HM is also compact in $C[t_0, \infty)$.

By the Schauder fixed point theorem, there exists a continuous function $u \in M$ such that $u = Hu$, then the function

$$x(t) = \exp\left\{-\int_{t_0}^{t} u(s)ds\right\} \tag{2.24}$$

is a bounded positive solution of (2.16). Moreover, since u is nonnegative, this solution is nonincreasing on $[t_0, \infty)$. □

Theorem 2.11. *Suppose that the conditions of Theorem 2.10 hold,*

$$\int_{t_0}^{\infty} \sum_{k=1}^{m} a_k(s)ds = \infty, \tag{2.25}$$

and x is a nonoscillatory solution of (2.16), *then* $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t) > 0$ for $t \geq t_0$, then $\dot{x}(t) \leq 0$ for $t \geq t_0$. Hence, $x(t)$ is nonincreasing and thus has a finite limit. If $\lim_{t\to\infty} x(t) = d > 0$, then $x(t) > d$ for any t, and thus $\dot{x}(t) \leq -d\sum_{k=1}^m a_k(t)$ which implies $\lim_{t\to\infty} x(t) = -\infty$. This contradicts to the assumption that $x(t)$ is positive, and therefore $\lim_{t\to\infty} x(t) = 0$. □

Let us note that we cannot guarantee any (exponential or polynomial) rate of convergence to zero even for constant coefficients a_k, as the following example demonstrates.

Example 2.12. Consider the equation $\dot{x}(t) + x(h(t)) = 0$, where $h(t) = t^{t\ln t}$, $t \geq 3$, $x(3) = 1/\ln 3$. Then, $x(t) = 1/(\ln t)$ is the solution which tends to zero slower than t^{-r} for any $r > 0$.

Consider now the advanced equation with positive and negative coefficients

$$\dot{x}(t) - \sum_{k=1}^{m} \left[a_k(t)x(h_k(t)) - b_k(t)x(g_k(t))\right] = 0, \quad t \geq 0. \tag{2.26}$$

Theorem 2.13. *Suppose that $a_k(t)$ and $b_k(t)$ are Lebesgue measurable locally essentially bounded functions, $a_k(t) \geq b_k(t) \geq 0$, $h_k(t)$ and $g_k(t)$ are Lebesgue measurable functions, $h_k(t) \geq g_k(t) \geq t$, and inequality* (2.2) *has a nonnegative solution, then* (2.26) *has a nonoscillatory solution; moreover, it has a positive nondecreasing and a negative nonincreasing solutions.*

Proof. Let u_0 be a nonnegative solution of (2.2) and denote

$$u_{n+1}(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_n(s)ds\right\}\right), \quad t \geq t_0,\ n \geq 0. \tag{2.27}$$

We have $u_0 \geq 0$, and by (2.2),

$$\begin{aligned} u_0 &\geq \sum_{k=1}^{m} a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} \\ &\geq \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_0(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_0(s)ds\right\}\right) = u_1(t). \end{aligned} \tag{2.28}$$

Since $a_k(t) \geq b_k(t) \geq 0$ and $t \leq g_k(t) \leq h_k(t)$, then $u_1(t) \geq 0$.

Next, let us assume that $0 \leq u_n(t) \leq u_{n-1}(t)$. The assumptions of the theorem imply $u_{n+1} \geq 0$. Let us demonstrate that $u_{n+1}(t) \leq u_n(t)$. This inequality has the form

$$\begin{aligned} &\sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_n(s)ds\right\}\right) \\ &\quad\leq \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u_{n-1}(s)ds\right\}\right), \end{aligned} \tag{2.29}$$

which is equivalent to

$$\begin{aligned} &\sum_{k=1}^{m}\exp\left\{\int_t^{h_k(t)} u_n(s)ds\right\}\left(a_k(t) - b_k(t)\exp\left\{-\int_{g_k(t)}^{h_k(t)} u_n(s)ds\right\}\right) \\ &\quad\leq \sum_{k=1}^{m}\exp\left\{\int_t^{h_k(t)} u_{n-1}(s)ds\right\}\left(a_k(t) - b_k(t)\exp\left\{-\int_{g_k(t)}^{h_k(t)} u_{n-1}(s)ds\right\}\right). \end{aligned} \tag{2.30}$$

This inequality is evident for any $0 \leq u_n(t) \leq u_{n-1}(t)$, $a_k(t) \geq 0$, and $b_k \geq 0$; thus, we have $u_{n+1}(t) \leq u_n(t)$.

By the Lebesgue convergence theorem, there is a pointwise limit $u(t) = \lim_{n\to\infty} u_n(t)$ satisfying

$$u(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{h_k(t)} u(s)ds\right\} - b_k(t)\exp\left\{\int_t^{g_k(t)} u(s)ds\right\}\right), \quad t \geq t_0, \tag{2.31}$$

$u(t) \geq 0$, $t \geq t_0$. Then, the function

$$x(t) = x(t_0)\exp\left\{\int_{t_0}^{t} u(s)ds\right\}, \quad t \geq t_0 \tag{2.32}$$

is a positive nondecreasing solution of (2.26) for any $x(t_0) > 0$ and is a negative nonincreasing solution of (2.26) for any $x(t_0) < 0$. □

Corollary 2.14. *Let $a_k(t)$ and $b_k(t)$ be Lebesgue measurable locally essentially bounded functions satisfying $a_k(t) \geq b_k(t) \geq 0$, and let $h_k(t)$ and $g_k(t)$ be Lebesgue measurable functions, where $h_k(t) \geq g_k(t) \geq t$. Assume in addition that inequality* (2.7) *holds. Then,* (2.26) *has a nonoscillatory solution.*

Consider now the equation with constant deviations of advanced arguments

$$\dot{x}(t) - \sum_{k=1}^{m}[a_k(t)x(t+\tau_k) - b_k(t)x(t+\sigma_k)] = 0, \tag{2.33}$$

where a_k, b_k are continuous functions, $\tau_k \geq 0$, $\sigma_k \geq 0$.

Denote $A_k = \sup_{t\geq t_0} a_k(t)$, $a_k = \inf_{t\geq t_0} a_k(t)$, $B_k = \sup_{t\geq t_0} b_k(t)$, $b_k = \inf_{t\geq t_0} b_k(t)$.

Theorem 2.15. *Suppose that $a_k \geq 0$, $b_k \geq 0$, $A_k < \infty$, and $B_k < \infty$.*

If there exists a number $\lambda_0 < 0$ such that

$$\sum_{k=1}^{m}\left(a_k e^{\lambda_0\tau_k} - B_k\right) \geq \lambda_0, \tag{2.34}$$

$$\sum_{k=1}^{m}\left(A_k - b_k e^{\lambda_0\sigma_k}\right) \leq 0, \tag{2.35}$$

then (2.33) *has a nonoscillatory solution; moreover, it has a positive nonincreasing and a negative nondecreasing solutions.*

Proof. In the space $C[t_0, \infty)$, consider the set $M = \{u \mid \lambda_0 \leq u \leq 0\}$ and the operator

$$(Hu)(t) = \sum_{k=1}^{m}\left(a_k(t)\exp\left\{\int_t^{t+\tau_k} u(s)ds\right\} - b_k(t)\exp\left\{\int_t^{t+\sigma_k} u(s)ds\right\}\right). \tag{2.36}$$

For $u \in M$, we have from (2.34) and (2.35)

$$\begin{aligned}(Hu)(t) &\leq \sum_{k=1}^{m}\left(A_k - b_k e^{\lambda_0\sigma_k}\right) \leq 0,\\ (Hu)(t) &\geq \sum_{k=1}^{m}\left(a_k e^{\lambda_0\tau_k} - B_k\right) \geq \lambda_0.\end{aligned} \tag{2.37}$$

Hence, $HM \subset M$.

Consider the integral operator

$$(Tu)(t) := \int_t^{t+\delta} u(s)ds, \quad \delta > 0. \tag{2.38}$$

We will show that TM is a compact set in the space $C[t_0, \infty)$. For $u \in M$, we have

$$\|(Tu)(t)\|_{C[t_0,\infty)} \leq \sup_{t\geq t_0} \int_t^{t+\delta} |u(s)|ds \leq |\lambda_0|\delta. \tag{2.39}$$

Hence, the functions in the set TM are uniformly bounded in the space $C[t_0, \infty)$.

The equality $\int_{t_0}^{t_0+\delta} - \int_t^{t+\delta} = \int_{t_0}^{t} + \int_t^{t_0+\delta} - \int_t^{t_0+\delta} - \int_{t_0+\delta}^{t+\delta} = \int_{t_0}^{t} - \int_{t_0+\delta}^{t+\delta}$ implies

$$\begin{aligned} |(Tu)(t) - (Tu)(t_0)| &= \left| \int_t^{t+\delta} u(s) - \int_{t_0}^{t_0+\delta} u(s)ds \right| \\ &\leq \int_{t_0}^{t} |u(s)|ds + \int_{t_0+\delta}^{t+\delta} |u(s)|ds \leq 2|\lambda_0||t - t_0|. \end{aligned} \tag{2.40}$$

Hence, the set TM and so the set HM are compact in the space $C[t_0, \infty)$.

By the Schauder fixed point theorem, there exists a continuous function u satisfying $\lambda_0 \leq u \leq 0$ such that $u = Hu$; thus, the function

$$x(t) = x(t_0) \exp\left\{ \int_{t_0}^{t} u(s)ds \right\}, \quad t \geq t_0 \tag{2.41}$$

is a positive nonincreasing solution of (2.33) for any $x(t_0) > 0$ and is a negative nondecreasing solution of (2.26) for any $x(t_0) < 0$. □

Let us remark that (2.35) for any $\lambda_0 < 0$ implies $\sum_{k=1}^{m}(A_k - b_k) < 0$.

Corollary 2.16. *Let* $\sum_{k=1}^{m}(A_k - b_k) < 0$, $\sum_{k=1}^{m} A_k > 0$, *and for*

$$\lambda_0 = \frac{\ln\left(\sum_{k=1}^{m} A_k / \sum_{k=1}^{m} b_k\right)}{\max_k \sigma_k}, \tag{2.42}$$

the inequality

$$\sum_{k=1}^{m} \left(a_k e^{\lambda_0 \tau_k} - B_k \right) \geq \lambda_0 \tag{2.43}$$

holds, then (2.33) *has a bounded positive solution.*

Proof. The negative number λ_0 defined in (2.42) is a solution of both (2.34) and (2.35); by definition, it satisfies (2.35), and (2.43) implies (2.34). □

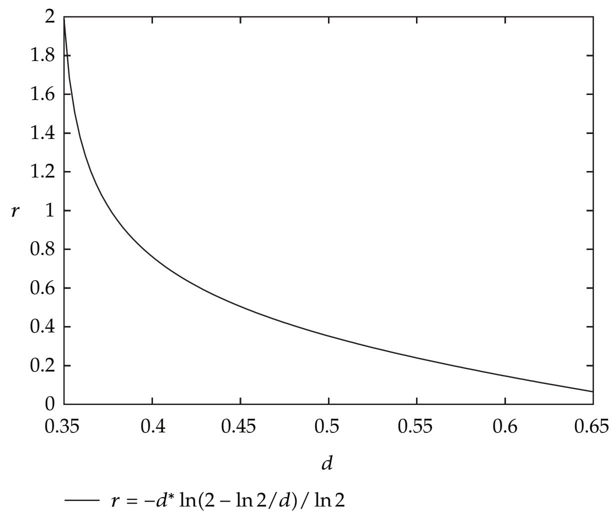

Figure 1: The domain of values (d, r) satisfying inequality (2.47). If the values of advances d and r are under the curve, then (2.44) has a positive solution.

Example 2.17. Consider the equation with constant advances and coefficients

$$\dot{x}(t) - ax(t+r) + bx(t+d) = 0, \tag{2.44}$$

where $0 < a < b$, $d > 0$, $r \geq 0$. Then, $\lambda_0 = (1/d)\ln(a/b)$ is the minimal value of λ for which inequality (2.35) holds; for (2.44), it has the form $a - be^{\lambda d} \leq 0$.

Inequality (2.34) for (2.44) can be rewritten as

$$f(\lambda) = ae^{\lambda r} - b - \lambda \geq 0, \tag{2.45}$$

where the function $f(x)$ decreases on $(-\infty, -\ln(ar)/r]$ if $\tau > 0$ and for any negative x if $r = 0$; besides, $f(0) < 0$. Thus, if $f(\lambda_1) < 0$ for some $\lambda_1 < 0$, then $f(\lambda) < 0$ for any $\lambda \in [\lambda_1, 0)$. Hence, the inequality

$$f(\lambda_0) = a\left(\frac{a}{b}\right)^{r/d} - b - \frac{1}{d}\ln\left(\frac{a}{b}\right) \geq 0 \tag{2.46}$$

is necessary and sufficient for the conditions of Theorem 2.15 to be satisfied for (2.44).

Figure 1 demonstrates possible values of advances d and r, such that Corollary 2.16 implies the existence of a positive solution in the case $1 = a < b = 2$. Then, (2.46) has the form $0.5^{r/d} \geq 2 - (\ln 2)/d$, which is possible only for $d > 0.5\ln 2 \approx 0.347$ and for these values is equivalent to

$$r \leq \frac{-d\ln(2 - \ln 2/d)}{\ln 2}. \tag{2.47}$$

3. Comments and Open Problems

In this paper, we have developed nonoscillation theory for advanced equations with variable coefficients and advances. Most previous nonoscillation results deal with either oscillation or constant deviations of arguments. Among all cited papers, only [8] has a nonoscillation condition (Theorem 2.11) for a partial case of (2.1) (with $h_k(t) = t + \tau_k$), which in this case coincides with Corollary 2.4. The comparison of results of the present paper with the previous results of the authors was discussed in the introduction.

Finally, let us state some open problems and topics for research.

(1) Prove or disprove:
if (2.1), with $a_k(t) \geq 0$, has a nonoscillatory solution, then (2.26) with positive and negative coefficients also has a nonoscillatory solution.

As the first step in this direction, prove or disprove that if $h(t) \geq t$ and the equation

$$\dot{x}(t) - a^{+}(t)x(h(t)) = 0 \tag{3.1}$$

has a nonoscillatory solution, then the equation

$$\dot{x}(t) - a(t)x(h(t)) = 0 \tag{3.2}$$

also has a nonoscillatory solution, where $a^{+}(t) = \max\{a(t), 0\}$.

If these conjectures are valid, obtain comparison results for advanced equations.

(2) Deduce nonoscillation conditions for (2.1) with oscillatory coefficients. Oscillation results for an equation with a constant advance and an oscillatory coefficient were recently obtained in [22].

(3) Consider advanced equations with positive and negative coefficients when the numbers of positive and negative terms do not coincide.

(4) Study existence and/or uniqueness problem for the initial value problem or boundary value problems for advanced differential equations.

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Research Article

On Nonseparated Three-Point Boundary Value Problems for Linear Functional Differential Equations

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For a system of linear functional differential equations, we consider a three-point problem with nonseparated boundary conditions determined by singular matrices. We show that, to investigate such a problem, it is often useful to reduce it to a parametric family of two-point boundary value problems for a suitably perturbed differential system. The auxiliary parametrised two-point problems are then studied by a method based upon a special kind of successive approximations constructed explicitly, whereas the values of the parameters that correspond to solutions of the original problem are found from certain numerical determining equations. We prove the uniform convergence of the approximations and establish some properties of the limit and determining functions.

1. Introduction

The aim of this paper is to show how a suitable parametrisation can help when dealing with nonseparated three-point boundary conditions determined by singular matrices. We construct a suitable numerical-analytic scheme allowing one to approach a three-point boundary value problem through a certain iteration procedure. To explain the term, we recall that, formally, the methods used in the theory of boundary value problems can be characterised as analytic, functional-analytic, numerical, or numerical-analytic ones.

While the analytic methods are generally used for the investigation of qualitative properties of solutions such as the existence, multiplicity, branching, stability, or dichotomy and generally use techniques of calculus (see, e.g., [1–11] and the references in [12]), the functional-analytic ones are based mainly on results of functional analysis and topological degree theory and essentially use various techniques related to operator equations in abstract spaces [13–26]. The numerical methods, under the assumption on the existence of solutions, provide practical numerical algorithms for their approximation [27, 28]. The numerical construction of approximate solutions is usually based on an idea of the shooting method and may face certain difficulties because, as a rule, this technique requires some global regularity conditions, which, however, are quite often satisfied only locally.

Methods of the so-called numerical-analytic type, in a sense, combine, advantages of the mentioned approaches and are usually based upon certain iteration processes constructed explicitly. Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of a solution and its approximate construction. In the theory of nonlinear oscillations, numerical-analytic methods of this kind had apparently been first developed in [20, 29–31] for the investigation of periodic boundary value problems. Appropriate versions were later developed for handling more general types of nonlinear boundary value problems for ordinary and functional-differential equations. We refer, for example, to the books [12, 32–34], the handbook [35], the papers [36–50], and the survey [51–57] for related references.

For a boundary value problem, the numerical-analytic approach usually replaces the problem by the Cauchy problem for a suitably perturbed system containing some artificially introduced vector parameter z, which most often has the meaning of an initial value of the solution and the numerical value of which is to be determined later. The solution of Cauchy problem for the perturbed system is sought for in an analytic form by successive approximations. The functional “perturbation term,” by which the modified equation differs from the original one, depends explicitly on the parameter z and generates a system of algebraic or transcendental “determining equations” from which the numerical values of z should be found. The solvability of the determining system, in turn, may by checked by studying some of its approximations that are constructed explicitly.

For example, in the case of the two-point boundary value problem

$$x'(t) = f(t, x(t)), \quad t \in [a, b], \tag{1.1}$$

$$Ax(a) + Dx(b) = d, \tag{1.2}$$

where $x : [a, b] \to \mathbb{R}^n, -\infty < a < b < +\infty,\ d \in \mathbb{R}^n,\ \det D \neq 0$, the corresponding Cauchy problem for the modified parametrised system of integrodifferential equations has the form [12]

$$\begin{gathered} x'(t) = f(t, x(t)) + \frac{1}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right) - \frac{1}{b-a}\int_a^b f(s, x(s))\,ds, \quad t \in [a, b], \\ x(a) = z, \end{gathered} \tag{1.3}$$

where $\mathbb{1}_n$ is the unit matrix of dimension n and the parameter $z \in \mathbb{R}^n$ has the meaning of initial value of the solution at the point a. The expression

$$\frac{1}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right) - \frac{1}{b-a}\int_a^b f(s, x(s))\,ds \tag{1.4}$$

in (1.3) plays the role of a "perturbation term" and its choice is, of course, not unique. The solution of problem (1.3) is sought for in an analytic form by the method of successive approximations similar to the Picard iterations. According to the formulas

$$\begin{aligned} x_{m+1}(t,z) := z + \int_a^t \left(f(s, x_m(s,z))ds - \frac{1}{b-a}\int_a^b f(\tau, x_m(\tau,z))d\tau \right) ds \\ + \frac{t-a}{b-a}\left(D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z\right), \quad m = 0,1,2,\ldots, \end{aligned} \tag{1.5}$$

where $x_0(t,z) := z$ for all $t \in [a,b]$ and $z \in \mathbb{R}^n$, one constructs the iterations $x_m(\cdot,z)$, $m = 1,2,\ldots$, which depend upon the parameter z and, for arbitrary its values, satisfy the given boundary conditions (1.2): $Ax_m(a,z) + Dx_m(b,z) = d$, $z \in \mathbb{R}^n$, $m = 1,2,\ldots$. Under suitable assumptions, one proves that sequence (1.5) converges to a limit function $x_\infty(\cdot,z)$ for any value of z.

The procedure of passing from the original differential system (1.1) to its "perturbed" counterpart and the investigation of the latter by using successive approximations (1.5) leads one to the system of determining equations

$$D^{-1}d - \left(D^{-1}A + \mathbb{1}_n\right)z - \int_a^b f(s, x_\infty(s,z))ds = 0, \tag{1.6}$$

which gives those numerical values $z = z_*$ of the parameter that correspond to solutions of the given boundary value problem (1.1), (1.10). The form of system (1.6) is, of course, determined by the choice of the perturbation term (1.4); in some other related works, auxiliary equations are constructed in a different way (see, e.g., [42]). It is clear that the complexity of the given equations and boundary conditions has an essential influence both on the possibility of an efficient construction of approximate solutions and the subsequent solvability analysis.

The aim of this paper is to extend the techniques used in [46] for the system of n linear functional differential equations of the form

$$x'(t) = P_0(t)x(t) + P_1(t)x(\beta(t)) + f(t), \quad t \in [0,T], \tag{1.7}$$

subjected to the inhomogeneous three-point Cauchy-Nicoletti boundary conditions

$$\begin{aligned} x_1(0) = x_{10}, \ldots,\ x_p(0) = x_{p0}, \\ x_{p+1}(\xi) = d_{p+1}, \ldots,\ x_{p+q}(\xi) = d_{p+q}, \\ x_{p+q+1}(T) = d_{p+q+1}, \ldots,\ x_n(T) = d_n, \end{aligned} \tag{1.8}$$

with $\xi \in (0,T)$ is given and $x = \mathrm{col}(x_1,\ldots,x_n)$, to the case where the system of linear functional differential equations under consideration has the general form

$$x'(t) = (lx)(t) + f(t), \quad t \in [a,b], \tag{1.9}$$

and the three-point boundary conditions are non-separated and have the form

$$Ax(a) + Bx(\xi) + Cx(b) = d, \tag{1.10}$$

where A, B, and C are singular matrices, $d = \operatorname{col}(d_1, \ldots, d_n)$. Here, $l = (l_k)_{k=1}^n : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ is a bounded linear operator and $f \in L_1([a,b], \mathbb{R}^n)$ is a given function.

It should be noted that, due to the singularity of the matrices that determine the boundary conditions (1.10), certain technical difficulties arise which complicate the construction of successive approximations.

The following notation is used in the sequel:

$C([a,b], \mathbb{R}^n)$ is the Banach space of the continuous functions $[a,b] \to \mathbb{R}^n$ with the standard uniform norm;

$L_1([a,b], \mathbb{R}^n)$ is the usual Banach space of the vector functions $[a,b] \to \mathbb{R}^n$ with Lebesgue integrable components;

$\mathcal{L}(\mathbb{R}^n)$ is the algebra of all the square matrices of dimension n with real elements;

$r(Q)$ is the maximal, in modulus, eigenvalue of a matrix $Q \in \mathcal{L}(\mathbb{R}^n)$;

$\mathbb{1}_k$ is the unit matrix of dimension k;

$\mathbb{0}_{i,j}$ is the zero matrix of dimension $i \times j$;

$\mathbb{0}_i = \mathbb{0}_{i,i}$.

2. Problem Setting and Freezing Technique

We consider the system of n linear functional differential equations (1.9) subjected to the nonseparated inhomogeneous three-point boundary conditions of form (1.10). In the boundary value problem (1.1), (1.10), we suppose that $-\infty < a < b < \infty$, $l = (l_k)_{k=1}^n : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n)$ is a bounded linear operator, $f : [\mathrm{a}, b] \to \mathbb{R}^n$ is an integrable function, $d \in \mathbb{R}^n$ is a given vector, A, B, and C are singular square matrices of dimension n, and C has the form

$$C = \begin{pmatrix} V & W \\ \mathbb{0}_{n-q,q} & \mathbb{0}_{n-q} \end{pmatrix}, \tag{2.1}$$

where V is nonsingular square matrix of dimension $q < n$ and W is an arbitrary matrix of dimension $q \times (n-q)$. The singularity of the matrices determining the boundary conditions (1.10) causes certain technical difficulties. To avoid dealing with singular matrices in the boundary conditions and simplify the construction of a solution in an analytic form, we use a two-stage parametrisation technique. Namely, we first replace the three-point boundary conditions by a suitable parametrised family of two-point inhomogeneous conditions, after which one more parametrisation is applied in order to construct an auxiliary perturbed differential system. The presence of unknown parameters leads one to a certain system of determining equations, from which one finds those numerical values of the parameters that correspond to the solutions of the given three-point boundary value problem.

We construct the auxiliary family of two-point problems by "freezing" the values of certain components of x at the points ξ and b as follows:

$$\operatorname{col}(x_1(\xi), \dots, x_n(\xi)) = \lambda, \\ \operatorname{col}\left(x_{q+1}(b), \dots, x_n(b)\right) = \eta, \tag{2.2}$$

where $\lambda = \operatorname{col}(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ and $\eta = col(\eta_1, \dots, \eta_{n-q}) \in \mathbb{R}^{n-q}$ are vector parameters. This leads us to the parametrised two-point boundary condition

$$Ax(a) + Dx(b) = d - B\lambda - N_q\eta, \tag{2.3}$$

where

$$N_q := \begin{pmatrix} \mathbb{0}_{q,n-q} \\ \mathbb{1}_{n-q} \end{pmatrix} \tag{2.4}$$

and the matrix D is given by the formula

$$D := \begin{pmatrix} V & W \\ \mathbb{0}_{n-q,q} & \mathbb{1}_{n-q} \end{pmatrix} \tag{2.5}$$

with a certain rectangular matrix W of dimension $q \times (n-q)$. It is important to point out that the matrix D appearing in the two-point condition (2.3) is non-singular.

It is easy to see that the solutions of the original three-point boundary value problem (1.1), (1.10) coincide with those solutions of the two-point boundary value problem (1.1), (2.3) for which the additional condition (2.2) is satisfied.

Remark 2.1. The matrices A and B in the boundary conditions (1.10) are arbitrary and, in particular, may be singular. If the number r of the linearly independent boundary conditions in (1.10) is less than n, that is, the rank of the $(n \times 3n)$-dimensional matrix $[A, B, C]$ is equal to r, then the boundary value problem (1.1), (1.10) may have an $(n-r)$-parametric family of solutions.

We assume that throughout the paper the operator l determining the system of equations (1.9) is represented in the form

$$l = l^0 - l^1, \tag{2.6}$$

where $l^j = (l_k^j)_{k=1}^n : C([a,b], \mathbb{R}^n) \to L_1([a,b], \mathbb{R}^n), j = 0, 1$, are bounded linear operators posi-tive in the sense that $(l_k^j u)(t) \geq 0$ for a.e. $t \in [a,b]$ and any $k = 1, 2, \dots, n$, $j = 0, 1$, and $u \in C([a,b], \mathbb{R}^n)$ such that $\min_{t \in [a,b]} u_k(t) \geq 0$ for all $k = 1, 2, \dots, n$. We also put $\widehat{l}_k := l_k^0 + l_k^1$, $k = 1, 2, \dots, n$, and

$$\widehat{l} := l^0 + l^1. \tag{2.7}$$

3. Auxiliary Estimates

In the sequel, we will need several auxiliary statements.

Lemma 3.1. *For an arbitrary essentially bounded function $u : [a,b] \to \mathbb{R}$, the estimates*

$$\left|\int_a^t \left(u(\tau) - \frac{1}{b-a}\int_a^b u(s)ds\right)d\tau\right| \leq \alpha(t)\left(\operatorname*{ess\,sup}_{s\in[a,b]} u(s) - \operatorname*{ess\,inf}_{s\in[a,b]} u(s)\right) \tag{3.1}$$

$$\leq \frac{b-a}{4}\left(\operatorname*{ess\,sup}_{s\in[a,b]} u(s) - \operatorname*{ess\,inf}_{s\in[a,b]} u(s)\right) \tag{3.2}$$

are true for all $t \in [a,b]$, where

$$\alpha(t) := (t-a)\left(1 - \frac{t-a}{b-a}\right), \quad t \in [a,b]. \tag{3.3}$$

Proof. Inequality (3.1) is established similarly to [58, Lemma 3] (see also [12, Lemma 2.3]), whereas (3.2) follows directly from (3.1) if the relation

$$\max_{t\in[a,b]} \alpha(t) = \frac{1}{4}(b-a) \tag{3.4}$$

is taken into account. □

Let us introduce some notation. For any $k = 1, 2, \ldots, n$, we define the n-dimensional row-vector e_k by putting

$$e_k := (\underbrace{0, 0, \ldots, 0,}_{k-1} 1, 0, \ldots, 0). \tag{3.5}$$

Using operators (2.7) and the unit vectors (3.5), we define the matrix-valued function $K_l : [a,b] \to \mathcal{L}(\mathbb{R}^n)$ by setting

$$K_l := \left[\widehat{l}e_1^*, \widehat{l}e_2^*, \ldots, \widehat{l}e_n^*\right]. \tag{3.6}$$

Note that, in (3.6), $\widehat{l}e_i^*$ means the value of the operator $\widehat{l}$ on the constant vector function is equal identically to e_i^*, where e_i^* is the vector transpose to e_i. It is easy to see that the components of K_l are Lebesgue integrable functions.

Lemma 3.2. *The componentwise estimate*

$$|(lx)(t)| \leq K_l(t) \max_{s\in[a,b]} |x(s)|, \quad t \in [a,b], \tag{3.7}$$

is true for any $x \in C([a,b], \mathbb{R}^n)$, where $K_l : [a,b] \to \mathcal{L}(\mathbb{R}^n)$ is the matrix-valued function given by formula (3.6).

Proof. Let $x = (x_k)_{k=1}^n$ be an arbitrary function from $C([a,b],\mathbb{R}^n)$. Then, recalling the notation for the components of l, we see that

$$lx = \sum_{i=1}^{n} e_i^* l_i x. \tag{3.8}$$

On the other hand, due to (3.5), we have $x = \sum_{k=1}^n e_k^* x_k$ and, therefore, by virtue of (3.8) and (2.6),

$$lx = \sum_{i=1}^{n} e_i^* l_i x = \sum_{i=1}^{n} e_i^* l_i \left(\sum_{k=1}^{n} e_k^* x_k \right) = \sum_{i=1}^{n} e_i^* \left(\sum_{k=1}^{n} \left(l_i^0 e_k^* x_k - l_i^1 e_k^* x_k \right) \right). \tag{3.9}$$

On the other hand, the obvious estimate

$$\sigma x_k(t) \le \max_{s\in[a,b]} |x_k(s)|, \quad t \in [a,b],\ k = 1,2,\ldots,n,\ \sigma \in \{-1,1\}, \tag{3.10}$$

and the positivity of the operators l^j, $j = 0,1$, imply

$$l_i^j(\sigma x_k)(t) = \sigma\left(l_i^j x_k\right)(t) \le l_i^j \max_{s\in[a,b]} |x_k(s)| \tag{3.11}$$

for a.e. $t \in [a,b]$ and any $k, j = 1,2,\ldots,n$, $\sigma \in \{-1,1\}$. This, in view of (2.7) and (3.9), leads us immediately to estimate (3.7). □

4. Successive Approximations

To study the solution of the auxiliary two-point parametrised boundary value problem (1.9), (2.3) let us introduce the sequence of functions $x_m : [a,b] \times \mathbb{R}^{3n-q} \to \mathbb{R}^n$, $m \ge 0$, by putting

$$\begin{aligned} x_{m+1}(t,z,\lambda,\eta) &:= \varphi_{z,\lambda,\eta}(t) + \int_a^t \left((l x_m(\cdot,z,\lambda,\eta))(s) + f(s) \right) ds \\ &\quad - \frac{t-a}{b-a} \int_a^b \left((l x_m(\cdot,z,\lambda,\eta))(s) + f(s) \right) ds, \quad m = 0,1,2,\ldots, \\ x_0(t,z,\lambda,\eta) &:= \varphi_{z,\lambda,\eta}(t) \end{aligned} \tag{4.1}$$

for all $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$, where

$$\varphi_{z,\lambda,\eta}(t) := z + \frac{t-a}{b-a}\left(D^{-1}(d - B\lambda + N_q \eta) - \left(D^{-1}A + \mathbb{1}_n \right) z \right). \tag{4.2}$$

In the sequel, we consider x_m as a function of t and treat the vectors z, λ, and η as parameters.

Lemma 4.1. *For any $m \geq 0$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$, the equalities*

$$\begin{gathered} x_m(a,z,\lambda,\eta) = z, \\ Ax_m(a,z,\lambda,\eta) + Dx_m(b,z,\lambda,\eta) = d - B\lambda + N_q\eta, \end{gathered} \tag{4.3}$$

are true.

The proof of Lemma 4.1 is carried out by straightforward computation. We emphasize that the matrix D appearing in the two-point condition (2.3) is non-singular. Let us also put

$$(\mathcal{M}y)(t) := \left(1 - \frac{t-a}{b-a}\right)\int_a^t y(s)ds + \frac{t-a}{b-a}\int_t^b y(s)ds, \quad t \in [a,b], \tag{4.4}$$

for an arbitrary $y \in L_1([a,b],\mathbb{R}^n)$. It is clear that $\mathcal{M} : L_1([a,b],\mathbb{R}^n) \to C([a,b],\mathbb{R}^n)$ is a positive linear operator. Using the operator $\mathcal{M}$, we put

$$Q_l := \left[\mathcal{M}(K_l e_1^*), \mathcal{M}(K_l e_2^*), \ \ldots, \mathcal{M}(K_l e_n^*)\right], \tag{4.5}$$

where K_l is given by formula (3.6). Finally, define a constant square matrix Q_l of dimension n by the formula

$$Q_l := \max_{t \in [a,b]} Q_l(t). \tag{4.6}$$

We point out that, as before, the maximum in (4.6) is taken componentwise (one should remember that, for $n > 1$, a point $t_* \in [a,b]$ such that $Q_l = Q_l(t_*)$ may not exist).

Theorem 4.2. *If the spectral radius of the matrix Q_l satisfies the inequality*

$$r(Q_l) < 1, \tag{4.7}$$

then, for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$:

(1) *the sequence of functions* (4.1) *converges uniformly in $t \in [a,b]$ for any fixed $(z,\lambda,\eta) \in \mathbb{R}^{3n-q}$ to a limit function*

$$x_\infty(t,z,\lambda,\eta) = \lim_{m \to +\infty} x_m(t,z,\lambda,\eta); \tag{4.8}$$

(2) *the limit function $x_\infty(\cdot,z,\lambda,\eta)$ possesses the properties*

$$\begin{gathered} x_\infty(a,z,\lambda,\eta) = z, \\ Ax_\infty(a,z,\lambda,\eta) + Dx_\infty(b,z,\lambda,\eta) = d - B\lambda + N_q\eta; \end{gathered} \tag{4.9}$$

(3) *function* (4.8) *is a unique absolutely continuous solution of the integro-functional equation*

$$x(t) = z + \frac{t-a}{b-a}\Big(D^{-1}(d - B\lambda + N_q\eta) - \Big(D^{-1}A + \mathbb{1}_n\Big)z\Big) + \int_a^t ((lx)(s) + f(s))ds - \frac{t-a}{b-a}\int_a^b ((lx)(s) + f(s))ds, \quad t \in [a,b]; \tag{4.10}$$

(4) *the error estimate*

$$\max_{t\in[a,b]}\left|x_\infty(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)\right| \le \frac{b-a}{4} Q_l^m(\mathbb{1}_n - Q_l)^{-1}\omega(z,\lambda,\eta) \tag{4.11}$$

holds, where $\omega : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ *is given by the equality*

$$\omega(z,\lambda,\eta) := \operatorname*{ess\,sup}_{s\in[a,b]}\big((l\varphi_{z,\lambda,\eta})(s) + f(s)\big) - \operatorname*{ess\,inf}_{s\in[a,b]}\big((l\varphi_{z,\lambda,\eta})(s) + f(s)\big). \tag{4.12}$$

In (3.6), (4.11) and similar relations, the signs $|\cdot|$, $\le$, $\ge$, as well as the operators "max", "ess sup", "ess inf", and so forth, applied to vectors or matrices are understood componentwise.

Proof. The validity of assertion 1 is an immediate consequence of the formula (4.1). To obtain the other required properties, we will show, that under the conditions assumed, sequence (4.1) is a Cauchy sequence in the Banach space $C([a,b],\mathbb{R}^n)$ equipped with the standard uniform norm. Let us put

$$r_m(t,z,\lambda,\eta) := x_{m+1}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta) \tag{4.13}$$

for all $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, $\eta \in \mathbb{R}^{n-q}$, $t \in [a,b]$, and $m \ge 0$. Using Lemma 3.2 and taking equality (3.4) into account, we find that (4.1) yields

$$\begin{aligned} \left|x_1(t,z,\lambda,\eta) - x_0(t,z,\lambda,\eta)\right| &= \left|\int_a^t [(l\varphi_{z,\lambda,\eta})(s) + f(s)]ds - \frac{t-a}{b-a}\int_a^b [(l\varphi_{z,\lambda,\eta})(s) + f(s)]ds\right| \\ &\le \alpha(t)\omega(z,\lambda,\eta) \\ &\le \frac{b-a}{4}\omega(z,\lambda,\eta), \end{aligned} \tag{4.14}$$

for arbitrary fixed z, λ, and η, where α is the function given by (3.3) and $\omega(\cdot)$ is defined by formula (4.12).

According to formulae (4.1), for all $t \in [a,b]$, arbitrary fixed z, λ, and η and $m = 1,2,\ldots$ we have

$$\begin{aligned}
r_m(t,z,\lambda,\eta) &= \int_a^t l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\
&\quad - \frac{t-a}{b-a}\int_a^b l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\
&= \left(1 - \frac{t-a}{b-a}\right)\int_a^t l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds \\
&\quad - \frac{t-a}{b-a}\int_t^b l(x_m(\cdot,z,\lambda,\eta) - x_{m-1}(\cdot,z,\lambda,\eta))(s)ds.
\end{aligned} \tag{4.15}$$

Equalities (4.13) and (4.15) imply that for all $m = 1,2,\ldots$, arbitrary fixed z,λ,η and $t \in [a,b]$,

$$\begin{aligned}
|r_m(t,z,\lambda,\eta)| &\leq \left(1 - \frac{t-a}{b-a}\right)\int_a^t |l\ (r_{m-1}(\cdot,z,\lambda,\eta)(s))|ds \\
&\quad + \frac{t-a}{b-a}\int_t^b |l(r_{m-1}(\cdot,z,\lambda,\eta))(s)|ds.
\end{aligned} \tag{4.16}$$

Applying inequality (3.7) of Lemma 3.2 and recalling formulae (4.5) and (4.6), we get

$$\begin{aligned}
|r_m(t,z,\lambda,\eta)| &\leq \left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|ds \\
&\quad + \frac{t-a}{b-a}\int_t^b K_l(s)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|ds \\
&= \left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds + \frac{t-a}{b-a}\int_t^b K_l(s)ds\right)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)| \\
&= Q_l(t)\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)| \\
&\leq Q_l\max_{\tau\in[a,b]}|r_{m-1}(\tau,z,\lambda,\eta)|.
\end{aligned} \tag{4.17}$$

Using (4.17) recursively and taking (4.14) into account, we obtain

$$\begin{aligned}
|r_m(t,z,\lambda,\eta)| &\leq Q_l^m\max_{\tau\in[a,b]}|r_0(\tau,z,\lambda,\eta)| \\
&\leq \frac{b-a}{4}Q_l^m\omega(z,\lambda,\eta),
\end{aligned} \tag{4.18}$$

for all $m \geq 1$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$. Using (4.18) and (4.13), we easily obtain that, for an arbitrary $j \in \mathbb{N}$,

$$\begin{aligned}\left|x_{m+j}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)\right| &= \big|\left(x_{m+j}(t,z,\lambda,\eta) - x_{m+j-1}(t,z,\lambda,\eta)\right) \\ &\quad + \left(x_{m+j-1}(t,z,\lambda,\eta) - x_{m+j-2}(t,z,\lambda,\eta)\right) + \cdots \\ &\quad + \left(x_{m+1}(t,z,\lambda,\eta) - x_{\mathrm{m}}(t,z,\lambda,\eta)\right)\big| \\ &\leq \sum_{i=0}^{j-1} \left|r_{m+i}(t,z,\lambda,\eta)\right| \\ &\leq \frac{b-a}{4} \sum_{i=0}^{j-1} Q_l^{m+i} \omega(z,\lambda,\eta).\end{aligned} \tag{4.19}$$

Therefore, by virtue of assumption (4.7), it follows that

$$\begin{aligned}\left|x_{m+j}(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)\right| \leq{} & \frac{b-a}{4}\, Q_l^m \sum_{i=0}^{+\infty} Q_l^i \omega(z,\lambda,\eta) \\ ={} & \frac{b-a}{4}\, Q_l^m (\mathbb{1}_n - Q_l)^{-1} \omega(z,\lambda,\eta)\end{aligned} \tag{4.20}$$

for all $m \geq 1$, $j \geq 1$, $t \in [a,b]$, $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$. We see from (4.20) that (4.1) is a Cauchy sequence in the Banach space $C([a,b],\mathbb{R}^n)$ and, therefore, converges uniformly in $t \in [a,b]$ for all $(z,\lambda,\eta) \in \mathbb{R}^{3n-q}$:

$$\lim_{m \to \infty} x_m(t,z,\lambda,\eta) = x_\infty(t,z,\lambda,\eta), \tag{4.21}$$

that is, assertion 2 holds. Since all functions $x_m(t,z,\lambda,\eta)$ of the sequence (4.1) satisfy the boundary conditions (2.3), by passing to the limit in (2.3) as $m \to +\infty$ we show that the function $x_\infty(\cdot,z,\lambda,\eta)$ satisfies these conditions.

Passing to the limit as $m \to \infty$ in (4.1), we show that the limit function is a solution of the integro-functional equation (4.10). Passing to the limit as $j \to \infty$ in (4.20) we obtain the estimate

$$\left|x_\infty(t,z,\lambda,\eta) - x_m(t,z,\lambda,\eta)\right| \leq \frac{b-a}{4} Q_l^m (\mathbb{1}_n - Q_l)^{-1} \omega(z,\lambda,\eta) \tag{4.22}$$

for a.e. $t \in [a,b]$ and arbitrary fixed z, λ, η, and $m = 1,2,\ldots$. This completes the proof of Theorem 4.2.

We have the following simple statement. □

Proposition 4.3. *If, under the assumptions of Theorem 4.2, one can specify some values of z, λ, and η, such that the limit function $x_\infty(\cdot, z, \lambda, \eta)$ possesses the property*

$$D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z = \int_a^b ((lx_\infty(\cdot, z, \lambda, \eta))(s) + f(s))ds = 0, \tag{4.23}$$

then, for these $z, \lambda,$ and η, it is also a solution of the boundary value problem (1.9), (2.3).

Proof. The proof is a straightforward application of the above theorem. □

5. Some Properties of the Limit Function

Let us first establish the relation of the limit function $x_\infty(\cdot, z, \lambda, \eta)$ to the auxiliary two-point boundary value problem (1.9), (2.3). Along with system (1.9), we also consider the system with a constant forcing term in the right-hand side

$$x'(t) = (lx)(t) + f(t) + \mu, \quad t \in [a, b], \tag{5.1}$$

and the initial condition

$$x(a) = z, \tag{5.2}$$

where $\mu = \mathrm{col}(\mu_1, \ldots, \mu_n)$ is a control parameter.

We will show that for arbitrary fixed $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in^{n-q}$, the parameter μ can be chosen so that the solution $x(\cdot, z, \lambda, \eta, \mu)$ of the initial value problem (5.1), (5.2) is, at the same time, a solution of the two-point parametrised boundary value problem (5.1), (2.3).

Proposition 5.1. *Let $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$ be arbitrary given vectors. Assume that condition* (4.7) *is satisfied. Then a solution $x(\cdot)$ of the initial value problem* (5.1), (5.2) *satisfies the boundary conditions* (2.3) *if and only if $x(\cdot)$ coincides with $x_\infty(\cdot, z, \lambda, \eta)$ and*

$$\mu = \mu_{z,\lambda,\eta}, \tag{5.3}$$

where

$$\begin{aligned} \mu_{z,\lambda,\eta} := {} & \frac{1}{b-a}\left(D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z\right) \\ & - \frac{1}{b-a}\int_a^b [(lx_\infty(\cdot, z, \lambda, \eta)(s) + f(s))]ds \end{aligned} \tag{5.4}$$

and $x_\infty(\cdot, z, \lambda, \eta)$ is the limit function of sequence (4.1).

Proof. The assertion of Proposition 5.1 is obtained by analogy to the proof of [50, Theorem 4.2]. Indeed, let $z \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^n$, and $\eta \in \mathbb{R}^{n-q}$ be arbitrary.

If μ is given by (5.3), then, due to Theorem 4.2, the function $x_\infty(\cdot, z, \lambda, \eta)$ has properties (4.9) and satisfies equation (4.10), whence, by differentiation, equation (5.1) with the above-mentioned value of μ is obtained. Thus, $x_\infty(\cdot, z, \lambda, \eta)$ is a solution of (5.1), (5.2) with μ of form (5.3) and, moreover, this function satisfies the two-point boundary conditions (2.3).

Let us fix an arbitrary $\mu \in \mathbb{R}^n$ and assume that the initial value problem (5.1), (5.2) has a solution y satisfies the two-point boundary conditions (2.3). Then

$$y(t) = z + \int_a^t \left[(ly)(s) + f(s) \right] ds + \mu(t-a), \tag{5.5}$$

for all $t \in [a, b]$. By assumption, y satisfies the two-point conditions (2.3) and, therefore, (5.5) yields

$$\begin{aligned} Ay(a) + Dy(b) &= Az + D\left(z + \int_a^b ((ly)(s) + f(s))(s) ds + \mu(b-a) \right) \\ &= d - B\lambda + N_q \eta, \end{aligned} \tag{5.6}$$

whence we find that μ can be represented in the form

$$\mu = \frac{1}{b-a} D^{-1} \left(d - B\lambda + N_q \eta - (A+D)z - \int_a^b ((ly)(s) + f(s))(s) ds \right). \tag{5.7}$$

On the other hand, we already know that the function $x_\infty(\cdot, z, \lambda, \eta)$, satisfies the two-point conditions (2.3) and is a solution of the initial value problem (5.1), (5.2) with $\mu = \mu_{z,\lambda,\eta}$, where the value $\mu_{z,\lambda,\eta}$ is defined by formula (5.4). Consequently,

$$x_\infty(t, z, \lambda, \eta) = z + \int_a^t \left[(l x_\infty(\cdot, z, \lambda, \eta)(s) + f(s)) \right] ds + \mu_{z,\lambda,\eta}(t-a), \quad t \in [a, b]. \tag{5.8}$$

Putting

$$h(t) := y(t) - x_\infty(t, z, \lambda, \eta), \quad t \in [a, b], \tag{5.9}$$

and taking (5.5), (5.8) into account, we obtain

$$h(t) = \int_a^t (lh)(s) ds + \left(\mu - \mu_{z,\lambda,\eta} \right)(t-a), \quad t \in [a, b]. \tag{5.10}$$

Recalling the definition (5.4) of $\mu_{z,\lambda,\eta}$ and using formula (5.7), we obtain

$$\begin{aligned} \mu - \mu_{z,\lambda,\eta} &= \frac{1}{b-a}\int_a^b l(x_\infty(\cdot,z,\lambda,\eta) - y)(s)ds \\ &= -\frac{1}{b-a}\int_a^b (lh)(s)ds, \end{aligned} \tag{5.11}$$

and, therefore, equality (5.10) can be rewritten as

$$\begin{aligned} h(t) &= \int_a^t (lh)(s)ds - \frac{t-a}{b-a}\int_a^b (lh)(s)ds \\ &= \left(1 - \frac{t-a}{b-a}\right)\int_a^t (lh)(s)ds - \frac{t-a}{b-a}\int_t^b (lh)(s)ds, \quad t \in [a,b]. \end{aligned} \tag{5.12}$$

Applying Lemma 3.2 and recalling notation (4.6), we get

$$\begin{aligned} |h(t)| &\le \left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds + \frac{t-a}{b-a}\int_t^b K_l(s)ds\right)\max_{\tau\in[a,b]}|h(\tau)| \\ &\le Q_l \max_{\tau\in[a,b]}|h(\tau)| \end{aligned} \tag{5.13}$$

for an arbitrary $t \in [a,b]$. By virtue of condition (4.7), inequality (5.13) implies that

$$\max_{\tau\in[a,b]}|h(\tau)| \le Q_l^m \max_{\tau\in[a,b]}|h(\tau)| \longrightarrow 0 \tag{5.14}$$

as $m \to +\infty$. According to (5.9), this means that y coincides with $x_\infty(\cdot,z,\lambda,\eta)$, and, therefore, by (5.11), $\mu = \mu_{z,\lambda,\eta}$, which brings us to the desired conclusion. □

We show that one can choose certain values of parameters $z = z_*,\ \lambda = \lambda_*,\ \eta = \eta_*$ for which the function $x_\infty(\cdot,\ z_*,\ \lambda_*,\ \eta_*)$ is the solution of the original three-point boundary value problem (1.9), (1.10). Let us consider the function $\Delta : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ given by formula

$$\Delta(z,\lambda,\eta) := g(z,\lambda,\eta) - \int_a^b ((lx_\infty(\cdot,z,\lambda,\eta))(s) + f(s))ds \tag{5.15}$$

with

$$g(z,\lambda,\eta) := D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z \tag{5.16}$$

for all z, λ, and η, where x_∞ is the limit function (4.8).

The following statement shows the relation of the limit function (4.8) to the solution of the original three-point boundary value problem (1.9), (1.10).

Theorem 5.2. *Assume condition* (4.7). *Then the function* $x_\infty(\cdot, z, \lambda, \eta)$ *is a solution of the three-point boundary value problem* (1.9), (1.10) *if and only if the triplet* z, λ, η *satisfies the system of* $3n - q$ *algebraic equations*

$$\Delta(z, \lambda, \eta) = 0, \tag{5.17}$$

$$e_1 x_\infty(\xi, z, \lambda, \eta) = \lambda_1, \; e_2 x_\infty(\xi, z, \lambda, \eta) = \lambda_2, \; \ldots, \; e_n x_\infty(\xi, z, \lambda, \eta) = \lambda_n, \tag{5.18}$$

$$e_{q+1} x_\infty(b, z, \lambda, \eta) = \eta_1, \; e_{q+2} x_\infty(b, z, \lambda, \eta) = \eta_2, \ldots, \; e_{q+\infty} x_\infty(b, z, \lambda, \eta) = \eta_{n-q}. \tag{5.19}$$

Proof. It is sufficient to apply Proposition 5.1 and notice that the differential equation in (5.1) coincides with (1.9) if and only if the triplet (z, λ, η) satisfies (5.17). On the other hand, (5.18) and (5.19) bring us from the auxiliary two-point parametrised conditions to the three-point conditions (1.10). □

Proposition 5.3. *Assume condition* (4.7). *Then, for any* (z^j, λ^j, η^j), $j = 0, 1$, *the estimate*

$$\max_{t \in [a,b]} \left| x_\infty\left(t, z^0, \lambda^0, \eta^0\right) - x_\infty\left(t, z^1, \lambda^1, \eta^1\right) \right| \le (\mathbb{1}_n - Q_l)^{-1} v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right) \tag{5.20}$$

holds, where

$$v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right) := \max_{t \in [a,b]} \left| \varphi_{z^0, \lambda^0, \eta^0}(t) - \varphi_{z^1, \lambda^1, \eta^1}(t) \right|. \tag{5.21}$$

Proof. Let us fix two arbitrary triplets (z^j, λ^j, η^j), $j = 0, 1$, and put

$$u_m(t) := x_m\left(t, z^0, \lambda^0, \eta^0\right) - x_m\left(t, z^1, \lambda^1, \eta^1\right), \quad t \in [a, b]. \tag{5.22}$$

Consider the sequence of vectors c_m, $m = 0, 1, \ldots$, determined by the recurrence relation

$$c_m := c_0 + Q_l c_{m-1}, \quad m \ge 1, \tag{5.23}$$

with

$$c_0 := v\left(z^0, \lambda^0, \eta^0, z^1, \lambda^1, \eta^1\right). \tag{5.24}$$

Let us show that

$$\max_{t \in [a,b]} |u_m(t)| \le c_m \tag{5.25}$$

for all $m \geq 0$. Indeed, estimate (5.25) is obvious for $m = 0$. Assume that

$$\max_{t\in[a,b]} |u_{m-1}(t)| \leq c_{m-1}. \tag{5.26}$$

It follows immediately from (4.1) that

$$\begin{aligned} u_m(t) &= \varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t) + \int_a^t (lu_{m-1})(s)ds - \frac{t-a}{b-a}\int_a^b (lu_{m-1})(s)ds \\ &= \varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t) \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t (lu_{m-1})(s)ds - \frac{t-a}{b-a}\int_t^b (lu_{m-1})(s)ds, \end{aligned} \tag{5.27}$$

whence, by virtue of (5.21), estimate (3.7) to Lemma 3.2, and assumption (5.26),

$$\begin{aligned} |u_m(t)| &\leq \left|\varphi_{z^0,\lambda^0,\eta^0}(t) - \varphi_{z^1,\lambda^1,\eta^1}(t)\right| \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t |(lu_{m-1})(s)|ds + \frac{t-a}{b-a}\int_t^b |(lu_{m-1})(s)|ds \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) \\ &\quad + \left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds \max_{t\in[a,b]}|u_{m-1}(t)| + \frac{t-a}{b-a}\int_t^b K_l(s)ds \max_{t\in[a,b]}|u_{m-1}(t)| \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) + \left(\left(1 - \frac{t-a}{b-a}\right)\int_a^t K_l(s)ds + \frac{t-a}{b-a}\int_t^b K_l(s)ds\right)c_{m-1} \\ &\leq v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right) + Q_l c_{m-1}, \end{aligned} \tag{5.28}$$

which estimate, in view of (5.23) and (5.24), coincides with the required inequality (5.25). Thus, (5.25) is true for any m. Using (5.23) and (5.25), we obtain

$$\begin{aligned} \max_{t\in[a,b]}|u_m(t)| &\leq c_0 + Q_l c_{m-1} = c_0 + Q_l c_0 + Q_l^2 c_{m-2} = \cdots \\ &= \sum_{k=0}^{m-1} Q_l^k c_0 + Q_l^m c_0. \end{aligned} \tag{5.29}$$

Due to assumption (4.7), $\lim_{m\to+\infty} Q_l^m = 0$. Therefore, passing to the limit in (5.29) as $m \to +\infty$ and recalling notation (5.22), we obtain the estimate

$$\max_{t\in[a,b]}\left|x_*\left(t,z^0,\lambda^0,\eta^0\right) - x_*\left(t,z^1,\lambda^1,\eta^1\right)\right| \leq \sum_{k=0}^{+\infty} Q_l^k c_0 = (\mathbb{1}_n - Q_l)^{-1} c_0, \tag{5.30}$$

which, in view of (5.24), coincides with (5.20). □

Now we establish some properties of the "determining function" $\Delta : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ given by equality (5.15).

Proposition 5.4. *Under condition* (3.10), *formula* (5.15) *determines a well-defined function* $\Delta : \mathbb{R}^{3n-q} \to \mathbb{R}^n$, *which, moreover, satisfies the estimate*

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\le \left|G[z^0-z^1,\lambda^0-\lambda^1,\eta^0-\eta^1]^*\right| \\ &\quad + R_l\max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}G[z^0-z^1,\lambda^0-\lambda^1,\eta^0-\eta^1]^*\right|,\end{aligned} \tag{5.31}$$

for all (z^j,λ^j,η^j), $j=0,1$, *where the* $(n\times n)$*-matrices* G *and* R_l *are defined by the equalities*

$$\begin{aligned} G &:= D^{-1}\left[A+D,B,N_q\right], \\ R_l &:= \int_a^b K_l(s)ds\ (\mathbb{1}_n-Q_l)^{-1}. \end{aligned} \tag{5.32}$$

Proof. According to the definition (5.15) of Δ, we have

$$\begin{aligned}\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right) &= g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right) \\ &\quad -\int_a^b\left(l\left(x_\infty\left(\cdot,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\cdot,z^1,\lambda^1,\eta^1\right)\right)(s)\right)ds,\end{aligned} \tag{5.33}$$

whence, due to Lemma 3.2,

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\le \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad +\int_a^b\left|l\left(x_\infty\left(\cdot,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\cdot,z^1,\lambda^1,\eta^1\right)\right)(s)\right|ds \\ &\le \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad +\int_a^b K_l(s)ds\max_{\tau\in[a,b]}\left|x_\infty\left(\tau,z^0,\lambda^0,\eta^0\right)-x_\infty\left(\tau,z^1,\lambda^1,\eta^1\right)(s)\right|.\end{aligned} \tag{5.34}$$

Using Proposition 5.3, we find

$$\begin{aligned}\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right| &\le \left|g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right| \\ &\quad+\int_a^b K_l(s)ds(\mathbb{1}_n-Q_l)^{-1}v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right).\end{aligned}\tag{5.35}$$

On the other hand, recalling (4.2) and (5.21), we get

$$v\left(z^0,\lambda^0,\eta^0,z^1,\lambda^1,\eta^1\right)=\max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}\left(g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)\right)\right|.\tag{5.36}$$

It follows immediately from (5.16) that

$$\begin{aligned}g\left(z^0,\lambda^0,\eta^0\right)-g\left(z^1,\lambda^1,\eta^1\right)&=-D^{-1}B\left(\lambda^0-\lambda^1\right)-D^{-1}N_q\left(\eta^0-\eta^1\right)-\left(D^{-1}A+\mathbb{1}_n\right)\left(z^0-z^1\right)\\ &=-D^{-1}\left[B\left(\lambda^0-\lambda^1\right)+N_q\left(\eta^0-\eta^1\right)+(A+D)\left(z^0-z^1\right)\right]\\ &=D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}.\end{aligned}\tag{5.37}$$

Therefore, (5.35) and (5.36) yield the estimate

$$\begin{aligned}&\left|\Delta\left(z^0,\lambda^0,\eta^0\right)-\Delta\left(z^1,\lambda^1,\eta^1\right)\right|\\ &\quad\le\left|D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}\right|\\ &\quad+\int_a^b K_l(s)ds(\mathbb{1}_n-Q_l)^{-1}\max_{t\in[a,b]}\left|z^0-z^1+\frac{t-a}{b-a}D^{-1}\left[A+D,B,N_q\right]\begin{pmatrix}z^0-z^1\\ \lambda^0-\lambda^1\\ \eta^0-\eta^1\end{pmatrix}\right|,\end{aligned}\tag{5.38}$$

which, in view of (5.32), coincides with (5.31). □

Properties stated by Propositions 5.3 and 5.4 can be used when analysing conditions guaranteeing the solvability of the determining equations.

6. On the Numerical-Analytic Algorithm of Solving the Problem

Theorems 4.2 and 5.2 allow one to formulate the following numerical-analytic algorithm for the construction of a solution of the three-point boundary value problem (1.9), (1.10).

(1) For any vector $z \in \mathbb{R}^n$, according to (4.1), we analytically construct the sequence of functions $x_m(\cdot, z, \lambda, \eta)$ depending on the parameters z, λ, η and satisfying the auxiliary two-point boundary condition (2.3).

(2) We find the limit $x_\infty(\cdot, z, \lambda, \eta)$ of the sequence $x_m(\cdot, z, \lambda, \eta)$ satisfying (2.3).

(3) We construct the algebraic determining system (5.17), (5.18), and (5.19) with respect $3n - q$ scalar variables.

(4) Using a suitable numerical method, we (approximately) find a root

$$z_* \in \mathbb{R}^n, \qquad \lambda_* \in \mathbb{R}^n, \qquad \eta_* \in \mathbb{R}^{n-q} \tag{6.1}$$

of the determining system (5.17), (5.18), and (5.19).

(5) Substituting values (6.1) into $x_\infty(\cdot, z, \lambda, \eta)$, we obtain a solution of the original three-point boundary value problem (1.9), (1.10) in the form

$$x(t) = x_\infty(t, z_*, \lambda_*, \eta_*), \quad t \in [a, b]. \tag{6.2}$$

This solution (6.2) can also be obtained by solving the Cauchy problem

$$x(a) = z_* \tag{6.3}$$

for (1.9).

The fundamental difficulty in the realization of this approach arises at point (2) and is related to the analytic construction of the function $x_\infty(\cdot, z, \lambda, \eta)$. This problem can often be overcome by considering certain approximations of form (4.1), which, unlike the function $x_\infty(\cdot, z, \lambda, \eta)$, are known in the analytic form. In practice, this means that we fix a suitable $m \geq 1$, construct the corresponding function $x_m(\cdot, z, \lambda, \eta)$ according to relation (4.1), and define the function $\Delta_m : \mathbb{R}^{3n-q} \to \mathbb{R}^n$ by putting

$$\Delta_m(z, \lambda, \eta) := D^{-1}(d - B\lambda + N_q\eta) - \left(D^{-1}A + \mathbb{1}_n\right)z - \int_a^b \left[(lx_m(\cdot, z, \lambda, \eta)(s) + f(s))\right] ds, \tag{6.4}$$

for arbitrary z, λ, and η. To investigate the solvability of the three-point boundary value problem (1.9), (1.10), along with the determining system (5.17), (5.18), and (5.19), one considers the mth approximate determining system

$$\begin{gathered} \Delta_m(z, \lambda, \eta) = 0, \\ e_1 x_m(\xi, z, \lambda, \eta) = \lambda_1, e_2 x_m(\xi, z, \lambda, \eta) = \lambda_2, \dots, e_n x_m(\xi, z, \lambda, \eta) = \lambda_n, \\ e_{q+1} x_m(b, z, \lambda, \eta) = \eta_1, \dots, e_n x_m(b, z, \lambda, \eta) = \eta_{n-q}, \end{gathered} \tag{6.5}$$

where e_i, $i = 1, 2, \ldots, n$, are the vectors given by (5.15).

It is natural to expect (and, in fact, can be proved) that, under suitable conditions, the systems (5.17), (5.18), (5.19), and (6.5) are "close enough" to one another for m sufficiently large. Based on this circumstance, existence theorems for the three-point boundary value problem (1.9), (1.10) can be obtained by studying the solvability of the approximate determining system (6.5) (in the case of periodic boundary conditions, see, e.g., [35]).

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Research Article

On Stability of Linear Delay Differential Equations under Perron's Condition

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The stability of the zero solution of a system of first-order linear functional differential equations with nonconstant delay is considered. Sufficient conditions for stability, uniform stability, asymptotic stability, and uniform asymptotic stability are established.

1. Introduction

We begin with a classical result for the linear system

$$x' = A(t)x, \tag{L1}$$

where A is an $n \times n$ matrix function defined and continuous on $[0, \infty)$. By $C_B[0, \infty)$, we will denote the set of bounded functions defined and continuous on $[0, \infty)$ and by $|\cdot|$ the Euclidean norm.

In 1930, Perron first formulated the following definition being named after him.

Definition 1.1 (see [1]). System (L1) is said to satisfy Perron's condition (P1) if, for any given vector function $f \in C_B[0, \infty)$, the solution $x(t)$ of

$$x' = A(t)x + f(t), \quad x(0) = 0 \tag{N1}$$

is bounded.

The following theorem by Bellman [2] is well known.

Theorem 1.2 (see [2]). *If* (P1) *holds and* $|A(t)| \leq M_1$ *for some positive number* M_1*, then the zero solution of* (L1) *is uniformly asymptotically stable.*

The proof is accomplished by making use of the basic properties of a fundamental matrix, the Banach-Steinhaus theorem, and the adjoint system

$$x' = -A^T(t)x, \tag{1.1}$$

where A^T denotes the transpose of A.

It is shown by an example in [3] that Theorem 1.2 may not be valid if the function f appearing in (N1) is replaced by a constant vector. However, such a theorem is later obtained in [4] under a *Perron-like* condition.

Theorem 1.2 is extended by Halanay [5] to linear delay systems of the form

$$x'(t) = A(t)x(t) + B(t)x(t-\tau), \tag{L2}$$

where A, B are $n \times n$ matrix functions defined and continuous on $[0,\infty)$ and τ is a positive real number.

Definition 1.3. System (L2) is said to satisfy Perron's condition (P2) if for any given vector function $f \in C_B[0,\infty)$, the solution $x(t)$ of

$$x'(t) = A(t)x(t) + B(t)x(t-\tau) + f(t) \tag{N2}$$

satisfying $x(t) = 0$, $t \leq 0$, is bounded.

Theorem 1.4 (see [5]). *If* (P2) *holds,* $|A(t)| \leq M_1$*, and* $|B(t)| \leq M_2$ *for some positive numbers* M_1 *and* M_2*, then the zero solution of* (L2) *is uniformly asymptotically stable.*

The method used to prove Theorem 1.4 is similar to Bellman's except that the adjoint system

$$y'(t) = -A^T(t)y(t) - B^T(t+\tau)y(t+\tau) \tag{1.2}$$

is not constructed with respect to an inner product but the functional

$$F(x,y)(t) = \int_t^{t+\tau} y^T(s)B(s)x(s-\tau)ds + x^T(t)y(t). \tag{1.3}$$

For some extensions to impulsive differential equations, we refer the reader in particular to [6, 7].

In this paper, we consider the more general linear delay system

$$x'(t) = A(t)x(t) + B(t)x(g(t)), \tag{1.4}$$

where A and B are $n \times n$ matrix functions defined and continuous on $[0, \infty)$ and g is a continuously differentiable increasing function defined on $[0, \infty)$ satisfying $g(t) < t$ and $g'(t) \leq 1$. We set $h := g^{-1}$. Obviously, $h \in C^1[0, \infty)$ and increases on $[0, \infty)$ and $h(t) > t$.

Perron's condition takes the following form.

Definition 1.5. System (1.4) is said to satisfy Perron's condition (P) if, for any given vector function $f \in C_B[0, \infty)$, the solution $x(t)$ of

$$x'(t) = A(t)x(t) + B(t)x\bigl(g(t)\bigr) + f(t) \tag{1.5}$$

satisfying $x(t) = 0$, $t \leq 0$ is bounded.

A natural question is whether the zero solution of (1.4) is uniformly asymptotically stable under Perron's condition (P). It turns out that the answer depends on the delay function g.

The paper is organized as follows. In Section 2, we only state our results; the proofs are included in Section 5. We define an adjoint system and give a variation of parameters formula in Section 3 to be needed in proving the main results. Section 4 contains also some lemmas concerning Perron's condition and a relation useful for changing the order of integration.

2. Stability Theorems

The conclusion obtained by Bellman and Halanay for systems (L1) and (L2), respectively, is quite strong. We are only able to prove the stability of the zero solution for more general equation (1.4) under Perron's condition. To get uniform stability or asymptotic stability or uniform asymptotic stability, we impose restrictions on the delay function.

For our purpose, we denote

$$\begin{aligned} h_*(t) &:= h(t) - t, \quad t \geq 0, \\ g_*(t, t_0) &:= \sup_{r \in [h(t_0), t]} \bigl\{r - g(r)\bigr\}, \quad t, t_0 \geq 0. \end{aligned} \tag{2.1}$$

Theorem 2.1. *Let* (P) *hold. If there are positive numbers M_1 and M_2 such that*

$$|A(t)| \leq M_1, \quad |B(t)| \leq M_2 \quad \forall t \geq 0, \tag{2.2}$$

then the zero solution of (1.4) *is stable.*

Theorem 2.2. *Let* (*P*) *hold. If* (2.2) *is satisfied and if there exists a positive real number M_3 such that*

$$h_*(t) \leq M_3 \quad \forall t \geq 0, \tag{2.3}$$

then the zero solution of (1.4) *is uniformly stable.*

Theorem 2.3. *Let* (P) *hold. If* (2.2) *and*

$$\limsup_{t\to\infty}\frac{g_*(t,t_0)}{t-t_0}=0 \quad \textit{for each } t_0\geq 0 \tag{2.4}$$

are satisfied, then the zero solution of (1.4) *is asymptotically stable.*

Theorem 2.4. *Let* (P) *hold. If* (2.2), (2.3), *and*

$$\limsup_{t\to\infty}\frac{g_*(t,t_0)}{t-t_0}=0 \quad \textit{uniformly for } t_0\geq 0 \tag{2.5}$$

are satisfied, then the zero solution of (1.4) *is uniformly asymptotically stable.*

Remark 2.5. Note that if $g(t)=t-\tau$, then $h(t)=t+\tau$ and hence the conditions (2.3), (2.4), and (2.5) are automatically satisfied. In this case, all theorems become equivalent, that is, the zero solution is uniformly asymptotically stable. Thus, the results obtained by Bellman and Halanay are recovered.

3. Variation of Parameters Formula

To establish a variation of parameters formula to represent the solutions of (1.5), one needs an adjoint system. The following lemma helps to define the adjoint of (1.4).

Lemma 3.1. *Let* $x(t)$ *be a solution of* (1.4). *If* $y(t)$ *is a solution of*

$$y'(t)=-A^T(t)y(t)-B^T(h(t))y(h(t))h'(t), \tag{3.1}$$

then

$$\frac{d}{dt}F\big(x(t),y(t)\big)=0, \tag{3.2}$$

where

$$F(x,y)(t)=\int_t^{h(t)} y^T(s)B(s)x\big(g(s)\big)ds+x^T(t)y(t). \tag{3.3}$$

Proof. Verify directly. □

Definition 3.2. The system (3.1) is said to be adjoint to system (1.4).

It is easy to see that the adjoint of system (3.1) is system (1.4); thus the systems are mutually adjoint to each other.

Lemma 3.3. *Let $Y(t,s)$ be a matrix solution of* (3.1) *for $t<s$ satisfying $Y(s,s)=I$ and $Y(t,s)=0$ for $t>s$. Then $x(t)$ is a solution of* (1.5) *if and only if*

$$x(t)=Y^T(s,t)x(s)+\int_{g(s)}^{s} Y^T(h(\beta),t)B(h(\beta))x(\beta)h'(\beta)d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta. \tag{3.4}$$

Proof. Replacing t by β in (1.5) and then integrating the resulting equation multiplied by $Y^T(\beta,t)$ over $\beta\in[s,t]$, we have

$$\begin{aligned}
&\int_s^t Y^T(\beta,t)A(\beta)x(\beta)d\beta+\int_s^t Y^T(\beta,t)B(\beta)x(g(\beta))d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta\\
&\quad=\int_s^t Y^T(\beta,t)x'(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)-\int_s^t\left[\frac{\partial}{\partial\beta}Y^T(\beta,t)\right]x(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)+\int_s^t\left[Y^T(\beta,t)A(\beta)+Y^T(h(\beta),t)B(h(\beta))h'(\beta)\right]x(\beta)d\beta\\
&\quad=x(t)-Y^T(s,t)x(s)+\int_s^t Y^T(\beta,t)A(\beta)x(\beta)d\beta+\int_{h(s)}^{h(t)} Y^T(\beta,t)B(\beta)x(g(\beta))d\beta.
\end{aligned} \tag{3.5}$$

Comparing both sides and using

$$\int_t^{h(t)} Y^T(\beta,t)B(\beta)x(g(\beta))d\beta=0, \tag{3.6}$$

which is true in view of $Y(\beta,t)=0$ for $\beta>t$, we get

$$x(t)=Y^T(s,t)x(s)-\int_{h(s)}^{s} Y^T(\beta,t)B(\beta)x(g(\beta))d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta \tag{3.7}$$

and hence

$$x(t)=Y^T(s,t)x(s)+\int_{g(s)}^{s} Y^T(h(\beta),t)B(h(\beta))x(\beta)h'(\beta)d\beta+\int_s^t Y^T(\beta,t)f(\beta)d\beta. \tag{3.8}$$

□

It is not difficult to see from (3.4) that if $X(t,s)$ is a matrix solution of (1.4) for $t>s$ satisfying $X(s,s)=I$ and $X(t,s)=0$ for $t<s$, then

$$X(t,s)=Y^T(s,t). \tag{3.9}$$

Using this relation in Lemma 3.3 leads to the following variation of parameters formula.

Lemma 3.4. *Let $X(t,s)$ be a matrix solution of* (1.4) *for $t > s$ satisfying $X(s,s) = I$ and $X(t,s) = 0$ for $t < s$. Then $x(t)$ is a solution of* (1.5) *if and only if*

$$x(t) = X(t,s)x(s) + \int_{g(s)}^{s} X(t,h(\beta))B(h(\beta))x(\beta)h'(\beta)d\beta + \int_{s}^{t} X(t,\beta)f(\beta)d\beta. \tag{3.10}$$

4. Auxiliary Results

Lemma 4.1. *If* (P) *holds, then there is a positive number K_1 such that*

$$\int_0^t |X(t,s)|ds \le K_1 \quad \forall t > 0. \tag{4.1}$$

Proof. The proof follows as in [5]. We provide only the steps for the reader's convenience.

Define

$$\begin{aligned}(Sf)(t) &= \int_0^t X(t,\beta)f(\beta)d\beta, \quad f \in C_B[0,\infty),\\ S_k(f) &= \int_0^{t_k} X(t_k,\beta)f(\beta)d\beta, \quad f \in C_B[0,\infty),\end{aligned} \tag{4.2}$$

for each rational number t_k, $k \in \mathbb{N}$.

In view of (P), the family of continuous linear operators $\{S_k\}$ from $C_B[0,\infty)$ to $C_B[0,\infty)$ is pointwise-bounded. For the space of bounded continuous functions $C_B[0,\infty)$, the usual sup norm $\|\cdot\|$ is used.

By the Banach-Steinhaus theorem, the family is uniformly bounded. Thus, there is a positive number M such that $\|S_k(f)\| \le M\|f\|$ for every $f \in C_B[0,\infty)$.

As the rational numbers are dense in the real numbers, for each t there is t_k such that $t_k \to t$ as $k \to \infty$ and so

$$\left|\int_0^t X(t,\beta)f(\beta)d\beta\right| \le M\|f\| \quad \forall f \in C_B[0,\infty). \tag{4.3}$$

The final step involves choosing a sequence of functions and using a limiting process. □

Lemma 4.2. *If* (2.2) *and* (4.1) *are true, then there is a positive number K_2 such that*

$$|Y(s,t)| \le K_2 \quad \forall\, 0 \le s < t. \tag{4.4}$$

Proof. From (3.1), we have

$$Y(s,t) = I + \int_s^t A^T(\beta)Y(\beta,t)d\beta + \int_s^t B^T(h(\beta))Y(h(\beta),t)h'(\beta)d\beta. \tag{4.5}$$

Hence, by using (4.1), we see that for all $0 \le s < t$,

$$|Y(s,t)| \le 1 + M_1 K_1 + M_2 K_1 =: K_2. \tag{4.6}$$

□

Lemma 4.3. *Let $G(r,t)$ be a continuous function satisfying $G(r,t) = 0$ for $r > t$. Then*

$$\int_{t_0}^{t} \left[\int_{s}^{h(s)} G(r,t)dr \right] ds = \int_{h(t_0)}^{t} (r - g(r))G(r,t)dr + \int_{t_0}^{h(t_0)} (r - t_0)G(r,t)dr. \tag{4.7}$$

5. Proofs of Theorems

Let $t_0 \ge 0$ be given. For a given continuous vector function ϕ defined on $[g(t_0), t_0]$, let $x(t) = x(t, t_0, \phi)$ denote the solution of (1.4) satisfying

$$x(t) = \phi(t), \quad t \le t_0. \tag{5.1}$$

As usual,

$$\|\phi\|_g = \sup_{t \in [g(t_0), t_0]} |\phi(t)|. \tag{5.2}$$

Proof of Theorem 2.1. From Lemma 3.3, we may write

$$x(t) = Y^T(t_0, t)\phi(t_0) + \int_{g(t_0)}^{t_0} Y^T(h(\beta), t)B(h(\beta))\phi(\beta)h'(\beta)d\beta. \tag{5.3}$$

In view of Lemma 4.2, it follows that

$$|x(t)| \le (K_2 + (h(t_0) - t_0)K_2 M_2)\|\phi\|_g. \tag{5.4}$$

Hence, the zero solution is stable. □

Proof of Theorem 2.2. Using (2.3) in (5.4), we get

$$|x(t)| \le K_3 \|\phi\|_g, \quad K_3 = K_2 + K_2 M_2 M_3, \tag{5.5}$$

from which the uniform stability follows. □

Proof of Theorem 2.3. By Theorem 2.1, the zero solution is stable. We need to show the attractivity property.

From Lemma 3.3, for $s \ge t_0$, we can write

$$x(t, t_0, \phi) = Y^T(s, t)x(s, t_0, \phi) + \int_{g(s)}^{s} G(h(\beta), t)x(\beta, t_0, \phi)h'(\beta)d\beta, \tag{5.6}$$

where

$$G(s,t) = Y^T(s,t)B(s). \tag{5.7}$$

Integrating with respect to s from t_0 to t, we have

$$(t-t_0)x(t,t_0,\phi) = \int_{t_0}^{t}\left[Y^T(s,t)x(s,t_0,\phi) + \int_{s}^{h(s)} G(r,t)x(g(r),t_0,\phi)dr\right]ds. \tag{5.8}$$

We change the order of integration by employing Lemma 4.3. After some rearrangements, we obtain

$$\begin{aligned}(t-t_0)x(t,t_0,\phi) &= \int_{t_0}^{t} Y^T(s,t)x(s,t_0,\phi)ds + \int_{h(t_0)}^{t}(s-g(s))G(s,t)x(g(s),t_0,\phi)ds \\ &\quad + \int_{t_0}^{h(t_0)}(s-t_0)G(s,t)x(g(s),t_0,\phi)ds.\end{aligned} \tag{5.9}$$

It follows that

$$(t-t_0)\left|x(t,t_0,\phi)\right| \le K_1K_3\|\phi\|_g + g_*(t,t_0)M_2K_1\|\phi\|_g + h_*(t_0)M_2K_1\|\phi\|_g. \tag{5.10}$$

In view of condition (2.4), we see from (5.10) that

$$\lim_{t\to\infty}\left|x(t,t_0,\phi)\right| = 0. \tag{5.11}$$

□

Proof of Theorem 2.4. By Theorem 2.2, the zero solution is uniformly stable. From (5.10) and (2.3), we have

$$(t-t_0)\left|x(t,t_0,\phi)\right| \le K_1K_3\|\phi\|_g + g_*(t,t_0)M_2K_1\|\phi\|_g + M_3M_2K_1\|\phi\|_g. \tag{5.12}$$

Using condition (2.4) in the above inequality, we see that the zero solution is uniformly asymptotically stable as $t \to \infty$. □

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Research Article

On the Reducibility for a Class of Quasi-Periodic Hamiltonian Systems with Small Perturbation Parameter

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We consider the following real two-dimensional nonlinear analytic quasi-periodic Hamiltonian system $\dot{x} = J\nabla_x H$, where $H(x,t,\varepsilon) = (1/2)\beta(x_1^2 + x_2^2) + F(x,t,\varepsilon)$ with $\beta \neq 0, \partial_x F(0,t,\varepsilon) = O(\varepsilon)$ and $\partial_{xx}F(0,t,\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. Without any nondegeneracy condition with respect to ε, we prove that for most of the sufficiently small ε, by a quasi-periodic symplectic transformation, it can be reduced to a quasi-periodic Hamiltonian system with an equilibrium.

1. Introduction

We first give some definitions and notations for our problem. A function $f(t)$ is called a quasi-periodic function with frequencies $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ if $f(t) = F(\omega_1 t, \omega_2 t, \ldots, \omega_l t)$ with $\theta_i = \omega_i t$, where $F(\theta_1, \theta_2, \ldots, \theta_l)$ is 2π periodic in all the arguments $\theta_j,\ j = 1,2,\ldots,l$. If $F(\theta)$ $(\theta = (\theta_1, \theta_2, \ldots, \theta_l))$ is analytic on $D_\rho = \{\theta \in C^l/2\pi Z^l \mid |\operatorname{Im}\theta_i| \leq \rho,\ i = 1,2,\ldots,l\}$, we call $f(t)$ analytic quasi-periodic on D_ρ. If all $q_{ij}(t)$ $(i,j = 1,2\ldots,n)$ are analytic quasi-periodic on D_ρ, then the matrix function $Q(t) = (q_{ij}(t))_{1\leq i,j\leq n}$ is called analytic quasi-periodic on D_ρ.

If $f(t)$ is analytic quasi-periodic on D_ρ, we can write it as Fourier series:

$$f(t) = \sum_{k\in Z^l} f_k e^{\mathrm{i}\langle k,\omega\rangle t}. \tag{1.1}$$

Define a norm of f by $\|f\|_\rho = \sum_{k\in Z^l} |f_k| e^{|k|\rho}$. It follows that $|f_k| \leq \|f\|_\rho e^{-|k|\rho}$. If the matrix function $Q(t)$ is analytic quasi-periodic on D_ρ, we define the norm of Q by $\|Q\|_\rho = n \times \max_{1\leq i,j\leq n}\|q_{ij}\|_\rho$. It is easy to verify $\|Q_1 Q_2\|_\rho \leq \|Q_1\|_\rho \|Q_2\|_\rho$. The average of $Q(t)$ is denoted by $[Q] = ([q_{ij}])_{1\le i,j\le n}$, where

$$[q_{ij}] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} q_{ij}(t)dt. \tag{1.2}$$

For the existence of the above limit, see [1].

Denote

$$D(r,\rho,\varepsilon_0) = \left\{(x,\theta,\varepsilon)\in C^n\times\left(\frac{C^l}{2\pi Z^l}\right)\times C \mid |x|\le r,\ \theta\in D_\rho,\ |\varepsilon|\le\varepsilon_0\right\}, \tag{1.3}$$

where $x = (x_1, x_2, \ldots, x_n)$ and $|x| = |x_1| + |x_2| + \cdots + |x_n|$.

Let $f(x,t,\varepsilon)$ be analytic quasi-periodic of t and analytic in x and ε on $D(r,\rho,\varepsilon_0)$. Then $f(x,t,\varepsilon)$ can be expanded as

$$f(x,t,\varepsilon) = \sum_{m=0}^{\infty}\sum_{k\in Z^l} f_{mk}(x)\varepsilon^m e^{\mathrm{i}\langle k,\omega\rangle t}. \tag{1.4}$$

Define a norm by

$$\|f\|_{D(r,\rho,\varepsilon_0)} = \sum_{m=0}^{\infty}\sum_{k\in Z^l}|f_{mk}|_r\varepsilon_0^m e^{\rho|k|}, \tag{1.5}$$

where $|f_{mk}|_r = \sup_{|x|\le r}|f_{mk}(x)|$. Note that

$$\|f_1\cdot f_2\|_{D(r,\rho,\varepsilon_0)} \le \|f_1\|_{D(r,\rho,\varepsilon_0)}\cdot\|f_2\|_{D(r,\rho,\varepsilon_0)}. \tag{1.6}$$

Problems

The reducibility on the linear differential system has been studied for a long time. The well-known Floquet theorem tells us that if $A(t)$ is a T-periodic matrix, then the linear system $\dot{x} = A(t)x$ is always reducible to the constant coefficient one by a T-periodic change of variables. However, this cannot be generalized to the quasi-periodic system. In [2], Johnson and Sell considered the quasi-periodic system $\dot{x} = A(t)x$, where $A(t)$ is a quasi-periodic matrix. Under some "full spectrum" conditions, they proved that $\dot{x} = A(t)x$ is reducible. That is, there exists a quasi-periodic nonsingular transformation $x = \phi(t)y$, where $\phi(t)$ and $\phi(t)^{-1}$ are quasi-periodic and bounded, such that $\dot{x} = A(t)x$ is transformed to $\dot{y} = By$, where B is a constant matrix.

In [3], Jorba and Simó considered the reducibility of the following linear system:

$$\dot{x} = (A + \varepsilon Q(t))x, \quad x\in R^n, \tag{1.7}$$

where A is an $n \times n$ constant matrix with n different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ and $Q(t)$ is analytic quasi-periodic with respect to t with frequencies $\omega = (\omega_1, \omega_2, \dots, \omega_l)$. Here ε is a small perturbation parameter. Suppose that the following nonresonance conditions hold:

$$\left| \langle k, \omega \rangle \sqrt{-1} + \lambda_i - \lambda_j \right| \geq \frac{\alpha}{|k|^{\tau}}, \tag{1.8}$$

for all $k \in Z^l \setminus \{0\}$, where $\alpha > 0$ is a small constant and $\tau > l - 1$. Assume that $\lambda_j^0(\varepsilon)$ $(j = 1, 2, \dots, n)$ are eigenvalues of $A + \varepsilon[Q]$. If the following non-degeneracy conditions hold:

$$\frac{d}{d\varepsilon}\left(\lambda_i^0(\varepsilon) - \lambda_j^0(\varepsilon)\right)\Big|_{\varepsilon=0} \neq 0, \quad \forall i \neq j, \tag{1.9}$$

then authors proved that for sufficiently small $\varepsilon_0 > 0$, there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that for $\varepsilon \in E$, the system (1.7) is reducible. Moreover, $\operatorname{meas}((0, \varepsilon_0) \setminus E) = o(\varepsilon_0)$.

Some related problems were considered by Eliasson in [4, 5]. In the paper [4], to study one-dimensional linear Schrödinger equation

$$\frac{d^2 q}{dt^2} + Q(\omega t) q = Eq, \tag{1.10}$$

Eliasson considered the following equivalent two-dimensional quasi-periodic Hamiltonian system:

$$\dot{p} = (E - Q(\omega t)) q, \quad \dot{q} = p, \tag{1.11}$$

where Q is an analytic quasi-periodic function and E is an energy parameter. The result in [4] implies that for almost every sufficiently large E, the quasi-periodic system (1.11) is reducible. Later, in [5] the author considered the almost reducibility of linear quasi-periodic systems. Recently, the similar problem was considered by Her and You [6]. Let $C^{\omega}(\Lambda, gl(m, C))$ be the set of $m \times m$ matrices $A(\lambda)$ depending analytically on a parameter λ in a closed interval $\Lambda \subset R$. In [6], Her and You considered one-parameter families of quasi-periodic linear equations

$$\dot{x} = \left(A(\lambda) + g(\omega_1 t, \dots, \omega_l t, \lambda)\right) x, \tag{1.12}$$

where $A \in C^{\omega}(\Lambda, gl(m, C))$, and g is analytic and sufficiently small. They proved that under some nonresonance conditions and some non-degeneracy conditions, there exists an open and dense set $\mathcal{A}$ in $C^{\omega}(\Lambda, gl(m, C))$, such that for each $A \in \mathcal{A}$, the system (1.12) is reducible for almost all $\lambda \in \Lambda$.

In 1996, Jorba and Simó extended the conclusion of the linear system to the nonlinear case. In [7], Jorba and Simó considered the quasi-periodic system

$$\dot{x} = (A + \varepsilon Q(t)) x + \varepsilon g(t) + h(x, t), \quad x \in R^n, \tag{1.13}$$

where A has n different nonzero eigenvalues λ_i. They proved that under some nonresonance conditions and some non-degeneracy conditions, there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that the system (1.13) is reducible for $\varepsilon \in E$.

In [8], the authors found that the non-degeneracy condition is not necessary for the two-dimensional quasi-periodic system. They considered the two-dimensional nonlinear quasi-periodic system:

$$\dot{x} = Ax + f(x, t, \varepsilon), \quad x \in R^2, \tag{1.14}$$

where A has a pair of pure imaginary eigenvalues $\pm\sqrt{-1}\omega_0$ with $\omega_0 \neq 0$ satisfying the nonresonance conditions

$$|\langle k, \omega \rangle| \geq \frac{\alpha}{|k|^\tau}, \qquad |\langle k, \omega \rangle - 2\omega_0| \geq \frac{\alpha}{|k|^\tau} \tag{1.15}$$

for all $k \in Z^l \setminus \{0\}$, where $\alpha > 0$ is a small constant and $\tau > l - 1$. Assume that $f(0, t, \varepsilon) = O(\varepsilon)$ and $\partial_x f(0, t, \varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. They proved that either of the following two results holds:

(1) for $\forall \varepsilon \in (0, \varepsilon_0)$, the system (1.14) is reducible to $\dot{y} = By + O(y)$ as $y \to 0$;

(2) there exists a nonempty Cantor subset $E \subset (0, \varepsilon_0)$, such that for $\varepsilon \in E$ the system (1.14) is reducible to $\dot{y} = By + O(y^2)$ as $y \to 0$.

Note that the result (1) happens when the eigenvalue of the perturbed matrix of A in KAM steps has nonzero real part. But the authors were interested in the equilibrium of the transformed system and obtained a small quasi-periodic solution for the original system.

Motivated by [8], in this paper we consider the Hamiltonian system and we have a better result.

2. Main Results

Theorem 2.1. *Consider the following real two-dimensional Hamiltonian system*

$$\dot{x} = J\nabla_x H, \quad x \in R^2, \tag{2.1}$$

where $H(x, t, \varepsilon) = (1/2)\beta(x_1^2 + x_2^2) + F(x, t, \varepsilon)$ *with* $\beta \neq 0$, $F(x, t, \varepsilon)$ *is analytic quasi-periodic with respect to* t *with frequencies* $\omega = (\omega_1, \omega_2, \ldots, \omega_l)$ *and real analytic with respect to* x *and* ε *on* $D(r, \rho, \varepsilon_0)$, *and*

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.2}$$

Here $\varepsilon \in (0, \varepsilon_0)$ *is a small parameter. Suppose that* $\partial_x F(0, t, \varepsilon) = O(\varepsilon)$ *and* $\partial_{xx} F(0, t, \varepsilon) = O(\varepsilon)$ *as* $\varepsilon \to 0$. *Moreover, assume that* β *and* ω *satisfy*

$$|\langle k, \omega \rangle| \geq \frac{\alpha_0}{|k|^\tau}, \tag{2.3}$$

$$\left|\langle k, \omega \rangle - 2\beta\right| \geq \frac{\alpha_0}{|k|^\tau} \tag{2.4}$$

for all $k \in Z^l \setminus \{0\}$, *where* $\alpha_0 > 0$ *is a small constant and* $\tau > l - 1$.

Then there exist a sufficiently small $\varepsilon_* \in (0, \varepsilon_0]$ *and a nonempty Cantor subset* $E_* \subset (0, \varepsilon_*)$, *such that for* $\varepsilon \in E_*$, *there exists an analytic quasi-periodic symplectic transformation* $x = \phi_*(t) y + \psi_*(t)$ *on* $D_{\rho/2}$ *with the frequencies* ω, *which changes* (2.1) *into the Hamiltonian system* $\dot{y} = J \nabla_y H_*$, *where* $H_*(y, t, \varepsilon) = 1/2 \beta_*(\varepsilon)(y_1^2 + y_2^2) + F_*(y, t, \varepsilon)$, *where* $F_*(y, t, \varepsilon) = O(y^3)$ *as* $y \to 0$. *Moreover,* $\operatorname{meas}((0, \varepsilon_*) \setminus E_*) = o(\varepsilon_*)$ *as* $\varepsilon_* \to 0$. *Furthermore,* $\beta_*(\varepsilon) = \beta + O(\varepsilon)$ *and* $\|\phi_* - Id\|_{\rho/2} + \|\psi_*\|_{\rho/2} = O(\varepsilon)$, *where* Id *is the 2-order unit matrix.*

3. The Lemmas

The proof of Theorem 2.1 is based on KAM-iteration. The idea is the same as [7, 8]. When the non-degeneracy conditions do not happen, the small parameter ε is not involved in the nonresonance conditions. So without deleting any parameter, the KAM step will be valid. Once the non-degeneracy conditions occur at some step, they will be kept for ever and we can apply the results with the non-degeneracy conditions. Thus, after infinite KAM steps, the transformed system is convergent to a desired form.

We first give some lemmas. Let $R = (r_{ij})_{1 \leq i, j \leq 2}$ be a Hamiltonian matrix. Then we have $r_{11} + r_{22} = 0$. Define a matrix $R_A = (1/2) d J$ with $d = r_{12} - r_{21}$. Let

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \sqrt{-1} & -\sqrt{-1} \end{pmatrix}. \tag{3.1}$$

It is easy to verify

$$\begin{gathered} B^{-1} R_A B = \frac{1}{2} \operatorname{diag}\left(\sqrt{-1} d, -\sqrt{-1} d\right), \\ B^{-1}(R - R_A) B = \frac{1}{2} \begin{pmatrix} 0 & \sigma' - \sqrt{-1}\kappa' \\ \sigma' + \sqrt{-1}\kappa' & 0 \end{pmatrix}, \end{gathered} \tag{3.2}$$

where $\sigma' = 2 r_{11}$ and $\kappa' = r_{21} + r_{12}$.

In the same way as in [7, 8], in KAM steps we need to solve linear homological equations. For this purpose we need the following lemma.

Lemma 3.1. *Consider the following equation of the matrix:*

$$\dot{P} = AP - PA + R(t), \tag{3.3}$$

where $A = \beta(\varepsilon)J$ *with* $|\beta(\varepsilon)| > \mu$, $\mu > 0$ *is a constant, and* $R(t) = (r_{ij}(t))_{1\le i,j\le 2}$ *is a real analytic quasi-periodic Hamiltonian matrix on* D_ρ *with frequencies* ω. *Suppose* $\beta(\varepsilon)$ *and* R *are smooth with respect to* ε *and* $|\varepsilon\beta'(\varepsilon)| \le c_0$ *for* $\varepsilon \in E \subset (0, \varepsilon_*)$, *where* c_0 *is a constant. Note that here and below the dependence of* ε *is usually implied and one does not write it explicitly for simplicity. Assume* $[R]_A = 0$, *where* $[R]$ *is the average of* R. *Suppose that for* $\varepsilon \in E$, *the small divisors conditions* (2.3) *and the following small divisors conditions hold:*

$$\left|\langle k, \omega\rangle - 2\beta(\varepsilon)\right| \ge \frac{\alpha}{|k|^{\tau'}}, \tag{3.4}$$

where $\tau' > 2\tau + l$. *Let* $0 < s < \rho$ *and* $\rho_1 = \rho - s$. *Then there exists a unique real analytic quasi-periodic Hamiltonian matrix P(t) with frequencies* ω, *which solves the homological linear equation* (3.3) *and satisfies*

$$\|P\|_{\rho_1} \le \frac{c}{\alpha s^{v}}\|R\|_\rho, \qquad \|\varepsilon\partial_\varepsilon P\|_{\rho_1} \le \frac{c}{\alpha^2 s^{v'}}\left(\|R\|_\rho + \|\varepsilon\partial_\varepsilon R\|_\rho\right), \tag{3.5}$$

where $v = \tau' + l$, $v' = 2\tau' + l$ *and* $c > 0$ *is a constant.*

Remark 3.2. The subset E of $(0, \varepsilon_*)$ is usually a Cantor set and so the derivative with respect to ε should be understood in the sense of Whitney [9].

Proof. Let $\overline{P} = B^{-1}PB$, where B is defined by (3.1). Similarly, define $\overline{A}$, $\overline{R}$, $\overline{R}_A$. Then (3.3) becomes

$$\dot{\overline{P}} = \overline{A}\,\overline{P} - \overline{P}\,\overline{A} + \overline{R}(t), \tag{3.6}$$

where

$$\overline{A} = \operatorname{diag}\left(\sqrt{-1}\beta, -\sqrt{-1}\beta\right). \tag{3.7}$$

Moreover, $\overline{R}_A$ and $\overline{R} - \overline{R}_A$ have the same forms as (3.2) and (3.2), respectively

Noting that $[R]_A = 0$, we have $[\overline{R}]_A = 0$. Write $\overline{P} = (\overline{p}_{ij})_{i,j}$ and $\overline{R} = (\overline{r}_{ij})_{i,j}$. Obviously, we have $\overline{r}_{11} = -\overline{r}_{22}$ with $[\overline{r}_{ii}] = 0$.

Insert the Fourier series of $\overline{P}$ and $\overline{R}$ into (3.6). Then it follows that $\overline{p}_{ii}^0 = 0$, $\overline{p}_{ii}^k = \overline{r}_{ii}^k/(\langle k, \omega\rangle\sqrt{-1})$ for $k \ne 0$, and

$$\overline{p}_{ij}^k = \frac{\overline{r}_{ij}^k}{\sqrt{-1}\left(\langle k, \omega\rangle \pm 2\beta\right)} \quad \text{for } i \ne j. \tag{3.8}$$

Since $\overline{R}$ is analytic on D_ρ, we have $|\overline{R}_k| \le \|\overline{R}\|_\rho e^{-|k|\rho}$. So it follows

$$\left\|\overline{P}\right\|_{\rho-s} \le \sum_{k\in Z^l}\left|\overline{P}_k\right| e^{|k|\,(\rho-s)} \le \frac{c}{\alpha s^{v}}\|R\|_\rho. \tag{3.9}$$

Note that here and below we always use c to indicate constants, which are independent of KAM steps.

Since A and $R(t)$ are real matrices, it is easy to obtain that $P(t)$ is also a real matrix. Obviously, it follows that $\overline{p}_{11} = -\overline{p}_{22}$ and the trace of the matrix $\overline{P}$ is zero. So is the trace of P. Thus, P is a Hamiltonian matrix.

Now we estimate $\|\varepsilon\partial P/\partial\varepsilon\|_{\rho_1}$. We only consider $\overline{p}_{12}$ and $\overline{p}_{21}$ since $\overline{p}_{11}$ and $\overline{p}_{22}$ are easy.

For $i \neq j$ we have

$$\frac{d\overline{p}_{ij}^{k}(\varepsilon)}{d\varepsilon} = \frac{\pm 2\beta'(\varepsilon)\overline{r}_{ij}^{k} - (\langle k,\omega\rangle \pm 2\beta)\overline{r}_{ij}^{k\,\prime}(\varepsilon)}{-\sqrt{-1}(\langle k,\omega\rangle \pm 2\beta)^{2}}. \tag{3.10}$$

Then, in the same way as above we obtain the estimate for $\|\varepsilon(\partial P/\partial\varepsilon)\|_{\rho_1}$. □

The following lemma will be used for the zero order term in KAM steps.

Lemma 3.3. *Consider the equation*

$$\dot{x} = Ax + g(t), \tag{3.11}$$

where A is the same as in Lemma 3.1, and g is real analytic quasi-periodic in t on D_ρ with frequencies ω and smooth with respect to ε. Suppose that the small divisors conditions (3.4) *hold. Then there exists a unique real analytic quasi-periodic solution $x(t)$ with frequencies ω, which satisfies*

$$\|x\|_{\rho_1} \leq \frac{c}{\alpha s^{v}}\|g\|_{\rho}, \qquad \left\|\varepsilon\frac{\partial x}{\partial\varepsilon}\right\|_{\rho_1} \leq \frac{c}{\alpha^{2}s^{v'}}\left(\|g\|_{\rho} + \left\|\varepsilon\frac{\partial g}{\partial\varepsilon}\right\|_{\rho}\right), \tag{3.12}$$

where s,ρ_1,v, v' are defined in Lemma 3.1.

Proof. Similarly, let $\overline{x} = B^{-1}x, \overline{A} = B^{-1}AB$ and $\overline{g}(t) = B^{-1}g(t)$. Then (3.11) becomes

$$\dot{\overline{x}} = \overline{A}\,\overline{x} + \overline{g}(t), \tag{3.13}$$

where $\overline{A} = \operatorname{diag}(\sqrt{-1}\beta, -\sqrt{-1}\beta)$. Expanding $\overline{x} = (\overline{x}_1, \overline{x}_2)$ and $\overline{g} = (\overline{g}_1, \overline{g}_2)$ into Fourier series and using (3.13), we have

$$\overline{x}_i^{k} = \frac{\overline{g}_i^{k}}{\sqrt{-1}\left(\langle k,\omega\rangle + (-1)^{i}\beta\right)}. \tag{3.14}$$

Using $2k$ in place of k in (3.4), we have

$$\left|\langle k,\omega\rangle-\beta(\varepsilon)\right|\geq\frac{\alpha}{2|k|^{\tau'}}.\tag{3.15}$$

Thus, in the same way as the proof of Lemma 3.1, we can estimate $\|x\|_{\rho_1}$ and $\|\varepsilon\partial_\varepsilon x\|_{\rho_1}$. We omit the details. □

The following lemma is used in the estimate of Lebesgue measure for the parameter ε in the case of non-degeneracy.

Lemma 3.4. *Let $\psi(\varepsilon)=\sigma\varepsilon^N+\varepsilon^N f(\varepsilon)$, where N is a positive integer and f satisfies that $f(\varepsilon)\to 0$ as $\varepsilon\to 0$ and $|f'(\varepsilon)|\leq c$ for $\varepsilon\in(0,\varepsilon_*)$. Let $\phi(\varepsilon)=\langle k,\omega\rangle-2\beta-\psi(\varepsilon)$. Let*

$$O=\left\{\varepsilon\in(0,\varepsilon_*)\mid\left|\phi(\varepsilon)\right|\geq\frac{\alpha}{|k|^{\tau'}},\ \forall k\neq 0\right\},\tag{3.16}$$

where $\tau'\geq 2\tau+l$, $\alpha\leq(1/2)\alpha_0,\sigma\neq 0$. Suppose that the small condition (2.4) *holds. Then when ε_* is sufficiently small, one has*

$$\operatorname{meas}(0,\varepsilon_*)\setminus O\leq c\frac{\alpha}{\alpha_0^2}\varepsilon_*^{N+1},\tag{3.17}$$

where c is a constant independent of $\alpha_0,\alpha,\varepsilon_*$

Proof. Let

$$O_k=\left\{\varepsilon\in(0,\varepsilon_*)\mid\left|\phi(\varepsilon)\right|<\frac{\alpha}{|k|^{\tau'}}\right\}.\tag{3.18}$$

By assumption, if ε_* is sufficient small, we have that $|\psi(\varepsilon)|\leq 2\sigma\varepsilon^N$ and $|\psi'(\varepsilon)|\geq(\sigma/2)\varepsilon^{N-1}$ for $\varepsilon\in(0,\varepsilon_*)$. If $\varepsilon^N\leq\alpha_0/(4\sigma|k|^\tau)$, by (2.4) we have

$$\left|\phi(\varepsilon)\right|\geq\left|\langle k,\omega\rangle-2\beta\right|-\left|\psi(\varepsilon)\right|\geq\frac{\alpha}{|k|^{\tau'}}.\tag{3.19}$$

Thus, we only consider the case that $\varepsilon_*^N\geq\varepsilon^N\geq(\alpha_0/(4\sigma|k|^\tau))$. We have $|k|\geq(\alpha_0/(4\sigma\varepsilon_*^N))^{1/\tau}=K$. Since

$$\left|\phi'(\varepsilon)\right|=\left|\psi'(\varepsilon)\right|\geq\frac{\sigma}{2}\varepsilon^{N-1}\geq\frac{\alpha_0}{8|k|^\tau\varepsilon_*},\tag{3.20}$$

we have $\text{meas}(O_k) \le ((2\alpha)/|k|^{\tau'}) \times ((8|k|^{\tau}\varepsilon_*)/\alpha_0) = (16\alpha\varepsilon_*)/(|k|^{\tau'-\tau}\alpha_0)$. So

$$\begin{aligned}\text{meas}((0,\varepsilon_*)\setminus 0) &\le \sum_{|k|\ge K}\text{meas}(O_k) \le \frac{16\alpha}{\alpha_0}\varepsilon_*\sum_{|k|\ge K}\frac{1}{|k|^{\tau'-\tau}}\\ &\le \frac{c\alpha}{\alpha_0}\varepsilon_* K^{l-\tau'+\tau} \le \frac{c\alpha}{\alpha_0^2}\varepsilon_*^{N+1},\end{aligned} \tag{3.21}$$

where c is a constant independent of α_0, α, and ε_*. □

Below we give a lemma with the non-degeneracy conditions.

Lemma 3.5. *Consider the real nonlinear Hamiltonian system $\dot{x} = J\nabla_x H$, where*

$$H(x,t,\varepsilon) = \frac{1}{2}\beta\left(x_1^2 + x_2^2\right) + F(x,t,\varepsilon) \quad \text{with } \beta \ne 0. \tag{3.22}$$

Suppose that $F(x,t,\varepsilon)$ is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D(r,\rho,\varepsilon_0)$. Let $f(x,t,\varepsilon) = J\nabla_x F(x,t,\varepsilon)$. Assume that $f(0,t,\varepsilon) = O(\varepsilon^{2m_0})$ and $\partial_x f(0,t,\varepsilon) = O(\varepsilon^{m_0})$ as $\varepsilon \to 0$, where m_0 is a positive integer. Let $Q(t,\varepsilon) = \partial_x f(0,t,\varepsilon) = \sum_{k\ge m_0} Q_k(t)\varepsilon^k$. Suppose there exists $m_0 \le k \le 2m_0 - 1$ such that $[Q_k]_A \ne 0$ and the nonresonance conditions (2.3) *and* (2.4) *hold. Then, for sufficiently small $\varepsilon_* > 0$, there exists a nonempty Cantor subset $E_* \subset (0,\varepsilon_*)$, such that for $\varepsilon \in E_*$, there exists a quasi-periodic symplectic transformation $x = \phi_*(t)y + \psi_*(t)$ with the frequencies ω, which changes the Hamiltonian system to $\dot{y} = J\nabla_y H_*$, where*

$$H_*(y,t,\varepsilon) = \frac{1}{2}\beta_*(\varepsilon)\left(y_1^2 + y_2^2\right) + F_*(y,t,\varepsilon), \tag{3.23}$$

where $F_*(y,t,\varepsilon) = O(y^3)$ as $y \to 0$. Moreover, $\text{meas}((0,\varepsilon_*)\setminus E_*) = O(\varepsilon_*^{m_0+1})$ as $\varepsilon_* \to 0$. Furthermore, $\beta_*(\varepsilon) = \beta + O(\varepsilon^{m_0})$ and $\|\phi_* - Id\|_{\rho/2} + \|\psi_*\|_{\rho/2} = O(\varepsilon^{m_0})$.

Proof

KAM Step

The proof is based on a modified KAM iteration. In spirit, it is very similar to [7, 8]. The important thing is to make symplectic transformations so that the Hamiltonian structure can be preserved. Note that $[Q_k]_A \ne 0$ for some $m_0 \le k \le 2m_0 - 1$ is a non-degeneracy condition.

Consider the following Hamiltonian system

$$\dot{x} = Ax + f(x,t,\varepsilon), \tag{3.24}$$

where $A = \beta(\varepsilon)J$ and f is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D = D(r,\rho,\varepsilon_*)$.

Let $\|f\|_D \leq \alpha r\widetilde{\varepsilon}$ and $\|\varepsilon\partial_\varepsilon f\|_D \leq \alpha r\widetilde{\varepsilon}$. Let $Q(t,\varepsilon) = \partial_x f(0,t,\varepsilon), g(t,\varepsilon) = f(0,t,\varepsilon)$ and

$$h(x,t,\varepsilon) = f(x,t,\varepsilon) - g(t,\varepsilon) - Q(t,\varepsilon)x. \tag{3.25}$$

Then h is the higher-order term of f. Moreover, the matrix $Q(t,\varepsilon)$ is Hamiltonian. Let $[Q]_A = \widehat{\beta}(\varepsilon)J$.

The system (3.24) is written as

$$\dot{x} = (A_+ + R(t,\varepsilon))x + g(t,\varepsilon) + h(x,t,\varepsilon), \tag{3.26}$$

where $A_+ = A + [Q]_A = \beta_+(\varepsilon)J$ and $R = Q - [Q]_A$. By assumption we have

$$\|g\|_\rho \leq \alpha r\widetilde{\varepsilon}, \qquad \|Q\|_\rho \leq \alpha\widetilde{\varepsilon}, \qquad \|h\|_D \leq 3\alpha r\widetilde{\varepsilon}. \tag{3.27}$$

Moreover, we have

$$\|\varepsilon\partial_\varepsilon g\|_\rho \leq \alpha r\widetilde{\varepsilon}, \qquad \|\varepsilon\partial_\varepsilon Q\|_\rho \leq \alpha\widetilde{\varepsilon}, \qquad \|\varepsilon\partial_\varepsilon h\|_D \leq 3\alpha r\widetilde{\varepsilon}. \tag{3.28}$$

Now we want to construct the symplectic change of variables $x = T'y = e^{P(t)}y$ to (3.26), where P is a Hamiltonian matrix to be defined later. Then we have

$$\begin{aligned}\dot{y} =& \left(e^{-P}(A_+ + R - \dot{P})e^{P} + e^{-P}\left(\dot{P}e^{P} - \frac{d}{dt}e^{P(t)}\right)\right)y \\ &+ e^{-P}g(t,\varepsilon) + e^{-P}h\left(e^{P}y,t,\varepsilon\right).\end{aligned} \tag{3.29}$$

Let $W = e^{P} - I - P$ and $\widetilde{W} = e^{-P} - I - P$. Then the system (3.29) becomes

$$\dot{y} = (A_+ + R - \dot{P} + A_+P - PA_+)y + Q'y + e^{-P}g(t,\varepsilon) + e^{-P}h\left(e^{P}y,t,\varepsilon\right), \tag{3.30}$$

where

$$\begin{aligned}Q' =& -P(R-\dot{P}) + (R-\dot{P})P - P(A_+ + R - \dot{P})P \\ &- P(A_+ + R - \dot{P})W + (A_+ + R - \dot{P})W \\ &+ \widetilde{W}(A_+ + R - \dot{P})e^{P} + e^{-P}\left(\dot{P}e^{P} - \frac{d}{dt}e^{P}\right).\end{aligned} \tag{3.31}$$

We would like to have

$$\dot{P} - A_+P + PA_+ = R, \tag{3.32}$$

where $R = Q - [Q]_A$. Suppose the small divisors conditions (2.3) hold. Let $E_+ \subset (0, \varepsilon_*)$ be a subset such that for $\varepsilon \in E_+$ the small divisors conditions hold:

$$\left|\langle k, \omega\rangle - 2\beta_+(\varepsilon)\right| \geq \frac{\alpha_+}{|k|^{\tau'}}, \quad \forall k \in Z^l \setminus \{0\}, \tag{3.33}$$

where $\tau' > 2\tau + l$. By Lemma 3.1, we have a quasi-periodic Hamiltonian matrix $P(t)$ with frequencies ω to solve the above equation with the following estimates:

$$\begin{gathered}\|P\|_{\rho-s} \leq \frac{c\|Q\|_\rho}{\alpha_+ s^v} \leq \frac{c\widetilde{\varepsilon}}{s^v}, \\ \left\|\varepsilon\frac{\partial P}{\partial \varepsilon}\right\|_{\rho-s} \leq \frac{c}{\alpha_+^2 s^{v'}}\left(\|Q\|_\rho + \left\|\varepsilon\frac{\partial Q}{\partial \varepsilon}\right\|_\rho\right) \leq \frac{c\widetilde{\varepsilon}}{\alpha_+ s^{v'}},\end{gathered} \tag{3.34}$$

where $v = \tau' + l,\ v' = 2\tau' + l$ and $c > 0$ is a constant. Then the system (3.30) becomes

$$\dot{y} = A_+ y + f'(y, t, \varepsilon), \tag{3.35}$$

where $f' = Q'y + e^{-P}g(t,\varepsilon) + e^{-P}h(e^P y, t, \varepsilon)$.

By Lemma 3.3, let us denote by $\underline{x}$ the solution of $\dot{x} = A_+ x + g'(t,\varepsilon)$ on $D_{\rho-2s}$, where $g' = e^{-P}g(t,\varepsilon)$. Then, by Lemma 3.3 we have

$$\begin{gathered}\|\underline{x}\|_{\rho-2s} \leq \frac{c\|g\|_{\rho-s}}{\alpha_+ s^v} \leq \frac{cr\widetilde{\varepsilon}}{s^v}, \\ \left\|\varepsilon\frac{\partial \underline{x}}{\partial \varepsilon}\right\|_{\rho-2s} \leq \frac{c}{\alpha_+^2 s^{v'}}\left(\|g\|_{\rho-s} + \left\|\varepsilon\frac{\partial g}{\partial \varepsilon}\right\|_{\rho-s}\right) \leq \frac{cr\widetilde{\varepsilon}}{\alpha_+ s^{v'}}.\end{gathered} \tag{3.36}$$

Under the symplectic change of variables $y = T''x_+ = \underline{x} + x_+$, the Hamiltonian system (3.35) is changed to

$$\dot{x}_+ = A_+ x_+ + f_+(x_+, t, \varepsilon), \tag{3.37}$$

where $A_+ = \beta_+ J$ and

$$f_+ = Q' \cdot T'' + e^{-P}h \circ T' \circ T''. \tag{3.38}$$

Let the symplectic transformation $T = T' \circ T''$. Then $x = Tx_+ = \phi(t)x_+ + \psi(t)$, where $\phi(t) = e^{P(t)}$ and $\psi(t) = e^{P(t)}\underline{x}(t)$. It is easy to obtain that if $\|P\|_{\rho-2s} \le 1/2$, then

$$\begin{aligned} \|\phi - I\|_{\rho-2s} &\le \frac{c\tilde{\varepsilon}}{s^{v}}, \qquad \|\varepsilon\partial_\varepsilon\phi\|_{\rho-2s} \le \frac{c\tilde{\varepsilon}}{\alpha_+ s^{v'}}, \\ \|\psi\|_{\rho-2s} &\le \frac{cr\tilde{\varepsilon}}{s^{v}}, \qquad \|\varepsilon\partial_\varepsilon\psi\|_{\rho-2s} \le \frac{cr\tilde{\varepsilon}}{\alpha_+ s^{v'}}. \end{aligned} \tag{3.39}$$

Under the symplectic change of variables $x = Tx_+$, the Hamiltonian system (3.24) becomes (3.37).

Below we give the estimates for A_+ and f_+. Obviously, it follows that $A_+(\varepsilon) - A = [Q]_A = \widehat{\beta}(\varepsilon)J$ and

$$\left|\beta_+(\varepsilon) - \beta(\varepsilon)\right| = \left|\widehat{\beta}(\varepsilon)\right| \le c\alpha\tilde{\varepsilon}, \qquad \left|\varepsilon\left(\beta_+'(\varepsilon) - \beta'(\varepsilon)\right)\right| = \left|\varepsilon\widehat{\beta}'(\varepsilon)\right| \le c\alpha\tilde{\varepsilon}. \tag{3.40}$$

By (3.38) we have

$$f_+(x_+, t, \varepsilon) = Q'(t)\left(x_+ + \underline{x}(t)\right) + e^{-P(t)}h\left(e^{P(t)}\left(x_+ + \underline{x}(t)\right), t, \varepsilon\right). \tag{3.41}$$

Let $\rho_+ = \rho - 2s$, and $r_+ = \eta r$ with $\eta \le 1/8$. If $c\tilde{\varepsilon}/\alpha_+ s^{v+v'} \le \eta$, it follows that $\|\underline{x}\|_{\rho-2s} \le (1/8)r$. Let $D_+ = D(r_+, s_+, \varepsilon_*)$. Note that Q' and h only consist of high-order terms of P and x, respectively. It is easy to see $|e^{P(t)}(x_+ + \underline{x}(t))| \le 4\eta r \le r$. By all the estimates (3.27), (3.28), (3.34), and (3.36), and using usual technique of KAM estimate, we have

$$\begin{aligned} \|f_+\|_{D_+} &\le \frac{c\tilde{\varepsilon}^2}{s^{2v}}\eta r + c\alpha r\tilde{\varepsilon}\eta^2 \le \left(\frac{c\tilde{\varepsilon}}{s^{2v}} + c\alpha\eta\right) r_+\tilde{\varepsilon}, \\ \|\varepsilon\partial_\varepsilon f_+\|_{D_+} &\le \frac{c\tilde{\varepsilon}^2}{\alpha_+ s^{v+v'}}\eta r + c\alpha r\tilde{\varepsilon}\eta^2 \le \left(\frac{c\tilde{\varepsilon}}{\alpha s^{v+v'}} + c\alpha\eta\right) r_+\tilde{\varepsilon}. \end{aligned} \tag{3.42}$$

Let $\alpha_+ = \alpha/2$ and $\eta = c\tilde{\varepsilon}/(\alpha^2 s^{v+v'})$. Then we have

$$\|f_+\|_{D_+} \le c\alpha_+ r_+ \eta\tilde{\varepsilon} = \alpha_+ r_+ \tilde{\varepsilon}_+, \qquad \tilde{\varepsilon}_+ = c\eta\tilde{\varepsilon}. \tag{3.43}$$

Similarly, we have

$$\|\varepsilon\partial_\varepsilon f_+\|_{D_+} \le \alpha_+ r_+ \tilde{\varepsilon}_+. \tag{3.44}$$

Note that KAM steps only make sense for the small parameter ε satisfying small divisors conditions. However, by Whitney's extension theorem, for convenience all the functions are supposed to be defined for ε on $[0, \varepsilon_*]$.

KAM Iteration

Now we can give the iteration procedure in the same way as in [7] and prove its convergence.

At the initial step, let $f_0 = f$. Let $f(x,t,\varepsilon) = f(0,t,\varepsilon) + \partial_x f(0,t,\varepsilon)x + h(x,t,\varepsilon)$. By assumption, if ε_* is sufficiently small, we have that for all $\varepsilon \in [0,\varepsilon_*]$

$$\begin{aligned} \left|f(0,t,\varepsilon)\right| &\le c\varepsilon^{2m_0}, & \left|\partial_x f(0,t,\varepsilon)\right| &\le c\varepsilon^{m_0}, \\ \left|\varepsilon\partial_\varepsilon f(0,t,\varepsilon)\right| &\le c\varepsilon^{2m_0}, & \left|\varepsilon\partial_\varepsilon\partial_x f(0,t,\varepsilon)\right| &\le c\varepsilon^{m_0}. \end{aligned} \tag{3.45}$$

Moreover,

$$|h(x,t,\varepsilon)| \le c|x|^2, \quad |\varepsilon\partial_\varepsilon h(x,t,\varepsilon)| \le c|x|^2, \quad \forall |x| \le \varepsilon^{m_0}, \quad \forall \varepsilon \in [0,\varepsilon_*]. \tag{3.46}$$

Let $r_0 = \varepsilon^{m_0}$, $\rho_0 = \rho$, $s_0 = \rho_0/8$, $D_0 = D(r_0,\rho_0,\varepsilon_*)$, and $\tilde{\varepsilon}_0 = c\varepsilon^{m_0}/\alpha_0$. Then we have

$$\left|f_0\right|_{D_0} \le \alpha_0 r_0 \tilde{\varepsilon}_0, \qquad \left|\varepsilon\partial_\varepsilon f_0\right|_{D_0} \le \alpha_0 r_0 \tilde{\varepsilon}_0. \tag{3.47}$$

For $n \ge 1$, let

$$\begin{aligned} \alpha_n &= \frac{\alpha_{n-1}}{2}, & s_n &= \frac{s_{n-1}}{2}, & \rho_n &= \rho_{n-1} - 2s_{n-1}, \\ \eta_{n-1} &= \frac{c\tilde{\varepsilon}_{n-1}}{\alpha_{n-1}^2 s_{n-1}^{v+v'}}, & r_n &= \eta_{n-1} r_{n-1}, & \tilde{\varepsilon}_n &= c\eta_{n-1}\tilde{\varepsilon}_{n-1}. \end{aligned} \tag{3.48}$$

Then we have a sequence of quasi-periodic symplectic transformations $\{T_n\}$ satisfying $T_n x = \phi_n(t)x + \varphi_n(t)$ with

$$\left\|\phi_n - I\right\|_{\rho_{n+1}} \le \frac{c\tilde{\varepsilon}_n}{s_n^v}, \qquad \left\|\varphi_n\right\|_{\rho_{n+1}} \le \frac{c r_n \tilde{\varepsilon}_n}{s_n^v}. \tag{3.49}$$

Let $T^n = T_0 \circ T_1 \cdots \circ T_{n-1}$. Then under the transformation $x = T^n y$ the Hamiltonian system $\dot{x} = A_0 x + f_0(x,t,\varepsilon)$ is changed to $\dot{y} = A_n y + f_n(y,t,\varepsilon)$.

Moreover, $A_n(\varepsilon) = \beta_n(\varepsilon)J$ satisfies $A_{n+1} - A_n = [Q_n]_A$ and

$$\left|\beta_{n+1}(\varepsilon) - \beta_n(\varepsilon)\right| \le c\alpha_n\tilde{\varepsilon}_n, \qquad \left|\varepsilon\left(\beta'_{n+1}(\varepsilon) - \beta'_n(\varepsilon)\right)\right| \le c\alpha_n\tilde{\varepsilon}_n, \tag{3.50}$$

$$\left\|f_n\right\|_{D_n} \le \alpha_n r_n \tilde{\varepsilon}_n. \tag{3.51}$$

Convergence

By the above definitions we have $\eta_n/\eta_{n-1} = c\tilde{\varepsilon}_n/\tilde{\varepsilon}_{n-1} = c\eta_{n-1}$. Thus, we have $\eta_n \le c\eta_{n-1}^2$ and so $c\eta_n \le (c\eta_{n-1})^2 \le (c\eta_0)^{2^n}$. Note that $\eta_0 = c\tilde{\varepsilon}_0/(\alpha_0^2 s_0^{v+v'}) \le c\varepsilon^{m_0}/(\alpha_0^2\rho_0^{v+v'})$. Suppose that ε_* is sufficiently small such that for $0 < \varepsilon < \varepsilon_*$ we have $c\eta_0 \le 1/2$. T_n are affine, so are T^n

with $T^n x = \phi^n(t)x + \psi^n(t)$. By the estimates (3.49) it is easy to prove that $\phi^n(t)$ and $\psi^n(t)$ are convergent and so T^n is actually convergent on the domain $D(r/2, \rho/2)$. Let $T^n \to T_*$ and $T_* x = \phi_*(t)x + \psi_*(t)$. It is easy to see that the estimates for ϕ_* and ψ_* in Theorem 2.1 hold.

