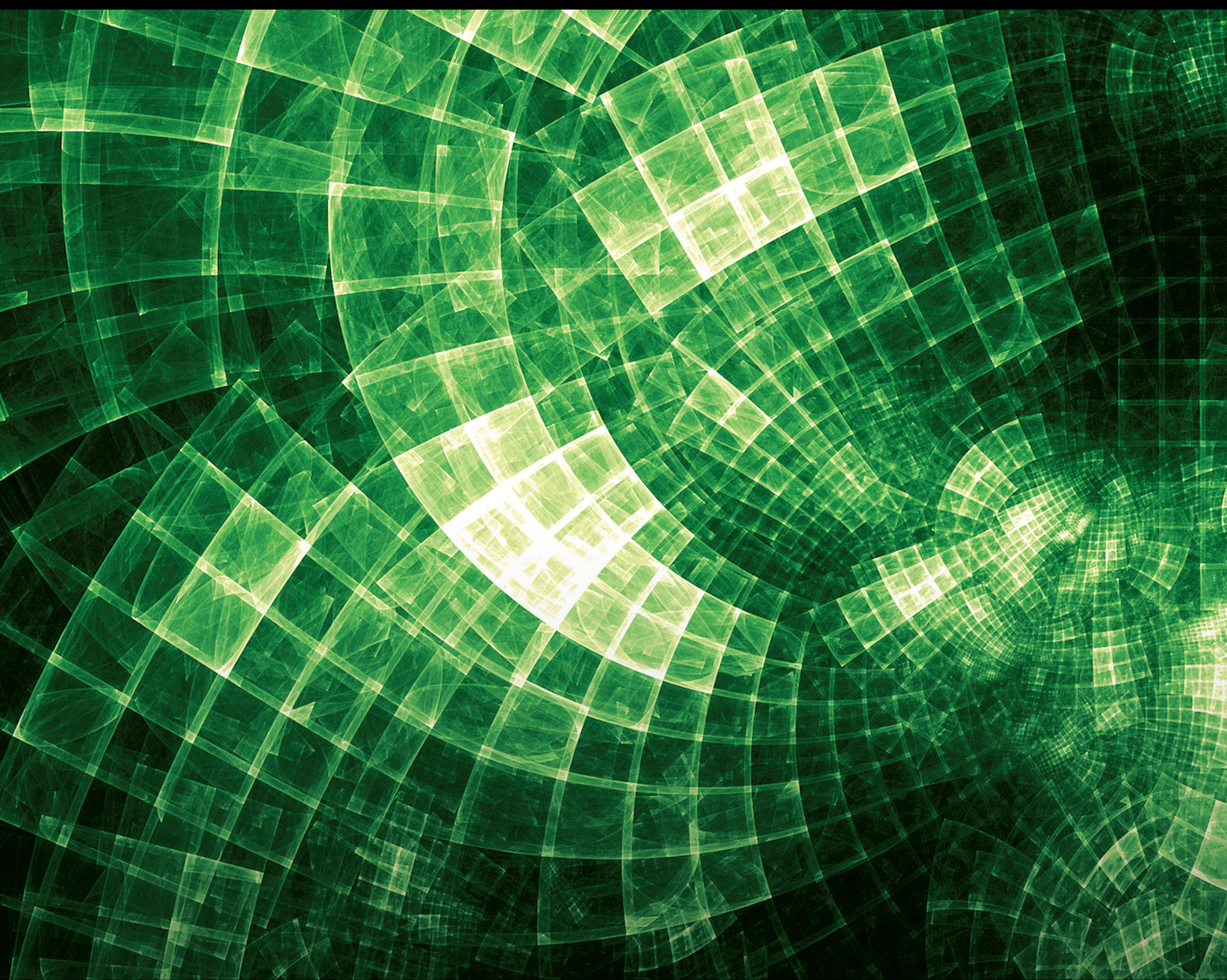
Journal of Mathematics

Decision Making Based on Intuitionistic Fuzzy Sets and their Generalizations

Lead Guest Editor: Tahir Mahmood
Guest Editors: Harish Garg and Lemnaouar Zedam



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Contents

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Retraction

Retracted: Correlation Coefficient and Entropy Measures Based on Complex Dual Type-2 Hesitant Fuzzy Sets and Their Applications

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

References

[1] T. Mahmood, Z. Ali, H. Garg, L. Zedam, and R. Chinram, "Correlation Coefficient and Entropy Measures Based on Complex Dual Type-2 Hesitant Fuzzy Sets and Their Applications," *Journal of Mathematics*, vol. 2021, Article ID 2568391, 34 pages, 2021.

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Retraction

Retracted: A Hybrid BSC-DEA Model with Indeterminate Information

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[1] M. Jaberi Hafshjani, S. E. Najafi, F. Hosseinzadeh Lotfi, and S. M. Hajimolana, "A Hybrid BSC-DEA Model with Indeterminate Information," *Journal of Mathematics*, vol. 2021, Article ID 8867135, 14 pages, 2021.

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Retraction

Retracted: The Neutro-Stability Analysis of Neutrosophic Cubic Sets with Application in Decision Making Problems

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

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We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] M. A. Al Shumrani, M. Gulistan, and S. Khan, "The Neutro-Stability Analysis of Neutrosophic Cubic Sets with Application in Decision Making Problems," *Journal of Mathematics*, vol. 2020, Article ID 8835019, 16 pages, 2020.

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Retraction

Retracted: A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System

Journal of Mathematics

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Journal of Mathematics has retracted the article titled "A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System" [1] due to concerns that the peer review process has been compromised.

Following an investigation conducted by the Hindawi Research Integrity team [2], significant concerns were identified with the peer reviewers assigned to this article; the investigation has concluded that the peer review process was compromised. We therefore can no longer trust the peer review process, and the article is being retracted with the agreement of the Chief Editor.

The authors do not agree to the retraction.

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[1] S. Abdullah, S. Khan, M. Qiyas, and R. Chinram, "A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System," *Journal of Mathematics*, vol. 2021, Article ID 8819517, 9 pages, 2021.

[2] L. Ferguson, "Advancing Research Integrity Collaboratively and with Vigour," 2022, https://www.hindawi.com/post/advancing-research-integrity-collaboratively-and-vigour/.

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Research Article

Complex Neutrosophic Soft Matrices Framework: An Application in Signal Processing

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In this paper, we introduce the concept of complex neutrosophic soft matrices. We define some basic operations including complement, union, and intersection on these matrices. We extend the concept of complex neutrosophic soft sets to complex neutrosophic soft matrices and prove related properties. Moreover, we develop an algorithm using complex neutrosophic soft matrices and apply it in signal processing.

1. Introduction

The models of real-life problems in almost every field of science like mathematics, physics, operations research, medical sciences, engineering, computer science, artificial intelligence, and management sciences are mostly full of complexities. Many theories have been developed to overcome these uncertainties; one among those theories is fuzzy set theory. Zadeh was the first who gave the concept of a fuzzy set in 1965 [1]. Fuzzy sets are the generalizations or extensions of crisps sets.

In order to add the concept of nonmembership term to the definition of fuzzy set, the concept of an intuitionistic fuzzy set was introduced by Atanassov in 1986 [2], where he added the concept of nonmembership term to the definition of fuzzy set. The intuitionistic fuzzy set is characterized by a membership function μ and a nonmembership function v with ranges [0, 1]. The intuitionistic fuzzy set is the generalization of a fuzzy set. An intuitionistic fuzzy set can be applied in several fields including modeling, medical diagnosis, and decision-making. [3] Molodtsov introduced the concept of a soft set in 1999 and developed the fundamental results related to this theory. Basic operations including complement, union, and intersection are also defined on this set. Molodtsov used soft sets for applications in games, probability, and operational theories [3–6]. In 2018, Smarandache generalized the soft set to the hypersoft set by transforming the classical uniargument function F into a multiargument function [7]. Maji et al. [8] introduced the concept of fuzzy soft sets by combining soft sets and fuzzy sets and applied them in decision-making problems [9]. In [10], Cagman and Enginolu used soft matrix theory for applications in decision-making problems.

The concept of neutrosophy was introduced by Smarandache [11] in 1998. A neutrosophic set is characterized by a truth membership function T, an indeterminacy function I, and a falsity membership function F. A neutrosophic set is a mathematical framework which generalizes the concept of a classical set, fuzzy set, intuitionistic fuzzy set, and interval valued fuzzy set [12]. In [13], Nabeeh introduced a method that can promote a personal selection process by integrating the neutrosophic analytical hierarchy process to show the proper solution among distinct options with order preference technique similar to an ideal solution (TOPSIS). In [14], Baset introduced a concept of a neutrosophy technique called type 2 neutrosophic numbers. By combining type 2 neutrosophic numbers and TOPSIS, they suggested a novel method T2NN-TOPSIS which has a lot of applications in group decision-making. They worked on a multicriteria group decision-making technique of the analytical network process method and Visekriterijusmska Optmzacija I Kommpromisno Resenje method under neutrosophic environment that deals high-order imprecision and incomplete information [15].

The largest number set is a complex set which is introduced by Gauss in 1795 and is the extension of a real number set. According to same fashion, a complex fuzzy set is extension to a fuzzy set as here the range set is extended from interval $[0, 1]$ to a closed disc of radius one in complex plane. The degree of membership a complex fuzzy set is not restricted to a value in $[0, 1]$; it is extended to a complex value lies in a disc of radius one in the complex plane.

Complex fuzzy sets are not simply a linear extension of conventional fuzzy sets; complex fuzzy sets allow a natural extension of fuzzy set theory to problems that are either very difficult or impossible to address with one-dimensional grades of membership. It is an obvious fact that uncertainty, indeterminacy, inconsistency, and incompleteness in data are periodic in nature. In order to address this difficulty, in 2002, Daniel Ramot was the first who gave the concept of a complex fuzzy set. The concept of a complex neutrosophic set was introduced in [16].

The complex fuzzy set C is described as membership function, with range in closed unit disc in the complex plane. The complex-valued membership function $\phi_s(x)$ is defined as $\phi_s(x) = t_s(x)e^{i.\eta_s(x)}$ that assigns a complex value of membership to any x in U (universal set) such that $t_s(x)$ and $\eta_s(x)$ both are real-valued with $t_s(x)$ is fuzzy set and $i = \sqrt{-1}$, where $t_s(x)$ is called amplitude term and $\eta_s(x)$ is called phase term.

Physically the complex fuzzy set is used for representing the complex fuzzy solar activity (solar maximum and solar minimum) through the measurement of sunspot number and is also used in signal processing. The complex neutrosophic set is the generalization of a complex fuzzy set and a neutrosophic set. The complex neutrosophic set is characterized by complex-valued truth membership function, complex-valued indeterminate function, and complex-valued falsehood function. In short, a complex neutrosophic set is more generalized because it is not only the generalization of all the current frameworks but also describes the information in a complete and comprehensive way.

A fuzzy set with its generalizations, like intuitionistic fuzzy sets, interval valued fuzzy sets, and cubic sets, represents uncertainties in models of the one-dimensional phenomenon while a complex fuzzy set is the only generalization of a fuzzy set which deals with the models of real-life problems with the two-dimensional and periodic phenomenon. A complex fuzzy set is more applicable because of its nature and can be used more widely in all branches of sciences. Since it is similar to that of a Fourier transform, more explicitly it is a particular sort of Fourier transform with the only restriction on the range which is a complex unit disc. A Fourier transform is used in signals and systems; that is, a Fourier transform is the mathematical tool for representing both continuous and discrete signals. Taking advantage of a complex fuzzy set, being a specific form of Fourier transform, it can be used to represent signals in a particular region of consideration. A neutrosophic set is the generalization of a fuzzy set which deals with the problems containing uncertainties of truthfulness, falsehood, and neutrality. The complex neutrosophic set has three major parts, that is, truth, intermediate, and falsehood membership functions. The truth membership function is totally the same as that of a complex fuzzy set while intermediate and falsehood membership functions are the new additions to it. Thus, a complex neutrosophic set can be applied more widely compared with other fuzzy sets.

In the vast area of science and technology, matrices play an important role. Classical matrix theory cannot solve all models of the daily life problems. In order to overcome these difficulties, Yang and Ji in [17] initiated a matrix representation of a fuzzy soft set and successfully applied the proposed notion of a fuzzy soft matrix in certain decision-making problems.

This work is basically the extension of the work of Ramot et al. [18], Alkouri and Saleh [19], Cai [20, 21], and Zhang et al. [22] to neutrosophic sets. Here, in this paper, we extend the concept by defining the complex neutrosophic fuzzy soft set and then the complex neutrosophic fuzzy soft matrix (CNFSM). Further, we discuss some basic operations on CNFSM and finally we develop an algorithm using these matrices and apply it in signal processing.

Soft matrices are widely used in signals and systems, decision-making problems, and medical diagnosis. This article has two aims. In the first part, we present theoretical foundations of the complex neutrosophic fuzzy soft matrices. These theoretical foundations provide basic notions and operations on complex neutrosophic soft matrices such as complex neutrosophic fuzzy soft zero matrix, complex neutrosophic fuzzy soft universal matrix, complex neutrosophic fuzzy soft submatrices, union of complex neutrosophic fuzzy soft matrices, intersection of complex neutrosophic fuzzy soft matrices, and complement of complex neutrosophic fuzzy soft matrices. Then, we introduce some fundamental results and discuss main strategies for applications of this concept in signals and systems, as well as a coherent discussion of the theory of complex neutrosophic fuzzy soft matrices. The aim of these new concepts is to provide a modern method with mathematical procedure to identify a reference signal out of large number of signals received by a digital receiver. The complex neutrosophic fuzzy soft matrix is the generalization of the fuzzy soft matrix, complex fuzzy soft matrix, and Pythagorean fuzzy soft matrix. The degree of membership function, nonmembership function, and phase terms are all applied to each entry of the matrix which give more fruitful results for a better choice in signals and systems along with other fields such as decision-making problems, medical diagnosis, and pattern recognition. These applied contexts provide solid evidence of the wide applications of the complex neutrosophic fuzzy soft matrices approach to signals and systems and decision-making problems.

2. Preliminaries

Here, we begin with a numerical example of a complex neutrosophic set which is already defined above.

Example 1. Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Then, the complex neutrosophic set S in X is given as

$$S = \frac{\left(0.6e^{j0.3}, e^{j\pi/2}, 0.3e^{j0.6}\right)}{x_1} + \frac{\left(0.4e^{j0}, 0.9e^{j\pi/4}, 0.4e^{j\pi/4}\right)}{x_2} + \frac{\left(0.5e^{j2\pi/3}, 0.2e^{j0.2}, 0.7e^{j\pi/3}\right)}{x_3}. \quad (1)$$

Definition 1 (fuzzy set (FS) [1]). Fuzzy set is defined by an arbitrary mapping from a nonempty set X to the unit interval $[0,1]$, i.e., $f: X \longrightarrow [0,1]$. The set of all fuzzy subsets of X is denoted by $F(X)$, i.e., $F(X) = \{f: f$ is a function from Xinto $[0,1]\}$.

Soft set theory is a generalization of fuzzy set theory, which was proposed by Molodtsov in 1999.

Definition 2 (soft set (SS) [3]). Let U be the universal set, E be the set of parameters, and $A \subseteq E$ and $P(U)$ be the power set of U, then a soft set F_A is defined by a mapping.

$f_A: E \longrightarrow P(U)$ such that $f_A(x) = \phi$ if $x \notin A$.

In other words, we can say that soft set F_A over U is the parameterized family of subsets of U, that is, $F_A = \{(x, f_A(x)): x \in E, f_A(x) \in P(U)\}$.

Definition 3 (fuzzy soft set (FSS) [8]). Let U be the universe of discourse, E be the set of parameters, and $A \subseteq E$, then a fuzzy soft set G_A is defined by a mapping: $g_A: E \longrightarrow P'(U)$ where $P'(U)$ is the collection of all fuzzy subsets of U, such that $g_A(x) = \phi$ if $x \notin A$.

In other words, we can say that fuzzy soft set G_A over U is the parameterized family of fuzzy subsets of U, that is, $G_A = \{(x, g_A(x)): x \in E, g_A(x) \in P'(U)\}$.

Definition 4 (intuitionistic fuzzy set (IFS) [2]). An intuitionistic fuzzy set I on a nonempty set U (universal set) is defined by the set of triplets given by

$$I = \{(x, \mu_I(x), \gamma_I(x)): x \in U\}. \quad (2)$$

Here, $\mu_I(x)$ and $\gamma_I(x)$ both are functions from U to $[0,1]$ as $\mu_I(x): U \longrightarrow [0,1]$ and $\gamma_I(x): U \longrightarrow [0,1]$. Here, $\mu_I(x)$ represents the degree of membership and $\gamma_I(x)$ represents the degree of nonmembership of each element $x \in U$ to the set I, respectively, also $0 \leq \mu_I(x) + \gamma_I(x) \leq 2$, for all $x \in U$.

Definition 5 (complex fuzzy set (CFS) [18]). The complex fuzzy set S on universe of discourse X is described as complex-valued membership function $\mu_S(x)$ that assigns value of membership of the form $r_s(x)e^{jw_s(x)}$ to any element $x \in X$, where $j = \sqrt{-1}$, $\mu_S(x)$ involves two real-valued $r_s(x)$ and $w_s(x)$, with $r_s(x) \in [0,1]$.

Mathematically, $S = \{(x, \mu_s(x)): x \in X\}$.

Definition 6 (complex intuitionistic fuzzy set (CIFS) [19]). The complex intuitionistic fuzzy set CI on a nonempty set U (universal set) is defined by the set of triplets given by $CI = \{(x, \mu_{CI}(x), \gamma_{CI}(x)): x \in U\}$. Here, $\mu_{CI}(x) = r_{CI}(x)e^{jw_{CI}(x)}$ and $\gamma_{CI}(x) = l_{CI}(x)e^{jm_{CI}(x)}$ both are functions from U to closed unit disc in the complex plane and also $\mu_{CI}(x)$ represents the degree of membership and $\gamma_{CI}(x)$ represents the degree of nonmembership of each element $x \in U$ to the set CI, respectively, and also $0 \leq r_{CI}(x) + l_{CI}(x) \leq 2$, for all $x \in U$.

Definition 7 (complex neutrosophic fuzzy set (CNFS) [16]). The complex neutrosophicfuzzy set N on a nonempty set U (universal set) is defined by the set as $N = \{(x, T_N(x), I_N(x), F_N(x): x \in U)\}$. Here, $T_N(x) = r_N(x) e^{jw_N(x)}$, $I_N(x) = l_N(x)e^{jm_N(x)}$, and $F_N(x) = p_N(x)e^{jq_N(x)}$ are the complex-valued functions from U to the closed unit disc in the complex plane where $T_N(x)$ describes complex-valued truth membership function, $I_N(x)$ describes complex-valued indeterminate membership function, and $F_N(x)$ describes complex-valued falsehood membership function of each element $x \in U$ to the set N, respectively, and also $0 \leq r_N(x) + l_N(x) + p_N(x) \leq 3$, for all $x \in U$.

3. Complex Neutrosophic Fuzzy Soft Matrix Theory

In this section, we introduced a new concept of complex neutrosophic fuzzy soft matrices. We defined the operations of union, intersection, compliment, and submatrices. We defined zero and universal matrices. Moreover, we proved some related results.

Definition 8 (complex neutrosophic fuzzy soft matrix (CNFSM)). Consider a universal set $U = \{u_1, u_2, u_3, \ldots, u_m\}$ and set of parameters $E = \{e_1, e_2, e_3, \ldots, e_n\}$ such that $A \subseteq E$ and (c_A, A) be a complex neutrosophic fuzzy soft set over (U, E). Then, the CNFSS (c_A, A) in matrix form is represented by $A_{m\times n} = [a_{ij}]_{m\times n}$ or $A^{\cdot} = [a_{ij}]$ where $i = 1, 2, 3, \ldots, m$ and $j = 1, 2, 3, \ldots, n$.

$$\text{Here, } a_{ij} = \begin{cases} \left|\mu_j(u_i)\right| = \left(\left|\mu_j^T(u_i)\right|, \left|\mu_j^I(u_i)\right|, \left|\mu_j^F(u_i)\right|\right), & \text{if } e_j \in A, \\ (0,0,0) & \text{if } e_j \notin A. \end{cases} \quad (3)$$

Now, $(\mu_j^T(u_i), \mu_j^I(u_i), \mu_j^F(u_i))$ represents degrees of membership of truth, intermediate, and falsehood on u_i. Throughout this paper, we will use the abbreviation $CNFSM_{m\times n}$ for complex neutrosophic fuzzy soft matrix over U. Following is the example of a complex neutrosophic fuzzy soft matrix.

Example 2. Let $U = \{u_1, u_2, u_3\}$ be a universal set representing the three firms, $E = \{e_1\,(\text{costly}), e_2\ (\text{beautiful}), e_3\ (\text{luxurious})\}$ be the parameters set, and $A = \{e_1, e_2\} \subseteq E$. Then, CNFSS (c_A, A) over the universal set U is given by

$$\begin{aligned}(c_A, A) = &\Big\{c_A(e_1) = \Big\{\Big(u_1, \Big(\big|0.3e^{j\pi}\big|, \big|0.6e^{j\pi/2}\big|, \big|e^{j\pi}\big|\Big)\Big)\Big\}, \Big(u_2, \Big(\big|0.7e^{j\pi/4}\big|, \big|0.8e^{j\pi/4}\big|, \big|0.5e^{j\pi/2}\big|\Big)\Big), \Big(u_3, \Big(\big|0.9e^{j\pi}\big|, \big|0.1e^{j\pi/6}\big|, \big|0.2e^{j\pi/2}\big|\Big)\Big), c_A(e_2)\\ &= \Big\{\Big(u_1, \Big(\big|0.1e^{j\pi/3}\big|, \big|0.2e^{j\pi/6}\big|, \big|0.1e^{j\pi}\big|\Big)\Big), \Big(u_2, \Big(\big|0.3e^{j\pi/2}\big|, \big|0.9e^{j\pi/2}\big|, \big|0.9e^{j\pi/4}\big|\Big)\Big), \Big(u_3, \Big(\big|0.5e^{j\pi/3}\big|, \big|0.5e^{j\pi}\big|, \big|0.6e^{j\pi/3}\big|\Big)\Big)\Big\}\Big\}.\end{aligned} \tag{4}$$

Here,

$$\begin{aligned}
0.3e^{j\pi} &= 0.3(\cos\pi + j\sin\pi) = 0.3(-1+0) = -0.3\\
\big|0.3e^{j\pi}\big| &= |-0.3| = 0.3\ 0.6\\
6e^{j\pi/2} &= 0.6\Big(\cos\Big(\frac{\pi}{2}\Big) + j\sin\Big(\frac{\pi}{2}\Big)\Big) = 0.6(0+j) = 0.6j\\
\big|0.6e^{j\pi/2}\big| &= |0.6j| = \sqrt{0.36} = 0.6\\
e^{j\pi} &= \cos\pi + j\sin\pi = -1 + 0 = -1\\
\big|e^{j\pi}\big| &= |-1| = 1, 0.7e^{j\pi/4} = 0.7\Big(\cos\Big(\frac{\pi}{4}\Big) + j\sin\Big(\frac{\pi}{4}\Big)\Big) = 0.7\Big(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\Big)\\
&= 0.7(0.707 + j0.707) = 0.494 + j0.494\\
\big|0.7e^{j\pi/4}\big| &= |0.494 + j0.494| = \sqrt{0.244 + 0.244} = 0.69,\\
0.8e^{j\pi/4} &= 0.8\Big(\cos\Big(\frac{\pi}{4}\Big) + j\sin\Big(\frac{\pi}{4}\Big)\Big) = 0.8\Big(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\Big)\\
&= 0.8(0.707 + j0.707) = 0.5656 + j0.5656,\\
|0.8e^{j\pi/4}| &= |0.5656 + j0.5656| = \sqrt{0.319 + 0.319} = 0.790.5\\
0.5e^{j\pi/2} &= 0.5\Big(\cos\Big(\frac{\pi}{2}\Big) + j\sin\Big(\frac{\pi}{2}\Big)\Big) = 0.5j\\
\big|0.5e^{j\pi/2}\big| &= |0.5j| = \sqrt{0.25} = 0.5,\\
0.9e^{j\pi} &= 0.9(\cos\pi + j\sin\pi) = 0.9(-1) = -0.9\\
\big|0.9e^{j\pi}\big| &= |-0.9| = 0.9\\
0.1e^{j\pi/6} &= 0.1\Big(\cos\Big(\frac{\pi}{6}\Big) + j\sin\Big(\frac{\pi}{6}\Big)\Big) = 0.1(0.866 + j0.5) = 0.0866 + j0.05\\
\big|0.1e^{j\pi/6}\big| &= |0.0866 + j0.05| = \sqrt{0.0074 + 0.0025} = 0.099,
\end{aligned}$$

$$\begin{aligned}
0.2e^{j\pi/2} &= 0.2\left(\cos\left(\frac{\pi}{2}\right) + j\ \sin\left(\frac{\pi}{2}\right)\right) = 0.2\,(0 + j) = 0.2j\\
\left|0.2e^{j\pi/2}\right| &= |0.2j| = \sqrt{0.04} = 0.2,\\
0.1e^{j\pi/3} &= 0.1\left(\cos\left(\frac{\pi}{3}\right) + j\ \sin\left(\frac{\pi}{3}\right)\right) = 0.1\,(0.5 + j0.866) = 0.05 + j0.0866\\
\left|0.1e^{j\pi/3}\right| &= |0.05 + j0.0866| = \sqrt{0.0025 + 0.0074} = 0.090.2e^{j\pi/6}\\
&= 0.2\left(\cos\left(\frac{\pi}{6}\right) + j\ \sin\left(\frac{\pi}{6}\right)\right) = 0.2\,(0.866 + j0.5) = 0.1732 + j0.1\\
\left|0.2e^{j\pi/6}\right| &= |0.1732 + j0.1| = \sqrt{0.029 + 0.01} = 0.19,\\
0.1e^{j\pi} &= 0.1\,(\cos\,\pi + j\ \sin\,\pi) = 0.1\,(-1 + 0) = -0.1\\
\left|0.1e^{j\pi}\right| &= |-0.1| = 0.1,\\
0.3e^{j\pi/2} &= 0.3\left(\cos\left(\frac{\pi}{2}\right) + j\ \sin\left(\frac{\pi}{2}\right)\right) = 0.3j\\
\left|0.3e^{j\pi/2}\right| &= |0.3j| = \sqrt{0.09} = 0.3\\
0.9e^{j\pi/2} &= 0.9\left(\cos\,\pi/2 + j\ \sin\left(\frac{\pi}{2}\right)\right) = 0.9j\\
\left|0.9e^{j\pi/2}\right| &= |0.9j| = \sqrt{0.81} = 0.9,\\
0.9e^{j\pi/4} &= 0.9\left(\cos\left(\frac{\pi}{4}\right) + j\ \sin\left(\frac{\pi}{4}\right)\right) = 0.9\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)\\
&= 0.9\,(0.707 + j0.707) = 0.636 + j0.636\\
\left|0.9e^{j\pi/4}\right| &= |0.636 + j0.636| = \sqrt{0.404 + 0.404} = 0.898,\\
0.5e^{j\pi/3} &= 0.5\left(\cos\left(\frac{\pi}{3}\right) + j\ \sin\left(\frac{\pi}{3}\right)\right) = 0.5\,(0.5 + j0.866) = 0.25 + j0.433\\
\left|0.5e^{j\pi/3}\right| &= |0.25 + j0.433| = \sqrt{0.0625 + 0.187} = 0.499\\
0.5e^{j\pi} &= 0.5\,(\cos\,\pi + j\ \sin\,\pi) = 0.5\,(-1) = -0.5\\
\left|0.5e^{j\pi}\right| &= |-0.5| = 0.5,\\
0.6e^{j\pi/3} &= 0.6\left(\cos\left(\frac{\pi}{3}\right) + j\ \sin\left(\frac{\pi}{3}\right)\right) = 0.6\,(0.5 + j0.866) = 0.3 + j0.519\\
\left|0.6e^{j\pi/3}\right| &= |0.3 + j0.519| = \sqrt{0.09 + 0.269} = 0.599.
\end{aligned} \tag{5}$$

Now, the abovementioned CNFSS (c_A, A) in matrix form is given by

$$A = \begin{bmatrix} (0.3, 0.6, 1) & (0.09, 0.19, 0.1) & (0, 0, 0)\\ (0.69, 0.79, 0.5) & (0.3, 0.9, 0.898) & (0, 0, 0)\\ (0.9, 0.099, 0.2) & (0.499, 0.5, 0.599) & (0, 0, 0) \end{bmatrix}. \tag{6}$$

Definition 9 (complex neutrosophic fuzzy soft zero natrix). Let $[a_{ij}] \in CNFSM_{m\times n}$, then $[a_{ij}]$ is called complex neutrosophic fuzzy soft zero matrix if $(a_{ij}, r_{ij}, l_{ij}) = (0, 0, 0)$, for all i and j, and is denoted by $[0]$.

Example 3

$$[0] = \begin{bmatrix} (0, 0, 0) & (0, 0, 0) & (0, 0, 0)\\ (0, 0, 0) & (0, 0, 0) & (0, 0, 0)\\ (0, 0, 0) & (0, 0, 0) & (0, 0, 0) \end{bmatrix}. \tag{7}$$

Definition 10 (complex neutrosophic fuzzy soft universal matrix). Let $[a_{ij}] \in \mathrm{CNFSM}_{m\times n}$, then $[a_{ij}]$ is called complex neutrosophic fuzzy soft universal matrix if $(a_{ij}, r_{ij}, l_{ij}) = (1, 1, 1)$, for all i and j, and is represented by $[1]$.

$$[1]=\begin{bmatrix}(1,1,1) & (1,1,1) & (1,1,1)\\ (1,1,1) & (1,1,1) & (1,1,1)\\ (1,1,1) & (1,1,1) & (1,1,1)\end{bmatrix}. \quad (8)$$

Definition 11 (complex neutrosophic fuzzy soft submatrices). Let $A_{m\times n}$ and $B_{m\times n}$ be two CNFSMs, then

(i) $A_{m\times n}$ is a CNFS submatrix of $B_{m\times n}$ and is denoted by $A_{m\times n}\sqsubseteq B_{m\times n}$ if $a_{ij}=(a_{ij},a'_{ij},a''_{ij})\preccurlyeq b_{ij}=(b_{ij},b'_{ij},b''_{ij})$, that is, $(a_{ij}\preccurlyeq b_{ij},a''_{ij}\preccurlyeq b'_{ij},a''_{ij}\preccurlyeq b''_{ij})$, for all $a_{ij}\in A_{m\times n},b_{ij}\in B_{m\times n}$

(ii) $A_{m\times n}$ is a proper *CNFS* submatrix of $B_{m\times n}$ and is denoted by $A_{m\times n}\sqsubset B_{m\times n}$ if $a_{ij}=(a_{ij},a'_{ij},a''_{ij})\prec b_{ij}=(b_{ij},b'_{ij},b''_{ij})$, that is, $(a_{ij}\prec b_{ij},a'_{ij}\prec b'_{ij},a''_{ij}\prec b''_{ij})$, for all $a_{ij}\in A_{m\times n}$, $b_{ij}\in B_{m\times n}$ and for at least one entry $a_{ij}\prec b_{ij}$, that is, $(a_{ij}\prec b_{ij},a'_{ij}\prec b'_{ij},a''_{ij}\prec b''_{ij})$

(iii) Two *CNFSMs* $A_{m\times n}$ and $B_{m\times mn}$ are equal and are denoted by $A_{m\times n}=B_{m\times n}$, if $a_{ij}=(a_{ij},a'_{ij},a''_{ij})=b_{ij}=(b_{ij},b'_{ij},b''_{ij})$, that is. $(a_{ij}=b_{ij},a'_{ij}=b'_{ij},a''_{ij}=b''_{ij})$, for all $a_{ij}\in A_{m\times n}$, $b_{ij}\in B_{m\times n}$

Example 4. Let

$$A_{2\times 2}=\begin{bmatrix}(0.2,0.4,0.1) & (0.1,0.5,0.2)\\ (0.3,0.7,0.3) & (0.5,0.4,0.4)\end{bmatrix},$$
$$B_{2\times 2}=\begin{bmatrix}(0.2,0.4,0.1) & (0.3,0.7,0.9)\\ (0.3,0.7,0.3) & (0.7,0.5,0.7)\end{bmatrix}. \quad (9)$$

So, we can write that $A_{2\times 2}\sqsubset B_{2\times 2}$. Moreover, $A\sqsubset B$.

Definition 12. (union/intersection and compliment of complex neutrosophic fuzzy soft matrices).

Let $A_{m\times n}$ and $B_{m\times n}$ be two CNFSM, then the CNFSM $C_{m\times n}$ is called

(i) Union of $A_{m\times n}$ and $B_{m\times n}$ and is denoted by $A_{m\times n}\sqcup B_{m\times n}$ if $C_{m\times n}=\max\{A_{m\times n},B_{m\times n}\}$, for all i and j, that is, $c_{ij}=(\max(a_{ij},b_{ij}),\min(a'_{ij},b'_{ij}),\min(a''_{ij},b''_{ij}))$ where $c_{ij}=(c_{ij},c'_{ij},c''_{ij})$

(ii) Intersection of $A_{m\times n}$ and $B_{m\times n}$ is denoted by $A_{m\times n}\sqcap B_{m\times n}$ if $C_{m\times n}=\min\{A_{m\times n},B_{m\times n}\}$, for all i and j, that is, $c_{ij}=(\min(a_{ij},b_{ij}),\max(a'_{ij},b'_{ij}),\max(a''_{ij},b''_{ij}))$, where $c_{ij}=(c_{ij},c'_{ij},c''_{ij})$

(iii) Complement of $A_{m\times n}$ is denoted by $A'_{m\times n}$ if $C_{m\times n}=1-A_{m\times n}$, for all i and j, that is, $c_{ij}=(1-a_{ij},1-a'_{ij},1-a''_{ij})$, where $c_{ij}=(c_{ij},c'_{ij},c''_{ij})$

Example 5. Assume that

$$A_{2\times 2}=\begin{bmatrix}(0.3,0.6,1) & (0.65,0,0.6)\\ (0.3,0.9,0) & (0.8,0.7,0.9)\end{bmatrix},$$
$$B_{2\times 2}=\begin{bmatrix}(0.49,0.5,0.4) & (0.2,0,0.3)\\ (0.1,0.9,0.3) & (0,0,0)\end{bmatrix},$$

then,

$$A_{2\times 2}\sqcup B_{2\times 2}=\begin{bmatrix}(0.49,0.5,0.4) & (0.65,0,0.3)\\ (0.3,0.9,0) & (0.8,0,0)\end{bmatrix}, \quad (10)$$
$$A_{2\times 2}\sqcap B_{2\times 2}=\begin{bmatrix}(0.3,0.6,1) & (0.2,0,0.6)\\ (0.1,0.9,0.3) & (0,0.7,0.9)\end{bmatrix},$$
$$A'_{2\times 2}=\begin{bmatrix}(0.7,0.4,0) & (0.35,1,0.4)\\ (0.7,0.1,1) & (0.2,0.3,0.1)\end{bmatrix}.$$

Proposition 1. *Let $A_{m\times n}$ be a CNFSM, then*

$$\begin{aligned}&\text{(i) } ((A_{m\times n})')'=A_{m\times n},\\ &\text{(ii) } [0]'=[1].\end{aligned} \quad (11)$$

Proof. It follows from definition. □

Proposition 2. *Let $A_{m\times n}$, $B_{m\times n}$, and $C_{m\times n}$ be three CNFSMs, then*

$$\begin{aligned}&\text{(i) } A_{m\times n}=B_{m\times n} \text{ and } B_{m\times n}=C_{m\times n}\Longrightarrow A_{m\times n}=C_{m\times n},\\ &\text{(ii) } A_{m\times n}\sqsubseteq B_{m\times n} \text{ and } B_{m\times n}\sqsubseteq A_{m\times n}\Longrightarrow A_{m\times n}=B_{m\times n}.\end{aligned} \quad (12)$$

Proof. It follows from definition. □

Proposition 3. *Let $A_{m\times n}$ and $B_{m\times n}$ be two CNFSMs, then*

$$A_{m\times n}\sqsubseteq B_{m\times n} \text{ and } B_{m\times n}\sqsubseteq C_{m\times n}\Longrightarrow A_{m\times n}\sqsubseteq C_{m\times n}. \quad (13)$$

Proof. It follows from definition. □

Proposition 4. *Let $A_{m\times n}$ and $B_{m\times n}$ be two CNFSMs, then*

$$\begin{aligned}&\text{(i) } A_{m\times n}\sqcup B_{m\times n}=B_{m\times n}\sqcup A_{m\times n},\\ &\text{(ii) } A_{m\times n}\sqcap B_{m\times n}=B_{m\times n}\sqcap A_{m\times n},\\ &\text{(iii) } (A_{m\times n}\sqcup B_{m\times n})\sqcup C_{m\times n}=A_{m\times n}\sqcup(B_{m\times n}\sqcup C_{m\times n}),\\ &\text{(iv) } (A_{m\times n}\sqcap B_{m\times n})\sqcap C_{m\times n}=A_{m\times n}\sqcap(B_{m\times n}\sqcap C_{m\times n}),\\ &\text{(v) } A_{m\times n}\sqcup(B_{m\times n}\sqcap C_{m\times n})=(A_{m\times n}\sqcup B_{m\times n})\sqcap(A_{m\times n}\sqcup C_{m\times n}),\\ &\text{(vi) } A\sqcap(B_{m\times n}\sqcup C_{m\times n})=(A_{m\times n}\sqcap B_{m\times n})\sqcup(A_{m\times n}\sqcap C_{m\times n}).\end{aligned} \quad (14)$$

Proof

$$\begin{aligned}
\text{(i) } A_{m\times n}\sqcup B_{m\times n} &= \max\left(A_{m\times n}, B_{m\times n}\right)\\
&= \max\left(B_{m\times n}, A_{m\times n}\right)\\
&= B_{m\times n}\sqcup A_{m\times n},\\
\text{(ii) } A_{m\times n}\sqcap B_{m\times n} &= \min\left(A_{m\times n}, B_{m\times n}\right)\\
&= \min\left(B_{m\times n}, A_{m\times n}\right)\\
&= B_{m\times n}\sqcap A_{m\times n},\\
\text{(iii) } \left(A_{m\times n}\sqcup B_{m\times n}\right)\sqcup C_{m\times n} &= \max\left(\left(A_{m\times n}\sqcup B_{m\times n}\right), C_{m\times n}\right)\\
&= \max\left(\max\left(A_{m\times n}, B_{m\times n}\right), C_{m\times n}\right)\\
&= \max\left(A_{m\times n}, \max\left(B_{m\times n}, C_{m\times n}\right)\right)\\
&= \max\left(A_{m\times n}, \left(B_{m\times n}\sqcup C_{m\times n}\right)\right)\\
&= A_{m\times n}\sqcup\left(B_{m\times n}\sqcup C_{m\times n}\right),\\
\text{(iv) } \left(A_{m\times n}\sqcap B_{m\times n}\right)\sqcap C_{m\times n} &= \min\left(\left(A_{m\times n}\sqcap B_{m\times n}\right), C_{m\times n}\right)\\
&= \min\left(\min\left(A_{m\times n}, B_{m\times n}\right), C_{m\times n}\right)\\
&= \min\left(A_{m\times n}, \min\left(B_{m\times n}, C_{m\times n}\right)\right)\\
&= \min\left(A_{m\times n}, \left(B_{m\times n}\sqcap C_{m\times n}\right)\right)\\
&= A_{m\times n}\sqcap\left(B_{m\times n}\sqcap C_{m\times n}\right).\\
\text{(v) } A_{m\times n}\sqcup\left(B_{m\times n}\sqcap C_{m\times n}\right) &= \max\left(A_{m\times n}, \left(B_{m\times n}\sqcap C_{m\times n}\right)\right)\\
&= \max\left(A_{m\times n}, \min\left(B_{m\times n}, C_{m\times n}\right)\right)\\
&= \min\left(\max\left(A_{m\times n}, B_{m\times n}\right), \max\left(A_{m\times n}, C_{m\times n}\right)\right)\\
&= \min\left(\left(A_{m\times n}\sqcup B_{m\times n}\right), \left(A_{m\times n}\sqcup C_{m\times n}\right)\right)\\
&= \left(A_{m\times n}\sqcup B_{m\times n}\right)\sqcap\left(A_{m\times n}\sqcup C_{m\times n}\right),\\
\text{(vi) } A_{m\times n}\sqcap\left(B_{m\times n}\sqcup C_{m\times n}\right) &= \min\left(A_{m\times n}, \left(B_{m\times n}\sqcup C_{m\times n}\right)\right)\\
&= \min\left(A_{m\times n}, \max\left(B_{m\times n}, C_{m\times n}\right)\right)\\
&= \max\left(\min\left(A_{m\times n}, B_{m\times n}\right), \min\left(A_{m\times n}, C_{m\times n}\right)\right)\\
&= \max\left(\left(A_{m\times n}\sqcap B_{m\times n}\right), \left(A_{m\times n}\sqcap C_{m\times n}\right)\right)\\
&= \left(A_{m\times n}\sqcap B_{m\times n}\right)\sqcup\left(A_{m\times n}\sqcap C_{m\times n}\right).
\end{aligned} \tag{15}$$

□

Proposition 5. *Let $A_{m\times n}$ and $B_{m\times n}$ be two CNFSMs, then the De-Morgan laws are valid:*

$$\begin{aligned}
&\text{(i) } \left(A_{m\times n}\sqcup B_{m\times n}\right)' = \left(A_{m\times n}\right)'\sqcap\left(B_{m\times n}\right)'\\
&(ii)\ \left(A_{m\times n}\sqcap B_{m\times n}\right)' = \left(A_{m\times n}\right)'\sqcup\left(B_{m\times n}\right)'.
\end{aligned} \tag{16}$$

Proof.

$$\begin{aligned}
\text{(i) } \left(A_{m\times n}\sqcup B_{m\times n}\right)' &= \left[\max\left(A_{m\times n}, B_{m\times n}\right)\right]'\\
&= \left[1-\max\left(A_{m\times n}, B_{m\times n}\right)\right]\\
&= \left[\min\left(1-A_{m\times n}, 1-B_{m\times n}\right)\right]\\
&= \left[A_{m\times n}\right]'\sqcap\left[B_{m\times n}\right]',\\
\left(A_{m\times n}\sqcap B_{m\times n}\right)' &= \left[\min\left(A_{m\times n}, B_{m\times n}\right)\right]'\\
&= \left[1-\min\left(A_{m\times n}, B_{m\times n}\right)\right]\\
&= \left[\max\left(1-A_{m\times n}, 1-B_{m\times n}\right)\right]\\
&= \left[A_{m\times n}\right]'\sqcup\left[B_{m\times n}\right]'.
\end{aligned} \tag{17}$$

□

4. Complex Neutrosophic Fuzzy Soft Decision-Making Method

Now, we are going to discuss real-life applications of newly defined $\text{CNFSM}_{m\times n}$. We will show how our theoretical concepts and results can be applied to the real-life phenomenon. Specifically, we will show that $\text{CNFSM}_{m\times n}$ explains how to get a better and clear signal for identification with a given reference signal. Before moving towards the algorithm, we will define the fuzzy soft (FS) max-min decision-making method (FSMmDM) by using FS max-min decision function and also define here the optimum FS on universal set U.

Definition 13 (fuzzy soft (FS) max-min decision-making function [10]). Let $[c_{ip}]\in SM_{m\times n^2}$, $I_k=\{p\colon \text{thereexisti}, c_{ip}\neq 0, (k-1)n<p\leq kn\}$, for all $k\in I=\{1,2,3,\ldots,n\}$. Then, soft max-min decision function, denoted Mm, is defined as follows:

$$Mm\colon SM_{m\times n^2}\longrightarrow SMm_{m\times 1},\ Mm\left[c_{ip}\right]=\left[\max_{k\in I}\{t_k\}\right], \tag{18}$$

where

$$t_k=\begin{pmatrix} \min\limits_{p\in I_k}\left\{c_{ip}\right\}, & \text{if } I_k\neq\{\},\\ 0, & \text{if } I_k=\{\}.\end{pmatrix} \tag{19}$$

The one column soft matrix $Mm[c_{ip}]$ is called max-min soft decision-making matrix.

Definition 14 (see [10]). Let $U=\{u_1,u_2,\ldots,u_m\}$ be a universal set and $Mm[c_{ip}]=[d_{i1}]$. Then, a subset of U can be obtained by using $[d_{i1}]$ as in the following way $\text{opt}_{[d_{i1}]}(U)=\{u_i\colon u_i\in U, d_{i1}=1\}$, which is called an optimum set on U.

4.1. Decision-Making Algorithm

step 1. Suppose that $\mathbf{M}$ different signals $S_1(t')$, $S_2(t'),\ldots,S_M(t')$ are detected and sampled by a receiver and let $U=\{S_1(t'),S_2(t'),\ldots,S_M(t')\}$. Each of these signals is sampled N times. Let $S_m(r')$ denote the r'th sample $(1\leq r'\leq N)$ of the mth signal $(1\leq m\leq M)$. Now, we know that each signal has its Fourier transform. So, each received signal can be expressed as summation of its Fourier components as

$$\begin{aligned}
S_m(r') &= \left(\frac{1}{N}\right)\sum_{n=1}^{N} C_{m,n}e^{i2\pi(n-1)(r'-1)/N}, \text{then}\\
\left|S_m(r')\right| &= \left(\frac{1}{N}\right)\sum_{n=1}^{N}\left|C_{m,n}\right|\cdot\left|e^{i2\pi(n-1)(r'-1)/N}\right|,
\end{aligned} \tag{20}$$

where $C_{m,n}\,(1\leq n\leq N)$ represents complex Fourier coefficients of S_m. The above expression can also be rewritten as follows:

$|S_m(r')| = (1/N)\sum_{n=1}^{N}|B_{m,n}|\cdot|e^{i(2\pi(n-1)(r'-1)+N\beta_{m,n})/N}|$, where $C_{m,n} = B_{m,n}e^{i\beta_{m,n}}$, with $B_{m,n}, \beta_{m,n}$ real-valued and $B_{m,n} \geq 0$, for all n, where $1 \leq n \leq N$.

step 2. The above given signals are expressed as in matrix form as $A = [|S_m(r')|]_{N\times M}$, that is, express N samples of each signal (total M signals) in columns:

$$A = \begin{bmatrix} \left(S_1^T(1), S_1^I(1), S_1^F(1)\right) & \left(S_2^T(1), S_2^I(1), S_2^F(1)\right) & \cdots & \left(S_M^T(1), S_M^I(1), S_M^F(1)\right) \\ \left(S_1^T(2), S_1^I(2), S_1^F(2)\right) & \left(S_2^T(2), S_2^I(2), S_2^F(2)\right) & \cdots & \left(S_M^T(2), S_M^I(2), S_M^F(2)\right) \\ \cdot & \cdot & \cdots & \cdot \\ \left(S_1^T(N), S_1^I(N)\cdot S_1^F(N)\right) & \left(S_2^T(N), S_2^I(N), S_2^F(N)\right) & \cdots & \left(S_M^T(N), S_M^I(N), S_M^F(N)\right) \end{bmatrix}. \quad (21)$$

step 3. Similarly, we will construct another matrix by the signals $S_m^*(r)$.

$$B = \begin{bmatrix} \left(S_1^{*T}(1), S_1^{*I}(1), S_1^{*F}(1)\right) & \left(S_2^{*T}(1), S_2^{*I}(1), S_2^{*F}(1)\right) & \cdots & \left(S_M^{*T}(1), S_M^{*I}(1), S_M^{*F}(1)\right) \\ \left(S_1^{*T}(2), S_1^{*I}(2), S_1^{*F}(2)\right) & \left(S_2^{*T}(2), S_2^{*I}(2), S_2^{*F}(2)\right) & \cdots & \left(S_M^{*T}(2), S_M^{*I}(2), S_M^{*F}(2)\right) \\ \cdot & \cdot & \cdots & \cdot \\ \left(S_1^{*T}(N), S_1^{*I}(N), S_1^{*F}(N)\right) & \left(S_2^{*T}(N), S_2^{*I}(N), S_2^{*F}(N)\right) & \cdots & \left(S_M^{*T}(N), S_M^{*I}(N), S_M^{*F}(N)\right) \end{bmatrix}. \quad (22)$$

step 4. Multiply matrices A and B using usual multiplication of matrices. In this multiplication, the truth value of the entry of the first matrix will be multiplied by the truth value of the entry of the second matrix. The intermediate and false values of the entries are multiplied similarly.

step 5. The complex neutrosophic fuzzy soft max-min decision-making matrix (CNFSMmDM) is found by taking minimum of truth, intermediate memberships, and maximum of falsehood membership values of each column, and we will get a column matrix $[d_{i1}]$, where $1 \leq i \leq M$.

step 6. An optimum set $\mathrm{opt}_{Mm[AB]}(U)$ on U is found, that is,

$$\left\{\left(\max\{|S_j^T(i)|, \max\left\{\left|S_j^I(u_i)\right|\right\}, \min\left|S_j^F(u_i)\right|\right), \text{for } 1 \leq j \leq M \text{ and } 1 \leq i \leq N\right\}. \quad (23)$$

5. Applications

Step 1. Assume that u_1, u_2, and u_3 be any three signals received by a digital receiver from any source. Each signal is a triplet of numbers. The first number of triplet represents the truth value, second represents the intermediate value, and the third represents the false value corresponding to each signal. Now, each of these signals is sampled three times. Let R be the given known reference signal. Each signal is compared with the reference signal in order to get the high degree of resemblance with the reference signal R. Now, we obtain the matrix A by setting the signals along column and their three times sampling along row. Similarly, we will obtain the matrix B.

step 2. Matrices A and B are given by

$$A = \begin{bmatrix} (0.7, 0.4, 0.5) & (0.6, 0.7, 1) & (0.8, 1, 0.7) \\ (0.8, 0.5, 0.3) & (0.2, 0, 0.9) & (0.5, 0.8, 0.4) \\ (0.4, 0, 0.8) & (0.8, 0.4, 0.6) & (0, 0.3, 0.9) \end{bmatrix}. \quad (24)$$

step 3

$$B = \begin{bmatrix} (0.4, 0.4, 0) & (0.6, 0.7, 0.4) & (0.1, 0.3, 0) \\ (0.3, 0.7, 0.7) & (0.4, 0.9, 0.4) & (0.1, 0.6, 0.4) \\ (0.2, 0.4, 0.5) & (0.4, 0.5, 0.3) & (0.8, 0.5, 0.8) \end{bmatrix}. \quad (25)$$

step 4. Now, we will calculate the product of above defined matrices by usual multiplication of matrices. In this multiplication, the truth value of the entry of the first matrix will be multiplied by the truth value of the entry of the second matrix. Similarly, the intermediate and false values of the entries are multiplied.

$$AB = \begin{bmatrix} (0.62, 0.69, 0.42) & (0.98, 0.96, 0.45) & (0.77, 0.59, 0.6) \\ (0.48, 0.52, 0.83) & (0.76, 0.75, 0.6) & (0.5, 0.55, 0.68) \\ (0.4, 0.4, 0.87) & (0.56, 0.51, 0.83) & (0.12, 0.39, 0.96) \end{bmatrix}. \quad (26)$$

step 5. We calculate $\mathrm{CNFSMmDM}[AB] = [d_{i1}]$, for all $i = 1, 2, 3$, where d_{i1} is defined as $d_{i1} = \min\{t_{k1}\} = \min\{t_{11}, t_{21}, t_{31}\}$ for all $k = 1, 2, 3$.

$$\begin{aligned} d_{11} &= \min\{t_{k1}\} = \min\{t_{11}, t_{21}, t_{31}\} \\ &= \min\{(0.62, 0.69, 0.42), (0.48, 0.52, 0.83), (0.4, 0.4, 0.87)\} = (0.4, 0.4, 0.42), \\ d_{21} &= \min\{t_{k2}\} = \min\{t_{12}, t_{22}, t_{32}\} \\ &= \min\{(0.98, 0.96, 0.45), (0.76, 0.75, 0.6), (0.56, 0.51, 0.83)\} = (0.56, 0.51, 0.45), \\ d_{31} &= \min\{t_{k3}\} = \min\{t_{13}, t_{23}, t_{33}\} \\ &= \min\{(0.77, 0.59, 0.6), (0.5, 0.55, 0.68), (0.12, 0.39, 0.96)\} = (0.12, 0.39, 0.6). \end{aligned} \tag{27}$$

We obtain CNFSMmDM as follows:

$$\text{CNFSMmDM}[AB] = [d_{i1}] = \begin{bmatrix} (0.4, 0.4, 0.42) \\ (0.56, 0.51, 0.45) \\ (0.12, 0.39, 0.6) \end{bmatrix}. \tag{28}$$

Step 6. Finally, we find out an optimum set on U as follows: $\text{opt}_{Mm[AB]}(U) = u_2$. So, the signal which is identified as a reference signal is the signal u_2.

6. Conclusion

This paper consists of CNFSM and different types of complex neutrosophic soft matrices with examples. We introduced some new operations on complex neutrosophic fuzzy soft matrices and explore related properties. Further, we constructed a complex neutrosophic soft decision-making algorithm with the help of these matrices and used it in signal processing. We hope that our finding will help in enhancing the study on complex neutrosophic soft theory and will open a new direction for applications especially in decision sciences. In future, we will define some new operations on complex neutrosophic fuzzy soft sets and will introduce some new algorithms for signals and other related decision-making in social sciences. Specifically, we will use complex fuzzy sets and complex neutrosophic fuzzy sets in signal processing for modeling of continuous signals.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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Research Article

Algorithms for a Generalized Multipolar Neutrosophic Soft Set with Information Measures to Solve Medical Diagnoses and Decision-Making Problems

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The aim of this paper is to propose the generalized version of the multipolar neutrosophic soft set with operations and basic properties. Here, we define the AND, OR, Truth-Favorite, and False-Favorite operators along with their properties. Also, we define the necessity and possibility of operations for them. Later on, to extend it to solve the decision-making problems, we define some information measures such as distance, similarity, and correlation coefficient for the generalized multipolar neutrosophic soft set. Several desirable properties and their relationship between them are derived. Finally, based on these information measures, a decision-making algorithm is stated under the neutrosophic environment to tackle the uncertain and vague information. The applicability of the proposed algorithm is demonstrated through a case study of the medical-diagnosis and the decision-making problems. A comparative analysis with several existing studies reveals the effectiveness of the approach.

1. Introduction

Uncertainty plays a dynamic part in numerous fields of life such as modeling, medical, and engineering fields. However, a general question of how we can express and use the uncertainty concept in mathematical modeling is raised. A lot of researchers in the world proposed and recommended different approaches to use uncertainty theory. First of all, Zadeh developed the notion of fuzzy sets [1] to solve those problems which contain uncertainty and vagueness. It is observed that in some cases circumstances cannot be handled by fuzzy sets; to overcome such types of situations, Turksen [2] gave the idea of the interval-valued fuzzy set (IVFS). In some cases, we must deliberate membership unbiassed as the nonmembership values for the suitable representation of an object in uncertain and indeterminate conditions that could not be handled by fuzzy sets or by IVFS. To overcome these difficulties, Atanassov presented the notion of intuitionistic fuzzy sets (IFSs) in [3]. The theory that was presented by Atanassov only deals with the insufficient data considering both membership and nonmembership values; however, the IFSs theory cannot handle the overall incompatible as well as imprecise information. To address such incompatible as well as imprecise records, the idea of the neutrosophic set (NS) was developed by Smarandache [4]. A general mathematical tool was proposed by Molodtsov [5] to deal with indeterminate, fuzzy, and not clearly defined substances known as a soft set (SS). Maji et al. [6] extended the work on SS and defined some operations and their properties. Maji et al. [7] utilized the SS theory for decision-making. Ali et al. [8] revised the Maji approach to SS and developed some new operations with their properties. De Morgan's Law on SS theory was proved in [9] by using different operators.

Maji [10] offered the idea of a neutrosophic soft set (NSS) with necessary operations and properties. The idea of the possibility NSS was developed by Karaaslan [11] and introduced a possibility of neutrosophic soft decision-making method to solve those problems which contain uncertainty based on And-product. Broumi [12] developed the generalized NSS with some operations and properties and used the proposed concept for decision-making. To solve MCDM problems with single-valued neutrosophic numbers (SVNNs) presented by Deli and Subas in [13], they constructed the concept of cut sets of SVNNs. On the basis of the correlation of IFS, the term correlation coefficient (CC) of SVNSs [14] was introduced. Ye [15] presented the simplified NSs introduced with some operational laws and aggregation operators such as weighted arithmetic and weighted geometric average operators. Therein, a multicriteria decision-making (MCDM) method was constructed based on proposed aggregation operators. Masooma et al. [16] progressed a new concept by combining the multipolar fuzzy set and neutrosophic set, which is known as the multipolar neutrosophic set. They also established various characterizations and operations with examples. Dey et al. [17] developed the grey relational projection method based on NSS to solve MADM complications. Pramanik et al. [18] extended the VIKOR technique to solve MAGDM problems under a bipolar neutrosophic set environment. Pramanik et al. [19] established the TOPSIS technique to solve MADM problems utilizing single-valued neutrosophic soft expert sets. Pramanik et al. [20] developed three different hybrid projection measures projection, bidirectional projection, and hybrid projection measures between bipolar neutrosophic sets.

Peng et al. [21] established the probability multivalued neutrosophic set by combining the multivalued neutrosophic set and probability distribution and used it for decision-making problems. Kamal et al. [22] proposed the idea of mPNSS with some important operations and properties; they also used the developed technique for decision-making. Garg [23] developed the MCDM method based on weighted cosine similarity measures under an intuitionistic fuzzy environment and used the proposed technique for pattern recognition and medical diagnoses. To measure the relative strength of IFS, Garg and Kumar [24] presented some new similarity measures. They also formulated a connection number for set pair analysis (SPA) and developed some new similarity measures and weighted similarity measures based on defined SPA. Garg and Rani [25] extended the IFS technique to complex intuitionistic fuzzy sets (CIFS) and developed the correlation and weighted correlation coefficient under the CIFS environment. To measure the relation between two Pythagorean fuzzy sets (PFS), Garg [26] proposed a novel CC and WCC and presented the numerical examples of pattern recognition and medical diagnoses to verify the validity of the proposed measures. Zulqarnain et al. [27] developed the aggregation operators for Pythagorean fuzzy soft sets and proposed a decision-making methodology using their developed aggregation operators. They also utilized their established decision-making technique for the selection of suppliers in green supply chain management. Zulqarnain et al. [28] extended the TOPSIS technique under Pythagorean fuzzy soft environment. Nguyen et al. [29] defined some similarity measures for PFS by using the exponential function for the membership and nonmembership degrees with its several properties and relations. Peng and Garg [30] presented some diverse types of similarity measures for PFS with multiple parameters. Wang and Li [31] introduced Pythagorean fuzzy interaction power Bonferroni mean (PBM) operators for solving MADM issues. Wang et al. [32] proposed the Pythagorean fuzzy interactive Hamacher power aggregation operators for assessment of express service quality with entropy weight. Saeed et al. [33] established the concept of mPNSS with its properties and operators; they also developed the distance-based similarity measures and used the proposed similarity measures for decision-making and medical diagnoses.

Gerstenkorn and Mafiko [34] proposed the functional measuring of the interrelation of IFSs, which is known nowadays as correlation, and developed its coefficient with properties. To measure the interrelation of fuzzy numbers, Yu [35] established the CC of fuzzy numbers. Evaluating the CC for fuzzy data had been developed by Chiang and Lin [36]. Hung and Wu [37] proposed the centroid method to calculate the CC of IFSs and extended the proposed method to interval-valued intuitionistic fuzzy sets (IVIFSs). Hong [38] and Mitchell [39] also established the CC for IFSs and IVIFSs, respectively. Ye [40] extended the work on IFSs and developed the CC of a single-valued neutrosophic set and developed a decision-making method for similarity measure. Xue et al. [41] developed the CC on a single-valued neutrosophic set and proposed a decision-making method for pattern recognition. Zulqarnain et al. [42] utilized the neutrosophic TOPSIS in the production industry for supplier selection. Garg and Arora [43] introduced the correlation measures on intuitionistic fuzzy soft sets and constructed the TOPSIS technique on developed correlation measures. In Iryna et al.'s work [44], an algorithm has been proposed to handle uncertainty in fault diagnoses by using single-valued neutrosophic sets. Faruk [45] established CC between possibility NSS and proved some properties. He also developed CC for a single-valued neutrosophic refined soft set, and it was used for clustering analysis [46]. A correlation measure of neutrosophic refined sets has been developed, which is the extension of the correlation measure of neutrosophic sets and intuitionistic fuzzy multisets [47].

In this era, professionals consider that the real life is moving in the direction of multipolarity. Thus, it projects as no surprise that multipolarity in information performs a significant part in flourishing numerous fields of science as well as technology. In neurobiology, multipolar neurons in the brain gather a good deal of information from other neurons. In information technology, multipolar technology could be used to control extensive structures. The motivation of the present research is extended and hybrid work is given step by step in the complete article. We demonstrate that different hybrid structures containing fuzzy sets are converted into the special privilege of mPNSS under whatsoever appropriate circumstances. The concept of a neutrosophic environment to a multipolar neutrosophic soft set is novel. We tend to discuss the effectiveness, flexibility, quality, and favorable position of our planned work and algorithms. The present research will be the most generalized form and is used to assemble data in considerable and appropriate medical, engineering, artificial intelligence, agriculture, and other everyday life complications. In the future, the present work might be gone competently for other approaches and different types of hybrid structures.

The remainder of the paper is organized as follows: in Section 2, we recollected some basic definitions which are used in the following sequel such as NS, SS, NSS, and multipolar neutrosophic set. In Section 3, we proposed the generalized version of mPNSS with its properties and operations, and we also developed the Truth-Favorite, False-Favorite, AND, and OR operators in this section. In Section 4, distance-based similarity measures have been developed by using Hamming distance and Euclidean distance between two generalized multipolar neutrosophic soft sets (GmPNSS). In Section 5, the idea of CC and WCC with their properties has been established. Finally, we use the developed distance-based similarity measures and CC for medical diagnoses and decision-making in Section 6. We also present the comparative study of our proposed similarity measures and CC with some already existing techniques in Section 7.

2. Preliminaries

In this section, we recollect some basic concepts such as neutrosophic set, soft set, neutrosophic soft set, and m-polar neutrosophic soft set, which are used in the following sequel.

Definition 1 (see [4]).

Let $\mathscr{U}$ be a universe and let $\mathscr{A}$ be an NS on $\mathscr{U}$ defined as $\mathscr{A} = \{u, (u_{\mathscr{A}}(u), v_{\mathscr{A}}(u), w_{\mathscr{A}}(u)): u \in \mathscr{U}\}$, where u, v, w: $\mathscr{U} \longrightarrow 0^-, 1^+$ and $0^- \le u_{\mathscr{A}}(u) + v_{\mathscr{A}}(u) + w_{\mathscr{A}}(u) \le 3^+$.

Definition 2 (see [5]).

Let $\mathscr{U}$ be the universal set and let $\mathscr{E}$ be the set of attributes concerning $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the power set of $\mathscr{U}$ and $\mathscr{A} \subseteq \mathscr{E}$. A pair $(\mathscr{F}, \mathscr{A})$ is called a soft set over $\mathscr{U}$ and its mapping is given as

$$\mathscr{F}: \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{1}$$

It is also defined as

$$(\mathscr{F}, \mathscr{A}) = \{\mathscr{F}(e) \in \mathscr{P}(\mathscr{U}): e \in \mathscr{E}, \mathscr{F}(e) = \varnothing \text{ if } e \notin \mathscr{A}\}. \tag{2}$$

Definition 3 (see [10]).

Let $\mathscr{U}$ be the universal set and let $\mathscr{E}$ be the set of attributes concerning $\mathscr{U}$. Let $\mathscr{P}(\mathscr{U})$ be the set of neutrosophic values of $\mathscr{U}$ and $\mathscr{A} \subseteq \mathscr{E}$. A pair $(\mathscr{F}, \mathscr{A})$ is called a neutrosophic soft set over $\mathscr{U}$ and its mapping is given as

$$\mathscr{F}: \mathscr{A} \longrightarrow \mathscr{P}(\mathscr{U}). \tag{3}$$

Definition 4 (see [16]).

Let $\mathscr{U}$ be the universal set and let $\mathscr{E}$ be the set of attributes concerning $\mathscr{U}$; then $\mathscr{F}_{\mathscr{E}}$ is said to be a multipolar neutrosophic set if $\mathscr{F}_{\mathscr{E}} = \{u, (s_i \bullet u_e(u), s_i \bullet v_e(u), s_i \bullet w_e(u)): u \in U, e \in E, i = 1, 2, 3, \ldots, m\}$, where $s_i \bullet u_{\mathscr{E}}, s_i \bullet v_{\mathscr{E}}, s_i \bullet w_{\mathscr{E}}: \mathscr{U} \longrightarrow [0, 1]$, and $0 \le s_i \bullet u_{\mathscr{E}}(u) + s_i \bullet v_{\mathscr{E}}(u) + s_i \bullet w_{\mathscr{E}}(u) \le 3$; $i = 1, 2, 3, \ldots, m$. u_e, v_e, and w_e represent the truth, indeterminacy, and falsity of the considered alternative.

3. Generalized Multipolar Neutrosophic Soft Set (GmPNSS) with Operators and Properties

In this section, we develop the concept of GmPNSS and introduce aggregate operators on GmPNSS with their properties.

Definition 5. Let $\boldsymbol{\mathscr{U}}$ and E be universal and set of attributes, respectively, and $\boldsymbol{\mathscr{A}} \subseteq E$, if there exists a mapping Φ such that

$$\Phi: \mathscr{A} \longrightarrow \text{GmPNSS}^{\mathscr{U}}, \tag{4}$$

then $(\Phi, \mathscr{A})$ is called GmPNSS over $\mathscr{U}$ defined as follows:

$$Y_k = (\Phi, \mathscr{A}) = \left\{\left(e, \left(u, \Phi_{\mathscr{A}(e)}(u)\right)\right): e \in E, u \in \mathscr{U}\right\}, \tag{5}$$

where $\Phi_{\mathscr{A}}(e) = \{u, (s_i \bullet u_{\mathscr{A}(e)}(u), s_i \bullet v_{\mathscr{A}(e)}(u), s_i \bullet w_{\mathscr{A}(e)}(u)): u \in \mathscr{U}, e \in E; i \in 1, 2, 3, \ldots, m\}$, and $0 \le s_i \bullet u_{\mathscr{A}(e)}(u) + s_i \bullet v_{\mathscr{A}(e)}(u) + s_i \bullet w_{\mathscr{A}(e)}(u) \le 3$ for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathscr{U}$.

Definition 6. Let $\Upsilon_{\mathscr{A}}$ and Υ_B be two GmPNSS over $\mathscr{U}$; then $\Upsilon_{\mathscr{A}}$ is called a multipolar neutrosophic soft subset of Υ_B, if

$$\begin{aligned} s_i \bullet u_{\mathscr{A}(e)}(u) &\le s_i \bullet u_{B(e)}(u), \\ s_i \bullet v_{\mathscr{A}(e)}(u) &\le s_i \bullet v_{B(e)}(u), \\ s_i \bullet w_{\mathscr{A}(e)}(u) &\ge s_i \bullet w_{B(e)}(u), \end{aligned} \tag{6}$$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathscr{U}$.

Definition 7. Let $\Upsilon_{\mathscr{A}}$ and Υ_B be two GmPNSS over $\mathscr{U}$, then $\Upsilon_{\mathscr{A}} = \Upsilon_B$, if

$$\begin{aligned} s_i \bullet u_{\mathscr{A}(e)}(u) &\le s_i \bullet u_{B(e)}(u), \\ s_i \bullet u_{B(e)}(u) &\le s_i \bullet u_{\mathscr{A}(e)}(u), \\ s_i \bullet v_{\mathscr{A}(e)}(u) &\ge s_i \bullet v_{B(e)}(u), \\ s_i \bullet v_{B(e)}(u) &\ge s_i \bullet v_{\mathscr{A}(e)}(u), \\ s_i \bullet w_{\mathscr{A}(e)}(u) &\ge s_i \bullet w_{B(e)}(u), \\ s_i \bullet w_{B(e)}(u) &\ge s_i \bullet w_{\mathscr{A}(e)}(u), \end{aligned} \tag{7}$$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathscr{U}$.

Definition 8. Let $\mathscr{F}_{\mathscr{A}}$ be a GmPNSS over $\mathscr{U}$, then empty GmPNSS can be represented as $\mathscr{F}_{\breve{0}}$ and defined as follows:

$$\mathscr{F}_{\breve{0}} = \{e, (u, (0, 1, 1), (0, 1, 1), \ldots, (0, 1, 1)) : e \in E, u \in \mathscr{U}\}. \tag{8}$$

Definition 9. Let $\mathscr{F}_{\mathscr{A}}$ be a GmPNSS over $\mathscr{U}$, then universal GmPNSS can be represented as $\mathscr{F}_{\breve{E}}$ and defined as follows:

$$\mathscr{F}_{\breve{E}} = \{e, (u, (1, 0, 0), (1, 0, 0), \ldots, (1, 0, 0)): e \in E, u \in \mathscr{U}\}. \tag{9}$$

Definition 10. Let $\mathscr{F}_{\mathscr{A}}$ be a GmPNSS over $\mathscr{U}$, then the complement of GmPNSS is defined as follows:

$$\mathscr{F}^c_{\mathscr{A}}(e) = \left\{\left(e, \left(u, s_i \bullet w_{\mathscr{A}(e)}(u), 1 - s_i \bullet v_{\mathscr{A}(e)}(u), s_i \bullet u_{\mathscr{A}(e)}(u)\right)\right): u \in \mathscr{U}\right\}, \tag{10}$$

for all $i \in 1, 2, 3, \ldots, m$; $e \in E$ and $u \in \mathscr{U}$.