Using the estimate for f_n and Cauchy's estimate, we have $|f_n(0,t,\varepsilon)| \le \alpha_n r_n \widetilde{\varepsilon}_n \to 0$ and $|\partial_x f_n(0,t,\varepsilon)| \le \alpha_n \widetilde{\varepsilon}_n \to 0$ as $n \to \infty$. Let $f_n \to f_*$. Then it follows that $f_*(x,t,\varepsilon) = O(x^2)$.

By the estimates (3.50) for β_n we have $\beta_n \to \beta_*$. Thus, by the quasi-periodic symplectic transformation $x = T_* y$, the original system is changed to $\dot{y} = A_* y + f_*(y,t,\varepsilon)$ with $A_* = \beta_* J$.

Estimate of Measure

Let

$$E_n = \left\{ \varepsilon \in (0, \varepsilon_*) \mid \left| \langle \omega, k \rangle - 2\beta_n(\varepsilon) \right| \ge \frac{\alpha_n}{|k|^{\tau'}} \right\}. \tag{3.52}$$

Note that $\beta_n = \beta_1 + \psi$, where $\psi = \sum_{j=1}^{n-1} \beta_{j+1} - \beta_j, \beta_1 = \beta + \widehat{\beta}$, and $\widehat{\beta} J = [Q]_A$. Note that $\widetilde{\varepsilon}_1 = c\widetilde{\varepsilon}_0^2 / (\alpha_0^2 s_0^{v+v'})$ and $\widetilde{\varepsilon}_0 = c\varepsilon^{m_0}/\alpha_0$. By the estimates (3.50), we have $\psi(\varepsilon) = O(\varepsilon^{2m_0})$ and $\varepsilon\psi'(\varepsilon) = O(\varepsilon^{2m_0})$. By assumption, $[Q]_A$ is analytic with respect to ε and there exists $m_0 \le N \le 2m_0 - 1$ such that $[Q]_A = \delta\varepsilon^N + O(\varepsilon^{N+1})$ with $\delta \ne 0$. Thus, $\beta_1(\varepsilon) = \beta + \delta\varepsilon^N + O(\varepsilon^{N+1})$. By Lemma 3.4, we have $\mathrm{meas}((0,\varepsilon_*) - E_n) \le c(\alpha_n/\alpha_0^2)\varepsilon_*^{N+1}$. Let $E_* = \bigcap_{n\ge1} E_n$. By $\alpha_n = \alpha_0/2^n$, it follows that $\mathrm{meas}((0,\varepsilon_*) - E_*) \le c\varepsilon_*^{N+1}/\alpha_0$. Thus Lemma 3.5 is proved. □

4. Proof of Theorem 2.1

As we pointed previously, once the non-degeneracy conditions are satisfied in some KAM step, the proof is complete by Lemma 3.5. If the non-degeneracy conditions never happen, the small parameter ε does not involve into the small divisors and so the systems are analytic in ε. To prepare for KAM iteration, we need a preliminary step to change the original system to a suitable form.

Preliminary Step

We first give the preliminary KAM step. Let

$$\dot{x} = Ax + f(x,t,\varepsilon), \tag{4.1}$$

where $A = \beta J$ and $f = J\nabla_x F$. By Lemma 3.3, denote by $\underline{x}$ the solution of $\dot{x} = Ax + f(0,t,\varepsilon)$ on $D_{3\rho/4}$. Under the change of variables $x = T_0 x_+ = \underline{x} + x_+$, the Hamiltonian system (2.1) becomes

$$\dot{x}_+ = Ax_+ + f_1(x_+,t,\varepsilon), \tag{4.2}$$

where $f_1(x_+,t,\varepsilon) = f(\underline{x} + x_+,t,\varepsilon) - f(0,t,\varepsilon)$ satisfying $f_1(0,t,\varepsilon) = O(\varepsilon^2)$ and $\partial_{x_+} f_1(0,t,\varepsilon) = O(\varepsilon)$.

KAM Step

The next step is almost the same as the proof of Lemma 3.5 and even more simple. In the KAM iteration, we only need consider the case that the non-degeneracy condition never happens. In this case, the normal frequency has no shift, which is equivalent to $A_n = A$ for all $n \geq 1$ in the KAM steps in the above nondegenerate case. Moreover, the small divisors conditions are always the initial ones as (2.3) and (2.4) and are independent of the small parameter ε. Thus, we need not delete any parameter. Moreover, the analyticity in ε remains in the KAM steps, which makes the estimate easier. At the first step, we consider $\dot{x} = Ax + f_1(x,t,\varepsilon)$. In the same way as the case of nondegenerate case, let $r_1 = \varepsilon, \rho_1 = 3\rho/4, \varepsilon_1 = \varepsilon_0, D_1 = D(r_1,\rho_1,\varepsilon_1)$, and $\widetilde{\varepsilon}_1 = c\varepsilon/\alpha_0$. Then we have $\|f_1\|_{D_1} \leq \alpha_0 r_1 \widetilde{\varepsilon}_1$.

At nth step, we consider the Hamiltonian system

$$\dot{x} = Ax + f_n(x,t,\varepsilon), \tag{4.3}$$

where f_n is analytic quasi-periodic with respect to t with frequencies ω and real analytic with respect to x and ε on $D_n = D(r_n,\rho_n,\varepsilon_n)$. Moreover, $\|f_n\|_{D_n} \leq \alpha_0 r_n \widetilde{\varepsilon}_n$. Suppose

$$Q_n(t,\varepsilon) = \partial_x f_n(0,t,\varepsilon) = O\left(\varepsilon^{2^{n-1}}\right), \quad f_n(0,t,\varepsilon) = O\left(\varepsilon^{2^n}\right). \tag{4.4}$$

Since Q_n is analytic with respect to ε, it follows that

$$Q_n = \sum_{k=2^{n-1}}^{\infty} Q_n^k \varepsilon^k. \tag{4.5}$$

Truncating the above power series of ε, we let

$$R_n(t,\varepsilon) = \sum_{k=2^{n-1}}^{2^n-1} Q_n^k \varepsilon^k, \qquad \widetilde{Q}_n = Q_n - R_n. \tag{4.6}$$

Because the non-degeneracy conditions do not happen in KAM steps, we must have $[R_n]_A = 0$. In the same way as the proof of Lemma 3.5, we have a quasi-periodic symplectic transformation T_n with $T_n x = \phi_n(t)x + \psi_n(t)$ satisfying (3.49). Let $T^n = T_1 \circ T_2 \cdots \circ T_{n-1}$.

By the transformation $x = T^n y$, the system (4.3) is changed to

$$\dot{y} = Ay + f_{n+1}(y,t,\varepsilon), \tag{4.7}$$

where $f_{n+1} = \widetilde{Q}_n \cdot T_n'' + Q_n' \cdot T_n'' + e^{-P_n} \cdot h_n \circ T_n = \widetilde{Q}_n(\underline{x}_n + y) + Q_n'(\underline{x}_n + y) + e^{-P_n} h_n(e^{P_n}(\underline{x}_n + y))$.

The last two terms can be estimated similarly as those of (3.41). Note that

$$\widetilde{Q}_n = Q_n - R_n = \sum_{k \geq 2^n} Q_n^k \varepsilon^k \tag{4.8}$$

only consists of the higher order terms of ε. So, in the same way as [8, 10], we use the technique of shriek of the domain interval of ε to estimate the first term.

Let $r_1 = \varepsilon, \rho_1 = 3\rho/4, \varepsilon_1 = \varepsilon_0$ and $s_1 = \rho/16$.

Define $s_{n+1} = s_n/2, \rho_{n+1} = \rho_n - 2s_n, \eta_n = (1/8)e^{-(4/3)^n},\ r_{n+1} = \eta_n r_n, \delta_n = 1-(2/3)^n$ and $\varepsilon_{n+1} = \delta_n \varepsilon_n$. Let $D_{n+1} = D(r_{n+1}, \rho_{n+1}, \varepsilon_{n+1})$.

If $c\tilde{\varepsilon}_n/s_n^{2v} \le \eta_n < (1/8)$, it follows that

$$\|f_{n+1}\|_{D_{n+1}} \le \left(\alpha_0 \tilde{\varepsilon}_n e^{-(4/3)^n} + \left(\frac{c\tilde{\varepsilon}_n}{s_n^v}\right)^2\right)\eta_n r_n + c\alpha_0 r_n \tilde{\varepsilon}_n \eta_n^2 \le \alpha_0 r_{n+1}\tilde{\varepsilon}_{n+1}, \tag{4.9}$$

where $\tilde{\varepsilon}_{n+1} = c\eta_n\tilde{\varepsilon}_n$. Moreover, it is easy to see

$$\partial_x f_{n+1}(0,t,\varepsilon) = O\left(\varepsilon^{2^n}\right), \quad f_{n+1}(0,t,\varepsilon) = O\left(\varepsilon^{2^{n+1}}\right). \tag{4.10}$$

Now we verify $c\tilde{\varepsilon}_n/s_n^{2v} \le \eta_n < 1/8$. Let $G_n = c\tilde{\varepsilon}_n/s_n^{2v}$. By $G_n = ce^{-(4/3)^{n-1}}16^v G_{n-1}$, it follows that

$$G_n = (c16^v)^{n-1} e^{-[(4/3)^{n-1}+(4/3)^{n-2}+\cdots+(4/3)^1]}\, G_1 = (c16^v)^{n-1} e^4 e^{-4(4/3)^{n-1}} G_1. \tag{4.11}$$

Note that $G_1 = c\tilde{\varepsilon}_1/s_1^{2v}$. If $\tilde{\varepsilon}_1$ is sufficiently small, we have $c\tilde{\varepsilon}_n/s_n^{2v} = G_n \le \eta_n$.

Note that $(cr_n\tilde{\varepsilon}_n/s_n^v) \to 0$ and $(c\tilde{\varepsilon}_n/(\eta_n s_n^v)) \to 0$ as $n \to \infty$, and $\tilde{\varepsilon}_n \le cs_n^{2v}G_n$. Let $\varepsilon_* = \prod_{n\ge1}(1-(2/3)^n)\varepsilon_0$. Thus, in the same way as before we can prove the convergence of the KAM iteration for all $\varepsilon \in (0,\varepsilon_*)$ and obtain the result of Theorem 2.1. We omit the details.

Remark 4.1. As suggested by the referee, we can also introduce an outer parameter to consider the Hamiltonian function $H(x,t,\varepsilon) = \langle\omega, I\rangle + (1/2)(\beta_* + \sigma(\varepsilon))(x_1^2+x_2^2) + F(x,t,\varepsilon)$, where (θ, I) are the angle variable and the action variable and $x = (x_1, x_2)$ are a pair of normal variables. In the same way as in [11], $\sigma(\varepsilon)$ is the modified term of the normal frequency. Then by some technique as in [11–13], we can also prove Theorem 2.1.

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Research Article

The Optimization of Solutions of the Dynamic Systems with Random Structure

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The paper deals with the class of jump control systems with semi-Markov coefficients. The control system is described as the system of linear differential equations. Every jump of the random process implies the random transformation of solutions of the considered system. Relations determining the optimal control to minimize the functional are derived using Lyapunov functions. Necessary conditions of optimization which enables the synthesis of the optimal control are established as well.

1. The Statement of the Problem

The optimal control theory as mathematical optimization method for deriving control policies plays an important role in the development of the modern mathematical control theory. The optimal control deals with the problem of finding such a control law for a given system that a certain optimality criterion is achieved. The background for the optimization method can be found in the work of Lev Pontryagin with his well-known Pontryagin's maximum principle. The optimal control has been applied in diverse fields, such as economics, bioengineering, process control, and many others. Some real-life problems are described by a continuous-time or discrete-time linear system of differential equations, but a lot of them are described by dynamic systems with random jumping changes, for example economics systems. The general theory of random structure systems can be found in the work of Artemiev and Kazakov [1]. The optimization of linear systems with random parameters are considered in many works, for example in [2–12]. Particularly, the original results concerning the stabilization of the systems with random coefficients and a random process are derived using moment equations and Lyapunov functions in [4]. These results create a more convenient technique for applying the method in practice using suitable software for engineering or economics investigation. Our aim is the expansion of the achieved results to a new class of systems of linear differential equations with semi-Markov coefficients and random transformation of solutions performed simultaneously with jumps of semi-Markov process. We will focus on using the particular values of Lyapunov functions for the calculation of coefficients of the control vector which minimize the quality criterion. We will also establish the necessary conditions of the optimal solution which enables the synthesis of the optimal control for the considered class of systems.

Let us consider the linear control system

$$\frac{dX(t)}{dt} = A(t,\xi(t))X(t) + B(t,\xi(t))U(t) \tag{1.1}$$

on the probability basis $(\Omega, \mathfrak{T}, \mathbf{P}, \mathbf{F} \equiv \{\mathbf{F}_t : t \geq 0\})$ and together with (1.1) we consider the initial conditions

$$X(0) = \varphi(\omega), \quad \varphi : \Omega \longrightarrow \mathbb{R}^n. \tag{1.2}$$

The coefficients of the system are semi-Markov coefficients defined by the transition intensities $q_{\alpha k}(t),\ \alpha, k = 1, 2, \ldots, n$, from state θ_k to state θ_α. We suppose that the vectors $U(t)$ belong to the set of control U and the functions $q_{\alpha k}(t),\ \alpha, k = 1, 2, \ldots, n$, satisfy the conditions [13]:

$$q_{\alpha k}(t) \geq 0, \quad \int_0^\infty q_k(t)dt = 1, \qquad q_k(t) \equiv \sum_{\alpha=1}^{\infty} q_{\alpha k}(t). \tag{1.3}$$

Definition 1.1. Let the matrices $Q(t,\xi(t))$, $L(t,\xi(t))$ with semi-Markov elements be symmetric and positive definite. The cost functional

$$J = \int_0^\infty \langle X^*(t)Q(t,\xi(t))X(t) + U^*(t)L(t,\xi(t))U(t)\rangle dt, \tag{1.4}$$

defined on the space $C^1 \times U$, where $\langle\cdot\rangle$ denotes mathematical expectation, is called *the quality criterion*.

Definition 1.2. Let $S(t,\xi(t))$ be a matrix with semi-Markov elements. The control vector

$$U(t) = S(t,\xi(t))X(t) \tag{1.5}$$

which minimizes the quality criterion $J(X,U)$ with respect to the system (1.1) is called *the optimal control*.

If we denote

$$\begin{gathered} G(t,\xi(t)) \equiv A(t,\xi(t)) + B(t,\xi(t))S(t,\xi(t)), \\ H(t,\zeta(t)) \equiv Q(t,\zeta(t)) + S^*(t,\zeta(t))L(t,\zeta(t))S(t,\zeta(t)), \end{gathered} \tag{1.6}$$

then the system (1.1) can be rewritten to the form

$$\frac{dX(t)}{dt} = G(t, \xi(t))X(t), \tag{1.7}$$

and the functional (1.4) to the form

$$J = \int_0^\infty \langle X^*(t)H(t, \xi(t))X(t)\rangle dt. \tag{1.8}$$

We suppose also that, together with every jump of random process $\xi(t)$ in time t_j, the solutions of the system (1.7) submit to the random transformation

$$X(t_j + 0) = C_{sk}X(t_j - 0), \quad s, k = 1, 2, \ldots, n, \tag{1.9}$$

if the conditions $\xi(t_j + 0) = \theta_s$, $\xi(t_j - 0) = \theta_k$ hold.

Definition 1.3. Let $a_k(t)$, $k = 1, \ldots, n$, $t \geq 0$ be a selection of n different positive functions. If $\xi(t_j + 0) = \theta_s$, $\xi(t_j - 0) = \theta_k$, $s, k = 1, \ldots, n$, and for $t_j \leq t \leq t_{j+1}$ the equality $a(t, \xi(t) = \theta_s) = a_s(t - t_j)$ holds, then the function $a(t, \xi(t))$ is called *semi-Markov function*.

The application of semi-Markov functions makes it possible to use the concept of stochastic operator. In fact, the semi-Markov function $a(t, \xi(t))$ is an operator of the semi-Markov process $\xi(t)$, because the value of the semi-Markov function $a(t, \xi(t))$ is defined not only by the values t and $\xi(t)$, but it is also necessary to specify the function $a_s(t)$, $t \geq 0$ and the value of the jump of the process $\xi(t)$ in time t_j which precedes the moment of time t.

Our task is the construction of Lyapunov function for the new class of systems of linear differential equations with semi-Markov coefficients and then applying the function to solve the optimization problem which minimizes the quality criterion.

2. Auxiliary Results

In the proof of Theorem 3.1 in Section 3, we will employ two results concerning the construction of the Lyapunov function and the construction of the optimal control for the system of linear differential equations in a deterministic case. We will derive these auxiliary results in this part.

2.1. The Construction of the Lyapunov Function

Let us consider the system of linear differential equations

$$\frac{dX(t)}{dt} = A(t, \xi(t))X(t) \tag{2.1}$$

associated to the system (1.1).

Let us define a quadratic form

$$w(t,x,\xi(t)) = x^* B(t,\xi(t))x, \qquad B(t,\xi(t)) > 0, \tag{2.2}$$

where elements of the matrix $B(t,\xi(t))$ are the semi-Markov processes. The matrix $B(t,\xi(t))$ is defined by such a set of n different symmetric and positive definite matrices $B_k(t)$, $t \geq 0$, $k = 1,\dots,n$, that the equality $\xi(t) = \theta_s$ for $t_j \leq t \leq t_j + 1$ implies

$$B(t,\xi(t)) = B_s(t - t_j), \quad s = 1,2,\dots,n. \tag{2.3}$$

Our purpose in this section is to express the value of the functional

$$v = \int_0^\infty \langle w(t, X(t), \xi(t))\rangle dt \tag{2.4}$$

in a convenient form, which can help us to prove the L_2-stability of the trivial solution of the system (2.1).

At first, we introduce the particular Lyapunov functions

$$v_k(x) = \int_0^\infty \langle w(t, X(t), \xi(t)) \mid X(t) = x,\ \xi(0) = \theta_k\rangle dt, \quad k = 1,2,\dots,n. \tag{2.5}$$

If we can find the values of the particular Lyapunov functions in the form $v_k(x) = x^* C_k x$, $k = 1,2,\dots,n$, then value of the functional v can be expressed by the formula

$$v = \int_{E_n} \sum_{k=1}^{n} v_k(x) f_k(0,x)dx = \sum_{k=1}^{n} \int_{E_n} C_k \circ xx^* f_k(0,x)dx = \sum_{k=1}^{n} C_k \circ D_k(0), \tag{2.6}$$

where the scalar value

$$N \circ S = \sum_{k=1}^{l} \sum_{j=1}^{m} v_{kj} s_{kj} \tag{2.7}$$

is called the scalar product of the two matrices $N = (v_{kj})$, $S = (s_{kj})$ and has the property [14]

$$\frac{D(N \circ S)}{DS} = N. \tag{2.8}$$

The first auxiliary result contains two equivalent, necessary, and sufficient conditions for the L_2-stability (see in [4]) of the trivial solution of the system (2.1) and one sufficient condition for the stability of the solutions.

Theorem 2.1. *The trivial solution of the system* (2.1) *is* L_2*-stable if and only if any of the next two equivalent conditions hold:*

(1) *the system of equations*

$$C_k = H_k + \int_0^{\infty} \sum_{s=1}^{n} q_{sk}(t) N_k^*(t) C_{sk}^* C_s C_{sk} N_k(t) dt, \quad k = 1, 2, \ldots, n \tag{2.9}$$

has a solution $C_k > 0$, $k = 1, 2, \ldots, n$ *for* $H_k > 0$, $k = 1, 2, \ldots, n$,

(2) *the sequence of the approximations*

$$\begin{aligned} C_k^{(0)} &= 0, \\ C_k^{(j+1)} &= H_k + \int_0^{\infty} \sum_{s=1}^{n} q_{sk}(t) N_k^*(t) C_{sk}^* C_s^{(j)} C_{sk} N_k(t) dt, \quad k = 1, 2, \ldots, n, \; j = 0, 1, 2, \end{aligned} \tag{2.10}$$

converges.

Moreover, the solutions of the system (2.1) *are* L_2*-stabile, if there exist symmetric and positive definite matrices* $C_k > 0$, $k = 1, 2, \ldots, n$, *such that the property*

$$C_k - \int_0^{\infty} \sum_{s=1}^{n} q_{sk}(t) N_k^*(t) C_{sk}^* C_s C_{sk} N_k(t) dt > 0, \quad k = 1, 2, \ldots, n \tag{2.11}$$

holds.

Proof. We will construct a system of equations, which will define the particular Lyapunov functions $v_k(x)$, $k = 1, 2, \ldots, n$. Let us introduce the auxiliary semi-Markov functions

$$u_k(t, x) = \langle w(t, X(t), \xi(t)) \mid X(0) = x, \; \xi(0) = \theta_k \rangle, \quad k = 1, 2, \ldots, n. \tag{2.12}$$

For the state $\xi(t) = \theta_k$, $t \geq 0$ of the random process $\xi(t)$, the equalities

$$X(t) = N_k(t) x, \quad X(0) = x \tag{2.13}$$

are true. Simultaneously, with the jumps of the random process $\xi(t)$, the jumps of solutions of (2.1) occurred, so in view of (2.12), we derive the equations

$$u_k(t, x) = \psi_k(t) w_k(t, N_k(t) x) + \int_0^{t} \sum_{s=1}^{n} q_{sk}(\tau) u_s(t - \tau, C_{sk} N_k(\tau) x) d\tau, \quad k = 1, 2, \ldots, n. \tag{2.14}$$

Further, if we introduce denoting

$$u_k(t, x) = x^* u_k(t) x, \quad k = 1, 2, \ldots, n, \tag{2.15}$$

then (2.14) can be rewritten as the system of integral equations for the matrix $u_k(t)$ in the form

$$\begin{aligned} u_k(t) &= \varphi_k(t)N_k^*(t)B_k(t)N_k(t) \\ &\quad + \int_0^t \sum_{s=1}^n q_{sk}(\tau)N_k^*(\tau)C_{sk}^* u_s(t-\tau)C_{sk}N_k(\tau)d\tau, \quad k = 1,2,\dots,n. \end{aligned} \tag{2.16}$$

We define matrices $C_k,\ k = 1,2,\dots,n$ and functions $v_k(t),\ k = 1,2,\dots,n$, with regard to (2.5) and (2.12), by formulas

$$C_k = \int_0^\infty u_k(t)dt, \qquad v_k(x) = \int_0^\infty u_k(t,x)dt. \tag{2.17}$$

Integrating the system (2.16) from 0 to ∞, we get the system

$$\begin{aligned} C_k &= \int_0^\infty \varphi_k(t)N_k^*(t)B_k(t)N_k(t)dt \\ &\quad + \int_0^\infty \sum_{s=1}^n q_{sk}(\tau)N_k^*(\tau)C_{sk}^* C_s C_{sk}N_k(\tau)d\tau, \quad k = 1,2,\dots,n. \end{aligned} \tag{2.18}$$

Similarly, integrating the system of (2.14), we get the system of equations determining the particular Lyapunov functions

$$v_k(x) = \int_0^\infty \varphi_k(t)w_k(t,N_k(t)x)dt + \int_0^\infty \sum_{s=1}^n q_{sk}(t)v_k(C_{sk}N_k(t)x)dt. \tag{2.19}$$

Let us denote

$$H_k = \int_0^\infty \varphi_k(t)N_k^*(t)B_k(t)N_k(t)dt, \quad k = 1,2,\dots,n. \tag{2.20}$$

If there exist such positive constants λ_1, λ_2 that

$$\lambda_1 E \le B_k(t) \le \lambda_2 E, \tag{2.21}$$

or equivalent conditions

$$\lambda_1\|x\|^2 \le x^* B_k(t)x \le \lambda_2\|x\|^2 \tag{2.22}$$

hold, then the matrices H_k, $k = 1,2,\dots,n$ are symmetric and positive definite. Using (2.17), the system (2.18) can be rewritten to the form

$$C_k = H_k + \sum_{s=1}^n L_{sk}^* C_s, \quad k = 1,2,\dots,n. \tag{2.23}$$

It is easy to see that the system (2.23) is conjugated to the system (2.9). Therefore, the existence of a positive definite solution $C_k > 0,\ k = 1, 2, \ldots, n$ of the system (2.23) is equivalent to the existence of a positive definite solution $B_k > 0,\ k = 1, 2, \ldots, n$ and it is equivalent to L_2-stability of the solution of the system (2.1). On the other hand, if the existence of the particular Lyapunov functions $v_k(x),\ k = 1, 2, \ldots, n$ in (2.5) implies L_2-stability of the solutions of the system (2.1), then, in view of conditions (2.22) and the convergence of the integral (2.17), we get the inequality

$$\int_0^\infty \langle w(t, X(t), \xi(t)) \rangle dt \geq \int_0^\infty \left\langle \|X\|^2 \right\rangle dt. \tag{2.24}$$

The theorem is proved. □

Remark 2.2. If the system of linear differential equations (2.1) is a system with piecewise constant coefficients and the function $w(t, X(t), \xi(t))$ has the form

$$w(t, X(t), \xi(t)) = x^* B(\xi(t)) x, \qquad B_k \equiv B(\theta_k), \quad k = 1, 2, \ldots, n, \tag{2.25}$$

then the system (2.18) can be written in the form

$$C_k = \int_0^\infty \psi_k(t) e^{A_k^* t} B_k e^{A_k t} dt + \int_0^\infty \sum_{s=1}^{n} q_{sk}(t) e^{A_k^* t} C_{sk}^* C_s C_{sk} e^{A_k t} dt, \quad k = 1, 2, \ldots, n. \tag{2.26}$$

Particularly, if the semi-Markov process $\xi(t)$ is identical with a Markov process, then the system (2.26) has the form

$$C_k = \int_0^\infty e^{a_{kk} t} e^{A_k^* t} B_k e^{A_k t} dt + \int_0^\infty \sum_{\substack{s=1 \\ s \neq k}}^{n} a_{sk} e^{a_{kk} t} e^{A_k^* t} C_{sk}^* C_s C_{sk} e^{A_k t} dt, \quad k = 1, 2, \ldots, n, \tag{2.27}$$

or, more simply

$$C_k = \int_0^\infty e^{a_{kk} t} e^{A_k^* t} \left(B_k + \sum_{\substack{s=1 \\ s \neq k}}^{n} a_{sk} C_{sk}^* C_s C_{sk} \right) e^{A_k t} dt, \quad k = 1, 2, \ldots, n. \tag{2.28}$$

Moreover, under the assumption that the integral in (2.28) converges, the system (2.28) is equivalent to the system of matrices equations

$$(E a_{kk} + A_k^*) C_k + C_k A_K + B_k + \sum_{\substack{s=1 \\ s \neq k}}^{n} a_{sk} C_{sk}^* C_s C_{sk} = 0, \quad k = 1, 2, \ldots, n, \tag{2.29}$$

which can be written as the system

$$A_k^* C_k + C_k A_K + B_k + \sum_{\substack{s=1 \\ s \neq k}}^{n} a_{sk} C_{sk}^* C_s C_{sk} = 0, \quad k = 1, 2, \ldots, n, \tag{2.30}$$

if $C_{kk} = E, \; k = 1, 2, \ldots, n$.

Example 2.3. Let the semi-Markov process $\xi(t)$ take two states θ_1, θ_2 and let it be identical with the Markov process described by the system of differential equations

$$\begin{aligned} \frac{dp_1(t)}{dt} &= -\lambda p_1(t) + \lambda p_2(t), \\ \frac{dp_2(t)}{dt} &= \lambda p_1(t) - \lambda p_2(t). \end{aligned} \tag{2.31}$$

We will consider the L_2-stability of the solutions of the differential equation

$$\frac{dx(t)}{dt} = a(\xi(t))x(t), \qquad a(\theta_k) \equiv a_k, \tag{2.32}$$

constructing a system of the type (2.26) related to (2.32). The system is

$$c_1 = 1 + \int_0^{\infty} e^{2a_2 t} \lambda e^{-\lambda t} c_2 dt, \qquad c_2 = 1 + \int_0^{\infty} e^{2a_1 t} \lambda e^{-\lambda t} c_1 dt, \tag{2.33}$$

and its solution is

$$c_1 = \frac{(\lambda - a_1)(\lambda - 2a_2)}{2a_1 a_2 - \lambda(a_1 + a_2)}, \qquad c_2 = \frac{(\lambda - a_2)(\lambda - 2a_1)}{2a_1 a_2 - \lambda(a_1 + a_2)}. \tag{2.34}$$

The trivial solution of (2.32) is L_2-stable, if $c_1 > 0$ and $c_2 > 0$. Let the intensities of semi-Markov process $\xi(t)$ satisfy the conditions

$$q_{11}(t) \approx 0, \qquad q_{22}(t) \approx 0, \qquad q_{21}(t) - \lambda e^{-\lambda t} \approx 0, \qquad q_{12}(t) - \lambda e^{-\lambda t} \approx 0. \tag{2.35}$$

Then, using the Theorem 2.1, the conditions

$$\begin{aligned} 1 - c_1 \int_0^{\infty} q_{11}(t) e^{2a_1 t} dt - c_2 \int_0^{\infty} \left(q_{21}(t) - \lambda e^{-\lambda t} \right) e^{2a_2 t} dt > 0, \\ 1 - c_1 \int_0^{\infty} \left(q_{12}(t) - \lambda e^{-\lambda t} \right) e^{2a_1 t} dt - c_2 \int_0^{\infty} q_{22}(t) e^{2a_2 t} dt > 0 \end{aligned} \tag{2.36}$$

are sufficient conditions for the L_2-stability of solutions of (2.32).

2.2. The Construction of an Optimal Control for the System of Linear Differential Equations in the Deterministic Case

Let us consider the deterministic system of the linear equations

$$\frac{dX(t)}{dt} = A(t)X(t) + B(t)U(t) \tag{2.37}$$

in the boundary field G, where $X \in \mathbb{R}^m$, $U \in \mathbb{R}^l$, and together with (2.37) we consider the initial conditions

$$X(t) = x_0. \tag{2.38}$$

We assume that the vector $U(t)$ belongs to the control set U. The quality criterion has the form of the quadratic functional

$$\begin{gathered} I(t) = \frac{1}{2}\int_t^{\infty} [X^*(\tau)C(\tau)X(\tau) + U^*(\tau)\mathbf{D}(\tau)C(\tau)]d\tau, \\ C^*(t) = C(t), \qquad \mathbf{D}^*(t) = \mathbf{D}(t) \end{gathered} \tag{2.39}$$

in the space $\mathbb{C}^1(G) \times U$. The control vector

$$U(t) = S(t)X(t), \quad \dim S(t) = l \times m, \tag{2.40}$$

which minimizes the quality criterion (2.39) is called the optimal control.

The optimization problem is the problem of finding the optimal control (2.40) from all feasible control U, or, in fact, it is the problem of finding the equation to determine $S(t)$, $\dim S(t) = l \times m$.

Theorem 2.4. *Let there exist the optimal control* (2.40) *for the system of* (2.37). *Then the control equations*

$$S = -\mathbf{D}^{-1}(t)B^*(t)\mathbf{\Psi}^*, \quad \mathbf{\Psi}^* = K(t)X(t), \tag{2.41}$$

where the matrix $K(t)$ satisfies the Riccati equation

$$\frac{dK(t)}{dt} = -C(t) - K(t)A(t) - A^*(t)K(t) + K^*(t)B(t)\mathbf{D}^{-1}(t)B^*(t)K(t), \tag{2.42}$$

determines the synthesis of the optimal control.

Proof. Let the control for the system (2.37) have the form (2.40), where the matrix $S(t)$ is unknown. Then, the minimum value of the quality criterion (2.39) is

$$\min_{S(t)} I(t) = \frac{1}{2}X^*(t)K(t)X(t) \equiv \nu(t, X(t)). \tag{2.43}$$

Under assumption that the vector $X(t)$ is known and using Pontryagin's maximum principle [1, 15], the minimum of the quality criterion (2.39) is written as

$$\min_{S(t)} I(t) = \frac{1}{2}\boldsymbol{\Psi}(t)X(t), \quad \tau \geq t, \tag{2.44}$$

where

$$\boldsymbol{\Psi}(t) = \frac{Dv(t,x)}{Dx} = X^* K(t) \tag{2.45}$$

is the row-vector. If we take Hamiltonian function [15] of the form

$$H(t,x,U,\boldsymbol{\Psi}) = \boldsymbol{\Psi}(A(t)x + B(t)U) + \frac{1}{2}(x^* C x + U^* \mathbf{D} U), \quad U = Sx, \tag{2.46}$$

the necessary condition for optimality is

$$\frac{\partial H}{\partial s_{kj}} = 0, \quad k = 1,2,\ldots,l, \ j = 1,2,\ldots,m, \tag{2.47}$$

where s_{kj} are elements of the matrix S. The scalar value

$$\frac{dH}{dS} = \left\| \frac{\partial H}{\partial s_{kj}} \right\|, \quad k = 1,2,\ldots,l, \ j = 1,2,\ldots,m, \tag{2.48}$$

is called derivative of the matrix H with respect to the matrix S.

Employing the scalar product of the two matrices in our calculation, the Hamiltonian function (2.46) can be rewritten into the form

$$H = \boldsymbol{\Psi} A(t)x + \frac{1}{2}x^* C(t)x + B^*(t)\boldsymbol{\Psi}^* x^* \circ S + \frac{1}{2}\mathbf{D}(t) \cdot Sxx^* \circ S, \tag{2.49}$$

and its derivative with respect to the matrix S is

$$\frac{dH}{dS} = B^*(t)\boldsymbol{\Psi}^* x^* + \mathbf{D}(t)Sxx^* = 0. \tag{2.50}$$

Because the equality (2.50) holds for any value of x, the expression of the vector control U has the form

$$U = Sx = -\mathbf{D}^{-1}(t)B^*(t)\boldsymbol{\Psi}^* = -\mathbf{D}^{-1}(t)B^*(t)K(t)x, \tag{2.51}$$

which implies

$$S = -\mathbf{D}^{-1}(t)B^*(t)\boldsymbol{\Psi}^*. \tag{2.52}$$

If we put the expression of matrix S to (2.49), we obtain a new expression for the Hamiltonian function

$$H = \mathbf{\Psi}(t)A(t)x + \frac{1}{2}x^{*}C(t)x - \frac{1}{2}\mathbf{\Psi}B(t)\mathbf{D}^{-1}(t)B^{*}(t)\mathbf{\Psi}^{*}, \tag{2.53}$$

for which the canonical system of linear differential equations

$$\frac{dx}{dt} = \frac{DH}{D\mathbf{\Psi}}, \qquad \frac{d\mathbf{\Psi}}{dt} = \frac{DH}{Dx} \tag{2.54}$$

has the form

$$\begin{aligned} \frac{dx}{dt} &= A(t)x - B(t)\mathbf{D}^{-1}(t)B^{*}(t)\mathbf{\Psi}^{*}, \\ \frac{d\mathbf{\Psi}^{*}}{dt} &= -C(t)x - A^{*}(t)\mathbf{\Psi}^{*}. \end{aligned} \tag{2.55}$$

In the end, we define the matrix $K(t)$ as the integral manifolds of solutions of the system equations

$$\mathbf{\Psi}^{*} = K(t)X(t). \tag{2.56}$$

If we derive the system (2.56) with respect to t regarding the system (2.55) and extract the vector $\mathbf{\Psi}^{*}$, then we obtain the matrix differential equation (2.40). This equation is known as Riccati equation in literature, see for example in [16, 17]. The solution $K_T(t)$ of (2.42) satisfying the initial condition

$$K_T(t) = 0, \qquad T > 0 \tag{2.57}$$

determines the minimum of the functional

$$\min_{S(\tau)} \int_t^T [X^{*}(\tau)C(\tau)X(\tau) + U^{*}(\tau)\mathbf{D}(\tau)U(\tau)]d\tau = \frac{1}{2}X^{*}(t)K_T(t)X(t), \tag{2.58}$$

and $K(t)$ can be obtained as the limit of the sequence $\{K_T(t)\}_{T=1}^{\infty}$ of the successive approximations $K_T(t)$:

$$K(t) = \lim_{T\to\infty} K_T(t). \tag{2.59}$$

□

Remark 2.5. Similar results can be obtained from the Bellman equation [18], where the function $v(t,x)$ satisfie

$$\min_{S(t)}\left\{\frac{\partial v(t,x)}{\partial t} + \frac{Dv(t,x)}{Dx}[A(t) + B(t)S(t)]x + \frac{1}{2}x^{*}C(t)x + \frac{1}{2}x^{*}S^{*}(t)\mathbf{D}(t)S(t)x\right\} = 0. \tag{2.60}$$

3. The Main Result

Theorem 3.1. *Let the coefficients of the control system* (1.1) *be the semi-Markov functions and let them be defined by the equations*

$$\frac{dX_k(t)}{dt} = G_k(t)X_k(t), \quad G_k(t) \equiv A_k(t) + B_k(t)S_k(t), \quad k = 1, \ldots, n. \tag{3.1}$$

Then the set of the optimal control is a nonempty subset of the control U, which is identical with the family of the solutions of the system

$$U_s(t) = L_s^{-1}(t)B_s^*(t)R_s(t)X_s(t), \quad s = 1, \ldots, n, \tag{3.2}$$

where the matrix $R_s(t)$ is defined by the system of Riccati type of differential equations

$$\begin{aligned} \frac{dR_s(t)}{dt} &= -Q_s(t) - A_s^*(t)R_s(t) - R_s(t)A_s(t) \\ &\quad + R_s(t)B_s(t)L_s^{-1}(t)B_s^*(t)R_s(t) - \frac{\Psi_s'}{\Psi_s(t)}R_s(t) \\ &\quad - \sum_{k=1}^{n} \frac{q_{ks}(t)}{\Psi_s(t)} C_{ks}^* R_k(0) C_{ks}, \quad s = 1, \ldots, n. \end{aligned} \tag{3.3}$$

3.1. The Proof of Main Result Using Lyapunov Functions

It should be recalled that the coefficients of the systems (1.1), (1.7) and of the functionals (1.4), (1.8) have the form

$$\begin{gathered} A(t, \xi(t)) = A_s(t - t_j), \qquad B(t, \xi(t)) = B_s(t - t_j), \\ Q(t, \xi(t)) = Q_s(t - t_j), \qquad L(t, \xi(t)) = L_s(t - t_j), \qquad S(t, \xi(t)) = S_s(t - t_j), \end{gathered} \tag{3.4}$$

if $t_j \le t < t_{j+1}$, $\xi(t) = \theta_s$. In addition to this, we have

$$\begin{gathered} G(t, \xi(t)) = G_s(t - t_j) \equiv A_s(t - t_j) + B_s(t - t_j)S_s(t - t_j), \\ H(t, \xi(t)) = H_s(t - t_j) \equiv Q_s(t - t_j) + S_s^*(t - t_j)L_s(t - t_j)S_s(t - t_j). \end{gathered} \tag{3.5}$$

The formula

$$V = \sum_{k=1}^{n} C_k \circ D_k(0) = \sum_{k=1}^{n} \int_{E_m} \nu_k(x) f_k(0, x)\,dx \tag{3.6}$$

is useful for the calculation of the particular Lyapunov functions $v_k(x) \equiv x^* C_k x$, $k = 1, \ldots, n$ of the functional (1.8). We get

$$\begin{aligned} v_k(x) &\equiv x^* C_k x \\ &= \int_0^\infty \langle X^*(t) H(t, \xi(t)) X(t) \mid X(0) = x,\ \xi(0) = \theta_k \rangle dt, \quad k = 1, 2, \ldots, n, \end{aligned} \tag{3.7}$$

or, the more convenient form

$$\begin{aligned} v_k(x) &\equiv x^* C_k x \\ &= \int_0^\infty \left[X_k^*(t) \left(\Psi_k(t) Q_k(t) + \sum_{s=1}^{n} q_{sk}(t) C_{sk}^* C_s C_{sk} \right) U_k^*(t) \Psi_k(t) L_k(t) U_k(t) \right] dt, \\ &\qquad k = 1, 2, \ldots, n. \end{aligned} \tag{3.8}$$

Then the system (3.1) has the form

$$\frac{dX_k(t)}{dt} = A_k(t) X_k(t) + B_k(t) U_k(t), \quad U_k(t) \equiv S_k(t) X_k(t),\ k = 1, \ldots, n. \tag{3.9}$$

Let us assume that for the control system (1.1) the optimal control exists in the form (1.5) independent of the initial value $X(0)$. Regarding the formula (3.6), there exist minimal values of the particular Lyapunov functions $v_k(x)$, $k = 1, \ldots, n$, which are associated with the optimal control. It also follows from the fact that the functions $v_k(x)$, $k = 1, \ldots, n$ are particular values of the functional (3.6). Finding the minimal values $v_k(x)$, $k = 1, \ldots, n$ by choosing the optimal control $U_k(x)$ is a well-studied problem, for the main results see [16]. It is significant that all matrices C_s, $s = 1, \ldots, n$ of the integrand in the formula (3.8) are constant matrices, hence, solving the optimization problem they can be considered as matrices of parameters.

Therefore, the problem to find the optimal control (1.5) for the system (1.1) can be transformed to n problems to find the optimal control for the deterministic system (3.9), which is equivalent to the system of linear differential equations of type (2.37).

3.2. The Proof of the Main Result Using Lagrange Functions

In this part, we get one more proof of the Theorem 3.1 using the Lagrange function.

We are looking for the optimal control which reaches the minimum of quality criterion

$$x^* C x = \int_0^T [(X^*(t) Q A) X(t) + U^*(t) L(t) U(t)] dt. \tag{3.10}$$

Let us introduce the Lagrange function

$$I = \int_0^T \left[X^*(t) Q(t) X(t) + U^*(t) L(t) U(t) + 2Y^*(t) \left(A(t) X(t) + B(t) U(t) - \frac{dX(t)}{dt} \right) \right] dt, \tag{3.11}$$

where $Y(t)$ is the column-vector of Lagrange multipliers. In accordance with Pontryagin's maximum principle, we put the first variations of the functionals ∂I_x, ∂I_y equal to zero and we obtain the system of linear differential equations

$$\begin{aligned}\frac{dX(t)}{dt} &= A(t)X(t) - B(t)L^{-1}(t)B^*(t)Y(t),\\ \frac{dY(t)}{dt} &= -Q(t)X(t) - A^*(t)Y(t).\end{aligned} \tag{3.12}$$

Then the optimal control $U(t)$ can be expressed by

$$U(t) = L^{-1}(t)B^*(t)Y(t), \quad Y(T) = 0. \tag{3.13}$$

The synthesis of the optimal control needs to find the integral manifolds of the solutions of the system (3.12) in the form

$$Y(t) = K(t)X(t), \quad K(T) = 0. \tag{3.14}$$

According to the theory of integral manifolds [19] we construct the differential matrix equations of the Riccati type

$$\frac{dK(t)}{dt} = -Q(t) - A^*(t)K(t)A(t) - K(t)B(t)L^{-1}(t)B^*(t)K(t). \tag{3.15}$$

for the matrix $K(t)$. Integrating them from time $t = T$ to time $t = 0$ and using the initial condition $K(T) = 0$ we obtain Lagrange functions for the optimal control

$$U(t) = -L^{-1}(t)B^*(t)K(t)X(t). \tag{3.16}$$

We will prove that

$$\int_t^T [X^*(\tau)Q(\tau)X(\tau) + U^*(\tau)L(\tau)U(\tau)]d\tau = X^*(t)K(t)X(t). \tag{3.17}$$

Differentiating the equality (3.17) with respect to t we obtain the matrix equation

$$\begin{aligned}-X^*(t)Q(t)X(t) - U^*(t)L(t)U(t) = X^*(t)\frac{dK(t)}{dt}X(t) &+ X^*(t)K(t)(A(t)X(t) + B(t)U(t))\\ &+ (X^*(t)A^*(t) + U^*(t)B^*(t))K(t)X(t),\end{aligned} \tag{3.18}$$

and extracting the optimal control $U(t)$ we obtain differential equation for $K(t)$ identical with (3.15). The equality $K(t) = K^*(t)$ follows from the positive definite matrices $Q(t)$, $L(t)$ for $t < T$. Therefore, from (3.17) we get $K(t) = 0$; moreover, from (3.10) it follows that $C = K(0)$.

Applying the formulas (3.15), (3.16) to the system (3.8) with minimal functionals (3.9), the expression for the optimal control can be found in the form

$$U_s(t) = -\Psi_s^{-1}(t)L_s^{-1}(t)B_s^*(t)K_s(t)X_s(t), \quad s = 1,2,\dots,n, \tag{3.19}$$

where symmetric matrices $K_s(t)$ satisfy the matrix system of differential equations

$$\begin{aligned}\frac{dK_s(t)}{dt} &= -\Psi_s(t) - Q_s(t) - A_s^*(t)K_s(t) - \sum_{k=1}^{n} q_{ks}C_{ks}^* C_k C_{ks} \\ &\quad + K_s(t)B_s(t)\Psi_s^{-1}(t)L_s^{-1}(t)B_s(t)K_s(t) \quad s = 1,2,\dots,n.\end{aligned} \tag{3.20}$$

The systems (3.9), (3.20) define the necessary condition such that the solutions of the systems (1.4) will be optimal. In addition to this, the system (3.8) defines the matrices $S_k(t)$, $k = 1,2,\dots,n$, of the optimal control in the form

$$S_k(t) = -\Psi_k^{-1}(t)L_k^{-1}(t)B_k^*(t)K_k(t), \quad k = 1,2,\dots,n. \tag{3.21}$$

We define matrices C_s from the system equations (3.20) in the view of

$$C_s = K_s(0), \quad s = 1,2,\dots,n. \tag{3.22}$$

In regards to

$$R_s(t) = -\Psi_s^{-1}(t)K_s(t), \quad \Psi_s(0) = 1, \quad C_s = R_s(0), \quad s = 1,2,\dots,n, \tag{3.23}$$

it can makes the system (3.20) simpler. Then the system (3.20) takes the form (3.3), and formula (3.2) defines the optimal control.

Remark 3.2. If the control system (1.1) is deterministic, then $q_{ks}(t) \equiv 0$, $\Psi_s(t) \equiv 0$, $k, s = 1,2,\dots,n$ and the system (3.3) is identical to the system of the Riccati type equations (3.15).

4. Particular Cases

The optimal control $U(t)$ for the system (1.1) has some special properties, and the equations determining it are different from those given in the previous section in case the coefficients of the control system (1.1) have special properties or intensities $q_{sk}(t)$ satisfy some relations or some other special conditions are satisfied. Some of these cases will be formulated as corollaries.

Corollary 4.1. *Let the control system* (1.1) *with piecewise constant coefficients have the form*

$$\frac{dX(t)}{dt} = A(\xi(t))X(t) + B(\xi(t))U(t). \tag{4.1}$$

Then the quadratics functional

$$V = \int_0^{\infty} \langle X^*(t)Q(\xi(t))X(t) + U^*(t)L(\xi(t))U(t)\rangle dt \tag{4.2}$$

determines the optimal control in the form

$$U(t) = S(t,\xi(t))X(t), \tag{4.3}$$

where

$$S(t,\xi(t)) = S_k\left(t - t_j\right), \tag{4.4}$$

and the matrices $S_k(t)$ satisfy the equations

$$S_k(t) = -L^{-1}B_k^* R_k(t), \quad k = 1,2,\ldots,n \tag{4.5}$$

if $t_j \le t < t_{j+1}$, $\xi(t) = \theta_k$.

The matrices $R_k(t)$, $k = 1,2,\ldots,n$ are the solutions of the systems of the Riccati-type equations:

$$\begin{aligned}\frac{dR_k(t)}{dt} &= -Q_k - A_k^* R_k(t) - R_k(t)A_k \\ &\quad + R_k(t)B_k L_k^{-1} B_k^* R_k(t) - \frac{\Psi'_k(t)}{\Psi_k(t)} R_k(t) \\ &\quad - \sum_{s=1}^{n} \frac{q_{sk}(t)}{\Psi_k(t)} C_{sk}^* R_s(0) C_{sk}, \quad k = 1,\ldots,n.\end{aligned} \tag{4.6}$$

Remark 4.2. In the corollary we mention piecewise constant coefficients of the control system (4.1). The coefficients of the functional (4.2) will be piecewise as well, but the optimal control is nonstationary.

Corollary 4.3. *Assume that*

$$\frac{\Psi'_k(t)}{\Psi_k(t)} = \text{const}, \quad \frac{q_{sk}(t)}{\Psi_k(t)} = \text{const}, \quad k,s = 1,2,\ldots,n. \tag{4.7}$$

Then the optimal control $U(t)$ will be piecewise constant.

Taking into consideration that the optimal control is piecewise constant, we find out that the matrices $R_k(t)$, $k = 1,2,\ldots,n$ in (4.5) are constant, which implies the form of the system (4.6) is changed to the form

$$Q_k + A_k^* R_k + R_k A_k - R_k B_k L_k^{-1} B_k^* R_k + \frac{\Psi'_k(t)}{\Psi_k(t)} R_k(t) + \sum_{s=1}^{n} \frac{q_{sk}(t)}{\Psi_k(t)} C_{sk}^* R_k C_{sk} = 0, \quad k = 1,\ldots,n. \tag{4.8}$$

The system (4.8) has constant solutions R_k, $k = 1, 2, \ldots, n$, if conditions (4.7) hold. Moreover, if the random process $\xi(t)$ is a Markov process then the conditions (4.7) have the form

$$\frac{\Psi_k'(t)}{\Psi_k(t)} = a_{kk} = \text{const}, \quad \frac{q_{sk}(t)}{\Psi_k(t)} = a_{sk} = \text{const}, \quad k, s = 1, 2, \ldots, n, \ k \neq s, \tag{4.9}$$

and the system (4.8) transforms to the form

$$Q_k + A_k^* R_k + R_k A_k - R_k B_k L_k^{-1} B_k^* R_k + \sum_{s=1}^{n} a_{sk} C_{sk}^* R_s C_{sk} = 0, \quad k = 1, \ldots, n \tag{4.10}$$

for which the optimal control is

$$U(t) = S(\xi(t)) X(t), \qquad S(\theta_k) \equiv S_k, \quad S_k = -L_k^{-1} B_k^* R_k, \quad k = 1, 2, \ldots, n. \tag{4.11}$$

Corollary 4.4. *Let the state θ_s of the semi-Markov process $\xi(t)$ be no longer than $T_s > 0$. Then the system* (3.8) *has the form*

$$\begin{aligned} v_k(x) &\equiv x^* C_k x \\ &= \int_0^{T_s} \left(X_k^*(t) \left(\Psi_k(t) Q_k(t) + \sum_{s=1}^{n} q_{sk}(t) C_{sk}^* C_s C_{sk} \right) X_k(t) + U_k^*(t) \Psi_k(t) L_k(t) U_k(t) \right) dt, \\ &\qquad k = 1, 2, \ldots, n. \end{aligned} \tag{4.12}$$

Because

$$K_s(T_s) = \Psi_s(t) R_s(t), \quad s = 1, 2, \ldots, n, \tag{4.13}$$

then

$$K_s(T_s) = 0, \quad s = 1, 2, \ldots, n. \tag{4.14}$$

In this case, the search for the matrix $K_s(t)$, $s = 1, 2, \ldots, n$ in concrete tasks is reduced to integration of the matrix system of differential equations (3.15) on the interval $[0, T_s]$ with initial conditions (4.14). In view of $\Psi_s(T_s) = 0$, $s = 1, 2, \ldots, n$, we can expect that every equation (3.15) has a singular point $t = T_s$. If $\Psi_s(t)$ has simple zero at the point $t = T_s$, then the system (4.6) meets the necessary condition

$$\Psi_s(T_s) R_s(T_s) + \sum_{k=1}^{n} q_{sk}(T_s) C_{ks}^* R_s(0) C_{ks} = 0, \quad s = 1, \ldots, n \tag{4.15}$$

for boundary of matrix $R_s(t)$ in the singular points.

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Research Article

Oscillation Criteria for Certain Second-Order Nonlinear Neutral Differential Equations of Mixed Type

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Some oscillation criteria are established for the second-order nonlinear neutral differential equations of mixed type $[(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))^\gamma]'' = q_1(t)x^\gamma(t-\sigma_1)+q_2(t)x^\gamma(t+\sigma_2)$, $t \ge t_0$, where $\gamma \ge 1$ is a quotient of odd positive integers. Our results generalize the results given in the literature.

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order nonlinear neutral differential equation of mixed type

$$\left[\left(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2)\right)^\gamma\right]'' = q_1(t)x^\gamma(t-\sigma_1)+q_2(t)x^\gamma(t+\sigma_2), \quad t \ge t_0. \tag{1.1}$$

Throughout this paper, we will assume the following conditions hold.

(A_1) p_i, τ_i, and σ_i, $i = 1,2$, are positive constants;

(A_2) $q_i \in C([t_0,\infty),[0,\infty))$, $i = 1,2$.

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \ge t_0$ which has the property that $(x(t)+p_1x(t-\tau_1)+p_2x(t+\tau_2))^\gamma \in C^2([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Neutral functional differential equations have numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1].

Recently, many results have been obtained on oscillation of nonneutral continuous and discrete equations and neutral functional differential equations, we refer the reader to the papers [2–35], and the references cited therein.

Philos [2] established some Philos-type oscillation criteria for the second-order linear differential equation

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0, \quad t \geq t_0. \tag{1.2}$$

In [3–5], the authors gave some sufficient conditions for oscillation of all solutions of second-order half-linear differential equation

$$\left(r(t)\left|x'(t)\right|^{\gamma-1}x'(t)\right)' + q(t)|x(\tau(t))|^{\gamma-1}x(\tau(t)) = 0, \quad t \geq t_0 \tag{1.3}$$

by employing a Riccati substitution technique.

Zhang et al. [15] examined the oscillation of even-order neutral differential equation

$$\left[x(t) + p(t)x(\tau(t))\right]^{(n)} + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0. \tag{1.4}$$

Some oscillation criteria for the following second-order quasilinear neutral differential equation

$$\left(r(t)\left|z'(t)\right|^{\gamma-1}z'(t)\right)' + q(t)|x(\sigma(t))|^{\gamma-1}x(\sigma(t)) = 0, \quad \text{for } z(t) = x(t) + p(t)x(\tau(t)),\ t \geq t_0 \tag{1.5}$$

were obtained by [12–17].

However, there are few results regarding the oscillatory properties of neutral differential equations with mixed arguments, see the papers [20–24]. In [25], the authors established some oscillation criteria for the following mixed neutral equation:

$$\left(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2)\right)'' = q_1(t)x(t-\sigma_1) + q_2(t)x(t+\sigma_2), \quad t \geq t_0; \tag{1.6}$$

here q_1 and q_2 are nonnegative real-valued functions. Grace [26] obtained some oscillation theorems for the odd order neutral differential equation

$$\left(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2)\right)^{(n)} = q_1x(t-\sigma_1) + q_2x(t+\sigma_2), \quad t \geq t_0, \tag{1.7}$$

where $n \geq 1$ is odd. Grace [27] and Yan [28] obtained several sufficient conditions for the oscillation of solutions of higher-order neutral functional differential equation of the form

$$(x(t) + cx(t-h) + Cx(t+H))^{(n)} + qx(t-g) + Qx(t+G) = 0, \quad t \geq t_0, \tag{1.8}$$

where q and Q are nonnegative real constants.

Clearly, (1.6) is a special case of (1.1). The purpose of this paper is to study the oscillation behavior of (1.1).

In the sequel, when we write a functional inequality without specifying its domain of validity we assume that it holds for all sufficiently large t.

2. Main Results

In the following, we give our results.

Theorem 2.1. *Assume that $\sigma_i > \tau_i$, $i = 1, 2$. If*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2} (t + \sigma_2 - \tau_2 - s) Q_2(s) ds > \left(2^{\gamma-1}\right)^2 \left(1 + p_1^\gamma + \frac{p_2^\gamma}{2^{\gamma-1}}\right), \tag{2.1}$$

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (s - t + \sigma_1 - \tau_1) Q_1(s) ds > \left(2^{\gamma-1}\right)^2 \left(1 + p_1^\gamma + \frac{p_2^\gamma}{2^{\gamma-1}}\right), \tag{2.2}$$

where

$$Q_i(t) = \min\{q_i(t-\tau_1), q_i(t), q_i(t+\tau_2)\}, \tag{2.3}$$

for $i = 1, 2$, then every solution of (1.1) oscillates.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t-\tau_1) > 0$, $x(t+\tau_2) > 0$, $x(t-\sigma_1) > 0$, and $x(t+\sigma_2) > 0$ for all $t \geq t_1$. Setting

$$\begin{aligned} z(t) &= \left(x(t) + p_1 x(t-\tau_1) + p_2 x(t+\tau_2)\right)^\gamma, \\ y(t) &= z(t) + p_1^\gamma z(t-\tau_1) + \frac{p_2^\gamma}{2^{\gamma-1}} z(t+\tau_2). \end{aligned} \tag{2.4}$$

Thus $z(t) > 0$, $y(t) > 0$, and

$$z''(t) = q_1(t) x^\gamma(t-\sigma_1) + q_2(t) x^\gamma(t+\sigma_2) \geq 0. \tag{2.5}$$

Then, $z'(t)$ is of constant sign, eventually. On the other hand,

$$\begin{aligned} y''(t) &= q_1(t)x^{\gamma}(t-\sigma_1) + q_2(t)x^{\gamma}(t+\sigma_2) \\ &\quad + p_1^{\gamma}q_1(t-\tau_1)x^{\gamma}(t-\tau_1-\sigma_1) + p_1^{\gamma}q_2(t-\tau_1)x^{\gamma}(t-\tau_1+\sigma_2) \\ &\quad + \frac{p_2^{\gamma}}{2^{\gamma-1}}q_1(t+\tau_2)x^{\gamma}(t+\tau_2-\sigma_1) \\ &\quad + \frac{p_2^{\gamma}}{2^{\gamma-1}}q_2(t+\tau_2)x^{\gamma}(t+\tau_2+\sigma_2). \end{aligned} \tag{2.6}$$

Note that $g(u) = u^{\gamma}, \gamma \geq 1, u \in (0,\infty)$ is a convex function. Hence, by the definition of convex function, we obtain

$$a^{\gamma} + b^{\gamma} \geq \frac{1}{2^{\gamma-1}}(a+b)^{\gamma}. \tag{2.7}$$

Using inequality (2.7), we get

$$\begin{aligned} &x^{\gamma}(t-\sigma_1) + p_1^{\gamma}x^{\gamma}(t-\tau_1-\sigma_1) \geq \frac{1}{2^{\gamma-1}}\left(x(t-\sigma_1) + p_1x(t-\tau_1-\sigma_1)\right)^{\gamma}, \\ &\frac{1}{2^{\gamma-1}}\left(x(t-\sigma_1) + p_1x(t-\tau_1-\sigma_1)\right)^{\gamma} + \frac{p_2^{\gamma}}{2^{\gamma-1}}x^{\gamma}(t+\tau_2-\sigma_1) \\ &\quad \geq \frac{1}{\left(2^{\gamma-1}\right)^2}\left(x(t-\sigma_1) + p_1x(t-\tau_1-\sigma_1) + p_2x(t+\tau_2-\sigma_1)\right)^{\gamma} = \frac{z(t-\sigma_1)}{\left(2^{\gamma-1}\right)^2}. \end{aligned} \tag{2.8}$$

Similarly, we obtain

$$x^{\gamma}(t+\sigma_2) + p_1^{\gamma}x^{\gamma}(t-\tau_1+\sigma_2) + \frac{p_2^{\gamma}}{2^{\gamma-1}}x^{\gamma}(t+\tau_2+\sigma_2) \geq \frac{z(t+\sigma_2)}{\left(2^{\gamma-1}\right)^2}. \tag{2.9}$$

Thus, from (2.6), we have

$$y''(t) \geq \frac{1}{\left(2^{\gamma-1}\right)^2}\left(Q_1(t)z(t-\sigma_1) + Q_2(t)z(t+\sigma_2)\right). \tag{2.10}$$

In the following, we consider two cases.

Case 1. Assume that $z'(t) > 0$. Then, $y'(t) > 0$. In view of (2.10), we see that

$$y''(t+\tau_2) \geq \frac{1}{\left(2^{\gamma-1}\right)^2}Q_2(t+\tau_2)z(t+\tau_2+\sigma_2). \tag{2.11}$$

Applying the monotonicity of z, we find

$$\begin{aligned} y(t+\sigma_2) &= z(t+\sigma_2) + p_1^{\gamma} z(t-\tau_1+\sigma_2) + \frac{p_2^{\gamma}}{2^{\gamma-1}} z(t+\tau_2+\sigma_2) \\ &\le \left(1 + p_1^{\gamma} + \frac{p_2^{\gamma}}{2^{\gamma-1}}\right) z(t+\tau_2+\sigma_2). \end{aligned} \tag{2.12}$$

Combining the last two inequalities, we obtain the inequality

$$y''(t+\tau_2) \ge \frac{Q_2(t+\tau_2)}{\left(2^{\gamma-1}\right)^2 \left(1 + p_1^{\gamma} + p_2^{\gamma}/2^{\gamma-1}\right)} y(t+\sigma_2). \tag{2.13}$$

Therefore, y is a positive increasing solution of the differential inequality

$$y''(t) \ge \frac{Q_2(t)}{\left(2^{\gamma-1}\right)^2 \left(1 + p_1^{\gamma} + p_2^{\gamma}/2^{\gamma-1}\right)} y(t-\tau_2+\sigma_2). \tag{2.14}$$

However, by [11], condition (2.1) contradicts the existence of a positive increasing solution of inequality (2.14).

Case 2. Assume that $z'(t) < 0$. Then, $y'(t) < 0$. In view of (2.10), we see that

$$y''(t-\tau_1) \ge \frac{1}{\left(2^{\gamma-1}\right)^2} Q_1(t-\tau_1) z(t-\tau_1-\sigma_1). \tag{2.15}$$

Applying the monotonicity of z, we find

$$\begin{aligned} y(t-\sigma_1) &= z(t-\sigma_1) + p_1^{\gamma} z(t-\tau_1-\sigma_1) + p_2^{\gamma} \frac{1}{2^{\gamma-1}} z(t+\tau_2-\sigma_1) \\ &\le \left(1 + p_1^{\gamma} + \frac{p_2^{\gamma}}{2^{\gamma-1}}\right) z(t-\tau_1-\sigma_1). \end{aligned} \tag{2.16}$$

Combining the last two inequalities, we obtain the inequality

$$y''(t-\tau_1) \ge \frac{Q_1(t-\tau_1)}{\left(2^{\gamma-1}\right)^2 \left(1 + p_1^{\gamma} + p_2^{\gamma}/2^{\gamma-1}\right)} y(t-\sigma_1). \tag{2.17}$$

Therefore, y is a positive decreasing solution of the differential inequality

$$y''(t) \ge \frac{Q_1(t)}{\left(2^{\gamma-1}\right)^2 \left(1 + p_1^{\gamma} + p_2^{\gamma}/2^{\gamma-1}\right)} y(t+\tau_1-\sigma_1). \tag{2.18}$$

However, by [11], condition (2.2) contradicts the existence of a positive decreasing solution of inequality (2.18). □

Remark 2.2. When $\gamma = 1$, Theorem 2.1 involves results of [25, Theorem 1].

Theorem 2.3. *Let $\beta_i = (\sigma_i - \tau_i)/2 > 0$, $i = 1, 2$. Suppose that, for $i = 1, 2$, there exist functions*

$$a_i \in C^1[t_0, \infty), \qquad a_i(t) > 0, \qquad (-1)^i a_i'(t) \leq 0, \tag{2.19}$$

such that

$$Q_i(t) \geq \left(2^{\gamma-1}\right)^2 \left(1 + p_1^\gamma + \frac{p_2^\gamma}{2^{\gamma-1}}\right) a_i(t) a_i\left(t + (-1)^i \beta_i\right), \tag{2.20}$$

where Q_i are as in (2.3) *for $i = 1, 2$. If the first-order differential inequality*

$$v'(t) + (-1)^{i+1} a_i\left(t + (-1)^i \beta_i\right) v\left(t + (-1)^i \beta_i\right) \geq 0 \tag{2.21}$$

has no eventually negative solution for $i = 1$ and no eventually positive solution for $i = 2$, then (1.1) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(t - \tau_1) > 0$, $x(t + \tau_2) > 0$, $x(t - \sigma_1) > 0$, and $x(t + \sigma_2) > 0$ for all $t \geq t_1$. Define z and y as in Theorem 2.1. Proceeding as in the proof of Theorem 2.1, we get (2.10).

In the following, we consider two cases.

Case 1. Assume that $z'(t) > 0$. Clearly, $y'(t) > 0$. Then, just as in Case 1 of Theorem 2.1, we find that y is a positive increasing solution of inequality (2.14). Let $b_2(t) = y'(t) + a_2(t) y(t + \beta_2)$. Then $b_2(t) > 0$. Using (2.19) and (2.20), we obtain

$$\begin{aligned}
&b_2'(t) - \frac{a_2'(t)}{a_2(t)} b_2(t) - a_2(t) b_2(t + \beta_2) \\
&\quad = y''(t) - \frac{a_2'(t)}{a_2(t)} y'(t) - a_2(t) a_2(t + \beta_2) y(t + 2\beta_2) \\
&\quad \geq y''(t) - a_2(t) a_2(t + \beta_2) y(t + 2\beta_2) \\
&\quad \geq y''(t) - \frac{Q_2(t)}{(2^{\gamma-1})^2 \left(1 + p_1^\gamma + \left(p_2^\gamma / 2^{\gamma-1}\right)\right)} y(t - \tau_2 + \sigma_2) \geq 0.
\end{aligned} \tag{2.22}$$

Define $b_2(t) = a_2(t) v(t)$. Then, v is a positive solution of (2.21) for $i = 2$, which is a contradiction.

Case 2. Assume that $z'(t) < 0$. Clearly, $y'(t) < 0$. Then, just as in Case 2 of Theorem 2.1, we find that y is a positive decreasing solution of inequality (2.18). Let $b_1(t) = y'(t) - a_1(t)y(t-\beta_1)$. Then $b_1(t) < 0$. Using (2.19) and (2.20), we obtain

$$\begin{aligned}
&b_1'(t) - \frac{a_1'(t)}{a_1(t)}b_1(t) + a_1(t)b_1(t-\beta_1)\\
&= y''(t) - \frac{a_1'(t)}{a_1(t)}y'(t) - a_1(t)a_1(t-\beta_1)y(t-2\beta_1)\\
&\geq y''(t) - a_1(t)a_1(t-\beta_1)y(t-2\beta_1)\\
&\geq y''(t) - \frac{Q_1(t)}{(2^{\gamma-1})^2\left(1+p_1^\gamma+p_2^\gamma/2^{\gamma-1}\right)}y(t+\tau_1-\sigma_1) \geq 0.
\end{aligned} \tag{2.23}$$

Define $b_1(t) = a_1(t)v(t)$. Then, v is a negative solution of (2.21) for $i = 1$. This contradiction completes the proof of the theorem. □

Remark 2.4. When $\gamma = 1$, Theorem 2.3 involves results of [25, Theorem 2].

From Theorem 2.3 and the results given in [12], we have the following oscillation criterion for (1.1).

Corollary 2.5. *Let $\beta_i = (\sigma_i - \tau_i)/2 > 0$, $i = 1, 2$. Assume that* (2.19) *and* (2.20) *hold for $i = 1, 2$. If*

$$\liminf_{t\to\infty} \int_{t-\beta_1}^{t} a_1(s-\beta_1)ds > \frac{1}{e}, \tag{2.24}$$

$$\liminf_{t\to\infty} \int_{t}^{t+\beta_2} a_2(s+\beta_2)ds > \frac{1}{e}, \tag{2.25}$$

then (1.1) *is oscillatory.*

Proof. It is known (see [12]) that condition (2.24) is sufficient for inequality (2.21) (for $i = 1$) to have no eventually negative solution. On the other hand, condition (2.25) is sufficient for inequality (2.21) (for $i = 2$) to have no eventually positive solution. □

For an application of our results, we give the following example.

Example 2.6. Consider the second-order differential equation

$$\left[\left(x(t) + p_1x(t-\tau_1) + p_2x(t+\tau_2)\right)^\gamma\right]'' = q_1x^\gamma(t-\sigma_1) + q_2x^\gamma(t+\sigma_2), \quad t \geq t_0, \tag{2.26}$$

where $q_i > 0$ are constants and $\sigma_i > \tau_i$ for $i = 1, 2$.

It is easy to see that $Q_i(t) = q_i,\ i = 1, 2$. Assume that $\varepsilon > 0$. Let $a_i(t) = (2 + \varepsilon)/(\mathrm{e}(\sigma_i - \tau_i)),\ i = 1, 2$. Clearly, (2.19) holds. If

$$q_i > \left[\frac{2}{(\mathrm{e}(\sigma_i - \tau_i))}\right]^2 \left(2^{\gamma-1}\right)^2 \left(1 + p_1^{\gamma} + \frac{p_2^{\gamma}}{2^{\gamma-1}}\right) \tag{2.27}$$

for $i = 1, 2$, then (2.20) holds. Moreover, we see that

$$\begin{gathered}\liminf_{t\to\infty} \int_{t-\beta_1}^{t} a_1(s - \beta_1)\mathrm{d}s = \frac{2+\varepsilon}{2\mathrm{e}} > \frac{1}{\mathrm{e}}, \\ \liminf_{t\to\infty} \int_{t}^{t+\beta_2} a_2(s + \beta_2)\mathrm{d}s = \frac{2+\varepsilon}{2\mathrm{e}} > \frac{1}{\mathrm{e}}.\end{gathered} \tag{2.28}$$

Hence by applying Corollary 2.5, we find that (2.26) is oscillatory.

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Research Article

Oscillation Criteria for a Class of Second-Order Neutral Delay Dynamic Equations of Emden-Fowler Type

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We establish some new oscillation criteria for the second-order neutral delay dynamic equations of Emden-Fowler type, $[a(t)(x(t)+r(t)x(\tau(t)))^{\Delta}]^{\Delta}+p(t)x^{\gamma}(\delta(t))=0$, on a time scale unbounded above. Here $\gamma>0$ is a quotient of odd positive integers with a and p being real-valued positive functions defined on $\mathbb{T}$. Our results in this paper not only extend and improve the results in the literature but also correct an error in one of the references.

1. Introduction

The study of dynamic equations on time scales, which goes back to its founder Hilger [1], is an area of mathematics that has recently received a lot of attention. It was partly created in order to unify the study of differential and difference equations. Many results concerning differential equations are carried over quite easily to corresponding results for difference equations, while other results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice—once for differential equations and once again for difference equations.

The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see Kac and Cheung [2]), that is, when

$\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$, and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$, where $q > 1$. Many other interesting time scales exist, and they give rise to many applications (see [3]). Dynamic equations on a time scale have an enormous potential for applications such as in population dynamics. For example, it can model insect populations that are continuous while in season, die out in, for example, winter, while their eggs are incubating or dormant, and then hatch in a new season, giving rise to a nonoverlapping population (see [3]). There are applications of dynamic equations on time scales to quantum mechanics, electrical engineering, neural networks, heat transfer, and combinatorics. A recent cover story article in New Scientist [4] discusses several possible applications. Several authors have expounded on various aspects of this new theory; see the survey paper by Agarwal et al. [5] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [3], summarizes and organizes much of time scale calculus; see also the book by Bohner and Peterson [6] for advances results of dynamic equations on time scales.

In recent years, there has been much research activity concerning the oscillation and nonoscillation of solutions of various dynamic equations on time scales unbounded above and neutral differential equations; we refer the reader to the papers [7–19]. Some authors are especially interested in obtaining sufficient conditions for the oscillation or nonoscillation of solutions of first and second-order linear and nonlinear neutral functional dynamic equations on time scales; we refer to the articles [20–28].

Agarwal et al. [7] considered the second-order delay dynamic equations

$$x^{\Delta\Delta}(t) + p(t)x(\tau(t)) = 0, \quad t \in \mathbb{T} \tag{1.1}$$

and established some sufficient conditions for oscillation of (1.1). Şahiner [11] studied the second-order nonlinear delay dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T} \tag{1.2}$$

and obtained some sufficient conditions for oscillation by employing Riccati transformation technique. Zhang and Zhu [13] examined the second-order dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(t-\tau)) = 0, \quad t \in \mathbb{T}, \tag{1.3}$$

and by using comparison theorems, they proved that oscillation of (1.3) is equivalent to the oscillation of the nonlinear dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \quad t \in \mathbb{T} \tag{1.4}$$

and established some sufficient conditions for oscillation by applying the results established in [15]. Erbe et al. [16] investigated the oscillation of the second-order nonlinear delay dynamic equations

$$\left(r(t)x^{\Delta}(t)\right)^{\Delta} + p(t)f(x(\tau(t))) = 0, \quad t \in \mathbb{T} \tag{1.5}$$

and by employing the generalized Riccati technique, they established some new sufficient conditions which ensure that every solution of (1.5) oscillates or converges to zero. Mathsen et al. [20] investigated the first-order neutral delay dynamic equations

$$\left[y(t)-r(t)y(\tau(t))\right]^{\Delta}+p(t)y(\delta(t))=0,\quad t\in\mathbb{T} \tag{1.6}$$

and established some new oscillation criteria which as a special case involve some well-known oscillation results for first-order neutral delay differential equations. Zhu and Wang [21] studied the nonoscillatory solutions to neutral dynamic equations

$$\left[y(t)+p(t)y(g(t))\right]^{\Delta}+f(t,x(h(t)))=0,\quad t\in\mathbb{T} \tag{1.7}$$

and gave a classification scheme for the eventually positive solutions of (1.7). Agarwal et al. [22], Şahíner [23], Saker et al. [24–26], Wu et al. [27], and Zhang and Wang [28] considered the second-order nonlinear neutral delay dynamic equations

$$\left(r(t)\left(\left(y(t)+p(t)y(\tau(t))\right)^{\Delta}\right)^{\gamma}\right)^{\Delta}+f(t,y(\delta(t)))=0,\quad t\in\mathbb{T}, \tag{1.8}$$

where $\gamma>0$ is a quotient of odd positive integers, the delay function τ and δ satisfy $\tau:\mathbb{T}\to\mathbb{T}$ and $\delta:\mathbb{T}\to\mathbb{T}$ for all $t\in\mathbb{T}$, and r and p are real-valued positive functions defined on $\mathbb{T}$, and

(h_1) $r(t)>0$, $\int_{t_0}^{\infty}(1/r(t))^{1/\gamma}\,\Delta t=\infty$, and $0\le p(t)<1$;

(h_2) $f:\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ is continuous function such that $uf(u)>0$ for all $u\neq 0$, and there exists a nonnegative function q defined on $\mathbb{T}$ such that $|f(t,u)|\ge q(t)|u|^{\gamma}$.

By employing different Riccati transformation technique, the authors established some oscillation criteria for all solutions of (1.8).

Recently, some authors have been interested in obtaining sufficient conditions for the oscillation and nonoscillation of solutions of Emden-Fowler type dynamic equations on time scales, differential equations, and difference equations; see, for example, [29–47].

Han et al. [32] studied the second-order Emden-Fowler delay dynamic equations

$$x^{\Delta\Delta}(t)+p(t)x^{\gamma}(\tau(t))=0,\quad t\in\mathbb{T} \tag{1.9}$$

and established some sufficient conditions for oscillation of (1.9) and extended the results given in [7].

Saker [34] studied the second-order superlinear neutral delay dynamic equation of Emden-Fowler type

$$\left[a(t)\left(y(t)+r(t)y(\tau(t))\right)^{\Delta}\right]^{\Delta}+p(t)\left|y(\delta(t))\right|^{\gamma}\operatorname{sign}y(\delta(t))=0 \tag{1.10}$$

on a time scale $\mathbb{T}$.

The author assumes that

(A_1) $\gamma > 1$;

(A_2) the delay functions τ and δ satisfy $\tau : \mathbb{T} \to \mathbb{T}$, $\delta : \mathbb{T} \to \mathbb{T}$, $\tau(t) \le t, \delta(t) \le t$ for all $t \in \mathbb{T}$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$;

(A_3) a, r and p are positive rd-continuous functions defined on $\mathbb{T}$ such that $a^{\Delta}(t) \ge 0$, $\int_{t_0}^{\infty}(\Delta t/a(t)) = \infty$, and $0 \le r(t) < 1$.

The main result for the oscillation of (1.10) in [34] is the following.

Theorem 1.1 (see, [34, Theorem 3.1]). *Assume that* (A_1)–(A_3) *hold. Furthermore, assume that*

$$\int_{t_0}^{\infty} p(t)(1 - r(\delta(t)))^{\gamma}\delta^{\gamma}(t)\Delta t = \infty, \tag{1.11}$$

and there exists a Δ*-differentiable function* η *such that for all constants* $M > 0$,

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\eta(s)p(s)(1 - r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma} - \frac{a(s)\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s = \infty. \tag{1.12}$$

Then every solution of (1.10) *is oscillatory.*

We note that in [34], the author gave an open problem, that is, how to establish oscillation criteria for (1.10) when $\gamma < 1$.

In [35], the author examined the oscillation of the second-order neutral delay dynamic equations

$$(x(t) - rx(\tau(t)))^{\Delta\Delta} + H(t, x(h_1(t))) = 0, \quad t \in \mathbb{T}. \tag{1.13}$$

The author assumes that

(H_1) τ and $h_1 \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{T}), \tau(t) < t$, $\tau(t) \to \infty$ as $t \to \infty$, $h_1(t) < t$, $h_1(t) \to \infty$ as $t \to \infty$, and $0 \le r < 1$;

(H_2) $H \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ for each $t \in \mathbb{T}$ which are nondecreasing in u, and $H(t, u) > 0$, for $u > 0$;

(H_3) $|H(t, u)| \ge \alpha(t)|u|^{\lambda}$, where $\alpha(t) \ge 0$, and $0 \le \lambda = p/q < 1$ with p, q being odd integers.

The main result for the oscillation of (1.13) in [35] is the following.

Theorem 1.2 (see, [35, Theorem 3.4]). *Assume that (H_1)–(H_3) hold. If for all sufficiently large* $t_1 \ge t_0$,

$$\int_{t_1}^{\infty}\alpha(s)(\tau(h_1(s)))^{\lambda}\Delta s = \infty, \tag{1.14}$$

then (1.13) *oscillates.*

We find that the conclusion of this theorem is wrong. The following is a counter example of this theorem.

Counter Example. Consider the second-order differential equation

$$\left(x(t)-\frac{1}{3}x\left(\frac{t}{3}\right)\right)''+\left(\frac{1}{27}e^{-1/3}-e^{-1/3}e^{-2t/3}\right)x^{1/3}(t-1)=0,\quad t\geq t_0. \tag{1.15}$$

Let $\alpha(t)=e^{-1/3}/27-e^{-1/3}e^{-2t/3}$, $r(t)=1/3$, $\tau(t)=t/3$, and $h_1(t)=t-1$, $\lambda=1/3$. For all sufficiently large $t_1\geq t_0$, we find that

$$\int_{t_1}^{\infty}\alpha(s)(\tau(h_1(s)))^{\lambda}\Delta s=\int_{t_1}^{\infty}\alpha(s)(\tau(h_1(s)))^{\lambda}\mathrm{d}s=\int_{t_1}^{\infty}\left(\frac{1}{27}e^{-1/3}-e^{-1/3}e^{-2s/3}\right)\left(\frac{s-1}{3}\right)^{1/3}\mathrm{d}s. \tag{1.16}$$

It is easy to see that

$$\begin{gathered}\int_{t_1}^{\infty}\frac{1}{27}e^{-1/3}\left(\frac{s-1}{3}\right)^{1/3}\mathrm{d}s=\infty,\\ \int_{t_1}^{\infty}e^{-2s/3}\left(\frac{s-1}{3}\right)^{1/3}\mathrm{d}s\leq\int_{t_1}^{\infty}e^{-2s/3}s^{1/3}\mathrm{d}s.\end{gathered} \tag{1.17}$$

Integrating by parts, we obtain

$$\int_{t_1}^{\infty}e^{-2s/3}s^{1/3}\mathrm{d}s=-t_1{}^{1/3}\left(\frac{3}{2}e^{-2t_1/3}\right)+\frac{1}{2}\int_{t_1}^{\infty}e^{-2s/3}s^{-2/3}\mathrm{d}s<\infty. \tag{1.18}$$

Hence

$$\int_{t_1}^{\infty}\alpha(s)(\tau(h_1(s)))^{\lambda}\mathrm{d}s=\infty. \tag{1.19}$$

Therefore, by the above theorem, (1.15) is oscillatory. However, $x(t)=\mathrm{e}^{-t}$ is a positive solution of (1.15). Therefore, the above theorem is wrong. Tracing the error to its source, we find that the following false assertion was used in the proof of the aforementioned theorem.

Assertion A

If x is an eventually positive solution of (1.13) , then $z(t)=x(t)-r(t)x(\tau(t))$ is eventually positive.

Abdalla [37] studied the second-order superlinear neutral delay differential equations

$$\left[a(t)\left(y(t)+r(t)y(\tau(t))\right)'\right]'+p(t)\left|y(\delta(t))\right|^{\gamma}\operatorname{sign}y(\delta(t))=0,\quad t\in[t_0,\infty). \tag{1.20}$$

Most of the oscillation criteria are unsatisfactory since additional assumptions have to be imposed on the unknown solutions. Also, the author proved that if

$$\int_{t_0}^{\infty} \frac{\mathrm{d}t}{a(t)} = \int_{t_0}^{\infty} p(t)\mathrm{d}t = \infty, \tag{1.21}$$

then every solution of (1.20) oscillates for every $r(t) > 0$, but one can easily see that this result cannot be applied when $p(t) = t^{-\alpha}$ for $\alpha > 1$.

Lin [38] considered the second-order nonlinear neutral differential equations

$$\left[x(t) - p(t)x(t-\tau)\right]'' + q(t)f(x(t-\sigma)) = 0, \quad t \geq 0, \tag{1.22}$$

where $0 \leq p(t) \leq 1,\ q(t) \geq 0,\ \tau,\ \sigma > 0$. The author investigated the oscillation for (1.22) when f is superlinear.

Wong [46, 47] studied the second-order neutral differential equations

$$\left[y(t) - py(t-\tau)\right]'' + q(t)f\left(y(t-\sigma)\right) = 0, \quad t \geq 0, \tag{1.23}$$

$q \in C[0,\infty),\ q(t) \geq 0,\ f \in C^1(-\infty,\infty),\ yf(y) > 0$ whenever $y \neq 0,\ f'(y) \geq 0$ for all y, and $0 < p < 1,\ \tau > 0,\ \sigma > 0$ are constants.

The main results for the oscillation of (1.23) in [46, 47] are the following.

Theorem 1.3 (see, [46, 47, Theorem 1]). *Suppose that f is superlinear. Then a solution of* (1.23) *is either oscillatory or tends to zero if and only if*

$$\int^{\infty} tq(t)\mathrm{d}t = \infty. \tag{1.24}$$

Theorem 1.4 (see, [46, 47, Theorem 2]). *Suppose that f is sublinear and in addition satisfies*

$$f(uv) \geq f(u)f(v), \quad uv \geq 0. \tag{1.25}$$

Then a solution of (1.23) *is either oscillatory or tends to zero if and only if*

$$\int^{\infty} f(t)q(t)\mathrm{d}t = \infty. \tag{1.26}$$

Li and Saker [40] investigated the second-order sublinear neutral delay difference equations

$$\Delta\left(a_n\Delta\left(x_n + p_n x_{n-\tau}\right)\right) + q_n x_{n-\sigma}^{\gamma} = 0, \tag{1.27}$$

where $0 < \gamma < 1$ is a quotient of odd positive integers, $a_n > 0,\ \Delta a_n \geq 0, \sum_{n=0}^{\infty} 1/a_n = \infty,\ 0 \leq p_n < 1$, for all $n \geq 0$ and $q_n \geq 0$.

The main result for the oscillation of (1.27) in [40] is the following.

Theorem 1.5 (see, [40, Theorem 2.1]). *Assume that there exists a positive sequence $\{\rho_n\}$ such that for every $\alpha \geq 1$,*

$$\limsup_{n\to\infty}\sum_{l=0}^{n}\left[\rho_l Q_l - \frac{a_{l-\sigma}(\alpha(l+1-\sigma))^{1-\gamma}(\Delta\rho_l)^2}{4\gamma\rho_l}\right] = \infty, \tag{1.28}$$

where $Q_n = q_n(1-p_{n-\sigma})^{\gamma}$. Then every solution of (1.27) *oscillates.*

Yildiz and Öcalan [41] studied the higher-order sublinear neutral delay difference equations of the type

$$\Delta^m\left(y_n + p_n y_{n-l}\right) + q_n y_{n-k}^{\alpha} = 0, \quad n \in \mathbb{N}, \tag{1.29}$$

where $0 < \alpha < 1$ is a ratio of odd positive integers. The authors established some oscillation criteria of (1.29).

The main results for the oscillation of (1.29) when $m = 2$ in [41] are the following.

Theorem 1.6 (see, [41, Theorem 2.1(a), $m = 2$]). *Assume that $0 \leq p_n < 1$, and*

$$\sum_{n=0}^{\infty} q_n\left[(1-p_{n-k})n\right]^{\alpha} = \infty. \tag{1.30}$$

Then all solutions of (1.29) *are oscillatory.*

Theorem 1.7 (see, [41, Theorem 2.2, $m = 2$]). *Assume that $-1 < -p_2 \leq p_n \leq 0$, where $p_2 > 0$ is a constant, and*

$$\sum_{n=0}^{\infty} q_n n^{\alpha} = \infty. \tag{1.31}$$

Then every solution of (1.29) *either oscillates or tends to zero as $n \to \infty$.*

Cheng [42] considered the oscillation of the second-order nonlinear neutral difference equations

$$\Delta\left(p_n(\Delta(x_n + c_n x_{n-\tau}))^{\gamma}\right) + q_n x_{n-\sigma}^{\beta} = 0 \tag{1.32}$$

and established some oscillation criteria of (1.32) by means of Riccati transformation techniques.

Following this trend, in this paper, we are concerned with oscillation of the second-order neutral delay dynamic equations of Emden-Fowler type

$$\left[a(t)(x(t) + r(t)x(\tau(t)))^{\Delta}\right]^{\Delta} + p(t)x^{\gamma}(\delta(t)) = 0, \quad t \in \mathbb{T}. \tag{1.33}$$

As we are interested in oscillatory behavior, we assume throughout this paper that the given time scales $\mathbb{T}$ are unbounded above; that is, it is a time scale interval of the form $[t_0, \infty)$ with $t_0 \in \mathbb{T}$.

We assume that $\gamma > 0$ is a quotient of odd positive integers, the delay functions τ and δ satisfy $\tau : \mathbb{T} \to \mathbb{T}$, $\delta : \mathbb{T} \to \mathbb{T}$, $\tau(t) \leq t$, $\delta(t) \leq t$ for all $t \in \mathbb{T}$, and $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$; a, r and p are real-valued rd-continuous functions defined on $\mathbb{T}$, $a(t) > 0$, $p(t) > 0$, $\int_{t_0}^{\infty} \Delta t/a(t) = \infty$.

We note that if $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = t$, $\mu(t) = 0$, $x^{\Delta}(t) = x'(t)$, and (1.33) becomes the second-order nonlinear delay differential equation

$$\left[a(t)(x(t) + r(t)x(\tau(t)))'\right]' + p(t)x^{\gamma}(\delta(t)) = 0, \quad t \in \mathbb{R}. \tag{1.34}$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t+1$, $\mu(t) = 1$, $x^{\Delta}(t) = \Delta x(t) = x(t+1) - x(t)$, and (1.33) becomes the second-order nonlinear delay differential equation

$$\Delta[a(t)\Delta(x(t) + r(t)x(\tau(t)))] + p(t)x^{\gamma}(\delta(t)) = 0, \quad t \in \mathbb{Z}. \tag{1.35}$$

In the case of $\gamma > 1$, (1.33) is the prototype of a wide class of nonlinear dynamic equations called Emden-Fowler sublinear dynamic equations, and if $\gamma < 1$, (1.33) is the prototype of dynamic equations called Emden-Fowler sublinear dynamic equations. It is interesting to study (1.33) because the continuous version, that is, (1.34), has several physical applications; see, for example, [1, 39], and when t is a discrete variable, it is (1.35), and it is also important in applications.

2. Main Results

In this section, we give some new oscillation criteria of (1.33). In order to prove our main results, we will use the formula

$$\left((x(t))^{\gamma}\right)^{\Delta} = \gamma \int_0^1 [hx^{\sigma}(t) + (1-h)x(t)]^{\gamma-1}x^{\Delta}(t)\mathrm{d}h, \tag{2.1}$$

which is a simple consequence of Keller's chain rule [3, Theorem 1.90]. Also, we need the following auxiliary results.

For the sake of convenience, we assume that

$$z(t) = x(t) + r(t)x(\tau(t)), \quad R(t,t_*) = a(t)\int_{t_*}^{t} \frac{\Delta s}{a(s)}, \quad \alpha(t,t_*) = \frac{\int_{t_*}^{\delta(t)} \Delta s/a(s)}{\int_{t_*}^{t} \Delta s/a(s)}, \quad t_* \geq t_0. \tag{2.2}$$

Lemma 2.1. *Assume that* (1.11) *holds,* $a^{\Delta}(t) \geq 0$, *and* $0 \leq r(t) < 1$. *Then an eventually positive solution* x *of* (1.33) *eventually satisfies that*

$$z(t) \geq tz^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) < 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad \frac{z(t)}{t} \text{ is nonincreasing.} \tag{2.3}$$

Proof. From (1.11), the proof is similar to that of Saker et al. [24, Lemma 2.1], so it is omitted. □

Lemma 2.2. *Assume that*

$$\int_{t_0}^{\infty} p(t)\delta^{\gamma}(t)\Delta t = \infty, \tag{2.4}$$

$a^{\Delta}(t) \geq 0$, $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then an eventually positive solution* x *of* (1.33) *eventually satisfies that*

$$z(t) \geq t z^{\Delta}(t) > 0, \quad z^{\Delta\Delta}(t) < 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad \frac{z(t)}{t} \text{ is nonincreasing}, \tag{2.5}$$

or $\lim_{t\to\infty} x(t) = 0$.

Proof. Let x be an eventually positive solution of (1.33). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1$. Assume that $\lim_{t\to\infty} x(t) \neq 0$, that is, $\limsup_{t\to\infty} x(t) > 0$. Then, we have to show that (2.5) holds. It follows from (1.33) that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} = -p(t)x^{\gamma}(\delta(t)) < 0, \quad t \geq t_1, \tag{2.6}$$

which implies that az^{Δ} is nonincreasing on $[t_1, \infty)_{\mathbb{T}}$. Since the function a is nondecreasing, z^{Δ} must be nonincreasing on $[t_1, \infty)_{\mathbb{T}}$, that is, z^{Δ} is eventually either positive or negative. In both cases, z is eventually monotonic, so that z has a limit at infinity (finite or infinite). This implies that $\lim_{t\to\infty} z(t) \neq 0$; that is, z is eventually positive (see [19, Lemma 3]). Then we proceed as in the proof of [24, Lemma 2.1] to obtain (2.5). The proof is complete. □

Lemma 2.3. *Assume that* $0 \leq r(t) < 1$. *Further,* x *is an eventually positive solution of* (1.33). *Then there exists a* $t_* \geq t_0$ *such that for* $t \geq t_*$,

$$z^{\Delta}(t) > 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad z(t) \geq R(t, t_*)z^{\Delta}(t), \quad z(\delta(t)) \geq \alpha(t, t_*)z(t). \tag{2.7}$$

Proof. Let x be an eventually positive solution of (1.33). Then there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1$. It follows from (1.33) that (2.6) holds. From (2.6), we know that $a(t)z^{\Delta}(t)$ is an eventually decreasing function. We claim that $z^{\Delta}(t) > 0$ eventually. Otherwise, if there exists a $t_2 \geq t_1$ such that $z^{\Delta}(t) < 0$, by (2.6), we have

$$a(t)z^{\Delta}(t) \leq a(t_2)z^{\Delta}(t_2) = b < 0, \quad t \geq t_2. \tag{2.8}$$

Thus

$$z^{\Delta}(t) \leq b\frac{1}{a(t)}. \tag{2.9}$$

Integrating the above inequality from t_2 to t leads to $\lim_{t\to\infty} z(t) = -\infty$, which contradicts $z(t) > 0$. Hence, $z^{\Delta}(t) > 0$ on $[t_2, \infty)_{\mathbb{T}}$. Therefore,

$$z(t) > z(t) - z(t_2) = \int_{t_2}^{t} \frac{a(s)z^{\Delta}(s)}{a(s)} \Delta s \geq \left(a(t)z^{\Delta}(t)\right) \int_{t_2}^{t} \frac{\Delta s}{a(s)}, \tag{2.10}$$

which yields

$$z(t) \geq \left(a(t) \int_{t_2}^{t} \frac{\Delta s}{a(s)}\right) z^{\Delta}(t). \tag{2.11}$$

Since $a(t)z^{\Delta}(t)$ is strictly decreasing, we have

$$z(t) - z(\delta(t)) = \int_{\delta(t)}^{t} \frac{a(s)z^{\Delta}(s)}{a(s)} \Delta s \leq a(\delta(t))z^{\Delta}(\delta(t)) \int_{\delta(t)}^{t} \frac{\Delta s}{a(s)}, \tag{2.12}$$

and so

$$\frac{z(t)}{z(\delta(t))} \leq 1 + \frac{a(\delta(t))z^{\Delta}(\delta(t))}{z(\delta(t))} \int_{\delta(t)}^{t} \frac{\Delta s}{a(s)}. \tag{2.13}$$

Also, we have that for large t,

$$z(\delta(t)) \geq z(\delta(t)) - z(t_2) = \int_{t_2}^{\delta(t)} \frac{a(s)z^{\Delta}(s)}{a(s)} \Delta s \geq a(\delta(t))z^{\Delta}(\delta(t)) \int_{t_2}^{\delta(t)} \frac{\Delta s}{a(s)}, \tag{2.14}$$

so we obtain

$$\frac{a(\delta(t))z^{\Delta}(\delta(t))}{z(\delta(t))} \leq \left(\int_{t_2}^{\delta(t)} \frac{\Delta s}{a(s)}\right)^{-1}. \tag{2.15}$$

Therefore, from (2.13), we have

$$z(\delta(t)) \geq \alpha(t, t_2) z(t). \tag{2.16}$$

This completes the proof. □

Lemma 2.4. *Assume that* $-1 < -r_0 \leq r(t) \leq 0$, $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then an eventually positive solution* x *of* (1.33) *satisfies that, for sufficiently large* $t_* \geq t_0$,

$$z^{\Delta}(t) > 0, \quad \left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0, \quad z(t) \geq R(t, t_*) z^{\Delta}(t), \quad z(\delta(t)) \geq \alpha(t, t_*) z(t), \quad t \geq t_*, \tag{2.17}$$

or $\lim_{t\to\infty} x(t) = 0$.

Proof. The proof is similar to that of the proof Lemmas 2.2 and 2.3, so we omit the details. □

Theorem 2.5. *Assume that* (1.11) *holds,* $a^{\Delta}(t) \geq 0$, *and* $0 \leq r(t) < 1$. *Then every solution of* (1.33) *oscillates if the inequality*

$$y^{\Delta}(t) + A(t)y^{\gamma}(\delta(t)) \leq 0, \tag{2.18}$$

where

$$A(t) = p(t)(1 - r(\delta(t)))^{\gamma} \frac{(\delta(t))^{\gamma}}{(a(\delta(t)))^{\gamma}}, \tag{2.19}$$

has no eventually positive solution.

Proof. Suppose to the contrary that (1.33) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\delta(t)) > 0$ for all $t \geq t_1$. From Lemma 2.1, there is some $t_2 \geq t_1$ such that

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) - r(t)z(\tau(t)) \geq (1 - r(t))z(t), \quad t \geq t_2. \tag{2.20}$$

From (1.33), there exists a $t_3 \geq t_2$ such that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)(1 - r(\delta(t)))^{\gamma}(z(\delta(t)))^{\gamma} \leq 0, \quad t \geq t_3. \tag{2.21}$$

By Lemma 2.1, there exists a $t_4 \geq t_3$ such that

$$z(\delta(t)) \geq \delta(t)z^{\Delta}(\delta(t)). \tag{2.22}$$

Substituting the last inequality in (2.21) we obtain for $t \geq t_4$ that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)(1 - r(\delta(t)))^{\gamma}(\delta(t))^{\gamma}\left(z^{\Delta}(\delta(t))\right)^{\gamma} \leq 0. \tag{2.23}$$

Set $y(t) = a(t)z^{\Delta}(t)$. Then from (2.23), y is positive and satisfies the inequality (2.18), and this contradicts the assumption of our theorem. Thus every solution of (1.33) oscillates. This completes the proof. □

By [41, Lemma 1.1] and Theorem 2.5 in this paper, we have the following result.

Corollary 2.6. *If* $\mathbb{T} = \mathbb{Z}$, $a(t) = 1$, $\delta(t) = t - l$, l *is a positive integer, and* $0 \leq r(t) < 1$, *then every solution of* (1.33) *oscillates if*

$$\sum_{t=n_0}^{\infty} t^{\gamma} p(t)(1 - r(\delta(t)))^{\gamma} = \infty. \tag{2.24}$$

Theorem 2.7. *Assume that* (2.4) *holds, and* $a^{\Delta}(t) \geq 0$, $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if the inequality*

$$y^{\Delta}(t) + B(t)y^{\gamma}(\delta(t)) \leq 0, \tag{2.25}$$

where

$$B(t) = p(t)\frac{(\delta(t))^{\gamma}}{(a(\delta(t)))^{\gamma}}, \tag{2.26}$$

has no eventually positive solution.

Proof. Suppose to the contrary that (1.33) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1$.

From Lemma 2.2, if (i) holds, there is some $t_2 \geq t_1$ such that

$$x(t) = z(t) - r(t)x(\tau(t)) \geq z(t) > 0, \quad t \geq t_2. \tag{2.27}$$

From (1.33), there exists a $t_3 \geq t_2$ such that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)(z(\delta(t)))^{\gamma} \leq 0, \quad t \geq t_3. \tag{2.28}$$

By Lemma 2.2, there exists a $t_3 \geq t_2$ such that

$$z(\delta(t)) \geq \delta(t)z^{\Delta}(\delta(t)). \tag{2.29}$$

Substituting the last inequality in (2.28), we obtain for $t \geq t_3$ that

$$\left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)(\delta(t))^{\gamma}\left(z^{\Delta}(\delta(t))\right)^{\gamma} \leq 0. \tag{2.30}$$

Set $y(t) = a(t)z^{\Delta}(t)$. Then from (2.30), y is positive and satisfies the inequality (2.25), and this contradicts the assumption of our theorem.

If (ii) holds, by Lemma 2.2, we have $\lim_{t\to\infty} x(t) = 0$. This completes the proof. □

By [41, Lemma 1.1] and Theorem 2.7 in this paper, we have the following result.

Corollary 2.8. *Assume that* $\mathbb{T} = \mathbb{Z}$, $a(t) = 1$, $\delta(t) = t - l$, l *is a positive integer,* $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\sum_{t=n_0}^{\infty} t^{\gamma} p(t) = \infty. \tag{2.31}$$

Remark 2.9. Theorems 2.5 and 2.7 reduce the question of (1.33) to the absence of eventually positive solution (the oscillatory) of the differential inequalities (2.18) and (2.25).

Remark 2.10. From Theorem 2.5, Theorem 2.7, and the results given in [7–9, 12, 14], we can obtain some oscillation criteria for (1.33) in the case when $\gamma = 1,\ a^{\Delta}(t) \geq 0$.

Theorem 2.11. *Assume that* (1.11) *holds,* $\gamma < 1,\ a^{\Delta}(t) \geq 0$, *and* $0 \leq r(t) < 1$. *Then every solution of* (1.33) *oscillates if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^{\gamma}} (1 - r(\delta(s)))^{\gamma} (\delta(s))^{\gamma} \Delta s = \infty. \tag{2.32}$$

Proof. We assume that (1.33) has a nonoscillatory solution such that $x(t) > 0,\ x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1 \geq t_0$. By proceeding as in the proof of Theorem 2.5, we get (2.21). By Lemma 2.1, note that $(a(t)z^{\Delta}(t))^{\Delta} < 0$, and from Keller's chain rule, we obtain

$$\begin{aligned}\left(\left(a(t)z^{\Delta}(t)\right)^{1-\gamma}\right)^{\Delta} &= (1-\gamma)\int_0^1 \left[h\left(a(t)z^{\Delta}(t)\right)^{\sigma} + (1-h)a(t)z^{\Delta}(t)\right]^{-\gamma}\left(a(t)z^{\Delta}(t)\right)^{\Delta} \mathrm{d}h \\ &\leq (1-\gamma)\int_0^1 \left[ha(t)z^{\Delta}(t) + (1-h)a(t)z^{\Delta}(t)\right]^{-\gamma}\left(a(t)z^{\Delta}(t)\right)^{\Delta} \mathrm{d}h \\ &= (1-\gamma)\left(a(t)z^{\Delta}(t)\right)^{-\gamma}\left(a(t)z^{\Delta}(t)\right)^{\Delta} < 0,\end{aligned} \tag{2.33}$$

so

$$\left(a(t)z^{\Delta}(t)\right)^{-\gamma}\left(a(t)z^{\Delta}(t)\right)^{\Delta} \geq \frac{\left((a(t)z^{\Delta}(t))^{1-\gamma}\right)^{\Delta}}{1-\gamma}. \tag{2.34}$$

Using (2.21), we have

$$\begin{aligned} 0 &\geq \frac{(a(t)z^{\Delta}(t))^{\Delta} + p(t)(1-r(\delta(t)))^{\gamma}(z(\delta(t)))^{\gamma}}{(a(t)z^{\Delta}(t))^{\gamma}} \\ &= \left(a(t)z^{\Delta}(t)\right)^{-\gamma}\left(a(t)z^{\Delta}(t)\right)^{\Delta} + p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{z(\delta(t))}{a(t)z^{\Delta}(t)}\right)^{\gamma} \\ &\geq \frac{\left((a(t)z^{\Delta}(t))^{1-\gamma}\right)^{\Delta}}{1-\gamma} + \frac{p(t)}{(a(\delta(t)))^{\gamma}}(1-r(\delta(t)))^{\gamma}(\delta(t))^{\gamma}. \end{aligned} \tag{2.35}$$

Hence,

$$\frac{p(t)}{(a(\delta(t)))^{\gamma}}(1-r(\delta(t)))^{\gamma}(\delta(t))^{\gamma} \leq \frac{\left((a(t)z^{\Delta}(t))^{1-\gamma}\right)^{\Delta}}{\gamma-1}. \tag{2.36}$$

Upon integration we arrive at

$$\int_{t_1}^{t} \frac{p(s)}{(a(\delta(s)))^{\gamma}}(1-r(\delta(s)))^{\gamma}(\delta(s))^{\gamma}\Delta s \le \int_{t_1}^{t} \frac{\left(\left(a(s)z^{\Delta}(s)\right)^{1-\gamma}\right)^{\Delta}}{\gamma-1}\Delta s \le \frac{\left(a(t_1)z^{\Delta}(t_1)\right)^{1-\gamma}}{1-\gamma}. \quad (2.37)$$

This contradicts (2.32) and finishes the proof. □

Theorem 2.12. *Assume that* (2.4) *holds, and* $\gamma < 1$, $a^{\Delta}(t) \ge 0$, $-1 < -r_0 \le r(t) \le 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^{\gamma}}(\delta(s))^{\gamma}\Delta s = \infty. \quad (2.38)$$

Proof. By Lemma 2.2, the proof is similar to that of the proof of Theorem 2.11, so we omit the details. □

Theorem 2.13. *Assume that* $\gamma < 1$ *and* $0 \le r(t) < 1$. *Then every solution of* (1.33) *oscillates if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^{\gamma}}(1-r(\delta(s)))^{\gamma}(R(\delta(s),t_*))^{\gamma}\Delta s = \infty \quad (2.39)$$

holds for all sufficiently large t_*.

Proof. By Lemma 2.3, the proof is similar to that of the proof Theorem 2.11, so we omit the details. □

Theorem 2.14. *Assume that* $\gamma < 1$, $-1 < -r_0 \le r(t) \le 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\int_{t_0}^{\infty} \frac{p(s)}{(a(\delta(s)))^{\gamma}}(R(\delta(s),t_*))^{\gamma}\Delta s = \infty \quad (2.40)$$

holds for all sufficiently large t_*.

Proof. By using Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.11, so we omit the details. □

Theorem 2.15. *Assume that* (1.11) *holds,* $\gamma \ge 1$, $a^{\Delta}(t) \ge 0$, *and* $0 \le r(t) < 1$. *Then every solution of* (1.33) *oscillates if*

$$\limsup_{t\to\infty}\left\{\frac{t}{a(t)}\int_{t}^{\infty} p(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}\Delta s\right\} = \infty. \quad (2.41)$$

Proof. Suppose to the contrary that (1.33) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \ge t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and

$x(\delta(t)) > 0$ for all $t \geq t_1$. By proceeding as in the proof of Theorem 2.5, we get (2.21). Thus from Lemma 2.1, we have for $T \geq t \geq t_1$,

$$\int_t^T p(s)(1-r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq -\int_t^T \left(a(s)z^\Delta(s)\right)^\Delta \Delta s = a(t)z^\Delta(t) - a(T)z^\Delta(T), \quad (2.42)$$

and hence

$$\int_t^T p(s)(1-r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \leq a(t)z^\Delta(t). \quad (2.43)$$

This and Lemma 2.1 provide, for sufficiently large $t \in \mathbb{T}$,

$$\begin{aligned} z(t) \geq tz^\Delta(t) &\geq \frac{t}{a(t)} \int_t^\infty p(s)(1-r(\delta(s)))^\gamma (z(\delta(s)))^\gamma \Delta s \\ &\geq \frac{t}{a(t)} \int_t^\infty p(s)(1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{s}\right)^\gamma z^\gamma(s)\Delta s \\ &\geq z^\gamma(t)\left\{\frac{t}{a(t)} \int_t^\infty p(s)(1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{s}\right)^\gamma \Delta s\right\}. \end{aligned} \quad (2.44)$$

So

$$\left\{\frac{t}{a(t)} \int_t^\infty p(s)(1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{s}\right)^\gamma \Delta s\right\} \leq \left(\frac{1}{z(t)}\right)^{\gamma-1}. \quad (2.45)$$

We note that $\gamma \geq 1$ and $z^\Delta(t) > 0$ imply

$$\left\{\frac{t}{a(t)} \int_t^\infty p(s)(1-r(\delta(s)))^\gamma \left(\frac{\delta(s)}{s}\right)^\gamma \Delta s\right\} \leq \left(\frac{1}{z(t_1)}\right)^{\gamma-1}. \quad (2.46)$$

This contradicts (2.41) and completes the proof. □

Theorem 2.16. *Assume that* (2.4) *holds, and* $\gamma \geq 1$, $a^\Delta(t) \geq 0$, $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\limsup_{t\to\infty}\left\{\frac{t}{a(t)} \int_t^\infty p(s)\left(\frac{\delta(s)}{s}\right)^\gamma \Delta s\right\} = \infty. \quad (2.47)$$

Proof. By using Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.15, so we omit the details. □

Theorem 2.17. *Assume that* $\gamma \geq 1,\ 0 \leq r(t) < 1$. *Then every solution of* (1.33) *oscillates if*

$$\limsup_{t\to\infty}\left\{\frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(\alpha(s,t_*))^\gamma\Delta s\right\}=\infty \tag{2.48}$$

holds for all sufficiently large t_*.

Proof. Suppose to the contrary that (1.33) has a nonoscillatory solution x. We may assume without loss of generality that there exists $t_1 \geq t_0$ such that $x(t) > 0,\ x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1$. By proceeding as in the proof of Theorem 2.5, we obtain (2.21). Thus from Lemma 2.3, we have, for $T \geq t \geq t_1$,

$$\int_t^T p(s)(1-r(\delta(s)))^\gamma(z(\delta(s)))^\gamma\Delta s \leq -\int_t^T \left(a(s)z^\Delta(s)\right)^\Delta\Delta s = a(t)z^\Delta(t)-a(T)z^\Delta(T), \tag{2.49}$$

and hence

$$\int_t^T p(s)(1-r(\delta(s)))^\gamma(z(\delta(s)))^\gamma\Delta s \leq a(t)z^\Delta(t). \tag{2.50}$$

This and Lemma 2.3 provide, for sufficiently large $t \in \mathbb{T}$,

$$\begin{aligned}
z(t) \geq R(t,t_*)z^\Delta(t) &\geq \frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(z(\delta(s)))^\gamma\Delta s\\
&\geq \frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(\alpha(s,t_*))^\gamma z^\gamma(s)\Delta s\\
&\geq z^\gamma(t)\left\{\frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(\alpha(s,t_*))^\gamma\Delta s\right\}.
\end{aligned} \tag{2.51}$$

So

$$\left\{\frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(\alpha(s,t_*))^\gamma\Delta s\right\} \leq \left(\frac{1}{z(t)}\right)^{\gamma-1}. \tag{2.52}$$

We note that $\gamma \geq 1$ and $z^\Delta(t) > 0$ imply

$$\left\{\frac{R(t,t_*)}{a(t)}\int_t^\infty p(s)(1-r(\delta(s)))^\gamma(\alpha(s,t_*))^\gamma\Delta s\right\} \leq \left(\frac{1}{z(t_1)}\right)^{\gamma-1}. \tag{2.53}$$

This contradicts (2.48) and completes the proof. □

Theorem 2.18. *Assume that* (2.4) *holds, and* $\gamma \geq 1$, $a^{\Delta}(t) \geq 0$, $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\limsup_{t\to\infty}\left\{\frac{R(t,t_*)}{a(t)}\int_t^{\infty} p(s)(\alpha(s,t_*))^{\gamma}\Delta s\right\} = \infty \tag{2.54}$$

holds for all sufficiently large t_*.

Proof. By using Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.17, so we omit the details. □

Theorem 2.19. *Assume that* (1.11) *holds,* $\gamma > 1$, $a^{\Delta}(t) \geq 0$, *and* $0 \leq r(t) < 1$. *Then every solution of* (1.33) *oscillates if*

$$\int_{t_0}^{\infty} \sigma(s)\frac{p(s)}{a(s)}(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s = \infty. \tag{2.55}$$

Proof. We assume that (1.33) has a nonoscillatory solution such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1 \geq t_0$. By proceeding as in the proof of Theorem 2.5, we get (2.21). Define the function

$$\omega(t) = \frac{ta(t)z^{\Delta}(t)}{z^{\gamma}(t)}, \quad t \geq t_1. \tag{2.56}$$

By Lemma 2.1, $\omega(t) > 0$. We calculate

$$\omega^{\Delta}(t) = \left\{a(t)z^{\Delta}(t) + \sigma(t)\left(a(t)z^{\Delta}(t)\right)^{\Delta}\right\}\left(z^{-\gamma}(t)\right)^{\sigma} + ta(t)z^{\Delta}(t)\left(z^{-\gamma}(t)\right)^{\Delta}. \tag{2.57}$$

From (2.21), we have

$$\omega^{\Delta}(t) \leq a(t)z^{\Delta}(t)\left(z^{-\gamma}(t)\right)^{\sigma} - \sigma(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{z(\delta(t))}{z(\sigma(t))}\right)^{\gamma} + ta(t)z^{\Delta}(t)\left(z^{-\gamma}(t)\right)^{\Delta}, \tag{2.58}$$

and by Lemma 2.1, we have

$$\omega^{\Delta}(t) \leq a(t)z^{\Delta}(t)\left(z^{-\gamma}(t)\right)^{\sigma} - \sigma(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma}, \tag{2.59}$$

because $(z^{-\gamma}(t))^{\Delta} \leq 0$ due to Keller's chain rule. Since

$$\begin{aligned}\left((z(t))^{1-\gamma}\right)^{\Delta} &= (1-\gamma)\int_0^1 \left[hz^{\sigma}(t) + (1-h)z(t)\right]^{-\gamma} z^{\Delta}(t)\mathrm{d}h \\ &\leq (1-\gamma)\int_0^1 \left[hz^{\sigma}(t) + (1-h)z^{\sigma}(t)\right]^{-\gamma} z^{\Delta}(t)\mathrm{d}h = (1-\gamma)\left(z^{\sigma}(t)\right)^{-\gamma} z^{\Delta}(t),\end{aligned} \tag{2.60}$$

thus

$$\omega^{\Delta}(t) \le a(t)\frac{\left((z(t))^{1-\gamma}\right)^{\Delta}}{1-\gamma} - \sigma(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\delta(t)}{\sigma(t)}\right)^{\gamma}. \tag{2.61}$$

Upon integration we arrive at

$$\begin{aligned}
&\int_{t_1}^{t} \sigma(s)\frac{p(s)}{a(s)}(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s \\
&\quad\le \int_{t_1}^{t}\left\{\frac{\left((z(s))^{1-\gamma}\right)^{\Delta}}{1-\gamma} - \frac{\omega^{\Delta}(s)}{a(s)}\right\}\Delta s \\
&\quad= \frac{(z(t))^{1-\gamma}}{1-\gamma} - \frac{(z(t_1))^{1-\gamma}}{1-\gamma} - \int_{t_1}^{t}\frac{\omega^{\Delta}(s)}{a(s)}\Delta s \\
&\quad= \frac{(z(t))^{1-\gamma}}{1-\gamma} - \frac{(z(t_1))^{1-\gamma}}{1-\gamma} - \frac{\omega(t)}{a(t)} + \frac{\omega(t_1)}{a(t_1)} + \int_{t_1}^{t}\omega^{\sigma}(s)\left(\frac{1}{a(s)}\right)^{\Delta}\Delta s.
\end{aligned} \tag{2.62}$$

Noting that $(1/a(t))^{\Delta} \le 0$, we have

$$\int_{t_1}^{t} \sigma(s)\frac{p(s)}{a(s)}(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s \le \frac{(z(t_1))^{1-\gamma}}{\gamma-1} + \frac{\omega(t_1)}{a(t_1)}. \tag{2.63}$$

This contradicts (2.55) and finishes the proof. □

Theorem 2.20. *Assume that* (2.4) *holds, and* $\gamma > 1$, $a^{\Delta}(t) \ge 0$, $-1 < -r_0 \le r(t) \le 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *Then every solution of* (1.33) *either oscillates or tends to zero as* $t \to \infty$ *if*

$$\int_{t_0}^{\infty} \sigma(s)p(s)\left(\frac{\delta(s)}{\sigma(s)}\right)^{\gamma}\Delta s = \infty. \tag{2.64}$$

Proof. By using Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.19, so we omit the details. □

In the following, we use a Riccati transformation technique to establish new oscillation criteria for (1.33).

Theorem 2.21. *Assume that* $\gamma \ge 1$, *and* $0 \le r(t) < 1$. *Furthermore, suppose that there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_*, *and for all constants* $M > 0$, *for* $t_1 \geq t_*$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\eta(s)p(s)(1-r(\delta(s)))^{\gamma}(\alpha(s,t_*))^{\gamma} - \frac{a(s)\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}\eta(s)} \right] \Delta s = \infty. \tag{2.65}$$

Then every solution of (1.33) *oscillates.*

Proof. We assume that (1.33) has a nonoscillatory solution such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1 \geq t_0$. By proceeding as in the proof of Theorem 2.5, we get (2.21). Define the function ω by the Riccati substitution

$$\omega(t) = \eta(t)\frac{a(t)z^{\Delta}(t)}{z^{\gamma}(t)}, \quad t \geq t_1. \tag{2.66}$$

Then $\omega(t) > 0$. By the product rule and then the quotient rule

$$\begin{aligned} \omega^{\Delta}(t) &= \left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{\eta(t)}{z^{\gamma}(t)}\right]^{\Delta} + \frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta} \\ &= \frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta} + \left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{z^{\gamma}(t)\eta^{\Delta}(t) - \eta(t)(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}\right]. \end{aligned} \tag{2.67}$$

In view of (2.21) and (2.66), we have

$$\omega^{\Delta}(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{z(\delta(t))}{z(t)}\right)^{\gamma} + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\eta(t)\left(a(t)z^{\Delta}(t)\right)^{\sigma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}. \tag{2.68}$$

By the chain rule and $\gamma \geq 1$, we obtain

$$(z^{\gamma}(t))^{\Delta} \geq \gamma z^{\gamma-1}(t)z^{\Delta}(t) \geq \gamma M^{\gamma-1}z^{\Delta}(t), \tag{2.69}$$

where $M = z(t_1) > 0$. In view of $(a(t)z^{\Delta}(t))^{\Delta} < 0$, we have

$$a(t)z^{\Delta}(t) \geq \left(a(t)z^{\Delta}(t)\right)^{\sigma}, \tag{2.70}$$

and by Lemma 2.3, we see that

$$\omega^{\Delta}(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^{\gamma}(\alpha(t,t_*))^{\gamma} + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\gamma M^{\gamma-1}\eta(t)}{a(t)\left(\eta^{\sigma}(t)\right)^2}(\omega^{\sigma}(t))^2. \tag{2.71}$$

Integrating (2.71) from t_1 to t, we obtain

$$\begin{aligned}
&\int_{t_1}^{t} \eta(s)p(s)(1-r(\delta(s)))^{\gamma}(\alpha(s,t_*))^{\gamma}\Delta s \\
&\quad \leq -\int_{t_1}^{t} \omega^{\Delta}(s)\Delta s \\
&\qquad + \int_{t_1}^{t} \frac{\eta^{\Delta}(s)}{\eta^{\sigma}(s)}\omega^{\sigma}(s)\Delta s - \int_{t_1}^{t} \frac{\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^{\sigma}(s))^2}(\omega^{\sigma}(s))^2\Delta s.
\end{aligned} \tag{2.72}$$

Hence

$$\int_{t_1}^{t}\left[\eta(s)p(s)(1-r(\delta(s)))^{\gamma}(\alpha(s,t_*))^{\gamma} - \frac{a(s)(\eta^{\Delta}(s))^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s \leq \omega(t_1), \tag{2.73}$$

which contradicts condition (2.65). The proof is complete. □

Theorem 2.22. *Assume that* $\gamma \geq 1$, $-1 < -r_0 \leq r(t) \leq 0$, *and* $\lim_{t\to\infty} r(t) = r_1 > -1$. *If there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_*, *and for all constants* $M > 0$, *for* $t_1 \geq t_*$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\eta(s)p(s)(\alpha(s,t_*))^{\gamma} - \frac{a(s)(\eta^{\Delta}(s))^2}{4\gamma M^{\gamma-1}\eta(s)}\right]\Delta s = \infty, \tag{2.74}$$

then every solution of (1.33) *either oscillates or tends to zero as* $t \to \infty$.

Proof. By Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.21, so we omit the details. □

Theorem 2.23. *Assume that* (1.11) *holds,* $\gamma \leq 1$, $a^{\Delta}(t) \geq 0$, *and* $0 \leq r(t) < 1$. *Furthermore, suppose that there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_1, *and for all constants* $M > 0$,

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\eta(s)p(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma} - \frac{a(s)(\eta^{\Delta}(s))^2}{4\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)}\right]\Delta s = \infty. \tag{2.75}$$

Then every solution of (1.33) *oscillates.*

Proof. We assume that (1.33) has a nonoscillatory solution such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \geq t_1 \geq t_0$. By proceeding as in the proof of Theorem 2.5, we obtain (2.21).