Proposition 1. *If $\mathscr{F}_{\mathscr{A}}$ is a GmPNSS, then*

(1) $(\mathscr{F}^c_{\mathscr{A}})^c = \mathscr{F}_{\mathscr{A}}$

(2) $(\mathscr{F}_{\breve{0}})^c = \mathscr{F}_{\breve{E}}$

(3) $(\mathscr{F}_{\breve{E}})^c = \mathscr{F}_{\breve{0}}$

Proof. Let

$$\mathscr{F}_{\mathscr{A}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet u_{\mathscr{A}(e)}(u), s_i \bullet v_{\mathscr{A}(e)}(u), s_i \bullet w_{\mathscr{A}e}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{11}$$

Then, by using Definition 10, we get

$$\mathscr{F}^c_{\mathscr{A}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet w_{\mathscr{A}(e)}(u), 1 - s_i \bullet v_{\mathscr{A}(e)}(u), s_i \bullet u_{\mathscr{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{12}$$

Again, by using Definition 10,

$$\begin{aligned}
(\mathscr{F}^c_{\mathscr{A}}(e))^c &= \left\{\left(e, \left(u, \left(s_i \bullet u_{\mathscr{A}(e)}(u), 1 - \left(1 - s_i \bullet v_{\mathscr{A}(e)}(u)\right), s_i \bullet w_{\mathscr{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\},\\
(\mathscr{F}^c_{\mathscr{A}}(e))^c &= \left\{\left(e, \left(u, \left(s_i \bullet u_{\mathscr{A}(e)}(u), s_i \bullet v_{\mathscr{A}(e)}(u), s_i \bullet w_{\mathscr{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\},\\
(\mathscr{F}^c_{\mathscr{A}}(e))^c &= \mathscr{F}_{\mathscr{A}}(e).
\end{aligned} \tag{13}$$

□

Proof. Let $\mathscr{F}_{\breve{0}}$ be an empty GmPNSS over $\mathscr{U}$.

$$\mathscr{F}_{\breve{0}} = \{e, (u, (0, 1, 1), (0, 1, 1), \ldots, (0, 1, 1)) : e \in E, u \in \mathscr{U}\}. \tag{14}$$

Utilizing Definition 10,

$$\left(\mathscr{F}_{\breve{0}}\right)^{c}=\{e,(u,\ (1,0,0),\ (1,0,0),\ldots,\ (1,0,0)):e\in E,\ u\in\mathscr{U}\},\tag{15}$$

$$\left(\mathscr{F}_{\breve{0}}\right)^{c}=\mathscr{F}_{\breve{E}}.\tag{16}$$

Similarly, we can prove 3. □

Definition 11. Let $\mathscr{F}_{\mathscr{A}(e)}$ and $\mathscr{G}_{B(e)}$ be two GmPNSS over $\mathscr{U}$. Then,

$$\mathscr{F}_{\mathscr{A}(e)}\cup\mathscr{G}_{B(e)}=\left\{\left(e,\left(u,\begin{pmatrix}\max\left\{s_i\bullet u_{\mathscr{A}(e)}(u),\ s_i\bullet u_{B(e)}(u)\right\}\\ \min\left\{s_i\bullet v_{\mathscr{A}(e)}(u),\ s_i\bullet v_{B(e)}(u)\right\}\\ \min\left\{s_i\bullet w_{\mathscr{A}(e)}(u),\ s_i\bullet w_{B(e)}(u)\right\}\end{pmatrix}\right)\right):u\in U,\ e\in E;i\in 1,2,3,\ldots,m\right\}.\tag{17}$$

Proposition 2. *Let $\mathscr{F}_{\breve{A}}$, $\mathscr{G}_{\breve{B}}$, and $\mathscr{H}_{\breve{C}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $\mathscr{F}_{\breve{A}}\cup\mathscr{F}_{\breve{A}}=\mathscr{F}_{\breve{A}}$

(2) $\mathscr{F}_{\breve{A}}\cup\mathscr{F}_{\breve{0}}=\mathscr{F}_{\breve{A}}$

(3) $\mathscr{F}_{\breve{A}}\cup\mathscr{G}_{\breve{B}}=\mathscr{G}_{\breve{B}}\cup\mathscr{F}_{\breve{A}}$

(4) $(\mathscr{F}_{\breve{A}}\cup\mathscr{G}_{\breve{B}})\cup\mathscr{H}_{\breve{C}}=\mathscr{F}_{\breve{A}}\cup(\mathscr{G}_{\breve{B}}\cup\mathscr{H}_{\breve{C}})$

Proof. Let

$$\mathscr{F}_{\breve{A}}(e)=\left\{\left(e,\left(u,\left(s_i\bullet u_{\breve{A}(e)}(u),\ s_i\bullet v_{\breve{A}(e)}(u),\ s_i\bullet w_{\breve{A}(e)}(u)\right)\right)\right):\ u\in U,\ e\in E;i\in 1,2,3,\ldots,m\right\},\tag{18}$$

be a GmPNSS. Then,

$$\mathscr{F}_{\breve{A}}(e)\cup\mathscr{F}_{\breve{A}}(e)=\left\{\left(e,\left(u,\begin{pmatrix}\max\left\{s_i\bullet u_{\breve{A}(e)}(u),\ s_i\bullet u_{\breve{A}(e)}(u)\right\}\\ \min\left\{s_i\bullet v_{\breve{A}(e)}(u),\ s_i\bullet v_{\breve{A}(e)}(u)\right\}\\ \min\left\{s_i\bullet w_{\breve{A}(e)}(u),\ s_i\bullet w_{\breve{A}(e)}(u)\right\}\end{pmatrix}\right)\right):u\in U,\ e\in E;i\in 1,2,3,\ldots,m\right\},\tag{19}$$

$$\begin{aligned}\mathscr{F}_{\breve{A}}(e)\cup\mathscr{F}_{\breve{A}}(e)&=\left\{\left(e,\left(u,\left(s_i\bullet u_{\breve{A}(e)}(u),\ s_i\bullet v_{\breve{A}(e)}(u),\ s_i\bullet w_{\breve{A}(e)}(u)\right)\right)\right):\ u\in U,\ e\in E;i\in 1,2,3,\ldots,m\right\},\\ \mathscr{F}_{\breve{A}}\cup\mathscr{F}_{\breve{A}}&=\mathscr{F}_{\breve{A}}.\end{aligned}\tag{20}$$

By using Definition 11, we can easily prove the remaining properties. □

Definition 12. Let $\mathscr{F}_{\mathscr{A}(e)}$ and $\mathscr{G}_{B(e)}$ be GmPNSS over $\mathscr{U}$. Then,

$$\mathscr{F}_{\mathscr{A}(e)}\cap\mathscr{G}_{B(e)}=\left\{\left(e,\left(u,\begin{pmatrix}\min\left\{s_i\bullet u_{\mathscr{A}(e)}(u),\ s_i\bullet u_{B(e)}(u)\right\}\\ \max\left\{s_i\bullet v_{\mathscr{A}(e)}(u),\ s_i\bullet v_{B(e)}(u)\right\}\\ \max\left\{s_i\bullet w_{\mathscr{A}(e)}(u),\ s_i\bullet w_{B(e)}(u)\right\}\end{pmatrix}\right)\right):u\in U,\ e\in E;i\in 1,2,3,\ldots,m\right\}.\tag{21}$$

Proposition 3. *Let $\mathscr{F}_{\breve{A}}$, $\mathscr{G}_{\breve{B}}$, and $\mathscr{H}_{\breve{C}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $\mathscr{F}_{\breve{A}}\cap\mathscr{F}_{\breve{A}}=\mathscr{F}_{\breve{A}}$

(2) $\mathscr{F}_{\breve{A}}\cap\mathscr{F}_{\breve{0}}=\mathscr{F}_{\breve{A}}$

(3) $\mathscr{F}_{\breve{A}}\cap\mathscr{F}_{\breve{E}}=\mathscr{F}_{\breve{E}}$

(4) $\mathscr{F}_{\breve{A}}\cap\mathscr{G}_{\breve{B}}=\mathscr{G}_{\breve{B}}\cap\mathscr{F}_{\breve{A}}$

(5) $(\mathscr{F}_{\check{A}} \cap \mathscr{G}_{\check{B}}) \cap \mathscr{H}_{\check{C}} = \mathscr{F}_{\check{A}} \cap (\mathscr{G}_{\check{B}} \cap \mathscr{H}_{\check{C}})$

Proof. By using Definition 12, the proof is easy. □

Proposition 4. *Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $(\mathscr{F}_{\check{A}(e)} \cup \mathscr{G}_{\check{B}(e)})^C = \mathscr{F}^C_{\check{A}(e)} \cap \mathscr{G}^C_{\check{B}(e)}$
(2) $(\mathscr{F}_{\check{A}(e)} \cap G_{B(e)})^C = \mathscr{F}^C_{\check{A}(e)} \cup \mathscr{G}^C_{\check{B}(e)}$

Proof. We know that

$$\mathscr{F}_{\check{A}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{A}(e)}(u), s_i \bullet v_{\check{A}(e)}(u), s_i \bullet w_{\check{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{22}$$

$$\mathscr{G}_{\check{B}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{B}(e)}(u), s_i \bullet v_{\check{B}(e)}(u), s_i \bullet w_{\check{B}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{23}$$

are two GmPNSS.

By using Definition 11,

$$\mathscr{F}_{\check{A}}(e) \cup \mathscr{G}_{\check{B}(e)} = \left\{\left(e, \left(u, \begin{pmatrix} \max\left\{s_i \bullet u_{\check{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \\ \min\left\{s_i \bullet v_{\check{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \min\left\{s_i \bullet w_{\check{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{24}$$

Now, by using Definition 10,

$$\left(\mathscr{F}_{\check{A}}(e) \cup \mathscr{G}_{\check{B}(e)}\right)^c = \left\{\left(e, \left(u, \begin{pmatrix} \min\left\{s_i \bullet w_{\check{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ 1 - \min\left\{s_i \bullet v_{\check{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \max\left\{s_i \bullet u_{\check{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{25}$$

Now,

$$\mathscr{F}_{\check{A}}(e)^C = \left\{\left(e, \left(u, \left(s_i \bullet w_{\check{A}(e)}(u), 1 - s_i \bullet v_{\check{A}(e)}(u), s_i \bullet u_{\check{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{26}$$

$$\mathscr{G}^C_{\check{B}(e)} = \left\{\left(e, \left(u, \left(s_i \bullet w_{B(e)}(u), 1 - s_i \bullet v_{B(e)}(u), s_i \bullet u_{B(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{27}$$

By using Definition 12,

$$\mathscr{F}_{\check{A}}(e)^C \cap \mathscr{G}^C_{\check{B}(e)} = \left\{\left(e, \left(u, \begin{pmatrix} \min\left\{s_i \bullet w_{\check{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ \max\left\{1 - s_i \bullet v_{\check{A}(e)}(u), 1 - s_i \bullet v_{B(e)}(u)\right\} \\ \max\left\{s_i \bullet u_{\check{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots m\right\}, \tag{28}$$

$$\mathscr{F}_{\check{A}}(e)^C \cap \mathscr{G}^C_{\check{B}(e)} = \left\{\left(e, \left(u, \begin{pmatrix} \min\left\{s_i \bullet w_{\check{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ 1 - \min\left\{s_i \bullet v_{\check{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \max\left\{s_i \bullet u_{\check{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{29}$$

Hence,

$$\mathscr{F}_{\check{A}}(e) \cup \mathscr{G}_{\check{B}(e)}{}^C = \mathscr{F}_{\check{A}}(e)^C \cap \mathscr{G}^C_{\check{B}(e)}. \tag{30}$$

□

Proof. We know that

$$\mathcal{F}_{\breve{A}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet v_{\breve{A}(e)}(u), s_i \bullet w_{\breve{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{31}$$

$$\mathcal{G}_{\breve{B}}(e) = \left\{\left(e, \left(u, \left(s_i \bullet u_{\breve{B}(e)}(u), s_i \bullet v_{\breve{B}(e)}(u), s_i \bullet w_{\breve{B}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{32}$$

are two GmPNSS. Utilizing Definition 12,

$$\mathcal{F}_{\breve{A}(e)} \cap G_{B(e)} = \left\{\left(e, \left(u, \begin{pmatrix} \min\left\{s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \\ \max\left\{s_i \bullet v_{\breve{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \max\left\{s_i \bullet w_{\breve{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{33}$$

By Definition 10,

$$\left(\mathcal{F}_{\breve{A}(e)} \cap G_{B(e)}\right)^C = \left\{\left(e, \left(u, \begin{pmatrix} \max\left\{s_i \bullet w_{\breve{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ 1 - \max\left\{s_i \bullet v_{\breve{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \min\left\{s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{34}$$

Now,

$$\mathcal{F}_{\breve{A}}(e)^C = \left\{\left(e, \left(u, \left(s_i \bullet w_{\breve{A}(e)}(u), 1 - s_i \bullet v_{\breve{A}(e)}(u), s_i \bullet u_{\breve{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \tag{35}$$

$$\mathcal{G}_{\breve{B}(e)}^C = \left\{\left(e, \left(u, \left(s_i \bullet w_{B(e)}(u), 1 - s_i \bullet v_{B(e)}(u), s_i \bullet u_{B(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{36}$$

By using Definition 11,

$$\mathcal{F}_{\breve{A}}(e)^C \cup \mathcal{G}_{\breve{B}(e)}^C = \left\{\left(e, \left(u, \begin{pmatrix} \max\left\{s_i \bullet w_{\breve{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ \min\left\{1 - s_i \bullet v_{\breve{A}(e)}(u), 1 - s_i \bullet v_{B(e)}(u)\right\} \\ \min\left\{s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots m\right\}, \tag{37}$$

$$\mathcal{F}_{\breve{A}}(e)^C \cup \mathcal{G}_{\breve{B}(e)}^C = \left\{\left(e, \left(u, \begin{pmatrix} \max\left\{s_i \bullet w_{\breve{A}(e)}(u), s_i \bullet w_{B(e)}(u)\right\} \\ 1 - \max\left\{s_i \bullet v_{\breve{A}(e)}(u), s_i \bullet v_{B(e)}(u)\right\} \\ \min\left\{s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet u_{B(e)}(u)\right\} \end{pmatrix}\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{38}$$

Hence,

$$\left(\mathcal{F}_{\breve{A}(e)} \cap G_{B(e)}\right)^C = \mathcal{F}_{\breve{A}(e)}^C \cup \mathcal{G}_{\breve{B}(e)}^C. \tag{39}$$

□

Proposition 5. *Let $\mathscr{F}_{\check{A(e)}}$, $\mathscr{G}_{\check{B(e)}}$, and $\mathscr{H}_{\check{C(e)}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $\mathscr{F}_{\check{A(e)}} \cup (\mathscr{G}_{\check{B(e)}} \cap \mathscr{H}_{\check{C(e)}}) = (\mathscr{F}_{\check{A(e)}} \cup \mathscr{G}_{\check{B(e)}}) \cap (\mathscr{F}_{\check{A(e)}} \cup \mathscr{H}_{\check{C(e)}})$

(2) $\mathscr{F}_{\check{A(e)}} \cap (\mathscr{G}_{\check{B(e)}} \cup \mathscr{H}_{\check{C(e)}}) = (\mathscr{F}_{\check{A(e)}} \cap \mathscr{G}_{\check{B(e)}}) \cup (\mathscr{F}_{\check{A(e)}} \cap \mathscr{H}_{\check{C(e)}})$

(3) $\mathscr{F}_{\check{A(e)}} \cup (\mathscr{F}_{\check{A(e)}} \cap \mathscr{G}_{\check{B(e)}}) = \mathscr{F}_{\check{A(e)}}$

(4) $\mathscr{F}_{\check{A(e)}} \cap (\mathscr{F}_{\check{A(e)}} \cup \mathscr{G}_{\check{B(e)}}) = \mathscr{F}_{\check{A(e)}}$

Proof. We know that

$$\mathscr{G}_{\check{B(e)}} \cap \mathscr{H}_{\check{C(e)}} = \left\{ \left(e, \left(u, \begin{pmatrix} \min\{s_i \bullet u_{\check{B(e)}}(u), s_i \bullet u_{\check{C(e)}}(u)\} \\ \max\{s_i \bullet v_{\check{B(e)}}(u), s_i \bullet v_{\check{C(e)}}(u)\} \\ \max\{s_i \bullet w_{\check{B(e)}}(u), s_i \bullet w_{\check{C(e)}}(u)\} \end{pmatrix} \right) \right) : u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}, \tag{40}$$

$$\mathscr{F}_{\check{A(e)}} \cup \left(\mathscr{G}_{\check{B(e)}} \cap \mathscr{H}_{\check{C(e)}}\right) = \left\{ \left(e, \left(u, \begin{pmatrix} \max\{s_i \bullet u_{\check{A(e)}}(u), \min\{s_i \bullet u_{\check{B(e)}}(u), s_i \bullet u_{\check{C(e)}}(u)\}\} \\ \min\{s_i \bullet v_{\check{A(e)}}(u), \max\{s_i \bullet v_{\check{B(e)}}(u), s_i \bullet v_{\check{C(e)}}(u)\}\} \\ \min\{s_i \bullet w_{\check{A(e)}}(u), \max\{s_i \bullet w_{\check{B(e)}}(u), s_i \bullet w_{\check{C(e)}}(u)\}\} \end{pmatrix} \right) \right) : u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}, \tag{41}$$

$$\mathscr{F}_{\check{A(e)}} \cap \mathscr{G}_{\check{B(e)}} = \left\{ \left(e, \left(u, \begin{pmatrix} \min\{s_i \bullet u_{\check{A(e)}}(u), s_i \bullet u_{\check{B(e)}}(u)\} \\ \max\{s_i \bullet v_{\check{A(e)}}(u), s_i \bullet v_{\check{B(e)}}(u)\} \\ \max\{s_i \bullet w_{\check{A(e)}}(u), s_i \bullet w_{\check{B(e)}}(u)\} \end{pmatrix} \right) \right) : u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}, \tag{42}$$

$$\mathscr{F}_{\check{A(e)}} \cap \mathscr{H}_{\check{C(e)}} = \left\{ \left(e, \left(u, \begin{pmatrix} \min\{s_i \bullet u_{\check{A(e)}}(u), s_i \bullet u_{\check{C(e)}}(u)\} \\ \max\{s_i \bullet v_{\check{A(e)}}(u), s_i \bullet v_{\check{C(e)}}(u)\} \\ \max\{s_i \bullet w_{\check{A(e)}}(u), s_i \bullet w_{\check{C(e)}}(u)\} \end{pmatrix} \right) \right) : u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}, \tag{43}$$

$$\left(\mathscr{F}_{\check{A(e)}} \cap \mathscr{G}_{\check{B(e)}}\right) \cup \left(\mathscr{F}_{\check{A(e)}} \cap \mathscr{H}_{\check{C(e)}}\right) = \left\{ \left(e, \left(u, \begin{pmatrix} \max\{\min\{s_i \bullet u_{\widetilde{A(e)}}(u), s_i \bullet u_{\widetilde{B(e)}}(u)\}, \min\{s_i \bullet u_{\widetilde{B(e)}}(u), s_i \bullet u_{\widetilde{C(e)}}(u)\}\} \\ \min\{\max\{s_i \bullet v_{\widetilde{A(e)}}(u), s_i \bullet v_{\widetilde{B(e)}}(u)\}, \max\{s_i \bullet v_{\widetilde{B(e)}}(u), s_i \bullet v_{\widetilde{C(e)}}(u)\}\} \\ \min\{\max\{s_i \bullet w_{\widetilde{A(e)}}(u), s_i \bullet w_{\widetilde{B(e)}}(u)\}, \max\{s_i \bullet w_{\widetilde{B(e)}}(u), s_i \bullet w_{\widetilde{C(e)}}(u)\}\} \end{pmatrix} \right) \right) u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}, \tag{44}$$

$$\left(\mathscr{F}_{\check{A(e)}} \cap \mathscr{G}_{\check{B(e)}}\right) \cup \left(\mathscr{F}_{\check{A(e)}} \cap \mathscr{H}_{\check{C(e)}}\right) = \left\{ \left(e, \left(u, \begin{pmatrix} \max\{s_i \bullet u_{\check{A(e)}}(u), \min\{s_i \bullet u_{\check{B(e)}}(u), s_i \bullet u_{\check{C(e)}}(u)\}\} \\ \min\{s_i \bullet v_{\check{A(e)}}(u), \max\{s_i \bullet v_{\check{B(e)}}(u), s_i \bullet v_{\check{C(e)}}(u)\}\} \\ \min\{s_i \bullet w_{\check{A(e)}}(u), \max\{s_i \bullet w_{\check{B(e)}}(u), s_i \bullet w_{\check{C(e)}}(u)\}\} \end{pmatrix} \right) \right) : u \in U, e \in E; i \in 1, 2, 3, \ldots, m \right\}. \tag{45}$$

Hence,

$$\mathscr{F}_{\check{A}(e)} \cup \left(\mathscr{G}_{\check{B}(e)} \cap \mathscr{H}_{\check{C}(e)}\right) = \left(\mathscr{F}_{\check{A}(e)} \cup \mathscr{G}_{\check{B}(e)}\right) \cap \left(\mathscr{F}_{\check{A}(e)} \cup \mathscr{H}_{\check{C}(e)}\right). \tag{46}$$

Similarly, we can prove other results. □

Definition 13. Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be GmPNSS, then their extended union is defined as

$$u\left(\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}\right) = \begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet u_{B(e)}(u), & \text{if } e \in B - A, \\ \max\left\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \tag{47}$$

$$v\left(\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}\right) = \begin{cases} s_i \bullet v_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet v_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet v_{A(e)}(u), s_i \bullet v_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \tag{48}$$

$$w\left(\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}\right) = \begin{cases} s_i \bullet w_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet w_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet w_{A(e)}(u), s_i \bullet w_{B(e)}(u)\right\}, & \text{if } e \in A \cap B. \end{cases} \tag{49}$$

Example 1. Assume that $\mathscr{U} = \{u_1, u_2\}$ is a universe of discourse and let $E = \{x_1, x_2, x_3, x_4\}$ be a set of attributes and $A = \{x_1, x_2\}$ and $B = \{x_2, x_3\} \subseteq E$. Consider $\mathscr{F}_{\check{A}(e)}$ and $\mathscr{G}_{\check{B}(e)}$ $\in$ G3-PNSS over $\mathscr{U}$ can be represented as follows:

$$\mathscr{F}_{\check{A}} = \left\{ \begin{array}{c} \left(x_1, \{u_1, (.5, .2, .1), (.3, .1, .2), (.6, .7, .8)\right) \\ \left(u_2, (.2, .3, .1), (.2, .1, .1), (.8, .6, .6)\right) \\ \left(x_2, \{u_1, (.3, .1, .3), (0, .1, .3), (.5, .3, .5)\right) \\ \left(u_2, (.2, .2, .5), (.3, .1, .5), (.6, .5, .6)\right) \end{array} \right\}, \tag{50}$$

$$\mathscr{G}_{\check{B}} = \left\{ \begin{array}{c} \left(x_2, \{u_1, (.4, .3, .2), (.2, .3, .4), (.7, .4, .5)\right) \\ \left(u_2, (.1, .5, .1), (.3, .2, .2), (.5, .7, .4)\right) \\ \left(x_3, \{u_1, (.2, .3, .2), (.1, .2, .2), (.4, .4, .5)\right) \\ \left(u_2, (.1, .1, .4), (.3, .3, 1), (.5, .3, .1)\right) \end{array} \right\}. \tag{51}$$

Then,

$$\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}} = \left\{ \begin{array}{c} \left(x_1, \{u_1, (.5, .2, .1), (.3, .1, .2), (.6, .7, .8)\right) \\ \left(u_2, (.2, .3, .1), (.2, .1, .1), (.8, .6, .6)\right) \\ \left(x_2, \{u_1, (.4, .1, .2), (.2, .1, .3), (.7, .3, .5)\right) \\ \left(u_2, (.2, .2, .1), (.3, .1, .2), (.6, .5, .4)\right) \\ \left(x_3, \{u_1, (.2, .3, .2), (.1, .2, .2), (.4, .4, .5)\right) \\ \left(u_2, (.1, .1, .4), (.3, .3, 1), (.5, .3, .1)\right) \end{array} \right\}. \tag{52}$$

Definition 14. Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be GmPNSS; then their extended union is defined as

$$u\left(\mathscr{F}_{\check{A}} \cap_{\varepsilon} \mathscr{G}_{\check{B}}\right) = \begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet u_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \tag{53}$$

$$v\left(\mathscr{F}_{\check{A}} \cap_{\varepsilon} \mathscr{G}_{\check{B}}\right) = \begin{cases} s_i \bullet v_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet v_{B(e)}(u), & \text{if } e \in B - A, \\ \max\left\{s_i \bullet v_{A(e)}(u), s_i \bullet v_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \tag{54}$$

$$w\left(\mathcal{F}_{\check{A}} \cap_{\varepsilon} \mathcal{G}_{\check{B}}\right) = \begin{cases} s_i \bullet w_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet w_{B(e)}(u), & \text{if } e \in B - A, \\ \max\left\{s_i \bullet w_{A(e)}(u), s_i \bullet w_{B(e)}(u)\right\}, & \text{if } e \in A \cap B. \end{cases} \tag{55}$$

Definition 15. Let $\mathcal{F}_{\check{A}}$ and $\mathcal{G}_{\check{B}}$ be GmPNSS, then their difference is defined as follows:

$$\mathcal{F}_{\check{A}} \backslash \mathcal{G}_{\check{B}} = \left\{\left(e, \left(u, \min\left\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\right\}, \max\left\{s_i \bullet v_{A(e)}(u), 1 - s_i \bullet v_{B(e)}(u)\right\}, \max\left\{s_i \bullet w_{A(e)}(u), s_i \bullet w_{B(e)}(u)\right\}\right)\right): u \in U;\ i \in 1, 2, 3, \ldots, m\right\}. \tag{56}$$

Definition 16. Let $\mathcal{F}_{\check{A}}$ and $\mathcal{G}_{\check{B}}$ be GmPNSS, then their addition is defined as follows:

$$\mathcal{F}_{\check{A}} + \mathcal{G}_{\check{B}} = \left\{\left(e, \left(u, \min\left\{s_i \bullet u_{A(e)}(u) + s_i \bullet u_{B(e)}(u), 1\right\}, \min\left\{s_i \bullet v_{A(e)}(u) + s_i \bullet v_B(e)(u), 1\right\}, \min\left\{s_i \bullet w_A(u) + s_i \bullet w_B(u), 1\right\}\right): u \in U;\ i \in 1, 2, 3, \ldots, m\right)\right\}. \tag{57}$$

Definition 17. Let $\mathcal{F}_{\check{A}}$ be a GmPNSS, then its scalar multiplication is represented as $\mathcal{F}_{\check{A}}(e).\check{a}$, where $\check{a} \in [0, 1]$ and it is defined as follows:

$$\mathcal{F}_{\check{A}} \cdot \check{a} = \{e, (u, \min\{s_i \bullet u_{A(e)}(u).\check{a}, 1\}, \min\{s_i \bullet v_{A(e)}(u).\check{a}, 1\}, \min\{s_i \bullet w_{A(e)}(u).\check{a}, 1\}: u \in U)\}. \tag{58}$$

Definition 18. Let $\mathcal{F}_{\check{A}}$ be a GmPNSS, then its scalar division is represented as $\mathcal{F}_{\check{A}}/\check{a}$, where $\check{a} \in [0, 1]$ and it is defined as follows:

$$\mathcal{F}_{\check{A}}/\check{a} = \left\{e, \left(u, \min\left\{s_i \bullet \frac{u_{A(e)}(u)}{\check{a}}, 1\right\}, \min\left\{s_i \bullet \frac{v_{A(e)}(u)}{\check{a}}, 1\right\}, \min\left\{s_i \bullet \frac{w_{A(e)}(u)}{\check{a}}, 1\right\}: u \in U\right)\right\}. \tag{59}$$

Definition 19. Let $\mathcal{F}_{\check{A}}$ be a GmPNSS over $\mathcal{U}$, then Truth-Favorite operator on $\mathcal{F}_{\check{A}}$ can be represented by $\widetilde{\Delta}\mathcal{F}_{\check{A}}$ and it is defined as follows:

$$\widetilde{\Delta}\mathcal{F}_{\check{A}} = \left(\left\{e, u, \min\left\{s_i \bullet u_{A(e)}(u) + s_i \bullet v_{A(e)}(u), 1\right\}, 0, s_i \bullet w_{A(e)}(u)\right\}: u \in U; i \in 1, 2, 3, \ldots, m\right). \tag{60}$$

Proposition 6. *Let $\mathscr{F}_{\breve{A}}$ and $\mathscr{G}_{\breve{B}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $\tilde{\Delta}\tilde{\Delta}\mathscr{F}_{\breve{A}} = \tilde{\Delta}\mathscr{F}_{\breve{A}}$
(2) $\tilde{\Delta}(\mathscr{F}_{\breve{A}} \cup \mathscr{G}_{\breve{B}}) \subseteq \tilde{\Delta}\mathscr{F}_{\breve{A}} \cup \tilde{\Delta}\mathscr{G}_{\breve{B}}$
(3) $\tilde{\Delta}(\mathscr{F}_{\breve{A}} \cap \mathscr{G}_{\breve{B}}) \subseteq \tilde{\Delta}\mathscr{F}_{\breve{A}} \cap \tilde{\Delta}\mathscr{G}_{\breve{B}}$
(4) $\tilde{\Delta}(\mathscr{F}_{\breve{A}} + \mathscr{G}_{\breve{B}}) = \tilde{\Delta}\mathscr{F}_{\breve{A}} + \tilde{\Delta}\mathscr{G}_{\breve{B}}$

The proof of the above proposition is easily obtained by using Definitions 11, 12, 16, and 19.

Definition 20. Let $\mathscr{F}_{\breve{A}}$ be a GmPNSS over $\mathscr{U}$, then False-Favorite operator on $\mathscr{F}_{\breve{A}}$ can be represented by $\tilde{\nabla}\mathscr{F}_{\breve{A}}$ and it is defined as follows:

$$\tilde{\nabla}\mathscr{F}_{\breve{A}} = \left\{ e, \left(u, s_i \bullet u_{A(e)}(u), 0, \min\left\{s_i \bullet w_{A(e)}(u) + s_i \bullet v_{A(e)}(u), 1\right\}: u \in U; i \in 1, 2, 3, \ldots, m\right) \right\}. \tag{61}$$

Proposition 7. *Let $\mathscr{F}_{\breve{A}}$ and $\mathscr{G}_{\breve{B}}$ be GmPNSS over $\mathscr{U}$. Then,*

(1) $\tilde{\nabla}\tilde{\nabla}\mathscr{F}_{\breve{A}} = \tilde{\nabla}\mathscr{F}_{\breve{A}}$
(2) $\tilde{\nabla}(\mathscr{F}_{\breve{A}} \cup \mathscr{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathscr{F}_{\breve{A}} \cup \tilde{\nabla}\mathscr{G}_{\breve{B}}$
(3) $\tilde{\nabla}(\mathscr{F}_{\breve{A}} \cap \mathscr{G}_{\breve{B}}) \subseteq \tilde{\nabla}\mathscr{F}_{\breve{A}} \cap \tilde{\nabla}\mathscr{G}_{\breve{B}}$
(4) $\tilde{\nabla}(\mathscr{F}_{\breve{A}} + \mathscr{G}_{\breve{B}}) = \tilde{\nabla}\mathscr{F}_{\breve{A}} + \tilde{\nabla}\mathscr{G}_{\breve{B}}$

The proof of the above proposition is easily obtained by using Definitions 11, 12, 16, and 20.

Definition 21. Let $\mathscr{F}_{\breve{A}}$ and $\mathscr{G}_{\breve{B}}$ be GmPNSS; then their AND operator is represented by $\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}}$ and it is defined as follows: $\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}} = \daleth_{A\times B}$, where $\daleth_{A\times B}(x, y) = \mathscr{F}_{A(\breve{e})}(x) \cap \mathscr{G}_{B(\breve{e})}(y)$ for all $(x, y) \in A \times B$.

Definition 22. Let $\mathscr{F}_{\breve{A}}$ and $\mathscr{G}_{\breve{B}}$ be GmPNSS; then their OR operator is represented by $\mathscr{F}_{\breve{A}} \vee \mathscr{G}_{\breve{B}}$ and it is defined as follows: $\mathscr{F}_{\breve{A}} \vee \mathscr{G}_{\breve{B}} = \daleth_{A\times B}$, where $\daleth_{A\times B}(x, y) = \mathscr{F}_{A(\breve{e})}(x) \cup \mathscr{G}_{B(\breve{e})}(y)$ for all $(x, y) \in A \times B$.

Example 2. Reconsider Example 1.

$$\mathscr{F}_{\breve{A}(x)} = \left\{ \begin{array}{l} (x_1, \{u_1, (.5, .2, .1), (.3, .1, .2), (.6, .7, .8)) \; (u_2, (.2, .3, .1), (.2, .1, .1), (.8, .6, .6)) \\ (x_2, \{u_1, (.3, .1, .3), (0, .1, .3), (.5, .3, .5)) \; (u_2, (.2, .2, .5), (.3, .1, .5), (.6, .5, .6)) \end{array} \right\}, \tag{62}$$

$$\mathscr{G}_{\breve{B}(x)} = \left\{ \begin{array}{l} (x_2, \{u_1, (.4, .3, .2), (.2, .3, .4), (.7, .4, .5)) \; (u_2, (.1, .5, .1), (.3, .2, .2), (.5, .7, .4)) \\ (x_3, \{u_1, (.2, .3, .2), (.1, .2, .2), (.4, .4, .5)) \; (u_2, (.1, .1, .4), (.3, .3, 1), (.5, .3, .1)) \end{array} \right\}, \tag{63}$$

$$\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}} = \left\{ \begin{array}{l} (x_1, x_2), (u_1, (.4,.3,.2), (.2,.3,.4), (.6,.7,.8)), (u_2, (.1,.5,.1), (.2,.2,.2), (.5,.7,.6)), \\ (x_1, x_3), (u_1, (.2,.3,.2), (.1,.2,.1), (.4,.7,.8)), (u_2, (.1,.3,.4), (.2,.3,.1), (.5,.6,.6)), \\ (x_2, x_2), (u_1, (.3,.1,.3), (.0,.1,.3), (.5,.3,.5)), (u_2, (.2,.2,.5), (.3,.1,.5), (.6,.5,.6)), \\ (x_2, x_3), (u_1, (.2,.1,.3), (.0,.2,.3), (.4,.4,.5)), (u_2, (.1,.2,.5), (.3,.3,.5), (.5,.5,.6)) \end{array} \right\}. \tag{64}$$

Proposition 8. *Let $\mathscr{F}_{\breve{A}}$, $\mathscr{G}_{\breve{B}}$, and $\mathscr{H}_{\breve{C}}$ be GmPNSS. Then,*

(1) $\mathscr{F}_{\breve{A}} \vee \mathscr{G}_{\breve{B}} = \mathscr{G}_{\breve{B}} \vee \mathscr{F}_{\breve{A}}$
(2) $\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}} = \mathscr{G}_{\breve{B}} \wedge \mathscr{F}_{\breve{A}}$
(3) $\mathscr{F}_{\breve{A}} \vee (\mathscr{G}_{\breve{B}} \vee \mathscr{H}_{\breve{C}}) = (\mathscr{F}_{\breve{A}} \vee \mathscr{G}_{\breve{B}}) \vee \mathscr{H}_{\breve{C}}$
(4) $\mathscr{F}_{\breve{A}} \wedge (\mathscr{G}_{\breve{B}} \wedge \mathscr{H}_{\breve{C}}) = (\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}}) \wedge \mathscr{H}_{\breve{C}}$
(5) $(\mathscr{F}_{\breve{A}} \vee \mathscr{G}_{\breve{B}})^c = \mathscr{F}^c(\breve{A}) \wedge \mathscr{G}^c(\breve{B})$
(6) $(\mathscr{F}_{\breve{A}} \wedge \mathscr{G}_{\breve{B}})^c = \mathscr{F}^c(\breve{A}) \vee \mathscr{G}^c(\breve{B})$

Proof. We can prove this easily by using Definitions 10, 21, and 22. □

Definition 23. Let $\mathscr{F}_{\breve{A}}$ be a GmPNSS; then necessity operation on GmPNSS is represented by $\oplus\mathscr{F}_{\breve{A}}$ and defined as follows:

$$\oplus\mathscr{F}_{\breve{A}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\breve{A}(e)}(u), s_i \bullet v_{\breve{A}(e)}(u), 1 - s_i \bullet u_{\breve{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \tag{65}$$

Definition 24. Let $\mathscr{F}_{\check{A}}$ be a GmPNSS; then possibility operation on GmPNSS is represented by $\otimes\mathscr{F}_{\check{A}}$ and defined as follows:

$$\otimes\mathscr{F}_{\check{A}} = \left\{\left(e, \left(u, \left(1 - s_i \bullet w_{\check{A}}(u), s_i \bullet v_{\check{A}(e)}(u), s_i \bullet w_{\check{A}}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \quad (66)$$

Proposition 9. *Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be two GmPNSS. Then,*

(1) $\oplus(\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}) = \oplus\mathscr{G}_{\check{B}} \cup_{\varepsilon} \oplus\mathscr{F}_{\check{A}}$

(2) $\oplus(\mathscr{F}_{\check{A}} \cap_{\varepsilon} \mathscr{G}_{\check{B}}) = \oplus\mathscr{G}_{\check{B}} \cap_{\varepsilon} \oplus\mathscr{F}_{\check{A}}$

Proof. We know that

$$\mathscr{F}_{\check{A}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{A}(e)}(u), s_i \bullet v_{\check{A}(e)}(u), s_i \bullet w_{\check{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \quad (67)$$

$$\mathscr{G}_{\check{B}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{B}(e)}(u), s_i \bullet v_{\check{B}(e)}(u), s_i \bullet w_{\check{B}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \quad (68)$$

are two GmPNSS.

Let $\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}} = \mathscr{H}_{\check{C}}$.

$$u(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet u_{B(e)}(u), & \text{if } e \in B - A, \\ \max\left\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \quad (69)$$

$$v(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet v_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet v_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet v_{A(e)}(u), s_i \bullet v_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \quad (70)$$

$$w(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet w_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet w_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet w_{A(e)}(u), s_i \bullet w_{B(e)}(u)\right\}, & \text{if } e \in A \cap B. \end{cases} \quad (71)$$

By using Definition 23,

$$\oplus u(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet u_{B(e)}(u), & \text{if } e \in B - A, \\ \max\left\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \quad (72)$$

$$\oplus v(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet v_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet v_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet v_{A(e)}(u), s_i \bullet v_{B(e)}(u)\right\}, & \text{if } e \in A \cap B, \end{cases} \quad (73)$$

$$\oplus w(\mathscr{H}_{\check{C}}) = \begin{cases} s_i \bullet w_{A(e)}(u), & \text{if } e \in A - B, \\ s_i \bullet w_{B(e)}(u), & \text{if } e \in B - A, \\ \min\left\{s_i \bullet w_{A(e)}(u), s_i \bullet w_{B(e)}(u)\right\}, & \text{if } e \in A \cap B. \end{cases} \quad (74)$$

Assume that $\oplus\mathscr{G}_{\check{B}} \cup_{\varepsilon} \oplus\mathscr{F}_{\check{A}} = \aleph$, where $\oplus\mathscr{F}_{\check{A}}$ and $\oplus\mathscr{G}_{\check{B}}$ are given as follows by using the definition of necessity operation:

$$\oplus\mathscr{F}_{\check{A}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{A}(e)}(u), s_i \bullet v_{\check{A}(e)}(u), 1 - s_i \bullet u_{\check{A}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}, \quad (75)$$

$$\oplus\mathscr{G}_{\check{B}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{B}(e)}(u), s_i \bullet v_{\check{B}(e)}(u), 1 - s_i \bullet u_{\check{B}(e)}(u)\right)\right)\right): u \in U, e \in E; i \in 1, 2, 3, \ldots, m\right\}. \quad (76)$$

Then, by using Definition 13,

$$u(\aleph)=\begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e\in A-B,\\ s_i \bullet u_{B(e)}(u), & \text{if } e\in B-A,\\ \max\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\}, & \text{if } e\in A\cap B,\end{cases} \quad (77)$$

$$v(\aleph)=\begin{cases} s_i \bullet v_{A(e)}(u), & \text{if } e\in A-B,\\ s_i \bullet v_{B(e)}(u), & \text{if } e\in B-A,\\ \min\{s_i \bullet v_{A(e)}(u), s_i \bullet v_{B(e)}(u)\}, & \text{if } e\in A\cap B,\end{cases} \quad (78)$$

$$w(\aleph)=\begin{cases} s_i \bullet u_{A(e)}(u), & \text{if } e\in A-B,\\ s_i \bullet u_{B(e)}(u), & \text{if } e\in B-A,\\ 1-\min\{s_i \bullet u_{A(e)}(u), s_i \bullet u_{B(e)}(u)\}, & \text{if } e\in A\cap B.\end{cases} \quad (79)$$

Consequently, $\oplus(\mathscr{H}_{\check{C}})$ and $\aleph$ are the same, so

$$\oplus(\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}) = \oplus\mathscr{G}_{\check{B}} \cup_{\varepsilon} \oplus\mathscr{F}_{\check{A}}. \quad (80)$$

Similarly, we can prove 2. □

Proposition 10. *Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be two GmPNSS. Then,*

(1) $\otimes, (\mathscr{F}_{\check{A}} \cup_{\varepsilon} \mathscr{G}_{\check{B}}) = \otimes, \mathscr{G}_{\check{B}} \cup_{\varepsilon} \otimes, \mathscr{F}_{\check{A}}$

(2) $\otimes, (\mathscr{F}_{\check{A}} \cap_{\varepsilon} \mathscr{G}_{\check{B}}) = \otimes, \mathscr{G}_{\check{B}} \cap_{\varepsilon} \otimes, \mathscr{F}_{\check{A}}$

Proof. The proof is similar to that of Proposition 9. □

Proposition 11. *Let $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ be GmPNSS, then we have the following:*

(1) $\oplus(\mathscr{F}_{\check{A}} \wedge \mathscr{G}_{\check{B}}) = \oplus\mathscr{F}_{\check{A}} \wedge \oplus\mathscr{G}_{\check{B}}$

(2) $\oplus(\mathscr{F}_{\check{A}} \vee \mathscr{G}_{\check{B}}) = \oplus\mathscr{F}_{\check{A}} \vee \oplus\mathscr{G}_{\check{B}}$

(3) $\otimes(\mathscr{F}_{\check{A}} \wedge \mathscr{G}_{\check{B}}) = \otimes\mathscr{F}_{\check{A}} \wedge \otimes\mathscr{G}_{\check{B}}$

(4) $\otimes(\mathscr{F}_{\check{A}} \vee \mathscr{G}_{\check{B}}) = \otimes\mathscr{F}_{\check{A}} \vee \otimes\mathscr{G}_{\check{B}}$

Proof. We know that $\mathscr{F}_{\check{A}}$ and $\mathscr{G}_{\check{B}}$ are GmPNSS:

$$\mathscr{F}_{\check{A}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{A}(e)}(u), s_i \bullet v_{\check{A}(e)}(u), s_i \bullet w_{\check{A}(e)}(u)\right)\right)\right): u\in U, e\in E; i\in 1,2,3,\ldots,m\right\}, \quad (81)$$

$$\mathscr{G}_{\check{B}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{B}(e)}(u), s_i \bullet v_{\check{B}(e)}(u), s_i \bullet w_{\check{B}(e)}(u)\right)\right)\right): u\in U, e\in E; i\in 1,2,3,\ldots,m\right\}. \quad (82)$$

Let $(\mathscr{F}_{\check{A}} \wedge \mathscr{G}_{\check{B}}) = \mathscr{H}_{\check{C}}$, where $\check{C} = \check{A}\times\check{B}$.

$$\mathscr{H}_{\check{C}} = \left\{(e_i, e_j), \left[u, \min\left\{\begin{matrix} s_i \bullet u_{\check{A}(e)}(u)\\ s_i \bullet u_{\check{B}(e)}(u)\end{matrix}\right\}, \max\left\{\begin{matrix} s_i \bullet v_{\check{A}(e)}(u)\\ s_i \bullet v_{\check{B}(e)}(u)\end{matrix}\right\}, \max\left\{\begin{matrix} s_i \bullet w_{\check{A}(e)}(u)\\ s_i \bullet w_{\check{B}(e)}(u)\end{matrix}\right\}\right]\right\}. \quad (83)$$

By using Definition 23,

$$\oplus\mathscr{H}_{\check{C}} = \left\{(e_i, e_j), \left[u, \min\left\{\begin{matrix} s_i \bullet u_{\check{A}(e)}(u)\\ s_i \bullet u_{\check{B}(e)}(u)\end{matrix}\right\}, \max\left\{\begin{matrix} s_i \bullet v_{\check{A}(e)}(u)\\ s_i \bullet v_{\check{B}(e)}(u)\end{matrix}\right\}, 1-\min\left\{\begin{matrix} s_i \bullet u_{\check{A}(e)}(u)\\ s_i \bullet u_{\check{B}(e)}(u)\end{matrix}\right\}\right]\right\}. \quad (84)$$

We have

$$\oplus\mathscr{F}_{\check{A}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{A}(e)}(u), s_i \bullet v_{\check{A}(e)}(u), 1-s_i \bullet u_{\check{A}(e)}(u)\right)\right)\right): u\in U, e\in E; i\in 1,2,3,\ldots,m\right\}, \quad (85)$$

$$\oplus\mathscr{G}_{\check{B}} = \left\{\left(e, \left(u, \left(s_i \bullet u_{\check{B}(e)}(u), s_i \bullet v_{\check{B}(e)}(u), 1-s_i \bullet u_{\check{B}(e)}(u)\right)\right)\right): u\in U, e\in E; i\in 1,2,3,\ldots,m\right\}. \quad (86)$$

By using Definition 21, we get

$$\oplus\mathcal{F}_{\breve{A}}\wedge\oplus\mathcal{G}_{\breve{B}} = \{(e_i, e_j), [u, \min\{s_i\bullet u_{\breve{A}(e)}(u), s_i\bullet u_{\breve{B}(e)}(u)\}, \max\{s_i\bullet v_{\breve{A}(e)}(u), s_i\bullet v_{\breve{B}(e)}(u)\}, \max\{(1 - s_i\bullet u_{\breve{A}(e)}(u)), (1 - s_i\bullet u_{\breve{B}(e)}(u))\}]\}, \tag{87}$$

$$\oplus\mathcal{F}_{\breve{A}}\wedge\oplus\mathcal{G}_{\breve{B}} = \{(e_i, e_j), [u, \min\{s_i\bullet u_{\breve{A}(e)}(u), s_i\bullet u_{\breve{B}(e)}(u)\}, \max\{s_i\bullet v_{\breve{A}(e)}(u), s_i\bullet v_{\breve{B}(e)}(u)\}, 1 - \min\{s_i\bullet u_{\breve{A}(e)}(u), s_i\bullet u_{\breve{B}(e)}(u)\}]\}. \tag{88}$$

So $\oplus(\mathcal{F}_{\breve{A}}\wedge\mathcal{G}_{\breve{B}}) = \oplus\mathcal{F}_{\breve{A}}\wedge\oplus\mathcal{G}_{\breve{B}}$.

Similar to Assertion 1, we can prove 2, 3, and 4. □

4. Distance and Similarity Measure of Generalized Multipolar Neutrosophic Soft Set

In this section, we introduce the Hamming distance and Euclidean distance between two GmPNSS and develop the similarity measure by using these distances.

Definition 25. $\mathcal{U}$ and E are a universal set and a set of attributes, respectively; assume that GmPNSS($\mathcal{U}$) represents the collection of all GmPNSS. Consider $(\Phi_{\mathcal{F}}, E)$ as well as $(\varphi_{\mathcal{G}}, E) \in$ GmPNSS and there exists a mapping $\Phi_{\mathcal{F}}, \varphi_{\mathcal{G}}: E \longrightarrow$ GmPNSS($\mathcal{U}$); after that, we tend to establish the distances between $(\Phi_{\mathcal{F}}, E)$ and $(\varphi_{\mathcal{G}}, E)$ as follows.

4.1. Hamming Distance

$$d^{H}_{\text{GmPNSS}}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left|s_i\bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left|s_i\bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)\right\}. \tag{89}$$

4.2. Normalized Hamming Distance

$$d^{\text{NH}}_{\text{GmPNSS}}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left|s_i\bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right) + \left(\left|s_i\bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)\right\}. \tag{90}$$

4.3. Euclidean Distance

$$d^{E}_{\text{GmPNSS}}(\Phi_{\mathcal{F}}(e), \varphi_{\mathcal{G}}(e)) = \left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet u_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \left(\left|s_i\bullet v_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet v_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2 + \left(\left|s_i\bullet w_{\Phi_{\mathcal{F}}}(u_j) - s_i\bullet w_{\varphi_{\mathcal{G}}}(u_j)\right|\right)^2\right\}\right)^{1/2}. \tag{91}$$

4.4. Normalized Euclidean Distance

$$d^{\mathrm{NE}}_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}}(e),\varphi_{\mathscr{G}}(e)\right)=\left(\frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet u_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^2+\left(\left|s_i\bullet v_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet v_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^2+\left(\left|s_i\bullet w_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet w_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^2\right\}\right)^{1/2}. \tag{92}$$

4.5. Weighted Distance

$$d^{w}_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}}(e),\varphi_{\mathscr{G}}(e)\right)=\left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}w_i\left\{\left(\left|s_i\bullet u_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet u_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet v_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet v_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet w_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet w_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r\right\}\right\}\right)^{1/r}. \tag{93}$$

where $r > 0$ and $w=(w_1, w_2, w_3, \ldots, w_n)^T$ is a weight vector of e_i $(i=1, 2, 3, \ldots, n)$. If $r=1$ and $r=2$, then equation (5) becomes the weighted hamming and weighted Euclidean distances, respectively.

Definition 26. $\mathscr{U}$ and E are a universal set and a set of attributes, respectively, and $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ are two GmPNSS($\mathscr{U}$). Then similarity measure based on Definition 25 between $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ is defined as follows:

$$S_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}},\varphi_{\mathscr{G}}\right)=\frac{1}{1+d\left(\Phi_{\mathscr{F}},\varphi_{\mathscr{G}}\right)}. \tag{94}$$

Another similarity measure between $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ is defined as

$$S_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}},\varphi_{\mathscr{G}}\right)=e^{-\beta\, d\left(\Phi_{\mathscr{F}},\varphi_{\mathscr{G}}\right)}, \tag{95}$$

where β is a steepness measure and a positive real number.

Definition 27. $\mathscr{U}$ and E are a universal set and a set of attributes, respectively, and $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ are two GmPNSS($\mathscr{U}$). Then, the distances between $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ are defined as follows:

$$d_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}}(e),\varphi_{\mathscr{G}}(e)\right)=\left(\frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet u_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet v_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet v_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet w_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet w_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r\right\}\right)^{1/r}, \tag{96}$$

$$d_{\mathrm{GmPNSS}}\left(\Phi_{\mathscr{F}}(e),\varphi_{\mathscr{G}}(e)\right)=\left(\frac{1}{2mp}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i\bullet u_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet u_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet v_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet v_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r+\left(\left|s_i\bullet w_{\Phi_{\mathscr{F}}}(u_j)-s_i\bullet w_{\varphi_{\mathscr{G}}}(u_j)\right|\right)^r\right\}\right)^{1/r}. \tag{97}$$

where $r>0$, and equations (8) and (9) are reduced to equations (1) and (2), respectively, if $r=1$. Similarly, if $r=2$, then equations (8) and (9) are reduced to equations (3) and (4), respectively.

Definition 28. Similarity measure between two GmPNSS $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ based on the weighted distance of $(\Phi_{\mathscr{F}}, E)$ and $(\varphi_{\mathscr{G}}, E)$ is defined as follows:

$$S_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}) = \frac{1}{1 + d^{w}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}})}. \tag{98}$$

Definition 29. Let $\Phi_{\mathscr{F}}$ and $\varphi_{\mathscr{G}}$ be GmPNSS over the universal set; then $\Phi_{\mathscr{F}}$ and $\varphi_{\mathscr{G}}$ are said to be α-similar if and only if $S_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}) \geq \alpha$ for $\alpha \in (0, 1)$. If $S_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}) > 1/2$, then we can say that $\Phi_{\mathscr{F}}$ and $\varphi_{\mathscr{G}}$ are significantly similar.

5. Correlation Coefficient of Generalized Multipolar Neutrosophic Soft Set

In this section, we propose the concept of correlation coefficient and weighted correlation coefficient of GmPNSS with some properties.

Definition 30. Let

$$\mathscr{F}_{\breve{A}} = \{u_k, (s_i \bullet u_{\breve{A}}(u_k), s_i \bullet v_{\breve{A}}(u_k), s_i \bullet w_{\breve{A}}(u_k)) \colon u_k \in U; i \in 1, 2, 3, \ldots, m\}, \tag{99}$$

$$G_{\breve{B}} = \{u_k, (s_i \bullet u_{\breve{B}}(u_k), s_i \bullet v_{\breve{B}}(u_k), s_i \bullet w_{\breve{B}}(u_k)) \colon u_k \in U; i \in 1, 2, 3, \ldots, m\}, \tag{100}$$

be two GmPNSS over a set of parameters $E = \{x_1, x_2, x_3, \ldots, x_n\}$.

Then, informational neutrosophic energies of two GmPNSS can be expressed as follows:

$$\varepsilon_{\text{GmPNSS}}(F_{\breve{A}}) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k) \right)^2 \right), \tag{101}$$

$$\varepsilon_{\text{GmPNSS}}(G_{\breve{B}}) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k) \right)^2 \right). \tag{102}$$

Definition 31. The correlation of two GmPNSS can be presented as follows:

$$\zeta_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left\{ \left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k) s_i \bullet w_{\breve{B}_j}(u_k) \right) \colon i \in 1, 2, 3, \ldots, m \right\}. \tag{103}$$

Definition 32. Let $F_{\breve{A}}$ and $G_{\breve{B}}$ be two GmPNSS; then the CC between them can be defined as follows:

$$R_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) = \frac{\zeta_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}})}{\sqrt{\varepsilon_{\text{GmPNSS}}(F_{\breve{A}}, F_{\breve{A}}) \cdot \varepsilon_{\text{GmPNSS}}(G_{\breve{B}}, G_{\breve{B}})}}, \tag{104}$$

$$R_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) = \frac{\sum_{j=1}^{z} \sum_{k=1}^{t} \left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k) s_i \bullet w_{\breve{B}_j}(u_k) \right)}{\sqrt{\sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{A}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{A}_j}(u_k) \right)^2 \right)} \sqrt{\sum_{j=1}^{z} \sum_{k=1}^{t} \left(\left(s_i \bullet u_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet v_{\breve{B}_j}(u_k) \right)^2 + \left(s_i \bullet w_{\breve{B}_j}(u_k) \right)^2 \right)}}. \tag{105}$$

Proposition 12. *Let $F_{\breve{A}}$ and $G_{\breve{B}}$ be two GmPNSS; then the CC $R_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}})$ between them satisfies the following properties:*

(1) $0 \le R_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) \le 1$

(2) $R_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) = R_{GmPNSS}(G_{\breve{B}}, F_{\breve{A}})$

(3) *If $F_{\breve{A}} = G_{\breve{B}}$, that is, $s_i \bullet u_{\breve{A}_j}(u_k) = s_i \bullet u_{\breve{B}_j}(u_k), s_i \bullet v_{\breve{A}_j}(u_k) = s_i \bullet v_{\breve{B}_j}(u_k)$, and $s_i \bullet w_{\breve{A}_j}(u_k) = s_i \bullet w_{\breve{B}_j}(u_k)$ for all j, k, where $i \in 1, 2, 3, \ldots, m$, then $R_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) = 1$*

Proof. $R_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) \ge 0$ is trivial, so we just need to prove that $R_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) \le 1$.

We know that

$$
\begin{aligned}
\zeta_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) &= \sum_{j=1}^{z}\sum_{k=1}^{t}\left(s_i \cdot u_{\breve{A}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k) + s_i \cdot v_{\breve{A}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k) + s_i \cdot w_{\breve{A}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)\right) \\
&= \sum_{j=1}^{z}\left(s_i \cdot u_{\breve{A}_j}(u_1) s_i \cdot u_{\breve{B}_j}(u_1) + s_i \cdot v_{\breve{A}_j}(u_1) s_i \cdot v_{\breve{B}_j}(u_1) + s_i \cdot w_{\breve{A}_j}(u_1) s_i \cdot w_{\breve{B}_j}(u_1)\right) \\
&\quad + \sum_{j=1}^{z}\left(s_i \cdot u_{\breve{A}_j}(u_2) s_i \cdot u_{\breve{B}_j}(u_2) + s_i \cdot v_{\breve{A}_j}(u_2) s_i \cdot v_{\breve{B}_j}(u_2) + s_i \cdot w_{\breve{A}_j}(u_2) s_i \cdot w_{\breve{B}_j}(u_2)\right) + \\
&\quad \vdots \\
&\quad + \sum_{j=1}^{z}\left(s_i \cdot u_{\breve{A}_j}(u_t) s_i \cdot u_{\breve{B}_j}(u_t) + s_i \cdot v_{\breve{A}_j}(u_t) s_i \cdot v_{\breve{B}_j}(u_t) + s_i \cdot w_{\breve{A}_j}(u_t) s_i \cdot w_{\breve{B}_j}(u_t)\right) \\
&= \\
&\left\{\begin{array}{c}
\left(s_i \cdot u_{\breve{A}_1}(u_1) s_i \cdot u_{\breve{B}_1}(u_1) + s_i \cdot v_{\breve{A}_1}(u_1) s_i \cdot v_{\breve{B}_1}(u_1) + s_i \cdot w_{\breve{A}_1}(u_1) s_i \cdot w_{\breve{B}_1}(u_1)\right) + \\
\left(s_i \cdot u_{\breve{A}_2}(u_1) s_i \cdot u_{\breve{B}_2}(u_1) + s_i \cdot v_{\breve{A}_2}(u_1) s_i \cdot v_{\breve{B}_2}(u_1) + s_i \cdot w_{\breve{A}_2}(u_1) s_i \cdot w_{\breve{B}_2}(u_1)\right) + \cdots + \\
\left(s_i \cdot u_{\breve{A}_z}(u_1) s_i \cdot u_{\breve{B}_z}(u_1) + s_i \cdot v_{\breve{A}_z}(u_1) s_i \cdot v_{\breve{B}_z}(u_1) + s_i \cdot w_{\breve{A}_z}(u_1) s_i \cdot w_{\breve{B}_z}(u_1)\right) +
\end{array}\right\} \\
&+ \\
&\left\{\begin{array}{c}
\left(s_i \cdot u_{\breve{A}_1}(u_2) s_i \cdot u_{\breve{B}_1}(u_2) + s_i \cdot v_{\breve{A}_1}(u_2) s_i \cdot v_{\breve{B}_1}(u_2) + s_i \cdot w_{\breve{A}_1}(u_2) s_i \cdot w_{\breve{B}_1}(u_2)\right) + \\
\left(s_i \cdot u_{\breve{A}_2}(u_2) s_i \cdot u_{\breve{B}_2}(u_2) + s_i \cdot v_{\breve{A}_2}(u_2) s_i \cdot v_{\breve{B}_2}(u_2) + s_i \cdot w_{\breve{A}_2}(u_2) s_i \cdot w_{\breve{B}_2}(u_2)\right) + \cdots + \\
\left(s_i \cdot u_{\breve{A}_z}(u_2) s_i \cdot u_{\breve{B}_z}(u_2) + s_i \cdot v_{\breve{A}_z}(u_2) s_i \cdot v_{\breve{B}_z}(u_2) + s_i \cdot w_{\breve{A}_z}(u_2) s_i \cdot w_{\breve{B}_z}(u_2)\right) +
\end{array}\right\} \\
&+ \cdots + \\
&\left\{\begin{array}{c}
\left(s_i \cdot u_{\breve{A}_1}(u_k) s_i \cdot u_{\breve{B}_1}(u_k) + s_i \cdot v_{\breve{A}_1}(u_k) s_i \cdot v_{\breve{B}_1}(u_k) + s_i \cdot w_{\breve{A}_1}(u_k) s_i \cdot w_{\breve{B}_1}(u_k)\right) + \\
\left(s_i \cdot u_{\breve{A}_2}(u_k) s_i \cdot u_{\breve{B}_2}(u_k) + s_i \cdot v_{\breve{A}_2}(u_k) s_i \cdot v_{\breve{B}_2}(u_k) + s_i \cdot w_{\breve{A}_2}(u_k) s_i \cdot w_{\breve{B}_2}(u_k)\right) + \cdots + \\
\left(s_i \cdot u_{\breve{A}_z}(u_k) s_i \cdot u_{\breve{B}_z}(u_k) + s_i \cdot v_{\breve{A}_z}(u_k) s_i \cdot v_{\breve{B}_z}(u_k) + s_i \cdot w_{\breve{A}_z}(u_k) s_i \cdot w_{\breve{B}_z}(u_k)\right) +
\end{array}\right\} \\
&= \sum_{j=1}^{z}\left(s_i \cdot u_{\breve{A}_j}(u_1) s_i \cdot u_{\breve{B}_j}(u_1) + s_i \cdot u_{\breve{A}_j}(u_2) s_i \cdot u_{\breve{B}_j}(u_2) + \cdots + s_i \cdot u_{\breve{A}_j}(u_t) s_i \cdot u_{\breve{B}_j}(u_t)\right) \\
&\quad + \sum_{j=1}^{z}\left(s_i \cdot v_{\breve{A}_j}(u_1) s_i \cdot v_{\breve{B}_j}(u_1) + s_i \cdot v_{\breve{A}_j}(u_2) s_i \cdot v_{\breve{B}_j}(u_2) + \cdots + s_i \cdot v_{\breve{A}_j}(u_t) s_i \cdot v_{\breve{B}_j}(u_t)\right) \\
&\quad + \sum_{j=1}^{z}\left(s_i \cdot w_{\breve{A}_j}(u_1) s_i \cdot w_{\breve{B}_j}(u_1) + s_i \cdot w_{\breve{A}_j}(u_2) s_i \cdot w_{\breve{B}_j}(u_2) + \cdots + s_i \cdot w_{\breve{A}_j}(u_t) s_i \cdot w_{\breve{B}_j}(u_t)\right).
\end{aligned} \tag{106}
$$

By using Cauchy-Schwarz inequality, we get

$$\begin{aligned}
\left(\zeta_{\mathrm{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)\right)^2 \leq\ & \left\{\begin{array}{c}
\sum_{j=1}^{z}\left(\left(s_i \cdot u_{\breve{A}_j}(u_1)\right)^2+\left(s_i \cdot u_{\breve{A}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot u_{\breve{A}_j}(u_t)\right)^2\right)+ \\
\sum_{j=1}^{z}\left(\left(s_i \cdot v_{\breve{A}_j}(u_1)\right)^2+\left(s_i \cdot v_{\breve{A}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot v_{\breve{A}_j}(u_t)\right)^2\right)+\cdots+ \\
\sum_{j=1}^{z}\left(\left(s_i \cdot w_{\breve{A}_j}(u_1)\right)^2+\left(s_i \cdot w_{\breve{A}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot w_{\breve{A}_j}(u_t)\right)^2\right)
\end{array}\right\} \\
& \times\left\{\begin{array}{c}
\sum_{j=1}^{z}\left(\left(s_i \cdot u_{\breve{B}_j}(u_1)\right)^2+\left(s_i \cdot u_{\breve{B}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot u_{\breve{B}_j}(u_t)\right)^2\right)+ \\
\sum_{j=1}^{z}\left(\left(s_i \cdot v_{\breve{B}_j}(u_1)\right)^2+\left(s_i \cdot v_{\breve{B}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot v_{\breve{B}_j}(u_t)\right)^2\right)+\cdots+ \\
\sum_{j=1}^{z}\left(\left(s_i \cdot w_{\breve{B}_j}(u_1)\right)^2+\left(s_i \cdot w_{\breve{B}_j}(u_2)\right)^2+\cdots+\left(s_i \cdot w_{\breve{B}_j}(u_t)\right)^2\right)
\end{array}\right\} \\
=\ & \left\{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{A}_j}(u_t)\right)^2+\left(s_i \cdot v_{\breve{A}_j}(u_t)\right)^2+\left(s_i \cdot w_{\breve{A}_j}(u_t)\right)^2\right)\right\} \\
& \times\left\{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_t)\right)^2+\left(s_i \cdot v_{\breve{B}_j}(u_t)\right)^2+\left(s_i \cdot w_{\breve{B}_j}(u_t)\right)^2\right)\right\} \\
=\ & \varepsilon_{\mathrm{GmPNSS}}\left(F_{\breve{A}}\right) \bullet \varepsilon_{\mathrm{GmPNSS}}\left(G_{\breve{B}}\right).
\end{aligned} \tag{107}$$

Therefore, $\left(\zeta_{\mathrm{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)\right)^2 \leq \varepsilon_{\mathrm{GmPNSS}}\left(F_{\breve{A}}\right) \cdot \varepsilon_{\mathrm{GmPNSS}}\left(G_{\breve{B}}\right)$. Hence, by using Definition 32, we get $R_{\mathrm{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) \leq 1$, so $0 \leq R_{\mathrm{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) \leq 1$. □

Proof. The proof is obvious. □

Proof. We know that

$$R_{\mathrm{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)=\frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(s_i \cdot u_{\breve{B}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k)+s_i \cdot v_{\breve{B}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k)+s_i \cdot w_{\breve{B}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)\right)}{\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2+\left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2+\left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)}\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2+\left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2+\left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)}}. \tag{108}$$

As we know that $s_i \cdot u_{\breve{A}_j}(u_k)=s_i \cdot u_{\breve{B}_j}(u_k)$, $s_i \cdot v_{\breve{A}_j}(u_k)=s_i \cdot v_{\breve{B}_j}(u_k)$, and $s_i \cdot w_{\breve{A}_j}(u_k)=s_i \cdot w_{\breve{B}_j}(u_k)$, for all j, k, by using Definition 32, we have

$$R_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)}{\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)}\sqrt{\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)}}. \tag{109}$$

Hence, $R_{\text{GmPNSS}}(F_{\breve{A}}, G_{\breve{B}}) = 1$. □

Definition 33. Let $F_{\breve{A}}$ and $G_{\breve{B}}$ be two GmPNSS; then the CC between them also can be defined as follows:

$$R^1_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\zeta_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)}{\max\left\{\varepsilon_{\text{GmPNSS}}\left(F_{\breve{A}}, F_{\breve{A}}\right), \varepsilon_{\text{GmPNSS}}\left(G_{\breve{B}}, G_{\breve{B}}\right)\right\}}, \tag{110}$$

$$R^1_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\sum_{j=1}^{z}\sum_{k=1}^{t} s_i \cdot u_{\breve{A}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k) + s_i \cdot v_{\breve{A}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k) + s_i \cdot w_{\breve{A}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)}{\max\left\{\begin{array}{l}\sum_{j=1}^{z}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{A}_j}(u_k)\right)^2\right), \\ \sum_{j=1}^{s}\sum_{k=1}^{t}\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)\end{array}\right\}}. \tag{111}$$

Proposition 13. *Let $F_{\breve{A}}$ and $G_{\breve{B}}$ are two GmPNSS; then the CC $R^1_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}})$ between them satisfies the following properties:*

(1) $0 \leq R^1_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) \leq 1$

(2) $R^1_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) = R^1_{GmPNSS}(G_{\breve{B}}, F_{\breve{A}})$

(3) If $F_{\breve{A}} = G_{\breve{B}}$, that is, $s_i \cdot u_{\breve{A}_j}(u_k) = s_i \cdot u_{\breve{B}_j}(u_k)$, $s_i \cdot v_{\breve{A}_j}(u_k) = s_i \cdot v_{\breve{B}_j}(u_k)$, and $s_i \cdot w_{\breve{A}_j}(u_k) = s_i \cdot w_{\breve{B}_j}(u_k)$, for all i, j, where $i \in 1, 2, 3, \ldots, m$, then $R^1_{GmPNSS}(F_{\breve{A}}, G_{\breve{B}}) = 1$

Proof. The proof is easy according to Definition 33.

Nowadays, considering that the weight of GmPNSS is very necessary for practical applications, the result of decisions may vary, whenever decision-makers adjust the different weight to every element in the universe of discourse. Consequently, it is particularly significant to plan the weight before decision-making. Let $\dot{\omega} = \{\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3, \ldots, \dot{\omega}_t\}$ be a weight vector for experts such as $\dot{\omega}_k > 0$ and $\sum_{k=1}^{t}\dot{\omega}_k = 1$, and let $\delta = \{\delta_1, \delta_2, \delta_3, \ldots, \delta_n\}$ be a weight vector for parameters such as $\delta_j > 0$ and $\sum_{j=1}^{z}\delta_j = 1$. In the following, we develop the WCC between GmPNSS by extending Definitions 32 and 33. □

Definition 34. For two GmPNSS $F_{\breve{A}}$ and $G_{\breve{B}}$, the WCC between them can be defined as follows:

$$R_{\text{GWmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\zeta_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)}{\sqrt{\varepsilon_{\text{GmPNSS}}\left(F_{\breve{A}}, F_{\breve{A}}\right)\varepsilon_{\text{GmPNSS}}\left(G_{\breve{B}}, G_{\breve{B}}\right)}}, \tag{112}$$

$$R_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(s_i \cdot u_{\breve{A}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k) + s_i \cdot v_{\breve{A}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k) + s_i \cdot w_{\breve{A}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)\right)\right)}{\left(\begin{array}{l}\sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i \cdot u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{A}_j}(u_k)\right)^2\right)\right)} \\ \sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)\right)}\end{array}\right)}. \tag{113}$$

Definition 35. Let $F_{\breve{A}}$ and $G_{\breve{B}}$ be two GmPNSS, then the WCC between them can be defined as follows:

$$R^1_{\text{GWmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\zeta_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right)}{\max\left\{\varepsilon_{\text{GmPNSS}}\left(F_{\breve{A}}, F_{\breve{A}}\right), \varepsilon_{\text{GmPNSS}}\left(G_{\breve{B}}, G_{\breve{B}}\right)\right\}}, \tag{114}$$

$$R^1_{\text{GmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) = \frac{\sum_{j=1}^{z} \delta_j\left(\sum_{k=1}^{t} \dot{\omega}_k\left(s_i \cdot u_{\breve{A}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k) + s_i \cdot v_{\breve{A}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k) + s_i \cdot w_{\breve{A}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)\right)\right)}{\max\left\{\begin{array}{l}\sum_{j=1}^{z} \delta_j\left(\sum_{k=1}^{t} \dot{\omega}_k\left(\left(s_i \cdot u_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{A}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{A}_j}(u_k)\right)^2\right),\right) \\ \sum_{j=1}^{z} \delta_j\left(\sum_{k=1}^{t} \dot{\omega}_k\left(\left(s_i \cdot u_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot v_{\breve{B}_j}(u_k)\right)^2 + \left(s_i \cdot w_{\breve{B}_j}(u_k)\right)^2\right)\right)\end{array}\right\}}. \tag{115}$$

If we consider $\dot{\omega} = \{1/t, t1/tn, q \ldots h_{,1/t\}}$ and $\delta = \{1/z, t1/zn, q \ldots h_{,1/z\}}$, then $R_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}})$ and $R^1_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}})$ are reduced to $R_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}})$ and $R^1_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}})$, respectively, defined in Definitions 32 and 33.