Define the function ω by the Riccati substitution as (2.66). Then $\omega(t) > 0$. By the product rule and then the quotient rule

$$\begin{aligned}\omega^{\Delta}(t) &= \left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{\eta(t)}{z^{\gamma}(t)}\right]^{\Delta} + \frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta} \\ &= \frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta} + \left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{z^{\gamma}(t)\eta^{\Delta}(t) - \eta(t)(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}\right].\end{aligned} \tag{2.76}$$

In view of (2.21) and (2.66), we have

$$\omega^{\Delta}(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{z(\delta(t))}{z(t)}\right)^{\gamma} + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\eta(t)\left(a(t)z^{\Delta}(t)\right)^{\sigma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}. \tag{2.77}$$

From the chain rule and $\gamma \leq 1$, we get

$$(z^{\gamma}(t))^{\Delta} \geq \gamma z^{\gamma-1}(\sigma(t))z^{\Delta}(t). \tag{2.78}$$

Noting that $z(t)/t$ is nonincreasing, and there exists a constant $M > 0$, such that $z(t) \leq Mt$, hence we have

$$(z^{\gamma}(t))^{\Delta} \geq \gamma z^{\gamma-1}(\sigma(t))z^{\Delta}(t) \geq \gamma M^{\gamma-1}(\sigma(t))^{\gamma-1}z^{\Delta}(t). \tag{2.79}$$

In view of $(a(t)z^{\Delta}(t))^{\Delta} < 0$, we have

$$a(t)z^{\Delta}(t) \geq \left(a(t)z^{\Delta}(t)\right)^{\sigma}, \tag{2.80}$$

and by Lemma 2.1, we see that

$$\omega^{\Delta}(t) \leq -\eta(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{\delta(t)}{t}\right)^{\gamma} + \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\gamma M^{\gamma-1}(\sigma(t))^{\gamma-1}\eta(t)}{a(t)\left(\eta^{\sigma}(t)\right)^{2}}(\omega^{\sigma}(t))^{2}. \tag{2.81}$$

Integrating (2.81) from t_1 to t, we obtain

$$\begin{aligned}&\int_{t_1}^{t}\eta(s)p(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}\Delta s \\ &\quad\leq -\int_{t_1}^{t}\omega^{\Delta}(s)\Delta s + \int_{t_1}^{t}\frac{\eta^{\Delta}(s)}{\eta^{\sigma}(s)}\omega^{\sigma}(s)\Delta s - \int_{t_1}^{t}\frac{\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)}{a(s)\left(\eta^{\sigma}(s)\right)^{2}}(\omega^{\sigma}(s))^{2}\Delta s.\end{aligned} \tag{2.82}$$

Hence

$$\int_{t_1}^{t}\left[\eta(s)p(s)(1-r(\delta(s)))^{\gamma}\left(\frac{\delta(s)}{s}\right)^{\gamma}-\frac{a(s)\left(\eta^{\Delta}(s)\right)^{2}}{4\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)}\right]\Delta s\le\omega(t_1),\tag{2.83}$$

which contradicts condition (2.75). The proof is complete. □

Theorem 2.24. *Assume that* (2.4) *holds,* $\gamma \le 1$*,* $a^{\Delta}(t) \ge 0$*,* $-1 < -r_0 \le r(t) \le 0$*, and* $\lim_{t\to\infty} r(t) = r_1 > -1$*. If there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_1*, and for all constants* $M > 0$*,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\eta(s)p(s)\left(\frac{\delta(s)}{s}\right)^{\gamma}-\frac{a(s)\left(\eta^{\Delta}(s)\right)^{2}}{4\gamma M^{\gamma-1}(\sigma(s))^{\gamma-1}\eta(s)}\right]\Delta s=\infty,\tag{2.84}$$

then every solution of (1.33) *either oscillates or tends to zero as* $t \to \infty$*.*

Proof. By Lemma 2.2 and (2.28), the proof is similar to that of the proof of Theorem 2.23, so we omit the details. □

Theorem 2.25. *Assume that* $\gamma \le 1$*,* $a^{\Delta}(t) \le 0$*, and* $0 \le r(t) < 1$*. Furthermore, suppose that there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_**, and for all constants* $M > 0$*, for* $t_1 \ge t_*$*,*

$$\limsup_{t\to\infty}\int_{t_1}^{t}\left[\eta(s)p(s)(1-r(\delta(s)))^{\gamma}(\alpha(s,t_*))^{\gamma}-\frac{a(s)(\sigma(s))^{1-\gamma}\left(\eta^{\Delta}(s)\right)^{2}}{4\gamma M^{\gamma-1}(a(\sigma(s)))^{1-\gamma}\eta(s)}\right]\Delta s=\infty.\tag{2.85}$$

Then every solution of (1.33) *oscillates.*

Proof. We assume that (1.33) has a nonoscillatory solution such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\delta(t)) > 0$ for all $t \ge t_1 \ge t_0$. By proceeding as in the proof of Theorem 2.5, we have (2.21). Define the function ω by the Riccati substitution as (2.66). Then $\omega(t) > 0$. By the product rule and then the quotient rule

$$\begin{aligned}\omega^{\Delta}(t)&=\left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{\eta(t)}{z^{\gamma}(t)}\right]^{\Delta}+\frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta}\\&=\frac{\eta(t)}{z^{\gamma}(t)}\left(a(t)z^{\Delta}(t)\right)^{\Delta}+\left(a(t)z^{\Delta}(t)\right)^{\sigma}\left[\frac{z^{\gamma}(t)\eta^{\Delta}(t)-\eta(t)(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}\right].\end{aligned}\tag{2.86}$$

In view of (2.21) and (2.66), we have

$$\omega^{\Delta}(t)\le-\eta(t)p(t)(1-r(\delta(t)))^{\gamma}\left(\frac{z(\delta(t))}{z(t)}\right)^{\gamma}+\frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t)-\frac{\eta(t)\left(a(t)z^{\Delta}(t)\right)^{\sigma}(z^{\gamma}(t))^{\Delta}}{z^{\gamma}(t)(z^{\sigma}(t))^{\gamma}}.\tag{2.87}$$

By the chain rule and $\gamma \leq 1$, we obtain

$$(z^{\gamma}(t))^{\Delta} \geq \gamma z^{\gamma-1}(\sigma(t))z^{\Delta}(t), \tag{2.88}$$

and noting that $(a(t)z^{\Delta}(t))^{\Delta} < 0$ and there exists a constant $L > 0$ such that $a(t)z^{\Delta}(t) \leq L$, so

$$z(t) = z(t_1) + \int_{t_1}^{t} z^{\Delta}(s)\Delta s \leq z(t_1) + \int_{t_1}^{t} \frac{L}{a(s)}\Delta s. \tag{2.89}$$

From $a^{\Delta}(t) \leq 0$, there exists a positive constant M such that

$$z(t) \leq z(t_1) + \frac{L}{a(t)}(t - t_1) = \frac{z(t_1)a(t) + L(t - t_1)}{a(t)} \leq \frac{Mt}{a(t)}. \tag{2.90}$$

Hence

$$(z^{\gamma}(t))^{\Delta} \geq \gamma z^{\gamma-1}(\sigma(t))z^{\Delta}(t) \geq \gamma M^{\gamma-1}\left(\frac{\sigma(t)}{a(\sigma(t))}\right)^{\gamma-1} z^{\Delta}(t). \tag{2.91}$$

In view of $(a(t)z^{\Delta}(t))^{\Delta} < 0$, we have

$$a(t)z^{\Delta}(t) \geq \left(a(t)z^{\Delta}(t)\right)^{\sigma}, \tag{2.92}$$

and by Lemma 2.3, we see that

$$\begin{aligned} \omega^{\Delta}(t) \leq{}& -\eta(t)p(t)(1 - r(\delta(t)))^{\gamma}(\alpha(t, t_*))^{\gamma} \\ &+ \frac{\eta^{\Delta}(t)}{\eta^{\sigma}(t)}\omega^{\sigma}(t) - \frac{\gamma M^{\gamma-1}\eta(t)}{a(t)(\eta^{\sigma}(t))^2}\left(\frac{\sigma(t)}{a(\sigma(t))}\right)^{\gamma-1}(\omega^{\sigma}(t))^2. \end{aligned} \tag{2.93}$$

Integrating (2.93) from t_1 to t, we obtain

$$\begin{aligned} &\int_{t_1}^{t} \eta(s)p(s)(1 - r(\delta(s)))^{\gamma}(\alpha(s, t_*))^{\gamma}\Delta s \\ &\quad \leq -\int_{t_1}^{t} \omega^{\Delta}(s)\Delta s + \int_{t_1}^{t} \frac{\eta^{\Delta}(s)}{\eta^{\sigma}(s)}\omega^{\sigma}(s)\Delta s - \int_{t_1}^{t} \frac{\gamma M^{\gamma-1}\eta(s)}{a(s)(\eta^{\sigma}(s))^2}\left(\frac{\sigma(s)}{a(\sigma(s))}\right)^{\gamma-1}(\omega^{\sigma}(s))^2\Delta s. \end{aligned} \tag{2.94}$$

Thus

$$\int_{t_1}^{t}\left[\eta(s)p(s)(1 - r(\delta(s)))^{\gamma}(\alpha(s, t_*))^{\gamma} - \frac{a(s)(\sigma(s))^{1-\gamma}(\eta^{\Delta}(s))^2}{4\gamma M^{\gamma-1}(a(\sigma(s)))^{1-\gamma}\eta(s)}\right]\Delta s \leq \omega(t_1), \tag{2.95}$$

which contradicts condition (2.85). The proof is complete. □

Theorem 2.26. *Assume that* $\gamma \le 1$, $a^{\Delta}(t) \le 0$, $-1 < -r_0 \le r(t) \le 0$, *and* $\lim_{t\to\infty} r(t) = r > -1$. *If there exists a positive* Δ*-differentiable function* η *such that for all sufficiently large* t_*, *and for all constants* $M > 0$, *for* $t_1 \ge t_*$,

$$\limsup_{t\to\infty} \int_{t_1}^{t} \left[\eta(s)p(s)(\alpha(s,t_*))^{\gamma} - \frac{a(s)(\sigma(s))^{1-\gamma}\left(\eta^{\Delta}(s)\right)^2}{4\gamma M^{\gamma-1}(a(\sigma(s)))^{1-\gamma}\eta(s)} \right] \Delta s = \infty, \tag{2.96}$$

then every solution of (1.33) *either oscillates or tends to zero as* $t \to \infty$.

Proof. By Lemma 2.4 and (2.28), the proof is similar to that of the proof of Theorem 2.25, so we omit the details. □

3. Conclusions

In this paper, we consider the oscillation of second-order Emden-Fowler neutral delay dynamic equations (1.33). In some sense, our results extend and improve the results in [7, 32, 34, 35, 40, 41]. For example, Theorems 2.5, 2.11, 2.13, and 2.23 give some answers for the open problem posed by [34] since these results can be applied to (1.33) when $\gamma < 1$, Theorems 2.7, 2.12, 2.14, 2.16, 2.18, 2.20, 2.22, 2.24, and 2.26 correct an error in [35]. Theorem 2.15 includes the results of [7, Theorem 4.4], [32, Theorem 3.1], Theorem 2.11 includes the result of [32, Theorem 3.5], Theorem 2.11 and Corollary 2.6 include the result of [41, Theorem 2.1(a), $m = 2$], Corollary 2.8 includes result of [41, Theorem 2.2, $m = 2$], Theorem 2.13 does not require the conditions $a^{\Delta}(t) \ge 0$, so it improves the results of [40], and Theorems 2.17 and 2.21 improve the results in [34] since these results can be applied when $a^{\Delta}(t) \le 0$.

The main results in this paper require that $\int_{t_0}^{\infty} \Delta t/a(t) = \infty$; it would be interesting to find another method to study (1.33) when $\int_{t_0}^{\infty} \Delta t/a(t) < \infty$. Additional examples may also be given; due to the limited space, we leave this to the interested reader.

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Research Article

Oscillation Criteria for Second-Order Superlinear Neutral Differential Equations

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Some oscillation criteria are established for the second-order superlinear neutral differential equations $(r(t)|z'(t)|^{\alpha-1}z'(t))' + f(t, x(\sigma(t))) = 0$, $t \ge t_0$, where $z(t) = x(t) + p(t)x(\tau(t))$, $\tau(t) \ge t$, $\sigma(t) \ge t$, $p \in C([t_0,\infty),[0,p_0])$, and $\alpha \ge 1$. Our results are based on the cases $\int_{t_0}^{\infty} 1/r^{1/\alpha}(t)\mathrm{d}t = \infty$ or $\int_{t_0}^{\infty} 1/r^{1/\alpha}(t)\mathrm{d}t < \infty$. Two examples are also provided to illustrate these results.

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order superlinear differential equation

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)' + f(t, x(\sigma(t))) = 0, \quad t \ge t_0, \tag{1.1}$$

where $\alpha \ge 1$ is a constant, $z(t) = x(t) + p(t)x(\tau(t))$.

Throughout this paper, we will assume the following hypotheses:

(A_1) $r \in C^1([t_0,\infty),\mathbb{R})$,$r(t) > 0$ for $t \ge t_0$;

(A_2) $p \in C([t_0,\infty),[0,p_0])$, where p_0 is a constant;

(A_3) $\tau \in C^1([t_0,\infty),\mathbb{R})$, $\tau'(t) \ge \tau_0 > 0$, $\tau(t) \ge t$;

(A_4) $\sigma \in C([t_0,\infty),\mathbb{R})$, $\sigma(t) \ge t, \tau \circ \sigma = \sigma \circ \tau$;

(A_5) $f(t,u) \in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$, and there exists a function $q \in C([t_0,\infty),[0,\infty))$ such that

$$f(t,u)\operatorname{sign} u \geq q(t)|u|^{\alpha}, \quad \text{for } u \neq 0,\ t \geq t_0. \tag{1.2}$$

By a solution of (1.1), we mean a function $x \in C([T_x,\infty),\mathbb{R})$ for some $T_x \geq t_0$ which has the property that $r(t)|z'(t)|^{\alpha-1}z'(t) \in C^1([T_x,\infty),\mathbb{R})$ and satisfies (1.1) on $[T_x,\infty)$. We consider only those solutions x which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. As is customary, a solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[t_0,\infty)$, otherwise, it is called nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

We note that neutral differential equations find numerous applications in electric networks. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines which rise in high-speed computers where the lossless transmission lines are used to interconnect switching circuits; see [1].

In the last few years, there are many studies that have been made on the oscillation and asymptotic behavior of solutions of discrete and continuous equations; see, for example, [2–28]. Agarwal et al. [5], Chern et al. [6], Džurina and Stavroulakis [7], Kusano and Yoshida [8], Kusano and Naito [9], Mirzov [10], and Sun and Meng [11] observed some similar properties between

$$\left(r(t)\left|x'(t)\right|^{\alpha-1}x'(t)\right)' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0 \tag{1.3}$$

and the corresponding linear equations

$$\left(r(t)x'(t)\right)' + q(t)x(t) = 0. \tag{1.4}$$

Baculíková [12] established some new oscillation results for (1.3) when $\alpha = 1$. In 1989, Philos [13] obtained some Philos-type oscillation criteria for (1.4).

Recently, many results have been obtained on oscillation and nonoscillation of neutral differential equations, and we refer the reader to [14–35] and the references cited therein. Liu and Bai [16], Xu and Meng [17, 18], Dong [19], Baculíková and Lacková [20], and Jiang and Li [21] established some oscillation criteria for (1.3) with neutral term under the assumptions $\tau(t) \leq t$, $\sigma(t) \leq t$,

$$R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)}\mathrm{d}s \longrightarrow \infty \text{ as } t \longrightarrow \infty, \tag{1.5}$$

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\mathrm{d}t < \infty. \tag{1.6}$$

Saker and O'Regan [24] studied the oscillatory behavior of (1.1) when $0 \leq p(t) < 1$, $\tau(t) \leq t$ and $\sigma(t) > t$.

Han et al. [26] examined the oscillation of second-order nonlinear neutral differential equation

$$\left(r(t)\left[x(t) + p(t)x(\tau(t))\right]'\right)' + q(t)f(x(\sigma(t))) = 0, \quad t \geq t_0, \tag{1.7}$$

where $\tau(t) \le t$, $\sigma(t) \le t$, $\tau'(t) = \tau_0 > 0$, $0 \le p(t) \le p_0 < \infty$, and the authors obtained some oscillation criteria for (1.7).

However, there are few results regarding the oscillatory problem of (1.1) when $\tau(t) \ge t$ and $\sigma(t) \ge t$. Our aim in this paper is to establish some oscillation criteria for (1.1) under the case when $\tau(t) \ge t$ and $\sigma(t) \ge t$.

The paper is organized as follows. In Section 2, we will establish an inequality to prove our results. In Section 3, some oscillation criteria are obtained for (1.1). In Section 4, we give two examples to show the importance of the main results.

All functional inequalities considered in this paper are assumed to hold eventually, that is, they are satisfied for all t large enough.

2. Lemma

In this section, we give the following lemma, which we will use in the proofs of our main results.

Lemma 2.1. *Assume that $\alpha \ge 1$, $a, b \in \mathbb{R}$. If $a \ge 0$, $b \ge 0$, then*

$$a^\alpha + b^\alpha \ge \frac{1}{2^{\alpha-1}}(a+b)^\alpha \tag{2.1}$$

holds.

Proof. (i) Suppose that $a = 0$ or $b = 0$. Obviously, we have (2.1). (ii) Suppose that $a > 0$, $b > 0$. Define the function g by $g(u) = u^\alpha$, $u \in (0, \infty)$. Then $g''(u) = \alpha(\alpha-1)u^{\alpha-2} \ge 0$ for $u > 0$. Thus, g is a convex function. By the definition of convex function, for $\lambda = 1/2$, $a, b \in (0, \infty)$, we have

$$g\left(\frac{a+b}{2}\right) \le \frac{g(a)+g(b)}{2}, \tag{2.2}$$

that is,

$$a^\alpha + b^\alpha \ge \frac{1}{2^{\alpha-1}}(a+b)^\alpha. \tag{2.3}$$

This completes the proof. □

3. Main Results

In this section, we will establish some oscillation criteria for (1.1).

First, we establish two comparison theorems which enable us to reduce the problem of the oscillation of (1.1) to the research of the first-order differential inequalities.

Theorem 3.1. *Suppose that* (1.5) *holds. If the first-order neutral differential inequality*

$$\left[u(t)+\frac{(p_0)^{\alpha}}{\tau_0}u(\tau(t))\right]'+\frac{1}{2^{\alpha-1}}Q(t)(R(\sigma(t))-R(t_1))^{\alpha}u(\sigma(t))\leq 0 \tag{3.1}$$

has no positive solution for all sufficiently large t_1*, where* $Q(t)=\min\{q(t),q(\tau(t))\}$*, then every solution of* (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for all $t\geq t_1$. Then $z(t)>0$ for $t\geq t_1$. In view of (1.1), we obtain

$$\left(r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right)'\leq -q(t)x^{\alpha}(\sigma(t))\leq 0,\quad t\geq t_1. \tag{3.2}$$

Thus, $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function. Now we have two possible cases for $z'(t)$: (i) $z'(t)<0$ eventually, (ii) $z'(t)>0$ eventually.

Suppose that $z'(t)<0$ for $t\geq t_2\geq t_1$. Then, from (3.2), we get

$$r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\leq r(t_2)\left|z'(t_2)\right|^{\alpha-1}z'(t_2),\quad t\geq t_2, \tag{3.3}$$

which implies that

$$z(t)\leq z(t_2)-r^{1/\alpha}(t_2)\left|z'(t_2)\right|\int_{t_2}^{t}r^{-1/\alpha}(s)\mathrm{d}s. \tag{3.4}$$

Letting $t\to\infty$, by (1.5), we find $z(t)\to -\infty$, which is a contradiction. Thus

$$z'(t)>0 \tag{3.5}$$

for $t\geq t_2$.

By applying (1.1), for all sufficiently large t, we obtain

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)'+q(t)x^{\alpha}(\sigma(t))+(p_0)^{\alpha}q(\tau(t))x^{\alpha}(\sigma(\tau(t)))+\frac{(p_0)^{\alpha}}{\tau'(t)}\left(r(\tau(t))\left(z'(\tau(t))\right)^{\alpha}\right)'\leq 0. \tag{3.6}$$

Using inequality (2.1), (3.2), (3.5), $\tau\circ\sigma=\sigma\circ\tau$, and the definition of z, we conclude that

$$\left(r(t)\left(z'(t)\right)^{\alpha}\right)'+\frac{(p_0)^{\alpha}}{\tau_0}r(\tau(t))\left(z'(\tau(t))^{\alpha}\right)'+\frac{1}{2^{\alpha-1}}Q(t)z^{\alpha}(\sigma(t))\leq 0. \tag{3.7}$$

It follows from (3.2) and (3.5) that $u(t) = r(t)(z'(t))^{\alpha} > 0$ is decreasing and then

$$z(t) \geq \int_{t_2}^{t} \frac{\left(r(s)(z'(s))^{\alpha}\right)^{1/\alpha}}{r^{1/\alpha}(s)} \mathrm{d}s \geq u^{1/\alpha}(t) \int_{t_2}^{t} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s = u^{1/\alpha}(t)(R(t) - R(t_2)). \tag{3.8}$$

Thus, from (3.7) and the above inequality, we find

$$\left[u(t) + \frac{(p_0)^{\alpha}}{\tau_0} u(\tau(t))\right]' + \frac{1}{2^{\alpha-1}} Q(t)(R(\sigma(t)) - R(t_2))^{\alpha} u(\sigma(t)) \leq 0. \tag{3.9}$$

That is, inequality (3.1) has a positive solution u; this is a contradiction. The proof is complete. □

Theorem 3.2. *Suppose that* (1.5) *holds. If the first-order differential inequality*

$$\eta'(t) + \frac{\tau_0}{2^{\alpha-1}\left(\tau_0 + (p_0)^{\alpha}\right)} Q(t)(R(\sigma(t)) \ - R(t_1))^{\alpha} \eta(\sigma(t)) \leq 0 \tag{3.10}$$

has no positive solution for all sufficiently large t_1, where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain that $u(t) = r(t)(z'(t))^{\alpha}$ is decreasing, and it satisfies inequality (3.1). Set $\eta(t) = u(t) + (p_0)^{\alpha} u(\tau(t))/\tau_0$. From $\tau(t) \geq t$, we get

$$\eta(t) = u(t) + \frac{(p_0)^{\alpha}}{\tau_0} u(\tau(t)) \leq \left(1 + \frac{(p_0)^{\alpha}}{\tau_0}\right) u(t). \tag{3.11}$$

In view of the above inequality and (3.1), we see that

$$\eta'(t) + \frac{\tau_0}{2^{\alpha-1}\left(\tau_0 + (p_0)^{\alpha}\right)} Q(t)(R(\sigma(t)) - R(t_1))^{\alpha} \eta(\sigma(t)) \leq 0. \tag{3.12}$$

That is, inequality (3.10) has a positive solution η; this is a contradiction. The proof is complete. □

Next, using Riccati transformation technique, we obtain the following results.

Theorem 3.3. *Suppose that* (1.5) *holds. Moreover, assume that there exists $\rho \in C^1([t_0, \infty), (0, \infty))$ such that*

$$\limsup_{t \to \infty} \int_{t_0}^{t} \left[\frac{\rho(s) Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}} \left(1 + \frac{(p_0)^{\alpha}}{\tau_0}\right) \frac{r(s)\left(\rho'_{+}(s)\right)^{\alpha+1}}{(\rho(s))^{\alpha}}\right] \mathrm{d}s = \infty \tag{3.13}$$

holds, where Q is defined as in Theorem 3.1, $\rho'_+(t) = \max\{0, \rho'(t)\}$. *Then every solution of* (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2)–(3.7).

Define a Riccati substitution

$$\omega(t) = \rho(t)\frac{r(t)(z'(t))^{\alpha}}{(z(t))^{\alpha}}, \quad t \geq t_2. \tag{3.14}$$

Thus $\omega(t) > 0$ for $t \geq t_2$. Differentiating (3.14) we find that

$$\omega'(t) = \rho'(t)\frac{r(t)(z'(t))^{\alpha}}{(z(t))^{\alpha}} + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha}z^{\alpha-1}(t)z'(t)}{(z(t))^{2\alpha}}. \tag{3.15}$$

Hence, by (3.14) and (3.15), we see that

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t). \tag{3.16}$$

Similarly, we introduce another Riccati substitution

$$\upsilon(t) = \rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}}, \quad t \geq t_2. \tag{3.17}$$

Then $\upsilon(t) > 0$ for $t \geq t_2$. From (3.2), (3.5), and $\tau(t) \geq t$, we have

$$z'(t) \geq \left(\frac{r(\tau(t))}{r(t)}\right)^{1/\alpha} z'(\tau(t)). \tag{3.18}$$

Differentiating (3.17), we find

$$\upsilon'(t) = \rho'(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}z^{\alpha-1}(t)z'(t)}{(z(t))^{2\alpha}}. \tag{3.19}$$

Therefore, by (3.17), (3.18), and (3.19), we see that

$$\upsilon'(t) \leq \frac{\rho'(t)}{\rho(t)}\upsilon(t) + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t). \tag{3.20}$$

Thus, from (3.16) and (3.20), we have

$$\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}\upsilon'(t)\le\rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)'+\left((p_0)^\alpha/\tau_0\right)\left(r(\tau(t))(z'(\tau(t)))^\alpha\right)'}{(z(t))^\alpha}$$
$$+\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)+\frac{(p_0)^\alpha}{\tau_0}\frac{\rho'(t)}{\rho(t)}\upsilon(t)\tag{3.21}$$
$$-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t).$$

It follows from (3.5), (3.7), and $\sigma(t)\ge t$ that

$$\begin{aligned}\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}\upsilon'(t)\le{}&-\frac{1}{2^{\alpha-1}}\rho(t)Q(t)+\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)\\&+\frac{(p_0)^\alpha}{\tau_0}\frac{\rho'_+(t)}{\rho(t)}\upsilon(t)-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\upsilon^{(\alpha+1)/\alpha}(t).\end{aligned}\tag{3.22}$$

Integrating the above inequality from t_2 to t, we obtain

$$\begin{aligned}&\omega(t)-\omega(t_2)+\frac{(p_0)^\alpha}{\tau_0}\upsilon(t)-\frac{(p_0)^\alpha}{\tau_0}\upsilon(t_2)\\&\quad\le-\int_{t_2}^t\frac{1}{2^{\alpha-1}}\rho(s)Q(s)\mathrm{d}s+\int_{t_2}^t\left[\frac{\rho'_+(s)}{\rho(s)}\omega(s)-\frac{\alpha}{\rho^{1/\alpha}(s)r^{1/\alpha}(s)}\omega^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s\\&\qquad+\int_{t_2}^t\frac{(p_0)^\alpha}{\tau_0}\left[\frac{\rho'_+(s)}{\rho(s)}\upsilon(s)-\frac{\alpha}{\rho^{1/\alpha}(s)r^{1/\alpha}(s)}\upsilon^{(\alpha+1)/\alpha}(s)\right]\mathrm{d}s.\end{aligned}\tag{3.23}$$

Define

$$A:=\left[\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\right]^{\alpha/(\alpha+1)}\omega(t),\qquad B:=\left[\frac{\rho'_+(t)}{\rho(t)}\frac{\alpha}{\alpha+1}\left[\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\right]^{-\alpha/(\alpha+1)}\right]^\alpha.\tag{3.24}$$

Using inequality

$$\frac{\alpha+1}{\alpha}AB^{1/\alpha}-A^{(\alpha+1)/\alpha}\le\frac{1}{\alpha}B^{(\alpha+1)/\alpha},\quad\text{for } A\ge0,\ B\ge0\ \text{ are constants},\tag{3.25}$$

we have

$$\frac{\rho'_+(t)}{\rho(t)}\omega(t)-\frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}\omega^{(\alpha+1)/\alpha}(t)\le\frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\rho(t)^\alpha}.\tag{3.26}$$

Similarly, we obtain

$$\frac{\rho'_+(t)}{\rho(t)}v(t) - \frac{\alpha}{\rho^{1/\alpha}(t)r^{1/\alpha}(t)}v^{(\alpha+1)/\alpha}(t) \le \frac{1}{(\alpha+1)^{\alpha+1}}\frac{r(t)\left(\rho'_+(t)\right)^{\alpha+1}}{\rho(t)^{\alpha}}. \tag{3.27}$$

Thus, from (3.23), we get

$$\begin{aligned} &\omega(t) - \omega(t_2) + \frac{(p_0)^{\alpha}}{\tau_0}v(t) - \frac{(p_0)^{\alpha}}{\tau_0}v(t_2) \\ &\quad\le -\int_{t_2}^{t}\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\frac{r(s)(\rho'_+(s))^{\alpha+1}}{\rho(s)^{\alpha}}\right]\mathrm{d}s, \end{aligned} \tag{3.28}$$

which contradicts (3.13). This completes the proof. □

As an immediate consequence of Theorem 3.3 we get the following.

Corollary 3.4. *Let assumption* (3.13) *in Theorem 3.3 be replaced by*

$$\begin{aligned} &\limsup_{t\to\infty}\int_{t_0}^{t}\rho(s)Q(s)\mathrm{d}s = \infty, \\ &\limsup_{t\to\infty}\int_{t_0}^{t}\frac{r(s)\left(\rho'_+(s)\right)^{\alpha+1}}{\left(\rho(s)\right)^{\alpha}}\mathrm{d}s < \infty. \end{aligned} \tag{3.29}$$

Then every solution of (1.1) *oscillates.*

From Theorem 3.3 by choosing the function ρ, appropriately, we can obtain different sufficient conditions for oscillation of (1.1), and if we define a function ρ by $\rho(t) = 1$, and $\rho(t) = t$, we have the following oscillation results.

Corollary 3.5. *Suppose that* (1.5) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}Q(s)\mathrm{d}s = \infty, \tag{3.30}$$

where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

Corollary 3.6. *Suppose that* (1.5) *holds. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left[\frac{sQ(s)}{2^{\alpha-1}} - \frac{1}{(\alpha+1)^{\alpha+1}}\left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\frac{r(s)}{s^{\alpha}}\right]\mathrm{d}s = \infty, \tag{3.31}$$

where Q is defined as in Theorem 3.1, then every solution of (1.1) *oscillates.*

In the following theorem, we present a Philos-type oscillation criterion for (1.1). First, we introduce a class of functions $\mathfrak{R}$. Let

$$\mathbb{D}_0 = \{(t,s) : t > s \geq t_0\}, \qquad \mathbb{D} = \{(t,s) : t \geq s \geq t_0\}. \tag{3.32}$$

The function $H \in C(\mathbb{D}, \mathbb{R})$ is said to belong to the class $\mathfrak{R}$ (defined by $H \in \mathfrak{R}$, for short) if

(i) $H(t,t) = 0$, for $t \geq t_0$, $H(t,s) > 0$, for $(t,s) \in \mathbb{D}_0$;

(ii) H has a continuous and nonpositive partial derivative $\partial H(t,s)/\partial s$ on D_0 with respect to s.

We assume that $\varsigma(t)$ and $\rho(t)$ for $t \geq t_0$ are given continuous functions such that $\rho(t) > 0$ and differentiable and define

$$\begin{gathered}\theta(t) = \frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}, \psi(t) = \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\}, \\ \phi(t,s) = \frac{r(s)\rho(s)}{(\alpha+1)^{\alpha+1}}\left(\theta(s) + \frac{\partial H(t,s)/\partial s}{H(t,s)}\right)^{\alpha+1}.\end{gathered} \tag{3.33}$$

Now, we give the following result.

Theorem 3.7. *Suppose that* (1.5) *holds and α is a quotient of odd positive integers. Moreover, let $H \in \mathfrak{R}$ be such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \left(1 + \frac{(p_0)^\alpha}{\tau_0}\right)(\psi(s) + \phi(t,s))\right] \mathrm{d}s = \infty \tag{3.34}$$

holds, where Q is defined as in Theorem 3.1. Then every solution of (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2)–(3.7). Define the Riccati substitution ω by

$$\omega(t) = \rho(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right], \quad t \geq t_2 \geq t_1. \tag{3.35}$$

Then, we have

$$\begin{aligned}\omega'(t) &= \rho'(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right] + \rho(t)\left[\frac{r(t)(z'(t))^\alpha}{(z(t))^\alpha} + r(t)\varsigma(t)\right]' \\ &= \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)'}{(z(t))^\alpha} - \alpha\rho(t)\frac{r(t)(z'(t))^{\alpha+1}}{(z(t))^{\alpha+1}}.\end{aligned} \tag{3.36}$$

Using (3.35), we get

$$\omega'(t) = \frac{\rho'(t)}{\rho(t)}\omega(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha\rho(t)}{r^{1/\alpha}(t)}\left[\frac{\omega(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha}. \tag{3.37}$$

Let

$$A = \frac{\omega(t)}{\rho(t)}, \qquad B = r(t)\varsigma(t). \tag{3.38}$$

By applying the inequality (see [21, 24])

$$A^{(1+\alpha)/\alpha} - (A-B)^{1+\alpha/\alpha} \leq B^{1/\alpha}\left[\left(1+\frac{1}{\alpha}\right)A - \frac{1}{\alpha}B\right], \quad \text{for } \alpha = \frac{\text{odd}}{\text{odd}} \geq 1, \tag{3.39}$$

we see that

$$\left[\frac{\omega(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha} \geq \frac{\omega^{(1+\alpha)/\alpha}(t)}{\rho^{(1+\alpha)/\alpha}(t)} + \frac{1}{\alpha}(r(t)\varsigma(t))^{(1+\alpha)/\alpha} - \frac{\alpha+1}{\alpha}\frac{(r(t)\varsigma(t))^{1/\alpha}}{\rho(t)}\omega(t). \tag{3.40}$$

Substituting (3.40) into (3.37), we have

$$\begin{aligned}\omega'(t) \leq{}& \left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}\right]\omega(t) + \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\} \\ &+ \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t).\end{aligned} \tag{3.41}$$

That is,

$$\omega'(t) \leq \theta(t)\omega(t) + \psi(t) + \rho(t)\frac{\left(r(t)(z'(t))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t). \tag{3.42}$$

Next, define another Riccati transformation u by

$$u(t) = \rho(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right], \quad t \geq t_2 \geq t_1. \tag{3.43}$$

Then, we have

$$u'(t) = \rho'(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right] + \rho(t)\left[\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}}{(z(t))^{\alpha}} + r(t)\varsigma(t)\right]'$$
$$= \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{r(\tau(t))(z'(\tau(t)))^{\alpha}z'(t)}{(z(t))^{\alpha+1}}. \tag{3.44}$$

From (3.2), (3.5), and $\tau(t) \geq t$, we have that (3.18) holds. Hence, we obtain

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)}{(z(t))^{\alpha}} - \alpha\rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)^{(1+\alpha)/\alpha}}{r^{1/\alpha}(t)(z(t))^{\alpha+1}}. \tag{3.45}$$

Using (3.43), we get

$$u'(t) \leq \frac{\rho'(t)}{\rho(t)}u(t) + \rho(t)[r(t)\varsigma(t)]' + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha\rho(t)}{r^{1/\alpha}(t)}\left[\frac{u(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha}. \tag{3.46}$$

Let

$$A = \frac{u(t)}{\rho(t)}, \qquad B = r(t)\varsigma(t). \tag{3.47}$$

By applying the inequality (3.39), we see that

$$\left[\frac{u(t)}{\rho(t)} - r(t)\varsigma(t)\right]^{(1+\alpha)/\alpha} \geq \frac{u^{(1+\alpha)/\alpha}(t)}{\rho^{(1+\alpha)/\alpha}(t)} + \frac{1}{\alpha}(r(t)\varsigma(t))^{(1+\alpha)/\alpha} - \frac{\alpha+1}{\alpha}\frac{(r(t)\varsigma(t))^{1/\alpha}}{\rho(t)}u(t). \tag{3.48}$$

Substituting (3.48) into (3.46), we have

$$\begin{aligned} u'(t) \leq{}& \left[\frac{\rho'(t)}{\rho(t)} + (\alpha+1)(\varsigma(t))^{1/\alpha}\right]u(t) + \rho(t)\left\{[r(t)\varsigma(t)]' - r(t)(\varsigma(t))^{(1+\alpha)/\alpha}\right\} \\ &+ \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t). \end{aligned} \tag{3.49}$$

That is,

$$u'(t) \leq \theta(t)u(t) + \psi(t) + \rho(t)\frac{\left(r(\tau(t))(z'(\tau(t)))^{\alpha}\right)'}{(z(t))^{\alpha}} - \frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t). \tag{3.50}$$

By (3.42) and (3.50), we find

$$\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}u'(t)\le\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(t)+\rho(t)\frac{\left(r(t)(z'(t))^\alpha\right)'+\left((p_0)^\alpha/\tau_0\right)\left(r(\tau(t))(z'(\tau(t)))^\alpha\right)'}{(z(t))^\alpha}$$
$$+\theta(t)\omega(t)-\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t)+\frac{(p_0)^\alpha}{\tau_0}\theta(t)u(t)$$
$$-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t). \tag{3.51}$$

In view of the above inequality, (3.5), (3.7), and $\sigma(t)\ge t$, we get

$$\omega'(t)+\frac{(p_0)^\alpha}{\tau_0}u'(t)\le\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(t)-\frac{\rho(t)Q(t)}{2^{\alpha-1}}+\theta(t)\omega(t)-\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}\omega^{(1+\alpha)/\alpha}(t)$$
$$+\frac{(p_0)^\alpha}{\tau_0}\theta(t)u(t)-\frac{(p_0)^\alpha}{\tau_0}\frac{\alpha}{r^{1/\alpha}(t)\rho^{1/\alpha}(t)}u^{(1+\alpha)/\alpha}(t), \tag{3.52}$$

which follows that

$$\begin{aligned}&\int_{t_2}^{t}H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}}-\left(1+\frac{(p_0)^\alpha}{\tau_0}\right)\varphi(s)\right]\mathrm{d}s\\&\quad\le-\int_{t_2}^{t}H(t,s)\omega'(s)\mathrm{d}s+\int_{t_2}^{t}H(t,s)\theta(s)\omega(s)\mathrm{d}s\\&\quad-\int_{t_2}^{t}H(t,s)\frac{\alpha\omega^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s-\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)u'(s)\mathrm{d}s\\&\quad+\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)\theta(s)u(s)\mathrm{d}s-\frac{(p_0)^\alpha}{\tau_0}\int_{t_2}^{t}H(t,s)\frac{\alpha u^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s.\end{aligned} \tag{3.53}$$

Using the integration by parts formula and $H(t,t)=0$, we have

$$\begin{aligned}\int_{t_2}^{t}H(t,s)\omega'(s)\mathrm{d}s&=-H(t,t_2)\omega(t_2)-\int_{t_2}^{t}\frac{\partial H(t,s)}{\partial s}\omega(s)\mathrm{d}s,\\\int_{t_2}^{t}H(t,s)u'(s)\mathrm{d}s&=-H(t,t_2)u(t_2)-\int_{t_2}^{t}\frac{\partial H(t,s)}{\partial s}u(s)\mathrm{d}s.\end{aligned} \tag{3.54}$$

So, by (3.53), we obtain

$$\begin{aligned}
&\int_{t_2}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)\varphi(s)\right]\mathrm{d}s \\
&\quad\le H(t,t_2)\omega(t_2) + \frac{(p_0)^{\alpha}}{\tau_0}H(t,t_2)u(t_2) \\
&\qquad+\int_{t_2}^{t} H(t,s)\left[\theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}\right]\omega(s)\mathrm{d}s - \int_{t_2}^{t} H(t,s)\frac{\alpha\omega^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s \\
&\qquad+\frac{(p_0)^{\alpha}}{\tau_0}\int_{t_2}^{t} H(t,s)\left[\theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}\right]u(s)\mathrm{d}s - \frac{(p_0)^{\alpha}}{\tau_0}\int_{t_2}^{t} H(t,s)\frac{\alpha u^{(1+\alpha)/\alpha}(s)}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}\mathrm{d}s.
\end{aligned} \tag{3.55}$$

Using the inequality

$$By - Ay^{(\alpha+1)/\alpha} \le \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}\frac{B^{\alpha+1}}{A^{\alpha}}, \tag{3.56}$$

where

$$A = \frac{\alpha}{r^{1/\alpha}(s)\rho^{1/\alpha}(s)}, \qquad B = \theta(s)+\frac{\partial H(t,s)/\partial s}{H(t,s)}, \tag{3.57}$$

we have

$$\int_{t_2}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)(\varphi(s)+\phi(t,s))\right]\mathrm{d}s \le H(t,t_2)\omega(t_2) + \frac{(p_0)^{\alpha}}{\tau_0}H(t,t_2)u(t_2) \tag{3.58}$$

due to (3.55), which yields that

$$\frac{1}{H(t,t_2)}\int_{t_2}^{t} H(t,s)\left[\frac{\rho(s)Q(s)}{2^{\alpha-1}} - \left(1+\frac{(p_0)^{\alpha}}{\tau_0}\right)(\varphi(s)+\phi(t,s))\right]\mathrm{d}s \le \omega(t_2) + \frac{(p_0)^{\alpha}}{\tau_0}u(t_2), \tag{3.59}$$

which contradicts (3.34). The proof is complete. □

From Theorem 3.7, we can obtain different oscillation conditions for all solutions of (1.1) with different choices of H; the details are left to the reader.

Theorem 3.8. *Assume that* (1.6) *and* (3.30) *hold. Furthermore, assume that* $0 \le p(t) \le p_1 < 1$. *If*

$$\int_{t_0}^{\infty}\left[\frac{1}{r(s)}\int_{t_0}^{s} q(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}s = \infty, \tag{3.60}$$

then every solution x of (1.1) *oscillates or* $\lim_{t\to\infty}x(t) = 0$.

Proof. Let x be a nonoscillatory solution of (1.1). Without loss of generality, we assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for all $t \geq t_1$. Then $z(t) > 0$ for $t \geq t_1$. Proceeding as in the proof of Theorem 3.1, we obtain (3.2). Thus $r(t)|z'(t)|^{\alpha-1}z'(t)$ is decreasing function, and there exists a $t_2 \geq t_1$ such that $z'(t) > 0$, $t \geq t_2$ or $z'(t) < 0$, $t \geq t_2$.

Case 1. Assume that $z'(t) > 0$, for $t \geq t_2$. Proceeding as in the proof of Theorem 3.3 and setting $\rho(t) = t$, we can obtain a contradiction with (3.31).

Case 2. Assume that $z'(t) < 0$, for $t \geq t_2$. Then there exists a finite limit

$$\lim_{t\to\infty} z(t) = l, \tag{3.61}$$

where $l \geq 0$. Next, we claim that $l = 0$. If not, then for any $\epsilon > 0$, we have $l < z(t) < l + \epsilon$, eventually. Take $0 < \epsilon < l(1-p_1)/p_1$. We calculate

$$x(t) = z(t) - p(t)x(\tau(t)) > l - p_1 z(\tau(t)) > l - p_1(l+\epsilon) = m(l+\epsilon) > mz(t), \tag{3.62}$$

where

$$m = \frac{l}{l+\epsilon} - p_1 = \frac{l(1-p_1) - \epsilon p_1}{l+\epsilon} > 0. \tag{3.63}$$

From (3.2) and (3.62), we have

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' \geq q(t)x^{\alpha}(\sigma(t)) \geq (ml)^{\alpha}q(t). \tag{3.64}$$

Integrating the above inequality from t_2 to t, we get

$$r(t)\left(-z'(t)\right)^{\alpha} - r(t_2)\left(-z'(t_2)\right)^{\alpha} \geq (ml)^{\alpha}\int_{t_2}^{t} q(s)\mathrm{d}s, \tag{3.65}$$

which implies

$$z'(t) \leq -ml\left[\frac{1}{r(t)}\int_{t_2}^{t} q(s)\mathrm{d}s\right]^{1/\alpha}. \tag{3.66}$$

Integrating the above inequality from t_2 to t, we have

$$z(t) \leq z(t_2) - ml\int_{t_2}^{t}\left[\frac{1}{r(s)}\int_{t_2}^{s} q(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}s, \tag{3.67}$$

which yields $z(t) \to -\infty$; this is a contradiction. Hence, $\lim_{t\to\infty} z(t) = 0$. Note that $0 < x(t) \leq z(t)$. Then, $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

4. Examples

In this section, we will give two examples to illustrate the main results.

Example 4.1. Consider the following linear neutral equation:

$$(x(t) + 2x(t + (2n - 1)\pi))'' + x(t + (2m - 1)\pi) = 0, \quad \text{for } t \geq t_0, \tag{4.1}$$

where n and m are positive integers.

Let

$$r(t) = 1, \quad p(t) = 2, \quad \tau(t) = t + (2n - 1)\pi, \quad q(t) = 1, \quad \sigma(t) = t + (2m - 1)\pi. \tag{4.2}$$

Hence, $Q(t) = 1$. Obviously, all the conditions of Corollary 3.5 hold. Thus by Corollary 3.5, every solution of (4.1) is oscillatory. It is easy to verify that $x(t) = \sin t$ is a solution of (4.1).

Example 4.2. Consider the following linear neutral equation:

$$\left(e^{2t}\left(x(t) + \frac{1}{2}x(t + 3)\right)'\right)' + \left(e^{2t+1} + \frac{1}{2}e^{2t-2}\right)x(t + 1) = 0, \quad \text{for } t \geq t_0, \tag{4.3}$$

where n and m are positive integers.

Let

$$r(t) = e^{2t}, \qquad p(t) = \frac{1}{2}, \qquad q(t) = e^{2t+1} + e^{2t-2}/2, \quad \alpha = 1. \tag{4.4}$$

Clearly, all the conditions of Theorem 3.8 hold. Thus by Theorem 3.8, every solution of (4.3) is either oscillatory or $\lim_{t\to\infty} x(t) = 0$. It is easy to verify that $x(t) = e^{-t}$ is a solution of (4.3).

Remark 4.3. Recent results cannot be applied to (4.1) and (4.3) since $\tau(t) \geq t$ and $\sigma(t) \geq t$.

Remark 4.4. Using the method given in this paper, we can get other oscillation criteria for (1.1); the details are left to the reader.

Remark 4.5. It would be interesting to find another method to study (1.1) when $\tau \circ \sigma \not\equiv \sigma \circ \tau$.

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Research Article

Oscillation of Second-Order Neutral Functional Differential Equations with Mixed Nonlinearities

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We study the following second-order neutral functional differential equation with mixed nonlinearities $(r(t)|(u(t)+p(t)u(t-\sigma))'|^{\alpha-1}(u(t)+p(t)u(t-\sigma))')' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) + q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0$, where $\gamma > \alpha > \beta > 0$, $\int_{t_0}^{\infty}(1/r^{1/\alpha}(t))\mathrm{d}t < \infty$. Oscillation results for the equation are established which improve the results obtained by Sun and Meng (2006), Xu and Meng (2006), Sun and Meng (2009), and Han et al. (2010).

1. Introduction

This paper is concerned with the oscillatory behavior of the second-order neutral functional differential equation with mixed nonlinearities

$$\begin{aligned}\left(r(t)\left|(u(t)+p(t)u(t-\sigma))'\right|^{\alpha-1}(u(t)+p(t)u(t-\sigma))'\right)' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t))\\ +q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0, \quad t \geq t_0,\end{aligned} \tag{1.1}$$

where $\gamma > \alpha > \beta > 0$ are constants, $r \in C^1([t_0,\infty),(0,\infty))$, $p \in C([t_0,\infty),[0,1))$, $q_i \in C([t_0,\infty),\mathbb{R})$, $i = 0,1,2$, are nonnegative, $\sigma \geq 0$ is a constant. Here, we assume that there exists $\tau \in C^1([t_0,\infty),\mathbb{R})$ such that $\tau(t) \leq \tau_i(t)$, $\tau(t) \leq t$, $\lim_{t\to\infty}\tau(t) = \infty$, and $\tau'(t) > 0$ for $t \geq t_0$.

One of our motivations for studying (1.1) is the application of this type of equations in real word life problems. For instance, neutral delay equations appear in modeling of networks containing lossless transmission lines, in the study of vibrating masses attached to an elastic bar; see the Euler equation in some variational problems, in the theory of automatic control and in neuromechanical systems in which inertia plays an important role. We refer the reader to Hale [1] and Driver [2], and references cited therein.

Recently, there has been much research activity concerning the oscillation of second-order differential equations [3–8] and neutral delay differential equations [9–20]. For the particular case when $p(t) = 0$, (1.1) reduces to the following equation:

$$\left(r(t)|u(t)|^{\alpha-1}u(t)\right)' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) + q_1(t)|u(\tau_1(t))|^{\beta-1}u(\tau_1(t)) + q_2(t)|u(\tau_2(t))|^{\gamma-1}u(\tau_2(t)) = 0, \quad t \ge t_0. \tag{1.2}$$

Sun and Meng [6] established some oscillation criteria for (1.2), under the condition

$$\int_{t_0}^{\infty} \frac{1}{r^{1/\alpha}(t)}\mathrm{d}t < \infty, \tag{1.3}$$

they only obtained the sufficient condition [6, Theorem 5], which guarantees that every solution u of (1.2) oscillates or tends to zero.

Sun and Meng [7] considered the oscillation of second-order nonlinear delay differential equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + q_0(t)|u(\tau_0(t))|^{\alpha-1}u(\tau_0(t)) = 0, \quad t \ge t_0 \tag{1.4}$$

and obtained some results for oscillation of (1.4), for example, under the case (1.3), they obtained some results which guarantee that every solution u of (1.4) oscillates or tends to zero, see [7, Theorem 2.2].

Xu and Meng [10] discussed the oscillation of the second-order neutral delay differential equation

$$\left(r(t)\left|(u(t)+p(t)u(t-\tau))'\right|^{\alpha-1}(u(t)+p(t)u(t-\tau))'\right)' + q(t)f(u(\sigma(t))) = 0, \quad t \ge t_0 \tag{1.5}$$

and established the sufficient condition [10, Theorem 2.3], which guarantees that every solution u of (1.5) oscillates or tends to zero.

Han et al. [11] examined the oscillation of second-order neutral delay differential equation

$$\left(r(t)\psi(u(t))\left|(u(t)+p(t)u(t-\tau))'\right|^{\alpha-1}(u(t)+p(t)u(t-\tau))'\right)' + q(t)f(u(\sigma(t))) = 0, \quad t \ge t_0 \tag{1.6}$$

and established some sufficient conditions for oscillation of (1.6) under the conditions (1.3) and

$$\sigma(t) \leq t - \tau. \tag{1.7}$$

The condition (1.7) can be restrictive condition, since the results cannot be applied on the equation

$$\left(e^{2t}\left(u(t) + \frac{1}{2}u(t-2)\right)'\right)' + \lambda\left(e^{2t} + \frac{1}{2}e^{2t+2}\right)u(t-1) = 0, \quad t \geq t_0. \tag{1.8}$$

The aim of this paper is to derive some sufficient conditions for the oscillation of solutions of (1.1). The paper is organized as follows. In Section 2, we establish some oscillation criteria for (1.1) under the assumption (1.3). In Section 3, we will give three examples to illustrate the main results. In Section 4, we give some conclusions for this paper.

2. Main Results

In this section, we give some new oscillation criteria for (1.1).

Below, for the sake of convenience, we denote

$$z(t) := u(t) + p(t)u(t-\sigma), \qquad R(t) := \int_{t_0}^{t} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s,$$

$$\xi(t) := r^{1/\alpha}(\tau(t)) \int_{t_1}^{t} \left(\frac{1}{r(\tau(s))}\right)^{1/\alpha} \tau'(s) \mathrm{d}s,$$

$$Q_0(t) := (1 - p(\tau_0(t)))^{\alpha} q_0(t), \qquad Q_1(t) := (1 - p(\tau_1(t)))^{\beta} q_1(t),$$

$$Q_2(t) := (1 - p(\tau_2(t)))^{\gamma} q_2(t),$$

$$\zeta_0(t) := q_0(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\alpha}, \qquad \zeta_1(t) := q_1(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\beta},$$

$$\zeta_2(t) := q_2(t)\left(\frac{1}{1 + p(\rho(t))}\right)^{\gamma},$$

$$h_0(t) := q_0(t)\left(\frac{1}{1 + p(t)}\right)^{\alpha}, \qquad h_1(t) := q_1(t)\left(\frac{1}{1 + p(t)}\right)^{\beta},$$

$$h_2(t) := q_2(t)\left(\frac{1}{1 + p(t)}\right)^{\gamma},$$

$$\delta(t) := \int_{\rho(t)}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s, \qquad \pi(t) := \int_{t}^{\infty} \frac{1}{r^{1/\alpha}(s)} \mathrm{d}s, \qquad k_1 := \frac{\gamma-\beta}{\gamma-\alpha}, \qquad k_2 := \frac{\gamma-\beta}{\alpha-\beta},$$

$$\varphi(t) := q_0(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\alpha} + q_1(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\beta} + q_2(t)\left(\frac{\delta(t)}{1+p(\rho(t))}\right)^{\gamma}. \tag{2.1}$$

Theorem 2.1. *Assume that* (1.3) *holds, $p'(t) \geq 0$, and there exists $\rho \in C^1([t_0,\infty),\mathbb{R})$, such that $\rho(t) \geq t$, $\rho'(t) > 0$, $\tau_i(t) \leq \rho(t) - \sigma, i = 0,1,2$. If for all sufficiently large t_1,*

$$\int^{\infty} \left\{ R^{\alpha}(\tau(t))\left[Q_0(t) + [k_1 Q_1(t)]^{1/k_1}[k_2 Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)} \right\} \mathrm{d}t = \infty, \tag{2.2}$$

$$\int^{\infty} \left\{ \left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1} \frac{\rho'(t)}{\delta(t) r^{1/\alpha}(\rho(t))} \right\} \mathrm{d}t = \infty, \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that

$$z(t) > 0, \quad z'(t) > 0, \qquad \left[r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right]' \leq 0, \tag{2.4}$$

or

$$z(t) > 0, \quad z'(t) < 0, \qquad \left[r(t)\left|z'(t)\right|^{\alpha-1}z'(t)\right]' \leq 0. \tag{2.5}$$

If (2.4) holds, we have

$$r(t)\left(z'(t)\right)^{\alpha} \leq r(\tau(t))\left(z'(\tau(t))\right)^{\alpha}, \qquad t \geq t_1. \tag{2.6}$$

From the definition of z, we obtain

$$u(t) = z(t) - p(t)u(t-\sigma) \geq z(t) - p(t)z(t-\sigma) \geq \left(1-p(t)\right)z(t). \tag{2.7}$$

Define

$$\omega(t) = R^{\alpha}(\tau(t))\frac{r(t)(z'(t))^{\alpha}}{(z(\tau(t)))^{\alpha}}, \qquad t \geq t_1. \tag{2.8}$$

Then, $\omega(t) > 0$ for $t \geq t_1$. Noting that $z'(t) > 0$, we get $z(\tau_i(t)) \geq z(\tau(t))$ for $i = 0, 1, 2$. Thus, from (1.1), (2.7), and (2.8), it follows that

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(t)(z'(t))^\alpha}{(z(\tau(t)))^\alpha} - R^\alpha(\tau(t))(1-p(\tau_0(t)))^\alpha q_0(t)\\ &- R^\alpha(\tau(t))\Big[(1-p(\tau_1(t)))^\beta q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^\gamma q_2(t)z^{\gamma-\alpha}(\tau(t))\Big]\\ &- \alpha R^\alpha(\tau(t))\frac{r(t)(z'(t))^\alpha}{(z(\tau(t)))^{\alpha+1}}z'(\tau(t))\tau'(t).\end{aligned} \tag{2.9}$$

By (2.4), (2.9), and $\tau'(t) > 0$, we get

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(t)(z'(t))^\alpha}{(z(\tau(t)))^\alpha} - R^\alpha(\tau(t))(1-p(\tau_0(t)))^\alpha q_0(t)\\ &- R^\alpha(\tau(t))\Big[(1-p(\tau_1(t)))^\beta q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^\gamma q_2(t)z^{\gamma-\alpha}(\tau(t))\Big].\end{aligned} \tag{2.10}$$

In view of (2.4), (2.6), and (2.10), we have

$$\begin{aligned}\omega'(t) \leq{}& \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))}{r^{1/\alpha}(\tau(t))}\frac{r(\tau(t))(z'(\tau(t)))^\alpha}{(z(\tau(t)))^\alpha} - R^\alpha(\tau(t))(1-p(\tau_0(t)))^\alpha q_0(t)\\ &- R^\alpha(\tau(t))\Big[(1-p(\tau_1(t)))^\beta q_1(t)z^{\beta-\alpha}(\tau(t)) + (1-p(\tau_2(t)))^\gamma q_2(t)z^{\gamma-\alpha}(\tau(t))\Big].\end{aligned} \tag{2.11}$$

By (2.4), we obtain

$$\begin{aligned}z(\tau(t)) &= z(\tau(t_1)) + \int_{t_1}^t z'(\tau(s))\tau'(s)\mathrm{d}s\\ &= z(\tau(t_1)) + \int_{t_1}^t \left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\left[r(\tau(s))(z'(\tau(s)))^\alpha\right]^{1/\alpha}\tau'(s)\mathrm{d}s\\ &\geq r^{1/\alpha}(\tau(t))z'(\tau(t))\int_{t_1}^t \left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s,\end{aligned} \tag{2.12}$$

that is,

$$z(\tau(t)) \geq \xi(t)z'(\tau(t)). \tag{2.13}$$

Set

$$a := \left[k_1Q_1(t)z^{\beta-\alpha}(\tau(t))\right]^{1/k_1}, \quad b := \left[k_2Q_2(t)z^{\gamma-\alpha}(\tau(t))\right]^{1/k_2}, \quad p := k_1, \quad q := k_2. \tag{2.14}$$

Using Young's inequality

$$|ab| \leq \frac{1}{p}|a|^p + \frac{1}{q}|b|^q, \quad a, b \in \mathbb{R}, \ p > 1, \ q > 1, \ \frac{1}{p} + \frac{1}{q} = 1, \tag{2.15}$$

we have

$$Q_1(t) z^{\beta-\alpha}(\tau(t)) + Q_2(t) z^{\gamma-\alpha}(\tau(t)) \geq [k_1 Q_1(t)]^{1/k_1} [k_2 Q_2(t)]^{1/k_2}. \tag{2.16}$$

Hence, by (2.11), (2.13), and (2.16), we obtain

$$\omega'(t) \leq \frac{\alpha \tau'(t) R^{\alpha-1}(\tau(t)) r^{1-1/\alpha}(\tau(t))}{\xi^\alpha(t)} - R^\alpha(\tau(t)) \Big[Q_0(t) + [k_1 Q_1(t)]^{1/k_1} [k_2 Q_2(t)]^{1/k_2} \Big]. \tag{2.17}$$

Integrating (2.17) from t_1 to t, we get

$$0 < \omega(t) \leq \omega(t_1), \tag{2.18}$$

$$-\int_{t_1}^{t} \left\{ R^\alpha(\tau(s)) \Big[Q_0(s) + [k_1 Q_1(s)]^{1/k_1} [k_2 Q_2(s)]^{1/k_2} \Big] - \frac{\alpha \tau'(s) R^{\alpha-1}(\tau(s)) r^{1-1/\alpha}(\tau(s))}{\xi^\alpha(s)} \right\} ds. \tag{2.19}$$

Letting $t \to \infty$ in (2.19), we get a contradiction to (2.2). If (2.5) holds, we define the function υ by

$$\upsilon(t) = \frac{r(t)(-z'(t))^{\alpha-1} z'(t)}{z^\alpha(\rho(t))}, \quad t \geq t_1. \tag{2.20}$$

Then, $\upsilon(t) < 0$ for $t \geq t_1$. It follows from $[r(t)|z'(t)|^{\alpha-1} z'(t)]' \leq 0$ that $r(t)|z'(t)|^{\alpha-1} z'(t)$ is nonincreasing. Thus, we have

$$r^{1/\alpha}(s) z'(s) \leq r^{1/\alpha}(t) z'(t), \quad s \geq t. \tag{2.21}$$

Dividing (2.21) by $r^{1/\alpha}(s)$ and integrating it from $\rho(t)$ to l, we obtain

$$z(l) \leq z(\rho(t)) + r^{1/\alpha}(t) z'(t) \int_{\rho(t)}^{l} \frac{ds}{r^{1/\alpha}(s)}, \quad l \geq \rho(t). \tag{2.22}$$

Letting $l \to \infty$ in the above inequality, we obtain

$$0 \leq z(\rho(t)) + r^{1/\alpha}(t) z'(t) \delta(t), \quad t \geq t_1, \tag{2.23}$$

that is,

$$r^{1/\alpha}(t)\delta(t)\frac{z'(t)}{z(\rho(t))} \geq -1, \quad t \geq t_1. \tag{2.24}$$

Hence, by (2.20), we have

$$-1 \leq v(t)\delta^{\alpha}(t) \leq 0, \quad t \geq t_1. \tag{2.25}$$

Differentiating (2.20), we get

$$v'(t) = \frac{\left(r(t)(-z'(t))^{\alpha-1}z'(t)\right)' z^{\alpha}(\rho(t)) - \alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(\rho(t))z'(\rho(t))\rho'(t)}{z^{2\alpha}(\rho(t))}, \tag{2.26}$$

by the above equality and (1.1), we obtain

$$\begin{aligned} v'(t) = {} & -q_0(t)\frac{u^{\alpha}(\tau_0(t))}{z^{\alpha}(\rho(t))} - q_1(t)\frac{u^{\beta}(\tau_1(t))}{z^{\alpha}(\rho(t))} - q_2(t)\frac{u^{\gamma}(\tau_2(t))}{z^{\alpha}(\rho(t))} \\ & - \frac{\alpha r(t)(-z'(t))^{\alpha-1}z'(t)z^{\alpha-1}(\rho(t))z'(\rho(t))\rho'(t)}{z^{2\alpha}(\rho(t))}. \end{aligned} \tag{2.27}$$

Noticing that $p'(t) \geq 0$, from [10, Theorem 2.3], we see that $u'(t) \leq 0$ for $t \geq t_1$, so by $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0, 1, 2$, we have

$$\begin{aligned} \frac{u^{\alpha}(\tau_0(t))}{z^{\alpha}(\rho(t))} &= \left(\frac{u(\tau_0(t))}{u(\rho(t)) + p(\rho(t))u(\rho(t)-\sigma)}\right)^{\alpha} \\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_0(t))) + p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_0(t)))}\right)^{\alpha} \\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\alpha}, \\ \frac{u^{\beta}(\tau_1(t))}{z^{\alpha}(\rho(t))} &= \left(\frac{u(\tau_1(t))}{u(\rho(t)) + p(\rho(t))u(\rho(t)-\sigma)}\right)^{\beta} z^{\beta-\alpha}(\rho(t)) \\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_1(t))) + p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_1(t)))}\right)^{\beta} z^{\beta-\alpha}(\rho(t)) \\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\beta} z^{\beta-\alpha}(\rho(t)), \end{aligned}$$

$$\begin{aligned}\left(u^{\gamma}(\tau_2(t))/z^{\alpha}(\rho(t))\right) &= \left(\frac{u(\tau_2(t))}{u(\rho(t))+p(\rho(t))u(\rho(t)-\sigma)}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t)) \\ &= \left(\frac{1}{(u(\rho(t))/u(\tau_2(t)))+p(\rho(t))(u(\rho(t)-\sigma)/u(\tau_2(t)))}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t)) \\ &\geq \left(\frac{1}{1+p(\rho(t))}\right)^{\gamma} z^{\gamma-\alpha}(\rho(t)).\end{aligned} \tag{2.28}$$

On the other hand, from $(r(t)(-z'(t))^{\alpha-1}z'(t))' \leq 0$, $\rho(t) \geq t$, we obtain

$$z'(\rho(t)) \leq \frac{r^{1/\alpha}(t)}{r^{1/\alpha}(\rho(t))} z'(t). \tag{2.29}$$

Thus, by (2.20) and (2.27), we get

$$v'(t) \leq -\left[\zeta_0(t)+\zeta_1(t)z^{\beta-\alpha}(\rho(t))+\zeta_2(t)z^{\gamma-\alpha}(\rho(t))\right] - \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha}. \tag{2.30}$$

Set

$$a := \left[k_1\zeta_1(t)z^{\beta-\alpha}(\rho(t))\right]^{1/k_1}, \quad b := \left[k_2\zeta_2(t)z^{\gamma-\alpha}(\rho(t))\right]^{1/k_2},\ p := k_1,\ q := k_2. \tag{2.31}$$

Using Young's inequality (2.15), we obtain

$$\zeta_1(t)z^{\beta-\alpha}(\rho(t))+\zeta_2(t)z^{\gamma-\alpha}(\rho(t)) \geq [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}. \tag{2.32}$$

Hence, from (2.30), we have

$$v'(t) \leq -\left[\zeta_0(t)+[k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right] - \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha}, \tag{2.33}$$

that is,

$$v'(t) + \left[\zeta_0(t)+[k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right] + \frac{\alpha\rho'(t)}{r^{1/\alpha}(\rho(t))}(-v(t))^{(\alpha+1)/\alpha} \leq 0, \quad t \geq t_1. \tag{2.34}$$

Multiplying (2.34) by $\delta^{\alpha}(t)$ and integrating it from t_1 to t implies that

$$\delta^{\alpha}(t)v(t)-\delta^{\alpha}(t_1)v(t_1)+\alpha\int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha-1}(s)v(s)\mathrm{d}s$$
$$+\int_{t_1}^{t}\left[\zeta_0(s)+[k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha}(s)\mathrm{d}s \tag{2.35}$$
$$+\alpha\int_{t_1}^{t}\frac{\delta^{\alpha}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s\le 0.$$

Set $p:=(\alpha+1)/\alpha$, $q:=\alpha+1$, and

$$a:=(\alpha+1)^{\alpha/(\alpha+1)}\delta^{\alpha^2/(\alpha+1)}(t)v(t),\qquad b:=\frac{\alpha}{(\alpha+1)^{\alpha/(\alpha+1)}}\delta^{-1/(\alpha+1)}(t). \tag{2.36}$$

Using Young's inequality (2.15), we get

$$-\alpha\delta^{\alpha-1}(t)v(t)\le\alpha\delta^{\alpha}(t)(-v(t))^{(\alpha+1)/\alpha}+\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(t)}. \tag{2.37}$$

Thus,

$$-\frac{\alpha\rho'(t)\delta^{\alpha-1}(t)v(t)}{r^{1/\alpha}(\rho(t))}\le\alpha\rho'(t)\frac{\delta^{\alpha}(t)(-v(t))^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\rho(t))}+\rho'(t)\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\delta(t)r^{1/\alpha}(\rho(t))}. \tag{2.38}$$

Therefore, (2.35) yields

$$\delta^{\alpha}(t)v(t)\le\delta^{\alpha}(t_1)v(t_1),$$
$$-\int_{t_1}^{t}\left\{\left[\zeta_0(s)+[k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha}(s)-\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(s)}{\delta(s)r^{1/\alpha}(\rho(s))}\right\}\mathrm{d}s. \tag{2.39}$$

Letting $t\to\infty$ in the above inequality, by (2.3), we get a contradiction with (2.25). This completes the proof of Theorem 2.1. □

From Theorem 2.1, when $\rho(t)=t$, we have the following result.

Corollary 2.2. *Assume that* (1.3) *holds,* $p'(t)\ge 0$, *and* $\tau_i(t)\le t-\sigma$, $i=0,1,2$. *If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int^{\infty}\left\{\left[h_0(t)+[k_1h_1(t)]^{1/k_1}[k_2h_2(t)]^{1/k_2}\right]\pi^{\alpha}(t)\ -\left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{1}{\pi(t)r^{1/\alpha}(t)}\right\}\mathrm{d}t=\infty, \tag{2.40}$$

then (1.1) *is oscillatory.*

Theorem 2.3. *Assume that* (1.3) *holds, $p'(t) \geq 0$, and there exists $\rho \in C^1([t_0,\infty),\mathbb{R})$, such that $\rho(t) \geq t$, $\rho'(t) > 0$, $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0,1,2$. If for all sufficiently large t_1 such that* (2.2) *holds and*

$$\int^{\infty}\left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha+1}(t)\mathrm{d}t = \infty, \tag{2.41}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that (2.4) or (2.5) holds.

If (2.4) holds, proceeding as in the proof of Theorem 2.1, we obtain a contradiction with (2.2).

If (2.5) holds, we proceed as in the proof of Theorem 2.1, then we get (2.25) and (2.34). Multiplying (2.34) by $\delta^{\alpha+1}(t)$ and integrating it from t_1 to t implies that

$$\begin{aligned}
&\delta^{\alpha+1}(t)v(t) - \delta^{\alpha+1}(t_1)v(t_1) + (\alpha+1)\int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha}(s)v(s)\mathrm{d}s \\
&\quad + \int_{t_1}^{t}\left[\zeta_0(s) + [k_1\zeta_1(s)]^{1/k_1}[k_2\zeta_2(s)]^{1/k_2}\right]\delta^{\alpha+1}(s)\mathrm{d}s \\
&\quad + \alpha\int_{t_1}^{t}\frac{\delta^{\alpha+1}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s \leq 0.
\end{aligned} \tag{2.42}$$

In view of (2.25), we have $-v(t)\delta^{\alpha+1}(t) \leq \delta(t) < \infty$, $t \to \infty$. From (1.3), we get

$$\begin{gathered}
\int_{t_1}^{t} -r^{-1/\alpha}(\rho(s))\rho'(s)\delta^{\alpha}(s)v(s)\mathrm{d}s \leq \int_{t_1}^{t} r^{-1/\alpha}(\rho(s))\rho'(s)\mathrm{d}s = \int_{\rho(t_1)}^{\rho(t)} r^{-1/\alpha}(u)\mathrm{d}u < \infty, \quad t \longrightarrow \infty, \\
\int_{t_1}^{t}\frac{\delta^{\alpha+1}(s)\rho'(s)}{r^{1/\alpha}(\rho(s))}(-v(s))^{(\alpha+1)/\alpha}\mathrm{d}s \leq \int_{\rho(t_1)}^{\rho(t)} r^{-1/\alpha}(u)\mathrm{d}u < \infty, \quad t \longrightarrow \infty.
\end{gathered} \tag{2.43}$$

Letting $t \to \infty$ in (2.42) and using the last inequalities, we obtain

$$\int^{\infty}\left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha+1}(t)\mathrm{d}t < \infty, \tag{2.44}$$

which contradicts (2.41). This completes the proof of Theorem 2.3. □

From Theorem 2.3, when $\rho(t) = t$, we have the following result.

Corollary 2.4. *Assume that* (1.3) *holds,* $p'(t) \geq 0$*,* $\tau_i(t) \leq t - \sigma$*,* $i = 0, 1, 2$*. If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int^{\infty} \left[h_0(t) + [k_1 h_1(t)]^{1/k_1} [k_2 h_2(t)]^{1/k_2}\right] \pi^{\alpha+1}(t) \mathrm{d}t = \infty, \tag{2.45}$$

then (1.1) *is oscillatory.*

Theorem 2.5. *Assume that* (1.3) *holds,* $p'(t) \geq 0$*, and there exists* $\rho \in C^1([t_0, \infty), \mathbb{R})$*, such that* $\rho(t) \geq t$*,* $\rho'(t) > 0$*,* $\tau_i(t) \leq \rho(t) - \sigma$*,* $i = 0, 1, 2$*. If for all sufficiently large* t_1 *such that* (2.2) *holds and*

$$\int_{t_1}^{\infty} r^{-1/\alpha}(v) \left[\int_{t_1}^{v} \varphi(u) \mathrm{d}u\right]^{1/\alpha} \mathrm{d}v = \infty, \tag{2.46}$$

then (1.1) *is oscillatory.*

Proof. Suppose to the contrary that u is a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(t) > 0$ for all large t. The case of $u(t) < 0$ can be considered by the same method. From (1.1) and (1.3), we can easily obtain that there exists a $t_1 \geq t_0$ such that (2.4) or (2.5) holds.

If (2.4) holds, proceeding as in the proof of Theorem 2.1, we obtain a contradiction with (2.2).

If (2.5) holds, we proceed as in the proof of Theorem 2.1, and we get (2.21). Dividing (2.21) by $r^{1/\alpha}(s)$ and integrating it from $\rho(t)$ to l, letting $l \to \infty$, yields

$$z(\rho(t)) \geq -r^{1/\alpha}(t) z'(t) \int_{\rho(t)}^{\infty} r^{-1/\alpha}(s) \mathrm{d}s = -r^{1/\alpha}(t) z'(t) \delta(t) \geq -r^{1/\alpha}(t_1) z'(t_1) \delta(t) := a\delta(t). \tag{2.47}$$

By (1.1), we have

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' = q_0(t) u^{\alpha}(\tau_0(t)) + q_1(t) u^{\beta}(\tau_1(t)) + q_2(t) u^{\gamma}(\tau_2(t)). \tag{2.48}$$

Noticing that $p'(t) \geq 0$, from [10, Theorem 2.3], we see that $u'(t) \leq 0$ for $t \geq t_1$, so by $\tau_i(t) \leq \rho(t) - \sigma$, $i = 0, 1, 2$, we get

$$\begin{aligned} \frac{u(\tau_i(t))}{z(\rho(t))} &= \frac{u(\tau_i(t))}{u(\rho(t)) + p(\rho(t)) u(\rho(t) - \sigma)} \\ &= \frac{1}{(u(\rho(t))/u(\tau_i(t))) + p(\rho(t))(u(\rho(t) - \sigma)/u(\tau_i(t)))} \geq \frac{1}{1 + p(\rho(t))}. \end{aligned} \tag{2.49}$$

Hence, we obtain

$$\left(r(t)\left(-z'(t)\right)^{\alpha}\right)' \geq b\varphi(t), \tag{2.50}$$

where $b = \min\{a^{\alpha}, a^{\beta}, a^{\gamma}\}$. Integrating the above inequality from t_1 to t, we have

$$r(t)\left(-z'(t)\right)^{\alpha} \geq r(t_1)\left(-z'(t_1)\right)^{\alpha} + b\int_{t_1}^{t}\varphi(u)\mathrm{d}u \geq b\int_{t_1}^{t}\varphi(u)\mathrm{d}u. \tag{2.51}$$

Integrating the above inequality from t_1 to t, we obtain

$$z(t_1) - z(t) \geq b^{1/\alpha}\int_{t_1}^{t} r^{-1/\alpha}(v)\left[\int_{t_1}^{v}\varphi(u)\mathrm{d}u\right]^{1/\alpha}\mathrm{d}v, \tag{2.52}$$

which contradicts (2.46). This completes the proof of Theorem 2.5. □

3. Examples

In this section, three examples are worked out to illustrate the main results.

Example 3.1. Consider the second-order neutral delay differential equation (1.8), where $\lambda > 0$ is a constant.

Let $r(t) = \mathrm{e}^{2t}$, $p(t) = 1/2$, $\sigma = 2$, $q_0(t) = \lambda(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2})/2$, $\alpha = 1$, $\tau_0(t) = t - 1$, $q_1(t) = q_2(t) = 0$, and $\tau(t) = \tau_0(t)$, then

$$R(t) = \int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s = \frac{\left(\mathrm{e}^{-2t_0} - \mathrm{e}^{-2t}\right)}{2},$$

$$\xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s = \frac{\left(\mathrm{e}^{2(t-t_1)} - 1\right)}{2}, \tag{3.1}$$

$$Q_0(t) = \frac{q_0(t)}{2} = \frac{\lambda\left(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2}\right)}{4}, \qquad \zeta_0(t) = \frac{2q_0(t)}{3} = \frac{\lambda\left(2\mathrm{e}^{2t} + \mathrm{e}^{2t+2}\right)}{3}.$$

Setting $\rho(t) = t + 1$, we have $\tau_0(t) = t - 1 \leq \rho(t) - \sigma$, $\delta(t) = \mathrm{e}^{-2t-2}/2$. Therefore, for all sufficiently large t_1,

$$\int^{\infty}\left\{R^{\alpha}(\tau(t))\left[Q_0(t) + [k_1Q_1(t)]^{1/k_1}[k_2Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)}\right\}\mathrm{d}t = \infty,$$

$$\int^{\infty}\left\{\left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(t)}{\delta(t)r^{1/\alpha}(\rho(t))}\right\}\mathrm{d}t$$

$$= \int^{\infty}\frac{\lambda\left(2\mathrm{e}^{-2} + 1\right) - 3}{6}\mathrm{d}t = \infty \tag{3.2}$$

if $\lambda > 3/(2\mathrm{e}^{-2} + 1)$. Hence, by Theorem 2.1, (1.8) is oscillatory when $\lambda > 3/(2\mathrm{e}^{-2} + 1)$.

Note that [11, Theorem 2.1] and [11, Theorem 2.2] cannot be applied in (1.8), since $\tau_0(t) > t-2$. On the other hand, applying [11, Theorem 3.2] to that (1.8), we obtain that (1.8) is oscillatory if $\lambda > 3/(\mathrm{e}^{-2}+2\mathrm{e}^{-4})$. So our results improve the results in [11].

Example 3.2. Consider the second-order neutral delay differential equation

$$\left(\mathrm{e}^t\left(u(t)+\frac{1}{2}u\left(t-\frac{\pi}{4}\right)\right)'\right)' + 12\sqrt{65}\mathrm{e}^t u\left(t-\frac{1}{8}\arcsin\frac{\sqrt{65}}{65}\right) = 0, \quad t \geq t_0. \tag{3.3}$$

Let $r(t) = \mathrm{e}^t$, $p(t) = 1/2$, $\sigma = \pi/4$, $q_0(t) = 12\sqrt{65}\mathrm{e}^t$, $q_1(t) = q_2(t) = 0$, $\alpha = 1$, $\tau_0(t) = t - (\arcsin\sqrt{65}/65)/8$, $\rho(t) = t+\pi/4$, and $\tau(t) = t-\pi/4$, then

$$\begin{gathered}
R(t) = \int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s = \mathrm{e}^{-t_0} - \mathrm{e}^{-t}, \qquad \xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s = \mathrm{e}^{t-t_1}-1, \\
Q_0(t) = \frac{q_0(t)}{2} = 6\sqrt{65}\mathrm{e}^t, \qquad \zeta_0(t) = \frac{2q_0(t)}{3} = 8\sqrt{65}\mathrm{e}^t, \qquad \delta(t) = \mathrm{e}^{-t-\pi/4}.
\end{gathered} \tag{3.4}$$

Therefore, for all sufficiently large t_1,

$$\begin{aligned}
&\int^{\infty}\left\{R^{\alpha}(\tau(t))\left[Q_0(t) + [k_1Q_1(t)]^{1/k_1}[k_2Q_2(t)]^{1/k_2}\right] - \frac{\alpha\tau'(t)R^{\alpha-1}(\tau(t))r^{1-1/\alpha}(\tau(t))}{\xi^{\alpha}(t)}\right\}\mathrm{d}t = \infty, \\
&\int^{\infty}\left\{\left[\zeta_0(t) + [k_1\zeta_1(t)]^{1/k_1}[k_2\zeta_2(t)]^{1/k_2}\right]\delta^{\alpha}(t) - \left(\frac{\alpha}{\alpha+1}\right)^{\alpha+1}\frac{\rho'(t)}{\delta(t)r^{1/\alpha}(\rho(t))}\right\}\mathrm{d}t \\
&\quad = \int^{\infty}\left(8\sqrt{65}\mathrm{e}^{-\pi/4} - \frac{1}{4}\right)\mathrm{d}t = \infty.
\end{aligned} \tag{3.5}$$

Hence, by Theorem 2.1, (3.3) oscillates. For example, $u(t) = \sin 8t$ is a solution of (3.3).

Example 3.3. Consider the second-order neutral differential equation

$$\left(\mathrm{e}^t z'(t)\right)' + \mathrm{e}^{2\lambda_* t}u(\lambda_0 t) + q_1(t)u^{1/3}(\lambda_1 t) + q_2(t)u^{5/3}(\lambda_2 t) = 0, \quad t \geq t_0, \tag{3.6}$$

where $z(t) = u(t) + u(t-1)/2$, $\lambda_i > 0$ for $i = 0,1,2$, are constants, $q_1(t) > 0$, $q_2(t) > 0$ for $t \geq t_0$.

Let $r(t) = \mathrm{e}^t$, $\sigma = 1$, $q_0(t) = \mathrm{e}^{2\lambda_* t}$, $\lambda_* = \max\{\lambda_0, \lambda_1, \lambda_2\}$, $\tau_i(t) = \lambda_i t$, $\tau(t) = \lambda t$, $0 < \lambda < \min\{\lambda_0, \lambda_1, \lambda_2, 1\}$, $\rho(t) = \lambda_* t + 1$, $\alpha = 1$, $\beta = 1/3$, and $\gamma = 5/3$, then $k_1 = k_2 = 2$,

$$\begin{gathered}
R(t) = \int_{t_0}^{t}\frac{1}{r^{1/\alpha}(s)}\mathrm{d}s = \mathrm{e}^{-t_0} - \mathrm{e}^{-t}, \\
\xi(t) = r^{1/\alpha}(\tau(t))\int_{t_1}^{t}\left(\frac{1}{r(\tau(s))}\right)^{1/\alpha}\tau'(s)\mathrm{d}s = \mathrm{e}^{\lambda(t-t_1)} - 1, \qquad \delta(t) = \mathrm{e}^{-\lambda_* t-1}.
\end{gathered} \tag{3.7}$$

It is easy to see that (2.2) and (2.41) hold for all sufficiently large t_1. Hence, by Theorem 2.3, (3.6) is oscillatory.

4. Conclusions

In this paper, we consider the oscillatory behavior of second-order neutral functional differential equation (1.1). Our results can be applied to the case when $\tau_i(t) > t$, $i = 0, 1, 2$; these results improve the results given in [6, 7, 10, 11].

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Research Article

Oscillation of Second-Order Sublinear Impulsive Differential Equations

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Oscillation criteria obtained by Kusano and Onose (1973) and by Belohorec (1969) are extended to second-order sublinear impulsive differential equations of Emden-Fowler type: $x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0$, $t \neq \theta_k$; $\Delta x'(t)|_{t=\theta_k} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0$; $\Delta x(t)|_{t=\theta_k} = 0$, $(0 < \alpha < 1)$ by considering the cases $\tau(t) \leq t$ and $\tau(t) = t$, respectively. Examples are inserted to show how impulsive perturbations greatly affect the oscillation behavior of the solutions.

1. Introduction

We deal with second-order sublinear impulsive differential equations of the form

$$\begin{gathered}
x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, \quad t \neq \theta_k, \\
\Delta x'(t)\big|_{t=\theta_k} + q_k|x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)) = 0, \\
\Delta x(t)|_{t=\theta_k} = 0,
\end{gathered} \tag{1.1}$$

where $0 < \alpha < 1$, $t \geq t_0$, and $k \geq k_0$ for some $t_0 \in \mathbb{R}_+$ and $k_0 \in \mathbb{N}$, $\{\theta_k\}$ is a strictly increasing unbounded sequence of positive real numbers,

$$\Delta z(t)|_{t=\theta} := z(\theta^+) - z(\theta^-), \quad z(\theta^\mp) := \lim_{t \to \theta^\mp} z(t). \tag{1.2}$$

Let $\mathrm{PLC}(J, R)$ denote the set of all real-valued functions u defined on J such that u is continuous for all $t \in J$ except possibly at $t = \theta_k$ where $u(\theta_k^\pm)$ exists and $u(\theta_k) := u(\theta_k^-)$.

We assume in the sequel that

(a) $p \in \mathrm{PLC}([t_0, \infty), \mathbb{R})$,

(b) $\{q_k\}$ is a sequence of real numbers,

(c) $\tau \in C([t_0, \infty), \mathbb{R}_+)$, $\tau(t) \le t$, $\lim_{t\to\infty}\tau(t) = \infty$.

By a solution of (1.1) on an interval $J \subset [t_0, \infty)$, we mean a function $x(t)$ which is defined on J such that $x, x', x'' \in \mathrm{PLC}(J)$ and which satisfies (1.1). Because of the requirement $\Delta x(t)|_{t=\theta_k} = 0$ every solution of (1.1) is necessarily continuous.

As usual we assume that (1.1) has solutions which are nontrivial for all large t. Such a solution of (1.1) is called oscillatory if it has no last zero and nonoscillatory otherwise.

In case there is no impulse, (1.1) reduces to Emden-Fowler equation with delay

$$x''(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) = 0, \quad 0 < \alpha < 1, \tag{1.3}$$

and without delay

$$x'' + p(t)|x|^{\alpha-1}x = 0, \quad 0 < \alpha < 1. \tag{1.4}$$

The problem of oscillation of solutions of (1.3) and (1.4) has been considered by many authors. Kusano and Onose [1] see also [2, 3] proved the following necessary and sufficient condition for oscillation of (1.3).

Theorem 1.1. *If $p(t) \ge 0$, then a necessary and sufficient condition for every solution of (1.3) to be oscillatory is that*

$$\int^{\infty} [\tau(t)]^{\alpha} p(t)dt = \infty. \tag{1.5}$$

The condition $p(t) \ge 0$ is required only for the sufficiency part, and no similar criteria is available for $p(t)$ changing sign, except in the case $\tau(t) = t$. Without imposing a sign condition on $p(t)$, Belohorec [4] obtained the following sufficient condition for oscillation of (1.4).

Theorem 1.2. *If*

$$\int^{\infty} t^{\beta} p(t)dt = \infty \tag{1.6}$$

for some $\beta \in [0, \alpha]$, then every solution of (1.4) is oscillatory.

Compared to the large body of papers on oscillation of differential equations, there is only little known about the oscillation of impulsive differential equations; see [5–7] for equations with delay and [8–13] for equations without delay. For some applications of such equations, we may refer to [14–18]. The books [19, 20] are good sources for a general theory of impulsive differential equations.

The object of this paper is to extend Theorems 1.1 and 1.2 to impulsive differential equations of the form (1.1). The results show that the impulsive perturbations may greatly change the oscillatory behavior of the solutions. A nonoscillatory solution of (1.3) or (1.4) may become oscillatory under impulsive perturbations.

The following two lemmas are crucial in the proof of our main theorems. The first lemma is contained in [21] and the second one is extracted from [22].

Lemma 1.3. *If each A_i is continuous on $[a,b]$, then*

$$\int_a^b \sum_{s\leq\theta_i<b} A_i(s)ds = \sum_{a\leq\theta_i<b}\int_a^{\theta_i} A_i(s)ds. \tag{1.7}$$

Lemma 1.4. Fix $J=[a,b]$, *let* $u,\lambda\in C(J,\mathbb{R}_+)$, $h\in C(\mathbb{R}_+,\mathbb{R}_+)$, *and* $c\in\mathbb{R}_+$, *and let* $\{\lambda_k\}$ *a sequence of positive real numbers. If* $u(J)\subset I\subset\mathbb{R}_+$ *and*

$$u(t)\leq c+\int_a^t \lambda(s)h(u(s))ds+\sum_{a<\theta_k<t}\lambda_k h(u(\theta_k)),\quad t\in J, \tag{1.8}$$

then

$$u(t)\leq G^{-1}\left\{G(c)+\int_a^t\lambda(s)ds+\sum_{a<\theta_k<t}\lambda_k\right\},\quad t\in[a,\beta), \tag{1.9}$$

where

$$\begin{gathered} G(u)=\int_{u_0}^{u}\frac{dx}{h(x)},\quad u,u_0\in I,\\ \beta=\sup\left\{\nu\in J: G(c)+\int_a^t\lambda(s)ds+\sum_{a<\theta_k<t}\lambda_k\in G(I),\ a\leq t\leq\nu\right\}. \end{gathered} \tag{1.10}$$

2. The Main Results

We first establish a necessary and sufficient condition for oscillation of solutions of (1.1) when $\tau(t)\leq t$.

Theorem 2.1. *If*

$$\int^{\infty}[\tau(t)]^{\alpha}|p(t)|dt+\sum^{\infty}[\tau(\theta_k)]^{\alpha}|q_k|<\infty, \tag{2.1}$$

then (1.1) *has a solution $x(t)$ satisfying*

$$\lim_{t\to\infty}\frac{x(t)}{t}=a\neq 0. \tag{2.2}$$

Proof. Choose $t_1 \geq \max\{1, t_0\}$. In view of Lemma 1.3 by integrating (1.1) twice from t_0 to t, we obtain

$$\begin{aligned} x(t) = x(t_1) - x'(t_1)(t - t_1) - \sum_{t_1 \leq \theta_k < t} q_k |x(\tau(\theta_k))|^{\alpha-1} x((\tau(\theta_k)))(t - \theta_k) \\ - \int_{t_1}^{t} (t-s) p(s) |x(\tau(s))|^{\alpha-1} x((\tau(s))) ds, \quad t \geq t_1. \end{aligned} \tag{2.3}$$

Set

$$u(t) = c + \sum_{t_1 \leq \theta_k < t} |q_k| |x(\tau(\theta_k))|^{\alpha} + \int_{t_1}^{t} |p(s)| |x(\tau(s))|^{\alpha}\, ds, \quad t \geq t_1, \tag{2.4}$$

where $c = |x(t_1)| + |x'(t_1)|$. Then

$$|x(t)| \leq t u(t), \quad t \geq t_1. \tag{2.5}$$

Let $t_2 \geq t_1$ be such that $\tau(t) \geq t_1$ for all $t \geq t_2$. Replacing t by $\tau(t)$ in (2.5) and using the increasing character of $u(t)$, we see that

$$|x(\tau(t))| \leq \tau(t) u(t), \quad t \geq t_2. \tag{2.6}$$

From (2.4), we also see that

$$u'(t) = |p(t)| |x(\tau(t))|^{\alpha}, \quad t \neq \theta_k, \tag{2.7}$$

$$\Delta u(t)|_{t=\theta_k} = |q_k| |x(\tau(\theta_k))|^{\alpha} \tag{2.8}$$

for $t \geq t_2$ and $\theta_k \geq t_2$. Now, in view of (2.6) and (2.8), an integration of (2.7) from t_2 to t leads to

$$u(t) \leq c + \int_{t_2}^{t} |p(s)| [\tau(s)]^{\alpha} [u(s)]^{\alpha} ds + \sum_{t_2 \leq \theta_k < t} |q_k| [\tau(\theta_k)]^{\alpha} [u(\theta_k)]^{\alpha}. \tag{2.9}$$

Applying Lemma 1.4 with

$$h(x) = x^{\alpha}, \qquad \lambda(s) = |p(s)| [\tau(s)]^{\alpha}, \qquad \lambda_k = |q_k| [\tau(\theta_k)]^{\alpha}, \tag{2.10}$$

we easily see that

$$u(t) \leq G^{-1}\left\{ G(c) + \int_{t_2}^{t} |p(s)| [\tau(s)]^{\alpha} ds + \sum_{t_2 \leq \theta_k < t} |q_k| [\tau(\theta_k)]^{\alpha} \right\}. \tag{2.11}$$

Since

$$G(u) = \frac{u^{1-\alpha}}{1-\alpha} - \frac{u_0^{1-\alpha}}{1-\alpha}, \qquad G^{-1}(u) = \left[(1-\alpha)u + u_0^{1-\alpha}\right]^{1/(1-\alpha)}, \tag{2.12}$$

the inequality (2.11) becomes

$$u(t) \le \left[c^{1-\alpha} + (1-\alpha)\int_{t_1}^{t} \left|p(s)\right|[\tau(s)]^{\alpha} ds + (1-\alpha)\sum_{t_1 \le \theta_k < t} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right]^{1/(1-\alpha)}, \tag{2.13}$$

from which, on using (2.1), we have

$$u(t) \le c_1, \quad t \ge t_2, \tag{2.14}$$

where

$$c_1 = \left[c^{1-\alpha} + (1-\alpha)\int_{t_1}^{\infty} \left|p(s)\right|[\tau(s)]^{\alpha} ds + (1-\alpha)\sum_{t_1 \le \theta_k < \infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right]^{1/(1-\alpha)}. \tag{2.15}$$

In view of (2.5), (2.6), and (2.14) we see that

$$|x(t)| \le c_1 t, \quad |x(\tau(t))| \le c_1 \tau(t), \quad t \ge t_2. \tag{2.16}$$

To complete the proof it suffices to show that $x'(t)$ approaches a nonzero limit as t tends to ∞. To see this we integrate (1.1) from t_2 to t to get

$$x'(t) = x'(t_1) - \int_{t_2}^{t} p(s)|x(\tau(s))|^{\alpha-1}x(\tau(s))ds - \sum_{t_2 \le \theta_k < t} q_k |x(\tau(\theta_k))|^{\alpha-1}x(\tau(\theta_k)). \tag{2.17}$$

Employing (2.16) we have

$$\begin{gathered} \int_{t_2}^{\infty} \left|p(s)x(\tau(s))\right|^{\alpha} ds \le c_1^{\alpha}\int_{t_2}^{\infty}\left|p(s)\right|[\tau(s)]^{\alpha} ds < \infty, \\ \sum_{t_2 \le \theta_k < \infty} \left|q_k x(\tau(\theta_k))\right|^{\alpha} \le c_1^{\alpha} \sum_{t_2 \le \theta_k < \infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha} < \infty. \end{gathered} \tag{2.18}$$

Therefore, $\lim_{t\to\infty} x'(t) = L$ exists. Clearly, we can make $L \ne 0$ by requiring that

$$x'(t_2) > c_1^{\alpha}\left[\int_{t_2}^{\infty} \left|p(s)\right|[\tau(s)]^{\alpha} ds + \sum_{t_2 \le \theta_k < \infty} \left|q_k\right|[\tau(\theta_k)]^{\alpha}\right], \tag{2.19}$$

which is always possible by arranging t_2. □

Theorem 2.2. *Suppose that p and $\{q_k\}$ are nonnegative. Then every solution of* (1.1) *is oscillatory if and only if*

$$\int^{\infty}[\tau(t)]^{\alpha}p(t)dt+\sum^{\infty}[\tau(\theta_k)]^{\alpha}q_k=\infty. \tag{2.20}$$

Proof. Let (2.20) fail to hold. Then, by Theorem 2.1 we see that there is a solution $x(t)$ which satisfies (2.2). Clearly, such a solution is nonoscillatory. This proves the necessity.

To show the sufficiency, suppose that (2.20) is valid but there is a nonoscillatory solution $x(t)$ of (1.1). We may assume that $x(t)$ is eventually positive; the case $x(t)$ being eventually negative is similar. Clearly, there exists $t_1\ge t_0$ such that $x(\tau(t))>0$ for all $t\ge t_1$. From (1.1), we have that

$$x''(t)\le 0 \quad \text{for } t\ge t_1,\ t\neq\theta_k. \tag{2.21}$$

Thus, $x'(t)$ is decreasing on every interval not containing $t=\theta_k$. From the impulse conditions in (1.1), we also have $\Delta x'(\theta_k)\le 0$. Therefore, we deduce that $x'(t)$ is nondecreasing on $[t_1,\infty)$.

We may claim that $x'(t)$ is eventually positive. Because if $x'(t)<0$ eventually, then $x(t)$ becomes negative for large values of t. This is a contradiction.

It is now easy to show that

$$x(t)\ge (t-t_1)x'(t), \quad t\ge t_1. \tag{2.22}$$

Therefore,

$$x(t)\ge \frac{t}{2}x'(t), \quad t\ge t_2=2t_1. \tag{2.23}$$

Let $t_3\ge t_2$ be such that $\tau(t)\ge t_2$ for $t\ge t_3$. Using (2.23) and the nonincreasing character of $x'(t)$, we have

$$x(\tau(t))\ge \frac{\tau(t)}{2}x'(t), \quad t\ge t_3, \tag{2.24}$$

and so, by (1.1),

$$x''(t)+2^{-\alpha}p(t)[\tau(t)]^{\alpha}\left[x'(t)\right]^{\alpha}\le 0, \quad t\neq\theta_k. \tag{2.25}$$

Dividing (2.25) by $[x'(t)]^{\alpha}$ and integrating from t_3 to t, we obtain

$$\begin{aligned}&\sum_{t_3\le\theta_k<t}\left\{\left[x'(\theta_k)\right]^{1-\alpha}-\left[x'(\theta_k)-q_k[x(\tau(\theta_k))]^{\alpha}\right]^{1-\alpha}\right\}\\&\quad+\left[x'(t)\right]^{1-\alpha}-\left[x'(t_3)\right]^{1-\alpha}+(1-\alpha)2^{-\alpha}\int_{t_3}^{t}[\tau(t)]^{\alpha}p(s)ds\le 0\end{aligned} \tag{2.26}$$

which clearly implies that

$$\sum_{t_3 \le \theta_k < t} a_k + (1-\alpha)2^{-\alpha} \int_{t_3}^{t} [\tau(t)]^{\alpha} p(s) ds \le \left[x'(t_3)\right]^{1-\alpha}, \tag{2.27}$$

where

$$a_k = \left[x'(\theta_k)\right]^{1-\alpha} \left[1 - \left(1 - \frac{q_k [x(\tau(\theta_k))]^{\alpha}}{x'(\theta_k)}\right)\right]^{1-\alpha}. \tag{2.28}$$

Since $1-(1-u)^{1-\alpha} \ge (1-\alpha)u$ for $u \in (0,\infty)$ and $0<\alpha<1$, by taking

$$u = \frac{q_k [x(\tau(\theta_k))]^{\alpha}}{x'(\theta_k)}, \tag{2.29}$$

we see from (2.28) that

$$a_k \ge (1-\alpha) \frac{q_k [x(\tau(\theta_k))]^{\alpha}}{[x'(\theta_k)]^{\alpha}}. \tag{2.30}$$

But, (2.24) gives

$$x(\tau(\theta_k)) \ge \frac{\tau(\theta_k)}{2} x'(\tau(\theta_k)) \ge \frac{\tau(\theta_k)}{2} x'(\theta_k), \tag{2.31}$$

and hence

$$a_k \ge (1-\alpha)2^{-\alpha} [\tau(\theta_k)]^{\alpha} q_k. \tag{2.32}$$

Finally, (2.27) and (2.32) result in

$$\int_{t_3}^{\infty} [\tau(t)]^{\alpha} p(t) dt + \sum_{t_3 < \theta_k < \infty} [\tau(\theta_k)]^{\alpha} q_k < \infty, \tag{2.33}$$

which contradicts (2.20). The proof is complete. □

Example 2.3. Consider the impulsive delay differential equation

$$\begin{gathered} x''(t) + (t-1)^{-2} |x(t-1)|^{-1/2} x(t-1) = 0, \quad t \ne k, \\ \Delta x'(t)\big|_{t=k} + (k-1)^{-1} |x(k-1)|^{-1/2} x(k-1) = 0, \\ \Delta x(t)|_{t=k} = 0, \end{gathered} \tag{2.34}$$

where $t \ge 2$ and $i \ge 2$.

We see that $\tau(t) = t - 1$, $\alpha = 1/2$, $p(t) = (t-1)^{-2}$, and $q_k = (k-1)^{-1}$, $\theta_k = k$. Since

$$\int^{\infty} (t-1)^{-3/2} dt + \sum^{\infty} (k-1)^{-1/2} = \infty, \tag{2.35}$$

applying Theorem 2.2 we conclude that every solution of (2.34) is oscillatory.

We note that if the equation is not subject to any impulse condition, then, since

$$\int^{\infty} (t-1)^{-5/2} dt < \infty, \tag{2.36}$$

the equation

$$x''(t) + (t-1)^{-2} |x(t-1)|^{-1/2} x(t-1) = 0 \tag{2.37}$$

has a nonoscillatory solution by Theorem 1.1.

Let us now consider (1.1) when $\tau(t) = t$. That is,

$$\begin{gathered} x'' + p(t)|x|^{\alpha-1}x = 0, \quad t \neq \theta_k, \\ \Delta x'\big|_{t=\theta_k} + q_k |x|^{\alpha-1} x = 0, \\ \Delta x|_{t=\theta_k} = 0, \end{gathered} \tag{2.38}$$

where $0 < \alpha < 1$ and p q_k are given by (a) and (b).

The following theorem is an extension of Theorem 1.2. Note that no sign condition is imposed on $p(t)$ and $\{q_k\}$.

Theorem 2.4. *If*

$$\int^{\infty} t^{\beta} p(t) dt + \sum^{\infty} \theta_k^{\beta} q_k = \infty \tag{2.39}$$

for some $\beta \in [0, \alpha]$, then every solution of (2.38) *is oscillatory.*

Proof. Assume on the contrary that (2.38) has a nonoscillatory solution $x(t)$ such that $x(t) > 0$ for all $t \geq t_0$ for some $t_0 \geq 0$. The proof is similar when $x(t)$ is eventually negative. We set

$$w(t) = \left(t^{-1} x(t)\right)^{1-\alpha}, \quad t \geq t_0. \tag{2.40}$$

It is not difficult to see that

$$w'(t) = (\alpha - 1) t^{\alpha-2} [x(t)]^{1-\alpha} + (1-\alpha) t^{\alpha-1} [x(t)]^{-\alpha} x'(t), \quad t \neq \theta_k, \tag{2.41}$$

and hence

$$\Delta w'\big|_{t=\theta_k} = (1-\alpha)q_k\theta_k^{\alpha-1}. \tag{2.42}$$

From (2.41), we have

$$\begin{aligned} t^{\beta-1-\alpha}\left(t^2 w'(t)\right)' &= (1-\alpha)t^{\beta}x''(t)x^{-\alpha}(t) \\ &\quad - \alpha(1-\alpha)t^{\beta-2}x^{-\alpha-1}(t)\left[tx'(t)-x(t)\right]^2, \end{aligned} \tag{2.43}$$

and so

$$t^{\beta-1-\alpha}\left(t^2 w'(t)\right)' \le (1-\alpha)t^{\beta}p(t), \quad t \ne \theta_k. \tag{2.44}$$

In view of (2.42), by a straightforward integration of (2.44), we have

$$\begin{aligned} \int_{t_0}^{t} s^{\beta-1-\alpha}\left(s^2 w'(s)\right)' ds &= s^{\beta-1-\alpha}s^2 w'(s)\Big|_{t_0}^{t} - \sum_{t_0\le\theta_k<t}\Delta\left(t^{\beta-\alpha+1}w'(t)\right)\Big|_{t=\theta_k} \\ &\quad - \int_{t_0}^{t}(\beta-1-\alpha)s^{\beta-\alpha}w'(s)ds \\ &= t^{\beta-\alpha+1}w'(t) - t_0^{\beta-\alpha+1}w'(t_0) - \sum_{t_0\le\theta_k<t}(1-\alpha)q_k\theta_k^{\beta} \\ &\quad -(\beta-\alpha-1)\left[s^{\beta-\alpha}w(s)\right]\Big|_{t_0}^{t} \\ &\quad + (\beta-\alpha)(\beta-\alpha-1)\int_{t_0}^{t} s^{\beta-1-\alpha}w(s)ds, \end{aligned} \tag{2.45}$$

which combined with (2.44) leads to

$$\begin{aligned} t^{\beta-\alpha+1}w'(t) &\le t_0^{\beta-\alpha+1}w'(t_0) - (\beta-\alpha+1)t_0^{\beta-\alpha}w(t_0) \\ &\quad + (1-\alpha)\left[\sum_{t_0\le\theta_k<t}\theta_k^{\beta}q_k + \int_{t_0}^{t}s^{\beta}p(s)ds\right]. \end{aligned} \tag{2.46}$$

Finally, by using (2.39) in the last inequality, we see that there is a $t_1 > t_0$ such that

$$w'(t) \le -t^{\alpha-\beta-1}, \quad t \ge t_1, \tag{2.47}$$

which, however, implies that $w(t) \to -\infty$ as $t \to \infty$, a contradiction with $x(t) > 0$. The proof is complete. □

Example 2.5. Consider the impulsive differential equation

$$
\begin{gathered}
x'' + t^{-7/3}|x|^{-1/2}x = 0, \quad t \neq k, \\
\Delta x'\big|_{t=k} + k^{-1/6}|x|^{-1/2}x = 0, \\
\Delta x|_{t=k} = 0,
\end{gathered}
\tag{2.48}
$$

where $t \geq 1$ and $i \geq 1$.

We have that $p(t) = t^{7/3}$, $\alpha = 1/2$, and $q_k = k^{-1/6}$, $\theta_k = k$. Taking $\beta = 1/3$ we see from (2.38) that

$$
\int^{\infty} t^{-2} dt + \sum^{\infty} k^{-1/3} = \infty. \tag{2.49}
$$

Since the conditions of Theorem 2.4 are satisfied, every solution of (2.48) is oscillatory.

Note that if the impulses are absent, then, since

$$
\int^{\infty} t^{-2} dt < \infty, \tag{2.50}
$$

the equation

$$
x'' + t^{-7/3}|x|^{-1/2}x = 0 \tag{2.51}
$$

is oscillatory by Theorem 1.2.

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Research Article

Oscillatory Periodic Solutions for Two Differential-Difference Equations Arising in Applications

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We study the existence of oscillatory periodic solutions for two nonautonomous differential-difference equations which arise in a variety of applications with the following forms: $\dot{x}(t) = -f(t, x(t-r))$ and $\dot{x}(t) = -f(t, x(t-s)) - f(t, x(t-2s))$, where $f \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x, and $r, s > 0$ are two given constants. By using a symplectic transformation constructed by Cheng (2010) and a result in Hamiltonian systems, the existence of oscillatory periodic solutions of the above-mentioned equations is established.

1. Introduction and Statement of Main Results

Furumochi [1] studied the following equation:

$$\dot{x}(t) = a - \sin(x(t-r)), \tag{1.1}$$

with $t \geq 0, a \geq 0, r > 0$, which models phase-locked loop control of high-frequency generators and is widely applied in communication systems. Obviously, (1.1) is a special case of the following differential-difference equations:

$$\dot{x}(t) = -\alpha f(x(t-r)), \tag{1.2}$$

where α is a real parameter. In fact, a lot of differential-difference equations occurring widely in applications and describing many interesting types of phenomena can also be written in the form of (1.2) by making an appropriate change of variables. For example, the following differential-difference equation:

$$\dot{x}(t) = -\alpha x(t-1)(1+x(t)) \tag{1.3}$$

arises in several applications and has been studied by many researchers. Equation (1.3) was first considered by Cunningham [2] as a nonlinear growth model denoting a mathematical description of a fluctuating population. Subsequently, (1.3) was proposed by Wright [3] as occurring in the application of probability methods to the theory of asymptotic prime number density. Jones [4] states that (1.3) may also describe the operation of a control system working with potentially explosive chemical reactions, and quite similar equations arise in economic studies of business cycles. Moreover, (1.3) and its similar ones were studied in [5] on ecology.

For (1.3), we make the following change of variables:

$$y = \ln(1+x). \tag{1.4}$$

Then, (1.3) can be changed to the form of (1.2)

$$\dot{y}(t) = -f\left(y(t-1)\right), \tag{1.5}$$

where $f(y) = \alpha(e^y - 1)$.

Although (1.2) looks very simple on surface, Saupe's results [6] of a careful numerical study show that (1.2) displays very complex dynamical behaviour. Moreover, little of them has been proved to the best of the author's knowledge.

Due to a variety of applications, (1.2) attracts many authors to study it. In 1970s and 1980s of the last century, there has been a great deal of research on problems of the existence of periodic solutions [1, 4, 7–10], slowly oscillating solutions [11], stability of solutions [12–14], homoclinic solutions [15], and bifurcations of solutions [6, 16, 17] to (1.2).

Since, generally, the main tool used to conclude the existence of periodic solutions is various fixed-point theorems, here we want to mention Kaplan and Yorke's work on the existence of oscillatory periodic solutions of (1.5) in [7]. In [7], they considered the following equations:

$$\begin{gathered}\dot{x}(t) = -f(x(t-1)),\\ \dot{x}(t) = -f(x(t-1)) - f(x(t-2)),\end{gathered} \tag{1.6}$$

where f is continuous, $xf(x) > 0$ for $x \neq 0$, and f satisfies some asymptotically linear conditions at 0 and ∞. The authors introduced a new technique for establishing the existence of oscillatory periodic solutions of (1.6). They reduced the search for periodic solutions of (1.6) to the problem of finding periodic solutions for a related systems of ordinary differential equations. We will give more details about the reduction method in Section 2.

In 1990s of the last century and at the beginning of this century, some authors [18–21] applied Kaplan and Yorke's original ideas in [7] to study the existence and multiplicity of periodic solutions of (1.2) with more than two delays. See also [22, 23] for some other methods.

The previous work mainly focuses on the autonomous differential-difference equation (1.2). However, some papers [13, 24] contain some interesting nonautonomous differential difference equations arising in economics and population biology where the delay r of (1.2) depends on time t instead of a positive constant. Motivated by the lack of more results on periodic solutions for nonautonomous differential-difference equations, in the present paper, we study the following equations:

$$\dot{x}(t) = -f(t, x(t-r)), \tag{1.7}$$

$$\dot{x}(t) = -f(t, x(t-s)) - f(t, x(t-2s)), \tag{1.8}$$

where $f(t,x) \in C(\mathbb{R} \times \mathbb{R}, \mathbb{R})$ is odd with respect to x and $r = \pi/2$, $s = \pi/3$. Here, we borrow the terminology "oscillatory periodic solution" for (1.7) and (1.8) since $f(t,x)$ is odd with respect to x.

Now, we state our main results as follows.

Theorem 1.1. *Suppose that $f(t,x) \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ is odd with respect to x and r-periodic with respect to t. Suppose that*

$$\lim_{x \to 0} \frac{f(t,x)}{x} = \omega_0(t), \qquad \lim_{x \to \infty} \frac{f(t,x)}{x} = \omega_\infty(t) \tag{1.9}$$

exist. Write $\alpha_0 = (1/r)\int_0^r \omega_0(t)dt$ and $\alpha_\infty = (1/r)\int_0^r \omega_\infty(t)dt$. Assume that

(H_1) $\alpha_0 \neq \pm k$, $\alpha_\infty \neq \pm k$, for all $k \in \mathbb{Z}^+$,

(H_2) there exists at least an integer k_0 with $k_0 \in \mathbb{Z}^+$ such that

$$\min\{\alpha_0, \alpha_\infty\} < \pm k_0 < \max\{\alpha_0, \alpha_\infty\}, \tag{1.10}$$

then (1.7) *has at least one nontrivial oscillatory periodic solution x satisfying $x(t) = -x(t-\pi)$.*

Theorem 1.2. *Suppose that $f(t,x) \in C(\mathbb{R}\times\mathbb{R}, \mathbb{R})$ is odd with respect to x and s-periodic with respect to t. Let $\omega_0(t)$ and $\omega_\infty(t)$ be the two functions defined in Theorem 1.1. Write $\beta_0 = (1/s)\int_0^s \omega_0(t)dt$ and $\beta_\infty = (1/s)\int_0^s \omega_\infty(t)dt$. Assume that*

(H_3) $\beta_0, 3\beta_0 \neq \pm k$, $\beta_\infty, 3\beta_\infty \neq \pm k$, for all $k \in \mathbb{Z}^+$,

(H_4) there exists at least an integer k_0 with $k_0 \in \mathbb{Z}^+$ such that

$$\min\{\beta_0, \beta_\infty\} < \pm k_0 < \max\{\beta_0, \beta_\infty\} \tag{1.11}$$

or

$$\min\{\beta_0, \beta_\infty\} < \pm\frac{k_0}{3} < \max\{\beta_0, \beta_\infty\}, \tag{1.12}$$

then (1.8) *has at least one nontrivial oscillatory periodic solution x satisfying $x(t) = -x(t-\pi)$.*

Remark 1.3. Theorems 1.1 and 1.2 are concerned with the existence of periodic solutions for nonautonomous differential-difference equations (1.7) and (1.8). Therefore, our results generalize some results obtained in the references. We will use a symplectic transformation constructed in [25] and a theorem of [26] to prove our main results.

2. Proof of the Main Results

Consider the following nonautonomous Hamiltonian system:

$$\dot{z}(t) = J\nabla_z H(t,z), \tag{2.1}$$

where $J = \begin{pmatrix} 0 & -I_N \\ I_N & 0 \end{pmatrix}$ is the standard symplectic matrix, I_N is the identity matrix in $\mathbb{R}^N$, $\nabla_z H(t,z)$ denotes the gradient of $H(t,z)$ with respect to z, and $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ is the Hamiltonian function. Suppose that there exist two constant symmetric matrices h_0 and h_∞ such that

$$\begin{gathered} \nabla_z H(t,z) - h_0 z = o(|z|), \quad \text{as } |z| \longrightarrow 0, \\ \nabla_z H(t,z) - h_\infty z = o(|z|), \quad \text{as } |z| \longrightarrow \infty. \end{gathered} \tag{2.2}$$

We call the Hamiltonian system (2.1) asymptotically linear both at 0 and ∞ with constant coefficients h_0 and h_∞ because of (2.2).

Now, we show that the reduction method in [7] can be used to study oscillatory periodic solutions of (1.7) and (1.8). More precisely, let $x(t)$ be any solution of (1.7) satisfying $x(t) = -x(t-2r)$. Let $x_1(t) = x(t), x_2(t) = x(t-r)$, then $X(t) = (x_1(t), x_2(t))^\top$ satisfies

$$\frac{d}{dt}X(t) = A_2\Phi_1(t, X(t)), \quad \text{where } A_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{2.3}$$

and $\Phi_1(t,X) = (f(t,x_1), f(t,x_2))^\top$. What is more, if $X(t)$ is a solution of (2.3) with the following symmetric structure

$$x_1(t) = -x_2(t-r), \qquad x_2(t) = x_1(t-r), \tag{2.4}$$

then $x(t) = x_1(t)$ gives a solution to (1.7) with the property $x(t) = -x(t-2r)$. Thus, solving (1.7) within the class of the solutions with the symmetry $x(t) = -x(t-2r)$ is equivalent to finding solutions of (2.3) with the symmetric structure (2.4).

Since A_2 is indeed the standard symplectic matrix in the plane $\mathbb{R}^2$, the system (2.3) can be written as the following Hamiltonian system:

$$\dot{y}(t) = A_2 \nabla_y H^*(t,y), \tag{2.5}$$

where $H^*(t,y) = \int_0^{y_1} f(t,x)dx + \int_0^{y_2} f(t,x)dx$ for each $y = (y_1, y_2)^\top \in \mathbb{R}^2$.

From the assumptions of Theorem 1.1, we have

$$\begin{aligned} f(t,x) &= \omega_0(t)x + o(|x|) \quad \text{as } |x| \longrightarrow 0, \\ f(t,x) &= \omega_\infty(t)x + o(|x|) \quad \text{as } |x| \longrightarrow \infty. \end{aligned} \tag{2.6}$$

Hence, the gradient of the Hamiltonian function $H^*(t,y)$ satisfies

$$\begin{aligned} \nabla_y H^*(t,y) &= \omega_0(t)y + o(|y|) \quad \text{as } |y| \longrightarrow 0, \\ \nabla_y H^*(t,y) &= \omega_\infty(t)y + o(|y|) \quad \text{as } |y| \longrightarrow \infty. \end{aligned} \tag{2.7}$$

By (2.7), according to [25], there is a symplectic transformation $y = \Psi_1(t,z)$ under which the Hamiltonian system (2.5) can be transformed to the following Hamiltonian system:

$$\dot{z}(t) = A_2 \nabla_z \widetilde{H}(t,z), \tag{2.8}$$

satisfying

$$\begin{aligned} \nabla_z \widetilde{H}(t,z) &= \alpha_0 I_2 z + o(|z|) \quad \text{as } |z| \longrightarrow 0, \\ \nabla_z \widetilde{H}(t,z) &= \alpha_\infty I_2 z + o(|z|) \quad \text{as } |z| \longrightarrow \infty, \end{aligned} \tag{2.9}$$

where α_0 and α_∞ are two constants defined in Theorem 1.1.

By (2.9), we have the following.

Lemma 2.1. *The Hamiltonian system* (2.8) *is asymptotically linear both at 0 and ∞ with constant coefficients $\alpha_0 I_2$ and $\alpha_\infty I_2$.*

Let $x(t)$ be any solution of (1.8) satisfying $x(t) = -x(t-3s)$. Let $x_1(t) = x(t)$, $x_2(t) = x(t-s)$, and $x_3(t) = x(t-2s)$, then $Y(t) = (x_1(t), x_2(t), x_3(t))^\top$ satisfies

$$\frac{d}{dt} Y(t) = A_3 \Phi_2(t, Y(t)), \quad \text{where } A_3 = \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}, \tag{2.10}$$

and $\Phi_2(t,Y) = (f(t,x_1), f(t,x_2), f(t,x_3))^\top$.

Following the ideas in [18], (2.10) can be reduced to a two-dimensional Hamiltonian system

$$\dot{y}(t) = A_2 \nabla_y H^{**}(t,y), \tag{2.11}$$

where $H^{**}(t,y) = \int_0^{y_1} f(t,x)dx + \int_0^{y_2} f(t,x)dx + \int_0^{y_2 - y_1} f(t,x)dx$ for each $y = (y_1, y_2)^\top \in \mathbb{R}^2$.

From the assumptions of Theorem 1.1, (2.6), the gradient of the Hamiltonian function $H^{**}(t,y)$ satisfies

$$\begin{aligned} \nabla_y H^{**}(t,y) &= \omega_0(t)My + o(|y|) \quad \text{as } |y| \longrightarrow 0, \\ \nabla_y H^{**}(t,y) &= \omega_\infty(t)My + o(|y|) \quad \text{as } |y| \longrightarrow \infty, \end{aligned} \tag{2.12}$$

where $M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$ is a symmetric positive definite matrix.

It follows from (2.12) and [25] that there exists a symplectic transformation $y = \Psi_2(t,z)$ under which the Hamiltonian system (2.11) can be changed to the following Hamiltonian system:

$$\dot{z}(t) = A_2 \nabla_z \widehat{H}(t,z), \tag{2.13}$$

satisfying

$$\begin{aligned} \nabla_z \widehat{H}(t,z) &= \beta_0 Mz + o(|z|) \quad \text{as } |z| \longrightarrow 0, \\ \nabla_z \widehat{H}(t,z) &= \beta_\infty Mz + o(|z|) \quad \text{as } |z| \longrightarrow \infty, \end{aligned} \tag{2.14}$$

where β_0 and β_∞ are two constants defined in Theorem 1.2.

Then, (2.14) yields the following.

Lemma 2.2. *The Hamiltonian system* (2.13) *is asymptotically linear both at 0 and ∞ with constant coefficients $\beta_0 M$ and $\beta_\infty M$.*

Remark 2.3. In order to find periodic solutions of (1.7) and (1.8), we only need to seek periodic solutions of the Hamiltonian systems (2.8) and (2.13) with the symmetric structure (2.4), respectively.

In the rest of this paper, we will work in the Hilbert space $E = W^{1/2,2}(S^1, \mathbb{R}^2)$, which consists of all $z(t)$ in $L^2(S^1, \mathbb{R}^2)$ whose Fourier series

$$z(t) = a_0 + \sum_{k=1}^{+\infty} (a_k \cos kt + b_k \sin kt) \tag{2.15}$$

satisfies

$$|a_0|^2 + \frac{1}{2}\sum_{k=1}^{+\infty} k\left(|a_k|^2 + |b_k|^2\right) < +\infty. \tag{2.16}$$

The inner product on E is defined by

$$\langle z_1, z_2 \rangle = \left(a_0^{(1)}, a_0^{(2)}\right) + \frac{1}{2}\sum_{k=1}^{\infty} k\left[\left(a_k^{(1)}, a_k^{(2)}\right) + \left(b_k^{(1)}, b_k^{(2)}\right)\right], \tag{2.17}$$

where $z_i = a_0^{(i)} + \sum_{k=1}^{+\infty}(a_k^{(i)} \cos kt + b_k^{(i)} \sin kt)$ $(i = 1, 2)$, the norm $\|z\|^2 = \langle z, z\rangle$, and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^2$.

In order to obtain solutions of (2.8) with the symmetric structure (2.4), we define a matrix T_2 with the following form:

$$T_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.18}$$

Then, by T_2, for any $z(t) \in E$, define an action δ_1 on z by

$$\delta_1 z(t) = T_2 z(t - r). \tag{2.19}$$

Then by a direct computation, we have that $\delta_1^2 z(t) = -z(t - 2r) = -z(t - \pi)$, $\delta_1^4 z(t) = z(t)$, and $G = \{\delta_1, \delta_1^2, \delta_1^3, \delta_1^4\}$ is a compact group action over E. If $\delta_1 z(t) = z(t)$ holds, then through a straightforward check, we have that $z(t)$ has the symmetric structure (2.4).

Lemma 2.4. *Write $SE = \{z \in E : \delta_1 z(t) = z(t)\}$, then SE is a subspace of E with the following form:*

$$\begin{aligned} SE = \Bigg\{ z(t) = \sum_{k=1}^{\infty}&(a_{2k-1}\cos(2k-1)t + b_{2k-1}\sin(2k-1)t) : \\ &a_{2k-1,1} = (-1)^{k+1} b_{2k-1,2}, b_{2k-1,1} = (-1)^k a_{2k-1,2} \Bigg\}, \end{aligned} \tag{2.20}$$

where $a_{2k-1} = (a_{2k-1,1}, a_{2k-1,2})^{\top}$ and $b_{2k-1} = (b_{2k-1,1}, b_{2k-1,2})^{\top}$.

Proof. Write $z(t) = (z_1(t), z_2(t))^{\top}$, where $z_1(t) = a_{0,1} + \sum_{k=1}^{+\infty}(a_{k,1}\cos kt + b_{k,1}\sin kt)$, $z_2(t) = a_{0,2} + \sum_{k=1}^{+\infty}(a_{k,2}\cos kt + b_{k,2}\sin kt)$. By $\delta_1 z = z$ and the definition of the action δ_1, we have

$$(z_1(t), z_2(t))^{\top} = \left(-z_2\left(t - \frac{\pi}{2}\right), z_1\left(t - \frac{\pi}{2}\right)\right)^{\top}, \tag{2.21}$$

which yields

$$\begin{aligned} &a_{0,1} + \sum_{k=1}^{+\infty}(a_{k,1}\cos kt + b_{k,1}\sin kt) \\ &\quad = \begin{cases} -a_{0,2} - \displaystyle\sum_{n=1}^{+\infty}(-1)^n [a_{2n,2}\cos 2nt + b_{2n,2}\sin 2nt], & \text{for } k = 2n \text{ is even}, \\ -a_{0,2} - \displaystyle\sum_{n=1}^{+\infty}(-1)^{n-1}[a_{2n-1,2}\sin(2n-1)t - b_{2n-1,2}\cos(2n-1)t], & \text{for } k = 2n-1 \text{ is odd}. \end{cases} \end{aligned} \tag{2.22}$$

Then, we have

$$\begin{gathered} a_{0,1} = -a_{0,2}, \qquad a_{2n,1} = (-1)^{n+1} a_{2n,2}, \qquad b_{2n,1} = (-1)^{n+1} b_{2n,2}, \\ a_{2n-1,1} = (-1)^{n+1} b_{2n-1,2}, \qquad b_{2n-1,1} = (-1)^{n} a_{2n-1,2}. \end{gathered} \tag{2.23}$$

Similarly, by $z_2(t) = z_1(t - (\pi/2))$, one has

$$\begin{gathered} a_{0,2} = a_{0,1}, \qquad a_{2n,2} = (-1)^{n} a_{2n,1}, \qquad b_{2n,2} = (-1)^{n} b_{2n,1}, \\ a_{2n-1,2} = (-1)^{n} b_{2n-1,1}, \qquad b_{2n-1,2} = (-1)^{n-1} a_{2n-1,1}. \end{gathered} \tag{2.24}$$

Therefore, $a_{0,2} = a_{0,1} = 0$, $a_{2n,1} = (-1)^{n+1} a_{2n,2} = (-1)^{n+1}(-1)^{n} a_{2n,1}$, that is, $a_{2n,1} = 0$. Similarly, $a_{2n,2} = b_{2n,1} = b_{2n,2} = 0$. Thus, for $z(t) \in \mathrm{SE}$,

$$z(t) = \sum_{k=1}^{\infty} \left[a_{2k-1} \cos(2k-1)t + b_{2k-1} \sin(2k-1)t \right], \tag{2.25}$$

where $a_{2k-1,1} = (-1)^{k+1} b_{2k-1,2}$, $b_{2k-1,1} = (-1)^{k} a_{2k-1,2}$.

Moreover, for any $z_1(t), z_2(t) \in \mathrm{SE}$,

$$\begin{aligned} \delta_1(z_1 + z_2) &= T_2(z_1(t-r) + z_2(t-r)) \\ &= T_2(z_1(t-r)) + T_2(z_2(t-r)) \\ &= \delta_1 z_1 + \delta_1 z_2. \end{aligned} \tag{2.26}$$

And for any $c \in \mathbb{R}$, $\delta_1(cz(t)) = T_2 cz(t-r) = cT_2 z(t-r) = c\delta_1 z(t)$. Thus, SE is a subspace of E. This completes the proof of Lemma 2.4. □

For the Hamiltonian system (2.13), we define another action matrix T_2^* with the following form:

$$T_2^* = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2.27}$$

Then, by T_2^*, for any $z(t) \in E$, define an action δ_2 on z by

$$\delta_2 z(t) = T_2^* z(t-s). \tag{2.28}$$

Then, by a direct computation, we have that $\delta_2^3 z(t) = -z(t-3s) = -z(t-\pi)$, $\delta_2^6 z(t) = z(t)$ and $G = \{\delta_2, \delta_2^2, \delta_2^3, \delta_2^4, \delta_2^5, \delta_2^6\}$ is a compact group action over E. If $\delta_2 z(t) = z(t)$ holds, then through a direct check, we have that $z(t)$ has the symmetric structure (2.4).

Remark 2.5. By $\delta_2^3 z(t) = -z(t-3s) = -z(t-\pi)$ and the definition of δ_2, the set $\{z \in E : \delta_2 z(t) = z(t)\}$ has the same structure (2.20), where the relation between the Fourier coefficients of the first component z_1 and the second component z_2 is slightly different with the elements in $\{z \in E : \delta_1 z(t) = z(t)\}$. We denote it also by SE which is a subspace of E.

Denote by $M^-(h)$, $M^+(h)$, and $M^0(h)$ the number of the negative, the positive, and the zero eigenvalues of a symmetric matrix h, respectively. For a constant symmetric matrix h, we define our index as

$$\begin{gathered} i^-(h) = \sum_{k=1}^{\infty} \left(M^-(T_k(h) - 2) \right), \\ i^0(h) = \sum_{k=1}^{\infty} M^0(T_k(h)), \end{gathered} \tag{2.29}$$

where

$$T_k(h) = \begin{pmatrix} -h & -kJ \\ kJ & -h \end{pmatrix}. \tag{2.30}$$

Observe that for k large enough, $M^-(T_k(h)) = 2$ and $M^0(T_k(h)) = 0$. In fact, write

$$T_k(h) = \begin{pmatrix} -h & -kJ \\ kJ & -h \end{pmatrix} = k \begin{pmatrix} 0 & J^\top \\ J & 0 \end{pmatrix} - \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}. \tag{2.31}$$

Notice that $-J = J^\top$. If $k > 0$ is sufficiently large, then $M^- = M^+ = 2$, which are the indices of the first matrix in (2.31). Furthermore, if k decreases, these indices can change only at those values of k, for which the matrix $T_k(h)$ is singular, that is, $M^0(T_k(h)) \neq 0$. This happens exactly for those values of $k \in \mathbb{R}$ for which ik is a pure imaginary eigenvalue of Jh. Indeed assume $(\xi_1, \xi_2) \in \mathbb{R}^2 \times \mathbb{R}^2$ is an eigenvector of $T_k(h)$ with eigenvalue 0, then by $J^\top = -J$, one has $h\xi_1 + kJ\xi_2 = 0$ and $h\xi_2 - kJ\xi_1 = 0$. Thus, $h(\xi_1 + i\xi_2) = kJ(i\xi_1 - \xi_2) = ikJ(\xi_1 + i\xi_2)$; therefore, $Jh(\xi_1 + i\xi_2) = -ik(\xi_1 + i\xi_2)$. Therefore, $\pm ik \in \sigma(Jh)$, as claimed. Hence, $i^-(h)$ and $i^0(h)$ are well defined.

The following theorem of [26] on the existence of periodic solutions for the Hamiltonian system (2.1) will be used in our discussion.

Theorem A. *Let $H \in C^1(\mathbb{R} \times \mathbb{R}^{2N}, \mathbb{R})$ be 2π-periodic in t and satisfy* (2.2)*. If $i^0(h_0) = i^0(h_\infty) = 0$ and $i^-(h_0) \neq i^-(h_\infty)$, then the Hamiltonian system* (2.1) *has at least one nontrivial periodic solution.*

Now, we claim the following.

Lemma 2.6. *If z is a solution of the Hamiltonian system* (2.8) ((2.13)) *in SE, then $y = \Psi_1(t, z)$ $(y = \Psi_2(t, z))$ is the solution of the Hamiltonian system* (2.5) ((2.11)) *with the symmetric structure* (2.4)*, respectively.*

Proof. By Lemma 2.4, any $z \in \mathrm{SE}$ has the structure (2.4). We only need to show $\delta_1 y = y$ or $\delta_2 y = y$, that is, $T_2\Psi_1(t,z) = \Psi_1(t,T_2 z)$ or $T_2^*\Psi_2(t,z) = \Psi_2(t,T_2^* z)$, which can be verified directly by the constructions of the symplectic transformations $\Psi_1(t,z)$ and $\Psi_2(t,z)$, respectively. Please see [25] for details. □

We denote the matrix αI_2 by α for convenience. We prove the following lemma.

Lemma 2.7. (1) *Suppose that (*H_1*) and (*H_3*) hold, then* $i^0(\alpha_0) = i^0(\alpha_\infty) = i^0(\beta_0 M) = i^0(\beta_\infty M) = 0$.

(2) *Suppose that (*H_1*) and (*H_2*) hold, then* $i^-(\alpha_0) \neq i^-(\alpha_\infty)$.

(3) *Suppose that (*H_3*) and (*H_4*) hold, then* $i^-(\beta_0 M) \neq i^-(\beta_\infty M)$.

Proof. For any $\alpha, \beta \in \mathbb{R}$, let $\sigma(T_k(\alpha))$ and $\sigma(T_k(\beta M))$ denote the spectra of $T_k(\alpha)$ and $T_k(\beta M)$, respectively. Denote by λ and γ the elements of $\sigma(T_k(\alpha))$ and $\sigma(T_k(\beta M))$, respectively, then

$$\begin{aligned}
\det(\lambda I_4 - T_k(\alpha)) &= \det\Big((\lambda+\alpha)^2 I_2 - k^2 I_2\Big) \\
&= \det((\lambda+\alpha)I_2 - kI_2)\det((\lambda+\alpha)I_2 + kI_2), \\
\det\big(\gamma I_4 - T_k(\beta M)\big) &= \det\Big(\big(\gamma I_2 + \beta M\big)^2 - k^2 I_2\Big) \\
&= \det\big(\big(\gamma I_2 + \beta M\big) - kI_2\big)\det\big(\big(\gamma I_2 + \beta M\big) + kI_2\big) \\
&= \det\Big(\big(\gamma + 2\beta - k\big)^2 - \beta^2\Big)\det\Big(\big(\gamma + 2\beta + k\big)^2 - \beta^2\Big).
\end{aligned} \tag{2.32}$$

The above computation of determinant shows that

$$\sigma(T_k(\alpha)) = \{\lambda = \pm k - \alpha : k \in \mathbb{Z}^+\}, \tag{2.33}$$

$$\sigma\big(T_k(\beta M)\big) = \big\{\gamma = \pm k - \beta, \pm k - 3\beta : k \in \mathbb{Z}^+\big\}. \tag{2.34}$$

Case 1. From (2.33), if $\alpha_0 \neq \pm k$, for all $k \in \mathbb{Z}^+$, then $\lambda \neq 0$, where λ is the eigenvalue of $T_k(\alpha_0)$. That means $M^0(T_k(\alpha_0)) = 0$ for $k \geq 1$. Thus, $i^0(\alpha_0) = \sum_{k=1}^{\infty} M^0(T_k(\alpha_0)) = 0$. Similarly, we have $i^0(\alpha_\infty) = i^0(\beta_0 M) = i^0(\beta_\infty M) = 0$.

Case 2. Without loss of generality, we suppose that $\alpha_0 < \alpha_\infty$. By the conditions *(*H_1*)* and *(*H_2*)*,

$$\alpha_0 < k_0 < \alpha_\infty. \tag{2.35}$$

Since $\alpha_0 < k_0$, by (2.33), $M^-(T_{k_0}(\alpha_0)) \leq 2$. By $-k_0 < k_0 < \alpha_\infty$ and (2.33), $M^-(T_{k_0}(\alpha_\infty)) = 4$, that is,

$$M^-(T_{k_0}(\alpha_0)) + 2 \leq M^-(T_{k_0}(\alpha_\infty)). \tag{2.36}$$

For each $k \neq k_0$ and from (2.33), one can check easily that $M^-(T_k(\alpha_0)) \leq M^-(T_k(\alpha_\infty))$. Hence, one has $\sum_{k=1}^{\infty}(M^-(T_k(\alpha_0)) - 2) < \sum_{k=1}^{\infty}(M^-(T_k(\alpha_\infty)) - 2)$, since $M^-(T_k(\alpha)) = 2$ for k large enough. This yields that $i^-(\alpha_0) < i^-(\alpha_\infty)$. Then, property (2) holds.

Case 3. By the conditions *(H$_3$)* and *(H$_4$)*, without loss of generality, we suppose that $\beta_0 < \beta_\infty$ and

$$\beta_0 < k_0 < \beta_\infty. \tag{2.37}$$

Since $\beta_0 < k_0$, by (2.34), $M^-(T_{k_0}(\beta_0 M)) \leq 3$. By $-k_0 < k_0 < \beta_\infty < 3\beta_\infty$ and (2.34), one has $M^-(T_{k_0}(\beta_\infty M)) = 4$, that is,

$$M^-\left(T_{k_0}(\beta_0 M)\right) + 1 \leq M^-\left(T_{k_0}(\beta_\infty M)\right). \tag{2.38}$$

For each $k \neq k_0$ and from (2.34), it is easy to see that $k - \beta_\infty < k - \beta_0$ and $k - 3\beta_\infty < k - 3\beta_0$. Then, by the definition of $M^-(T_k(\beta M))$, we have $M^-(T_k(\beta_0 M)) \leq M^-(T_k(\beta_\infty M))$. Therefore, we have

$$\sum_{k=1}^{\infty}\left(M^-\left(T_k(\beta_0 M)\right) - 2\right) < \sum_{k=1}^{\infty}\left(M^-\left(T_k(\beta_\infty M)\right) - 2\right), \tag{2.39}$$

since $M^-(T_k(\beta M)) = 2$ for k large enough. This implies that $i^-(\beta_0 M) < i^-(\beta_\infty M)$. Then, property (3) holds. □

Now, we are ready to prove the main results. We first give the proof of Theorem 1.1.

Proof of Theorem 1.1. Solutions of (2.8) in SE are indeed nonconstant classic 2π-periodic solutions with the symmetric structure (2.4), and hence they give solutions of (1.7) with the property $x(t - \pi) = -x(t)$. Therefore, we will seek solutions of (2.8) in SE.

Now, Theorem 1.1 follows from Lemmas 2.1, 2.6, and 2.7 and Theorem A. □

Proof of Theorem 1.2. Obviously, Theorem 1.2 follows from Lemmas 2.2, 2.6, and 2.7 and Theorem A. □

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Research Article

Periodic Problems of Difference Equations and Ergodic Theory

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The necessary and sufficient conditions for solvability of the family of difference equations with periodic boundary condition were obtained using the notion of relative spectrum of linear bounded operator in the Banach space and the ergodic theorem. It is shown that when the condition of existence is satisfied, then such periodic solutions are built using the formula for the generalized inverse operator to the linear limited one.

1. The Problem and The Main Statement

The problem of existence of periodic solutions for classes of equations is well known. Though it is hard to mention all the contributors in a single paper, we would like to mark out well-developed Floke theory [1], which is used in analysis of linear differential equation systems by the means of monodromy matrix. Operator analogy of such theory in noncritical case (when there is single solution) for differential equations in Banach space was developed by Daletskyi and Krein [2].

This paper is dedicated to obtaining analogous conditions for a family of difference equations in Banach space and to building representations of corresponding solutions. The proposed approach allows obtaining solutions for both critical and noncritical cases. Note that this problem can be approached using well-developed pseudoinverse techniques in theory of boundary value problems [3]. In this paper we firstly build a new representation of the pseudoinverse operator based on results of ergodic theory, and then we provide the necessary and sufficient conditions that guarantee the existence of the corresponding solutions.

Let $\mathbf{B}$-complex Banach space with norm $\|\cdot\|$ and zero-element $\overline{0}$; $\mathcal{L}(\mathbf{B})$-Banach space of bounded linear operators from $\mathbf{B}$ to $\mathbf{B}$. In this paper we consider existence of periodic solutions of the equation

$$x_{n+1} = \lambda A_{n+1} x_n + h_{n+1}, \quad n \geqslant 0, \tag{1.1}$$

with periodicity condition

$$x_0 = x_m, \tag{1.2}$$

where $A_n \in \mathcal{L}(\mathbf{B})$, $A_{n+m} = A_n$, for all $n \geqslant 0$, λ is a complex parameter, and $\{h_n\}_{n=0}^{\infty}$ is a sequence in $\mathbf{B}$. The solution of the corresponding homogeneous equation to (1.1) has the following form [4]:

$$x_m(\lambda) = \Phi(m, n, \lambda) x_n(\lambda), \quad m \geqslant n, \tag{1.3}$$

where

$$\Phi(m, n, \lambda) = \lambda^{m-n} A_{m+1} A_m \cdots A_{n+1}, \quad m > n \tag{1.4}$$

is evolution operator for problem (1.1); $\Phi(m, m, \lambda) = E$, where E is identity operator. Let us remark that $U(m, \lambda) = \Phi(m, 0, \lambda)$, $U(0, \lambda) = E$ and $U(k+n, \lambda) = U(k, \lambda) U(n, \lambda)$. Operator $U(m, \lambda)$ is traditionally called monodromy operator.

We can represent [4] the solution (1.1) with arbitrary initial condition $x(0, \lambda) = x_0$, $x_0 \in \mathbf{B}$ in the form

$$x_k(\lambda) = \Phi(k, 0, \lambda) x_0 + g(k, \lambda), \tag{1.5}$$

where

$$g(k, \lambda) = \sum_{i=0}^{k} \Phi(k, i, \lambda) h_i. \tag{1.6}$$

If we substitute this representation in boundary condition (1.2), we obtain operator equation

$$x_0(\lambda) - x_m(\lambda) = x_0 - \Phi(m, 0, \lambda) x_0 - g(m, \lambda) = \overline{0}. \tag{1.7}$$

According to notations, we get operator equation

$$(E - U(m, \lambda)) x_0 = g(m, \lambda). \tag{1.8}$$

Boundary value problem (1.1), (1.2) has periodic solution if and only if operator equation (1.8) is solvable.

Following the paper [5], point λ is called *right stable point* if monodromy operator satisfies inequality $\{\|U^n(m,\lambda)\| \leqslant c,\, n \geqslant 0\}$.

Denote $\rho_{NS}(E - U(m,\lambda)) = \{\lambda \in \mathbb{C} : R(E - U(m,\lambda)) = \overline{R(E - U(m,\lambda))}\}$ (this set coincides with the set of all $\lambda \in \mathbb{C}$ such that operator $E - U(m,\lambda)$ is normally solvable). It follows easily that resolvent set $\rho(E - U(m,\lambda))$ of the operator $E - U(m,\lambda)$ lies in $\rho_{NS}(E - U(m,\lambda))$.

In the sequel we assume that $\mathbf{B}$ is reflexive for simplicity [6].

The main result of this paper is contained in Theorem 1.1.

Theorem 1.1. *Let $\lambda \in \rho_{NS}(E - U(m,\lambda))$ be right stability point for* (1.1). *Then*

(a) *boundary value problem* (1.1), (1.2) *has solutions if and only if sequence $\{h_n\}_{n\in\mathbb{Z}_+}$, $h_n \in \mathbf{B}$ satisfies condition*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{i=0}^{m} U^k(m,\lambda)\Phi(m,i,\lambda)h_i}{n} = 0, \tag{1.9}$$

(b) *under condition* (1.9), *solutions of boundary value problem* (1.1), (1.2) *have the following form:*

$$x_n = U(n,\lambda)\lim_{k\to\infty}\frac{\sum_{m=1}^{k} U^m(k,\lambda)}{k}c + U(n,\lambda)G(n,\lambda)[h_n], \tag{1.10}$$

where c is an arbitrary element of Banach space $\mathbf{B}$, $G(n,\lambda)$-generalized Green operator of boundary value (1.1), (1.2), *which is defined by equality*

$$\begin{aligned} G(n,\lambda)[h_n] = {} & \sum_{k=0}^{\infty}(1-\mu)^k\left\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(m,\lambda)-U_0(\lambda))^l\right\}^{k+1}\sum_{i=0}^{m}\Phi(m,i,\lambda)h_i \\ & - U_0(\lambda)\sum_{i=0}^{m}\Phi(m,i,\lambda)h_i + \sum_{i=0}^{n}\Phi(n,i,\lambda)h_i. \end{aligned} \tag{1.11}$$

2. Auxiliary Result

Let us formulate and prove a number of auxiliary lemmas, which entail the theorem.

Lemma 2.1. *If $\lambda \in \rho_{NS}(E - U(m,\lambda))$, then boundary value problem* (1.1), (1.2) *is solvable if and only if sequence h_n satisfies the condition*

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n}\sum_{i=0}^{m} U^k(m,\lambda)\Phi(m,i,\lambda)h_i}{n} = 0. \tag{2.1}$$

Proof. From the assumption above it follows that the conditions of statistical ergodic theorem hold [6]. Then

$$R(E-U(m,\lambda)) = \left\{ x \in \mathbf{B} : \lim_{n\to\infty} U_n(m,\lambda)x = \overline{0}, U_n(m,\lambda) = \frac{\sum_{k=1}^{n} U^k(m,\lambda)}{n} \right\}. \tag{2.2}$$

It follows from the equation above that element $g(m,\lambda)$ lies in value set of the operator $E - U(m,\lambda)$ if and only if

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} U^k(m,\lambda)}{n} \sum_{i=0}^{m} \Phi(m,i,\lambda)h_i = 0, \tag{2.3}$$

which proves the lemma. □

Consider the following consequences of the assumptions above for further reasoning. Suppose that $\lambda \in \rho_{NS}(E-U(m,\lambda))$ and λ is right stable point of the monodromy operator, such that λ define eigenspace $N(E-U(m,\lambda))$, which coincides with the values set of operator $U_0(\lambda)x = \lim_{n\to\infty} U_n(m,\lambda)x$. This operator satisfies the following conditions [6]:

$$\text{(i)}\quad U_0(\lambda) = U_0^2(\lambda), \qquad \text{(ii)}\quad U_0(\lambda) = U(m,\lambda)U_0(\lambda), \qquad \text{(iii)}\quad U_0(\lambda) = U_0(\lambda)U(m,\lambda). \tag{2.4}$$

Lemma 2.2. *Operator* $E - U(m,\lambda) + U_0(\lambda) : \mathbf{B} \to \mathbf{B}$ *has bounded inverse of the form*

$$(E - U(m,\lambda) + U_0(\lambda))^{-1} = \sum_{k=0}^{\infty} (\mu-1)^k \left\{ \sum_{l=0}^{\infty} \mu^{-l-1} (U(m,\lambda) - U_0(\lambda))^l \right\}^{k+1}, \tag{2.5}$$

for all $\mu > 1 : |1-\mu| < 1/\|R_\mu\|$.

Proof. Let us show that $\operatorname{Ker}(I - U(m,\lambda) + U_0(\lambda)) = \overline{0}$. Indeed, if $x \in \operatorname{Ker}(I - U(m,\lambda) + U_0(\lambda))$, then

$$(I - U(m,\lambda) + U_0(\lambda))x = \overline{0}. \tag{2.6}$$

Since $(I - U(m,\lambda))x \in \operatorname{Im}(I - U(m,\lambda))$ and $U_0(\lambda)x \in \operatorname{Ker}(I - U(m,\lambda))$ [6], subspaces $\operatorname{Im}(I - U(m,\lambda))$ and $\operatorname{Ker}(I - U(m,\lambda))$ intersect only at zero point, and condition (2.6) is satisfied if and only if $(I - U(m,\lambda))x = \overline{0}$ and $U_0(\lambda)x = \overline{0}$. This is possible if and only if $x = \overline{0}$. Let us show that $\operatorname{Im}(I - U(m,\lambda) + U_0(\lambda)) = \mathbf{B}$. Note [6] $\mathbf{B} = \operatorname{Ker}(I - U(m,\lambda)) \oplus \operatorname{Im}(I - U(m,\lambda)) = \operatorname{Im}(U_0(\lambda)) \oplus Im(I - U(m,\lambda))$. It follows from the last decomposition that any element $x \in \mathbf{B}$ has the form

$(I - U(m,\lambda))y + U_0(\lambda)z$, where $y, z \in \mathbf{B}$, which proves that $\operatorname{Im}(I - U(m,\lambda) + U_0(\lambda)) = \mathbf{B}$. Hence according to the Banach theorem [6] original operator has inverse since it bijectively maps $\mathbf{B}$ to itself. Therefore point $\mu = 1$ is regular [6] for the operator $\mu I - U(m,\lambda) + U_0(\lambda)$. Since powers of the operator $U(m,\lambda)$ are uniformly bounded and spectral radius $r_{U(m,\lambda)} \leqslant 1$ ($\sqrt[n]{\|U(m,\lambda)^n\|} \leqslant \sqrt[n]{c}$, then $r_{U(m,\lambda)} = \lim_{n\to\infty}\sqrt[n]{\|U(m,\lambda)^n\|} \leqslant \lim_{n\to\infty}\sqrt[n]{c} = 1$). It is well known [6] that resolvent set of a bounded operator is open. Number $\mu = 1 \in \rho(U(m,\lambda) - U_0(\lambda))$; thus there exist a neighborhood of μ such that each point from the neighborhood belongs to resolvent set. For any point $\mu > r_{(U(m,\lambda)-U_0(\lambda))}$ that belongs to the neighborhood there exists a resolvent [6], which has the form of converging in the norm series

$$R_\mu := R_\mu(U(m,\lambda) - U_0(\lambda)) = \sum_{l=0}^{\infty}\mu^{-l-1}(U(m,\lambda) - U_0(\lambda))^l. \tag{2.7}$$

Using the analyticity of the resolvent and well-known identity for points $\mu > 1$ such that $|1-\mu| < 1/(\|R_\mu(U(m,\lambda) - U_0(\lambda))\|)$, we obtain

$$R_1 = \sum_{k=0}^{\infty}(\mu-1)^k R_\mu^{k+1}. \tag{2.8}$$

Finally, by substituting the series in the equation above, we get (2.5), which proves the lemma. □

Let us introduce some notation first before proving next statement.

Definition 2.3. Operator $L^- \in \mathcal{L}(\mathbf{B})$ is called generalized inverse for operator $L \in \mathcal{L}(\mathbf{B})$ [3] if the following conditions hold:

$$(1)\quad LL^-L = L, \qquad (2)\quad L^-LL^- = L^-. \tag{2.9}$$

Lemma 2.4. *Operator $E - U(m,\lambda)$ is generalized inverse and*

$$(E - U(m,\lambda))^- = (E - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda), \tag{2.10}$$

or in the form of converging operator series

$$(E - U(m,\lambda))^- = \sum_{k=0}^{\infty}(\mu-1)^k\left\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(m,\lambda) - U_0(\lambda))^l\right\}^{k+1} - U_0(\lambda), \tag{2.11}$$

for all $\mu > 1 : |1-\mu| < 1/\|R_\mu\|$.

Proof. It suffices to check conditions (1) and (2) of the Definition 2.3. We use both representations (2.10), (2.11) and the expression (2.4) for operator $U_0(\lambda)$. Consider the following product:

$$
\begin{aligned}
&(I-U(m,\lambda))\Big((I-U(m,\lambda)+U_0(\lambda))^{-1}-U_0(\lambda)\Big)(I-U(\lambda))\\
&\quad=((I-U(m,\lambda)+U_0(\lambda))\;\;-U_0(\lambda))\times\Big((I-U(m,\lambda)+U_0(\lambda))^{-1}-U_0(\lambda)\Big)(I-U(m,\lambda))\\
&\quad=\Big(I-U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}-(I-U(m,\lambda)+U_0(\lambda))U_0(\lambda)+U_0(\lambda)^2\Big)\\
&\qquad\times(I-U(m,\lambda))\\
&\quad=\Big(I-U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}\Big)\times(I-U(m,\lambda))\\
&\quad=\Big(I-U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}\Big)((I-U(m,\lambda)+U_0(\lambda))-U_0(\lambda))\\
&\quad=I-U(m,\lambda)+U_0(\lambda)-U_0(\lambda)-U_0(\lambda)+U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}U_0(\lambda)\\
&\quad=I-U(m,\lambda)-U_0(\lambda)+U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}U_0(\lambda).
\end{aligned}
\tag{2.12}
$$

Note that $U_0(\lambda)(U(m,\lambda)-U_0(\lambda))^l=0$ for any $l\in\mathbb{N}$ (this directly follows from (2.4) using formula of binominal coefficient). Now, prove that

$$
\begin{aligned}
U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}U_0(\lambda)&=U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}\\
&=(I-U(m,\lambda)+U_0(\lambda))^{-1}U_0(\lambda)\\
&=U_0(\lambda).
\end{aligned}
\tag{2.13}
$$

Indeed

$$
\begin{aligned}
U_0(\lambda)(I-U(m,\lambda)+U_0(\lambda))^{-1}U_0(\lambda)&=\sum_{k=0}^{\infty}(\mu-1)^kU_0(\lambda)\left\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(m,\lambda)-U_0(\lambda))^l\right\}^{k+1}U_0(\lambda)\\
&=\sum_{k=0}^{\infty}\Bigg(\left(\mu^{-1}\right)^{k+1}(\mu-1)^kU_0(\lambda)+(\mu-1)^kU_0(\lambda)\\
&\qquad\times\left\{\sum_{l=1}^{\infty}\mu^{-l-1}(U(m,\lambda)-U_0(\lambda))^l\right\}^{k+1}\Bigg)U_0(\lambda)\\
&=\sum_{k=0}^{+\infty}\mu^{-k-1}(\mu-1)^kU_0(\lambda)\\
&=\frac{1}{\mu}\sum_{k=0}^{+\infty}\left(\frac{\mu-1}{\mu}\right)^kU_0(\lambda)\\
&=\frac{1}{\mu}\frac{1}{1-(\mu-1)/\mu}U_0(\lambda)\\
&=U_0(\lambda).
\end{aligned}
\tag{2.14}
$$

Thus

$$I - U(m,\lambda) - U_0(\lambda) + U_0(\lambda)(I - U(m,\lambda) + U_0(\lambda))^{-1}U_0(\lambda) = I - U(m,\lambda). \tag{2.15}$$

We have that the operator $I - U(m,\lambda)$ satisfies condition (1) of the Definition 2.3. Let us check condition (2)

$$\begin{aligned}
&\left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right)(I - U(m,\lambda))\left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right) \\
&\quad = \left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right)((I - U(m,\lambda) + U_0(\lambda)) - U_0(\lambda)) \\
&\qquad \times \left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right) \\
&\quad = \left(I - U_0(\lambda)(I - U(m,\lambda) + U_0(\lambda)) - (I - U(m,\lambda) + U_0(\lambda))^{-1}U_0(\lambda) + U_0(\lambda)^2\right) \\
&\qquad \times \left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right) \\
&\quad = \left(I - (I - U(m,\lambda) + U_0(\lambda))^{-1}U_0(\lambda)\right)\left((I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda)\right) \\
&\quad = (I - U(m,\lambda) + U_0(\lambda))^{-1} - (I - U(m,\lambda) + U_0(\lambda))^{-1}U_0(\lambda)(I - U(m,\lambda) + U_0(\lambda))^{-1} \\
&\qquad - U_0(\lambda) + (I - U(m,\lambda) + U_0(\lambda))^{-1}U_0(\lambda) \\
&\quad = (I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda) - U_0(\lambda) + U_0(\lambda) \\
&\quad = (I - U(m,\lambda) + U_0(\lambda))^{-1} - U_0(\lambda).
\end{aligned} \tag{2.16}$$

□

3. Proof of Theorem 1.1

According to general theory of linear equations solvability [3], we obtain that the problem (1.1), (1.2) is solvable for sets $\{h_n\}_n \in \mathbb{Z}_+$ that satisfy the condition

$$U_0(\lambda)g(m,\lambda) = 0. \tag{3.1}$$

This condition along with Lemma 2.1 is equivalent to represantion (a) of the Theorem 1.1.

Under such a condition, all solutions of the problem (1.1), (1.2) have the form

$$\begin{aligned}
x_n &= U(n,\lambda)U_0(\lambda)c + U(n,\lambda)(I - U(m,\lambda))^{-}g(m,\lambda) + g(n,\lambda) \\
&= U(n,\lambda)U_0(\lambda)c + U(n,\lambda)\sum_{k=0}^{\infty}(\mu - 1)^k\left\{\sum_{l=0}^{\infty}\mu^{-l-1}(U(m,\lambda) - U_0(\lambda))^l\right\}^{k+1} g(m,\lambda) \\
&\quad - U(n,\lambda)U_0(\lambda)g(m,\lambda) + g(n,\lambda),
\end{aligned} \tag{3.2}$$

which along with notations introduced is equivalent to representation (b) of the theorem.

4. Comments and Examples

Remark 4.1. Suppose B is Hilbert space, in such case we can show that formulas (2.10), (2.11) give us the representation for the Moore-Penrose pseudoinverse [7, 8] for $E - U(m,\lambda)$ with $U_0(\lambda)$ being self-adjoint operator (orthogonal projector) [6].

Remark 4.2. Supposing $A_k^{-1} \in \mathcal{L}(\mathbf{B}) \in L(B)$ exist for all $k = \overline{0, m-1}$, then the following equation holds: $\Phi(k,i,\lambda) = U(k,\lambda)U^{-1}(i,\lambda), k > i$. This allows representing the solutions of (1.1), (1.2) using only the family of operators $U(n,\lambda)$ and their inverse.

Let us illustrate the statements proved above on example of two-dimensional systems.
(1) Consider equation

$$\overrightarrow{x}_{n+1} = \lambda A_{n+1}\overrightarrow{x}_n + \overrightarrow{h}_{n+1}, \quad n \geqslant 0 \tag{4.1}$$

with periodicity condition

$$\overrightarrow{x}_3 = \overrightarrow{x}_0, \tag{4.2}$$

where $\overrightarrow{x}_n = (x_n^1, x_n^2)^T, x_n^1, x_n^2 \in \mathbb{R}$, $\overrightarrow{h}_n = ((3\sqrt{3}r)/4\pi, 0)^T$,

$$A_n = \begin{pmatrix} -\dfrac{1}{2} & -\dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix}, \quad \forall n \geqslant 0. \tag{4.3}$$

It is easy to see that

$$\overrightarrow{x}_3 = \lambda^3\overrightarrow{x}_0 + g(3,\lambda), \tag{4.4}$$

where

$$g(3,\lambda) = \left(\frac{-3\sqrt{3}r\lambda - 3\sqrt{3}r\lambda^2 + 6\sqrt{3}r}{8\pi}, \frac{9r\lambda - 9r\lambda^2}{8\pi}\right)^T. \tag{4.5}$$

Then the following hold for all $k \geqslant 0$

$$U(3k+1,\lambda) = \lambda^{3k+1}A_2, \qquad U(3k+2,\lambda) = \lambda^{3k+2}\begin{pmatrix} -\dfrac{1}{2} & \dfrac{\sqrt{3}}{2} \\ \dfrac{\sqrt{3}}{2} & -\dfrac{1}{2} \end{pmatrix}, \qquad U(3k+3,\lambda) = \lambda^{3k+3}E. \tag{4.6}$$

By substituting periodicity condition (4.2) into (4.4) we obtain an equation depending on $\overrightarrow{x}_0$:

$$\left(1 - \lambda^3\right)\overrightarrow{x}_0 = g(3,\lambda). \tag{4.7}$$

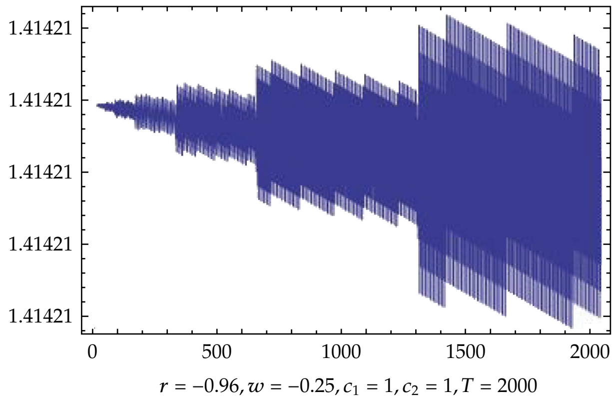


Figure 1

Consider the case when $\lambda = 1$. In such case (4.7) turns into $0\overrightarrow{x}_0 = (0,0)^T$ which holds for arbitrary initial vector $\overrightarrow{x}_0 \in \mathbb{R}^2$. Obviously $U^n(1,1) = U(n,1)$ and $U_0(1) = E$. According to Theorem 1.1, all periodic solutions of (4.1) have the form

$$\begin{pmatrix} x_n^1(c_1,c_2) \\ x_n^2(c_1,c_2) \end{pmatrix} = \begin{pmatrix} \cos\dfrac{2\pi}{3}n & \sin\dfrac{2\pi}{3}n \\ -\sin\dfrac{2\pi}{3}n & \cos\dfrac{2\pi}{3}n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \dfrac{3r}{2\pi}\sin\dfrac{2\pi}{3}n \\ 0 \end{pmatrix}, \tag{4.8}$$

for all $\overrightarrow{c} = (c_1,c_2)^T \in \mathbb{R}^2$.

(2) We can search for periodic solutions of any period w in previous problem. They have common view

$$\overrightarrow{x}_n(c_1,c_2,w,r) = \begin{pmatrix} \cos\dfrac{2\pi}{w}n & \sin\dfrac{2\pi}{w}n \\ -\sin\dfrac{2\pi}{w}n & \cos\dfrac{2\pi}{w}n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} \dfrac{rw}{2\pi}\sin\dfrac{2\pi}{w}n \\ 0 \end{pmatrix}, \tag{4.9}$$

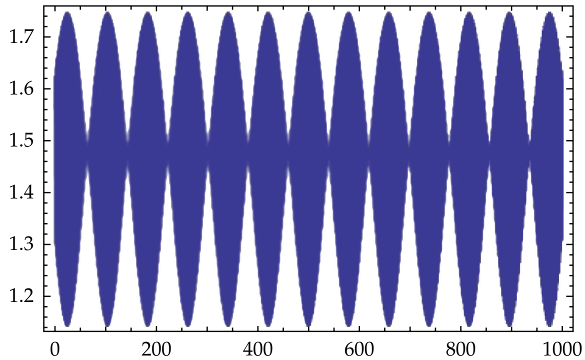
where c_1, c_2, w, r are parameters.

To illustrate complexity of the set we did the following.

Recall that the length of vector $\overrightarrow{x}_n$ is $\ell\overrightarrow{x}_n = \sqrt{(\overrightarrow{x}_n^1)^2 + (\overrightarrow{x}_n^2)^2}$. System (4.9) was implemented using the Wolfram Mathematica 7 framework. x-axis corresponds to time, while y-axis corresponds to the length of the vector. The length of the vector was calculated in the integer moments of time n. The points obtained in such way were connected in a piecewise linear way. The results obtained for particular values of the parameters are depicted on the following figures.

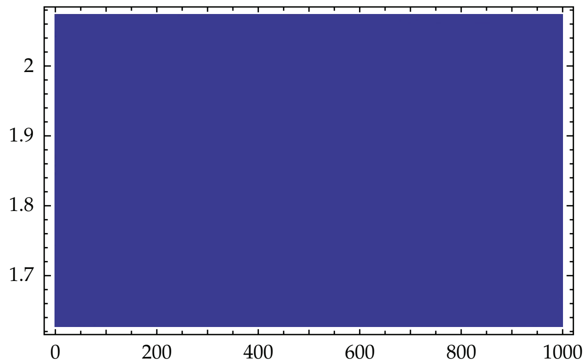
We can see how the trajectory of vector length densely fills rectangle or turns into a line (Figures 3 and 4). Figures 1, 2, 5, and 6 demonstrate that the trajectory can fill structured sets. The structure depicted on Figure 1 resembles fractal.

This allows us to conclude that behavior of the system is rather complex; it can undergo unpredictable changes with the slightest variations of a single parameter. We must admit that effects described need further theoretical investigation.



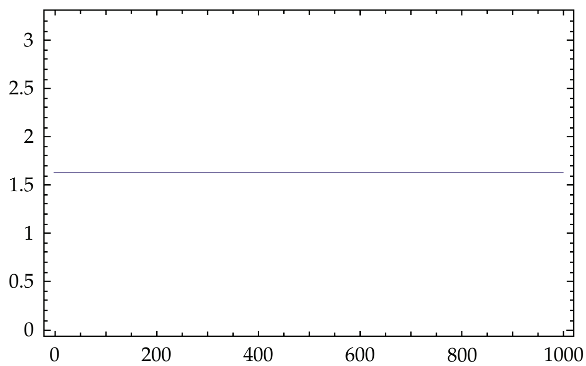
$r = -0.96, w = -3.95, c_1 = -1.23, c_2 = -1.06, T = 1000$

Figure 2



$r = -0.96, w = 4, c_1 = -1.23, c_2 = -1.06, T = 1000$

Figure 3



$r = -0.96, w = 0.5, c_1 = -1.23, c_2 = -1.06, T = 1000$

Figure 4

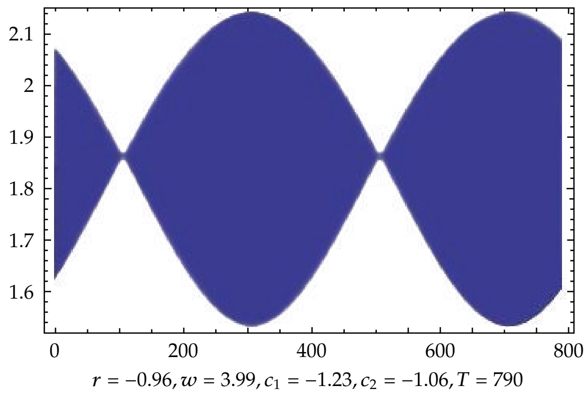


Figure 5

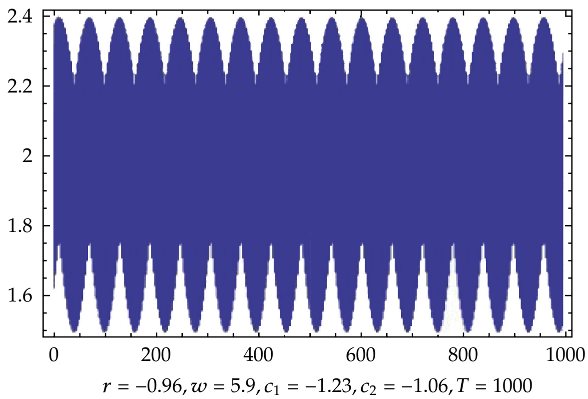


Figure 6

Acknowledgment

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Research Article

Positive Solutions to Boundary Value Problems of Nonlinear Fractional Differential Equations

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We study the existence of positive solutions for the boundary value problem of nonlinear fractional differential equations $D_{0^+}^{\alpha}u(t) + \lambda f(u(t)) = 0$, $0 < t < 1$, $u(0) = u(1) = u'(0) = 0$, where $2 < \alpha \leq 3$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, λ is a positive parameter, and $f : (0, +\infty) \to (0, +\infty)$ is continuous. By the properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As an application, some examples are presented to illustrate the main results.

1. Introduction

Fractional differential equations have been of great interest recently. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications; see [1–4]. It should be noted that most of papers and books on fractional calculus are devoted to the solvability of linear initial fractional differential equations on terms of special functions.

Recently, there are some papers dealing with the existence of solutions (or positive solutions) of nonlinear initial fractional differential equations by the use of techniques of nonlinear analysis (fixed-point theorems, Leray-Schauder theory, Adomian decomposition method, etc.); see [5–11]. In fact, there has the same requirements for boundary conditions. However, there exist some papers considered the boundary value problems of fractional differential equations; see [12–19].

Yu and Jiang [19] examined the existence of positive solutions for the following problem:

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0, \end{gathered} \tag{1.1}$$

where $2 < \alpha \leq 3$ is a real number, $f \in C([0,1] \times [0,+\infty), (0,+\infty))$, and $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional differentiation. By using the properties of the Green function, they obtained some existence criteria for one or two positive solutions for singular and nonsingular boundary value problems by means of the Krasnosel'skii fixed point theorem and a mixed monotone method.

To the best of our knowledge, there is very little known about the existence of positive solutions for the following problem:

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + \lambda f(u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0, \end{gathered} \tag{1.2}$$

where $2 < \alpha \leq 3$ is a real number, $D_{0^+}^{\alpha}$ is the Riemann-Liouville fractional derivative, λ is a positive parameter and $f : (0,+\infty) \to (0,+\infty)$ is continuous.

On one hand, the boundary value problem in [19] is the particular case of problem (1.2) as the case of $\lambda = 1$. On the other hand, as Yu and Jiang discussed in [19], we also give some existence results by the fixed point theorem on a cone in this paper. Moreover, the purpose of this paper is to derive a λ-interval such that, for any λ lying in this interval, the problem (1.2) has existence and multiplicity on positive solutions.

In this paper, by analogy with boundary value problems for differential equations of integer order, we firstly give the corresponding Green function named by fractional Green's function and some properties of the Green function. Consequently, the problem (1.2) is reduced to an equivalent Fredholm integral equation. Finally, by the properties of the Green function and Guo-Krasnosel'skii fixed point theorem on cones, the eigenvalue intervals of the nonlinear fractional differential equation boundary value problem are considered, some sufficient conditions for the nonexistence and existence of at least one or two positive solutions for the boundary value problem are established. As an application, some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we give some background materials from fractional calculus theory to facilitate analysis of problem (1.2). These materials can be found in the recent literature; see [19–21].

Definition 2.1 (see [20]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0,+\infty) \to \mathbb{R}$ is given by

$$D_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_0^t \frac{f(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.1}$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α, provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (see [20]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{2.2}$$

provided that the right side is pointwise defined on $(0, +\infty)$.

From the definition of the Riemann-Liouville derivative, we can obtain the following statement.

Lemma 2.3 (see [20]). *Let $\alpha > 0$. If we assume $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0^+}^{\alpha} u(t) = 0 \tag{2.3}$$

has $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}$, $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, as unique solutions, where N is the smallest integer greater than or equal to α.

Lemma 2.4 (see [20]). *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0,1) \cap L(0,1)$. Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \tag{2.4}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$, where N is the smallest integer greater than or equal to α.

In the following, we present the Green function of fractional differential equation boundary value problem.

Lemma 2.5 (see [19]). *Let $h \in C[0,1]$ and $2 < \alpha \leq 3$. The unique solution of problem*

$$\begin{gathered} D_{0^+}^{\alpha} u(t) + h(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = u'(0) = 0 \end{gathered} \tag{2.5}$$

is

$$u(t) = \int_0^1 G(t,s) h(s) ds, \tag{2.6}$$

where

$$G(t,s)=\begin{cases}\dfrac{t^{\alpha-1}(1-s)^{\alpha-1}-(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1,\\ \dfrac{t^{\alpha-1}(1-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1.\end{cases} \tag{2.7}$$

Here $G(t,s)$ *is called the Green function of boundary value problem* (2.5).

The following properties of the Green function play important roles in this paper.

Lemma 2.6 (see [19]). *The function* $G(t,s)$ *defined by* (2.7) *satisfies the following conditions:*

(1) $G(t,s)=G(1-s,1-t)$, *for* $t,s\in(0,1)$;

(2) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\le\Gamma(\alpha)G(t,s)\le(\alpha-1)s(1-s)^{\alpha-1}$, *for* $t,s\in(0,1)$;

(3) $G(t,s)>0$, *for* $t,s\in(0,1)$;

(4) $t^{\alpha-1}(1-t)s(1-s)^{\alpha-1}\le\Gamma(\alpha)G(t,s)\le(\alpha-1)(1-t)t^{\alpha-1}$, *for* $t,s\in(0,1)$.

The following lemma is fundamental in the proofs of our main results.

Lemma 2.7 (see [21]). *Let* X *be a Banach space, and let* $P\subset X$ *be a cone in* X. *Assume* Ω_1, Ω_2 *are open subsets of* X *with* $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$, *and let* $S:P\to P$ *be a completely continuous operator such that, either*

$(A1)$ $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_2$ *or*

$(A2)$ $\|Sw\|\ge\|w\|$, $w\in P\cap\partial\Omega_1$, $\|Sw\|\le\|w\|$, $w\in P\cap\partial\Omega_2$.

Then S *has a fixed point in* $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

For convenience, we set $q(t)=t^{\alpha-1}(1-t)$, $k(s)=s(1-s)^{\alpha-1}$; then

$$q(t)k(s)\le\Gamma(\alpha)G(t,s)\le(\alpha-1)k(s). \tag{2.8}$$

3. Main Results

In this section, we establish the existence of positive solutions for boundary value problem (1.2).

Let Banach space $E=C[0,1]$ be endowed with the norm $\|u\|=\max_{0\le t\le 1}|u(t)|$. Define the cone $P\subset E$ by

$$P=\left\{u\in E: u(t)\ge\frac{q(t)}{\alpha-1}\|u\|,\ t\in[0,1]\right\}. \tag{3.1}$$

Suppose that u is a solution of boundary value problem (1.2). Then

$$u(t)=\lambda\int_0^1 G(t,s)f(u(s))ds,\quad t\in[0,1]. \tag{3.2}$$

We define an operator $A_\lambda : P \to E$ as follows:

$$(A_\lambda u)(t) = \lambda \int_0^1 G(t,s) f(u(s))ds, \quad t \in [0,1]. \tag{3.3}$$

By Lemma 2.6, we have

$$\begin{aligned} \|A_\lambda u\| &\le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s)f(u(s))ds, \\ (A_\lambda u)(t) &\ge \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(t)k(s)f(u(s))ds \\ &\ge \frac{q(t)}{\alpha-1}\|A_\lambda u\|. \end{aligned} \tag{3.4}$$

Thus, $A_\lambda(P) \subset P$.

Then we have the following lemma.

Lemma 3.1. *$A_\lambda : P \to P$ is completely continuous.*

Proof. The operator $A_\lambda : P \to P$ is continuous in view of continuity of $G(t,s)$ and $f(u(t))$. By means of the Arzela-Ascoli theorem, $A_\lambda : P \to P$ is completely continuous.

For convenience, we denote

$$\begin{gathered} F_0 = \lim_{u\to 0^+} \sup \frac{f(u)}{u}, \qquad F_\infty = \lim_{u\to +\infty} \sup \frac{f(u)}{u}, \\ f_0 = \lim_{u\to 0^+} \inf \frac{f(u)}{u}, \qquad f_\infty = \lim_{u\to +\infty} \inf \frac{f(u)}{u}, \\ C_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s)ds, \\ C_2 = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha-1)} q(s)k(s)ds, \\ C_3 = \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha-1)} k(s)ds. \end{gathered} \tag{3.5}$$

□

Theorem 3.2. *If there exists $l \in (0,1)$ such that $q(l) f_\infty C_2 > F_0 C_1$ holds, then for each*

$$\lambda \in \left((q(l) f_\infty C_2)^{-1}, (F_0 C_1)^{-1} \right), \tag{3.6}$$

the boundary value problem (1.2) *has at least one positive solution. Here we impose $(q(l) f_\infty C_2)^{-1} = 0$ if $f_\infty = +\infty$ and $(F_0 C_1)^{-1} = +\infty$ if $F_0 = 0$.*

Proof. Let λ satisfy (3.6) and $\varepsilon > 0$ be such that

$$\left(q(l)\left(f_\infty - \varepsilon\right)C_2\right)^{-1} \leq \lambda \leq ((F_0 + \varepsilon)C_1)^{-1}. \tag{3.7}$$

By the definition of F_0, we see that there exists $r_1 > 0$ such that

$$f(u) \leq (F_0 + \varepsilon)u, \quad \text{for } 0 < u \leq r_1. \tag{3.8}$$

So if $u \in P$ with $\|u\| = r_1$, then by (3.7) and (3.8), we have

$$\begin{aligned}\|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (\alpha-1)k(s)f(u(s))ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 (\alpha-1)k(s)(F_0+\varepsilon)r_1 ds \\ &= \lambda(F_0+\varepsilon)r_1C_1 \\ &\leq r_1 = \|u\|.\end{aligned} \tag{3.9}$$

Hence, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_1. \tag{3.10}$$

Let $r_3 > 0$ be such that

$$f(u) \geq \left(f_\infty - \varepsilon\right)u, \quad \text{for } u \geq r_3. \tag{3.11}$$

If $u \in P$ with $\|u\| = r_2 = \max\{2r_1, r_3\}$, then by (3.7) and (3.11), we have

$$\begin{aligned}\|A_\lambda u\| &\geq A_\lambda u(l) \\ &= \lambda\int_0^1 G(l,s)f(u(s))ds \\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 q(l)k(s)f(u(s))ds \\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 q(l)k(s)\left(f_\infty-\varepsilon\right)u(s)ds \\ &\geq \frac{\lambda}{\Gamma(\alpha)}\int_0^1 \frac{q(l)}{\alpha-1}q(s)k(s)\left(f_\infty-\varepsilon\right)\|u\|ds \\ &= \lambda q(l)C_2\left(f_\infty-\varepsilon\right)\|u\| \geq \|u\|.\end{aligned} \tag{3.12}$$

Thus, if we set $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.13}$$

Now, from (3.10), (3.13), and Lemma 2.7, we guarantee that A_λ has a fixed-point $u \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$, and clearly u is a positive solution of (1.2). The proof is complete. □

Theorem 3.3. *If there exists $l \in (0,1)$ such that $q(l)C_2 f_0 > F_\infty C_1$ holds, then for each*

$$\lambda \in \left(\left(q(l) f_0 C_2\right)^{-1}, \left(F_\infty C_1\right)^{-1}\right), \tag{3.14}$$

the boundary value problem (1.2) *has at least one positive solution. Here we impose $(q(l) f_0 C_2)^{-1} = 0$ if $f_0 = +\infty$ and $(F_\infty C_1)^{-1} = +\infty$ if $F_\infty = 0$.*

Proof. Let λ satisfy (3.14) and $\varepsilon > 0$ be such that

$$\left(q(l)\left(f_0 - \varepsilon\right)C_2\right)^{-1} \leq \lambda \leq \left(\left(F_\infty + \varepsilon\right)C_1\right)^{-1}. \tag{3.15}$$

From the definition of f_0, we see that there exists $r_1 > 0$ such that

$$f(u) \geq \left(f_0 - \varepsilon\right)u, \quad \text{for } 0 < u \leq r_1. \tag{3.16}$$

Further, if $u \in P$ with $\|u\| = r_1$, then similar to the second part of Theorem 3.2, we can obtain that $\|A_\lambda u\| \geq \|u\|$. Thus, if we choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$, then

$$\|A_\lambda u\| \geq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.17}$$

Next, we may choose $R_1 > 0$ such that

$$f(u) \leq (F_\infty + \varepsilon)u, \quad \text{for } u \geq R_1. \tag{3.18}$$

We consider two cases.

Case 1. Suppose f is bounded. Then there exists some $M > 0$, such that

$$f(u) \leq M, \quad \text{for } u \in (0, +\infty). \tag{3.19}$$

We define $r_3 = \max\{2r_1, \lambda M C_1\}$, and $u \in P$ with $\|u\| = r_3$, then

$$\begin{aligned} \|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(u(s)) ds \\ &\leq \frac{\lambda M}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) ds \\ &\leq \lambda M C_1 \\ &\leq r_3 \leq \|u\|. \end{aligned} \tag{3.20}$$

Hence,

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in P_{r_3} = \{u \in P : \|u\| \leq r_3\}. \tag{3.21}$$

Case 2. Suppose f is unbounded. Then there exists some $r_4 > \max\{2r_1, R_1\}$, such that

$$f(u) \leq f(r_4), \quad \text{for } 0 < u \leq r_4. \tag{3.22}$$

Let $u \in P$ with $\|u\| = r_4$. Then by (3.15) and (3.18), we have

$$\begin{aligned} \|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(u(s)) ds \\ &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s)(F_\infty + \varepsilon)\|u\| ds \\ &\leq \lambda C_1 (F_\infty + \varepsilon)\|u\| \\ &\leq \|u\|. \end{aligned} \tag{3.23}$$

Thus, (3.21) is also true.

In both Cases 1 and 2, if we set $\Omega_2 = \{u \in E : \|u\| < r_2 = \max\{r_3, r_4\}\}$, then

$$\|A_\lambda u\| \leq \|u\|, \quad \text{for } u \in P \cap \partial\Omega_2. \tag{3.24}$$

Now that we obtain (3.17) and (3.24), it follows from Lemma 2.7 that A_λ has a fixed-point $u \in P \cap (\overline{\Omega_2} \setminus \Omega_1)$ with $r_1 \leq \|u\| \leq r_2$. It is clear u is a positive solution of (1.2). The proof is complete. □

Theorem 3.4. *Suppose there exist $l \in (0,1)$, $r_2 > r_1 > 0$ such that $q(l) > (\alpha-1)r_1/r_2$, and f satisfy*

$$\min_{(q(l)/(\alpha-1))r_1 \leq u \leq r_1} f(u) \geq \frac{r_1}{\lambda(\alpha-1)q(l)C_3}, \qquad \max_{0 \leq u \leq r_2} f(u) \leq \frac{r_2}{\lambda C_1}. \tag{3.25}$$

Then the boundary value problem (1.2) *has a positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$.*

Proof. Choose $\Omega_1 = \{u \in E : \|u\| < r_1\}$; then for $u \in P \cap \partial\Omega_1$, we have

$$\begin{aligned}
\|A_\lambda u\| &\geq A_\lambda u(l) \\
&= \lambda \int_0^1 G(l,s) f(u(s)) ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l) k(s) f(u(s)) ds \\
&\geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l) k(s) \min_{(q(l)/(\alpha-1)) r_1 \leq u \leq r_1} f(u(s)) ds \\
&\geq \lambda(\alpha-1) q(l) C_3 \frac{r_1}{\lambda(\alpha-1) q(l) C_3} \\
&= r_1 = \|u\|.
\end{aligned} \tag{3.26}$$

On the other hand, choose $\Omega_2 = \{u \in E : \|u\| < r_2\}$, then for $u \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
\|A_\lambda u\| &\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) f(u(s)) ds \\
&\leq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1) k(s) \max_{0 \leq u \leq r_2} f(u(s)) ds \\
&\leq \lambda C_1 \frac{r_2}{\lambda C_1} \\
&= r_2 = \|u\|.
\end{aligned} \tag{3.27}$$

Thus, by Lemma 2.7, the boundary value problem (1.2) has a positive solution $u \in P$ with $r_1 \leq \|u\| \leq r_2$. The proof is complete. □

For the reminder of the paper, we will need the following condition.

(H) $(\min_{u \in [(q(l)/(\alpha-1))r,\, r]} f(u))/r > 0$, where $l \in (0,1)$.

Denote

$$\lambda_1 = \sup_{r>0} \frac{r}{C_1 \max_{0 \leq u \leq r} f(u)}, \tag{3.28}$$

$$\lambda_2 = \inf_{r>0} \frac{r}{C_3 \min_{(q(l)/(\alpha-1)) r \leq u \leq r} f(u)}. \tag{3.29}$$

In view of the continuity of $f(u)$ and (H), we have $0 < \lambda_1 \leq +\infty$ and $0 \leq \lambda_2 < +\infty$.

Theorem 3.5. *Assume (H) holds. If $f_0 = +\infty$ and $f_\infty = +\infty$, then the boundary value problem* (1.2) *has at least two positive solutions for each $\lambda \in (0, \lambda_1)$.*

Proof. Define

$$a(r) = \frac{r}{C_1 \max_{0 \le u \le r} f(u)}. \tag{3.30}$$

By the continuity of $f(u)$, $f_0 = +\infty$ and $f_\infty = +\infty$, we have that $a(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r \to 0} a(r) = \lim_{r \to +\infty} a(r) = 0. \tag{3.31}$$

By (3.28), there exists $r_0 \in (0, +\infty)$, such that

$$a(r_0) = \sup_{r > 0} a(r) = \lambda_1; \tag{3.32}$$

then for $\lambda \in (0, \lambda_1)$, there exist constants c_1, c_2 $(0 < c_1 < r_0 < c_2 < +\infty)$ with

$$a(c_1) = a(c_2) = \lambda. \tag{3.33}$$

Thus,

$$f(u) \le \frac{c_1}{\lambda C_1}, \quad \text{for } u \in [0, c_1], \tag{3.34}$$

$$f(u) \le \frac{c_2}{\lambda C_1}, \quad \text{for } u \in [0, c_2]. \tag{3.35}$$

On the other hand, applying the conditions $f_0 = +\infty$ and $f_\infty = +\infty$, there exist constants d_1, d_2 $(0 < d_1 < c_1 < r_0 < c_2 < d_2 < +\infty)$ with

$$\frac{f(u)}{u} \ge \frac{1}{q^2(l)\lambda C_3}, \quad \text{for } u \in (0, d_1) \cup \left(\frac{q(l)}{\alpha - 1} d_2, +\infty\right). \tag{3.36}$$

Then

$$\min_{(q(l)/(\alpha-1))d_1 \le u \le d_1} f(u) \ge \frac{d_1}{\lambda(\alpha - 1)q(l)C_3}, \tag{3.37}$$

$$\min_{(q(l)/(\alpha-1))d_2 \le u \le d_2} f(u) \ge \frac{d_2}{\lambda(\alpha - 1)q(l)C_3}. \tag{3.38}$$

By (3.34) and (3.37), (3.35) and (3.38), combining with Theorem 3.4 and Lemma 2.7, we can complete the proof. □

Corollary 3.6. *Assume* (H) *holds. If* $f_0 = +\infty$ *or* $f_\infty = +\infty$*, then the boundary value problem* (1.2) *has at least one positive solution for each* $\lambda \in (0, \lambda_1)$.

Theorem 3.7. *Assume* (H) *holds. If* $f_0 = 0$ *and* $f_\infty = 0$, *then for each* $\lambda \in (\lambda_2, +\infty)$, *the boundary value problem* (1.2) *has at least two positive solutions.*

Proof. Define

$$b(r) = \frac{r}{C_3 \min_{(q(l)/(\alpha-1))r \le u \le r} f(u)}. \tag{3.39}$$

By the continuity of $f(u)$, $f_0 = 0$ and $f_\infty = 0$, we easily see that $b(r) : (0, +\infty) \to (0, +\infty)$ is continuous and

$$\lim_{r \to 0} b(r) = \lim_{r \to +\infty} b(r) = +\infty. \tag{3.40}$$

By (3.29), there exists $r_0 \in (0, +\infty)$, such that

$$b(r_0) = \inf_{r>0} b(r) = \lambda_2. \tag{3.41}$$

For $\lambda \in (\lambda_2, +\infty)$, there exist constants d_1, d_2 $(0 < d_1 < r_0 < d_2 < +\infty)$ with

$$b(d_1) = b(d_2) = \lambda. \tag{3.42}$$

Therefore,

$$\begin{gathered} f(u) \ge \frac{d_1}{\lambda(\alpha-1)q(l)C_3}, \quad \text{for } u \in \left[\frac{q(l)}{\alpha-1}d_1, d_1\right], \\ f(u) \ge \frac{d_2}{\lambda(\alpha-1)q(l)C_3}, \quad \text{for } u \in \left[\frac{q(l)}{\alpha-1}d_2, d_2\right]. \end{gathered} \tag{3.43}$$

On the other hand, using $f_0 = 0$, we know that there exists a constant c_1 $(0 < c_1 < d_1)$ with

$$\frac{f(u)}{u} \le \frac{1}{\lambda C_1}, \quad \text{for } u \in (0, c_1), \tag{3.44}$$

$$\max_{0 \le u \le c_1} f(u) \le \frac{c_1}{\lambda C_1}. \tag{3.45}$$

In view of $f_\infty = 0$, there exists a constant $c_2 \in (d_2, +\infty)$ such that

$$\frac{f(u)}{u} \le \frac{1}{\lambda C_1}, \quad \text{for } u \in (c_2, +\infty). \tag{3.46}$$

Let

$$M = \max_{0 \le u \le c_2} f(u), \quad c_2 \ge \lambda C_1 M. \tag{3.47}$$

It is easily seen that

$$\max_{0\le u\le c_2} f(u) \le \frac{c_2}{\lambda C_1}. \tag{3.48}$$

By (3.45) and (3.48), combining with Theorem 3.4 and Lemma 2.7, the proof is complete. □

Corollary 3.8. *Assume* (H) *holds. If* $f_0 = 0$ *or* $f_\infty = 0$*, then for each* $\lambda \in (\lambda_2, +\infty)$*, the boundary value problem* (1.2) *has at least one positive solution.*

By the above theorems, we can obtain the following results.

Corollary 3.9. *Assume* (H) *holds. If* $f_0 = +\infty$, $f_\infty = d$, *or* $f_\infty = +\infty$, $f_0 = d$, *then for any* $\lambda \in (0, (dC_1)^{-1})$*, the boundary value problem* (1.2) *has at least one positive solution.*

Corollary 3.10. *Assume* (H) *holds. If* $f_0 = 0, f_\infty = d$, *or if* $f_\infty = 0$, $f_0 = d$, *then for any* $\lambda \in ((q(l)dC_2)^{-1}, +\infty)$*, the boundary value problem* (1.2) *has at least one positive solution.*

Remark 3.11. For the integer derivative case $\alpha = 3$, Theorems 3.2–3.7 also hold; we can find the corresponding existence results in [22].

4. Nonexistence

In this section, we give some sufficient conditions for the nonexistence of positive solution to the problem (1.2).

Theorem 4.1. *Assume* (H) *holds. If* $F_0 < +\infty$ *and* $F_\infty < \infty$*, then there exists a* $\lambda_0 > 0$ *such that for all* $0 < \lambda < \lambda_0$*, the boundary value problem* (1.2) *has no positive solution.*

Proof. Since $F_0 < +\infty$ and $F_\infty < +\infty$, there exist positive numbers m_1, m_2, r_1, and r_2, such that $r_1 < r_2$ and

$$\begin{aligned} f(u) &\le m_1 u, \quad \text{for } u \in [0, r_1], \\ f(u) &\le m_2 u, \quad \text{for } u \in [r_2, +\infty). \end{aligned} \tag{4.1}$$

Let $m = \max\{m_1, m_2, \max_{r_1\le u\le r_2}\{f(u)/u\}\}$. Then we have

$$f(u) \le mu, \quad \text{for } u \in [0, +\infty). \tag{4.2}$$

Assume $v(t)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $0 < \lambda < \lambda_0 := (mC_1)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \le \frac{\lambda}{\Gamma(\alpha)} \int_0^1 (\alpha-1)k(s) f(v(s))ds \le \frac{m\lambda}{\Gamma(\alpha)} \|v\| \int_0^1 (\alpha-1)k(s)ds < \|v\|, \tag{4.3}$$

which is a contradiction. Therefore, (1.2) has no positive solution. The proof is complete. □

Theorem 4.2. *Assume (H) holds. If $f_0 > 0$ and $f_\infty > 0$, then there exists a $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$, the boundary value problem* (1.2) *has no positive solution.*

Proof. By $f_0 > 0$ and $f_\infty > 0$, we know that there exist positive numbers $n_1,\ n_2,\ r_1$, and r_2, such that $r_1 < r_2$ and

$$\begin{aligned} f(u) &\geq n_1 u, \quad \text{for } u \in [0, r_1], \\ f(u) &\geq n_2 u, \quad \text{for } u \in [r_2, +\infty). \end{aligned} \tag{4.4}$$

Let $n = \min\{n_1, n_2, \min_{r_1 \leq u \leq r_2}\{f(u)/u\}\} > 0$. Then we get

$$f(u) \geq nu, \quad \text{for } u \in [0, +\infty). \tag{4.5}$$

Assume $v(t)$ is a positive solution of (1.2). We will show that this leads to a contradiction for $\lambda > \lambda_0 := (q(l)nC_2)^{-1}$. Since $A_\lambda v(t) = v(t)$ for $t \in [0,1]$,

$$\|v\| = \|A_\lambda v\| \geq \frac{\lambda}{\Gamma(\alpha)} \int_0^1 q(l)k(s)f(v(s))ds > \|v\|, \tag{4.6}$$

which is a contradiction. Thus, (1.2) has no positive solution. The proof is complete. □

5. Examples

In this section, we will present some examples to illustrate the main results.

Example 5.1. Consider the boundary value problem

$$\begin{gathered} D_{0^+}^{5/2} u(t) + \lambda u^a = 0, \quad 0 < t < 1, \ a > 1, \\ u(0) = u(1) = u'(0) = 0. \end{gathered} \tag{5.1}$$

Since $\alpha = 5/2$, we have

$$\begin{aligned} C_1 &= \frac{1}{\Gamma(\alpha)} \int_0^1 (\alpha - 1)k(s)ds = \frac{1}{\Gamma(5/2)} \int_0^1 \frac{3}{2} s(1-s)^{3/2} ds = 0.1290, \\ C_2 &= \frac{1}{\Gamma(\alpha)} \int_0^1 \frac{1}{(\alpha-1)} q(s)k(s)ds = \frac{1}{\Gamma(5/2)} \int_0^1 \frac{2}{3} s^{5/2}(1-s)^{5/2} ds = 0.0077. \end{aligned} \tag{5.2}$$

Let $f(u) = u^a,\ a > 1$. Then we have $F_0 = 0$, $f_\infty = +\infty$. Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (5.1) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 5.2. Discuss the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda u^b = 0, \quad 0 < t < 1, \ 0 < b < 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.3}$$

Since $\alpha = 5/2$, we have $C_1 = 0.1290$ and $C_2 = 0.0077$. Let $f(u) = u^b$, $0 < b < 1$. Then we have $F_\infty = 0$, $f_0 = +\infty$. Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_0 > F_\infty C_1$ holds. Thus, by Theorem 3.3, the boundary value problem (5.3) has a positive solution for each $\lambda \in (0, +\infty)$.

Example 5.3. Consider the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda \frac{(200u^2 + u)(2 + \sin u)}{u + 1} = 0, \quad 0 < t < 1, \ a > 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.4}$$

Since $\alpha = 5/2$, we have $C_1 = 0.129$ and $C_2 = 0.0077$. Let $f(u) = (200u^2 + u)(2 + \sin u)/(u + 1)$. Then we have $F_0 = f_0 = 2$, $F_\infty = 600$, $f_\infty = 200$, and $2u < f(u) < 600u$.

(i) Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_\infty > F_0 C_1$ holds. Thus, by Theorem 3.2, the boundary value problem (5.4) has a positive solution for each $\lambda \in (3.6937, 3.8759)$.

(ii) By Theorem 4.1, the boundary value problem (5.4) has no positive solution for all $\lambda \in (0, 0.0129)$.

(iii) By Theorem 4.2, the boundary value problem (5.4) has no positive solution for all $\lambda \in (369.369, +\infty)$.

Example 5.4. Consider the boundary value problem

$$D_{0^+}^{5/2}u(t) + \lambda \frac{(u^2 + u)(2 + \sin u)}{150u + 1} = 0, \quad 0 < t < 1, \ a > 1,$$
$$u(0) = u(1) = u'(0) = 0. \tag{5.5}$$

Since $\alpha = 5/2$, we have $C_1 = 0.129$ and $C_2 = 0.0077$. Let $f(u) = (u^2 + u)(2 + \sin u)/(150u + 1)$. Then we have $F_0 = f_0 = 2$, $F_\infty = 1/50$, $f_\infty = 1/150$, and $u/150 < f(u) < 2u$.

(i) Choose $l = 1/2$. Then $q(1/2) = \sqrt{2}/8 = 0.1768$. So $q(l)C_2 f_0 > F_\infty C_1$ holds. Thus, by Theorem 3.3, the boundary value problem (5.5) has a positive solution for each $\lambda \in (369.369, 387.5968)$.

(ii) By Theorem 4.1, the boundary value problem (5.5) has no positive solution for all $\lambda \in (0, 3.8759)$.

(iii) By Theorem 4.2, the boundary value problem (5.5) has no positive solution for all $\lambda \in (110810.6911, +\infty)$.

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Research Article

Properties of Third-Order Nonlinear Functional Differential Equations with Mixed Arguments

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The aim of this paper is to offer sufficient conditions for property (B) and/or the oscillation of the third-order nonlinear functional differential equation with mixed arguments $[a(t)[x''(t)]^{\gamma}]' = q(t)f(x[\tau(t)])+p(t)h(x[\sigma(t)])$. Both cases $\int^{\infty} a^{-1/\gamma}(s)\mathrm{d}s = \infty$ and $\int^{\infty} a^{-1/\gamma}(s)\mathrm{d}s < \infty$ are considered. We deduce properties of the studied equations via new comparison theorems. The results obtained essentially improve and complement earlier ones.

1. Introduction

We are concerned with the oscillatory and certain asymptotic behavior of all solutions of the third-order functional differential equations

$$\left[a(t)\left[x''(t)\right]^{\gamma}\right]' = q(t)f(x[\tau(t)]) + p(t)h(x[\sigma(t)]). \tag{E}$$

Throughout the paper, it is assumed that $a, q, p \in C([t_0,\infty))$, $\tau, \sigma \in C^1([t_0,\infty))$, $f, h \in C((-\infty,\infty))$, and

(H_1) γ is the ratio of two positive odd integers,

(H_2) $a(t)$, $q(t)$, $p(t)$ are positive,

(H_3) $\tau(t) \le t$, $\sigma(t) \ge t$, $\tau'(t) > 0$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \infty$,

(H_4) $f^{1/\gamma}(x)/x \ge 1$, $xh(x) > 0$, $f'(x) \ge 0$, and $h'(x) \ge 0$ for $x \neq 0$,

(H_5) $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$ and $-h(-xy) \ge h(xy) \ge h(x)h(y)$ for $xy > 0$.

By a solution of (E), we mean a function $x(t) \in C^2([T_x, \infty))$, $T_x \geq t_0$, which has the property $a(t)(x''(t))^\gamma \in C^1([T_x, \infty))$ and satisfies (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and, otherwise, it is nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Recently, (E) and its particular cases (see [1–17]) have been intensively studied. The effort has been oriented to provide sufficient conditions for every (E) to satisfy

$$\lim_{t \to \infty} |x(t)| = \infty \tag{1.1}$$

or to eliminate all nonoscillatory solutions. Following [6, 8, 13, 15], we say that (E) has property (B) if each of its nonoscillatory solutions satisfies (1.1).

We will discuss both cases

$$\int_{t_0}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s < \infty, \tag{1.2}$$

$$\int_{t_0}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s = \infty. \tag{1.3}$$

We will establish suitable comparison theorems that enable us to study properties of (E) regardless of the fact that (1.3) or (1.2) holds. We will compare (E) with the first-order advanced/delay equations, in the sense that the oscillation of these first-order equations yields property (B) or the oscillation of (E).

In the paper, we are motivated by an interesting result of Grace et al. [10], where the oscillation criteria for (E) are discussed. This result has been complemented by Baculíková et al. [5]. When studying properties of (E), the authors usually reduce (E) onto the corresponding differential inequalities

$$\begin{aligned} \left[a(t)\left[x''(t)\right]^\gamma\right]' &\geq q(t) f(x[\tau(t)]), \\ \left[a(t)\left[x''(t)\right]^\gamma\right]' &\geq p(t) h(x[\sigma(t)]), \end{aligned} \tag{E_σ}$$

and further study only properties of these inequalities. Therefore, the criteria obtained withhold information either from delay argument $\tau(t)$ and the corresponding functions $q(t)$ and $f(u)$ or from advanced argument $\sigma(t)$ and the corresponding functions $p(t)$ and $h(u)$. In the paper, we offer a technique for obtaining new criteria for property (B) and the oscillation of (E) that involve both arguments $\tau(t)$ and $\sigma(t)$. Consequently, our results are new even for the linear case of (E) and properly complement and extend earlier ones presented in [1–17].

Remark 1.1. All functional inequalities considered in this paper are assumed to hold eventually; that is, they are satisfied for all t large enough.

2. Main Results

The following results are elementary but useful in what comes next.

Lemma 2.1. *Assume that $A \geq 0$, $B \geq 0$, $\alpha \geq 1$. Then,*

$$(A+B)^{\alpha} \geq A^{\alpha} + B^{\alpha}. \tag{2.1}$$

Proof. If $A = 0$ or $B = 0$, then (2.1) holds. For $A \neq 0$, setting $x = B/A$, condition (2.1) takes the form $(1+x)^{\alpha} \geq 1 + x^{\alpha}$, which is for $x > 0$ evidently true. □

Lemma 2.2. *Assume that $A \geq 0$, $B \geq 0$, $0 < \alpha \leq 1$. Then,*

$$(A+B)^{\alpha} \geq \frac{A^{\alpha} + B^{\alpha}}{2^{1-\alpha}}. \tag{2.2}$$

Proof. We may assume that $0 < A < B$. Consider a function $g(u) = u^{\alpha}$. Since $g''(u) < 0$ for $u > 0$, function $g(u)$ is concave down; that is,

$$g\left(\frac{A+B}{2}\right) \geq \frac{g(A) + g(B)}{2} \tag{2.3}$$

which implies (2.2). □

The following result presents a useful relationship between an existence of positive solutions of the advanced differential inequality and the corresponding advanced differential equation.

Lemma 2.3. *Suppose that $p(t)$, $\sigma(t)$, and $h(u)$ satisfy (H_2), (H_3), and (H_4), respectively. If the first-order advanced differential inequality*

$$z'(t) - p(t)h(z(\sigma(t))) \geq 0 \tag{2.4}$$

has an eventually positive solution, so does the advanced differential equation

$$z'(t) - p(t)h(z(\sigma(t))) = 0. \tag{2.5}$$

Proof. Let $z(t)$ be a positive solution of (2.4) on $[t_1, \infty)$. Then, $z(t)$ satisfies the inequality

$$z(t) \geq z(t_1) + \int_{t_1}^{t} p(s)h(z(\sigma(s)))\mathrm{d}s. \tag{2.6}$$

Let

$$\begin{aligned} y_1(t) &= z(t), \\ y_n(t) &= z(t_1) + \int_{t_1}^{t} p(s)h\big(y_{n-1}(\sigma(s))\big)\mathrm{d}s, \qquad n = 2,3\ldots. \end{aligned} \tag{2.7}$$

It follows from the definition of $y_n(t)$ and (H_4) that the sequence $\{y_n\}$ has the property

$$z(t) = y_1(t) \geq y_2(t) \geq \cdots \geq z(t_1), \qquad t \geq t_1. \tag{2.8}$$

Hence, $\{y_n\}$ converges pointwise to a function $y(t)$, where $z(t) \geq y(t) \geq z(t_1)$. Let $h_n(t) = p(t)h(y_n(\sigma(t)))$, $n = 1, 2, \ldots$, then $h_1(t) \geq h_2(t) \geq \cdots \geq 0$. Since $h_1(t)$ is integrable on $[t_1, t]$ and $\lim_{n\to\infty} h_n(t) = p(t)h(y(\sigma(t)))$, it follows by Lebesgue's dominated convergence theorem that

$$y(t) = z(t_1) + \int_{t_1}^{t} p(s)h(y(\sigma(s)))\mathrm{d}s. \tag{2.9}$$

Thus, $y(t)$ satisfies (2.5). □

We start our main results with the classification of the possible nonoscillatory solutions of (E).

Lemma 2.4. *Let $x(t)$ be a nonoscillatory solution of (E). Then, $x(t)$ satisfies, eventually, one of the following conditions*

(I)

$$x(t)x'(t) > 0, \qquad x(t)x''(t) > 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0, \tag{2.10}$$

(II)

$$x(t)x'(t) > 0, \qquad x(t)x''(t) < 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0, \tag{2.11}$$

and if (1.2) *holds, then also*

(III)

$$x(t)x'(t) < 0, \qquad x(t)x''(t) > 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0. \tag{2.12}$$

Proof. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0$. It follows from (E) that $[a(t)[x''(t)]^{\gamma}] > 0$, eventually. Thus, $x''(t)$ and $x'(t)$ are of fixed sign for $t \geq t_1$, t_1 large enough. At first, we assume that $x''(t) < 0$. Then, either $x'(t) > 0$ or $x'(t) < 0$, eventually. But $x''(t) < 0$ together with $x'(t) < 0$ imply that $x(t) < 0$. A contradiction, that is, Case (II) holds.

Now, we suppose that $x''(t) > 0$, then either Case (I) or Case (III) holds. On the other hand, if (1.3) holds, then Case (III) implies that $a(t)[x''(t)]^{\gamma} \geq c > 0$, $t \geq t_1$. Integrating from t_1 to t, we have

$$x'(t) - x'(t_1) \geq c^{1/\gamma} \int_{t_1}^{t} a^{-1/\gamma}(s)\mathrm{d}s, \tag{2.13}$$

which implies that $x'(t) \to \infty$ as $t \to \infty$, and we deduce that Case (III) may occur only if (1.2) is satisfied. The proof is complete. □

Remark 2.5. It follows from Lemma 2.4 that if (1.3) holds, then only Cases (I) and (II) may occur.

In the following results, we provide criteria for the elimination of Cases (I)–(III) of Lemma 2.4 to obtain property (B)/oscillation of (E).

Let us denote for our further references that

$$P(t) = \int_t^\infty a^{-1/\gamma}(u)\left(\int_u^\infty p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u, \tag{2.14}$$

$$Q(t) = \int_t^\infty a^{-1/\gamma}(u)\left(\int_u^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u. \tag{2.15}$$

Theorem 2.6. *Let $0 < \gamma \le 1$. Assume that $x(t)$ is a nonoscillatory solution of (E). If the first-order advanced differential equation*

$$z'(t) - P(t)e^{-\int_{t_1}^t Q(s)\mathrm{d}s}\ h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right)h^{1/\gamma}(z[\sigma(t)]) = 0 \tag{E_1}$$

is oscillatory, then Case (II) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of (E), satisfying Case (II) of Lemma 2.4. We may assume that $x(t) > 0$ for $t \ge t_0$. Integrating (E) from t to ∞, one gets

$$-a(t)[x''(t)]^\gamma \ge \int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s + \int_t^\infty p(s)h(x[\sigma(s)])\mathrm{d}s. \tag{2.16}$$

On the other hand, the substitution $\tau(s) = u$ gives

$$\begin{aligned}\int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s &= \int_{\tau(t)}^\infty \frac{q(\tau^{-1}(u))}{\tau'(\tau^{-1}(u))}f(x(u))\mathrm{d}u \\ &\ge \int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}f(x(s))\mathrm{d}s.\end{aligned} \tag{2.17}$$

Using (2.17) in (2.16), we find

$$-x''(t) \ge a^{-1/\gamma}(t)\left(\int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}f(x(s))\mathrm{d}s + \int_t^\infty p(s)h(x[\sigma(s)])\mathrm{d}s\right)^{1/\gamma}. \tag{2.18}$$

Taking into account the monotonicity of $x(t)$, it follows from Lemma 2.1 that

$$
\begin{aligned}
-x''(t) \geq{}& \frac{f^{1/\gamma}(x(t))}{a^{1/\gamma}(t)}\left(\int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma} \\
&+\frac{h^{1/\gamma}(x[\sigma(t)])}{a^{1/\gamma}(t)}\left(\int_t^\infty p(s)\mathrm{d}s\right)^{1/\gamma},
\end{aligned}
\tag{2.19}
$$

where we have used (H_3) and (H_4). An integration from t to ∞ yields

$$
\begin{aligned}
x'(t) \geq{}& \int_t^\infty \frac{f^{1/\gamma}(x(u))}{a^{1/\gamma}(u)}\left(\int_u^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\
&+\int_t^\infty \frac{h^{1/\gamma}(x[\sigma(u)])}{a^{1/\gamma}(u)}\left(\int_u^\infty p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\
\geq{}& f^{1/\gamma}(x(t))Q(t)+h^{1/\gamma}(x[\sigma(t)])P(t).
\end{aligned}
\tag{2.20}
$$

Regarding (H_4), it follows that $x(t)$ is a positive solution of the differential inequality

$$
x'(t)-Q(t)x(t) \geq P(t)h^{1/\gamma}(x[\sigma(t)]).
\tag{2.21}
$$

Applying the transformation

$$
x(t)=w(t)e^{\int_{t_1}^t Q(s)\mathrm{d}s},
\tag{2.22}
$$

we can easily verify that $w(t)$ is a positive solution of the advanced differential inequality

$$
w'(t)-P(t)e^{-\int_{t_1}^t Q(s)\mathrm{d}s}\, h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right)h^{1/\gamma}(w[\sigma(t)]) \geq 0.
\tag{2.23}
$$

By Lemma 2.3, we conclude that the corresponding differential equation (E_1) has also a positive solution. A contradiction. Therefore, $x(t)$ cannot satisfy Case (II). □

Remark 2.7. It follows from the proof of Theorem 2.8 that if at least one of the following conditions is satisfied:

$$\begin{gathered}\int_{t_0}^{\infty} p(s)\mathrm{d}s = \infty, \\ \int_{t_0}^{\infty} \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s = \infty, \\ \int_{t_0}^{\infty} a^{-1/\gamma}(u)\left(\int_u^{\infty} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u = \infty, \\ \int_{t_0}^{\infty} a^{-1/\gamma}(u)\left(\int_u^{\infty} \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u = \infty,\end{gathered} \tag{2.24}$$

then any nonoscillatory solution $x(t)$ of (E) cannot satisfy Case (II). Therefore, we may assume that the corresponding integrals in (2.14)-(2.15) are convergent.

Now, we are prepared to provide new criteria for property (B) of (E) and also the rate of divergence of all nonoscillatory solutions.

Theorem 2.8. *Let* (1.3) *hold and* $0 < \gamma \le 1$. *Assume that* (E_1) *is oscillatory. Then,* (E) *has property (B) and, what is more, the following rate of divergence for each of its nonoscillatory solutions holds:*

$$|x(t)| \ge c\int_{t_1}^{t} a^{-1/\gamma}(s)(t-s)\mathrm{d}s, \qquad c > 0. \tag{2.25}$$

Proof. Let $x(t)$ be a positive solution of (E). It follows from Lemma 2.4 and Remark 2.5 that $x(t)$ satisfies either Case (I) or (II). But Theorem 2.6 implies that the Case (II) cannot hold. Therefore, $x(t)$ satisfies Case (I), which implies (1.1); that is, (E) has property (B). On the other hand, there is a constant $c > 0$ such that

$$a(t)\left(x''(t)\right)^{\gamma} \ge c^{\gamma}. \tag{2.26}$$

Integrating twice from t_1 to t, we have

$$x(t) \ge c\int_{t_1}^{t}\left(\int_{t_1}^{u} a^{-1/\gamma}(s)\mathrm{d}s\right)\mathrm{d}u = c\int_{t_1}^{t} a^{-1/\gamma}(s)(t-s)\mathrm{d}s, \tag{2.27}$$

which is the desired estimate. □

Employing an additional condition on the function $h(x)$, we get easily verifiable criterion for property (B) of (E).

Corollary 2.9. *Let* $0 < \gamma \leq 1$ *and* (1.3) *hold. Assume that*

$$h^{1/\gamma}(x)/x \geq 1, \qquad |x| \geq 1, \tag{2.28}$$

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u > \frac{1}{e}. \tag{2.29}$$

Then, (E) *has property (B).*

Proof. First note that (2.29) implies

$$\int_{t_0}^{\infty} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u = \infty. \tag{2.30}$$

By Theorem 2.8, it is sufficient to show that (E_1) is oscillatory. Assume the converse, let (E_1) have an eventually positive solution $z(t)$. Then, $z'(t) > 0$ and so $z(\sigma(t)) > c > 0$. Integrating (E_1) from t_1 to t, we have in view of (2.28)

$$\begin{aligned} z(t) &\geq \int_{t_1}^{t} P(u) e^{-\int_{t_1}^{u} Q(s)\mathrm{d}s} h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(u)} Q(s)\mathrm{d}s}\right) h^{1/\gamma}(z[\sigma(u)])\mathrm{d}u \\ &\geq h^{1/\gamma}(c) \int_{t_1}^{t} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u. \end{aligned} \tag{2.31}$$

Using (2.30) in the previous inequalities, we get $z(t) \to \infty$ as $t \to \infty$. Therefore, $z(t) \geq 1$, eventually. Now, using (2.28) in (E_1), one can verify that $z(t)$ is a positive solution of the differential inequality

$$z'(t) - P(t) e^{\int_t^{\sigma(t)} Q(s)\mathrm{d}s} z(\sigma(t)) \geq 0. \tag{2.32}$$

But, by [14, Theorem 2.4.1], condition (2.29) ensures that (2.32) has no positive solutions. This is a contradiction, and we conclude that (E) has property (B). □

Example 2.10. Consider the third-order nonlinear differential equation with mixed arguments

$$\left(t^{1/3}\left(x''(t)\right)^{1/3}\right)' = \frac{a}{t^{4/3}} x^{1/3}(\lambda t) + \frac{b}{t^{4/3}} x^{\beta}(\omega t), \tag{E_{x1}}$$

where $a, b > 0$, $0 < \lambda < 1$, $\omega > 1$, and $\beta \geq 1/3$ is a ratio of two positive odd integers. Since

$$P(t) = \frac{27b^3}{t}, \qquad Q(t) = \frac{27a^3\lambda}{t}, \tag{2.33}$$

Corollary 2.9 implies that (E_{x1}) has property (B) provided that

$$b^3 \omega^{27a^3\lambda} \ln \omega > \frac{1}{27\mathrm{e}}. \tag{2.34}$$

Moreover, by Theorem 2.8, the rate of divergence of every nonoscillatory solution of (E_{x1}) is

$$|x(t)| \geq ct\ln t, \qquad c > 0. \tag{2.35}$$

For $\beta = 1/3$ and $\delta > 1$ satisfying $\delta^{1/3}(\delta-1)^{4/3} = 3a\lambda^{\delta/3} + 3b\omega^{\delta/3}$, one such solution is t^{δ}.

Now, we turn our attention to the case when $\gamma \geq 1$.

Theorem 2.11. *Let $\gamma \geq 1$. Assume that $x(t)$ is a nonoscillatory solution of (E). If the first-order advanced differential equation*

$$z'(t) - 2^{(1-\gamma)/\gamma}P(t)e^{[-2^{(1-\gamma)/\gamma}\int_{t_1}^{t}Q(s)\mathrm{d}s]}h^{1/\gamma}\left(e^{2^{(1-\gamma)/\gamma}\int_{t_1}^{\sigma(t)}Q(s)\mathrm{d}s}\right)h^{1/\gamma}(z[\sigma(t)]) = 0 \tag{E_2}$$

is oscillatory, then Case (II) cannot hold.

Proof. Let $x(t)$ be an eventually positive solution of (E), satisfying Case (II) of Lemma 2.4. Then, (2.18) holds. Lemma 2.2, in view of the monotonicity of $x(t)$, (H_3), and (H_4), implies

$$\begin{aligned}
-x''(t) \geq{}& \frac{f^{1/\gamma}(x(t))}{2^{(\gamma-1)/\gamma}a^{1/\gamma}(t)}\left(\int_t^{\infty}\frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma} \\
&+ \frac{h^{1/\gamma}(x[\sigma(t)])}{2^{(\gamma-1)/\gamma}a^{1/\gamma}(t)}\left(\int_t^{\infty}p(s)\mathrm{d}s\right)^{1/\gamma}.
\end{aligned} \tag{2.36}$$

An integration from t to ∞ yields

$$\begin{aligned}
x'(t) \geq{}& \int_t^{\infty}\frac{f^{1/\gamma}(x(u))}{2^{(\gamma-1)/\gamma}a^{1/\gamma}(u)}\left(\int_u^{\infty}\frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\
&+ \int_t^{\infty}\frac{h^{1/\gamma}(x[\sigma(u)])}{2^{(\gamma-1)/\gamma}a^{1/\gamma}(u)}\left(\int_u^{\infty}p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\
\geq{}& f^{1/\gamma}(x(t))2^{(1-\gamma)/\gamma}Q(t) + h^{1/\gamma}(x[\sigma(t)])2^{(1-\gamma)/\gamma}P(t).
\end{aligned} \tag{2.37}$$

Noting (H_4), we see that $x(t)$ is a positive solution of the differential inequality

$$x'(t) \geq 2^{(1-\gamma)/\gamma}Q(t)x(t) + 2^{(1-\gamma)/\gamma}P(t)h^{1/\gamma}(x[\sigma(t)]). \tag{2.38}$$

Setting

$$x(t) = w(t)e^{[2^{(1-\gamma)/\gamma}\int_{t_1}^{t}Q(s)\mathrm{d}s]}, \tag{2.39}$$

one can see that $w(t)$ is a positive solution of the advanced differential inequality

$$w'(t) - 2^{(1-\gamma)/\gamma} P(t) e^{\left[-2^{(1-\gamma)/\gamma} \int_{t_1}^{t} Q(s)\mathrm{d}s\right]} h^{1/\gamma}\left(e^{2^{(1-\gamma)/\gamma} \int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right) h^{1/\gamma}(w[\sigma(t)]) \geq 0. \tag{2.40}$$

By Lemma 2.3, we deduce that the corresponding differential equation (E_2) has also a positive solution. A contradiction. Therefore, $x(t)$ cannot satisfy Case (II). □

The following result is obvious.

Theorem 2.12. *Let* (1.3) *hold and* $\gamma \geq 1$*. Assume that* (E_2) *is oscillatory. Then,* (E) *has property (B) and, what is more, each of its nonoscillatory solutions satisfies* (2.25)*.*

Now, we present easily verifiable criterion for property (B) of (E).

Corollary 2.13. *Let* (1.3) *and* (2.28) *hold and* $\gamma \geq 1$*. If*

$$\liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u) e^{\left[2^{(1-\gamma)/\gamma} \int_{u}^{\sigma(u)} Q(s)\mathrm{d}s\right]} \mathrm{d}u > \frac{2^{(\gamma-1)/\gamma}}{e}, \tag{2.41}$$

then (E) *has property (B).*

Proof. The proof is similar to the proof of Corollary 2.9 and so it can be omitted. □

Remark 2.14. Theorems 2.6, 2.8, 2.11, and 2.12 and Corollaries 2.9 and 2.13 provide criteria for property (B) that include both delay and advanced arguments and all coefficients and functions of (E). Our results are new even for the linear case of (E).

Remark 2.15. It is useful to notice that if we apply the traditional approach to (E), that is, if we replace (E) by the corresponding differential inequality (E_σ), then conditions (2.29) of Corollary 2.9 and (2.41) of Corollary 2.13 would take the forms

$$\liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u)\mathrm{d}u > \frac{1}{e}, \qquad \liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u)\mathrm{d}u > \frac{2^{(\gamma-1)/\gamma}}{e}, \tag{2.42}$$

respectively, which are evidently second to (2.29) and (2.41).

Example 2.16. Consider the third-order nonlinear differential equation with mixed arguments

$$\left(t(x''(t))^3\right)' = \frac{a}{t^6} x^3(\lambda t) + \frac{b}{t^6} x^\beta(\omega t), \tag{E_{x2}}$$

where $a, b > 0$, $0 < \lambda < 1$, $\beta \geq 3$ is a ratio of two positive odd integers and $\omega > 1$. It is easy to see that conditions (2.14) and (2.15) for (E_{x2}) reduce to

$$P(t) = \frac{b^{1/3}}{5^{1/3} t}, \qquad Q(t) = \frac{\lambda^{5/3} a^{1/3}}{5^{1/3} t}, \tag{2.43}$$

respectively. It follows from Corollary 2.13 that (E_{x2}) has property (B) provided that

$$b^{1/3}\left[\omega^{\lambda^{5/3}a^{1/3}/2^{2/3}5^{1/3}}\right]\ln\omega \geq \frac{2^{2/3}5^{1/3}}{e}. \tag{2.44}$$

Moreover, (2.25) provides the following rate of divergence for every nonoscillatory solution of (E_{x2}):

$$|x(t)| \geq ct^{5/3}, \qquad c > 0. \tag{2.45}$$

Now, we eliminate Case (I) of Lemma 2.4, to get the oscillation of (E).

Theorem 2.17. *Let $x(t)$ be a nonoscillatory solution of (E). Assume that there exists a function $\xi(t) \in C^1([t_0,\infty))$ such that*

$$\xi'(t) \geq 0, \quad \xi(t) < t, \qquad \eta(t) = \sigma(\xi(\xi(t))) > t. \tag{2.46}$$

If the first-order advanced differential equation

$$z'(t) - \left\{\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}h^{1/\gamma}(z[\eta(t)]) = 0 \tag{E_3}$$

is oscillatory, then Case (I) cannot hold.

Proof. Let $x(t)$ be an eventually positive solution of (E), satisfying Case (I). It follows from (E) that

$$\left[a(t)[x''(t)]^{\gamma}\right]' \geq p(t)h(x[\sigma(t)]). \tag{2.47}$$

Integrating from $\xi(t)$ to t, we have

$$\begin{aligned} a(t)[x''(t)]^{\gamma} - a(\xi(t))[x''(\xi(t))]^{\gamma} &\geq \int_{\xi(t)}^{t} p(s)h(x[\sigma(s)])\mathrm{d}s \\ &\geq h(x[\sigma(\xi(t))])\int_{\xi(t)}^{t} p(s)\mathrm{d}s. \end{aligned} \tag{2.48}$$

Therefore,

$$x''(t) \geq h^{1/\gamma}(x[\sigma(\xi(t))])a^{-1/\gamma}(t)\left(\int_{\xi(t)}^{t} p(s)\mathrm{d}s\right)^{1/\gamma}. \tag{2.49}$$

An integration from $\xi(t)$ to t yields

$$\begin{aligned} x'(t) &\geq \int_{\xi(t)}^{t} h^{1/\gamma}(x[\sigma(\xi(u))])a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\ &\geq h^{1/\gamma}(x[\eta(t)])\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u. \end{aligned} \tag{2.50}$$

Consequently, $x(t)$ is a positive solution of the advanced differential inequality

$$x'(t) - \left\{\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}h^{1/\gamma}(x[\eta(t)]) \geq 0. \tag{2.51}$$

Hence, by Lemma 2.3, we conclude that the corresponding differential equation (E_3) also has a positive solution, which contradicts the oscillation of (E_3). Therefore, $x(t)$ cannot satisfy Case (I). □

Combining Theorem 2.17 with Theorems 2.6 and 2.11, we get two criteria for the oscillation of (E).

Theorem 2.18. *Let* (1.3) *hold and* $0 < \gamma \leq 1$. *Assume that both of the first-order advanced equations* (E_1) *and* (E_3) *are oscillatory, then* (E) *is oscillatory.*

Proof. Assume that (E) has a nonoscillatory solution. It follows from Remark 2.5 that $x(t)$ satisfies either Case (I) or (II). But both cases are excluded by the oscillation of (E_1) and (E_3). □

Corollary 2.19. *Let* $0 < \gamma \leq 1$. *Assume that* (1.3), (2.28), (2.29), *and* (2.46) *hold. If*

$$\liminf_{t\to\infty}\int_{t}^{\eta(t)}\left\{\int_{\xi(v)}^{v} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}\mathrm{d}v > \frac{1}{e}, \tag{2.52}$$

then (E) *is oscillatory.*

Proof. Conditions (2.29) and (2.52) guarantee the oscillation of (E_1) and (E_3), respectively. The assertion now follows from Theorem 2.18. □

Example 2.20. We consider once more the third-order differential equation (E_{x1}) with the same restrictions as in Example 2.10. We set $\xi(t) = \alpha_0 t$, where $\alpha_0 = (1+\sqrt{\omega})/2\sqrt{\omega}$. Then condition (2.52) takes the form

$$b^3\frac{(1-\alpha_0)\left(1-\alpha_0^{1/3}\right)^3}{\alpha_0^2}\ln\left(\omega\alpha_0^2\right) > \frac{1}{27e}, \tag{2.53}$$

which by Corollary 2.19, implies the oscillation of (E_{x1}).

The following results are obvious.

Theorem 2.21. *Let* (1.3) *hold and* $\gamma \geq 1$. *Assume that both of the first-order advanced equations* (E_2) *and* (E_3) *are oscillatory, then* (E) *is oscillatory.*

Corollary 2.22. *Let* $\gamma \geq 1$. *Assume that* (1.3), (2.28), (2.41), (2.46), *and* (2.52) *hold. Then* (E) *is oscillatory.*

Example 2.23. We recall again the differential equation (E_{x2}) with the same assumptions as in Example 2.16. We set $\xi(t) = \alpha_0 t$ with $\alpha_0 = (1+\sqrt{\omega})/2\sqrt{\omega}$. Then condition (2.52) reduces to

$$b^{1/3}\frac{(1-\alpha_0)\left(1-\alpha_0^5\right)^{1/3}}{\alpha_0^{8/3}}\ln\left(\omega\alpha_0^2\right) > \frac{5^{1/3}}{e}, \tag{2.54}$$

which, by Corollary 2.22, guarantees the oscillation of (E_{x2}).

The following result is intended to exclude Case (III) of Lemma 2.4.

Theorem 2.24. *Let* $x(t)$ *be a nonoscillatory solution of* (E). *Assume that* (1.2) *holds. If the first-order delay differential equation*

$$z'(t) + \left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{1/\gamma}\left(\int_{t}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s\right) f^{1/\gamma}(z[\tau(t)]) = 0. \tag{E_4}$$

is oscillatory, then Case (III) cannot hold.

Proof. Let $x(t)$ be a positive solution of (E), satisfying Case (III) of Lemma 2.4. Using that $a(t)[x''(t)]^{\gamma}$ is increasing, we find that

$$\begin{aligned} -x'(t) &\geq \int_{t}^{\infty} x''(s)\mathrm{d}s = \int_{t}^{\infty}\left(a^{1/\gamma}(s)x''(s)\right)a^{-1/\gamma}(s)\mathrm{d}s \\ &\geq a(t)^{1/\gamma}x''(t)\int_{t}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s. \end{aligned} \tag{2.55}$$

Integrating the inequality $[a(t)[x''(t)]^{\gamma}]' \geq q(t)f(x[\tau(t)])$ from t_1 to t, we have

$$a(t)\left[x''(t)\right]^{\gamma} \geq \int_{t_1}^{t} q(s)f(x[\tau(s)]\mathrm{d}s) \geq f(x[\tau(t)])\int_{t_1}^{t} q(s)\mathrm{d}s. \tag{2.56}$$

Thus,

$$a^{1/\gamma}(t)x''(t) \geq f^{1/\gamma}(x[\tau(t)])\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{1/\gamma}. \tag{2.57}$$

Combining (2.57) with (2.55), we find

$$0 \geq x'(t) + \left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{1/\gamma} \left(\int_{t}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s\right) f^{1/\gamma}(x[\tau(t)]). \tag{2.58}$$

It follows from [16, Theorem 1] that the corresponding differential equation (E_4) also has a positive solution. A contradiction. For that reason, $x(t)$ cannot satisfy Case (III). □

The following results are immediate.

Theorem 2.25. *Let* (1.2) *hold and* $0 < \gamma \leq 1$. *Assume that both of the first-order advanced equations* (E_1) *and* (E_4) *are oscillatory, then* (E) *has property (B).*

Theorem 2.26. *Let* (1.2) *hold and* $0 < \gamma \leq 1$. *Assume that all of the three first-order advanced equations* (E_1), (E_3), *and* (E_4) *are oscillatory, then* (E) *is oscillatory.*

Theorem 2.27. *Let* (1.2) *hold and* $\gamma \geq 1$. *Assume that both of the first-order advanced equations* (E_2) *and* (E_4) *are oscillatory, then* (E) *has property (B).*

Theorem 2.28. *Let* (1.2) *hold and* $\gamma \geq 1$. *Assume that all of the three first-order advanced equations* (E_2), (E_3), *and* (E_4) *are oscillatory, then* (E) *is oscillatory.*

3. Summary

In this paper, we have presented new comparison theorems for deducing the property (B)/oscillation of (E) from the oscillation of a set of the suitable first-order delay/advanced differential equation. We were able to present such criteria for studied properties that employ all coefficients and functions included in studied equations. Our method essentially simplifies the examination of the third-order equations, and, what is more, it supports backward the research on the first-order delay/advanced differential equations. Our results here extend and complement latest ones of Grace et al. [10], Agarwal et al. [1–3], Cecchi et al. [6], Parhi and Pardi [15], and the present authors [4, 8]. The suitable illustrative examples are also provided.

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Research Article

The Lie Group in Infinite Dimension

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A Lie group acting on finite-dimensional space is generated by its infinitesimal transformations and conversely, any Lie algebra of vector fields in finite dimension generates a Lie group (the first fundamental theorem). This classical result is adjusted for the infinite-dimensional case. We prove that the (local, C^∞ smooth) action of a Lie group on infinite-dimensional space (a manifold modelled on $\mathbb{R}^\infty$) may be regarded as a limit of finite-dimensional approximations and the corresponding Lie algebra of vector fields may be characterized by certain finiteness requirements. The result is applied to the theory of generalized (or higher-order) infinitesimal symmetries of differential equations.

1. Preface

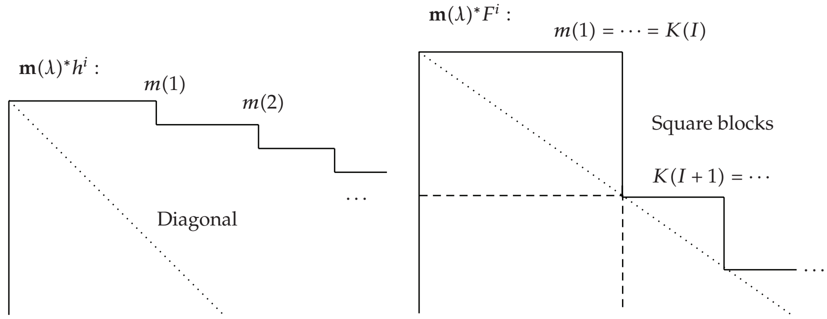
In the symmetry theory of differential equations, the *generalized* (*or*: *higher-order, Lie-Bäcklund*) *infinitesimal symmetries*

$$Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_I^j \frac{\partial}{\partial w_I^j} \quad \left(i = 1, \dots, n;\ j = 1, \dots, m;\ I = i_1 \cdots i_n;\ i_1, \dots, i_n = 1, \dots, n\right), \tag{1.1}$$

where the coefficients

$$z_i = z_i\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right), \qquad z_I^j = z_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right) \tag{1.2}$$

are functions of independent variables x_i, dependent variables w^j and a finite number of jet variables $w_I^j = \partial^n w^j / \partial x_{i_1} \cdots \partial x_{i_n}$ belong to well-established concepts. However, in spite of



(a) (b)

Figure 1

this matter of fact, they cause an unpleasant feeling. Indeed, such vector fields as a rule do not generate any one-parameter group of transformations

$$\overline{x}_i = G_i\left(\lambda; \dots, x_{i'}, w^{j'}_{I'}, \dots\right), \qquad \overline{w}^j_I = G^j_I\left(\lambda; \dots, x_{i'}, w^{j'}_{I'}, \dots\right) \tag{1.3}$$

in the underlying infinite-order jet space since the relevant Lie system

$$\frac{\partial G_i}{\partial \lambda} = z_i\left(\dots, G_{i'}, G^{j'}_{I'}, \dots\right), \quad \frac{\partial G^j_I}{\partial \lambda} = z^j_I\left(\dots, G_{i'}, G^{j'}_{I'}, \dots\right) \quad \left(G_i|_{\lambda=0} = x_i,\, G^j_I\Big|_{\lambda=0} = w^j_I\right) \tag{1.4}$$

need not have any reasonable (locally unique) solution. Then Z is a mere formal concept [1–7] not related to any true transformations and the term "infinitesimal symmetry Z" is misleading, no Z-symmetries of differential equations in reality appear.

In order to clarify the situation, we consider one-parameter groups of local transformations in $\mathbb{R}^\infty$. We will see that they admit "finite-dimensional approximations" and as a byproduct, the relevant infinitesimal transformations may be exactly characterized by certain "finiteness requirements" of purely algebraical nature. With a little effort, the multidimensional groups can be easily involved, too. This result was briefly discussed in [8, page 243] and systematically mentioned at several places in monograph [9], but our aim is to make some details more explicit in order to prepare the necessary tools for systematic investigation of *groups of generalized symmetries*. We intend to continue our previous articles [10–13] where the algorithm for determination of all *individual generalized symmetries* was already proposed.

For the convenience of reader, let us transparently describe the crucial approximation result. We consider transformations (2.1) of a local one-parameter group in the space $\mathbb{R}^\infty$ with coordinates $h^1, h^2, \dots$. Equations (2.1) of transformations $\mathbf{m}(\lambda)$ can be schematically represented by Figure 1(a).

We prove that in appropriate new coordinate system $F^1, F^2, \dots$ on $\mathbb{R}^\infty$, the same transformations $\mathbf{m}(\lambda)$ become block triangular as in Figure 1(b). It follows that a certain hierarchy of finite-dimensional subspaces of $\mathbb{R}^\infty$ is preserved which provides the "approximation" of $\mathbf{m}(\lambda)$. The infinitesimal transformation $Z = \mathrm{d}\mathbf{m}(\lambda)/\mathrm{d}\lambda|_{\lambda=0}$ clearly preserves the same hierarchy which provides certain algebraical "finiteness" of Z.

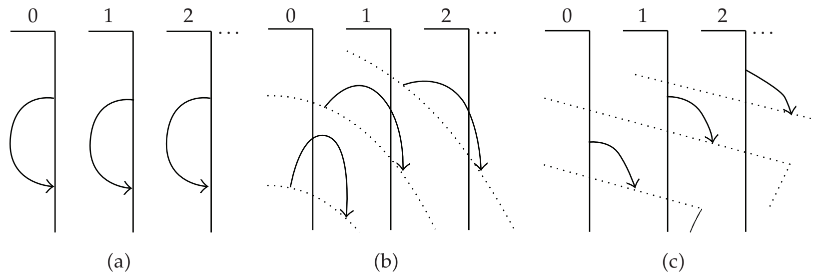


Figure 2

If the primary space $\mathbb{R}^\infty$ is moreover equipped with an appropriate structure, for example, the contact forms, it turns into the jet space and the results concerning the transformation groups on $\mathbb{R}^\infty$ become the theory of higher-order symmetries of differential equations. Unlike the common point symmetries which occupy a number of voluminous monographs (see, e.g., [14, 15] and extensive references therein) this higher-order theory was not systematically investigated yet. We can mention only the isolated article [16] which involves a direct proof of the "finiteness requirements" for one-parameter groups (namely, the result (ι) of Lemma 5.4 below) with two particular examples and monograph [7] involving a theory of generalized infinitesimal symmetries in the formal sense.

Let us finally mention the intentions of this paper. In the *classical theory of point or Lie's contact-symmetries* of differential equations, the order of derivatives is preserved (Figure 2(a)). Then the common Lie's and Cartan's methods acting in finite dimensional spaces *given ahead of calculations* can be applied. On the other extremity, the *generalized symmetries* need not preserve the order (Figure 2(c)) and even any finite-dimensional space and then the common classical methods fail. For the favourable intermediate case of *groups of generalized symmetries*, the invariant finite-dimensional subspaces exist, however, they are *not known in advance* (Figure 2(b)). We believe that the classical methods can be appropriately adapted for the latter case, and this paper should be regarded as a modest preparation for this task.

2. Fundamental Approximation Results

Our reasonings will be carried out in the space $\mathbb{R}^\infty$ with coordinates $h^1, h^2, \ldots$ [9] and we introduce the structural family $\mathcal{F}$ of all real-valued, locally defined and C^∞-smooth functions $f = f(h^1, \ldots, h^{m(f)})$ depending on a finite number of coordinates. In future, such functions will contain certain C^∞-smooth real parameters, too.

We are interested in (local) groups of transformations $\mathbf{m}(\lambda)$ in $\mathbb{R}^\infty$ defined by formulae

$$\mathbf{m}(\lambda)^* h^i = H^i\left(\lambda; h^1, \ldots, h^{m(i)}\right), \quad -\varepsilon^i < \lambda < \varepsilon^i,\ \varepsilon^i > 0\ (i = 1, 2, \ldots), \tag{2.1}$$

where $H^i \in \mathcal{F}$ if the parameter λ is kept fixed. We suppose

$$\mathbf{m}(0) = \text{id.}, \qquad \mathbf{m}(\lambda + \mu) = \mathbf{m}(\lambda)\mathbf{m}(\mu) \tag{2.2}$$

whenever it makes a sense. An open and common definition domain for all functions H^i is tacitly supposed. (In more generality, a common definition domain for *every finite number* of functions H^i is quite enough and the germ and sheaf terminology would be more adequate for our reasonings, alas, it looks rather clumsy.)

Definition 2.1. For every $I = 1, 2, \ldots$ and $0 < \varepsilon < \min\{\varepsilon^1, \ldots, \varepsilon^I\}$, let $\mathcal{F}(I, \varepsilon) \subset \mathcal{F}$ be the subset of all composed functions

$$F = F\Big(\ldots, \mathbf{m}(\lambda_j)^* h^i, \ldots\Big) = F\Big(\ldots, H^i\Big(\lambda_j; h^1, \ldots, h^{m(i)}\Big), \ldots\Big), \tag{2.3}$$

where $i = 1, \ldots, I$; $-\varepsilon < \lambda_j < \varepsilon$; $j = 1, \ldots, J = J(I) = \max\{m(1), \ldots, m(I)\}$ and F is arbitrary C^∞-smooth function (of IJ variables). In functions $F \in \mathcal{F}(I, \varepsilon)$, variables $\lambda_1, \ldots, \lambda_J$ are regarded as mere parameters.

Functions (2.3) will be considered on open subsets of $\mathbb{R}^\infty$ where the rank of the Jacobi $(IJ \times J)$-matrix

$$\left(\frac{\partial}{\partial h^{j'}} H^i\Big(\lambda_j; h^1, \ldots, h^{m(i)}\Big)\right) \quad (i = 1, \ldots, I;\ j, j' = 1, \ldots, J) \tag{2.4}$$

of functions $H^i(\lambda_j; h^1, \ldots, h^{m(i)})$ locally attains the maximum (for appropriate choice of parameters). This rank and therefore the subset $\mathcal{F}(I, \varepsilon) \subset \mathcal{F}$ does not depend on ε as soon as $\varepsilon = \varepsilon(I)$ is close enough to zero. *This is supposed from now on and we may abbreviate* $\mathcal{F}(I) = \mathcal{F}(I, \varepsilon)$.

We deal with highly nonlinear topics. Then the definition domains cannot be kept fixed in advance. Our results will be true *locally*, near *generic points*, on certain *open everywhere dense subsets* of the underlying space $\mathbb{R}^\infty$. With a little effort, the subsets can be exactly characterized, for example, by locally constant rank of matrices, functional independence, existence of implicit function, and so like. We follow the common practice and as a rule omit such routine details from now on.

Lemma 2.2 (approximation lemma). *The following inclusion is true:*

$$\mathbf{m}(\lambda)^* \mathcal{F}(I) \subset \mathcal{F}(I). \tag{2.5}$$

Proof. Clearly

$$\mathbf{m}(\lambda)^* H^i(\lambda_j; \ldots) = \mathbf{m}(\lambda)^* \mathbf{m}(\lambda_j)^* h^i = \mathbf{m}(\lambda + \lambda_j)^* h^i = H^i(\lambda + \lambda_j; \ldots) \tag{2.6}$$

and therefore

$$\mathbf{m}(\lambda)^* F = F\Big(\ldots, H^i\Big(\lambda + \lambda_j; h^1, \ldots, h^{m(i)}\Big), \ldots\Big) \in \mathcal{F}(I). \tag{2.7}$$

□

Denoting by $K(I)$ the rank of matrix (2.4), there exist *basical functions*

$$F^k = F^k\left(\dots, H^i\left(\lambda_j; h^1, \dots, h^{m(i)}\right), \dots\right) \in \mathcal{F}(I) \quad (k = 1, \dots, K(I)) \tag{2.8}$$

such that $\operatorname{rank}(\partial F^k / \partial h^{j'}) = K(I)$. Then a function $f \in \mathcal{F}$ lies in $\mathcal{F}(I)$ if and only if $f = \overline{f}(F^1, \dots, F^{K(I)})$ is a composed function. In more detail

$$F = \overline{F}\left(\lambda_1, \dots, \lambda_J; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \tag{2.9}$$

is such a composed function if we choose $f = F$ given by (2.3). Parameters $\lambda_1, \dots, \lambda_J$ occurring in (2.3) are taken into account here. It follows that

$$\frac{\partial F}{\partial \lambda_j} = \frac{\partial \overline{F}}{\partial \lambda_j}\left(\lambda_1, \dots, \lambda_J; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (j = 1, \dots, J) \tag{2.10}$$

and analogously for the higher derivatives.

In particular, we also have

$$H^i\left(\lambda; h^1, \dots, h^{m(i)}\right) = \overline{H}^i\left(\lambda; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (i = 1, \dots, I) \tag{2.11}$$

for the choice $F = H^i(\lambda; \dots)$ in (2.9) whence

$$\frac{\partial^r H^i}{\partial \lambda^r} = \frac{\partial^r \overline{H}^i}{\partial \lambda^r}\left(\lambda; F^1, \dots, F^{K(I)}\right) \in \mathcal{F}(I) \quad (i = 1, \dots, I;\ r = 0, 1, \dots). \tag{2.12}$$

The basical functions can be taken from the family of functions $H^i(\lambda; \dots)$ $(i = 1, \dots, I)$ for appropriate choice of *various* values of λ. Functions (2.12) are enough as well even for a *fixed* value λ, for example, for $\lambda = 0$, see Theorem 3.2 below.

Lemma 2.3. *For any basical function, one has*

$$\mathbf{m}(\lambda)^* F^k = \overline{F}^k\left(\lambda; F^1, \dots, F^{K(I)}\right) \quad (k = 1, \dots, K(I)). \tag{2.13}$$

Proof. $F^k \in \mathcal{F}(I)$ implies $\mathbf{m}(\lambda)^* F^k \in \mathcal{F}(I)$ and (2.9) may be applied with the choice $F = \mathbf{m}(\lambda)^* F^k$ and $\lambda_1 = \cdots = \lambda_J = \lambda$. □

Summary 1. Coordinates $h^i = H^i(0; \dots)$ $(i = 1, \dots, I)$ were included into the subfamily $\mathcal{F}(I) \subset \mathcal{F}$ which is transformed into itself by virtue of (2.13). So we have a one-parameter group acting on $\mathcal{F}(I)$. One can even choose $F^1 = h^1, \dots, F^I = h^I$ here and then, if I is large enough, formulae (2.13) provide a "finite-dimensional approximation" of the primary mapping $\mathbf{m}(\lambda)$. The block-triangular structure of the infinite matrix of transformations $\mathbf{m}(\lambda)$ mentioned in Section 1 appears if $I \to \infty$ and the system of functions $F^1, F^2, \dots$ is succesively completed.

3. The Infinitesimal Approach

We introduce the vector field

$$Z = \sum z^i \frac{\partial}{\partial h^i} = \frac{\mathrm{d}\mathbf{m}(\lambda)}{\mathrm{d}\lambda}\bigg|_{\lambda=0} \quad \left(z^i = \frac{\partial H^i}{\partial \lambda}\left(0; h^1, \dots, h^{m(i)}\right);\ i = 1, 2, \dots \right), \tag{3.1}$$

the *infinitesimal transformation* ($\mathcal{IT}$) of group $\mathbf{m}(\lambda)$. Let us recall the celebrated Lie system

$$\begin{aligned} \frac{\partial}{\partial \lambda}\mathbf{m}(\lambda)^* h^i &= \frac{\partial H^i}{\partial \lambda}(\lambda; \dots) = \frac{\partial H^i}{\partial \mu}(\lambda + \mu; \dots)\bigg|_{\mu=0} \\ &= \frac{\partial}{\partial \mu}\mathbf{m}(\lambda + \mu)^* h^i\bigg|_{\mu=0} = \mathbf{m}(\lambda)^* \frac{\partial}{\partial \mu}\mathbf{m}(\mu)^* h^i\bigg|_{\mu=0} = \mathbf{m}(\lambda)^* Z h^i = \mathbf{m}(\lambda)^* z^i. \end{aligned} \tag{3.2}$$

In more explicit (and classical) transcription

$$\frac{\partial H^i}{\partial \lambda}\left(\lambda; h^1, \dots, h^{m(i)}\right) = z^i\left(H^1\left(\lambda; h^1, \dots, h^{m(1)}\right), \dots, H^{m(i)}\left(\lambda; h^1, \dots, h^{m(m(i))}\right)\right). \tag{3.3}$$

One can also check the general identity

$$\frac{\partial^r}{\partial \lambda^r}\mathbf{m}(\lambda)^* f = \mathbf{m}(\lambda)^* Z^r f \quad (f \in \mathcal{F};\ r = 0, 1, \dots) \tag{3.4}$$

by a mere routine induction on r.