Proposition 14. *Let $F_{\breve{A}}$ and $G_{\breve{B}}$ be two GmPNSS; then the CC $R_{GWmPNSS}\,(F_{\breve{A}}, G_{\breve{B}})$ between them satisfies the following properties:*

(1) $0 \leq R_{GWmPNSS}\,(F_{\breve{A}}, G_{\breve{B}}) \leq 1$

(2) $R_{GWmPNSS}\,(F_{\breve{A}}, G_{\breve{B}}) = R_{GWmPNSS}\,(G_{\breve{B}}, F_{\breve{A}})$

(3) If $F_{\breve{A}} = G_{\breve{B}}$, *that is,* $s_i \cdot u_{\breve{A}_j}(u_k) = s_i \cdot u_{\breve{B}_j}(u_k)$, $s_i \cdot v_{\breve{A}_j}(u_k) = s_i \cdot v_{\breve{B}_j}(u_k)$, *and* $s_i \cdot w_{\breve{A}_j}(u_k) = s_i \cdot w_{\breve{B}_j}(u_k)$, *for all j, k, where* $i \in 1, 2, 3, \ldots, m$, *then* $R_{GWmPNSS}\,(F_{\breve{A}}, G_{\breve{B}}) = 1$

Proof. $R_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}}) \geq 0$ is trivial, so we just need to prove that $R_{\text{GWmPNSS}}\,(F_{\breve{A}}, G_{\breve{B}}) \leq 1$.

We know that

$$\begin{aligned}
\zeta_{\text{GWmPNSS}}\left(F_{\breve{A}}, G_{\breve{B}}\right) &= \sum_{j=1}^{z} \delta_j\left(\sum_{k=1}^{t} \dot{\omega}_k\left(s_i \cdot u_{\breve{A}_j}(u_k) s_i \cdot u_{\breve{B}_j}(u_k) + s_i \cdot v_{\breve{A}_j}(u_k) s_i \cdot v_{\breve{B}_j}(u_k) + s_i \cdot w_{\breve{A}_j}(u_k) s_i \cdot w_{\breve{B}_j}(u_k)\right)\right) \\
&= \sum_{j=1}^{z} \delta_j\left(\dot{\omega}_1\left(s_i \cdot u_{\breve{A}_j}(u_1) s_i \cdot u_{\breve{B}_j}(u_1) + s_i \cdot v_{\breve{A}_j}(u_1) s_i \cdot v_{\breve{B}_j}(u_1) + s_i \cdot w_{\breve{A}_j}(u_1) s_i \cdot w_{\breve{B}_j}(u_1)\right)\right) \\
&\quad + \sum_{j=1}^{z} \delta_j\left(\dot{\omega}_1\left(s_i \cdot u_{\breve{A}_j}(u_2) s_i \cdot u_{\breve{B}_j}(u_2) + s_i \cdot v_{\breve{A}_j}(u_2) s_i \cdot v_{\breve{B}_j}(u_2) + s_i \cdot w_{\breve{A}_j}(u_2) s_i \cdot w_{\breve{B}_j}(u_2)\right)\right) \\
&\qquad + \\
&\qquad \vdots \\
&\qquad + \\
&\quad \sum_{j=1}^{z} \delta_j\left(\dot{\omega}_t\left(s_i \cdot u_{\breve{A}_j}(u_t) s_i \cdot u_{\breve{B}_j}(u_t) + s_i \cdot v_{\breve{A}_j}(u_t) s_i \cdot v_{\breve{B}_j}(u_t) + s_i \cdot w_{\breve{A}_j}(u_t) s_i \cdot w_{\breve{B}_j}(u_t)\right)\right) \\
&= \left\{\begin{array}{c}
\delta_1\left(\dot{\omega}_1\left(s_i \cdot u_{\breve{A}_1}(u_1) s_i \cdot u_{\breve{B}_1}(u_1) + s_i \cdot v_{\breve{A}_1}(u_1) s_i \cdot v_{\breve{B}_1}(u_1) + s_i \cdot w_{\breve{A}_1}(u_1) s_i \cdot w_{\breve{B}_1}(u_1)\right)\right) + \\
\delta_2\left(\dot{\omega}_1\left(s_i \cdot u_{\breve{A}_2}(u_1) s_i \cdot u_{\breve{B}_2}(u_1) + s_i \cdot v_{\breve{A}_2}(u_1) s_i \cdot v_{\breve{B}_2}(u_1) + s_i \cdot w_{\breve{A}_2}(u_1) s_i \cdot w_{\breve{B}_2}(u_1)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\dot{\omega}_1\left(s_i \cdot u_{\breve{A}_z}(u_1) s_i \cdot u_{\breve{B}_z}(u_1) + s_i \cdot v_{\breve{A}_z}(u_1) s_i \cdot v_{\breve{B}_z}(u_1) + s_i \cdot w_{\breve{A}_z}(u_1) s_i \cdot w_{\breve{B}_z}(u_1)\right)\right)
\end{array}\right\} \\
&\quad + \left\{\begin{array}{c}
\delta_1\left(\dot{\omega}_2\left(s_i \cdot u_{\breve{A}_1}(u_2) s_i \cdot u_{\breve{B}_1}(u_2) + s_i \cdot v_{\breve{A}_1}(u_2) s_i \cdot v_{\breve{B}_1}(u_2) + s_i \cdot w_{\breve{A}_1}(u_2) s_i \cdot w_{\breve{B}_1}(u_2)\right)\right) + \\
\delta_2\left(\dot{\omega}_2\left(s_i \cdot u_{\breve{A}_2}(u_2) s_i \cdot u_{\breve{B}_2}(u_2) + s_i \cdot v_{\breve{A}_2}(u_2) s_i \cdot v_{\breve{B}_2}(u_2) + s_i \cdot w_{\breve{A}_2}(u_2) s_i \cdot w_{\breve{B}_2}(u_2)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\dot{\omega}_2\left(s_i \cdot u_{\breve{A}_z}(u_2) s_i \cdot u_{\breve{B}_z}(u_2) + s_i \cdot v_{\breve{A}_z}(u_2) s_i \cdot v_{\breve{B}_z}(u_2) + s_i \cdot w_{\breve{A}_z}(u_2) s_i \cdot w_{\breve{B}_z}(u_2)\right)\right)
\end{array}\right\} +
\end{aligned}$$

$$
\begin{array}{l}
\vdots \\
+ \\
\left\{\begin{array}{c}
\delta_1\left(\dot{\omega}_t\left(s_i \cdot u_{\breve{A}_1}(u_t) s_i \cdot u_{\breve{B}_1}(u_t) + s_i \cdot v_{\breve{A}_1}(u_t) s_i \cdot v_{\breve{B}_1}(u_t) + s_i \cdot w_{\breve{A}_1}(u_t) s_i \cdot w_{\breve{B}_1}(u_t)\right)\right) + \\
\delta_2\left(\dot{\omega}_t\left(s_i \cdot u_{\breve{A}_2}(u_t) s_i \cdot u_{\breve{B}_2}(u_t) + s_i \cdot v_{\breve{A}_2}(u_t) s_i \cdot v_{\breve{B}_2}(u_t) + s_i \cdot w_{\breve{A}_2}(u_t) s_i \cdot w_{\breve{B}_2}(u_t)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\dot{\omega}_t\left(s_i \cdot u_{\breve{A}_z}(u_t) s_i \cdot u_{\breve{B}_z}(u_t) + s_i \cdot v_{\breve{A}_z}(u_t) s_i \cdot v_{\breve{B}_z}(u_t) + s_i \cdot w_{\breve{A}_z}(u_t) s_i \cdot w_{\breve{B}_z}(u_t)\right)\right)
\end{array}\right\} \\
= \\
\left\{\begin{array}{c}
\delta_1\left(\sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{A}_1}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{B}_1}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{A}_1}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{B}_1}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{A}_1}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{B}_1}(u_1)\right)\right) + \\
\delta_2\left(\sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{A}_2}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{B}_2}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{A}_2}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{B}_2}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{A}_2}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{B}_2}(u_1)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{A}_z}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot u_{\breve{B}_z}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{A}_z}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot v_{\breve{B}_z}(u_1)\right) + \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{A}_z}(u_1)\right) * \sqrt{\dot{\omega}_1}\left(s_i \cdot w_{\breve{B}_z}(u_1)\right)\right)
\end{array}\right\} \\
+
\end{array}
\tag{116}
$$

$$
\begin{array}{l}
\left\{\begin{array}{c}
\delta_1\left(\sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{A}_1}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{B}_1}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{A}_1}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{B}_1}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{A}_1}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{B}_1}(u_2)\right)\right) + \\
\delta_2\left(\sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{A}_2}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{B}_2}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{A}_2}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{B}_2}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{A}_2}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{B}_2}(u_2)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{A}_z}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot u_{\breve{B}_z}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{A}_z}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot v_{\breve{B}_z}(u_2)\right) + \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{A}_z}(u_2)\right) * \sqrt{\dot{\omega}_2}\left(s_i \cdot w_{\breve{B}_z}(u_2)\right)\right)
\end{array}\right\} \\
+ \\
\vdots \\
+ \\
\left\{\begin{array}{c}
\delta_1\left(\sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{A}_1}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{B}_1}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{A}_1}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{B}_1}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{A}_1}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{B}_1}(u_t)\right)\right) + \\
\delta_2\left(\sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{A}_2}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{B}_2}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{A}_2}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{B}_2}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{A}_2}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{B}_2}(u_t)\right)\right) + \\
\vdots \\
+ \\
\delta_z\left(\sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{A}_z}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot u_{\breve{B}_z}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{A}_z}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot v_{\breve{B}_z}(u_t)\right) + \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{A}_z}(u_t)\right) * \sqrt{\dot{\omega}_t}\left(s_i \cdot w_{\breve{B}_z}(u_t)\right)\right)
\end{array}\right\}
\end{array}
$$

$$
\begin{aligned}
&= \\
&\left\{\begin{array}{c}
\left(\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{A}_1}(u_1)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{B}_1}(u_1)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{A}_1}(u_1)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{B}_1}(u_1)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{A}_1}(u_1)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{B}_1}(u_1)\right)\right)+ \\
\left(\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{A}_2}(u_1)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{B}_2}(u_1)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{A}_2}(u_1)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{B}_2}(u_1)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{A}_2}(u_1)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{B}_2}(u_1)\right)\right)+ \\
\vdots \\
+ \\
\left(\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{A}_z}(u_1)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot u_{\breve{B}_z}(u_1)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{A}_z}(u_1)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot v_{\breve{B}_z}(u_1)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{A}_z}(u_1)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_1}\left(s_i\cdot w_{\breve{B}_z}(u_1)\right)\right)
\end{array}\right\} \\
&+ \\
&\left\{\begin{array}{c}
\left(\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{A}_1}(u_2)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{B}_1}(u_2)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{A}_1}(u_2)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{B}_1}(u_2)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{A}_1}(u_2)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{B}_1}(u_2)\right)\right)+ \\
\left(\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{A}_2}(u_2)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{B}_2}(u_2)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{A}_2}(u_2)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{B}_2}(u_2)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{A}_2}(u_2)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{B}_2}(u_2)\right)\right)+ \\
\vdots \\
+ \\
\left(\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{A}_z}(u_2)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot u_{\breve{B}_z}(u_2)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{A}_z}(u_2)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot v_{\breve{B}_z}(u_2)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{A}_z}(u_2)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_2}\left(s_i\cdot w_{\breve{B}_z}(u_2)\right)\right)
\end{array}\right\} \\
&+ \\
&\vdots \\
&+ \\
&\left\{\begin{array}{c}
\left(\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{A}_1}(u_t)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{B}_1}(u_t)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{A}_1}(u_t)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{B}_1}(u_t)\right)+\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{A}_1}(u_t)\right)*\sqrt{\delta_1}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{B}_1}(u_t)\right)\right)+ \\
\left(\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{A}_2}(u_t)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{B}_2}(u_t)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{A}_2}(u_t)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{B}_2}(u_t)\right)+\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{A}_2}(u_t)\right)*\sqrt{\delta_2}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{B}_2}(u_t)\right)\right)+ \\
\vdots \\
+ \\
\left(\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{A}_z}(u_t)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot u_{\breve{B}_z}(u_t)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{A}_z}(u_t)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot v_{\breve{B}_z}(u_t)\right)+\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{A}_z}(u_t)\right)*\sqrt{\delta_z}\sqrt{\dot{\omega}_t}\left(s_i\cdot w_{\breve{B}_z}(u_t)\right)\right)
\end{array}\right\}.
\end{aligned}
\tag{117}
$$

By using Cauchy–Schwarz inequality, we get

$$
\begin{aligned}
&\left(\zeta_{\mathrm{GWmPNSS}}\left(F_{\breve{A}},G_{\breve{B}}\right)\right)^2\leq \\
&\left\{\begin{array}{c}
\left\{\begin{array}{c}
\left(\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{A}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{A}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{A}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{A}_2}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{A}_2}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{A}_2}(u_1)\right)^2\right)+ \\
\left(\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{A}_z}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{A}_z}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{A}_z}(u_1)\right)^2\right)
\end{array}\right\}+ \\
\left\{\begin{array}{c}
\left(\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{A}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{A}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{A}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{A}_2}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{A}_2}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{A}_2}(u_2)\right)^2\right)+ \\
\left(\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{A}_z}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{A}_z}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{A}_z}(u_2)\right)^2\right)
\end{array}\right\}+ \\
\vdots \\
+ \\
\left\{\begin{array}{c}
\left(\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot u_{\breve{A}_1}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot v_{\breve{A}_1}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot w_{\breve{A}_1}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot u_{\breve{A}_2}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot v_{\breve{A}_2}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot w_{\breve{A}_2}(u_t)\right)^2\right)+ \\
\left(\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot u_{\breve{A}_z}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot v_{\breve{A}_z}(u_t)\right)^2+\delta_1\dot{\omega}_{\mathrm{t}}\left(s_i\cdot w_{\breve{A}_z}(u_t)\right)^2\right)
\end{array}\right\}
\end{array}\right\}
\end{aligned}
$$

$$\times\left\{\begin{array}{c}\left\{\begin{array}{c}\left(\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{B}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{B}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{B}_1}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{B}_2}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{B}_2}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{B}_2}(u_1)\right)^2\right)+\\ \left(\delta_1\dot{\omega}_1\left(s_i\cdot u_{\breve{B}_z}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot v_{\breve{B}_z}(u_1)\right)^2+\delta_1\dot{\omega}_1\left(s_i\cdot w_{\breve{B}_z}(u_1)\right)^2\right)\end{array}\right\}+\\ \left\{\begin{array}{c}\left(\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{B}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{B}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{B}_1}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{B}_2}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{B}_2}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{B}_2}(u_2)\right)^2\right)+\\ \left(\delta_1\dot{\omega}_2\left(s_i\cdot u_{\breve{B}_z}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot v_{\breve{B}_z}(u_2)\right)^2+\delta_1\dot{\omega}_2\left(s_i\cdot w_{\breve{B}_z}(u_2)\right)^2\right)\end{array}\right\}+\\ \vdots\\ +\\ \left\{\begin{array}{c}\left(\delta_1\dot{\omega}_t\left(s_i\cdot u_{\breve{B}_1}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot v_{\breve{B}_1}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot w_{\breve{B}_1}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot u_{\breve{B}_2}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot v_{\breve{B}_2}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot w_{\breve{B}_2}(u_t)\right)^2\right)+\\ \left(\delta_1\dot{\omega}_t\left(s_i\cdot u_{\breve{B}_z}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot v_{\breve{B}_z}(u_t)\right)^2+\delta_1\dot{\omega}_t\left(s_i\cdot w_{\breve{B}_z}(u_t)\right)^2\right)\end{array}\right\}\end{array}\right\} \tag{118}$$

$$\begin{aligned}\left(\zeta_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}},\mathcal{G}_{\check{B}}\right)\right)^2 &\leq \sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\bullet u_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet v_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet w_{\check{A}_j}(u_k)\right)^2\right)\right)\\ &\quad\times\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\bullet u_{\check{B}_j}(u_k)\right)^2+\left(s_i\bullet v_{\check{B}_j}(u_k)\right)^2+\left(s_i\bullet w_{\check{B}_j}(u_k)\right)^2\right)\right)\\ &= \mathscr{E}_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}}\right)\cdot\mathscr{E}_{\mathrm{GWmPNSS}}\left(\mathcal{G}_{\check{B}}\right).\end{aligned} \tag{119}$$

Therefore, $\left(\zeta_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}},\mathcal{G}_{\check{B}}\right)\right)^2 \leq \mathscr{E}_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}}\right)\cdot \mathscr{E}_{\mathrm{GWmPNSS}}\left(\mathcal{G}_{\check{B}}\right)$. Hence, by using Definition 34, we get $\mathscr{R}_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}},\mathcal{G}_{\check{B}}\right)\leq 1$, so $0\leq \mathscr{R}_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}},\mathcal{G}_{\check{B}}\right)\leq 1$. □

Proof. The proof is obvious. □

Proof. Utilizing Definition 34,

$$R_{\mathrm{GWmPNSS}}\left(F_{\breve{A}},G_{\breve{B}}\right)=\frac{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(s_i\cdot u_{\breve{A}_j}(u_k)s_i\cdot u_{\breve{B}_j}(u_k)+s_i\cdot v_{\breve{A}_j}(u_k)s_i\cdot v_{\breve{B}_j}(u_k)+s_i\cdot w_{\breve{A}_j}(u_k)s_i\cdot w_{\breve{B}_j}(u_k)\right)\right)}{\left(\begin{array}{c}\sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\cdot u_{\breve{A}_j}(u_k)\right)^2+\left(s_i\cdot v_{\breve{A}_j}(u_k)\right)^2+\left(s_i\cdot w_{\breve{A}_j}(u_k)\right)^2\right)\right)}\\ \sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\cdot u_{\breve{B}_j}(u_k)\right)^2+\left(s_i\cdot v_{\breve{B}_j}(u_k)\right)^2+\left(s_i\cdot w_{\breve{B}_j}(u_k)\right)^2\right)\right)}\end{array}\right)}. \tag{120}$$

As we know that $s_i\bullet u_{\check{A}_j}(u_k)=s_i\bullet u_{\check{B}_j}(u_k)$, $s_i\bullet v_{\check{A}_j}(u_k)=s_i\bullet v_{\check{B}_j}(u_k)$, and $s_i\bullet w_{\check{A}_j}(u_k)=s_i\bullet w_{\check{B}_j}(u_k)$, for all j, k, by using Definition 34, we have

$$\mathscr{R}_{\mathrm{GWmPNSS}}\left(\mathcal{F}_{\check{A}},\mathcal{G}_{\check{B}}\right)=\frac{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\bullet u_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet v_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet w_{\check{A}_j}(u_k)\right)^2\right)\right)}{\left(\begin{array}{c}\sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\bullet u_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet v_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet w_{\check{A}_j}(u_k)\right)^2\right)\right)}\\ \sqrt{\sum_{j=1}^{z}\delta_j\left(\sum_{k=1}^{t}\dot{\omega}_k\left(\left(s_i\bullet u_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet v_{\check{A}_j}(u_k)\right)^2+\left(s_i\bullet w_{\check{A}_j}(u_k)\right)^2\right)\right)}\end{array}\right)}. \tag{121}$$

Hence, $\mathscr{R}_{\text{GWmPNSS}}\left(\mathscr{F}_{\check{A}}, \mathscr{G}_{\check{B}}\right) = 1$. □

6. Applications of Similarity Measures and Correlation Coefficient of GmPNSS in Medical Diagnoses and Decision-Making

In this section, we proposed the algorithm for GmPNSS by using developed similarity measures and CC. We also used the proposed methods for medical diagnoses and decision-making in real-life problems.

6.1. Application of Similarity Measure in Medical Diagnoses. We develop the algorithm of GmPNSS for similarity measure and use the developed similarity measure for medical diagnoses by using the proposed algorithm, shown in Figure 1.

6.1.1. Algorithm for Similarity Measure of GmPNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the GmPNSS according to experts.

Step 3. Construct GmPNSS $\varphi^t_{\mathscr{G}}$ for the evaluation of different decision-makers, where $t = 1, 2, \ldots, m$.

Step 4. Find the distance between two GmPNSS by using the distance formula:

$$d^H_{\text{GmPNSS}}\left(\Phi_{\mathscr{F}}(e), \varphi_{\mathscr{G}}(e)\right) = \frac{1}{2m}\left\{\sum_{i=1}^{m}\sum_{j=1}^{p}\left(\left|s_i \bullet u_{\Phi_{\mathscr{F}}}\left(u_j\right) - s_i \bullet u_{\varphi_{\mathscr{G}}}\left(u_j\right)\right|\right) + \left(\left|s_i \bullet v_{\Phi_{\mathscr{F}}}\left(u_j\right) - s_i \bullet v_{\varphi_{\mathscr{G}}}\left(u_j\right)\right|\right) + \left(\left|s_i \bullet w_{\Phi_{\mathscr{F}}}\left(u_j\right) - s_i \bullet w_{\varphi_{\mathscr{G}}}\left(u_j\right)\right|\right)\right\}. \tag{122}$$

Step 5. Compute the similarity measure between two GmPNSS by utilizing the following formula:

$$S_{\text{GmPNSS}}\left(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}\right) = \frac{1}{1 + d\left(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}\right)}. \tag{123}$$

Step 6. Analyze the result.

The flow chart of the presented algorithm can be seen in Figure 1.

6.2. Problem Formulation and Application of Similarity Measure and CC of GmPNSS for Disease Diagnoses. The general proposed algorithm can be used in diagnosis complications. In the light of scientific discipline, a numerical example is presented to deal the diagnostic difficulties. This planned algorithm may be obtained from immoderate medical disease diagnosis complications. We consider typhoid disease as a diagnosis problem, so regardless of whether a well-advised patient has typhoid or not, as many containing the overall signs and symptoms of typhoid are going to be compatible as well as other diseases such as malaria. For a verbal description of the disease, we tend to dispensed similarity measures along the GmPNSS structure to attain an insured person as well as high-fidelity consequences. The general m-polar anatomical structure offers us a record of medical experts rating for the extraordinary disease.

6.2.1. Application of Similarity Measure. Now, we consider the universal set as follows: $\mathscr{U} = \{u_1 = \text{typhoid}, u_2 = \text{nontyphoid}\}$ and E is a set of parameters consisting of symptoms of typhoid disease such as $E = \{x_1 = \text{flu}, x_2 = \text{body pain}, x_3 = \text{headache}\}$. Consider $\mathscr{F}$ and $\mathscr{G} \subseteq E$; then we construct the G3-PNSS of $\mathscr{F}$ and $\mathscr{G}$ such as $\Phi_{\mathscr{F}}(x)$ and $\varphi_{\mathscr{G}}(x)$ according to experts as given in Tables 1 and 2.

Compute distances between $\Phi_{\mathscr{F}}(x)$ and $\varphi_{\mathscr{G}}(x)$ by using Definition 25 given as follows:

$$d^H_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}(e), \varphi_{\mathscr{G}}(e)\right) = 0.6183$$
$$d^{\text{NH}}_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}(e), \varphi_{\mathscr{G}}(e)\right) = 0.3092$$
$$d^E_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}(e), \varphi_{\mathscr{G}}(e)\right) = 0.7749$$
$$d^{\text{NE}}_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}(e), \varphi_{\mathscr{G}}(e)\right) = 0.5481$$

By using Hamming distance, we will find the similarity measure between $\Phi_{\mathscr{F}}(e)$ and $\varphi_{\mathscr{G}}(e)$ given as follows:

$$S_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}\right) = 0.6179 > 0.5.$$

According to the above calculation, $S_{\text{G3-PNSS}}\left(\Phi_{\mathscr{F}}, \varphi_{\mathscr{G}}\right) \geq 0.5$, so G3-PNSS of $\Phi_{\mathscr{F}}$ and $\varphi_{\mathscr{G}}$ are significantly similar, which shows that the patient suffers from typhoid.

6.3. Applications of Correlation Coefficient in Medical Diagnoses. We develop the algorithm of GmPNSS for CC and use the developed CC for medical diagnoses by developing an algorithm.

6.3.1. Algorithm for Correlation Coefficient of GmPNSS

Step 1. Pick out the set containing parameters.

Step 2. Construct the GmPNSS according to experts.

Step 3. Find the informational neutrosophic energies of any two GmPNSS.

Step 4. Calculate the correlation between two GmPNSS by using the following formula:

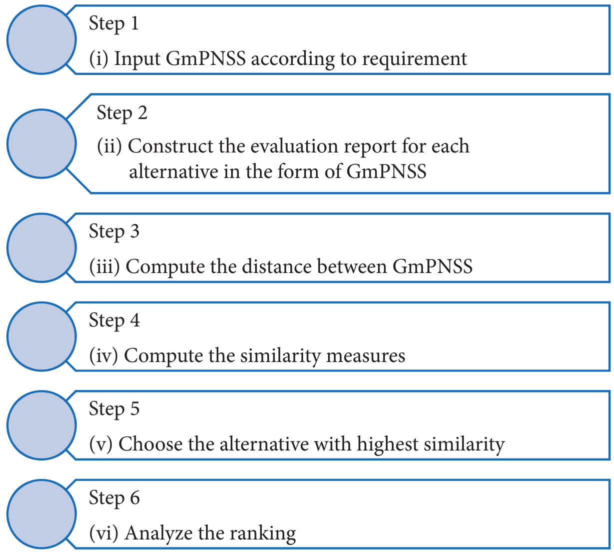

Figure 1: Flow chart of presented algorithm for GmPNSS based on the similarity measure.

Table 1: G3-PNSS of $\mathscr{F}_{\breve{A}}$ according to experts.

$\Phi_{\mathscr{F}}(x)$	x_1	x_2	x_3
u_1	(.69, .52, .61), (.37, .44, .23), (.46, .37, .29)	(.54, .63, .55), (.48, .44, .26), (.63, .47, .59)	(.34, .47, .27), (.46, .48, .37), (.75, .58, .69)
u_2	(.43, .66, .62), (.48, .45, .53), (.47, .52, .36)	(.17, .23, .29), (.37, .41, .47), (.53, .59, .61)	(.58, .53, .55), (.37, .35, .32), (.65, .63, .59)

Table 2: G3-PNSS of $\mathscr{G}_{\breve{B}}$ according to experts.

$\varphi_{\mathscr{G}}(x)$	x_1	x_2	x_3
u_1	(.63, .57, .54), (.47, .46, .32), (.62, .75, .67)	(.45, .71, .50), (.50, .43, .26), (.61, .50, .47)	(.27, .38, .24), (.58, .37, .47), (.65, .69, .70)
u_2	(.47, .59, .69), (.53, .50, .60), (.43, .58, .32)	(.15, .25, .25), (.32, .40, .43), (.53, .60, .60)	(.47, .46, .64), (.44, .40, .30), (.61, .60, .68)

$$\zeta_{\text{GmPNSS}}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right) = \sum_{j=1}^{z} \sum_{k=1}^{t} \left(s_i \bullet u_{\breve{A}_j}(u_k) s_i \bullet u_{\breve{B}_j}(u_k) + s_i \bullet v_{\breve{A}_j}(u_k) s_i \bullet v_{\breve{B}_j}(u_k) + s_i \bullet w_{\breve{A}_j}(u_k)\, s_i \bullet w_{\breve{B}_j}(u_k) \colon i \in 1, 2, 3, \ldots, m \right). \tag{124}$$

Step 5. Calculate the CC between any two GmPNSS by using the following formula:

$$\mathscr{R}_{\text{GmPNSS}}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right) = \frac{\zeta_{\text{GmPNSS}}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right)}{\sqrt{\mathscr{E}_{\text{GmPNSS}}\left(\mathscr{F}_{\breve{A}}, \mathscr{F}_{\breve{A}}\right) . \mathscr{E}_{\text{GmPNSS}}\left(\mathscr{G}_{\breve{B}}, \mathscr{G}_{\breve{B}}\right)}}. \tag{125}$$

Step 6. Analyze the results.

The flow chart of the presented algorithm can be seen in Figure 2.

6.3.2. Application of Correlation Coefficient. We use the proposed algorithm for medical diagnoses. For this, we consider that $\mathscr{F}_{\breve{A}}$ and $\mathscr{G}_{\breve{B}}$ are G3-PNSS which are described in Section 6.2.1 in Tables 1 and 2, respectively. By using equation (12), we can find CC against the values of the universal set given as follows: $\mathscr{R}_{\text{G3-PNSS}(u_1)}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right) = 0.9967$ and $\mathscr{R}_{\text{G3-PNSS}(u_2)}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right) = 0.9925$. By the above calculation, we analyze the results and get $\mathscr{R}_{\text{G3-PNSS}(u_1)}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right) > \mathscr{R}_{\text{G3-PNSS}(u_2)}\left(\mathscr{F}_{\breve{A}}, \mathscr{G}_{\breve{B}}\right)$, which shows that patient suffers from typhoid.

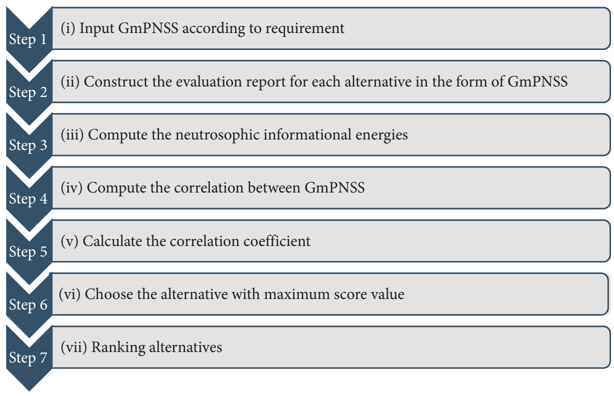

Figure 2: Flow chart of the presented algorithm based on the correlation coefficient.

6.4. Problem Formulation and Application of GmPNSS for Decision-Making. Department of the scientific discipline of some university U will have one scholarship for a post-doctoral position. Several applicants apply for scholarship but only four $S = \{S_1, S_2, S_3, S_4\}$ applicants received the interview call for evaluation based on their CGPA (cumulative grade points average). The president of the university hires a committee of four experts $X = \{X_1, X_2, X_3, X_4\}$ for the selection of the postdoctoral scholars. First of all, the committee decides the set of parameters such as $E = \{x_1, x_2, x_3\}$, where x_1, x_2, and x_3 represent the research papers, research quality, and communication skills for the selection of postdoctoral scholars. The experts evaluate the scholars under defined parameters and forward the performance evaluation to the president of the university. Finally, the president of the university scrutinizes the one best scholar based on the expert's evaluation for the postdoctoral scholarship.

6.4.1. Application of GmPNSS for Decision-Making. Assume that $S = \{S_1, S_2, S_3, S_4\}$ is a set of scholars who are shortlisted for interview and $E = \{x_1 = \text{research paper}, x_2 = \text{research quality}, x_3 = \text{interview}\}$ is a set of parameters for the selection of scholarship. Let $\mathscr{F}$ and $\mathscr{G} \subseteq E$; then we construct the G3-PNSS $\Phi_{\mathscr{F}}(x)$ according to the requirement of the scientific discipline department.

Now we will construct the G3-PNSS $\varphi^t_{\mathscr{G}}$ according to four experts, where $t = 1, 2, 3, 4$.

By using equation (3), we calculate the Euclidean distance between $\Phi_{\mathscr{F}}$ and $\varphi^t_{\mathscr{G}}$ as follows:

$$d^E_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^1_{\mathscr{G}}) = 1.32$$
$$d^E_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^2_{\mathscr{G}}) = 1.3185$$
$$d^E_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^3_{\mathscr{G}}) = 0.4598$$
$$d^E_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^4_{\mathscr{G}}) = 1.1132$$

Similarity measures of $\Phi_{\mathscr{F}}$ and $\varphi^t_{\mathscr{G}}$ can be calculated as follows:

$$S_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^1_{\mathscr{G}}) = 0.4310$$
$$S_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^2_{\mathscr{G}}) = 0.4313$$
$$S_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^3_{\mathscr{G}}) = 0.6850$$
$$S_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^4_{\mathscr{G}}) = 0.4732$$

According to the proposed similarity measure, ranking of the alternatives is $S_3 > S_4 > S_2 > S_1$, and it is clear that $S_{\text{G3-PNSS}}(\Phi_{\mathscr{F}}, \varphi^3_{\mathscr{G}}) = 0.6850 > 0.5$, which shows that $\Phi_{\mathscr{F}}$ and $\varphi^3_{\mathscr{G}}$ are significantly similar to G3-PNSS. So S_3 is the best scholar for the postdoctoral position. Graphical representation of alternatives ranking can be seen in Figure 3.

6.4.2. Solution by Using Algorithm 2. Now, by using Tables 3–7, we can find the correlation coefficient for each alternative by using equation (12) given as $\mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^1_{\mathscr{G}}) = .8374$, $\mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^2_{\mathscr{G}}) = .7821$, $\mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^3_{\mathscr{G}}) = .9462$, and $\mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^4_{\mathscr{G}}) = .9422$. This shows that $\mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^3_{\mathscr{G}}) > \mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^4_{\mathscr{G}}) > \mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^1_{\mathscr{G}}) > \mathscr{R}_{\text{GmPNSS}}(\Phi_{\mathscr{F}}, \varphi^2_{\mathscr{G}})$. Hence, S_3 is the best scholar for a postdoctoral position. In Figure 3, we can see the graphical representation of alternatives ranking.

7. Result Discussion and Comparative Analysis

In the following section, we will discuss the effectiveness, naivety, flexibility, and advantages of the proposed methods and algorithms. We also conducted a brief comparative analysis of the following: suggested methods and existing methods.

7.1. Advantages and Flexibility of the Proposed Approach. The recommended technique is effective and applicable to all forms of input data. Here, we introduce two novel algorithms based on GmPNSS: one is CC, and the other is similarity measures. Both algorithms are effective and can provide the best results in MCDM problems. The recommended algorithm is simple and easy to understand, can deepen understanding, and is suitable for many types of choices and indicators. Developed algorithms are flexible and easy to change to suit different situations, inputs, and outputs. There are subtle differences between the rankings of the suggested methods because different techniques have different ranking methods, so they can be afforded according to their considerations.

7.1.1. Superiority of the Proposed Method. Through this research and comparative analysis, we have concluded that the results obtained by the proposed method are more general than the prevailing methods. However, in the decision-making process, compared with the existing decision-making methods, it contains more information to deal with the uncertainty in the data. Moreover, the mixed structure of many FS has become a special case of GmPNSS, by adding some suitable conditions. Among them, the information related to the object can be expressed more accurately and empirically, so it is a convenient tool for combining inaccurate and uncertain information in the decision-making process. Therefore, our proposed method is effective, flexible, simple, and superior to other hybrid structures of fuzzy sets.

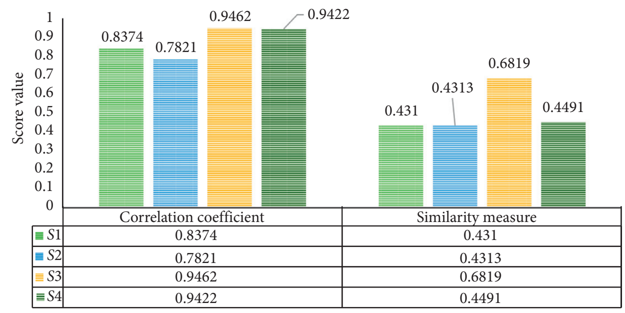

Figure 3: Alternative final score value with the proposed algorithms.

Table 3: Construction of G3-PNSS of all scholars according to department requirement.

$\Phi_{\mathscr{F}}(x)$	x_1	x_2	x_3
$\mathbf{X}_1$	(.82, .55, .63), (.55, .46, .28), (.43, .38, .60)	(.43, .68, .86), (.47, .67, .56), (.42, .51, .33)	(.73, .48, .53), (.87, .43, .77), (.76, .53, .62)
$\mathbf{X}_2$	(.50, .62, .52), (.93, .57, .80), (.66, .48, .52)	(.77, .54, .81), (.75, .54, .72), (.53, .54, .69)	(.64, .48, .59), (.32, .58, .22), (.94, .64, .62)
$\mathbf{X}_3$	(.29, .25, .41), (.73, .34, .32), (.64, .44, .56)	(.36, .45, .27), (.47, .65, .21), (.61, .37, .39)	(.57, .25, .41), (.72, .55, .29), (.64, .31, .34)
$\mathbf{X}_4$	(.91, .50, .16), (.30, .24, .63), (.16, .55, .20)	(.69, .52, .61), (.37, .44, .23), (.46, .37, .29)	(.39, .35, .67), (.47, .24, .32), (.40, .71, .56)

Table 4: G3-PNSS evaluation report according to experts of S_1.

$\varphi^1_{\mathscr{G}}$	x_1	x_2	x_3
$\mathbf{X}_1$	(.13, .15, .22), (.89, .78, .83), (.77, .82, .91)	(.91, .50, .16), (.30, .24, .63), (.16, .55, .20)	(.69, .52, .61), (.37, .44, .23), (.46, .37, .29)
$\mathbf{X}_2$	(.79, .84, .93), (.36, .18, .26), (.21, .24, .16)	(.39, .35, .67), (.47, .24, .32), (.40, .71, .56)	(.76, .62, .41), (.36, .49, .79), (.53, .59, .91)
$\mathbf{X}_3$	(.07, .23, .32), (.12, .18, .20), (.74, .79, .88)	(.70, .22, .11), (.67, .43, .53), (.41, .57, .49)	(.87, .58, .66), (.77, .22, .56), (.57, .33, .29)
$\mathbf{X}_4$	(.23, .12, .17), (.25, .16, .22), (.14, .16, .18)	(.74, .62, .66), (.67, .41, .93), (.85, .67, .99)	(.27, .29, .61), (.71, .43, .21), (.47, .70, .89)

Table 5: G3-PNSS evaluation report according to experts of S_2.

$\varphi^2_{\mathscr{G}}$	x_1	x_2	x_3
$\mathbf{X}_1$	(.16, .20, .27), (.83, .87, .89), (70, .75, .86)	(.91, .50, .16), (.30, .24, .63), (.16, .55, .20)	(.69, .52, .61), (.37, .44, .23), (.46, .37, .29)
$\mathbf{X}_2$	(.13, .21, .24), (.18, .20, .20), (.70, .84, .90)	(.39, .35, .67), (.47, .24, .32), (.40, .71, .56)	(.76, .62, .41), (.36, .49, .79), (.53, .59, .91)
$\mathbf{X}_3$	(.20, .16, .27), (.29, .17, .26), (.14, .15, .12)	(.70, .22, .11), (.67, .43, .53), (.41, .57, .49)	(.87, .58, .66), (.77, .22, .56), (.57, .33, .29)
$\mathbf{X}_4$	(.88, .81, .90), (.40, .20, .26), (.22, .27, .17)	(.74, .62, .66), (.67, .41, .93), (.85, .67, .99)	(.27, .29, .61), (.71, .43, .21), (.47, .70, .89)

Table 6: G3-PNSS evaluation report according to experts of S_3.

$\varphi^3_{\mathscr{G}}$	x_1	x_2	x_3
$\mathbf{X}_1$	(.77, .49, .61), (.71, .43, .21), (.47, .40, .69)	(.47, .59, .76), (.67, .62, .56), (.57, .43, .29)	(.70, .54, .61), (.79, .44, .63), (.61, .41, .51)
$\mathbf{X}_2$	(.60, .32, .32), (.77, .49, .83), (.76, .32, .59)	(.76, .62, .61), (.56, .49, .79), (.53, .59, .81)	(.69, .62, .67), (.57, .74, .43), (.86, .47, .79)
$\mathbf{X}_3$	(.60, .22, .21), (.67, .43, .53), (.49, .57, .49)	(.29, .72, .41), (.30, .66, .29), (.56, .32, .39)	(.74, .52, .66), (.67, .41, .93), (.85, .47, .59)
$\mathbf{X}_4$	(.74, .26, .37), (.49, .41, .63), (.44, .35, .32)	(.41, .66, .51), (.39, .27, .36), (.41, .51, .21)	(.60, .16, .47), (.31, .17, .24), (.54, .35, .24)

Table 7: G3-PNSS evaluation report according to experts of S_4.

$\varphi^4_{\mathscr{G}}$	x_1	x_2	x_3
$\mathbf{X}_1$	(.23, .13, .22), (.31, .25, .43), (.19, .22, .27)	(.43, .68, .86), (.47, .67, .56), (.42, .51, .33)	(.82, .55, .63), (.55, .46, .28), (.43, .38, .60)
$\mathbf{X}_2$	(.10, .13, .11), (.91, .84, .69), (.31, .30, .28)	(.27, .29, .61), (.71, .43, .21), (.47, .70, .89)	(.50, .62, .52), (.93, .57, .80), (.66, .48, .52)
$\mathbf{X}_3$	(.70, .22, .11), (.67, .43, .53), (.41, .57, .49)	(.70, .22, .11), (.67, .43, .53), (.41, .57, .49)	(.36, .45, .27), (.47, .65, .21), (.61, .37, .39)
$\mathbf{X}_4$	(.45, .16, .27), (.91, .67, .23), (.64, .88, .10)	(.67, .81, .17), (.21, .54, .71), (.41, .54, .21)	(.20, .76, .47), (.39, .17, .46), (.41, .53, .22)

Table 8: Comparative analysis between some existing techniques and the proposed approach.

	Set	Truthiness	Indeterminacy	Falsity	Multipolarity	Loss of information
Chen et al. [48]	mPFS	✓	×	×	✓	×
Xu et al. [49]	IFS	✓	×	✓	×	×
Zhang et al. [50]	IFS	✓	×	✓	×	✓
Ali et al. [51]	BPNSS	✓	✓	✓	×	×
Proposed approach	GmPNSS	✓	✓	✓	✓	×

Table 9: Comparison between GmPNSS and some existing studies.

Method	Alternative final ranking	Optimal choice
Masooma et al. [16]	$S_3 > S_2 > S_1 > S_4$	S_3
Saeed et al. [33]	$S_3 > S_4 > S_2 > S_1$	S_3
Riaz et al. [52]	$S_3 > S_2 > S_1 > S_4$	S_3
Kamal et al. [22]	$S_3 > S_4 > S_2 > S_1$	S_3
Proposed algorithm 1	$S_3 > S_4 > S_2 > S_1$	S_3
Proposed algorithm 2	$S_3 > S_4 > S_1 > S_2$	S_3

It turns out that this is a contemporary issue. Why do we have to express novel algorithms based on the current novel structure? Many indications are compared with other existing methods, and the recommended methods are surely competent. We remember the following fact: the mixed structures have some limitations in IFS, picture fuzzy sets, FS, hesitation fuzzy sets, NS, and other fuzzy sets, so complete information about the situation cannot be provided. But our m-polar model GmPNSS can be the most suitable for MCDM because it can deal with truth, indeterminacy, and falsity. Due to the exaggerated multipolar neutrosophy, these three degrees are independent of each other and provide a lot of information about alternative norms. Other similarity measures of available hybrid structures are converted into special cases of GmPNSS. A comparative analysis of some existing techniques is listed in Table 8. Therefore, compared with intuitionistic, neutrosophy, hesitant, image, and ambiguity substitution, this model is more versatile and can easily resolve complications. The similarity measures established for GmPNSS become better than the existing similarity measures for MCDM.

7.1.2. Discussion. By using the technique of Chen et al. [48], we deal with the multipolar information of fuzzy sets, but, with this method, we cannot deal with the indeterminacy and falsity objects of alternatives. By using the methodologies of Xu et al. [49] and Zhang et al. [50], we cannot deal with the multipolar information and uncertainty part of the alternative. But, on the other hand, the methodology we established involves the truthiness, indeterminacy, and falsity of alternatives with multiple data. Therefore, the technique we developed is more efficient and can provide better results for decision-makers through various information. Ali et al.'s method [51] dealt with the truthiness, indeterminacy, and falsity levels of alternatives, but these techniques cannot manage multiple data. Instead, the method we developed is an advanced technique that can handle alternatives with multiple types of information. It can be seen in Table 8.

7.1.3. Comparative Analysis. In this article, we propose two types of algorithms. First, an algorithm is proposed based on the correlation coefficient, and the other is based on similarity measures for GmPNSS. Next, both algorithms are utilized to solve practical problems in real life, that is, for the selection of a postdoctoral position. The graphical representation of results obtained by both algorithms is shown in Figure 3. The results show that the proposed technique is effective and practical. Finally, the ranking of all alternatives using the existing methodologies gives the same final decision; that is, the "postdoctoral" position is selected as S_3. All rankings are also calculated by applying existing methods with the same case study. The proposed method is also compared with other existing methods: Saeed et al. [33], Masooma et al. [16], Riaz et al. [52], and Kamal et al. [22]. The comparison results are listed in Table 9, which shows the final ranking of the top 4 alternatives. It can be observed that the best selections made by the proposed methods are compared with the already established methods which are expressive in themselves and approve the reliability and validity of the proposed method.

8. Conclusion

In this paper, we study the mPNSS and propose a generalized version of mPNSS with some basic operations and properties. We also develop the AND operator, OR operator, Truth-Favorite operator, and False-Favorite operator with properties and examples. The concepts of necessity and possibility operations with their properties are developed in this research. The distance-based similarity measures on GmPNSS are established by using the Hamming and Euclidean distances with their properties, and a decision-making approach is presented to solve multicriteria decision-making problems. We also established the correlation coefficient and the weighted correlation coefficient of GmPNSS with the decision-making technique. Furthermore, a numerical illustration has been described to solve the MCDM problem by using the proposed decision-making approaches for medical diagnoses and decision-making. A comparative analysis is presented to verify the validity and demonstration of the proposed method. Finally, the suggested techniques showed higher stability and practicality for decision-makers in the decision-making process. Based on the results obtained, it is concluded that the proposed method is most suitable for solving the MCDM problem in today's life. The presented technique is unable to handle the scenario when the information of truth, falsity, and indeterminacy is given in intervals. In the future, the concept of mPNSS will be extended to interval-valued mPNSS and the developed techniques to other fields, such as mathematical programming, cluster analysis, and big data analysis.

Data Availability

No data were used in this manuscript.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: A Hybrid BSC-DEA Model with Indeterminate Information

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] M. Jaberi Hafshjani, S. E. Najafi, F. Hosseinzadeh Lotfi, and S. M. Hajimolana, "A Hybrid BSC-DEA Model with Indeterminate Information," *Journal of Mathematics*, vol. 2021, Article ID 8867135, 14 pages, 2021.

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Research Article

A Hybrid BSC-DEA Model with Indeterminate Information

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Strategy is the main source of long-term growth for organizations, and if it is not successfully implemented, even if appropriate ones are adopted, the process is futile. The balanced scorecard which focuses on four aspects such as growth and learning, internal processes, customer, and financial is considered as a comprehensive framework for assessing performance and the progress of the strategy. Moreover, the data envelopment analysis is one of the best mathematical methods to compute the efficiency of organizations. The combination of these two techniques is a significant quantitative measurement with respect to the organization's performance. However, in the real world, determinate and indeterminate information exists. Henceforth, the indeterminate issues are inescapable and must be considered in the performance evaluation. Neutrosophic number is a helpful tool for dealing with information that is indeterminate and incomplete. In this paper, we propose a new model of data envelopment analysis in the neutrosophic number environment. Furthermore, we attempt to combine the new model with the balanced scorecard to rank different decision-making units. Finally, the proposed method is illustrated by an empirical study involving 20 banking branches. The results show the effectiveness of the proposed method and indicate that the model has practical outcomes for decision-makers.

1. Introduction

All organizations whether governmental or private require an effective performance assessment for development, growth, and sustainability in the competitive world of today. In other words, senior executive managers have always been seeking a solution to ensure that their strategies are executed and, hence, have selected performance assessment methods as tools to implement their strategies.

The balanced scorecard (BSC) has been introduced as a comprehensive framework for performance assessment and advancement of strategy, which balances the short- and long-term goals, financial and nonfinancial measures, internal and external performance, internal and external stakeholders, and the occurring progressive and nonprogressive performance indexes. BSC is a proven framework that describes and operates the organization's strategy [1].

Data envelopment analysis (DEA) is a mathematical programming for measuring the relative efficiencies of homogeneous decision-making units (DMUs) without knowing production functions, just by utilizing input and output information [2, 3]. The first models in DEA are the CCR and BCC models in which the efficiency of each DMU obtained as the maximum of a ratio of weighted outputs to weighted inputs subject to that of the similar ratio for all DMUs is less than or equal to one [2, 3]. DEA technique has just been effectively connected in various cases such as broadcasting companies [4], banking institutions [5–8], R&D organizations [9, 10], health care services [11], manufacturing [12, 13], telecommunication [14], and supply chain management [15].

One of the disadvantages of the BSC is the lack of a quantitative measurement of the organization's performance using the mathematical method. Therefore, the integrated BSC-DEA approach can be used to provide a mathematical model of performance measurement for macrogoals, which is complete than the separate models [16]. In the hybrid BSC-DEA technique, BSC is utilized as a tool for designing the assessment indexes for performance, whereas the DEA is used as a tool for performance evaluation. This approach has drawn the attention of several researchers within a very short period of time.

Eilat et al. [17–19] used the BSC-DEA method for the first time in R&D projects. Min et al. [20] applied the BSC model and the DEA technique for the efficiency of Korean hotels. Chen et al. [21, 22], based on the four perspectives of the BSC, with the help of the quantitative DEA tools, carried out the efficiency evaluation of semiconductor industries and cooperative credit banks in Taiwan. Macedo et al. [23] applied a hybrid BSC-DEA model to performance measurement of the bank branches in Brazil with 6 indexes. García-Valderrama et al. [24] proposed a framework for the analysis of the relationships between the four perspectives of the BSC, and utilizing DEA developed several different models of efficiency. Chiang et al. [25] attempted to develop an integrated framework to encompass the BSC and DEA for measuring management performance and selected auto and commercial bank industries as the targets for empirical investigation. Amado et al. [26] presented the development of a conceptual framework which aims to assess DMUs from multiple perspectives. The proposed conceptual framework combines the BSC method with DEA by using various interconnected models which try to encapsulate four perspectives of BSC. Wu and Liao [27] proposed an integrated DEA-BSC model to evaluate the operational efficiency of airlines. To adapt this model, 38 major airlines in the world were selected to assess their relative performance. In [28], the information technology (IT) project has been evaluated by using a hybrid DEA-BSC model. This approach uses BSC as a comprehensive framework for defining IT project evaluation criteria and uses DEA as a nonparametric technique for ranking IT projects. For illustrations of the other researches which have been executed in relative to assessing the efficiency of organizations by utilizing the DEA and the BSC, refer to [29–32].

However, data in real world are imprecise and vague, and one of the main tools for description of this kind of data is fuzzy number. Since Zadeh [33] presented fuzzy sets (FS), fuzzy theory has been applied effectively in an extensive variety of subject fields [34–37]. Some researchers also considered the BSC-DEA models under fuzzy environment [38–42]. Since the fuzzy set considers only the degree of membership and has not the degree of nonmembership, Atanassov [43] made an enhancement to overcome this weakness and presented the intuitionist fuzzy set (IFS) consisting of the degree of membership and the degree of nonmembership. There are various models of DEA with IFSs (see [44–46]). However, the IFS did not consider the degree of indeterminacy. We know that the incomplete, indeterminate, and inconsistent information in real life often exists. Because of an absence of data, estimation mistakes, or the limited attention and knowledge of decision-makers, in numerous circumstances, the obtained information might be partial determinacy and/or partial indeterminacy. Fuzzy and intuitionist fuzzy sets cannot therefore represent data with both determined and indefinite data.

To express this kind of information, Smarandache [47–49] originally established the neutrosophic logic, which generalizes the concept of the classic set, fuzzy set, interval-valued fuzzy set, and intuitionistic fuzzy set. This logic divided into two categories of the neutrosophic sets (NSs) and the neutrosophic numbers (NNs).

The neutrosophic sets (NSs) are represented by a truth-membership degree, an indeterminacy-membership degree, and a falsity-membership degree and have some subclasses such as single valued neutrosophic sets [50–60], interval neutrosophic sets [61–65], and simplified neutrosophic sets [64, 66–68]. A neutrosophic set A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]0^-, 1^+[$; that is, $T_A(x)\colon X \longrightarrow]0^-, 1^+[$, $I_A(x)\colon X \longrightarrow]0^-, 1^+[$,and $F_A(x)\colon X \longrightarrow]0^-, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$, so

$$0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+. \qquad (1)$$

The neutrosophic number (NN) introduces a concept of indeterminacy, denoted by $A = m + nI\,(m, n \in R)$, and consists of its determinate part m and its indeterminate part nI. In the worst scenario, A can be unknown, *i.e.*, $A = nI$. However, when there is no indeterminacy related to A, in the best scenario, there is only its determinate part, *i.e.*, $A = m$. Smarandache also refined the NNs by decomposition of the indeterminacy I into different types of indeterminacies such as $I_1, I_2, \ldots, I_n$, and extended the neutrosophic number to the refined neutrosophic number [69, 70].

It is worth mentioning that the neutrosophic sets (NSs) cannot deal with decision-making problems with neutrosophic numbers, as NSs and NNs are two different branches in neutrosophic theory and indicate different forms and concepts of information.

It is clear that the NNs are very practical for conveying information about indeterminate evaluations in complex decision-making problems. For example, Ye [71] provided a neutrosophic number tool for a multiple attribute group decision-making (MAGDM) problem with NNs. He presented a de-neutrosophication method and a possibility degree ranking method for NNs as a methodological support for group decision-making problems. Additionally, Ye [72] developed a bidirectional projection measure of NNs for MAGDM problems. Under a NN environment, Chen and Ye [73] presented a projection model of NNs and its decision-making method for the selecting problems of clay-bricks. Kong et al. [74] presented a distance measure and cosine similarity measure between NNs and applied it to the misfire fault diagnosis of gasoline engines. Furthermore, Smarandache [75] introduced the concept of a neutrosophic linguistic number (NLN) in symbolic neutrosophic theory. Based on this concept, Ye [76] proposed basic operational laws of NLNs. Zhang et al. [77] proposed an extend TODIM method to handle multiple attribute group decision-making problems in which the evaluation information is expressed by NNs. Zheng et al. [78] presented some aggregation operators based on NNs, which are used to handle MAGDM problems. Furthermore, under this environment, Liu and Liu [79] proposed some generalized weighted power aggregation operators that are used to deal with MAGDM problems more effectively. Jiang and Ye [80] defined a new concept of neutrosophic number functions for the objective functions and constraints in engineering optimization design problems with determinate and indeterminate information and obtained a general NN optimization model of truss structure design. To overcome the complex calculation and difficult solution problems in methods of [80], Ye [81] proposed an improved NN optimization method and applied it to a three-bar planar truss structural design with indeterminate information. Furthermore, Ye et al. [82] using the neutrosophic number functions investigated the anisotropy and scale effect of indeterminate joint roughness coefficient (JRC), which is a quite crucial parameter for determining the shear strength in rock mechanics. Recently, some scholars also under NN environment proposed some models for various optimization problems such as linear programming [83], multiobjective programming [84], nonlinear programming [85], and bilevel linear programming problem [86].

The first model of DEA with NS was established by Edalatpanah [87], and additional investigations have been accessible in [88–95]. However, these models are formulated solely for NSs. In real-life situations, some inputs/outputs in DEA may also be indeterminate and inconsistent, and considering neutrosophic number for each input/output of DMUs helps decision-makers to obtain a better interpretation of information. In addition, by using the NNs in DEA, analysts can obtain a better representation of reality through considering all aspects of the decision-making process. Unfortunately, in the current literature, there is no study of data envelopment analysis (DEA) models and also BSC-DEA methodology in the NN environment. It is clear that the conventional fuzzy sets cannot express neutrosophic DEA with both determinate and indeterminate information. Therefore, it is necessary to propose a new method based on the neutrosophic numbers to BSC-DEA methodology. The main purposes of this paper are as follows: (1) to develop a new model for DEA within the NN environment and (2) to combine the new model with BSC to rank different decision-making units. There are usually many qualitative ways to evaluate an organization's performance. One of these tools is the balanced scorecard approach that separates the organization from the operational point of view. By examining the organization using this segmentation, one can usually gain an understanding of organizational performance, but in quantitative terms, approaches are always needed to evaluate activities separately and to provide accurate benchmarks for different decisions. Data envelopment analysis approach is one of the tools that can be helpful and provides a little understanding of the various points of the scorecard. Understanding the need and the precise relationship between these two concepts illustrates the importance of the subject and led us to present this hybrid model for ranking the decision-making units in the organization.

This hybrid model is presented in an innovative way and demonstrates the significant relationship between the qualitative concepts in the BSC and the quantitative concepts in data envelopment analysis for the purpose of decision-making strategy and ultimately enhancing organizational performance.

The rest of the paper is organized as follows: Section 2 presents some essential concepts regarding neutrosophic numbers and BSC and DEA models. Section 3 proposes a new model of DEA in neutrosophic number environment. Section 4 explains a hybrid BSC-DEA model with NNs. An empirical study involving 20 banking branches and conclusions are given in Sections 5 and 6, respectively.

2. Preliminary Concepts

In this section, we present several basic discussion concerning neutrosophic numbers, balance scorecard, and data envelopment analysis.

2.1. Neutrosophic Number Concept. A neutrosophic number (NN) is represented by $A = m + nI$ $(m, n \in \mathbb{R})$, where m and nI are determinate and indeterminate parts, respectively; for example, consider a NN as $A = 3 + 4I$. Then, it indicates that its determinate value is 3, and its indeterminate value is $4I$. Assume that the indeterminacy I is considered as such a possible interval [0, 2], and then, it is equivalent to $A = [3, 11]$, where A is within the interval $[3, 11]$. For the best case, we have $nI = 0$ and A can be expressed as the determinate part $A = m$, whereas in the worst case $m = 0$ and A expressed as the indeterminate part, $A = nI$. For convenience, let $\aleph$ be the set of all NNs, and then, a NN is denoted by $A = m + nI = [m + n(\inf(I)), m + n(\sup(I))]$ for $I \subseteq [\inf(I), \sup(I)]$ and $A \in \aleph$.

Definition 1 (see [80, 83]).

Let $A_1 = m_1 + n_1 I$ and $A_2 = m_2 + n_2 I$ for $m_i, n_i \in R$, $A_i \in \aleph$, and $I \in [I^L, I^U]$ be two NNs, then they contain the following arithmetic laws:

$$\begin{aligned}
&\text{(i) } A_1 + A_2 = (m_1 + m_2) + (n_1 + n_2)I,\\
&\text{(ii) } A_1 - A_2 = (m_1 - m_2) + (n_1 - n_2)I,\\
&\text{(iii) } A_1 \times A_2 = m_1 m_2 + (m_1 n_2 + m_2 n_1)I + n_1 n_2 I^2.
\end{aligned} \tag{2}$$

Definition 2 (see [81]). A NN function with n variables and $\aleph$ domain is defined as $f(x, I)\colon \aleph^n \longrightarrow \aleph$, where, $x = [x_1, \ldots, x_n]^T \in \aleph^n$ and $I \in [I^L, I^U]$. Moreover, I is indeterminacy and $f(x, I)$ can be an NN linear/ nonlinear function; for example, $f(x, I)\colon (5 + 4I)x_1 + (1 + 2I)x_2 + 3I$ for $x = [x_1, x_2]^T \in \aleph^2$ is an NN linear function.

2.2. Balance Scorecard (BSC). Kaplan and Norton proposed the BSC model as a method to evaluate the performance of an organization. The traditional performance assessment systems are more prominently based on financial indexes, whereas successful companies rely not only on financial indicators to evaluate their performance but they also considered their performance from three other BSC perspectives; i.e., customer, internal processes, learning, and growth [96, 97]. The BSC method is a performance measurement framework that provides a complete overview of an organization's performance with a set of financial and nonfinancial scales. The BSC model has been utilized effectively in manufacturing, service, nonprofitable, and government organizations. Many applications for a balanced scorecard have defined from a business perspective [98, 99]. In Figure 1, the four aspects of the balanced scorecard have been depicted.

According to Figure 2, we must create value for our customers (the customer perspective) in order to achieve financial (the financial perspective). This would not be feasible unless we excel in our operational processes and adapt them to the needs of our customers (the internal processes perspective). It is not possible to gain excellence in operation and process of value unless we create the right work environment for employees and strengthen the innovation and creativity in learning and growth (the learning and growth perspective).

2.3. Data Envelopment Analysis (DEA). Data envelopment analysis (DEA) is a linear programming method for assessing the efficiency and productivity of decision-making units (DMUs). In the traditional DEA literature, various well-known DEA approaches can be found such as CCR and BCC models [2, 3]. The efficiency of a DMU is established as the ratio of sum weighted output to sum weighted input, subjected to happen between one and zero. The outputs and inputs are known, and the weighted value of the inputs and outputs is selected in such a manner that the efficiency of that DMU is maximized.

Let us assume that n DMU's are present as {DMUj: $j = 1$, ..., n}, which utilize m inputs x_{ij} $(i = 1, 2, \ldots, m)$ to produce s outputs x_{ij} $(i = 1, 2, \ldots, m)$. Here, $u_r\,(r = 1, 2, \ldots, s)$ and $v_i\,(i = 1, 2, \ldots, m)$ are the weights of the ith input and rth output. Then, the CCR model is as follows:

$$
\begin{aligned}
\theta_p^* &= \max \sum_{r=1}^{s} u_r y_{rp},\\
\text{s.t}:&\\
&\sum_{i=1}^{m} v_i x_{ip} = 1,\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, \quad \forall j,\\
&u_r, v_i \ge 0, \quad \forall r, i.
\end{aligned} \tag{3}
$$

We solve model (3) n-times to work out the efficiency of n DMUs. If $\theta_p^* = 1$, we say that the DMUp is efficient; otherwise, it is inefficient.

3. New Model of DEA in NN Environment

In this section, we propose a new model of DEA in the neutrosophic number environment. Let us consider the CCR model (3) under the environment of the neutrosophic number. Then, we have

$$
\begin{aligned}
\theta_p^* &= \max \sum_{r=1}^{s} u_r \tilde{y}_{rp},\\
\text{s.t}:&\\
&\sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1,\\
&\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \le 0, \quad \forall j,\\
&u_r, v_i \ge 0, \quad \forall r, i,
\end{aligned} \tag{4}
$$

where $\tilde{x}_{ij} = x_{ij} + \eta_{ij} I\,(i = 1, 2, \ldots, m)$ and $\tilde{y}_{rj} = y_{rj} + \gamma_{rj} I$ $(r = 1, 2, \ldots, s)$are neutrosophic numbers of the input and output for the jth DMU and also $I \subseteq [\inf(I), \sup(I)]$. We propose a new model to solve (4).

Theorem 1 shows the feasibility and boundedness of model (4).

Theorem 1. *Model (4) is always feasible and bounded. Furthermore, its optimal objective function is 1.*

Proof. With the solution $u^t = (0, \ldots, 0)$ and $v^t = (0, \ldots, (1/\tilde{x}_{ip}), \ldots, 0)$, it is easy to see that model (11) is always feasible. Thus, regardless of the values of inputs and outputs, there is always at least one feasible solution for model (11). On the other hand, by this solution, we have

$$
\begin{aligned}
&\sum_{i=1}^{m} v_i \tilde{x}_{ip} = 1,\\
&\sum_{r=1}^{s} u_r \tilde{y}_{rj} \le 1.
\end{aligned} \tag{5}
$$

Because the above solution is feasible along with the objective function of model (4) is maximization, the best value regarding the objective function is certainly equal to 1. □

4. A Hybrid BSC-DEA Model with Neutrosophic Numbers

In this section, we attempt to combine the new neutrosophic DEA model proposed in Section 3 with the BSC to rank different DMUs. Since the BSC model evaluates the performance of an organization in the grounds of macrogoals and model of DEA with neutrosophic numbers is also a method to measure efficiency or performance with indeterminate information, therefore, by combining the two-abovementioned measuring methods, the performance is measured and aligned with strategic goals. In the hybrid of DEA and BSC models, the BSC is utilized as a tool for the assessment of performance indexes and the neutrosophic DEA model is used as a tool to evaluate the efficiency in this model. The entire structure of the hybrid DEA-BSC model is shown in Figure 3.

Figure 4 also denotes the four aspects of the performance of the BSC with specific organizational strategies, and in each of the domains, the relative indexes have been defined.

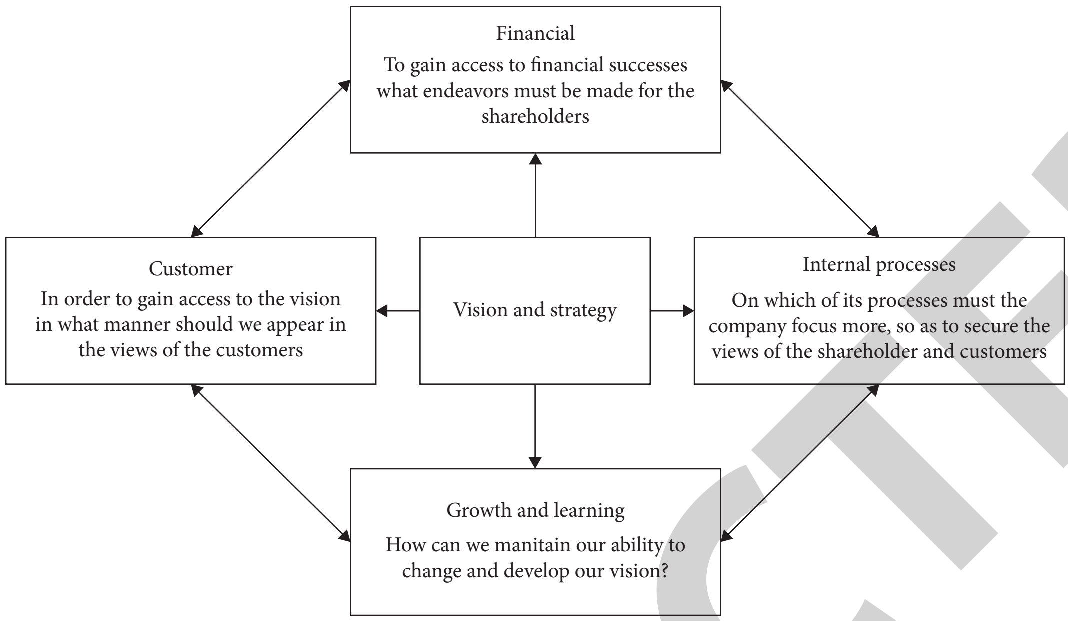

Figure 1: Transforming the vision and strategy into the four BSC aspects.

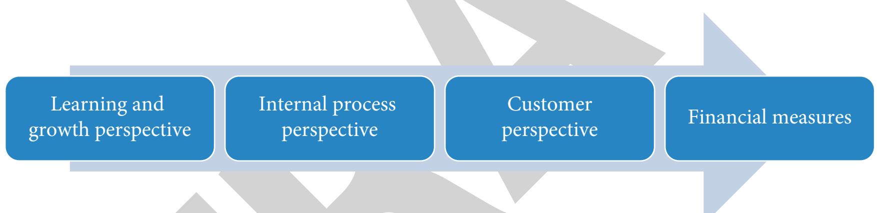

Figure 2: Causal correlation in BSC.

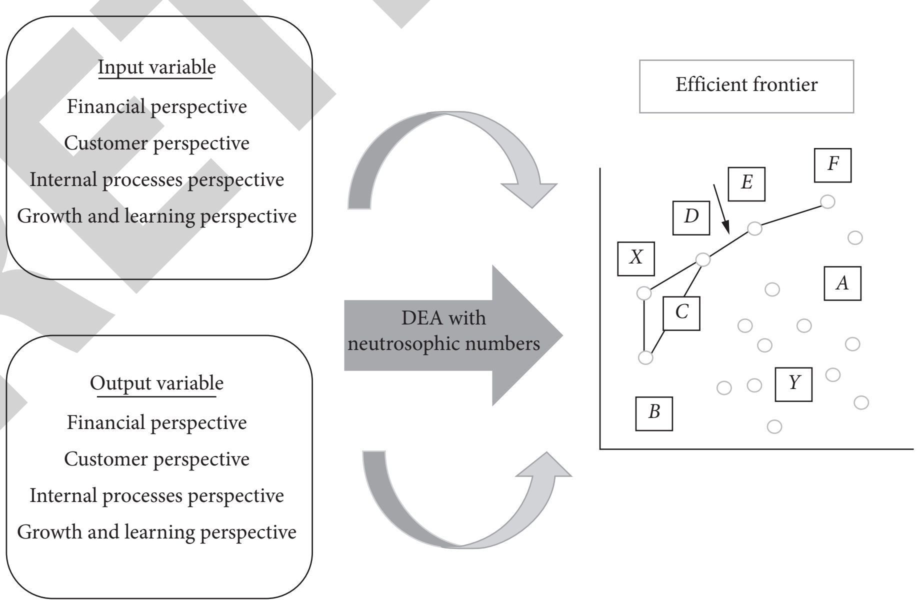

Figure 3: Hybrid BSC-DEA model with neutrosophic numbers.

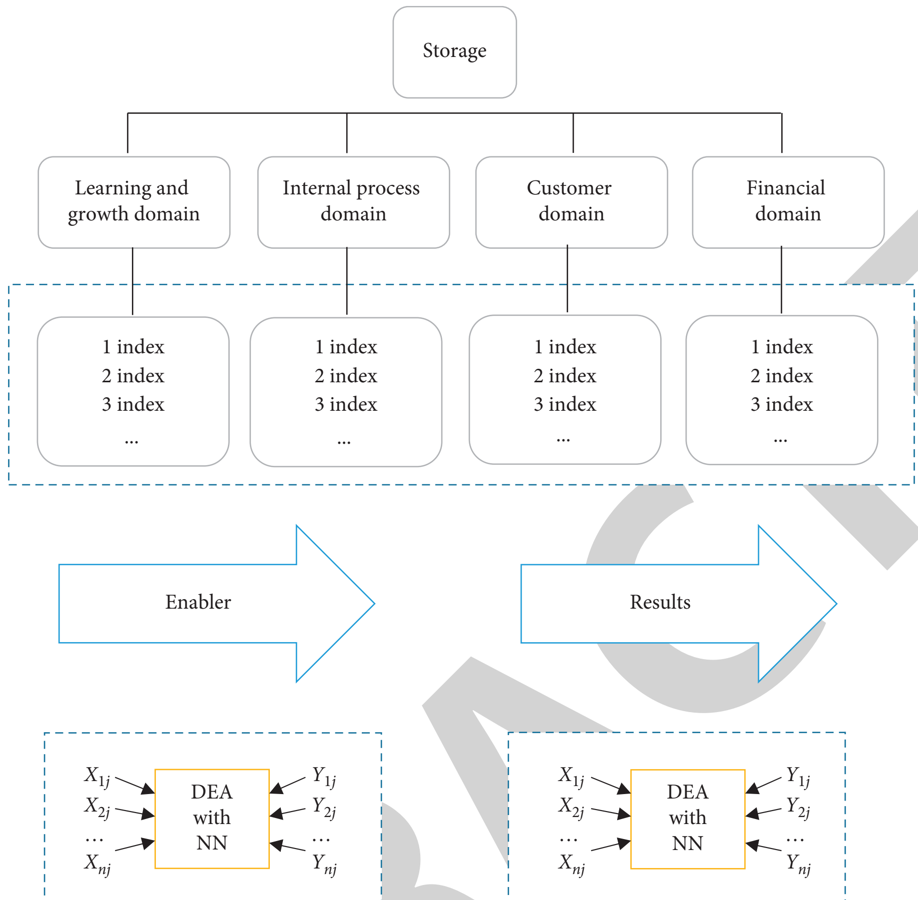

FIGURE 4: A hybrid BSC-DEA strategy with neutrosophic numbers.

A number of these indexes could be inputs and or outputs. Following Figure 4, we have defined two areas of "Enablers" and "Results."

The *enabler's field*, which consists of the two parts of "learning and growth" and "internal process," is an area in which every organization should invest in order to build strong and motivated personnel alongside a secure process. The *results' field* which includes "customer" and "financial" is the indicator that provides the benefit of customers to realize the financial goals. There is no doubt that the profit and loss of the organization are determined in the field of *results*, but creating a margin of profit is surely the yield of efforts and investment in the field of *enablers*.

The steps of indexing and performance evaluation using the two techniques BSC and DEA can be expressed as follows:

(i) Organization identification: at this stage, the goals and strategies of the organization are identified, and using the BSC technique, indicators are created in a balanced manner with different perspectives.

(ii) Performance evaluation: the indicators evaluated by the BSC are classified in two areas of enablers and results, and each area is classified into both input and output groups and used by the neutrosophic DEA in horizontal assessment (overtime periods) or vertical evaluation (in comparison with similar DMUs).

(iii) The correction and improvement of path design: by the neutrosophic DEA, the path to correction and improvement for each indicator is determined.

(iv) Determination of target goals for the next period.

The goals of the indicators that set by the neutrosophic DEA are set as the goals of the indicators for the next period of BSC implementation. In this method, each time the BSC is executed, *i.e.*, at each period when the organization's data are entered into the BSC system, and the results are presented, the neutrosophic DEA model assesses the organization, and the objectives of the indices are determined in the next period. If the goals are met, the organization will achieve the desired and expected efficiency. In the next period of performance evaluation, the organization's condition is compared with the expected conditions from the previous period and the new efficiency is determined. This method is executed periodically, and after each implementation, the manager is expected to lead the organization into the desired optimal efficiency.

5. Numerical Experiment

In Section 4, phases that must be considered in the designing of a hybrid BSC and DEA system were explained. In this section, a case study of this combined system, which has been executed on 20 branches of one of the Iranian banks, will be described. It is worth emphasizing that, due to the privacy policies, the names of these branches are not shared. Furthermore, for each branch of the bank, we gather the related data from the records unit of the bank, the statistical center of Iran, the reliable library, online resources, and the judgments of some experts. After collecting data, we found that the information is sometimes inconsistent, indeterminate, and incomplete. The investigation revealed that several reforms of the mentioned bank and other issues have led to considerable uncertainty and indeterminacy about the data. As a result, we identified them as NNs. According to the presented model, information has been gathered in the two domains of the enablers and the results. Tables 1 and 2 show the indicators and their information in these two domains. As can be seen, the percentage of banking services and the growth rate of services have grown dramatically.

According to Algorithm 1, we can obtain the relative efficiency of DMUs. In this paper, we consider $\lambda = 0, 0.5, 1$ and $I = [0,1.2]$. For example, in the enablers' stage, the relative efficiency of DMU1 can be used as follows: $\theta_1^* = \max z_1 = (47 + 3I)u_1 + (3.5 + 2I)u_2 + (83.5 + I)u_3$,

$$
\begin{aligned}
&\text{s.t}: \\
&(12 + 2I)v_1 + 12.2v_2 + (50 + I)v_3 = 1, \\
&(47 + 3I)u_1 + (3.5 + 2I)u_2 + (83.5 + I)u_3 - \\
&(12 + 2I)v_1 - 12.2v_2 - (50 + I)v_3 \leq 0, \\
&(45 + I)u_1 + (3.4 + 4I)u_2 + (80.4 + 3I)u_3 - \\
&(63 + 3I)v_1 - (10.6 + I)v_2 - 16.2v_3 \leq 0, \\
&(43.8 + 3I)u_1 + 6.2u_2 + (80.7 + I)u_3 - \\
&(14 + I)v_1 - 10.7v_2 - (65.5 + 5I)v_3 \leq 0, \\
&(65 + 2I)u_1 + (8.5 + I)u_2 + 93u_3 - \\
&(15.3 + 6I)v_1 - (11.6 + I)v_2 - 32.5v_3 \leq 0, \\
&(40.4 + I)u_1 + (5.2 + 2I)u_2 + 84.6u_3 - \\
&(17.8 + I)v_1 - (11.3 + I)v_2 - (38 + 3I)v_3 \leq 0, \\
&(65.8 + I)u_1 + (3.7 + 2I)u_2 + (88.2 + I)u_3 - \\
&(14.8 + I)v_1 - 10.6v_2 - (38 + 3I)v_3 \leq 0, \\
&(47.3 + 3I)u_1 + (8.5 + 2I)u_2 + (91.1 + 4I)u_3 - \\
&18.4v_1 - (13.2 + I)v_2 - 67v_3 \leq 0, \\
&(55.4 + 4I)u_1 + (8.2 + 2I)u_2 + (83.6 + I)u_3 - \\
&16.9v_1 - 12.5v_2 - (65.8 + 2I)v_3 \leq 0, \\
&(58 + I)u_1 + 3.7u_2 + (76 + 4I)u_3 - \\
&(21.5 + I)v_1 - 11.9v_2 - 73.5v_3 \leq 0, \\
&(54.7 + I)u_1 + (8.4 + 4I)u_2 + (79.8 + 5I)u_3 - \\
&12.2v_1 - (10.5 + 2I)v_2 - (65 + I)v_3 \leq 0, \\
&(69.2 + 6I)u_1 + (4.6 + I)u_2 + (96.3 + 3I)u_3 - \\
&(19.7 + 4I)v_1 - (10.7 + 3I)v_2 - (60 + I)v_3 \leq 0, \\
&(64 + I)u_1 + (4.3 + I)u_2 + (94 + I)u_3 - \\
&(15.3 + 2I)v_1 - 12.2v_2 - 87v_3 \leq 0, \\
&(58.3 + 2I)u_1 + (5.9 + 3I)u_2 + (96.2 + 2I)u_3 - \\
&(18.7 + I)v_1 - 14v_2 - (71.2 + I)v_3 \leq 0, \\
&(55.7 + 2I)u_1 + (5.5 + 5I)u_2 + (81 + I)u_3 - \\
&(25 + 4I)v_1 - (14 + 3I)v_2 - (78.6 + 2I)v_3 \leq 0, \\
&47u_1 + (9.4 + 3I)u_2 + (84.1 + 5I)u_3 - \\
&(19.3 + 3I)v_1 - 10.8v_2 - (65.6 + 2I)v_3 \leq 0, \\
&(67.1 + I)u_1 + (5.3 + 4I)u_2 + (85.7 + I)u_3 - \\
&(18 + I)v_1 - (12 + 5I)v_2 - 72.8v_3 \leq 0, \\
&(59.5 + 2I)u_1 + (6.9 + 6I)u_2 + (90 + 4I)u_3 - \qquad (6) \\
&(17.8 + 4I)v_1 - (11 + 4I)v_2 - 64v_3 \leq 0, \\
&(65.3 + I)u_1 + (7 + I)u_2 + (86.4 + I)u_3 - \\
&(20 + 2I)v_1 - (11 + I)v_2 - (62 + I)v_3 \leq 0, \\
&(49 + I)u_1 + (4.1 + I)u_2 + (91.4 + 2I)u_3 - \\
&(22 + 2I)v_1 - (12.5 + 3I)v_2 - (74.5 + I)v_3 \leq 0, \\
&(65 + I)u_1 + (6.4 + 2I)u_2 + (95 + I)u_3 - \\
&(15.4 + I)v_1 - (18 + 4I)v_2 - (70 + I)v_3 \leq 0, \\
&I - 1.2\lambda = 0, \\
&u_1, u_2, u_3, , v_1, v_2, v_3 \geq 0.
\end{aligned}
$$

Now, by solving above problem, we can see that, for all values of λ, the relative efficiency of DMU1 is one. Furthermore, the relative efficiency of all DMUs for $\lambda = 0, 0.5, 1$ and $I = [0,1.2]$ was calculated, and the results are obtained in Tables 3 and 4 .

For better understanding, in Figure 5, we show the relative efficiency of DMUs for $I = [0,1.2]$ and different λ.

Form Tables 3–4 and Figure 5, it can be seen that obtaining the optimal results depends on the investment and effort in the enablers sector; that is, until the "learning and growth" and "internal processes" sections do not work well, gaining success is undoubtedly impossible. However, for the success of an organization, planning should be done in the two areas of enablers and results, but it can be clearly stated that the "results" sector requires appropriate measures in the field of enablers. In other words, efficiency in the field of results depends on the efficiency of enablers. Looking at Figure 5, we can infer the following:

(i) The DMUs 1,4,10, and 15, which were efficient in enablers sector, were also able to be efficient in the results section, using the capabilities they gained. It can be said that the efficiency condition in the field of results is efficiency in enablers sector.

(ii) Other DMUs that were not efficient in enablers sector could not be efficient in the results.

(iii) The DMUs 3, 5, 12, 16, 17, 18, and 20, despite the great efforts and obtaining privileges close to the efficient DMUs in the field of results, could not be efficient due to weaknesses in the enablers sector. It can be predicted that these DMUs will be efficient in the results sector if they are efficient in the field of

Table 1: Input and output of enablers.

DMUs	Inputs				Outputs	
	Motivational costs (%)	Increasing expertise of employees (%)	Employee satisfaction (%)	Banking services (%)	Improvement of computer software (%)	Increasing speed of service (%)
1	$12+2I$	12.2	$50+I$	$47+3I$	$3.5+2I$	$83.5+I$
2	16.2	$10.6+I$	$63+3I$	$45+I$	$3.4+4I$	$80.4+3I$
3	$14+I$	10.7	$65.5+5I$	$43.8+3I$	6.2	$80.7+I$
4	$15.3+6I$	$11.6+I$	32.5	$65+2I$	$8.5+I$	93
5	$17.8+I$	$11.3+I$	$38+3I$	$40.4+I$	$5.2+2I$	84.6
6	$14.8+I$	10.6	$78+I$	$65.8+I$	$3.7+2I$	$88.2+I$
7	18.4	$13.2+I$	67	$47.3+3I$	$8.5+2I$	$91.1+4I$
8	16.9	12.5	$65.8+2I$	$55.4+4I$	$8.2+2I$	$83.6+I$
9	$21.5+I$	11.9	73.5	$58+I$	3.7	$76+4I$
10	12.2	$10.5+2I$	$65+I$	$54.7+I$	$8.4+4I$	$79.8+5I$
11	$19.7+4I$	$10.7+3I$	$60+I$	$69.2+6I$	$4.6+I$	$96.3+3I$
12	$15.3+2I$	12.2	87	$64+I$	$4.3+I$	$94+I$
13	$18.7+I$	14	$71.2+I$	$58.3+2I$	$5.9+3I$	$96.2+2I$
14	$25+4I$	$14+3I$	$78.6+2I$	$55.7+2I$	$5.5+5I$	$81+I$
15	$19.3+3I$	10.8	$65.6+2I$	47	$9.4+3I$	$84.1+5I$
16	$18+I$	$12+5I$	72.8	$67.1+I$	$5.3+4I$	$85.7+I$
17	$17.8+4I$	$11+4I$	64	$59.5+2I$	$6.9+6I$	$90+4I$
18	$20+2I$	$11+I$	$62+I$	$65.3+I$	$7+I$	$86.4+I$
19	$22+2I$	$12.5+3I$	$74.5+I$	$49+I$	$4.1+I$	$91.4+2I$
20	$15.4+I$	$18+4I$	$70+I$	$65+I$	$6.4+2I$	$95+I$

Table 2: Input and output of results part.

DMUs	Inputs		Outputs		
	Improvement of operational processes (%)	Customer acquisition rate (%)	Customer satisfaction (%)	Profit margin (%)	Returns to investment (%)
1	$4.2+I$	$20+2I$	$41+I$	$6.5+6I$	$6.9+4I$
2	$5.5+I$	$23.1+2I$	$31.4+I$	$5.8+4I$	$5.6+3I$
3	$7.6+3I$	$21.4+I$	47.1	$4.9+6I$	$7.5+3I$
4	3.3	$22.6+2I$	$46.4+I$	$6.4+7I$	$7.9+3I$
5	$5.4+2I$	$17.9+3I$	$29+I$	$4.8+4I$	$4+2I$
6	4.5	$28.1+2I$	43.2	$7.5+6I$	$8.9+2I$
7	7.2	20.9	$36.8+I$	$4.4+3I$	$6.1+2I$
8	$5.4+I$	$18.7+2I$	$39.4+2I$	$5.7+2I$	$3.8+3I$
9	$6.5+I$	$28.4+2I$	$54.3+I$	$5.4+3I$	$5.7+2I$
10	$5.8+I$	$17.6+2I$	$42+2I$	$6.1+8I$	$7.5+3I$
11	$6.2+I$	$23+4I$	$36.6+6I$	$5.9+3I$	$8.1+2I$
12	4.6	$19.4+I$	$51.3+2I$	$5.8+3I$	$6.2+I$
13	$5.9+I$	$24+2I$	49.2	$6.2+3I$	$5.1+4I$
14	7.1	$46.2+2I$	57.5	$4.5+I$	$3.5+I$
15	6	$27.2+I$	22.6	$8+I$	$6.8+3I$
16	$3.5+3I$	$22.4+2I$	$47.5+I$	$5.8+2I$	$7.4+3I$
17	$5.1+I$	$17.3+I$	$45.3+I$	$5.6+I$	$4.6+2I$
18	$4.4+3I$	$21.7+I$	$34.6+2I$	$6.6+I$	$7.1+2I$
19	$46+I$	$28.4+I$	$48.9+4I$	$6.9+4I$	$7+I$
20	$4+2I$	$18+2I$	$52.2+2I$	$5.5+I$	$6.3+3I$

enablers. Meanwhile, the behaviour of the DMU18 is interesting.

(iv) Before calculations, it was anticipated that inefficiencies in enablers sector would have the most ineffective outcomes in the results sector. These contents are seen in DMUs 8, 9, 13, 14, 17, and 18 with the efficiency level in the field of results, lower than the efficiency in the enablers sector.

To validate the proposed efficiencies, the proposed efficiencies are compared with the efficiencies of crisp CCR (model (3), or in our model when $I(\lambda)=0$) that are given in Figure 5. In this figure, the efficiencies of DMUs are found to be smaller by our model compared to crisp CCR. It is interesting that DMU11 and DMU18 are efficient in crisp DEA, but they are inefficient using a new model. Therefore, the new neutrosophic DEA is more realistic rather than crisp CCR. Also, crisp CCR and the new neutrosophic DEA may

Step 1. Consider the DEA model that the inputs and outputs of each DMU are neutrosophic numbers.
Step 2. Using NN function, transform model (2) into the following model:

$$\theta_p^* = \max f_p(u, I),$$
s.t :
$$g_p(v, I) = 1,$$
$$f_j(u, I) - g_j(v, I) \le 0, \quad j = 1, \ldots, n,$$
$$u = [u_1, \ldots, u_s]^T,$$
$$v = [v_1, \ldots, v_m]^T \ge 0,$$

where $f_j(u, I) = \sum_{r=1}^{s} u_r(y_{rj} + \gamma_{rj} I)$and $g_j(v, I) = \sum_{i=1}^{m} v_i(x_{ij} + \eta_{ij} I)$.

Step 3. Consider $I \subseteq [\inf(I), \sup(I)]$ and using de-neutrosophication model of [77]; for $\lambda \in [0,1]$, set $I(\lambda) = (1-\lambda)\inf(I) + \lambda \sup(I)$; then, for $I(\lambda) \in [0,1]$, transform model (3) into the following model:

$$\theta_p^* = \max f_p(u, I(\lambda))$$
s.t :
$$g_p(v, I(\lambda)) = 1,$$
$$f_j(u, I(\lambda)) - g_j(v, I(\lambda)) \le 0, \quad j = 1, \ldots, n,$$
$$u = [u_1, \ldots, u_s]^T,$$
$$v = [v_1, \ldots, v_m]^T \ge 0.$$

Step 4. Obtain the corresponding optimal solutions of $u$and v for $I(\lambda) = 0, 0.5, 1$ that are considered as the minimum, the moderate, and the maximum indeterminacy, respectively, in the DEA problem (3).

Algorithm 1

Table 3: The relative efficiency of DMUs for enablers' stage.

DMUs		λ	
	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
DMU1	1.0000	1.0000	1.0000
DMU2	0.8936	0.9140	0.9370
DMU3	0.9400	0.9534	0.9522
DMU4	1.0000	1.0000	1.0000
DMU5	0.9115	0.9196	0.9445
DMU6	1.0000	1.0000	1.0000
DMU7	0.8624	0.8987	0.9356
DMU8	0.8496	0.9072	0.9614
DMU9	0.7560	0.8137	0.8345
DMU10	1.0000	1.0000	1.0000
DMU11	1.0000	1.0000	0.9476
DMU12	0.9773	0.9622	0.9487
DMU13	0.8480	0.8984	0.9344
DMU14	0.6766	0.6831	0.6852
DMU15	1.0000	1.0000	1.0000
DMU16	0.9156	0.8867	0.9050
DMU17	0.9788	0.9090	0.9716
DMU18	1.0000	0.9983	0.9556
DMU19	0.8238	0.8060	0.7681
DMU20	0.9569	0.9622	0.9659

Table 4: The relative efficiency of DMUs for results' stage.

DMUs	λ		
	$\lambda = 0$	$\lambda = 0.5$	$\lambda = 1$
DMU1	1.0000	1.0000	1.0000
DMU2	0.8103	0.8477	0.8657
DMU3	0.8571	0.8898	0.9065
DMU4	1.0000	1.0000	1.0000
DMU5	0.8361	0.8403	0.8569
DMU6	1.0000	1.0000	1.0000
DMU7	0.7932	0.8100	0.7930
DMU8	0.9102	0.6600	0.6595
DMU9	0.5880	0.5434	0.5151
DMU10	1.0000	1.0000	1.0000
DMU11	1.0000	0.9078	0.7900
DMU12	0.9062	0.7606	0.7104
DMU13	0.7856	0.6722	0.7220
DMU14	0.3990	0.3264	0.2853
DMU15	1.0000	1.0000	1.0000
DMU16	0.9337	0.8832	0.8368
DMU17	0.9512	0.6678	0.6853
DMU18	1.0000	0.9133	0.8574
DMU19	0.7699	0.7066	0.6278
DMU20	0.9569	0.9080	0.8919

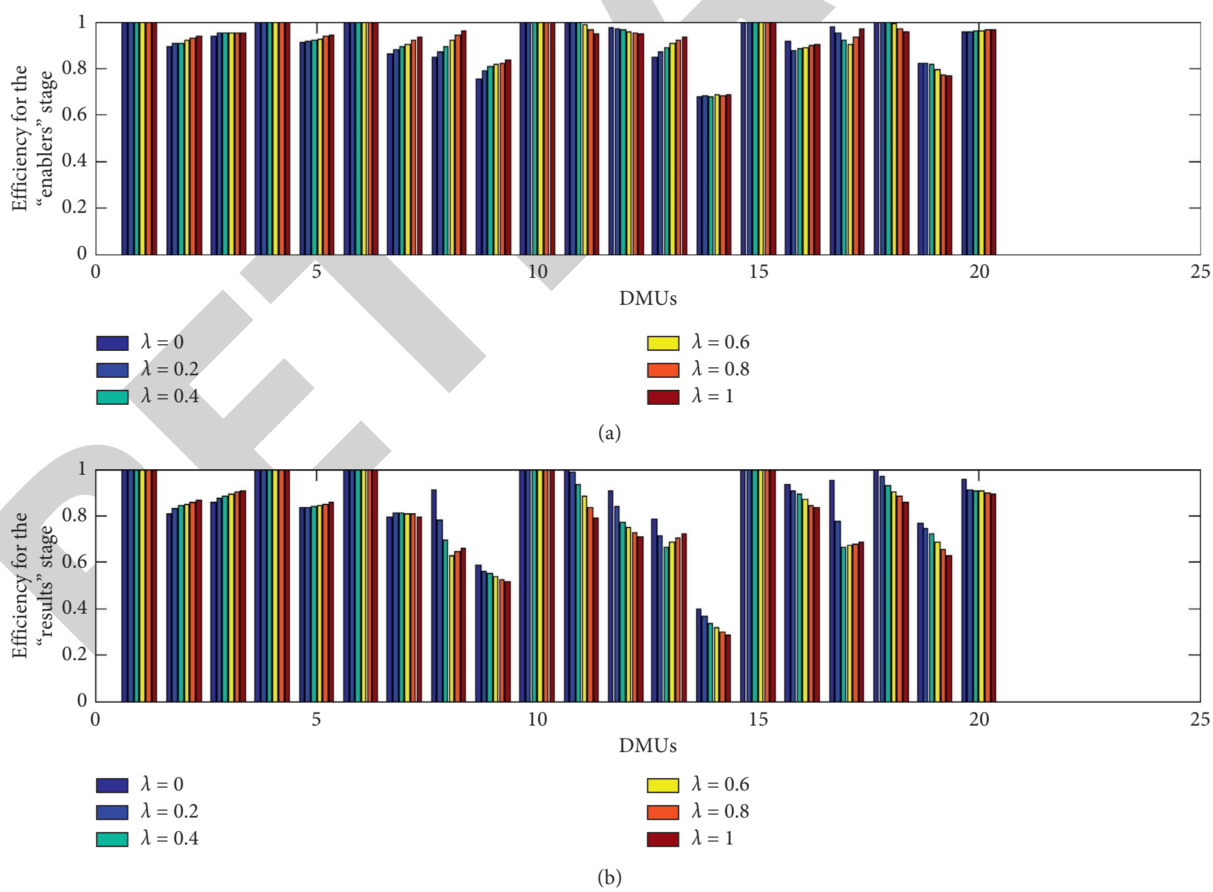

Figure 5: The relative efficiency of DMUs for enablers and results stages.

give the same efficiencies for certain data. However, crisp CCR model does not deal with the uncertain, indeterminate, and incongruous information. Therefore, the new model is more efficient rather than crisp CCR.

6. Conclusions

Specifying the various performance evaluation models and the appropriate utilization of these models in organizations is a crucial issue. In this paper, we proposed a new model of DEA in neutrosophic number environment and combined this model with BSC to rank different decision-making units. Finally, the proposed method is illustrated by an empirical study involving 20 banking branches. The results provide a more realistic framework and consider various aspects of indeterminate information. Moreover, although the new model and results presented here demonstrate the effectiveness of our approach, it could also be considered in other types of DEA models such as network DEA and their applications to banks, supplier selection, police stations, hospitals, tax offices, prisons, schools, and universities. However, developing data envelopment analysis models based on the plithogenic set, which is a generalization of neutrosophic set, and other perspectives of neutrosophic set is another area for further studies. As future researches, we intend to study these problems.

Abbreviations

BSC:	Balanced scorecard
DEA:	Data envelopment analysis
DMU:	Decision-making units
CCR model:	Charnes, Cooper, and Rhodes model
BCC model:	Banker, Charnes, and Cooper model
MAGDM:	Multiple attribute group decision-making
JRC:	Joint roughness coefficient
FS:	Fuzzy set
IFS:	Intuitionistic fuzzy set
NS:	Neutrosophic set
NN:	Neutrosophic number
NLN:	Neutrosophic linguistic number.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Retraction

Retracted: Correlation Coefficient and Entropy Measures Based on Complex Dual Type-2 Hesitant Fuzzy Sets and Their Applications

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Manipulated or compromised peer review

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] T. Mahmood, Z. Ali, H. Garg, L. Zedam, and R. Chinram, "Correlation Coefficient and Entropy Measures Based on Complex Dual Type-2 Hesitant Fuzzy Sets and Their Applications," *Journal of Mathematics*, vol. 2021, Article ID 2568391, 34 pages, 2021.

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Research Article

Correlation Coefficient and Entropy Measures Based on Complex Dual Type-2 Hesitant Fuzzy Sets and Their Applications

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The theory of complex dual type-2 hesitant fuzzy sets (CDT-2HFSs) is a blend of two different modifications of fuzzy sets (FSs), called complex fuzzy sets (CFSs) and dual type-2 hesitant fuzzy sets (DT-2HFSs). CDT-2HFS is a proficient technique to cope with unpredictable and awkward information in realistic decision problems. CDT-2HFS is composed of the grade of truth and the grade of falsity, and the grade of truth (also for grade of falsity) contains the grade of primary and secondary parts in the form of polar coordinates with the condition that the sum of the maximum of the real part (also for the imaginary part) of the primary grade (also for the secondary grade) cannot exceed the unit interval [0, 1]. The aims of this manuscript are to discover the novel approach of CDT-2HFS and its operational laws. These operational laws are also justified with the help of an example. Additionally, based on a novel CDT-2HFS, we explored the correlation coefficient (CC) and entropy measures (EMs), and their special cases are also discussed. TOPSIS method based on CDT-2HFS is also explored. Then, we applied our explored measures based on CDT-2HFSs in the environment of the TOPSIS method, medical diagnosis, pattern recognition, and clustering algorithm to cope with the awkward and complicated information in realistic decision issues. Finally, some numerical examples are given to examine the proficiency and validity of the explored measures. Comparative analysis, advantages, and graphical interpretation of the explored measures with some other existing measures are also discussed.

1. Introduction

The present decision-making is one of the genuinely basic movements in individuals' everyday life, the reason for existing of which is to rank the limited arrangement of options regarding that they are so solid to the choice maker(s). Multiattribute decision-making (MADM) is a part of decision-making and is viewed as an intellectual-based human movement. People unavoidably are confronted with different decision-making issues, which include numerous fields [1–3]. The idea of the fuzzy set (FS) proposed by Zadeh [4] modified the method of measuring the vulnerability/fuzziness. Before the development of the FS hypothesis by Zadeh [4], the likelihood hypothesis was the customary instrument to quantify the vulnerability. Be that as it may, to gauge the vulnerability utilizing likelihood, it ought to have been communicated as exact numbers which are its primary constraints. The obscure terms, for instance, "without doubt" and "marginally," could not be measured utilizing the likelihood hypothesis. To gauge the vulnerability/fuzziness related to such unclear terms, the FS hypothesis has ended up being a successful apparatus. In the FS hypothesis, every component relating to a specific universe of talk has been appointed an enrollment degree lying somewhere in the range of 0 and 1, which indicates its level of belongingness to the set being referred to called FS. By the goodness of its reasonableness in genuine issues, FSs increased a lot of prevalence with analysts around the world. Endeavors were made to additionally sum up the idea by numerous creators to make it progressively versatile for viable issues.

Notwithstanding, in certain issues including etymological factors, for example, exceptionally low, low, medium, high, and extremely high, the assurance of the participation capacity may not be simple; that is, in an issue, dubious participation capacity might be experienced. To survive such circumstances, the idea of type-2 FSs (T-2FSs) was presented by Zadeh [5], as a distinction from common FSs. Many researchers have utilized T-2FSs in different areas [6–8]. The tale structures, which are speculations and expansions of the FSs, have been proposed by numerous analysts since Zadeh presented the FSs. The fundamental motivation behind these structures is to take out vulnerabilities and to guarantee that specialists settle on choices in a way that is without blunder or with not many mistakes. One of these structures is the idea of hesitant FS (HFS) characterized by Torra [9]. Feng et al. [10] presented the type-2 hesitant fuzzy set (T-2HFS). The idea of dual HFS (DHFS) was first characterized as a speculation of the HFSs characterized by Zhu et al. [11]. A DHFS is distinguished as two distinct capacities called enrollment and nonmembership capacity. This structure permits the leader to make more adaptable, precise, and reasonable remarks about the components under the reluctant zone. In this manner, it limits the blunder edge by giving more solid outcomes than the current structures, as HFSs and interval-valued HFSs. Alcantud et al. [12] characterized the idea of the double broadened HFSs and applied it to a decision-making issue under dual extended hesitant fuzzy data.

As for the above existing examinations, it has been dissected that they have researched the decision-making issues under the FS, IFS, or its speculations, which are just ready to manage the vulnerability and dubiousness existing in the information. These models cannot speak to the fractional obliviousness of the information and its changes at a given period of time. Be that as it may, in complex informational collections, vulnerability and ambiguity in the information happen simultaneously with changes to the stage (periodicity) of the information. Instances of complex informational indexes incorporate a lot of information that is created from clinical research, just as government databases for biometric and facial acknowledgments, sound, and pictures, all of which may contain a lot of deficient, dubious, and ambiguous data. To deal with these kinds of issues, the theory of complex FS (CFS) was discovered by Ramot et al. [13]. CFS contains the grade of membership in the form of a complex number belonging to a unit disc in a complex plane. Various scholars utilized CFS in different fields [14–16].

Correlation examination shows a direct connection between two sets and it has a very significant spot for dynamics. In this way, numerous researchers in various fields have considered the relationship coefficients. Additionally, the FS and its speculations have a significant job in dynamics, so CCs have drawn in the consideration of scientists examining the FS and its speculations. For instance, Chiang and Lin [17] and Chaudhuri and Bhattacharya [18] examined the correlation between two FSs. Gerstenkorn and Mańko [19] worked the relationship and CC of the intuitionistic FSs (IFSs). The entropy of FSs is a proportion of fuzziness between FSs. De Luca and Termini [20] first presented the aphorism development for the entropy of FSs concerning Shannon's likelihood entropy. Yager [21] characterized fuzziness proportions of FSs as far as a need of differentiation between the FS and its nullification based on Lp standard. Kosko [22] gave a proportion of fuzziness between FSs utilizing a proportion of separation between the FS and its closest set to the separation between the FS and its farthest set. Xuecheng [23] gave some aphorism definitions of entropy and furthermore characterized a σ-entropy. Pal and Pal [24] proposed exponential entropy. Meanwhile Fan and Ma [25] gave some new fuzzy entropy equations. The technique for establishing order preference by similarity to the ideal solution (TOPSIS) technique as a strategy for building up request inclination by likeness to the perfect arrangement, started by Hwang and Yoon [26], is one of the best and beneficial methods for decision-making. The basic idea of TOPSIS strategy is to pick the elective that has the briefest good way from the positive perfect arrangement (PIS) and the greatest good way from the negative perfect arrangement (NIS). There exists a tremendous writing including study and utilization of TOPSIS hypothesis in a wide scope of MCDM just as multicriteria group decision-making (MCGDM) issues [27–29].

Dual type-2 hesitant fuzzy set contains the grade of truth and the grade of falsity in the form of the subset of the unit interval with the condition that the sum of the maximum of the truth grade and the maximum of the falsity grade cannot exceed the unit interval. The complex dual type-2 hesitant fuzzy set is a generalization of the dual type-2 hesitant fuzzy set, in which the amplitude term provides the extent of belonging of an object, while the phase term describes the periodicity. These phase terms distinguish the complex dual type-2 hesitant fuzzy set from the traditional dual type-2 hesitant fuzzy set theories. In dual type-2 hesitant fuzzy set theory, the data are managed with the compensation of only the degree of belonging, while the part of periodicity is completely ignored. Hence, this may result in the loss of information during the decision-making processes in some certain cases. To further illustrate the concept of phase terms, we take an example. Suppose that a person wants to purchase a car under crucial factors such as its model and its production date. Since the model of each car moves with the evolution of the production dates, to make a selection or decision regarding choosing the optimal car is a decision-making process taking these two factors into account simultaneously. Moreover, it is quite obvious that such types of problems cannot be modeled accurately with traditional theories. However, complex dual type-2 hesitant fuzzy set theory is well suited for such classes of problems, where the amplitude terms may be used to provide a decision about the model of a car, while the phase term concerns its production dates. Henceforth, a complex dual type-2 hesitant fuzzy set is a more generalized continuation of the existing theories, such as type-2 hesitant fuzzy sets and dual type-2 hesitant fuzzy sets.

When a decision-maker gives $(0.4e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.2)})$ and $(0.41e^{i2\pi(0.31)}, 0.31e^{i2\pi(0.21)})$ for the grade of complex-valued supporting and the grade of complex-valued supporting against in the form of primary and secondary information with the condition that the sum of the maximum of the real part (also for the imaginary part) of the complex-valued supporting (also for supporting against) grade for primary (also for secondary) information cannot exceed the unit interval, There exist notions like FSs, T-2FSs, HFSs, DHFSs, CFSs, and DT-2HFSs. Handling such kind of issues is very difficult, but when a decision-maker provides such kind of information in the form of the finite subset of unit interval, then it is very complicated for a decision-maker to handle it. For coping with such kind of issues, in this manuscript, the novel approach of CDT-2HFS, which is a mixture of two different modifications of FS, that is, CFS and DT-2HFS, is explored. CDT-2HFS is a proficient technique to cope with unpredictable and awkward information in realistic decision problems. CDT-2HFS composes the grade of truth and the grade of falsity, and the grade truth (also for falsity grade) contains the grade of primary and secondary parts in the form of polar coordinates with the condition that the sum of the maximum of the real part (also for the imaginary part) of the primary grade (also for the secondary grade) cannot exceed the unit interval. The aims of this manuscript are to discover the novel approach of CDT-2HFS and its operational laws. These operational laws are also justified with the help of an example. Additionally, based on a novel CDT-2HFS, we explored the correlation coefficient (CC) and entropy measures (EMs), and their special cases are discussed. TOPSIS method based on CDT-2HFS is also explored. Then, we applied our explored measures based on CDT-2HFSs in the environment of the TOPSIS method, medical diagnosis, pattern recognition, and clustering algorithm to cope with awkward and complicated information in realistic decision issues. Finally, four numerical examples are resolved to examine the proficiency and validity of the explored measures. Comparative analysis, advantages, and graphical interpretation of the explored measures with some other existing measures are also discussed.

The aims of this manuscript are summarized as follows: in Section 2, we review some basic notions like FSs, T-2FSs, HFSs, DHFSs, CFSs, and their basic laws. In Section 3, the theory of CDT-2HFS, which is a mixture of two different modifications of FS, that is, CFS and DT-2HFS, is presented. CDT-2HFS is a proficient technique to cope with unpredictable and awkward information in realistic decision problems. CDT-2HFS is composed of the grade of truth and the grade of falsity, and the grade truth (also for falsity grade) contains the grade of primary and secondary parts in the form of polar coordinates with the condition that the sum of the maximum of the real part (also for the imaginary part) of the primary grade (also for secondary grade) cannot exceed the unit interval. The aims of this manuscript were to discover the novel approach of CDT-2HFS and its operational laws. These operational laws are also justified with the help of examples. In Sections 4 and 5, based on a novel CDT-2HFS, we explored the correlation coefficient (CC) and entropy measures (EMs), and their special cases are discussed. In Section 6, TOPSIS method based on CDT-2HFS is also explored. Then, we applied our explored measures based on CDT-2HFSs in the environment of TOPSIS method, medical diagnosis, pattern recognition, and clustering algorithm to cope with awkward and complicated information in realistic decision issues. Finally, four numerical examples are resolved to examine the proficiency and validity of the explored measures. Comparative analysis, advantages, and graphical interpretation of the explored measures with some other existing measures are also discussed. The conclusion of this paper is discussed in Section 7.

2. Preliminaries

Basic notions of FSs, T-2FSs, HFSs, DHFSs, CFSs, and their operational laws are briefly reviewed in this study. Throughout this manuscript, the symbol $\mathcal{X}_{\mathrm{UNI}}$ denotes the fixed set.

Definition 1 (see [4]). A FS is an object of the form

$$\mathcal{Q}_{\mathrm{FS}} = \left\{\left(\tilde{x}, M_{\mathcal{Q}_{\mathrm{FS}}}(\tilde{x})\right): \quad \tilde{x} \in \mathcal{X}_{\mathrm{UNI}}\right\}, \tag{1}$$

where $M_{\mathcal{Q}_{\mathrm{FS}}}$ represents the grade of supporting with the condition that $0 \le M_{\mathcal{Q}_{FS}} \le 1$.

Definition 2 (see [5]). A T-2FS is an object of the form

$$\mathcal{Q}_{\mathrm{T-2FS}} = \left\{\left((\tilde{x}, \tilde{x}'), M_{\mathcal{Q}_{t-2\mathrm{FS}}}(\tilde{x}, \tilde{x}')\right): \quad \forall \tilde{x} \in \mathcal{X}_{\mathrm{UNI}}, \tilde{x}' \in J_{\tilde{x}} \subseteq [0,1]\right\}, \tag{2}$$

where $M_{\mathcal{Q}_{t-2\mathrm{FS}}}(\tilde{x}, \tilde{x}')$ represents the grade of type-2 supporting with the condition that $0 \le M_{\mathcal{Q}_{t-2\mathrm{FS}}}(\tilde{x}, \tilde{x}') \le 1$.

Definition 3 (see [9]). A HFS is an object of the form

$$\mathcal{Q}_{\mathrm{HFS}} = \left\{\left(\tilde{x}, M_{\mathcal{Q}_{\mathrm{HFS}}}(\tilde{x})\right): \quad \tilde{x} \in \mathcal{X}_{\mathrm{UNI}}\right\}, \tag{3}$$

where $M_{\mathcal{Q}_{\mathrm{HFS}}}$ represents the grade of supporting in the form of the subset of the unit interval, with the condition that $0 \le \mathrm{Max}(M_{\mathcal{Q}_{\mathrm{HFS}}}) \le 1$, whenever $M_{\mathcal{Q}_{\mathrm{HFS}}} \subseteq [0,1]$.

Definition 4 (see [11]). A DHFS is an object of the form

$$\mathcal{Q}_{\mathrm{DHFS}} = \left\{\left(\tilde{x}, M_{\mathcal{Q}_{\mathrm{DHFS}}}(\tilde{x}), N_{\mathcal{Q}_{\mathrm{DHFS}}}(\tilde{x})\right): \quad \tilde{x} \in \mathcal{X}_{\mathrm{UNI}}\right\}, \tag{4}$$

where $M_{\mathcal{Q}_{\mathrm{DHFS}}}$ and $N_{\mathcal{Q}_{\mathrm{DHFS}}}$ represent the grade of supporting and the grade of supporting against with the condition that $0 \le \max(M_{\mathcal{Q}_{\mathrm{DHFS}}}) + \max(N_{\mathcal{Q}_{\mathrm{DHFS}}}) \le 1$, whenever $M_{\mathcal{Q}_{\mathrm{DHFS}}}, N_{\mathcal{Q}_{\mathrm{DHFS}}} \subseteq [0,1]$.

Additionally, we defined some operational laws based on DHFSs. For any two DHFSs $\mathcal{Q}_{\mathrm{DHFS-1}} = (M^{6(1)}_{\mathcal{Q}_{\mathrm{DHFS-1}}}(\tilde{x}), N^{6(1)}_{\mathcal{Q}_{\mathrm{DHFS-1}}}(\tilde{x}))$ and $\mathcal{Q}_{\mathrm{DHFS-1}} = (M^{6(1)}_{\mathcal{Q}_{\mathrm{DHFS-2}}}(\tilde{x}), N^{6(1)}_{\mathcal{Q}_{\mathrm{DHFS-2}}}(\tilde{x}))$, we have

$$\begin{aligned}\mathcal{Q}_{\text{DHFS}-1} \cup \mathcal{Q}_{\text{DHFS}-2} &= \left(\max\left(M^{6(1)}_{\mathcal{Q}_{\text{DHFS}-1}}(\tilde{x}), M^{6(1)}_{\mathcal{Q}_{\text{DHFS}-2}}(\tilde{x})\right), \min\left(N^{6(1)}_{\mathcal{Q}_{\text{DHFS}-1}}(\tilde{x}), N^{6(1)}_{\mathcal{Q}_{\text{DHFS}-2}}(\tilde{x})\right)\right),\\ \mathcal{Q}_{\text{DHFS}-1} \cap \mathcal{Q}_{\text{DHFS}-2} &= \left(\min\left(M^{6(1)}_{\mathcal{Q}_{\text{DHFS}-1}}(\tilde{x}), M^{6(1)}_{\mathcal{Q}_{\text{DHFS}-2}}(\tilde{x})\right), \max\left(N^{6(1)}_{\mathcal{Q}_{\text{DHFS}-1}}(\tilde{x}), N^{6(1)}_{\mathcal{Q}_{\text{DHFS}-2}}(\tilde{x})\right)\right).\end{aligned} \tag{5}$$

Definition 5 (see [13]). A CFS is an object of the form

$$\mathcal{Q}_{\text{CFS}} = \left\{\left(\tilde{x}, M_{\mathcal{Q}_{\text{CFS}}}(\tilde{x})\right): \quad \tilde{x} \in \mathcal{X}_{\text{UNI}}\right\}, \tag{6}$$

where $M_{\mathcal{Q}_{\text{CFS}}} = M_{\mathcal{Q}_{\text{CFRP}}}(\tilde{x})e^{i2\pi(M_{\mathcal{Q}_{\text{CEIP}}}(\tilde{x}))}$ represents the grade of complex-valued supporting with the condition that $0 \leq M_{\mathcal{Q}_{\text{CFRP}}}, M_{\mathcal{Q}_{\text{CFIP}}} \leq 1$.

3. Complex Dual Type-2 Hesitant Fuzzy Sets

Based on the existing drawbacks [30], in this study, we discovered the new theory of CDT-2HFSs and their operational laws. The presented operational laws are also justified with the help of some examples.

Definition 6. A CDT-2HFS is an object of the form

$$\mathcal{Q}_{\text{CDTH}} = \left\{\left(\tilde{x}, \left(M_{\mathcal{Q}_{\text{CDTH}}}(\tilde{x}), N_{\mathcal{Q}_{\text{CDTH}}}(\tilde{x})\right)\right): \quad \tilde{x} \in \mathcal{X}_{\text{UNI}}\right\}. \tag{7}$$

where $M_{\mathcal{Q}_{\text{CDTH}}}(\tilde{x}) = \{(M^{6}_{\mathcal{Q}_{\text{CDTHRP}-P}}(j)(\tilde{x}) \cdot e^{i2\pi(M^{6(j)}_{\mathcal{Q}_{\text{CDTHRP}-P}}(\tilde{x}))}, M^{6(j)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x}) \cdot e^{i2\pi(M^{6(j)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x}))}): \quad j = 1, 2, 3, \ldots, n\}$ and $N_{\mathcal{Q}_{\text{CDTH}}}(\tilde{x}) = \{(N^{6(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}}(\tilde{x}) \cdot e^{i2\pi(M^{6(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}}(\tilde{x}))}, N^{6(k)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x}) \cdot e^{i2\pi(M^{6(k)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x}))}): \quad k = 1, 2, 3, \ldots, m\}$ represent the grade of complex-valued supporting and the grade of complex-valued supporting against in the form of complex type-2 hesitant fuzzy elements (CT-2HFEs) with the following conditions: $0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P}}) + \max N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}} \leq 1, 0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}}) \leq 1, 0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S}}) \leq 1$ and $0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S}}) \leq 1$. The complex dual type-2 hesitant fuzzy set is expressed by

$$\mathcal{Q}_{\text{CDTH}} = \left\{\left(\begin{array}{c}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P}}(\tilde{x}) \cdot e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P}}(\tilde{x})\right)}, M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x}) \cdot e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S}}(\tilde{x})\right)}\right),\\ \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}}(\tilde{x}) \cdot e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P}}(\tilde{x})\right)}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S}}(\tilde{x})e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S}}(\tilde{x})\right)}\right)\end{array}\right), \quad j, k = 1, 2, 3, \ldots, n, m\right\}. \tag{8}$$

Additionally, we defined some operational laws based on CDT-2HFEs. For any two CDT-2HFEs $\mathcal{Q}_{\text{CDTH}-1} = ((M^{\text{б}(1)}_{Q_{\text{CDTHRP}-P-1}}(\tilde{x}) \cdot e^{i2\pi}(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x})), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}) \cdot e^{i2\pi}(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}))), (N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}) \cdot e^{i2\pi(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}))}, N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}) \cdot e^{i2\pi(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}))}))$ and $\mathcal{Q}_{\text{CDTH}-2} = ((M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}) \cdot e^{i2\pi(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}))}, M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}) \cdot e^{i2\pi}(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}))), (N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}) \cdot e^{i2\pi}(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x})), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}) \cdot e^{i2\pi(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}))}))$, we have

$$\mathcal{Q}_{\text{CDTH}-1} \cup \mathcal{Q}_{\text{CDTH}-2} = \left\{\left(\begin{array}{c}\left(\begin{array}{c}\max\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x})\right)e^{i2\pi\left(\max\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x})\right)\right)},\\ \max\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x})\right)e^{i2\pi\left(\max\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x})\right)\right)}\end{array}\right),\\ \left(\begin{array}{c}\min\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x})\right)e^{i2\pi\left(\min\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDHIP}-S-2}}(\tilde{x})\right)\right)},\\ \min\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x})\right)e^{i2\pi\left(\min\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x})\right)\right)}\end{array}\right)\end{array}\right)\right\}, \tag{9}$$

$$\mathcal{Q}_{\text{CDTH}-1} \cap \mathcal{Q}_{\text{CDTH}-2} = \left\{ \left(\begin{pmatrix} \min\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x})\right) e^{i2\pi\left(\min\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x})\right)\right)}, \\ \min\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x})\right) e^{i2\pi\left(\min\left(M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}), M^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x})\right)\right)} \end{pmatrix}, \begin{pmatrix} \max\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x})\right) e^{i2\pi\left(\max\left(N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x})\right)\right)}, \\ \max\left(N^{\text{б}1}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}), N^{\text{б}1}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x})\right) e^{i2\pi\left(\max\left(N^{\text{б}1}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}), N^{\text{б}(1)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x})\right)\right)} \end{pmatrix} \right) \right\}. \tag{10}$$

Example 1. For any two CDT-2HFSs, all their entries in the form of complex numbers are stated as follows:

$$\begin{aligned} \mathcal{Q}_{\text{CDTH}-1} &= \begin{pmatrix} \left\{\left(0.1e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}\right), \left(0.2e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}\right), \left(0.3e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.5)}\right)\right\}, \\ \left\{\left(0.01e^{i2\pi(0.02)}, 0.02e^{i2\pi(0.03)}\right), \left(0.03e^{i2\pi(0.04)}, 0.05e^{i2\pi(0.06)}\right)\right\} \end{pmatrix}, \\ \mathcal{Q}_{\text{CDTH}-1} &= \left\{ \begin{array}{c} \left\{\left(0.7e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.5)}\right), \left(0.6e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.4)}\right), \left(0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.3)}\right)\right\}, \\ \left\{\left(0.07e^{i2\pi(0.08)}, 0.09e^{i2\pi(0.01)}\right), \left(0.01e^{i2\pi(0.03)}, 0.22e^{i2\pi(0.03)}\right)\right\} \end{array} \right\}. \end{aligned} \tag{11}$$

Then, by using equations (9) and (10), we get

$$\begin{aligned} \mathcal{Q}_{\text{CDTH}-1} \cup \mathcal{Q}_{\text{CDTH}-2} &= \begin{pmatrix} \left\{\left(0.7e^{i2\pi(0.6)}, 0.6e^{i2\pi(0.5)}\right), \left(0.6e^{i2\pi(0.5)}, 0.5e^{i2\pi(0.4)}\right), \left(0.5e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.5)}\right)\right\}, \\ \left\{\left(0.01e^{i2\pi(0.02)}, 0.02e^{i2\pi(0.01)}\right), \left(0.01e^{i2\pi(0.03)}, 0.05e^{i2\pi(0.03)}\right)\right\} \end{pmatrix}, \\ \mathcal{Q}_{\text{CDTH}-1} \cap \mathcal{Q}_{\text{CDTH}-2} &= \begin{pmatrix} \left\{\left(0.1e^{i2\pi(0.2)}, 0.2e^{i2\pi(0.3)}\right), \left(0.2e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.4)}\right), \left(0.3e^{i2\pi(0.4)}, 0.4e^{i2\pi(0.3)}\right)\right\}, \\ \left\{\left(0.07e^{i2\pi(0.08)}, 0.09e^{i2\pi(0.03)}\right), \left(0.03e^{i2\pi(0.04)}, 0.22e^{i2\pi(0.06)}\right)\right\} \end{pmatrix}. \end{aligned} \tag{12}$$

The explored notions, which are stated in the form of equations (7), (9), and (10), are more proficient and more modified than the existing drawbacks; for instance, if we choose the imaginary part of equations (7), (9), and (10) to be zero, then equations (7), (9), and (10) convert it for DT-2HFS [30].

4. Correlation Coefficient for Complex Dual Type-2 Hesitant Fuzzy Sets

The aim of this study is to present the novel correlation, correlation coefficient (CC), maximum-based CC (MCC), weighted CC (WCC), and maximum-based WCC (MWCC). The special cases of the explored measures are also explored.

Definition 7. For any two CDT-2HFSs, $\mathcal{Q}_{\text{CDTH}-1} = \{((M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i) \cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i) \cdot e^{i2\pi}(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))), (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i) \cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))})), j, k = 1, 2, 3, \ldots, n, m\}$ and $\mathcal{Q}_{\text{CDTH}-2} = \{((M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i) \cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i))}, M^{\text{б}}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(j)(\tilde{x}_i) \cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i)))}, (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i))}, N^{\text{б}}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(k)(\tilde{x}_i) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i))})), j, k = 1, 2, 3, \ldots, \tilde{l}_{\tilde{x}_i}, \sim l_{\tilde{x}_i}\}$, the correlation is of the form

$$\varsigma_{\mathrm{CDTHF-c}}\left(\mathcal{Q}_{\mathrm{CDTH-1}},\mathcal{Q}_{\mathrm{CDTH-2}}\right)=\frac{1}{4n}\sum_{i=1}^{n}\begin{pmatrix}\frac{1}{\widetilde{l}_{\widetilde{x}_i}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)+\\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-2}}\left(\widetilde{x}_i\right)\right)+\\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)\right)+\\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)\end{pmatrix}. \quad (13)$$

Definition 8. For any two CDT-2HFSs, $\mathcal{Q}_{\mathrm{CDTH-1}}=\left\{\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}(\widetilde{x}_i)\cdot e^{i2\pi}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}(\widetilde{x}_i)\right),M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}(\widetilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-1}}(\widetilde{x}_i))}\right),\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}(\widetilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}(\widetilde{x}_i))},N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}(\widetilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-1}}(\widetilde{x}_i))}\right)\right),j,k=1,2,3,\ldots,n,m\right\}$ and $\mathcal{Q}_{\mathrm{CDTH-2}}\left\{\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}(\widetilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}(\widetilde{x}_i))},M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}(\widetilde{x}_i)\cdot e^{i2\pi}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-2}}(\widetilde{x}_i)\right)\right)\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}(\widetilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}(\widetilde{x}_i))},N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}(\widetilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-2}}(\widetilde{x}_i))}\right)\right),j,k=1,2,3,\ldots,\widetilde{l}_{\widetilde{x}_i},\sim l_{\widetilde{x}_i}\right\}$, the correlation coefficient is of the form

$$\varsigma_{\mathrm{CDTHF-}cc}\left(\mathcal{Q}_{\mathrm{CDTH-1}},\mathcal{Q}_{\mathrm{CDTH-2}}\right)=\frac{\varsigma_{\mathrm{CDTHF-}c}\left(\mathcal{Q}_{\mathrm{CDTH-1}},\mathcal{Q}_{\mathrm{CDTH-2}}\right)}{\varsigma_{\mathrm{CDTHF-}c}\left(\mathcal{Q}_{\mathrm{CDTH-1}},\mathcal{Q}_{\mathrm{CDTH-1}}\right)^{(1/2)}-\varsigma_{\mathrm{CDTHF-}c}\left(\mathcal{Q}_{\mathrm{CDTH-1}},\mathcal{Q}_{\mathrm{CDTH-2}}\right)^{(1/2)}}$$

$$=\frac{\sum_{i=1}^{n}\begin{pmatrix}\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-1}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-2}}\left(\widetilde{x}_i\right)\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}\left(\widetilde{x}_i\right)N^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)\right)\end{pmatrix}}{\left(\left(\sum_{i=1}^{n}\begin{pmatrix}\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-1}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-1}}\left(\widetilde{x}_i\right)\right)^2\right)\end{pmatrix}\right)^{(1/2)}\times\left(\sum_{i=1}^{n}\begin{pmatrix}\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHIP-}S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\ \left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\mathrm{CDTHIP-}P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(j)}_{\mathcal{Q}_{\mathrm{CDTHRP-}S-2}}\left(\widetilde{x}_i\right)\right)^2\right)\end{pmatrix}\right)^{(1/2)}\right)}. \quad (14)$$

Proposition 1. *For any two CDT-2HFSs, QCDTH-1 and QCDTH-2, the CC among CDT-2HFSs satisfies the following axioms:*

(1) $0 \leq \text{Ç}_{\text{CDTHF-cc}}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) \leq 1$

(2) $\text{Ç}_{CDTHF-cc}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) = 1 \Leftrightarrow \mathcal{Q}_{CDTH-1} = \mathcal{Q}_{CDTH-2}$

(3) $\text{Ç}_{CDTHF-cc}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) = \text{Ç}_{CDTHF-cc}\left(\mathcal{Q}_{CDTH-2}, \mathcal{Q}_{CDTH-1}\right)$

Proof. We prove the three above conditions by using equation (14). By using the inequality, it is clear that $0 \leq \text{Ç}_{\text{CDTHF-cc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right)$; then we only prove that $\text{Ç}_{\text{CDTHF-cc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right) \leq 1$. For this, we choose that

$$\text{Ç}_{\text{CDTHF}-c}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right) = \begin{pmatrix} \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right) \right) + \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right) \right) + \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right) \right) + \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right) \right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_1\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_1\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_1\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_1\right) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_n\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_n\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_n\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_n\right) \right) \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_1\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_1\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_1\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_1\right) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{x_i}} \left(M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_n\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_n\right) + M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_n\right) M^{\delta(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_n\right) \right) \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_1\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_1\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_1\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_1\right) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_n\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_n\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_n\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_n\right) \right) \\ \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_1\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_1\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_1\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_1\right) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\widetilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{x_i}} \left(N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_n\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_n\right) + N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_n\right) N^{\delta(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_n\right) \right) \end{pmatrix}. \tag{15}$$

By using the Cauchy–Schwarz inequality, $(x_1\, y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \quad (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots + y_n^2)$, we have

$$\left(\text{Ç}_{CDTHF-c}\left(\mathcal{Q}_{CDTH-1},\mathcal{Q}_{CDTH-2}\right)\right)^2 \leq \begin{pmatrix} \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(j)}{}^2\left(\widetilde{x}_1\right)+M_{\mathcal{Q}_{CDTHRP-S-1}}^{\sigma(j)}{}^2\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(j)}{}^2\left(\widetilde{x}_n\right)+M_{\mathcal{Q}_{CDTHRP-S-1}}^{\sigma(j)}{}^2\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(j)2}\left(\widetilde{x}_1\right)+M_{\mathcal{Q}_{CDTHIP-S-1}}^{\sigma(j)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)+M_{\mathcal{Q}_{CDTHIP-S-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)+N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)+N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{Q_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)+N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)+N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)\right) \end{pmatrix}$$

$$\times \begin{pmatrix} \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{j=1}^{\widetilde{L}_{x_1}}\left(M_{\mathcal{Q}_{CDTHRP-P-2}}^{\sigma(j)2}\left(\widetilde{x}_1\right)+M_{\mathcal{Q}_{CDTHRP-S-2}}^{\sigma(j)2}\left(\widetilde{x}_1\right)\right) \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)+M_{\mathcal{Q}_{CDTHRP-S-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(j)2}\left(\widetilde{x}_1\right)+M_{\mathcal{Q}_{CDTHIP-S-1}}^{\sigma(j)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)+M_{\mathcal{Q}_{CDTHIP-S-1}}^{\sigma(j)2}\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)+N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)+N_{\mathcal{Q}_{CDTHRP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)\right) \\ \frac{1}{\widetilde{L}_{\widetilde{x}_1}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)+N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_1\right)\right)+ \\ \cdots+\frac{1}{\widetilde{L}_{\widetilde{x}_n}}\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)+N_{\mathcal{Q}_{CDTHIP-P-1}}^{\sigma(k)2}\left(\widetilde{x}_n\right)\right) \end{pmatrix} = \text{Ç}_{CDTHF-c}\left(\mathcal{Q}_{CDTH-1},\mathcal{Q}_{CDTH-1}\right).\text{Ç}_{CDTHF-c}\left(\mathcal{Q}_{CDTH-2},\mathcal{Q}_{CDTH-2}\right). \tag{16}$$

$Ç_{CDTHF-c}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) \leq (Ç_{CDTHF-c}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-1}))^{(1/2)}(Ç_{CDTHF-c}(\mathcal{Q}_{CDTH-2}, \mathcal{Q}_{CDTH-2}))^{(1/2)}$; thus $0 \leq Ç_{CDTHF-cc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) \leq 1$. Further, we prove the second part by using (13)fd14. By hypothesis, it is given that $\mathcal{Q}_{CDTH-1} = \mathcal{Q}_{CDTH-2}$, and then $M^{\text{б}}_{\mathcal{Q}_{CDTHRP-P-1}}(j)(\tilde{x}_i) = M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i), M^{\text{б}}_{\mathcal{Q}_{CDTHIP-P-1}}(j)(\tilde{x}_i) = M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)$, $M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i) = M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i), M^{\text{б}(j)}_{Q_{CDTHIP-S-1}}(\tilde{x}_i) = M^{\text{б}(j)}_{Q_{C\,DT\,HRP-S-2}}(\tilde{x}_i), M^{\text{б}(j)}_{Q_{C\,DT\,HIP-S-1}}(\tilde{x}_i) = M^{\text{б}(j)}_{Q_{CDTHIP-S-2}}(\tilde{x}_i)$, $N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}} = N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}, N_{\mathcal{Q}_{CDTHRP-P-1}}{}^{\text{б}(k)}(\tilde{x}_i) = N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)$, and $N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-1}} = N^{\text{б}}_{\mathcal{Q}_{CDTHRP-S-2}}(k), N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i) = N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)$; then, by using (13)fd14, we get $Ç_{CDTHF-cc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) = 1$. Additionally, we prove the third condition such that

$$
\begin{aligned}
Ç_{CDTHF-cc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) &= \frac{\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)\right)
\end{pmatrix}}{\left(\begin{pmatrix}\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)^2\right) + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)\right)^2\right)
\end{pmatrix}\end{pmatrix}^{(1/2)} \times \begin{pmatrix}\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)^2\right) + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)\right)^2\right)
\end{pmatrix}\end{pmatrix}^{(1/2)}\right)} \\
&= \frac{\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)\right)
\end{pmatrix}}{\left(\begin{pmatrix}\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-2}}(\tilde{x}_i)^2\right) + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-2}}(\tilde{x}_i)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-2}}(\tilde{x}_i)\right)^2\right)
\end{pmatrix}\end{pmatrix}^{(1/2)} \times \begin{pmatrix}\sum_{i=1}^{n}\begin{pmatrix}
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{j=1}^{\tilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-P-1}}(\tilde{x}_i)^2\right) + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-1}}(\tilde{x}_i)\right)^2\right)+\\
(1/\tilde{l}_{\tilde{x}_i})\sum_{k=1}^{\tilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-P-1}}(\tilde{x}_i)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{CDTHIP-S-1}}(\tilde{x}_i)\right)^2\right)
\end{pmatrix}\end{pmatrix}^{(1/2)}\right)} = Ç_{CDTHF-cc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2})
\end{aligned}
\tag{17}
$$

Definition 9. For any two CDT-2HFSs, $\mathcal{Q}_{\text{CDTH}-1} = \left\{\left(\begin{array}{c}(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))}),\\ (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))})\end{array}\right), j,k = 1,2,3,\ldots,n,m\right\}$ and $\mathcal{Q}_{\text{CDTH}-2} = \left\{\left(\begin{array}{c}(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, M^{\text{б}(j)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\cdot e^{i2\pi(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))}),\\ (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i))}, N^{\text{б}(k)}_{\mathcal{Q}_{CDTHRP-S-2}}(\tilde{x}_i)\cdot e^{i2\pi(N^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i))})\end{array}\right), j,k = 1,2,3,\ldots,\widetilde{L}_{x_i}, \widehat{l}_{\tilde{x}_i}\right\}$, the maximum-based correlation coefficient (MCC) is of the form

$$
\begin{aligned}
\text{Ç}_{\text{CDTHF}-\text{mcc}}(\mathcal{Q}_{\text{CDTH}-1},\mathcal{Q}_{\text{CDTH}-2}) &= \frac{\text{Ç}_{\text{CDTHF}-c}(\mathcal{Q}_{\text{CDTH}-1},\mathcal{Q}_{CDTH-2})}{\max\left(\text{Ç}_{\text{CDTHF}-c}(\mathcal{Q}_{\text{CDTH}-1},\mathcal{Q}_{\text{CDTH}-1})^{(1/2)},\text{Ç}_{\text{CDTHF}-c}(\mathcal{Q}_{\text{CDTH}-1},\mathcal{Q}_{\text{CDTH}-2})^{(1/2)}\right)}\\
&= \frac{\sum_{i=1}^{n}\left(\begin{array}{c}
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}_i)\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i)\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}_i)\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i)\right)
\end{array}\right)}{\max\left(\begin{array}{c}
\left(\sum_{i=1}^{n}\left(\begin{array}{c}
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\tilde{x}_i)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\tilde{x}_i)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\tilde{x}_i)\right)^2\right)
\end{array}\right)\right)^{(1/2)},\\
\left(\sum_{i=1}^{n}\left(\begin{array}{c}
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\tilde{x}_i)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\tilde{x}_i)\right)^2\right)+\\
\left(1/\widetilde{l}_{\tilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\tilde{x}_i)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\tilde{x}_i)\right)^2\right)
\end{array}\right)\right)
\end{array}\right)}.
\end{aligned}
\tag{18}
$$

Proposition 2. *For any two CDT-2HFSs, QCDTH-1 and QCDTH-2, the MCC among CDT-2HFSs satisfies the following axioms:*

(1) $0 \le \text{Ç}_{CDTH-mcc}(\mathcal{Q}_{CDTH-1},\mathcal{Q}_{CDTH-2}) \le 1$

(2) $\text{Ç}_{CDTHF-mcc}(\mathcal{Q}_{CDTH-1},\mathcal{Q}_{CDTH-2}) = 1 \Leftrightarrow \mathcal{Q}_{CDTH-1} = \mathcal{Q}_{CDTH-2}$

(3) $\text{Ç}_{CDTHF-mcc}(\mathcal{Q}_{CDTH-1},\mathcal{Q}_{CDTH-2}) = \text{Ç}_{CDTHF-mcc}(\mathcal{Q}_{CDTH-2},\mathcal{Q}_{CDTH-1})$

Proof. We prove the three above conditions by using equation (18). By using the inequality, it is clear that

$0 \leq \text{Ç}_{\text{CDTH-mcc}}(\mathcal{Q}_{\text{CDTH-1}}, \mathcal{Q}_{\text{CDTH-2}})$; then, we only prove that $\text{Ç}_{\text{CDTH-mcc}}(\mathcal{Q}_{\text{CDTH-1}}, \mathcal{Q}_{\text{CDTH-2}}) \leq 1$. For this, we choose that

$$\text{Ç}_{\text{CDTHF-}c}(\mathcal{Q}_{\text{CDTH-1}}, \mathcal{Q}_{\text{CDTH-2}}) = \begin{pmatrix} \frac{1}{\widetilde{l}_{\tilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_i) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}S\text{-}1}}(\tilde{x}_i) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}S\text{-}2}}(\tilde{x}_i) \right) + \\ \frac{1}{\widetilde{l}_{\tilde{x}_i}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_i) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_i) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}1}}(\tilde{x}_i) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}2}}(\tilde{x}_i) \right) + \\ \frac{1}{\widetilde{l}_{\tilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_i) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_i) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_i) \right) + \\ \frac{1}{\widetilde{l}_{\tilde{x}_i}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_i) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_i) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_i) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_i) \right) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\widetilde{l}_{\tilde{x}_1}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_1) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_1) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2\text{-}S\text{-}1}}(\tilde{x}_1) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2\text{-}S\text{-}2}}(\tilde{x}_1) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\tilde{x}_n}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_n) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2\text{-}P\text{-}2}}(\tilde{x}_n) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2\text{-}S\text{-}1}}(\tilde{x}_n) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2\text{-}S\text{-}2}}(\tilde{x}_n) \right) \\ \frac{1}{\widetilde{l}_{\tilde{x}_1}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_1) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_1) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}1}}(\tilde{x}_1) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}2}}(\tilde{x}_1) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\tilde{x}_n}} \sum_{j=1}^{\widetilde{L}_{\tilde{x}_i}} \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_n) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_n) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}1}}(\tilde{x}_n) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-}2}}(\tilde{x}_n) \right) \\ \frac{1}{\widetilde{l}_{\tilde{x}_1}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_1) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_1) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_1) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_1) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\tilde{x}_n}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_n) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_n) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}1}}(\tilde{x}_n) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-}2}}(\tilde{x}_n) \right) \\ \frac{1}{\widetilde{l}_{\tilde{x}_1}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_1) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_1) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_1) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_1) \right) + \\ \cdots + \frac{1}{\widetilde{l}_{\tilde{x}_n}} \sum_{k=1}^{\widetilde{L}_{\tilde{x}_i}} \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_n) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_n) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}1}}(\tilde{x}_n) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-}2}}(\tilde{x}_n) \right) \end{pmatrix}. \tag{19}$$

By using the Cauchy–Schwarz inequality, $(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^2 \le (x_1^2 + x_2^2 + \cdots + x_n^2)(y_1^2 + y_2^2 + \cdots y_n^2)$, we have

$$
\begin{aligned}
\left(\varsigma_{\text{CDTHF-c}}\left(\mathcal{Q}_{\text{CDTH-1}}, \mathcal{Q}_{\text{CDTH-2}}\right)\right)^2 \le &\begin{pmatrix}
\frac{1}{\tilde{l}_{x_1}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_1\right) + M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHRP-}S\text{-1}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_n\right) + M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHRP-}S\text{-1}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_1\right) + M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-1}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_n\right) + M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-1}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_1\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_n\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-1}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_1\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_n\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-1}}}\left(\tilde{x}_n\right)\right)
\end{pmatrix} \\
&\times \begin{pmatrix}
\frac{1}{\tilde{l}_{x_1}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{Q_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_1\right) + M^{\text{б}(j)2}_{Q_{\text{CDTHRP-}S\text{-2}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{Q_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_n\right) + M^{\text{б}(j)2}_{Q_{\text{CDTHRP-}S\text{-2}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{Q_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_1\right) + M^{\text{б}(j)2}_{Q_{\text{CDTHIP-}S\text{-2}}}\left(\tilde{x}_1\right)\right) + \\
.. + \frac{1}{\tilde{l}_{x_n}} \sum_{j=1}^{\tilde{L}_{x_i}} \left(M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_n\right) + M^{\text{б}(j)2}_{\mathcal{Q}_{\text{CDTHIP-}S\text{-2}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_1\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_n\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHRP-}P\text{-2}}}\left(\tilde{x}_n\right)\right) \\
\frac{1}{\tilde{l}_{x_1}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_1\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_1\right)\right) + \\
\cdots + \frac{1}{\tilde{l}_{x_n}} \sum_{k=1}^{\tilde{L}_{x_i}} \left(N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_n\right) + N^{\text{б}(k)2}_{\mathcal{Q}_{\text{CDTHIP-}P\text{-2}}}\left(\tilde{x}_n\right)\right)
\end{pmatrix} = \varsigma_{\text{CDTHF-}c}\left(\mathcal{Q}_{\text{CDTH-1}}, \mathcal{Q}_{\text{CDTH-1}}\right).\varsigma_{\text{CDTHF-}c}\left(\mathcal{Q}_{\text{CDTH-2}}, \mathcal{Q}_{\text{CDTH-2}}\right).
\end{aligned}
\tag{20}
$$

$\text{Ç}_{\text{CDTHF}-c}(\text{Q}_{\text{CDTH}-1}, \text{Q}_{\text{CDTH}-2}) \leq (\text{Ç}_{\text{CDTHF}-c}(\text{Q}_{\text{CDTH}-1}, \text{Q}_{\text{CDTH}-1}))^{(1/2)} (\text{Ç}_{\text{CDTHF}-c}(\text{Q}_{\text{CDTH}-2}, \text{Q}_{\text{CDTH}-2}))^{(1/2)}$; thus $0 \leq \text{Ç}_{\text{CDTHF}-\text{mcc}}(\text{Q}_{\text{CDTH}-1}, \text{Q}_{\text{CDTH}-2}) \leq 1$. Furthermore, we prove the second part by using equation (18). By hypothesis, it is given that $\text{Q}_{\text{CDTH}-1} = \text{Q}_{\text{CDTH}-2}$; then $M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i) = M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i)$, $M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i) = M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i)$, $M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\widetilde{x}_i) = M^{\sigma(j)}_{\text{Q}_{\text{CDTHRP}-S-2}}(\widetilde{x}_i)$, $M^{\sigma(j)}_{\text{Q}_{\text{CDTHIP}-S-1}}(\widetilde{x}_i) = M^{\sigma(j)}_{\text{Q}_{\text{CDTHIP}-S-2}}(\widetilde{x}_i)$, $N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}} = N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}$, $N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i) = N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i)$, and $N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}} = N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}$, $N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\widetilde{x}_i) = N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\widetilde{x}_i)$; then, by using equation (18), we get $\text{Ç}_{\text{CDTHF}-\text{mcc}}(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}) = 1$. Additionally, we prove the third condition such that

$$\text{Ç}_{\text{CDTHF}-\text{mcc}}(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}) = \frac{\sum_{i=1}^{n}\begin{pmatrix} (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i)M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i) + M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\widetilde{x}_i)M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\widetilde{x}_i)\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i)M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i) + M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\widetilde{x}_i)M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\widetilde{x}_i)\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i)N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i) + N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i)N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i)\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i)N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i) + N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i)N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i)\right) \end{pmatrix}}{\max\begin{pmatrix} \left(\sum_{i=1}^{n}\begin{pmatrix} (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i)\right)^2 + \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\widetilde{x}_i)\right)^2\right) + \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i)\right)^2 + \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\widetilde{x}_i)\right)^2\right) + \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}(\widetilde{x}_i)\right)^2 + \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}(\widetilde{x}_i)\right)^2\right) + \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}(\widetilde{x}_i)\right)^2 + \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}(\widetilde{x}_i)\right)^2\right) \end{pmatrix}\right)^{(1/2)} \\ \left(\sum_{i=1}^{n}\begin{pmatrix} (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i)\right)^2 + \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\widetilde{x}_i)\right)^2\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i)\right)^2 + \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\widetilde{x}_i)\right)^2\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}(\widetilde{x}_i)\right)^2 + \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(\widetilde{x}_i)\right)^2\right) \\ (1/\widetilde{L}_{\widetilde{x}_i})\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}(\widetilde{x}_i)\right)^2 + \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}(\widetilde{x}_i)\right)^2\right) \end{pmatrix}\right)^{(1/2)} \end{pmatrix}} =$$

$$
\frac{\sum_{i=1}^{n}\begin{pmatrix}
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right)\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right)M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right)\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)\right)
\end{pmatrix}}{\max\begin{pmatrix}
\left(\sum_{i=1}^{n}\begin{pmatrix}
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right)\right)^2\right)+\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right)\right)^2\right)
\end{pmatrix}\right)^{(1/2)}\\
\left(\sum_{i=1}^{n}\begin{pmatrix}
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right)\right)^2\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{L}_{x_i}}\left(\left(M^{\text{б}(j)}_{Q_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right)\right)^2\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right)\right)^2\right)\\
\left(1/\widetilde{l}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{L}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)\right)^2+\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right)\right)^2\right)
\end{pmatrix}\right)^{(1/2)}
\end{pmatrix}} = \text{Ç}_{\text{CDTHF}-\text{mcc}}\left(Q_{\text{CDTH}-1}, Q_{\text{CDTH}-2}\right). \tag{21}
$$

Definition 10. For any two CDT-2HFSs, $\mathcal{Q}_{\text{CDTH}-1} = \left\{\begin{pmatrix}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)\right)}, M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right).e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right)\right)}\right),\\ \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_i\right)\right)}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_i\right)\right)}\right)\end{pmatrix}, j, k = 1, 2, 3, \ldots, n, m\right\}$ and $\mathcal{Q}_{\text{CDTH}-2} = \left\{\begin{pmatrix}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)\right)}, M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right)\right)}\right),\\ \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_i\right)\right)}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_i\right)\cdot e^{i2\pi\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_i\right)\right)}\right)\end{pmatrix}, j, k = 1, 2, 3, \ldots, \widetilde{l}_{\widetilde{x}_i}, \sim l_{\widetilde{x}_i}\right\}$, the weighted correlation coefficient is of the form

$$
\text{Ç}_{\text{CDTHF}-\text{wcc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right) = \frac{\text{Ç}_{\text{CDTHF}-\text{wc}}\left(\mathcal{Q}_{\text{CDTH}-1} - \mathcal{Q}_{\text{CDTH}-2}\right)}{\text{Ç}_{\text{CDTHF}-\text{wc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-1}\right)^{(1/2)} \times \text{Ç}_{\text{CDTHF}-\text{wc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right)^{(1/2)}}
$$

$$
= \frac{\sum_{i=1}^{n}\Omega_i \begin{pmatrix} \left(1/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}} M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_\iota\right) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_\iota\right) + \\ \left(1/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}} M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_\iota\right) + M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right) M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_\iota\right) + \\ \left(1/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}} N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_\iota\right) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_\iota\right) + \\ \left(1/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}} N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_\iota\right) + N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right) N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_\iota\right) \end{pmatrix}}{\begin{pmatrix} \left(\sum_{i=1}^{n} \begin{pmatrix} \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\ell}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\ell}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \end{pmatrix}\right)^{(1/2)} \times \\ \sum_{i=1}^{n} \begin{pmatrix} \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\ell}_{x_i}}\left(\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\ell}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \\ \left(\Omega_i/\widetilde{\ell}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\ell}_{x_i}}\left(\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}\left(\widetilde{x}_\iota\right)\right)^2 + \left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)^2\right) + \end{pmatrix}^{(1/2)} \end{pmatrix}}, \tag{22}
$$

where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ represents weight vector with the condition that $\sum_{i=1}^{n}\Omega_i = 1$, $\Omega_i \in [0, 1]$.