Lemma 3.1 (finiteness lemma). *For all $r \in \mathbb{N}$, $Z^r \mathcal{F}(I) \subset \mathcal{F}(I)$.*

Proof. Clearly

$$ZF = \mathbf{m}(\lambda)^* ZF|_{\lambda=0} = \frac{\partial}{\partial \lambda}\mathbf{m}(\lambda)^* F\bigg|_{\lambda=0} \in \mathcal{F}(I) \tag{3.5}$$

for any function (2.3) by virtue of (2.10): induction on r. □

Theorem 3.2 (finiteness theorem). *Every function $F \in \mathcal{F}(I)$ admits (locally, near generic points) the representation*

$$F = \widetilde{F}\left(\dots, \frac{\partial^r H^i}{\partial \lambda^r}\left(0; h^1, \dots, h^{m(i)}\right), \dots\right) \tag{3.6}$$

in terms of a composed function where $i = 1, \dots, I$ and $\widetilde{F}$ is a $\mathbb{C}^\infty$-smooth function of a finite number of variables.

Proof. Let us temporarily denote

$$H_r^i = \frac{\partial^r H^i}{\partial \lambda^r}(\lambda;\ldots) = \frac{\partial^r}{\partial \lambda^r}\mathbf{m}(\lambda)^* h^i, \qquad h_r^i = H_r^i(0;\ldots) = Z^r h^i, \tag{3.7}$$

where the second equality follows from (3.4) with $f = h^i$, $\lambda = 0$. Then

$$H_r^i = \mathbf{m}(\lambda)^* h_r^i = \mathbf{m}(\lambda)^* Z^r h^i \tag{3.8}$$

by virtue of (3.4) with general λ.

If $j = j(i)$ is large enough, there does exist an identity $h_{j+1}^i = G^i(h_0^i,\ldots,h_j^i)$. Therefore

$$\frac{\partial^{j+1} H^i}{\partial \lambda^{j+1}} = H_{j+1}^i = G^i\left(H_0^i,\ldots,H_j^i\right) = G^i\left(H^i,\ldots,\frac{\partial^j H^i}{\partial \lambda^j}\right) \tag{3.9}$$

by applying $\mathbf{m}(\lambda)^*$. This may be regarded as ordinary differential equation with initial values

$$H^i\Big|_{\lambda=0} = h_0^i,\ldots,\left.\frac{\partial^j H^i}{\partial \lambda^j}\right|_{\lambda=0} = h_j^i. \tag{3.10}$$

The solution $H^i = \widetilde{H}^i(\lambda; h_0^i,\ldots,h_j^i)$ expressed in terms of initial values reads

$$H^i\left(\lambda; h^1,\ldots,h^{m(i)}\right) = \widetilde{H}^i\left(\lambda; H^i\left(0;h^1,\ldots,h^{m(i)}\right),\ldots,\frac{\partial^j H^i}{\partial \lambda^j}\left(0;h^1,\ldots,h^{m(i)}\right)\right) \tag{3.11}$$

in full detail. If λ is kept fixed, this is exactly the identity (3.6) for the particular case $F = H^i(\lambda; h^1,\ldots,h^{m(i)})$. The general case follows by a routine. □

Definition 3.3. Let $\mathbb{G}$ be the set of (local) vector fields

$$Z = \sum z^i \frac{\partial}{\partial h^i} \quad \left(z^i \in \mathcal{F},\ \text{infinite sum}\right) \tag{3.12}$$

such that every family of functions $\{Z^r h^i\}_{r\in\mathbb{N}}$ (i fixed but arbitrary) can be expressed in terms of a finite number of coordinates.

Remark 3.4. Neither $\mathbb{G} + \mathbb{G} \subset \mathbb{G}$ nor $[\mathbb{G},\mathbb{G}] \subset \mathbb{G}$ as follows from simple examples. However, $\mathbb{G}$ is a *conical set* (over $\mathcal{F}$): if $Z \in \mathbb{G}$ then $fZ \in \mathbb{G}$ for any $f \in \mathcal{F}$. Easy direct proof may be omitted here.

Summary 2. If Z is $\mathcal{IT}$ of a group then all functions $Z^r h^i$ ($i = 1,\ldots,I$; $r = 0,1,\ldots$) are included into family $\mathcal{F}(I)$ hence $Z \in \mathbb{G}$. The converse is clearly also true: every vector field $Z \in \mathbb{G}$ generates a local Lie group since the Lie system (3.3) admits finite-dimensional approximations in spaces $\mathcal{F}(I)$.

Let us finally reformulate the last sentence in terms of basical functions.

Theorem 3.5 (approximation theorem). *Let $Z \in \mathbb{G}$ be a vector field locally defined on $\mathbb{R}^\infty$ and $F^1, \dots, F^{K(I)} \in \mathcal{F}$ be a maximal functionally independent subset of the family of all functions*

$$Z^r h^i \quad (i = 1, \dots, I;\ r = 0, 1, \dots). \tag{3.13}$$

Denoting $ZF^k = \overline{F}^k(F^1, \dots, F^{K(I)})$, then the system

$$\frac{\partial}{\partial \lambda}\mathbf{m}(\lambda)^* F^k = \mathbf{m}(\lambda)^* Z F^k = \overline{F}^k\left(\mathbf{m}(\lambda)^* F^1, \dots, \mathbf{m}(\lambda)^* F^{K(I)}\right) \quad (k = 1, \dots, K(I)) \tag{3.14}$$

may be regarded as a "finite-dimensional approximation" to the Lie system (3.3) *of the one-parameter local group* $\mathbf{m}(\lambda)$ *generated by Z.*

In particular, assuming $F^1 = h^1, \dots, F^I = h^I$, then the the initial portion

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{m}(\lambda)^* F^i = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbf{m}(\lambda)^* h^i = \frac{\mathrm{d}}{\mathrm{d}\lambda}H^i = z^i\left(H^1, \dots, H^{m(i)}\right) \quad (i = 1, \dots, I) \tag{3.15}$$

of the above system transparently demonstrates the approximation property.

4. On the Multiparameter Case

The following result does not bring much novelty and we omit the proof.

Theorem 4.1. *Let $Z_1, \dots, Z_d$ be commuting local vector fields in the space $\mathbb{R}^\infty$. Then $Z_1, \dots, Z_d \in \mathbb{G}$ if and only if the vector fields $Z = a_1 Z_1 + \cdots + a_d Z_d$ $(a_1, \dots, a_d \in \mathbb{R})$ locally generate an abelian Lie group.*

In full non-Abelian generality, let us consider a (local) multiparameter group formally given by the same equations (2.1) as above where $\lambda = (\lambda_1, \dots, \lambda_d) \in \mathbb{R}^d$ are parameters close to the zero point $0 = (0, \dots, 0) \in \mathbb{R}^d$. The rule (2.2) is generalized as

$$\mathbf{m}(0) = \mathrm{id.}, \qquad \mathbf{m}(\varphi(\lambda, \mu)) = \mathbf{m}(\lambda)\mathbf{m}(\mu), \tag{4.1}$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$, $\mu = (\mu_1, \dots, \mu_d)$ and $\varphi = (\varphi_1, \dots, \varphi_d)$ determine the composition of parameters. Appropriately adapting the space $\mathcal{F}(I)$ and the concept of basical functions $F^1, \dots, F^{K(I)}$, Lemma 2.2 holds true without any change.

Passing to the infinitesimal approach, we introduce vector fields $Z_1, \dots, Z_d$ which are $\mathcal{IT}$ of the group. We recall (without proof) the Lie equations [17]

$$\frac{\partial}{\partial \lambda_j}\mathbf{m}(\lambda)^* f = \sum a_i^j(\lambda)\mathbf{m}(\lambda)^* Z_j f \quad (f \in \mathcal{F};\ j = 1, \dots, d) \tag{4.2}$$

with the initial condition $\mathbf{m}(0) = \mathrm{id}$. Assuming $Z_1, \dots, Z_d$ linearly independent over $\mathbb{R}$, coefficients $a_i^j(\lambda)$ may be arbitrarily chosen and the solution $\mathbf{m}(\lambda)$ always is a group transformation (the first fundamental theorem). If basical functions $F^1, \ldots, F^{K(I)}$ are inserted for f, we have a finite-dimensional approximation which is self-contained in the sense that

$$Z_j F^k = \tilde{F}_j^k \left(F^1, \ldots, F^{K(I)} \right) \quad (j = 1, \ldots, d; \; k = 1, \ldots, K(I)) \tag{4.3}$$

are composed functions in accordance with the definition of the basical functions.

Let us conversely consider a Lie algebra of local vector fields $Z = a_1 Z_1 + \cdots + a_d Z_d$ $(a_i \in \mathbb{R})$ on the space $\mathbb{R}^\infty$. Let moreover $Z_1, \ldots, Z_d \in \mathbb{G}$ *uniformly* in the sense that there is a *universal space* $\mathcal{F}(I)$ with $\mathcal{L}_{Z_i} \mathcal{F}(I) \subset \mathcal{F}(I)$ for all $i = 1, \ldots, d$. Then the Lie equations may be applied and we obtain reasonable finite-dimensional approximations.

Summary 3. Theorem 4.1 holds true even in the non-Abelian and multidimensional case if the inclusions $Z_1, \ldots, Z_d \in \mathbb{G}$ are uniformly satisfied.

As yet we have closely simulated the primary one-parameter approach, however, the results are a little misleading: the uniformity requirement in Summary 3 may be completely omitted. This follows from the following result [9, page 30] needless here and therefore stated without proof.

Theorem 4.2. *Let $\mathcal{K}$ be a finite-dimensional submodule of the module of vector fields on $\mathbb{R}^\infty$ such that $[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}$. Then $\mathcal{K} \subset \mathbb{G}$ if and only if there exist generators (over $\mathcal{F}$) of submodule $\mathcal{K}$ that are lying in $\mathbb{G}$.*

5. Symmetries of the Infinite-Order Jet Space

The previous results can be applied to the groups of generalized symmetries of *partial differential equations*. Alas, some additional technical tools cannot be easily explained at this place, see the concluding Section 11 below. So we restrict ourselves to the *trivial differential equations*, that is, to the groups of generalized symmetries in the total *infinite-order jet space* which do not require any additional preparations.

Let $\mathbf{M}(m,n)$ be the jet space of n-dimensional submanifolds in $\mathbb{R}^{m+n}$ [9–13]. We recall the familiar (local) jet coordinates

$$x_i, w_I^j \quad (I = i_1 \ldots i_r; \; i, i_1, \ldots, i_r = 1, \ldots, n; \; r = 0, 1, \ldots; \; j = 1, \ldots, m). \tag{5.1}$$

Functions $f = f(\ldots, x_i, w_I^j, \ldots)$ on $\mathbf{M}(m,n)$ are C^∞-smooth and depend on a finite number of coordinates. The jet coordinates serve as a mere technical tool. The true jet structure is given just by the *module* $\Omega(m,n)$ *of contact forms*

$$\omega = \sum a_I^j \omega_I^j \quad \left(\text{finite sum}, \omega_I^j = \mathrm{d}w_I^j - \sum w_{Ii}^j \mathrm{d}x_i \right) \tag{5.2}$$

or, equivalently, by the *"orthogonal" module* $\mathcal{H}(m,n) = \Omega^\perp(m,n)$ *of formal derivatives*

$$D = \sum a_i D_i \left(D_i = \frac{\partial}{\partial x_i} + \sum w_{Ii}^j \frac{\partial}{\partial w_I^j}; \; i = 1, \ldots, n; D \rfloor \omega_I^j = \omega_I^j(D) = 0 \right). \tag{5.3}$$

Let us state useful formulae

$$\mathrm{d}f = \sum D_i f \,\mathrm{d}x_i + \sum \frac{\partial f}{\partial w_I^j}\omega_I^j, \qquad D_i \rfloor \mathrm{d}\omega_I^j = \omega_{Ii}^j, \qquad \mathcal{L}_{D_i}\omega_I^j = \omega_{Ii}^j, \tag{5.4}$$

where $\mathcal{L}_{D_i} = D_i \rfloor \mathrm{d} + \mathrm{d}D_i \rfloor$ denotes the Lie derivative.

We are interested in (local) one-parameter groups of transformations $\mathbf{m}(\lambda)$ given by certain formulae

$$\mathbf{m}(\lambda)^* x_i = G_i\left(\lambda; \dots, x_{i'}, w_{I'}^{j'}, \dots\right), \qquad \mathbf{m}(\lambda)^* w_I^j = G_I^j\left(\lambda; \dots, x_{i'}, w_{I'}^{j'}, \dots\right) \tag{5.5}$$

and in vector fields

$$Z = \sum z_i\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right)\frac{\partial}{\partial x_i} + \sum z_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right)\frac{\partial}{\partial w_I^j} \tag{5.6}$$

locally defined on the jet space $\mathbf{M}(m,n)$; see also (1.1) and (1.2).

Definition 5.1. We speak of a *group of morphisms* (5.5)of the jet structure if the inclusion $\mathbf{m}(\lambda)^*\Omega(m,n) \subset \Omega(m,n)$ holds true. We speak of a *(universal) variation* (5.6) of the jet structure if$\mathcal{L}_Z\Omega(m,n) \subset \Omega(m,n)$. If a variation (5.6) moreover generates a group, speaks of a *(generalized or higher-order) infinitesimal symmetry* of the jet structure.

So we intentionally distinguish between true infinitesimal transformations generating a group and the formal concepts; this point of view and the terminology are not commonly used in the current literature.

Remark 5.2. A few notes concerning this unorthodox terminology are useful here. In actual literature, the vector fields (5.6) are as a rule decomposed into the "trivial summand D" and the so-called "evolutionary form V" of the vector field Z, explicitly

$$Z = D + \mathrm{V} \quad \left(D = \sum z_i D_i \in \mathcal{H}(m,n), V = \sum Q_I^j \frac{\partial}{\partial w_I^j},\ Q_I^j = z_I^j - \sum w_{Ii}^j z_i\right). \tag{5.7}$$

The summand D is usually neglected in a certain sense [3–7] and the "essential" summand V is identified with the evolutional system

$$\frac{\partial w_I^j}{\partial \lambda} = Q_I^j\left(\dots, x_{i'}, w_{I'}^{j'}, \dots\right) \quad \left(w_I^j = \frac{\partial^n w^j}{\partial x_{i_1} \cdots \partial x_{i_n}}(\lambda, x_1, \dots, x_n)\right) \tag{5.8}$$

of partial differential equations (the finite subsystem with $I = \phi$ empty is enough here since the remaining part is a mere prolongation). This evolutional system is regarded as a "virtual flow" on the "space of solutions" $w^j = w^j(x_1, \dots, x_n)$, see [7, especially page 11]. In more generality, some differential constraints may be adjoint. However, in accordance with the ancient classical tradition, functions $\delta w^j = \partial w^j/\partial\lambda$ are just the *variations*. (There is only one novelty: in classical theory, δw^j are introduced only *along a given solution* while the vector fields Z are "universally" defined on the space.) In this "evolutionary approach", the properties of the primary vector field Z are utterly destroyed. It seems that the true sense of this approach lies in the applications to the topical soliton theory. However, then the evolutional system is always completed with boundary conditions and embedded into some normed functional spaces in order to ensure the existence of global "true flows". This is already quite a different story and we return to our topic.

In more explicit terms, morphisms (5.5) are characterized by the (implicit) recurrence

$$\sum G^j_{Ii} D_{i'} G_i = D_{i'} G^j_I \quad (i' = 1, \ldots, n), \tag{5.9}$$

where $\det(D_{i'}G_i) \neq 0$ is supposed and vector field (5.6) is a variation if and only if

$$z^j_{Ii} = D_i z^j_I - \sum w^j_{Ii'} D_i z_{i'}. \tag{5.10}$$

Recurrence (5.9) easily follows from the inclusion $\mathbf{m}(\lambda)^* \omega^j_I \in \Omega(m,n)$ and we omit the proof. Recurrence (5.10) follows from the identity

$$\begin{aligned} \mathcal{L}_Z \omega^j_I &= \mathcal{L}_Z\left(\mathrm{d}w^j_I - \sum w^j_{Ii}\mathrm{d}x_i\right) = \mathrm{d}z^j_I - \sum z^j_{Ii}\mathrm{d}x_i - \sum w^j_{Ii}\mathrm{d}z_i \\ &\cong \left(\sum D_{i'} z^j_I - \sum z^j_{Ii'} - \sum w^j_{Ii} D_{i'} z_i\right)\mathrm{d}x_{i'} \ (\mathrm{mod}\ \Omega(m,n)) \end{aligned} \tag{5.11}$$

and the inclusion $\mathcal{L}_Z \omega^j_I \in \Omega(m,n)$. The obvious formula

$$\mathcal{L}_Z \omega^j_I = \sum \left(\frac{\partial z^j_I}{\partial w^{j'}_{I'}} - \sum w^j_{Ii} \frac{\partial z_i}{\partial w^{j'}_{I'}} \right) \omega^{j'}_{I'} \tag{5.12}$$

appearing on this occasion also is of a certain sense, see Theorem 5.5 and Section 10 below. It follows that the initial functions G_i, G^j, z_i, z^j (empty $I = \phi$) may be in principle arbitrarily prescribed in advance. This is the familiar *prolongation procedure* in the jet theory.

Remark 5.3. Recurrence (5.10) for the variation Z can be succintly expressed by $\omega^j_{Ii}(Z) = D_i \omega^j_I(Z)$. This remarkable formula admits far going generalizations, see concluding Examples 11.3 and 11.4 below.

Let us recall that a vector field (5.6) generates a group (5.5) if and only if $Z \in \mathbb{G}$ hence if and only if every family

$$\{Z^r x_i\}_{r\in\mathbb{N}}, \qquad \left\{Z^r w^j_I\right\}_{r\in\mathbb{N}} \tag{5.13}$$

can be expressed in terms of a finite number of jet coordinates. We conclude with simple but practicable remark: due to jet structure, the *infinite number of conditions* (5.13) can be replaced by a *finite number of requirements* if Z is a variation.

Lemma 5.4. *Let* (5.6) *be a variation of the jet structure. Then the inclusion* $Z \in \mathbb{G}$ *is equivalent to any of the requirements*

(ι) *every family of functions*

$$\{Z^r x_i\}_{r\in\mathbb{N}}, \quad \left\{Z^r w^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m) \tag{5.14}$$

can be expressed in terms of a finite number of jet coordinates,

($\iota\iota$) *every family of differential forms*

$$\left\{\mathcal{L}_Z^r \mathrm{d}x_i\right\}_{r\in\mathbb{N}}, \quad \left\{\mathcal{L}_Z^r \mathrm{d}w^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m) \tag{5.15}$$

involves only a finite number of linearly independent terms,

($\iota\iota\iota$) *every family of differential forms*

$$\left\{\mathcal{L}_Z^r \mathrm{d}x_i\right\}_{r\in\mathbb{N}}, \quad \left\{\mathcal{L}_Z^r \mathrm{d}w_I^j\right\}_{r\in\mathbb{N}} \quad (i = 1,\ldots,n;\ j = 1,\ldots,m;\ \text{arbitrary } I) \tag{5.16}$$

involves only a finite number of linearly independent terms.

Proof. Inclusion $Z \in \mathbb{G}$ is defined by using the families (5.13) and this trivially implies (ι) where only the empty multi-indice $I = \phi$ is involved. Then (ι) implies ($\iota\iota$) by using the rule $\mathcal{L}_Z \mathrm{d}f = \mathrm{d}Zf$. Assuming ($\iota\iota$), we may employ the commutative rule

$$[D_i, Z] = D_i Z - Z D_i = \sum a_i^{i'} D_{i'} \quad \left(a_i^{i'} = D_i z_{i'}\right) \tag{5.17}$$

in order to verify identities of the kind

$$\mathcal{L}_Z \mathrm{d}w_i^j = \mathcal{L}_Z \mathrm{d}D_i w^j = \mathcal{L}_Z \mathcal{L}_{D_i} \mathrm{d}w^i = \mathcal{L}_{D_i} \mathcal{L}_Z \mathrm{d}w^i - \sum a_i^{i'} \mathcal{L}_{D_{i'}} w^j \tag{5.18}$$

and in full generality identities of the kind

$$\mathcal{L}_Z^k \mathrm{d}w_I^j = \sum a_{I,k}^{I'} \mathcal{L}_{D_{I'}} \mathcal{L}_Z^{k'} \mathrm{d}w^j \quad \left(\text{sum with } k' \le k,\ \left|I'\right| \le |I|\right) \tag{5.19}$$

with unimportant coefficients, therefore ($\iota\iota\iota$) follows. Finally ($\iota\iota\iota$) obviously implies the primary requirement on the families (5.13). □

This is not a whole story. The requirements can be expressed only in terms of the structural contact forms. With this final result, the algorithms [10–13] for determination of all *individual morphisms* can be closely simulated in order to obtain the algorithm for the determination of all *groups* $\mathbf{m}(\lambda)$ *of morphisms*, see Section 10 below.

Theorem 5.5 (technical theorem). *Let* (5.6) *be a variation of the jet space. Then* $Z \in \mathbb{G}$ *if and only if every family*

$$\left\{ \mathcal{L}_Z^r \omega^j \right\}_{r \in \mathbb{N}} \quad (j = 1, \ldots, m) \tag{5.20}$$

involves only a finite number of linearly independent terms.

Some nontrivial preparation is needful for the proof. Let Θ be a finite-dimensional module of 1-forms (on the space $\mathbf{M}(m,n)$ but the underlying space is irrelevant here). Let us consider vector fields X such that $\mathcal{L}_{fX}\Theta \subset \Theta$ for all functions f. Let moreover Adj Θ be the module of all forms φ satisfying $\varphi(X) = 0$ for all such X. Then Adj Θ has a basis consisting of total differentials of certain functions $f^1, \ldots, f^K$ (the Frobenius theorem), and there is a basis of module Θ which can be expressed in terms of functions $f^1, \ldots, f^K$. Alternatively saying, (an appropriate basis of) the Pfaffian system $\vartheta = 0$ ($\vartheta \in \Theta$) can be expressed only in terms of functions $f^1, \ldots, f^K$. This result frequently appears in Cartan's work, but we may refer only to [9, 18, 19] and to the appendix below for the proof.

Module Adj Θ is intrinsically related to Θ: if a mapping $\mathbf{m}$ preserves Θ then $\mathbf{m}$ preserves Adj Θ. In particular, assuming

$$\mathbf{m}(\lambda)^*\Theta \subset \Theta, \quad \text{then } \mathbf{m}(\lambda)^*\text{Adj } \Theta \subset \text{Adj } \Theta \tag{5.21}$$

is true for a group $\mathbf{m}(\lambda)$. In terms of $\mathcal{IT}$ of the group $\mathbf{m}(\lambda)$, we have equivalent assertion

$$\mathcal{L}_Z\Theta \subset \Theta \text{ implies } \mathcal{L}_Z\text{Adj } \Theta \subset \text{Adj } \Theta \tag{5.22}$$

and therefore $\mathcal{L}_Z^r\text{Adj } \Theta \subset \text{Adj } \Theta$ for all r. The preparation is done.

Proof. Let Θ be the module generated by all differential forms $\mathcal{L}_Z^r \omega^j$ ($j = 1, \ldots, m$; $r = 0, 1, \ldots$). Assuming finite dimension of module Θ, we have module Adj Θ and clearly $\mathcal{L}_Z\Theta \subset \Theta$ whence $\mathcal{L}_Z^r\text{Adj } \Theta \subset \text{Adj } \Theta$ ($r = 0, 1, \ldots$). However Adj Θ involves both the differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$ (see below) and the forms $\omega^1, \ldots, \omega^m$. Point ($u$) of previous Lemma 5.4 implies $Z \in \mathbb{G}$. The converse is trivial.

In order to finish the proof, let us on the contrary assume that Adj Θ *does not* contain all differentials $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Alternatively saying, the Pfaffian system $\vartheta = 0$ ($\vartheta \in \Theta$) can be expressed in terms of certain functions $f^1, \ldots, f^K$ such that $\mathrm{d}f^1 = \cdots = \mathrm{d}f^K = 0$ does not imply $\mathrm{d}x_1 = \cdots = \mathrm{d}x_n = 0$. On the other hand, it follows clearly that maximal solutions of the Pfaffian system can be expressed only in terms of functions $f^1, \ldots, f^K$ and therefore we *do not need all* independent variables $x_1, \ldots, x_n$. This is however a contradiction: the Pfaffian system consists of contact forms and involves the equations $\omega^1 = \cdots = \omega^n = 0$. All independent variables are needful if we deal with the common classical solutions $w^j = w^j(x_1, \ldots, x_n)$. □

The result can be rephrased as follows.

Theorem 5.6. *Let* $\Omega_0 \subset \Omega(m,n)$ *be the submodule of all zeroth-order contact forms* $\omega = \sum a^j \omega^j$ *and* Z *be a variation of the jet structure. Then* $Z \in \mathbb{G}$ *if and only if* $\dim \oplus \mathcal{L}_Z^r \Omega_0 < \infty$.

6. On the Multiparameter Case

Let us temporarily denote by $\mathbb{V}$ the family of all infinitesimal variations (5.6) of the jet structure. Then $\mathbb{V}+\mathbb{V}\subset\mathbb{V}$, $c\mathbb{V}\subset\mathbb{V}$ ($c\in\mathbb{R}$), $[\mathbb{V},\mathbb{V}]\subset\mathbb{V}$, and it follows that $\mathbb{V}$ is an infinite-dimensional Lie algebra (coefficients in $\mathbb{R}$). On the other hand, if $Z\in\mathbb{V}$ and $fZ\in\mathbb{V}$ for certain $f\in\mathcal{F}$ then $f\in\mathbb{R}$ is a constant. (Briefly saying: *the conical variations of the total jet space do not exist*. We omit easy direct proof.) It follows that only the common Lie algebras over $\mathbb{R}$ are engaged if we deal with morphisms of the jet spaces $\mathbf{M}(m,n)$.

Theorem 6.1. *Let $\mathcal{G}\subset\mathbb{V}$ be a finite-dimensional Lie subalgebra. Then $\mathcal{G}\subset\mathbb{G}$ if and only if there exists a basis of $\mathcal{G}$ that is lying in $\mathbb{G}$.*

The proof is elementary and may be omitted. Briefly saying, Theorem 4.2 (coefficients in $\mathcal{F}$) turns into quite other and much easier Theorem 6.1 (coefficients in $\mathbb{R}$).

7. The Order-Preserving Groups in Jet Space

Passing to particular examples from now on, we will briefly comment some well-known classical results for the sake of completeness.

Let $\Omega_l\subset\Omega(m,n)$ be the submodule of all contact forms $\omega=\sum a_I^j\omega_I^j$ (sum with $|I|\le l$) of the order l at most. A morphism (5.5) and the infinitesimal variation (5.6) are called *order preserving* if

$$\mathbf{m}(\lambda)^*\Omega_l\subset\Omega_l,\qquad \mathcal{L}_Z\Omega_l\subset\Omega_l, \tag{7.1}$$

respectively, for a certain $l=0,1,\ldots$ (equivalently: for all $l\in\mathbb{N}$, see Lemmas 9.1 and 9.2 below). Due to the fundamental Lie-Bäcklund theorem [1, 3, 6, 10–13], this is possible only in the *pointwise case* or in the *Lie's contact transformation case*. In quite explicit terms: assuming (7.1) then either functions G_i, G^j, z_i, z^j (empty $I=\phi$) in formulae (5.5) and (5.6) are functions only of the zeroth-order jet variables $x_{i'}$, $w^{j'}$ or, in the second case, we have $m=1$ and all functions G_i, G^1, G_i^1, z_i, z^1, z_i^1 contain only the zeroth- and first-order variables $x_{i'}$, w^1, $w_{i'}^1$.

A somewhat paradoxically, short proofs of this fundamental result are not easily available in current literature. We recall a tricky approach here already applied in [10–13], to the case of the order-preserving morphisms. The approach is a little formally improved and appropriately adapted to the infinitesimal case.

Theorem 7.1 (infinitesimal Lie-Bäcklund). *Let a variation Z preserve a submodule $\Omega_l\subset\Omega(m,n)$ of contact forms of the order l at most for a certain $l\in\mathbb{N}$. Then $Z\in\mathbb{G}$ and either Z is an infinitesimal point transformation or $m=1$ and Z is the infinitesimal Lie's contact transformation.*

Proof. We suppose $\mathcal{L}_Z\Omega_l\subset\Omega_l$. Then $\mathcal{L}_Z^r\Omega_0\subset\mathcal{L}_Z^r\Omega_l\subset\Omega_l$ therefore $Z\in\mathbb{G}$ by virtue of Theorem 5.5. Moreover $\mathcal{L}_Z\Omega_{l-1}\subset\Omega_{l-1},\ldots,\mathcal{L}_Z\Omega_0\subset\Omega_0$ by virtue of Lemma 9.2 below. So we have

$$\mathcal{L}_Z\omega^j=\sum a^{jj'}\omega^{j'}\quad (j,j'=1,\ldots,m). \tag{7.2}$$

Assuming $m = 1$, then (7.2) turns into the classical definition of Lie's infinitesimal contact transformation. Assume $m \geq 2$. In order to finish the proof we refer to the following result which implies that Z is indeed an infinitesimal point transformation. □

Lemma 7.2. *Let Z be a vector field on the jet space $\mathbf{M}(m,n)$ satisfying* (7.2) *and $m \geq 2$. Then*

$$Zx_i = z_i\left(\dots, x_{i'}, w^{j'}, \dots\right), \quad Zw^j = z^j\left(\dots, x_{i'}, w^{j'}, \dots\right) \quad (i = 1, \dots, n;\ j = 1, \dots, m) \tag{7.3}$$

are functions only of the point variables.

Proof. Let us introduce module Θ of $(m+2n)$-forms generated by all forms of the kind

$$\begin{aligned} &\omega^1 \wedge \cdots \wedge \omega^m \wedge \left(\mathrm{d}\omega^{j_1}\right)^{n_1} \wedge \left(\mathrm{d}\omega^{j_k}\right)^{n_k} \\ &\quad = \mathrm{d}w^1 \wedge \cdots \mathrm{d}w^m \wedge \mathrm{d}x_1 \wedge \cdots \mathrm{d}x_n \wedge \sum \pm\, \mathrm{d}w^{j'_1}_{i_1} \wedge \cdots \wedge \mathrm{d}w^{j'_n}_{i_n}, \end{aligned} \tag{7.4}$$

where $\sum n_k = n$. Clearly $\Theta = (\Omega_0)^m \wedge (\mathrm{d}\Omega_0)^n$. The inclusions

$$\mathcal{L}_Z\Omega_0 \subset \Omega_0, \qquad \mathcal{L}_Z\mathrm{d}\Omega_0 = \mathrm{d}\mathcal{L}_Z\Omega_0 + \Omega_0 \subset \mathrm{d}\Omega_0 + \Omega_0 \tag{7.5}$$

are true by virtue of (7.2) and imply $\mathcal{L}_Z\Theta \subset \Theta$.

Module Θ vanishes when restricted to certain hyperplanes, namely, just to the hyperplanes of the kind

$$\vartheta = \sum a_i \mathrm{d}x_i + \sum a^j \mathrm{d}w^j = 0 \tag{7.6}$$

(use $m \geq 2$ here). This is expressed by $\Theta \wedge \vartheta = 0$ and it follows that

$$0 = \mathcal{L}_Z(\Theta \wedge \vartheta) = \mathcal{L}_Z\Theta \wedge \vartheta + \Theta \wedge \mathcal{L}_Z\vartheta = \Theta \wedge \mathcal{L}_Z\vartheta. \tag{7.7}$$

Therefore $\mathcal{L}_Z\vartheta$ again is such a hyperplane: $\mathcal{L}_Z\vartheta \cong 0$ (mod all $\mathrm{d}x_i$ and $\mathrm{d}w^j$). On the other hand,

$$\mathcal{L}_Z\vartheta \cong \sum a_i \mathrm{d}z_i + \sum a^j \mathrm{d}z^j \quad \left(\text{mod all } \mathrm{d}x_i \text{ and } \mathrm{d}w^j\right) \tag{7.8}$$

and it follows that $\mathrm{d}z_i$, $\mathrm{d}z^j \cong 0$. □

There is a vast literature devoted to the pointwise transformations and symmetries so that any additional comments are needless. On the other hand, the contact transformations are more involved and less popular. They explicitly appear on rather peculiar and dissimilar occasions in actual literature [20, 21]. However, in reality the groups of Lie contact transformations are latently involved in the classical calculus of variations and provide the core of the Hilbert-Weierstrass extremality theory of variational integrals.

8. Digression to the Calculus of Variations

We establish the following principle.

Theorem 8.1 (metatheorem). *The geometries of nondegenerate local one-parameter groups of Lie contact transformations ($\mathcal{CT}$) and of nondegenerate first-order one-dimensional variational integrals ($\mathcal{VI}$) are identical. In particular, the orbits of a given $\mathcal{CT}$ group are extremals of appropriate $\mathcal{VI}$ and conversely.*

Proof. The $\mathcal{CT}$ groups act in the jet space $\mathbf{M}(1,n)$ equipped with the contact module $\Omega(1,n)$. Then the abbreviations

$$w_I = w_I^1, \quad \omega_I = \omega_I^1 = \mathrm{d}w_I - \sum w_{Ii}\mathrm{d}x_i \quad Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_I^1 \frac{\partial}{\partial w_I} \tag{8.1}$$

are possible. Let us recall the classical approach [22, 23]. The Lie contact transformations defined by certain formulae

$$\mathbf{m}^* x_i = G_i(\cdot), \quad \mathbf{m}^* w = G^1(\cdot), \quad \mathbf{m}^* w_i = G_i^1(\cdot) \quad ((\cdot) = (x_1,\dots,x_n,w,w_1,\dots,w_n)) \tag{8.2}$$

preserve the Pfaffian equation $\omega = \mathrm{d}w - \sum w_i \mathrm{d}x_i = 0$ or (equivalently) the submodule $\Omega_0 \subset \Omega(1,n)$ of zeroth-order contact forms. Explicit formulae are available in literature. We are interested in one-parameter local $\mathcal{CT}$ groups of transformations $\mathbf{m}(\lambda)(-\varepsilon < \lambda < \varepsilon)$ which are "nondegenerate" in a sense stated below and then the explicit formulae are not available yet. On the other hand, our $\mathcal{VI}$ with smooth Lagrangian Ł

$$\int Ł(t, y_1,\dots,y_n,y_1',\dots,y_n')dt \qquad \left(y_i = y_i(t),\ ' = \frac{d}{dt},\ \det\left(\frac{\partial^2 Ł}{\partial y_i' \partial y_j'}\right) \neq 0\right) \tag{8.3}$$

to appear later, involves variables from quite other jet space $\mathbf{M}(n,1)$ with coordinates denoted t (the independent variable), $y_1,\dots,y_n$ (the dependent variables) and higher-order jet variables like y_i', y_i'' and so on.

We are passing to the topic proper. Let us start in the space $\mathbf{M}(1,n)$ with $\mathcal{CT}$ groups. One can check that vector field (5.6) is infinitesimal $\mathcal{CT}$ if and only if

$$Z = -\sum Q_{w_i}\frac{\partial}{\partial x_i} + \left(Q - \sum w_i Q_{w_i}\right)\frac{\partial}{\partial w} + \sum (Q_{x_i} + w_i Q_w)\frac{\partial}{\partial w_i} + \cdots, \tag{8.4}$$

where the function $Q = Q(x_1,\dots,x_n,w,w_1,\dots,w_n)$ may be arbitrarily chosen.

"Hint: we have, by definition

$$\mathcal{L}_Z \omega = Z \rfloor \mathrm{d}\omega + \mathrm{d}\omega(Z) = \sum (z_i \omega_i - \omega_i(Z)\mathrm{d}x_i) + \mathrm{d}Q \in \Omega_0, \tag{8.5}$$

where $Q = Q(x_1,\dots,x_n,w,w_1,\dots,w_n,\dots) = \omega(Z) = z^1 - \sum w_i z_i$,

$$\mathrm{d}Q = \sum D_i Q \mathrm{d}x_i + \frac{\partial Q}{\partial w}\omega + \sum \frac{\partial Q}{\partial w_i}\omega_i \tag{8.6}$$

whence immediately $z_i = -\partial Q/\partial w_i$, $z^1 = Q + \sum w_i z_i = Q - \sum w_i \cdot \partial Q/\partial w_i$, $\partial Q/\partial w_I = 0$ if $|I| \geq 1$ and formula (8.4) follows."

Alas, the corresponding Lie system (not written here) is not much inspirational. Let us however consider a function $w = w(x_1, \ldots, x_n)$ implicitly defined by an equation $V(x_1, \ldots, x_n, w) = 0$. We may suppose that the transformed function $\mathbf{m}(\lambda)^* w$ satisfies the equation

$$V\left(x_1, \ldots, x_n, \mathbf{m}(\lambda)^* w\right) = \lambda \tag{8.7}$$

without any loss of generality. In infinitesimal terms

$$1 = \frac{\partial(V-\lambda)}{\partial\lambda} = Z(V-\lambda) = -\sum Q_{w_i} V_{x_i} + \left(Q - \sum w_i Q_{w_i}\right) V_w. \tag{8.8}$$

However $w_i = \partial w/\partial x_i = -V_{x_i}/V_w$ may be inserted here, and we have the crucial *Jacobi equation*

$$1 = Q\left(x_1, \ldots, x_n, w, -\frac{V_{x_1}}{V_w}, \ldots, -\frac{V_{x_n}}{V_w}\right) V_w \tag{8.9}$$

(not involving V) which can be uniquely rewritten as the *Hamilton-Jacobi* ($\mathcal{HJ}$) *equation*

$$V_w + \mathcal{H}(x_1, \ldots, x_n, w, p_1, \ldots, p_n) \quad (p_i = V_{x_i}) \tag{8.10}$$

in the "nondegenerate" case $\sum Q_{w_i} V_{x_i} \neq 1$. Let us recall the characteristic curves [22, 23] of the $\mathcal{HJ}$ equation given by the system

$$\frac{\mathrm{d}w}{1} = \frac{\mathrm{d}x_i}{\mathcal{H}_{p_i}} = -\frac{\mathrm{d}p_i}{\mathcal{H}_{x_i}} = \frac{\mathrm{d}V}{-\mathcal{H} + \sum p_i \mathcal{H}_{p_i}}. \tag{8.11}$$

The curves may be interpreted as the orbits of the group $\mathbf{m}(\lambda)$. (Hint: look at the well-known classical construction of the solution V of the Cauchy problem [22, 23] in terms of the characteristics. The initial Cauchy data are transferred just along the characteristics, i.e., along the group orbits.) Assume moreover the additional condition $\det(\partial^2\mathcal{H}/\partial p_i \partial p_j) \neq 0$. We may introduce variational integral (8.3) with the Lagrange function Ł given by the familiar identities

$$Ł + \mathcal{H} = \sum p_i y_i' \tag{8.12}$$

with interrelations

$$t = w, \quad y_i = x_i, \quad y_i' = \mathcal{H}_{p_i}, \quad p_i = Ł_{y_i'} \quad (i = 1, \ldots, n) \tag{8.13}$$

between variables t, y_i, y_i' of the space $\mathbf{M}(n,1)$ and variables x_i, w, w_i of the space $\mathbf{M}(1,n)$. Since (8.11) may be regarded as a Hamiltonian system for the extremals of $\mathcal{VI}$, the metatheorem is clarified. □

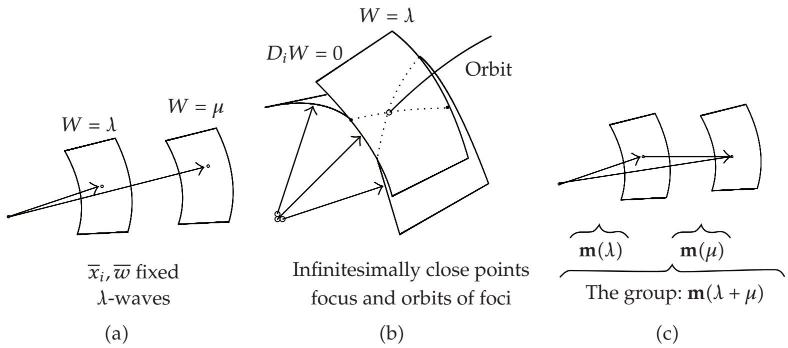


Figure 3

Remark 8.2. Let us recall the Mayer fields of extremals for the $\mathcal{VJ}$ since they provide the true sense of the above construction. The familiar Poincaré-Cartan form

$$\breve{\varphi} = \text{Ł}\mathrm{d}t + \sum \text{Ł}_{y_i'}(\mathrm{d}y_i - y_i'\mathrm{d}t) = -\mathscr{H}\mathrm{d}t + \sum p_i \mathrm{d}y_i \tag{8.14}$$

is restricted to appropriate subspace $y_i' = g_i(t, y_1, \ldots, y_n)$ ($i = 1, \ldots, n$; the *slope field*) in order to become a total differential

$$\breve{\varphi}\big|_{y_i'=g_i} = \mathrm{d}V(t, y_1, \ldots, y_n) = V_t\mathrm{d}t + \sum V_{y_i}\mathrm{d}y_i \tag{8.15}$$

of the *action* V. We obtain the requirements $V_t = -\mathscr{H}$, $V_{y_i} = p_i$ identical with (8.10). In geometrical terms: *transformations of a hypersurface* $V = 0$ *by means of* $\mathcal{CT}$ *group may be identified with the level sets* $V = \lambda$ $(\lambda \in \mathbb{R})$ *of the action of a Mayer fields of extremals.*

The last statement is in accordance with (8.11) where

$$\mathrm{d}V = \left(-\mathscr{H} + \sum p_i \mathscr{H}_{p_i}\right)\mathrm{d}w = \left(-\mathscr{H} + \sum p_i y_i'\right)\mathrm{d}t = \text{Ł}\mathrm{d}t, \tag{8.16}$$

use the identifications (8.13) of coordinates. This is the classical definition of the action V in a Mayer field. We have moreover clarified the additive nature of the level sets $V = \lambda$: roughly saying, the composition with $V = \mu$ provides $V = \lambda + \mu$ (see Figure 3(c)) and this is caused by the additivity of the integral $\int \text{Ł}\ dt$ calculated along the orbits.

On this occasion, the wave enveloping approach to $\mathcal{CT}$ groups is also worth mentioning.

Lemma 8.3 (see [10–13]). *Let* $W(\overline{x}_1, \ldots, \overline{x}_n, \overline{w}, x_1, \ldots, x_n, w)$ *be a function of* $2n + 2$ *variables. Assume that the system* $W = D_1 W = \cdots = D_n W = 0$ *admits a unique solution*

$$\overline{x}_i = F_i(\ldots, x_i', w, w_i', \ldots), \qquad \overline{w} = F^1(\ldots, x_i', w, w_i', \ldots) \tag{8.17}$$

by applying the implicit function theorem and analogously the system $W = \overline{D}_1 W = \cdots = \overline{D}_n W = 0$ *(where* $\overline{D}_i = \partial/\partial\overline{x}_i + \sum \overline{w}_i \partial/\partial\overline{w}$*) admits a certain solution*

$$x_i = \overline{F}_i(\ldots, \overline{x}_{i'}, \overline{w}, \overline{w}_{i'}, \ldots), \qquad w = \overline{F}^1(\ldots, \overline{x}_{i'}, \overline{w}, \overline{w}_{i'}, \ldots). \tag{8.18}$$

Then $\mathbf{m}^* x_i = F_i$, $\mathbf{m}^* w = F^1$ *provides a Lie* $\mathcal{CT}$ *and* $(\mathbf{m}^{-1})^* \overline{x}_i = \overline{F}_i$, $(\mathbf{m}^{-1})^* \overline{w} = \overline{F}^1$ *is the inverse.*

In more generality, if function W in Lemma 8.3 moreover depends on a parameter λ, we obtain a mapping $\mathbf{m}(\lambda)$ which is a certain $\mathcal{CT}$ involving a parameter λ and the inverse $\mathbf{m}(\lambda)^{-1}$. *In favourable case* (see below) *this* $\mathbf{m}(\lambda)$ *may be even a* $\mathcal{CT}$ *group*. The geometrical sense is as follows. Equation $W = 0$ with $\overline{x}_i$, $\overline{w}$ kept fixed represents a wave in the space x_i, w (Figure 3(a)).

The total system $W = D_1 W = \cdots = D_n W = 0$ provides the intersection (envelope) of infinitely close waves (Figure 3(b)) with the resulting transform, the focus point $\mathbf{m}$ (or $\mathbf{m}(\lambda)$ if the parameter λ is present). The reverse waves with the role of variables interchanged gives the inversion. Then the group property holds true if the waves can be composed (Figure 3(c)) within the parameters λ, μ, but this *need not* be in general the case.

Let us eventually deal with the condition ensuring the group composition property. Without loss of generality, we may consider the λ-depending wave

$$W(\overline{x}_1, \ldots, \overline{x}_n, \overline{w}, x_1, \ldots, x_n, w) - \lambda = 0. \tag{8.19}$$

If $\overline{x}_i$, $\overline{w}$ are kept fixed, the previous results may be applied. We obtain a group if and only if the $\mathcal{HJ}$ equation (8.10) holds true, therefore

$$W_w + \mathcal{H}(x_1, \ldots, x_n, w, W_{x_1}, \ldots, W_{x_n}) = 0. \tag{8.20}$$

The existence of such function $\mathcal{H}$ means that functions $W_w, W_{x_1}, \ldots, W_{x_n}$ *of dashed variables* are functionally dependent whence

$$\det\begin{pmatrix} W_{w\overline{w}} & W_{w\overline{x}_{i'}} \\ W_{x_i\overline{w}} & W_{x_i\overline{x}_{i'}} \end{pmatrix} = 0, \qquad \det\left(W_{x_i\overline{x}_{i'}}\right) \neq 0. \tag{8.21}$$

The symmetry $\overline{x}_i, \overline{w} \leftrightarrow x_i, w$ is not surprising here since the change $\lambda \leftrightarrow -\lambda$ provides the inverse mapping: equations

$$W(\ldots, \overline{x}_i, \overline{w}, \ldots, x_i, w) = \lambda, \qquad W(\ldots, x_i, w, \ldots, \overline{x}_i, \overline{w}) = -\lambda \tag{8.22}$$

are equivalent. In particular, it follows that

$$W(\ldots, \overline{x}_i, \overline{w}, \ldots, x_i, w) = -W(\ldots, x_i, w, \ldots, \overline{x}_i, \overline{w}), \qquad W(\ldots, x_i, w, \ldots, x_i, w) = 0 \tag{8.23}$$

and the wave $W - \lambda = 0$ corresponds to the Mayer central field of extremals.

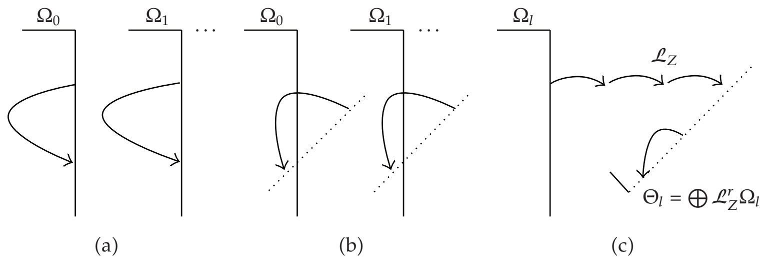


Figure 4

Summary 4. Conditions (8.21) ensure the existence of$\mathcal{HJ}$ equation (8.20) for the λ-wave (8.19) and therefore the group composition property of waves (8.19) in the nondegenerate case $\det(\partial^2\mathcal{H}/\partial p_i \partial p_j) \neq 0$.

Remark 8.4. A reasonable theory of Mayer fields of extremals and Hamilton-Jacobi equations can be developed also for the constrained variational integrals (the Lagrange problem) within the framework of jet spaces, that is, without the additional Lagrange multipliers [9, Chapter 3]. It follows that there do exist certain groups of generalized Lie's contact transformations with differential constraints.

9. On the Order-Destroying Groups in Jet Space

We recall that in the order-preserving case, the filtration

$$\Omega(m,n)_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega(m,n) = \cup \Omega_l \tag{9.1}$$

of module $\Omega(m,n)$ is preserved (Figure 4(a)). It follows that certain invariant submodules $\Omega_l \subset \Omega(m,n)$ are a priori prescribed which essentially restricts the store of the symmetries (the Lie-Bäcklund theorem). The order-destroying groups also preserve certain submodules of $\Omega(m,n)$ due to approximation results, however, they are not known in advance (Figure 4(b)) and appear after certain saturation (Figure 4(c)) described in technical theorem 5.1.

The saturation is in general a toilsome procedure. It may be simplified by applying two simple principles.

Lemma 9.1 (going-up lemma). *Let a group of morphisms* $\mathbf{m}(\lambda)$ *preserve a submodule* $\Theta \subset \Omega(m,n)$*. Then also the submodule*

$$\Theta + \sum \mathcal{L}_{D_i}\Theta \subset \Omega(m,n) \tag{9.2}$$

is preserved.

Proof. We suppose $\mathcal{L}_Z\Theta \subset \Theta$. Then

$$\mathcal{L}_Z\left(\Theta + \sum \mathcal{L}_{D_i}\Theta\right) = \mathcal{L}_Z\Theta + \left(\mathcal{L}_{D_i}\mathcal{L}_Z\Theta - \sum D_i z_{i'} \mathcal{L}_{D_{i'}}\Theta\right) \subset \Theta + \sum \mathcal{L}_{D_i}\Theta \tag{9.3}$$

by using the commutative rule (5.17). □

Lemma 9.2 (going-down lemma). *Let the group of morphisms* $\mathbf{m}(\lambda)$ *preserve a submodule* $\Theta \subset \Omega(m,n)$. *Let* $\Theta' \subset \Theta$ *be the submodule of all* $\omega \in \Theta$ *satisfying* $\mathcal{L}_{D_i}\omega \in \Theta$ $(i = 1,\ldots,n)$. *Then* Θ' *is preserved, too.*

Proof. Assume $\omega \in \Theta'$ hence $\mathcal{L}_{D_i}\omega \in \Theta$. Then $\mathcal{L}_{D_i}\mathcal{L}_Z\omega = \mathcal{L}_Z\mathcal{L}_{D_i}\omega + \mathcal{L}_{\sum D_i z_{i'} \cdot D_{i'}}\omega \in \Theta$ hence $\mathcal{L}_Z\omega \in \Theta'$ and Θ' is preserved. □

We are passing to illustrative examples.

Example 9.3. Let us consider the vector field (the variation of jet structure)

$$Z = \sum z_I^j \frac{\partial}{\partial w_I^j} \quad \left(z_I^j = D_I z^j,\ D_I = D_{i_1}\cdots D_{i_n}\right), \tag{9.4}$$

see (5.6) and (5.10) for the particular case $z_i = 0$. Then $Z^r x_i = 0$ $(i = 1,\ldots,n)$ and the *sufficient* requirement $Z^2 w^j = 0$ $(j = 1,\ldots,m)$ ensures $Z \in \mathbb{G}$, see (ι) of Lemma 5.4. We will deal with the linear case where

$$z^j = \sum a_{i'}^{jj'} w_{i'}^{j'} \quad \left(a_{i'}^{jj'} \in \mathbb{R}\right) \tag{9.5}$$

is supposed. Then

$$Z^2 w^j = Z z^j = \sum a_{i'}^{jj'} z_{i'}^{j'} = \sum a_{i'}^{jj'} a_{i''}^{j'j''} w_{i'i''}^{j''} = 0 \tag{9.6}$$

identically if and only if

$$\sum_{j'}\left(a_{i'}^{jj'} a_{i''}^{j'j''} + a_{i''}^{jj'} a_{i'}^{j'j''}\right) = 0 \quad \left(i', i'' = 1,\ldots,n;\ j, j', j'' = 1,\ldots,m\right). \tag{9.7}$$

This may be expressed in terms of matrix equations

$$A_i A_{i'} = 0 \quad \left(i, i' = 1,\ldots,n;\ A_i = \left(a_i^{jj'}\right)\right) \tag{9.8}$$

or, in either of more geometrical transcriptions

$$A^2 = 0, \qquad \operatorname{Im} A \subset \operatorname{Ker} A \quad \left(A = \sum \lambda_i A_i,\ \lambda_i \in \mathbb{R}\right), \tag{9.9}$$

where A is regarded as (a matrix of an) operator acting in m-dimensional linear space and depending on parameters $\lambda_1,\ldots,\lambda_n$. We do not know explicit solutions A in full generality, however, solutions A such that Ker A does not depend on the parameters $\lambda_1,\ldots,\lambda_n$ can be easily found (and need not be stated here). The same approach can be applied to the more general *sufficient* requirement $Z^r w^j = 0$ $(j = 1,\ldots,m;$ fixed $r)$ ensuring $Z \in \mathbb{G}$. If $r \geq n$, the requirement is *equivalent* to the inclusion $Z \in \mathbb{G}$.

Example 9.4. Let us consider vector field (5.6) where $z^1 = \cdots = z^m = 0$. In more detail, we take

$$Z = \sum z_i \frac{\partial}{\partial x_i} + \sum z_i^j \frac{\partial}{\partial w_i^j} + \cdots \quad \left(z_i^j = -\sum w_{i'}^j D_i z_{i'} \right). \tag{9.10}$$

Then $Z^r w^j = 0$ and we have to deal with functions $Z^r x_i$ in order to ensure the inclusion $Z \in \mathbb{G}$. This is a difficult task. Let us therefore suppose

$$z_1 = z\left(\ldots, x_{i'}, w^{j'}, w_1^{j'}, \ldots \right), \qquad z_k = c_k \in \mathbb{R} \quad (k = 2, \ldots, n). \tag{9.11}$$

Then $Zx_k = 0$ $(k = 2, \ldots, n)$ and

$$Z^2 x_1 = Zz = \sum \frac{\partial z}{\partial x_i} z_i + \sum \frac{\partial z}{\partial w_1^j} z_1^j, \tag{9.12}$$

where

$$z_1^j = -w_1^j D_1 z = -w_1^j \left(\frac{\partial z}{\partial x_1} + \sum \frac{\partial z}{\partial w^{j'}} w_1^{j'} + \sum \frac{\partial z}{\partial w_1^{j'}} w_{11}^{j'} \right). \tag{9.13}$$

The second-order summand

$$Z^2 x_1 = \cdots + \sum \frac{\partial z}{\partial w_1^j} z_1^j = \cdots - \sum \frac{\partial z}{\partial w_1^j} w_1^j \frac{\partial z}{\partial w_1^{j'}} w_{11}^{j'} \tag{9.14}$$

identically vanishes for the choice

$$z = f\left(\ldots, x_{i'}, w^{j'}, u^l, \ldots \right) \quad \left(u^l = \frac{w_1^l}{w_1^1};\ l = 2, \ldots, m \right) \tag{9.15}$$

as follows by direct verification. Quite analogously

$$Zu^l = Z \frac{w_1^l}{w_1^1} = z_1^l \frac{1}{w_1^1} - z_1^1 \frac{w_1^l}{(w_1^1)^2} = \left(-w_1^l \frac{1}{w_1^1} + w_1^1 \frac{w_1^l}{(w_1^1)^2} \right) D_1 z = 0. \tag{9.16}$$

It follows that all functions $Z^r x_i$, $Z^r w^j$ can be expressed in terms of the *finite family* of functions x_i $(i = 1, \ldots, n)$, w^j $(j = 1, \ldots, m)$, u^l $(l = 2, \ldots, m)$ and therefore $Z \in \mathbb{G}$.

Remark 9.5. On this occasion, let us briefly mention the groups generated by vector fields Z of the above examples. The Lie system of the vector field (9.4) and (9.5) reads

$$\frac{\mathrm{d}G_i}{\mathrm{d}\lambda} = 0, \qquad \frac{\mathrm{d}G^j}{\mathrm{d}\lambda} = \sum a_{i'}^{jj'} G_{i'}^{j'} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m), \tag{9.17}$$

where we omit the prolongations. It is resolved by

$$G_i = x_i, \quad G^j = w^j + \lambda \sum a^{jj'}_{i'} w^{j'}_{i'} \quad (i = 1, \ldots, n;\ j = 1, \ldots, m) \tag{9.18}$$

as follows either by direct verification or, alternatively, from the property $Z^2 x_i = Z z_i = 0$ ($i = 1, \ldots, n$) which implies

$$\frac{\mathrm{d} \sum a^{jj'}_{i'} G^{j'}_{i'}}{\mathrm{d}\lambda} = 0, \qquad \sum a^{jj'}_{i'} G^{j'}_{i'} = \sum a^{jj'}_{i'} G^{j'}_{i'} \Big|_{\lambda=0} = \sum a^{jj'}_{i'} w^{j'}_{i'}. \tag{9.19}$$

Quite analogously, the Lie system of the vector field (9.10), (9.11), (9.15) reads

$$\frac{\mathrm{d}G_1}{\mathrm{d}\lambda} = f\left(\ldots, G_{i'}, G^{j'}, \frac{G^{l'}_1}{G^1_1}, \ldots\right), \quad \frac{\mathrm{d}G_k}{\mathrm{d}\lambda} = c_k, \quad \frac{\mathrm{d}G^j}{\mathrm{d}\lambda} = 0 \quad (k = 2, \ldots, n;\ j = 1, \ldots, m) \tag{9.20}$$

and may be completed with the equations

$$\frac{\mathrm{d}\left(G^l_1 / G^1_1\right)}{\mathrm{d}\lambda} = 0 \quad (l = 2, \ldots, m) \tag{9.21}$$

following from (9.16). This provides a classical self-contained system of ordinary differential equations where the common existence theorems can be applied.

The above Lie systems admit many nontrivial *first integrals* $F \in \mathcal{F}$, that is, functions F that are constant on the orbits of the group. Conditions $F = 0$ may be interpreted as differential equations in the total jet space, and the above transformation groups turn into the *external generalized symmetries* of such differential equations, see Section 11 below.

10. Towards the Main Algorithm

We briefly recall the algorithm [10–13] for determination of all individual automorphisms $\mathbf{m}$ of the jet space $\mathbf{M}(m, n)$ in order to compare it with the subsequent calculation of vector field $Z \in \mathbb{G}$.

Morphisms $\mathbf{m}$ of the jet structure were defined by the property $\mathbf{m}^*\Omega(m, n) \subset \Omega(m, n)$. The inverse $\mathbf{m}^{-1}$ exists if and only if

$$\Omega_0 \subset \mathbf{m}^*\Omega(m, n), \quad \text{equivalently } \Omega_0 \subset \mathbf{m}^*\Omega_l \quad (l = l(\mathbf{m})) \tag{10.1}$$

for appropriate term $\Omega_{l(\mathbf{m})}$ of filtration (9.1). However

$$\mathbf{m}^*\Omega_{l+1} = \mathbf{m}^*\Omega_l + \sum \mathcal{L}_{D_i} \mathbf{m}^*\Omega_l \tag{10.2}$$

and it follows that criterion (10.1) can be verified by repeated use of operators $\mathcal{L}_{D_i}$. In more detail, we start with equations

$$\mathbf{m}^*\omega^j = \sum a^{jj'}_{I'}\omega^{j'}_{I'} \quad \left(= \mathrm{d}\mathbf{m}^*w^j - \sum \mathbf{m}^*w^j_i \mathrm{d}\mathbf{m}^*x_i\right) \tag{10.3}$$

with uncertain coefficients. Formulae (10.3) determine the module $\mathbf{m}^*\Omega_0$. Then we search for lower-order contact forms, especially forms from Ω_0, lying in $\mathbf{m}^*\Omega_l$ with the use of (10.2). Such forms are ensured if certain *linear relations among coefficients exist*. The calculation is finished on a certain level $l = l(\mathbf{m})$ and this is the *algebraic part* of the algorithm. With this favourable choice of coefficients $a^{jj'}_{I'}$, functions $\mathbf{m}^*x_i$, $\mathbf{m}^*w^j$ (and therefore the invertible morphism $\mathbf{m}$) can be determined by inspection of the bracket in (10.3). This is the *analytic part* of algorithm.

Let us turn to the infinitesimal theory. Then the main technical tool is the rule (5.17) in the following transcription:

$$\mathcal{L}_Z\mathcal{L}_{D_i} = \mathcal{L}_{D_i}\mathcal{L}_Z - \sum D_i z_{i'}\mathcal{L}_{D_{i'}} \tag{10.4}$$

or, when applied to basical forms

$$\mathcal{L}_Z\omega^j_{Ii} = \mathcal{L}_{D_i}\mathcal{L}_Z\omega^j_I - \sum D_i z_{i'}\omega^j_{Ii'}. \tag{10.5}$$

We are interested in vector fields $Z \in \mathbb{G}$. They satisfy the recurrence (5.10) together with requirements

$$\dim \oplus \mathcal{L}^r_Z\Omega_0 < \infty, \quad \text{equivalently } \mathcal{L}^r_Z\Omega_0 \subset \Omega_{l(Z)} \quad (r = 0, 1, \ldots) \tag{10.6}$$

for appropriate $l(Z) \in \mathbb{N}$. Due to the recurrence (10.5) these requirements can be effectively investigated. In more detail, we start with equations

$$\mathcal{L}_Z\omega^j = \sum a^{jj'}_{I'}\omega^{j'}_{I'} \quad \left(= \mathrm{d}z^j - \sum z^j_i \mathrm{d}x_i - \sum w^j_i \mathrm{d}z_i\right). \tag{10.7}$$

Formulae (10.7) determine module $\mathcal{L}_Z\Omega_0$. Then, choosing $l(Z) \in \mathbb{N}$, operator $\mathcal{L}_Z$ is to be repeatedly applied and requirements (10.6) provide certain *polynomial relations for the coefficients* by using (10.5). This is the *algebraical part* of the algorithm. With such coefficients $a^{jj'}_{I'}$ available, functions $z_i = \mathcal{L}_Z x_i$, $z^j = \mathcal{L}_Z w^j$ (and therefore the vector field $Z \in \mathbb{G}$) can be determined by inspection of the bracket in (10.7) or, alternatively, with the use of formulae (5.12) for the particular case $I = \phi$ empty

$$\mathcal{L}_Z\omega^j = \sum\left(\frac{\partial z^j}{\partial w^{j'}_{I'}} - \sum w^j_i \frac{\partial z_i}{\partial w^{j'}_{I'}}\right)\omega^{j'}_{I'}. \tag{10.8}$$

This is the *analytic part* of the algorithm.

Altogether taken, the algorithm is not easy and the conviction [7, page 121] that the "exhaustive description of integrable C-fields (fields $Z \in \mathbb{Z}$ in our notation) is given in [16]" is disputable. We can state only one optimistic result at this place.

Theorem 10.1. *The jet spaces* $\mathbf{M}(1, n)$ *do not admit any true generalized infinitesimal symmetries* $Z \in \mathbb{G}$.

Proof. We suppose $m = 1$ and then (10.7) reads

$$\mathcal{L}_Z \omega^1 = \sum a^{11}_{I'} \omega^1_{I'} = \cdots + a^{11}_{I''} \omega^1_{I''} \quad \left(a^{11}_{I''} \neq 0 \right), \tag{10.9}$$

where we state a summand of maximal order. Assuming $I'' = \phi$, the Lie-Bäcklund theorem can be applied and we do not have the true generalized symmetry Z. Assuming $I'' \neq \phi$, then

$$\mathcal{L}^r_Z \omega^1 = \cdots + a^{11}_{I''} \omega^1_{I'' \ldots I''} \quad (r \text{ terms } I'') \tag{10.10}$$

by using rule (10.5) where the last summand may be omitted. It follows that (10.6) is not satisfied hence $Z \notin \mathbb{G}$. □

Example 10.2. We discuss the simplest possible but still a nontrivial particular example. Assume $m = 2$, $n = 1$ and $l(Z) = 1$. Let us abbreviate

$$x = x_1, \quad D = D_1, \quad Z = z\frac{\partial}{\partial x} + \sum z^j_I \frac{\partial}{\partial w^j_I} \quad (j = 1, 2;\ I = 1 \cdots 1). \tag{10.11}$$

Then, due to $l(Z) = 1$, requirement (10.6) reads

$$\mathcal{L}^r_Z \Omega_0 \subset \Omega_1 \quad (r = 0, 1, \ldots). \tag{10.12}$$

In particular (if $r = 1$) we have (10.7) written here in the simplified notation

$$\mathcal{L}_Z \omega^j = a^{j1}\omega^1 + a^{j2}\omega^2 + b^{j1}\omega^1_1 + b^{j2}\omega^2_1 \quad (j = 1, 2). \tag{10.13}$$

The next requirement ($r = 2$) *implies the* (only seemingly) *stronger inclusion*

$$\mathcal{L}^2_Z \Omega_0 \subset \mathcal{L}_Z \Omega_0 + \Omega_0 \tag{10.14}$$

which already ensures (10.12) for all r and therefore $Z \in \mathbb{G}$ (easy). *We suppose* (10.14) *from now on*.

"Hint for proof of (10.14): assuming (10.12) and moreover the equality

$$\mathcal{L}^2_Z \Omega_0 + \mathcal{L}_Z \Omega_0 + \Omega_0 = \Omega_1, \tag{10.15}$$

it follows that

$$\mathcal{L}_Z\Omega_1 \subset \mathcal{L}_Z^3\Omega_0 + \mathcal{L}_Z^2\Omega_0 + \mathcal{L}_Z\Omega_0 \subset \Omega_1 \tag{10.16}$$

and Lie-Bäcklund theorem can be applied whence $\mathcal{L}_Z\Omega_0 \subset \Omega_0$, $l(Z) = 0$ which we exclude. It follows that necessarily

$$\dim\left(\mathcal{L}_Z^2\Omega_0 + \mathcal{L}_Z\Omega_0 + \Omega_0\right) < \dim\Omega_1 = 4. \tag{10.17}$$

On the other hand $\dim(\mathcal{L}_Z\Omega_0 + \Omega_0) \geq 3$ and the inclusion (10.14) follows."

After this preparation, we are passing to the proper algebra. Clearly

$$\mathcal{L}_Z^2\omega^j = \cdots + b^{j1}\mathcal{L}_Z\omega_1^1 + b^{j2}\mathcal{L}_Z\omega_1^2 = \cdots + b^{j1}\left(b^{11}\omega_{11}^1 + b^{12}\omega_{11}^2\right) + b^{j2}\left(b^{21}\omega_{11}^1 + b^{22}\omega_{11}^2\right) \tag{10.18}$$

by using the commutative rule (10.5). Due to "weaker" inclusion (10.12) with $r = 2$, we obtain identities

$$b^{j1}b^{11} + b^{j2}b^{21} = 0, \quad b^{j1}b^{12} + b^{j2}b^{22} = 0 \quad (j = 1, 2). \tag{10.19}$$

Omitting the trivial solution, they are satisfied if either

$$b^{11} + b^{22} = 0, \qquad b^{12} = cb^{11}, \qquad b^{11} + cb^{21} = 0 \tag{10.20}$$

for appropriate factor c (where $b^{11} \neq 0$ and either $b^{12} \neq 0$ or $b^{21} \neq 0$ is supposed) or

$$b^{11} = b^{22} = 0, \quad \text{either } b^{12} = 0 \quad \text{or } b^{21} = 0. \tag{10.21}$$

We deal only with the (more interesting) identities (10.20) here. Then

$$\begin{aligned} \mathcal{L}_Z\omega^1 &= a^{11}\omega^1 + a^{12}\omega^2 - cb\left(\omega_1^1 + c\omega_1^2\right), \\ \mathcal{L}_Z\omega^2 &= a^{21}\omega^1 + a^{22}\omega^2 + b\left(\omega_1^1 + c\omega_1^2\right) \end{aligned} \tag{10.22}$$

(abbreviation $b = b^{21}$) by inserting (10.20) into (10.13). It follows that

$$\mathcal{L}_Z\left(\omega^1 + c\omega^2\right) = a^1\omega^1 + a^2\omega^2 \quad \left(a^1 = a^{11} + ca^{21}, a^2 = a^{12} + ca^{22} + Zc\right). \tag{10.23}$$

It may be seen by direct calculation of $\mathcal{L}_Z^2\omega^2$ that the "stronger" inclusion (10.14) is equivalent to the identity $ca^1 = a^2$, that is,

$$\mathcal{L}_Z\left(\omega^1 + c\omega^2\right) = a\left(\omega^1 + c\omega^2\right) \tag{10.24}$$

(abbreviation $a = a^1$). Alternatively, (10.24) can be proved by using Lemma 9.2.

"Hint: denoting $\Theta = \mathcal{L}_Z\Omega_0 + \Omega_0$, (10.14) implies $\mathcal{L}_Z\Theta \subset \Theta$. Moreover $\mathcal{L}_D(\omega^1 + c\omega^2) \in \Theta$ by using (10.22). Lemma 9.2 can be applied: $\omega^1 + c\omega^2 \in \Theta'$ and Θ' involves just all multiples of form $\omega^1 + c\omega^2$. Therefore $\mathcal{L}_Z(\omega^1 + c\omega^2) \in \Theta'$ is a multiple of $\omega^1 + c\omega^2$."

The algebraical part is concluded. We have congruences

$$\mathcal{L}_Z\omega^1 \cong -cb\left(\omega_1^1 + c\omega_1^2\right), \quad \mathcal{L}_Z\omega^2 \cong b\left(\omega_1^1 + c\omega_1^2\right) \pmod{\Omega_0} \tag{10.25}$$

and equality

$$\mathcal{L}_Z\omega^1 + c\mathcal{L}_Z\omega^2 + Zc\omega^2 = a\left(\omega^1 + c\omega^2\right). \tag{10.26}$$

If Z is a variation then these three conditions together ensure the "stronger inclusion" (10.14) hence $Z \in \mathbb{G}$.

We turn to analysis. Abbreviating

$$Z_{I'}^{jj'} = \frac{\partial z^j}{\partial w_{I'}^{j'}} - w_1^j \frac{\partial z}{\partial w_{I'}^{j'}} \quad (j, j' = 1, 2;\ I' = 1 \cdots 1) \tag{10.27}$$

and employing (10.8), the above conditions (10.25) and (10.26) read

$$\begin{gathered}\sum Z_{I'}^{1j'}\omega_{I'}^{j'} = -cb\left(\omega_1^1 + c\omega_1^2\right), \quad \sum Z_{I'}^{2j'}\omega_{I'}^{j'} = b\left(\omega_1^1 + c\omega_1^2\right) \quad \left(\left|I'\right| \geq 1\right), \\ \sum\left(Z_{I'}^{1j'} + cZ_{I'}^{2j'}\right)\omega_{I'}^{j'} + Zc\omega^2 = a\left(\omega^1 + c\omega^2\right).\end{gathered} \tag{10.28}$$

We compare coefficients of forms ω_I^j on the level $s = |I'|$

$$s = 0\colon\ Z^{11} + cZ^{21} = a, \qquad Z^{12} + cZ^{22} + Zc = ac, \tag{10.29}$$

$$s = 1\colon\ Z_1^{11} = -cb, \qquad Z_1^{12} = -(c)^2 b, \qquad Z_1^{21} = b, \qquad Z_1^{22} = bc, \qquad Z_1^{1j} + cZ_1^{2j} = 0, \tag{10.30}$$

$$s \geq 2\colon\ Z_{I'}^{jj'} = 0, \qquad Z_{I'}^{1j'} + cZ_{I'}^{2j'} = 0. \tag{10.31}$$

We will successively delete the coefficients a, b, c in order to obtain interrelations only for variables $Z_{I'}^{jj'}$. Clearly

$$\begin{gathered}s = 0\colon\ Z^{12} + cZ^{22} + Zc = \left(Z^{11} + cZ^{21}\right)c, \\ s = 1\colon\ Z_1^{11} + Z_1^{22} = 0, \qquad Z_1^{11}Z_1^{22} = Z_1^{12}Z_1^{21},\end{gathered} \tag{10.32}$$

and we moreover have three compatible equations

$$c = -\frac{Z_1^{11}}{Z_1^{21}} = -\frac{Z_1^{12}}{Z_1^{22}}, \qquad (c)^2 = -\frac{Z_1^{12}}{Z_1^{21}} \tag{10.33}$$

for the coefficient c. To cope with levels $s \geq 2$, we introduce functions

$$Q^j = \omega^j(Z) = z^j - w_1^j z \quad (j = 1, 2). \tag{10.34}$$

Then substitution into (10.27) with the help of (10.31) gives

$$\frac{\partial Q^j}{\partial w_{I'}^{j'}} = 0 \quad \left(j, j' = 1, 2;\ \left|I'\right| \geq 2\right). \tag{10.35}$$

It follows moreover easily that

$$Z_1^{jj'} = \frac{\partial Q^j}{\partial w_1^{j'}} \quad (j \neq j'), \qquad Z_1^{jj} = z + \frac{\partial Q^j}{\partial w_1^j}, \qquad Z^{jj'} = \frac{\partial Q^j}{\partial w^{j'}} \tag{10.36}$$

and we have *the final differential equations*

$$s = 0\colon \frac{\partial Q^1}{\partial w^2} + c\frac{\partial Q^2}{\partial w^2} + Zc = \left(\frac{\partial Q^1}{\partial w^1} + c\frac{\partial Q^2}{\partial w^1}\right)c, \tag{10.37}$$

$$s = 1\colon 2z + \frac{\partial Q^1}{\partial w_1^1} + \frac{\partial Q^2}{\partial w_1^2} = 0, \quad \left(z + \frac{\partial Q^1}{\partial w_1^1}\right)\left(z + \frac{\partial Q^2}{\partial w_1^2}\right) = \frac{\partial Q^1}{\partial w_1^2}\frac{\partial Q^2}{\partial w_1^1} \tag{10.38}$$

for the unknown functions

$$z = z\left(x, w^1, w^2, w_1^1, w_1^2\right), \qquad Q^j = Q^j\left(x, w^1, w^2, w_1^1, w_1^2\right). \tag{10.39}$$

The coefficient c is determined by (10.33) and (10.36) in terms of functions Q^j. This concludes the analytic part of the algorithm since trivially $z^j = w_1^j z + Q^j$ and the vector field Z is determined.

The system is compatible: particular solutions with functions Q^j quadratic in jet variables and $c = \text{const.}$ can be found as follows. Assume

$$Q^j = A^j\left(w_1^1\right)^2 + 2B^j w_1^1 w_1^2 + C^j\left(w_1^2\right)^2 \quad (j = 1, 2) \tag{10.40}$$

with constant coefficients $A^j, B^j, C^j \in \mathbb{R}$. We also suppose $c \in \mathbb{R}$ and then (10.37) is trivially satisfied.

On the other hand, (10.33) provide the requirements

$$z + \frac{\partial Q^1}{\partial w_1^1} + c\frac{\partial Q^2}{\partial w_1^1} = \frac{\partial Q^1}{\partial w_1^2} + c\left(z + \frac{\partial Q^2}{\partial w_1^2}\right) = \frac{\partial Q^1}{\partial w_1^2} + (c)^2\frac{\partial Q^2}{\partial w_1^1} = 0 \tag{10.41}$$

by using (10.36). If we put

$$z = -\frac{\partial Q^1}{\partial w_1^1} - \frac{\partial Q^2}{\partial w_1^2} = -\left(A^1 + B^1\right)w_1^1 - \left(B^1 + C^2\right)w_1^2, \tag{10.42}$$

then (10.38) is satisfied (a clumsy direct verification).

The above requirements turn to a system of six homogeneous linear equations (not written here) for the six constants A^j, B^j, C^j $(j = 1, 2)$ with determinant $\Delta = c^2(c^2 - 8)$ if the values z, Q^1, Q^2 are inserted and the coefficients of w_1^1 and w_1^2 are compared. The roots $c = 0$ and $c = \pm 2\sqrt{2}$ of the equation $\Delta = 0$ provide rather nontrivial infinitesimal transformation Z, however, we can state only the simplest result for the trivial root $c = 0$ for obvious reason. It reads

$$Q^1 = A^1\left(w_1^1\right)^2, \quad Q^2 = A^2\left(w_1^1\right)^2, \quad z = -A^1 w_1^1, \quad z^1 = 0, \quad z^2 = w_1^1\left(A^2 w_1^1 + A^1 w_1^2\right), \tag{10.43}$$

where A^1, A^2 are arbitrary constants.

Remark 10.3. It follows that investigation of vector fields $Z \in \mathbb{G}$ cannot be regarded for easy task and some new powerful methods are necessary, for example, better use of differential forms (involutive systems) with pseudogroup symmetries of the problem (moving frames).

11. A Few Notes on the Symmetries of Differential Equations

The *external theory* deals with (systems of) differential equations ($\mathfrak{DE}$) that are firmly localized in the jet spaces. This is the common approach and it runs as follows. A given finite system of $\mathfrak{DE}$ is infinitely prolonged in order to ensure the compatibility. In general, this prolongation is a toilsome and delicate task, in particular the "singular solutions" are tacitly passed over. The prolongation procedure is expressed in terms of jet variables and as a result a *fixed subspace* of the (infinite-order) jet space appears which represents the $\mathfrak{DE}$ under consideration. Then the *external symmetries* [2, 3, 6, 7] are such symmetries of the ambient jet space which preserve the subspace. In this sense we may speak of *classical symmetries* (point and contact transformations) and *higher-order symmetries* (which destroy the order of derivatives).

The *internal theory* of $\mathfrak{DE}$ is irrelevant to the jet localization, in particular to the choice of the hierarchy of independent and dependent variables. This point of view is due to E. Cartan and actually the congenial term "diffiety" was introduced in [6, 7]. Alas, these diffieties were defined as objects *locally identical with appropriate external* $\mathfrak{DE}$ restricted to the corresponding subspace of the ambient total jet space. This can hardly be regarded as a coordinate-free (or jet theory-free) approach since the model objects (*external* $\mathfrak{DE}$) and the intertwining mappings (*higher-order symmetries*) essentially need the use of the above hard jet theory mechanisms and concepts.

In reality, the final result of prolongation, the infinitely prolonged $\mathfrak{DE}$, can be alternatively characterized by three simple axioms as follows [8, 9, 24–27].

Let $\mathbf{M}$ be a space *modelled on* $\mathbb{R}^\infty$ (local coordinates $h^1, h^2, \dots$ as in Sections 1 and 2 above). Denote by $\mathcal{F}(\mathbf{M})$ the *structural module* of all smooth functions f on $\mathbf{M}$ (locally depending on a finite number $m(f)$ of coordinates). Let $\Phi(\mathbf{M})$, $\mathfrak{T}(\mathbf{M})$ be the $\mathcal{F}(\mathbf{M})$-modules of all differential 1-forms and vector fields on $\mathbf{M}$, respectively. For every submodule $\Omega \subset \Phi(\mathbf{M})$, we have the "orthogonal" submodule $\Omega^\perp = \mathcal{H} \subset \mathfrak{T}(\mathbf{M})$ of all $X \in \mathcal{H}$ such that $\Omega(X) = 0$.

Then an $\mathcal{F}(\mathbf{M})$-submodule $\Omega \subset \Phi(\mathbf{M})$ is called a *diffiety* if the following three requirements are locally satisfied.

($\mathcal{A}$) Ω is of codimension $n < \infty$, equivalent $\mathcal{H}$ is of dimension $n < \infty$.
Here n is the *number of independent variables*. The independent variables provide the complementary module to Ω in $\Phi(\mathbf{M})$ which is not prescribed in advance.

($\mathcal{B}$) $d\Omega \cong 0 \pmod{\Omega}$, equivalent $\mathcal{L}_{\mathcal{H}}\Omega \subset \Omega$, equivalently: $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$.
This *Frobenius condition* ensures the classical *passivity requirement*: we deal with the compatible infinite prolongation of differential equations.

($\mathcal{C}$) There exists filtration $\Omega_* : \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega = \cup\Omega_l$ by finite-dimensional submodules $\Omega_l \subset \Omega$ such that

$$\mathcal{L}_{\mathcal{H}}\Omega_l \subset \Omega_{l+1} \quad (\text{all } l), \quad \Omega_{l+1} = \Omega_l + \mathcal{L}_{\mathcal{H}}\Omega_l \quad (l \text{ large enough}). \tag{11.1}$$

This condition may be expressed in terms of a $\odot\mathcal{H}$-polynomial algebra on the graded module $\oplus\, \Omega_l/\Omega_{l-1}$ (the *Noetherian property*) and ensures the *finite number of dependent variables*. Filtration Ω_* may be capriciously modified. In particular, various localizations of Ω in jet spaces $\Omega(m, n)$ can be easily obtained.

The *internal symmetries* naturally appear. For instance, a vector field $Z \in \mathfrak{T}(\mathbf{M})$ is called a (*universal) variation* of diffiety Ω if $\mathcal{L}_Z\Omega \subset \Omega$ and *infinitesimal symmetry* if moreover Z generates a local group, that is, if and only if $Z \in \mathbb{G}$.

Theorem 11.1 (technical theorem). *Let Z be a variation of diffiety Ω. Then $Z \in \mathbb{G}$ if and only if there is a finite-dimensional $\mathcal{F}(M)$-submodule $\Theta \subset \Omega$ such that*

$$\oplus\mathcal{L}^r_{\mathcal{H}}\Theta = \Omega, \qquad \dim \oplus\mathcal{L}^r_Z\Theta < \infty. \tag{11.2}$$

This is exactly counterpart to Theorem 5.6: submodule $\Theta \subset \Omega$ stands here for the previous submodule $\Omega_0 \subset \Omega(m, n)$. We postpone the proof of Theorem 11.1 together with applications to some convenient occasion.

Remark 11.2. There may exist conical symmetries Z of a diffiety Ω, however, they are all lying in $\mathcal{H}$ and generate just the *Cauchy characteristics* of the diffiety [9, page 155].

We conclude with two examples of internal theory of underdetermined ordinary differential equations. The reasonings to follow can be carried over quite general diffieties without any change.

Example 11.3. Let us deal with the *Monge equation*

$$\frac{dx}{dt} = f\left(t, x, y, \frac{dy}{dt}\right). \tag{11.3}$$

The prolongation can be represented as the Pfaffian system

$$\mathrm{d}x - f(t, x, y, y')\mathrm{d}t = 0, \quad \mathrm{d}y - y'\mathrm{d}t = 0, \quad \mathrm{d}y' - y''\mathrm{d}t = 0, \ldots. \tag{11.4}$$

Within the framework of diffieties, we introduce space $\mathbf{M}$ with coordinates

$$t, x_0, y_0, y_1, y_2, \ldots \tag{11.5}$$

and submodule $\Omega \subset \Phi(\mathbf{M})$ with generators

$$\mathrm{d}x_0 - f\mathrm{d}t, \quad (\omega_r =)\mathrm{d}y_r - y_{r+1}\mathrm{d}t \quad (r = 0, 1, \ldots;\ f = f(t, x_0, y_0, y_1)). \tag{11.6}$$

Clearly $\mathscr{H} = \Omega^{\perp} \subset \mathcal{T}(\mathbf{M})$ is one-dimensional subspace including the vector field

$$D = \frac{\partial}{\partial t} + f\frac{\partial}{\partial x_0} + \sum y_{r+1}\frac{\partial}{\partial y_r}. \tag{11.7}$$

One can easily find that we have a diffiety. ($\mathcal{A}$ and $\mathcal{B}$ are trivially satisfied. The *common order preserving filtrations* where Ω_l involves $\mathrm{d}x_0 - f\mathrm{d}t$ and ω_r with $r \le l$ is enough for $\mathcal{C}$.)

We introduce a new (*standard* [9]) *filtration* $\overline{\Omega}_*$ where the submodule $\overline{\Omega}_l \subset \Omega$ is generated by the forms

$$\vartheta_0 = \mathrm{d}x_0 - f\mathrm{d}t - \frac{\partial f}{\partial y_1}\omega_0, \omega_r \quad (r \le l - 1). \tag{11.8}$$

This is indeed a filtration since

$$\begin{aligned} \mathcal{L}_D\vartheta_0 = \mathrm{d}f - Df\mathrm{d}t - D\frac{\partial f}{\partial y_1}\cdot\omega_0 - \frac{\partial f}{\partial y_1}\omega_1 &= \frac{\partial f}{\partial x_0}(\mathrm{d}x_0 - f\mathrm{d}t) + \left(\frac{\partial f}{\partial y_0} - D\frac{\partial f}{\partial y_1}\right)\omega_0 \\ &= \frac{\partial f}{\partial x_0}\vartheta_0 + A\omega_0 \quad \left(A = \frac{\partial f}{\partial y_0} + \frac{\partial f}{\partial x_0}\frac{\partial f}{\partial y_1} - D\frac{\partial f}{\partial y_1}\right) \end{aligned} \tag{11.9}$$

and (trivially) $\mathcal{L}_D\omega_r = \omega_{r+1}$. Assuming $A \neq 0$ from now on (this is satisfied if $f_{y_1y_1} \neq 0$) *every module* $\overline{\Omega}_l$ *is generated by the forms* $\vartheta_r = \mathcal{L}_D^r\vartheta_0$ $(r \le l)$.

The forms ϑ_r satisfy the recurrence $\mathcal{L}_D\vartheta_r = \vartheta_{r+1}$. Then the formula

$$\vartheta_{r+1} = \mathcal{L}_D\vartheta_r = D\rfloor\mathrm{d}\vartheta_r + \mathrm{d}\vartheta_r(D) = D\rfloor\mathrm{d}\vartheta_r \tag{11.10}$$

implies the congruence $\mathrm{d}\vartheta_r \cong \mathrm{d}t \wedge \vartheta_{r+1}$ (mod $\Omega \wedge \Omega$). Let

$$Z = z\frac{\partial}{\partial t} + z^0\frac{\partial}{\partial x_0} + \sum z_r\frac{\partial}{\partial y_r} \tag{11.11}$$

be a variation of Ω in the common sense $\mathcal{L}_Z\Omega \subset \Omega$. This inclusion is equivalent to the congruence

$$\mathcal{L}_Z\vartheta_r = Z\rfloor \mathrm{d}\vartheta_r + \mathrm{d}\vartheta_r(Z) \cong -\vartheta_{r+1}(Z)\mathrm{d}t + D\vartheta_r(Z)\mathrm{d}t = 0\ (\mathrm{mod}\ \Omega) \tag{11.12}$$

whence to the recurrence

$$\vartheta_{r+1}(Z) = D\vartheta_r(Z) \tag{11.13}$$

quite analogous to the recurrence (5.10), see Remark 5.3. It follows that the functions

$$z = Zt = \mathrm{d}t(Z), \qquad g = \vartheta_0(Z) \tag{11.14}$$

can be quite arbitrarily chosen. Then functions $\vartheta_r(Z) = D^r g$ are determined and we obtain *quite explicit formulae for the variation* Z. In more detail

$$\begin{aligned} g = \vartheta_0(Z) &= \left(\mathrm{d}x_0 - f\mathrm{d}t - \frac{\partial f}{\partial y_1}\omega_0\right)(Z) = z^0 - fz - \frac{\partial f}{\partial y_1}(z_0 - y_1 z), \\ Dg = \vartheta_1(Z)&\left(\frac{\partial f}{\partial x_0}\vartheta_0 + A\omega_0\right)(Z) = \frac{\partial f}{\partial x_0}g + A(z_0 - y_1 z) \end{aligned} \tag{11.15}$$

and these equations determine coefficients z^0 and z_0 in terms of functions z and g. Coefficients z_r ($r \geq 1$) follow by prolongation (not stated here). If moreover

$$\dim\{\mathcal{L}_Z^r\vartheta_0\}_{r\in\mathbb{N}} < \infty \tag{11.16}$$

we have infinitesimal symmetry $Z \in \mathbb{G}$, see Theorem 11.1.

Example 11.4. Let us deal with the *Hilbert-Cartan equation* [3]

$$\frac{dy}{dt} = \left(\frac{d^2x}{dt^2}\right)^2. \tag{11.17}$$

Passing to the diffiety, we introduce space $\mathbf{M}$ with coordinates

$$t, x_0, x_1, y_0, y_1, y_2, \ldots \tag{11.18}$$

and submodule $\Omega \subset \Phi(\mathbf{M})$ generated by forms

$$\mathrm{d}x_0 - x_1\mathrm{d}t, \quad \mathrm{d}x_1 - \sqrt{y_1}\mathrm{d}t, \quad (\omega_r =)\mathrm{d}y_r - y_{r+1}\mathrm{d}t \quad (r = 0, 1, \ldots). \tag{11.19}$$

The submodule $\mathcal{H} = \Omega^{\perp} \subset \mathfrak{T}(\mathbf{M})$ is generated by the vector field

$$D = \frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_0} + \sqrt{y_1}\frac{\partial}{\partial x_1} + \sum y_{r+1}\frac{\partial}{\partial y_r}. \tag{11.20}$$

We introduce the form

$$\vartheta_0 = \mathrm{d}x_0 - x_1\mathrm{d}t + B\left\{\mathrm{d}x_1 - \sqrt{y_1}\mathrm{d}t - \frac{1}{2\sqrt{y_1}}\omega_0\right\} \quad \left(B = \frac{1/\sqrt{y_1}}{D(1/\sqrt{y_1})}\right) \tag{11.21}$$

and moreover the forms

$$\begin{aligned}
&\vartheta_1 = \mathcal{L}_D\vartheta_0 = (1 + DB)\{\cdots\},\\
&\vartheta_2 = \mathcal{L}_D\vartheta_1 = D^2B\{\cdots\} - C\omega_0 \quad \left(C = (1 + DB)D\frac{1}{2\sqrt{y_1}}\right),\\
&\vartheta_3 = \cdots + C\omega_1,\\
&\vartheta_4 = \cdots + C\omega_2,\\
&\quad\vdots
\end{aligned} \tag{11.22}$$

Assuming $C \neq 0$, we have a standard filtration $\overline{\Omega}_*$ where the submodules $\overline{\Omega}_l \subset \Omega$ are generated by forms ϑ_r $(r \leq l)$. Explicit formulae for variations

$$Z = z\frac{\partial}{\partial t} + z^0\frac{\partial}{\partial x_0} + z^1\frac{\partial}{\partial x_1} + \sum z_r\frac{\partial}{\partial y_r} \tag{11.23}$$

can be obtained analogously as in Example 11.3 (and are omitted here). Functions z and $g = \vartheta_0(Z)$ can be arbitrarily chosen. Condition (11.16) ensures $Z \in \mathbb{G}$.

Appendix

For the convenience of reader, we survey some results [9, 18, 19] on the modules Adj. Our reasonings are carried out in the space $\mathbb{R}^n$ and will be true *locally* near *generic points*.

Let Θ be a given module of 1-forms and $A(\Theta)$ the module of all vector fields X such that $\mathcal{L}_{fX}\Theta \subset \Theta$ for all functions f, see [9]. Clearly

$$\mathcal{L}_{[X,Z]}\Theta = (\mathcal{L}_X\mathcal{L}_Z - \mathcal{L}_Z\mathcal{L}_X)\Theta \subset \Theta \quad (X, Z \in A(\Theta)) \tag{A.1}$$

and it follows that identity

$$f[X,Y] = [X,Z] + Xf \cdot Y \quad \big(X,Y \in A(\Theta);\ Z = fY\big) \tag{A.2}$$

implies $\mathcal{L}_{f[X,Y]}\Theta \subset \Theta$ whence $[A(\Theta), A(\Theta)] \subset A(\Theta)$.

Let Θ be of a finite dimension I. The Frobenius theorem can be applied, and it follows that module Adj $\Theta = A(\Theta)^{\perp}$ (of all forms φ satisfying $\varphi(A(\Theta)) = 0$) has a certain basis $\mathrm{d}f^1, \dots, \mathrm{d}f^K$ $(K \geq I)$.

On the other hand, identity

$$\mathcal{L}_{fX}\vartheta = \quad fX\rfloor \mathrm{d}\vartheta + \mathrm{d}\big(f\vartheta(X)\big) = f\mathcal{L}_X\vartheta + \vartheta(X)\vartheta \tag{A.3}$$

implies that $X \in A(\Theta)$ if and only if

$$\vartheta(X) = 0,\ X\rfloor \mathrm{d}\vartheta \in \Theta \quad (\vartheta \in \Theta) \tag{A.4}$$

which is the classical definition, see [2]. In particular $\Theta \subset$ Adj Θ so we may suppose the generators

$$\vartheta^i = \mathrm{d}f^i + g^i_{I+1}\mathrm{d}f^{I+1} + \cdots + g^i_K \mathrm{d}f^K \in \Theta \quad (i = 1, \dots, I) \tag{A.5}$$

of module Θ. Recall that $Xf^k = 0$ $(k = 1, \dots, K;\ X \in A(\Theta))$ whence

$$\mathcal{L}_X\vartheta^i = Xg^i_{I+1}\mathrm{d}f^{I+1} + \cdots + Xg^i_K\mathrm{d}f^K \in \Theta \tag{A.6}$$

and this implies $Xg^i_{I+1} = \cdots = Xg^i_K = 0$. It follows that

$$\mathrm{d}g^i_{I+1}, \dots, \mathrm{d}g^i_K \in \operatorname{Adj}\Theta \quad (i = 1, \dots, I) \tag{A.7}$$

and therefore all coefficients g^i_k depend only on variables $f^1, \dots, f^K$.

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Research Article

The Local Strong and Weak Solutions for a Nonlinear Dissipative Camassa-Holm Equation

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Using the Kato theorem for abstract differential equations, the local well-posedness of the solution for a nonlinear dissipative Camassa-Holm equation is established in space $C([0,T), H^s(R)) \cap C^1([0,T), H^{s-1}(R))$ with $s > 3/2$. In addition, a sufficient condition for the existence of weak solutions of the equation in lower order Sobolev space $H^s(R)$ with $1 \le s \le 3/2$ is developed.

1. Introduction

Camassa and Holm [1] used the Hamiltonian method to derive a completely integrable wave equation

$$u_t - u_{xxt} + 2ku_x + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1.1}$$

by retaining two terms that are usually neglected in the small amplitude, shallow water limit. Its alternative derivation as a model for water waves can be found in Constantin and Lannes [2] and Johnson [3]. Equation (1.1) also models wave current interaction [4], while Dai [5] derived it as a model in elasticity (see Constantin and Strauss [6]). Moreover, it was pointed out in Lakshmanan [7] that the Camassa-Holm equation (1.1) could be relevant to the modeling of tsunami waves (see Constantin and Johnson [8]).

In fact, a huge amount of work has been carried out to investigate the dynamic properties of (1.1). For $k = 0$, (1.1) has traveling wave solutions of the form $c\,e^{-|x-ct|}$, called peakons, which capture the main feature of the exact traveling wave solutions of greatest height of the governing equations (see [9–11]). For $k > 0$, its solitary waves are stable solitons [6, 11]. It was shown in [12–14] that the inverse spectral or scattering approach was a powerful tool to handle Camassa-Holm equation. Equation (1.1) is a completely integrable infinite-dimensional Hamiltonian system (in the sense that for a large class of initial data, the flow is equivalent to a linear flow at constant speed [15]). It should be emphasized that (1.1) gives rise to geodesic flow of a certain invariant metric on the Bott-Virasoro group (see [16, 17]), and this geometric illustration leads to a proof that the Least Action Principle holds. It is worthwhile to mention that Xin and Zhang [18] proved that the global existence of the weak solution in the energy space $H^1(R)$ without any sign conditions on the initial value, and the uniqueness of this weak solution is obtained under some conditions on the solution [19]. Coclite et al. [20] extended the analysis presented in [18, 19] and obtained many useful dynamic properties to other equations (also see [21–24]). Li and Olver [25] established the local well-posedness in the Sobolev space $H^s(R)$ with $s > 3/2$ for (1.1) and gave conditions on the initial data that lead to finite time blowup of certain solutions. It was shown in Constantin and Escher [26] that the blowup occurs in the form of breaking waves, namely, the solution remains bounded but its slope becomes unbounded in finite time. After wave breaking, the solution can be continued uniquely either as a global conservative weak solution [21] or a global dissipative solution [22]. For peakons, these possibilities are explicitly illustrated in the paper [27]. For other methods to handle the problems relating to various dynamic properties of the Camassa-Holm equation and other shallow water models, the reader is referred to [10, 28–32] and the references therein.

In this paper, motivated by the work in [25, 33], we study the following generalized Camassa-Holm equation

$$u_t - u_{txx} + 2ku_x + au^m u_x = 2u_x u_{xx} + uu_{xxx} + \beta\partial_x\left[(u_x)^N\right], \tag{1.2}$$

where $m \geq 1$ and $N \geq 1$ are natural numbers, and a, k, and β are arbitrary constants. Obviously, (1.2) reduces to (1.1) if we set $a = 3$, $m = 1$, and $\beta = 0$. Actually, Wu and Yin [34] consider a nonlinearly dissipative Camassa-Holm equation which includes a nonlinearly dissipative term $L(u)$, where L is a differential operator or a quasidifferential operator. Therefore, we can regard the term $\beta\partial_x[(u_x)^N]$ as a nonlinearly dissipative term for the dissipative Camassa-Holm equation (1.2).

Due to the term $\beta\partial_x[(u_x)^N]$ in (1.2), the conservation laws in previous works [10, 25] for (1.1) lose their powers to obtain some bounded estimates of the solution for (1.2). A new conservation law different from those presented in [10, 25] will be established to prove the local existence and uniqueness of the solution to (2.3) subject to initial value $u_0(x) \in H^s(R)$ with $s > 3/2$. We should address that all the generalized versions of the Camassa-Holm equation in previous works (see [17, 25, 34]) do not involve the nonlinear term $\partial_x[(u_x)^N]$. Lai and Wu [33] only studied a generalized Camassa-Holm equation in the case where $\beta \geq 0$ and N is an odd number. Namely, (1.2) with $\beta < 0$ and arbitrary positive integer N was not investigated in [33].

The main tasks of this paper are two-fold. Firstly, by using the Kato theorem for abstract differential equations, we establish the local existence and uniqueness of solutions for (1.2) with any β and arbitrary positive integer N in space $C([0,T), H^s(R)) \bigcap C^1([0,T), H^{s-1}(R))$ with $s > 3/2$. Secondly, it is shown that the existence of weak solutions in lower order Sobolev space $H^s(R)$ with $1 \leq s \leq 3/2$. The ideas of proving the second result come from those presented in Li and Olver [25].

2. Main Results

Firstly, we give some notation.

The space of all infinitely differentiable functions $\phi(t,x)$ with compact support in $[0,+\infty)\times R$ is denoted by C_0^∞. $L^p = L^p(R)$ $(1 \le p < +\infty)$ is the space of all measurable functions h such that $\|h\|_{L^p}^p = \int_R |h(t,x)|^p dx < \infty$. We define $L^\infty = L^\infty(R)$ with the standard norm $\|h\|_{L^\infty} = \inf_{m(e)=0} \sup_{x\in R\setminus e} |h(t,x)|$. For any real number s, $H^s = H^s(R)$ denotes the Sobolev space with the norm defined by

$$\|h\|_{H^s} = \left(\int_R \left(1+|\xi|^2\right)^s \left|\widehat{h}(t,\xi)\right|^2 d\xi\right)^{1/2} < \infty, \tag{2.1}$$

where $\widehat{h}(t,\xi) = \int_R e^{-ix\xi} h(t,x)dx$.

For $T > 0$ and nonnegative number s, $C([0,T); H^s(R))$ denotes the Frechet space of all continuous H^s-valued functions on $[0,T)$. We set $\Lambda = (1-\partial_x^2)^{1/2}$.

In order to study the existence of solutions for (1.2), we consider its Cauchy problem in the form

$$\begin{aligned} u_t - u_{txx} &= -2ku_x - \frac{a}{m+1}\left(u^{m+1}\right)_x + 2u_x u_{xx} + uu_{xxx} + \beta\partial_x\left[(u_x)^N\right] \\ &= -ku_x - \frac{a}{m+1}\left(u^{m+1}\right)_x + \frac{1}{2}\partial_x^3 u^2 - \frac{1}{2}\partial_x\left(u_x^2\right) + \beta\partial_x\left[(u_x)^N\right], \\ u(0,x) &= u_0(x), \end{aligned} \tag{2.2}$$

which is equivalent to

$$\begin{gathered} u_t + uu_x = \Lambda^{-2}\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x + \Lambda^{-2}(uu_x) - \frac{1}{2}\Lambda^{-2}\partial_x\left(u_x^2\right) + \beta\Lambda^{-2}\partial_x\left[(u_x)^N\right], \\ u(0,x) = u_0(x). \end{gathered} \tag{2.3}$$

Now, we state our main results.

Theorem 2.1. *Let $u_0(x) \in H^s(R)$ with $s > 3/2$. Then problem* (2.2) *or problem* (2.3) *has a unique solution $u(t,x) \in C([0,T); H^s(R)) \bigcap C^1([0,T); H^{s-1}(R))$ where $T > 0$ depends on $\|u_0\|_{H^s(R)}$.*

Theorem 2.2. *Suppose that $u_0(x) \in H^s$ with $1 \le s \le 3/2$ and $\|u_{0x}\|_{L^\infty} < \infty$. Then there exists a $T > 0$ such that* (1.2) *subject to initial value $u_0(x)$ has a weak solution $u(t,x) \in L^2([0,T], H^s)$ in the sense of distribution and $u_x \in L^\infty([0,T]\times R)$.*

3. Local Well-Posedness

We consider the abstract quasilinear evolution equation

$$\frac{dv}{dt} + A(v)v = f(v), \quad t \ge 0, \; v(0) = v_0. \tag{3.1}$$

Let X and Y be Hilbert spaces such that Y is continuously and densely embedded in X, and let $Q : Y \to X$ be a topological isomorphism. Let $L(Y,X)$ be the space of all bounded linear operators from Y to X. If $X = Y$, we denote this space by $L(X)$. We state the following conditions in which ρ_1, ρ_2, ρ_3, and ρ_4 are constants depending on $\max\{\|y\|_Y, \|z\|_Y\}$.

(i) $A(y) \in L(Y,X)$ for $y \in X$ with

$$\|(A(y) - A(z))w\|_X \leq \rho_1 \|y - z\|_X \|w\|_Y, \quad y, z, w \in Y, \tag{3.2}$$

and $A(y) \in G(X, 1, \beta)$ (i.e., $A(y)$ is quasi-m-accretive), uniformly on bounded sets in Y.

(ii) $QA(y)Q^{-1} = A(y) + B(y)$, where $B(y) \in L(X)$ is bounded, uniformly on bounded sets in Y. Moreover,

$$\|(B(y) - B(z))w\|_X \leq \rho_2 \|y - z\|_Y \|w\|_X, \quad y, z \in Y, \ w \in X. \tag{3.3}$$

(iii) $f : Y \to Y$ extends to a map from X into X is bounded on bounded sets in Y, and satisfies

$$\begin{aligned} \|f(y) - f(z)\|_Y &\leq \rho_3 \|y - z\|_Y, \quad y, z \in Y, \\ \|f(y) - f(z)\|_X &\leq \rho_4 \|y - z\|_X, \quad y, z \in Y. \end{aligned} \tag{3.4}$$

Kato Theorem (see [35])

Assume that (i), (ii), and (iii) hold. If $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$, and a unique solution v to problem (3.1) such that

$$v = v(\cdot, v_0) \in C([0,T); Y) \bigcap C^1([0,T); X). \tag{3.5}$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is a continuous map from Y to the space

$$C([0,T); Y) \bigcap C^1([0,T); X). \tag{3.6}$$

For problem (2.3), we set $A(u) = u\partial_x$, $Y = H^s(R)$, $X = H^{s-1}(R)$, $\Lambda = (1 - \partial_x^2)^{1/2}$,

$$f(u) = \Lambda^{-2}\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x + \Lambda^{-2}(uu_x) - \frac{1}{2}\Lambda^{-2}\partial_x\left(u_x^2\right) + \beta\Lambda^{-2}\partial_x\left[(u_x)^N\right], \tag{3.7}$$

and $Q = \Lambda$. In order to prove Theorem 2.1, we only need to check that $A(u)$ and $f(u)$ satisfy assumptions (i)–(iii).

Lemma 3.1. *The operator $A(u) = u\partial_x$ with $u \in H^s(R)$, $s > 3/2$ belongs to $G(H^{s-1}, 1, \beta)$.*

Lemma 3.2. *Let $A(u) = u\partial_x$ with $u \in H^s$ and $s > 3/2$. Then $A(u) \in L(H^s, H^{s-1})$ for all $u \in H^s$. Moreover,*

$$\|(A(u) - A(z))w\|_{H^{s-1}} \le \rho_1 \|u - z\|_{H^{s-1}} \|w\|_{H^s}, \quad u, z, w \in H^s(R). \tag{3.8}$$

Lemma 3.3. *For $s > 3/2$, $u, z \in H^s$ and $w \in H^{s-1}$, it holds that $B(u) = [\Lambda, u\partial_x]\Lambda^{-1} \in L(H^{s-1})$ for $u \in H^s$ and*

$$\|(B(u) - B(z))w\|_{H^{s-1}} \le \rho_2 \|u - z\|_{H^s} \|w\|_{H^{s-1}}. \tag{3.9}$$

Proofs of the above Lemmas 3.1–3.3 can be found in [29] or [31].

Lemma 3.4 (see [35]). *Let r and q be real numbers such that $-r < q \le r$. Then*

$$\begin{gathered} \|uv\|_{H^q} \le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r > \frac{1}{2}, \\ \|uv\|_{H^{r+q-1/2}} \le c\|u\|_{H^r}\|v\|_{H^q}, \quad \text{if } r < \frac{1}{2}. \end{gathered} \tag{3.10}$$

Lemma 3.5. *Let $u, z \in H^s$ with $s > 3/2$, then $f(u)$ is bounded on bounded sets in H^s and satisfies*

$$\|f(u) - f(z)\|_{H^s} \le \rho_3 \|u - z\|_{H^s}, \tag{3.11}$$

$$\|f(u) - f(z)\|_{H^{s-1}} \le \rho_4 \|u - z\|_{H^{s-1}}. \tag{3.12}$$

Proof. Using the algebra property of the space H^{s_0} with $s_0 > 1/2$, we have

$$\begin{aligned} &\|f(u) - f(z)\|_{H^s} \\ &\quad \le c\Bigg[\left\|\Lambda^{-2}\left(\left[-ku - \frac{a}{m+1}\left(u^{m+1}\right)\right]_x - \left[-kz - \frac{a}{m+1}\left(z^{m+1}\right)\right]_x\right)\right\|_{H^s} \\ &\qquad + \left\|\Lambda^{-2}(uu_x - zz_x)\right\|_{H^s} + \left\|\Lambda^{-2}\partial_x\left(u_x^2 - z_x^2\right)\right\|_{H^s} + \left\|\Lambda^{-2}\partial_x\left[(u_x)^N\right] - \Lambda^{-2}\partial_x\left[(z_x)^N\right]\right\|_{H^s}\Bigg] \\ &\quad \le c\Big[\|u - z\|_{H^{s-1}} + \left\|u^{m+1} - z^{m+1}\right\|_{H^{s-1}} + \|uu_x - zz_x\|_{H^{s-1}} + \left\|u_x^2 - z_x^2\right\|_{H^{s-1}} \\ &\qquad + \left\|(u_x)^N - (z_x)^N\right\|_{H^{s-1}}\Big] \\ &\quad \le c\|u - z\|_{H^s}\left[1 + \sum_{j=0}^{m}\|u\|_{H^s}^{m-j}\|z\|_{H^s}^{j} + \|u\|_{H^s} + \|z\|_{H^s} + \sum_{j=0}^{N-1}\|u_x\|_{H^{s-1}}^{N-j}\|z_x\|_{H^{s-1}}^{j}\right] \\ &\quad \le \rho_3\|u - z\|_{H^s}, \end{aligned} \tag{3.13}$$

from which we obtain (3.11).

Applying Lemma 3.4, $uu_x = (1/2)(u^2)_x$, $s > 3/2$, $\|u\|_{L^\infty} \le c\|u\|_{H^{s-1}}$ and $\|u_x\|_{L^\infty} \le c\|u\|_{H^s}$, we get

$$\begin{aligned}
&\|f(u) - f(z)\|_{H^{s-1}} \\
&\quad \le c\Bigg[\|u - z\|_{H^{s-2}} + \left\|u^{m+1} - z^{m+1}\right\|_{H^{s-2}} + \left\|u^2 - z^2\right\|_{H^{s-2}} \\
&\qquad + \|(u_x - z_x)(u_x + z_x)\|_{H^{s-2}} + \left\|(u_x - z_x)\sum_{j=0}^{N-1} u_x^{N-1-j} z_x^j\right\|_{H^{s-2}}\Bigg] \\
&\quad \le c\|u - z\|_{H^{s-1}}\Bigg[1 + \sum_{j=0}^{m} \|u\|_{H^{s-1}}^{m-j}\|z\|_{H^{s-1}}^{j} + \|u\|_{H^{s-1}} + \|z\|_{H^{s-1}} \\
&\qquad + \|u\|_{H^s} + \|z\|_{H^s} + \sum_{j=0}^{N-1} \|u_x\|_{H^{s-1}}^{N-j}\|z_x\|_{H^{s-1}}^{j}\Bigg] \\
&\quad \le \rho_4\|u - z\|_{H^{s-1}},
\end{aligned} \tag{3.14}$$

which completes the proof of (3.12). □

Proof of Theorem 2.1. Using the Kato Theorem, Lemmas 3.1–3.3, and 3.5, we know that system (2.2) or problem (2.3) has a unique solution

$$u(t, x) \in C([0, T); H^s(R)) \bigcap C^1\left([0, T); H^{s-1}(R)\right). \tag{3.15}$$

□

4. Existence of Weak Solutions

For $s \ge 2$, using the first equation of system (2.2) derives

$$\frac{d}{dt}\int_R\left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1}d\tau\right)dx = 0, \tag{4.1}$$

from which we have the conservation law

$$\int_R\left(u^2 + u_x^2 + 2\beta\int_0^t u_x^{N+1}d\tau\right)dx = \int_R\left(u_0^2 + u_{0x}^2\right)dx. \tag{4.2}$$

Lemma 4.1 (Kato and Ponce [36]). *If $r > 0$, then $H^r \bigcap L^\infty$ is an algebra. Moreover,*

$$\|uv\|_r \le c(\|u\|_{L^\infty}\|v\|_r + \|u\|_r\|v\|_{L^\infty}), \tag{4.3}$$

where c is a constant depending only on r.

Lemma 4.2 (Kato and Ponce [36]). *Let $r > 0$. If $u \in H^r \bigcap W^{1,\infty}$ and $v \in H^{r-1} \bigcap L^\infty$, then*

$$\|[\Lambda^r, u]v\|_{L^2} \le c\left(\|\partial_x u\|_{L^\infty}\left\|\Lambda^{r-1}v\right\|_{L^2} + \|\Lambda^r u\|_{L^2}\|v\|_{L^\infty}\right). \tag{4.4}$$

Lemma 4.3. *Let $s \ge 2$ and the function $u(t,x)$ is a solution of problem (2.2) and the initial data $u_0(x) \in H^s(R)$. Then the following inequality holds*

$$\|u\|_{L^\infty} \le \|u\|_{H^1} \le \|u_0\|_{H^1} e^{|\beta|\int_0^t \|u_x\|_{L^\infty}^{N-1} d\tau}. \tag{4.5}$$

For $q \in (0, s-1]$, there is a constant c, which only depends on m, N, k, a, and β, such that

$$\begin{aligned}\int_R \left(\Lambda^{q+1}u\right)^2 dx \le \int_R \left(\Lambda^{q+1}u_0\right)^2 dx + c\int_0^t \|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2\left(1 + \|u\|_{L^\infty}^{m-1}\right)d\tau \\ + c\int_0^t \|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}d\tau.\end{aligned} \tag{4.6}$$

For $q \in [0, s-1]$, there is a constant c, which only depends on m, N, k, a, and β, such that

$$\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(1 + \left(1 + \|u\|_{L^\infty}^{m-1}\right)\|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.7}$$

Proof. Using $\|u\|_{H^1}^2 = \int_R (u^2 + u_x^2)dx$ and (4.2) derives (4.5).

Using $\partial_x^2 = -\Lambda^2 + 1$ and the Parseval equality gives rise to

$$\int_R \Lambda^q u \Lambda^q \partial_x^3\left(u^2\right)dx = -2\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx + 2\int_R (\Lambda^q u)\Lambda^q(uu_x)dx. \tag{4.8}$$

For $q \in (0, s-1]$, applying $(\Lambda^q u)\Lambda^q$ to both sides of the first equation of system (2.3) and integrating with respect to x by parts, we have the identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_R \left((\Lambda^q u)^2 + (\Lambda^q u_x)^2\right)dx = -a\int_R (\Lambda^q u)\Lambda^q(u^m u_x)dx \\ - \int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx + \frac{1}{2}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx \\ + \int_R (\Lambda^q u)\Lambda^q(uu_x)dx - \beta\int_R \Lambda^q u_x \Lambda^q\left[(u_x)^N\right]dx.\end{aligned} \tag{4.9}$$

We will estimate the terms on the right-hand side of (4.9) separately. For the first term, by using the Cauchy-Schwartz inequality and Lemmas 4.1 and 4.2, we have

$$\begin{aligned}\int_R (\Lambda^q u)\Lambda^q(u^m u_x)dx &= \int_R (\Lambda^q u)[\Lambda^q(u^m u_x) - u^m\Lambda^q u_x]dx + \int_R (\Lambda^q u)u^m\Lambda^q u_x dx \\ &\leq c\|u\|_{H^q}\left(m\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}\|u\|_{H^q} + \|u_x\|_{L^\infty}\|u\|_{L^\infty}^{m-1}\|u\|_{H^q}\right) \\ &\quad + \frac{1}{2}\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}\|\Lambda^q u\|_{L^2}^2 \\ &\leq c\|u\|_{H^q}^2\|u\|_{L^\infty}^{m-1}\|u_x\|_{L^\infty}.\end{aligned} \tag{4.10}$$

Using the above estimate to the second term yields

$$\int_R \left(\Lambda^{q+1}u\right)\Lambda^{q+1}(uu_x)dx \leq c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}. \tag{4.11}$$

For the third term, using the Cauchy-Schwartz inequality and Lemma 4.1, we obtain

$$\begin{aligned}\int_R (\Lambda^q u_x)\Lambda^q\left(u_x^2\right)dx &\leq \|\Lambda^q u_x\|_{L^2}\left\|\Lambda^q\left(u_x^2\right)\right\|_{L^2} \\ &\leq c\|u\|_{H^{q+1}}(\|u_x\|_{L^\infty}\|u_x\|_{H^q} + \|u_x\|_{L^\infty}\|u_x\|_{H^q}) \\ &\leq c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}.\end{aligned} \tag{4.12}$$

For the last term in (4.9), using Lemma 4.1 repeatedly results in

$$\begin{aligned}\left|\int_R (\Lambda^q u_x)\Lambda^q(u_x)^N dx\right| &\leq \|u_x\|_{H^q}\left\|u_x^N\right\|_{H^q} \\ &\leq c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}.\end{aligned} \tag{4.13}$$

It follows from (4.9) to (4.13) that there exists a constant c depending only on m, N and the coefficients of (1.2) such that

$$\frac{1}{2}\frac{d}{dt}\int_R \left[(\Lambda^q u)^2 + (\Lambda^q u_x)^2\right]dx \leq c\|u_x\|_{L^\infty}\|u\|_{H^{q+1}}^2\left(1 + \|u\|_{L^\infty}^{m-1}\right) + c\|u\|_{H^{q+1}}^2\|u_x\|_{L^\infty}^{N-1}. \tag{4.14}$$

Integrating both sides of the above inequality with respect to t results in inequality (4.6).

To estimate the norm of u_t, we apply the operator $(1-\partial_x^2)^{-1}$ to both sides of the first equation of system (2.3) to obtain the equation

$$u_t = \left(1-\partial_x^2\right)^{-1}\left[-2ku_x + \partial_x\left(-\frac{a}{m+1}u^{m+1} + \frac{1}{2}\partial_x^2\left(u^2\right) - \frac{1}{2}u_x^2\right) + \beta\partial_x\left[(u_x)^N\right]\right]. \tag{4.15}$$

Applying $(\Lambda^q u_t)\Lambda^q$ to both sides of (4.15) for $q \in (0, s-1]$ gives rise to

$$\int_R (\Lambda^q u_t)^2 dx = \int_R (\Lambda^q u_t)\Lambda^{q-2}\left[\partial_x\left(-2ku - \frac{a}{m+1}u^{m+1} + \frac{1}{2}\partial_x^2\left(u^2\right) - \frac{1}{2}u_x^2\right) + \beta\partial_x\left[(u_x)^N\right]\right] d\tau. \tag{4.16}$$

For the right-hand side of (4.16), we have

$$\begin{aligned}
&\int_R (\Lambda^q u_t)\Lambda^{q-2}(-2ku_x)dx \le c\|u_t\|_{H^q}\|u\|_{H^q},\\
&\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(-\frac{a}{m+1}u^{m+1} - \frac{1}{2}u_x^2\right)dx\\
&\quad\le c\|u_t\|_{H^q}\left(\int_R \left(1+\xi^2\right)^{q-1} \times \left[\int_R \left[-\frac{a}{m+1}\widehat{u^m}(\xi-\eta)\widehat{u}(\eta) - \frac{1}{2}\widehat{u_x}(\xi-\eta)\widehat{u_x}(\eta)\right]d\eta\right]^2\right)^{1/2}\\
&\quad\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}\left(1+\|u\|_{L^\infty}^{m-1}\right).
\end{aligned} \tag{4.17}$$

Since

$$\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x^2(uu_x)dx = -\int (\Lambda^q u_t)\Lambda^q(uu_x)dx + \int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx, \tag{4.18}$$

using Lemma 4.1, $\|uu_x\|_{H^q} \le c\|(u^2)_x\|_{H^q} \le c\|u\|_{L^\infty}\|u\|_{H^{q+1}}$ and $\|u\|_{L^\infty} \le \|u\|_{H^1}$, we have

$$\begin{aligned}
\int (\Lambda^q u_t)\Lambda^q(uu_x)dx &\le c\|u_t\|_{H^q}\|uu_x\|_{H^q}\\
&\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}},\\
\int (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q(uu_x)dx &\le c\|u_t\|_{H^q}\|u\|_{H^1}\|u\|_{H^{q+1}}.
\end{aligned} \tag{4.19}$$

Using the Cauchy-Schwartz inequality and Lemma 4.1 yields

$$\left|\int_R (\Lambda^q u_t)\left(1-\partial_x^2\right)^{-1}\Lambda^q\partial_x\left(u_x^N\right)dx\right| \le c\|u_t\|_{H^q}\|u_x\|_{L^\infty}^{N-1}\|u\|_{H^{q+1}}. \tag{4.20}$$

Applying (4.17)–(4.20) into (4.16) yields the inequality

$$\|u_t\|_{H^q} \le c\|u\|_{H^{q+1}}\left(1 + \left(1+\|u\|_{L^\infty}^{m-1}\right)\|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\right). \tag{4.21}$$

This completes the proof of Lemma 4.3. □

Defining

$$\phi(x)=\begin{cases} e^{1/(x^2-1)}, & |x|<1,\\ 0, & |x|\geq 1,\end{cases} \tag{4.22}$$

and setting $\phi_\varepsilon(x)=\varepsilon^{-1/4}\phi(\varepsilon^{-1/4}x)$ with $0<\varepsilon<1/4$ and $u_{\varepsilon 0}=\phi_\varepsilon \star u_0$, we know that $u_{\varepsilon 0}\in C^\infty$ for any $u_0\in H^s(R)$ and $s>0$.

It follows from Theorem 2.1 that for each ε the Cauchy problem

$$\begin{gathered} u_t-u_{txx}=\partial_x\left(-2ku-\frac{a}{m+1}u^{m+1}\right)+\frac{1}{2}\partial_x^3\left(u^2\right)-\frac{1}{2}\partial_x\left(u_x^2\right)+\beta\partial_x\left[(u_x)^N\right],\\ u(0,x)=u_{\varepsilon 0}(x),\quad x\in R,\end{gathered} \tag{4.23}$$

has a unique solution $u_\varepsilon(t,x)\in C^\infty([0,T);H^\infty)$.

Lemma 4.4. *Under the assumptions of problem* (4.23)*, the following estimates hold for any ε with $0<\varepsilon<1/4$ and $s>0$*

$$\begin{gathered} \|u_{\varepsilon 0x}\|_{L^\infty}\leq c_1\|u_{0x}\|_{L^\infty},\\ \|u_{\varepsilon 0}\|_{H^q}\leq c_1,\quad \text{if } q\leq s,\\ \|u_{\varepsilon 0}\|_{H^q}\leq c_1\varepsilon^{(s-q)/4},\quad \text{if } q>s,\\ \|u_{\varepsilon 0}-u_0\|_{H^q}\leq c_1\varepsilon^{(s-q)/4},\quad \text{if } q\leq s,\\ \|u_{\varepsilon 0}-u_0\|_{H^s}=o(1),\end{gathered} \tag{4.24}$$

where c_1 is a constant independent of ε.