Proposition 3. *For any two CDT-2HFSs, QCDTH-1 and QCDTH-2, the CC among CDT-2HFSs satisfies the following axioms:*

(1) $0 \leq Ç_{CDTHF-wcc}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) \leq 1$

(2) $Ç_{CDTHF-wcc}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) = 1 \Leftrightarrow \mathcal{Q}_{CDTH-1} = \mathcal{Q}_{CDTH-2}$

(3) $Ç_{CDTHF-wcc}\left(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}\right) = Ç_{CDTHF-wcc}\left(\mathcal{Q}_{CDTH-2}, \mathcal{Q}_{CDTH-1}\right)$

Proof. The proof is straightforward.

Definition 11. For any two CDT-2HFSs, $\mathcal{Q}_{\text{CDTH}-1} = \left\{\begin{pmatrix} (M^{\text{б}(j)}_{Q_{\text{CDTHRP}-P-1}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-P-1}}(\widetilde{x}_\iota))}, M^{\text{б}(j)}_{Q_{\text{CDTHRP}-S-1}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-S-1}}(\widetilde{x}_\iota))}), \\ (N^{\text{б}(k)}_{Q_{\text{CDTHRP}-P-1}}(\widetilde{\widetilde{x}}_\iota) \cdot e^{i2\pi (M^{\text{б}(k)}_{Q_{\text{CDTHIP}-P-1}}(\widetilde{x}_\iota))}, N^{\text{б}(k)}_{Q_{\text{CDTHRP}-S-1}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(k)}_{Q_{\text{CDTHIP}-S-1}}(\widetilde{x}_\iota))}) \end{pmatrix}, j, k = 1, 2, 3, \ldots, n, m\right\}$ and $\mathcal{Q}_{\text{CDTH}-2} = \left\{\begin{pmatrix} (M^{\text{б}(j)}_{Q_{\text{CDTHRP}-P-2}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-P-2}}(\widetilde{x}_\iota))}, M^{\text{б}(j)}_{Q_{\text{CDTHRP}-S-2}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-S-2}}(\widetilde{x}_\iota))}), \\ (N^{\text{б}(j)}_{Q_{\text{CDTHRP}-P-2}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-P-2}}(\widetilde{x}_\iota))}, N^{\text{б}(j)}_{Q_{\text{CDTHRP}-S-2}}(\widetilde{x}_\iota) \cdot e^{i2\pi (M^{\text{б}(j)}_{Q_{\text{CDTHIP}-S-2}}(\widetilde{x}_\iota))}) \end{pmatrix}, j, k = 1, 2, 3, \ldots, \widetilde{\ell}_{\widetilde{x}_i}, \widetilde{\ell}_{\widetilde{x}_i}\right\}$, the maximum-based weighted correlation coefficient (MCC) is of the form

$$
Ç_{\text{CDTHF-mwcc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right) = \frac{Ç_{\text{CDTHF-wc}}\left(\mathcal{Q}_{\text{CDTH}-1} - \mathcal{Q}_{\text{CDTH}-2}\right)}{\max\left(Ç_{\text{CDTHF-wc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-1}\right)^{(1/2)} \times Ç_{\text{CDTHF-wc}}\left(\mathcal{Q}_{\text{CDTH}-1}, \mathcal{Q}_{\text{CDTH}-2}\right)^{(1/2)}\right)}
$$

$$= \frac{\sum_{i=1}^{n}\Omega_i\begin{pmatrix}\left(1/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}} M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-P-1}}\left(\tilde{x}_\iota\right)M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-P-2}}\left(\tilde{x}_\iota\right)+M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-S-1}}\left(\tilde{x}_\iota\right)M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-S-2}}\left(\tilde{x}_\iota\right)+\\ \left(1/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}} M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-P-1}}\left(\tilde{x}_\iota\right)M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-P-2}}\left(\tilde{x}_\iota\right)+M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-S-1}}\left(\tilde{x}_\iota\right)M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-S-2}}\left(\tilde{x}_\iota\right)+\\ \left(1/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}} N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-P-1}}\left(\tilde{x}_\iota\right)N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-P-2}}\left(\tilde{x}_\iota\right)+N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-S-1}}\left(\tilde{x}_\iota\right)N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-S-2}}\left(\tilde{x}_\iota\right)+\\ \left(1/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}} N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-P-1}}\left(\tilde{x}_\iota\right)N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-P-2}}\left(\tilde{x}_\iota\right)+N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-S-1}}\left(\tilde{x}_\iota\right)N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-S-2}}\left(\tilde{x}_\iota\right)\end{pmatrix}}{\max\begin{pmatrix}\left(\sum_{i=1}^{n}\begin{pmatrix}\left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-P-1}}\left(\tilde{x}_\iota\right)\right)^2+\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-P-1}}\left(\tilde{x}_\iota\right)\right)^2+\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\ell}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-P-1}}\left(\tilde{x}_\iota\right)\right)^2+\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\ell}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-P-1}}\left(\tilde{x}_\iota\right)\right)^2+\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\end{pmatrix}\right)^{(1/2)}\times\\ \sum_{i=1}^{n}\begin{pmatrix}\left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-P-2}}\left(\tilde{x}_\iota\right)\right)^2+\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHRP-S-2}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\ell}_{x_i}}\left(\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-P-2}}\left(\tilde{x}_\iota\right)\right)^2+\left(M^{\sigma(j)}_{\mathcal{Q}_{CDTHIP-S-2}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\ell}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-P-2}}\left(\tilde{x}_\iota\right)\right)^2+\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHRP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\\ \left(\Omega_i/\ell_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\ell}_{x_i}}\left(\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-P-2}}\left(\tilde{x}_\iota\right)\right)^2+\left(N^{\sigma(k)}_{\mathcal{Q}_{CDTHIP-S-1}}\left(\tilde{x}_\iota\right)\right)^2\right)+\end{pmatrix}^{(1/2)}\end{pmatrix}}. \tag{23}$$

where $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ represents weight vector with the condition that $\sum_{i=1}^{n}\Omega_i = 1, \Omega_i \in [0,1]$.

Proposition 4. *For any two CDT-2HFSs, QCDTH-1 and QCDTH-2, the MCC among CDT-2HFSs satisfies the following axioms:*

(1) $0 \leq \c{C}_{CDTHF-mwcc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) \leq 1$

(2) $\c{C}_{CDTHF-mwcc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) = 1 \Leftrightarrow \mathcal{Q}_{CDTH-1} = \mathcal{Q}_{CDTH-2}$

(3) $\c{C}_{CDTHF-mwcc}(\mathcal{Q}_{CDTH-1}, \mathcal{Q}_{CDTH-2}) = \c{C}_{CDTHF-mwcc}(\mathcal{Q}_{CDTH-2}, \mathcal{Q}_{CDTH-1})$

Proof. The proof is straightforward.

The explored notions, which are stated in the form of equations (13)–(23), are more proficient and more modified than the existing drawbacks; for instance, if we choose the imaginary part of equations (13)–(23) to be zero, then equations (13)–(23) convert it for DT-2HFS.

5. Entropy Measures for Complex Dual Type-2 Hesitant Fuzzy Sets

The aim of this study is to present the novel of two types of entropy measures (EMs). The special cases of the explored measures are also explored.

Definition 12. For any two CDT-2HFSs $\mathcal{Q}_{CDTH-1} = \left\{\begin{pmatrix}\left(M^{\sigma(j)}_{Q_{CDTHRP-P-1}}(\tilde{x}_\iota)\cdot e^{i2\pi\left(M^{\sigma(j)}_{Q_{CDTHIP-P-1}}(\tilde{x}_\iota)\right)}, M^{\sigma(j)}_{Q_{CDTHRP-S-1}}(\tilde{x}_\iota)\cdot e^{i2\pi\left(M^{\sigma(j)}_{Q_{CDTHIP-S-1}}(\tilde{x}_\iota)\right)}\right),\\ \left(N^{\sigma(k)}_{Q_{CDTHRP-P-1}}(\tilde{x}_\iota)\cdot e^{i2\pi\left(M^{\sigma(k)}_{Q_{CDTHIP-P-1}}(\tilde{x}_\iota)\right)}, N^{\sigma(k)}_{Q_{CDTHRP-S-1}}(\tilde{x}_\iota)\cdot e^{i2\pi\left(M^{\sigma(k)}_{Q_{CDTHIP-S-1}}(\tilde{x}_\iota)\right)}\right)\end{pmatrix}, j, k = 1,2,3,\ldots,n,m\right\}$, the two EMs are defined by

$$\mathcal{J}_1\left(\mathcal{Q}_{\text{CDTH}-1}\right)=\frac{1}{n}\sum_{i=1}^{n}\left\{\left\{\begin{array}{c}\sin\left(\frac{\begin{array}{c}2+\left(1/\iota_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)-\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)-\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)\end{array}}{8}\right)\\ \sin\left(\frac{\pi\times\left(\begin{array}{c}2+\left(1/\iota_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}\left(\widetilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}}\left(\widetilde{x}_\iota\right)\right)-\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\iota_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}\left(\widetilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}\left(\widetilde{x}_\iota\right)\right)\end{array}\right)}{8}\right)-1\end{array}\right\}\right\}, \tag{24}$$

$$\mathcal{J}_1\left(\mathcal{Q}_{\text{CDTH}-1}\right)=\frac{1}{n}\sum_{i=1}^{n}\left\{\left\{\begin{matrix}\cos\left(\dfrac{\pi\times\begin{pmatrix}2+\left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(M_{\mathcal{Q}_{\text{CDTHRP}-P-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)+M_{\mathcal{Q}_{\text{CDTHRP}-S-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(N_{\mathcal{Q}_{\text{CDTHRP}-P-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)+N_{\mathcal{Q}_{\text{CDTHRP}-S-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)\right)-\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(M_{\mathcal{Q}_{\text{CDTHIP}-P-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)+M_{\mathcal{Q}_{\text{CDTHIP}-S-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(N_{\mathcal{Q}_{\text{CDTHIP}-P-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)+N_{\mathcal{Q}_{\text{CDTHIP}-S-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)\right)\end{pmatrix}}{8}\right)\\ \cos\left(\dfrac{\pi\times\begin{pmatrix}2+\left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(M_{\mathcal{Q}_{\text{CDTHRP}-P-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)+M_{\mathcal{Q}_{\text{CDTHRP}-S-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(N_{\mathcal{Q}_{\text{CDTHRP}-P-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)+N_{\mathcal{Q}_{\text{CDTHRP}-S-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)\right)-\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{j=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(M_{\mathcal{Q}_{\text{CDTHIP}-P-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)+M_{\mathcal{Q}_{\text{CDTHIP}-S-1}}^{\text{б}(j)}\left(\widetilde{x}_\iota\right)\right)+\\ \left(1/\widetilde{\iota}_{\widetilde{x}_i}\right)\sum_{k=1}^{\widetilde{\iota}_{\widetilde{x}_i}}\left(N_{\mathcal{Q}_{\text{CDTHIP}-P-1}}^{\text{б}(jk)}\left(\widetilde{x}_\iota\right)+N_{\mathcal{Q}_{\text{CDTHIP}-S-1}}^{\text{б}(k)}\left(\widetilde{x}_\iota\right)\right)\end{pmatrix}}{8}\right)-1\end{matrix}\right\}\right\}. \tag{25}$$

The two EMs based on CDT-2HFSs has the following properties:

(1) If $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}} = M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}} = 1$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}} = N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}} = 0$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}} = M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}} = 1$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}} = N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}} = 0$ or $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}} = M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}} = 1, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}} = N^{\text{б}}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}(k) = 0$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}} = M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}} = 1$, $N^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}} = N^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}} = 0$

(2) If $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}} \leq M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-1}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-1}} \geq N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-2}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-2}}$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}} \leq M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}$, $M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-1}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-1}} \geq N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-2}}$, $N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-2}}$ then, $\mathcal{J}_1(\mathcal{Q}_{\text{CDTH}-1}) \leq \mathcal{J}_1(\mathcal{Q}_{\text{CDTH}-2})$, and if we change $\leq$ into $\geq$, then $\mathcal{J}_1(\mathcal{Q}_{\text{CDTH}-1}) \geq \mathcal{J}_1(\mathcal{Q}_{\text{CDTH}-2})$

(3) $\mathcal{J}_1(\mathcal{Q}_{\text{CDTH}-1}) = \mathcal{J}_1(\mathcal{Q}^c_{\text{CDTH}-1})$ and $\mathcal{J}_2(\mathcal{Q}_{CDTH-1}) = \mathcal{J}_2(\mathcal{Q}^c_{CDTH-1})$.

The explored notions, which are stated in the form of (23) and (24)fd25, are more proficient and more modified than the existing drawbacks; for instance, if we choose the imaginary part of (23) and (24)fd25 to be zero, then (23) and (24)fd25 convert it for DT-2HFS.

6. TOPSIS Method Based on CDT-2HFSs

Basically, a novel TOPSIS method using CC and EM is provided to handle the MADM problems based on Cq-ROFS. Previously, TOPSIS method was proposed based on sample SMs, but in our proposed work we considered the CC and EM. The DM cannot accurately examine the proximity of each alternative to ideal solution in some particular cases. So, we replace the TOPSIS method with the CC instead of DM to check the efficacy and effectiveness of the proposed work.

6.1. Problem Description. Consider an MADM problem, whose m alternatives and n attributes are denoted by $C = \{c_1, c_2, c_3, \ldots, c_m\}$ and $U = (u_1, u_2, u_3 \ldots, u_n)$, respectively, with respect to weight vectors represented by $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ with the conditions that $\Omega \in 0, 1$ and $\sum_{i=1}^n \Omega_i = 1$. Each attribute of each alternative is simplified using CDT-2HFSs $\mathcal{Q}_{\text{CDTH}-z} = \left\{ \begin{pmatrix} (M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-z}}(\tilde{x}_\iota) \cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-z}}(\tilde{x}_\iota))}, M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-z}}(\tilde{x}_\iota) \cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-z}}(\tilde{x}_\iota))}), \\ (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-z}}(\tilde{x}_\iota) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-z}}(\tilde{x}_\iota))}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-z}}(\tilde{x}_\iota) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-z}}(\tilde{x}_\iota))}) \end{pmatrix}, j, k = 1, 2, 3, \ldots, n, m1, 2, 3, \ldots, n, m \right\}$, $z = 1, 2, 3, \ldots, n$, satisfying the following conditions: $0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-P}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P}}) \leq 1, 0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-P}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P}}) \leq 1$, $0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHRP}-S}}) + \max(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S}}) \leq 1$ and $0 \leq \max(M^{\text{б}(j)}_{\mathcal{Q}_{\text{CDTHIP}-S}}) + \max(N_{\mathcal{Q}_{\text{CDTHIP}-P}}{}^{\text{б}(k)}) \leq 1$. All the attributes values of the alternatives are in the form of CDT-2HF decision matrix (CDT-2HFDM); that is, $[\mathcal{Q}_{\text{CDTH}-yz}]_{m\times n} = \left\{ \begin{pmatrix} (M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}(\tilde{x}_\iota) \cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}(\tilde{x}_\iota))}, M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}(\tilde{x}_\iota) \cdot e^{i2\pi(M^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}(\tilde{x}_\iota))}), \\ (N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}(\tilde{x}_\iota) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}(\tilde{x}_\iota))}, N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}(\tilde{x}_\iota) \cdot e^{i2\pi(N^{\text{б}(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}(\tilde{x}_\iota))}) \end{pmatrix}, j, k = 1, 2, 3, \ldots, n, m1, 2, 3, \ldots, n, m \right\}$, $y, z = 1, 2, 3, \ldots, m, n$.

Due to tension, time limitations of the decision-makers (DMs), and complication of problems, it is awkward to give the weight information of attribute in advance. To handle such type of issue, we compute the weights of attributes and consider the proposed EM examining the weight of each attribute as

$$\Omega_j = \frac{1 - \mathcal{H}_j}{n - \sum_{i=1}^n \mathcal{H}_j}, \tag{26}$$

where $\mathcal{H}_j \in [0, 1]$, $j = 1, 2, 3, \ldots, n$, is defined as

$$\mathscr{H}_j = \frac{1}{n}\sum_{i=1}^{n}\left\{\left\{\begin{array}{c}\sin\left(\dfrac{\pi\times\left(\begin{array}{c}2+\left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right)-\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right)+\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right)-\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right)\end{array}\right)}{8}\right)+\\ \sin\left(\dfrac{\pi\times\left(\begin{array}{c}2-\left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right)+\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right)-\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{j=1}^{\tilde{\iota}_{x_i}}\left(M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right)+M^{\text{б}(j)}_{\mathscr{Q}_{\text{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right)+\\ \left(1/\tilde{\iota}_{\tilde{x}_i}\right)\sum_{k=1}^{\tilde{\iota}_{x_i}}\left(N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right)+N^{\text{б}(k)}_{\mathscr{Q}_{\text{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right)\end{array}\right)}{8}\right)-1\end{array}\right\}\right\}, \tag{27}$$

$$\mathcal{H}_j = \frac{1}{n}\sum_{i=1}^{n}\left\{\left\{\begin{array}{l}\cos\left(\dfrac{\pi \times \left(\begin{array}{c} 2 + \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{j=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right) + M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right) - \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{k=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right) + N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right) + \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{j=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right) + M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right) - \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{k=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right) + N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right) \end{array}\right)}{8}\right) + \\ \cos\left(\dfrac{\pi \times \left(\begin{array}{c} 2 - \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{j=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right) + M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right) + \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{k=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHRP}-P-i}}\left(\tilde{x}_\iota\right) + N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHRP}-S-i}}\left(\tilde{x}_\iota\right)\right) - \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{j=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right) + M^{\sigma(j)}_{\mathcal{Q}_{\mathrm{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right) + \\ \left(1/\overleftarrow{\ell}_{\tilde{x}_i}\right)\sum_{k=1}^{\overleftarrow{\ell}_{\tilde{x}_i}}\left(N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHIP}-P-i}}\left(\tilde{x}_\iota\right) + N^{\sigma(k)}_{\mathcal{Q}_{\mathrm{CDTHIP}-S-i}}\left(\tilde{x}_\iota\right)\right) \end{array}\right)}{8}\right) - 1 \end{array}\right\}\right\} \tag{28}$$

When we consider that the imaginary part of (26) and (27) will be zero, (26) and (27) will be converted for DT-2HFS.

Procedure for MADM problem based on the above analysis by considering the proposed CDT-2HF TOPSIS method using CC and EM is explained below.

6.2. Application. The steps of the CDT-2HF TOPSIS method using WCC are as follows:

(i) Step 1: some decision-making problems also contain benefits and cost types of informations, so for this, we normalized the decision matrix by considering the following formula. We have

$$r_{yz} = \left(\begin{array}{l} \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right), \\ \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right) \end{array} \right) =$$

$$\cdot \begin{cases} \left(\begin{array}{l} \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right), \\ \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right) \end{array} \right) & \text{for benefit types of attributes,} \\ \left(\begin{array}{l} \left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right), \\ \left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}_\iota\right)\right)}, M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}_\iota\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}_\iota\right)\right)} \right) \end{array} \right) & \text{for cost types of attributes.} \end{cases} \tag{29}$$

Step 2: by using (25), we examine the weight vector of the attributes.

Step 3: by using (30) and (31), we examine the PIS and NIS among the alternatives.

$$\begin{aligned} R^{+} &= \left(r^{+}_{i1}, r^{+}_{i2}, r^{+}_{i3}, \ldots, r^{+}_{in}\right), \\ &\cdot r^{+}_{ij} \left(\begin{array}{l} \left(\max M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}\right)\right)}, \max M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}\right)\right)} \right), \\ \left(\min N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}\right)\right)}, \min N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}\right)\right)} \right) \end{array} \right), \end{aligned} \tag{30}$$

$$\begin{aligned} R^{-} &= \left(r^{-}_{i1}, r^{-}_{i2}, r^{-}_{i3}, \ldots, r^{-}_{in}\right), \\ &\cdot r^{-}_{ij} \left(\begin{array}{l} \left(\min M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}\right)\right)}, \min M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(M^{\sigma(j)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}\right)\right)} \right), \\ \left(\max N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-P-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-P-yz}}\left(\widetilde{x}\right)\right)}, \max N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHRP}-S-yz}}\left(\widetilde{x}\right) \cdot e^{i2\pi\left(N^{\sigma(k)}_{\mathcal{Q}_{\text{CDTHIP}-S-yz}}\left(\widetilde{x}\right)\right)} \right) \end{array} \right). \end{aligned} \tag{31}$$

Step 4: by using (21), we examine the CDT-2HF PIS, and we have

$$Ç_{\text{CDTHF-wcc-}y}\left(r_{yz}, R^{+}\right). \tag{32}$$

We also examine complex CDT-2HF NIS by using (21), and we have

$$Ç_{\text{CDTHF-wcc-}y}\left(r_{yz}, R^{-}\right). \tag{33}$$

Step 5: by using (34), we examine the closeness of each of the alternatives, and we have

$$\mathscr{P}_i = \frac{Ç_{\text{CDTHF-wcc-}y}\left(r_{yz}, R^{+}\right)}{Ç_{\text{CDTHF-wcc-}y}\left(r_{yz}, R^{+}\right) + Ç_{\text{CDTHF-wcc-}y}\left(r_{yz}, R^{-}\right)} \longrightarrow . \tag{34}$$

Step 6: we rank all alternatives and examine the best optimal one.

Step 7: the end.

Example 2. The company of intranet is usually attacked by malicious intrusions. To enhance the security of the intranet, the company plans to purchase the firewall production and put it between the intranet and extranet for blocking illegal access. Basically, there are four types of firewall productions given to be considered, which are detailed as follows: $C = \{c_1, c_2, c_3, c_4\}$. When choosing the firewall production, the company pays attention to the factors detailed as follows: $u_1 \longrightarrow$ the promotion, $u_2 \longrightarrow$ configuration simplicity, $u_3 \longrightarrow$ Security level, and $u_4 \longrightarrow$ maintenance sever level, the weight vector of which is denoted and defined by $\Omega_j \in [0, 1]$, $\sum_{j=1}^{n} \Omega_j = 1$, and $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)^T$. To examine the firewall production with respect to their factors, we consider the following matrix, and the decision matrix is given in the form of Table 1.

The steps of the proposed complex dual type-2 hesitant fuzzy TOPSIS method are as follows:

(i) Step 1: some decision-making problems also contain benefits and cost types of informations, so for this, we normalized the decision matrix by considering (29), but the considered information cannot be normalized. So, we have used the information available in Table 1 and go to step 2.

(ii) Step 2: by using (26), we examine the weight vector of the attributes.

$$\Omega = \{0.342, 0.155, 0.067, 0.2, 0.236\}^T. \tag{35}$$

(iii) Step 3: by using (30) and (31), we examine the PIS and NIS among the alternatives.

$$R^{+} = \left\{ \begin{array}{l} \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.62e^{i2\pi(0.62)}, \\ 0.12e^{i2\pi(0.12)} \end{pmatrix}, \begin{pmatrix} 0.22e^{i2\pi(0.22)}, \\ 0.24e^{i2\pi(0.24)} \end{pmatrix}, \begin{pmatrix} 0.21e^{i2\pi(0.21)}, \\ 0.25e^{i2\pi(0.25)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.23e^{i2\pi(0.23)}, \\ 0.23e^{i2\pi(0.23)} \end{pmatrix}, \left(0.1e^{i2\pi(0.1)}, 0.23e^{i2\pi(0.23)}\right) \right\} \end{array} \right], \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.5e^{i2\pi(0.5)}, \\ 0.04e^{i2\pi(0.04)} \end{pmatrix}, \begin{pmatrix} 0.31e^{i2\pi(0.31)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}, \begin{pmatrix} 0.04e^{i2\pi(0.04)}, \\ 0.43e^{i2\pi(0.43)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.12e^{i2\pi(0.12)} \\ 0.1e^{i2\pi(0.1)} \end{pmatrix}, \left(0.01e^{i2\pi(0.01)}, 0.01e^{i2\pi(0.01)}\right) \right\} \end{array} \right] \\ \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.32e^{i2\pi(0.32)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}, \begin{pmatrix} 0.33e^{i2\pi(0.33)}, \\ 0.43e^{i2\pi(0.34)} \end{pmatrix}, \begin{pmatrix} 0.4e^{i2\pi(0.43)}, \\ 0.5e^{i2\pi(0.5)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.12e^{i2\pi(0.12)} \end{pmatrix}, \left(0.2e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right) \right\} \end{array} \right], \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.31e^{i2\pi(0.31)}, \\ 0.32e^{i2\pi(0.32)} \end{pmatrix}, \begin{pmatrix} 0.32e^{i2\pi(0.32)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}, \begin{pmatrix} 0.33e^{i2\pi(0.33)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.24e^{i2\pi(0.6)}, \\ 0.23e^{i2\pi(0.23)} \end{pmatrix}, \left(0.2e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.1)}\right) \right\} \end{array} \right] \end{array} \right\},$$

$$R^{-} = \left\{ \begin{array}{l} \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.36e^{i2\pi(0.36)}, \\ 0.12e^{i2\pi(0.12)} \end{pmatrix}, \begin{pmatrix} 0.12e^{i2\pi(0.12)}, \\ 0.14e^{i2\pi(0.24)} \end{pmatrix}, \begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.15e^{i2\pi(0.15)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.33e^{i2\pi(0.33)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}, \left(0.34e^{i2\pi(0.34)}, 0.31e^{i2\pi(0.31)}\right) \right\} \end{array} \right], \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.15e^{i2\pi(0.15)}, \\ 0.01e^{i2\pi(0.01)} \end{pmatrix}, \begin{pmatrix} 0.31e^{i2\pi(0.31)}, \\ 0.13e^{i2\pi(0.13)} \end{pmatrix}, \begin{pmatrix} 0.04e^{i2\pi(0.04)}, \\ 0.14e^{i2\pi(0.14)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.32e^{i2\pi(0.32)} \\ 0.31^{i2\pi(0.31)} \end{pmatrix}, \left(0.04e^{i2\pi(0.04)}, 0.4e^{i2\pi(0.4)}\right) \right\} \end{array} \right] \\ \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.12e^{i2\pi(0.12)}, \\ 0.31e^{i2\pi(0.31)} \end{pmatrix}, \begin{pmatrix} 0.13e^{i2\pi(0.13)}, \\ 0.14e^{i2\pi(0.14)} \end{pmatrix}, \begin{pmatrix} 0.14e^{i2\pi(0.4)}, \\ 0.25e^{i2\pi(0.25)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.31e^{i2\pi(0.31)}, \\ 0.32e^{i2\pi(0.32)} \end{pmatrix}, \left(0.32e^{i2\pi(0.2)}, 0.33e^{i2\pi(0.33)}\right) \right\} \end{array} \right], \left[\begin{array}{c} \left\{ \begin{pmatrix} 0.1e^{i2\pi(0.1)}, \\ 0.2e^{i2\pi(0.2)} \end{pmatrix}, \begin{pmatrix} 0.2e^{i2\pi(0.2)}, \\ 0.3e^{i2\pi(0.3)} \end{pmatrix}, \begin{pmatrix} 0.3e^{i2\pi(0.3)}, \\ 0.4e^{i2\pi(0.4)} \end{pmatrix} \right\}, \\ \left\{ \begin{pmatrix} 0.41e^{i2\pi(0.41)}, \\ 0.33e^{i2\pi(0.33)} \end{pmatrix}, \left(0.32e^{i2\pi(0.32)}, 0.31e^{i2\pi(0.31)}\right) \right\} \end{array} \right] \end{array} \right\}. \tag{36}$$

Table 1: Original decision matrix, all items of which are in the form of complex numbers.

Symbols	u_1	u_2
c_1	$\left\{\left\{\left(0.6e^{i2\pi(0.6)}, 0.1e^{i2\pi(0.1)}\right), \left(0.2e^{i2\pi(0.2)}, 0.4e^{i2\pi(0.4)}\right), \left(0.1e^{i2\pi(0.1)}, 0.5e^{i2\pi(0.5)}\right)\right\}, \left\{\left(0.3e^{i2\pi(0.3)}, 0.3e^{i2\pi(0.3)}\right), \left(0.1e^{i2\pi(0.1)}, 0.3e^{i2\pi(0.3)}\right)\right\}\right\}$	$\left\{\left\{\left(0.5e^{i2\pi(0.5)}, 0.01e^{i2\pi(0.01)}\right), \left(0.1e^{i2\pi(0.1)}, 0.3e^{i2\pi(0.3)}\right), \left(0.01e^{i2\pi(0.01)}, 0.4e^{i2\pi(0.4)}\right)\right\}, \left\{\left(0.2e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.1)}\right), \left(0.01e^{i2\pi(0.01)}, 0.03e^{i2\pi(0.03)}\right)\right\}\right\}$
c_2	$\left\{\left\{\left(0.61e^{i2\pi(0.61)}, 0.11e^{i2\pi(0.11)}\right), \left(0.12e^{i2\pi(0.12)}, 0.14e^{i2\pi(0.14)}\right), \left(0.11e^{i2\pi(0.11)}, 0.15e^{i2\pi(0.51)}\right)\right\}, \left\{\left(0.13e^{i2\pi(0.13)}, 0.31e^{i2\pi(0.13)}\right), \left(0.11e^{i2\pi(0.11)}, 0.13e^{i2\pi(0.13)}\right)\right\}\right\}$	$\left\{\left\{\left(0.15e^{i2\pi(0.15)}, 0.02e^{i2\pi(0.02)}\right), \left(0.11e^{i2\pi(0.11)}, 0.13e^{i2\pi(0.13)}\right), \left(0.02e^{i2\pi(0.02)}, 0.14e^{i2\pi(0.14)}\right)\right\}, \left\{\left(0.12e^{i2\pi(0.12)}, 0.11e^{i2\pi(0.11)}\right), \left(0.02e^{i2\pi(0.02)}, 0.01e^{i2\pi(0.01)}\right)\right\}\right\}$
c_3	$\left\{\left\{\left(0.62e^{i2\pi(0.62)}, 0.12e^{i2\pi(0.12)}\right), \left(0.22e^{i2\pi(0.22)}, 0.24e^{i2\pi(0.24)}\right), \left(0.21e^{i2\pi(0.21)}, 0.25e^{i2\pi(0.25)}\right)\right\}, \left\{\left(0.23e^{i2\pi(0.23)}, 0.23e^{i2\pi(0.23)}\right), \left(0.21e^{i2\pi(0.21)}, 0.23e^{i2\pi(0.23)}\right)\right\}\right\}$	$\left\{\left\{\left(0.25e^{i2\pi(0.25)}, 0.03e^{i2\pi(0.03)}\right), \left(0.21e^{i2\pi(0.21)}, 0.23e^{i2\pi(0.23)}\right), \left(0.03e^{i2\pi(0.3)}, 0.24e^{i2\pi(0.24)}\right)\right\}, \left\{\left(0.22e^{i2\pi(0.22)}, 0.21e^{i2\pi(0.21)}\right), \left(0.02e^{i2\pi(0.2)}, 0.02e^{i2\pi(0.02)}\right)\right\}\right\}$
$c4$	$\left\{\left\{\left(0.36e^{i2\pi(0.36)}, 0.31e^{i2\pi(0.13)}\right), \left(0.32e^{i2\pi(0.32)}, 0.34e^{i2\pi(0.34)}\right), \left(0.31e^{i2\pi(0.31)}, 0.35e^{i2\pi(0.35)}\right)\right\}, \left\{\left(0.33e^{i2\pi(0.33)}, 0.33e^{i2\pi(0.33)}\right), \left(0.31e^{i2\pi(0.31)}, 0.33e^{i2\pi(0.33)}\right)\right\}\right\}$	$\left\{\left\{\left(0.35e^{i2\pi(0.35)}, 0.04e^{i2\pi(0.04)}\right), \left(0.31e^{i2\pi(0.31)}, 0.33e^{i2\pi(0.33)}\right), \left(0.04e^{i2\pi(0.04)}, 0.43e^{i2\pi(0.34)}\right)\right\}; \left\{\left(0.32e^{i2\pi(0.32)}, 0.31e^{i2\pi(0.31)}\right), \left(0.04e^{i2\pi(0.04)}, 0.04e^{i2\pi(0.04)}\right)\right\}\right\}$
	u_3	u_4
c_1	$\left\{\left\{\left(0.2e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right), \left(0.3e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.4)}\right), \left(0.4e^{i2\pi(0.4)}, 0.5e^{i2\pi(0.5)}\right)\right\}, \left\{\left(0.01e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.2)}\right), \left(0.2e^{i2\pi(0.2)}, 0.3e^{i2\pi(0.3)}\right)\right\}\right\}$	$\left\{\left\{\left(0.1e^{i2\pi(0.1)}, 0.2e^{i2\pi(0.6)}\right), \left(0.2e^{i2\pi(0.6)}, 0.3e^{i2\pi(0.3)}\right), \left(0.3e^{i2\pi(0.3)}, 0.4e^{i2\pi(0.4)}\right)\right\}, \left\{\left(0.4e^{i2\pi(0.4)}, 0.3e^{i2\pi(0.3)}\right), \left(0.2e^{i2\pi(0.2)}, 0.1e^{i2\pi(0.1)}\right)\right\}\right\}$
c_2	$\left\{\left\{\left(0.12e^{i2\pi(0.12)}, 0.31e^{i2\pi(0.31)}\right), \left(0.13e^{i2\pi(0.13)}, 0.14e^{i2\pi(0.14)}\right), \left(0.14e^{i2\pi(0.41)}, 0.51e^{i2\pi(0.51)}\right)\right\}, \left\{\left(0.11e^{i2\pi(0.11)}, 0.12e^{i2\pi(0.12)}\right), \left(0.21e^{i2\pi(0.12)}, 0.31e^{i2\pi(0.31)}\right)\right\}\right\}$	$\left\{\left\{\left(0.11e^{i2\pi(0.11)}, 0.21e^{i2\pi(0.21)}\right), \left(0.21e^{i2\pi(0.21)}, 0.31e^{i2\pi(0.31)}\right), \left(0.31e^{i2\pi(0.31)}, 0.41e^{i2\pi(0.41)}\right)\right\}, \left\{\left(0.41e^{i2\pi(0.41)}, 0.31e^{i2\pi(0.31)}\right), \left(0.21e^{i2\pi(0.21)}, 0.11e^{i2\pi(0.11)}\right)\right\}\right\}$
c_3	$\left\{\left\{\left(0.22e^{i2\pi(0.22)}, 0.32e^{i2\pi(0.23)}\right), \left(0.23e^{i2\pi(0.23)}, 0.24e^{i2\pi(0.24)}\right), \left(0.24e^{i2\pi(0.42)}, 0.25e^{i2\pi(0.25)}\right)\right\}, \left\{\left(0.21e^{i2\pi(0.21)}, 0.22e^{i2\pi(02.2)}\right), \left(0.22e^{i2\pi(0.22)}, 0.23e^{i2\pi(0.23)}\right)\right\}\right\}$	$\left\{\left\{\left(0.21e^{i2\pi(0.21)}, 0.22e^{i2\pi(0.22)}\right), \left(0.22e^{i2\pi(0.22)}, 0.23e^{i2\pi(0.23)}\right), \left(0.23e^{i2\pi(0.23)}, 0.24e^{i2\pi(0.24)}\right)\right\}, \left\{\left(0.24e^{i2\pi(0.24)}, 0.23e^{i2\pi(0.23)}\right), \left(0.22e^{i2\pi(0.22)}, 0.12e^{i2\pi(0.21)}\right)\right\}\right\}$
c_4	$\left\{\left\{\left(0.32e^{i2\pi(0.32)}, 0.33e^{i2\pi(0.33)}\right), \left(0.33e^{i2\pi(0.33)}, 0.43e^{i2\pi(0.34)}\right), \left(0.34e^{i2\pi(0.43)}, 0.35e^{i2\pi(0.35)}\right)\right\}, \left\{\left(0.31e^{i2\pi(0.31)}, 0.32e^{i2\pi(0.32)}\right), \left(0.32e^{i2\pi(0.32)}, 0.33e^{i2\pi(0.33)}\right)\right\}\right\}$	$\left\{\left\{\left(0.31e^{i2\pi(0.31)}, 0.32e^{i2\pi(0.32)}\right), \left(0.32e^{i2\pi(0.32)}, 0.33e^{i2\pi(0.33)}\right), \left(0.33e^{i2\pi(0.33)}, 0.34e^{i2\pi(0.34)}\right)\right\}, \left\{\left(0.34e^{i2\pi(0.34)}, 0.33e^{i2\pi(0.33)}\right), \left(0.32e^{i2\pi(0.32)}, 0.31e^{i2\pi(0.31)}\right)\right\}\right\}$

(iv) Step 4: by using (22), we examine the complex dual type-2 hesitant fuzzy PIS, and we have

$$\begin{aligned} K_1\left(r_{1j}, R^+\right) &= 0.7782,\\ K_2\left(r_{2j}, R^+\right) &= 0.6645,\\ K_3\left(r_{3j}, R^+\right) &= 0.6612,\\ K_4\left(r_{4j}, R^+\right) &= 0.5534. \end{aligned} \tag{37}$$

We also examine complex dual type-2 hesitant NIS by using equation (34), and we have

$$\begin{aligned} K_1\left(r_{1j}, R^-\right) &= 0.7146,\\ K_2\left(r_{2j}, R^-\right) &= 0.5627,\\ K_3\left(r_{3j}, R^-\right) &= 0.5537,\\ K_4\left(r_{4j}, R^-\right) &= 0.5718. \end{aligned} \tag{38}$$

(v) Step 5: by using (32), we examine the closeness of each of the alternatives, and we have

$$\mathscr{P}_1 = 0.5213,$$
$$\mathscr{P}_2 = 0.5415,$$
$$\mathscr{P}_3 = 0.5442,$$
$$\mathscr{P}_4 = 0.4918. \tag{39}$$

(v) Step 6: we rank all alternatives and examine the best optimal one.

$$\mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1 \geq \mathscr{P}_4. \tag{40}$$

(vii) Step 7: the end. The comparative analysis of the explored measure with existing measures [30] is discussed in the form of Table 2.

6.3. Medical Diagnosis. In this study, we explored the idea of the algorithm which is taken from [31] which is very effective and meaningful for explored works. The new algorithm utilizes the complex dual type-2 hesitant fuzzy similarity and entropy measures which obtain excellent results in application.

Problem statement: suppose that five patients, namely, Lil, Jones, Deby, Ramot, and Inas, visit a given laboratory for medical diagnosis. They are observed to have the following symptoms: heart pain, temperature, cough, liver pain, and kidney pain. That is, the set of patients QCDTH is as follows:

$$\mathcal{Q}_{\text{CDTH}} = \{\text{viz, Lil, Jones, Deby, Ramot, Inas}\}, \tag{41}$$

and the set of symptoms X is as follows:

$$X = \{x_1\,(\text{heart pain}), x_2\,(\text{temperature}), x_3\,(\text{cough}), x_4\,(\text{liver pain}),\ x_5\,(\text{kidney pain})\}. \tag{42}$$

Then we will find which patient has which kind of disease. The information related to this problem is given in Example 3, which is discussed below.

Example 3. Suppose a set of diagnoses $\mathcal{Q}_{\text{CDTH}} = \left\{ \begin{array}{c} \mathcal{Q}_{\text{CDTH}-1}\,(\text{Heart problem}), \mathcal{Q}_{CDTH-2}\,(\text{Fever}), \mathcal{Q}_{\text{CDTH}-3}\,(\text{Flu}), \\ \mathcal{Q}_{\text{CDTH}-4}\,(\text{Liver problem}), \mathcal{Q}_{CDTH-5}\,(\text{Kidney problem}) \end{array} \right\}$ and a set of symptoms $X = \left\{ \begin{array}{c} x_1\,(\text{heart problem}), x_2\,(\text{temperature}), \\ x_3\,(\text{cough}), x_4\,(\text{liver pain}), x_5\,(\text{kidney pain}) \end{array} \right\}$. Suppose a sick person according to all symptoms is represented by the CDT-2HFSs given as follows:

$$\mathcal{Q}_{CDTH} = \left\{ \begin{array}{c} \left(0.91e^{i2\pi(0.91)}, 0.92e^{i2\pi(0.91)}\right), \left(0.93e^{i2\pi(0.93)}, 0.94e^{i2\pi(0.94)}\right), \left(0.95e^{i2\pi(0.95)}\right), 0.86e^{i2\pi(0.85)} \\ \left(0.002e^{i2\pi(0.003)}, 0.004e^{i2\pi(0.003)}\right), \left(0.003e^{i2\pi(0.003)}, 0.001e^{i2\pi(0.001)}\right) \end{array} \right\}. \tag{43}$$

All diagnoses $\mathcal{Q}_{\text{CDTH}-i}\,(i = 1, 2, 3, 4, 5)$ that can be represented as CDT-2HFSs according to all symptoms are given below:

$$\begin{aligned}
\mathcal{Q}_{\text{CDTH}-1}\,(\text{heart problem}) &= \left\{ \begin{array}{c} \left\{\left(0.81e^{i2\pi(0.71)}, 0.76e^{i2\pi(0.57)}\right), \left(0.82e^{i2\pi(0.72)}, 0.77e^{i2\pi(0.57)}\right), \left(0.85e^{i2\pi(0.75)}, 0.8e^{i2\pi(0.61)}\right)\right\}, \\ \left\{\left(0.07e^{i2\pi(0.06)}, 0.06e^{i2\pi(0.07)}\right), \left(0.02e^{i2\pi(0.02)}, 0.07e^{i2\pi(0.08)}\right)\right\} \end{array} \right\}, \\
\mathcal{Q}_{\text{CDTH}-1}\,(\text{fever}) &= \left\{ \begin{array}{c} \left\{\left(0.82e^{i2\pi(0.72)}, 0.77e^{i2\pi(0.58)}\right), \left(0.83e^{i2\pi(0.73)}, 0.78e^{i2\pi(0.59)}\right), \left(0.86e^{i2\pi(0.76)}, 0.81e^{i2\pi(0.62)}\right)\right\}, \\ \left\{\left(0.08e^{i2\pi(0.07)}, 0.07e^{i2\pi(0.08)}\right), \left(0.03e^{i2\pi(0.03)}, 0.08e^{i2\pi(0.05)}\right)\right\} \end{array} \right\}, \\
Q_{CDTH-4}\,(\text{flu}) &= \left\{ \begin{array}{c} \left\{\left(0.83e^{i2\pi(0.73),}, 0.78e^{i2\pi(0.59)}\right), \left(0.837e^{i2\pi(0.74)}, 0.79e^{i2\pi(0.6)}\right), \left(0.87e^{i2\pi(0.77)}, 0.82e^{i2\pi(0.63)}\right)\right\}, \\ \left\{\left(0.01e^{i2\pi(0.01),}, 0.03e^{i2\pi(0.06)}\right), \left(0.04e^{i2\pi(0.01),}, 0.00e^{i2\pi(0.00)}\right)\right\} \end{array} \right\}, \\
Q_{CDTH-4}\,(\text{liver problem}) &= \left\{ \begin{array}{c} \left\{\left(0.83e^{i2\pi(0.73),}, 0.78e^{i2\pi(0.59)}\right), \left(0.84e^{i2\pi(0.74)}, 0.79e^{i2\pi(0.6)}\right), \left(0.87e^{i2\pi(0.77)}, 0.82e^{i2\pi(0.63)}\right)\right\}, \\ \left\{\left(0.03e^{i2\pi(0.04),}, 0.05e^{i2\pi(0.06)}\right), \left(0.06e^{i2\pi(0.03),}, 0.02e^{i2\pi(0.08)}\right)\right\} \end{array} \right\}, \\
Q_{CDTH-4}\,(\text{kidney problem}) &= \left\{ \begin{array}{c} \left\{\left(0.84e^{i2\pi(0.74),}, 0.77e^{i2\pi(0.6)}\right), \left(0.84e^{i2\pi(0.75)}, 0.8e^{i2\pi(0.61)}\right), \left(0.88e^{i2\pi(0.78)}, 0.83e^{i2\pi(0.64)}\right)\right\}, \\ \left\{\left(0.04e^{i2\pi(0.03),}, 0.05e^{i2\pi(0.05)}\right), \left(0.06e^{i2\pi(0.02),}, 0.03e^{i2\pi(0.07)}\right)\right\} \end{array} \right\}.
\end{aligned} \tag{44}$$

Table 2: Comparison of the explored work with existing work.

Method	Score values	Ranking
Karaaslan et al. [30]	$\mathscr{P}_1 = 0.5249, \mathscr{P}_2 = 0.5417, \mathscr{P}_3 = 0.5439, \mathscr{P}_4 = 0.4923$	$\mathscr{P}_1 \geq \mathscr{P}_2 \geq \mathscr{P}_1 \geq \mathscr{P}_4$
Explored approach	$\mathscr{P}_1 = 0.5213, \mathscr{P}_2 = 0.5415, \mathscr{P}_3 = 0.5442, \mathscr{P}_4 = 0.4918$	$\mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1 \geq \mathscr{P}_4$

The geometrical representation of the explored measures, which are discussed in Table 2, is described with the help of Figure 1.

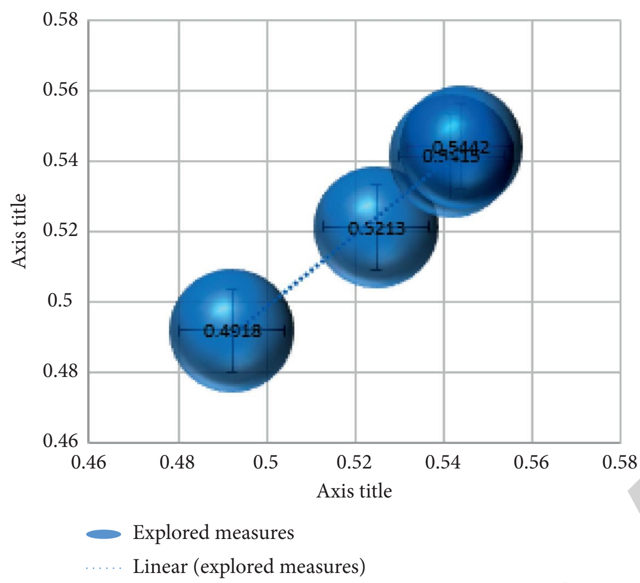

Figure 1: Graphical representation for the information of Table 2.

The aim of this work is to examine the best alternative from the family of alternatives by using the measures. The information of the resultant values of the explored measure and some existing measures is stated in Table 3. The, $\c{C}_{\mathrm{CDTH-mcc-i}}(\mathrm{Q}_{\mathrm{CDTH-1}}, \mathrm{Q}_{\mathrm{CDTH-2}}) = \mathrm{P}_i, i = 1, 2, 3, 4, 5,$ are follow as.

6.4. Pattern Recognition. The instruments of likeness measures have applications in design classification. In such a marvel, the class of an obscure example or item is discovered utilizing some likeness estimating devices and a few inclinations of leaders. In this segment, the likeness estimates that grew so far in Section 3 are applied to an example acknowledgment (building design acknowledgment) issue, where the class of an obscure structure material should have been assessed. The outcomes got utilizing the similitude proportions of CHFSs are then examined for portrayal of the benefits of proposed work and the constraints of existing work. To clarify the marvel, an illustrative model adjusted from [31] is discussed.

To evaluate the proficiency of the explored measures, we adopt the pattern recognition model form [31]. The purpose of this application is to find the reliability and skill of the presented measures; we solve a numerical example that contains the CDT-2HFNs and utilized it in the environment of pattern recognition.

Example 4. We consider five knowns with their class labels being represented as follows: $P_{CQ-1}, P_{CQ-2}, P_{CQ-3}, P_{CQ-4}$, and P_{CQ-5} and $\mathrm{Q}_{\mathrm{CDTH-1}}, \mathrm{Q}_{\mathrm{CDTH-2}}, \mathrm{Q}_{\mathrm{CDTH-3}}, \mathrm{Q}_{\mathrm{CDTH-4}}$, and $\mathrm{Q}_{\mathrm{CDTH-5}}$. The information of the above patterns is in the form of CDT-2HFNs for universal set $X_{\mathrm{UNI}} = \{x_1, x_2, x_3, x_4, x_5\}$, which is stated as follows:

$$
\begin{aligned}
\mathrm{Q}_{\mathrm{CDTH-1}} &= \left\{ \begin{array}{c} \left\{\left(0.8e^{i2\pi(0.7)}, 0.75e^{i2\pi(0.56)}\right), \left(0.81e^{i2\pi(0.71)}, 0.76e^{i2\pi(0.57)}\right), \left(0.84e^{i2\pi(0.74)}, 0.79e^{i2\pi(0.60)}\right)\right\}, \\ \left\{\left(0.08e^{i2\pi(0.07)}, 0.07e^{i2\pi(0.08)}\right), \left(0.03e^{i2\pi(0.03)}, 0.08e^{i2\pi(0.09)}\right)\right\} \end{array} \right\}, \\
\mathrm{Q}_{\mathrm{CDTH-2}} &= \left\{ \begin{array}{c} \left\{\left(0.81e^{i2\pi(0.71)}, 0.76e^{i2\pi(0.57)}\right), \left(0.82e^{i2\pi(0.72)}, 0.77e^{i2\pi(0.58)}\right), \left(0.85e^{i2\pi(0.75)}, 0.8e^{i2\pi(0.61)}\right)\right\}, \\ \left\{\left(0.09e^{i2\pi(0.08)}, 0.08e^{i2\pi(0.09)}\right), \left(0.04e^{i2\pi(0.04)}, 0.09e^{i2\pi(0.06)}\right)\right\} \end{array} \right\}, \\
\mathrm{Q}_{\mathrm{CDTH-3}} &= \left\{ \begin{array}{c} \left\{\left(0.82e^{i2\pi(0.72)}, 0.77e^{i2\pi(0.58)}\right), \left(0.83e^{i2\pi(0.73)}, 0.78e^{i2\pi(0.59)}\right), \left(0.86e^{i2\pi(0.76)}, 0.81e^{i2\pi(0.62)}\right)\right\}, \\ \left\{\left(0.02e^{i2\pi(0.03)}, 0.04e^{i2\pi(0.05)}\right), \left(0.05e^{i2\pi(0.02)}, 0.01e^{i2\pi(0.07)}\right)\right\} \end{array} \right\}, \\
\mathrm{Q}_{\mathrm{CDTH-4}} &= \left\{ \begin{array}{c} \left\{\left(0.83e^{i2\pi(0.73)}, 0.78e^{i2\pi(0.59)}\right), \left(0.84e^{i2\pi(0.74)}, 0.79e^{i2\pi(0.6)}\right), \left(0.87e^{i2\pi(0.77)}, 0.82e^{i2\pi(0.63)}\right)\right\}, \\ \left\{\left(0.03e^{i2\pi(0.03)}, 0.05e^{i2\pi(0.06)}\right), \left(0.06e^{i2\pi(0.03)}, 0.02e^{i2\pi(0.08)}\right)\right\} \end{array} \right\}, \\
\mathrm{Q}_{\mathrm{CDTH-5}} &= \left\{ \begin{array}{c} \left\{\left(0.84e^{i2\pi(0.74)}, 0.77^{i2\pi(0.6)}\right), \left(0.84e^{i2\pi(0.75)}, 0.8^{i2\pi(0.61)}\right), \left(0.88e^{i2\pi(0.78)}, 0.83^{i2\pi(0.64)}\right)\right\}, \\ \left\{\left(0.04e^{i2\pi(0.03)}, 0.05^{i2\pi(0.05)}\right), \left(0.06e^{i2\pi(0.02)}, 0.03^{i2\pi(0.07)}\right)\right\} \end{array} \right\}.
\end{aligned} \tag{45}
$$

Table 3: Comparison of the explored work with existing work.

Method	Measures	Ranking
Karaaslan et al. [30]	$\mathscr{P}_1 = 0.803, \mathscr{P}_2 = 0.853, \mathscr{P}_3 = 0.889, \mathscr{P}_4 = 0.902, \mathscr{P}_5 = 0.943$	$\mathscr{P}_5 \geq \mathscr{P}_4 \geq \mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1$
Explored approach, equation (18)	$\mathscr{P}_1 = 0.834, \mathscr{P}_2 = 0.851, \mathscr{P}_3 = 0.888, \mathscr{P}_4 = 0.916, \mathscr{P}_5 = 0.951$	$\mathscr{P}_5 \geq \mathscr{P}_4 \geq \mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1$

The geometrical representation of the explored measures, which is discussed in Table 3, is described with the help of Figure 2.

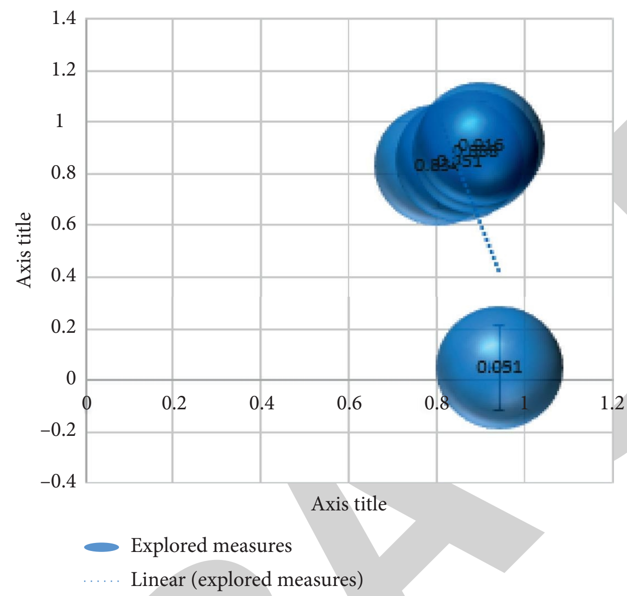

Figure 2: Graphical representation for the information of Table 3.

Their unknown pattern is stated as follows:

$$Q_{\text{CDTH}} = \left\{ \begin{array}{c} \left(0.9e^{i2\pi(0.9)}, 0.91e^{i2\pi(0.91)}\right), \left(0.92e^{i2\pi(0.92)}, 0.93e^{i2\pi(0.93)}\right)\left(0.94e^{i2\pi(0.94)}, 0.85e^{i2\pi(0.85)}\right) \\ \left(0.02e^{i2\pi(0.03)}, 0.04e^{i2\pi(0.03)}\right), \left(0.03e^{i2\pi(0.03)}, 0.01e^{i2\pi(0.01)}\right) \end{array} \right\}. \tag{46}$$

The aim of this work is to examine the best alternative from the family of alternatives by using the measures. The information of the resultant values of the explored measure and some existing measures is stated in Table 4. The, $\text{Ç}_{\text{CDTHF-mcc-i}}(Q_{\text{CCDTH}-1}, Q_{\text{CDTH}-2}) = \mathscr{P}_i$, $i = 1, 2, 3, 4, 5$, are follow as.

The geometrical representation of the explored measures, which is discussed in Table 4, is described with the help of Figure 3.

6.5. Clustering Algorithm Based on CDT-2HFSs. The aim of this study is to present the clustering algorithm based on the novel approach of CDT-2HFSs to examine the reliability and proficiency of the explored approach. For this, we choose the set of alternatives and their attributes with weight vectors, whose expressions are in the form of $A_{\text{AL}} = \{A_{\text{AL}-1}, A_{\text{AL}-2}, \ldots, A_{\text{AL}-m}\}, C_{\text{AT}} = \{C_{\text{AT}-1}, C_{\text{AT}-2}, \ldots, C_{\text{AT}-n}\}$, and $\Omega = (\Omega_1, \Omega_2, \ldots, \Omega_n)^T$ and $\Omega_j \in [0,1], \sum_{j=1}^{n} \Omega_j = 1$. The technique of the clustering algorithm is summarized as follows:

Step 1: construct the decision matrix, all entities of which are in the form of CDT-2HFSs.

Step 2: construct the correlation matrix by using (12). The correlation matrix is expressed by $\text{Ç}_{\text{CDTHF-cc}} = \text{Ç}_{\text{CDTHF-cc-}yz}(Q_{\text{CDTH-}y}, Q_{\text{CDTH-}z})$ where $(\text{Ç}_{\text{CDTHF-cc}})_{m\times m} = C$.

Step 3: we checked whether the correlation matrix C satisfies $C^2 \subseteq C$, where $C = C \circ C = (\text{Ç}_{\text{CDTHF-cc-}yz})_{m\times m}, \text{Ç}_{\text{CDTHF-cc-}yz} = \max_z\{\min\{\text{Ç}_{\text{CDTHF-cc-}yx}, \text{Ç}_{\text{CDTHF-cc-}xz}\}\}$. If C does not satisfy condition $C^2 \subseteq C$, then the equivalent correlation matrix C^{2^k} will be formed: $C \longrightarrow C^2 \longrightarrow C^4 \longrightarrow \cdots \longrightarrow C^{2^k} \longrightarrow \cdots$ until $C^{2^{k+1}}$.

Step 4: furthermore, we examine the λ-cutting matrix by using

Table 4: Comparison of the explored work with existing work.

Method	Measures	Ranking
Karaaslan and Ozlu [30]	$\mathscr{P}_1 = 0.814, \mathscr{P}_2 = 0.862, \mathscr{P}_3 = 0.892, \mathscr{P}_4 = 0.911, \mathscr{P}_5 = 0951$	$\mathscr{P}_5 \geq \mathscr{P}_4 \geq \mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1$
Explored approach equation (18)	$\mathscr{P}_1 = 0.845, \mathscr{P}_2 = 0.866, \mathscr{P}_3 = 0.897, \mathscr{P}_4 = 0.913, \mathscr{P}_5 = 0.947$	$\mathscr{P}_5 \geq \mathscr{P}_4 \geq \mathscr{P}_3 \geq \mathscr{P}_2 \geq \mathscr{P}_1$

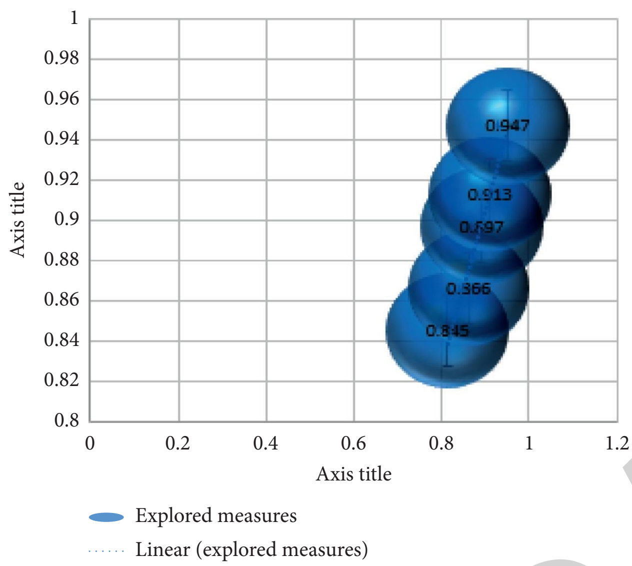

Figure 3: Graphical representation for the information of Table 4.

$$C_\lambda = \left(\lambda Ç_{\text{CDTHF-cc-}yz}\right)_{m\times m} = \begin{cases} 0, & Ç_{\text{CDTHF-cc-}yz} < \lambda, \\ 1, & Ç_{\text{CDTHF-cc-}yz} \geq \lambda, \end{cases} \tag{47}$$

where $\lambda \in [0,1]$ denotes the confidence level.

Step 5: we build up all possible classifications based on λ-cutting matrix. If all elements of the yth line (column) in C_λ are the same as the corresponding elements of the zth line (column) in C_λ, then the CDT-2HFS D_y and D_z are of the same type. For simplicity, we draw the graphical shape of the explored clustering algorithm, which is stated with the help of Table 5.

Example 5. (see [30]). Consider a speculation organization that needs to put a total cash in the most ideal choice and along these lines organization officials decide five choices by considering different standards to recognize the best choice to put away the cash: (a) $\mathscr{A}_{\text{AL}-5}$ is a structure organization; (b) $\mathscr{A}_{\text{AL}-2}$ is an airplane organization; (c) $\mathscr{A}_{\text{AL}-3}$ is a food organization; (d) $\mathscr{A}_{\text{AL}-4}$ is an electronic things organization; (e) $\mathscr{A}_{\text{AL}-5}$ is a cowhide organization; (f) $\mathscr{A}_{\text{AL}-6}$ is a vehicle organization; (g) $\mathscr{A}_{\text{AL}-7}$ is a correspondence organization; (h) $\mathscr{A}_{\text{AL}-8}$ is a product organization; (I) $\mathscr{A}_{\text{AL}-9}$ is a paper creation organization; (j) $\mathscr{A}_{\text{AL}-10}$ is a plastic creation organization. The speculation organization must make a choice as indicated by the five rules: (a) x_1 is the transportation; (b) x_2 is the work; (c) x_3 is an ecological effect; (d) $x4$ is the vicinity to crude material; (e) x_5 is the experience. The loads of standards x_1, x_2, x_3, x_4 and x_5 are given by $\Omega_1 = 0.2, \Omega_2 = 0.25, \Omega_3 = 0.1, \Omega_4 = 0.3, \Omega_5 = 0.15,$ individually. The 10 choices are assessed under the standards by etymological evaluations yielded in Table 6 and given by decision-makers. The technique of the clustering algorithm is summarized as follows:

Step 1: we construct the decision matrix, all entities of which are in the form of CDT-2HFSs; see Table 5.

Step 2: we construct the correlation matrix by using equation (22). The correlation matrix is expressed by $Ç_{\text{CDTHF-cc}} = Ç_{\text{CDTHF-cc-yz}}(Q_{\text{CDTH-}y}, Q_{\text{CDTH-}z})$, where $(Ç_{\text{CDTHF-cc}})_{m\times m} = C$, such that

$$Ç_{\text{CDTHF-cc}}{}^2 = \begin{bmatrix} 1 & 0.79 & 0.85 & 0.81 & 0.8 & 0.72 & 0.75 & 0.66 & 0.67 & 0.7 \\ 0.79 & 1 & 0.83 & 0.81 & 0.75 & 0.72 & 0.65 & 0.65 & 0.64 & 0.69 \\ 0.85 & 0.83 & 1 & 0.89 & 0.89 & 0.88 & 0.82 & 0.75 & 0.74 & 0.78 \\ 0.81 & 0.81 & 0.89 & 1 & 0.87 & 0.8 & 0.76 & 0.72 & 0.69 & 0.79 \\ 0.8 & 0.75 & 0.89 & 0.87 & 1 & 0.85 & 0.83 & 0.73 & 0.75 & 0.8 \\ 0.72 & 0.72 & 0.88 & 0.8 & 0.85 & 1 & 0.87 & 0.79 & 0.83 & 0.82 \\ 0.75 & 0.65 & 0.82 & 0.76 & 0.83 & 0.87 & 1 & 0.84 & 0.83 & 0.85 \\ 0.66 & 0.65 & 0.75 & 0.75 & 0.73 & 0.79 & 0.84 & 1 & 0.92 & 0.89 \\ 0.67 & 0.64 & 0.74 & 0.69 & 0.75 & 0.83 & 0.83 & 0.92 & 1 & 0.86 \\ 0.7 & 0.69 & 0.78 & 0.79 & 0.8 & 0.82 & 0.85 & 0.89 & 0.86 & 1 \end{bmatrix}. \tag{48}$$

Step 3: we checked whether the correlation matrix C satisfies $C^2 \subseteq C$, where $C^2 = C \circ C = (Ç_{CDTHF-cc-yz})_{m\times m}$, $Ç_{CDTHF-cc-yz} = \max_z\{\min\{Ç_{CDTHF-cc-yx}, Ç_{CDTHF-cc-xz}\}\}$.

$$Ç_{CDTHF-cc}{}^{2} = \begin{bmatrix} 1 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.82 & 0.75 & 0.75 & 0.8 \\ 0.83 & 1 & 0.83 & 0.83 & 0.83 & 0.83 & 0.82 & 0.75 & 0.75 & 0.79 \\ 0.85 & 0.83 & 1 & 0.89 & 0.89 & 0.88 & 0.87 & 0.82 & 0.83 & 0.82 \\ 0.85 & 0.83 & 0.89 & 1 & 0.89 & 0.88 & 0.83 & 0.79 & 0.8 & 0.8 \\ 0.85 & 0.83 & 0.89 & 0.89 & 1 & 0.88 & 0.85 & 0.83 & 0.83 & 0.83 \\ 0.85 & 0.83 & 0.88 & 0.88 & 0.88 & 1 & 0.87 & 0.84 & 0.83 & 0.85 \\ 0.82 & 0.82 & 0.87 & 0.83 & 0.85 & 0.87 & 1 & 0.85 & 0.85 & 0.85 \\ 0.75 & 0.75 & 0.82 & 0.79 & 0.83 & 0.84 & 0.85 & 1 & 0.92 & 0.89 \\ 0.75 & 0.75 & 0.83 & 0.8 & 0.83 & 0.83 & 0.85 & 0.92 & 1 & 0.89 \\ 0.8 & 0.79 & 0.82 & 0.8 & 0.83 & 0.85 & 0.85 & 0.89 & 0.89 & 1 \end{bmatrix}, \tag{49}$$

where C does not satisfy the condition C2 ⊆C; then the equivalent correlation matrix C^{2^k} will be formed: $C \longrightarrow C^2 \longrightarrow C^4 \longrightarrow \cdots \longrightarrow C^{2^k} \longrightarrow \cdots$ until $C^{2^k} = C^{2^{k+1}}$. Then

$$Ç_{CDTHF-cc}{}^{4} = \begin{bmatrix} 1 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.84 & 0.83 & 0.85 \\ 0.83 & 1 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 \\ 0.85 & 0.83 & 1 & 0.89 & 0.89 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.89 & 1 & 0.89 & 0.88 & 0.87 & 0.84 & 0.83 & 0.85 \\ 0.85 & 0.83 & 0.89 & 0.89 & 1 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.88 & 0.88 & 0.88 & 1 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.87 & 0.87 & 0.87 & 0.87 & 1 & 0.85 & 0.85 & 0.85 \\ 0.84 & 0.83 & 0.85 & 0.84 & 0.85 & 0.85 & 0.85 & 1 & 0.92 & 0.89 \\ 0.83 & 0.83 & 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.92 & 1 & 0.89 \\ 0.8 & 0.79 & 0.82 & 0.8 & 0.83 & 0.85 & 0.85 & 0.89 & 0.89 & 1 \end{bmatrix},$$

$$Ç_{CDTHF-cc}{}^{8} = \begin{bmatrix} 1 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 \\ 0.83 & 1 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 \\ 0.85 & 0.83 & 1 & 0.89 & 0.89 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.89 & 1 & 0.89 & 0.88 & 0.87 & 0.84 & 0.83 & 0.85 \\ 0.85 & 0.83 & 0.89 & 0.89 & 1 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.88 & 0.88 & 0.88 & 1 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.87 & 0.87 & 0.87 & 0.87 & 1 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 1 & 0.92 & 0.89 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.92 & 1 & 0.89 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.89 & 0.89 & 1 \end{bmatrix},$$

$$\text{Ç}^{16}_{\text{CDTHF}-\text{cc}} = \begin{bmatrix} 1 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 \\ 0.83 & 1 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 & 0.83 \\ 0.85 & 0.83 & 1 & 0.89 & 0.89 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.89 & 1 & 0.89 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.89 & 0.89 & 1 & 0.88 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.88 & 0.88 & 0.88 & 1 & 0.87 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.87 & 0.87 & 0.87 & 0.87 & 1 & 0.85 & 0.85 & 0.85 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 1 & 0.92 & 0.89 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.92 & 1 & 0.89 \\ 0.85 & 0.83 & 0.85 & 0.85 & 0.85 & 0.85 & 0.85 & 0.89 & 0.89 & 1 \end{bmatrix} = \text{Ç}_{\text{CDTHF}-\text{cc}}{}^{8}. \quad (50)$$

Therefore, we get $\text{Ç}^{16}_{\text{CDTHF}-\text{cc}} = \text{Ç}_{\text{CDTHF}-\text{cc}}{}^{8}$.

Step 4: furthermore, we examine the λ-cutting matrix by using

$$C_{\lambda} = \left(\lambda \text{Ç}_{\text{CDTHF}-\text{cc}-yz}\right)_{m\times m} = \begin{cases} 0, & \text{Ç}_{C\,DT\,HF-cc-yz} < \lambda, \\ 1, & \text{Ç}_{C\,DT\,HF-cc-yz} \geq \lambda, \end{cases} \quad (51)$$

where $\lambda \in 0, 1$ denotes the confidence level, such that

$$C_{0<\lambda\leq 0.83} = \begin{bmatrix} 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \\ 1&1&1&1&1&1&1&1&1&1 \end{bmatrix}, C_{0.83<\lambda\leq 0.85} = \begin{bmatrix} 1&0&1&1&1&1&1&1&1&1 \\ 0&1&0&0&0&0&0&0&0&0 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \\ 1&0&1&1&1&1&1&1&1&1 \end{bmatrix},$$

$$C_{0.85<\lambda\leq 0.87} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \end{bmatrix}, C_{0.87<\lambda\leq 0.88} = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&1&1&1&1&1&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \\ 0&0&0&0&0&0&0&1&1&1 \end{bmatrix},$$

Table 5: All possible classifications based on λ-cutting matrices.

Classifications	Representations	Limitations
$\mathscr{A}_{\mathrm{AL-i}}$ of the same characteristic	$\left\{\begin{matrix}\mathscr{A}_{\mathrm{AL-1}}, \mathscr{A}_{\mathrm{AL-2}}, \mathscr{A}_{\mathrm{AL-3}}, \mathscr{A}_{\mathrm{AL-4}}, \mathscr{A}_{\mathrm{AL-5}}, \\ \mathscr{A}_{\mathrm{AL-6}}, \mathscr{A}_{\mathrm{AL-7}}, \mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\end{matrix}\right\}$	$0 < \lambda \leq 0.83$
$\mathscr{A}_{\mathrm{AL-i}}$ of the two characteristics	$\left\{\begin{matrix}\mathscr{A}_{\mathrm{AL-1}}, \mathscr{A}_{\mathrm{AL-2}}, \mathscr{A}_{\mathrm{AL-3}}, \mathscr{A}_{\mathrm{AL-4}}, \mathscr{A}_{\mathrm{AL-5}}, \\ \mathscr{A}_{\mathrm{AL-6}}, \mathscr{A}_{\mathrm{AL-7}}, \mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\end{matrix}\right\}, \{\mathscr{A}_{\mathrm{AL-2}}\}$	$0.83 < \lambda \leq 0.85$
$\mathscr{A}_{\mathrm{AL-i}}$ of the three characteristics	$\{\mathscr{A}_{\mathrm{AL-1}}\}, \left\{\begin{matrix}\mathscr{A}_{\mathrm{AL-3}}, \mathscr{A}_{\mathrm{AL-4}}, \mathscr{A}_{\mathrm{AL-5}}, \\ \mathscr{A}_{\mathrm{AL-6}}, \mathscr{A}_{\mathrm{AL-7}}\end{matrix}\right\}, \{\mathscr{A}_{\mathrm{AL-2}}\}, \{\mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\}$	$0.85 < \lambda \leq 0.87$
$\mathscr{A}_{\mathrm{AL-i}}$ of the five characteristics	$\{\mathscr{A}_{\mathrm{AL-1}}\}, \left\{\begin{matrix}\mathscr{A}_{\mathrm{AL-3}}, \mathscr{A}_{\mathrm{AL-4}}, \mathscr{A}_{\mathrm{AL-5}}, \\ \mathscr{A}_{\mathrm{AL-6}}\end{matrix}\right\}, \{\mathscr{A}_{\mathrm{AL-2}}\}, \{\mathscr{A}_{\mathrm{AL-7}}\}\{\mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\}$	$0.87 < \lambda \leq 0.88$
$\mathscr{A}_{\mathrm{AL-i}}$ of the six characteristics	$\{\mathscr{A}_{\mathrm{AL-1}}\}, \{\mathscr{A}_{\mathrm{AL-3}}, \mathscr{A}_{\mathrm{AL-4}}, \mathscr{A}_{\mathrm{AL-5}},\}, \{\mathscr{A}_{\mathrm{AL-2}}\}, \{\mathscr{A}_{\mathrm{AL-6}}\}, \{\mathscr{A}_{\mathrm{AL-7}}\}\{\mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\}$	$0.88 < \lambda \leq 0.89$
$\mathscr{A}_{\mathrm{AL-i}}$ of the nine characteristics	$\{\mathscr{A}_{\mathrm{AL-1}}\}, \{\mathscr{A}_{\mathrm{AL-3}}\}\{\mathscr{A}_{\mathrm{AL-4}}\}, \{\mathscr{A}_{\mathrm{AL-5}}\}\{\mathscr{A}_{\mathrm{AL-2}}\}, \{\mathscr{A}_{\mathrm{AL-6}}\}, \{\mathscr{A}_{\mathrm{AL-7}}\}\{\mathscr{A}_{\mathrm{AL-8}}, \mathscr{A}_{\mathrm{AL-9}}, \mathscr{A}_{\mathrm{AL-10}}\}$	$0.89 < \lambda \leq 0.92$
$\mathscr{A}_{\mathrm{AL-i}}$ of the ten characteristics	$\{\mathscr{A}_{\mathrm{AL-1}}\}, \{\mathscr{A}_{\mathrm{AL-2}}\}, \{\mathscr{A}_{\mathrm{AL-3}}\}, \{\mathscr{A}_{\mathrm{AL-4}}\}, \{\mathscr{A}_{\mathrm{AL-5}}\}, \{\mathscr{A}_{\mathrm{AL-6}}\}, \{\mathscr{A}_{\mathrm{AL-7}}\}, \{\mathscr{A}_{\mathrm{AL-8}}\}, \{\mathscr{A}_{\mathrm{AL-9}}\}, \{\mathscr{A}_{\mathrm{AL-10}}\}$	$0.92 < \lambda \leq 1$

$$
C_{0.88<\lambda\leq 0.89} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}, C_{0.89<\lambda\leq 0.92} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1
\end{bmatrix},
$$

$$
C_{0.92<\lambda\leq 1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}. \tag{52}
$$

Step 5: we build up all possible classifications based on λ-cutting matrix. If all elements of the yth line (column) in C_λ are the same as the corresponding elements of the zth line (column) in C_λ, then the CDT-2HFS D_y and D_z are of the same type. For simplicity, we draw the graphical shape of the explored clustering algorithm, which is stated with the help of Table 5.

Therefore, from the above analysis, we get the result that the explored notions and their measures are more powerful and more proficient than exiting measures. CDT-2HFS is a proficient technique to cope with unpredictable and awkward information in realistic decision problems. CDT-2HFS is composed of the grade of truth and the grade of falsity, and the grade of truth (also for falsity grade) contains the grade of primary and secondary parts in the form of polar coordinates with the condition that the sum of the maximum of the real part (also for imaginary part) of the primary grade (also for secondary grade) cannot exceed the unit interval.

Table 6: Common evaluations of alternatives performed by decision-makers.

	x_1	x_2	x_3	x_4	x_5
$\mathscr{A}_{AL-1}$	{{(0.6, 0.1), (0.2, 0.8), (0.1, 0.5)}, {(0.3, 0.6), (0.1, 0.3)}}	{{(0.59, 0.1), (0.4, 0.72), (0.3, 0.2)}, {(0.4, 0.5), (0.3, 0.1)}}	{{(0.7, 0.2), (0.6, 0.1), (0.2, 0.4)}, {(0.3, 0.3), (0.2, 0.0)}}	{{(0.3, 0.1), (0.25, 0.65), (0.2, 0.2)}, {(0.7, 0.3), (0.66, 0.60)}}	{{(0.8, 0.1), (0.27, 0.89), (0.14, 0.5)}, {(0.2, 0.6), (0.1, 0.34)}}
$\mathscr{A}_{AL-2}$	{{(0.35, 0.4), (0.2, 0.9), (0.1, 0.1)}, {(0.3, 0.5), (0.2, 0.8)}}	{{(0.4, 0.5), (0.1, 0.9), (0.1, 0.62)}, {(0.5, 0.3), (0.4, 0.3)}}	{{(0.7, 0.2), (0.6, 0.1), (0.2, 0.4)}, {(0.3, 0.3), (0.2, 0.0)}}	{{(0.35, 0.45), (0.3, 0.1), (0.1, 0.3)}, {(0.55, 0.1), (0.3, 0.2)}}	{{(0.45, 0.45), (0.2, 0.1), (0.1, 0.4)}, {(0.55, 0.1), (0.5, 0.2)}}
$\mathscr{A}_{AL-3}$	{{(0.65, 0.12), (0.21, 0.7), (0.2, 0.3)}, {(0.35, 0.1), (0.34, 0.065)}}	{{(0.7, 0.), (0.35, 0.71), (0.3, 0.2)}, {(0.3, 0.1), (0.2, 0.5)}}	{{(0.7, 0.6), (0.6, 0.1), (0.2, 0.4)}, {(0.3, 0.3), (0.15, 0.0)}}	{{(0.4, 0.1), (0.25, 0.75), (0.2, 0.2)}, {(0.6, 0.3), (0.5, 0.6)}}	{{(0.35, 0.1), (0.35, 0.1), (0.25, 0.15)}, {(0.6, 0.3), (0.3, 0.6)}}
$\mathscr{A}_{AL-4}$	{{(0.85, 0.12), (0.21, 0.7), (0.2, 0.3)}, {(0.15, 0.1), (0.14, 0.65)}}	{{(0.3, 0.2), (0.15, 0.71), (0.15, 0.1)}, {(0.3, 0.1), (0.2, 0.5)}}	{{(0.5, 0.5), (0.2, 0.4), (0.2, 0.1)}, {(0.3, 0.3), (0.15, 0.0)}}	{{(0.4, 0.1), (0.25, 0.75), (0.2, 0.2)}, {(0.5, 0.6), (0.45, 0.3)}}	{{(0.5, 0.4), (0.48, 0.1), (0.15, 0.75)}, {(0.5, 0.6), (0.35, 0.3)}}
$\mathscr{A}_{AL-5}$	{{(0.25, 0.12), (0.21, 0.7), (0.1, 0.3)}, {(0.75, 0.1), (0.14, 0.65)}}	{{(0.6, 0.2), (0.45, 0.71), (0.35, 0.1)}, {(0.3, 0.1), (0.2, 0.5)}}	{{(0.5, 0.8), (0.3, 0.1), (0.14, 0.4)}, {(0.45, 0.0), (0.3, 0.3)}}	{{(0.57, 0.1), (0.25, 0.75), (0.23, 0.2)}, {(0.45, 0.6), (0.35, 0.3)}}	{{(0.37, 0.1), (0.25, 0.15), (0.23, 0.25)}, {(0.35, 0.9), (0.33, 0.6)}}
$\mathscr{A}_{AL-6}$	{{(0.35, 0.12), (0.11, 0.9), (0.1, 0.8)}, {(0.55, 0.1), (0.14, 0.65)}}	{{(0.6, 0.3), (0.35, 0.71), (0.25, 0.1)}, {(0.4, 0.1), (0.1, 0.9)}}	{{(0.5, 0.8), (0.3, 0.1), (0.14, 0.4)}, {(0.35, 0.1), (0.3, 0.3)}}	{{(0.57, 0.1), (0.25, 0.75), (0.23, 0.2)}, {(0.35, 0.3), (0.13, 0.9)}}	{{(0.15, 0.65), (0.15, 0.2), (0.15, 0.1)}, {(0.85, 0.3), (0.13, 0.9)}}
$\mathscr{A}_{AL-7}$	{{(0.45, 0.12), (0.11, 0.75), (0.1, 0.15)}, {(0.45, 0.1), (0.14, 0.65)}}	{{(0.9, 0.3), (0.25, 0.71), (0.25, 0.1)}, {(0.1, 0.1), (0.0, 0.9)}}	{{(0.5, 0.8), (0.3, 0.9), (0.3, 0.3)}, {(0.35, 0.1), (0.2, 0.9)}}	{{(0.85, 0.1), (0.35, 0.75), (0.25, 0.1)}, {(0.15, 0.3), (0.13, 0.9)}}	{{(0.75, 0.1), (0.25, 0.75), (0.25, 0.1)}, {(0.25, 0.3), (0.23, 0.9)}}
$\mathscr{A}_{AL-8}$	{{(0.95, 0.82), (0.11, 0.75), (0.1, 0.55)}, {(0.05, 0.10), (0.14, 0.65)}}	{{(0.9, 0.3), (0.65, 0.1), (0.25, 0.71)}, {(0.1, 0.7), (0.0, 0.9)}}	{{(0.4, 0.8), (0.3, 0.6), (0.2, 0.3)}, {(0.35, 0.1), (0.2, 0.9)}}	{{(0.55, 0.1), (0.45, 0.2), (0.05, 0.75)}, {(0.15, 0.3), (0.13, 0.9)}}	{{(0.65, 0.2), (0.55, 0.1), (0.4, 0.45)}, {(0.25, 0.3), (0.03, 0.9)}}
$\mathscr{A}_{AL-9}$	{{(0.55, 0.22), (0.21, 0.75), (0.0, 0.95)}, {(0.45, 0.1), (0.24, 0.15)}}	{{(0.67, 0.1), (0.5, 0.3), (0.25, 0.71)}, {(0.3, 0.9), (0.23, 0.70)}}	{{(0.4, 0.8), (0.3, 0.6), (0.2, 0.3)}, {(0.35, 0.1), (0.2, 0.9)}}	{{(0.55, 0.1), (0.45, 0.2), (0.05, 0.75)}, {(0.15, 0.3), (0.13, 0.9)}}	{{(0.65, 0.55), (0.35, 0.2), (0.35, 0.1)}, {(0.25, 0.3), (0.23, 0.9)}}
$\mathscr{A}_{AL-10}$	{{(0.83, 0.82), (0.21, 0.75), (0.1, 0.65)}, {(0.17, 0.1), (0.04, 0.55)}}	{{(0.9, 0.2), (0.25, 0.71), (0.25, 0.1)}, {(0.1, 0.7), (0.0, 0.9)}}	{{(0.7, 0.8), (0.35, 0.65), (0.22, 0.23)}, {(0.25, 0.1), (0.2, 0.9)}}	{{(0.65, 0.25), (0.45, 0.2), (0.45, 0.1)}, {(0.15, 0.35), (0.13, 0.5)}}	{{(0.45, 0.25), (0.35, 0.1), (0.25, 0.2)}, {(0.43, 0.7), (0.11, 0.35)}}

7. Conclusion

The theory of CDT-2HFS is a mixture of two different modifications of FS, called CFS and DT-2HFS. CDT-2HFS is a proficient technique to cope with unpredictable and awkward information in realistic decision problems. The intention of this manuscript is to determine the novel methodology of CDT-2HFSs and to discuss their operational laws. These operational laws are also defensible with the assistance of examples. Furthermore, based on novel CDT-2HFS, we reconnoitered the CC and EMs, and their special cases are also discussed. TOPSIS method based on CDT-2HFS is also explored. Then, we applied our explored measures based on CDT-2HFSs in the environment of TOPSIS method, medical diagnosis, pattern recognition, and clustering algorithm to cope with awkward and complicated information in realistic decision issues. Finally, some numerical examples are given and discussed to examine the proficiency and validity of the explored measures. Comparative analysis, advantages, and graphical interpretation of the explored measures with some other existing measures are also discussed.

In the future, the concept of complex dual type-2 hesitant fuzzy sets can be applied to group MADM problems. Moreover, the problems discussed in this manuscript can be discussed in the environment of complex q-rung orthopair fuzzy sets [32–39], T-spherical fuzzy sets [40, 41], and some others [42–45].

Data Availability

The data used in this article are artificial and hypothetical, and anyone can use these data without prior permission by just citing this article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System

Journal of Mathematics

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Journal of Mathematics has retracted the article titled "A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System" [1] due to concerns that the peer review process has been compromised.

Following an investigation conducted by the Hindawi Research Integrity team [2], significant concerns were identified with the peer reviewers assigned to this article; the investigation has concluded that the peer review process was compromised. We therefore can no longer trust the peer review process, and the article is being retracted with the agreement of the Chief Editor.

The authors do not agree to the retraction.

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[1] S. Abdullah, S. Khan, M. Qiyas, and R. Chinram, "A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System," *Journal of Mathematics*, vol. 2021, Article ID 8819517, 9 pages, 2021.
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Research Article

A Novel Approach Based on Sine Trigonometric Picture Fuzzy Aggregation Operators and Their Application in Decision Support System

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Picture fuzzy sets (PFSs) are one of the fundamental concepts for addressing uncertainties in decision problems, and they can address more uncertainties compared to the existing structures of fuzzy sets; thus, their implementation was more substantial. The well-known sine trigonometric function maintains the periodicity and symmetry of the origin in nature and thus satisfies the expectations of the decision-maker over the multiple parameters. Taking this feature and the significances of the PFSs into consideration, the main objective of the article is to describe some reliable sine trigonometric laws (STLs) for PFSs. Associated with these laws, we develop new average and geometric aggregation operators to aggregate the picture fuzzy numbers. Also, we characterized the desirable properties of the proposed operators. Then, we presented a group decision-making strategy to address the multiple attribute group decision-making (MAGDM) problem using the developed aggregation operators and demonstrated this with a practical example. To show the superiority and the validity of the proposed aggregation operations, we compared them with the existing methods and concluded from the comparison and sensitivity analysis that our proposed technique is more effective and reliable.

1. Introduction

Multiple attribute group decision-making (MAGDM) method is one of the most relevant and evolving topics explaining how to choose the finest alternative with community of decision-makers (DMs) with some attributes. There are two relevant tasks in this system. The first is to define the context in which the values of the various parameters are effectively calculated, while the second is to summarize the information described. Traditionally, the information describing the objects is taken mostly to be deterministic or crisp in nature. With the increasing complexity of a system on a daily basis, however, it is difficult to aggregate the data, from the logbook, resources, and experts, in the crisp form. Therefore, [1] developed the core concept of fuzzy set (FS) and also [2] worked on it and further developed a new idea of intuitionistic fuzzy set (IFS), [3] developed the Pythagorean fuzzy sets (PyFSs), and [4] defined the idea of hesitant fuzzy sets, which are used by scholars to communicate the information clearly. In IFS, it is observed that each object has two membership grades, positive $(\breve{E})$ and negative $(\breve{Z})$, which satisfy the condition $0 \le \breve{E} + \breve{Z} \le 1$, and, for all $\breve{E}, \breve{Z}$ is lying in the closed interval 0 and 1. However, in the Pythagorean fuzzy sets, this constraint is relaxed from $\breve{E} + \breve{Z} \le 1$ to $\breve{E}^2 + \breve{Z}^2 \le 1$ for $\breve{E}, \breve{Z} \in [0, 1]$. Using this concept, many researchers have successfully addressed the above two critical tasks and discretion of the techniques under the different aspects. Verma and Sharma [5] proposed a new measure of inaccuracy with its application to multicriteria decision-making under intuitionistic fuzzy environment. Some of the basic results of IFSs and Pythagorean fuzzy sets are the operational laws [6, 7], some exponential operational laws [8], some distance or similarity measures [9, 10], and some information entropy [11]. Many researchers [12–17], under IFS, defined some basic aggregation operators (AOs), such as average and geometric, interactive, and Hamacher AOs. Meanwhile, for Pythagorean fuzzy sets, some basic operators are proposed by Peng and Yang [18]. To solve the MAGDM problems, Garg [19, 20] presented some basic concept of Einstein aggregation operators. Some extended aggregation operators are dependent on intuitionistic and Pythagorean fuzzy information, including the TOPSIS technique based on IF [21] and Pythagorean fuzzy set [22], partitioned Bonferroni mean [23], and Maclaurin symmetric mean [24, 25]. Apart from this, Yager et al. [26] intuitively developed the idea of q-rung orthopair fuzzy sets (q-ROFSs). Gao et al. [27] developed the basic idea of the continuities and differential of q-ROFSs. Peng et al. [28] presented the exponential and logarithm operational laws for q-ROFNs. Liu and Wang [29] developed weighted average and geometric aggregation operators for q-ROFNs.

Meanwhile, the ideas of IFSs and Pythagorean FSs are widely studied and implemented in various fields. But their ability to express the information is still limited. Thus, it was still difficult for the decision-makers (DMs) and their corresponding information to convey the information in such sets. To overcome this information, the notion of the picture fuzzy sets (PFSs) was defined by Cuong and Kreinovich [30]. Thus, it was clearly noticed that the PFS is the extended form of the IFS to accommodate some more ambiguities. In picture fuzzy sets, each object was observed by defining three grades of the member named membership $\breve{E}$, neutral $\breve{R}$, and nonmembership $\breve{Z}$ with the constraint that $\breve{E} + \breve{R} + \breve{Z} \leq 1$, for $\breve{E}, \breve{R}, \breve{Z} \in [0, 1]$. The definition of the PFS will convey the opinions of experts like "yes," "abstain," "no," and "refusal" while avoiding missing evaluation details and encouraging the reliability of the acquired data with the actual environment for decision-making. Although the concept of PFSs is widely studied and applied in different fields and their extension focuses on the basic operational laws, which is the important aspect of the PFS as well as aggregation operators (AOs), which are an effective tool by the help of these AOs, we obtain raking of the alternatives by providing the comprehensive values to the alternatives. Wei [31] developed some operations of the PFS. Son [32] developed measuring analogousness in PFSs. Apart from these, several other kinds of the AOs of the PFSs have been developed such as logarithmic PF aggregation operators, which were presented by Khan et al. [33], Wang et al. [34] presented PF normalized projection based VIKOR method, and Wang et al. [35] developed PF Muirhead mean operators. Wei et al. [36] defined the idea of some q-ROF Maclaurin symmetric mean operators. Wang et al. [37] introduced a similarity measure of q-ROFSs. Wei et al. [38] developed bidirectional projection method for PFSs. Ashraf et al. [39–41] developed the idea of different approaches to MAGDM problems, picture fuzzy linguistic sets and exponential Jensen PF divergence measure, respectively. Khan et al. [42] presented PF aggregation based on Einstein operation. Qiyas et al. [43] presented linguistic PF Dombi aggregation operators.

Among the above aspects, it is very clear that operational laws are a main role model for any aggregation process. In that direction, recently, Khan et al. [33] defined the new concept about logarithmic operation laws for PFSs. Besides these mathematical logarithmic functions, another important feature is the sine trigonometry feature, which plays a main role during the fusion of the information. In this way, taking into consideration the advantages and usefulness of the sine trigonometric function, some new sine trigonometric operational laws need to be developed for PFSs and their behavior needs to be studied. Consequently, the paper's purpose is to develop some new operation laws for PFSs and also introduce the MAGDM algorithm for managing the information for PFSs evaluation, as well as describing several more sophisticated operational laws for PFSs in addition to a novel entropy to remove the weight of the attributes to prevent subjective and objective aspects. Some more generalized functional aggregation operators are presented with the help of the defined sine trigonometric operational laws (STOLs) for PFNs, and many basic relations between the developed AOs are discussed; also, a novel MAGDM technique depending on the developed operators to solve the group decision-making problems is presented. Finally, the proposed approach is compared with the existing methods. So, the goals and the motivations of this paper are as follows:

(1) The paper presents some more advanced operational laws for PFSs by combining the features of the ST and PFNs.

(2) A novel entropy is presented to extract the attributes' weight for avoiding the influence of subjective and objective aspects.

(3) Some more generalized functional AOs are presented with the help of the defined STOLs for PFNs. Also, the several fundamental relations between the proposed AOs are derived to show their significance.

(4) A novel MAGDM method based on the proposed operators to solve the group decision-making problems is presented. The consistency of the proposed method is confirmed through these examples, and their evaluations are carried out in detail.

In Section 2 of the article, we can define some ideas related to PFSs. In Section 3, we define the new PFS operational laws based on sine trigonometric functions and their properties. In Section 4, we present a series of AOs along with their required properties, based on sine trigonometric operational laws. Section 5 provides the basic connection between the developed AOs. In Section 6, using the new aggregation operators, we introduce a new MAGDM approach and give detailed steps. Examples are given in Section 7 to validate the new method and comparative analysis is carried out by the current method. Finally, the work is concluded in Section 8.

2. Preliminaries

Some fundamental ideas about picture fuzzy set (PFS) on the universal set $\breve{U}$ are discussed in this portion.

$\overline{S}(\breve{I}_1) > \overline{S}(\breve{I}_2)$, then $\breve{I}_1 > \breve{I}_2$, and if the score function, that is,
$\overline{S}(\breve{I}_1) = \overline{S}(\breve{I}_2)$, and $\overline{H}(\breve{I}_1) > \overline{H}(\breve{I}_2)$, then $\breve{I}_1 > \breve{I}_2$;
if $\overline{H}(\breve{I}_1) = \overline{H}(\breve{I}_2)$, then $\breve{I}_1 = \breve{I}_2$.

Definition 1 (see [31]). Let $\breve{U}$ be the nonempty fixed sets. Then, the set

$$\breve{I} = \left(\breve{u}, \breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \frac{\breve{Z}_{\breve{I}}(\breve{u})}{\breve{u} \in \breve{U}} \right). \quad (1)$$

is said to be a picture fuzzy set (PFS), where $\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{I}_{\breve{I}}(\breve{u}) \in [0,1]$ are called the grade of membership, positive, neutral, and negative, of the elements $\breve{u} \in \breve{U}$ to the set $\breve{I}$, respectively, where the following constraint has been fulfilled by $\breve{E}(\breve{u}), \breve{R}(\breve{u}), \breve{I}(\breve{u})$ for all $\breve{u} \in \breve{U}$:

$$0 \leq \breve{E}(\breve{u}) + \breve{R}(\breve{u}) + \breve{Z}(\breve{u}) \leq 1. \quad (2)$$

Definition 2 (see [31]). Let three PFNs be $\breve{I} = (\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u}))\breve{I}_1 = (\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u}))$, and $\breve{I}_2 = (\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u}))$. Also $\ddot{\varpi} > 0$ is any scalar. Then,

$$\begin{aligned}
\breve{I}^{c} &= \left[\breve{Z}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{E}_{\breve{I}}(\breve{u})\right],\\
\breve{I}_1 \wedge \breve{I}_2 &= \left[\min\left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{E}_{\breve{I}_2}(\breve{u})\right), \max\left(\breve{R}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u})\right), \max\left(\breve{Z}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right],\\
\breve{I}_1 \vee \breve{I}_2 &= \left[\max\left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{E}_{\breve{I}_2}(\breve{u})\right), \min\left(\breve{R}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u})\right), \min\left(\breve{Z}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right],\\
\breve{I}_1 \oplus \breve{I}_2 &= \left[\breve{E}_{\breve{I}_1}(\breve{u}) + \breve{E}_{\breve{I}_2}(\breve{u}) - \breve{E}_{\breve{I}_1}(\breve{u}).\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}).\breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u}) \cdot \breve{Z}_{\breve{I}_2}(\breve{u})\right],\\
\breve{I}_1 \otimes \breve{I}_2 &= \left[\breve{E}_{\breve{I}_1}(\breve{u}) \cdot \breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}) + \breve{R}_{\breve{I}_2}(\breve{u}) - \breve{R}_{\breve{I}_1}(\breve{u}) \cdot \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u}) + \breve{Z}_{\breve{I}_2}(\breve{u}) - \breve{Z}_{\breve{I}_1}(\breve{u}) \cdot \breve{Z}_{\breve{I}_2}(\breve{u})\right],\\
\ddot{\varpi} \cdot \breve{I} &= \left[1 - \left(1 - \breve{E}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}, \left(\breve{R}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}, \left(\breve{Z}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}\right],\\
(\breve{I})^{\ddot{\varpi}} &= \left[\left(\breve{E}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}, 1 - \left(1 - \breve{R}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}, 1 - \left(1 - \breve{Z}_{\breve{I}}(\breve{u})\right)^{\ddot{\varpi}}\right].
\end{aligned} \quad (3)$$

Definition 3 (see [44]). Let all the PFNs $\breve{I} = (\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u}))$. The score and accuracy functions are then described as follows:

$$\begin{aligned}
\overline{S}(\breve{I}) &= \breve{E}_{\breve{I}}(\breve{u}) - \breve{R}_{\breve{I}}(\breve{u}) - \breve{Z}_{\breve{I}}(\breve{u}), \quad \overline{S}(\breve{I}) \in [-1,1],\\
\overline{H}(\breve{I}) &= \breve{E}_{\breve{I}}(\breve{u}) + \breve{R}_{\breve{I}}(\breve{u}) + \breve{Z}_{\breve{I}}(\breve{u}), \quad \overline{H}(\breve{I}) \in [0,1].
\end{aligned} \quad (4)$$

Definition 4. (see [44]). Let two PFNs be $\breve{I}_1 = (\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u}))$ and $\breve{I}_2 = (\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u}))$. Then, the rules for comparison can be defined as follows: if the score function, that is,

3. New Sine Trigonometric Operational Laws (STOLs) for PFSs

We will define some operational laws for PFNs in this portion. First, the sine trigonometric PFSs are defined.

Definition 5. Let the PFS be $\breve{I} = (\breve{E}(\breve{u}), \breve{R}(\breve{u}), \breve{Z}(\breve{u}))$. Then, we define STOLs of a picture fuzzy set as

$$\sin \breve{I} = \langle \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}\right)\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_{\breve{I}}\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\breve{I}}\right)\rangle, \quad 0 < \breve{E}_{\breve{I}} \leq 1. \quad (5)$$

From the above definition, it is clear that $\sin \breve{I}$ is also a PFS and also satisfied the following conditions of the PFS as the membership, neutral, and nonmembership degrees of PFS are defined, respectively:

$$\begin{aligned}
&\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}\right)\right): \breve{U} \longrightarrow [0,1], \quad \text{such that } 0 \leq \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}\right)\right) \leq 1,\\
&2\sin^2\left(\frac{\pi}{4}\breve{R}_{\breve{I}}\right): \breve{U} \longrightarrow [0,1], \quad \text{such that } 0 \leq 2\sin^2\left(\frac{\pi}{4}\breve{R}_{\breve{I}}\right) \leq 1,\\
&2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\breve{I}}\right): \breve{U} \longrightarrow [0,1], \quad \text{such that } 0 \leq 2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\breve{I}}\right) \leq 1.
\end{aligned} \quad (6)$$

Therefore,

$$\sin \breve{I} = \left\langle \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}\right)\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_{\breve{I}}\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\breve{I}}\right)\right\rangle \tag{7}$$

is a PFS.

Definition 6. Let $\breve{I} = (\breve{E}_{\breve{I}}, \breve{R}_{\breve{I}}, \breve{Z}_{\breve{I}})$ be a PFN, if

$$\sin \breve{I} = \left\langle \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}\right)\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_{\breve{I}}\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\breve{I}}\right)\right\rangle, \quad 0 < \breve{E}_{\breve{I}} \leq 1, \tag{8}$$

is known as sine trigonometric (ST) operator and its value is known as sine trigonometric PFN.

Definition 7. Let the collection of PFNs be $\breve{I} = (\breve{E}, \breve{R}, \breve{Z})$, $\breve{I}_1 = (\breve{E}_1, \breve{R}_1, \breve{Z}_1)$, and $\breve{I}_2 = (\breve{E}_2, \breve{R}_2, \breve{Z}_2)$. Then, we define the following operational laws where $\ddot{\omega} > 0$ is any scalar:

$$\begin{aligned}
\sin \breve{I}_1 \oplus \sin \breve{I}_2 &= \left[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right).\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\right],\\
\sin \breve{I}_1 \otimes \sin \breve{I}_2 &= \left[\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_1\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\breve{E}_2\right)\right), 1 - \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right), 1 - \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\right],\\
\ddot{\omega} \cdot \sin \breve{I} &= \left[1 - \left(1 - \sin\left(\left(\frac{\pi}{2}\breve{E}\right)\right)\right)^{\ddot{\omega}}, \left(2\sin^2\left(\frac{\pi}{4}\breve{R}\right)\right)^{\ddot{\omega}}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)^{\ddot{\omega}}\right],\\
(\sin \breve{I})^{\ddot{\omega}} &= \left[\left(\sin\left(\frac{\pi}{2}\left(\breve{E}\right)\right)\right)^{\ddot{\omega}}, 1 - \left(2\sin^2\left(\frac{\pi}{4}\breve{R}\right)\right)^{\ddot{\omega}} 1 - \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)^{\ddot{\omega}}\right].
\end{aligned} \tag{9}$$

3.1. Some Basic Properties of STOLs of PFNs. Some fundamental properties of sine trigonometric PFNs are discussed in this portion, using the sine trigonometric operational laws (STOLs).

Theorem 1. *Let a collection of PFNs be* $\breve{I}_{\hat{J}} = (\breve{E}_{\hat{J}}, \breve{R}_{\hat{J}}, \breve{Z}_{\hat{J}})$, *where* $\hat{J} = 1, 2, 3$. *Then,*

$$\begin{aligned}
\sin \breve{I}_1 \oplus \sin \breve{I}_2 &= \sin \breve{I}_2 \oplus \sin \breve{I}_1,\\
\sin \breve{I}_1 \otimes \sin \breve{I}_2 &= \sin \breve{I}_2 \otimes \sin \breve{I}_1,\\
\left(\sin \breve{I}_1 \oplus \sin \breve{I}_2\right) \oplus \sin \breve{I}_3 &= \sin \breve{I}_1 \oplus \left(\sin \breve{I}_2 \oplus \sin \breve{I}_3\right),\\
\left(\sin \breve{I}_1 \otimes \sin \breve{I}_2\right) \otimes \sin \breve{I}_3 &= \sin \breve{I}_1 \otimes \left(\sin \breve{I}_2 \otimes \sin \breve{I}_3\right).
\end{aligned} \tag{10}$$

Proof. Here, we solve the first two parts using the STOLs (sine trigonometric operation laws) defined in Definition 7, and the proof of the other two parts is similar to the first parts, so we omit it here; we get

$$\begin{aligned}
\sin \breve{I}_1 \oplus \sin \breve{I}_2 &= \left[\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right] \oplus \left[\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right],\\
&= \left[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right).\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\right],\\
&= \left[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right).\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)\right],\\
&= \left[\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right] \oplus \left[\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right],\\
&= \sin \breve{I}_2 \oplus \sin \breve{I}_1.
\end{aligned} \tag{11}$$

Therefore, from the above,

$$\begin{aligned}
\sin \breve{I}_1 \oplus \sin \breve{I}_2 &= \sin \breve{I}_2 \oplus \sin \breve{I}_1,\\
\sin \breve{I}_1 \otimes \sin \breve{I}_2 &= \left[\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right] \otimes \left[\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right],\\
&= \left[\sin\left(\frac{\pi}{2}\breve{E}_1\right)\cdot \sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\cdot 2\sin^2\left(\frac{\pi}{4}\left(\breve{R}_2\right)\right), 2\sin^2\frac{\pi}{4}\left(\breve{Z}_1\right)\cdot 2\sin^2\frac{\pi}{4}\left(\breve{Z}_2\right)\right],\\
&= \left[\sin\left(\frac{\pi}{2}\breve{E}_2\right)\cdot \sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\cdot 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\cdot 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right],\\
&= \left[\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right] \otimes \left[\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right],\\
&= \sin \breve{I}_1 \otimes \sin \breve{I}_2.
\end{aligned} \tag{12}$$

Therefore, from the above solution,

$$\sin \breve{I}_1 \otimes \sin \breve{I}_2 = \sin \breve{I}_2 \otimes \sin \breve{I}_1. \tag{13}$$

□

Theorem 2. *Let a collection of PFNs be* $\breve{I} = (\breve{E}, \breve{R}, \breve{Z})$ *and* $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$*, where* $\widehat{J} = 1, 2$*. Also let* $\widetilde{N}, \widetilde{N}_1, \widetilde{N}_2 > 0$ *be the real number; then*

$$\begin{aligned}
&\widetilde{N}\cdot\left(\sin \breve{I}_1 \oplus \sin \breve{I}_2\right) = \widetilde{N}\cdot \sin \breve{I}_1 \oplus \widetilde{N}\cdot \sin \breve{I}_2,\\
&\left(\sin \breve{I}_1 \otimes \sin \breve{I}_2\right)^{\widetilde{N}} = \left(\sin \breve{I}_1\right)^{\widetilde{N}} \otimes \left(\sin \breve{I}_2\right)^{\widetilde{N}},\\
&\widetilde{N}_1\cdot \sin \breve{I} \oplus \widetilde{N}_2\cdot \sin \breve{I} = (\widetilde{N}_1 \oplus \widetilde{N}_2)\cdot \sin \breve{I},\\
&(\sin \breve{I})^{\widetilde{N}_1} \otimes (\sin \breve{I})^{\widetilde{N}_2} = (\sin \breve{I})^{\widetilde{N}_1 \oplus \widetilde{N}_2},\\
&\left((\sin \breve{I})^{\widetilde{N}_1}\right)^{\widetilde{N}_2} = (\sin \breve{I})^{\widetilde{N}_1 \cdot \widetilde{N}_2}.
\end{aligned} \tag{14}$$

Proof. Here, we will prove the first part of the above theorem only by using the STOLs defined in Definition 7, while the rest can be proven similarly. But,

$$\begin{aligned}
\sin \breve{I}_1 &= \left(\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right),\\
\sin \breve{I}_2 &= \left(\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right),
\end{aligned} \tag{15}$$

and, by using the STOLs, we have

$$\begin{aligned}
\sin \breve{I}_1 \oplus \sin \breve{I}_2 = \Bigg[1 - &\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right)\\
&.\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right)\\
&.\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\Bigg],
\end{aligned} \tag{16}$$

but it is given in statement of the theorem that $\breve{R} > 0$; again, by using Definition 6, we have

$$\begin{aligned}
\widetilde{N}\cdot\left(\sin \breve{I}_1 \oplus \sin \breve{I}_2\right) &= \Bigg[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right)^{\widetilde{N}}\\
&\quad .\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right)^{\widetilde{N}}, \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right)^{\widetilde{N}}\\
&\quad .\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right)^{\widetilde{N}}, \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)^{\widetilde{N}}\\
&\quad .\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)^{\widetilde{N}}\Bigg],\\
&= \Bigg[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right)^{\widetilde{N}},\\
&\quad \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right)^{\widetilde{N}}, \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)^{\widetilde{N}}\Bigg]\\
&\quad \oplus\Bigg[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right)^{\widetilde{N}},\\
&\quad \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right)^{\widetilde{N}}, \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)^{\widetilde{N}}\Bigg]\\
&= \widetilde{N}\cdot \sin \breve{I}_1 \oplus \widetilde{N}\cdot \sin \breve{I}_2.
\end{aligned} \tag{17}$$

□

Corollary 1. *Let a collection of two PFNs be $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, where $\widehat{J} = 1, 2$, such that $\breve{E}_1 \geq \breve{E}_2, \breve{R}_1 \leq \breve{R}_2$, and $\breve{Z}_1 \leq \breve{Z}_2$. Then show that $\sin \breve{I}_1 \geq \sin \breve{I}_2$.*

Proof. Let $\breve{I}_1 = (\breve{E}_1, \breve{R}_1, \breve{Z}_1)$ and $\breve{I}_2 = (\breve{E}_2, \breve{R}_2, \breve{Z}_2)$ be the PFNs with condition $\breve{E}_1 \geq \breve{E}_2$, since in the closed interval $[0, \pi/2]$ sine is an increasing function; thus, we have $\sin((\pi/2)\breve{E}_1) \geq \sin((\pi/2)\breve{E}_1)$. But also, given that $\breve{R}_1 \leq \breve{R}_2$ which implies that $(1 - \breve{R}_1) \geq (1 - \breve{R}_2)$, since in closed interval $[0, \pi/2]$ sine is an increasing function, we have $\sin((\pi/2)(1 - \breve{R}_1)) \geq \sin((\pi/2)(1 - \breve{R}_2))$, which implies that $2\sin^2((\pi/4)\breve{R}_1) \leq 2\sin^2((\pi/4)\breve{R}_2)$. Similarly, $\breve{Z}_1 \leq \breve{Z}_1$, which implies that $(1 - \breve{Z}_1) \geq (1 - \breve{Z}_1)$, since in closed interval $[0, \pi/2]$ sine is an increasing function; thus, we have $2\sin^2((\pi/4)\breve{Z}_1) \leq 2\sin^2((\pi/4)\breve{Z}_1)$; hence, we get

$$\left(\sin\left(\frac{\pi}{2}\breve{E}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right) \geq \left(\sin\left(\frac{\pi}{2}\breve{E}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right), 2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right), \tag{18}$$

and, therefore, we get the required result by using Definition 7:

$$\sin \breve{I}_1 \geq \sin \breve{I}_2. \tag{19}$$

□

4. Sine Trigonometric Aggregation Operators

We have described a number of aggregation operators in this portion of the article on the basis of sine trigonometric operational laws (STOLs).

Definition 8. Let a collection of PFNs be $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, where $\widehat{J} = 1, \ldots, n$. Then, the mapping ST − PFWA: $\Psi^n \longrightarrow \Psi$ is known as the sine trigonometric picture fuzzy weighted average (ST − PFWA) operator, if

$$\text{ST} - \text{PFWA}\left(\breve{I}_1, \ldots, \breve{I}_n\right) = \ddot{\omega}_1 \cdot \sin \breve{I}_1 \oplus \cdots \oplus \ddot{\omega}_n \cdot \sin \breve{I}_n, \tag{20}$$

where $\ddot{\omega}_{\widehat{J}}$ are the weighted vectors of $\sin \breve{I}_{\widehat{J}} (\widehat{J} = 1, \ldots, n)$ which fulfilled the criteria of $\ddot{\omega}_{\widehat{J}} > 0$ and $\sum_{\widehat{J}=1}^{n} \ddot{\omega}_{\widehat{J}} = 1$.

Theorem 3. *Let a collection of PFNs be $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, where $\widehat{J} = 1, \ldots, n$. Then, the aggregated value is also a PFN by utilizing the ST − PFWA operator and is given by*

$$\text{ST} - \text{PFWA}\left(\breve{I}_1, \ldots, \breve{I}_n\right) = \left[1 - \prod_{\widehat{J}=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{21}$$

Proof. By using the process of mathematical induction, we prove the said theorem. Because $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$ is a PFN for each $\widehat{J}$, which implies that $\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}} \in [0, 1]$ and also $\breve{E}_{\widehat{J}} + \breve{R}_{\widehat{J}} + \breve{Z}_{\widehat{J}} \leq 1$, the following mathematical induction steps were then performed.

Step 1. Now, for $n = 2$, we get $\text{ST} - \text{PFWA}(\breve{I}_1, \breve{I}_2) = \ddot{\omega}_1 \cdot \sin \breve{I}_1 \oplus \ddot{\omega}_2 \cdot \sin \breve{I}_2$, where

$$\begin{aligned}
\ddot{\omega}_1 \cdot \sin \breve{I}_1 &= \left(1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right)^{\ddot{\omega}_1}, \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right)^{\ddot{\omega}_1}, \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right)^{\ddot{\omega}_1}\right),\\
\ddot{\omega}_2 \cdot \sin \breve{I}_2 &= \left(1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right)^{\ddot{\omega}_2}, \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right)^{\ddot{\omega}_2}, \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)^{\ddot{\omega}_2}\right),
\end{aligned} \tag{22}$$

and hence, by using the definition [7], we get

$$\ddot{\omega}_1 \cdot \sin \breve{I}_1 \oplus \ddot{\omega}_2 \cdot \sin \breve{I}_2 = \left[1 - \prod_{\widehat{J}=1}^{2}\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, \prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\check{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, \prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\check{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{23}$$

Step 2. Now say it is true for $n = k$.

$$\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right) = \left[1-\prod_{\widehat{J}=1}^{k}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{24}$$

Step 3. Now, we prove that this is true for $n = k + 1$:

$$\begin{aligned}\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_{k+1}\right) &= \ddot{\omega}_1\cdot\sin\breve{I}_1\oplus\cdots\oplus\ddot{\omega}_n\cdot\sin\breve{I}_n\oplus\ddot{\omega}_{k+1}\cdot\sin\breve{I}_{k+1}\\ &=\left[1-\prod_{\widehat{J}=1}^{k}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]\\ &\oplus\left[1-\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{k+1}\right)\right)^{\ddot{\omega}_{k+1}},\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{k+1}\right)\right)^{\ddot{\omega}_{k+1}},\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{k+1}\right)\right)^{\ddot{\omega}_{k+1}}\right],\end{aligned} \tag{25}$$

and, again, by using Definition 7, we obtain

$$\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_{k+1}\right) = \left[1-\prod_{\widehat{J}=1}^{k+1}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k+1}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{k+1}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{26}$$

Hence, $n = k + 1$ holds. Then, the statement is valid for all n through the principal of mathematical induction. □

Property 1. If all collection of PFNs $\breve{I}_{\widehat{J}} = \breve{I}$, where $\breve{I}$ is another PFN $(\widehat{J} = 1,\ldots,n)$, then

$$\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right) = \sin\breve{I}. \tag{27}$$

Proof. Let $\breve{I} = (\breve{E},\breve{R},\breve{Z})$ be a PFN, such that $\breve{I}_{\widehat{J}} = \breve{I}$. Then, by using Theorem 4, we get

$$\begin{aligned}\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right) &= \left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right],\\ &=\left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right],\\ &=\left[1-\left(1-\sin\left(\frac{\pi}{2}\breve{E}\right)\right)^{\sum_{\widehat{J}=1}n\ddot{\omega}_{\widehat{J}}},\left(2\sin^2\left(\frac{\pi}{4}\breve{R}\right)\right)^{\sum_{\widehat{J}=1}n\ddot{\omega}_{\widehat{J}}},\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}\right)\right)^{\sum_{\widehat{J}=1}n\ddot{\omega}_{\widehat{J}}}\right],\\ &=\left[1-\left(1-\sin\left(\frac{\pi}{2}\breve{E}\right)\right),\left(2\sin^2\left(\frac{\pi}{4}\breve{R}\right)\right)\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}\right)\right)\right],\\ &=\sin\breve{I}.\end{aligned} \tag{28}$$

□

Property 2. If $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, where we let $\widehat{J} = 1, \ldots, n$, $\breve{I}^{-} = (\min_{\widehat{J}}\{\breve{E}_{\widehat{J}}\}, \max_{\widehat{J}}\{\breve{R}_{\widehat{J}}\}, \max_{\widehat{J}}\{\breve{Z}_{\widehat{J}}\})$, and $\breve{I}^{+} = (\max_{\widehat{J}}\{\breve{E}_{\widehat{J}}\}, \min_{\widehat{J}}\{\breve{R}_{\widehat{J}}\}, \min_{\widehat{J}} \min_{\widehat{J}}\{\breve{Z}_{\widehat{J}}\})$ be PFNs, then

$$\sin \breve{I}^{-} \leq \mathrm{ST-PFWA}\left(\breve{I}_1, \breve{I}_2, \ldots, \breve{I}_n\right) \leq \sin \breve{I}^{+}. \tag{29}$$

Proof. Since, for any $\widehat{J}$, $\min_{\widehat{J}}\{\breve{E}_{\widehat{J}}\} \leq \breve{E}_{\widehat{J}} \leq \max_{\widehat{J}}\{\breve{E}_{\widehat{J}}\}$, $\min_{\widehat{J}}\{\breve{R}_{\widehat{J}}\} \leq \breve{R}_{\widehat{J}} \leq \max_{\widehat{J}}\{\breve{R}_{\widehat{J}}\}$, and $\min_{\widehat{J}}\{\breve{Z}_{\widehat{J}}\} \leq \breve{Z}_{\widehat{J}} \leq \max_{\widehat{J}}\{\breve{Z}_{\widehat{J}}\}$, this implies that $\breve{I}^{-} \leq \breve{I}_{\widehat{J}} \leq \breve{I}^{+}$. Assume that $\mathrm{ST-PFWA}(\breve{I}_1, \breve{I}_2, \ldots, \breve{I}_n) = \sin \breve{I} = (\breve{E}_{\breve{I}}, \breve{R}_{\breve{I}}, \breve{Z}_{\breve{I}})$, $\sin \breve{I}^{+} = (\breve{E}_{\breve{I}^{+}}, \breve{R}_{\breve{I}^{-}}, \breve{Z}_{\breve{I}^{-}})$, and $\sin \breve{I}^{-} = (\breve{E}_{\breve{I}^{-}}, \breve{R}_{\breve{I}^{+}}, \breve{Z}_{\breve{I}^{+}})$. Then, by the monotonicity of the sine trigonometric function, we have

$$\begin{aligned}
\breve{E}_{\breve{I}} &= 1 - \prod_{\widehat{J}=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \geq 1 - \prod_{\widehat{J}=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\min_{\widehat{J}}\left\{\breve{E}_{\widehat{J}}\right\}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \\
&= 1 - \left(1 - \sin\left(\frac{\pi}{2}\min\left\{\breve{E}_{\widehat{J}}\right\}\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}} \\
&= \sin\left(\frac{\pi}{2}\min\left\{\breve{E}_{\widehat{J}}\right\}\right) \\
&= \breve{E}_{\breve{I}^{-}}, \\
\breve{R}_{\breve{I}} &= \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \geq \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\left(\min_{\widehat{J}}\left\{\breve{R}_{\widehat{J}}\right\}\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}} \\
&= \left(2\sin^2\left(\frac{\pi}{4}\left(\min\left\{\breve{R}_{\widehat{J}}\right\}\right)\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}} \\
&= \left(2\sin^2\left(\frac{\pi}{4}\left(\min_{\widehat{J}}\left\{\breve{R}_{\widehat{J}}\right\}\right)\right)\right) \\
&= \breve{R}_{\breve{I}^{-}}, \\
\breve{Z}_{\breve{I}} &= \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \geq \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\left(\min_{\widehat{J}}\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}} \\
&= \left(2\sin^2\left(\frac{\pi}{4}\left(\min\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}} \\
&= \left(2\sin^2\left(\frac{\pi}{4}\left(\min_{\widehat{J}}\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right) \\
&= \breve{Z}_{\breve{I}^{-}},
\end{aligned} \tag{30}$$

and also

$$
\begin{aligned}
\breve{E}_{\breve{I}} &= 1 - \prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\widehat{J}}\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}} \le 1 - \prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\max_{\widehat{J}}\left\{\breve{E}_{\widehat{J}}\right\}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\\
&= 1-\left(1-\sin\left(\frac{\pi}{2}\max\left\{\breve{E}_{\widehat{J}}\right\}\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}}\\
&= \sin\left(\frac{\pi}{2}\max_{\widehat{J}}\left\{\breve{E}_{\widehat{J}}\right\}\right)\\
&= \breve{E}_{\breve{I}^{+}},\\
\breve{R}_{\breve{I}} &= \prod_{\widehat{J}=1}^{n}\left(2\sin^{2}\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \le \prod_{\widehat{J}=1}^{n}\left(2\sin^{2}\left(\frac{\pi}{4}\left(\max_{\widehat{J}}\left\{\breve{R}_{\widehat{J}}\right\}\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}}\\
&= \left(2\sin^{2}\left(\frac{\pi}{4}\left(\max\left\{\check{R}_{\widehat{J}}\right\}\right)\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}}\\
&= \left(2\sin^{2}\left(\frac{\pi}{4}\left(\max_{\widehat{J}}\left\{\breve{R}_{\widehat{J}}\right\}\right)\right)\right) = \breve{R}_{\breve{I}^{+}},\\
\breve{Z}_{\breve{I}} &= \prod_{\widehat{J}=1}^{n}\left(2\sin^{2}\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \ge \prod_{\widehat{J}=1}^{n}\left(2\sin^{2}\left(\frac{\pi}{4}\left(\max_{\widehat{J}}\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}}\\
&= \left(2\sin^{2}\left(\frac{\pi}{4}\left(\max\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right)^{\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}} = \left(2\sin^{2}\left(\frac{\pi}{4}\left(\max_{\widehat{J}}\left\{\breve{Z}_{\widehat{J}}\right\}\right)\right)\right) = \breve{Z}_{\breve{I}^{+}}.
\end{aligned}
\tag{31}
$$

Based on score function in Definition 3, we get

$$
\begin{aligned}
S(\sin\breve{I}) &= \breve{E}_{\breve{I}} - \breve{R}_{\breve{I}} - \breve{Z}_{\breve{I}} \le \breve{E}_{\breve{I}^{+}} - \breve{R}_{\breve{I}^{-}} - \breve{Z}_{\breve{I}^{-}} = S\left(\sin\breve{I}^{+}\right),\\
S(\sin\breve{I}) &= \breve{E}_{\breve{I}} - \breve{R}_{\breve{I}} - \breve{Z}_{\breve{I}} \ge \breve{E}_{\breve{I}^{-}} - \breve{R}_{\breve{I}^{+}} - \breve{Z}_{\breve{I}^{+}} = S\left(\sin\breve{I}^{-}\right).
\end{aligned}
\tag{32}
$$

Hence, $S(\sin\breve{I}^{-}) \le S(\sin\breve{I}) \le S(\sin\breve{I}^{+})$. Now, we explain three cases:

If $S(\sin\breve{I}^{-}) \le S(\sin\breve{I}) \le S(\sin\breve{I}^{+})$, then the result holds.

If $S(\sin\breve{I}^{+}) = S(\sin\breve{I})$, then $\breve{E}_{\breve{I}} - \breve{R}_{\breve{I}} - \breve{Z}_{\breve{I}} = \breve{E}_{\breve{I}^{+}} - \breve{R}_{\breve{I}^{+}} - \breve{Z}_{\breve{I}^{+}}$, which implies that $\breve{E}_{\breve{I}} = \breve{E}_{\breve{I}^{+}}, \breve{R}_{\breve{I}} = \breve{R}_{\breve{I}^{+}}$, and $\breve{Z}_{\breve{I}} = \breve{Z}_{\breve{I}^{+}}$ and $H(\sin\breve{I}^{+}) = H(\sin\breve{I})$.

If $S(\sin\breve{I}^{-}) = S(\sin\breve{I})$, then $\breve{E}_{\breve{I}} - \breve{R}_{\breve{I}} - \breve{Z}_{\breve{I}} = \breve{E}_{\breve{I}^{-}} - \breve{R}_{\breve{I}^{-}} - \breve{Z}_{\breve{I}^{-}}$, which implies that $\breve{E}_{\breve{I}} = \breve{E}_{\breve{I}^{-}}, \breve{R}_{\breve{I}} = \breve{R}_{\breve{I}^{-}}$, and $\breve{Z}_{\breve{I}} = \breve{Z}_{\breve{I}^{-}}$ and $H(\sin\breve{I}^{-}) = H(\sin\breve{I})$; therefore, by combining all these cases, we get

$$
\sin\breve{I}^{-} \le \mathrm{ST-PFWA}\left(\breve{I}_1, \breve{I}_2, \ldots, \breve{I}_n\right) \le \sin\breve{I}^{+}. \tag{33}
$$

□

Property 3. Let the collection of PFNs be $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$ and $\breve{I}^{*}_{\widehat{J}} = (\breve{E}^{*}_{\widehat{J}}, \breve{R}^{*}_{\widehat{J}}, \breve{Z}^{*}_{\widehat{J}})$, where $\widehat{J} = 1, \ldots, n$. If $\breve{E}_{\widehat{J}} \le \breve{E}^{*}_{\widehat{J}}, \breve{R}_{\widehat{J}} \ge \breve{R}^{*}_{\widehat{J}}$, and $\breve{Z}_{\widehat{J}} \ge \breve{Z}^{*}_{\widehat{J}}$, then

$$
\mathrm{ST-PFWA}\left(\breve{I}_1, \breve{I}_2, \ldots, \breve{I}_n\right) \le \mathrm{ST-PFWA}\left(\breve{I}^{*}_1, \breve{I}^{*}_2, \ldots, \breve{I}^{*}_n\right). \tag{34}
$$

Proof. It follows from the above, so we omit it here. □

Definition 9. A sine trigonometric PF ordered weighted average (ST − PFOWA) operator is a mapping ST − PFOWA: $\Psi^{n} \longrightarrow \Psi$ such that weighted vector $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^{T}$, which fulfilled the criteria of $\ddot{\omega}_{\widehat{J}} > 0$ and $\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}} = 1$.

$$\text{ST}-\text{PFOWA}=\ddot{\omega}_1\cdot\sin\breve{I}_{\overline{O}(1)}\oplus\ddot{\omega}_2\cdot\sin\breve{I}_{\overline{O}(2)}\oplus\cdots\oplus\ddot{\omega}_n\sin\breve{I}_{\overline{O}(n)}, \tag{35}$$

where $(1,\ldots,n)$ is the permutation of $\overline{O}$, such that $\breve{I}_{\overline{O}(\widehat{J}-1)}\geq\breve{I}_{\overline{O}(\widehat{J})}$ for any $\widehat{J}$.

Theorem 4. *Let a collection of PFNs be $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$, where $\widehat{J}=1,\ldots,n$. Then, by utilizing the operator, that is, ST − PFOWA, the aggregated value is also a PFN and is given by*

$$\text{ST}-\text{PFHA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)=\left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{36}$$

where $(1,\ldots,n)$ is the permutation of $\overline{O}$, as $\breve{I}_{\overline{O}(\widehat{J}-1)}\geq\breve{I}_{\overline{O}(\widehat{J})}$ for any $\widehat{J}$ and $\ddot{I}_{\widehat{J}}=n\ddot{\omega}_{\widehat{J}}\breve{I}_{\widehat{J}}$.

Proof. The proof is the same as that of Theorem 4, so it is omitted here. □

Definition 10. A sine trigonometric PF hybrid average (ST − PFHA) operator is a mapping ST − PFHA: $\Psi^n\longrightarrow\Psi$ such that the associated vectors $\xi=(\xi_1,\xi_2,\ldots,\xi_n)^T$ which fulfilled the criteria of $\xi_{\widehat{J}}>0$ and $\sum_{\widehat{J}=1}^{n}\xi_{\widehat{J}}=1$.

$$\text{ST}-\text{PFHA}=\xi_1\cdot\sin\ddot{I}_{\overline{O}(1)}\oplus\xi_2\sin\ddot{I}_{\overline{O}(2)}\oplus\cdots\oplus\xi_n\sin\ddot{I}_{\overline{O}(n)}, \tag{37}$$

Theorem 5. *Let a collection of PFNs be $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$, where $\widehat{J}=1,\ldots,n$. Then, the aggregated value is also a PFN by utilizing the operator ST − PFHA and is given by*

$$\text{ST}-\text{PFHA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)=\left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\mathring{E}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\mathring{R}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\mathring{Z}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{38}$$

where the weight vectors are $\ddot{\omega}=(\ddot{\omega}_1,\ddot{\omega}_2,\ldots,\ddot{\omega}_n)^T$ of $\sin\breve{I}_{\widehat{J}}(\widehat{J}=1,\ldots,n)$, which fulfilled the criteria of $\ddot{\omega}_{\widehat{J}}>0$ and $\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}=1$.

Proof. The proof is the same as that of Theorem 4, so it is omitted here. □

Definition 11. Let a collection of PFNs be $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$, where $\widehat{J}=1,\ldots,n$. Then the mapping ST − PFWG: $\Psi^n\longrightarrow\Psi$ is known as the sine trigonometric picture fuzzy weighted geometric (ST − PFWG) operator, if

$$\text{ST}-\text{PFHA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)=\left(\sin\breve{I}_1\right)^{\omega_1}\otimes\cdots\otimes\left(\sin\breve{I}_n\right)^{\omega_n}, \tag{39}$$

Theorem 6. *Let a collection of PFNs be $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$, where $\widehat{J}=1,\ldots,n$. Then, the aggregated value is also a PFN by using the ST − PFWG operator and is given by*

$$\text{ST}-\text{PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right)=\left[\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \tag{40}$$

Proof. The proof is similar to that of Theorem 4, so it is omitted here. □

Definition 12. A ST − PFOWG is a mapping from Ψ^n to Ψ such that the weighted vectors $\ddot{\omega}=(\ddot{\omega}_1,\ddot{\omega}_2,\ldots,\ddot{\omega}_n)^T$ which fulfilled the criteria of $\ddot{\omega}_{\widehat{J}}>0$ and $\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}=1$.

$$\text{ST}-\text{PFOWG}=\left(\sin\breve{I}_{\overline{O}(1)}\right)^{\ddot{\omega}_1}\oplus\left(\sin\breve{I}_{\overline{O}(2)}\right)^{\ddot{\omega}_2}\oplus\cdots\oplus\left(\sin\breve{I}_{\overline{O}(n)}\right)^{\ddot{\omega}_n}, \tag{41}$$

where $\overline{O}$ is the permutation of $(1,\ldots,n)$ as $\breve{I}_{\overline{O}(\widehat{J}-1)}\geq\breve{I}_{\overline{O}(\widehat{J})}$ for any $\widehat{J}$.

Theorem 7. *Let a family of PFNs be* $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, *where* $\widehat{J} = 1, \ldots, n$. *Then, the aggregated value is also a PFN by using the* $ST-PFOWG$ *operator and is given by*

$$\mathrm{ST-PFOWG}\left(\breve{I}_1, \ldots, \breve{I}_n\right) = \left[\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, 1 - \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, 1 - \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \quad (42)$$

where $\overline{O}$ is the permutation of $(1, \ldots, n)$ as $\breve{I}_{\overline{O}(\widehat{J}-1)} \geq \breve{I}_{\overline{O}(\widehat{J})}$ for any $\widehat{J}$ and $\ddot{I}_{\widehat{J}} = n\ddot{\omega}_{\widehat{J}}\breve{I}_{\widehat{J}}$.

Proof. The proof is similar to that of Theorem 4. □

Definition 13. A sine trigonometric picture fuzzy hybrid geometric ($ST-PFHG$) operator is a mapping $ST-PFHG$: $\Psi^n \longrightarrow \Psi$, such that the associated vectors are $\xi = (\xi_1, \xi_2, \ldots, \xi_n)^T$, which fulfilled the conditions of $\xi_{\widehat{J}} > 0$ and $\sum_{\widehat{J}=1}^{n} \xi_{\widehat{J}} = 1$.

$$\mathrm{ST-PFHG} = \left(\sin \ddot{I}_{\overline{O}(1)}\right)^{\xi_1} \otimes \left(\sin \ddot{I}_{\overline{O}(2)}\right)^{\xi_2} \otimes \cdots \otimes \left(\sin \ddot{I}_{\overline{O}(n)}\right)^{\xi_n}, \quad (43)$$

Theorem 8. *Let a family of PFNs be* $\breve{I}_{\widehat{J}} = (\breve{E}_{\widehat{J}}, \breve{R}_{\widehat{J}}, \breve{Z}_{\widehat{J}})$, *where* $\widehat{J} = 1, \ldots, n$. *Then, by utilizing the operator, that is,* $ST-PFHG$, *the aggregated value is also a PFN and is given by*

$$\mathrm{ST-PFHG}\left(\breve{I}_1, \ldots, \breve{I}_n\right) = \left[\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\mathring{E}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, 1 - \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\mathring{R}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, 1 - \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\mathring{Z}_{\overline{O}(\widehat{J})}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right]. \quad (44)$$

Proof. The proof is the same as that of Theorem 4, so it is omitted here. □

Similar to $ST-PFWA$ operator, $ST-PFOWA$, $ST-PFHA$, $ST-PFWG$, $ST-PFOWG$, and $ST-PFHG$ operators satisfy some properties such as blondeness and monotonicity.

5. Fundamental Properties of the Proposed Aggregation Operators

In this section of the paper, we discuss many relations between the proposed aggregation operators and also discuss their fundamental properties.

Theorem 9. *For any two PFNs, that is,* $\breve{I}_1$ *and* $\breve{I}_2$, *we have* $\sin \breve{I}_1 \oplus \sin \breve{I}_2 \geq \sin \breve{I}_1 \otimes \sin \breve{I}_2$.

Proof. Let two PFNs be $\breve{I}_1 = (\breve{E}_1, \breve{R}_1, \breve{Z}_1)$ and $\breve{I}_2 = (\breve{E}_2, \breve{R}_2, \breve{Z}_2)$. Then, by using Definitions 6 and 7, we get

$$\begin{aligned}\sin \breve{I}_1 \oplus \sin \breve{I}_2 &= \left[1 - \left(1 - \sin\left(\frac{\pi}{2}\breve{E}_1\right)\right).\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{R}_1\right)\right)\right.\\ &\quad \left..\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right), \left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\right]\\ &= \left[1 - \prod_{\widehat{J}=1}^{2}\left(1 - \sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right), \prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right), \prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)\right],\end{aligned} \quad (45)$$

and also

$$\begin{aligned}\sin\breve{I}_1\otimes\sin\breve{I}_2 &= \left[\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_1\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\breve{E}_2\right)\right),1-\left(2\sin^2\left(\frac{\pi}{4}\left(\breve{R}_1\right)\right)\right)\right.\\ &\quad\left..\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_2\right)\right),1-\left(2\sin^2\left(\frac{\pi}{2}\breve{Z}_1\right)\right).\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_2\right)\right)\right]\\ &=\left[\prod_{\widehat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right),1-\prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right),1-\prod_{\widehat{J}=1}^{2}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)\right].\end{aligned}\tag{46}$$

Since, for any two nonnegative real numbers a and b, their arithmetic mean is greater than or equal to their geometric mean, $((a+b)/2)\geq ab$, and it follows that $a+b-ab\geq ab$. Thus, by taking $a=\sin((\pi/2)\breve{E}_1)$ and $b=\sin((\pi/2)\breve{E}_2)$, we have $1-(1-\sin((\pi/2)\ \breve{E}_1))\cdot 1-\sin((\pi/2)\breve{E}_2)\geq \sin((\pi/2)\breve{E}_1)\cdot(\sin((\pi/2)\breve{E}_2))$, which further gives that $1-\prod_{\widehat{J}=1}^{2}(1-\sin((\pi/2)\breve{E}_{\widehat{J}}))\geq\prod_{\widehat{J}=1}^{2}(\sin((\pi/2)\breve{E}_{\widehat{J}}))$. Similarly, we have obtained the other two as $\prod_{\widehat{J}=1}^{2}(2\sin^2((\pi/4)\breve{R}_{\widehat{J}}))\leq 1-\prod_{\widehat{J}=1}^{2}(2\sin^2((\pi/4)\breve{R}_{\widehat{J}}))$ and $\prod_{\widehat{J}=1}^{2}(2\sin^2((\pi/4)\breve{Z}_{\widehat{J}}))\leq 1-\prod_{\widehat{J}=1}^{2}(2\sin^2((\pi/4)\breve{Z}_{\widehat{J}}))$. Hence, by using Definition 7, we get

$$\sin\breve{I}_1\oplus\sin\breve{I}_2\geq\sin\breve{I}_1\otimes\sin\breve{I}_2.\tag{47}$$

□

Theorem 10. *For any PFN, that is,* $\breve{I}$, *and positive real number* $\dot{\tau}>0$, $\dot{\tau}\cdot\sin\breve{I}\geq(\sin\breve{I})^{\dot{\tau}}$ *if and only if* $\dot{\tau}\geq 1$ *and* $\dot{\tau}\cdot\sin\breve{I}\leq(\sin\breve{I})^{\dot{\tau}}$ *if and only if* $0<\dot{\tau}\leq 1$.