The proof of this Lemma can be found in Lai and Wu [33].

Lemma 4.5. *If $u_0(x)\in H^s(R)$ with $s\in[1,3/2]$ such that $\|u_{0x}\|_{L^\infty}<\infty$. Let $u_{\varepsilon 0}$ be defined as in system* (4.23)*. Then there exist two positive constants T and c, which are independent of ε, such that the solution u_ε of problem* (4.23) *satisfies $\|u_{\varepsilon x}\|_{L^\infty}\leq c$ for any $t\in[0,T)$.*

Proof. Using notation $u=u_\varepsilon$ and differentiating both sides of the first equation of problem (4.23) or (4.15) with respect to x give rise to

$$\begin{aligned} u_{tx}+\frac{1}{2}\partial_x^2u^2-\frac{1}{2}u_x^2&=2ku+\frac{a}{m+1}u^{m+1}-\frac{1}{2}u^2-\beta u_x^N\\ &\quad-\Lambda^{-2}\left[2ku+\frac{a}{m+1}u^{m+1}-\frac{1}{2}u^2+\frac{1}{2}u_x^2-\beta u_x^N\right].\end{aligned} \tag{4.25}$$

Letting $p>0$ be an integer and multiplying the above equation by $(u_x)^{2p+1}$ and then integrating the resulting equation with respect to x yield the equality

$$\begin{aligned} &\frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx + \frac{p}{2p+2}\int_R (u_x)^{2p+3}dx \\ &\quad = \int_R (u_x)^{2p+1}\left(2ku + \frac{a}{m+1}u^{m+1} - \frac{1}{2}u^2 - \beta u_x^N\right)dx \\ &\qquad - \int_R (u_x)^{2p+1}\Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]dx. \end{aligned} \tag{4.26}$$

Applying the Hölder's inequality yields

$$\begin{aligned} \frac{1}{2p+2}\frac{d}{dt}\int_R (u_x)^{2p+2}dx \le{}& \left\{|2k|\left(\int_R |u|^{2p+2}dx\right)^{1/(2p+2)} + \frac{a}{m+1}\left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)}\right. \\ &+ \frac{1}{2}\left(\int_R \left|u^2\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} \\ &\left.+\left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)}\right\}\left(\int_R |u_x|^{2p+2}dx\right)^{(2p+1)/(2p+2)} \\ &+ \frac{p}{2p+2}\|u_x\|_{L^\infty}\int_R |u_x|^{2p+2}dx, \end{aligned} \tag{4.27}$$

or

$$\begin{aligned} \frac{d}{dt}\left(\int_R (u_x)^{2p+2}dx\right)^{1/(2p+2)} \le{}& |2k|\left(\int_R |u|^{2p+2}dx\right)^{1/(2p+2)} + \frac{a}{m+1}\left(\int_R \left|u^{m+1}\right|^{2p+2}dx\right)^{1/(2p+2)} \\ &+ \frac{1}{2}\left(\int_R \left|u^2\right|^{2p+2}dx\right)^{1/(2p+2)} + \beta\left(\int_R \left|u_x^N\right|^{2p+2}dx\right)^{1/(2p+2)} \\ &+ \left(\int_R |G|^{2p+2}dx\right)^{1/(2p+2)} + \frac{p}{2p+2}\|u_x\|_{L^\infty}\left(\int_R |u_x|^{2p+2}dx\right)^{1/(2p+2)}, \end{aligned} \tag{4.28}$$

where

$$G = \Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]. \tag{4.29}$$

Since $\|f\|_{L^p} \to \|f\|_{L^\infty}$ as $p \to \infty$ for any $f \in L^\infty \bigcap L^2$, integrating both sides of the inequality (4.28) with respect to t and taking the limit as $p \to \infty$ result in the estimate

$$\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + \int_0^t c\left[\left(\|u\|_{L^\infty} + \left\|u^2\right\|_{L^\infty} + \left\|u^{m+1}\right\|_{L^\infty} + \beta\|u_x\|_{L^\infty}^N + \|G\|_{L^\infty}\right) + \frac{1}{2}\|u_x\|_{L^\infty}^2\right]d\tau. \tag{4.30}$$

Using the algebra property of $H^{s_0}(R)$ with $s_0 > 1/2$ yields ($\|u_\varepsilon\|_{H^{(1/2)+}}$ means that there exists a sufficiently small $\delta > 0$ such that $\|u_\varepsilon\|_{(1/2)+} = \|u_\varepsilon\|_{H^{1/2+\delta}}$)

$$\begin{aligned}
\|G\|_{L^\infty} &\leq c\|G\|_{H^{(1/2)+}} \\
&\leq c\left\|\Lambda^{-2}\left[2ku + \frac{a}{m+1}u^{m+1} - \frac{u^2}{2} + \frac{1}{2}u_x^2 - \beta u_x^N\right]\right\|_{H^{(1/2)+}} \\
&\leq c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \left\|\Lambda^{-2}(u_x^2)\right\|_{H^{(1/2)+}} + \left\|\Lambda^{-2}(u_x^N)\right\|_{H^{(1/2)+}}\right) \\
&\leq c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \left\|u_x^2\right\|_{H^0} + \left\|u_x^N\right\|_{H^0}\right) \\
&\leq c\left(\|u\|_{H^1} + \|u\|_{H^1}^2 + \|u\|_{H^1}^{m+1} + \|u_x\|_{L^\infty}\|u\|_{H^1} + \|u_x\|_{L^\infty}^{N-1}\|u\|_{H^1}\right) \\
&\leq ce^{c\int_0^t \|u_x\|_{L^\infty}^{N-1}d\tau}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right),
\end{aligned} \tag{4.31}$$

in which Lemma 4.3 is used. Therefore, we get

$$\int_0^t \|G\|_{L^\infty}d\tau \leq c\int_0^t e^{c\int_0^\tau \|u_x\|_{L^\infty}^{N-1}d\xi}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right)d\tau. \tag{4.32}$$

From (4.30) and (4.32), one has

$$\begin{aligned}
\|u_x\|_{L^\infty} \leq \|u_{0x}\|_{L^\infty} + c\int_0^t \Big[&\|u_x\|_{L^\infty}^2 + \|u_x\|_{L^\infty}^N + e^{c\int_0^t \|u_x\|_{L^\infty}^{N-1}d\tau} \\
&+ e^{c\int_0^\tau \|u_x\|_{L^\infty}^{N-1}d\xi}\left(1 + \|u_x\|_{L^\infty} + \|u_x\|_{L^\infty}^{N-1}\right)\Big]d\tau.
\end{aligned} \tag{4.33}$$

From Lemma 4.4, it follows from the contraction mapping principle that there is a $T > 0$ such that the equation

$$\begin{aligned}
\|W\|_{L^\infty} = \|u_{0x}\|_{L^\infty} + c\int_0^t \Big[&\|W\|_{L^\infty}^2 + \|W\|_{L^\infty}^N + e^{c\int_0^t \|W\|_{L^\infty}^{N-1}d\tau} \\
&+ e^{c\int_0^\tau \|W\|_{L^\infty}^{N-1}d\xi}\left(1 + \|W\|_{L^\infty} + \|W\|_{L^\infty}^{N-1}\right)\Big]d\tau
\end{aligned} \tag{4.34}$$

has a unique solution $W \in C[0,T]$. Using the Theorem presented at page 51 in [25] or Theorem 2 in Section 1.1 presented in [37] yields that there are constants $T > 0$ and $c > 0$

independent of ε such that $\|u_x\|_{L^\infty} \leq W(t)$ for arbitrary $t \in [0,T]$, which leads to the conclusion of Lemma 4.5.

Using Lemmas 4.3 and 4.5, notation $u_\varepsilon = u$ and Gronwall's inequality results in the inequalities

$$\|u_\varepsilon\|_{H^q} \leq C_T e^{C_T},$$
$$\|u_{\varepsilon t}\|_{H^r} \leq C_T e^{C_T}, \tag{4.35}$$

where $q \in (0,s]$, $r \in (0,s-1]$ and C_T depends on T. It follows from Aubin's compactness theorem that there is a subsequence of $\{u_\varepsilon\}$, denoted by $\{u_{\varepsilon_n}\}$, such that $\{u_{\varepsilon_n}\}$ and their temporal derivatives $\{u_{\varepsilon_n t}\}$ are weakly convergent to a function $u(t,x)$ and its derivative u_t in $L^2([0,T],H^s)$ and $L^2([0,T],H^{s-1})$, respectively. Moreover, for any real number $R_1 > 0$, $\{u_{\varepsilon_n}\}$ is convergent to the function u strongly in the space $L^2([0,T],H^q(-R_1,R_1))$ for $q \in [0,s)$ and $\{u_{\varepsilon_n t}\}$ converges to u_t strongly in the space $L^2([0,T],\ H^r(-R_1,R_1))$ for $r \in [0,s-1]$. Thus, we can prove the existence of a weak solution to (2.2). □

Proof of Theorem 2.2. From Lemma 4.5, we know that $\{u_{\varepsilon_n x}\}$ $(\varepsilon_n \rightarrow 0)$ is bounded in the space L^∞. Thus, the sequences $\{u_{\varepsilon_n}\}$ and $\{u_{\varepsilon_n x}\}$ are weakly convergent to u and u_x in $L^2[0,T], H^r(-R,R)$ for any $r \in [0,s-1)$, respectively. Therefore, u satisfies the equation

$$-\int_0^T\int_R u(g_t - g_{xxt})dx\,dt = \int_0^T\int_R\left[\left(2ku + \frac{a}{m+1}u^{m+1} + \frac{1}{2}\left(u_x^2\right)\right)g_x \right.$$
$$\left. -\frac{1}{2}u^2 g_{xxx} - \beta(u_x)^N g_x\right]dx\,dt, \tag{4.36}$$

with $u(0,x) = u_0(x)$ and $g \in C_0^\infty$. Since $X = L^1([0,T]\times R)$ is a separable Banach space and $\{u_{\varepsilon_n x}\}$ is a bounded sequence in the dual space $X^* = L^\infty([0,T]\times R)$ of X, there exists a subsequence of $\{u_{\varepsilon_n x}\}$, still denoted by $\{u_{\varepsilon_n x}\}$, weakly star convergent to a function v in $L^\infty([0,T]\times R)$. It derives from the $\{u_{\varepsilon_n x}\}$ weakly convergent to u_x in $L^2([0,T]\times R)$ that $u_x = v$ almost everywhere. Thus, we obtain $u_x \in L^\infty([0,T]\times R)$. □

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Research Article

Translation Invariant Spaces and Asymptotic Properties of Variational Equations

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We present a new perspective concerning the study of the asymptotic behavior of variational equations by employing function spaces techniques. We give a complete description of the dichotomous behaviors of the most general case of skew-product flows, without any assumption concerning the flow, the cocycle or the splitting of the state space, our study being based only on the solvability of some associated control systems between certain function spaces. The main results do not only point out new necessary and sufficient conditions for the existence of uniform and exponential dichotomy of skew-product flows, but also provide a clear chart of the connections between the classes of translation invariant function spaces that play the role of the input or output classes with respect to certain control systems. Finally, we emphasize the significance of each underlying hypothesis by illustrative examples and present several interesting applications.

1. Introduction

Starting from a collection of open questions related to the modeling of the equations of mathematical physics in the unified setting of dynamical systems, the study of their qualitative properties became a domain of large interest and with a wide applicability area. In this context, the interaction between the modern methods of pure mathematics and questions arising naturally from mathematical physics created a very active field of research (see [1–18] and the references therein). In recent years, some interesting unsolved problems concerning the long-time behavior of dynamical systems were identified, whose potential results would be of major importance in the process of understanding, clarifying, and solving some of the essential problems belonging to a wide range of scientific domains, among, we mention: fluid mechanics, aeronautics, magnetism, ecology, population dynamics, and so forth. Generally, the asymptotic behavior of the solutions of nonlinear evolution equations arising in mathematical physics can be described in terms of attractors, which are often studied by constructing the skew-product flows of the dynamical processes.

It was natural then to independently consider and analyze the asymptotic behavior of variational systems modeled by skew-product flows (see [3–5, 14–19]). In this framework, two of the most important asymptotic properties are described by uniform dichotomy and exponential dichotomy. Both properties focus on the decomposition of the state space into a direct sum of two closed invariant subspaces such that the solution on these subspaces (uniformly or exponentially) decays backward and forward in time, and the splitting holds at every point of the flow's domain. Precisely, these phenomena naturally lead to the study of the existence of stable and unstable invariant manifolds. It is worth mentioning that starting with the remarkable works of Coppel [20], Daleckii and Krein [21], and Massera and Schäffer [22] the study of the dichotomy had a notable impact on the development of the qualitative theory of dynamical systems (see [1–9, 13, 14, 17, 18, 23]).

A very important step in the infinite-dimensional asymptotic theory of dynamical systems was made by Van Minh et al. in [7] where the authors proposed a unified treatment of the stability, instability, and dichotomy of evolution families on the half-line via input-output techniques. Their paper carried out a beautiful connection between the classical techniques originating in the pioneering works of Perron [11] and Maĭzel [24] and the natural requests imposed by the development of the infinite-dimensional systems theory. They extended the applicability area of the so-called admissibility techniques developed by Massera and Schäffer in [22], from differential equations in infinite-dimensional spaces to general evolutionary processes described by propagators. The authors pointed out that instead of characterizing the behavior of a homogeneous equation in terms of the solvability of the associated inhomogeneous equation (see [20–22]) one may detect the asymptotic properties by analyzing the existence of the solutions of the associated integral system given by the variation of constants formula. These new methods technically moved the central investigation of the qualitative properties into a different sphere, where the study strongly relied on control-type arguments. It is important to mention that the control-type techniques have been also successfully used by Palmer (see [9]) and by Rodrigues and Ruas-Filho (see [13]) in order to formulate characterizations for exponential dichotomy in terms of the Fredholm Alternative. Starting with these papers, the interaction between control theory and the asymptotic theory of dynamical systems became more profound, and the obtained results covered a large variety of open problems (see, e.g., [1, 2, 12, 14–17, 23] and the references therein).

Despite the density of papers devoted to the study of the dichotomy in the past few years and in contrast with the apparent impression that the phenomenon is well understood, a large number of unsolved problems still raise in this topic, most of them concerning the variational case. In the present paper, we will provide a complete answer to such an open question. We start from a natural problem of finding suitable conditions for the existence of uniform dichotomy as well as of exponential dichotomy using control-type methods, emphasizing on the identification of the essential structures involved in such a construction, as the input-output system, the eligible spaces, the interplay between their main properties, the specific lines that make the differences between a necessary and a sufficient condition, and the proper motivation of each underlying condition.

In this paper, we propose an inedit link between the theory of function spaces and the dichotomous behavior of the solutions of infinite dimensional variational systems, which offers a deeper understanding of the subtle mechanisms that govern the control-type approaches in the study of the existence of the invariant stable and unstable manifolds.

We consider the general setting of variational equations described by skew-product flows, and we associate a control system on the real line. Beside obtaining new conditions for the existence of uniform or exponential dichotomy of skew-product flows, the main aim is to clarify the chart of the connections between the classes of translation invariant function spaces that play the role of the input class or of the output class with respect to the associated control system, proposing a merger between the functional methods proceeding from interpolation theory and the qualitative techniques from the asymptotic theory of dynamical systems in infinite dimensional spaces.

We consider the most general case of skew-product flows, without any assumption concerning the flow or the cocycle, without any invertibility property, and we work without assuming any initial splitting of the state space and without imposing any invariance property. Our central aim is to establish the existence of the dichotomous behaviors with all their properties (see Definitions 3.5 and 4.1) based only on the minimal solvability of an associated control system described at every point of the base space by an integral equation on the real line. First, we deduce conditions for the existence of uniform dichotomy of skew-product flows and we discuss the technical consequences implied by the solvability of the associated control system between two appropriate translation invariant spaces. We point out, for the first time, that an adequate solvability on the real line of the associated integral control system (see Definition 3.6) implies both the existence of the uniform dichotomy projections as well as their uniform boundedness. Next, the attention focuses on the exponential behavior on the stable and unstable manifold, preserving the solvability concept from the previous section and modifying the properties of the input and the output spaces. Thus, we deduce a clear overview on the representative classes of function spaces which should be considered in the detection of the exponential dichotomy of skew-product flows in terms of the solvability of associated control systems on the real line. The obtained results provide not only new necessary and sufficient conditions for exponential dichotomy, but also a complete diagram of the specific delimitations between the classes of function spaces which may be considered in the study of the exponential dichotomy compared with those from the uniform dichotomy case. Moreover, we point out which are the specific properties of the underlying spaces which make a difference between the sufficient hypotheses and the necessary conditions for the existence of exponential dichotomy of skew-product flows. Finally, we motivate our techniques by illustrative examples and present several interesting applications of the main theorems which generalize the input-output type results of previous research in this topic, among, we mention the well-known theorems due to Perron [11], Daleckii and Krein [21], Massera and Schäffer [22], Van Minh et al. [7], and so forth.

2. Banach Function Spaces: Basic Notations and Preliminaries

In this section, for the sake of clarity, we recall several definitions and properties of Banach function spaces, and, also, we establish the notations that will be used throughout the paper.

Let $\mathbb{R}$ denote the set of real numbers, let $\mathbb{R}_+ = \{t \in \mathbb{R} : t \geq 0\}$, and let $\mathbb{R}_- = \{t \in \mathbb{R} : t \leq 0\}$. For every $A \subset \mathbb{R}$, χ_A denotes the characteristic function of the set A. Let $\mathcal{M}(\mathbb{R}, \mathbb{R})$ be the linear space of all Lebesgue measurable functions $u : \mathbb{R} \to \mathbb{R}$ identifying the functions which are equal almost everywhere.

Definition 2.1. A linear subspace $B \subset \mathcal{M}(\mathbb{R}, \mathbb{R})$ is called *normed function space* if there is a mapping $|\cdot|_B : B \to \mathbb{R}_+$ such that the following properties hold:

(i) $|u|_B = 0$ if and only if $u = 0$ a.e.;

(ii) $|\alpha u|_B = |\alpha||u|_B$, for all $(\alpha, u) \in \mathbb{R} \times B$;

(iii) $|u + v|_B \le |u|_B + |v|_B$, for all $u, v \in B$;

(iv) if $|u(t)| \le |v(t)|$ a.e. $t \in \mathbb{R}$ and $v \in B$, then $u \in B$ and $|u|_B \le |v|_B$.

If $(B, |\cdot|_B)$ is complete, then B is called a *Banach function space*.

Remark 2.2. If $(B, |\cdot|_B)$ is a Banach function space and $u \in B$, then also $|u(\cdot)| \in B$.

Definition 2.3. A Banach function space $(B, |\cdot|_B)$ is said to be *invariant under translations* if for every $(u, t) \in B \times \mathbb{R}$ the function $u_t : \mathbb{R} \to \mathbb{R}$, $u_t(s) = u(s - t)$ belongs to B and $|u_t|_B = |u|_B$.

Let $\mathcal{C}_c(\mathbb{R}, \mathbb{R})$ be the linear space of all continuous functions $v : \mathbb{R} \to \mathbb{R}$ with compact support. We denote by $\mathcal{T}(\mathbb{R})$ the class of all Banach function spaces B which are invariant under translations, $\mathcal{C}_c(\mathbb{R}, \mathbb{R}) \subset B$ and

(i) for every $t > 0$ there is $c(t) > 0$ such that $\int_0^t |u(\tau)| d\tau \le c(t)|u|_B$, for all $u \in B$;

(ii) if $B \setminus L^1(\mathbb{R}, \mathbb{R}) \ne \emptyset$, then there is a continuous function $\gamma \in B \setminus L^1(\mathbb{R}, \mathbb{R})$.

Remark 2.4. Let $B \in \mathcal{T}(\mathbb{R})$. Then, the following properties hold:

(i) if $J \subset \mathbb{R}$ is a bounded interval, then $\chi_J \in B$.

(ii) if $u_n \to u$ in B, then there is a subsequence $(u_{k_n}) \subset (u_n)$ which converges to u a.e. (see, e.g., [25]).

Remark 2.5. Let $B \in \mathcal{T}(\mathbb{R})$. If $\nu > 0$ and $e_\nu : \mathbb{R} \to \mathbb{R}$ is defined by

$$e_\nu(t) = \begin{cases} e^{-\nu t}, & t \ge 0, \\ 0, & t < 0, \end{cases} \tag{2.1}$$

then it is easy to see that

$$e_\nu(t) = \sum_{n=0}^{\infty} e^{-\nu t} \chi_{[n,n+1)}(t) \le \sum_{n=0}^{\infty} e^{-\nu n} \chi_{[n,n+1)}(t), \quad \forall t \in \mathbb{R}. \tag{2.2}$$

It follows that $e_\nu \in B$ and $|e_\nu|_B \le |\chi_{[0,1)}|_B / (1 - e^{-\nu})$.

Example 2.6. (i) If $p \in [1, \infty)$, then $L^p(\mathbb{R}, \mathbb{R}) = \{u \in \mathcal{M}(\mathbb{R}, \mathbb{R}) : \int_{\mathbb{R}} |u(t)|^p dt < \infty\}$, with respect to the norm $\|u\|_p = (\int_{\mathbb{R}} |u(t)|^p dt)^{1/p}$, is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

(ii) The linear space $L^\infty(\mathbb{R}, \mathbb{R})$ of all measurable essentially bounded functions $u : \mathbb{R} \to \mathbb{R}$ with respect to the norm $\|u\|_\infty = \text{ess}\sup_{t \in \mathbb{R}} |u(t)|$ is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

Example 2.7 (Orlicz spaces). Let $\varphi : \mathbb{R}_+ \to \overline{\mathbb{R}}_+$ be a nondecreasing left continuous function which is not identically 0 or ∞ on $(0, \infty)$, and let $Y_\varphi(t) := \int_0^t \varphi(s) ds$. If $u \in \mathcal{M}(\mathbb{R}, \mathbb{R})$ let

$$M_\varphi(u) := \int_{\mathbb{R}} Y_\varphi(|u(s)|) ds. \tag{2.3}$$

The linear space $O_\varphi(\mathbb{R},\mathbb{R}) := \{u \in \mathcal{M}(\mathbb{R},\mathbb{R}) : \exists k > 0$ such that $M_\varphi(ku) < \infty\}$, with respect to the norm

$$|u|_\varphi := \inf\left\{k > 0 : M_\varphi\left(\frac{u}{k}\right) \leq 1\right\}, \tag{2.4}$$

is a Banach function space called the *Orlicz space* associated to φ. It is easy to see that $O_\varphi(\mathbb{R},\mathbb{R})$ is invariant under translations.

Remark 2.8. A remarkable example of Orlicz space is represented by $L^p(\mathbb{R},\mathbb{R})$, for every $p \in [1,\infty]$. This can be obtained for $\varphi(t) = pt^{p-1}$, if $p \in [1,\infty)$ and for

$$\varphi(t) = \begin{cases} 0, & t \in [0,1], \\ \infty, & t > 1, \end{cases} \quad \text{if } p = \infty. \tag{2.5}$$

Lemma 2.9. *If $\varphi(1) < \infty$, then $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{T}(\mathbb{R})$.*

Proof. Let $v \in \mathcal{C}_c(\mathbb{R},\mathbb{R})$. Then, there are $a,\ b \in \mathbb{R}, a < b$ such that $v(t) = 0$, for all $t \in \mathbb{R} \setminus (a,b)$. Since v is continuous on $[a,b]$, there is $M > 0$ such that $|v(t)| \leq M$, for all $t \in [a,b]$. Then, we have that

$$|v(t)| \leq M\chi_{[a,b]}(t), \quad \forall t \in \mathbb{R}. \tag{2.6}$$

We observe that

$$M_\varphi(\chi_{[a,b]}) = \int_{\mathbb{R}} Y_\varphi(\chi_{[a,b]}(\tau))d\tau = (b-a)Y_\varphi(1) \leq (b-a)\varphi(1) < \infty. \tag{2.7}$$

This implies that $\chi_{[a,b]} \in O_\varphi(\mathbb{R},\mathbb{R})$. Using (2.6), we deduce that $v \in O_\varphi(\mathbb{R},\mathbb{R})$. So, $\mathcal{C}_c(\mathbb{R},\mathbb{R}) \subset O_\varphi(\mathbb{R},\mathbb{R})$.

Since Y_φ is nondecreasing with $\lim_{t\to\infty} Y_\varphi(t) = \infty$, there is $q > 0$ such that $Y_\varphi(t) > 1$, for all $t \geq q$.

Let $t \geq 1$ and let $u \in O_\varphi(\mathbb{R},\mathbb{R}) \setminus \{0\}$. Taking into account that Y_φ is a convex function and using Jensen's inequality (see, e.g., [26]), we deduce that

$$Y_\varphi\left(\frac{1}{t}\int_0^t \frac{|u(\tau)|}{|u|_\varphi}d\tau\right) \leq \frac{1}{t}\int_0^t Y_\varphi\left(\frac{|u(\tau)|}{|u|_\varphi}\right)d\tau \leq M_\varphi\left(\frac{u}{|u|_\varphi}\right) \leq 1. \tag{2.8}$$

This implies that

$$\frac{1}{t}\int_0^t \frac{|u(\tau)|}{|u|_\varphi}d\tau \leq q, \quad \forall t \geq 1. \tag{2.9}$$

In addition, using (2.9), we have that

$$\int_0^t |u(\tau)|d\tau \le \int_0^1 |u(\tau)|d\tau \le q|u|_\varphi, \quad \forall t \in [0,1). \tag{2.10}$$

Taking $c : (0,\infty) \to (0,\infty)$, $c(t) = \max\{qt, q\}$, from relations (2.9) and (2.10), it follows that

$$\int_0^t |u(\tau)|d\tau \le c(t)|u|_\varphi, \quad \forall t \ge 0. \tag{2.11}$$

Since the function c does not depend on u, we obtain that $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{T}(\mathbb{R})$. □

Example 2.10. If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ defined by $\varphi(0) = 0$, $\varphi(t) = 1$, for $t \in (0,1]$ and $\varphi(t) = e^{t-1}$, for $t > 1$, then according to Lemma 2.9 we have that the Orlicz space $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{T}(\mathbb{R})$. Moreover, it is easy to see that $O_\varphi(\mathbb{R},\mathbb{R})$ is a proper subspace of $L^1(\mathbb{R},\mathbb{R})$.

Example 2.11. Let $p \in [1,\infty)$ and let $M^p(\mathbb{R},\mathbb{R})$ be the linear space of all $u \in \mathcal{M}(\mathbb{R},\mathbb{R})$ with $\sup_{t\in\mathbb{R}} \int_t^{t+1} |u(s)|^p ds < \infty$. With respect to the norm

$$\|u\|_{M^p} := \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |u(s)|^p ds \right)^{1/p}, \tag{2.12}$$

this is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$.

Remark 2.12. If $B \in \mathcal{T}(\mathbb{R})$, then $B \subset M^1(\mathbb{R},\mathbb{R})$.

Indeed, let $c(1) > 0$ be such that $\int_0^1 |u(\tau)|d\tau \le c(1)|u|_B$, for all $u \in B$. If $u \in B$ we observe that

$$\int_t^{t+1} |u(\tau)|d\tau = \int_0^1 |u_t(\xi)|d\xi \le c(1)|u_t|_B = c(1)|u|_B, \quad \forall t \in \mathbb{R}, \tag{2.13}$$

so $u \in M^1(\mathbb{R},\mathbb{R})$.

In what follows, we will introduce three remarkable subclasses of $\mathcal{T}(\mathbb{R})$, which will have an essential role in the study of the existence of dichotomy from the next sections. To do this, we first need the following.

Definition 2.13. Let $B \in \mathcal{T}(\mathbb{R})$. The mapping $F_B : (0,\infty) \to \mathbb{R}_+$, $F_B(t) = |\chi_{[0,t)}|_B$ is called *the fundamental function* of the space B.

Remark 2.14. If $B \in \mathcal{T}(\mathbb{R})$, then the fundamental function F_B is nondecreasing.

Notation 1. We denote by $Q(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ with the property that $\sup_{t>0} F_B(t) = \infty$.

Lemma 2.15. *If $\varphi(t) \in (0,\infty)$, for all $t > 0$, then $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{Q}(\mathbb{R})$.*

Proof. It is easy to see that Y_φ is strictly increasing, continuous with $Y_\varphi(0) = 0$ and $Y_\varphi(t) \geq (t-1)\varphi(1)$, for all $t > 1$, so $\lim_{t\to\infty} Y_\varphi(t) = \infty$. Hence, Y_φ is bijective.

Let $t > 0$. Since

$$M_\varphi\left(\frac{1}{k}\chi_{[0,t)}\right) = tY_\varphi\left(\frac{1}{k}\right), \quad \forall k > 0, \tag{2.14}$$

it follows that $M_\varphi((1/k)\chi_{[0,t)}) \leq 1$ if and only if $1/Y_\varphi^{-1}(1/t) \leq k$. This implies that

$$F_{O_\varphi(\mathbb{R},\mathbb{R})}(t) = \frac{1}{Y_\varphi^{-1}(1/t)}, \quad \forall t > 0. \tag{2.15}$$

Since $Y_\varphi^{-1}(0) = 0$, from (2.15), we obtain that $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{Q}(\mathbb{R})$. □

Another distinctive subclass of $\mathcal{T}(\mathbb{R})$ is introduced in the following.

Notation 2. Let $\mathcal{L}(\mathbb{R})$ denote the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ with the property that $B \setminus L^1(\mathbb{R},\mathbb{R}) \neq \emptyset$.

Remark 2.16. According to Remark 2.2, we have that if $B \in \mathcal{L}(\mathbb{R})$, then there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that $\gamma \in B \setminus L^1(\mathbb{R},\mathbb{R})$.

We will also see, in this paper, that the necessary conditions for the existence of exponential dichotomy should be expressed using another remarkable subclass of $\mathcal{T}(\mathbb{R})$—the rearrangement invariant spaces, see the definitions below.

Definition 2.17. Let $u, v \in \mathcal{M}(\mathbb{R},\mathbb{R})$. We say that u and v are *equimeasurable* if for every $t > 0$ the sets $\{s \in \mathbb{R} : |u(s)| > t\}$ and $\{s \in \mathbb{R} : |v(s)| > t\}$ have the same measure.

Definition 2.18. A Banach function space $(B, |\cdot|_B)$ is *rearrangement invariant* if for every equimeasurable functions u, v with $u \in B$, we have that $v \in B$ and $|u|_B = |v|_B$.

Notation 3. We denote by $\mathcal{R}(\mathbb{R})$ the class of all Banach function spaces $B \in \mathcal{T}(\mathbb{R})$ which are rearrangement invariant.

Remark 2.19. If $B \in \mathcal{R}(\mathbb{R})$, then B is an interpolation space between $L^1(\mathbb{R},\mathbb{R})$ and $L^\infty(\mathbb{R},\mathbb{R})$ (see [27, Theorem 2.2, page 106]).

Remark 2.20. The Orlicz spaces are rearrangement invariant (see [27]). Using Lemma 2.9, we deduce that if $\varphi(1) < \infty$, then $O_\varphi(\mathbb{R},\mathbb{R}) \in \mathcal{R}(\mathbb{R})$.

Lemma 2.21. *Let $B \in \mathcal{R}(\mathbb{R})$ and let $\nu > 0$. Then for every $u \in B$, the functions $\varphi_u, \psi_u : \mathbb{R} \to \mathbb{R}$ defined by*

$$\varphi_u(t) = \int_{-\infty}^{t} e^{-\nu(t-\tau)}u(\tau)d\tau, \qquad \psi_u(t) = \int_{t}^{\infty} e^{-\nu(\tau-t)}u(\tau)d\tau \tag{2.16}$$

belong to B. Moreover, there is $\gamma_{B,\nu} > 0$ which depends only on B and ν such that

$$\left|\varphi_u\right|_B \leq \gamma_{B,\nu}|u|_B, \qquad \left|\psi_u\right|_B \leq \gamma_{B,\nu}|u|_B, \quad \forall u \in B. \tag{2.17}$$

Proof. We consider the operators

$$\begin{aligned} Z : L^\infty(\mathbb{R},\mathbb{R}) &\longrightarrow L^\infty(\mathbb{R},\mathbb{R}), \quad (Z(u))(t) = \int_{-\infty}^{t} e^{-\nu(t-\tau)}u(\tau)d\tau, \\ W : L^\infty(\mathbb{R},\mathbb{R}) &\longrightarrow L^\infty(\mathbb{R},\mathbb{R}), \quad (W(u))(t) = \int_{t}^{\infty} e^{-\nu(\tau-t)}u(\tau)d\tau. \end{aligned} \tag{2.18}$$

We have that Z and W are correctly defined bounded linear operators. Moreover, the restrictions $Z_| : L^1(\mathbb{R},\mathbb{R}) \to L^1(\mathbb{R},\mathbb{R})$ and $W_| : L^1(\mathbb{R},\mathbb{R}) \to L^1(\mathbb{R},\mathbb{R})$ are correctly defined and bounded linear operators. Since $B \in \mathcal{R}(\mathbb{R})$, then, from Remark 2.19, we have that B is an interpolation space between $L^1(\mathbb{R},\mathbb{R})$ and $L^\infty(\mathbb{R},\mathbb{R})$. This implies that the restrictions $Z_{|B} : B \to B$ and $W_{|B} : B \to B$ are correctly defined and bounded linear operators. Setting $\gamma_{B,\nu} = \max\{\|Z_{|B}\|, \|W_{|B}\|\}$, the proof is complete. □

Notations

If X is a Banach space, for every Banach function space $B \in \mathcal{T}(\mathbb{R})$, we denote by $B(\mathbb{R},X)$ the space of all Bochner measurable functions $v : \mathbb{R} \to X$ with the property that the mapping $N_v : \mathbb{R} \to \mathbb{R}_+$, $N_v(t) = \|v(t)\|$ belongs to B. With respect to the norm

$$\|v\|_{B(\mathbb{R},X)} := |N_v|_B, \tag{2.19}$$

$B(\mathbb{R},X)$ is a Banach space. We also denote by $C_{0,c}(\mathbb{R},X)$ the linear space of all continuous functions $v : \mathbb{R} \to X$ with compact support contained in $(0,\infty)$. It is easy to see that $C_{0,c}(\mathbb{R},X) \subset B(\mathbb{R},X)$, for all $B \in \mathcal{T}(\mathbb{R})$.

3. Uniform Dichotomy for Skew-Product Flows

In this section, we start our investigation by studying the existence of by the upper and lower uniform boundedness of the solution in a uniform way on certain complemented subspaces. We will employ a control-type technique and we will show that the use of the function spaces, from the class $\mathcal{T}(\mathbb{R})$ introduced in the previous section, provides several interesting conclusions concerning the qualitative behavior of the solutions of variational equations.

Let X be a real or complex Banach space and let I_d denote the identity operator on X. The norm on X and on $\mathcal{B}(X)$—the Banach algebra of all bounded linear operators on X, will be denoted by $\|\cdot\|$. Let (Θ, d) be a metric space.

Definition 3.1. A continuous mapping $\sigma : \Theta \times \mathbb{R} \to \Theta$ is called a *flow* on Θ if $\sigma(\theta,0) = \theta$ and $\sigma(\theta, s+t) = \sigma(\sigma(\theta,s),t)$, for all $(\theta,s,t) \in \Theta \times \mathbb{R}^2$.

Definition 3.2. A pair $\pi = (\Phi,\sigma)$ is called a *skew-product flow* on $X \times \Theta$ if σ is a flow on Θ and the mapping $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ called *cocycle*, satisfies the following conditions:

(i) $\Phi(\theta,0) = I_d$ and $\Phi(\theta,t+s) = \Phi(\sigma(\theta,s),t)\Phi(\theta,s)$, for all $(\theta,t,s) \in \Theta \times \mathbb{R}_+^2$;

(ii) there are $M \geq 1$ and $\omega > 0$ such that $\|\Phi(\theta,t)\| \leq Me^{\omega t}$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$;

(iii) for every $(x,\theta) \in X \times \Theta$, the mapping $t \mapsto \Phi(\theta,t)x$ is continuous on $\mathbb{R}_+$.

Example 3.3 (Particular cases). The class described by skew-product flows generalizes the autonomous systems as well as the nonautonomous systems, as the following examples show:

(i) If $\Theta = \mathbb{R}$, then let $\tilde{\sigma}(\theta,t) = \theta + t$ and let $\{U(t,s)\}_{t\geq s}$ be an evolution family on the Banach space X. Setting $\Phi_U(\theta,t) := U(\theta+t,\theta)$, we observe that $\pi_U = (\Phi_U,\tilde{\sigma})$ is a skew-product flow.

(ii) Let $\{T(t)\}_{t\geq 0}$ be a $\mathcal{C}_0$-semigroup on the Banach space X and let Θ be a metric space.

(ii)$_1$ If σ is an arbitrary flow on Θ and $\Phi_T(\theta,t) := T(t)$, then $\pi_T = (\Phi_T,\sigma)$ is a skew-product flow.

(ii)$_2$ Let $\hat{\sigma} : \Theta \times \mathbb{R} \to \Theta$, $\hat{\sigma}(\theta,t) = \theta$ be the projection flow on Θ and let $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{B}(X)$ be a uniformly bounded family of projections such that $P(\theta)T(t) = T(t)P(\theta)$, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$. If $\Phi_P(\theta,t) := P(\theta)T(t)$, then $\pi_P = (\Phi_P,\hat{\sigma})$ is a skew-product flow.

Starting with the remarkable work of Foias et al. (see [19]), the qualitative theory of dynamical systems acquired a new perspective concerning the connections between bifurcation theory and the mathematical modeling of nonlinear equations. In [19], the authors proved that classical equations like Navier-Stokes, Taylor-Couette, and Bubnov-Galerkin can be modeled and studied in the unified setting of skew-product flows. In this context, it was pointed out that the skew-product flows often proceed from the linearization of nonlinear equations. Thus, classical examples of skew-product flows arise as operator solutions for variational equations.

Example 3.4 (The variational equation). Let Θ be a locally compact metric space and let σ be a flow on Θ. Let X be a Banach space and let $\{A(\theta) : D(A(\theta)) \subseteq X \to X : \theta \in \Theta\}$ be a family of densely defined closed operators. We consider the variational equation

$$\dot{x}(t) = A(\sigma(\theta,t))x(t), \quad (\theta,t) \in \Theta \times \mathbb{R}_+. \tag{A}$$

A cocycle $\Phi : \Theta \times \mathbb{R}_+ \to \mathcal{B}(X)$ is said to be *a solution of* (A) if for every $\theta \in \Theta$, there is a dense subset $D_\theta \subset D(A(\theta))$ such that for every initial condition $x_\theta \in D_\theta$ the mapping $t \mapsto x(t) := \Phi(\theta,t)x_\theta$ is differentiable on $\mathbb{R}_+$, for every $t \in \mathbb{R}_+ x(t) \in D(A(\sigma(\theta,t)))$ and the mapping $t \mapsto x(t)$ satisfies (A).

An important asymptotic behavior of skew-product flows is described by the uniform dichotomy, which relies on the splitting of the Banach space X at every point $\theta \in \Theta$ into a direct sum of two invariant subspaces such that on the first subspace the trajectory solution is uniformly stable, on the second subspace the restriction of the cocycle is reversible and also the trajectory solution is uniformly unstable on the second subspace. This is given by the following.

Definition 3.5. A skew-product flow $\pi = (\Phi, \sigma)$ is said to be *uniformly dichotomic* if there exist a family of projections $\{P(\theta)\}_{\theta\in\Theta} \subset \mathcal{B}(X)$ and a constant $K \geq 1$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$, all $x \in \operatorname{Range} P(\theta)$ and all $\theta \in \Theta$;

(iii) the restriction $\Phi(\theta, t)_| : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) $\|\Phi(\theta, t)y\| \geq (1/K)\|y\|$, for all $t \geq 0$, all $y \in \operatorname{Ker} P(\theta)$ and all $\theta \in \Theta$;

(v) $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$.

In what follows, our main attention will focus on finding suitable conditions for the existence of uniform dichotomy for skew-product flows. To do this, we will introduce an integral control system associated with a skew-product flow such that the input and the output spaces of the system belong to the general class $\mathcal{T}(\mathbb{R})$. We will emphasize that the class $\mathcal{T}(\mathbb{R})$ has an essential role in the study of the dichotomous behavior of variational equations.

Let I, O be two Banach function spaces with $I, O \in \mathcal{T}(\mathbb{R})$. Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. We associate with π the input-output control system $E_\pi = (E_\theta)_{\theta\in\Theta}$, where for every $\theta \in \Theta$

$$f(t) = \Phi(\sigma(\theta, s), t - s)f(s) + \int_s^t \Phi(\sigma(\theta, \tau), t - \tau)v(\tau)d\tau, \qquad \forall t \geq s, \tag{E_θ}$$

such that the input function $v \in C_{0,c}(\mathbb{R}, X)$ and the output function $f \in O(\mathbb{R}, X)$.

Definition 3.6. The pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is said to be *uniformly admissible* for the system (E_π) if there is $L > 0$ such that for every $\theta \in \Theta$, the following properties hold:

(i) for every $v \in C_{0,c}(\mathbb{R}, X)$ there exists $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ);

(ii) if $v \in C_{0,c}(\mathbb{R}, X)$ and $f \in O(\mathbb{R}, X)$ are such that the pair (f, v) satisfies (E_θ), then $\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}$.

Remark 3.7. (i) According to this admissibility concept, it is sufficient to choose all the input functions from the space $C_{0,c}(\mathbb{R}, X)$, and, thus, we point out that $C_{0,c}(\mathbb{R}, X)$ is in fact *the smaller possible input space* that can be used in the input-output study of the dichotomy.

(ii) It is also interesting to see that the norm estimation from (ii) reflects the presence (and implicitly the structure) of the space $I(\mathbb{R}, X)$. Actually, condition (ii) shows that the norm of each output function in the space $O(\mathbb{R}, X)$ is bounded by the norm of the input function in the space $I(\mathbb{R}, X)$ uniformly with respect to $\theta \in \Theta$.

(iii) In the admissibility concept, there is no need to require the uniqueness of the output function in the property (i), because this follows from condition (ii). Indeed, if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then from (ii) we deduce that for every $\theta \in \Theta$ and every $v \in C_{0,c}(\mathbb{R}, X)$ there exists *a unique* $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ).

In what follows we will analyze the implications of the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ with $I, O \in \mathcal{T}(\mathbb{R})$ concerning the asymptotic behavior of skew-product flows. With this purpose we introduce two category of subspaces (stable and unstable) and we will point out their role in the detection of the uniform dichotomy.

For every $(x,\theta) \in X \times \Theta$, we consider the function

$$\lambda_{x,\theta} : \mathbb{R} \longrightarrow X, \quad \lambda_{x,\theta}(t) = \begin{cases} \Phi(\theta,t)x, & t \geq 0, \\ 0, & t < 0, \end{cases} \tag{3.1}$$

called *the trajectory* determined by the vector x and the point $\theta \in \Theta$.

For every $\theta \in \Theta$, we denote by $\mathcal{F}(\theta)$ the linear space of all functions $\varphi : \mathbb{R} \to X$ with the property that

$$\varphi(t) = \Phi(\sigma(\theta,s), t-s)\varphi(s), \quad \forall s \leq t \leq 0. \tag{3.2}$$

For every $\theta \in \Theta$, we consider the *stable subset*

$$\mathcal{S}(\theta) = \{x \in X : \lambda_{x,\theta} \in O(\mathbb{R}, X)\} \tag{3.3}$$

and, respectively, the *unstable subset*

$$\mathcal{U}(\theta) = \{x \in X : \exists \varphi \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta) \text{ with } \varphi(0) = x\}. \tag{3.4}$$

Remark 3.8. It is easy to see that for every $\theta \in \Theta$, $\mathcal{S}(\theta)$, and $\mathcal{U}(\theta)$ are linear subspaces. Therefore, in all what follows, we will refer $\mathcal{S}(\theta)$ as the stable subspace and, respectively, $\mathcal{U}(\theta)$ as the unstable subspace, for each $\theta \in \Theta$.

Proposition 3.9. *For every $(\theta,t) \in \Theta \times \mathbb{R}_+$, the following assertions hold:*

(i) $\Phi(\theta,t)\mathcal{S}(\theta) \subseteq \mathcal{S}(\sigma(\theta,t))$;

(ii) $\Phi(\theta,t)\mathcal{U}(\theta) = \mathcal{U}(\sigma(\theta,t))$.

Proof. The property (i) is immediate. To prove the assertion (ii) let $M, \omega > 0$ be given by Definition 3.2(ii). Let $(\theta,t) \in \Theta \times (0,\infty)$. Let $x \in \mathcal{U}(\theta)$. Then, there is $\varphi \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta)$ with $\varphi(0) = x$. We set $y = \Phi(\theta,t)x$, and we consider

$$\psi : \mathbb{R} \longrightarrow X, \quad \psi(s) = \begin{cases} 0, & s > t, \\ \Phi(\theta,s)x, & s \in [0,t], \\ \varphi(s), & s < 0. \end{cases} \tag{3.5}$$

We observe that $\|\psi(s)\| \leq \|\varphi(s)\| + Me^{\omega t}\chi_{[0,t]}(s)\|x\|$, for all $s \in \mathbb{R}$, and since $\varphi \in O(\mathbb{R}, X)$, we deduce that $\psi \in O(\mathbb{R}, X)$. Using the fact that $\varphi \in \mathcal{F}(\theta)$, we obtain that

$$\psi(s) = \Phi(\sigma(\theta,\tau), s-\tau)\psi(\tau), \quad \forall \tau \leq s \leq t. \tag{3.6}$$

Then, we define the function $\delta : \mathbb{R} \to X$, $\delta(s) = \varphi(s+t)$ and since $O(\mathbb{R}, X)$ is invariant under translations, we deduce that $\delta \in O(\mathbb{R}, X)$. Moreover, from (3.6), it follows that

$$\delta(r) = \Phi(\sigma(\theta, \xi + t), r - \xi)\delta(\xi) = \Phi(\sigma(\sigma(\theta, t), \xi), r - \xi)\delta(\xi), \quad \forall \xi \leq r \leq 0. \tag{3.7}$$

The relation (3.7) implies that $\delta \in \mathcal{F}(\sigma(\theta, t))$, so $y = \delta(0) \in \mathcal{U}(\sigma(\theta, t))$.

Conversely, let $z \in \mathcal{U}(\sigma(\theta, t))$. Then, there is $h \in \mathcal{F}(\sigma(\theta, t)) \cap O(\mathbb{R}, X)$ with $h(0) = z$. Taking $q : \mathbb{R} \to X$, $q(s) = h(s-t)$, we have that $q \in O(\mathbb{R}, X)$ and

$$q(s) = \Phi(\sigma(\theta, \tau), s - \tau)q(\tau), \quad \forall \tau \leq s \leq t. \tag{3.8}$$

In particular, for $\tau \leq s \leq 0$, from (3.8), we deduce that $q \in \mathcal{F}(\theta)$. This implies that $q(0) \in \mathcal{U}(\theta)$. Then, $z = h(0) = q(t) = \Phi(\theta, t)q(0) \in \Phi(\theta, t)\mathcal{U}(\theta)$ and the proof is complete. □

Remark 3.10. From Proposition 3.9(ii), we have that for every $(\theta, t) \in \Theta \times \mathbb{R}_+$ the restriction $\Phi(\theta, t)_| : \mathcal{U}(\theta) \to \mathcal{U}(\sigma(\theta, t))$ is surjective. We also note that according to Proposition 3.9 one may deduce that, the stable subspace and the unstable subspace are candidates for the possible splitting of the main space X required by any dichotomous behavior.

In what follows, we will study the behavior of the cocycle on the stable subspace and also on the unstable subspace and we will deduce several interesting properties of these subspaces in the hypothesis that a pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ of spaces from the class $\mathcal{T}(\mathbb{R})$ is admissible for the control system associated with the skew-product flow.

Theorem 3.11 (The behavior on the stable subspace). *If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then the following assertions hold:*

(i) *there is $K > 0$ such that $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$, all $x \in \mathcal{S}(\theta)$ and all $\theta \in \Theta$;*

(ii) *$\mathcal{S}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.*

Proof. Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$.

(i) Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta)$. We consider the functions

$$v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\Phi(\theta, t)x,$$
$$f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \Phi(\theta, t)x, & t \geq 1, \\ \displaystyle\int_0^t \alpha(\tau)d\tau\, \Phi(\theta, t)x, & t \in [0, 1), \\ 0, & t < 0. \end{cases} \tag{3.9}$$

Then, $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and

$$\|f(t)\| \leq \|\lambda_{x,\theta}(t)\|, \quad \forall t \in \mathbb{R}. \tag{3.10}$$

Since $x \in \mathcal{S}(\theta)$, we have that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$. Then, from (3.10), we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then,

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{3.11}$$

From $\|v(t)\| \leq \alpha(t)Me^{\omega}\|x\|$, for all $t \in \mathbb{R}$, we obtain that $\|v\|_{I(\mathbb{R},X)} \leq Me^{\omega}|\alpha|_I\|x\|$.

Let $t \geq 2$. From

$$\|\Phi(\theta, t)x\| \leq Me^{\omega}\|\Phi(\theta, s)x\|, \quad \forall s \in [t-1, t), \tag{3.12}$$

it follows that

$$\|\Phi(\theta, t)x\|\chi_{[t-1,t)}(s) \leq Me^{\omega}\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.13}$$

Since O is invariant under translations, we deduce that

$$\|\Phi(\theta, t)x\|F_O(1) \leq Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{3.14}$$

Using relations (3.11) and (3.14), we have that

$$\|\Phi(\theta, t)x\| \leq M^2e^{2\omega}\frac{L|\alpha|_I}{F_O(1)}\|x\|, \quad \forall t \geq 2. \tag{3.15}$$

Since $\|\Phi(\theta, t)x\| \leq Me^{2\omega}\|x\|$, for all $t \in [0, 2)$, setting $K := \max\{(M^2e^{2\omega}L|\alpha|_I)/F_O(1), Me^{2\omega}\}$ we deduce that $\|\Phi(\theta, t)x\| \leq K\|x\|$, for all $t \geq 0$. Taking into account that K does not depend on θ or x, it follows that

$$\|\Phi(\theta, t)x\| \leq K\|x\|, \quad \forall t \geq 0, \ \forall x \in \mathcal{S}(\theta), \ \forall \theta \in \Theta. \tag{3.16}$$

(ii) Let $\theta \in \Theta$ and let $(x_n) \subset \mathcal{S}(\theta)$ with $x_n \underset{n\to\infty}{\longrightarrow} x$. For every $n \in \mathbb{N}$, we consider the sequence

$$\begin{gathered} v_n : \mathbb{R} \longrightarrow X, \quad v_n(t) = \alpha(t)\Phi(\theta, t)x_n, \\ f_n : \mathbb{R} \longrightarrow X, \quad f_n(t) = \begin{cases} \Phi(\theta, t)x_n, & t \geq 1, \\ \displaystyle\int_0^t \alpha(\tau)d\tau\, \Phi(\theta, t)x_n, & t \in [0, 1), \\ 0, & t < 0. \end{cases} \end{gathered} \tag{3.17}$$

We have that $v_n \in \mathcal{C}_{0c}(\mathbb{R}, X)$, for all $n \in \mathbb{N}$ and using similar arguments with those used in relation (3.10), we obtain that $f_n \in O(\mathbb{R}, X)$, for all $n \in \mathbb{N}$. An easy computation shows that the pair (f_n, v_n) satisfies (E_θ). Let $v : \mathbb{R} \to X$, $v(t) = \alpha(t)\Phi(\theta, t)x$. Then, $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$. According to our hypothesis there is, $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ).

Taking $u_n = v_n - v$ and $g_n = f_n - f$ we observe that $u_n \in C_{0c}(\mathbb{R}, X)$, $g_n \in O(\mathbb{R}, X)$, and the pair (g_n, u_n) satisfies (E_θ). This implies that

$$\|f_n - f\|_{O(\mathbb{R},X)} \leq L\|v_n - v\|_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{3.18}$$

From $\|v_n(t) - v(t)\| \leq \alpha(t)Me^{\omega}\|x_n - x\|$, for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, we deduce that

$$\|v_n - v\|_{I(\mathbb{R},X)} \leq Me^{\omega}|\alpha|_I\|x_n - x\|, \quad \forall n \in \mathbb{N}. \tag{3.19}$$

From (3.18) and (3.19), it follows that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $O(\mathbb{R}, X)$. From Remark 2.4(ii), we have that there is a subsequence (f_{k_n}) and a negligible set $A \subset \mathbb{R}$ such that $f_{k_n}(t) \underset{n\to\infty}{\longrightarrow} f(t)$, for all $t \in \mathbb{R} \setminus A$. In particular, it follows that there is $r > 1$ such that

$$f(r) = \lim_{n\to\infty} f_{k_n}(r) = \lim_{n\to\infty} \Phi(\theta, r)x_{k_n} = \Phi(\theta, r)x. \tag{3.20}$$

Because the pair (f, v) satisfies (E_θ), we obtain that

$$f(t) = \Phi(\sigma(\theta, r), t - r)f(r) = \Phi(\theta, t)x, \quad \forall t \geq r. \tag{3.21}$$

This shows that $f(t) = \lambda_{x,\theta}(t)$, for all $t \geq r$. Then, from

$$\|\lambda_{x,\theta}(t)\| \leq \|f(t)\| + Me^{\omega r}\|x\|\chi_{[0,r)}(t), \quad \forall t \in \mathbb{R}, \tag{3.22}$$

using the fact that $f \in O(\mathbb{R}, X)$ and Remark 2.4(i), we obtain that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$, so $x \in \mathcal{S}(\theta)$.

In conclusion, $\mathcal{S}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$. □

Theorem 3.12 (The behavior on the unstable subspace). *If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then the following assertions hold:*

(i) *there is $K > 0$ such that $\|\Phi(\theta, t)y\| \geq (1/K)\|y\|$, for all $t \geq 0$, all $y \in \mathcal{U}(\theta)$ and all $\theta \in \Theta$;*

(ii) *$\mathcal{U}(\theta)$ is a closed linear subspace, for all $\theta \in \Theta$.*

Proof. Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$.

(i) Let $\theta \in \Theta$ and let $y \in \mathcal{U}(\theta)$. Then, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. Let $t > 0$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(s) = -\alpha(s - t)\Phi(\theta, s)y, \\ f : \mathbb{R} \longrightarrow X, \quad f(s) = \begin{cases} \displaystyle\int_s^{\infty} \alpha(\tau - t)d\tau\, \Phi(\theta, s)y, & s \geq 0, \\ \varphi(s), & s < 0. \end{cases} \end{gathered} \tag{3.23}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. Let $m = \sup_{s\in[0,t+1]} \|f(s)\|$. Then, we have that

$$\|f(s)\| \leq \|\varphi(s)\| + m\chi_{[0,t+1]}(s), \quad \forall s \in \mathbb{R}. \tag{3.24}$$

From (3.24) and Remark 2.4(i), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then, according to our hypothesis, we have that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{3.25}$$

From $\|v(s)\| \leq \alpha(s-t)Me^{\omega}\|\Phi(\theta, t)y\|$, for all $s \in \mathbb{R}$, we obtain that

$$\|v\|_{I(\mathbb{R},X)} \leq |\alpha|_I Me^{\omega}\|\Phi(\theta, t)y\|. \tag{3.26}$$

Since $y = \varphi(0) = \Phi(\sigma(\theta, s), -s)\varphi(s)$, for all $s \in [-1, 0)$, we have that

$$\|y\|\chi_{[-1,0)}(s) \leq Me^{\omega}\|\varphi(s)\|\chi_{[-1,0)}(s) \leq Me^{\omega}\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.27}$$

Using the invariance under translations of the space O from relation (3.27), we obtain that

$$\|y\|F_O(1) \leq Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{3.28}$$

Taking $K = (M^2 e^{2\omega} L|\alpha|_I)/F_O(1)$ from relations (3.25), (3.26), and (3.28), it follows that $\|\Phi(\theta, t)y\| \geq (1/K)\|y\|$. Taking into account that K does not depend on t, y or θ, we conclude that

$$\|\Phi(\theta, t)y\| \geq \frac{1}{K}\|y\|, \quad \forall t \geq 0, \ \forall y \in \mathcal{U}(\theta), \ \forall \theta \in \Theta. \tag{3.29}$$

(ii) Let $\theta \in \Theta$ and let $(y_n) \subset \mathcal{U}(\theta)$ with $y_n \to y$. Then, for every $n \in \mathbb{N}$, there is $\varphi_n \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta)$ with $\varphi_n(0) = y_n$. For every $n \in \mathbb{N}$, we consider the functions

$$\begin{gathered} v_n : \mathbb{R} \longrightarrow X, \quad v_n(t) = -\alpha(t)\Phi(\theta, t)y_n, \\ f_n : \mathbb{R} \longrightarrow X, \quad f_n(t) = \begin{cases} \displaystyle\int_t^{\infty} \alpha(\tau)d\tau \, \Phi(\theta, t)y_n, & t \geq 0, \\ \varphi_n(t), & t < 0. \end{cases} \end{gathered} \tag{3.30}$$

We have that $v_n \in \mathcal{C}_{0c}(\mathbb{R}, X)$, and, using similar arguments with those used in relation (3.24), we deduce that $f_n \in O(\mathbb{R}, X)$, for all $n \in \mathbb{N}$. An easy computation shows that the pair (f_n, v_n) satisfies (E_θ). Let

$$v : \mathbb{R} \longrightarrow X, \quad v(t) = -\alpha(t)\Phi(\theta, t)y. \tag{3.31}$$

According to our hypothesis, there is $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ). In particular, this implies that $f \in \mathcal{F}(\theta)$. Moreover, for every $n \in \mathbb{N}$, the pair $(f_n - f, v_n - v)$ satisfies (E_θ). According to our hypothesis, it follows that

$$\|f_n - f\|_{O(\mathbb{R},X)} \le L\|v_n - v\|_{I(\mathbb{R},X)}, \quad \forall n \in \mathbb{N}. \tag{3.32}$$

We have that $\|v_n(t) - v(t)\| \le \alpha(t) M e^{\omega} \|y_n - y\|$, for all $t \in \mathbb{R}$ and all $n \in \mathbb{N}$, so

$$\|v_n - v\|_{I(\mathbb{R},X)} \le M e^{\omega} |\alpha|_I \|y_n - y\|, \quad \forall n \in \mathbb{N}. \tag{3.33}$$

From (3.32) and (3.33) it follows that $f_n \underset{n\to\infty}{\longrightarrow} f$ in $O(\mathbb{R}, X)$. Then, from Remark 2.4(ii), there is a subsequence $(f_{k_n}) \subset (f_n)$ and a negligible set $A \subset \mathbb{R}$ such that $f_{k_n}(t) \underset{n\to\infty}{\longrightarrow} f(t)$, for all $t \in \mathbb{R} \setminus A$. In particular, there is $h < 0$ such that $f_{k_n}(h) \underset{n\to\infty}{\longrightarrow} f(h)$. Since $f, f_{k_n} \in \mathcal{F}(\theta)$, we successively deduce that

$$y = \lim_{n\to\infty} y_{k_n} = \lim_{n\to\infty} f_{k_n}(0) = \lim_{n\to\infty} \Phi(\sigma(\theta, h), -h) f_{k_n}(h) = \Phi(\sigma(\theta, h), -h) f(h) = f(0). \tag{3.34}$$

This implies that $y \in \mathcal{U}(\theta)$, so $\mathcal{U}(\theta)$ is a closed linear subspace. □

Taking into account the above results it makes sense to study whether the uniform admissibility of a pair of function spaces from the class $\mathcal{T}(\mathbb{R})$ is a sufficient condition for the existence of the uniform dichotomy. Thus, the main result of this section is as follows.

Theorem 3.13 (Sufficient condition for uniform dichotomy). *Let $O, I \in \mathcal{T}(\mathbb{R})$ and let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then π is uniformly dichotomic.*

Proof. Let $L > 0$ be given by Definition 3.6. Let $M, \omega > 0$ be given by Definition 3.2. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau) d\tau = 1$. □

Step 1. We prove that $\mathcal{S}(\theta) \cap \mathcal{U}(\theta) = \{0\}$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $x \in \mathcal{S}(\theta) \cap \mathcal{U}(\theta)$. Then, there is $\varphi \in O(\mathbb{R}, X) \cap \mathcal{F}(\theta)$ with $\varphi(0) = x$. We consider the function

$$f : \mathbb{R} \to X, \qquad f(t) = \begin{cases} \Phi(\theta, t)x, & t \ge 0, \\ \varphi(t), & t < 0. \end{cases} \tag{3.35}$$

Then, $\|f(t)\| \le \|\varphi(t)\| + \|\lambda_{x,\theta}(t)\|$, for all $t \in \mathbb{R}$. This implies that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair $(f, 0)$ satisfies (E_θ). Then, according to our hypothesis, it follows that $\|f\|_{O(\mathbb{R},X)} = 0$, so $f(t) = 0$ a.e. $t \in \mathbb{R}$. Observing that f is continuous, we obtain that $f(t) = 0$, for all $t \in \mathbb{R}$. In particular, we have that $x = f(0) = 0$.

Step 2. We prove that $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$, for all $\theta \in \Theta$.

Let $\theta \in \Theta$ and let $x \in X$. Let $v : \mathbb{R} \to X$, $v(t) = \alpha(t)\Phi(\theta, t)x$. Then, $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$, so there is $f \in O(\mathbb{R}, X)$ such that the pair (f, v) satisfies (E_θ). In particular, this implies that $f \in \mathcal{F}(\theta)$, so $f(0) \in \mathcal{U}(\theta)$. In addition, we observe that

$$f(t) = \Phi(\theta,t)f(0) + \left(\int_0^t \alpha(\tau)d\tau\right)\Phi(\theta,t)x = \Phi(\theta,t)\big(f(0)+x\big), \quad \forall t \geq 1. \tag{3.36}$$

Setting $z_x = f(0) + x$ from (3.36), we have that $\lambda_{z_x,\theta}(t) = f(t)$, for all $t \geq 1$. It follows that

$$\|\lambda_{z_x,\theta}(t)\| \leq \|f(t)\| + Me^{\omega}\|z_x\|\chi_{[0,1)}(t), \quad \forall t \in \mathbb{R}. \tag{3.37}$$

From relation (3.37) and Remark 2.4(i) we obtain that $\lambda_{z_x,\theta} \in O(\mathbb{R}, X)$, so $z_x \in \mathcal{S}(\theta)$. This shows that $x = z_x - f(0) \in \mathcal{S}(\theta) + \mathcal{U}(\theta)$, so $\mathcal{S}(\theta) + \mathcal{U}(\theta) = X$.

According to Steps 1 and 2, Theorem 3.11(ii), and Theorem 3.12(ii), we deduce that

$$\mathcal{S}(\theta) \oplus \mathcal{U}(\theta) = X, \quad \forall \theta \in \Theta. \tag{3.38}$$

For every $\theta \in \Theta$ we denote by $P(\theta)$ the projection with the property that

$$\operatorname{Range} P(\theta) = \mathcal{S}(\theta), \qquad \operatorname{Ker} P(\theta) = \mathcal{U}(\theta). \tag{3.39}$$

Using Proposition 3.9 we obtain that

$$\Phi(\theta,t)P(\theta) = P(\sigma(\theta,t))\Phi(\theta,t), \quad \forall(\theta,t) \in \Theta \times \mathbb{R}_+. \tag{3.40}$$

Let $(\theta, t) \in \Theta \times \mathbb{R}_+$. From Proposition 3.9(ii), it follows that the restriction $\Phi(\theta,t)_| : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta,t))$ is correctly defined and surjective. According to Theorem 3.12(ii) we have that $\Phi(\theta,t)_|$ is also injective, so this is an isomorphism, for all $(\theta,t) \in \Theta \times \mathbb{R}_+$.

Step 3. We prove that $\sup_{\theta\in\Theta}\|P(\theta)\| < \infty$.

Let $\theta \in \Theta$ and let $x \in X$. Let $x_s^\theta = P(\theta)x$ and let $x_u^\theta = (I - P(\theta))x$. Since $x_u^\theta \in \operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = x_u^\theta$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\Phi(\theta,t)x, \\ f : \mathbb{R} \to X, \quad f(t) = \begin{cases} \Phi(\theta,t)x_s^\theta, & t \geq 1, \\ -\Phi(\theta,t)x_u^\theta + \left(\displaystyle\int_0^t \alpha(\tau)d\tau\right)\Phi(\theta,t)x, & t \in [0,1), \\ -\varphi(t), & t < 0. \end{cases} \end{gathered} \tag{3.41}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. From $x_s^\theta \in \operatorname{Range} P(\theta) = \mathcal{S}(\theta)$, we have that the function $\lambda_{x_s^\theta,\theta}$ belongs to $O(\mathbb{R}, X)$. Setting $m = \sup_{t\in[0,1]}\|f(t)\|$ and observing that

$$\|f(t)\| \le \|\varphi(t)\| + m\chi_{[0,1]}(t) + \|\lambda_{x_s^\theta,\theta}(t)\|, \quad \forall t \in \mathbb{R}, \tag{3.42}$$

from (3.42), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). This implies that

$$\|f\|_{O(\mathbb{R},X)} \le L\|v\|_{I(\mathbb{R},X)}. \tag{3.43}$$

Since $\varphi \in \mathcal{F}(\theta)$, we have that $x_u^\theta = \varphi(0) = \Phi(\sigma(\theta, s), -s)\varphi(s)$, for all $s \in [-1, 0)$. This implies that

$$\|x_u^\theta\| \le Me^\omega\|\varphi(s)\| = Me^\omega\|f(s)\|, \quad \forall s \in [-1, 0), \tag{3.44}$$

and we obtain that

$$\|x_u^\theta\|\chi_{[-1,0)}(s) \le Me^\omega\|f(s)\|, \quad \forall s \in \mathbb{R}. \tag{3.45}$$

Using the invariance under translations of the space O, from relation (3.45) we deduce that

$$\|x_u^\theta\|F_O(1) \le Me^\omega\|f\|_{O(\mathbb{R},X)}. \tag{3.46}$$

In addition, from

$$\|v(t)\| \le \alpha(t)Me^\omega\|x\|, \quad \forall t \in \mathbb{R}, \tag{3.47}$$

we obtain that

$$\|v\|_{I(\mathbb{R},X)} \le |\alpha|_I Me^\omega\|x\|. \tag{3.48}$$

Setting $\gamma := [L|\alpha|_I M^2 e^{2\omega}/F_O(1)]$ from relations (3.43), (3.46), and (3.48), we have that

$$\|(I - P(\theta))x\| = \|x_u^\theta\| \le \gamma\|x\|. \tag{3.49}$$

This implies that

$$\|P(\theta)x\| \le (1 + \gamma)\|x\|. \tag{3.50}$$

Taking into account that γ does not depend on θ or x, it follows that relation (3.50) holds, for all $\theta \in \Theta$ and all $x \in X$, so $\|P(\theta)\| \le 1 + \gamma$, for all $\theta \in \Theta$.

Finally, from Theorem 3.11(i) and Theorem 3.12(i), we conclude that π is uniformly dichotomic.

Remark 3.14. Relation (3.39) shows that the stable subspace and the instable subspace play a central role in the detection of the dichotomous behavior of a skew-product flow and gives a comprehensible motivation for their usual appellation.

4. Exponential Dichotomy of Skew-Product Flows

In the previous section, we have obtained sufficient conditions for the uniform dichotomy of a skew-product flow $\pi = (\Phi, \sigma)$ on $X \times \Theta$ in terms of the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the associated control system (E_π), where $O, I \in \mathcal{T}(\mathbb{R})$. The natural question arises: which are the additional (preferably minimal) hypotheses under which this admissibility may provide the existence of the *exponential* dichotomy? In this context, the main purpose of this section is to establish which are the most general classes of Banach function spaces where O or I may belong to, such that the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the control system (E_π) is a sufficient (and also a necessary) condition for the existence of exponential dichotomy.

Let X be a real or complex Banach space and let (Θ, d) be a metric space. Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$.

Definition 4.1. A skew-product flow $\pi = (\Phi, \sigma)$ is said to be *exponentially dichotomic* if there exist a family of projections $\{P(\theta)\}_{\theta \in \Theta} \subset \mathcal{B}(X)$ and two constants $K \geq 1$ and $\nu > 0$ such that the following properties hold:

(i) $\Phi(\theta, t)P(\theta) = P(\sigma(\theta, t))\Phi(\theta, t)$, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(ii) $\|\Phi(\theta, t)x\| \leq Ke^{-\nu t}\|x\|$, for all $t \geq 0$, all $x \in \operatorname{Range} P(\theta)$ and all $\theta \in \Theta$;

(iii) the restriction $\Phi(\theta, t)_{|} : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$ is an isomorphism, for all $(\theta, t) \in \Theta \times \mathbb{R}_+$;

(iv) $\|\Phi(\theta, t)y\| \geq (1/K)e^{\nu t}\|y\|$, for all $t \geq 0$, all $y \in \operatorname{Ker} P(\theta)$ and all $\theta \in \Theta$.

Before proceeding to the next steps, we need a technical lemma.

Lemma 4.2. *If a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta \in \Theta}$, then $\sup_{\theta \in \Theta}\|P(\theta)\| < \infty$.*

Proof. Let $K, \nu > 0$ be given by Definition 4.1 and let $M, \omega > 0$ be given by Definition 3.2. For every $(x, \theta) \in X \times \Theta$ and every $t \geq 0$, we have that

$$\begin{aligned} \frac{1}{K}e^{\nu t}\|(I - P(\theta))x\| &\leq \|\Phi(\theta, t)(I - P(\theta))x\| \leq Me^{\omega t}\|x\| + Ke^{-\nu t}\|P(\theta)x\| \\ &\leq \left(Me^{\omega t} + K\right)\|x\| + Ke^{-\nu t}\|(I - P(\theta))x\|, \end{aligned} \tag{4.1}$$

which implies that

$$\left(e^{2\nu t} - K^2\right)\frac{e^{-\nu t}}{K}\|(I - P(\theta))x\| \leq \left(Me^{\omega t} + K\right)\|x\|, \quad \forall t \geq 0,\ \forall (x, \theta) \in X \times \Theta. \tag{4.2}$$

Let $h > 0$ be such that $e^{2\nu h} - K^2 > 0$. Setting $\alpha := (e^{2\nu h} - K^2)e^{-\nu h}/K$ and $\delta := (Me^{\omega h} + K)$, it follows that $\|(I - P(\theta))x\| \leq (\delta/\alpha)\|x\|$, for all $(x, \theta) \in X \times \Theta$. This implies that $\|I - P(\theta)\| \leq \delta/\alpha$, for all $\theta \in \Theta$, so $\|P(\theta)\| \leq 1 + (\delta/\alpha)$, for all $\theta \in \Theta$, and the proof is complete. □

Remark 4.3. (i) Using Lemma 4.2, we deduce that if a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$, then π is uniformly dichotomic with respect to the same family of projections.

(ii) If a skew-product flow π is exponentially dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$, then this family is uniquely determined (see, e.g., [18], Remark 2.5).

Remark 4.4. In the description of any dichotomous behavior, the properties (i) and (iii) are inherent, because beside the splitting of the space ensured by the presence of the dichotomy projections, these properties reflect both the invariance with respect to the decomposition induced by each projection as well as the reversibility of the cocycle restricted to the kernel of each projection.

In this context, it is extremely important to note that if in the detection of the dichotomy one assumes from the very beginning that there exist a projection family such that the invariance property (i) and the reversibility condition (iii) hold, then the dichotomy concept is resumed to a stability property (ii) and to an instability condition (iv), which via (iii) will consist only of a double stability. Thus, if in the study of the dichotomy one considers (i) and (iii) as working hypotheses, then the entire investigation is reduced to a quasitrivial case of (double) stability.

In conclusion, in the study of the existence of (uniform or) exponential dichotomy, it is essential to determine conditions *which imply the existence of the projection family* and also the fulfillment of *all* the conditions from Definition 4.1.

Now let O, I be two Banach function spaces such that $O, I \in \mathcal{T}(\mathbb{R})$. According to the main result in the previous section (see Theorem 3.13), if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then π is uniformly dichotomic with respect to a family of projections $\{P(\theta)\}_{\theta\in\Theta}$ with the property that

$$\operatorname{Range} P(\theta) = \mathcal{S}(\theta), \qquad \operatorname{Ker} P(\theta) = \mathcal{U}(\theta), \quad \forall\theta \in \Theta. \tag{4.3}$$

In what follows, we will see that by imposing some conditions *either* on the output space O *or* on the input space I, the admissibility becomes a sufficient condition for the exponential dichotomy.

Theorem 4.5 (The behavior on the stable subspace). *Let O, I be two Banach function spaces such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then there are $K, \nu > 0$ such that*

$$\|\Phi(\theta, t)x\| \leq Ke^{-\nu t}\|x\|, \quad \forall t \geq 0,\ \forall x \in \operatorname{Range} P(\theta),\ \forall\theta \in \Theta. \tag{4.4}$$

Proof. Let $\delta > 0$ be such that

$$\|\Phi(\theta, t)x\| \leq \delta\|x\|, \quad \forall t \geq 0,\ \forall x \in \operatorname{Range} P(\theta),\ \forall\theta \in \Theta. \tag{4.5}$$

We prove that there is $h > 0$ such that

$$\|\Phi(\theta, h)x\| \leq \frac{1}{e}\|x\|, \quad \forall x \in \text{Range}\, P(\theta), \ \forall \theta \in \Theta. \tag{4.6}$$

Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2.

Case 1. Suppose that $O \in \mathcal{Q}(\mathbb{R})$. Let $\alpha : \mathbb{R} \to [0,2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0,1)$ such that $\int_0^1 \alpha(\tau) d\tau = 1$. Since $\sup_{t>0} F_O(t) = \infty$, there is $r > 0$ such that

$$F_O(r) \geq e\delta^2 L|\alpha|_I. \tag{4.7}$$

Let $\theta \in \Theta$ and let $x \in \text{Range}\, P(\theta)$. If $\Phi(\theta, 1)x \neq 0$, then we consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = \alpha(t)\frac{\Phi(\theta,t)x}{\|\Phi(\theta,t)x\|}, \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} a\Phi(\theta,t)x, & t \geq 1, \\ \displaystyle\int_0^t \frac{\alpha(\tau)}{\|\Phi(\theta,\tau)x\|} d\tau\, \Phi(\theta,t)x, & t \in [0,1], \\ 0, & t < 0, \end{cases} \end{gathered} \tag{4.8}$$

where

$$a := \int_0^1 \frac{\alpha(\tau)}{\|\Phi(\theta,\tau)x\|} d\tau. \tag{4.9}$$

We observe that f is continuous and

$$\|f(t)\| \leq a\|\lambda_{x,\theta}(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.10}$$

Since $x \in \text{Range}\, P(\theta) = \mathcal{S}(\theta)$, we have that $\lambda_{x,\theta} \in O(\mathbb{R}, X)$. Then using Remark 2.4(i), we deduce that $f \in O(\mathbb{R}, X)$. In addition, we have that $v \in C_{0c}(\mathbb{R}, X)$ and an easy computation shows that the pair (f, v) satisfies (E_θ). Then, according to our hypothesis, it follows that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.11}$$

Because $\|v(t)\| = \alpha(t)$, for all $t \in \mathbb{R}$, the relation (4.11) becomes

$$\|f\|_{O(\mathbb{R},X)} \leq L|\alpha|_I. \tag{4.12}$$

Using relation (4.5), we deduce that

$$\|\Phi(\theta, r+1)x\| \leq \delta\|\Phi(\theta,t)x\| = \frac{\delta}{a}\|f(t)\|, \quad \forall t \in [1, r+1), \tag{4.13}$$

so

$$\|\Phi(\theta, r+1)x\|\chi_{[1,r+1)}(t) \leq \frac{\delta}{a}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.14}$$

Using the invariance under translations of the space O from relation (4.14), we obtain that

$$\|\Phi(\theta, r+1)x\| F_O(r) \leq \frac{\delta}{a}\|f\|_{O(\mathbb{R},X)}. \tag{4.15}$$

Setting $h := r+1$ from relations (4.12) and (4.15), it follows that

$$\|\Phi(\theta, h)x\| F_O(r) \leq \frac{\delta L|\alpha|_I}{a}. \tag{4.16}$$

Moreover, from relation (4.5), we have that $\|\Phi(\theta,\tau)x\| \leq \delta\|x\|$, for all $\tau \in [0,1)$, so

$$a = \int_0^1 \frac{\alpha(\tau)}{\|\Phi(\theta,\tau)x\|} d\tau \geq \frac{1}{\delta\|x\|}. \tag{4.17}$$

From relations (4.7), (4.16), and (4.17), it follows that

$$\|\Phi(\theta, h)x\| \leq \frac{1}{e}\|x\|. \tag{4.18}$$

If $\Phi(\theta,1)x = 0$, then $\Phi(\theta,h)x = 0$, so the above relation holds. Taking into account that h does not depend on θ or x, we obtain that in this case, there is $h > 0$ such that relation (4.6) holds.

Case 2. Suppose that $I \in \mathcal{L}(\mathbb{R})$. In this situation, from Remark 2.16, we have that there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ such that $\gamma \in I \setminus L^1(\mathbb{R},\mathbb{R})$. Since the space I is invariant under translations, we may assume that there is $r > 1$ such that

$$\int_1^r \gamma(\tau) d\tau \geq \frac{eL\delta^2|\gamma|_I}{F_O(1)}. \tag{4.19}$$

Let $\beta : \mathbb{R} \to [0,1]$ be a continuous function with $\operatorname{supp}\beta \subset (0, r+1)$ and $\beta(t) = 1$, for all $t \in [1,r]$.

Let $\theta \in \Theta$ and let $x \in \operatorname{Range} P(\theta)$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = \beta(t)\gamma(t)\Phi(\theta,t)x, \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} q\Phi(\theta,t)x, & t \geq r+1, \\ \displaystyle\int_0^t \beta(\tau)\gamma(\tau)d\tau\, \Phi(\theta,t)x, & t \in [0, r+1), \\ 0, & t < 0, \end{cases} \end{gathered} \tag{4.20}$$

where

$$q = \int_0^{r+1} \beta(\tau)\gamma(\tau)d\tau. \tag{4.21}$$

We have that $v \in C_{0c}(\mathbb{R}, X)$, f is continuous, and $\|f(t)\| \le q\|\lambda_{x,\theta}(t)\|$, for all $t \in \mathbb{R}$. Using similar arguments with those used in relation (4.10), we deduce that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ). Then, we have that

$$\|f\|_{O(\mathbb{R},X)} \le L\|v\|_{I(\mathbb{R},X)}. \tag{4.22}$$

Using relation (4.5), we obtain that

$$\|v(t)\| \le \delta\gamma(t)\|x\|, \quad \forall t \in \mathbb{R}, \tag{4.23}$$

which implies that

$$\|v\|_{I(\mathbb{R},X)} \le \delta|\gamma|_I\|x\|. \tag{4.24}$$

In addition, from $\|\Phi(\theta, r+2)x\| \le \delta\|\Phi(\theta, t)x\|$, for all $t \in [r+1, r+2)$, we deduce that

$$\|\Phi(\theta, r+2)x\|\chi_{[r+1,r+2)}(t) \le \frac{\delta}{q}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.25}$$

Using the invariance under translations of the space O from relations (4.25), (4.22), and (4.24) we have that

$$q\|\Phi(\theta, r+2)x\|F_O(1) \le \delta\|f\|_{O(\mathbb{R},X)} \le L\delta^2|\gamma|_I\|x\|. \tag{4.26}$$

Since $q \ge \int_1^r \gamma(\tau)d\tau$, from relations (4.19), (4.21), and (4.26), it follows that

$$\|\Phi(\theta, r+2)x\| \le \frac{1}{e}\|x\|. \tag{4.27}$$

Setting $h = r+2$ and taking into account that h does not depend on θ or x, we obtain that relation (4.6) holds.

In conclusion, in both situations, there is $h > 0$ such that

$$\|\Phi(\theta, h)x\| \le \frac{1}{e}\|x\|, \quad \forall x \in \text{Range}\, P(\theta),\ \forall \theta \in \Theta. \tag{4.28}$$

Let $\nu := 1/h$ and let $K = \delta e$. Let $\theta \in \Theta$ and let $x \in \operatorname{Range} P(\theta)$. Let $t > 0$. Then, there are $k \in \mathbb{N}$ and $\tau \in [0, h)$ such that $t = kh + \tau$. Using relations (4.5) and (4.6), we successively deduce that

$$\|\Phi(\theta, t)x\| \leq \delta\|\Phi(\theta, kh)x\| \leq \delta e^{-k}\|x\| \leq Ke^{-\nu t}\|x\|. \tag{4.29}$$

□

Theorem 4.6 (The behavior on the unstable subspace). *Let O, I be two Banach function spaces such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. If the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then, there are $K, \nu > 0$ such that*

$$\|\Phi(\theta, t)y\| \geq \frac{1}{K}e^{\nu t}\|y\|, \quad \forall t \geq 0,\ \forall y \in \operatorname{Ker} P(\theta),\ \forall \theta \in \Theta. \tag{4.30}$$

Proof. Let $\delta > 0$ be such that

$$\|\Phi(\theta, t)y\| \geq \frac{1}{\delta}\|y\|, \quad \forall t \geq 0,\ \forall y \in \operatorname{Ker} P(\theta),\ \forall \theta \in \Theta. \tag{4.31}$$

Let $L > 0$ be given by Definition 3.6 and let $M, \omega > 0$ be given by Definition 3.2. We prove that there is $h > 0$ such that

$$\|\Phi(\theta, h)y\| \geq e\|y\|, \quad \forall y \in \operatorname{Ker} P(\theta),\ \forall \theta \in \Theta. \tag{4.32}$$

Case 1. Suppose that $O \in \mathcal{Q}(\mathbb{R})$. Let $\alpha : \mathbb{R} \to [0, 2]$ be a continuous function with $\operatorname{supp} \alpha \subset (0, 1)$ and $\int_0^1 \alpha(\tau)d\tau = 1$. In this case, there is $r > 0$ such that

$$F_O(r) \geq e\delta^2 L|\alpha|_I. \tag{4.33}$$

Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta) \setminus \{0\}$. Then, $\Phi(\theta, t)y \neq 0$, for all $t \geq 0$. Since $y \in \operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$, there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. We consider the functions

$$\begin{aligned} &v : \mathbb{R} \longrightarrow X, \quad v(t) = -\alpha(t - r)\frac{\Phi(\theta, t)y}{\|\Phi(\theta, t)y\|} \\ &f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \displaystyle\int_t^\infty \frac{\alpha(\tau - r)}{\|\Phi(\theta, \tau)y\|}d\tau\, \Phi(\theta, t)y, & t \geq r, \\ a\Phi(\theta, t)y, & t \in [0, r), \\ a\varphi(t), & t < 0, \end{cases} \end{aligned} \tag{4.34}$$

where

$$a := \int_r^{r+1} \frac{\alpha(\tau - r)}{\|\Phi(\theta, \tau)y\|}d\tau. \tag{4.35}$$

We have that $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and f is continuous. Moreover, from

$$\|f(t)\| \leq a\|\varphi(t)\| + aMe^{\omega(r+1)}\|y\|\chi_{[0,r+1)}(t), \quad \forall t \in \mathbb{R}, \tag{4.36}$$

we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ), so

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.37}$$

Observing that $\|v(t)\| = \alpha(t-r)$, for all $t \in \mathbb{R}$, the relation (4.37) becomes

$$\|f\|_{O(\mathbb{R},X)} \leq L|\alpha|_I. \tag{4.38}$$

From relation (4.31), we have that

$$\|\Phi(\theta, r+1)y\| \geq \frac{1}{\delta}\|\Phi(\theta, \tau)y\|, \quad \forall \tau \in [r, r+1]. \tag{4.39}$$

This implies that

$$a \geq \frac{1}{\delta\|\Phi(\theta, r+1)y\|}. \tag{4.40}$$

In addition, from relation (4.31), we have that

$$\|\Phi(\theta, t)y\| \geq \frac{1}{\delta}\|y\|, \quad \forall t \in [0, r) \tag{4.41}$$

which implies that

$$\|y\|\chi_{[0,r)}(t) \leq \delta\|\Phi(\theta, t)y\|\chi_{[0,r)}(t) \leq \frac{\delta}{a}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.42}$$

From relation (4.42), it follows that

$$\|y\|F_O(r) \leq \frac{\delta}{a}\|f\|_{O(\mathbb{R},X)}. \tag{4.43}$$

From relations (4.38), (4.40), and (4.43), we deduce that

$$\|y\|F_O(r) \leq \frac{\delta L|\alpha|_I}{a} \leq \delta^2 L|\alpha|_I\|\Phi(\theta, r+1)y\|. \tag{4.44}$$

From relations (4.44) and (4.33), we have that

$$\|\Phi(\theta, r+1)y\| \geq e\|y\|. \tag{4.45}$$

Setting $h := r + 1$ and taking into account that h does not depend on y or θ we obtain that relation (4.32) holds.

Case 2. Suppose that $I \in \mathcal{L}(\mathbb{R})$. In this situation, using Remark 2.16 and the translation invariance of the space I, we have that there is a continuous function $\gamma : \mathbb{R} \to \mathbb{R}_+$ with $\gamma \in I \setminus L^1(\mathbb{R}, \mathbb{R})$ and $r > 1$ such that

$$\int_1^r \gamma(\tau)d\tau \geq e^{\omega+1}\frac{LM\delta|\gamma|_I}{F_O(1)}. \tag{4.46}$$

Let $\beta : \mathbb{R} \to [0,1]$ be a continuous function with $\operatorname{supp}\beta \subset (0, r+1)$ and $\beta(t) = 1$, for all $t \in [1, r]$.

Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta)$. Since $\operatorname{Ker} P(\theta) = \mathcal{U}(\theta)$ there is $\varphi \in \mathcal{F}(\theta) \cap O(\mathbb{R}, X)$ with $\varphi(0) = y$. We consider the functions

$$\begin{gathered} v : \mathbb{R} \longrightarrow X, \quad v(t) = -\beta(t)\gamma(t)\Phi(\theta, t)y, \\ f : \mathbb{R} \longrightarrow X, \quad f(t) = \begin{cases} \displaystyle\int_t^\infty \beta(\tau)\gamma(\tau)d\tau\Phi(\theta, t)y, & t \geq 0, \\ q\varphi(t), & t < 0, \end{cases} \end{gathered} \tag{4.47}$$

where $q := \int_0^{r+1} \beta(\tau)\gamma(\tau)d\tau$. We have that $v \in C_{0c}(\mathbb{R}, X)$, and, using similar arguments with those from Case 1, we obtain that $f \in O(\mathbb{R}, X)$. An easy computation shows that the pair (f, v) satisfies (E_θ), so

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.48}$$

From (4.31), we have that $\|\Phi(\theta, r+1)y\| \geq (1/\delta)\|\Phi(\theta, t)y\|$, for all $t \in [0, r+1]$. This implies that

$$\|v(t)\| \leq \gamma(t)\delta\|\Phi(\theta, r+1)y\|, \quad \forall t \in \mathbb{R}, \tag{4.49}$$

so

$$\|v\|_{I(\mathbb{R},X)} \leq |\gamma|_I\delta\|\Phi(\theta, r+1)y\|. \tag{4.50}$$

Since $\varphi \in \mathcal{F}(\theta)$, we have that

$$\|y\| = \|\varphi(0)\| = \|\Phi(\sigma(\theta, t), -t)\varphi(t)\| \leq Me^\omega\|\varphi(t)\|, \quad \forall t \in [-1, 0). \tag{4.51}$$

From relation (4.51), it follows that

$$\|y\|\chi_{[-1,0)}(t) \leq Me^\omega\|\varphi(t)\|\chi_{[-1,0)}(t) \leq \frac{Me^\omega}{q}\|f(t)\|, \quad \forall t \in \mathbb{R}. \tag{4.52}$$

Using the translation invariance of the space O from (4.52), we obtain that

$$q\|y\|F_O(1) \le Me^{\omega}\|f\|_{O(\mathbb{R},X)}. \tag{4.53}$$

Since $q \ge \int_1^r \gamma(\tau)d\tau$, from relations (4.46), (4.48), (4.50) we deduce that

$$\|\Phi(\theta, r+1)y\| \ge e\|y\|. \tag{4.54}$$

Setting $h := r+1$ and since h does not depend on y or θ, we have that the relation (4.32) holds.

In conclusion, in both situations there is $h > 0$ such that

$$\|\Phi(\theta, h)y\| \ge e\|y\|, \quad \forall y \in \operatorname{Ker} P(\theta),\ \forall \theta \in \Theta. \tag{4.55}$$

Let $\nu = 1/h$ and let $K = \delta e$. Let $\theta \in \Theta$ and let $y \in \operatorname{Ker} P(\theta)$. Let $t > 0$. Then, there are $j \in \mathbb{N}$ and $s \in [0, h)$ such that $t = jh + s$. Using relations (4.31) and (4.32), we obtain that

$$\|\Phi(\theta, t)y\| \ge \frac{1}{\delta}\|\Phi(\theta, jh)y\| \ge \frac{1}{\delta}e^{j}\|y\| \ge \frac{1}{K}e^{\nu t}\|y\|. \tag{4.56}$$

□

According to the previous results we may formulate now a sufficient condition for the existence of the exponential dichotomy. Moreover, for the converse implication we will show that it sufficient to chose one of the spaces in the admissible pair from the class $\mathcal{R}(\mathbb{R})$. Thus, the main result of this section is as follows.

Theorem 4.7 (Necessary and sufficient condition for exponential dichotomy). *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and let O, I be two Banach function spaces with $O, I \in \mathcal{T}(\mathbb{R})$ such that either $O \in \mathcal{Q}(\mathbb{R})$ or $I \in \mathcal{L}(\mathbb{R})$. The following assertions hold:*

(i) if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π), then π is exponentially dichotomic.

(ii) if $I \subset O$ and one of the spaces I or O belongs to the class $\mathcal{R}(\mathbb{R})$, then π is exponentially dichotomic if and only if the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Proof. (i) This follows from Theorem 3.13, Theorem 4.5, and Theorem 4.6.

(ii) Since $I \subset O$, it follows that there is $\alpha > 0$ such that

$$|u|_O \le \alpha |u|_I, \quad \forall u \in I. \tag{4.57}$$

Necessity. Suppose that π is exponentially dichotomic with respect to the family of projections $\{P(\theta)\}_{\theta\in\Theta}$ and let $K, \nu > 0$ be two constants given by Definition 4.1. According to Lemma 4.2, we have that $q := \sup_{\theta\in\Theta}\|P(\theta)\| < \infty$. For every $(\theta, t) \in \Theta \times \mathbb{R}_+$ we denote by $\Phi(\theta, t)_|^{-1}$ the inverse of the operator $\Phi(\theta, t)_| : \operatorname{Ker} P(\theta) \to \operatorname{Ker} P(\sigma(\theta, t))$.

Let $\theta \in \Theta$ and let $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$. We consider the function $f_v : \mathbb{R} \to X$ given by

$$\begin{aligned} f_v(t) = &\int_{-\infty}^{t} \Phi(\sigma(\theta,\tau), t-\tau) P(\sigma(\theta,\tau)) v(\tau) d\tau \\ &- \int_{t}^{\infty} \Phi(\sigma(\theta,t), \tau - t)_{|}^{-1} (I - P(\sigma(\theta,\tau)))\ v(\tau) d\tau. \end{aligned} \tag{4.58}$$

We have that f_v is continuous, and a direct computation shows that the pair (f_v, v) satisfies (E_θ). In addition, we have that

$$\begin{aligned} \|f_v(t)\| \le qK &\int_{-\infty}^{t} e^{-\nu(t-\tau)} \|v(\tau)\| d\tau \\ &+ (1+q)K \int_{t}^{\infty} e^{-\nu(\tau - t)} \|v(\tau)\| d\tau, \quad \forall t \in \mathbb{R}. \end{aligned} \tag{4.59}$$

If $I \in \mathcal{R}(\mathbb{R})$, let $\gamma_{I,\nu} > 0$ be the constant given by Lemma 2.21. Then, from (4.59) and Lemma 2.21, it follows that $f_v \in I(\mathbb{R}, X)$ and

$$\|f_v\|_{I(\mathbb{R},X)} \le (1+2q) K \gamma_{I,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.60}$$

Then, from (4.57) and (4.60), we deduce that $f_v \in O(\mathbb{R}, X)$ and

$$\|f_v\|_{O(\mathbb{R},X)} \le \alpha(1+2q) K \gamma_{I,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.61}$$

If $O \in \mathcal{R}(\mathbb{R})$, let $\gamma_{O,\nu} > 0$ be the constant given by Lemma 2.21. Then, from (4.59), (4.57) and using Lemma 2.21, we successively obtain that $f_v \in O(\mathbb{R}, X)$ and

$$\|f_v\|_{O(\mathbb{R},X)} \le (1+2q) K \gamma_{O,\nu} \|v\|_{O(\mathbb{R},X)} \le \alpha(1+2q) K \gamma_{O,\nu} \|v\|_{I(\mathbb{R},X)}. \tag{4.62}$$

Let

$$\gamma := \begin{cases} \gamma_{I,\nu}, & \text{if } I \in \mathcal{R}(\mathbb{R}), \\ \gamma_{O,\nu}, & \text{if } I \notin \mathcal{R}(\mathbb{R}), \quad O \in \mathcal{R}(\mathbb{R}). \end{cases} \tag{4.63}$$

Then setting $L := \alpha(1+2q)K\gamma$ from relations (4.61) and (4.62), we have that

$$\|f_v\|_{O(\mathbb{R},X)} \le L \|v\|_{I(\mathbb{R},X)}. \tag{4.64}$$

Now let $v \in \mathcal{C}_{0c}(\mathbb{R}, X)$ and $f \in O(\mathbb{R}, X)$ be such that the pair (f, v) satisfies (E_θ). We set $\varphi := f - f_v$, and we have that $\varphi \in O(\mathbb{R}, X)$ and

$$\varphi(t) = \Phi(\sigma(\theta, s), t - s)\varphi(s), \quad \forall t \ge s. \tag{4.65}$$

Let $\varphi_1(t) = P(\sigma(\theta,t))\varphi(t)$, for all $t \in \mathbb{R}$ and let $\varphi_2(t) = (I - P(\sigma(\theta,t)))\varphi(t)$, for all $t \in \mathbb{R}$. Then from (4.65), we obtain that

$$\varphi_k(t) = \Phi(\sigma(\theta,s), t-s)\varphi_k(s), \quad \forall t \geq s, \forall k \in \{1,2\}. \tag{4.66}$$

Let $t_0 \in \mathbb{R}$. From (4.66), it follows that

$$\|\varphi_1(t_0)\| \leq K e^{-\nu(t_0-s)}\|\varphi_1(s)\| \leq qK e^{-\nu(t_0-s)}\|\varphi(s)\|, \ \forall s \leq t_0. \tag{4.67}$$

Since $\varphi \in O(\mathbb{R},X)$, from Remark 2.12 it follows that $\varphi \in M^1(\mathbb{R},X)$. Then, from (4.67), we have that

$$\begin{aligned}\|\varphi_1(t_0)\| &\leq qK\int_{s-1}^{s} e^{-\nu(t_0-\tau)}\|\varphi(\tau)\|d\tau \leq qKe^{-\nu(t_0-s)}\int_{s-1}^{s}\|\varphi(\tau)\|d\tau \\ &\leq qKe^{-\nu(t_0-s)}\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall s \leq t_0.\end{aligned} \tag{4.68}$$

For $s \to -\infty$ in (4.68), it follows that $\varphi_1(t_0) = 0$. In addition, from (4.66) we have that

$$\frac{1}{K}e^{\nu(t-t_0)}\|\varphi_2(t_0)\| \leq \|\varphi_2(t)\| \leq (1+q)\|\varphi(t)\|, \quad \forall t \geq t_0. \tag{4.69}$$

This implies that

$$\frac{1}{K}e^{\nu(t-t_0)}\|\varphi_2(t_0)\| \leq (1+q)\int_t^{t+1}\|\varphi(\tau)\|d\tau \leq (1+q)\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall t \geq t_0. \tag{4.70}$$

The relation (4.70) shows that

$$\|\varphi_2(t_0)\| \leq K(1+q)e^{-\nu(t-t_0)}\|\varphi\|_{M^1(\mathbb{R},X)}, \quad \forall t \geq t_0. \tag{4.71}$$

For $t \to \infty$ in (4.71), it follows that $\varphi_2(t_0) = 0$. This shows that $\varphi(t_0) = \varphi_1(t_0) + \varphi_2(t_0) = 0$. Since $t_0 \in \mathbb{R}$ was arbitrary, we deduce that $\varphi = 0$, so $f = f_v$. Then, from (4.64), we have that

$$\|f\|_{O(\mathbb{R},X)} \leq L\|v\|_{I(\mathbb{R},X)}. \tag{4.72}$$

Taking into account that L does not depend on $\theta \in \Theta$ or on $v \in C_{0c}(\mathbb{R},X)$, we finally conclude that the pair $(O(\mathbb{R},X), I(\mathbb{R},X))$ is uniformly admissible for the system (E_π).

Sufficiency follows from (i). □

Corollary 4.8. *Let $\pi = (\Phi,\sigma)$ be a skew-product flow on $\mathcal{E} = X \times \Theta$ and let V be a Banach function space with $V \in \mathcal{T}(\mathbb{R})$. Then, the following assertions hold:*

(i) *if the pair $(V(\mathbb{R},X), V(\mathbb{R},X))$ is uniformly admissible for the system (E_π), then, π is exponentially dichotomic;*

(ii) *if* $V \in \mathcal{R}(\mathbb{R})$*, then,* π *is exponentially dichotomic if and only if the pair* $(V(\mathbb{R}, X), V(\mathbb{R}, X))$ *is uniformly admissible for the system* (E_π)*.*

Proof. We prove that either $V \in \mathcal{Q}(\mathbb{R})$ or $V \in \mathcal{L}(\mathbb{R})$. Indeed, suppose by contrary that $V \notin \mathcal{Q}(\mathbb{R})$ and $V \notin \mathcal{L}(\mathbb{R})$. Then, $M := \sup_{t>0} F_V(t) < \infty$ and $V \subset L^1(\mathbb{R}, \mathbb{R})$. From $V \subset L^1(\mathbb{R}, \mathbb{R})$, it follows that there is $\gamma > 0$ such that

$$\|v\|_1 \leq \gamma |v|_V, \quad \forall v \in V. \tag{4.73}$$

In particular, from $v = \chi_{[0,t)}$ in relation (4.73), we deduce that

$$t \leq \gamma \left|\chi_{[0,t)}\right|_V = \gamma F_V(t) \leq \gamma M, \quad \forall t > 0, \tag{4.74}$$

which is absurd. This shows that the assumption is false, which shows that either $V \in \mathcal{Q}(\mathbb{R})$ or $V \in \mathcal{L}(\mathbb{R})$. By applying Theorem 4.7, we obtain the conclusion. □

5. Applications and Conclusions

We have seen in the previous section that in the study of the exponential dichotomy of variational equations the classes $\mathcal{Q}(\mathbb{R})$ and, respectively, $\mathcal{L}(\mathbb{R})$ have a crucial role in the identification of the appropriate function spaces in the admissible pair. Moreover, it was also important to point out that it is sufficient to impose conditions either on the input space or on the output space. In this context, the natural question arises if these conditions are indeed necessary and whether our hypotheses may be dropped. The aim of this section is to answer this question. With this purpose, we will present an illustrative example of uniform admissibility, and we will discuss the concrete implications concerning the existence of the exponential dichotomy.

Let X be a Banach space. We denote by $C_0(\mathbb{R}, X)$ the space of all continuous functions $u : \mathbb{R} \to X$ with $\lim_{t\to\infty} u(t) = \lim_{t\to-\infty} u(t) = 0$, which is a Banach space with respect to the norm

$$|||u||| := \sup_{t\in\mathbb{R}} \|u(t)\|. \tag{5.1}$$

We start with a technical lemma.

Lemma 5.1. *If* O *is a Banach function space with* $O \in \mathcal{T}(\mathbb{R}) \setminus \mathcal{Q}(\mathbb{R})$*, then,* $C_0(\mathbb{R}, \mathbb{R}) \subset O$*.*

Proof. Let $c := \sup_{t>0} F_O(t)$. Let $u \in C_0(\mathbb{R}, \mathbb{R})$. Then, there is an unbounded increasing sequence $(t_n) \subset (0, \infty)$ such that $|u(t)| \leq 1/(n+1)$, for all $|t| \geq t_n$ and all $n \in \mathbb{N}$. Setting $u_n = u\chi_{[-t_n,t_n)}$ we have that

$$\left|u_{n+p} - u_n\right|_O \leq \frac{\left|\chi_{[-t_{n+p},-t_n)}\right|_O}{n+1} + \frac{\left|\chi_{[t_n,t_{n+p})}\right|_O}{n+1} \leq \frac{2c}{n+1}, \quad \forall n \in \mathbb{N}, \ \forall p \in \mathbb{N}^*. \tag{5.2}$$

From relation (5.2), it follows that the sequence (u_n) is fundamental in O, so this is convergent, that is, there exists $v \in O$ such that $u_n \to v$ in O. According to Remark 2.4(ii), there exists a subsequence (u_{k_n}) such that $u_{k_n}(t) \to v(t)$ for a.e. $t \in \mathbb{R}$. This implies that $v(t) = u(t)$ for a.e. $t \in \mathbb{R}$, so $v = u$ in O. In conclusion, $u \in O$, and the proof is complete. □

In what follows, we present a concrete situation which illustrates the relevance of the hypotheses on the underlying function spaces considered in the admissible pair, for the study of the dichotomous behavior of skew-product flows.

Example 5.2. Let $X = \mathbb{R} \times \mathbb{R}$ which is a Banach space with respect to the norm $\|(x_1, x_2)\| = |x_1| + |x_2|$. Let $\Theta = \mathbb{R}$ and let $\sigma : \Theta \times \mathbb{R} \to \Theta$, $\sigma(\theta, t) = \theta + t$. We have that σ is a flow on Θ. Let

$$\varphi : \mathbb{R} \longrightarrow (0, \infty), \qquad \varphi(t) = \begin{cases} \dfrac{2}{t+1}, & t \geq 0, \\ 1 + e^{-t}, & t < 0. \end{cases} \tag{5.3}$$

For every $(\theta, t) \in \Theta \times \mathbb{R}_+$, we consider the operator

$$\Phi(\theta, t) : X \longrightarrow X, \quad \Phi(\theta, t)(x_1, x_2) = \left(\frac{\varphi(\theta + t)}{\varphi(\theta)} x_1, e^t x_2 \right). \tag{5.4}$$

It is easy to see that $\pi = (\Phi, \sigma)$ is a skew-product flow.

Now, let O, I be two Banach function spaces with $O, I \in \mathcal{T}(\mathbb{R})$ such that $O \notin \mathcal{Q}(\mathbb{R})$ and $I \notin \mathcal{L}(\mathbb{R})$. It follows that $I \subset L^1(\mathbb{R}, \mathbb{R})$, and, using Lemma 5.1, we obtain that $C_0(\mathbb{R}, \mathbb{R}) \subset O$. Then, there are $\alpha, \beta > 0$ such that

$$\begin{gathered} \|u\|_1 \leq \alpha |u|_I, \quad \forall u \in I, \\ \|u\|_O \leq \beta |||u|||, \quad \forall u \in C_0(\mathbb{R}, \mathbb{R}). \end{gathered} \tag{5.5}$$

Step 1. We prove that the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Let $\theta \in \Theta$ and let $v = (v_1, v_2) \in C_{0c}(\mathbb{R}, X)$ and let $h > 0$ be such that $\operatorname{supp} v \subset (0, h)$. We consider the function $f : \mathbb{R} \to X$ where $f = (f_1, f_2)$ and

$$f_1(t) = \int_{-\infty}^{t} \frac{\varphi(\theta + t)}{\varphi(\theta + \tau)} v_1(\tau) d\tau, \qquad f_2(t) = -\int_{t}^{\infty} e^{-(\tau - t)} v_2(\tau) d\tau, \quad \forall t \in \mathbb{R}. \tag{5.6}$$

We have that f is continuous and an easy computation shows that the pair (f, v) satisfies (E_θ). Since $\operatorname{supp} v \subset (0, h)$, we obtain that $f_1(t) = 0$, for all $t \leq 0$ and $f_2(t) = 0$, for all $t \geq h$. From

$$f_1(t) = \varphi(\theta + t) \int_0^h \frac{v_1(\tau)}{\varphi(\theta + \tau)} d\tau, \qquad \forall t \geq h, \tag{5.7}$$

we have that $\lim_{t \to \infty} f_1(t) = 0$. In addition, from

$$f_2(t) = -e^t \int_0^h e^{-\tau} v_2(\tau) d\tau, \quad \forall t \leq 0, \tag{5.8}$$

we deduce that $\lim_{t\to-\infty} f_2(t) = 0$. Thus, we obtain that $f \in \mathcal{C}_0(\mathbb{R}, X)$ so $f \in O(\mathbb{R}, X)$. Moreover, from

$$\begin{aligned} |f_1(t)| &\leq \int_{-\infty}^{t} |v_1(\tau)| d\tau \leq \|v_1\|_{L^1(\mathbb{R},\mathbb{R})}, \quad \forall t \in \mathbb{R}, \\ |f_2(t)| &\leq \int_{t}^{\infty} |v_2(\tau)| d\tau \leq \|v_2\|_{L^1(\mathbb{R},\mathbb{R})}, \quad \forall t \in \mathbb{R}, \end{aligned} \tag{5.9}$$

it follows that

$$|||f||| \leq \|v\|_{L^1(\mathbb{R},X)}. \tag{5.10}$$

From relations (5.5) and (5.10), we obtain that

$$\|f\|_{O(\mathbb{R},X)} \leq \alpha\beta \|v\|_{I(\mathbb{R},X)}. \tag{5.11}$$

Let $\tilde{f} \in O(\mathbb{R}, X)$ be such that the pair $(\tilde{f}, v)$ satisfies (E_θ) and let $g = \tilde{f} - f$. Then, $g \in O(\mathbb{R}, X)$ and $g(t) = \Phi(\sigma(\theta, s), t - s) g(s)$, for all $t \geq s$. More exactly, if $g = (g_1, g_2)$, then we have that

$$g_1(t) = \frac{\varphi(\theta + t)}{\varphi(\theta + s)} g_1(s), \quad \forall t \geq s, \tag{5.12}$$

$$g_2(t) = e^{t-s} g_2(s), \quad \forall t \geq s. \tag{5.13}$$

Since $g \in O(\mathbb{R}, X)$ from Remark 2.12, it follows that $g \in M^1(\mathbb{R}, X)$, so $g_1, g_2 \in M^1(\mathbb{R}, \mathbb{R})$.

Let $t_0 \in \mathbb{R}$. For every $s \leq t_0$ from relation (5.12), we have that

$$\frac{|g_1(t_0)|}{\varphi(\theta + t_0)} = \int_{s-1}^{s} \frac{|g_1(\tau)|}{\varphi(\theta + \tau)} d\tau \leq \frac{1}{\varphi(\theta + s)} \int_{s-1}^{s} |g_1(\tau)| d\tau \leq \frac{\|g_1\|_{M^1(\mathbb{R},\mathbb{R})}}{\varphi(\theta + s)}. \tag{5.14}$$

Since $\varphi(r) \to \infty$ as $r \to -\infty$, for $s \to -\infty$ in (5.14), we obtain that $g_1(t_0) = 0$. In addition, for every $t \geq t_0$ from relation (5.13) we have that

$$e^{-t_0} |g_2(t_0)| = \int_{t}^{t+1} e^{-\tau} |g_2(\tau)| d\tau \leq e^{-t} \int_{t}^{t+1} |g_2(\tau)| d\tau \leq e^{-t} \|g_2\|_{M^1(\mathbb{R},\mathbb{R})}. \tag{5.15}$$

For $t \to \infty$ in (5.15) we deduce that $g_2(t_0) = 0$. So, we obtain that $g(t_0) = 0$. Taking into account that $t_0 \in \mathbb{R}$ was arbitrary it follows that $g = 0$. This implies that $\tilde{f} = f$. Then, from relation (5.11) we have that

$$\left\|\tilde{f}\right\|_{O(\mathbb{R},X)} \leq \alpha\beta \|v\|_{I(\mathbb{R},X)}. \tag{5.16}$$

We set $L = \alpha\beta$, and, taking into account that L does not depend on θ or v, we conclude that the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).

Step 2. We prove that π is not exponentially dichotomic. Suppose by contrary that π is exponentially dichotomic with respect to the family of projections $\{P(\theta)\}_{\theta\in\Theta}$ and let $K, \nu > 0$ be two constants given by Definition 4.1. In this case, according to Proposition 2.1 from [18] we have that

$$\operatorname{Im} P(\theta) = \{x \in X : \Phi(\theta, t)x \longrightarrow 0 \text{ as } t \longrightarrow \infty\}, \quad \forall\theta \in \Theta. \tag{5.17}$$

This characterization implies that $\operatorname{Im} P(\theta) = \mathbb{R} \times \{0\}$, for all $\theta \in \Theta$. Then, from

$$\|\Phi(\theta, t)x\| \le Ke^{-\nu t}\|x\|, \quad \forall t \ge 0,\ \forall x \in \operatorname{Im} P(\theta),\ \forall\theta \in \Theta, \tag{5.18}$$

we obtain that

$$\frac{\varphi(\theta + t)}{\varphi(\theta)}|x_1| \le Ke^{-\nu t}|x_1|, \quad \forall x_1 \in \mathbb{R},\ \forall t \ge 0,\ \forall\theta \in \Theta, \tag{5.19}$$

which shows that

$$\frac{\varphi(\theta + t)}{\varphi(\theta)} \le Ke^{-\nu t}, \quad \forall t \ge 0,\ \forall\theta \in \Theta. \tag{5.20}$$

In particular, for $\theta = 0$, from (5.20), we have that

$$\frac{1}{t + 1} \le Ke^{-\nu t}, \quad \forall t \ge 0, \tag{5.21}$$

which is absurd. This shows that the assumption is false, so π is not exponentially dichotomic.

Remark 5.3. The above example shows that if I, O are two Banach function spaces from the class $\mathcal{T}(\mathbb{R})$ such that $O \notin \mathcal{Q}(\mathbb{R})$ and $I \notin \mathcal{L}(\mathbb{R})$, then the uniform admissibility of the pair $(O(\mathbb{R}, X), I(\mathbb{R}, X))$ for the system (E_π) *does not imply* the existence of the exponential dichotomy of π. This shows that the hypotheses of the main result from the previous section are indeed necessary and emphasizes the fact that in the study of the exponential dichotomy in terms of the uniform admissibility at least one of the output space or the input space should belong to, respectively, $\mathcal{Q}(\mathbb{R})$ or $\mathcal{L}(\mathbb{R})$.

Finally, we complete our study with several consequences of the main result, which will point out some interesting conclusions for some usual classes of spaces often used in control-type problems arising in qualitative theory of dynamical systems. We will also show that, in our approach, the input space can be successively minimized, and we will discuss several optimization directions concerning the admissibility-type techniques.

Remark 5.4. The input-output characterizations for the asymptotic properties of systems have a wider applicability area if the input space is as small as possible and the output space is very general. In our main result, given by Theorem 4.7, the input functions belong to the space $C_{0c}(\mathbb{R}, X)$ while the output space is a general Banach function space. By analyzing condition (ii) from Definition 3.6, we observe that the input-output characterization given by Theorem 4.7 becomes more flexible and provides a more competitive applicability spectrum when the norm on the input space is larger.

Another interesting aspect that must be noted is that the class $\mathcal{T}(\mathbb{R})$ is closed to finite intersections. Indeed, if $I_1, \ldots, I_n \in \mathcal{T}(\mathbb{R})$, then we may define $I := I_1 \cap I_2 \cap \cdots I_n$ with respect to the norm

$$|u|_I := \max\{|u|_{I_1}, |u|_{I_2}, \ldots, |u|_{I_n}\}, \tag{5.22}$$

which is a Banach function space which belongs to $\mathcal{T}(\mathbb{R})$. So, taking as input space a Banach function space which is obtained as an intersection of Banach function spaces from the class $\mathcal{T}(\mathbb{R})$ we will have a "larger" norm in our admissibility condition, and, thus the estimation will be more permissive and more general.

As a consequence of the aspects presented in the above remark we deduce the following corollaries.

Corollary 5.5. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. Let O_φ be an Orlicz space with $0 < \varphi(t) < \infty$, for all $t > 0$. Let $n \in \mathbb{N}^*$, let $O_{\varphi_1}, \ldots, O_{\varphi_n}$ be Orlicz spaces such that $\varphi_k(1) < \infty$, for all $k \in \{1, \ldots, n\}$ and let $I := O_{\varphi_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap O_{\varphi_n}(\mathbb{R}, \mathbb{R}) \cap O_\varphi(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(O_\varphi(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. From Lemma 2.15 and Remark 2.20, it follows that $O_\varphi \in \mathcal{Q}(\mathbb{R}) \cap \mathcal{R}(\mathbb{R})$. By applying Theorem 4.7, the proof is complete. □

Corollary 5.6. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$ and let $p \in [1, \infty)$. Let $n \in \mathbb{N}^*, q_1, \ldots, q_n \in [1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}, X), I(\mathbb{R}, X))$ is admissible for the system (E_π).*

Proof. This follows from Corollary 5.5. □

Corollary 5.7. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$ and let $p \in (1, \infty]$. Let $n \in \mathbb{N}^*, q_1, \ldots, q_n \in (1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R}) \cap L^p(\mathbb{R}, \mathbb{R})$. Then π is exponentially dichotomic if and only if the pair $(L^p(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. This follows from Theorem 4.7 by observing that $I \in \mathcal{L}(\mathbb{R})$. □

Remark 5.8. According to Remark 2.12, the largest space from the class $\mathcal{T}(\mathbb{R})$ is $M^1(\mathbb{R}, \mathbb{R})$. Thus, considering the output space $M^1(\mathbb{R}, \mathbb{R})$, in order to obtain optimal input-output characterizations for exponential dichotomy in terms of admissibility, it is sufficient to work with smaller and smaller input spaces.

Corollary 5.9. *Let $\pi = (\Phi, \sigma)$ be a skew-product flow on $X \times \Theta$. Let $n \in \mathbb{N}^*$, $q_1, \ldots, q_n \in (1, \infty]$ and $I = L^{q_1}(\mathbb{R}, \mathbb{R}) \cap \cdots \cap L^{q_n}(\mathbb{R}, \mathbb{R})$. Then, π is exponentially dichotomic if and only if the pair $(M^1(\mathbb{R}, X), I(\mathbb{R}, X))$ is uniformly admissible for the system (E_π).*

Proof. We observe that $I \in \mathcal{L}(\mathbb{R})$, and, from Remark 2.12, we have that $I \subset M^1(\mathbb{R}, \mathbb{R})$. By applying Theorem 4.7, we obtain the conclusion. □

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Research Article

Two-Parametric Conditionally Oscillatory Half-Linear Differential Equations

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We study perturbations of the nonoscillatory half-linear differential equation $(r(t)\Phi(x'))' + c(t)\Phi(x) = 0$, $\Phi(x) := |x|^{p-2}x$, $p > 1$. We find explicit formulas for the functions $\hat{r}$, $\hat{c}$ such that the equation $[(r(t) + \lambda\hat{r}(t))\Phi(x')]' + [c(t) + \mu\hat{c}(t)]\Phi(x) = 0$ is conditionally oscillatory, that is, there exists a constant γ such that the previous equation is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$. The obtained results extend the previous results concerning two-parametric perturbations of the half-linear Euler differential equation.

1. Introduction

Conditionally oscillatory equations play an important role in the oscillation theory of the Sturm-Liouville second-order differential equation

$$\left(r(t)x'\right)' + c(t)x = 0, \tag{1.1}$$

with positive continuous functions r, c. Equation (1.1) with λc instead of c is said to be *conditionally oscillatory* if there exists $\lambda_0 > 0$, the so-called *oscillation constant* of (1.1), such that this equation is oscillatory for $\lambda > \lambda_0$ and nonoscillatory for $\lambda < \lambda_0$. A typical example of a conditionally oscillatory equation is the Euler differential equation

$$x'' + \frac{\lambda}{t^2}x = 0, \tag{1.2}$$

which has the oscillation constant $\lambda_0 = 1/4$ as can be verified by a direct computation when looking for solutions of (1.2) in the form $x(t) = t^{\alpha}$. This leads to the classical Kneser (non)oscillation criterion which states that (1.1) with $r(t) \equiv 1$ is oscillatory provided

$$\liminf_{t\to\infty} t^2 c(t) > \frac{1}{4}, \tag{1.3}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^2 c(t) < \frac{1}{4}. \tag{1.4}$$

This shows that the potential $c(t) = t^{-2}$ is the border line between oscillation and nonoscillation. Note that the concept of conditional oscillation of (1.1) was introduced in [1].

The linear oscillation theory extends almost verbatim to the half-linear differential equation

$$\left(r(t)\Phi\left(x'\right)\right)' + c(t)\Phi(x) = 0, \quad \Phi(x) := |x|^{p-2}x, \ p > 1, \tag{1.5}$$

including the definition of conditional oscillation. The half-linear version of Euler equation (1.2) is the equation

$$\left(\Phi\left(x'\right)\right)' + \frac{\lambda}{t^p}\Phi(x) = 0, \tag{1.6}$$

which has the oscillation constant $\lambda_0 = \gamma_p := ((p-1)/p)^p$, and (non)oscillation criteria (1.3), (1.4) extend in a natural way to (1.5) with $r(t) \equiv 1$. A complementary concept to the conditional oscillation is the concept of strong (non)oscillation. Equation (1.5) with λc instead of c is said to be *strongly (non)oscillatory* if it is (non)oscillatory for every $\lambda > 0$. Sometimes, strongly oscillatory equations are regarded as conditionally oscillatory with the oscillation constant $\lambda_0 = 0$ and strongly nonoscillatory as conditionally oscillatory with the oscillation constant $\lambda_0 = \infty$. We refer to [2] for results along this line.

In our paper, we are motivated by a statement presented in [3, 4], where the two-parametric perturbation of the Euler differential equation with the critical coefficient

$$\left(\Phi\left(x'\right)\right)' + \frac{\gamma_p}{t^p}\Phi(x) = 0 \tag{1.7}$$

is investigated. It is shown there that the equation

$$\left[\left(1 + \frac{\lambda}{\log^2 t}\right)\Phi\left(x'\right)\right]' + \left[\frac{\gamma_p}{t^p} + \frac{\mu}{t^p \log^2 t}\right]\Phi(x) = 0 \tag{1.8}$$

is oscillatory if $\mu - \gamma_p \lambda > \mu_p := (1/2)((p-1)/p)^{p-1}$ and nonoscillatory in the opposite case. Note that an important role in proving the results of [4] is played by the fact that we know explicitly the solution $h(t) = t^{(p-1)/p}$ of (1.7).

Here, we treat the problem of conditional oscillation in the following general setting. We suppose that (1.5) is nonoscillatory and that h is its eventually positive solution. We find explicit formulas for the functions $\hat{r}$, $\hat{c}$ such that the equation

$$\left[(r(t) + \lambda \hat{r}(t))\Phi(x')\right]' + \left[c(t) + \mu \hat{c}(t)\right]\Phi(x) = 0 \tag{1.9}$$

is conditionally oscillatory, that is, there exists a constant γ such that (1.9) is oscillatory if $\mu - \lambda > \gamma$ and nonoscillatory if $\mu - \lambda < \gamma$.

The setup of the paper is as follows. In the next section, we present some statements of the half-linear oscillation theory. Section 3 is devoted to the so-called modified Riccati equation associated with (1.5) and (1.9). The main result of the paper, the construction of the functions $\hat{r}$, $\hat{c}$ such that (1.9) is two-parametric conditionally oscillatory, is presented in Section 4.

2. Auxiliary Results

As we have already mentioned in the previous section, the linear oscillation theory extends almost verbatim to half-linear equation (1.5). The word "almost" reflects the fact that not all linear methods can be extended to (1.5), some results for (1.5) are the same as those for (1.1), but to prove them, one has to use different methods than in the linear case. A typical method of this kind is the following transformation formula. If $f(t) \neq 0$ is a sufficiently smooth function and functions x, y are related by the formula $x = f(t)y$, then we have the identity

$$f(t)\left[\left(r(t)x'\right)' + c(t)x\right] = \left(R(t)y'\right)' + C(t)y, \tag{2.1}$$

where

$$R(t) = r(t)f^2(t), \qquad C(t) = f(t)\left[\left(r(t)f'(t)\right)' + c(t)f(t)\right]. \tag{2.2}$$

In particular, x is a solution of (1.1) if and only if y is a solution of the equation $(Ry')' + Cy = 0$. The transformation identity (2.1) *does not extend* to (1.5).