Proof. The proof is the same as that of Theorem 9. □

Lemma 1. *For* $a_{\widehat{J}}\geq 0$ *and* $b_{\widehat{J}}\geq 0$, *we have* $\prod_{\widehat{J}=1}^{n}a_{\widehat{J}}^{b_{\widehat{J}}}\leq\sum_{\widehat{J}=1}^{n}b_{\widehat{J}}\cdot a_{\widehat{J}}$ *and the equality holds if* $.a_1=a_2=\cdots=a_n$..

Lemma 2. *Let* $0\leq a,b\leq 1$, *and* $0\leq x\leq 1$; *then* $0\leq ax+b(1-x)\leq 1$.

Lemma 3. *Let* $0\leq a,b\leq 1$, *and* $1-(1-a)(1-b)\geq ab$.

Theorem 11. *For PFNs* $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$, *the operators ST − PFWA and ST − PFWG satisfy the inequality*

$$\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\geq\mathrm{ST-PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right),\tag{48}$$

where the equality holds if $\breve{I}_1=\breve{I}_2=\cdots=\breve{I}_n$.

Proof. For n, PFNs $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$ and normalized weight vector $\ddot{\omega}_{\widehat{J}}>0$; we have

$$\begin{aligned}\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)&=\left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right],\\ \mathrm{ST-PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right)&=\left[\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right].\end{aligned}\tag{49}$$

For $\ddot{\omega}_{\widehat{J}}>0$, $\sin((\pi/2)(\breve{E}_{\widehat{J}}))\in[0,1]$, and, by Lemma 3, we get

$$\begin{aligned}1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}&\geq 1-\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}\cdot\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)\\ &\geq 1-1+\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)\geq\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\end{aligned}\tag{50}$$

which implies that

$$1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\geq\prod_{\widehat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}. \tag{51}$$

For neutral and negative membership components, we have

$$\begin{aligned}\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}&\leq\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}.\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)\\&\leq 1-\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}.\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)\\&\leq 1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\end{aligned} \tag{52}$$

which implies that

$$\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\leq 1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}, \tag{53}$$

and, similarly, the negative grade is

$$\begin{aligned}\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}&\leq\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}.\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)\\&\leq 1-\sum_{\widehat{J}=1}^{n}\ddot{\omega}_{\widehat{J}}\cdot\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)\\&\leq 1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\end{aligned} \tag{54}$$

which implies that

$$\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\leq 1-\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}. \tag{55}$$

Hence, from all the above equations, we get

$$\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\geq\mathrm{ST-PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right). \tag{56}$$

□

Theorem 12. *Let $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})\,(\widehat{J}=1,\ldots,n)$ and $\breve{I}=(\breve{E},\breve{R},\breve{Z})$ be PFNs; then*

$$\begin{aligned}\mathrm{ST-PFWA}\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\oplus\breve{I}\right)&\geq\mathrm{ST-PFWA}\left(\breve{I}_1\otimes\cdots\oplus\breve{I}_n\otimes\breve{I}\right),\\\mathrm{ST-PFWG}\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\oplus\breve{I}\right)&\geq\mathrm{ST-PFWG}\left(\breve{I}_1\otimes\cdots\oplus\breve{I}_n\otimes\breve{I}\right),\\\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\oplus\sin\breve{I}&\geq\mathrm{ST-PFWA}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\oplus\sin\breve{I},\\\mathrm{ST-PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\otimes\sin\breve{I}&\geq\mathrm{ST-PFWG}\left(\breve{I}_1,\ldots,\breve{I}_n\right)\otimes\sin\breve{I}.\end{aligned} \tag{57}$$

Proof. Here, we prove only the first part, while the other parts can be deduced similarly; for this, let $\breve{I}_{\widehat{J}}=(\breve{E}_{\widehat{J}},\breve{R}_{\widehat{J}},\breve{Z}_{\widehat{J}})$ and $\breve{I}=(\breve{E},\breve{R},\breve{Z})$, since both $\breve{I}_{\widehat{J}}$ and $\breve{I}$ are PFNs.

$$\begin{aligned}\mathrm{ST-PFWA}\left(\breve{I}_1\oplus\breve{I}_2\oplus\cdots\oplus\breve{I}_n\oplus\breve{I}\right)&=\left[1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\widehat{J}}\right)(1-\breve{E})\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}}\right],\\\mathrm{ST-PFWA}\left(\breve{I}_1\otimes\cdots\otimes\breve{I}_n\otimes\breve{I}\right)&=1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\cdot\breve{E}\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\cdot\breve{R}\right)^{\ddot{\omega}_{\widehat{J}}},\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\cdot\breve{Z}\right)^{\ddot{\omega}_{\widehat{J}}}.\end{aligned} \tag{58}$$

For $\breve{E}_{\widehat{J}},\breve{E}\in[0,1]$ and from Lemma 3, we have $1-(1-\breve{E}_{\widehat{J}})(1-\breve{E})\geq\breve{E}_{\widehat{J}}.\breve{E}$. Since "sine" is an increasing function, we get $\sin((\pi/2)(1-(1-\breve{E}_{\widehat{J}})(1-\breve{E})))\geq\sin(\pi/2)(\breve{E}_{\widehat{J}}\cdot\breve{E})$, which gives that

$$\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\widehat{J}}\right)(1-\breve{E})\right)\right) \geq \sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\cdot\breve{E}$$
$$\Rightarrow 1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\widehat{J}}\right)(1-\breve{E})\right)\right) \leq 1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\cdot\breve{E}$$
$$\Rightarrow \prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\widehat{J}}\right)(1-\breve{E})\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}} \leq \prod_{\widehat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(\breve{E}_{\widehat{J}}\cdot\breve{E}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \tag{59}$$
$$\Rightarrow 1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\widehat{J}}\right)(1-\breve{E})\right)\right)\right)^{\ddot{\omega}_{\widehat{J}}} \geq 1-\prod_{\widehat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\widehat{J}}\right)\cdot\breve{E}\right)^{\ddot{\omega}_{\widehat{J}}}.$$

Similarly, for the neutral and negative grades, we get

$$\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \leq \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{R}_{\widehat{J}}\right)\cdot(\breve{R})\right)^{\ddot{\omega}_{\widehat{J}}},$$
$$\prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\right)^{\ddot{\omega}_{\widehat{J}}} \leq \prod_{\widehat{J}=1}^{n}\left(2\sin^2\left(\frac{\pi}{4}\breve{Z}_{\widehat{J}}\right)\cdot(\breve{Z})\right)^{\ddot{\omega}_{\widehat{J}}}. \tag{60}$$

Therefore, from the above equation, we get

$$\begin{aligned}&ST-PFWA\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\oplus\breve{I}\right)\\ &\quad\geq ST-PFWA\left(\breve{I}_1\otimes\cdots\oplus\breve{I}_n\otimes\breve{I}\right).\end{aligned} \tag{61}$$

□

6. Decision-Making Approach

This section provides a strategy, preceded by an illustrative example, to solve the decision-making problem. For this reason, assume m alternative $(\widehat{\gamma}_1, \widehat{\gamma}_2, \ldots, \widehat{\gamma}_m)$ that is evaluated by a group of DMs under the n different criteria $(\breve{G}_1, \breve{G}_2, \ldots, \breve{G}_n)$. That decision-maker tests $\widehat{\gamma}_i$ and $\breve{G}_j$ and gives their preferences in terms of PFNs $\alpha_{ij}^{(\kappa)} = (\breve{E}_{ij}^{(\kappa)}, \breve{R}_{ij}^{(\kappa)}, \breve{Z}_{ij}^{(\kappa)})$, where $i = 1(1)m$; $j = 1(1)n$; $\kappa = 1(1)\widehat{D}$. Then, the rating of each alternative $\widehat{\gamma}_i$ under G_j is expressed as

$$\widehat{\gamma}_i = \left[\left(\breve{G}_1, \alpha_{i1}\right), \left(\breve{G}_2, \alpha_{i2}\right), \ldots, \left(\breve{G}_n, \alpha_{in}\right)\right], \tag{62}$$

and let $\ddot{\omega}_j > 0$ be the normalized weight vector of criteria $\breve{G}_j$. The following steps are taken to calculate the best choice:

Step 1: in terms of decision matrix, summarize the values of each alternative $\widehat{D}^{(\kappa)} = \alpha_{ij}^{(\kappa)}$ with PFS information.

Step 2: aggregate the different preferences $\alpha_{ij}^{(\kappa)}$, $\kappa = 1, \ldots, d$, into $\alpha_{ij} = (\breve{E}_{ij}, \breve{R}_{ij}, \breve{Z}_{ij})$ by either operator.

Step 3: construct the normalized decision matrix $R = (r_{ij})$ from $\widehat{D} = (\alpha_{ij})$, where r_{ij} is computed as

$$r_{ij} = \begin{cases}\left(\breve{E}_{ij}, \breve{R}_{ij}, \breve{Z}_{ij}\right), & \text{if benefit type attributes,}\\ \left(\breve{Z}_{ij}, \breve{R}_{ij}, \breve{E}_{ij}\right), & \text{if cost type attributes.}\end{cases} \tag{63}$$

Step 4: if the weights of the attributes are known as before, then use them. Otherwise, we measure these by using the entropy principle. For this, the information entropy of criteria $\breve{G}_j$ is computed as

$$\begin{aligned}\overline{E}_j = \frac{1}{(\sqrt{2}-1)m}\sum_{j=1}^{m}\Bigg[&\sin\left(\frac{\pi}{4}\left(1+\breve{E}_{ij}-\breve{R}_{ij}-\breve{Z}_{ij}\right)\right)\\ &+\sin\left(\frac{\pi}{4}\left(1-\breve{E}_{ij}+\breve{R}_{ij}+\breve{Z}_{ij}\right)\right)-1\Bigg],\end{aligned} \tag{64}$$

where $1/(\sqrt{2}-1)m$ is a constant for assuring $0 \leq \overline{E}_j \leq 1$.

Based on it, the weights of the attributes are computed as $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$, where

$$\omega_j = \frac{1-\overline{E}_j}{n-\sum_{j=1}^{n}\overline{E}_j}. \tag{65}$$

Step 5: with weight vector ω and the proposed averaging or geometric PF aggregation operators, the collective values are obtained as $r_i = (\breve{E}_i, \breve{R}_i, \breve{Z}_i)$ for each alternative $\widehat{\gamma}_i$.

Step 6: find the score values of $r_i = (\breve{E}_i, \breve{R}_i, \breve{Z}_i)(i = 1, \ldots, m)$.

Step 7: grade all the possible alternatives $\widehat{\gamma}_i(i = 1, \ldots, m)$ and select the most desirable alternative(s).

7. Illustrative Example

In this portion, we discuss with an example the result of the defined MAGDM approach and compare its results with the existing approaches [38].

7.1. Application of the Proposed MAGDM Method. Assume that the five companies $\widehat{\gamma}_1, \widehat{\gamma}_2, \widehat{\gamma}_3, \widehat{\gamma}_4$, and $\widehat{\gamma}_5$ were assessed by three decision-makers $DM_{(1)}$, $DM_{(2)}$, and $DM_{(3)}$ for funding focused on four criteria, which are given as follows:

(1) $\breve{G}_1$ denotes the enterprise level of the management
(2) $\breve{G}_2$ denotes the growth ability of the business
(3) $\breve{G}_3$ denotes the economic benefit

Table 1: $DM_{(1)}$.

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$	$\breve{G}_4$
$\hat{\gamma}_1$	(0.64, 0.17, 0.19)	(0.49, 0.41, 0.10)	(0.60, 0.31, 0.09)	(0.25, 0.18, 0.57)
$\hat{\gamma}_2$	(0.56, 0.11, 0.33)	(0.59, 0.29, 0.12)	(0.55, 0.34, 0.11)	(0.62, 0.25, 0.13)
$\hat{\gamma}_3$	(0.46, 0.29, 0.25)	(0.62, 0.15, 0.23)	(0.51, 0.22, 0.27)	(0.62, 0.14, 0.24)
$\hat{\gamma}_4$	(0.50, 0.36, 0.14)	(0.71, 0.18, 0.11)	(0.46, 0.17, 0.37)	(0.59, 0.21, 0.20)
$\hat{\gamma}_5$	(0.12, 0.13, 0.75)	(0.43, 0.26, 0.31)	(0.41, 0.13, 0.46)	(0.30, 0.27, 0.43)

Table 2: $DM_{(2)}$.

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$	$\breve{G}_4$
$\hat{\gamma}_1$	(0.22, 0.10, 0.68)	(0.36, 0.27, 0.37)	(0.74, 0.16, 0.10)	(0.57, 0.30, 0.13)
$\hat{\gamma}_2$	(0.78, 0.12, 0.10)	(0.67, 0.14, 0.19)	(0.81, 0.10, 0.09)	(0.58, 0.33, 0.09)
$\hat{\gamma}_3$	(0.39, 0.27, 0.34)	(0.34, 0.32, 0.34)	(0.46, 0.41, 0.13)	(0.68, 0.21, 0.11)
$\hat{\gamma}_4$	(0.47, 0.19, 0.34)	(0.49, 0.18, 0.33)	(0.34, 0.26, 0.40)	(0.64, 0.27, 0.09)
$\hat{\gamma}_5$	(0.56, 0.26, 0.18)	(0.50, 0.35, 0.15)	(0.52, 0.35, 0.13)	(0.19, 0.11, 0.70)

(4) $\breve{G}_4$ denotes the corporate reputation

Assume that $\ddot{\omega} = (0.37, 0.41, 0.22)$ represents the experts weight information and assessment of the decision matrices using PFNs in the following Tables 1–3. The aim of this issue is to choose the best company to invest.

Step 1: the evaluations of all decision-makers are summarized in Tables 1–3.

Step 2: by taking the weight of the experts, that is, $\ddot{\omega} = (0.37, 0.41, 0.22)$, and then utilizing the ST–PFWA operator to achieve the collective data on each alternative, the results are shown in Table 4.

Step 3: almost all of the four attributes are just to be the benefit types; then normalization is not needed.

Step 4: we used the idea of the entropy in this step to obtain the values:

$$\begin{aligned}\overline{E}_1 &= 0.728313575,\\ \overline{E}_2 &= 0.697921077,\\ \overline{E}_3 &= 0.653517061,\\ \overline{E}_4 &= 0.664506725.\end{aligned} \tag{66}$$

By the help of this, we find the attributes $\omega = (0.216355366, 0.240558194, 0.275918987, 0.267167453)$.

Step 5: based on $\omega = (0.216355366, 0.240558194, 0.275918987, 0.267167453)$ and utilizing the ST–PFWA operator, the collective values of each alternative are gained as

$$\begin{aligned}\gamma_1 &= (0.905540446, 1.26702E-07, 3.5098E-08),\\ \gamma_2 &= (0.976969493, 3.90763E-08, 1.68931E-10),\\ \gamma_3 &= (0.904107853, 6.55832E-08, 1.24568E-07),\\ \gamma_4 &= (0.935101452, 4.68173E-08, 4.68173E-08),\\ \gamma_5 &= (0.790485389, 6.43519E-08, 7.16207E-06).\end{aligned} \tag{67}$$

Step 6: we can get the scores of each by using the definition

$$\begin{aligned}\overline{S}(\gamma_1) &= 0.905540284,\\ \overline{S}(\gamma_2) &= 0.976969454,\\ \overline{S}(\gamma_3) &= 0.904107663,\\ \overline{S}(\gamma_4) &= 0.935101326,\\ \overline{S}(\gamma_5) &= 0.790478163.\end{aligned} \tag{68}$$

Step 7: according to $\overline{S}(\gamma_2) > \overline{S}(\gamma_4) > \overline{S}(\gamma_1) > \overline{S}(\gamma_3) > \overline{S}(\gamma_5)$, the ranking order is $\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_1 > \hat{\gamma}_3 > \hat{\gamma}_5$. Hence, $\hat{\gamma}_2$ is the best alternative.

During Step 5 of the established method, the complete analysis of changing aggregation operators is analyzed, and their results are shown in Table 5.

We can therefore conclude from all the abovementioned computational process that the alternative $\hat{\gamma}_2$ is really the best option among the other options and therefore it is strongly recommended to choose the appropriate option. In Figure 1, we draw the graphical representation of all the alternatives ranked based on the score values and show that the alternative $\hat{\gamma}_2$ is the best one.

8. Comparative Analysis

In this section, we give some brief discussion on the comparison of the proposed method with some well-known related methods [33, 38, 45, 46].

8.1. Comparison with [38]. In the existing method, the bi-directional project methods for MAGDM problems with PFNs are discussed, but, in the proposed method, we defined the sine trigonometric entropy aggregation operators for MAGDM problem. The results of the MAGDM approach are listed in Table 6. It is concluded that the best alternative remains the same. Therefore, the suggested approach is more rational than the current one [38].

Table 3: $DM_{(3)}$.

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$	$\breve{G}_4$
$\hat{\gamma}_1$	$(0.53, 0.21, 0.26)$	$(0.51, 0.11, 0.38)$	$(0.55, 0.23, 0.22)$	$(0.34, 0.25, 0.41)$
$\hat{\gamma}_2$	$(0.61, 0.38, 0.01)$	$(0.54, 0.17, 0.29)$	$(0.65, 0.20, 0.15)$	$(0.77, 0.10, 0.13)$
$\hat{\gamma}_3$	$(0.58, 0.15, 0.27)$	$(0.19, 0.13, 0.68)$	$(0.61, 0.11, 0.28)$	$(0.25, 0.18, 0.57)$
$\hat{\gamma}_4$	$(0.42, 0.31, 0.27)$	$(0.58, 0.20, 0.22)$	$(0.81, 0.10, 0.09)$	$(0.52, 0.15, 0.33)$
$\hat{\gamma}_5$	$(0.26, 0.24, 0.50)$	$(0.27, 0.29, 0.44)$	$(0.34, 0.39, 0.27)$	$(0.52, 0.14, 0.34)$

Table 4: Aggregated values of experts by using the ST-PFWA operator.

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$	$\breve{G}_4$
$\hat{\gamma}_1$	$(0.684, 0.025, 0.138)$	$(0.615, 0.081, 0.063)$	$(0.857, 0.059, 0.016)$	$(0.656, 0.069, 0.099)$
$\hat{\gamma}_2$	$(0.875, 0.027, 0.011)$	$(0.823, 0.044, 0.037)$	$(0.892, 0.041, 0.014)$	$(0.849, 0.064, 0.015)$
$\hat{\gamma}_3$	$(0.665, 0.072, 0.101)$	$(0.638, 0.047, 0.139)$	$(0.724, 0.072, 0.049)$	$(0.801, 0.037, 0.054)$
$\hat{\gamma}_4$	$(0.674, 0.087, 0.065)$	$(0.813, 0.042, 0.049)$	$(0.748, 0.039, 0.094)$	$(0.807, 0.056, 0.032)$
$\hat{\gamma}_5$	$(0.546, 0.047, 0.170)$	$(0.635, 0.101, 0.075)$	$(0.643, 0.074, 0.071)$	$(0.180, 0.032, 0.289)$

Table 5: Score value and ranking of the different operators.

Operators	Score values					Ranking
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	$\hat{\gamma}_5$	
ST-PFWA	0.9055	0.9769	0.9041	0.9351	0.7905	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_1 > \hat{\gamma}_3 > \hat{\gamma}_5$
ST-PFOWA	0.8948	0.9760	0.8972	0.9314	0.7846	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFHA	0.9829	0.9991	0.9849	0.9943	0.9368	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFWG	0.8607	0.9719	0.8791	0.9186	0.7327	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFOWG	0.8525	0.9710	0.8715	0.9138	0.7260	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFHG	0.9705	0.9988	0.9763	0.9887	0.9245	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$

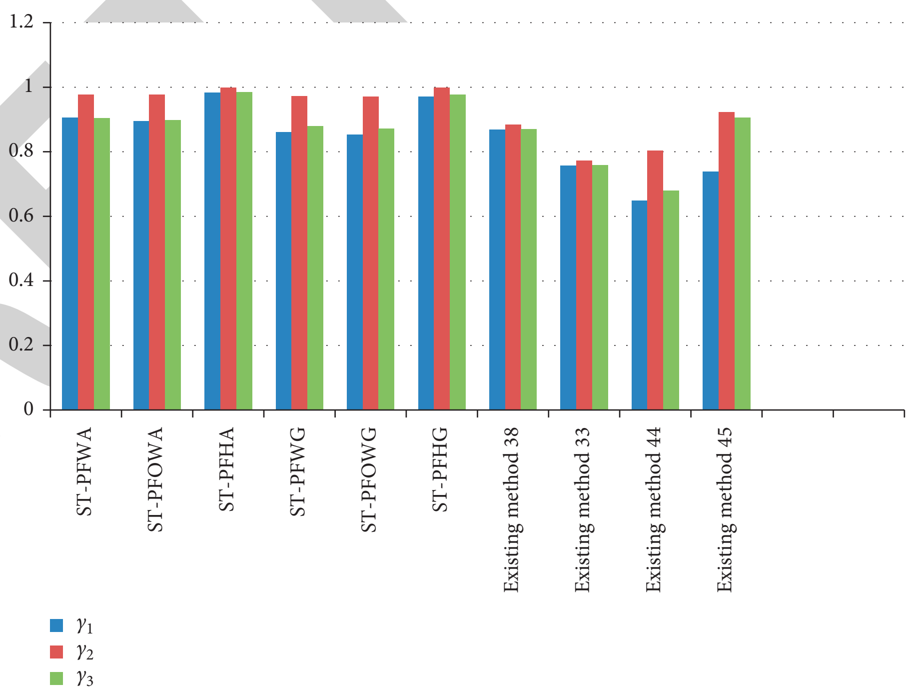

Figure 1: Graphical representation of the obtained results using different proposed operators.

Table 6: Score values and ranking of the proposed operators and existing operators.

Proposed operators	Score values					Ranking
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	$\hat{\gamma}_5$	
ST-PFWA	0.9055	0.9769	0.9041	0.9351	0.7905	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_1 > \hat{\gamma}_3 > \hat{\gamma}_5$
ST-PFOWA	0.8948	0.9760	0.8972	0.9314	0.7846	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFHA	0.9829	0.9991	0.9849	0.9943	0.9368	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFWG	0.8607	0.9719	0.8791	0.9186	0.7327	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFOWG	0.8525	0.9710	0.8715	0.9138	0.7260	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
ST-PFHG	0.9705	0.9988	0.9763	0.9887	0.9245	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
Existing method [38]	0.8681	0.8837	0.8690	0.8754	0.8600	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
Existing method [33]	0.7570	0.7726	0.7580	0.7350	0.7500	$\hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4 > \hat{\gamma}_5$
Existing method [45]	0.6480	0.8037	0.6790	0.7600	0.6300	$\hat{\gamma}_2 > \hat{\gamma}_4 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_5$
Existing method [46]	0.7382	0.9217	0.9060	0.8761	0.7702	$\hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_4 > \hat{\gamma}_5 > \hat{\gamma}_1$

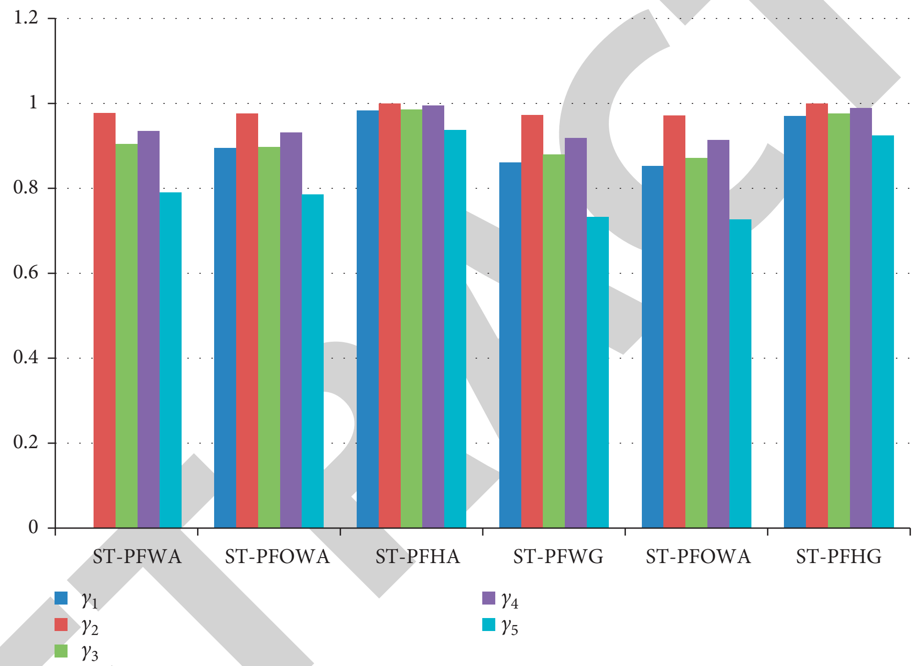

Figure 2: Graphical representation of the obtained result utilizing proposed operators and other existing methods.

Furthermore, we compare our proposed aggregation operators with some other existing approaches, which are proposed by [33, 45, 46], to deal with picture fuzzy quantities. Then, the calculating results are the same in ranking alternatives and the best alternative is also the same. Thus, these four methods with PFNs are conducted to further illustrate the advantages of the new approach.

We can therefore conclude from all the abovementioned comparative studies that the alternative $\hat{\gamma}_2$ is the best among the other options. In Figure 2, we draw the graphical representation of all the alternatives ranked based on the score values by using the proposed operators and existing operators and show that the alternative $\hat{\gamma}_2$ is the best one.

9. Conclusion

A research related to aggregation operators was investigated in this study by establishing some new sine trigonometric operation laws for PFSs. During decision-making problems, the well-defined operational laws play a major role. On the other hand, the sine trigonometric function has the features of periodicity as well as being symmetric about the origin and hence is more likely to satisfy the decision-maker's preference over the multiple time periods. We therefore describe some sine trigonometric operational laws for PFNs and study their properties in order to take these advantages and make a smoother and more important decision. We have defined various averaging and geometric aggregation operators on the basis of these operators to club decision maker's preference. The different elementary relations between the aggregation operators are studied and explained in detail. We developed a new MAGDM algorithm for group decision-making problems, in which goals are classified in terms of PFNs to enforce the proposed laws on decision-making problems. Further, we compute the weight of the attribute by combining the subjective and objective data in terms of the measure. The functionality of the proposed method is applied to an example, and superiority and feasibility of the approach are investigated in detail. A comparative study is often carried out with current works to verify its performance.

In the future, we will use the framework built on new multiattribute assessment models to tackle fuzziness and ambiguity in a variety of decision-making parameters, for example, advanced study of the similarity measures of intuitionistic fuzzy sets based on the set pair analysis theory, generalized intuitionistic fuzzy entropy-based approach for solving MADM problems with unknown attribute weights; intuitionistic fuzzy Hamacher aggregation operators with entropy weight and their applications to MCDM problems, and linguistic picture fuzzy Dombi aggregation operators and their application in a MAGDM problem.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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Research Article

Decision Support Technique Based on Spherical Fuzzy Yager Aggregation Operators and Their Application in Wind Power Plant Locations: A Case Study of Jhimpir, Pakistan

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The problem of energy crisis and environmental pollution has been mitigated by the generation and use of wind power; however, the choice of locations for wind power plants is a difficult task because the decision-making process includes political, socioeconomic, and environmental aspects. Thus, several adverse consequences have been created by the choice of suboptimal locations. The objective of this paper is to address the integrated qualitative and quantitative multicriteria decision-making framework for the selection of wind power plant locations. Spherical fuzzy sets are the latest extension of the ordinary fuzzy sets. The main characteristic of the spherical fuzzy sets is satisfying the condition that the squared sum of the positive, neutral, and negative grades must be at least zero and at most one. In this research, we establish novel operational laws based on the Yager t-norm and t-conorm under spherical fuzzy environments (SFE). Furthermore, based on these Yager operational laws, we develop list of novel aggregation operators under SFE. In addition, we design an algorithm to tackle the uncertainty to investigating the best wind power plant selection in four potential locations in Pakistan. A numerical example of wind power plant location problem is considered to show the supremacy and effectiveness of the proposed study. Also, a detailed comparison is constructed to evaluate the performance and validity of the established technique.

1. Introduction

One of the common and daily activity in humans' life is decision-making, aiming to choose the optimal alternative with respect to a list of attributes. Due to high capacity of decision-making to model the uncertainty of information, it has been widely studied and successfully applied to economics, management, and the other fields in recent years. Due to the uncertainty of decision information, utilizing fuzzy set theory to settle decision-making problem has become a hotspot in recent years. The concept of fuzzy set (FS) theory was firstly proposed by Zadeh [1], and, since then, the FSs have been widely used in many decision-making (DM) problems. FSs theory is a useful and appropriate approach to handle inaccurate and uncertain information in vague situations. Since the introduction of the FSs by Zadeh, they have been accepted and widespread in nearly all branches of science. Many extensions of ordinary FSs have been introduced by many researchers [2–7]. These extensions have been used frequently in the progress of DM problems in an uncertain environment. Some commonly used extensions of ordinary FSs will be explained in the following.

Intuitionistic FS is firstly established by Atanassov by adding the negative membership grades. Intuitionistic FS is the important generation of FS theory to tackle the uncertainty in complex real-life DM problems with more effective and efficient way. Many researchers contribute to intuitionistic FS theory; for example, Xu and Yager [8] introduced the geometric means-based aggregation operators (AOs) under intuitionistic FSs. Xu [9] established list of novel AOs to tackle the complex uncertainty under intuitionistic fuzzy settings. Wang and Liu [10] proposed the Einstein norm-based AOs using intuitionistic FSs. Yu and Xu [11] utilized prioritized concept to establish new AOs for intuitionistic FSs. Munezza et al. [12] presented the multicriteria DM system using cubic intuitionistic FS to tackle the uncertainty in location selection DM problem of small hydropower plant. Khan et al. [13] presented the novel DM approach under generalized intuitionistic fuzzy soft sets to tackle the incomplete information in daily life decision problems.

Although intuitionistic FS can deal with incomplete and uncertainty information, it cannot handle inconsistent information better in real situations. For example, in the work of Son [14], in the election of village director, the voting results can be divided into three categories: "vote for," "neutral voting," and "vote against." "Neutral voting" means that the voting paper is a white paper rejecting both agree and disagree for the candidate, but it still takes the vote. This example happened in reality, but intuitionistic FS could not handle it. In order to solve these problems, Cuong et al. [15, 16] proposed picture FS, which contains three aspects of information: yes, neutral, and no. It can deal with inconsistent information. Up to now, many outstanding contributions have been made in the research of picture FSs; for example, Wei [17] introduced some novel AOs for picture FS and discussed their applications in DM problems. Ashraf et al. [18] highlighted the deficiency in the existing operational laws and established novel improved AOs to tackle the uncertainty in complex real-life DM problems under picture fuzzy environment. Khan et al. [19] established the novel extension, generalized picture fuzzy soft sets, and discussed their DM applications. Khan et al. [20] established the novel AOs using logarithmic function and algebraic norm under picture fuzzy environment. Qiyas et al. [21] presented the linguistic information and algebraic norm-based novel AOs using picture FSs. Ashraf et al. [22] presented the cleaner production evaluation technique based on the cubic picture fuzzy AOs using distance information measures. Qiyas et al. [23] utilized linguistic variables to develop the list of AOs based on Dombi operational laws to tackle the DM problems of real word. Ashraf and Abdullah [24] introduced algebraic norm-based AOs under cubic picture FS and discussed their applications in decision problem.

Picture FS is an important generalization of FS theory, but, with the constant complexity of human knowledge modeling and theory development, picture FS will be invalid in some DM problems. Ashraf et al. [25, 26] introduced a new and more general concept spherical fuzzy set (spherical FS), which is an extension of FS by further slackening the condition that $0 \leq \mu^2(v) + \wp^2(v) + \partial^2(v) \leq 1$. We must also note that the acceptable spherical space provides more freedom for observers to express their belief in supporting membership. Therefore, spherical FSs express more extensive fuzzy information, while spherical FSs are more maneuverable and more appropriate for dealing with uncertainties information. However, spherical FSs have been successfully applied in some fields, especially in decision-making fields.

As aggregation operators have a strong role in DM problems, several researchers have made quite valuable contributions to introduce aggregation operators for spherical FS. Spherical aggregation operators based on algebraic norms [26] deal with uncertainty and inaccurate information in DM problems. Spherical FS representations of spherical fuzzy norms [27] are introduced under SF information. SF Dombi aggregation operators based on Dombi norm are introduced in [28]. SF logarithmic aggregation operators based on entropy are proposed in [29]. Linguistic SF aggregation operators are presented in [30] for SF information to tackle the uncertainty in DMP. GRA methodology based on spherical linguistic fuzzy Choquet integral is proposed [31] for SF information. Cosine similarity measures are presented in [32] to discuss the application in DMP. Application of SF distance measures is discussed in [33] to determine the child development influence environmental factors using SF information. In [34], the TOPSIS approach based on SF rough set was proposed and its application in DMP was discussed. Gü ndoğdu and Kahraman [35] established the TOPSIS methodology under spherical FSs and also proposed its applications. Ashraf and Abdullah [36] presented the emergency decision-making technique of coronavirus using the spherical FSs. Ashraf et al. [37] introduced the symmetric sum-based AOs under spherical FSs to tackle the uncertainty in daily life DM problems. Gundogdu and Kahraman [38] established the generalized methodology based on WASPAS under spherical FSs. Shishavan et al. [39] established the list of similarity measures to tackle the uncertainty in the form of spherical fuzzy environment. Gündoğdu and Kahraman [40] presented the new AHP technique to tackle the uncertainty in renewable energy and in [41] discussed the spherical fuzzy QFD technique to tackle the uncertainty in robot technology development problems.

It is evident that the abovementioned AOs are focused on the algebraic, Einstein, Dombi, and Hamacher norms under spherical FSs for the implementation of the combination process. Algebraic, Einstein, Dombi, and Hamacher product and sum are not only fundamental spherical FS operations that describe the union and the intersection of any two spherical FSs. A general union and intersection under SF information can be developed from a generalized norm; that is, instances of deferent-norms families may be used to execute the respective intersections and unions under spherical fuzzy environment. The Yager product and sum are good replacement of the algebraic, Einstein, Dombi, and Hamacher product for an intersection and union and are capable of delivering smooth estimates of the algebraic product and sum. However, there seems to be little work in the literature on aggregation approaches that use the Yager operations on FS theory to aggregate the fuzzy numbers. Akram and Shahzadi [42] introduced the q-rung orthopair

FS-based Yager AOs to tackle the DM problems. Akram et al. [43] presented the Yager norm-based AOs under complex Pythagorean FSs and discussed their application in DM problems. Shahzadi et al. [44] presented the DM approach based on Yager operational laws under Pythagorean information. Garg et al. [45] presented the DM problem of COVID-19 testing facility using Fermatean FS and Yager norm information.

From the above analysis, we note that, in many practical applications, various aggregation operators have been put forward and implemented. Although in practical problems many existing AOs are not able to address such specific cases, in some circumstances, many of these may result in unreasonable or counterintuitive results. Certain new regulations built without a simple function may have a complicated description. But generalized aggregation operators for SFSs continue to be an open subject that attracts the attention of many researchers. Therefore, in this article, our aim is to present some novel spherical fuzzy Yager operational laws based AOs to tackle the uncertainty in real-world DM problems with more effective and efficient way. The contribution and originality of this study are summarized as follows:

(i) Novel ranking methodology and Yager norm-based novel operational laws for spherical fuzzy sets are proposed
(ii) The new spherical fuzzy Yager averaging/geometric aggregation operators are proposed to aggregate the uncertainties in the form of spherical fuzzy environment
(iii) Decision-making algorithm is proposed to tackle the real-world DM problems
(iv) A real-life numerical application about wind power plant location selection problem in Pakistan is discussed to show the applicability of the proposed technique

The rest of this article shall be organized as set out below. Section 2 provides basic information concerning spherical FSs. Section 3 describes the Yager operations of spherical FSs. Section 4 proposes a new way to rank the spherical fuzzy number with more consistency. Section 5, presented as the cornerstone of this work, proposes novel spherical fuzzy Yager AOs based on the Yager norm, together with the associated proof of its properties. Section 6 introduces the novel methodology for interacting with the ambiguity in DM problems in order to pick the best alternative according to the list of attributes. Section 7 provides a numerical application about wind power plant location selection problem used to illustrate the designed MAGDM method and a comparative analysis with some existing frameworks of MAGDM is discussed in Section 8. The article is concluded in Section 9.

2. Preliminaries

Let us briefly recall in this segment the rudiments of FSs and spherical FSs. For the following review, these definitions will be included here.

Definition 1 (see [1]). A fuzzy set (FS) F in a universe set U is an object having the form

$$F = \{\langle v, \mu(v)\rangle | v \in U\}, \quad (1)$$

where $\mu(v) \in [0,1]$ is represented by the positive membership grade.

Definition 2 (see [46]). An intuitionistic FS F_s in a universe set U is an object having the form

$$F_s = \{\langle v, \mu(v), \partial(v)\rangle | v \in U\}, \quad (2)$$

where $\mu(v) \in [0,1]$ and $\partial(v) \in [0,1]$ are positive and negative membership grades, respectively. In addition, $0 \le \mu(v) + \partial(v) \le 1$, $\forall\, v \in U$.

Definition 3 (see [47]). A Pythagorean FS F_s in a universe set U is an object having the form

$$F_s = \{\langle v, \mu(v), \partial(v)\rangle | v \in U\}, \quad (3)$$

where $\mu(v) \in [0,1]$ and $\partial(v) \in [0,1]$ are positive and negative membership grades, respectively. In addition, $0 \le \mu^2(v) + \partial^2(v) \le 1$, $\forall\, v \in U$.

Definition 4 (see [15]). A picture FS F_s in a universe set U is an object having the form

$$F_s = \{\langle v, \mu(v), \wp(v), \partial(v)\rangle | v \in U\}, \quad (4)$$

where $\mu(v) \in [0,1]$, $\wp(v) \in [0,1]$, and $\partial(v) \in [0,1]$ are positive, neutral, and negative membership grades, respectively. In addition, $0 \le \mu(v) + \wp(v) + \partial(v) \le 1$, $\forall v \in U$.

Definition 5 (see [25, 26]). A spherical FS F_s in a universe set U is an object having the form

$$F_s = \{\langle v, \mu(v), \wp(v), \partial(v)\rangle | v \in U\}, \quad (5)$$

where $\mu(v) \in [0,1]$, $\wp(v) \in [0,1]$, and $\partial(v) \in [0,1]$ are positive, neutral, and negative membership grades, respectively. In addition, $0 \le \mu^2(v) + \wp^2(v) + \partial^2(v) \le 1$, $\forall\, v \in U$.

In what follows, we signify by SFS (U) the family of all spherical FSs. We shall signify the spherical fuzzy number (SFN) with the triplet $F_{ts} = (\mu(v), \wp(v), \partial(v))$ for simplicity.

Definition 6 (see [25]). Suppose that, for any $F_{ts_{(1)}}, F_{ts_{(2)}} \in$ SFS (U).

(1) $F_{ts_{(1)}} \subseteq F_{ts_{(2)}}$ if and only if $\mu_1 \le \mu_2, \wp_1 \le \wp_2$ and $\partial_1 \ge \partial_2$. Clearly, $F_{ts_{(1)}} = F_{ts_{(2)}}$ if $F_{ts_{(1)}} \subseteq F_{ts_{(2)}}$ and $F_{ts_{(2)}} \subseteq F_{ts_{(1)}}$.
(2) $F_{ts_{(1)}} \cap F_{ts_{(2)}} = \{\min(\mu_1, \mu_2), \min(\wp_1, \wp_2), \max(\partial_1, \partial_2)\}$.
(3) $F_{ts_{(1)}} \cup F_{ts_{(2)}} = \{\max(\mu_1, \mu_2), \min(\wp_1, \wp_2), \min(\partial_1, \partial_2)\}$.
(4) $F^c_{ts_{(1)}} = \{\partial_1, \wp_1, \mu_1\}$.

Definition 7 (see [25]). Let $F_{ts_{(1)}}, F_{ts_{(2)}} \in$ SFS (U) with $\varrho > 0$. The operating laws are as follows:

(1) $F_{ts_{(1)}} \otimes F_{ts_{(2)}} = \left\{\mu_1\mu_2, \wp_1\wp_2, \sqrt{\partial_1^2 + \partial_2^2 - \partial_1^2\partial_2^2}\right\}$

(2) $F_{ts_{(1)}} \oplus F_{ts_{(2)}} = \left\{ \sqrt{\mu_1^2 + \mu_2^2 - \mu_1^2\mu_2^2}, \wp_1\wp_2, \partial_1\partial_2 \right\}$

(3) $F_{ts_{(1)}}^{\varrho} = \left\{ \mu_1^{\varrho}, \wp_1^{\varrho}, \sqrt{1-(1-\partial_1^2)^{\varrho}} \right\}$

(4) $\varrho \cdot F_{ts_{(1)}} = \left\{ \sqrt{1-(1-\mu_1^2)^{\varrho}}, \wp_1^{\varrho}, \partial_1^{\varrho} \right\}$

Definition 8 (see [25, 48]). Let $F_{ts_{(1)}} = (\mu_1, \wp_1, \partial_1)$ and $F_{ts_{(2)}} = (\mu_2, \wp_2, \partial_2) \in \text{SFS}\ (U)$. $\widehat{Sc}(F_{ts_{(1)}}) = \mu_1^2 - \partial_1^2$ and $\widehat{Sc}(F_{ts_{(2)}}) = \mu_2^2 - \partial_2^2$ are the score values of SFNs. Also $\widehat{Ac}(F_{ts_{(1)}}) = \mu_1^2 + \wp_1^2 + \partial_1^2$ and $\widehat{Ac}(F_{ts_{(2)}}) = \mu_2^2 + \wp_2^2 + \partial_2^2$ are the accuracy values of SFNs. We have the following:

(a) $\widehat{Sc}(F_{ts_{(1)}}) < \widehat{Sc}(F_{ts_{(2)}}) \Longrightarrow F_{ts_{(1)}} < F_{ts_{(2)}}$

(b) $\widehat{Sc}(F_{ts_{(1)}}) = \widehat{Sc}(F_{ts_{(2)}}), \widehat{Ac}(F_{ts_{(1)}}) < \widehat{Ac}(F_{ts_{(2)}}) \Longrightarrow F_{ts_{(1)}} < F_{ts_{(2)}}$

(c) $\widehat{Sc}(F_{ts_{(1)}}) = \widehat{Sc}(F_{ts_{(2)}}), \widehat{Ac}(F_{ts_{(1)}}) = \widehat{Ac}(F_{ts_{(2)}}) \Longrightarrow F_{ts_{(1)}} = F_{ts_{(2)}}$

Definition 9 (see [26]). Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}\ (U)\ (g = 1, 2, 3, \ldots, n)$. Then, the weighted averaging AOs for SFN (U) are described as

$$\begin{aligned} \text{SFWA}\ (F_1, F_2, \ldots, F_n) &= \ell_1 F_1 \oplus \ell_2 F_2 \oplus \cdots \oplus \ell_n F_n \\ &= \sum_{g=1}^{n} \ell_g F_g \\ &= \left\{ \sqrt{1 - \prod_{g=1}^{n} \left(1-\mu_g^2\right)^{\ell_g}}, \prod_{g=1}^{n} \left(\wp_g\right)^{\ell_g}, \prod_{g=1}^{n} \left(\partial_g\right)^{\ell_g} \right\}, \end{aligned} \tag{6}$$

where the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

Definition 10 (see [26]). Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}\ (U)\ (g = 1, 2, 3, \ldots, n)$. Then, the weighted geometric AOs for SFN (U) are described as

$$\begin{aligned} \text{SFWG}\ (F_1, F_2, \ldots, F_n) &= F_1^{\ell_1} \otimes F_2^{\ell_2} \otimes \cdots \otimes F_n^{\ell_n} \\ &= \prod_{g=1}^{n} \left(F_g\right)^{\ell_g} \\ &= \left\{ \prod_{g=1}^{n} \left(\mu_g\right)^{\ell_g}, \prod_{g=1}^{n} \left(\wp_g\right)^{\ell_g}, \sqrt{1 - \prod_{g=1}^{n} \left(1-\partial_g^2\right)^{\ell_g}} \right\}, \end{aligned} \tag{7}$$

where the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

3. New Operating Laws for Spherical FS

Aggregation operators (AOs) play an essential part in combining data into one form and in tackling MCGDM problems. Aggregation facilitates the establishment of a number of choices in a system or a collection of objects that have come together or have been brought together. In recent years, AOs based on FSs and their different hybrid compositions have provided a great deal of attention and have become interesting because they can quickly execute functional areas of various regions. In this section, we propose the Yager norms-based novel operational laws for spherical FNs.

Definition 11 (see [49]). Suppose that, for any real numbers l and m, Yager's norms have the forms

(1) $\check{T}(l, m) = 1 - \min(1, ((1-l)^{\delta} + (1-m)^{\delta})^{1/\delta})$;

(2) $\widehat{S}(l, m) = \min(1, (l^{\delta} - m^{\delta})^{1/\delta}),\ \delta \in (0, \infty)$.

Definition 12. Let $F_{ts_{(1)}}, F_{ts_{(2)}} \in \text{SFS}\ (U)$ with $\varrho, \delta > 0$. The Yager operating laws (YOLs) are $F_{ts} = (\mu(v), \wp(v), \partial(v))$ and are described as follows:

(1) $$\begin{aligned} F_{ts_{(1)}} \otimes F_{ts_{(2)}} = \Big\{ &\sqrt{1 - \min(1, ((1-\mu_1^2)^{\delta} + (1-\mu_2^2)^{\delta})^{1/\delta})}, \\ &\sqrt{1 - \min(1, ((1-\wp_1^2)^{\delta} + (1-\wp_2^2)^{\delta})^{1/\delta})}, \\ &\sqrt{\min(1, (\partial_1^{2\delta} + \partial_2^{2\delta})^{1/\delta})} \Big\} \end{aligned}$$

(2) $$\begin{aligned} F_{ts_{(1)}} \oplus F_{ts_{(2)}} = \Big\{ &\sqrt{\min(1, (\mu_1^{2\delta} + \mu_2^{2\delta})^{1/\delta})}, \\ &\sqrt{1 - \min(1, ((1-\wp_1^2)^{\delta} + (1-\wp_2^2)^{\delta})^{1/\delta})}, \\ &\sqrt{1 - \min(1, ((1-\partial_1^2)^{\delta} + (1-\partial_2^2)^{\delta})^{1/\delta})} \Big\} \end{aligned}$$

(3) $F_{ts_{(1)}}^{\varrho} = \left\{ \sqrt{1 - \min(1, (\varrho(1-\mu_1^2)^\delta)^{1/\delta})}, \sqrt{1 - \min(1, (\varrho(1-\wp_1^2)^\delta)^{1/\delta})}, \sqrt{\min(1, (\varrho\partial_1^{2\delta})^{1/\delta})} \right\}$

(4) $\varrho \cdot F_{ts_{(1)}} = \left\{ \sqrt{\min(1, (\varrho\mu_1^{2\delta})^{1/\delta})}, \sqrt{1 - \min(1, (\varrho(1-\wp_1^2)^\delta)^{1/\delta})}, \sqrt{1 - \min(1, (\varrho(1-\partial_1^2)^\delta)^{1/\delta})} \right\}$

Theorem 1. *Let $F_{ts_{(1)}}, F_{ts_{(2)}} \in SFS\ (U)$ with $\varrho_1, \varrho_2 > 0$. Then,*

(1) $F_{ts_{(1)}} \oplus F_{ts_{(2)}} = F_{ts_{(2)}} \oplus F_{ts_{(1)}}$
(2) $F_{ts_{(1)}} \otimes F_{ts_{(2)}} = F_{ts_{(2)}} \otimes F_{ts_{(1)}}$
(3) $\varrho(F_{ts_{(1)}} \oplus F_{ts_{(2)}}) = \varrho F_{ts_{(1)}} \oplus \varrho F_{ts_{(2)}}$
(4) $(\varrho_1 + \varrho_2) F_{ts_{(1)}} = \varrho_1 F_{ts_{(1)}} \oplus \varrho_2 F_{ts_{(1)}}$
(5) $(F_{ts_{(1)}} \otimes F_{ts_{(2)}})^{\varrho} = F_{ts_{(1)}}^{\varrho} \otimes F_{ts_{(2)}}^{\varrho}$
(6) $F_{ts_{(1)}}^{\varrho_1} \otimes F_{ts_{(1)}}^{\varrho_2} = F_{ts_{(1)}}^{(\varrho_1+\varrho_2)}$

Proof. For any $F_{ts_{(1)}}, F_{ts_{(2)}} \in$ SFS (U) with $\varrho_1, \varrho_2 > 0$, we have

$$\begin{aligned}
F_{ts_{(1)}} \oplus F_{ts_{(2)}} &= \left\{ \sqrt{\min\left(1, \left(\mu_1^{2\delta} + \mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\wp_1^2\right)^\delta + \left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\partial_1^2\right)^\delta + \left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{\min\left(1, \left(\mu_2^{2\delta} + \mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\wp_2^2\right)^\delta + \left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\partial_2^2\right)^\delta + \left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= F_{ts_{(2)}} \oplus F_{ts_{(1)}}, \\
F_{ts_{(1)}} \otimes F_{ts_{(2)}} &= \left\{ \sqrt{1 - \min\left(1, \left(\left(1-\mu_1^2\right)^\delta + \left(1-\mu_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\wp_1^2\right)^\delta + \left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{\min\left(1, \left(\partial_1^{2\delta} + \partial_2^{2\delta}\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{1 - \min\left(1, \left(\left(1-\mu_2^2\right)^\delta + \left(1-\mu_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\wp_2^2\right)^\delta + \left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{\min\left(1, \left(\partial_2^{2\delta} + \partial_1^{2\delta}\right)^{1/\delta}\right)} \right\} \\
&= F_{ts_{(2)}} \otimes F_{ts_{(1)}}, \\
\varrho\left(F_{ts_{(1)}} \oplus F_{ts_{(2)}}\right) &= \varrho \cdot \left\{ \sqrt{\min\left(1, \left(\mu_1^{2\delta} + \mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\wp_1^2\right)^\delta + \left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1-\partial_1^2\right)^\delta + \left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{\min\left(1, \left(\varrho\mu_1^{2\delta} + \varrho\mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\wp_1^2\right)^\delta + \varrho\left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\partial_1^2\right)^\delta + \varrho\left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
\varrho F_{ts_{(1)}} \oplus \varrho F_{ts_{(2)}} &= \left\{ \sqrt{\min\left(1, \left(\varrho\mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&\quad \oplus \left\{ \sqrt{\min\left(1, \left(\varrho\mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{\min\left(1, \left(\varrho\mu_1^{2\delta} + \varrho\mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\wp_1^2\right)^\delta + \varrho\left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho\left(1-\partial_1^2\right)^\delta + \varrho\left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\}, \\
\Rightarrow \varrho\left(F_{ts_{(1)}} \oplus F_{ts_{(2)}}\right) &= \varrho F_{ts_{(1)}} \oplus \varrho F_{ts_{(2)}}. \\
\varrho_1 F_{ts_{(1)}} \oplus \varrho_2 F_{ts_{(1)}} &= \left\{ \sqrt{\min\left(1, \left(\varrho_1\mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_1\left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_1\left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&\quad \oplus \left\{ \sqrt{\min\left(1, \left(\varrho_2\mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_2\left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_2\left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{\min\left(1, \left((\varrho_1+\varrho_2)\mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left((\varrho_1+\varrho_2)\left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left((\varrho_1+\varrho_2)\left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= (\varrho_1+\varrho_2) F_{ts_{(1)}}. \\
\varrho_1 F_{ts_{(1)}} \oplus \varrho_2 F_{ts_{(1)}} &= \left\{ \sqrt{\min\left(1, \left(\varrho_1\mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_1\left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_1\left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&\quad \oplus \left\{ \sqrt{\min\left(1, \left(\varrho_2\mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_2\left(1-\wp_2^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\varrho_2\left(1-\partial_2^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= \left\{ \sqrt{\min\left(1, \left((\varrho_1+\varrho_2)\mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left((\varrho_1+\varrho_2)\left(1-\wp_1^2\right)^\delta\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left((\varrho_1+\varrho_2)\left(1-\partial_1^2\right)^\delta\right)^{1/\delta}\right)} \right\} \\
&= (\varrho_1+\varrho_2) F_{ts_{(1)}}.
\end{aligned} \tag{8}$$

Proofs of (5) and (6) are similar as above. □

4. A New Way to Rank SFNs

Here, we will construct a new procedure for the ranking of SFNs in the present section. This new framework is being used to rank SFNs and to choose the best alternative. From Definition 8, we know that Mahmood et al. [48] have given us a ranking method of SFNs. Therefore, this framework is sensitive to a slight change in the SFNs, as shown by the example below.

Example 1. Suppose the following: $F_{ts_{(\alpha 1)}} = (0.7, 0.5, 0.66)$, $F_{ts_{(\alpha 2)}} = (0.7, 0.5, 0.6599)$, $F_{ts_{(\beta 1)}} = (0.8, 0.3, 0.77)$, and $F_{ts_{(\beta 2)}} = (0.8, 0.3, 0.7699) \in SFS\ (U)$. Then, by Definition 8, we have

$$\begin{aligned} \widehat{Sc}\left(F_{ts_{(\alpha 1)}}\right) &= 0.055504, \widehat{Sc}\left(F_{ts_{(\alpha 2)}}\right) = 0.055634,\\ \widehat{Sc}\left(F_{ts_{(\beta 1)}}\right) &= 0.055467, \widehat{Sc}\left(F_{ts_{(\beta 2)}}\right) = 0.055644. \end{aligned} \tag{9}$$

Since $\widehat{Sc}(F_{ts_{(\alpha 1)}}) > \widehat{Sc}(F_{ts_{(\beta 1)}})$ and $\widehat{Sc}(F_{ts_{(\alpha 2)}}) < \widehat{Sc}(F_{ts_{(\beta 2)}})$, $F_{ts_{(\alpha 1)}} > F_{ts_{(\beta 1)}}$ and $F_{ts_{(\alpha 2)}} < F_{ts_{(\beta 2)}}$.

The findings in the above example demonstrate that the SFNs ranking, which is assessed on the basis of Definition 8, would absolutely change, even if the SFNs slightly change. A new ranking way is proposed in this article in order to overcome the shortcomings of the ranking framework based on Definition 8. The novel ranking framework is proposed in the following definition.

Definition 13. Let $F_{ts_{(1)}} = (\mu_1, \wp_1, \partial_1) \in SFS\ (U)$. Then, the score value $\widehat{Sc}$ is described as

$$\widehat{Sc}\left(F_{ts_{(1)}}\right) = \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\mu_1)^2 + 1 - \mu_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}, \tag{10}$$

where $\widehat{Sc}(F_{ts_{(1)}}) \in [0, 1]$.

Theorem 2. *Let $F_{ts_{(1)}} = (\mu_1, \wp_1, \partial_1) \in SFS\ (U)$. If $\mu_1 = \partial_1$, then $\widehat{Sc}(F_{ts_{(1)}}) = 1/2$; if $\mu_1 > \partial_1$, then $\widehat{Sc}(F_{ts_{(1)}}) > 1/2$, and if $\mu_1 < \partial_1$, then $\widehat{Sc}(F_{ts_{(1)}}) < 1/2$.*

Proof. Let $F_{ts_{(1)}} = (\mu_1, \wp_1, \partial_1) \in SFS\ (U)$. If $\mu_1 = \partial_1$, then

$$\widehat{Sc}\left(F_{ts_{(1)}}\right) = \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\mu_1)^2 + 1 - \mu_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} = \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} = \frac{1}{2}. \tag{11}$$

If $\mu_1 > \partial_1$, then

$$\widehat{Sc}\left(F_{ts_{(1)}}\right) = \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\mu_1)^2 + 1 - \mu_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} > \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} = \frac{1}{2}. \tag{12}$$

If $\mu_1 < \partial_1$, then

$$\widehat{Sc}\left(F_{ts_{(1)}}\right) = \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\mu_1)^2 + 1 - \mu_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} < \frac{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}}{\left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2} + \left((1-\partial_1)^2 + 1 - \partial_1^2\right)^{1/2}} = \frac{1}{2}. \tag{13}$$

The proof is completed. □

Theorem 3. *Let $F_{ts_{(1)}} = (\mu_1, \wp_1, \partial_1)$ and $F_{ts_{(2)}} = (\mu_2, \wp_2, \partial_2) \in SFS\ (U)$. If $\mu_1 = \mu_2$ and $\partial_1 < \partial_2$, then $\widehat{Sc}(F_{ts_{(1)}}) > \widehat{Sc}(F_{ts_{(2)}})$. Otherwise, if $\partial_1 = \partial_2$ and $\mu_1 > \mu_2$, then $\widehat{Sc}(F_{ts_{(1)}}) > \widehat{Sc}(F_{ts_{(2)}})$.*

Proof. Let $J = \widehat{Sc}(F_{ts_{(1)}})/\widehat{Sc}(F_{ts_{(2)}})$. According to Definition 13, we have

$$J=\frac{\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}/\left(\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}+\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\right)}{\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}/\left(\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}+\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\right)}$$
$$=\frac{\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\left[\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}+\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\right]}{\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\left[\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}+\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\right]} \tag{14}$$
$$=\frac{\left[\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]+\left[\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}.\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\right]}{\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}\right]+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\right]}.$$

Let $P=[((1-\partial_1)^2+1-\partial_1^2)^{1/2}.((1-\partial_2)^2+1-\partial_2^2)^{1/2}]$; then we have

$$J=\frac{P+\left[\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}\right]}. \tag{15}$$

If $\mu_1=\mu_2$ and $\partial_1<\partial_2$, then we have

$$\begin{aligned}J&=\frac{P+\left[\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}\right]}\\&>\frac{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}\\&=1.\end{aligned} \tag{16}$$

Thus, $J=(\widehat{S}c(F_{ts_{(1)}})/\widehat{S}c(F_{ts_{(2)}}))>1\Rightarrow\widehat{S}c(F_{ts_{(1)}})>\widehat{S}c(F_{ts_{(2)}})$. If $\partial_1=\partial_2$ and $\mu_1>\mu_2$, then we have

$$\begin{aligned}J&=\frac{P+\left[\left((1-\partial_1)^2+1-\partial_1^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_1)^2+1-\mu_1^2\right)^{1/2}\right]}\\&>\frac{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}{P+\left[\left((1-\partial_2)^2+1-\partial_2^2\right)^{1/2}\cdot\left((1-\mu_2)^2+1-\mu_2^2\right)^{1/2}\right]}\\&=1.\end{aligned} \tag{17}$$

Thus, $J=(\widehat{S}c(F_{ts_{(1)}})/\widehat{S}c(F_{ts_{(2)}}))>1\Rightarrow\widehat{S}c(F_{ts_{(1)}})>\widehat{S}c(F_{ts_{(2)}})$. The proof is completed.

However, it is also observed that, in certain situations, the score function that we have described cannot distinguish SFNs. For example, consider the following: $F_{ts_{(1)}}=(0.6,0.5,0.6)$ and $F_{ts_{(2)}}=(0.7,0.5,0.7)\in SFS(U)$. It is not difficult to get that $\widehat{S}c(F_{ts_{(1)}})=\widehat{S}c(F_{ts_{(2)}})=0.5$ according to Definition 13. To equate SFNs in these situations, the accuracy function is established and a novel framework for ordering SFNs is proposed. □

Definition 14. Let $F_{ts_{(1)}}=(\mu_1,\wp_1,\partial_1)$ and $F_{ts_{(2)}}=(\mu_2,\wp_2,\partial_2)\in SFS(U)$. $\widehat{S}c(F_{ts_{(1)}})$ and $\widehat{S}c(F_{ts_{(2)}})$ are the score values of $F_{ts_{(1)}}$ and $F_{ts_{(2)}}$, and $\widehat{A}c(F_{ts_{(1)}})$ and $\widehat{A}c(F_{ts_{(2)}})$ are the accuracy values of $F_{ts_{(1)}}$ and $F_{ts_{(2)}}$, respectively.

(1) If $\widehat{S}c(F_{ts_{(1)}})<\widehat{S}c(F_{ts_{(2)}})$, then $F_{ts_{(1)}}<F_{ts_{(2)}}$
(2) If $\widehat{S}c(F_{ts_{(1)}})=\widehat{S}c(F_{ts_{(2)}})$, then
 (a) if $\widehat{A}c(F_{ts_{(1)}})<\widehat{A}c(F_{ts_{(2)}})$, then $F_{ts_{(1)}}<F_{ts_{(2)}}$
 (b) $\widehat{A}c(F_{ts_{(1)}})=\widehat{A}c(F_{ts_{(2)}})$, then $F_{ts_{(1)}}=F_{ts_{(2)}}$.

Example 2. Consider the following: $F_{ts_{(\alpha1)}}=(0.7,0.5,0.66)$, $F_{ts_{(\alpha2)}}=(0.7,0.5,0.6599)$, $F_{ts_{(\beta1)}}=(0.8,0.3,0.77)$, and $F_{ts_{(\beta2)}}=(0.8,0.3,0.7699)\in SFS(U)$ (from Example 1). Then, by Definition 13, we have

$$\begin{aligned}\widehat{S}c\left(F_{ts_{(\alpha1)}}\right)&=0.055504,\quad \widehat{S}c\left(F_{ts_{(\alpha2)}}\right)=0.055634,\\ \widehat{S}c\left(F_{ts_{(\beta1)}}\right)&=0.055467,\quad \widehat{S}c\left(F_{ts_{(\beta2)}}\right)=0.055644.\end{aligned} \tag{18}$$

Since $\widehat{S}c(F_{ts_{(\alpha1)}})<\widehat{S}c(F_{ts_{(\beta1)}})$ and $\widehat{S}c(F_{ts_{(\alpha2)}})<\widehat{S}c(F_{ts_{(\beta2)}})$, $F_{ts_{(\beta1)}}>F_{ts_{(\alpha1)}}$ and $F_{ts_{(\beta2)}}>F_{ts_{(\alpha2)}}$.

Incorporating Examples 1 and 2, we can conclude that the new ranking framework is more reliable and less sensitive than in the previous process.

5. Aggregation Operators Based on Yager's Norms

This section presents some spherical fuzzy AOs using Yager OLs of SFNs.

5.1. Yager Weighted Averaging AOs

Definition 15. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in$ SFN (U) $(g = 1, 2, 3, \ldots, n)$. Then, Yager weighted averaging AOs for SFN (U) are described as

$$\mathrm{SFYWA}\left(F_1, F_2, \ldots, F_n\right) = \ell_1 F_1 \oplus \ell_2 F_2 \oplus \cdots \oplus \ell_n F_n = \sum_{g=1}^{n} \ell_g F_g, \tag{19}$$

where the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

Theorem 4. *Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in$ SFN (U) $(g = 1, 2, 3, \ldots, n)$ and the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$. The SFYWA AOs are a mapping $\mathcal{G}^n \longrightarrow \mathcal{G}$ such that*

$$\begin{aligned}
\mathrm{SFYWA}\left(F_1, F_2, \ldots, F_n\right) &= \sum_{g=1}^{n} \ell_g F_g \\
&= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\right. \\
&\quad \left.\sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}\right).
\end{aligned} \tag{20}$$

Proof. We prove Theorem 4 by applying mathematical induction on n. For each g, $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in$ SFN (U), which implies that $\mu_g, \wp_g, \partial_g \in [0, 1]$ and $\mu_g^2 + \wp_g^2 + \partial_g^2 \leq 1$.

Step 1: for $n = 2$, we get

$$\mathrm{SFYWA}\left(F_1, F_2\right) = \ell_1 F_1 \oplus \ell_2 F_2. \tag{21}$$

By Definition 12, we have

$$\begin{aligned}
\mathrm{SFYWA}\left(F_1, F_2\right) &= \ell_1 F_1 \oplus \ell_2 F_2 \\
&= \left\{\sqrt{\min\left(1, \left(\ell_1 \mu_1^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_1 \left(1 - \wp_1^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_1 \left(1 - \partial_1^2\right)^{\delta}\right)^{1/\delta}\right)}\right\} \\
&\oplus \left\{\sqrt{\min\left(1, \left(\ell_2 \mu_2^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_2 \left(1 - \wp_2^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_2 \left(1 - \partial_2^2\right)^{\delta}\right)^{1/\delta}\right)}\right\} \\
&= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{2} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{2} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\right. \\
&\quad \left.\sqrt{1 - \min\left(1, \left(\sum_{g=1}^{2} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}\right).
\end{aligned} \tag{22}$$

Step 2: suppose that equation (22) holds for $n = \kappa$; we have

$$\mathrm{SFYWA}\left(F_1, F_2, \ldots, F_\kappa\right) = \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}\right). \tag{23}$$

Step 3: now, we have to prove that equation (22) holds for $n = \kappa + 1$.

$$\begin{aligned}\mathrm{SFYWA}\left(F_1, F_2, \ldots F_{\kappa+1}\right) &= \sum_{g=1}^{\kappa} \ell_g F_g \oplus \ell_{\kappa+1} F_{\kappa+1} \\ &= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}\right) \\ &\oplus \left(\sqrt{\min\left(1, \left(\ell_{\kappa+1} \mu_{\kappa+1}^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_{\kappa+1} \left(1 - \wp_{\kappa+1}^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\ell_{\kappa+1} \left(1 - \partial_{\kappa+1}^2\right)^{\delta}\right)^{1/\delta}\right)}\right) \\ &= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{\kappa+1} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right), \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa+1} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right), \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{\kappa+1} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}}}\right).\end{aligned} \tag{24}$$

That is, when $n = z + 1$, equation (22) also holds.

Hence, equation (22) holds for any n. The proof is completed. □

Next, we give some properties of the proposed SFYWA aggregation operator.

Theorem 5. *Let* $F_g = \left(\mu_g(v), \wp_g(v), \partial_g(v)\right) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *such that* $F_g = F$. *Then,*

$$\mathrm{SFYWA}\left(F_1, F_2, \ldots, F_n\right) = F. \tag{25}$$

Proof. Since $F_g = F$ $(g = 1, 2, 3, \ldots, n)$, by Theorem 4, we get

$$\begin{aligned}\mathrm{SFYWA}\left(F_1, F_2, \ldots, F_n\right) &= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \mu_g^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \partial_g^2\right)^{\delta}\right)^{1/\delta}\right)}\right) \\ &= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \mu^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \partial^2\right)^{\delta}\right)^{1/\delta}\right)}\right) \\ &= \left(\sqrt{\min\left(1, \left(\mu^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1 - \wp^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\left(1 - \partial^2\right)^{\delta}\right)^{1/\delta}\right)}\right) \\ &= \left(\mu(v), \wp(v), \partial(v)\right) \\ &= F.\end{aligned} \tag{26}$$

The proof is completed. □

Theorem 6. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^- = \{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\}$, *and* $F_g^+ = \{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\} \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \text{SFYWA}(F_1, F_2, \ldots, F_n) \leq F_g^+. \quad (27)$$

Proof. The procedure is similar to the above theorem, so it is eliminated here. □

Theorem 7. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*, \wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\text{SFYWA}(F_1, F_2, \ldots, F_n) \leq \text{SFYWA}(F_1^*, F_2^*, \ldots, F_n^*). \quad (28)$$

Proof. The procedure is similar to the above theorem, so it is eliminated here. □

Definition 16. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}(U)$ $(g = 1, 2, 3, \ldots, n)$. Then, Yager ordered weighted averaging AOs for SFN (U) are described as

$$\begin{aligned}\text{SFYOWA}(F_1, F_2, \ldots, F_n) &= \ell_1 F_{v(1)} \oplus \ell_2 F_{v(2)} \oplus \cdots \oplus \ell_n F_{v(n)} \\ &= \sum_{g=1}^{n} \ell_g F_{v(g)},\end{aligned} \quad (29)$$

where $v(g)$ represented the ordered and $(v(1), v(2), v(3), \ldots, v(n))$ is a permutation of $(1, 2, 3, \ldots, n)$, subject to $\varepsilon_{v(g-1)} \geq \varepsilon_{v(g)}$ for all g. Also the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

Theorem 8. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *and the weights* $(\ell_1, \ell_2, \ldots, \ell_g)$ *of* F_g *have* $\ell_g \geq 0$ *and* $\sum_{g=1}^{n} \ell_g = 1$. *The SFYOWA AOs are a mapping* $\mathcal{G}^n \longrightarrow \mathcal{G}$ *such that*

$$\begin{aligned}\text{SFYOWA}(F_1, F_2, \ldots, F_n) &= \sum_{g=1}^{n} \ell_g F_{v(g)} \\ &= \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \mu_{v(g)}^{2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_{v(g)}^2\right)^{\delta}\right)^{1/\delta}\right)}, \right. \\ &\quad \left. \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \partial_{v(g)}^2\right)^{\delta}\right)^{1/\delta}\right)} \right).\end{aligned} \quad (30)$$

Proof. It follows from Theorem 4 similarly. □

Theorem 9. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *such that* $F_g = F$. *Then,*

$$\text{SFYOWA}(F_1, F_2, \ldots, F_n) = F. \quad (31)$$

Theorem 10. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^- = \{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\}$, *and* $F_g^+ = \{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\} \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \text{SFYOWA}(F_1, F_2, \ldots, F_n) \leq F_g^+. \quad (32)$$

Theorem 11. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*, \wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\text{SFYOWA}(F_1, F_2, \ldots, F_n) \leq \text{SFYOWA}(F_1^*, F_2^*, \ldots, F_n^*). \quad (33)$$

The proof of these theorems is similarly followed by Theorems 5–7.