To illustrate the meaning of this fact in the conditional oscillation of (1.1) and (1.5), suppose that (1.1) is nonoscillatory and let h be its so-called *principal solution* (see [5, Chapter XI]), that is, a solution such that $\int^\infty r^{-1}(t)h^{-2}(t)\mathrm{d}t = \infty$. We would like to find a function $\hat{c}$ such that the equation

$$\left(r(t)x'\right)' + \left(c(t) + \mu \hat{c}(t)\right)x = 0 \tag{2.3}$$

is conditionally oscillatory and to find its oscillation constant. The transformation $x = h(t)y$ transforms (1.1) into the one term equation $\left(r(t)h^2(t)y'\right)' = 0$ and the transformation of independent variable $s = \int^t r^{-1}(\tau)h^{-2}(\tau)\mathrm{d}\tau$ further to the equation $\mathrm{d}^2 y/\mathrm{d}s^2 = 0$. Now, from (1.2), we know that the "right" perturbation term in the last equation is $1/s^2$ with the oscillation constant $1/4$. Substituting back for s, we get the conditionally oscillatory equation

$$(R(t)y')' + \frac{\mu}{R(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^2} y = 0, \quad R(t) = r(t)h^2(t), \tag{2.4}$$

and the back transformation $y = h^{-1}(t)x$ results in the conditionally oscillatory equation

$$(r(t)x')' + \left[c(t) + \frac{\mu}{h^2(t)R(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^2}\right] x = 0, \tag{2.5}$$

with the oscillation constant $\mu_0 = 1/4$. The previous result is one of the main statements of [6], but it was proved there by a different method.

In the next section, we will show how to modify this method to be applicable to half-linear equations. At this moment, we present the result of [7] with the classical (i.e., one parametric) conditional oscillation of (1.5). Let h be a positive solution of (1.5) such that $h'(t) \neq 0$ for large t. We denote

$$R(t) := r(t)h^2(t)|h'(t)|^{p-2}, \qquad G(t) := r(t)h(t)\Phi(h'(t)), \tag{2.6}$$

$$\hat{c}(t) = \frac{1}{|h(t)|^p R(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^2}. \tag{2.7}$$

Theorem 2.1. *Suppose that* (1.5) *possesses a nonoscillatory solution h such that $h'(t) \neq 0$ for large t, and R, G are given by* (2.6)*. If*

$$\int^\infty \frac{\mathrm{d}t}{R(t)} = \infty, \qquad \liminf_{t\to\infty} |G(t)| > 0, \tag{2.8}$$

then the equation

$$(r(t)\Phi(x'))' + [c(t) + \mu\hat{c}(t)]\Phi(x) = 0 \tag{2.9}$$

is conditionally oscillatory, and its oscillation constant is $\mu_0 = 1/2q$, where q is the conjugate exponent to p, that is, $1/p + 1/q = 1$.

Note that in the linear case $p = 2$, the function $f(t) = h(t)\sqrt{\int^t r^{-1}(\tau)h^{-2}(\tau)\mathrm{d}\tau}$ is a solution of (2.9) with $\mu = \mu_0 = 1/4$. In the general half-linear case, an explicit solution of (2.9) is no longer known, but we are able to "estimate" this solution. The next statement, which is also taken from [7], presents a result along this line.

Theorem 2.2. *Suppose that* (2.8) *holds and let* $f(t) = h(t)(\int^t R^{-1}(s)\mathrm{d}s)^{1/p}$, *then a solution of* (2.9) *with* $\mu = 1/2q$ *is of the form*

$$x(t) = f(t)\left(1 + O\left(\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{-1}\right)\right), \tag{2.10}$$

and (suppressing the argument t)

$$\begin{aligned}
&f\left[(r\Phi(f'))' + \left(c + \frac{1}{2qh^p R\left(\int^t R^{-1}\right)^2}\right)\Phi(f)\right] \\
&\quad = -\frac{(p-1)(p-2)G'}{G^2\left(\int^t R^{-1}\right)} - \frac{(p-1)(p-2)}{3p^3G^3\left(\int^t R^{-1}\right)^2}\left[(p-3)G' + 2pr\left|h'\right|^p\right] \\
&\qquad + O\left(G^{-3}\left(\int^t R^{-1}\right)^{-3}\right)\left[\frac{G'}{pG^2} - \frac{(p^3-4p^2+11p-6)h'}{2p^3h} - \frac{1}{qR\left(\int^t R^{-1}\right)}\right],
\end{aligned} \tag{2.11}$$

as $t \to \infty$.

The last statement presented in this section is the so-called *reciprocity principle*. Let x be a solution of (1.5) and let $u := r\Phi(x')$ be its *quasiderivative*, then u is a solution of the reciprocal equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + r^{1-q}(t)\Phi^{-1}(u) = 0, \tag{2.12}$$

where $\Phi^{-1}(u) = |u|^{q-2}u$ is the inverse function of Φ.

3. Modified Riccati Equation

Suppose that λ and $\hat{r}$ in (1.9) are such that $r(t) + \lambda\hat{r}(t) > 0$. Let $x(t) \neq 0$ in an interval I be a solution of (1.9), and let $w = (r + \lambda\hat{r})\Phi(x'/x)$. Then, w solves in I the "standard" Riccati equation

$$w' + c(t) + \mu\hat{c}(t) + (p-1)\left[r(t) + \lambda\hat{r}(t)\right]^{1-q}|w|^q = 0. \tag{3.1}$$

More precisely, the following statement holds.

Lemma 3.1 ([8, Theorem 2.2.1]). *The following statements are equivalent:*

(i) *equation* (1.9) *is nonoscillatory;*

(ii) *equation* (3.1) *has a solution on an interval* $[T, \infty)$*;*

(iii) *there exists a continuously differentiable function w such that*

$$w' + c(t) + \mu\hat{c}(t) + (p-1)[r(t) + \lambda\hat{r}(t)]^{1-q}(t)|w|^q \leq 0 \tag{3.2}$$

on an interval $[T, \infty)$.

In the linear case, if x is a solution of (1.1), $x = f(t)y$, and $v = rf^2y'/y$ is the Riccati variable corresponding to the equation on the right-hand side in (2.1), then $v = f^2(w - w_f)$ where $w = rx'/x$, $w_f = rf'/f$. This suggests to investigate the function $v = f^p(w - w_f)$ in the half-linear case, and this leads to the *modified Riccati equation* introduced in the next statement which is taken from [4] with a modification from [3].

Lemma 3.2. *Suppose that f is a positive differentiable function, $w_f = (r + \lambda\hat{r})\Phi(f'/f)$, and w is a continuously differentiable function, and put $v = f^p(w - w_f)$, then the following identity holds:*

$$\begin{aligned} &f^p(t)\left[w' + c(t) + \mu\hat{c}(t) + (p-1)(r(t) + \lambda\hat{r}(t))^{1-q}|w|^q\right] \\ &\quad = v' + f(t)\left[\ell(f(t)) + \hat{\ell}(f(t))\right] + (p-1)(r(t) + \lambda\hat{r}(t))^{1-q} f^{-q}(t)\mathcal{G}(t,v), \end{aligned} \tag{3.3}$$

where

$$\ell(f) = (r(t)\Phi(f'))' + c(t)\Phi(f), \qquad \hat{\ell}(f) = \lambda(\hat{r}(t)\Phi(f'))' + \mu\hat{c}(t)\Phi(f), \tag{3.4}$$

$$\mathcal{G}(t,v) = |v + \Omega(t)|^q - q\Phi^{-1}(\Omega(t))v - |\Omega(t)|^q, \quad \Omega := (r + \lambda\hat{r})f\Phi(f'). \tag{3.5}$$

In particular, if w is a solution of (3.1)*, then v is a solution of the modified Riccati equation*

$$v' + f(t)\left[\ell(f(t)) + \hat{\ell}(f(t))\right] + (p-1)(r(t) + \lambda\hat{r}(t))^{1-q} f^{-q}(t)\mathcal{G}(t,v) = 0. \tag{3.6}$$

Conversely, if v is a solution of (3.6)*, then $w = w_f + f^{-p}v$ is a solution of* (3.1)*.*

Observe that in case $f \equiv 1$, the modified Riccati equation (3.6) reduces to the standard Riccati equation (3.1).

Next, we will investigate the function $\mathcal{G}$ in (3.5). First, we present a result from [4, Lemmas 5 and 6].

Lemma 3.3. *The function $\mathcal{G}$ defined in* (3.5) *has the following properties.*

(i) *$\mathcal{G}(t,v) \geq 0$ with the equality if and only if $v = 0$.*

(ii) *If $q \geq 2$, one has the inequality*

$$\mathcal{G}(t,v) \geq \frac{q}{2}|\Omega(t)|^{q-2}v^2. \tag{3.7}$$

Now, we concentrate on an estimate of the function $\mathcal{G}$ in case $q < 2$.

Lemma 3.4. *Suppose that* $q < 2$ *and* $\lim_{t\to\infty}|\Omega(t)| = \infty$, *then there is a constant* $\beta > 0$ *such that for* $v \in (-\infty, -v_0]$, $v_0 > 0$, *and large* t

$$\mathcal{G}(t,v) \geq \beta|\Omega(t)|^{q-2}|v|^q. \tag{3.8}$$

Proof. Consider the function

$$\mathcal{H}(t,v) = \begin{cases} \dfrac{\mathcal{G}(t,v)}{|v|^q}, & \text{for } v \neq 0, \\ 0, & \text{for } v = 0. \end{cases} \tag{3.9}$$

First of all,

$$\lim_{v\to\pm\infty}\mathcal{H}(t,v) = 1, \qquad \lim_{v\to 0}\mathcal{H}(t,v) = 0. \tag{3.10}$$

Now, we compute local extrema of $\mathcal{H}$ with respect to v. We have (suppressing the argument t)

$$\begin{aligned} \mathcal{H}_v &= \frac{1}{|v|^{2q}}\left\{\left[q\Phi^{-1}(v+\Omega) - q\Phi^{-1}(\Omega)\right]|v|^q - q\Phi^{-1}(v)\left[|v+\Omega|^q - q\Phi^{-1}(\Omega)v - |\Omega|^q\right]\right\} \\ &= \frac{q}{v^2\Phi^{-1}(v)}\left\{v\Phi^{-1}(v+\Omega) - v\Phi^{-1}(\Omega) - |v+\Omega|^q + q\Phi^{-1}(\Omega)v + |\Omega|^q\right\} \\ &= \frac{q}{v^2\Phi^{-1}(v)}\left\{-\Omega\Phi^{-1}(v+\Omega) + (q-1)\Phi^{-1}(\Omega)v + |\Omega|^q\right\}. \end{aligned} \tag{3.11}$$

Denote $\mathcal{N}(v)$ the function in braces on the last line of the previous computation. We have $\mathcal{N}(0) = 0$,

$$\begin{aligned} \mathcal{N}'(v) &= -(q-1)\Omega|v+\Omega|^{q-2} + (q-1)\Phi^{-1}(\Omega) \\ &= (q-1)\Omega\left[-|v+\Omega|^{q-2} + |\Omega|^{q-2}\right] \\ &= 0 \end{aligned} \tag{3.12}$$

if and only if $v = 0$ and $v = -2\Omega$, and

$$\mathcal{N}''(v) = -(q-1)(q-2)\Omega|v+\Omega|^{q-3}\operatorname{sgn}(v+\Omega). \tag{3.13}$$

This means that $v = 0$ is the local minimum and $v = -2\Omega$ is the local maximum of the function $\mathcal{N}$. Using this result, an examination of the graph of the function $\mathcal{H}$ shows that this function has the local minimum at $v = 0$ and a local maximum in the interval $(-\infty, -2\Omega)$ if $\Omega > 0$, and this maximum is in $(-2\Omega, \infty)$ if $\Omega < 0$. Next, denote v^* the value for which $\mathscr{H}(t, v^*) = 1$. Consequently, for any $v_0 > 0$, it follows from (3.10) that

$$\inf_{v\in(-\infty,-v_0]} \mathscr{H}(t,v) = \mathscr{H}(t,-\overline{v}) = \frac{1}{|\overline{v}|^q}\left[|\Omega-\overline{v}|^q + q\overline{v}\Phi^{-1}(\Omega) - |\Omega|^q\right], \tag{3.14}$$

where

$$\overline{v} = \begin{cases} -v^* & \text{if } -v_0 < v^* < 0, \\ v_0 & \text{otherwise.} \end{cases} \tag{3.15}$$

Next, we want to investigate the dependence of this infimum on Ω when $|\Omega| \to \infty$. To this end, we investigate the function $F(x) = |x-a|^q + qa\Phi^{-1}(x) - |x|^q$ for $x \to \pm\infty$, $a \in \mathbb{R}$ being a parameter. We have (using the expansion formula for $(1+x)^\alpha$)

$$\begin{aligned} F(x) &= \Phi^{-1}(x)\left\{\frac{|x-a|^q - |x|^q}{\Phi^{-1}(x)} + qa\right\} = \Phi^{-1}(x)\left\{x\left[\left(1-\frac{a}{x}\right)^q - 1\right] + qa\right\} \\ &= \Phi^{-1}(x)\left(\binom{q}{2}\frac{a^2}{x} + o\left(x^{-1}\right)\right) = a^2\binom{q}{2}|x|^{q-2}(1+o(1)), \end{aligned} \tag{3.16}$$

as $|x| \to \infty$. Consequently, if $\lim_{t\to\infty}|\Omega(t)| = \infty$, there exists a constant $\beta > 0$ such that (3.8) holds. □

Now, we are ready to formulate a complement of [9, Theorem 2] which is presented in that paper under the assumption that the function Ω is bounded.

Theorem 3.5. *Let f be a positive continuously differentiable function such that $f'(t) \neq 0$ for large t. Suppose that $\int^\infty \mathcal{R}^{-1}(t)\mathrm{d}t = \infty$, where $\mathcal{R} = (r+\lambda\hat{r})f^2|f'|^{p-2}$, $C(t) \geq 0$ for large t, and $\lim_{t\to\infty}|\Omega(t)| = \infty$, then all possible proper solutions (i.e., solutions which exist on some interval of the form $[T,\infty)$) of the equation*

$$v' + C(t) + (p-1)(r(t) + \lambda\hat{r}(t))^{1-q} f^{-q}(t)\mathcal{G}(t,v) = 0 \tag{3.17}$$

are nonnegative.

Proof. First consider the case $q < 2$. Let $v_0 > 0$ be arbitrary. By Lemma 3.4, there exists $T_0 \in \mathbb{R}$ and $\beta > 0$ such that for $t \geq T_0$ and $v \in (-\infty, -v_0]$,

$$(p-1)(r+\lambda\hat{r})^{1-q}f^{-q}\mathcal{G}(t,v) \geq \beta(p-1)(r+\lambda\hat{r})^{1-q}f^{-q}|\Omega|^{q-2}|v|^q = (p-1)\beta\frac{|v|^q}{\mathcal{R}}. \tag{3.18}$$

Suppose that v is the solution of (3.17) such that $v(t_0) = -v_0$ for some $t_0 \geq T_0$, then

$$v' + C(t) + (p-1)\beta\frac{|v|^q}{\mathcal{R}(t)} \leq 0, \tag{3.19}$$

for $t \geq t_0$ for which the solution v exists. Now, we use the same argument as in the proof of Theorem 2 in [9]. Consider the equation

$$z' + C(t) + (p-1)\beta \frac{|z|^q}{\mathcal{R}(t)} = 0. \tag{3.20}$$

This is the standard Riccati equation corresponding to the half-linear equation

$$\left(\mathcal{R}^{p-1}(t)\Phi(x')\right)' + \beta^{p-1}C(t)\Phi(x) = 0. \tag{3.21}$$

Assumptions of theorem imply, by [8, Corollary 4.2.1], that all proper solutions of (3.20) are nonnegative. It means that any solution of (3.20) which starts with a negative initial condition blows down to $-\infty$ in a finite time. Inequality (3.19) implies that if z is the solution of (3.20) satisfying $z(t_0) = v(t_0) = -v_0$, that is, z starts with the same initial value as the solution v of (3.17), then v decreases faster than z. In particular, if z blows down to $-\infty$ at a finite time, then v does as well. This means that all proper solutions of (3.17), if any, are nonnegative.

In case $q \geq 2$, we proceed in a similar way. We use (3.7) and we compare (3.17) with the equation

$$z' + C(t) + \frac{p}{2}\frac{z^2}{\mathcal{R}(t)} = 0, \tag{3.22}$$

which is the standard Riccati equation corresponding to the linear equation

$$(\mathcal{R}(t)x')' + \frac{p}{2}C(t)x = 0. \tag{3.23}$$

Then, reasoning in the same way as in case $q < 2$, we obtain the conclusion that all proper solutions of (3.17) are nonnegative also in this case. $\square$

4. Two-Parametric Conditional Oscillation

Recall that h is a positive solution of (1.5) such that $h'(t) \neq 0$ for large t, $g = r\Phi(h')$ is its quasiderivative, R, G are given by (2.6), and $\hat{c}$ is given by (2.7). Recall also that the quasiderivative g is a solution of the reciprocal equation (2.12), denote by

$$\widetilde{G} := c^{1-q} g \Phi^{-1}(g') = -rh\Phi(h'), \qquad \widetilde{R} := c^{1-q} g^2 |g'|^{q-2} = \frac{r^2|h'|^{2p-2}}{ch^{p-2}} \tag{4.1}$$

the "reciprocal" analogues of G and R, and define

$$\hat{r}(t) = \frac{1}{|h'(t)|^p \widetilde{R}(t)\left(\int^t \widetilde{R}^{-1}(s)\mathrm{d}s\right)^2}. \tag{4.2}$$

Our main result reads as follows.

Theorem 4.1. *Suppose that conditions* (2.8) *hold. Further, suppose that*

$$\lim_{t\to\infty}\frac{\hat{r}(t)}{r(t)}=0, \tag{4.3}$$

and that there exist limits

$$\lim_{t\to\infty}\frac{r(t)|h'(t)|^p}{c(t)h^p(t)}, \qquad \lim_{t\to\infty}\frac{\left(\hat{r}(t)\Phi\left(f'(t)\right)\right)'}{\hat{c}(t)\Phi\left(f(t)\right)}, \tag{4.4}$$

the second one being finite, where $f(t)=h(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{1/p}$. *If* $\mu-\lambda<1/2q$, *then* (1.9) *is nonoscillatory; if* $\mu-\lambda>1/2q$, *then it is oscillatory.*

Proof. First consider the case $\mu=0$ in (1.9), that is, we consider the equation

$$\left[(r(t)+\lambda\hat{r}(t))\Phi\left(x'\right)\right]'+c(t)\Phi(x)=0. \tag{4.5}$$

The quantities $\widetilde{G}$ and $\widetilde{R}$ defined in (4.1) satisfy

$$\begin{gathered}\widetilde{G}=-rh\Phi(h')=-G\,,\\ \widetilde{R}=\frac{r^2|h'|^{2p-2}}{ch^{p-2}}=-\frac{h(r\Phi(h'))^2}{(r\Phi(h'))'},\end{gathered} \tag{4.6}$$

hence, integrating by parts,

$$\begin{aligned}\int^t\widetilde{R}^{-1}(s)\mathrm{d}s&=-\int^t\frac{1}{h(s)}\frac{[r(s)\Phi(h'(s))]'}{[r(s)\Phi(h'(s))]^2}\mathrm{d}s\\ &=\frac{1}{h(t)r(t)\Phi(h'(t))}+\int^t\frac{h'(s)}{h^2(s)}\frac{1}{r(s)\Phi(h'(s))}\mathrm{d}s\\ &=\frac{1}{G(t)}+\int^t R^{-1}(s)\mathrm{d}s.\end{aligned} \tag{4.7}$$

Consequently, conditions (2.8) imply that corresponding conditions for $\widetilde{G}$ and $\widetilde{R}$ also hold. This means, in view of Theorem 2.1 (applied to the reciprocal equation (2.12)), that the equation

$$\left(c^{1-q}(t)\Phi^{-1}\left(u'\right)\right)'+\left[r^{1-q}(t)+\frac{\lambda}{|g(t)|^q\widetilde{R}(t)\left(\int^t\widetilde{R}^{-1}(s)\mathrm{d}s\right)^2}\right]\Phi^{-1}(u)=0 \tag{4.8}$$

is oscillatory for $\lambda>1/2p$ and nonoscillatory in the opposite case.

The reciprocal equation to (4.5) is the equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + (r(t)+\lambda\hat{r}(t))^{1-q}\Phi^{-1}(u) = 0. \tag{4.9}$$

Since (4.3) holds, we have

$$(r+\lambda\hat{r})^{1-q} = r^{1-q}\left(1+\frac{\lambda\hat{r}}{r}\right)^{1-q} = r^{1-q}\left(1+\frac{(1-q)\lambda\hat{r}}{r}+o\left(\frac{\hat{r}}{r}\right)\right), \tag{4.10}$$

as $t \to \infty$. Hence, we can rewrite (4.9) in the following form:

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + r^{1-q}(t)\left(1+\frac{(1-q)\lambda\hat{r}(t)}{r(t)}+o\left(\frac{\hat{r}(t)}{r(t)}\right)\right)\Phi^{-1}(u) = 0. \tag{4.11}$$

Let $\lambda > -1/2q$ what is equivalent to $\lambda(1-q) < 1/2p$, then, in view of (4.3), there exists $\tilde{\lambda}$ such that $\lambda(1-q) < \tilde{\lambda} < 1/2p$, hence, for large t,

$$r^{1-q}\left(1+\frac{\lambda(1-q)\hat{r}}{r}+o\left(\frac{\hat{r}}{r}\right)\right) < r^{1-q}\left(1+\frac{\tilde{\lambda}\hat{r}}{r}\right) = r^{1-q} + \frac{\tilde{\lambda}}{|g|^q\tilde{R}\left(\int^t \tilde{R}^{-1}(s)\mathrm{d}s\right)^2}. \tag{4.12}$$

This means that the equation

$$\left(c^{1-q}(t)\Phi^{-1}(u')\right)' + \left[r^{1-q}(t) + \frac{\tilde{\lambda}}{|g(t)|^q\tilde{R}(t)\left(\int^t \tilde{R}^{-1}(s)\mathrm{d}s\right)^2}\right]\Phi^{-1}(u) = 0 \tag{4.13}$$

is a majorant of (4.9) and this majorant is nonoscillatory by Theorem 2.1 applied to (4.8). So (4.9) is also nonoscillatory, and hence (4.5) is nonoscillatory as well. The same argument implies oscillation of (4.5) if $\lambda < -1/2q$.

Now, we turn our attention to the general case $\mu \neq 0$. Let $f := h(\int^t R^{-1}(s)\mathrm{d}s)^{1/p}$, and consider the term

$$f\left[\ell(f)+\hat{\ell}(f)\right] \tag{4.14}$$

appearing in the modified Riccati equation (3.6), where the operators $\ell, \hat{\ell}$ are defined by (3.4). In order to use the asymptotic formula from Theorem 2.2, we write $f[\ell(f)+\hat{\ell}(f)] = A+B$, where

$$\begin{gathered} A = f\left[(r\Phi(f'))' + \left(c+\frac{1}{2q}\hat{c}\right)\Phi(f)\right], \\ B = f\left[\lambda(\hat{r}\Phi(f'))' + \left(\mu-\frac{1}{2q}\right)\hat{c}\Phi(f)\right]. \end{gathered} \tag{4.15}$$

Let $L \in \mathbb{R}$ be the second limit in (4.4), that is,

$$(\widehat{r}\Phi(f'))' = L\widehat{c}\Phi(f)(1+o(1)) \quad \text{as } t \longrightarrow \infty. \tag{4.16}$$

The leading term in the expression A is const $G'G^{-2}(\int^t R^{-1}(s)\mathrm{d}s)^{-1}$ by Theorem 2.2, while, concerning the asymptotics of B,

$$B = f\widehat{c}\Phi(f)\left[L\lambda + \mu - \frac{1}{2q} + o(1)\right] = \frac{1}{R\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\left[L\lambda + \mu - \frac{1}{2q} + o(1)\right], \tag{4.17}$$

as $t \to \infty$. The existence of the first limit in (4.4) implies that there exists the limit

$$\begin{aligned}\lim_{t\to\infty}\frac{G'(t)G^{-2}(t)}{R^{-1}(t)} &= \lim_{t\to\infty}\frac{r(t)h^2(t)|h'(t)|^{p-2}\left(r(t)|h'(t)|^p - c(t)h^p(t)\right)}{(r(t)h(t)\Phi(h'(t)))^2}\\ &= 1 - \lim_{t\to\infty}\frac{c(t)h^p(t)}{r(t)|h'(t)|^p}.\end{aligned} \tag{4.18}$$

The limit in (4.18) must be 0, which follows from the l'Hospital rule and the fact that the integral of R^{-1} is divergent, while the integral of $G'G^{-2}$ is convergent by the second assumption in (2.8). This means that the term B dominates A; hence, $A(t) + B(t) > 0$ for large t if $L\lambda + \mu - 1/2q > 0$ and $A(t) + B(t) < 0$ for large t if $L\lambda + \mu - 1/2q < 0$.

Now, it remains to prove that these inequalities imply (non)oscillation of (1.9) and that $L = -1$.

To prove the nonoscillation, let $L\lambda + \mu - 1/2q < 0$, that is, $A(t) + B(t) < 0$ for large t, and let $\mathcal{G}$ be defined by (3.5). By Lemma 3.3(i) $v = 0$ is a solution of the inequality

$$v' + A(t) + B(t) + (p-1)(r(t) + \lambda\widehat{r}(t))^{1-q}f^{-q}(t)\mathcal{G}(t,v) \leq 0, \tag{4.19}$$

for large t, and by identity (3.3) in Lemma 3.2 we obtain that $w = (r + \lambda\widehat{r})\Phi(f'/f)$ satisfies the Riccati inequality (3.2), that is, (1.9) is nonoscillatory by Lemma 3.1(iii).

To prove the oscillation, let $L\lambda + \mu - 1/2q > 0$, that is, $A(t) + B(t) > 0$ for large t. Observe that for $t \to \infty$

$$\int^t f^p(s)\widehat{c}(s)\mathrm{d}s = \int^t \frac{1}{R(s)\left(\int^t R^{-1}(\tau)\mathrm{d}\tau\right)}\mathrm{d}s = \log\left(\int^t R^{-1}(s)\mathrm{d}s\right) \longrightarrow \infty, \tag{4.20}$$

and hence $\int^\infty B(t)\mathrm{d}t = \infty$, which consequently means that $\int^\infty (A(t) + B(t))\mathrm{d}t = \infty$. Here, we have used the fact that the integral of the leading term in A and also integrals of other terms in the asymptotic formula of Theorem 2.2 are convergent, see [7, page 161]. Suppose, on the contrary, that (1.9) is nonoscillatory. Then by Lemma 3.1, there exists a solution w of the associated Riccati equation (3.1) for large t and, by Lemma 3.2, the function $v = f^p(w - w_f)$, where $w_f = (r + \lambda\hat{r})\Phi(f'/f)$, is a solution of the modified Riccati equation (3.6) for large t. Integrating (3.6), we get

$$\begin{aligned} v(T) - v(t) = &\int_T^t (A(s) + B(s))\mathrm{d}s \\ &+ (p-1)\int_T^t (r(s) + \lambda\hat{r}(s))^{1-q} f^{-q}(s)\mathcal{G}(s, v(s))\mathrm{d}s. \end{aligned} \tag{4.21}$$

Now, we use Theorem 3.5. In view of (2.8) and (4.3), we have for $t \to \infty$,

$$\begin{aligned} |\Omega(t)| &= (r(t) + \lambda\hat{r}(t)) f(t)\left|\Phi(f'(t))\right| \\ &= r(t)(1 + o(1))h(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{1/p} \left|\Phi(h'(t))\right| \left(\int^t R^{-1}(s)\mathrm{d}s\right)^{(p-1)/p} \\ &\quad \times \left(1 + \frac{1}{pG(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\right)^{p-1} \\ &= |G(t)|\left(\int^t R^{-1}(s)\mathrm{d}s\right)(1 + o(1)) \longrightarrow \infty, \\ \mathcal{R}(t) &= (r(t) + \lambda\hat{r}(t)) f^2(t)\left|f'(t)\right|^{p-2} \\ &= r(t)(1 + o(1))h^2(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)^{2/p} |h'(t)|^{p-2} \left(\int^t R^{-1}(s)\mathrm{d}s\right)^{(p-2)/p} \\ &\quad \times \left(1 + \frac{1}{pG(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)}\right)^{p-2} \\ &= R(t)\left(\int^t R^{-1}(s)\mathrm{d}s\right)(1 + o(1)), \end{aligned} \tag{4.22}$$

and hence

$$\int^t \frac{\mathrm{d}s}{\mathcal{R}(s)} \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{4.23}$$

Consequently, $v(t) \geq 0$ by Theorem 3.5. This means that the left-hand side in (4.21) is bounded above as $t \to \infty$, while the right-hand side tends to ∞ which yields the required contradiction proving that (1.9) is oscillatory if $L\lambda + \mu > 1/2q$.

Finally, consider again the case $\mu = 0$. In that case, we proved in the first part of the proof that (1.9) is oscillatory or nonoscillatory depending on whether $\lambda < -1/2q$ or $\lambda > -1/2q$. This shows that the second limit in (4.4) must be -1. □

Remark 4.2. (i) From the proof of Theorem 4.1, it follows that if the first limit in (4.4) exists, then conditions (2.8) imply that this limit is 1, and the assumptions of the theorem imply that if the second limit in (4.4) exists and is finite, then it is -1.

(ii) Theorem 4.1 can be applied to the Euler equation (1.7), and one can obtain the same result for (1.8) as in [4, Corollary 3]. Indeed, in this case, we have $h(t) = t^{(p-1)/p}$, $r = 1$, $c(t) = \gamma_p t^{-p}$, where $\gamma_p = ((p-1)/p)^p$ and by a direct computation

$$G(t) = \left(\frac{p-1}{p}\right)^{p-1}, \qquad R(t) = \widetilde{R}(t) = \left(\frac{p-1}{p}\right)^{p-2} t, \tag{4.24}$$

hence,

$$\begin{aligned}
\widehat{c}(t) &= \left[\left(t^{(p-1)/p}\right)^p \left(\frac{p-1}{p}\right)^{p-2} t \left[\left(\frac{p}{p-1}\right)^{p-2} \log t\right]^2\right]^{-1} = \left(\frac{p}{p-1}\right)^{2-p} t^{-p} \log^{-2} t, \\
\widehat{r}(t) &= \left[\left(\frac{p-1}{p} t^{-1/p}\right)^p \left(\frac{p-1}{p}\right)^{p-2} t \left[\left(\frac{p}{p-1}\right)^{p-2} \log t\right]^2\right]^{-1} = \left(\frac{p}{p-1}\right)^2 \log^{-2} t,
\end{aligned} \tag{4.25}$$

which mean that conditions (2.8) and (4.3) are satisfied. Concerning the limits in (4.4), we have

$$r\left|h'(t)\right|^p = \left(\frac{p-1}{p}\right)^p t^{-1} = c(t) h^p(t), \tag{4.26}$$

that is, the first limit in (4.4) is 1. Next,

$$f(t) = \left(\frac{p}{p-1}\right)^{(p-2)/p} t^{(p-1)/p} \log^{1/p} t, \tag{4.27}$$

and consequently,

$$\begin{aligned}
\widehat{c}(t)\Phi(f(t)) &= \left(\frac{p}{p-1}\right)^{2-p} t^{-p} \log^{-2} t \left[\left(\frac{p}{p-1}\right)^{(p-2)/p} t^{(p-1)/p} \log^{1/p} t\right]^{p-1} \\
&= \left(\frac{p}{p-1}\right)^{-1+2/p} t^{-2+1/p} \log^{-1-1/p} t, \\
f'(t) &= \left(\frac{p}{p-1}\right)^{(p-2)/p} \left[\frac{p-1}{p} t^{-1/p} \log^{1/p} t + \frac{1}{p} t^{-1/p} \log^{1/p-1} t\right] \\
&= \left(\frac{p-1}{p}\right)^{2/p} t^{-1/p} \log^{1/p} t \left[1 + \frac{1}{p-1} \log^{-1} t\right].
\end{aligned} \tag{4.28}$$

Using this formula,

$$\hat{r}(t)\Phi(f'(t)) = \left(\frac{p}{p-1}\right)^{2/p} t^{-1+1/p}\log^{-1-1/p}t\left[1+\log^{-1}t+\frac{p-2}{2}\log^{-2}t+O\left(\log^{-2}t\right)\right], \quad (4.29)$$

and hence,

$$\begin{aligned}(\hat{r}(t)\Phi(f'(t)))' = \left(\frac{p}{p-1}\right)^{2/p}&\left[-\frac{p-1}{p}t^{-2+1/p}\log^{-1-1/p}t\left(1+O\left(\log^{-1}t\right)\right)\right.\\ &-\frac{p+1}{p}t^{-2+1/p}\log^{-2-1/p}t\left(1+O\left(\log^{-1}t\right)\right)\\ &\left.+t^{-2+1/p}\log^{-1-1/p}tO\left(\log^{-2}t\right)\right]\\ = -\left(\frac{p}{p-1}\right)^{2/p-1}&t^{-2+1/p}\log^{-1-1/p}t\left(1+O\left(\log^{-1}t\right)\right),\end{aligned} \quad (4.30)$$

as $t \to \infty$. This means that the second limit in (4.4) is -1. According to Theorem 4.1, we obtain that the equation

$$\left[\left(1+\lambda\left(\frac{p}{p-1}\right)^2\frac{1}{\log^2 t}\right)\Phi(x')\right]' + \left[\frac{\gamma_p}{t^p}+\mu\left(\frac{p}{p-1}\right)^{2-p}\frac{1}{t^p\log^2 t}\right]\Phi(x) = 0 \quad (4.31)$$

is nonoscillatory if $\mu-\lambda < 1/2q$ and oscillatory if $\mu-\lambda > 1/2q$. If we denote $\tilde{\lambda} = \lambda(p/(p-1))^2$ and $\tilde{\mu} = \mu(p/(p-1))^{2-p}$, we see that (1.8) (with $\tilde{\lambda}$, $\tilde{\mu}$ instead of λ, μ, resp.) is nonoscillatory if $\tilde{\mu}-\gamma_p\tilde{\lambda} < (1/2)((p-1)/p)^{p-1}$, and it is oscillatory if $\tilde{\mu}-\gamma_p\tilde{\lambda} > (1/2)((p-1)/p)^{p-1}$, that is, we have the statement from [4].

(iii) In [3], it is proved that (1.8) is nonoscillatory also in the limiting case $\mu - \gamma_p\lambda = \mu_p$. We conjecture that we have also the same situation in the general case, that is, (1.9) is nonoscillatory also in the case $\mu - \lambda = 1/2q$.

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Research Article

Uniqueness of Positive Solutions for a Class of Fourth-Order Boundary Value Problems

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The purpose of this paper is to investigate the existence and uniqueness of positive solutions for the following fourth-order boundary value problem: $y^{(4)}(t) = f(t, y(t))$, $t \in [0,1]$, $y(0) = y(1) = y'(0) = y'(1) = 0$. Moreover, under certain assumptions, we will prove that the above boundary value problem has a unique symmetric positive solution. Finally, we present some examples and we compare our results with the ones obtained in recent papers. Our analysis relies on a fixed point theorem in partially ordered metric spaces.

1. Introduction

The purpose of this paper is to consider the existence and uniqueness of positive solutions for the following fourth-order two-point boundary value problem:

$$\begin{aligned} &y^{(4)}(t) = f(t, y(t)), \quad t \in [0,1], \\ &y(0) = y(1) = y'(0) = y'(1) = 0, \end{aligned} \tag{1.1}$$

which describes the bending of an elastic beam clamped at both endpoints.

There have been extensive studies on fourth-order boundary value problems with diverse boundary conditions. Some of the main tools of nonlinear analysis devoted to the study of this type of problems are, among others, lower and upper solutions [1–4], monotone iterative technique [5–7], Krasnoselskii fixed point theorem [8], fixed point index [9–11], Leray-Schauder degree [12, 13], and bifurcation theory [14–16].

2. Background

In this section, we present some basic facts which are necessary for our results.

In our study, we will use a fixed point theorem in partially ordered metric spaces which appears in [17].

Let $\mathcal{M}$ denote the class of those functions $\beta : [0, \infty) \to [0, 1)$ satisfying the condition

$$\beta(t_n) \longrightarrow 1 \quad \text{implies } t_n \to 0. \tag{2.1}$$

Now, we recall the above mentioned fixed point theorem.

Theorem 2.1 (see 1, Theorem 2.1). *Let $(X, \leq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X, d) is a complete metric space. Let $T : X \to X$ be a nondecreasing mapping such that there exists an element $x_0 \in X$ with $x_0 \leq Tx_0$. Suppose that there exists $\beta \in \mathcal{M}$ such that*

$$d(Tx, Ty) \leq \beta(d(x, y)) \cdot d(x, y), \quad \textit{for any } x, y \in X \textit{ with } x \geq y. \tag{2.2}$$

Assume that either T is continuous or X is such that

$$\textit{if } (x_n) \textit{ is a nondecreasing sequence in } X \textit{ such that } x_n \longrightarrow x, \textit{ then } x_n \leq x \textit{ for all } n \in \mathbb{N}. \tag{2.3}$$

Besides, suppose that

$$\textit{for each } x, y \in X, \textit{ there exists } z \in X \textit{ which is comparable to } x \textit{ and } y. \tag{2.4}$$

Then T has a unique fixed point.

In our considerations, we will work with a subset of the classical Banach space $C[0, 1]$. This space will be considered with the standard metric

$$d(x, y) = \sup_{0 \leq t \leq 1} |x(t) - y(t)|. \tag{2.5}$$

This space can be equipped with a partial order given by

$$x, y \in C[0, 1], \quad x \leq y \Longleftrightarrow x(t) \leq y(t), \quad \text{for } t \in [0, 1]. \tag{2.6}$$

In [18], it is proved that $(C[0, 1], \leq)$ with the above mentioned metric satisfies condition (2.3) of Theorem 2.1. Moreover, for $x, y \in C[0, 1]$, as the function $\max(x, y) \in C[0, 1]$, $(C[0, 1], \leq)$ satisfies condition (2.4).

On the other hand, the boundary value problem (1.1) can be rewritten as the integral equation (see, e.g., [19])

$$y(t) = \int_0^1 G(t, s) f(s, u(s)) ds, \quad \text{for } t \in [0, 1], \tag{2.7}$$

where $G(t,s)$ is the Green's function given by

$$G(t,s)=\frac{1}{6}\begin{cases} t^2(1-s)^2[(s-t)+2(1-t)s], & 0\le t\le s\le 1,\\ s^2(1-t)^2[(t-s)+2(1-s)t], & 0\le s\le t\le 1.\end{cases} \tag{2.8}$$

Note that $G(t,s)$ satisfies the following properties:

(i) $G(t,s)$ is a continuous function on $[0,1]\times[0,1]$,

(ii) $G(0,s)=G(1,s)=0$, for $s\in[0,1]$,

(iii) $G(t,s)\ge 0$, for $t,s\in[0,1]$.

3. Main Results

Our starting point in this section is to present the class of functions $\mathcal{A}$ which we use later. By $\mathcal{A}$ we denote the class of functions $\phi:[0,\infty)\to[0,\infty)$ satisfying the following conditions:

(i) ϕ is nondecreasing,

(ii) for any $x>0$, $\phi(x)<x$,

(iii) $\beta(x)=\phi(x)/x\in\mathcal{M}$.

Examples of functions in $\mathcal{A}$ are $\phi(x)=\mu x$ with $0\le\mu<1$, $\phi(x)=x/(1+x)$ and $\phi(x)=\ln(1+x)$. In the sequel, we formulate our main result.

Theorem 3.1. *Consider problem* (1.1) *assuming the following hypotheses:*

(a) $f:[0,1]\times[0,\infty)\to[0,\infty)$ *is continuous,*

(b) $f(t,y)$ *is nondecreasing with respect to the second variable, for each* $t\in[0,1]$,

(c) *suppose that there exists* $0<\alpha\le 384$, *such that, for* $x,y\in[0,\infty)$ *with* $y\ge x$,

$$f(t,y)-f(t,x)\le\alpha\phi(y-x),\qquad \text{with } \phi\in\mathcal{A}. \tag{3.1}$$

Then, problem (1.1) *has a unique nonnegative solution.*

Proof. Consider the cone

$$P=\{x\in C[0,1]:x\ge 0\}. \tag{3.2}$$

Obviously, (P,d) with $d(x,y)=\sup\{|x(t)-y(t)|:t\in[0,1]\}$ is a complete metric space satisfying condition (2.3) and condition (2.4) of Theorem 2.1.

Consider the operator defined by

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\mathrm{d}s, \quad \text{for } x \in P, \tag{3.3}$$

where $G(t,s)$ is the Green's function defined in Section 2.

It is clear that T applies the cone P into itself since $f(t,x)$ and $G(t,s)$ are nonnegative continuous functions.

Now, we check that assumptions in Theorems 2.1 are satisfied.

Firstly, the operator T is nondecreasing.

Indeed, since f is nondecreasing with respect to the second variable, for $u, v \in P$, $u \geq v$ and $t \in [0,1]$, we have

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s \\ &\geq \int_0^1 G(t,s)f(s,v(s))\mathrm{d}s \\ &= (Tv)(t).\end{aligned} \tag{3.4}$$

On the other hand, a straightforward calculation gives us

$$\begin{gathered}\int_0^1 G(t,s)\mathrm{d}s = \int_0^t G(t,s)\mathrm{d}s + \int_t^1 G(t,s)\mathrm{d}s = \frac{t^2}{24} - \frac{t^3}{12} + \frac{t^4}{24}, \\ \max_{0\leq t\leq 1} \int_0^1 G(t,s)\mathrm{d}s = \max_{0\leq t\leq 1}\left(\frac{t^2}{24} - \frac{t^3}{12} + \frac{t^4}{24}\right) = \frac{1}{384}.\end{gathered} \tag{3.5}$$

Taking into account this fact and our hypotheses, for $u, v \in P$ and $u > v$, we can obtain the following estimate:

$$\begin{aligned}d(Tu,Tv) &= \sup_{0\leq t\leq 1}\left|(Tu)(t) - (Tv)(t)\right| \\ &= \sup_{0\leq t\leq 1}((Tu)(t) - (Tv)(t)) \\ &= \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\big(f(s,u(s)) - f(s,v(s))\big)\mathrm{d}s \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\alpha\phi(u(s) - v(s))\mathrm{d}s \\ &\leq \sup_{0\leq t\leq 1}\int_0^1 G(t,s)\alpha\phi(d(u,v))\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&= \alpha\phi(d(u,v)) \sup_{0\le t\le 1}\int_0^1 G(t,s)\mathrm{d}s \\
&= \alpha\phi(d(u,v)) \cdot \frac{1}{384} \\
&\le \phi(d(u,v)) \\
&= \frac{\phi(d(u,v))}{d(u,v)} \cdot d(u,v).
\end{aligned}
\tag{3.6}
$$

This gives us, for $u, v \in P$ and $u > v$,

$$
d(Tu, Tv) \le \beta(d(u,v)) \cdot d(u,v), \tag{3.7}
$$

where $\beta(x) = \phi(x)/x \in \mathcal{M}$.

Obviously, the last inequality is satisfied for $u = v$.

Therefore, the contractive condition appearing in Theorem 2.1 is satisfied for $u \ge v$. Besides, as f and G are nonnegative functions,

$$
T0 = \int_0^1 G(t,s) f(s,0)\mathrm{d}s \ge 0. \tag{3.8}
$$

Finally, Theorem 2.1 tells us that T has a unique fixed point in P, and this means that problem (1.1) has a unique nonnegative solution.

This finishes the proof. □

Now, we present a sufficient condition for the existence and uniqueness of positive solutions for our problem (1.1) (positive solution means $x(t) > 0$, for $t \in (0,1)$). The proof of the following theorem is similar to the proof of Theorem 3.6 of [8]. We present a proof for completeness.

Theorem 3.2. *Under assumptions of Theorem 3.1 and suppose that $f(t_0, 0) \neq 0$ for certain $t_0 \in [0,1]$, problem* (1.1) *has a unique positive solution.*

Proof. Consider the nonnegative solution $x(t)$ given by Theorem 3.1 of problem (1.1).

Notice that this solution satisfies

$$
x(t) = \int_0^1 G(t,s) f(s, x(s))\mathrm{d}s. \tag{3.9}
$$

Now, we will prove that x is a positive solution.

In contrary case, suppose that there exists $0 < t^* < 1$ such that $x(t^*) = 0$ and, consequently,

$$x(t^*) = \int_0^1 G(t^*, s) f(s, x(s)) \mathrm{d}s = 0. \tag{3.10}$$

Since $x \geq 0$, f is nondecreasing with respect to the second variable and $G(t, s) \geq 0$, we have

$$0 = x(t^*) = \int_0^1 G(t^*, s) f(s, x(\mathrm{s})) \mathrm{d}s \geq \int_0^1 G(t^*, s) f(s, 0) \mathrm{d}s \geq 0, \tag{3.11}$$

and this gives us

$$\int_0^1 G(t^*, s) f(s, 0) \mathrm{d}s = 0. \tag{3.12}$$

This fact and the nonnegative character of $G(t, s)$ and $f(t, x)$ imply

$$G(t^*, s) \cdot f(s, 0) = 0 \quad \mathrm{a \cdot e}\ (s). \tag{3.13}$$

As $G(t^*, s) \neq 0 \mathrm{a{\cdot}e}\ (s)$, because $G(t^*, s)$ is given by a polynomial, we obtain

$$f(s, 0) = 0 \quad \mathrm{a \cdot e}\ (s). \tag{3.14}$$

On the other hand, as $f(t_0, 0) \neq 0$ for certain $t_0 \in [0, 1]$ and $f(t_0, x) \geq 0$, we have that $f(t_0, 0) > 0$.

The continuity of f gives us the existence of a set $A \subset [0, 1]$ with $t_0 \in A$ and $\mu(A) > 0$, where μ is the Lebesgue measure, satisfying that $f(t, 0) > 0$ for any $t \in A$. This contradicts (3.14).

Therefore, $x(t) > 0$ for $t \in (0, 1)$.

This finishes the proof. □

Now, we present an example which illustrates our results.

Example 3.3. Consider the nonlinear fourth-order two-point boundary value problem

$$\begin{gathered} y^{(4)}(t) = c + \lambda \arctan(y(t)), \quad t \in (0, 1),\ c, \lambda > 0, \\ y(0) = y(1) = y'(0) = y'(1) = 0. \end{gathered} \tag{3.15}$$

In this case, $f(t, y) = c + \lambda \arctan y$. It is easily seen that $f(t, y)$ satisfies (a) and (b) of Theorem 3.1.

In order to prove that $f(t,y)$ satisfies (c) of Theorem 3.1, previously, we will prove that the function $\phi : [0,\infty) \to [0,\infty)$, defined by $\phi(x) = \arctan x$, satisfies

$$\phi(u) - \phi(v) \leq \phi(u-v) \quad \text{for } u \geq v. \tag{3.16}$$

In fact, put $\phi(u) = \arctan u = \alpha$ and $\phi(v) = \arctan v = \beta$ (notice that, as $u \geq v$ and ϕ is nondecreasing, $\alpha \geq \beta$). Then, from

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \cdot \tan\beta}, \tag{3.17}$$

as $\alpha, \beta \in [0, \pi/2)$, then $\tan\alpha, \tan\beta \in [0,\infty)$, we obtain

$$\tan(\alpha - \beta) \leq \tan\alpha - \tan\beta. \tag{3.18}$$

Applying ϕ to this inequality and taking into account the nondecreasing character of ϕ, we have

$$\alpha - \beta \leq \arctan(\tan\alpha - \tan\beta) \tag{3.19}$$

or, equivalently,

$$\phi(u) - \phi(v) = \arctan u - \arctan v \leq \arctan(u-v) = \phi(u-v). \tag{3.20}$$

This proves our claim.

In the sequel, we prove that $f(t,y)$ satisfies assumption (c) of Theorem 3.1.

In fact, for $y \geq x$ and $t \in [0,1]$, we can obtain

$$\begin{aligned} f(t,y) - f(t,x) &= \lambda(\arctan y - \arctan x) \\ &\leq \lambda \arctan(y-x). \end{aligned} \tag{3.21}$$

Now, we will prove that $\phi(x) = \arctan x$ belongs to $\mathcal{A}$. In fact, obviously ϕ takes $[0,\infty)$ into itself and, as $\phi'(x) = 1/(1+x^2)$, ϕ is nondecreasing. Besides, as the derivative of $\psi(x) = x - \phi(x)$ is $\psi'(x) = 1 - 1/(1+x^2) > 0$ for $x > 0$, ψ is strictly increasing, and, consequently, $\phi(x) < x$ for $x > 0$ (notice that $\psi(0) = 0$). Notice that if $\beta(x) = \phi(x)/x = \arctan x/x$ and $\beta(t_n) \to 1$, then (t_n) is a bounded sequence because, in contrary case, $t_n \to \infty$ and, thus, $\beta(t_n) \to 0$. Suppose that $t_n \nrightarrow 0$. Then, we can find $\epsilon > 0$ such that, for each $n \in \mathbb{N}$, there exists $p_n \geq n$ with $t_{p_n} \geq \epsilon$. The bounded character of (t_n) gives us the existence of a subsequence (t_{k_n}) of (t_{p_n}) with (t_{k_n}) convergent. Suppose that $t_{k_n} \to a$. From $\beta(t_n) \to 1$, we obtain $\arctan t_{k_n}/t_{k_n} \to \arctan a/a = 1$ and, as the unique solution of $\arctan x = x$ is $x_0 = 0$, we obtain $a = 0$. Thus, $t_{k_n} \to 0$, and this contradicts the fact that $t_{k_n} \geq \epsilon$ for any $n \in \mathbb{N}$. Therefore, $t_n \to 0$. This proves that $f(t,y)$ satisfies assumption (c) of Theorem 3.1. Finally, as $f(t,0) = c > 0$, Problem (3.15) has a unique positive solution for $0 < \lambda \leq 384$ by Theorems 3.1 and 3.2.

Remark 3.4. In Theorem 3.2, the condition $f(t_0, 0) \neq 0$ for certain $t_0 \in [0,1]$ seems to be a strong condition in order to obtain a positive solution for Problem (1.1), but when the solution is unique, we will see that this condition is very adjusted one. More precisely, under assumption that Problem (1.1) has a unique nonnegative solution $x(t)$, then

$$f(t,0) = 0 \quad \text{for } t \in [0,1] \text{ iff } x(t) \equiv 0. \tag{3.22}$$

In fact, if $f(t,0) = 0$ for $t \in [0,1]$, then it is easily seen that the zero function satisfies Problem (1.1) and the uniqueness of solution gives us $x(t) \equiv 0$.

The other implication is obvious since if the zero function is solution of Problem (1.1), then $0 = f(t,0)$ for any $t \in [0,1]$.

Remark 3.5. Notice that assumptions in Theorem 3.1 are invariant by continuous perturbations. More precisely, if $f(t,0) = 0$ for any $t \in [0,1]$ and f satisfies (a), (b), and (c) of Theorem 3.1, then $g(t,x) = a(t) + f(t,x)$, with $a : [0,1] \to [0,\infty)$ continuous and $a \neq 0$, satisfies assumptions of Theorem 3.2, and this means that the following boundary value problem

$$\begin{gathered} y^{(4)}(t) = g(t, y(t)), \quad t \in [0,1], \\ y(0) = y(1) = y'(0) = y'(1) = 0, \end{gathered} \tag{3.23}$$

has a unique positive solution.

4. Some Remarks

In this section, we compare our results with the ones obtained in recent papers. Recently, in [19], the authors present as main result the following theorem.

Theorem 4.1 (Theorem 3.1 of [19]). *Suppose that*

$(H1)$ $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous,*

$(H2)$ $f(t,y)$ *is nondecreasing in* y, *for each* $t \in [0,1]$,

$(H3)$ $f(t,y) = f(1-t,y)$ *for each* $(t,y) \in [0,1] \times [0,\infty)$.

Moreover, suppose that there exist positive numbers $a > b$ *such that*

$$\max_{0 \le t \le 1} f(t,a) \le a \cdot A, \qquad \min_{1/4 \le t \le 3/4} f\left(t, \frac{b}{16}\right) \ge b \cdot B, \tag{4.1}$$

where

$$A = \left(\max_{0 \le t \le 1} \int_0^1 G(t,s) \mathrm{d}s \right)^{-1}, \qquad B = \left(\max_{0 \le t \le 1} \int_{1/4}^{3/4} G(t,s) \mathrm{d}s \right)^{-1}, \tag{4.2}$$

with $G(t,s)$ being the Green's function defined in Section 2.Then, Problem (1.1) has at least one symmetric positive solution $y^* \in C[0,1]$ such that $b \le \|y^*\| \le a$ and, moreover, $y^* = \lim_{k\to\infty} T^k y_0$ in the uniform norm, where T is the operator defined by

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\mathrm{d}s, \quad \text{for } x \in C[0,1] \tag{4.3}$$

and y_0 is the function given by $y_0(t) = b \cdot q(t)$, for $t \in [0,1]$, with $q(t) = \min(t^2,(1-t)^2)$, for $t \in [0,1]$ (symmetric solution means a solution $y(t)$ satisfying $y(t) = y(1-t)$, for $t \in [0,1]$).

In what follows, we present a parallel result to Theorem 3.2 where we obtain uniqueness of a symmetric positive solution of Problem (1.1).

Theorem 4.2. *Adding assumption $(H3)$ of Theorem 4.1 to the hypotheses of Theorem 3.2, one obtains a unique symmetric positive solution of Problem (1.1).*

Proof. As in the proof of Theorem 3.1, instead of P, we consider the following set K

$$K = \{x \in C[0,1] : x \ge 0 \text{ and } x \text{ is symmetric}\}. \tag{4.4}$$

It is easily seen that K is a closed subset of $C[0,1]$. Thus, (K,d), where d is the induced metric given by

$$d(x,y) = \sup_{0\le t\le 1} |x(t) - y(t)|, \quad \text{for } x,y \in K, \tag{4.5}$$

is a complete metric space.

Moreover, K with the induced order by $(C[0,1],\le)$ satisfies condition (2.3) of Theorem 2.1, and it is easily proved that the function $\max(x,y) \in K$, for $x,y \in K$ and, consequently, $(K,\le)$, satisfies condition (2.4) of Theorem 2.1.

Now, as in Theorem 2.1, we consider the operator defined by

$$(Tx)(t) = \int_0^1 G(t,s) f(s,x(s))\mathrm{d}s, \quad \text{for } x \in K. \tag{4.6}$$

In the sequel, we prove that, under our assumptions, T applies K into itself.

In fact, suppose that x is symmetric, then for $t \in [0,1]$, we have

$$(Tx)(1-t) = \int_0^1 G(1-t,s) f(s,x(s))\mathrm{d}s. \tag{4.7}$$

Making the change of variables $s = 1 - u$, we obtain

$$\begin{aligned}(Tx)(1-t) &= -\int_1^0 G(1-t,1-u)f(1-u,x(1-u))\mathrm{d}u \\ &= \int_0^1 G(1-t,1-u)f(1-u,x(1-u))\mathrm{d}u.\end{aligned} \tag{4.8}$$

Now, it is easily seen that $G(t,s) = G(1-t,1-s)$ for $t,s \in [0,1]$ and taking into account assumption $(H3)$ of Theorem 4.1 and the symmetric character of x, we have

$$\begin{aligned}(Tx)(1-t) &= \int_0^1 G(t,u)f(u,x(1-u))\mathrm{d}u \\ &= \int_0^1 G(t,u)f(u,x(u))\mathrm{d}u \\ &= (Tx)(t).\end{aligned} \tag{4.9}$$

The rest of the proof follows the lines of Theorems 3.1 and 3.2.

This finishes the proof. □

Now, we present an example which illustrates Theorem 4.2.

Example 4.3. Consider the following problem

$$\begin{gathered}y^{(4)}(t) = c + \lambda\sin(\pi t)\arctan\bigl(y(t)\bigr), \quad t \in (0,1),\ c,\lambda > 0, \\ y(0) = y(1) = y'(0) = y'(1) = 0.\end{gathered} \tag{4.10}$$

In this case, $f(t,y) = c + \lambda\sin(\pi t)\arctan y$. It is easily checked that $f(t,y)$ satisfies (a) and (b) of Theorem 3.1 and $f(t,y) = f(1-t,y)$, for $(t,y) \in [0,1] \times [0,\infty)$.

On the other hand, taking into account Example 3.3, we can obtain, for $y \geq x$ and $t \in [0,1]$,

$$\begin{aligned}f(t,y) - f(t,x) &= \lambda\sin\pi t\bigl[\arctan y - \arctan x\bigr] \\ &\leq \lambda\sin\pi t\bigl[\arctan(y-x)\bigr] \\ &\leq \lambda\arctan(y-x).\end{aligned} \tag{4.11}$$

Finally, as it is proved in Example 3.3, $\phi(x) = \arctan x$ belongs to $\mathcal{A}$. Therefore, Theorem 4.2 tells us that Problem (4.10) has a unique symmetric positive solution for $0 < \lambda \leq 384$. In what follows, we prove that Problem (4.10) can be treated using Theorem 4.1. In fact, in this case, $f(t,y) = c+\lambda\sin(\pi t)\arctan y$. Moreover, $A = 384$ (see proof of Theorem 3.1);

it can be proved that $B = 531.61$. As we have seen in Example 4.3, $f(t, y)$ satisfies assumptions ($H1$), ($H2$), and ($H3$) of Theorem 4.1. Moreover,

$$\max_{0\le t\le 1} f(t,a) = f\left(\frac{1}{2}, a\right) = c + \lambda \arctan a,$$
$$\min_{1/4\le t\le 3/4} f\left(t, \frac{b}{16}\right) = f\left(\frac{1}{4}, \frac{b}{16}\right) = c + \lambda \sin\frac{\pi}{4} \arctan\left(\frac{b}{16}\right) = c + \lambda \frac{\sqrt{2}}{2} \arctan\left(\frac{b}{16}\right). \tag{4.12}$$

Consider the function $\varphi(a) = 384 \cdot a - (c + \lambda \arctan a)$, with $0 < \lambda \le 384$ and $a \in [0, \infty)$. Obviously, $\varphi(0) = -c < 0$ and, as $\lim_{a\to\infty}\varphi(a) = \infty$, we can find $a_0 > 0$ such that $\varphi(a_0) > 0$. This means that

$$c + \lambda \arctan a_0 \le 384 a_0. \tag{4.13}$$

On the other hand, we consider the function $\psi(b) = c + \lambda(\sqrt{2}/2)\arctan(b/16) - 531.61 \cdot b$, with $0 < \lambda \le 384$ and $b \in [0, \infty)$.

Then, as $\psi(0) = c > 0$ and ψ is a continuous function, we can find b_0 such that

$$\min_{1/4\le t\le 3/4} f\left(t, \frac{b_0}{16}\right) = c + \lambda \frac{\sqrt{2}}{2} \arctan\left(\frac{b_0}{16}\right) \ge b_0 \cdot 531.61. \tag{4.14}$$

Therefore, Problem (4.10) can be treated using Theorem 4.1, and we obtain the existence of a symmetric positive solution.

Our main contribution is the uniqueness of the solution.

In what follows, we present the following example which can be treated by Theorem 4.2 and Theorem 4.1 cannot be used.

Example 4.4. Consider the following problem which is a variant of Example 4.3:

$$y^{(4)}(t) = c(t) + \lambda \sin(\pi t)\arctan(y(t)), \quad t \in (0,1),\ \lambda > 0$$
$$y(0) = y(1) = y'(0) = y'(1) = 0, \tag{4.15}$$

where $c(t)$ is a symmetric positive function satisfying $c(1/4) = 0$, for example,

$$c(t) = \begin{cases} 1 - 4t, & 0 \le t \le \dfrac{1}{4}, \\ 0, & \dfrac{1}{4} \le t \le \dfrac{3}{4} \\ 4t - 3, & \dfrac{3}{4} \le t \le 1. \end{cases} \tag{4.16}$$

In this case, $f(t, y) = c(t) + \lambda \sin(\pi t) \arctan(y(t))$. Taking into account Example 4.3, it is easily proved that $f(t, y)$ satisfies assumptions of Theorem 4.2, and, consequently, Problem (4.15) has a unique symmetric positive solution for $0 < \lambda \le 384$.

Now, we prove that $f(t, y)$ does not satisfy assumptions of Theorem 4.1 and, consequently, Problem (4.15) cannot be treated using this theorem. In fact, in this case (notice that $c(1/4) = 0$),

$$\min_{1/4\le t\le 3/4} f\left(t, \frac{b}{16}\right) = f\left(\frac{1}{4}, \frac{b}{16}\right) = \lambda \sin\frac{\pi}{4} \arctan\left(\frac{b}{16}\right) = \lambda\frac{\sqrt{2}}{2} \arctan\left(\frac{b}{16}\right), \tag{4.17}$$

and we cannot find a positive number b_0 such that

$$\lambda\frac{\sqrt{2}}{2} \arctan\left(\frac{b_0}{16}\right) \ge b_0 \cdot 531.61, \quad \text{for } 0 < \lambda \le 384. \tag{4.18}$$

This proves that Problem (4.15) cannot be treated by Theorem 4.1.

Now, we compare our results with the ones obtained in [14]. In [14], the author studies positive solutions of the problem

$$\begin{gathered} u^{(iv)}(x) = \lambda f(u(x)), \quad x \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{gathered} \tag{4.19}$$

using theory of bifurcation.

His main result works with functions $f(u)$ satisfying

(i) $f(u) > 0$, for $u \le 0$,

(ii) $\lim_{u\to\infty} f(u)/u = \infty$,

(iii) $f'(0) \ge 0$,

(iv) $f''(u) > 0$, for $u > 0$,

and the author proves that there exists a critical λ_0 such that Problem (4.19) has exactly two, exactly one, or no symmetric positive solution depending on whether $0 < \lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$.

Our Example 3.3 cannot be treated by the results of [14], because, in this case, $f(u) = c + \lambda \arctan u$ and f does not satisfy assumptions (ii) and (iv) above mentioned.

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Research Article

Weighted Asymptotically Periodic Solutions of Linear Volterra Difference Equations

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A linear Volterra difference equation of the form $x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i)$, where $x : \mathbb{N}_0 \to \mathbb{R}$, $a : \mathbb{N}_0 \to \mathbb{R}$, $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$ and $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ is ω-periodic, is considered. Sufficient conditions for the existence of weighted asymptotically periodic solutions of this equation are obtained. Unlike previous investigations, no restriction on $\prod_{j=0}^{\omega-1} b(j)$ is assumed. The results generalize some of the recent results.

1. Introduction

In the paper, we study a linear Volterra difference equation

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \tag{1.1}$$

where $n \in \mathbb{N}_0 := \{0,1,2,\ldots\}$, $a : \mathbb{N}_0 \to \mathbb{R}$, $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$, and $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ is ω-periodic, $\omega \in \mathbb{N} := \{1,2,\ldots\}$. We will also adopt the customary notations

$$\sum_{i=k+s}^{k} \mathcal{O}(i) = 0, \qquad \prod_{i=k+s}^{k} \mathcal{O}(i) = 1, \tag{1.2}$$

where k is an integer, s is a positive integer, and "$\mathcal{O}$" denotes the function considered independently of whether it is defined for the arguments indicated or not.

In [1], the authors considered (1.1) under the assumption

$$\prod_{j=0}^{\omega-1} b(j) = 1, \tag{1.3}$$

and gave sufficient conditions for the existence of asymptotically ω-periodic solutions of (1.1) where the notion for an asymptotically ω-periodic function has been given by the following definition.

Definition 1.1. Let ω be a positive integer. The sequence $y : \mathbb{N}_0 \to \mathbb{R}$ is called ω-periodic if $y(n+\omega) = y(n)$ for all $n \in \mathbb{N}_0$. The sequence y is called asymptotically ω-periodic if there exist two sequences $u, v : \mathbb{N}_0 \to \mathbb{R}$ such that u is ω-periodic, $\lim_{n\to\infty} v(n) = 0$, and

$$y(n) = u(n) + v(n) \tag{1.4}$$

for all $n \in \mathbb{N}_0$.

In this paper, in general, we do not assume that (1.3) holds. Then, we are able to derive sufficient conditions for the existence of a weighted asymptotically ω-periodic solution of (1.1). We give a definition of a weighted asymptotically ω-periodic function.

Definition 1.2. Let ω be a positive integer. The sequence $y : \mathbb{N}_0 \to \mathbb{R}$ is called weighted asymptotically ω-periodic if there exist two sequences $u, v : \mathbb{N}_0 \to \mathbb{R}$ such that u is ω-periodic and $\lim_{n\to\infty} v(n) = 0$, and, moreover, if there exists a sequence $w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ such that

$$\frac{y(n)}{w(n)} = u(n) + v(n), \tag{1.5}$$

for all $n \in \mathbb{N}_0$.

Apart from this, when we assume

$$\prod_{k=0}^{\omega-1} b(k) = -1, \tag{1.6}$$

then, as a consequence of our main result (Theorem 2.2), the existence of an asymptotically 2ω-periodic solution of (1.1) is obtained.

For the reader's convenience, we note that the background for discrete Volterra equations can be found, for example, in the well-known monograph by Agarwal [2], as well as by Elaydi [3] or Kocić and Ladas [4]. Volterra difference equations were studied by many others, for example, by Appleby et al. [5], by Elaydi and Murakami [6], by Győri and Horváth [7], by Győri and Reynolds [8], and by Song and Baker [9]. For some results on periodic solutions of difference equations, see, for example, [2–4, 10–13] and the related references therein.

2. Weighted Asymptotically Periodic Solutions

In this section, sufficient conditions for the existence of weighted asymptotically ω-periodic solutions of (1.1) will be derived. The following version of Schauder's fixed point theorem given in [14] will serve as a tool used in the proof.

Lemma 2.1. *Let Ω be a Banach space and S its nonempty, closed, and convex subset and let T be a continuous mapping such that $T(S)$ is contained in S and the closure $\overline{T(S)}$ is compact. Then, T has a fixed point in S.*

We set

$$\beta(n) := \prod_{j=0}^{n-1} b(j), \quad n \in \mathbb{N}_0, \tag{2.1}$$

$$\mathcal{B} := \beta(\omega). \tag{2.2}$$

Moreover, we define

$$n^* := n - 1 - \omega \left\lfloor \frac{n-1}{\omega} \right\rfloor, \tag{2.3}$$

where $\lfloor\cdot\rfloor$ is the floor function (the greatest-integer function) and n^* is the "remainder" of dividing $n-1$ by ω. Obviously, $\{\beta(n^*)\}$, $n \in \mathbb{N}$ is an ω-periodic sequence.

Now, we derive sufficient conditions for the existence of a weighted asymptotically ω-periodic solution of (1.1).

Theorem 2.2 (Main result). *Let ω be a positive integer, $b : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ be ω-periodic, $a : \mathbb{N}_0 \to \mathbb{R}$, and $K : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{R}$. Assume that*

$$\begin{gathered} \sum_{i=0}^{\infty} \left| \frac{a(i)}{\beta(i+1)} \right| < \infty, \\ \sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| \frac{K(j,i)\beta(i)}{\beta(j+1)} \right| < 1, \end{gathered} \tag{2.4}$$

and that at least one of the real numbers in the left-hand sides of inequalities (2.4) *is positive.*

Then, for any nonzero constant c, there exists a weighted asymptotically ω-periodic solution $x : \mathbb{N}_0 \to \mathbb{R}$ of (1.1) *with $u, v : \mathbb{N}_0 \to \mathbb{R}$ and $w : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$ in representation* (1.5) *such that*

$$w(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor}, \qquad u(n) := c\beta(n^*+1), \qquad \lim_{n\to\infty} v(n) = 0, \tag{2.5}$$

that is,

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^*+1) + v(n), \quad n \in \mathbb{N}_0. \tag{2.6}$$

Proof. We will use a notation

$$M := \sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|, \tag{2.7}$$

whenever this is useful.

Case 1. First assume $c > 0$. We will define an auxiliary sequence of positive numbers $\{\alpha(n)\}$, $n \in \mathbb{N}_0$. We set

$$\alpha(0) := \frac{\sum_{i=0}^{\infty}|a(i)/(\beta(i+1))| + c\sum_{j=0}^{\infty}\sum_{i=0}^{j}|(K(j,i)\beta(i))/(\beta(j+1))|}{1-\sum_{j=0}^{\infty}\sum_{i=0}^{j}|(K(j,i)\beta(i))/(\beta(j+1))|}, \tag{2.8}$$

where the expression on the right-hand side is well defined due to (2.4). Moreover, we define

$$\alpha(n) := \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|, \tag{2.9}$$

for $n \geq 1$. It is easy to see that

$$\lim_{n\to\infty}\alpha(n) = 0. \tag{2.10}$$

We show, moreover, that

$$\alpha(n) \leq \alpha(0), \tag{2.11}$$

for any $n \in \mathbb{N}$. Let us first remark that

$$\alpha(0) = \sum_{i=0}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right|. \tag{2.12}$$

Then, due to the convergence of both series (see (2.4)), the inequality

$$\begin{aligned}\alpha(0) &= \sum_{i=0}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=0}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| \\ &\geq \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| = \alpha(n)\end{aligned} \tag{2.13}$$

obviously holds for every $n \in \mathbb{N}$ and (2.11) is proved.

Let B be the Banach space of all real bounded sequences $z : \mathbb{N}_0 \to \mathbb{R}$ equipped with the usual supremum norm $\|z\| = \sup_{n\in\mathbb{N}_0}|z(n)|$ for $z \in B$. We define a subset $S \subset B$ as

$$S := \{z \in B : c - \alpha(0) \le z(n) \le c + \alpha(0),\ n \in \mathbb{N}_0\}. \tag{2.14}$$

It is not difficult to prove that S is a nonempty, bounded, convex, and closed subset of B.

Let us define a mapping $T : S \to B$ as follows:

$$(Tz)(n) = c - \sum_{i=n}^{\infty}\frac{a(i)}{\beta(i+1)} - \sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K(j,i)\beta(i)}{\beta(j+1)}z(i), \tag{2.15}$$

for any $n \in \mathbb{N}_0$.

We will prove that the mapping T has a fixed point in S.

We first show that $T(S) \subset S$. Indeed, if $z \in S$, then $|z(n) - c| \le \alpha(0)$ for $n \in \mathbb{N}_0$ and, by (2.11) and (2.15), we have

$$|(Tz)(n) - c| \le \sum_{i=n}^{\infty}\left|\frac{a(i)}{\beta(i+1)}\right| + (c+\alpha(0))\sum_{j=n}^{\infty}\sum_{i=0}^{j}\left|\frac{K(j,i)\beta(i)}{\beta(j+1)}\right| = \alpha(n) \le \alpha(0). \tag{2.16}$$

Next, we prove that T is continuous. Let $z^{(p)}$ be a sequence in S such that $z^{(p)} \to z$ as $p \to \infty$. Because S is closed, $z \in S$. Now, utilizing (2.15), we get

$$\begin{aligned}\left|\left(Tz^{(p)}\right)(n) - (Tz)(n)\right| &= \left|\sum_{j=n}^{\infty}\sum_{i=0}^{j}\frac{K(j,i)\beta(i)}{\beta(j+1)}\left(z^{(p)}(i) - z(i)\right)\right| \\ &\le M\sup_{i\ge 0}\left|z^{(p)}(i) - z(i)\right| = M\left\|z^{(p)} - z\right\|, \quad n \in \mathbb{N}_0.\end{aligned} \tag{2.17}$$

Therefore,

$$\begin{gathered}\left\|Tz^{(p)} - Tz\right\| \le M\left\|z^{(p)} - z\right\|, \\ \lim_{p\to\infty}\left\|Tz^{(p)} - Tz\right\| = 0.\end{gathered} \tag{2.18}$$

This means that T is continuous.

Now, we show that $\overline{T(S)}$ is compact. As is generally known, it is enough to verify that every ε-open covering of $\overline{T(S)}$ contains a finite ε-subcover of $\overline{T(S)}$, that is, finitely many of these open sets already cover $\overline{T(S)}$ ([15], page 756 (12)). Thus, to prove that $\overline{T(S)}$ is compact, we take an arbitrary $\varepsilon > 0$ and assume that an open ε-cover $\mathcal{C}_\varepsilon$ of $\overline{T(S)}$ is given. Then, from (2.10), we conclude that there exists an $n_\varepsilon \in \mathbb{N}$ such that $\alpha(n) < \varepsilon/4$ for $n \ge n_\varepsilon$.

Suppose that $x_T^1 \in \overline{T(S)}$ is one of the elements generating the ε-cover $\mathcal{C}_\varepsilon$ of $\overline{T(S)}$. Then (as follows from (2.16)), for an arbitrary $x_T \in \overline{T(\mathrm{S})}$,

$$\left| x_T^1(n) - x_T(n) \right| < \varepsilon \tag{2.19}$$

if $n \geq n_\varepsilon$. In other words, the ε-neighborhood of $x_T^1 - c^*$:

$$\left\| x_T^1 - c^* \right\| < \varepsilon, \tag{2.20}$$

where $c^* = \{c, c, \ldots\} \in S$ covers the set $\overline{T(S)}$ on an infinite interval $n \geq n_\varepsilon$. It remains to cover the rest of $\overline{T(S)}$ on a finite interval for $n \in \{0, 1 \ldots, n_\varepsilon - 1\}$ by a finite number of ε-neighborhoods of elements generating ε-cover $\mathcal{C}_\varepsilon$. Supposing that x_T^1 itself is not able to generate such cover, we fix $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$ and split the interval

$$[c - \alpha(n), c + \alpha(n)] \tag{2.21}$$

into a finite number $h(\varepsilon, n)$ of closed subintervals

$$I_1(n), I_2(n), \ldots, I_{h(\varepsilon,n)}(n) \tag{2.22}$$

each with a length not greater then $\varepsilon/2$ such that

$$\begin{gathered} \bigcup_{i=1}^{h(\varepsilon,n)} I_i(n) = [c - \alpha(n), c + \alpha(n)], \\ \operatorname{int} I_i(n) \cap \operatorname{int} I_j(n) = \emptyset, \quad i, j = 1, 2, \ldots, h(\varepsilon, n), \ i \neq j. \end{gathered} \tag{2.23}$$

Finally, the set

$$\bigcup_{n=0}^{n_\varepsilon - 1} [c - \alpha(n), c + \alpha(n)] \tag{2.24}$$

equals

$$\bigcup_{n=0}^{n_\varepsilon - 1} \bigcup_{i=1}^{h(\varepsilon,n)} I_i(n) \tag{2.25}$$

and can be divided into a finite number

$$M_\varepsilon := \sum_{n=0}^{n_\varepsilon - 1} h(\varepsilon, n) \tag{2.26}$$

of different subintervals (2.22). This means that, at most, M_ε of elements generating the cover $\mathcal{C}_\varepsilon$ are sufficient to generate a finite ε-subcover of $\overline{T(S)}$ for $n \in \{0, 1, \ldots, n_\varepsilon - 1\}$. We remark that each of such elements simultaneously plays the same role as $x_T^1(n)$ for $n \geq n_\varepsilon$. Since $\varepsilon > 0$ can be chosen as arbitrarily small, $\overline{T(S)}$ is compact.

By Schauder's fixed point theorem, there exists a $z \in S$ such that $z(n) = (Tz)(n)$ for $n \in \mathbb{N}_0$. Thus,

$$z(n) = c - \sum_{i=n}^{\infty} \frac{a(i)}{\beta(i+1)} - \sum_{j=n}^{\infty} \sum_{i=0}^{j} \frac{\beta(i)}{\beta(j+1)} K(j,i) z(i), \tag{2.27}$$

for any $n \in \mathbb{N}_0$.

Due to (2.10) and (2.16), for fixed point $z \in S$ of T, we have

$$\lim_{n\to\infty} |z(n) - c| = \lim_{n\to\infty} |(Tz)(n) - c| \leq \lim_{n\to\infty} \alpha(n) = 0, \tag{2.28}$$

or, equivalently,

$$\lim_{n\to\infty} z(n) = c. \tag{2.29}$$

Finally, we will show that there exists a connection between the fixed point $z \in S$ and the existence of a solution of (1.1) which divided by $\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}$ provides an asymptotically ω-periodic sequence. Considering (2.27) for $z(n+1)$ and $z(n)$, we get

$$\Delta z(n) = \frac{a(n)}{\beta(n+1)} + \sum_{i=0}^{n} \frac{\beta(i)}{\beta(n+1)} K(n,i) z(i), \tag{2.30}$$

where $n \in \mathbb{N}_0$. Hence, we have

$$z(n+1) - z(n) = \frac{a(n)}{\beta(n+1)} + \frac{1}{\beta(n+1)} \sum_{i=0}^{n} \beta(i) K(n,i) z(i), \quad n \in \mathbb{N}_0. \tag{2.31}$$

Putting

$$z(n) = \frac{x(n)}{\beta(n)}, \quad n \in \mathbb{N}_0 \tag{2.32}$$

in (2.31), we get (1.1) since

$$\frac{x(n+1)}{\beta(n+1)} - \frac{x(n)}{\beta(n)} = \frac{a(n)}{\beta(n+1)} + \frac{1}{\beta(n+1)} \sum_{i=0}^{n} K(n,i) x(i), \quad n \in \mathbb{N}_0 \tag{2.33}$$

yields

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=0}^{n} K(n,i)x(i), \quad n \in \mathbb{N}_0. \tag{2.34}$$

Consequently, x defined by (2.32) is a solution of (1.1). From (2.29) and (2.32), we obtain

$$\frac{x(n)}{\beta(n)} = z(n) = c + o(1), \tag{2.35}$$

for $n \to \infty$ (where $o(1)$ is the Landau order symbol). Hence,

$$x(n) = \beta(n)(c + o(1)), \quad n \longrightarrow \infty. \tag{2.36}$$

It is easy to show that the function β defined by (2.1) can be expressed in the form

$$\beta(n) = \prod_{j=0}^{n-1} b(j) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} \cdot \beta(n^* + 1), \tag{2.37}$$

for $n \in \mathbb{N}_0$. Then, as follows from (2.36),

$$x(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} \cdot \beta(n^* + 1)(c + o(1)), \quad n \longrightarrow \infty, \tag{2.38}$$

or

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^* + 1) + \beta(n^* + 1)o(1), \quad n \longrightarrow \infty. \tag{2.39}$$

The proof is completed since the sequence $\{\beta(n^* + 1)\}$ is ω-periodic, hence bounded and, due to the properties of Landau order symbols, we have

$$\beta(n^* + 1)o(1) = o(1), \quad n \longrightarrow \infty, \tag{2.40}$$

and it is easy to see that the choice

$$u(n) := c\beta(n^* + 1), \qquad w(n) := \mathcal{B}^{\lfloor (n-1)/\omega \rfloor}, \quad n \in \mathbb{N}_0, \tag{2.41}$$

and an appropriate function $v : \mathbb{N}_0 \to \mathbb{R}$ such that

$$\lim_{n \to \infty} v(n) = 0 \tag{2.42}$$

finishes this part of the proof. Although for $n = 0$, there is no correspondence between formula (2.36) and the definitions of functions u and w, we assume that function v makes up for this.

Case 2. If $c < 0$, we can proceed as follows. It is easy to see that arbitrary solution $y = y(n)$ of the equation

$$y(n+1) = -a(n) + b(n)y(n) + \sum_{i=0}^{n} K(n,i)y(i) \tag{2.43}$$

defines a solution $x = x(n)$ of (1.1) since a substitution $y(n) = -x(n)$ in (2.43) turns (2.43) into (1.1). If the assumptions of Theorem 2.2 hold for (1.1), then, obviously, Theorem 2.2 holds for (2.43) as well. So, for an arbitrary $c > 0$, (2.43) has a solution that can be represented by formula (2.6), that is,

$$\frac{y(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c\beta(n^* + 1) + v(n), \quad n \in \mathbb{N}_0. \tag{2.44}$$

Or, in other words, (1.1) has a solution that can be represented by formula (2.44) as

$$\frac{x(n)}{\mathcal{B}^{\lfloor (n-1)/\omega \rfloor}} = c_0\beta(n^* + 1) + v^*(n), \quad n \in \mathbb{N}_0, \tag{2.45}$$

with $c_0 = -c$ and $v^*(n) = -v(n)$. In (2.45), $c_0 < 0$ and the function $v^*(n)$ has the same properties as the function $v(n)$. Therefore, formula (2.6) is valid for an arbitrary negative c as well. □

Now, we give an example which illustrates the case where there exists a solution to equation of the type (1.1) which is weighted asymptotically periodic, but is not asymptotically periodic.

Example 2.3. We consider (1.1) with

$$\begin{gathered} a(n) = (-1)^{n+1}\left(1 - \frac{1}{3^{n+1}}\right), \\ b(n) = 3(-1)^n, \\ K(n,i) = (-1)^{n+(i(i-1))/2}\frac{1}{3^{2i}}, \end{gathered} \tag{2.46}$$

that is, the equation

$$x(n+1) = (-1)^{n+1}\left(1 - \frac{1}{3^{n+1}}\right) + 3(-1)^n x(n) + \sum_{i=0}^{n} (-1)^{n+(i(i-1))/2}\frac{1}{3^{2i}}x(i). \tag{2.47}$$

The sequence $b(n)$ is 2-periodic and

$$\begin{gathered}
\beta(n) = \prod_{j=0}^{n-1} b(j) = (-1)^{n(n-1)/2} 3^n \ , \\
\mathcal{B} = \beta(\omega) = \beta(2) = -9, \\
\beta(n^* + 1) = -3 + 6(-1)^{n+1}, \\
\frac{a(n)}{\beta(n+1)} = (-1)^{(-n^2+n+2)/2} \left(\frac{1}{3^{n+1}} - \frac{1}{3^{2(n+1)}} \right), \\
\sum_{i=0}^{\infty} \left| \frac{a(i)}{\beta(i+1)} \right| < \infty, \\
\sum_{j=0}^{\infty} \sum_{i=0}^{j} \left| \frac{K(j,i)\beta(i)}{\beta(j+1)} \right| < \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \left| \frac{K(j,i)\beta(i)}{\beta(j+1)} \right| = \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{1}{3^{i+j+1}} \\
= \frac{1}{3} \left(\sum_{j=0}^{\infty} \frac{1}{3^j} \right) \left(\sum_{i=0}^{\infty} \frac{1}{3^i} \right) = \frac{1}{3} \cdot \frac{1}{1 - 1/3} \cdot \frac{1}{1 - 1/3} \\
= \frac{1}{3} \cdot \frac{3}{2} \cdot \frac{3}{2} = \frac{3}{4} < 1.
\end{gathered} \tag{2.48}$$

By virtue of Theorem 2.2, for any nonzero constant c, there exists a solution $x : \mathbb{N}_0 \rightarrow \mathbb{R}$ of (1.1) which is weighed asymptotically 2-periodic. Let, for example, $c = 2/3$. Then,

$$\begin{gathered}
w(n) = (-9)^{\lfloor (n-1)/2 \rfloor}, \\
u(n) = c\beta(n^* + 1) = \frac{2}{3} \left(-3 + 6(-1)^{n+1} \right) = -2 + 4(-1)^{n+1},
\end{gathered} \tag{2.49}$$

and the sequence $x(n)$ given by

$$\frac{x(n)}{(-9)^{\lfloor (n-1)/2 \rfloor}} = -2 + 4(-1)^{n+1} + v(n), \quad n \in \mathbb{N}_0, \tag{2.50}$$

or, equivalently,

$$x(n) = (-9)^{\lfloor (n-1)/2 \rfloor} \left(-2 + 4(-1)^{n+1} \right) + v(n), \quad n \in \mathbb{N}_0 \tag{2.51}$$

is such a solution. We remark that such solution is not asymptotically 2-periodic in the meaning of Definition 1.1.

It is easy to verify that the sequence $x^*(n)$ obtained from (2.51) if $v(n) = 0$, $n \in \mathbb{N}_0$, that is,

$$x^*(n) = (-9)^{\lfloor (n-1)/2 \rfloor} \left(-2 + 4(-1)^{n+1} \right) = \frac{2}{3} \cdot (-1)^{n(n-1)/2} \cdot 3^n, \quad n \in \mathbb{N}_0 \tag{2.52}$$

is a true solution of (2.47).

3. Concluding Remarks and Open Problems

It is easy to prove the following corollary.

Corollary 3.1. *Let Theorem 2.2 be valid. If, moreover, $|\mathcal{B}| < 1$, then every solution $x = x(n)$ of* (1.1) *described by formula* (2.6) *satisfies*

$$\lim_{n \to \infty} x(n) = 0. \tag{3.1}$$

If $|\mathcal{B}| > 1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\liminf_{n \to \infty} x(n) = -\infty \tag{3.2}$$

or/and

$$\limsup_{n \to \infty} x(n) = \infty. \tag{3.3}$$

Finally, if $\mathcal{B} > 1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\lim_{n \to \infty} x(n) = \infty, \tag{3.4}$$

and if $\mathcal{B} < -1$, then, for every solution $x = x(n)$ of (1.1) *described by formula* (2.6), *one has*

$$\lim_{n \to \infty} x(n) = -\infty. \tag{3.5}$$

Now, let us discuss the case when (1.6) holds, that is, when

$$\mathcal{B} = \prod_{j=0}^{\omega-1} b(j) = -1. \tag{3.6}$$

Corollary 3.2. *Let Theorem 2.2 be valid. Assume that $\mathcal{B} = -1$. Then, for any nonzero constant c, there exists an asymptotically 2ω-periodic solution $x = x(n)$, $n \in \mathbb{N}_0$ of* (1.1) *such that*

$$x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + z(n), \quad n \in \mathbb{N}_0, \tag{3.7}$$

with

$$u(n) := c\beta(n^* + 1), \qquad \lim_{n\to\infty} z(n) = 0. \tag{3.8}$$

Proof. Putting $\mathcal{B} = -1$ in Theorem 2.2, we get

$$x(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) + (-1)^{\lfloor (n-1)/\omega \rfloor} v(n), \tag{3.9}$$

with

$$u(n) := c\beta(n^* + 1), \qquad \lim_{n\to\infty} v(n) = 0. \tag{3.10}$$

Due to the definition of n^*, we see that the sequence

$$\{\beta(n^* + 1)\} = \{\beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), \ldots\}, \tag{3.11}$$

is an ω-periodic sequence. Since

$$\left\{ \left\lfloor \frac{n-1}{\omega} \right\rfloor \right\} = \left\{ -1, \underbrace{0, \ldots, 0}_{\omega}, \underbrace{1, \ldots, 1}_{\omega}, 2, \ldots \right\}, \tag{3.12}$$

for $n \in \mathbb{N}_0$, we have

$$\left\{ (-1)^{\lfloor (n-1)/\omega \rfloor} \right\} = \left\{ -1, \underbrace{1, \ldots, 1}_{\omega}, \underbrace{-1, \ldots, -1}_{\omega}, 1, \ldots \right\}. \tag{3.13}$$

Therefore, the sequence

$$\left\{ (-1)^{\lfloor (n-1)/\omega \rfloor} u(n) \right\} = c\{-\beta(\omega), \beta(1), \beta(2), \ldots, \beta(\omega), -\beta(1), -\beta(2), \ldots, -\beta(\omega), \ldots\} \tag{3.14}$$

is a 2ω-periodic sequence. Set

$$z(n) = (-1)^{\lfloor (n-1)/\omega \rfloor} v(n). \tag{3.15}$$

Then,

$$\lim_{n\to\infty} z(n) = 0. \tag{3.16}$$

The proof is completed. □

Remark 3.3. From the proof, we see that Theorem 2.2 remains valid even in the case of $c = 0$. Then, there exists an "asymptotically weighted ω-periodic solution" $x = x(n)$ of (1.1) as well. The formula (2.6) reduces to

$$x(n) = \mathcal{B}^{\lfloor (n-1)/\omega \rfloor} v(n) = o(1), \quad n \in \mathbb{N}_0, \tag{3.17}$$

since $u(n) = 0$. In the light of Definition 1.2, we can treat this case as follows. We set (as a singular case) $u \equiv 0$ with an arbitrary (possibly other than "ω") period and with $v = o(1)$, $n \to \infty$.

Remark 3.4. The assumptions of Theorem 2.2 [1] are substantially different from those of the present Theorem 2.2. However, it is easy to see that Theorem 2.2 [1] is a particular case of the present Theorem 2.2 if (1.3) holds, that is, if $\mathcal{B} = 1$. Therefore, our results can be viewed as a generalization of some results in [1].

In connection with the above investigations, some open problems arise.

Open Problem 1. The results of [1] are extended to systems of linear Volterra discrete equations in [16, 17]. It is an open question if the results presented can be extended to systems of linear Volterra discrete equations.

Open Problem 2. Unlike the result of Theorem 2.2 [1] where a parameter c can be arbitrary, the assumptions of the results in [16, 17] are more restrictive since the related parameters should satisfy certain inequalities as well. Different results on the existence of asymptotically periodic solutions were recently proved in [8]. Using an example, it is shown that the results in [8] can be less restrictive. Therefore, an additional open problem arises if the results in [16, 17] can be improved in such a way that the related parameters can be arbitrary and if the expected extension of the results suggested in Open Problem 1 can be given in such a way that the related parameters can be arbitrary as well.

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Research Article

Weyl-Titchmarsh Theory for Time Scale Symplectic Systems on Half Line

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We develop the Weyl-Titchmarsh theory for time scale symplectic systems. We introduce the $M(\lambda)$-function, study its properties, construct the corresponding Weyl disk and Weyl circle, and establish their geometric structure including the formulas for their center and matrix radii. Similar properties are then derived for the limiting Weyl disk. We discuss the notions of the system being in the limit point or limit circle case and prove several characterizations of the system in the limit point case and one condition for the limit circle case. We also define the Green function for the associated nonhomogeneous system and use its properties for deriving further results for the original system in the limit point or limit circle case. Our work directly generalizes the corresponding discrete time theory obtained recently by S. Clark and P. Zemánek (2010). It also unifies the results in many other papers on the Weyl-Titchmarsh theory for linear Hamiltonian differential, difference, and dynamic systems when the spectral parameter appears in the second equation. Some of our results are new even in the case of the second-order Sturm-Liouville equations on time scales.

1. Introduction

In this paper we develop systematically the Weyl-Titchmarsh theory for time scale symplectic systems. Such systems unify and extend the classical linear Hamiltonian differential systems and discrete symplectic and Hamiltonian systems, including the Sturm-Liouville differential and difference equations of arbitrary even order. As the research in the Weyl-Titchmarsh theory has been very active in the last years, we contribute to this development by presenting a theory which directly generalizes and unifies the results in several recent papers, such as [1–4] and partly in [5–14].

Historically, the theory nowadays called by Weyl and Titchmarsh started in [15] by the investigation of the second-order linear differential equation

$$(r(t)z'(t)) + q(t)z(t) = \lambda z(t), \quad t \in [0, \infty), \tag{1.1}$$

where $r, q : [0, \infty) \to \mathbb{R}$ are continuous, $r(t) > 0$, and $\lambda \in \mathbb{C}$, is a spectral parameter. By using a geometrical approach it was showed that (1.1) can be divided into two classes called the limit circle and limit point meaning that either all solutions of (1.1) are square integrable for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$ or there is a unique (up to a multiplicative constant) square-integrable solution of (1.1) on $[0, \infty)$. Analytic methods for the investigation of (1.1) have been introduced in a series of papers starting with [16]; see also [17]. We refer to [18–20] for an overview of the original contributions to the Weyl-Titchmarsh theory for (1.1); see also [21]. Extensions of the Weyl-Titchmarsh theory to more general equations, namely, to the linear Hamiltonian differential systems

$$z'(t) = [\lambda A(t) + B(t)]z(t), \quad t \in [0, \infty), \tag{1.2}$$

was initiated in [22] and developed further in [6, 8, 10, 11, 23–38].

According to [19], the first paper dealing with the parallel discrete time Weyl theory for second-order difference equations appears to be the work mentioned in [39]. Since then a long time elapsed until the theory of difference equations attracted more attention. The Weyl-Titchmarsh theory for the second-order Sturm-Liouville difference equations was developed in [22, 40, 41]; see also the references in [19]. For higher-order Sturm-Liouville difference equations and linear Hamiltonian difference systems, such as

$$\Delta x_k = A_k x_{k+1} + \left(B_k + \lambda W_k^{[2]}\right)u_k, \quad \Delta u_k = \left(C_k - \lambda W_k^{[1]}\right)x_{k+1} - A_k^* u_k, \quad k \in [0, \infty)_{\mathbb{Z}}, \tag{1.3}$$

where A_k, B_k, C_k, $W_k^{[1]}$, $W_k^{[2]}$ are complex $n \times n$ matrices such that B_k and C_k are Hermitian and $W_k^{[1]}$ and $W_k^{[2]}$ are Hermitian and nonnegative definite, the Weyl-Titchmarsh theory was studied in [9, 14, 42]. Recently, the results for linear Hamiltonian difference systems were generalized in [1, 2] to discrete symplectic systems

$$x_{k+1} = \mathcal{A}_k x_k + \mathcal{B}_k u_k, \quad u_{k+1} = \mathcal{C}_k x_k + \mathfrak{D}_k u_k + \lambda \mathcal{W}_k x_{k+1}, \quad k \in [0, \infty)_{\mathbb{Z}}, \tag{1.4}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathfrak{D}_k$, $\mathcal{W}_k$ are complex $n \times n$ matrices such that $\mathcal{W}_k$ is Hermitian and nonnegative definite and the $2n \times 2n$ transition matrix in (1.4) is symplectic, that is,

$$\mathcal{S}_k := \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathfrak{D}_k \end{pmatrix}, \quad \mathcal{S}_k^* \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}. \tag{1.5}$$

In the unifying theory for differential and difference equations—the theory of time scales—the classification of second-order Sturm-Liouville dynamic equations

$$y^{\Delta\Delta}(t) + q(t)y^{\sigma}(t) = \lambda y^{\sigma}(t), \quad t \in [a, \infty)_{\mathbb{T}}, \tag{1.6}$$

to be of the limit point or limit circle type is given in [4, 43]. These two papers seem to be the only ones on time scales which are devoted to the Weyl-Titchmarsh theory for the second order dynamic equations. Another way of generalizing the Weyl-Titchmarsh theory for continuous and discrete Hamiltonian systems was presented in [3, 5]. In these references the authors consider the linear Hamiltonian system

$$\begin{aligned} x^{\Delta}(t) &= A(t)x^{\sigma}(t) + [B(t) + \lambda W_2(t)]u(t), \\ u^{\Delta}(t) &= [C(t) - \lambda W_1(t)]x^{\sigma}(t) - A^*(t)u(t), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{1.7}$$

on the so-called Sturmian or general time scales, respectively. Here $f^{\Delta}(t)$ is the time scale Δ-derivative and $f^{\sigma}(t) := f(\sigma(t))$, where $\sigma(t)$ is the forward jump at t; see the time scale notation in Section 2.

In the present paper we develop the Weyl-Titchmarsh theory for more general linear dynamic systems, namely, the time scale symplectic systems

$$\begin{aligned} x^{\Delta}(t) &= \mathcal{A}(t)x(t) + \mathcal{B}(t)u(t), \\ u^{\Delta}(t) &= \mathcal{C}(t)x(t) + \mathfrak{D}(t)u(t) - \lambda \mathcal{W}(t)x^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}, \end{aligned} \tag{$\mathcal{S}_\lambda$}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathfrak{D}$, $\mathcal{W}$ are complex $n \times n$ matrix functions on $[a,\infty)_{\mathbb{T}}$, $\mathcal{W}(t)$ is Hermitian and nonnegative definite, $\lambda \in \mathbb{C}$, and the $2n \times 2n$ coefficient matrix in system $(\mathcal{S}_\lambda)$ satisfies

$$\mathcal{S}(t) := \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathfrak{D}(t) \end{pmatrix}, \quad \mathcal{S}^*(t)\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\mathcal{S}^*(t)\mathcal{J}\mathcal{S}(t) = 0, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1.8}$$

where $\mu(t) := \sigma(t) - t$ is the graininess of the time scale. The spectral parameter λ is only in the second equation of system $(\mathcal{S}_\lambda)$. This system was introduced in [44], and it naturally unifies the previously mentioned continuous, discrete, and time scale linear Hamiltonian systems (having the spectral parameter in the second equation only) and discrete symplectic systems into one framework. Our main results are the properties of the $M(\lambda)$ function, the geometric description of the Weyl disks, and characterizations of the limit point and limit circle cases for the time scale symplectic system $(\mathcal{S}_\lambda)$. In addition, we give a formula for the $L^2_{\mathcal{W}}$ solutions of a nonhomogeneous time scale symplectic system in terms of its Green function. These results generalize and unify in particular all the results in [1–4] and some results from [5–14]. The theory of time scale symplectic systems or Hamiltonian systems is a topic with active research in recent years; see, for example, [44–51]. This paper can be regarded not only as a completion of these papers by establishing the Weyl-Titchmarsh theory for time scale symplectic systems but also as a comparison of the corresponding continuous and discrete time results. The references to particular statements in the literature are displayed throughout the text. Many results of this paper are new even for (1.6), being a special case of system $(\mathcal{S}_\lambda)$. An overview of these new results for (1.6) will be presented in our subsequent work.

This paper is organized as follows. In the next section we recall some basic notions from the theory of time scales and linear algebra. In Section 3 we present fundamental properties of time scale symplectic systems with complex coefficients, including the important Lagrange identity (Theorem 3.5) and other formulas involving their solutions.

In Section 4 we define the time scale $M(\lambda)$-function for system $(\mathcal{S}_\lambda)$ and establish its basic properties in the case of the regular spectral problem. In Section 5 we introduce the Weyl disks and circles for system $(\mathcal{S}_\lambda)$ and describe their geometric structure in terms of contractive matrices in $\mathbb{C}^{n\times n}$. The properties of the limiting Weyl disk and Weyl circle are then studied in Section 6, where we also prove that system $(\mathcal{S}_\lambda)$ has at least n linearly independent solutions in the space $L^2_{\mathcal{W}}$ (see Theorem 6.7). In Section 7 we define the system $(\mathcal{S}_\lambda)$ to be in the limit point and limit circle case and prove several characterizations of these properties. In the final section we consider the system $(\mathcal{S}_\lambda)$ with a nonhomogeneous term. We construct its Green function, discuss its properties, and characterize the $L^2_{\mathcal{W}}$ solutions of this nonhomogeneous system in terms of the Green function (Theorem 8.5). A certain uniqueness result is also proven for the limit point case.

2. Time Scales

Following [52, 53], a time scale $\mathbb{T}$ is any nonempty and closed subset of $\mathbb{R}$. A bounded time scale can be therefore identified as $[a,b]_{\mathbb{T}} := [a,b] \cap \mathbb{T}$ which we call the time scale interval, where $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$. Similarly, a time scale which is unbounded above has the form $[a,\infty)_{\mathbb{T}} := [a,\infty) \cap \mathbb{T}$. The forward and backward jump operators on a time scale are denoted by $\sigma(t)$ and $\rho(t)$ and the graininess function by $\mu(t) := \sigma(t) - t$. If not otherwise stated, all functions in this paper are considered to be complex valued. A function f on $[a,b]_{\mathbb{T}}$ is called *piecewise rd-continuous*; we write $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ if the right-hand limit $f(t^+)$ exists finite at all right-dense points $t \in [a,b)_{\mathbb{T}}$, and the left-hand limit $f(t^-)$ exists finite at all left-dense points $t \in (a,b]_{\mathbb{T}}$ and f is continuous in the topology of the given time scale at all but possibly finitely many right-dense points $t \in [a,b)_{\mathbb{T}}$. A function f on $[a,\infty)_{\mathbb{T}}$ is *piecewise rd-continuous*; we write $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$ if $f \in \mathrm{C}_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ for every $b \in (a,\infty)_{\mathbb{T}}$. An $n \times n$ matrix-valued function f is called *regressive* on a given time scale interval if $I + \mu(t) f(t)$ is invertible for all t in this interval.

The time scale Δ-derivative of a function f at a point t is denoted by $f^\Delta(t)$; see [52, Definition 1.10]. Whenever $f^\Delta(t)$ exists, the formula $f^\sigma(t) = f(t) + \mu(t) f^\Delta(t)$ holds true. The product rule for the Δ-differentiation of the product of two functions has the form

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f^\sigma(t) g^\Delta(t) = f^\Delta(t) g^\sigma(t) + f(t) g^\Delta(t). \tag{2.1}$$

A function f on $[a,b]_{\mathbb{T}}$ is called *piecewise rd-continuously Δ-differentiable*; we write $f \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$; if it is continuous on $[a,b]_{\mathbb{T}}$, then $f^\Delta(t)$ exists at all except for possibly finitely many points $t \in [a,\rho(b)]_{\mathbb{T}}$, and $f^\Delta \in \mathrm{C}_{\mathrm{prd}}$ on $[a,\rho(b)]_{\mathbb{T}}$. As a consequence we have that the finitely many points t_i at which $f^\Delta(t_i)$ does not exist belong to $(a,b)_{\mathbb{T}}$ and these points t_i are necessarily right-dense and left-dense at the same time. Also, since at those points we know that $f^\Delta(t_i^+)$ and $f^\Delta(t_i^-)$ exist finite, we replace the quantity $f^\Delta(t_i)$ by $f^\Delta(t_i^\pm)$ in any formula involving $f^\Delta(t)$ for all $t \in [a,\rho(b)]_{\mathbb{T}}$. Similarly as above we define $f \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$. The time scale integral of a piecewise rd-continuous function f over $[a,b]_{\mathbb{T}}$ is denoted by $\int_a^b f(t)\Delta t$ and over $[a,\infty)_{\mathbb{T}}$ by $\int_a^\infty f(t)\Delta t$ provided this integral is convergent in the usual sense; see [52, Definitions 1.71 and 1.82].

Remark 2.1. As it is known in [52, Theorem 5.8] and discussed in [54, Remark 3.8], for a fixed $t_0 \in [a,b]_{\mathbb{T}}$ and a piecewise rd-continuous $n \times n$ matrix function $A(\cdot)$ on $[a,b]_{\mathbb{T}}$ which is regressive on $[a,t_0)_{\mathbb{T}}$, the initial value problem $y^{\Delta}(t) = A(t)y(t)$ for $t \in [a,\rho(b)]_{\mathbb{T}}$ with $y(t_0) = y_0$ has a unique solution $y(\cdot) \in C^1_{\mathrm{prd}}$ on $[a,b]_{\mathbb{T}}$ for any $y_0 \in \mathbb{C}^n$. Similarly, this result holds on $[a,\infty)_{\mathbb{T}}$.

Let us recall some matrix notations from linear algebra used in this paper. Given a complex square matrix M, by M^*, $M > 0$, $M \geq 0$, $M < 0$, $M \leq 0$, $\operatorname{rank} M$, $\operatorname{Ker} M$, $\operatorname{def} M$, we denote, respectively, the conjugate transpose, positive definiteness, positive semidefiniteness, negative definiteness, negative semidefiniteness, rank, kernel, and the defect (i.e., the dimension of the kernel) of the matrix M. Moreover, we will use the notation $\operatorname{Im}(M) := (M - M^*)/(2i)$ and $\operatorname{Re}(M) := (M + M^*)/2$ for the Hermitian components of the matrix M; see [55, pages 268-269] or [56, Fact 3.5.24]. This notation will be also used with $\lambda \in \mathbb{C}$, and in this case $\operatorname{Im}(\lambda)$ and $\operatorname{Re}(\lambda)$ represent the imaginary and real parts of λ.

Remark 2.2. If the matrix $\operatorname{Im}(M)$ is positive or negative definite, then the matrix M is necessarily invertible. The proof of this fact can be found, for example, in [2, Remark 2.6].

In order to simplify the notation we abbreviate $[f^{\sigma}(t)]^*$ and $[f^*(t)]^{\sigma}$ by $f^{\sigma *}(t)$. Similarly, instead of $[f^{\Delta}(t)]^*$ and $[f^*(t)]^{\Delta}$ we will use $f^{\Delta *}(t)$.

3. Time Scale Symplectic Systems

Let $\mathcal{A}(\cdot), \mathcal{B}(\cdot), \mathcal{C}(\cdot), \mathfrak{D}(\cdot), \mathcal{W}(\cdot)$ be $n \times n$ piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that $\mathcal{W}(t) \geq 0$ for all $t \in [a,\infty)_{\mathbb{T}}$; that is, $\mathcal{W}(t)$ is Hermitian and nonnegative definite, satisfying identity (1.8). In this paper we consider the linear system $(\mathcal{S}_\lambda)$ introduced in the previous section. This system can be written as

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t)z(t,\lambda) + \lambda \mathcal{J}\widetilde{\mathcal{W}}(t)z^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \qquad (\mathcal{S}_\lambda)$$

where the $2n \times 2n$ matrix $\widetilde{\mathcal{W}}(t)$ is defined and has the property

$$\widetilde{\mathcal{W}}(t) := \begin{pmatrix} \mathcal{W}(t) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathcal{J}\widetilde{\mathcal{W}}(t) = \begin{pmatrix} 0 & 0 \\ -\mathcal{W}(t) & 0 \end{pmatrix}. \qquad (3.1)$$

The system $(\mathcal{S}_\lambda)$ can be written in the equivalent form

$$z^{\Delta}(t,\lambda) = \mathcal{S}(t,\lambda)z(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}, \qquad (3.2)$$

where the matrix $\mathcal{S}(t,\lambda)$ is defined through the matrices $\mathcal{S}(t)$ and $\widetilde{\mathcal{W}}(t)$ from (1.8) and (3.1) by

$$\mathcal{S}(t,\lambda) := \mathcal{S}(t) + \lambda\mathcal{J}\widetilde{\mathcal{W}}(t)\left[I+\mu(t)\mathcal{S}(t)\right] = \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t)-\lambda\mathcal{W}(t)\left[I+\mu(t)\mathcal{A}(t)\right] & \mathcal{D}(t)-\lambda\mu(t)\mathcal{W}(t)\mathcal{B}(t) \end{pmatrix}. \tag{3.3}$$

By using the identity in (1.8), a direct calculation shows that the matrix function $\mathcal{S}(\cdot,\cdot)$ satisfies

$$\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}\left(t,\overline{\lambda}\right) + \mu(t)\mathcal{S}^*(t,\lambda)\mathcal{J}\mathcal{S}\left(t,\overline{\lambda}\right) = 0, \quad t\in[a,\infty)_{\mathbb{T}},\ \lambda\in\mathbb{C}. \tag{3.4}$$

Here $\mathcal{S}^*(t,\lambda) = [\mathcal{S}(t,\lambda)]^*$, and $\overline{\lambda}$ is the usual conjugate number to λ.

Remark 3.1. The name time scale *symplectic system* or *Hamiltonian system* has been reserved in the literature for the system of the form

$$z^{\Delta}(t) = \mathbb{S}(t)z(t), \quad t\in[a,\infty)_{\mathbb{T}}, \tag{3.5}$$

in which the matrix function $\mathbb{S}(\cdot)$ satisfies the identity in (1.8); see [44–47, 57], and compare also, for example, with [58–61]. Since for a fixed $\lambda,\nu\in\mathbb{C}$ the matrix $\mathcal{S}(t,\lambda)$ from (3.3) satisfies

$$\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}(t,\nu) + \mu(t)\mathcal{S}^*(t,\lambda)\mathcal{J}\mathcal{S}(t,\nu) = \left(\overline{\lambda}-\nu\right)\left[I+\mu(t)\mathcal{S}^*(t)\right]\widetilde{\mathcal{W}}(t)\left[I+\mu(t)\mathcal{S}(t)\right], \tag{3.6}$$

it follows that the system $(\mathcal{S}_\lambda)$ is a true time scale symplectic system according to the above terminology only for $\lambda\in\mathbb{R}$, while strictly speaking $(\mathcal{S}_\lambda)$ *is not* a time scale symplectic system for $\lambda\in\mathbb{C}\setminus\mathbb{R}$. However, since $(\mathcal{S}_\lambda)$ is a perturbation of the time scale symplectic system $(\mathcal{S}_0)$ and since the important properties of time scale symplectic systems needed in the presented Weyl-Titchmarsh theory, such as (3.4) or (3.8), are satisfied in an appropriate modification, we accept with the above understanding the same terminology for the system $(\mathcal{S}_\lambda)$ for any $\lambda\in\mathbb{C}$.

Equation (3.4) represents a fundamental identity for the theory of time scale symplectic systems $(\mathcal{S}_\lambda)$. Some important properties of the matrix $\mathcal{S}(t,\lambda)$ are displayed below. Note that formula (3.7) is a generalization of [46, equation (10.4)] to complex values of λ.

Lemma 3.2. *Identity* (3.4) *is equivalent to the identity*

$$\mathcal{S}\left(t,\overline{\lambda}\right)\mathcal{J} + \mathcal{J}\mathcal{S}^*(t,\lambda) + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\mathcal{J}\mathcal{S}^*(t,\lambda) = 0, \quad t\in[a,\infty)_{\mathbb{T}},\ \lambda\in\mathbb{C}. \tag{3.7}$$

In this case for any $\lambda \in \mathbb{C}$ we have

$$[I + \mu(t)\mathcal{S}^*(t,\lambda)]\mathcal{J}\left[I + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\right] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.8}$$

$$\left[I + \mu(t)\mathcal{S}\left(t,\overline{\lambda}\right)\right]\mathcal{J}[I + \mu(t)\mathcal{S}^*(t,\lambda)] = \mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}, \tag{3.9}$$

and the matrices $I + \mu(t)\mathcal{S}(t,\lambda)$ and $I + \mu(t)\mathcal{S}(t,\overline{\lambda})$ are invertible with

$$[I + \mu(t)\mathcal{S}(t,\lambda)]^{-1} = -\mathcal{J}\left[I + \mu(t)\mathcal{S}^*\left(t,\overline{\lambda}\right)\right]\mathcal{J}, \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.10}$$

Proof. Let $t \in [a,\infty)_{\mathbb{T}}$ and $\lambda \in \mathbb{C}$ be fixed. If t is right-dense, that is, $\mu(t) = 0$, then identity (3.4) reduces to $\mathcal{S}^*(t,\lambda)\mathcal{J} + \mathcal{J}\mathcal{S}(t,\overline{\lambda}) = 0$. Upon multiplying this equation by $\mathcal{J}$ from the left and right side, we get identity (3.7) with $\mu(t) = 0$. If t is right scattered, that is, $\mu(t) > 0$, then (3.4) is equivalent to (3.8). It follows that the determinants of $I + \mu(t)\mathcal{S}(t,\lambda)$ and $I + \mu(t)\mathcal{S}(t,\overline{\lambda})$ are nonzero proving that these matrices are invertible with the inverse given by (3.10). Upon multiplying (3.8) by the invertible matrices $[I + \mu(t)\mathcal{S}(t,\overline{\lambda})]\mathcal{J}$ from the left and $-[I + \mu(t)\mathcal{S}(t,\overline{\lambda})]^{-1}\mathcal{J}$ from the right and by using $\mathcal{J}^2 = -I$, we get formula (3.9), which is equivalent to (3.7) due to $\mu(t) > 0$. □

Remark 3.3. Equation (3.10) allows writing the system $(\mathcal{S}_\lambda)$ in the equivalent adjoint form

$$z^{\Delta}(t,\lambda) = \mathcal{J}\mathcal{S}^*\left(t,\overline{\lambda}\right)\mathcal{J}z^{\sigma}(t,\lambda), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.11}$$

System (3.11) can be found, for example, in [47, Remark 3.1(iii)] or [50, equation (3.2)] in the connection with optimality conditions for variational problems over time scales.

In the following result we show that (3.4) guarantees, among other properties, the existence and uniqueness of solutions of the initial value problems associated with $(\mathcal{S}_\lambda)$.

Theorem 3.4 (existence and uniqueness theorem). *Let $\lambda \in \mathbb{C}$, $t_0 \in [a,\infty)_{\mathbb{T}}$, and $z_0 \in \mathbb{C}^{2n}$ be given. Then the initial value problem $(\mathcal{S}_\lambda)$ with $z(t_0) = z_0$ has a unique solution $z(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$ on the interval $[a,\infty)_{\mathbb{T}}$.*

Proof. The coefficient matrix of system $(\mathcal{S}_\lambda)$, or equivalently of system (3.2), is piecewise rd-continuous on $[a,\infty)_{\mathbb{T}}$. By Lemma 3.2, the matrix $I+\mu(t)\mathcal{S}(t,\lambda)$ is invertible for all $t \in [a,\infty)_{\mathbb{T}}$, which proves that the function $\mathcal{S}(\cdot,\lambda)$ is regressive on $[a,\infty)_{\mathbb{T}}$. Hence, the result follows from Remark 2.1. □

If not specified otherwise, we use a common agreement that $2n$-vector solutions of system $(\mathcal{S}_\lambda)$ and $2n \times n$-matrix solutions of system $(\mathcal{S}_\lambda)$ are denoted by small letters and capital letters, respectively, typically by $z(\cdot,\lambda)$ or $\tilde{z}(\cdot,\lambda)$ and $Z(\cdot,\lambda)$ or $\widetilde{Z}(\cdot,\lambda)$.

Next we establish several identities involving solutions of system $(\mathcal{S}_\lambda)$ or solutions of two such systems with different spectral parameters. The first result is the Lagrange identity known in the special cases of continuous time linear Hamiltonian systems in [11, Theorem 4.1] or [8, equation (2.23)], discrete linear Hamiltonian systems in [9, equation (2.55)] or [14, Lemma 2.2], discrete symplectic systems in [1, Lemma 2.6] or [2, Lemma 2.3], and time scale linear Hamiltonian systems in [3, Lemma 3.5] and [5, Theorem 2.2].

Theorem 3.5 (Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n\times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then*

$$[z^*(t,\lambda)\mathcal{J}z(t,\nu)]^\Delta = \left(\overline{\lambda}-\nu\right)z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)z^\sigma(t,\nu), \quad t\in[a,\infty)_\mathbb{T}. \tag{3.12}$$

Proof. Formula (3.12) follows from the time scales product rule (2.1) by using the relation $z^\sigma(t,\lambda) = [I+\mu(t)\mathcal{S}(t,\lambda)]z(t,\lambda)$ and identity (3.6). □

As consequences of Theorem 3.5, we obtain the following.

Corollary 3.6. *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $z(\cdot,\nu)$ are $2n\times m$ solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, then for all $t \in [a,\infty)_\mathbb{T}$ we have*

$$z^*(t,\lambda)\mathcal{J}z(t,\nu) = z^*(a,\lambda)\mathcal{J}z(a,\nu) + \left(\overline{\lambda}-\nu\right)\int_a^t z^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)z^\sigma(s,\nu)\Delta s. \tag{3.13}$$

One can easily see that if $z(\cdot,\lambda)$ is a solution of system $(\mathcal{S}_\lambda)$, then $z(\cdot,\overline{\lambda})$ is a solution of system $(\mathcal{S}_{\overline{\lambda}})$. Therefore, Theorem 3.5 with $\nu=\overline{\lambda}$ yields a Wronskian-type property of solutions of system $(\mathcal{S}_\lambda)$.

Corollary 3.7. *Let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. For any $2n\times m$ solution $z(\cdot,\lambda)$ of systems $(\mathcal{S}_\lambda)$*

$$z^*(t,\lambda)\mathcal{J}z\left(t,\overline{\lambda}\right) \equiv z^*(a,\lambda)\mathcal{J}z\left(a,\overline{\lambda}\right), \quad \textit{is constant on } [a,\infty)_\mathbb{T}. \tag{3.14}$$

The following result gives another interesting property of solutions of system $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$.

Lemma 3.8. *Let $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. For any $2n\times m$ solutions $z(\cdot,\lambda)$ and $\tilde{z}(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$, the $2n\times 2n$ matrix function $K(\cdot,\lambda)$ defined by*

$$K(t,\lambda) := z(t,\lambda)\tilde{z}^*\left(t,\overline{\lambda}\right) - \tilde{z}(t,\lambda)z^*\left(t,\overline{\lambda}\right), \quad t\in[a,\infty)_\mathbb{T}, \tag{3.15}$$

satisfies the dynamic equation

$$K^\Delta(t,\lambda) = \mathcal{S}(t,\lambda)K(t,\lambda) + [I+\mu(t)\mathcal{S}(t,\lambda)]K(t,\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right), \quad t\in[a,\infty)_\mathbb{T}, \tag{3.16}$$

and the identities $K^*(t,\lambda) = -K(t,\overline{\lambda})$ and

$$K^\sigma(t,\lambda) = [I+\mu(t)\mathcal{S}(t,\lambda)]K(t,\lambda)\left[I+\mu(t)\mathcal{S}^*\left(t,\overline{\lambda}\right)\right], \quad t\in[a,\infty)_\mathbb{T}. \tag{3.17}$$

Proof. Having that $z(\cdot,\lambda)$ and $\tilde{z}(\cdot,\lambda)$ are solutions of system $(\mathcal{S}_\lambda)$, it follows that $z(\cdot,\overline{\lambda})$ and $\tilde{z}(\cdot,\overline{\lambda})$ are solutions of system $(\mathcal{S}_{\overline{\lambda}})$. The results then follow by direct calculations. □

Remark 3.9. The content of Lemma 3.8 appears to be new both in the continuous and discrete time cases. Moreover, when the matrix function $K(\cdot,\lambda) \equiv K(\lambda)$ is constant, identity (3.17) yields for any right-scattered $t \in [a,\infty)_{\mathbb{T}}$ that

$$\mathcal{S}(t,\lambda)K(\lambda) + K(\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right) + \mu(t)\mathcal{S}(t,\lambda)K(\lambda)\mathcal{S}^*\left(t,\overline{\lambda}\right) = 0. \tag{3.18}$$

It is interesting to note that this formula is very much like (3.7). More precisely, identity (3.7) is a consequence of (3.18) for the case of $K(\lambda) \equiv \mathcal{J}$.

Next we present properties of certain fundamental matrices $\Psi(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$, which are generalizations of the corresponding results in [46, Section 10.2] to complex λ. Some of these results can be proven under the weaker condition that the initial value of $\Psi(a,\lambda)$ does depend on λ and satisfies $\Psi^*(a,\lambda)\mathcal{J}\Psi(a,\overline{\lambda}) = \mathcal{J}$. However, these more general results will not be needed in this paper.

Lemma 3.10. *Let $\lambda \in \mathbb{C}$ be fixed. If $\Psi(\cdot,\lambda)$ is a fundamental matrix of system $(\mathcal{S}_\lambda)$ such that $\Psi(a,\lambda)$ is symplectic and independent of λ, then for any $t \in [a,\infty)_{\mathbb{T}}$ we have*

$$\Psi^*(t,\lambda)\mathcal{J}\Psi\left(t,\overline{\lambda}\right) = \mathcal{J}, \qquad \Psi^{-1}(t,\lambda) = -\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right)\mathcal{J}, \qquad \Psi(t,\lambda)\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right) = \mathcal{J}. \tag{3.19}$$

Proof. Identity (3.19)(i) is a consequence of Corollary 3.7, in which we use the fact that $\Psi(a,\lambda)$ is symplectic and independent of λ. The second identity in (3.19) follows from the first one, while the third identity is obtained from the equation $\Psi(t,\lambda)\Psi^{-1}(t,\lambda) = I$. □

Remark 3.11. If the fundamental matrix $\Psi(\cdot,\lambda) = (Z(\cdot,\lambda)\ \ \tilde{Z}(\cdot,\lambda))$ in Lemma 3.10 is partitioned into two $2n \times n$ blocks, then (3.19)(i) and (3.19)(iii) have, respectively, the form

$$Z^*(t,\lambda)\mathcal{J}Z\left(t,\overline{\lambda}\right) = 0, \qquad Z^*(t,\lambda)\mathcal{J}\tilde{Z}\left(t,\overline{\lambda}\right) = I, \qquad \tilde{Z}^*(t,\lambda)\mathcal{J}\tilde{Z}\left(t,\overline{\lambda}\right) = 0, \tag{3.20}$$

$$Z(t,\lambda)\tilde{Z}^*\left(t,\overline{\lambda}\right) - \tilde{Z}(t,\lambda)Z^*\left(t,\overline{\lambda}\right) = \mathcal{J}. \tag{3.21}$$

Observe that the matrix on the left-hand side of (3.21) represents a constant matrix $K(t,\lambda)$ from Lemma 3.8 and Remark 3.9.

Corollary 3.12. *Under the conditions of Lemma 3.10, for any $t \in [a,\infty)_{\mathbb{T}}$, we have*

$$\Psi^\sigma(t,\lambda)\mathcal{J}\Psi^*\left(t,\overline{\lambda}\right) = [I + \mu(t)\mathcal{S}(t,\lambda)]\mathcal{J}, \tag{3.22}$$

which in the notation of Remark 3.11 has the form

$$Z^\sigma(t,\lambda)\tilde{Z}^*\left(t,\overline{\lambda}\right) - \tilde{Z}^\sigma(t,\lambda)Z^*\left(t,\overline{\lambda}\right) = [I + \mu(t)\mathcal{S}(t,\lambda)]\mathcal{J}. \tag{3.23}$$

Proof. Identity (3.22) follows from the equation $\Psi^{\sigma}(t,\lambda) = [I + \mu(t)\mathcal{S}(t,\lambda)]\Psi(t,\lambda)$ by applying formula (3.19)(ii). $\square$

4. $M(\lambda)$-Function for Regular Spectral Problem

In this section we consider the regular spectral problem on the time scale interval $[a,b]_{\mathbb{T}}$ with some fixed $b \in (a,\infty)_{\mathbb{T}}$. We will specify the corresponding boundary conditions in terms of complex $n \times 2n$ matrices from the set

$$\Gamma := \left\{ \alpha \in \mathbb{C}^{n\times 2n},\ \alpha\alpha^* = I,\ \alpha\mathcal{J}\alpha^* = 0 \right\}. \tag{4.1}$$

The two defining conditions for $\alpha \in \mathbb{C}^{n\times 2n}$ in (4.1) imply that the $2n \times 2n$ matrix $(\alpha^* \quad -\mathcal{J}\alpha^*)$ is unitary and symplectic. This yields the identity

$$(\alpha^* \quad -\mathcal{J}\alpha^*) \begin{pmatrix} \alpha \\ \alpha\mathcal{J} \end{pmatrix} = I \in \mathbb{C}^{2n\times 2n}, \quad \text{that is,}\ \alpha^*\alpha - \mathcal{J}\alpha^*\alpha\mathcal{J} = I. \tag{4.2}$$

The last equation also implies, compare with [60, Remark 2.1.2], that

$$\operatorname{Ker}\alpha = \operatorname{Im}\mathcal{J}\alpha^*. \tag{4.3}$$

Let $\alpha, \beta \in \Gamma$ be fixed and consider the boundary value problem

$$(\mathcal{S}_\lambda), \quad \alpha z(a,\lambda) = 0, \quad \beta z(b,\lambda) = 0. \tag{4.4}$$

Our first result shows that the boundary conditions in (4.4) are equivalent with the boundary conditions phrased in terms of the images of the $2n \times 2n$ matrices

$$R_a := (\mathcal{J}\alpha^* \quad 0), \qquad R_b := (0 \quad -\mathcal{J}\beta^*), \tag{4.5}$$

which satisfy $R_a^*\mathcal{J}R_a = 0$, $R_b^*\mathcal{J}R_b = 0$, and $\operatorname{rank}(R_a^* \quad R_b^*) = 2n$.

Lemma 4.1. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. A solution $z(\cdot,\lambda)$ of system $(\mathcal{S}_\lambda)$ satisfies the boundary conditions in* (4.4) *if and only if there exists a unique vector $\xi \in \mathbb{C}^{2n}$ such that*

$$z(a,\lambda) = R_a\xi, \qquad z(b,\lambda) = R_b\xi. \tag{4.6}$$

Proof. Assume that (4.4) holds. Identity (4.3) implies the existence of vectors $\xi_a, \xi_b \in \mathbb{C}^n$ such that $z(a,\lambda) = -\mathcal{J}\alpha^*\xi_a$ and $z(b,\lambda) = -\mathcal{J}\beta^*\xi_b$. It follows that $z(\cdot,\lambda)$ satisfies (4.6) with $\xi := (-\xi_a^* \quad \xi_b^*)^*$. It remains to prove that ξ is unique such a vector. If $z(\cdot,\lambda)$ satisfies (4.6) and also $z(a,\lambda) = R_a\zeta$ and $z(b,\lambda) = R_b\zeta$ for some $\xi,\zeta \in \mathbb{C}^{2n}$, then $R_a(\xi-\zeta) = 0$ and $R_b(\xi-\zeta) = 0$. Hence, $\mathcal{J}\alpha^*(I \quad 0)(\xi-\zeta) = 0$ and $-\mathcal{J}\beta^*(0 \quad I)(\xi-\zeta) = 0$. If we multiply the latter two equalities by $\alpha\mathcal{J}$ and $\beta\mathcal{J}$, respectively, and use $\alpha\alpha^* = I = \beta\beta^*$, then we obtain $(I \quad 0)(\xi-\zeta) = 0$ and $(0 \quad I)(\xi-\zeta) = 0$.

This yields $\xi - \zeta = 0$, which shows that the vector ξ in (4.6) is unique. The opposite direction, that is, that (4.6) implies (4.4), is trivial. $\square$

Following the standard terminology, see, for example, [62, 63], a number $\lambda \in \mathbb{C}$ is an *eigenvalue* of (4.4) if this boundary value problem has a solution $z(\cdot, \lambda) \not\equiv 0$. In this case the function $z(\cdot, \lambda)$ is called the *eigenfunction* corresponding to the eigenvalue λ, and the dimension of the space of all eigenfunctions corresponding to λ (together with the zero function) is called the *geometric multiplicity* of λ.

Given $\alpha \in \Gamma$, we will utilize from now on the fundamental matrix $\Psi(\cdot, \lambda, \alpha)$ of system $(\mathcal{S}_\lambda)$ satisfying the initial condition from (4.4), that is,

$$\Psi^{\Delta}(t, \lambda, \alpha) = \mathcal{S}(t, \lambda)\Psi(t, \lambda, \alpha), \quad t \in [a, \rho(b)]_{\mathbb{T}}, \quad \Psi(a, \lambda, \alpha) = \begin{pmatrix} \alpha^* & -\mathcal{J}\alpha^* \end{pmatrix}. \tag{4.7}$$

Then $\Psi(a, \lambda, \alpha)$ does not depend on λ, and it is symplectic and unitary with the inverse $\Psi^{-1}(a, \lambda, \alpha) = \Psi^*(a, \lambda, \alpha)$. Hence, the properties of fundamental matrices derived earlier in Lemma 3.10, Remark 3.11, and Corollary 3.12 apply for the matrix function $\Psi(\cdot, \lambda, \alpha)$.

The following assumption will be imposed in this section when studying the regular spectral problem.

Hypothesis 4.2. For every $\lambda \in \mathbb{C}$, we have

$$\int_a^b \Psi^{\sigma *}(t, \lambda, \alpha)\widetilde{\mathcal{W}}(t)\Psi^{\sigma}(t, \lambda, \alpha)\Delta t > 0. \tag{4.8}$$

Condition (4.8) can be written in the equivalent form as

$$\int_a^b z^{\sigma *}(t, \lambda)\widetilde{\mathcal{W}}(t) z^{\sigma}(t, \lambda)\Delta t > 0, \tag{4.9}$$

for every nontrivial solution $z(\cdot, \lambda)$ of system $(\mathcal{S}_\lambda)$. Assumptions (4.8) and (4.9) are equivalent by a simple argument using the uniqueness of solutions of system $(\mathcal{S}_\lambda)$. The latter form (4.9) has been widely used in the literature, such as in the continuous time case in [8, Hypothesis 2.2], [30, equation (1.3)], [26, equation (2.3)], in the discrete time case in [9, Condition (2.16)], [14, equation (1.7)], [1, Assumption 2.2], [2, Hypothesis 2.4], and in the time scale Hamiltonian case in [3, Assumption 3] and [5, Condition (3.9)].

Following Remark 3.11, we partition the fundamental matrix $\Psi(\cdot, \lambda, \alpha)$ as

$$\Psi(\cdot, \lambda, \alpha) = \begin{pmatrix} Z(\cdot, \lambda, \alpha) & \widetilde{Z}(\cdot, \lambda, \alpha) \end{pmatrix}, \tag{4.10}$$

where $Z(\cdot, \lambda, \alpha)$ and $\widetilde{Z}(\cdot, \lambda, \alpha)$ are the $2n \times n$ solutions of system $(\mathcal{S}_\lambda)$ satisfying $Z(a, \lambda, \alpha) = \alpha^*$ and $\widetilde{Z}(a, \lambda, \alpha) = -\mathcal{J}\alpha^*$. With the notation

$$\Lambda(\lambda, \alpha, \beta) := \Psi(b, \lambda, \alpha)\Psi^*(a, \lambda, \alpha)R_a - R_b = \begin{pmatrix} -\widetilde{Z}(b, \lambda, \alpha) & \mathcal{J}\beta^* \end{pmatrix}, \tag{4.11}$$

we have the classical characterization of the eigenvalues of (4.4); see, for example, the continuous time in [64, Chapter 4], the discrete time in [14, Theorem 2.3, Lemma 2.4], [2, Lemma 2.9, Theorem 2.11], and the time scale case in [62, Lemma 3], [63, Corollary 1].

Proposition 4.3. *For $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$, we have with notation* (4.11) *the following.*

(i) *The number λ is an eigenvalue of* (4.4) *if and only if* $\det \Lambda(\lambda, \alpha, \beta) = 0$.

(ii) *The algebraic multiplicity of the eigenvalue λ, that is, the number* $\operatorname{def} \Lambda(\lambda, \alpha, \beta)$, *is equal to the geometric multiplicity of λ.*

(iii) *Under Hypothesis 4.2, the eigenvalues of* (4.4) *are real, and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle z(\cdot, \lambda), z(\cdot, \nu) \rangle_{\mathcal{W},b} := \int_a^b z^{\sigma *}(t, \lambda) \widetilde{\mathcal{W}}(t) z^{\sigma}(t, \nu) \Delta t. \tag{4.12}$$

Proof. The arguments are here standard, and we refer to [44, Section 5], [63, Corollary 1], [3, Theorem 3.6]. □

The next algebraic characterization of the eigenvalues of (4.4) is more appropriate for the development of the Weyl-Titchmarsh theory for (4.4), since it uses the matrix $\beta \widetilde{Z}(b, \lambda, \alpha)$ which has dimension n instead of using the matrix $\Lambda(\lambda, \alpha, \beta)$ which has dimension $2n$. Results of this type can be found in special cases of system $(\mathcal{S}_\lambda)$ in [8, Lemma 2.5], [11, Theorem 4.1], [9, Lemma 2.8], [14, Lemma 3.1], [1, Lemma 2.5], [3, Theorem 3.4], and [2, Lemma 3.1].

Lemma 4.4. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C}$ be fixed. Then λ is an eigenvalue of* (4.4) *if and only if* $\det \beta \widetilde{Z}(b, \lambda, \alpha) = 0$. *In this case the algebraic and geometric multiplicities of λ are equal to* $\operatorname{def} \beta \widetilde{Z}(b, \lambda, \alpha)$.

Proof. One can follow the same arguments as in the proof of the corresponding discrete symplectic case in [2, Lemma 3.1]. However, having the result of Proposition 4.3, we can proceed directly by the methods of linear algebra. In this proof we abbreviate $\Lambda := \Lambda(\lambda, \alpha, \beta)$ and $\widetilde{Z} := \widetilde{Z}(b, \lambda, \alpha)$. Assume that Λ is singular, that is, $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$ for some vectors $c, d \in \mathbb{C}^n$, not both zero. Then $\widetilde{Z}c = \mathcal{J}\beta^* d$, which yields that $\beta \widetilde{Z} c = 0$. If $c = 0$, then $\mathcal{J}\beta^* d = 0$, which implies upon the multiplication by $\beta \mathcal{J}$ from the left that $d = 0$. Since not both c and d can be zero, it follows that $c \neq 0$ and the matrix $\beta \widetilde{Z}$ is singular. Conversely, if $\beta \widetilde{Z} c = 0$ for some nonzero vector $c \in \mathbb{C}^n$, then $-\widetilde{Z}c + \mathcal{J}\beta^* d = 0$; that is, Λ is singular, with the vector $d := -\beta \mathcal{J} \widetilde{Z} c$. Indeed, by using identity (4.2) we have $\mathcal{J}\beta^* d = -\mathcal{J}\beta^* \beta \mathcal{J} \widetilde{Z} c = (I - \beta^* \beta) \widetilde{Z} c = \widetilde{Z} c$. From the above we can also see that the number of linearly independent vectors in $\operatorname{Ker} \beta \widetilde{Z}$ is the same as the number of linearly independent vectors in $\operatorname{Ker} \Lambda$. Therefore, by Proposition 4.3(ii), the algebraic and geometric multiplicities of λ as an eigenvalue of (4.4) are equal to $\operatorname{def} \beta \widetilde{Z}$. □

Since the eigenvalues of (4.4) are real, it follows that the matrix $\beta \widetilde{Z}(b, \lambda, \alpha)$ is invertible for every $\lambda \in \mathbb{C}$ except for at most n real numbers. This motivates the definition of the $M(\lambda)$-function for the regular spectral problem.

Definition 4.5 ($M(\lambda)$-function). Let $\alpha, \beta \in \Gamma$. Whenever the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible for some value $\lambda \in \mathbb{C}$, we define the *Weyl-Titchmarsh $M(\lambda)$-function* as the $n \times n$ matrix

$$M(\lambda) = M(\lambda, b) = M(\lambda, b, \alpha, \beta) := -\left[\beta\widetilde{Z}(b,\lambda,\alpha)\right]^{-1}\beta Z(b,\lambda,\alpha). \tag{4.13}$$

The above definition of the $M(\lambda)$-function is a generalization of the corresponding definitions for the continuous and discrete linear Hamiltonian and symplectic systems in [8, Definition 2.6], [9, Definition 2.9], [14, equation (3.10)], [1, page 2859], [2, Definition 3.2] and time scale linear Hamiltonian systems in [3, equation (4.1)]. The dependence of the $M(\lambda)$-function on b, α, and β will be suppressed in the notation, and $M(\lambda, b)$ or $M(\lambda, b, \alpha, \beta)$ will be used only in few situations when we emphasize the dependence on b (such as at the end of Section 5) or on α and β (as in Lemma 4.14). By [65, Corollary 4.5], see also [44, Remark 2.2], the $M(\cdot)$-function is an entire function in λ. Another important property of the $M(\lambda)$-function is established in the following.

Lemma 4.6. *Let $\alpha, \beta \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$M^*(\lambda) = M\left(\overline{\lambda}\right). \tag{4.14}$$

Proof. We abbreviate $Z(\lambda) := Z(b,\lambda,\alpha)$ and $\widetilde{Z}(\lambda) := \widetilde{Z}(b,\lambda,\alpha)$. By using the definition of $M(\lambda)$ in (4.13) and identity (3.21), we have

$$\begin{aligned} M^*(\lambda) - M\left(\overline{\lambda}\right) &= \left[\beta\widetilde{Z}\left(\overline{\lambda}\right)\right]^{-1}\beta\left[Z\left(\overline{\lambda}\right)\widetilde{Z}^*(\lambda) - \widetilde{Z}\left(\overline{\lambda}\right)Z^*(\lambda)\right]\beta^*\left[\beta\widetilde{Z}(\lambda)\right]^{*-1} \\ &\overset{(3.21)}{=} \left[\beta\widetilde{Z}\left(\overline{\lambda}\right)\right]^{-1}\beta\mathcal{J}\beta^*\left[\beta\widetilde{Z}(\lambda)\right]^{*-1} = 0, \end{aligned} \tag{4.15}$$

because $\beta \in \Gamma$. Hence, equality (4.14) holds true. □

The following solution plays an important role in particular in the results concerning the square integrable solutions of system $(\mathcal{S}_\lambda)$.

Definition 4.7 (Weyl solution). For any matrix $M \in \mathbb{C}^{n\times n}$, we define the so-called *Weyl solution* of system $(\mathcal{S}_\lambda)$ by

$$\mathcal{X}(\cdot,\lambda,\alpha,M) := \Psi(\cdot,\lambda,\alpha)\left(I \;\; M^*\right)^* = Z(\cdot,\lambda,\alpha) + \widetilde{Z}(\cdot,\lambda,\alpha)M, \tag{4.16}$$

where $Z(\cdot,\lambda,\alpha)$ and $\widetilde{Z}(\cdot,\lambda,\alpha)$ are defined in (4.10).

The function $\mathcal{X}(\cdot,\lambda,\alpha,M)$, being a linear combination of two solutions of system $(\mathcal{S}_\lambda)$, is also a solution of this system. Moreover, $\alpha\mathcal{X}(a,\lambda,\alpha,M) = I$, and, if $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible, then $\beta\widetilde{\mathcal{X}}(b,\lambda,\alpha,M) = \beta\widetilde{Z}(b,\lambda,\alpha)[M - M(\lambda)]$. Consequently, if we take $M := M(\lambda)$ in Definition 4.7, then $\beta\mathcal{X}(b,\lambda,\alpha,M(\lambda)) = 0$; that is, the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M(\lambda))$ satisfies the right endpoint boundary condition in (4.4).

Following the corresponding notions in [8, equation (2.18)], [9, equation (2.51)], [14, page 471], [1, page 2859], [2, equation (3.13)], [3, equation (4.2)], we define the Hermitian $n \times n$ matrix function $\mathcal{E}(M)$ for system $(\mathcal{S}_\lambda)$.

Definition 4.8. For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we define the matrix function

$$\mathcal{E} : \mathbb{C}^{n\times n} \longrightarrow \mathbb{C}^{n\times n}, \qquad \mathcal{E}(M) = \mathcal{E}(M,b) := i\delta(\lambda)\mathcal{X}^*(b,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(b,\lambda,\alpha,M), \tag{4.17}$$

where $\delta(\lambda) := \operatorname{sgn} \operatorname{Im}(\lambda)$.

For brevity we suppress the dependence of the function $\mathcal{E}(\cdot)$ on b and λ. In few cases we will need $\mathcal{E}(M)$ depending on b (as in Theorem 5.1 and Definition 6.2) and in such situations we will use the notation $\mathcal{E}(M,b)$. Since $(i\mathcal{J})^* = i\mathcal{J}$, it follows that $\mathcal{E}(M)$ is a Hermitian matrix for any $M \in \mathbb{C}^{n\times n}$. Moreover, from Corollary 3.6, we obtain the identity

$$\mathcal{E}(M) = -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t, \tag{4.18}$$

where we used the fact that

$$\mathcal{X}^*(a,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(a,\lambda,\alpha,M) \overset{(4.7)}{=} M - M^* = 2i\operatorname{Im}(M). \tag{4.19}$$

Next we define the Weyl disk and Weyl circle for the regular spectral problem. The geometric characterizations of the Weyl disk and Weyl circle in terms of the contractive or unitary matrices which justify the terminology "disk" or "circle" will be presented in Section 5.

Definition 4.9 (Weyl disk and Weyl circle). For a fixed $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the set

$$D(\lambda) = D(\lambda,b) := \left\{M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) \le 0\right\}, \tag{4.20}$$

is called the *Weyl disk*, and the set

$$C(\lambda) = C(\lambda,b) := \partial D(\lambda) = \left\{M \in \mathbb{C}^{n\times n},\ \mathcal{E}(M) = 0\right\}, \tag{4.21}$$

is called the *Weyl circle*.

The dependence of the Weyl disk and Weyl circle on b will be again suppressed. In the following result we show that the Weyl circle consists of precisely those matrices $M(\lambda)$ with $\beta \in \Gamma$. This result generalizes the corresponding statements in [8, Lemma 2.8], [9, Lemma 2.13], [14, Lemma 3.3], [1, Theorem 3.1], [2, Theorem 3.6], and [3, Theorem 4.2].

Theorem 4.10. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the Weyl circle $C(\lambda)$ if and only if there exists $\beta \in \Gamma$ such that $\beta\mathcal{X}(b,\lambda,\alpha,M) = 0$. In this case and under Hypothesis 4.2, we have with such a matrix β that $M = M(\lambda)$ as defined in* (4.13).

Proof. Assume that $M \in C(\lambda)$, that is, $\mathcal{E}(M) = 0$. Then, with the vector

$$\beta := \mathcal{X}^*(b)\mathcal{J} = \begin{pmatrix} I & M^* \end{pmatrix}\Psi^*(b,\lambda,\alpha)\mathcal{J} \in \mathbb{C}^{n\times 2n}, \tag{4.22}$$

where $\mathcal{X}(b)$ denotes $\mathcal{X}(b,\lambda,\alpha,M)$, we have

$$\beta\mathcal{X}(b) = \mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b) = \left[\frac{1}{(i\delta(\lambda))}\right]\mathcal{E}(M) = 0. \tag{4.23}$$

Moreover, $\operatorname{rank}\beta = n$, because the matrices $\Psi(b,\lambda,\alpha)$ and $\mathcal{J}$ are invertible and $\operatorname{rank}(I \;\; M^*) = n$. In addition, the identity $\mathcal{J}^* = \mathcal{J}^{-1}$ yields

$$\beta\mathcal{J}\beta^* = \mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b) \overset{(4.23)}{=} 0. \tag{4.24}$$

Now, if the condition $\beta\beta^* = I$ is not satisfied, then we replace β by $\widetilde{\beta} := (\beta\beta^*)^{-1/2}\beta$ (note that $\beta\beta^* > 0$, so that $(\beta\beta^*)^{-1/2}$ is well defined), and in this case

$$\begin{aligned}
\widetilde{\beta}\mathcal{X}(b) &= (\beta\beta^*)^{-1/2}\beta\mathcal{X}(b) \overset{(4.23)}{=} 0, \\
\widetilde{\beta}\mathcal{J}\widetilde{\beta}^* &= (\beta\beta^*)^{-1/2}\beta\mathcal{J}\beta^*(\beta\beta^*)^{-1/2} \overset{(4.24)}{=} 0, \\
\widetilde{\beta}\widetilde{\beta}^* &= (\beta\beta^*)^{-1/2}\beta\beta^*(\beta\beta^*)^{-1/2} = I.
\end{aligned} \tag{4.25}$$

Conversely, suppose that for a given $M \in \mathbb{C}^{n\times n}$ there exists $\beta \in \Gamma$ such that $\beta\mathcal{X}(b) = 0$. Then from (4.3) it follows that $\mathcal{X}(b) = \mathcal{J}\beta^* P$ for the matrix $P := -\beta\mathcal{J}\mathcal{X}(b) \in \mathbb{C}^{n\times n}$. Hence,

$$\mathcal{E}(M) = i\delta(\lambda)P^*\beta\mathcal{J}^*\mathcal{J}\mathcal{J}\beta^*P = i\delta(\lambda)P^*\beta\mathcal{J}\beta^*P = 0, \tag{4.26}$$

that is, $M \in C(\lambda)$. Finally, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, then by Proposition 4.3(iii) the number λ is not an eigenvalue of (4.4), which by Lemma 4.4 shows that the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible. The definition of the Weyl solution in (4.16) then yields

$$\beta Z(b,\lambda,\alpha) + \beta\widetilde{Z}(b,\lambda,\alpha)M = \beta\mathcal{X}(b,\lambda,\alpha,M) = 0, \tag{4.27}$$

which implies that $M = -\left[\beta\widetilde{Z}(b,\lambda,\alpha)\right]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. □

Remark 4.11. The matrix $P := -\beta\mathcal{J}\mathcal{X}(b,\lambda,\alpha,M) \in \mathbb{C}^{n\times n}$ from the proof of Theorem 4.10 is invertible. This fact was not needed in that proof. However, we show that P is invertible because this argument will be used in the proof of Lemma 4.14. First we prove that $\operatorname{Ker} P = \operatorname{Ker}\mathcal{X}(b,\lambda,\alpha,M)$. For if $Pd = 0$ for some $d \in \mathbb{C}^n$, then from identity (4.2) we get $\mathcal{X}(b,\lambda,\alpha,M)d = (I - \beta^*\beta)\mathcal{X}(b,\lambda,\alpha,M)d = \mathcal{J}\beta^*Pd = 0$. Therefore, $\operatorname{Ker} P \subseteq \operatorname{Ker}\mathcal{X}(b,\lambda,\alpha,M)$. The opposite inclusion follows by the definition of P. And since, by (4.16), $\operatorname{rank}\mathcal{X}(b,\lambda,\alpha,M) = \operatorname{rank}(I \;\; M^*)^* = n$, it follows that $\operatorname{Ker}\mathcal{X}(b,\lambda,\alpha,M) = \{0\}$. Hence, $\operatorname{Ker} P = \{0\}$ as well; that is, the matrix P is invertible.

The next result contains a characterization of the matrices $M \in \mathbb{C}^{n\times n}$ which lie "inside" the Weyl disk $D(\lambda)$. In the previous result (Theorem 4.10) we have characterized the elements of the boundary of the Weyl disk $D(\lambda)$, that is, the elements of the Weyl circle $C(\lambda)$, in terms of the matrices $\beta \in \Gamma$. For such β we have $\beta\mathcal{J}\beta^* = 0$, which yields $i\delta(\lambda)\beta\mathcal{J}\beta^* = 0$. Comparing with that statement we now utilize the matrices $\beta \in \mathbb{C}^{n\times 2n}$ which satisfy $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$. In the special cases of the continuous and discrete time, this result can be found in [8, Lemma 2.13], [9, Lemma 2.18], and [2, Theorem 3.13].

Theorem 4.12. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M satisfies $\mathcal{E}(M) < 0$ if and only if there exists $\beta \in \mathbb{C}^{n\times 2n}$ such that $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ and $\beta\mathcal{X}(b,\lambda,\alpha,M) = 0$. In this case and under Hypothesis 4.2, we have with such a matrix β that $M = M(\lambda)$ as defined in* (4.13) *and β may be chosen so that $\beta\beta^* = I$.*

Proof. For $M \in \mathbb{C}^{n\times n}$ consider on $[a,b]_{\mathbb{T}}$ the Weyl solution

$$\mathcal{X}(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M) = \begin{pmatrix} \mathcal{X}_1(\cdot) \\ \mathcal{X}_2(\cdot) \end{pmatrix}, \quad \text{with } n\times n \text{ blocks } \mathcal{X}_1(\cdot) \text{ and } \mathcal{X}_2(\cdot). \tag{4.28}$$

Suppose first that $\mathcal{E}(M) < 0$. Then the matrices $\mathcal{X}_j(b)$, $j \in \{1,2\}$, are invertible. Indeed, if one of them is singular, then there exists a nonzero vector $v \in \mathbb{C}^n$ such that $\mathcal{X}_1(b)v = 0$ or $\mathcal{X}_2(b)v = 0$. Then

$$v^*\mathcal{E}(M)v = i\delta(\lambda)v^*\mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b)v = i\delta(\lambda)v^*\left[\mathcal{X}_1^*(b)\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\mathcal{X}_1(b)\right]v = 0, \tag{4.29}$$

which contradicts $\mathcal{E}(M) < 0$. Now we set $\beta_1 := I$, $\beta_2 := -\mathcal{X}_1(b)\mathcal{X}_2^{-1}(b)$, and $\beta := (\beta_1 \;\; \beta_2)$. Then for this $2n \times n$ matrix β we have $\beta\mathcal{X}(b) = 0$ and, by a similar calculation as in (4.29),

$$\begin{aligned} \mathcal{E}(M) &= i\delta(\lambda)\mathcal{X}^*(b)\mathcal{J}\mathcal{X}(b) = i\delta(\lambda)\mathcal{X}_2^*(b)\left(\beta_2\beta_1^* - \beta_1\beta_2^*\right)\mathcal{X}_2(b) \\ &= 2\delta(\lambda)\mathcal{X}_2^*(b)\operatorname{Im}\left(\beta_1\beta_2^*\right)\mathcal{X}_2(b) = -i\delta(\lambda)\mathcal{X}_2^*(b)\beta\mathcal{J}\beta^*\mathcal{X}_2(b), \end{aligned} \tag{4.30}$$

where we used the equality $\beta\mathcal{J}\beta^* = 2i\operatorname{Im}(\beta_1\beta_2^*)$. Since $\mathcal{E}(M) < 0$ and $\mathcal{X}_2(b)$ is invertible, it follows that $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$. Conversely, assume that for a given matrix $M \in \mathbb{C}^{n\times n}$ there is $\beta = (\beta_1 \;\; \beta_2) \in \mathbb{C}^{n\times 2n}$ satisfying $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ and $\beta\mathcal{X}(b) = 0$. Condition $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ is equivalent to $\operatorname{Im}(\beta_1\beta_2^*) < 0$ when $\operatorname{Im}(\lambda) > 0$ and to $\operatorname{Im}(\beta_1\beta_2^*) > 0$ when $\operatorname{Im}(\lambda) < 0$. The positive or negative definiteness of $\operatorname{Im}(\beta_1\beta_2^*)$ implies the invertibility of β_1 and β_2; see Remark 2.2. Therefore, from the equality $\beta_1\mathcal{X}_1(b) + \beta_2\mathcal{X}_2(b) = \beta\mathcal{X}(b) = 0$, we obtain $\mathcal{X}_1(b) = -\beta_1^{-1}\beta_2\mathcal{X}_2(b)$, and so

$$\begin{aligned} \mathcal{E}(M) &= i\delta(\lambda)\left[\mathcal{X}_1^*(b)\mathcal{X}_2(b) - \mathcal{X}_2^*(b)\mathcal{X}_1(b)\right] \\ &= i\delta(\lambda)\mathcal{X}_2^*(b)\beta_1^{-1}\left(\beta_2\beta_1^* - \beta_1\beta_2^*\right)\beta_1^{*-1}\mathcal{X}_2(b) \\ &= -i\delta(\lambda)\mathcal{X}_2^*(b)\beta_1^{-1}\beta\mathcal{J}\beta^*\beta_1^{*-1}\mathcal{X}_2(b). \end{aligned} \tag{4.31}$$

The matrix $\mathcal{X}_2(b)$ is invertible, because if $\mathcal{X}_2(b)d = 0$ for some nonzero vector $d \in \mathbb{C}^n$, then $\mathcal{X}_1(b)d = -\beta_1^{-1}\beta_2\mathcal{X}_2(b)d = 0$, showing that $\operatorname{rank}\mathcal{X}(b) < n$. This however contradicts $\operatorname{rank}\mathcal{X}(b) = n$ which we have from the definition of the Weyl solution $\mathcal{X}(\cdot)$ in (4.16). Consequently, (4.31) yields through $i\delta(\lambda)\beta\mathcal{J}\beta^* > 0$ that $\mathcal{E}(M) < 0$.

If the matrix β does not satisfy $\beta\beta^* = I$, then we modify it according to the procedure described in the proof of Theorem 4.10. Finally, since $\lambda \in \mathbb{C} \setminus \mathbb{R}$, we get from Proposition 4.3(iii) and Lemma 4.4 that the matrix $\beta\widetilde{Z}(b,\lambda,\alpha)$ is invertible which in turn implies through the calculation in (4.27) that $M = -[\beta\widetilde{Z}(b,\lambda,\alpha)]^{-1}\beta Z(b,\lambda,\alpha) = M(\lambda)$. □

In the following lemma we derive some additional properties of the Weyl disk and the $M(\lambda)$-function. Special cases of this statement can be found in [8, Lemma 2.9], [33, Theorem 3.1], [9, Lemma 2.14], [14, Lemma 3.2(ii)], [1, Theorem 3.7], [2, Lemma 3.7], and [3, Theorem 4.13].

Theorem 4.13. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. For any matrix $M \in D(\lambda)$ we have*

$$\delta(\lambda)\operatorname{Im}(M) \geq |\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \geq 0. \tag{4.32}$$

In addition, under Hypothesis 4.2, we have $\delta(\lambda)\operatorname{Im}(M) > 0$.

Proof. By identity (4.18), for any matrix $M \in D(\lambda)$, we have

$$\begin{aligned} 2\delta(\lambda)\operatorname{Im}(M) &= -\mathcal{E}(M) + 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\ &\geq 2|\operatorname{Im}(\lambda)| \int_a^b \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t, \end{aligned} \tag{4.33}$$

which yields together with $\widetilde{\mathcal{W}}(t) \geq 0$ on $[a,\rho(b)]_{\mathbb{T}}$ the inequalities in (4.32). The last assertion in Theorem 4.13 is a simple consequence of Hypothesis 4.2. □

In the last part of this section we wish to study the effect of changing α, which is one of the parameters of the $M(\lambda)$-function and the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, when α varies within the set Γ. For this purpose we will use the $M(\lambda)$-function with all its arguments in the following two statements.

Lemma 4.14. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$M(\lambda,b,\alpha,\beta) = \left[\alpha\mathcal{J}\gamma^* + \alpha\gamma^* M(\lambda,b,\gamma,\beta)\right]\left[\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M(\lambda,b,\gamma,\beta)\right]^{-1}. \tag{4.34}$$

Proof. Let $M(b,\lambda,\alpha,\beta)$ and $M(b,\lambda,\gamma,\beta)$ be given via (4.13), and consider the Weyl solutions $\mathcal{X}_\alpha(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M(b,\lambda,\alpha,\beta))$ and $\mathcal{X}_\gamma(\cdot) := \mathcal{X}(\cdot,\lambda,\gamma,M(b,\lambda,\gamma,\beta))$ defined by (4.16) with $M = M(b,\lambda,\alpha,\beta)$ and $M = M(b,\lambda,\gamma,\beta)$, respectively. First we prove that the two Weyl solutions $\mathcal{X}_\alpha(\cdot)$ and $\mathcal{X}_\gamma(\cdot)$ differ by a constant nonsingular multiple. By definition, $\beta\mathcal{X}_\alpha(b) = 0$ and $\beta\mathcal{X}_\gamma(b) = 0$, which implies through (4.3) that $\mathcal{X}_\alpha(b) = \mathcal{J}\beta^* P_\alpha$ and $\mathcal{X}_\gamma(b) = \mathcal{J}\beta^* P_\gamma$

for some matrices $P_\alpha, P_\gamma \in \mathbb{C}^{n\times n}$, which are invertible by Remark 4.11. This implies that $\mathcal{X}_\alpha(b)P_\alpha^{-1} = \mathcal{J}\beta^* = \mathcal{X}_\gamma(b)P_\gamma^{-1}$. Consequently, $\mathcal{X}_\alpha(b) = \mathcal{X}_\gamma(b)P$, where $P := P_\gamma^{-1}P_\alpha$. By the uniqueness of solutions of system $(\mathcal{S}_\lambda)$, see Theorem 3.4, we obtain that $\mathcal{X}_\alpha(\cdot) = \mathcal{X}_\gamma(\cdot)P$ on $[a,b]_{\mathbb{T}}$. Upon the evaluation at $t = a$ we get

$$\Psi(a,\lambda,\alpha)\begin{pmatrix} I \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} = \Psi(a,\lambda,\gamma)\begin{pmatrix} I \\ M(\lambda,b,\gamma,\beta)\end{pmatrix}P. \tag{4.35}$$

Since the matrices $\Psi(a,\lambda,\alpha) = (\alpha^* \quad -\mathcal{J}\alpha^*)$ and $\Psi(a,\lambda,\gamma) = (\gamma^* \quad -\mathcal{J}\gamma^*)$ are unitary, it follows from (4.35) that

$$\begin{aligned}\begin{pmatrix} I \\ M(\lambda,b,\alpha,\beta)\end{pmatrix} &= \begin{pmatrix} \alpha \\ \alpha\mathcal{J}\end{pmatrix}(\gamma^* \quad -\mathcal{J}\gamma^*)\begin{pmatrix} I \\ M(\lambda,b,\gamma,\beta)\end{pmatrix}P \\ &= \begin{pmatrix} \alpha\gamma^* - \alpha\mathcal{J}\gamma^* M(\lambda,b,\gamma,\beta) \\ \alpha\mathcal{J}\gamma^* + \alpha\gamma^* M(\lambda,b,\gamma,\beta)\end{pmatrix}P.\end{aligned} \tag{4.36}$$

The first row above yields that $P = [\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M(\lambda,b,\gamma,\beta)]^{-1}$, while the second row is then written as identity (4.34). □

Corollary 4.15. *Let $\alpha,\beta,\gamma \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. With notation* (4.16) *and* (4.13) *we have*

$$\mathcal{X}(\cdot,\lambda,\alpha,M(\lambda,b,\alpha,\beta)) = \mathcal{X}(\cdot,\lambda,\gamma,M(\lambda,b,\gamma,\beta))\,[\alpha\gamma^* - \alpha\mathcal{J}\gamma^* M(\lambda,b,\gamma,\beta)]^{-1}. \tag{4.37}$$

Proof. The above identity follows from (4.35) and the formula for the matrix P from the end of the proof of Lemma 4.14. □

5. Geometric Properties of Weyl Disks

In this section we study the geometric properties of the Weyl disks as the point b moves through the interval $[a,\infty)_{\mathbb{T}}$. Our first result shows that the Weyl disks $D(\lambda,b)$ are nested. This statement generalizes the results in [11, Theorem 4.5], [66, Section 3.2.1], [9, equation (2.70)], [14, Theorem 3.1], [3, Theorem 4.4], and [5, Theorem 3.3(i)].

Theorem 5.1 (nesting property of Weyl disks). *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C}\setminus\mathbb{R}$. Then*

$$D(\lambda,b_2) \subseteq D(\lambda,b_1), \quad \textit{for every } b_1,b_2 \in [a,\infty)_{\mathbb{T}},\ b_1 < b_2. \tag{5.1}$$

Proof. Let $b_1, b_2 \in [a,\infty)_{\mathbb{T}}$ with $b_1 < b_2$, and take $M \in D(\lambda, b_2)$, that is, $\mathcal{E}(M, b_2) \le 0$. From identity (4.18) with $b = b_1$ and later with $b = b_2$ and by using $\widetilde{\mathcal{W}}(\cdot) \ge 0$, we have

$$\begin{aligned}
\mathcal{E}(M,b_1) &\overset{(4.18)}{=} -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^{b_1} \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\
&\le -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)| \int_a^{b_2} \mathcal{X}^{\sigma*}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^{\sigma}(t,\lambda,\alpha,M)\Delta t \\
&\overset{(4.18)}{=} \mathcal{E}(M,b_2) \le 0.
\end{aligned} \tag{5.2}$$

Therefore, by Definition 4.9, the matrix M belongs to $D(\lambda, b_1)$, which shows the result. □

Similarly for the regular case (Hypothesis 4.2) we now introduce the following assumption.

Hypothesis 5.2. There exists $b_0 \in (a,\infty)_{\mathbb{T}}$ such that Hypothesis 4.2 is satisfied with $b = b_0$; that is, inequality (4.8) holds with $b = b_0$ for every $\lambda \in \mathbb{C}$.

From Hypothesis 5.2 it follows by $\widetilde{\mathcal{W}}(\cdot) \ge 0$ that inequality (4.8) holds for every $b \in [b_0,\infty)_{\mathbb{T}}$.

For the study of the geometric properties of Weyl disks we will use the following representation:

$$\mathcal{E}(M,b) = i\delta(\lambda)\mathcal{X}^*(b,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(b,\lambda,\alpha,M) = \begin{pmatrix} I & M^* \end{pmatrix} \begin{pmatrix} \mathcal{F}(b,\lambda,\alpha) & \mathcal{G}^*(b,\lambda,\alpha) \\ \mathcal{G}(b,\lambda,\alpha) & \mathcal{H}(b,\lambda,\alpha) \end{pmatrix} \begin{pmatrix} I \\ M \end{pmatrix}, \tag{5.3}$$

of the matrix $\mathcal{E}(M,b)$, where we define on $[a,\infty)_{\mathbb{T}}$ the $n \times n$ matrices

$$\begin{aligned}
\mathcal{F}(\cdot,\lambda,\alpha) &:= i\delta(\lambda) Z^*(\cdot,\lambda,\alpha)\mathcal{J}Z(\cdot,\lambda,\alpha), \\
\mathcal{G}(\cdot,\lambda,\alpha) &:= i\delta(\lambda) \widetilde{Z}^*(\cdot,\lambda,\alpha)\mathcal{J}Z(\cdot,\lambda,\alpha), \\
\mathcal{H}(\cdot,\lambda,\alpha) &:= i\delta(\lambda) \widetilde{Z}^*(\cdot,\lambda,\alpha)\mathcal{J}\widetilde{Z}(\cdot,\lambda,\alpha).
\end{aligned} \tag{5.4}$$

Since $\mathcal{E}(M,b)$ is Hermitian, it follows that $\mathcal{F}(\cdot,\lambda,\alpha)$ and $\mathcal{H}(\cdot,\lambda,\alpha)$ are also Hermitian. Moreover, by (4.7), we have $\mathcal{H}(a,\lambda,\alpha) = 0$. In addition, if $b \in [b_0,\infty)_{\mathbb{T}}$, then Corollary 3.7 and Hypothesis 5.2 yield for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$

$$\mathcal{H}(b,\lambda,\alpha) = 2|\operatorname{Im}(\lambda)| \int_a^b \widetilde{Z}^{\sigma*}(t,\lambda,\alpha)\widetilde{\mathcal{W}}(t)\widetilde{Z}^{\sigma}(t,\lambda,\alpha)\Delta t > 0. \tag{5.5}$$

Therefore, $\mathcal{H}(b,\lambda,\alpha)$ is invertible (positive definite) for all $b \in [b_0,\infty)_{\mathbb{T}}$ and monotone nondecreasing as $b \to \infty$, with a consequence that $\mathcal{H}^{-1}(b,\lambda,\alpha)$ is monotone nonincreasing as $b \to \infty$. The following factorization of $\mathcal{E}(M,b)$ holds true; see also [2, equation (4.11)].

Lemma 5.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. With the notation* (5.4), *for any $M \in \mathbb{C}^{n\times n}$ and $b \in [a,\infty)_{\mathbb{T}}$ we have*

$$\begin{aligned}\mathcal{E}(M,b) &= \mathcal{F}(b,\lambda,\alpha) - \mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha)\\ &\quad + \left[\mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha) + M^*\right]\mathcal{H}(b,\lambda,\alpha)\left[\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) + M\right],\end{aligned} \tag{5.6}$$

whenever the matrix $\mathcal{H}(b,\lambda,\alpha)$ is invertible.

Proof. The result is shown by a direct calculation. □

The following identity is a generalization of its corresponding versions in [11, Lemma 4.3], [1, Lemma 3.3], [14, Proposition 3.2], [2, Lemma 4.2], [3, Lemma 4.6], and [5, Theorem 5.6].

Lemma 5.4. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. With the notation* (5.4), *for any $b \in [a,\infty)_{\mathbb{T}}$, we have*

$$\mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) - \mathcal{F}(b,\lambda,\alpha) = \mathcal{H}^{-1}\left(b,\overline{\lambda},\alpha\right), \tag{5.7}$$

whenever the matrices $\mathcal{H}(b,\lambda,\alpha)$ and $\mathcal{H}(b,\overline{\lambda},\alpha)$ are invertible.

Proof. In order to simplify and abbreviate the notation we introduce the matrices

$$\begin{aligned}&\mathcal{F} := \mathcal{F}(b,\lambda,\alpha), \qquad \mathcal{G} := \mathcal{G}(b,\lambda,\alpha), \qquad \mathcal{H} := \mathcal{H}(b,\lambda,\alpha),\\ &\widetilde{\mathcal{F}} := \mathcal{F}\left(b,\overline{\lambda},\alpha\right), \qquad \widetilde{\mathcal{G}} := \mathcal{G}\left(b,\overline{\lambda},\alpha\right), \qquad \widetilde{\mathcal{H}} := \mathcal{H}\left(b,\overline{\lambda},\alpha\right),\end{aligned} \tag{5.8}$$

and use the notation $Z(\lambda)$ and $\widetilde{Z}(\lambda)$ for $Z(b,\lambda,\alpha)$ and $\widetilde{Z}(b,\lambda,\alpha)$, respectively. Then, since $\mathcal{F}^* = \mathcal{F}$ and $\delta(\lambda)\delta(\overline{\lambda}) = -1$, we get the identities

$$\mathcal{G}^*\widetilde{\mathcal{F}} - \mathcal{F}^*\widetilde{\mathcal{G}} = Z^*(\lambda)\mathcal{J}\left[\widetilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\widetilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.21)}{=} Z^*(\lambda)\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.20)}{=} 0, \tag{5.9}$$

$$\mathcal{H}\widetilde{\mathcal{G}}^* - \mathcal{G}\mathcal{H}^* = \widetilde{Z}^*(\lambda)\mathcal{J}\left[\widetilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\widetilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}\widetilde{Z}\left(\overline{\lambda}\right) \overset{(3.21)}{=} \widetilde{Z}^*(\lambda)\mathcal{J}\widetilde{Z}\left(\overline{\lambda}\right) \overset{(3.20)}{=} 0, \tag{5.10}$$

$$\mathcal{G}\widetilde{\mathcal{G}} - \mathcal{H}\widetilde{\mathcal{F}} = \widetilde{Z}^*(\lambda)\mathcal{J}\left[Z(\lambda)\widetilde{Z}^*\left(\overline{\lambda}\right) - \widetilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right)\right]\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.21)}{=} -\widetilde{Z}^*(\lambda)\mathcal{J}Z\left(\overline{\lambda}\right) \overset{(3.20)}{=} I, \tag{5.11}$$

$$\mathcal{G}^*\widetilde{\mathcal{G}}^* - \mathcal{F}\widetilde{\mathcal{H}} = Z^*(\lambda)\mathcal{J}\left[\widetilde{Z}(\lambda)Z^*\left(\overline{\lambda}\right) - Z(\lambda)\widetilde{Z}^*\left(\overline{\lambda}\right)\right]\mathcal{J}\widetilde{Z}\left(\overline{\lambda}\right) \overset{(3.21)}{=} Z^*(\lambda)\mathcal{J}\widetilde{Z}\left(\overline{\lambda}\right) \overset{(3.20)}{=} I. \tag{5.12}$$

Hence, by using that $\widetilde{\mathcal{H}}$ is Hermitian, we see that

$$\widetilde{\mathcal{H}}^{-1} \overset{(5.12)}{=} \mathcal{G}^*\widetilde{\mathcal{G}}^*\widetilde{\mathcal{H}}^{-1} - \mathcal{F} = \mathcal{G}^*\widetilde{\mathcal{G}}^*\widetilde{\mathcal{H}}^{*-1} - \mathcal{F} \overset{(5.10)}{=} \mathcal{G}^*\mathcal{H}^{-1}\mathcal{G} - \mathcal{F}. \tag{5.13}$$

Identity (5.7) is now proven. □

Corollary 5.5. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, the matrix $\mathcal{H}(b,\lambda,\alpha)$ is invertible for every $b \in [b_0,\infty)_{\mathbb{T}}$, and for these values of b we have*

$$\mathcal{G}^*(b,\lambda,\alpha)\mathcal{H}^{-1}(b,\lambda,\alpha)\mathcal{G}(b,\lambda,\alpha) - \mathcal{F}(b,\lambda,\alpha) > 0. \tag{5.14}$$

Proof. Since $b \in [b_0,\infty)_{\mathbb{T}}$, then identity (5.5) yields that $\mathcal{H}(b,\lambda,\alpha) > 0$ and $\mathcal{H}(b,\overline{\lambda},\alpha) > 0$. Consequently, inequality (5.14) follows from (5.7) of Lemma 5.4. □

In the next result we justify the terminology for the sets $D(\lambda,b)$ and $C(\lambda,b)$ in Definition 4.9 to be called a “disk” and a “circle.” It is a generalization of [14, Theorem 3.1], [2, Theorem 5.4], [5, Theorem 3.3(iii)]; see also [66, Theorem 3.5], [26, pages 70-71], [8, page 3485], [14, Proposition 3.3], [1, Theorem 3.3], [3, Theorem 4.8]. Consider the sets $\mathcal{V}$ and $\mathcal{U}$ of contractive and unitary matrices in $\mathbb{C}^{n\times n}$, respectively, that is,

$$\mathcal{V} := \left\{V \in \mathbb{C}^{n\times n},\ V^*V \le I\right\}, \qquad \mathcal{U} := \partial\mathcal{V} = \left\{U \in \mathbb{C}^{n\times n},\ U^*U = I\right\}. \tag{5.15}$$

The set $\mathcal{V}$ is known to be closed (in fact compact, since $\mathcal{V}$ is bounded) and convex.

Theorem 5.6. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, for every $b \in [b_0,\infty)_{\mathbb{T}}$, the Weyl disk and Weyl circle have the representations*

$$D(\lambda,b) = \left\{P(\lambda,b) + R(\lambda,b)VR\left(\overline{\lambda},b\right),\ V \in \mathcal{V}\right\}, \tag{5.16}$$

$$C(\lambda,b) = \left\{P(\lambda,b) + R(\lambda,b)UR\left(\overline{\lambda},b\right),\ U \in \mathcal{U}\right\}, \tag{5.17}$$

where, with the notation (5.4),

$$P(\lambda,b) := -\mathcal{H}^{-1}(\lambda,b,\alpha)\mathcal{G}(\lambda,b,\alpha), \qquad R(\lambda,b) := \mathcal{H}^{-1/2}(\lambda,b,\alpha). \tag{5.18}$$

Consequently, for every $b \in [b_0,\infty)_{\mathbb{T}}$, the sets $D(\lambda,b)$ are closed and convex.

The representations of $D(\lambda,b)$ and $C(\lambda,b)$ in (5.16) and (5.17) can be written as $D(\lambda,b) = P(\lambda,b) + R(\lambda,b)\mathcal{V}R(\overline{\lambda},b)$ and $C(\lambda,b) = P(\lambda,b) + R(\lambda,b)\mathcal{U}R(\overline{\lambda},b)$. The importance of the matrices $P(\lambda,b)$ and $R(\lambda,b)$ is justified in the following.

Definition 5.7. For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $b \in [a,\infty)_{\mathbb{T}}$ such that $\mathcal{H}(\lambda,b,\alpha)$ and $\mathcal{H}(\overline{\lambda},b,\alpha)$ are positive definite, the matrix $P(\lambda,b)$ is called the center of the Weyl disk or the Weyl circle. The matrices $R(\lambda,b)$ and $R(\overline{\lambda},b)$ are called the *matrix radii* of the Weyl disk or the Weyl circle.

Proof of Theorem 5.6. By (5.5) and for any $b \in [b_0,\infty)_{\mathbb{T}}$, the matrices $\mathcal{H} := \mathcal{H}(\lambda,b,\alpha)$ and $\widetilde{\mathcal{H}} := \mathcal{H}(\overline{\lambda},b,\alpha)$ are positive definite, so that the matrices $P := P(\lambda,b)$, $R(\lambda) := R(\lambda,b)$, and

$R(\overline{\lambda}) := R(\overline{\lambda}, b)$ are well defined. By Definition 4.9, for $M \in D(\lambda, b)$, we have $\mathcal{E}(M, b) \le 0$, which in turn with notation (5.8) implies by Lemmas 5.3 and 5.4 that

$$-R^2\left(\overline{\lambda}\right) + (M^* - P^*)R^{-2}(\lambda)(M - P) \stackrel{(5.7)}{=} \mathcal{F} - \mathcal{G}^*\mathcal{H}^{-1}\mathcal{G} + \left(\mathcal{H}^{-1}\mathcal{G} + M\right)^*\mathcal{H}\left(\mathcal{H}^{-1}\mathcal{G} + M\right) = \mathcal{E}(M, b) \le 0. \tag{5.19}$$

Therefore, the matrix

$$V := R^{-1}(\lambda)(M - P)R^{-1}\left(\overline{\lambda}\right), \tag{5.20}$$

satisfies $V^*V \le I$. This relation between the matrices $M \in D(\lambda, b)$ and $V \in \mathcal{V}$ is bijective (more precisely, it is a homeomorphism), and the inverse to (5.20) is given by $M = P + R(\lambda)VR(\overline{\lambda})$. The latter formula proves that the Weyl disk $D(\lambda, b)$ has the representation in (5.16). Moreover, since by the definition $M \in C(\lambda, b)$ means that $\mathcal{E}(M, b) = 0$, it follows that the elements of the Weyl circle $C(\lambda, b)$ are in one-to-one correspondence with the matrices V defined in (5.20) which, similarly as in (5.19), now satisfy $V^*V = I$. Hence, the representation of $C(\lambda, b)$ in (5.17) follows. The fact that for $b \in [b_0, \infty)_{\mathbb{T}}$ the sets $D(\lambda, b)$ are closed and convex follows from the same properties of the set $\mathcal{V}$, being homeomorphic to $D(\lambda, b)$. □

6. Limiting Weyl Disk and Weyl Circle

In this section we study the limiting properties of the Weyl disk and Weyl circle and their center and matrix radii. Since under Hypothesis 5.2 the matrix function $\mathcal{H}(\cdot, \lambda, \alpha)$ is monotone nondecreasing as $b \to \infty$, it follows from the definition of $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ in (5.18) that the two matrix functions $R(\lambda, \cdot)$ and $R(\overline{\lambda}, \cdot)$ are monotone nonincreasing for $b \to \infty$. Furthermore, since $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ are Hermitian and positive definite for $b \in [b_0, \infty)_{\mathbb{T}}$, the limits

$$R_+(\lambda) := \lim_{b\to\infty} R(\lambda, b), \qquad R_+\left(\overline{\lambda}\right) := \lim_{b\to\infty} R\left(\overline{\lambda}, b\right), \tag{6.1}$$

exist and satisfy $R_+(\lambda) \ge 0$ and $R_+(\overline{\lambda}) \ge 0$. The index "+" in the above notation as well as in Definition 6.2 refers to the limiting disk at $+\infty$. In the following result we will see that the center $P(\lambda, b)$ also converges to a limiting matrix when $b \to \infty$. This is a generalization of [11, Theorem 4.7], [1, Theorem 3.5], [14, Proposition 3.5], [2, Theorem 4.5], and [3, Theorem 4.10].

Theorem 6.1. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, the center $P(\lambda, b)$ converges as $b \to \infty$ to a limiting matrix $P_+(\lambda) \in \mathbb{C}^{n\times n}$, that is,*

$$P_+(\lambda) := \lim_{b\to\infty} P(\lambda, b). \tag{6.2}$$

Proof. We prove that the matrix function $P(\lambda, \cdot)$ satisfies the Cauchy convergence criterion. Let $b_1, b_2 \in [b_0, \infty)_{\mathbb{T}}$ be given with $b_1 < b_2$. By Theorem 5.1, we have that $D(\lambda, b_2) \subseteq D(\lambda, b_1)$.

Therefore, by (5.16) of Theorem 5.6, for a matrix $M \in D(\lambda, b_2)$, there are (unique) matrices $V_1, V_2 \in \mathcal{V}$ such that

$$M = P(\lambda, b_j) + R(\lambda, b_j) V_j R\left(\overline{\lambda}, b_j\right), \quad j \in \{1,2\}. \tag{6.3}$$

Upon subtracting the two equations in (6.3), we get

$$P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2) V_2 R\left(\overline{\lambda}, b_2\right) = R(\lambda, b_1) V_1 R\left(\overline{\lambda}, b_1\right). \tag{6.4}$$

This equation, when solved for V_1 in terms of V_2, has the form

$$V_1 = R^{-1}(\lambda, b_1)\left[P(\lambda, b_2) - P(\lambda, b_1) + R(\lambda, b_2) V_2 R\left(\overline{\lambda}, b_2\right)\right] R^{-1}\left(\overline{\lambda}, b_1\right) =: T(V_2), \tag{6.5}$$

which defines a continuous mapping $T : \mathcal{V} \to \mathcal{V}$, $T(V_2) = V_1$. Since $\mathcal{V}$ is compact, it follows that the mapping T has a fixed point in $\mathcal{V}$, that is, $T(V) = V$ for some matrix $V \in \mathcal{V}$. Equation $T(V) = V$ implies that

$$\begin{aligned} P(\lambda, b_2) - P(\lambda, b_1) &= R(\lambda, b_1) V R\left(\overline{\lambda}, b_1\right) - R(\lambda, b_2) V R\left(\overline{\lambda}, b_2\right) \\ &= [R(\lambda, b_1) - R(\lambda, b_2)] V R\left(\overline{\lambda}, b_1\right) - R(\lambda, b_2) V \left[R\left(\overline{\lambda}, b_1\right) - R\left(\overline{\lambda}, b_2\right)\right]. \end{aligned} \tag{6.6}$$

Hence, by $\|V\| \le 1$, we have

$$\|P(\lambda, b_2) - P(\lambda, b_1)\| \le \|R(\lambda, b_1) - R(\lambda, b_2)\| \left\|R\left(\overline{\lambda}, b_1\right)\right\| + \|R(\lambda, b_2)\| \left\|R\left(\overline{\lambda}, b_1\right) - R\left(\overline{\lambda}, b_2\right)\right\|. \tag{6.7}$$

Since the functions $R(\lambda, \cdot)$ and $R(\overline{\lambda}, \cdot)$ are monotone nonincreasing, they are bounded; that is, for some $K > 0$, we have $\|R(\lambda, b)\| \le K$ and $\|R(\overline{\lambda}, b)\| \le K$ for all $b \in [b_0, \infty)_{\mathbb{T}}$.

Let $\varepsilon > 0$ be arbitrary. The convergence of $R(\lambda, b)$ and $R(\overline{\lambda}, b)$ as $b \to \infty$ yields the existence of $b_3 \in [b_0, \infty)_{\mathbb{T}}$ such that for every $b_1, b_2 \in [b_3, \infty)_{\mathbb{T}}$ with $b_1 < b_2$ we have

$$\|R(\nu, b_1) - R(\nu, b_2)\| \le \frac{\varepsilon}{(2K)}, \quad \nu \in \left\{\lambda, \overline{\lambda}\right\}. \tag{6.8}$$

Using estimate (6.8) in inequality (6.7) we obtain for $b_2 > b_1 \ge b_3$

$$\|P(\lambda, b_2) - P(\lambda, b_1)\| < \frac{\varepsilon}{(2K)} \cdot K + \frac{\varepsilon}{(2K)} \cdot K = \varepsilon. \tag{6.9}$$

This means that the limit $P_+(\lambda) \in \mathbb{C}^{n \times n}$ in (6.2) exists, which completes the proof. $\square$

By Theorems 5.1 and 5.6 we know that the Weyl disks $D(\lambda, b)$ are closed, convex, and nested as $b \to \infty$. Thereore the limit of $D(\lambda, b)$ as $b \to \infty$ is a closed, convex, and nonempty set. This motivates the following definition, which can be found in the special cases of system $(\mathcal{S}_\lambda)$ in [26, Theorem 3.3], [1, Theorem 3.6], [2, Definition 4.7], and [3, Theorem 4.12].

Definition 6.2 (limiting Weyl disk). Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then the set

$$D_+(\lambda) := \bigcap_{b \in [a,\infty)_{\mathbb{T}}} D(\lambda, b), \tag{6.10}$$

is called the *limiting Weyl disk*. The matrix $P_+(\lambda)$ from Theorem 6.1 is called the *center* of $D_+(\lambda)$ and the matrices $R_+(\lambda)$ and $R_+(\overline{\lambda})$ from (6.1) its *matrix radii*.

As a consequence of Theorem 5.6, we obtain the following characterization of the limiting Weyl disk.

Corollary 6.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Under Hypothesis 5.2, we have*

$$D_+(\lambda) = P_+(\lambda) + R_+(\lambda) \mathcal{V} R_+\left(\overline{\lambda}\right), \tag{6.11}$$

where $\mathcal{V}$ is the set of all contractive matrices defined in (5.15).

From now on we assume that Hypothesis 5.2 holds, so that the limiting center $P_+(\lambda)$ and the limiting matrix radii $R_+(\lambda)$ and $R_+(\overline{\lambda})$ of $D_+(\lambda)$ are well defined.

Remark 6.4. By means of the nesting property of the disks (Theorem 5.1) and Theorems 4.10 and 4.12, it follows that the elements of the limiting Weyl disk $D_+(\lambda)$ are of the form

$$M_+(\lambda) \in D_+(\lambda), \quad M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta(b)), \tag{6.12}$$

where $\beta(b) \in \mathbb{C}^{n \times 2n}$ satisfies $\beta(b)\beta^*(b) = I$ and $i\delta(\lambda)\beta(b)\mathcal{J}\beta^*(b) \geq 0$ for all $b \in [a, \infty)$. Moreover, from Lemma 4.6, we conclude that

$$M_+^*(\lambda) = M_+\left(\overline{\lambda}\right). \tag{6.13}$$

A matrix $M_+(\lambda)$ from (6.12) is called a *half-line Weyl-Titchmarsh $M(\lambda)$-function*. Also, as noted in [2, Section 4], see also [8, Theorem 2.18], the function $M_+(\lambda)$ is a Herglotz function with rank n and has a certain integral representation (which will not be needed in this paper).

Our next result shows another characterization of the elements of $D_+(\lambda)$ in terms of the Weyl solution $\mathcal{X}(\cdot, \alpha, \lambda, M)$ defined in (4.16). This is a generalization of [11, page 671], [26, equation (3.2)], [1, Theorem 3.8(i)], [2, Theorem 4.8], and [3, Theorem 4.15].

Theorem 6.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. The matrix M belongs to the limiting Weyl disk $D_+(\lambda)$ if and only if*

$$\int_a^\infty \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^\sigma(t,\lambda,\alpha,M)\Delta t \le \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{6.14}$$

Proof. By Definition 6.2, we have $M \in D_+(\lambda)$ if and only if $M \in D(\lambda, b)$, that is, $\mathcal{E}(M,b) \le 0$, for all $b \in [a,\infty)_\mathbb{T}$. Therefore, by formula (4.18), we get

$$\int_a^b \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^\sigma(t,\lambda,\alpha,M)\Delta t = \frac{\mathcal{E}(M,b)}{2|\operatorname{Im}(\lambda)|} + \frac{\delta(\lambda)\operatorname{Im}(M)}{|\operatorname{Im}(\lambda)|} \le \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}, \tag{6.15}$$

for every $b \in [a,\infty)_\mathbb{T}$, which is equivalent to inequality (6.14). □

Remark 6.6. In [1, Definition 3.4], the notion of a boundary of the limiting Weyl disk $D_+(\lambda)$ is discussed. This would be a "limiting Weyl circle" according to Definitions 4.9 and 6.2. The description of matrices $M \in \mathbb{C}^{n\times n}$ laying on this boundary follows from Theorems 6.5 and 4.10, giving for such matrices M the equality

$$\int_a^\infty \mathcal{X}^{\sigma *}(t,\lambda,\alpha,M)\widetilde{\mathcal{W}}(t)\mathcal{X}^\sigma(t,\lambda,\alpha,M)\Delta t = \frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}. \tag{6.16}$$

Condition (6.16) is also equivalent to

$$\lim_{t\to\infty} \mathcal{X}^*(t,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M) = 0. \tag{6.17}$$

This is because, by (4.19) and the Lagrange identity (Corollary 3.6),

$$\begin{aligned} &\mathcal{X}^*(t,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(t,\lambda,\alpha,M) \\ &\quad = 2i\operatorname{Im}(\lambda)\left[\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)} - \int_a^t \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s,\lambda,\alpha,M)\Delta s\right], \end{aligned} \tag{6.18}$$

for every $t \in [a,\infty)_\mathbb{T}$. From this we can see that the integral on the right-hand side above converges for $t \to \infty$ and (6.16) holds if and only if condition (6.17) is satisfied. Characterizations (6.16) and (6.17) of the matrices M on the boundary of the limiting Weyl disk $D_+(\lambda)$ generalize the corresponding results in [1, Theorems 3.8(ii) and 3.9]; see also [14, Theorem 6.3].

Consider the linear space of square integrable C^1_{prd} functions

$$L^2_\mathcal{W} = L^2_\mathcal{W}[a,\infty)_\mathbb{T} := \left\{z : [a,\infty)_\mathbb{T} \longrightarrow \mathbb{C}^{2n},\ z \in C^1_{\text{prd}},\ \|z(\cdot)\|_\mathcal{W} < \infty\right\}, \tag{6.19}$$

where we define

$$\|z(\cdot)\|_{\mathcal{W}} := \sqrt{\langle z(\cdot), z(\cdot)\rangle_{\mathcal{W}}}, \qquad \langle z(\cdot), \tilde{z}(\cdot)\rangle_{\mathcal{W}} := \int_a^\infty z^{\sigma *}(t)\widetilde{\mathcal{W}}(t)\tilde{z}^{\sigma}(t)\Delta t. \tag{6.20}$$

In the following result we prove that the space $L^2_{\mathcal{W}}$ contains the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$ when M belongs to the limiting Weyl disk $D_+(\lambda)$. This implies that there are at least n linearly independent solutions of system $(\mathcal{S}_\lambda)$ in $L^2_{\mathcal{W}}$. This is a generalization of [11, Theorem 5.1], [14, Theorem 4.1], [2, Theorem 4.10], and [5, page 716].

Theorem 6.7. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $M \in D_+(\lambda)$. The columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a linearly independent system of solutions of system $(\mathcal{S}_\lambda)$, each of which belongs to $L^2_{\mathcal{W}}$.*

Proof. Let $z_j(\cdot) := \mathcal{X}(\cdot,\lambda,\alpha,M)e_j$ for $j \in \{1,\dots,n\}$ be the columns of the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$, where e_j is the jth unit vector. We prove that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Assume that $\sum_{j=1}^n c_j z_j(\cdot) = 0$ on $[a,\infty)_{\mathbb{T}}$ for some $c_1,\dots,c_n \in \mathbb{C}$. Then $\mathcal{X}(\cdot,\lambda,\alpha,M)c = 0$, where $c := (c_1^*,\dots,c_n^*)^* \in \mathbb{C}^n$. It follows by (4.19) that

$$2ic^* \operatorname{Im}(M)c = c^*\mathcal{X}^*(a,\lambda,\alpha,M)\mathcal{J}\mathcal{X}(a,\lambda,\alpha,M)c = 0, \tag{6.21}$$

which implies the equality $c^*\delta(\lambda)\operatorname{Im}(M)c = 0$. Using that $M \in D_+(\lambda) \subseteq D(\lambda,b)$ for some $b \in [b_0,\infty)_{\mathbb{T}}$, we obtain from Theorem 4.13 that the matrix $\delta(\lambda)\operatorname{Im}(M)$ is positive definite. Hence, $c = 0$ so that the functions $z_1(\cdot),\dots,z_n(\cdot)$ are linearly independent. Finally, for every $j \in \{1,\dots,n\}$ we get from Theorem 6.5 the inequality

$$\|z_j(\cdot)\|^2_{\mathcal{W}} = \int_a^\infty z_j^{\sigma *}(t)\widetilde{\mathcal{W}}(t)z_j^{\sigma}(t)\Delta t \overset{(6.14)}{\leq} e_j^*\frac{\operatorname{Im}(M)}{\operatorname{Im}(\lambda)}e_j \leq \frac{\|\delta(\lambda)\operatorname{Im}(M)\|}{|\operatorname{Im}(\lambda)|} < \infty. \tag{6.22}$$

Thus, $z_j(\cdot) \in L^2_{\mathcal{W}}$ for every $j \in \{1,\dots,n\}$, and the proof is complete. □

Denote by $\mathcal{N}(\lambda)$ the linear space of all square integrable solutions of system $(\mathcal{S}_\lambda)$, that is,

$$\mathcal{N}(\lambda) := \left\{z(\cdot) \in L^2_{\mathcal{W}},\ z(\cdot) \text{ solves } (\mathcal{S}_\lambda)\right\}. \tag{6.23}$$

Then as a consequence of Theorem 6.7 we obtain the estimate

$$\dim \mathcal{N}(\lambda) \geq n, \quad \text{for each } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{6.24}$$

Next we discuss the situation when $\dim \mathcal{N}(\lambda) = n$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Lemma 6.8. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $\dim \mathcal{N}(\lambda) = n$. Then the matrix radii of the limiting Weyl disk $D_+(\lambda)$ satisfy $R_+(\lambda) = 0 = R_+(\overline{\lambda})$. Consequently, the set $D_+(\lambda)$ consists of the single matrix $M = P_+(\lambda)$, that is, the center of $D_+(\lambda)$, which is given by formula (6.2) of Theorem 6.1.*

Proof. With the matrix radii $R_+(\lambda)$ and $R_+(\overline{\lambda})$ of $D_+(\lambda)$ defined in (6.1) and with the Weyl solution $\mathcal{X}(\cdot,\lambda,\alpha,M)$ given by a matrix $M \in D_+(\lambda)$, we observe that the columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a basis of the space $\mathcal{N}(\lambda)$. Since the columns of the fundamental matrix $\Psi(\cdot,\lambda,\alpha) = (Z(\cdot,\lambda,\alpha)\ \ \widetilde{Z}(\cdot,\lambda,\alpha))$ span all solutions of system $(\mathcal{S}_\lambda)$, the definition of $\mathcal{X}(\cdot,\lambda,\alpha,M) = Z(\cdot,\lambda,\alpha) + \widetilde{Z}(\cdot,\lambda,\alpha)M$ yields that the columns of $\widetilde{Z}(\cdot,\lambda,\alpha)$ together with the columns of $\mathcal{X}(\cdot,\lambda,\alpha,M)$ form a basis of all solutions of system $(\mathcal{S}_\lambda)$. Hence, from $\dim \mathcal{N}(\lambda) = n$ and Theorem 6.7, we get that the columns of $\widetilde{Z}(\cdot,\lambda,\alpha)$ do not belong to $L^2_{\mathcal{W}}$. Consequently, by formula (5.5), the Hermitian matrix functions $\mathcal{H}(\cdot,\lambda,\alpha)$ and $\mathcal{H}(\cdot,\overline{\lambda},\alpha)$ defined in (5.4) are monotone nondecreasing on $[a,\infty)_{\mathbb{T}}$ without any upper bound; that is, their eigenvalues—being real—tend to ∞. Therefore, the functions $R(\lambda,\cdot)$ and $R(\overline{\lambda},\cdot)$ as defined in (5.18) have limits at ∞ equal to zero; that is, $R_+(\lambda) = 0$ and $R_+(\overline{\lambda}) = 0$. The fact that the set $D_+(\lambda) = \{P_+(\lambda)\}$ then follows from the characterization of $D_+(\lambda)$ in Corollary 6.3. □

In the final result of this section, we establish another characterization of the matrices M from the limiting Weyl disk $D_+(\lambda)$. In comparison with Theorem 6.5, we now use a similar condition to the one in Theorem 4.12 for the regular spectral problem. However, a stronger assumption than Hypothesis 5.2 is now required for this result to hold; compare with [9, Lemma 2.21] and [2, Theorem 4.16].

Hypothesis 6.9. For every $a_0, b_0 \in (a,\infty)_{\mathbb{T}}$ with $a_0 < b_0$ and for every $\lambda \in \mathbb{C}$, we have

$$\int_{a_0}^{b_0} \Psi^{\sigma *}(t,\lambda,\alpha)\widetilde{\mathcal{W}}(t)\Psi^{\sigma}(t,\lambda,\alpha)\Delta t > 0. \tag{6.25}$$

Under Hypothesis 6.9, the Weyl disks $D(\lambda,b)$ converge to the limiting disk "monotonically" as $b \to \infty$; that is, the limiting Weyl disk $D_+(\lambda)$ is "open" in the sense that all of its elements lie inside $D_+(\lambda)$. This can be interpreted in view of Theorem 4.12 as $\mathcal{E}(M,t) < 0$ for all $t \in [a,\infty)_{\mathbb{T}}$.

Theorem 6.10. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and $M \in \mathbb{C}^{n\times n}$. Under Hypothesis 6.9, the matrix $M \in D_+(\lambda)$ if and only if*

$$\mathcal{E}(M,t) < 0, \quad \forall t \in [a,\infty)_{\mathbb{T}}. \tag{6.26}$$

Proof. If condition (6.26) holds, then $M \in D_+(\lambda)$ follows from the definition of $D_+(\lambda)$. Conversely, suppose that $M \in D_+(\lambda)$, and let $t \in [a,\infty)_{\mathbb{T}}$ be given. Then for any $b \in (t,\infty)_{\mathbb{T}}$ we have by formula (4.18) that

$$\begin{aligned}\mathcal{E}(M,t) &= -2\delta(\lambda)\operatorname{Im}(M) + 2|\operatorname{Im}(\lambda)|\int_a^t \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\Delta s \\ &= \mathcal{E}(M,b) - 2|\operatorname{Im}(\lambda)|\int_t^b \mathcal{X}^{\sigma *}(s,\lambda,\alpha,M)\widetilde{\mathcal{W}}(s)\mathcal{X}^{\sigma}(s,\lambda,\alpha,M)\Delta s,\end{aligned} \tag{6.27}$$

where we used the property $\int_a^t f(s)\Delta s = \int_a^b f(s)\Delta s - \int_t^b f(s)\Delta s$. Since $M \in D_+(\lambda)$ is assumed, we have $M \in D(\lambda, b)$, that is, $\mathcal{E}(M, b) \leq 0$, while Hypothesis 6.9 implies the positivity of the integral over $[t, b]_{\mathbb{T}}$ in (6.27). Consequently, (6.27) yields that $\mathcal{E}(M, t) < 0$. □

Remark 6.11. If we partition the Weyl solution $\mathcal{X}(\cdot, \lambda) := \mathcal{X}(\cdot, \lambda, \alpha, M)$ into two $n \times n$ blocks $\mathcal{X}_1(\cdot, \lambda)$ and $\mathcal{X}_2(\cdot, \lambda)$ as in (4.28), then condition (6.26) can be written as

$$\delta(\lambda) \operatorname{Im}\left(\mathcal{X}_1^*(t, \lambda)\mathcal{X}_2(t, \lambda)\right) > 0, \quad \forall t \in [a, \infty)_{\mathbb{T}}. \tag{6.28}$$

Therefore, by Remark 2.2, the matrices $\mathcal{X}_1(t, \lambda)$ and $\mathcal{X}_2(t, \lambda)$ are invertible for all $t \in [a, \infty)_{\mathbb{T}}$. A standard argument then yields that the quotient $Q(\cdot, \lambda) := \mathcal{X}_2(\cdot, \lambda)\mathcal{X}_1^{-1}(\cdot, \lambda)$ satisfies the *Riccati matrix equation* (suppressing the argument t in the coefficients)

$$Q^{\Delta} - (\mathcal{C} + \mathcal{D}Q) + Q^{\sigma}(\mathcal{A} + \mathcal{B}Q) + \lambda \mathcal{W}\left[I + \mu(\mathcal{A} + \mathcal{B}Q)\right] = 0, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{6.29}$$

see [57, Theorem 3], [48, Section 6], and [49].

7. Limit Point and Limit Circle Criteria

Throughout this section we assume that Hypothesis 5.2 is satisfied. The results from Theorem 6.7 and Lemma 6.8 motivate the following terminology; compare with [4, page 75], [43, Definition 1.2] in the time scales scalar case $n = 1$, with [8, page 3486], [36, page 1668], [30, page 274], [38, Definition 3.1], [37, Definition 1], [67, page 2826] in the continuous case, and with [14, Definition 5.1], [2, Definition 4.12] in the discrete case.

Definition 7.1 (limit point and limit circle case for system $(\mathcal{S}_\lambda)$). The system $(\mathcal{S}_\lambda)$ is said to be in the *limit point case* at ∞ (or of the *limit point type*) if

$$\dim \mathcal{N}(\lambda) = n, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{7.1}$$

The system $(\mathcal{S}_\lambda)$ is said to be in the *limit circle case* at ∞ (or of the *limit circle type*) if

$$\dim \mathcal{N}(\lambda) = 2n, \quad \forall \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{7.2}$$

Remark 7.2. According to Remark 6.4 (in which $\beta(b) \equiv \beta$), the center $P_+(\lambda)$ of the limiting Weyl disk $D_+(\lambda)$ can be expressed in the limit point case as

$$P_+(\lambda) = M_+(\lambda) = \lim_{b \to \infty} M(\lambda, b, \alpha, \beta), \tag{7.3}$$

where $\beta \in \Gamma$ is arbitrary but fixed.

Next we establish the first main result of this section. Its continuous time version can be found in [30, Theorem 2.1], [11, Theorem 8.5] and the discrete time version in [9, Lemma 3.2], [2, Theorem 4.13].

Theorem 7.3. *Let the system* $(\mathcal{S}_\lambda)$ *be in the limit point or limit circle case, fix* $\alpha \in \Gamma$, *and let* $\lambda, \nu \in \mathbb{C} \setminus \mathbb{R}$. *Then*

$$\lim_{t\to\infty} \mathcal{X}_+^*(t,\lambda,\alpha,M_+(\lambda))\mathcal{J}\mathcal{X}_+(t,\nu,\alpha,M_+(\nu)) = 0, \tag{7.4}$$

where $\mathcal{X}_+(\cdot,\lambda,\alpha,M_+(\lambda))$ *and* $\mathcal{X}_+(\cdot,\nu,\alpha,M_+(\nu))$ *are the Weyl solutions of* $(\mathcal{S}_\lambda)$ *and* $(\mathcal{S}_\nu)$, *respectively, defined by* (4.16) *through the matrices* $M_+(\lambda)$ *and* $M_+(\nu)$, *which are determined by the limit in* (6.12).

Proof. For every $t \in [a,\infty)_\mathbb{T}$ and matrices $\beta(t) \in \mathbb{C}^{n\times 2n}$ such that $\beta(t)\beta^*(t) = I$ and $i\delta(\lambda)\beta(t)\mathcal{J}\beta^*(t) \geq 0$ and for $\kappa \in \{\lambda,\nu\}$, we define the matrix (compare with Definition 4.5)

$$M(\kappa,t,\alpha,\beta(t)) := -\left[\beta(t)\widetilde{Z}(t,\kappa,\alpha)\right]^{-1}\beta(t)Z(t,\kappa,\alpha). \tag{7.5}$$

Then, by Theorems 4.10 and 4.12, we have $M(\kappa,t,\alpha,\beta(t)) \in D(\kappa,t)$. Following the notation in (4.16), we consider the Weyl solutions $\mathcal{X}(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M(\kappa,t,\alpha,\beta(\cdot)))$. Similarly, let $\mathcal{X}_+(\cdot,\kappa) := \mathcal{X}(\cdot,\kappa,\alpha,M_+(\kappa))$ be the Weyl solutions corresponding to the matrices $M_+(\kappa) \in D_+(\kappa)$ from the statement of this theorem.

First assume that the system $(\mathcal{S}_\lambda)$ is of the limit point type. In this case, by Remark 7.2, we may take $\beta(t) \in \Gamma$ for all $t \in [a,\infty)_\mathbb{T}$. Hence, from Theorem 4.10, we get that $\beta(\cdot)\mathcal{X}(\cdot,\kappa) = 0$ on $[a,\infty)_\mathbb{T}$. By (4.3), for each $t \in [a,\infty)_\mathbb{T}$ and $\kappa \in \{\lambda,\nu\}$, there is a matrix $Q_\kappa(t) \in \mathbb{C}^{n\times n}$ such that $\mathcal{X}(\cdot,\kappa) = \mathcal{J}\beta^*(\cdot)Q_\kappa(\cdot)$ on $[a,\infty)_\mathbb{T}$. Hence, we have on $[a,\infty)_\mathbb{T}$

$$\begin{aligned} &\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+(t,\nu) + F(t,\lambda,\nu,\beta(t)) + G(t,\lambda,\nu,\beta(t)) \\ &\qquad = \mathcal{X}^*(t,\lambda)\mathcal{J}\mathcal{X}(t,\nu) = Q_\lambda^*(t)\beta(t)\mathcal{J}\beta^*(t)Q_\nu(t) = 0, \end{aligned} \tag{7.6}$$

where we define

$$\begin{aligned} F(t,\lambda,\nu,\beta(t)) &:= \mathcal{X}_+^*(t,\lambda)\mathcal{J}\widetilde{Z}(t,\nu,\alpha)\left[M(\nu,t,\alpha,\beta(t)) - M_+(\nu)\right], \\ G(t,\lambda,\nu,\beta(t)) &:= \left[M^*(\lambda,t,\alpha,\beta(t)) - M_+^*(\lambda)\right]\widetilde{Z}^*(t,\lambda,\alpha)\mathcal{J}\mathcal{X}(t,\nu). \end{aligned} \tag{7.7}$$

If we show that

$$\lim_{t\to\infty} F(t,\lambda,\nu,\beta(t)) = 0, \qquad \lim_{t\to\infty} G(t,\lambda,\nu,\beta(t)) = 0, \tag{7.8}$$

then (7.6) implies the result claimed in (7.4). First we prove the second limit in (7.8). Pick any $t \in [b_0,\infty)_\mathbb{T}$. By Theorem 5.6, Corollary 6.3, and $D_+(\lambda) \subseteq D(\lambda,t)$, we have

$$M(\lambda,t,\alpha,\beta(t)) = P(\lambda,t) + R(\lambda,t)U(t)R(\overline{\lambda},t), \qquad M_+(\lambda) = P(\lambda,t) + R(\lambda,t)V(t)R(\overline{\lambda},t), \tag{7.9}$$

where $U(t) \in \mathcal{U}$ and $V(t) \in \mathcal{V}$. Therefore,

$$M(\lambda, t, \alpha, \beta(t)) - M_+(\lambda) = R(\lambda, t)[U(t) - V(t)]R\left(\overline{\lambda}, t\right). \tag{7.10}$$

Since $\widetilde{Z}(\cdot, \lambda, \alpha)$ and $\mathcal{X}(\cdot, \nu)$ are, respectively, solutions of systems $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$ which satisfy $\widetilde{Z}^*(a, \lambda, \alpha)\mathcal{J}\mathcal{X}(a, \nu) = -I$, it follows from Corollary 3.6 that

$$\widetilde{Z}^*(t, \lambda, \alpha)\mathcal{J}\mathcal{X}(t, \nu) = -I + \left(\overline{\lambda} - \nu\right)\int_a^t \widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s. \tag{7.11}$$

Hence, we can write

$$G(t, \lambda, \nu, \beta(t)) = R\left(\overline{\lambda}, t\right)[U^*(t) - V^*(t)]R(\lambda, t)\left[\left(\overline{\lambda} - \nu\right)\int_a^t \widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s - I\right], \tag{7.12}$$

where we used the Hermitian property of $R(\lambda, t)$ and $R(\overline{\lambda}, t)$. Since we now assume that system $(\mathcal{S}_\lambda)$ is in the limit point case, we know from Lemma 6.8 that $\lim_{t\to\infty}R(\lambda, t) = 0$ and $\lim_{t\to\infty}R(\overline{\lambda}, t) = 0$. Therefore, in order to establish (7.8)(ii), it is sufficient to show that

$$R(\lambda, t)\int_a^t \widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}^\sigma(s, \nu)\Delta s, \tag{7.13}$$

is bounded for $t \in [b_0, \infty)_{\mathbb{T}}$. Let $\eta \in \mathbb{C}^n$ be a unit vector, and denote by $\mathcal{X}_j(\cdot, \nu) := \mathcal{X}(\cdot, \nu)e_j$ the jth column of $\mathcal{X}(\cdot, \nu)$ for $j \in \{1, \dots, n\}$. With the definition of $R(\lambda, \cdot)$ in (5.18) we have

$$\begin{aligned}
&\left|\int_a^t \eta^* R(\lambda, s)\widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\mathcal{X}_j^\sigma(s, \nu)\Delta s\right| \\
&\quad\le \int_a^t \left|\widetilde{\mathcal{W}}^{1/2}(s)\widetilde{Z}^{\sigma*}(s, \lambda, \alpha)R(\lambda, s)\eta\right|\left|\widetilde{\mathcal{W}}^{1/2}(s)\mathcal{X}_j^\sigma(s, \nu)\right|\Delta s \\
&\quad\overset{\text{C-S}}{\le} \left(\int_a^t \eta^* R(\lambda, s)\widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\widetilde{Z}^\sigma(s, \lambda, \alpha)R(\lambda, s)\eta\Delta s\right)^{1/2} \\
&\qquad\times\left(\int_a^t \mathcal{X}_j^{\sigma*}(s, \nu)\widetilde{\mathcal{W}}(s)\mathcal{X}_j^\sigma(s, \nu)\Delta s\right)^{1/2},
\end{aligned} \tag{7.14}$$

where the last step follows from the Cauchy-Schwarz inequality (C-S) on time scales. From (5.5) we obtain

$$\mathcal{H}^{-1/2}(t, \lambda, \alpha)\int_a^t \widetilde{Z}^{\sigma*}(s, \lambda, \alpha)\widetilde{\mathcal{W}}(s)\widetilde{Z}^\sigma(s, \lambda, \alpha)\Delta s\mathcal{H}^{-1/2}(t, \lambda, \alpha) = \frac{1}{2|\mathrm{Im}(\lambda)|}I, \tag{7.15}$$

so that the first term in the product in (7.14) is bounded by $1/\sqrt{2|\operatorname{Im}(\lambda)|}$. Moreover, from formula (4.18) we get that the second term in the product in (7.14) is bounded by the number $[e_j^* \operatorname{Im}(M(\nu,t,\alpha,\beta(t)))e_j]/\operatorname{Im}(\nu)$. Hence, upon recalling the limit in (6.12), we conclude that the product in (7.14) is bounded by

$$\frac{1}{2|\operatorname{Im}(\lambda)|} \cdot \frac{e_j^* \operatorname{Im}(M_+(\nu))e_j}{\operatorname{Im}(\nu)}, \tag{7.16}$$

which is independent of t. Consequently, the second limit in (7.8) is established. The first limit in (7.8) is then proven in a similar manner. The proof for the limit point case is finished.

If the system $(\mathcal{S}_\lambda)$ is in the limit circle case, then for $\kappa \in \{\lambda, \nu\}$ the columns of $\widetilde{Z}(\cdot,\kappa,\alpha)$ and $\mathcal{X}_+(\cdot,\kappa)$ belong to $L^2_{\mathcal{W}}$; hence, they are bounded in the $L^2_{\mathcal{W}}$ norm. In this case the limits in (7.8) easily follow from the limit (6.12) for $M_+(\kappa)$, $\kappa \in \{\lambda,\nu\}$. $\square$

In the next result we provide a characterization of the system $(\mathcal{S}_\lambda)$ being of the limit point type. Special cases of this statement can be found, for example, in [14, Theorem 6.12] and [2, Theorem 4.14].

Theorem 7.4. *Let $\alpha \in \Gamma$. The system $(\mathcal{S}_\lambda)$ is in the limit point case if and only if, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\overline{\lambda})$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$, respectively, we have*

$$z_1^*(t,\lambda)\mathcal{J}z_2\left(t,\overline{\lambda}\right) = 0, \quad \forall t \in [b_0,\infty)_{\mathbb{T}}. \tag{7.17}$$

Proof. Let $(\mathcal{S}_\lambda)$ be in the limit point case. Fix any $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and suppose that $z_1(\cdot,\lambda)$ and $z_2(\cdot,\overline{\lambda})$ are solutions of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_{\overline{\lambda}})$, respectively. Then, by Theorem 6.7 and Remark 6.4, there are vectors $\xi_1,\xi_2 \in \mathbb{C}^n$ such that $z_1(\cdot,\lambda) = \mathcal{X}_+(\cdot,\lambda)\xi_1$ and $z_2(\cdot,\overline{\lambda}) = \mathcal{X}_+(\cdot,\overline{\lambda})\xi_2$ on $[a,\infty)_{\mathbb{T}}$, where $\mathcal{X}_+(\cdot,\kappa) := \mathcal{X}_+(\cdot,\kappa,\alpha,M_+(\kappa))$ are the Weyl solutions corresponding to some matrices $M_+(\kappa) \in D_+(\kappa)$ for $\kappa \in \{\lambda,\overline{\lambda}\}$. In fact, by Lemma 6.8, the matrix $M_+(\kappa)$ is equal to the center of the disk $D_+(\kappa)$. It follows that for any $t \in [b_0,\infty)_{\mathbb{T}}$ equality

$$\begin{aligned}
&\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+\left(t,\overline{\lambda}\right) \\
&\quad \overset{(4.16)}{=} (I \;\; M_+^*(\lambda))\Psi^*(t,\lambda,\alpha)\mathcal{J}\Psi\left(t,\overline{\lambda},\alpha\right)\left(I \;\; M_+^*(\overline{\lambda})\right)^* \overset{(3.19)(\mathrm{i})}{=} M_+^*\left(\overline{\lambda}\right) - M_+^*(\lambda) \overset{(6.13)}{=} 0,
\end{aligned} \tag{7.18}$$

holds, so that (7.17) is established. Conversely, let $\nu \in \mathbb{C}\setminus\mathbb{R}$ be arbitrary but fixed, set $\lambda := \overline{\nu}$, and suppose that, for every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, condition (7.17) is satisfied. From Theorem 6.7 we know that for $M_+(\kappa) \in D_+(\kappa)$ the columns $\mathcal{X}_+^{[j]}(\cdot,\kappa)$, $j \in \{1,\ldots,n\}$, of the Weyl solution $\mathcal{X}_+(\cdot,\kappa)$ are linearly independent square integrable solutions of $(\mathcal{S}_\kappa)$, $\kappa \in \{\lambda,\nu\}$. Therefore, $\dim \mathcal{N}(\lambda) \geq n$, and $\dim \mathcal{N}(\nu) \geq n$. Moreover, by identity (3.19)(i), we have

$$\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+^{[j]}(t,\nu) = 0, \quad \forall t \in [b_0,\infty)_{\mathbb{T}}, \; j \in \{1,\ldots,n\}. \tag{7.19}$$

Let $z(\cdot,\nu)$ be any square integrable solution of system $(\mathcal{S}_\nu)$. Then, by our assumption (7.17),

$$\mathcal{X}_+^*(t,\lambda)\mathcal{J}z(t,\nu)=0,\quad \forall t\in[b_0,\infty)_{\mathbb{T}}. \tag{7.20}$$

From (7.19) and (7.20) it follows that the vectors $\mathcal{X}_+^{[j]}(a,\nu)$, $j\in\{1,\dots,n\}$, and $z(a,\nu)$ are solutions of the linear homogeneous system

$$\mathcal{X}_+^*(a,\lambda)\mathcal{J}\eta=0. \tag{7.21}$$

Since, by Theorem 6.7, the vectors $\mathcal{X}_+^{[j]}(a,\nu)$ for $j\in\{1,\dots,n\}$ represent a basis of the solution space of system (7.21), there exists a vector $\xi\in\mathbb{C}^n$ such that $z(a,\nu)=\mathcal{X}_+(a,\nu)\xi$. By the uniqueness of solutions of system $(\mathcal{S}_\nu)$ we then get $z(\cdot,\nu)=\mathcal{X}_+(\cdot,\nu)\xi$ on $[a,\infty)_{\mathbb{T}}$. Hence, the solution $z(\cdot,\nu)$ is square integrable and $\dim\mathcal{N}(\nu)=n$. Since $\nu\in\mathbb{C}\setminus\mathbb{R}$ was arbitrary, it follows that the system $(\mathcal{S}_\lambda)$ is in the limit point case. □

As a consequence of the above result, we obtain a characterization of the limit point case in terms of a condition similar to (7.17), but using a limit. This statement is a generalization of [30, Corollary 2.3], [9, Corollary 3.3], [14, Theorem 6.14], [2, Corollary 4.15], [1, Theorem 3.9], [3, Theorem 4.16].

Corollary 7.5. *Let $\alpha\in\Gamma$. The system $(\mathcal{S}_\lambda)$ is in the limit point case if and only if, for every $\lambda,\nu\in\mathbb{C}\setminus\mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$, respectively, we have*

$$\lim_{t\to\infty} z_1^*(t,\lambda)\mathcal{J}z_2(t,\nu)=0. \tag{7.22}$$

Proof. The necessity follows directly from Theorem 7.3. Conversely, assume that condition (7.22) holds for every $\lambda,\nu\in\mathbb{C}\setminus\mathbb{R}$ and every square integrable solutions $z_1(\cdot,\lambda)$ and $z_2(\cdot,\nu)$ of $(\mathcal{S}_\lambda)$ and $(\mathcal{S}_\nu)$. Fix $\lambda\in\mathbb{C}\setminus\mathbb{R}$, and set $\nu:=\bar{\lambda}$. By Corollary 3.7 we know that $z_1^*(\cdot,\lambda)\mathcal{J}z_2(\cdot,\nu)$ is constant on $[a,\infty)_{\mathbb{T}}$. Therefore, by using condition (7.22), we can see that identity (7.17) must be satisfied, which yields by Theorem 7.4 that the system $(\mathcal{S}_\lambda)$ is of the limit point type. □

8. Nonhomogeneous Time Scale Symplectic Systems

In this section we consider the nonhomogeneous time scale symplectic system

$$z^\Delta(t,\lambda)=\mathcal{S}(t,\lambda)z(t,\lambda)-\mathcal{J}\widetilde{\mathcal{W}}(t)f^\sigma(t),\quad t\in[a,\infty)_{\mathbb{T}}, \tag{8.1}$$

where the matrix function $\mathcal{S}(\cdot,\lambda)$ and $\widetilde{\mathcal{W}}(\cdot)$ are defined in (3.3) and (3.1), $f\in L^2_{\mathcal{W}}$, and where the associated homogeneous system $(\mathcal{S}_\lambda)$ is either of the limit point or limit circle type at ∞. Together with system (8.1) we consider a second system of the same form but with a different spectral parameter and a different nonhomogeneous term

$$y^\Delta(t,\nu)=\mathcal{S}(t,\nu)y(t,\nu)-\mathcal{J}\widetilde{\mathcal{W}}(t)g^\sigma(t),\quad t\in[a,\infty)_{\mathbb{T}}, \tag{8.2}$$

with $g\in L^2_{\mathcal{W}}$. The following is a generalization of Theorem 3.5 to nonhomogeneous systems.

Theorem 8.1 (Lagrange identity). *Let $\lambda, \nu \in \mathbb{C}$ and $m \in \mathbb{N}$ be given. If $z(\cdot,\lambda)$ and $y(\cdot,\nu)$ are $2n \times m$ solutions of systems* (8.1) *and* (8.2), *respectively, then*

$$\begin{aligned}&\left[z^*(t,\lambda)\mathcal{J}y(t,\nu)\right]^{\Delta}\\ &\quad= \left(\overline{\lambda}-\nu\right)z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)y^{\sigma}(t,\nu) - f^{\sigma*}(t)\widetilde{\mathcal{W}}(t)y^{\sigma}(t,\nu) + z^{\sigma*}(t,\lambda)\widetilde{\mathcal{W}}(t)g^{\sigma}(t), \quad t \in [a,\infty)_{\mathbb{T}}.\end{aligned} \tag{8.3}$$

Proof. Formula (8.3) follows by the product rule (2.1) with the aid of the relation

$$z^{\sigma}(t,\lambda) = \left[I + \mu(t)\mathcal{S}(t,\lambda)\right]z(t,\lambda) + \mu(t)\widetilde{\mathcal{W}}(t)f^{\sigma}(t), \tag{8.4}$$

and identity (3.6). □

For $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $t, s \in [a,\infty)_{\mathbb{T}}$, we define the function

$$G(t,s,\lambda,\alpha) := \begin{cases} \widetilde{Z}(t,\lambda,\alpha)\mathcal{X}_+^*\left(s,\overline{\lambda},\alpha\right), & \text{for } t \in [a,s)_{\mathbb{T}}, \\ \mathcal{X}_+(t,\lambda,\alpha)\widetilde{Z}^*\left(s,\overline{\lambda},\alpha\right), & \text{for } t \in [s,\infty)_{\mathbb{T}}, \end{cases} \tag{8.5}$$

where $\widetilde{Z}(\cdot,\lambda,\alpha)$ is the solution of system $(\mathcal{S}_\lambda)$ given in (4.10), that is, $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$, and $\mathcal{X}_+(\cdot,\lambda,\alpha) := \mathcal{X}(\cdot,\lambda,\alpha,M_+(\lambda))$ is the Weyl solution of $(\mathcal{S}_\lambda)$ as in (4.16) determined by a matrix $M_+(\lambda) \in D_+(\lambda)$. This matrix $M_+(\lambda) \in D_+(\lambda)$ is arbitrary but fixed throughout this section. By interchanging the order of the arguments t and s, we have

$$G(t,s,\lambda,\alpha) = \begin{cases} \mathcal{X}_+(t,\lambda,\alpha)\widetilde{Z}^*\left(s,\overline{\lambda},\alpha\right), & \text{for } s \in [a,t]_{\mathbb{T}}, \\ \widetilde{Z}(t,\lambda,\alpha)\mathcal{X}_+^*\left(s,\overline{\lambda},\alpha\right), & \text{for } s \in (t,\infty)_{\mathbb{T}}. \end{cases} \tag{8.6}$$

In the literature the function $G(\cdot,\cdot,\lambda,\alpha)$ is called a resolvent kernel, compare with [30, page 283], [32, page 15], [2, equation (5.4)], and in this section it will play a role of the Green function.

Lemma 8.2. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then*

$$\mathcal{X}_+(t,\lambda,\alpha)\widetilde{Z}^*\left(t,\overline{\lambda},\alpha\right) - \widetilde{Z}(t,\lambda,\alpha)\mathcal{X}_+^*\left(t,\overline{\lambda},\alpha\right) = \mathcal{J}, \quad \forall t \in [a,\infty)_{\mathbb{T}}. \tag{8.7}$$

Proof. Identity (8.7) follows by a direct calculation from the definition of $\mathcal{X}_+(\cdot,\lambda,\alpha)$ via (4.16) with a matrix $M_+(\lambda) \in D_+(\lambda)$ by using formulas (3.21) and (6.13). □

In the next lemma we summarize the properties of the function $G(\cdot,\cdot,\lambda,\alpha)$, which together with Proposition 8.4 and Theorem 8.5 justifies the terminology "Green function" of the system (8.1); compare with [68, Section 4]. A discrete version of the following result can be found in [2, Lemma 5.1].

Lemma 8.3. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. The function $G(\cdot, \cdot, \lambda, \alpha)$ has the following properties:*

(i) $G^*(t, s, \lambda, \alpha) = G(s, t, \overline{\lambda}, \alpha)$ *for every* $t, s \in [a, \infty)_{\mathbb{T}}$, $t \neq s$,

(ii) $G^*(t, t, \lambda, \alpha) = G(t, t, \overline{\lambda}, \alpha) - \mathcal{J}$ *for every* $t \in [a, \infty)_{\mathbb{T}}$,

(iii) $G(\sigma(t), \sigma(t), \lambda, \alpha) = [I + \mu(t)\mathcal{S}(t, \lambda)]G(t, \sigma(t), \lambda, \alpha) + \mathcal{J}$ *for every right-scattered point* $t \in [a, \infty)_{\mathbb{T}}$,

(iv) *for every* $t, s \in [a, \infty)_{\mathbb{T}}$ *such that* $t \notin \mathcal{T}(s)$, *the function* $G(\cdot, s, \lambda, \alpha)$ *solves the homogeneous system* $(\mathcal{S}_\lambda)$ *on the set* $\mathcal{T}(s)$, *where*

$$\mathcal{T}(s) := \{\tau \in [a, \infty)_{\mathbb{T}},\ \tau \neq \rho(s) \text{ if } s \text{ is left-scattered}\}, \tag{8.8}$$

(v) *the columns of* $G(\cdot, s, \lambda, \alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $s \in [a, \infty)_{\mathbb{T}}$, *and the columns of* $G(t, \cdot, \lambda, \alpha)$ *belong to* $L^2_{\mathcal{W}}$ *for every* $t \in [a, \infty)_{\mathbb{T}}$.

Proof. Condition (i) follows from the definition of $G(\cdot, s, \lambda, \alpha)$ in (8.5). Condition (ii) is a consequence of Lemma 8.2. Condition (iii) is proven from the definition of $G(\sigma(t), \sigma(t), \lambda, \alpha)$ in (8.5) by using Lemma 8.2 and $\widetilde{Z}(t, \lambda, \alpha) = \widetilde{Z}^\sigma(t, \lambda, \alpha) - \mu(t)\mathcal{S}(t, \lambda)\widetilde{Z}(t, \lambda, \alpha)$. Concerning condition (iv), the function $G(\cdot, s, \lambda, \alpha)$ solves the system $(\mathcal{S}_\lambda)$ on $[s, \infty)_{\mathbb{T}}$ because $\mathcal{X}_+(\cdot, \lambda, \alpha)$ solves this system on $[s, \infty)_{\mathbb{T}}$. If $s \in (a, \infty)_{\mathbb{T}}$ is left-dense, then $G(\cdot, s, \lambda, \alpha)$ solves $(\mathcal{S}_\lambda)$ on $[a, s)_{\mathbb{T}}$, since $\widetilde{Z}(\cdot, \lambda, \alpha)$ solves this system on $[a, s)_{\mathbb{T}}$. For the same reason $G(\cdot, s, \lambda, \alpha)$ solves $(\mathcal{S}_\lambda)$ on $[a, \rho(s))_{\mathbb{T}}$ if $s \in (a, \infty)_{\mathbb{T}}$ is left-scattered. Condition (v) follows from the definition of $G(\cdot, s, \lambda, \alpha)$ in (8.5) used with $t \geq s$ and from the fact that the columns of $\mathcal{X}_+(\cdot, \lambda, \alpha)$ belong to $L^2_{\mathcal{W}}$, by Theorem 6.7. The columns of $G(t, \cdot, \lambda, \alpha)$ then belong to $L^2_{\mathcal{W}}$ by part (i) of this lemma. ◻

Since by Lemma 8.3(v) the columns of $G(t, \cdot, \lambda, \alpha)$ belong to $L^2_{\mathcal{W}}$, the function

$$\widehat{z}(t, \lambda, \alpha) := -\int_a^\infty G(t, \sigma(s), \lambda, \alpha)\widetilde{\mathcal{W}}(s) f^\sigma(s)\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}, \tag{8.9}$$

is well defined whenever $f \in L^2_{\mathcal{W}}$. Moreover, by using (8.6), we can write $\widehat{z}(t, \lambda, \alpha)$ as

$$\begin{aligned} \widehat{z}(t, \lambda, \alpha) = {} & -\mathcal{X}_+(t, \lambda, \alpha)\int_a^t \widetilde{Z}^{\sigma *}\left(s, \overline{\lambda}, \alpha\right)\widetilde{\mathcal{W}}(s) f^\sigma(s)\Delta s \\ & - \widetilde{Z}(t, \lambda, \alpha)\int_t^\infty \mathcal{X}_+^{\sigma *}\left(s, \overline{\lambda}, \alpha\right)\widetilde{\mathcal{W}}(s) f^\sigma(s)\Delta s, \quad t \in [a, \infty)_{\mathbb{T}}. \end{aligned} \tag{8.10}$$

Proposition 8.4. *For* $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, *and* $f \in L^2_{\mathcal{W}}$, *the function* $\widehat{z}(\cdot, \lambda, \alpha)$ *defined in* (8.9) *solves the nonhomogeneous system* (8.1) *with the initial condition* $\alpha\widehat{z}(a, \lambda, \alpha) = 0$.

Proof. By the time scales product rule (2.1) when we Δ-differentiate expression (8.10), we have for every $t \in [a,\infty)_{\mathbb{T}}$ (suppressing the dependence on α in the the following calculation)

$$\begin{aligned}
\widehat{z}^{\Delta}(t,\lambda) &= -\mathcal{X}_{+}^{\Delta}(t,\lambda)\int_a^t \widetilde{Z}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s - \mathcal{X}_{+}^{\sigma}(t,\lambda)\widetilde{Z}^{\sigma *}\left(t,\overline{\lambda}\right)\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&\quad - \widetilde{Z}^{\Delta}(t,\lambda)\int_t^{\infty} \mathcal{X}_{+}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s + \widetilde{Z}^{\sigma}(t,\lambda)\mathcal{X}_{+}^{\sigma *}\left(t,\overline{\lambda}\right)\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&= \mathcal{S}(t,\lambda)\widehat{z}(t,\lambda) - \left[\mathcal{X}_{+}^{\sigma}(t,\lambda)\widetilde{Z}^{\sigma *}\left(t,\overline{\lambda}\right) - \widetilde{Z}^{\sigma}(t,\lambda)\mathcal{X}_{+}^{\sigma *}\left(t,\overline{\lambda}\right)\right]\widetilde{\mathcal{W}}(t)f^{\sigma}(t) \\
&\overset{(8.7)}{=} \mathcal{S}(t,\lambda)\widehat{z}(t,\lambda) - \mathcal{J}\widetilde{\mathcal{W}}(t)f^{\sigma}(t).
\end{aligned} \tag{8.11}$$

This shows that $\widehat{z}(\cdot,\lambda,\alpha)$ is a solution of system (8.1). From (8.10) with $t = a$, we get

$$\alpha\widehat{z}(a,\lambda,\alpha) = -\alpha\widetilde{Z}(a,\lambda,\alpha)\int_a^{\infty} \mathcal{X}_{+}^{\sigma *}\left(s,\overline{\lambda},\alpha\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s = 0, \tag{8.12}$$

where we used the initial condition $\widetilde{Z}(a,\lambda,\alpha) = -\mathcal{J}\alpha^*$ and $\alpha\mathcal{J}\alpha^* = 0$ coming from $\alpha \in \Gamma$. □

The following theorem provides further properties of the solution $\widehat{z}(\cdot,\lambda,\alpha)$ of system (8.1). It is a generalization of [10, Lemma 4.2], [11, Theorem 7.5], [2, Theorem 5.2] to time scales.

Theorem 8.5. *Let $\alpha \in \Gamma$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$, and $f \in L^2_{\mathcal{W}}$. Suppose that system $(\mathcal{S}_{\lambda})$ is in the limit point or limit circle case. Then the solution $\widehat{z}(\cdot,\lambda,\alpha)$ of system* (8.1) *defined in* (8.9) *belongs to $L^2_{\mathcal{W}}$ and satisfies*

$$\|\widehat{z}(\cdot,\lambda,\alpha)\|_{\mathcal{W}} \le \frac{1}{|\operatorname{Im}(\lambda)|}\|f\|_{\mathcal{W}}, \tag{8.13}$$

$$\lim_{t\to\infty} \mathcal{X}_{+}^{*}(t,\nu,\alpha)\mathcal{J}\widehat{z}(t,\lambda,\alpha) = 0, \quad \textit{for every } \nu \in \mathbb{C}\setminus\mathbb{R}. \tag{8.14}$$

Proof. To shorten the notation we suppress the dependence on α in all quantities appearing in this proof. Assume first that system $(\mathcal{S}_{\lambda})$ is in the limit point case. For every $r \in [a,\infty)_{\mathbb{T}}$ we define the function $f_r(\cdot) := f(\cdot)$ on $[a,r]_{\mathbb{T}}$ and $f_r(\cdot) := 0$ on $(r,\infty)_{\mathbb{T}}$ and the function

$$\widehat{z}_r(t,\lambda) := -\int_a^{\infty} G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f_r^{\sigma}(s)\Delta s = -\int_a^{r} G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.15}$$

For every $t \in [r,\infty)_{\mathbb{T}}$ we have as in (8.10) that

$$\widehat{z}_r(t,\lambda) = -\mathcal{X}_{+}(t,\lambda)g(r,\lambda), \quad g(r,\lambda) := \int_a^r \widetilde{Z}^{\sigma *}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.16}$$

Since by Theorem 6.7 the solution $\mathcal{X}_+(\cdot,\lambda)\in L^2_{\mathcal{W}}$, (8.16) shows that $\widehat{z}_r(\cdot,\lambda)$, being a multiple of $\mathcal{X}_+(\cdot,\lambda)$, also belongs to $L^2_{\mathcal{W}}$. Moreover, by Theorem 7.3,

$$\lim_{t\to\infty}\widehat{z}_r^*(t,\lambda)\mathcal{J}\widehat{z}_r(t,\lambda)\overset{(8.16)}{=}g^*(r,\lambda)\lim_{t\to\infty}\mathcal{X}_+^*(t,\lambda)\mathcal{J}\mathcal{X}_+(t,\lambda)g(r,\lambda)\overset{(7.4)}{=}0. \tag{8.17}$$

On the other hand, $\widehat{z}_r^*(a,\lambda)\mathcal{J}\widehat{z}_r(a,\lambda)=0$, and for any $t\in[a,\infty)_{\mathbb{T}}$ identity (8.3) implies

$$\begin{aligned}&\widehat{z}_r^*(t,\lambda)\mathcal{J}\widehat{z}_r(t,\lambda)\\&\quad=-2i\operatorname{Im}(\lambda)\int_a^t\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s+2i\operatorname{Im}\left(\int_a^t\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)f_r^{\sigma}(s)\Delta s\right).\end{aligned} \tag{8.18}$$

Combining (8.18), where $t\to\infty$, formula (8.17), and the definition on $f_r(\cdot)$ yields

$$\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}^2=\int_a^{\infty}\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s=\frac{1}{\operatorname{Im}(\lambda)}\operatorname{Im}\left(\int_a^r\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right). \tag{8.19}$$

By using the Cauchy-Schwarz inequality (C-S) on time scales and $\widetilde{\mathcal{W}}(\cdot)\geq 0$, we then have

$$\begin{aligned}\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}^2&=\frac{1}{2i\operatorname{Im}(\lambda)}\left[\int_a^r\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s-\int_a^r f^{\sigma*}(s)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s\right]\\&\leq\frac{1}{|\operatorname{Im}(\lambda)|}\left|\int_a^r\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right|\\&\overset{\text{C-S}}{\leq}\frac{1}{|\operatorname{Im}(\lambda)|}\left(\int_a^r\widehat{z}_r^{\sigma*}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s\right)^{1/2}\left(\int_a^r f^{\sigma*}(s)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s\right)^{1/2}\\&\leq\frac{1}{|\operatorname{Im}(\lambda)|}\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}\|f\|_{\mathcal{W}}.\end{aligned} \tag{8.20}$$

Since $\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}$ is finite by $\widehat{z}_r(\cdot,\lambda)\in L^2_{\mathcal{W}}$, we get from the above calculation that

$$\|\widehat{z}_r(\cdot,\lambda)\|_{\mathcal{W}}\leq\frac{1}{|\operatorname{Im}(\lambda)|}\|f\|_{\mathcal{W}}. \tag{8.21}$$

We will prove that (8.21) implies estimate (8.13) by the convergence argument. For any $t,r\in[a,\infty)_{\mathbb{T}}$ we observe that

$$\widehat{z}(t,\lambda)-\widehat{z}_r(t,\lambda)=-\int_r^{\infty}G(t,\sigma(s),\lambda)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.22}$$

Now we fix $q\in[a,r)_{\mathbb{T}}$. By the definition of $G(\cdot,\cdot,\lambda)$ in (8.5) we have for every $t\in[a,q]_{\mathbb{T}}$

$$\widehat{z}(t,\lambda)-\widehat{z}_r(t,\lambda)=-\widetilde{Z}(t,\lambda)\int_r^{\infty}\mathcal{X}_+^*\left(\sigma(s),\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.23}$$

Since the functions $\mathcal{X}_+(\cdot,\overline{\lambda})$ and $f(\cdot)$ belong to $L^2_{\mathcal{W}}$, it follows that the right-hand side of (8.23) converges to zero as $r \to \infty$ for every $t \in [a,q]_{\mathbb{T}}$. Hence, $\widehat{z}_r(\cdot,\lambda)$ converges to the function $\widehat{z}(\cdot,\lambda)$ uniformly on $[a,q]_{\mathbb{T}}$. Since $\widehat{z}(\cdot,\lambda) = \widehat{z}_r(\cdot,\lambda)$ on $[a,q]_{\mathbb{T}}$, we have by $\widetilde{\mathcal{W}}(\cdot) \geq 0$ and (8.21) that

$$\int_a^q \widehat{z}^{\sigma *}(s,\lambda)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s \leq \|\widehat{z}_r(\cdot,\lambda)\|^2_{\mathcal{W}} \overset{(8.21)}{\leq} \frac{1}{|\operatorname{Im}(\lambda)|^2}\|f\|^2_{\mathcal{W}}. \tag{8.24}$$

Since $q \in [a,\infty)_{\mathbb{T}}$ was arbitrary, inequality (8.24) implies the result in (8.13). In the limit circle case inequality (8.13) follows by the same argument by using the fact that all solutions of system $(\mathcal{S}_\lambda)$ belong to $L^2_{\mathcal{W}}$.

Now we prove the existence of the limit (8.14). Assume that the system $(\mathcal{S}_\lambda)$ is in the limit point case, and let $\nu \in \mathbb{C} \setminus \mathbb{R}$ be arbitrary. Following the argument in the proof of [30, Lemma 4.1] and [2, Theorem 5.2], we have from identity (8.3) that for any $r,t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}\mathcal{X}_+^*(t,\nu)\mathcal{J}\widehat{z}_r(t,\lambda) = \mathcal{X}_+^*(a,\nu)\mathcal{J}\widehat{z}_r(a,\lambda) + (\overline{\nu}-\lambda)\int_a^t \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s \\ + \int_a^t \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)f_r^{\sigma}(s)\Delta s.\end{aligned} \tag{8.25}$$

Since for $t \in [r,\infty)_{\mathbb{T}}$ equality (8.16) holds, it follows that

$$\lim_{t\to\infty}\mathcal{X}_+^*(t,\nu)\mathcal{J}\widehat{z}_r(t,\lambda) = -\lim_{t\to\infty}\mathcal{X}_+^*(t,\nu)\mathcal{J}\mathcal{X}_+(t,\lambda)g(r,\lambda) \overset{(7.4)}{=} 0. \tag{8.26}$$

Hence, by (8.25),

$$\mathcal{X}_+^*(a,\nu)\mathcal{J}\widehat{z}_r(a,\lambda) = (\lambda-\overline{\nu})\int_a^\infty \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}_r^{\sigma}(s,\lambda)\Delta s - \int_a^r \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.27}$$

By the uniform convergence of $\widehat{z}_r(\cdot,\lambda)$ to $\widehat{z}(\cdot,\lambda)$ on compact intervals, we get from (8.27) with $r \to \infty$ the equality

$$\mathcal{X}_+^*(a,\nu)\mathcal{J}\widehat{z}(a,\lambda) = (\lambda-\overline{\nu})\int_a^\infty \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s - \int_a^\infty \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s. \tag{8.28}$$

On the other hand, by (8.3), we obtain for every $t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned}\mathcal{X}_+^*(t,\nu)\mathcal{J}\widehat{z}(t,\lambda) = \mathcal{X}_+^*(a,\nu)\mathcal{J}\widehat{z}(a,\lambda) + (\overline{\nu}-\lambda)\int_a^t \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)\widehat{z}^{\sigma}(s,\lambda)\Delta s \\ + \int_a^t \mathcal{X}_+^{\sigma *}(s,\nu)\widetilde{\mathcal{W}}(s)f^{\sigma}(s)\Delta s.\end{aligned} \tag{8.29}$$

Upon taking the limit in (8.29) as $t \to \infty$ and using equality (8.28), we conclude that the limit in (8.14) holds true.

In the limit circle case, the limit in (8.14) can be proved similarly as above, because all the solutions of system $(\mathcal{S}_\lambda)$ now belong to $L^2_{\mathcal{W}}$. However, in this case, we can apply a direct argument to show that (8.14) holds. By formula (8.10) we get for every $t \in [a,\infty)_{\mathbb{T}}$

$$\begin{aligned} \mathcal{X}_+^*(t,\nu)\mathcal{J}\widehat{z}(t,\lambda) = &-\mathcal{X}_+^*(t,\nu)\mathcal{J}\mathcal{X}_+(t,\lambda)\int_a^t \widetilde{Z}^{\sigma*}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s \\ &- \mathcal{X}_+^*(t,\nu)\mathcal{J}\widetilde{Z}(t,\lambda)\int_t^\infty \mathcal{X}_+^{\sigma*}\left(s,\overline{\lambda}\right)\widetilde{\mathcal{W}}(s)f^\sigma(s)\Delta s. \end{aligned} \tag{8.30}$$

The limit of the first term in (8.30) is zero because $\mathcal{X}_+^*(t,\nu)\mathcal{J}\mathcal{X}_+(t,\lambda)$ tends to zero for $t \to \infty$ by (7.4), and it is multiplied by a convergent integral as $t \to \infty$. Since the columns of $\widetilde{Z}(\cdot,\lambda)$ belong to $L^2_{\mathcal{W}}$, the function $\mathcal{X}_+^*(\cdot,\nu)\mathcal{J}\widetilde{Z}(\cdot,\lambda)$ is bounded on $[a,\infty)_{\mathbb{T}}$, and it is multiplied by an integral converging to zero as $t \to \infty$. Therefore, formula (8.14) follows. □

In the last result of this paper we construct another solution of the nonhomogeneous system (8.1) satisfying condition (8.14) and such that it starts with a possibly nonzero initial condition at $t = a$. It can be considered as an extension of Theorem 8.5.

Corollary 8.6. *Let $\alpha \in \Gamma$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Assume that $(\mathcal{S}_\lambda)$ is in the limit point or limit circle case. For $f \in L^2_{\mathcal{W}}$ and $v \in \mathbb{C}^n$ we define*

$$\widetilde{z}(t,\lambda,\alpha) := \mathcal{X}_+(t,\lambda,\alpha)v + \widehat{z}(t,\lambda,\alpha), \quad \forall t \in [a,\infty)_{\mathbb{T}}, \tag{8.31}$$

where $\widehat{z}(\cdot,\lambda,\alpha)$ is given in (8.9). *Then $\widetilde{z}(\cdot,\lambda,\alpha)$ solves the system* (8.1) *with $\alpha\widetilde{z}(a,\lambda,\alpha) = v$,*

$$\|\widetilde{z}(\cdot,\lambda,\alpha)\|_{\mathcal{W}} \leq \frac{1}{|\mathrm{Im}(\lambda)|}\|f\|_{\mathcal{W}} + \|\mathcal{X}_+(\cdot,\lambda,\alpha)v\|_{\mathcal{W}}, \tag{8.32}$$

$$\lim_{t\to\infty}\mathcal{X}_+^*(t,\nu,\alpha)\mathcal{J}\widetilde{z}(t,\lambda,\alpha) = 0, \quad \text{for every } \nu \in \mathbb{C}\setminus\mathbb{R}. \tag{8.33}$$

In addition, if the system $(\mathcal{S}_\lambda)$ is in the limit point case, then $\widetilde{z}(\cdot,\lambda,\alpha)$ is the only $L^2_{\mathcal{W}}$ solution of (8.1) *satisfying $\alpha\widetilde{z}(a,\lambda,\alpha) = v$.*

Proof. As in the previous proof we suppress the dependence on α. Since the function $\mathcal{X}_+(\cdot,\lambda)v$ solves $(\mathcal{S}_\lambda)$, it follows from Proposition 8.4 that $\widetilde{z}(\cdot,\lambda,\alpha)$ solves the system (8.1) and $\alpha\widetilde{z}(a,\lambda) = \alpha\mathcal{X}_+(a,\lambda)v = v$. Next, $\widetilde{z}(\cdot,\lambda) \in L^2_{\mathcal{W}}$ as a sum of two $L^2_{\mathcal{W}}$ functions. The limit in (8.33) follows from the limit (8.14) of Theorem 8.5 and from identity (7.4), because

$$\lim_{t\to\infty}\mathcal{X}_+^*(t,\nu)\mathcal{J}\widetilde{z}(t,\lambda) = \lim_{t\to\infty}\left\{\mathcal{X}_+^*(t,\nu)\mathcal{J}\mathcal{X}_+(t,\lambda)v + \mathcal{X}_+^*(t,\nu)\mathcal{J}\widehat{z}(t,\lambda)\right\} = 0. \tag{8.34}$$

Inequality (8.32) is obtained from estimate (8.13) by the triangle inequality.

Now we prove the uniqueness of $\tilde{z}(\cdot,\lambda)$ in the case of $(\mathcal{S}_\lambda)$ being of the limit point type. If $z_1(\cdot,\lambda)$ and $z_2(\cdot,\lambda)$ are two $L^2_{\mathcal{W}}$ solutions of (8.1) satisfying $\alpha z_1(a,\lambda) = v = \alpha z_2(a,\lambda)$, then their difference $z(\cdot,\lambda) := z_1(\cdot,\lambda) - z_2(\cdot,\lambda)$ also belongs to $L^2_{\mathcal{W}}$ and solves system $(\mathcal{S}_\lambda)$ with $\alpha z(\cdot,\lambda) = 0$. Since $z(\cdot,\lambda) = \Psi(\cdot,\lambda)c$ for some $c \in \mathbb{C}^{2n}$, the initial condition $\alpha z(\cdot,\lambda) = 0$ implies through (4.7) that $z(\cdot,\lambda) = \tilde{Z}(\cdot,\lambda)d$ for some $d \in \mathbb{C}^n$. If $d \neq 0$, then $z(\cdot,\lambda) \notin L^2_{\mathcal{W}}$, because in the limit point case the columns of $\tilde{Z}(\cdot,\lambda)$ do not belong to $L^2_{\mathcal{W}}$, which is a contradiction. Therefore, $d = 0$ and the uniqueness of $\tilde{z}(\cdot,\lambda)$ is established. □

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