Definition 17. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}(U)$ $(g = 1, 2, 3, \ldots, n)$. Then, Yager hybrid weighted averaging AOs for SFN (U) are described as

$$\begin{aligned}\text{SFYHWA}(F_1, F_2, \ldots, F_n) &= \sigma_g F'_{v(1)} \oplus \sigma_g F'_{v(2)} \oplus \cdots \oplus \sigma_g F'_{v(n)} \\ &= \sum_{g=1}^{n} \sigma_g F'_{v(g)},\end{aligned} \quad (34)$$

where weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$ and gth biggest weighted value is $F_{v(g)}^{'} (F_{v(g)}^{'} = n\ell_g F_{v(g)} | g = 1, 2, \ldots, n)$ consequently by total order $(v(1), v(2), v(3), \ldots, v(n))$. Also, associated weights $(\sigma_1, \sigma_2, \ldots, \sigma_g)$ of F_g have $\sigma_g \geq 0$ and $\sum_{g=1}^{n} \sigma_g = 1$.

Theorem 12. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *and the weights* $(\ell_1, \ell_2, \ldots, \ell_g)$ *of* F_g *have* $\ell_g \geq 0$ *and* $\sum_{g=1}^{n} \ell_g = 1$. *The SFYHWA AOs are a mapping* $\mathcal{G}^n \longrightarrow \mathcal{G}$ *with associated weights* $(\sigma_1, \sigma_2, \ldots, \sigma_g)$ *of* F_g *having* $\sigma_g \geq 0$ *and* $\sum_{g=1}^{n} \sigma_g = 1$; *we have*

$$\text{SFYHWA}\left(F_1, F_2, \ldots, F_n\right) = \sum_{g=1}^{n} \sigma_g F_{v(g)}{}' = \left(\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \mu_{v(g)}^{\prime 2\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_{v(g)}^{\prime 2}\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \partial_{v(g)}^{\prime 2}\right)^{\delta}\right)^{1/\delta}\right)}\right). \tag{35}$$

Proof. It follows from Theorem 4 similarly. □

Theorem 13. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *such that* $F_g = F$. *Then,*

$$\text{SFYHWA}\left(F_1, F_2, \ldots, F_n\right) = F. \tag{36}$$

Theorem 14. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v))$, $F_g^- = \left\{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\right\}$, *and* $F_g^+ = \left\{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\right\} \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \text{SFYHWA}\left(F_1, F_2, \ldots, F_n\right) \leq F_g^+. \tag{37}$$

Theorem 15. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*, \wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\text{SFYHWA}\left(F_1, F_2, \ldots, F_n\right) \leq \text{SFYHWA}\left(F_1^*, F_2^*, \ldots, F_n^*\right). \tag{38}$$

The proof of these theorems is similarly followed by Theorems 5–7.

5.2. Yager Weighted Geometric AOs

Definition 18. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}(U)$ $(g = 1, 2, 3, \ldots, n)$. Then, Yager weighted geometric AOs for SFN (U) are described as

$$\text{SFYWG}\left(F_1, F_2, \ldots, F_n\right) = F_1^{\ell_1} \otimes F_2^{\ell_2} \otimes \cdots \otimes F_n^{\ell_n} = \prod_{g=1}^{n} \left(F_g\right)^{\ell_g}, \tag{39}$$

where the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

Theorem 16. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *and the weights* $(\ell_1, \ell_2, \ldots, \ell_g)$ *of* F_g *have* $\ell_g \geq 0$ *and* $\sum_{g=1}^{n} \ell_g = 1$. *The SFYWG AOs are a mapping* $\mathcal{G}^n \longrightarrow \mathcal{G}$ *such that*

$$\text{SFYWG}\left(F_1, F_2, \ldots, F_n\right) = \prod_{g=1}^{n} \left(F_g\right)^{\ell_g} = \left(\sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \mu_g^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_g^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \partial_g^{2\delta}\right)^{1/\delta}\right)}\right). \tag{40}$$

Proof. We prove Theorem 16 by applying mathematical induction on n. For each g, $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}(U)$, which implies that $\mu_g, \wp_g, \partial_g \in [0, 1]$ and $\mu_g^2 + \wp_g^2 + \partial_g^2 \leq 1$.

Step 1: for $n = 2$, we get

$$\text{SFYWG}\left(F_1, F_2\right) = F_1^{\ell_1} \otimes F_2^{\ell_2}. \tag{41}$$

By Definition 12, we have

$$\begin{aligned}
\text{SFYWG}\,(F_1,F_2) &= F_1^{\ell_1}\otimes F_2^{\ell_2}\\
&= \left\{\sqrt{1-\min\left(1,\left(\ell_1\left(1-\mu_1^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\ell_1\left(1-\wp_1^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{\min\left(1,\left(\ell_1\partial_1^{2\delta}\right)^{1/\delta}\right)}\right\}\\
&\oplus\left\{\sqrt{1-\min\left(1,\left(\ell_2\left(1-\mu_2^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\ell_2\left(1-\wp_2^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{\min\left(1,\left(\ell_2\partial_2^{2\delta}\right)^{1/\delta}\right)}\right\}\\
&= \left(\sqrt{1-\min\left(1,\left(\sum_{g=1}^{2}\ell_g\left(1-\mu_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\sum_{g=1}^{2}\ell_g\left(1-\wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{\min\left(1,\left(\sum_{g=1}^{2}\ell_g\partial_g^{2\delta}\right)^{1/\delta}\right)}\right).
\end{aligned} \tag{42}$$

Step 2: suppose that equation (42) holds for $n=\kappa$; we have

$$\begin{aligned}
\text{SFYWG}\,(F_1,F_2,\ldots,F_\kappa) = \Bigg(&\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\left(1-\mu_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\left(1-\wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\\
&\sqrt{\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\partial_g^{2\delta}\right)^{1/\delta}\right)}\Bigg).
\end{aligned} \tag{43}$$

Step 3: now, we have to prove that equation (42) holds for $n=\kappa+1$.

$$\begin{aligned}
\text{SFYWG}\,(F_1,F_2,\ldots F_{\kappa+1}) &= \prod_{g=1}^{\kappa}\left(F_g\right)^{\ell_g}\otimes\left(F_{\kappa+1}\right)^{\ell_{\kappa+1}}\\
&= \left(\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\left(1-\mu_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\left(1-\wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{\min\left(1,\left(\sum_{g=1}^{\kappa}\ell_g\partial_g^{2\delta}\right)^{1/\delta}\right)}\right)\\
&\oplus\left\{\sqrt{1-\min\left(1,\left(\ell_{\kappa+1}\left(1-\mu_{\kappa+1}^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\ell_{\kappa+1}\left(1-\wp_{\kappa+1}^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{\min\left(1,\left(\ell_{\kappa+1}\partial_{\kappa+1}^{2\delta}\right)^{1/\delta}\right)}\right\}\\
&= \Bigg(\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa+1}\ell_g\left(1-\mu_g^2\right)^{\delta}\right)^{1/\delta}\right)},\sqrt{1-\min\left(1,\left(\sum_{g=1}^{\kappa+1}\ell_g\left(1-\wp_g^2\right)^{\delta}\right)^{1/\delta}\right)},\\
&\quad\sqrt{\min\left(1,\left(\sum_{g=1}^{\kappa+1}\ell_g\partial_g^{2\delta}\right)^{1/\delta}\right)}\Bigg).
\end{aligned} \tag{44}$$

That is, when $n=z+1$, equation (42) also holds.

Hence, equation (42) holds for any n. The proof is completed. □

Theorem 17. *Let* $F_g=(\mu_g(v),\wp_g(v),\partial_g(v))\in SFN\,(U)$ $(g=1,2,3,\ldots,n)$ *such that* $F_g=F$. *Then,*

$$\mathrm{SFYWG}\left(F_1, F_2, \ldots, F_n\right) = F. \tag{45}$$

Proof. Since $F_g = F$ $(g = 1, 2, 3, \ldots, n)$, by Theorem 16, we get

$$\begin{aligned}
\mathrm{SFYWG}\left(F_1, F_2, \ldots, F_n\right) &= \left(\sqrt{1-\min\left(1, \left(\sum_{g=1}^{n} \ell_g\left(1-\mu^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1-\min\left(1, \left(\sum_{g=1}^{n} \ell_g\left(1-\wp^2\right)^{\delta}\right)^{1/\delta}\right)},\right. \\
&\left.\sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \partial^{2\delta}\right)^{1/\delta}\right)}\right) \\
&= \left(\sqrt{1-\min\left(1,\left(1-\mu^2\right)\right)}, \sqrt{1-\min\left(1,\left(1-\wp^2\right)\right)}, \sqrt{\min\left(1,\left(\partial^2\right)\right)}\right) \\
&= (\mu(v), \wp(v), \partial(v)) \\
&= F.
\end{aligned} \tag{46}$$

The proof is completed. □

Theorem 18. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^- = \left\{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\right\}$, *and* $F_g^+ = \left\{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\right\} \in SFN\ (U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \mathrm{SFYWG}\left(F_1, F_2, \ldots, F_n\right) \leq F_g^+. \tag{47}$$

Theorem 19. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN\ (U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*, \wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\mathrm{SFYWG}\left(F_1, F_2, \ldots, F_n\right) \leq \mathrm{SFYWG}\left(F_1^*, F_2^*, \ldots, F_n^*\right). \tag{48}$$

Definition 19. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \mathrm{SFN}\ (U)$ $(g = 1, 2, 3, \ldots, n)$. Then, Yager ordered weighted geometric AOs for SFN (U) are described as

$$\mathrm{SFYOWG}\left(F_1, F_2, \ldots, F_n\right) = \left(F_{v(1)}\right)^{\ell_1} \otimes \left(F_{v(2)}\right)^{\ell_2} \otimes \cdots \otimes \left(F_{v(n)}\right)^{\ell_n} = \prod_{g=1}^{n}\left(F_{v(g)}\right)^{\ell_g}, \tag{49}$$

where $v(g)$ represented the ordered and $(v(1), v(2), v(3), \ldots, v(n))$ is a permutation of $(1, 2, 3, \ldots, n)$, subject to $\varepsilon_{v(g-1)} \geq \varepsilon_{v(g)}$ for all g. Also the weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$.

Theorem 20. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN\ (U)$ $(g = 1, 2, 3, \ldots, n)$ *and the weights* $(\ell_1, \ell_2, \ldots, \ell_g)$ *of* F_g *have* $\ell_g \geq 0$ *and* $\sum_{g=1}^{n} \ell_g = 1$. *The SFYOWG AOs are a mapping* $\mathcal{G}^n \longrightarrow \mathcal{G}$ *such that*

$$\begin{aligned}
\mathrm{SFYOWG}\left(F_1, F_2, \ldots, F_n\right) = \prod_{g=1}^{n}\left(F_{v(g)}\right)^{\ell_g} &= \left(\sqrt{1-\min\left(1, \left(\sum_{g=1}^{n} \ell_g\left(1-\mu_{v(g)}^2\right)^{\delta}\right)^{1/\delta}\right)},\right. \\
&\left.\sqrt{1-\min\left(1, \left(\sum_{g=1}^{n} \ell_g\left(1-\wp_{v(g)}^2\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \partial_{v(g)}^{2\delta}\right)^{1/\delta}\right)}\right).
\end{aligned} \tag{50}$$

Proof. It follows from Theorem 16 similarly. □

Theorem 21. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN\ (U)$ $(g = 1, 2, 3, \ldots, n)$ *such that* $F_g = F$. *Then,*

$$\mathrm{SFYOWG}\left(F_1, F_2, \ldots, F_n\right) = F. \tag{51}$$

Theorem 22. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^- = \left\{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\right\}$, *and* $F_g^+ = \left\{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\right\} \in SFN\ (U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \mathrm{SFYOWG}\left(F_1, F_2, \ldots, F_n\right) \leq F_g^+. \tag{52}$$

Theorem 23. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*, \wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\text{SFYOWG}(F_1, F_2, \ldots, F_n) \leq \text{SFYOWG}(F_1^*, F_2^*, \ldots, F_n^*). \tag{53}$$

The proof of these theorems is similarly followed by Theorems 17–19.

Definition 20. Let $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in \text{SFN}(U)$ $(g = 1, 2, 3, \ldots, n)$. Then, Yager hybrid weighted geometric AOs for SFN (U) are described as

$$\text{SFYHWG}(F_1, F_2, \ldots, F_n) = \left(F_{v(1)}'\right)^{\sigma_g} \otimes \left(F_{v(2)}'\right)^{\sigma_g} \otimes \cdots \otimes \left(F_{v(n)}'\right)^{\sigma_g} = \prod_{g=1}^{n} \left(F_{v(g)}'\right)^{\sigma_g}, \tag{54}$$

where weights $(\ell_1, \ell_2, \ldots, \ell_g)$ of F_g have $\ell_g \geq 0$ and $\sum_{g=1}^{n} \ell_g = 1$ and gth biggest weighted value is $F_{v(g)}'(F_{v(g)}' = n\ell_g F_{v(g)} | g = 1, 2, \ldots, n)$ consequently by total order $(v(1), v(2), v(3), \ldots, v(n))$. Also, associated weights $(\sigma_1, \sigma_2, \ldots, \sigma_g)$ of F_g have $\sigma_g \geq 0$ and $\sum_{g=1}^{n} \sigma_g = 1$.

Theorem 24. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *and the weights* $(\ell_1, \ell_2, \ldots, \ell_g)$ *of* F_g *have* $\ell_g \geq 0$ *and* $\sum_{g=1}^{n} \ell_g = 1$. *The SFYHWG AOs are a mapping* $\mathcal{G}^n \longrightarrow \mathcal{G}$ *with associated weights* $(\sigma_1, \sigma_2, \ldots, \sigma_g)$ *of* F_g *having* $\sigma_g \geq 0$ *and* $\sum_{g=1}^{n} \sigma_g = 1$; *we have*

$$\text{SFYHWG}(F_1, F_2, \ldots, F_n) = \prod_{g=1}^{n} \left(F_{v(g)}'\right)^{\sigma_g} = \left(\sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \mu_{v(g)}^{'2}\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{1 - \min\left(1, \left(\sum_{g=1}^{n} \ell_g \left(1 - \wp_{v(g)}^{'2}\right)^{\delta}\right)^{1/\delta}\right)}, \sqrt{\min\left(1, \left(\sum_{g=1}^{n} \ell_g \partial_{v(g)}^{'2\delta}\right)^{1/\delta}\right)} \right). \tag{55}$$

Proof. It follows from Theorem 16 similarly. □

The proof of these theorems is similarly followed by Theorems 17–19.

Theorem 25. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$ *such that* $F_g = F$. *Then,*

$$\text{SFYHWG}(F_1, F_2, \ldots, F_n) = F. \tag{56}$$

Theorem 26. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^- = \{\min(\mu_g(v)), \min(\wp_g(v)), \max(\partial_g(v))\}$, *and* $F_g^+ = \{\max(\mu_g(v)), \min(\wp_g(v)), \min(\partial_g(v))\} \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *Then,*

$$F_g^- \leq \text{SFYHWG}(F_1, F_2, \ldots, F_n) \leq F_g^+. \tag{57}$$

Theorem 27. *Let* $F_g = (\mu_g(v), \wp_g(v), \partial_g(v)), F_g^* = (\mu_g^*(v), \wp_g^*(v), \partial_g^*(v)) \in SFN(U)$ $(g = 1, 2, 3, \ldots, n)$. *If* $\mu_g \leq \mu_g^*$, $\wp_g \leq \wp_g^*$ *and* $\partial_g \leq \partial_g^*$, *then*

$$\text{SFYHWG}(F_1, F_2, \ldots, F_n) \leq \text{SFYHWG}(F_1^*, F_2^*, \ldots, F_n^*). \tag{58}$$

6. Algorithm for Decision-Making Problems (DMPs)

In this section, we propose a framework for solving multiattribute DMPs under SF information. Consider a MAGDM with a set of m alternatives $\{ℶ_1, ℶ_2, \ldots, ℶ_g\}$ and let $\{\urcorner_1, \urcorner_2, \ldots, \urcorner_h\}$ be a set of attributes with weight vector $\ell = (\ell_1, \ell_2, \ldots, \ell_h)$, where $\ell_t \in [0, 1]$ and $\sum_{t=1}^{h} \ell_t = 1$. To assess the performance of kth alternative $ℶ_k$ under the t-th attribute $\urcorner_t$, let $\{\mathring{D}_1, \mathring{D}_2, \ldots, \mathring{D}_1\}$ be a set of decision-makers and let $\widehat{w} = (\widehat{w}_1, \widehat{w}_2, \ldots, \widehat{w}_{\widehat{j}})$ be the weighted vector of decision-makers with $\widehat{w}_s \in [0, 1]$ and $\sum_{s=1}^{\widehat{j}} \widehat{w}_s = 1$. The SF decision matrix can be written as

$$\begin{bmatrix} (\mu_{11}(v), \wp_{11}(v), \partial_{11}(v)) & (\mu_{12}(v), \wp_{12}(v), \partial_{12}(v)) & \cdots & (\mu_{1h}(v), \wp_{1h}(v), \partial_{1h}(v)) \\ (\mu_{21}(v), \wp_{21}(v), \partial_{21}(v)) & (\mu_{22}(v), \wp_{22}(v), \partial_{22}(v)) & \cdots & (\mu_{2h}(v), \wp_{2h}(v), \partial_{2h}(v)) \\ (\mu_{31}(v), \wp_{31}(v), \partial_{31}(v)) & (\mu_{32}(v), \wp_{32}(v), \partial_{32}(v)) & \cdots & (\mu_{3h}(v), \wp_{3h}(v), \partial_{3h}(v)) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{g1}(v), \wp_{g1}(v), \partial_{g1}(v)) & (\mu_{g2}(v), \wp_{g2}(v), \partial_{g2}(v)) & \cdots & (\mu_{gh}(v), \wp_{gh}(v), \partial_{gh}(v)) \end{bmatrix}, \tag{59}$$

where $\mu(v) \in [0,1]$, $\wp(v) \in [0,1]$, and $\partial(v) \in [0,1]$ are positive, neutral, and negative membership grades, respectively. In addition, $0 \leq \mu^2(v) + \wp^2(v) + \partial^2(v) \leq 1$, $\forall v \in U$. Key steps of the developed multiattribute group decision-making (MAGDM) problem are described as follows:

Step 1: construct the SF decision matrix based on the experts evaluations.

$$\begin{bmatrix} \left(\mu_{11}^{\hat{j}}(v), \wp_{11}^{\hat{j}}(v), \partial_{11}^{\hat{j}}(v)\right) & \left(\mu_{12}^{\hat{j}}(v), \wp_{12}^{\hat{j}}(v), \partial_{12}^{\hat{j}}(v)\right) & \cdots & \left(\mu_{1h}^{\hat{j}}(v), \wp_{1h}^{\hat{j}}(v), \partial_{1h}^{\hat{j}}(v)\right) \\ \left(\mu_{21}^{\hat{j}}(v), \wp_{21}^{\hat{j}}(v), \partial_{21}^{\hat{j}}(v)\right) & \left(\mu_{22}^{\hat{j}}(v), \wp_{22}^{\hat{j}}(v), \partial_{22}^{\hat{j}}(v)\right) & \cdots & \left(\mu_{2h}^{\hat{j}}(v), \wp_{2h}^{\hat{j}}(v), \partial_{2h}^{\hat{j}}(v)\right) \\ \left(\mu_{31}^{\hat{j}}(v), \wp_{31}^{\hat{j}}(v), \partial_{31}^{\hat{j}}(v)\right) & \left(\mu_{32}^{\hat{j}}(v), \wp_{32}^{\hat{j}}(v), \partial_{32}^{\hat{j}}(v)\right) & \cdots & \left(\mu_{3h}^{\hat{j}}(v), \wp_{3h}^{\hat{j}}(v), \partial_{3h}^{\hat{j}}(v)\right) \\ \vdots & \vdots & \ddots & \vdots \\ \left(\mu_{g1}^{\hat{j}}(v), \wp_{g1}^{\hat{j}}(v), \partial_{g1}^{\hat{j}}(v)\right) & \left(\mu_{g2}^{\hat{j}}(v), \wp_{g2}^{\hat{j}}(v), \partial_{g2}^{\hat{j}}(v)\right) & \cdots & \left(\mu_{gh}^{\hat{j}}(v), \wp_{gh}^{\hat{j}}(v), \partial_{gh}^{\hat{j}}(v)\right) \end{bmatrix}, \tag{60}$$

where $\hat{j}$ represents the number of experts.

Step 2: aggregate the individual decision matrices based on the aggregation operators to construct the aggregated matrix. Hence, the aggregated decision matrix is constructed as

$$\begin{bmatrix} (\mu_{11}(v), \wp_{11}(v), \partial_{11}(v)) & (\mu_{12}(v), \wp_{12}(v), \partial_{12}(v)) & \cdots & (\mu_{1h}(v), \wp_{1h}(v), \partial_{1h}(v)) \\ (\mu_{21}(v), \wp_{21}(v), \partial_{21}(v)) & (\mu_{22}(v), \wp_{22}(v), \partial_{22}(v)) & \cdots & (\mu_{2h}(v), \wp_{2h}(v), \partial_{2h}(v)) \\ (\mu_{31}(v), \wp_{31}(v), \partial_{31}(v)) & (\mu_{32}(v), \wp_{32}(v), \partial_{32}(v)) & \cdots & (\mu_{3h}(v), \wp_{3h}(v), \partial_{3h}(v)) \\ \vdots & \vdots & \ddots & \vdots \\ (\mu_{g1}(v), \wp_{g1}(v), \partial_{g1}(v)) & (\mu_{g2}(v), \wp_{g2}(v), \partial_{g2}(v)) & \cdots & (\mu_{gh}(v), \wp_{gh}(v), \partial_{gh}(v)) \end{bmatrix}. \tag{61}$$

Step 3: if the weights of the attribute are known as a prior, then use them. Otherwise, we will calculate them using the concept of spherical entropy measure. Spherical entropy measure is as follows:

$$\ell_j = \frac{1 + (1/h)\sum_{i=1}^{h}\left(\mu_{ij}\log\left(\mu_{ij}\right) + \wp_{ij}\log\left(\wp_{ij}\right) + \partial_{ij}\log\left(\partial_{ij}\right)\right)}{\sum_{j=1}^{g}\left(1 + (1/h)\sum_{i=1}^{h}\left(\mu_{ij}\log\left(\mu_{ij}\right) + \wp_{ij}\log\left(\wp_{ij}\right) + \partial_{ij}\log\left(\partial_{ij}\right)\right)\right)}. \tag{62}$$

Step 3: exploit the established aggregation operators to achieve the SFN F_t $(t = 1, 2, \ldots, g)$ for the alternatives $\gimel_k$, that is, the established operators to obtain the collective overall preference values of F_t $(t = 1, 2, \ldots, g)$ for the alternatives $\gimel_k$, where $\ell = (\ell_1, \ell_2, \ldots, \ell_h)$ is the weight vector of the attributes.

Step 4: after that, we compute the scores of all the overall values F_t $(t = 1, 2, \ldots, g)$ for the alternatives $\gimel_k$.

Step 5: according to Definition 13, rank the alternatives $\gimel_k$ $(k = 1, 2, \ldots, g)$ and select the best one having the greater value.

7. Application of Proposed Decision-Making Technique

In this section, a numerical application about wind power plant location selection problem is firstly used to illustrate the designed MAGDM method. Then a comparison between the presented Yager aggregation operators and the existing aggregation operators of spherical fuzzy numbers is carried out to show the characteristics and advantage of the presented AOs.

7.1. Practical Case Study. In this segment, a case study is provided to illustrate the effectiveness and reliability of the established decision-making approach.

The case study area was Jhimpir, a village in Thatta district of Sindh province in Pakistan, 120 kilometres northeast of Karachi. Jhimpir's geographical coordinates are $25^{\circ}\,1'\,0''$ North, $68^{\circ}\,1'\,0''$ East. Location of Jhimpir is shown in Figure 1.

The digital elevation model of Jhimpir is given in Figure 2.The required data were collected from numerous resources including governmental agencies, open sources, and related literature such as National Authority for Remote Sensing & Space Sciences, Pakistan Meteorological Authority, New and Renewable Energy Authority, Pakistan General Survey Authority, NASA's Prediction of Worldwide Energy Resources (POWER), United States Geological Survey, and Pakistan Environmental Affairs Agency.

Electricity plays an essential part in any nation's socioeconomic progress and social prosperity. Electricity energy should be regarded as the fundamental need for human development. In Pakistan, limited power generation is a major issue that directly restricts the country's growth. In a landmark achievement, the 50-megawatt Jhimpir wind power project has begun commercial operations as Pakistan gradually moves to ramp up renewable energy generation in line with the global trend and to bridge the domestic shortfall. The total cost of project is \$136 million. Completed in 2002, it has a total capacity of 50 MW. This wind corridor has a 50000-megawatt potential with average wind speeds over 7 meters per second. The government has announced upfront tariff and ROI of 17 percent, which is highest in the world. There are 14 projects in the pipeline, out of which 50 MW FFCEL project has achieved COD by mid-December 2012.

Pakistan's National Renewable Energy Laboratories (NREL) wind resource map has provided a major boost to the development of wind power in the wind corridor regions. These regions are the Karachi-Hyderabad region especially on hilltops, ridges in the northern Indus valley, wind corridor areas in western Pakistan, high mountainous regions, and hills and ridges in southwestern Pakistan. This potential area has now become the focal point of wind power 's near future development. The coastal belt of Pakistan has a wind corridor that is 60 km wide and 180 km long, as per the collected data. This corridor has an electricity generation potential of up to 50 000 MW of exploitable wind power. Here, we enlist the wind power energy projects and discuss their production in Table 1.

For our research, we used a dataset comprising topographic, geological, and climatic factors. Based on several literatures and case studies concerning wind farm site selection and local conditions, different criteria were reviewed by three experts and five locations $\{\gimel_1, \gimel_2, \gimel_3, \gimel_4, \gimel_5\}$ under four criteria were selected to evaluate the suitable sites for wind farms. The detailed description of the criteria is as follows:

(1) Natural Factors ($\urcorner_1$): in order to be effective, the position of the wind turbines is measured by the prevailing wind direction. In generating electricity from wind turbines, wind speed is a vital factor. Wind speed above certain rates is essential for producing wind energy [50, 51].

(2) Political Aspect ($\urcorner_2$): select the location that offers maximum output, minimizing project costs, giving the political point score to the government for installations of wind energy projects.

(3) Socioeconomic Factors ($\urcorner_3$): in order to minimize the cost of building wind farms and to reduce the cost of transporting electricity, wind farms should be located close to the existing transmission grids [52].

(4) Environmental Factors ($\urcorner_4$): wind farms in areas where they negligibly interfere with existing land use outside protected areas, artificial surfaces, wetlands, aquatic areas, and forestry areas should be installed [52]. It is necessary to keep all the mechanical parts of wind turbines away from the water. Wind turbine fins are lowered and disconnected to prevent harm to the components of the turbine.

The three experts were asked in this assessment to use spherical fuzzy information and their weights are $(0.314, 0.355, 0.331)^T$.

Step 1: three experts listed their evaluation information using the spherical fuzzy numbers in Tables 2–4

Step 2: aggregated SF information is evaluated using spherical fuzzy weighted averaging operators. The results are shown in Table 5.

Step 4: now, we find out the attribute weight vector using spherical fuzzy entropy measure as follows.

$$\ell = \{\ell_1 = 0.256, \ell_2 = 0.248, \ell_3 = 0.245, \ell_4 = 0.251\}. \tag{63}$$

Step 5: evaluate the overall perfumes of the alternatives; we utilized the proposed Yager aggregation operators as follows in Tables 6 and 7.

Step 6: compute the score value of each collective SF information of each alternative as follows in Table 8.

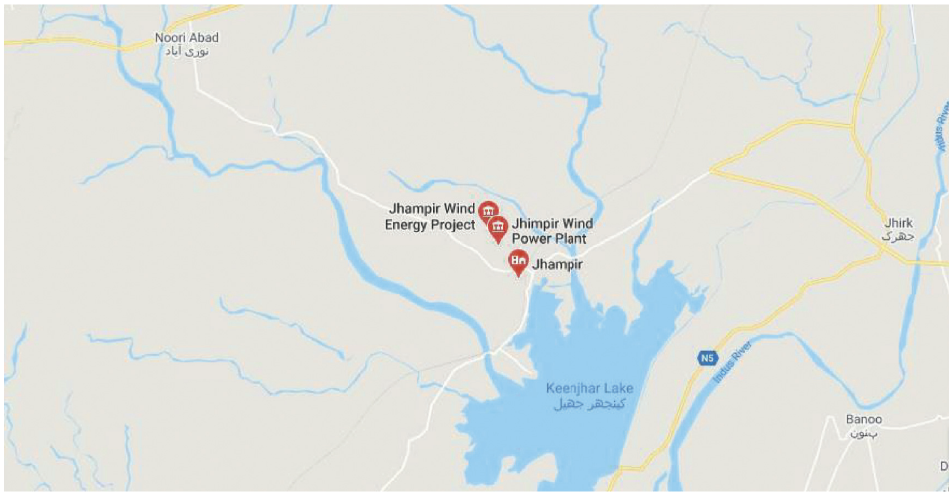

FIGURE 1: Location of Jhimpir.

FIGURE 2: Digital elevation model.

TABLE 1: Wind power projects and their production.

Station	Location	Capacity (MW)	In service date
FFC Energy Wind Project	Jhimpir, Sindh	49.5	2013
Zorlu Enerji Pakistan	Jhimpir, Sindh	56.4	2013
Three Gorges Pvt. Ltd.	Jhimpir, Sindh	150	2014
Sapphire Wind Power Pvt. Ltd.	Jhimpir, Sindh	52.8	2015
Yunus Energy Ltd.	Jhimpir, Sindh	50	2016
Metro Wind Power Co. Ltd.	Jhimpir, Sindh	50	2016
Gul Ahmed Wind Power Ltd.	Jhimpir, Sindh	50	2016
Master Wind Energy Ltd.	Jhimpir, Sindh	52.8	2016
ACT Wind Pvt. Ltd.	Jhimpir, Sindh	30	2016
Sachal Energy Wind Farm	Jhimpir, Sindh	50	2017
United Energy Pakistan Wind Ltd.	Jhimpir, Sindh	100	2017
Hawa Energy Ltd.	Jhimpir, Sindh	50	2018
Burj Capital Jhimpir Wind Power Limited	Jhimpir, Sindh	50	2018
Artistic Energy Pvt. Ltd.	Jhimpir, Sindh	49.3	2018
Tricon Boston Corporation	Jhimpir, Sindh	150	2018

TABLE 2: D_1.

	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\beth_1$	$(0.84, 0.34, 0.40)$	$(0.78, 0.39, 0.43)$	$(0.67, 0.50, 0.30)$	$(0.71, 0.21, 0.31)$
$\beth_2$	$(0.60, 0.11, 0.53)$	$(0.59, 0.35, 0.23)$	$(0.72, 0.31, 0.41)$	$(0.82, 0.25, 0.11)$
$\beth_3$	$(0.79, 0.19, 0.39)$	$(0.91, 0.21, 0.11)$	$(0.71, 0.41, 0.13)$	$(0.51, 0.25, 0.34)$
$\beth_4$	$(0.63, 0.51, 0.13)$	$(0.42, 0.33, 0.49)$	$(0.61, 0.43, 0.45)$	$(0.59, 0.37, 0.49)$
$\beth_5$	$(0.57, 0.36, 0.29)$	$(0.60, 0.15, 0.50)$	$(0.70, 0.32, 0.40)$	$(0.65, 0.44, 0.33)$

Table 3: D_2.

	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\beth_1$	(0.61, 0.15, 0.53)	(0.62, 0.35, 0.16)	(0.61, 0.35, 0.47)	(0.74, 0.17, 0.55)
$\beth_2$	(0.66, 0.11, 0.51)	(0.77, 0.23, 0.43)	(0.93, 0.08, 0.09)	(0.99, 0.06, 0.02)
$\beth_3$	(0.88, 0.09, 0.07)	(0.89, 0.06, 0.05)	(0.56, 0.17, 0.44)	(0.61, 0.13, 0.43)
$\beth_4$	(0.59, 0.32, 0.34)	(0.51, 0.48, 0.24)	(0.68, 0.53, 0.39)	(0.61, 0.21, 0.34)
$\beth_5$	(0.71, 0.31, 0.24)	(0.69, 0.41, 0.35)	(0.73, 0.44, 0.21)	(0.74, 0.49, 0.22)

Table 4: D_3.

	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\beth_1$	(0.85, 0.25.0.15)	(0.88, 0.23, 0.14)	(0.78, 0.38, 0.18)	(0.83, 0.39, 0.29)
$\beth_2$	(0.94, 0.04, 0.07)	(0.61, 0.19, 0.39)	(0.63, 0.18, 0.35)	(0.56, 0.49, 0.48)
$\beth_3$	(0.73, 0.13, 0.46)	(0.88, 0.39, 0.19)	(0.87, 0.35, 0.18)	(0.81, 0.13, 0.41)
$\beth_4$	(0.82, 0.12, 0.43)	(0.63, 0.21, 0.55)	(0.53, 0.33, 0.47)	(0.51, 0.23, 0.46)
$\beth_5$	(0.61, 0.33, 0.29)	(0.63, 0.41, 0.28)	(0.74, 0.34, 0.14)	(0.65, 0.32, 0.37)

Table 5: Aggregated SF set information.

	$\daleth_1$	$\daleth_2$	$\daleth_3$	$\daleth_4$
$\beth_1$	(0.788, 0.229, 0.319)	(0.785, 0.315, 0.208)	(0.696, 0.402, 0.297)	(0.767, 0.239, 0.371)
$\beth_2$	(0.807, 0.078, 0.279)	(0.674, 0.246, 0.342)	(0.818, 0.160, 0.227)	(0.919, 0.188, 0.097)
$\beth_3$	(0.814, 0.128, 0.223)	(0.893, 0.165, 0.099)	(0.748, 0.284, 0.223)	(0.677, 0.159, 0.393)
$\beth_4$	(0.702, 0.267, 0.271)	(0.533, 0.324, 0.395)	(0.615, 0.424, 0.433)	(0.573, 0.258, 0.421)
$\beth_5$	(0.639, 0.331, 0.271)	(0.644, 0.298, 0.363)	(0.724, 0.365, 0.224)	(0.685, 0.411, 0.296)

Table 6: Yager weighted averaging.

	SFYWA	SFYOWA	SFYHWA
$\beth_1$	(0.7620, 0.3015, 0.3038)	(0.7616, 0.3031, 0.3025)	(0.7732, 0.2710, 0.3144)
$\beth_2$	(0.8183, 0.1775, 0.2512)	(0.8187, 0.1785, 0.2505)	(0.8278, 0.1582, 0.2388)
$\beth_3$	(0.7950, 0.1916, 0.2541)	(0.7958, 0.1918, 0.2534)	(0.8039, 0.1957, 0.2268)
$\beth_4$	(0.6163, 0.3223, 0.3830)	(0.6163, 0.3230, 0.3830)	(0.6183, 0.3392, 0.3966)
$\beth_5$	(0.6752, 0.3529, 0.2920)	(0.6761, 0.3530, 0.2918)	(0.6771, 0.3680, 0.2828)

Table 7: Yager weighted geometric.

	SFYWG	SFYOWG	SFYHWG
$\beth_1$	(0.7580, 0.3015, 0.3144)	(0.7575, 0.3031, 0.3133)	(0.7710, 0.2710, 0.3236)
$\beth_2$	(0.7928, 0.1775, 0.2737)	(0.7928, 0.1785, 0.2735)	(0.8130, 0.1582, 0.2555)
$\beth_3$	(0.7741, 0.1916, 0.2918)	(0.7746, 0.1918, 0.2916)	(0.7900, 0.1957, 0.2535)
$\beth_4$	(0.6056, 0.3223, 0.3932)	(0.6056, 0.3230, 0.3932)	(0.6110, 0.3392, 0.4053)
$\beth_5$	(0.6720, 0.3529, 0.3012)	(0.6728, 0.3530, 0.3011)	(0.6743, 0.3680, 0.2885)

Table 8: Score values.

	$\check{s}\check{c}(\beth_1)c$	$\check{s}\check{c}(\beth_2)$	$\check{s}\check{c}(\beth_3)$	$\check{s}\check{c}(\beth_4)$	$\check{s}\check{c}(\beth_5)$
SFYWA	0.631061	0.669998	0.656079	0.559098	0.596208
SFYOWA	0.631075	0.670331	0.656609	0.559115	0.596561
SFYHWA	0.634867	0.677689	0.665076	0.557001	0.598448
SFYWG	0.627303	0.651881	0.639113	0.553654	0.593435
SFYOWG	0.627262	0.651912	0.639385	0.553665	0.593733
SFYHWG	0.632199	0.666162	0.653417	0.552859	0.596481

Table 9: Ranking of alternatives.

	Score ranking	Best alternative
SFYWA	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$
SFYOWA	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$
SFYHWA	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$
SFYWG	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$
SFYOWG	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$
SFYHWG	$\check{s}\check{c}(\beth_2) > \check{s}\check{c}(\beth_3) > \check{s}\check{c}(\beth_1) > \check{s}\check{c}(\beth_5) > \check{s}\check{c}(\beth_4)$	$\beth_2$

Table 10: Existing aggregation operators.

	SFSWA [37]	SFSOWA [37]	SFSHWA [37]	L – SFWA [29]
$\beth_1$	(0.76, 0.30, 0.30)	(0.75, 0.30, 0.30)	(0.77, 0.27, 0.31)	(0.91, 0.06, 0.06)
$\beth_2$	(0.80, 0.17, 0.25)	(0.81, 0.17, 0.25)	(0.82, 0.15, 0.24)	(0.95, 0.01, 0.03)
$\beth_3$	(0.78, 0.19, 0.25)	(0.78, 0.19, 0.25)	(0.79, 0.19, 0.22)	(0.94, 0.02, 0.03)
$\beth_4$	(0.60, 0.32, 0.38)	(0.61, 0.32, 0.38)	(0.79, 0.19, 0.22)	(0.70, 0.07, 0.10)
$\beth_5$	(0.67, 0.35, 0.29)	(0.67, 0.35, 0.29)	(0.66, 0.33, 0.29)	(0.82, 0.09, 0.06)

Table 11: Existing aggregation operators.

	L – SFOWA [29]	L – SFHWA [29]	L – SFWG [29]	L – SFOWG [29]
$\beth_1$	(0.91, 0.06, 0.06)	(0.99, 0.002, 0.004)	(0.91, 0.07, 0.07)	(0.91, 0.07, 0.07)
$\beth_2$	(0.96, 0.01, 0.03)	(0.99, 0.0001, 0.0003)	(0.92, 0.02, 0.05)	(0.92, 0.02, 0.05)
$\beth_3$	(0.94, 0.02, 0.03)	(0.99, 0.0003, 0.001)	(0.91, 0.03, 0.06)	(0.91, 0.03, 0.06)
$\beth_4$	(0.70, 0.07, 0.10)	(0.81, 0.003, 0.01)	(0.62, 0.08, 0.12)	(0.62, 0.08, 0.12)
$\beth_5$	(0.82, 0.09, 0.06)	(0.96, 0.007, 0.002)	(0.81, 0.10, 0.06)	(0.81, 0.10, 0.06)

Table 12: Overall ranking of the alternatives.

Existing operators	Ranking	Best alternative
SFSWA [37]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFSOWA [37]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFSHWA [37]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFWA [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFOWA [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFHWA [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFWG [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFOWG [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
L – SFHWG [29]	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$

Table 13: Overall ranking of the alternatives.

Proposed operators	Ranking	Best alternative
SFYWA	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFYOWA	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFYHWA	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFYWG	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFYOWG	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$
SFYHWG	$\breve{s}\breve{c}(\beth_2) > \breve{s}\breve{c}(\beth_3) > \breve{s}\breve{c}(\beth_1) > \breve{s}\breve{c}(\beth_5) > \breve{s}\breve{c}(\beth_4)$	$\beth_2$

Step 7: select the optimal alternative according to the maximum score value as follows in Table 9.

We can conclude from this above computational process that location $\beth_2$ is the best for the installation of the wind power plant among others, and therefore it is highly recommended.

8. Comparison Analysis

We provide some appropriate examples below to test the potential and efficacy of the established decision-making approach and to compare it with the recent findings.

The use of existing methods and different aggregation operators for computed aggregate information is shown in Tables 10 and 11.

Now, according to their aggregated data, we evaluate the ranking of the alternatives in Tables 12 and 13.

9. Discussion

From the outcomes of the proposed operators and the other existing methods, we conclude that ranking lists obtained from both the proposed method and the compared methods are the same. The Yager operators with the spherical fuzzy set environment represent a generalized and novel approach to tackle uncertainty in DM problems. The Yager operators with the spherical fuzzy environment are more flexible and effective to evaluate best alternative in real-word problems.

10. Conclusion

Spherical FS, which is a general extension of intuitionistic FS, picture FS, is more capable of dealing with incomplete and inconsistent information. Therefore, it is widely used in various fields. Spherical FS tackles the vagueness and uncertain information in real-world complex problems with more flexible and effective way. In addition, the Yager norms have a more generalized framework that works effectively to incorporate complex information. We are motivated by the deficiencies of the existing methods and the beneficial features of the Yager AOs to work towards improving a successful merger with SFNs.

In this study, under the spherical fuzzy model, we modified the multiskilled Yager AOs to integrate the benefits and flexibility of both theories. Later, we explore operational laws of SFN to construct spherical fuzzy AOs that comply with the principles of Yager operations. We have established the SFYWA, SFYOWA, SFYHWA, SFYWG, SFYOWG, and SFYHWG operators to aggregate the SFNs. Some of the main characteristics of the proposed operators have been studied, including idempotency, boundedness, and monotonicity.

The main objective of this study is to present a strategy to address MAGDM that includes spherical fuzzy evaluations based on the proposed operators. The theoretical basis of AOs needs to be carefully considered in preparation for their use in MAGDM. A practical example is provided to demonstrate the implementation of the established strategy for the selection of a suitable location for wind power stations. The comparison analysis of our proposed theory was conducted with the existing operators. The superiority of our proposed operators over the existing DM method has been highlighted. We examined the effect of different parameter values on the results of MAGDM issues. In short, this article creates a tool that has the rich properties of Yager AOs and the SF model's flexibility. We will expand our models to spherical hesitant fuzzy set environments in future research.

Data Availability

No data were used to support this study.

Ethical Approval

This article does not contain any studies with human participants or animals performed by any of the authors.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Retraction

Retracted: The Neutro-Stability Analysis of Neutrosophic Cubic Sets with Application in Decision Making Problems

Journal of Mathematics

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This article has been retracted by Hindawi following an investigation undertaken by the publisher [1]. This investigation has uncovered evidence of one or more of the following indicators of systematic manipulation of the publication process:

(1) Discrepancies in scope
(2) Discrepancies in the description of the research reported
(3) Discrepancies between the availability of data and the research described
(4) Inappropriate citations
(5) Incoherent, meaningless and/or irrelevant content included in the article
(6) Peer-review manipulation

The presence of these indicators undermines our confidence in the integrity of the article's content and we cannot, therefore, vouch for its reliability. Please note that this notice is intended solely to alert readers that the content of this article is unreliable. We have not investigated whether authors were aware of or involved in the systematic manipulation of the publication process.

Wiley and Hindawi regrets that the usual quality checks did not identify these issues before publication and have since put additional measures in place to safeguard research integrity.

We wish to credit our own Research Integrity and Research Publishing teams and anonymous and named external researchers and research integrity experts for contributing to this investigation.

The corresponding author, as the representative of all authors, has been given the opportunity to register their agreement or disagreement to this retraction. We have kept a record of any response received.

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[1] M. A. Al Shumrani, M. Gulistan, and S. Khan, "The Neutro-Stability Analysis of Neutrosophic Cubic Sets with Application in Decision Making Problems," *Journal of Mathematics*, vol. 2020, Article ID 8835019, 16 pages, 2020.

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Research Article

The Neutro-Stability Analysis of Neutrosophic Cubic Sets with Application in Decision Making Problems

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The neutrosophic cubic sets (NCSs) attained attraction of many researchers in the current time, so the need to discuss and study their stability was felt. Thus, in this article, we discuss the three types of stability of NCSs such as truth-stability, indeterminacy-stability, and falsity-stability. We define the left (resp., right) truth-left evaluative set, left (resp., right) indeterminacy-evaluative set, and left (resp., right) falsity-evaluative set. A new notion of stable NCSs, partially stable NCSs, and unstable NCSs is defined. We observe that every NCS needs not to be a stable NCS but each stable NCS must be an NCS, i.e., every internal NCS is a stable NCS but an external NCS may or may not be a stable NCS. We also discuss some conditions under which the left and right evaluative points of an external NCS becomes a neutrosophic bipolar fuzz set. We have provided the condition under which an external NCS becomes stable. Moreover, we discuss the truth-stable degree, indeterminacy-stable degree, and falsity-stable degree of NCSs. We have also defined an almost truth-stable set, almost indeterminacy-stable set, almost falsity-stable set, almost partially stable set, and almost stable set with examples. Application of stable NCSs is given with a numerical example at the end.

1. Introduction

The crisp set lost the stability as it covers the extremes only, which is not the ideal situation in every problem. To cover this gap, Zadeh [1] presented the idea of the fuzzy set (FS) in 1965 which is stable as compared to the crisp set. But, when there is a case to handle the negative characteristics, the fuzzy set (FS) too lost its stability. To cover this gap, Atanassov [2], in 1986, gave the idea of intutionistic fuzzy sets (IFSs) which are more stable than the fuzzy set. But, the problem with Atanassov's idea is that indeterminacy is lost and no proper attraction is given to it. Then, Smarandache [3] covered this gap by giving a new idea of a neutrosophic set which is a stable version other than the fuzzy set and intutionistic fuzzy sets. The neutrosophic set (NS) is the extension of the FS, IVFS, and IFS. In the NS, we deal with its three components, that is, truthfulness, indeterminate, and untruthfulness, and these three functions are independent completely. Neutrosophy gives us a support for a whole family of new mathematical theories with the abstraction of both classical and fuzzy counterparts. In real life and in scientific problems to apply the neutrosophic set, Wang et al. [4] introduced the new idea of a single-valued neutrosophic set (SVNS) and interval neutrosophic set (INS). These are subclasses of the NS, in which truthfulness, indeterminate, and untruthfulness were taken in a closed interval $[0, 1]$, see also [5]. On the other side, Zadeh [6] made another extension which is known as the interval-valued fuzzy set (IVFS), in which he described interval membership function. There are many real-life applications of the IVFS, i.e., Sambuc [7] in medical diagnosis in thyroidian, Gorzalczany in approximate reasoning, and Turksen [8, 9] in interval-valued logic. In 2012, the theme of the cubic set (CS) was used by Jun et al. [10]. CS is the combination of the IVFS and FS in the form of an ordered pair. These all are mathematical tools to determine the complications in our daily life. Jun et al. [11] gave the idea of the NCS. For application of NCSs, we refer to [12–17]. In 2017, the concept of stable cubic sets was introduced by Muhiuddin et al. [18]. In 2019 and 2020, Smarandache [19–21] generalized the classical algebraic structures to neutroalgebraic structures (or neutroalgebras) (whose operations and axioms are partially true, partially indeterminate, and partially false) as extensions of partial algebra and to antialgebraic structures (or antialgebras) (whose operations and axioms are totally false). Also, in general, he extended any classical structure, in no matter what field of knowledge, to a neutrostructure and an antistructure. Similarly, as alternatives to a classical theorem (that is true for all sets' elements) are the neutrotheorem (partially true, partially indeterminate, and partially false) and antitheorem (false for all sets' elements), respectively.

In this paper, we define different types of the stable neutrosophic cubic set with examples and some basic results. We also define the concept of almost stable neutrosophic cubic sets. At the end, we have provided an application of the presented theory.

2. Preliminaries

This section mainly recalls some basic concepts related to fuzzy sets [1], cubic sets [10], neutrosophic sets [3, 4], neutrosophic cubic sets [11], and evaluative structure of cubic sets [18]. For more detail of these sets, we refer the reader to [1, 3, 4, 10, 11, 18].

Definition 1 (see [1]). A mapping $p: U \longrightarrow [0,1]$ is called an FS, and $\tilde{p}$ (ů) is a membership function and denoted by p.

Definition 2 (see [10]). A structure $C = \{(\mathring{u}; \tilde{p}(\mathring{u}), p(\mathring{u}) | \mathring{u} \in U)\}$ is a cubic set in U in which $\tilde{p}$(ů) is IVF in U, and p (ů) is an FS in U. This is simply denoted by $C = (\tilde{p}, p)$. $C^{\mathring{u}}$ denotes the collection of cubic sets in U.

Definition 3 (see [3, 4]). A neutrosophic set is a structure

$$N = \{(\mathring{u}; T_N(\mathring{u}), I_N(\mathring{u}), F_N(\mathring{u}) | \mathring{u} \in U)\}, \tag{1}$$

in U. Here, $(T_N(\mathring{u}), I_N(\mathring{u}), F_N(\mathring{u}) \in [0,1])$ are three functions, known as truthfulness, indeterminate, and untruthfulness, respectively, simply denoted by $N = (T_N, I_N, F_N)$.

Definition 4 (see [11]). A structure

$$N_C = \left\{\left(\mathring{u}; \tilde{T}_{N_C}(\mathring{u}), \tilde{I}_{N_C}(\mathring{u}), \tilde{F}_{N_C}(\mathring{u}), T_{N_C}(\mathring{u}), I_{N_C}(\mathring{u}), F_{N_C}(\mathring{u}) | \mathring{u} \in U\right)\right\}, \tag{2}$$

is an NCS in X. Here,

$$\left(\tilde{T}_{N_C} = \left[T^L_{N_C}, T^U_{N_C}\right], \tilde{I}_{N_C} = \left[I^L_{N_C}, I^U_{N_C}\right], \tilde{F}_{N_C}\left[F^L_{N_C}, F^U_{N_C}\right]\right), \tag{3}$$

is an interval NS and $(T_{N_C}, I_{N_C}, F_{N_C})$ is an NS in X simply denoted by

$$\begin{aligned} N_C &= \left(\tilde{T}_{N_C}, \tilde{I}_{N_C}, \tilde{F}_{N_C}, T_{N_C}, I_{N_C}, F_{N_C}\right), \\ [0,0] &\leq \tilde{T}_{N_C} + \tilde{I}_{N_C} + \tilde{F}_{N_C} \leq [3,3], \\ 0 &\leq T_{N_C} + I_{N_C} + F_{N_C} \leq 1. \end{aligned} \tag{4}$$

Definition 5 (see [18]). A structure $C = \{(\mathring{u}; \tilde{p}(\mathring{u}), p(\mathring{u}) | \mathring{u} \in U)\}$ is a CS in U in which C (ů) is the evaluative structure defined as follows:

$$E_C = \{(\mathring{u}; E_C(\mathring{u}) | \mathring{u} \in U)\}, \tag{5}$$

where $E_C(\mathring{u}) = \langle l(E_C(\mathring{u})), r(E_C(\mathring{u}))\rangle$ with left evaluative point $l(E_C(\mathring{u})) = p(\mathring{u}) - \tilde{p}(\mathring{u})$ and right evaluative point $r(E_C(\mathring{u})) = p(\mathring{u})^+ - p(\mathring{u})$ at $\mathring{u} \in U$. We say that $E_C(\mathring{u})$ is the evaluative point of $C = (\tilde{p}, p)$ at $\mathring{u} \in U$.

3. Neutrostable Neutrosophic Cubic Sets

In this section, we provide the concepts of the truth-evaluative set, indeterminacy-evaluative set, falsity-evaluative set, stable truth-element, stable indeterminacy-element, stable falsity-element, and unstable element of the NCS. We also discuss some interesting results.

Definition 6. Let $p = \langle T_p, I_p, F_p, t_p, i_p, f_p\rangle$ be an NCS in U. Then,

(1) The truth-evaluative set of $p = \langle T_p, I_p, F_p, t_p, i_p, f_p\rangle$ is represented as

$$\begin{aligned} E_{Tp} &= \left\{\left(\mathring{u}, E_{Tp}(\mathring{u})\right) | \mathring{u} \in U\right\} \\ &= (\text{left truth} - \text{evaluative point}, \text{right truth} - \text{evaluative point}) \\ &= \left(l\left(E_{Tp}(\mathring{u})\right), r\left(E_{Tp}(\mathring{u})\right)\right) \\ &= \left(t(\mathring{u}) - T^-(\mathring{u}), T^+(\mathring{u}) - t(\mathring{u})\right). \end{aligned} \tag{6}$$

(2) The indeterminacy-evaluative set of $p = \langle T_p, I_p, F_p, t_p, i_p, f_p \rangle$ is represented as

$$\begin{aligned} E_{Ip} &= \left\{ \left(\mathring{u}, E_{Ip}(\mathring{u}) \right) | \mathring{u} \in U \right\} \\ &= (\text{left indeterminacy} - \text{evaluative point, right indeterminacy} - \text{evaluative point}) \\ &= \langle l\left(E_{Ip}(\mathring{u}) \right), r\left(E_{Ip}(\mathring{u}) \right) \rangle \\ &= \left(i(\mathring{u}) - I^{-}(\mathring{u}), I^{+}(\mathring{u}) - i(\mathring{u}) \right). \end{aligned} \tag{7}$$

(3) The falsity-evaluative set of $p = \langle T_p, I_p, F_p, t_p, i_p, f_p \rangle$ is represented as

$$\begin{aligned} E_{Fp} &= \left\{ \left(\mathring{u}, E_{Fp}(\mathring{u}) \right) | \mathring{u} \in U \right\} \\ &= (\text{left falsity} - \text{evaluative point, right falsity} - \text{evaluative point}) \\ &= \langle l\left(E_{Fp}(\mathring{u}) \right), r\left(E_{Fp}(\mathring{u}) \right) \rangle \\ &= \left(f(\mathring{u}) - F^{-}(\mathring{u}), F^{+}(\mathring{u}) - f(\mathring{u}) \right). \end{aligned} \tag{8}$$

The collection

$$E_{Lp}(\mathring{u}) = \left(l\left(E_{Tp}(\mathring{u}) \right), l\left(E_{Ip}(\mathring{u}) \right), l\left(E_{F\,p}(\mathring{u}) \right) \right), \tag{9}$$

is called the left evaluative point and the collection

$$E_{Rp}(\mathring{u}) = \left(r\left(E_{Tp}(\mathring{u}) \right), r\left(E_{Ip}(\mathring{u}) \right), r\left(E_{F\,p}(\mathring{u}) \right) \right), \tag{10}$$

is called the right evaluative point. We say that $E_{\beta}(\mathring{u}) = (E_{L\beta}(\mathring{u}), E_{R\beta}(\mathring{u}))$ is the evaluative point.

Example 1. Let $\beta = \{\langle \mathring{u}, T(\mathring{u}), I(\mathring{u}), F(\mathring{u}), t(\mathring{u}), i(\mathring{u}), f(\mathring{u}) \rangle | \mathring{u} \in I\}$ be an NCS in U. If

$$\begin{aligned} &\langle T(\mathring{u}), I(\mathring{u}), F(\mathring{u}), t(\mathring{u}), i(\mathring{u}), f(\mathring{u}) \rangle \\ &\quad = \langle [0.2, 0.4], [0.4, 0.6], [0.5, 0.7], (0.3, 0.2, 0.8) \rangle, \\ &\qquad \text{for all } \mathring{u} \in U, \end{aligned} \tag{11}$$

then $E_{T\beta} = \{0.1, 0.1\}, E_{I\beta} = \{-0.2, 0.4\}, E_{F\beta} = \{0.3, -0.1\}$. Thus,

$$\begin{aligned} E_{\beta}(\mathring{u}) &= \left(E_{L\beta}(\mathring{u}), E_{R\beta}(\mathring{u}) \right) \\ &= \{ (\mathring{u}, \langle 0.1, -0.2, 0.3, 0.1, 0.4, -0.1 \rangle) | \mathring{u} \in U \}. \end{aligned} \tag{12}$$

Remark 1. In Example 1, we observe that the left or right evaluative point of the NCS is not necessarily an NS. This motivates us to define the following terminologies.

Definition 7. Let $\beta = \langle T_{\beta}, I_{\beta}, F_{\beta}, t_{\beta}, i_{\beta}, f_{\beta} \rangle$ be an NCS in U with the evaluative set

$$E_{\beta} = \left\{ \left(\mathring{u}; \left(E_{L\beta}(\mathring{u}), E_{R\beta}(\mathring{u}) \right) \right) | \mathring{u} \in U \right\}. \tag{13}$$

An element ůϵU is called

(1) Truth stable element of U if

$$\begin{aligned} E_{T\beta} &= \left\{ \mathring{u}; t\left(l\left(S_{T\beta}(\mathring{u}) \right), r\left(S_{T\beta}(\mathring{u}) \right) \right) \right\} \\ &= \{ \mathring{u}; t\,(\text{left truth stable} - \text{element, right truth stable} - \text{element}) \} \\ &= \{ \mathring{u}; t\,(t(\mathring{u}) - T^{-}\,(\mathring{u}T^{+}(\mathring{u}) - t(\mathring{u}))n \geq q0 \}. \end{aligned} \tag{14}$$

(2) Indeterminacy stable element of U if

$$\begin{aligned} E_{I\beta} &= \left\{ \mathring{u}; t\left(l\left(S_{I\beta}(\mathring{u}) \right), r\left(S_{I\beta}(\mathring{u}) \right) \right) \right\} \\ &= \{ \mathring{u}; t\,(\text{left indeterminacy stable} - \text{element, right indeterminacy stable} - \text{element}) \}, \\ &= \{ \mathring{u}; t\,(i(\mathring{u}) - I^{-}(\mathring{u}), I^{+}(\mathring{u}) - i(\mathring{u}))n \geq q0 \} \end{aligned} \tag{15}$$

(3) Falsity stable element of U if

$$
\begin{aligned}
E_{F\beta} &= \left(l\left(S_{F\beta}(\mathring{u})\right), r\left(S_{F\beta}(\mathring{u})\right)\right) \\
&= \{\mathring{u}; t\,(\text{left stable falsity} - \text{element, right stable falsity} - \text{element})\} \\
&= \left(f(\mathring{u}) - F^{-}(\mathring{u}), F^{+}(\mathring{u}) - f(\mathring{u})\right) \geq 0.
\end{aligned} \tag{16}
$$

An element $\mathring{u}\in U$ is called stable if it satisfies conditions (1–3). The set of all stable elements of U is called stable cut of $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ in U and is denoted by S_β. We say that $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ is a stable neutrosophic set if $S_\beta = U$.

An element $\mathring{u}\in U$ is called partially stable if it partially satisfies conditions (1–3). The set of all partially stable elements of U is called partially stable cut of $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ in U and is denoted by P_β. We say that $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ is a partially stable neutrosophic set if $P_\beta \subset U$.

An element $\mathring{u}\in U$ is called antistable (unstable) if it does not satisfy conditions (1–3). The set of all unstable stable elements of U is called unstable stable cut of $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ in U and is denoted by U_β. We say that $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ is a unstable stable neutrosophic set if $U_{\backslash ss} \subseteq U$.

Thus, $U = S_\beta \cup P_\beta \cup U_\beta$.

Example 2. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an NCS in $U = \{0, a, b, c\}$ given by Table1.

Clearly, $\{0, a\}$ are stable elements of U and $\{b, c\}$ are unstable elements of U. Thus,

$$
\begin{aligned}
U &= \{a, b, c, d\} \\
&= S_\beta = \{0, a\} \cup P_\beta = \Phi \cup U_\beta = \{b, c\}.
\end{aligned} \tag{17}
$$

Example 3. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an NCS in $U = \{a, b\}$ given by Table 2.

Clearly, a and b are stable elements of U. Thus,

$$
\begin{aligned}
U &= \{a, b\} \\
&= S_\beta = \{a, b\} \cup P_\beta = \Phi \cup U_\beta = \Phi.
\end{aligned} \tag{18}
$$

Remark 2. Every internal NCS is a stable NCS, as shown in example 3. If an NCS is neither internal nor external, then we may have some stable elements with respect to the internal portion and some unstable elements with respect to the external portion as given in the Example 2. Thus, an external NCS may or may not be a stable NCS, as shown in Examples 4 and 5.

Example 4. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an external NCS in $U = \{a, b\}$ given by Table 3.

Then, clearly, a, b are unstable elements of U. Thus,

$$
\begin{aligned}
U &= \{a, b\} \\
&= S_\beta = \Phi \cup P_\beta = \{a, b\} \cup U_\beta = \Phi.
\end{aligned} \tag{19}
$$

Example 5. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an external NCS in $U = \{a, b\}$ given by Table 4.

Then, clearly, a, b are stable elements of U. Thus,

$$
\begin{aligned}
U &= \{a, b\} \\
&= S_\beta = \{a, b\}\Phi \cup P_\beta = \Phi \cup U_\beta = \Phi.
\end{aligned} \tag{20}
$$

Example 6. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an external NCS in $U = \{a\}$ given by Table 5.

Clearly, a is an unstable element of U. Thus, $U_\beta = \{a\} = U$. Hence, $U = S_\beta = \Phi \cup P_\beta = \Phi \cup U_\beta = \{a\}$.

Example 7. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an external NCS in $UU = \{a\}$ given by Table 6.

Clearly, a is an unstable element of U. Thus, $U_\beta = \{a\} = U$. Hence, $U = S_\beta = \Phi \cup P_\beta = \Phi \cup U_\beta = \{a\}$.

Example 8. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an NCS in $U = \{a, b, c\}$ given by Table 7.

Clearly, a and b are partially stable elements of U, so $P_\beta = \{a, b\} \subset U$ and c is the only stable element of U, so $S_\beta = \{c\}$. Also, there is no element which is unstable, so $U_\beta = \Phi$. Hence, $U = S_\beta \cup P_\beta \cup U_\beta$.

Remark 3

(1) If we have an external NCS which is unstable like in Example 6 such that

$$
\begin{aligned}
t(\mathring{u}) &> [T^{-}(\mathring{u}), T^{+}(\mathring{u})], i(\mathring{u}) \\
&> [I^{-}(\mathring{u}), I^{+}(\mathring{u})], f(\mathring{u}) > [F^{-}(\mathring{u}), F^{+}(\mathring{u})],
\end{aligned} \tag{21}
$$

then its right evaluative point becomes a neutrosophic bipolar fuzzy set.

(2) If we have an external NCS which is unstable like in example 7 such that

$$
\begin{aligned}
t(\mathring{u}) &< [T^{-}(\mathring{u}), T^{+}(\mathring{u})], i(\mathring{u}) \\
&< [I^{-}(\mathring{u}), I^{+}(\mathring{u})], f(\mathring{u}) < [F^{-}(\mathring{u}), F^{+}(\mathring{u})],
\end{aligned} \tag{22}
$$

then its left evaluative point becomes a neutrosophic bipolar fuzzy set.

Table 1: Neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
0	[0.3, 0.5]	[0.2, 0.4]	[0.2, 0.5]	0.4	0.3	0.4
a	[0.3, 0.5]	[0.3, 0.5]	[0.3, 0.6]	0.4	0.4	0.5
b	[0.6, 0.8]	[0.5, 0.6]	[0.4, 0.5]	0.5	0.4	0.3
c	[0.4, 0.8]	[0.5, 0.6]	[0.6, 0.7]	0.9	0.7	0.8

Table 2: Neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.1, 0.7]	[0.1, 0.6]	[0.2, 0.8]	0.6	0.5	0.7
b	[0.6, 0.8]	[0.6, 0.9]	[0.5, 0.7]	0.7	0.8	0.6

Table 3: Neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.1, 0.3]	[0.1, 0.4]	[0.3, 0.6]	0.4	0.5	0.7
b	[0.5, 0.8]	[0.6, 0.8]	[0.4, 0.6]	0.4	0.5	0.3

Table 4: External neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.2, 0.4]	[0.3, 0.5]	[0.3, 0.6]	0.2	0.3	0.3
b	[0.4, 0.8]	[0.6, 0.7]	[0.4, 0.5]	0.8	0.7	0.5

Table 5: External neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.3, 0.5]	[0.1, 0.4]	[0.4, 0.6]	0.8	0.5	0.7

Table 6: External neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.5, 0.6]	[0.3, 0.5]	[0.7, 0.9]	0.4	0.2	0.6

Table 7: Neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.7, 0.8]	[0.3, 0.5]	[0.6, 0.9]	0.7	0.8	0.2
b	[0.1, 0.5]	[0.6, 0.9]	[0.3, 0.8]	0.2	0.7	0.1
c	[0.1, 0.4]	[0.2, 0.5]	[0.3, 0.7]	0.3	0.4	0.5

(3) Every NCS needs not to be a stable NCS, but each stable NCS must be an NCS.

(4) Observing Example 5, we reached at Theorem 1.

Theorem 1. *If an external NCS $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ in U satisfies the condition*

$$(\forall \mathring{u} \in U) \begin{pmatrix} \left(T_\beta^-(\mathring{u}) = t_\beta(\mathring{u}), T_\beta^+(\mathring{u}) = t(\mathring{u})\right), \\ \left(I^-(\mathring{u}) = i(\mathring{u}), I^+(\mathring{u}) = i(\mathring{u})\right), \\ \left(F^-(\mathring{u}) = f(\mathring{u}), F^+(\mathring{u}) = f(\mathring{u})\right) \end{pmatrix}, \quad (23)$$

then $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ is a stable NCS.

Proof. Straightforward.

Remark 4. We observe that if β is both an internal and external NCS, then β is a stable NCS.

Theorem 2. *The complement of a stable NCS is also a stable NCS.*

Proof. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ be a stable NCS in U. Then,

$$U = S_\beta = \left\{\mathring{u} \in \mathring{u} | l\left(E_\beta(\mathring{u})\right) \geq 0, r\left(E_\beta(\mathring{u})\right) \geq 0\right\}. \tag{24}$$

Hence,

$$\begin{pmatrix} t(\mathring{u}) - T^-(\mathring{u}) \geq 0, \\ T^+(\mathring{u}) - t(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} i(\mathring{u}) - I^-(\mathring{u}) \geq 0, \\ I^+(\mathring{u}) - i(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} f(\mathring{u}) - F^-(\mathring{u}) \geq 0, \\ F^+(\mathring{u}) - f(\mathring{u}) \geq 0 \end{pmatrix}, \quad \forall \mathring{u} \in U. \tag{25}$$

It follows that

$$\begin{aligned}
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - t(\mathring{u})) - \left(1 - T^+(\mathring{u})\right) = T^+(\mathring{u}) - t(\mathring{u}) \geq 0, \\
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - i(\mathring{u})) - \left(1 - I^+(\mathring{u})\right) = I^+(\mathring{u}) - i(\mathring{u}) \geq 0, \\
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - f(\mathring{u})) - \left(1 - F^+(\mathring{u})\right) = F^+(\mathring{u}) - f(\mathring{u}) \geq 0, \\
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - T^-(\mathring{u})) - (1 - t(\mathring{u})) = t(\mathring{u}) - T^-(\mathring{u}) \geq 0, \\
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - I^-(\mathring{u})) - (1 - i(\mathring{u})) = i(\mathring{u}) - I^-(\mathring{u}) \geq 0, \\
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - F^-(\mathring{u})) - (1 - f(\mathring{u})) = f(\mathring{u}) - F^-(\mathring{u}) \geq 0.
\end{aligned} \tag{26}$$

Therefore, $\beta^C = \langle T_\beta^c, I_\beta^c, F_\beta^c, t_\beta^c, i_\beta^c, f_\beta^c \rangle$ is a stable NCS.

Theorem 3. *The complement of an unstable NCS is also an unstable NCS.*

Proof. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ be an unstable NCS in U. Then,

$$U = U_\beta = \left\{\mathring{u} \in \mathring{u} | l\left(E_\beta(\mathring{u})\right) < 0\right\} \cup \left\{\mathring{u} \in \mathring{u} | r\left(E_\beta(\mathring{u})\right) < 0\right\} \neq \Phi, \tag{27}$$

and so, there exist $\mathring{u} \in U$ such that

$$\begin{aligned} &(t(\mathring{u}) - T^-(\mathring{u}) < 0), (i(\mathring{u}) - I^-(\mathring{u}) < 0), \\ &(f(\mathring{u}) - F^-(\mathring{u}) < 0), \quad \forall \mathring{u} \in U, \end{aligned} \tag{28}$$

or

$$\begin{aligned} &\left(T^+(\mathring{u}) - t(\mathring{u}) < 0\right), \left(I^+(\mathring{u}) - i(\mathring{u}) < 0\right), \\ &\left(F^+(\mathring{u}) - f(\mathring{u}) < 0\right), \quad \forall \mathring{u} \in U. \end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - t(\mathring{u})) - \left(1 - T^+(\mathring{u})\right) = T^+(\mathring{u}) - t(\mathring{u}) < 0, \\
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - i(\mathring{u})) - \left(1 - I^+(\mathring{u})\right) = I^+(\mathring{u}) - i(\mathring{u}) < 0, \\
l\left(E_{\beta^C}(\mathring{u})\right) &= (1 - f(\mathring{u})) - \left(1 - F^+(\mathring{u})\right) = F^+(\mathring{u}) - f(\mathring{u}) < 0,
\end{aligned} \tag{30}$$

or

$$\begin{aligned}
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - T^-(\mathring{u})) - (1 - t(\mathring{u})) = t(\mathring{u}) - T^-(\mathring{u}) < 0, \\
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - I^-(\mathring{u})) - (1 - i(\mathring{u})) = i(\mathring{u}) - I^-(\mathring{u}) < 0, \\
r\left(E_{\beta^C}(\mathring{u})\right) &= (1 - F^-(\mathring{u})) - (1 - f(\mathring{u})) = f(\mathring{u}) - F^-(\mathring{u}) < 0.
\end{aligned} \tag{31}$$

Hence, $U_{\beta^C} \neq \Phi$, and therefore, $\beta^C = \langle T_\beta^c, I_\beta^c, F_\beta^c, t_\beta^c, i_\beta^c, f_\beta^c \rangle$ is an unstable NCS.

Example 9 illustrates Theorem 3.

Example 9. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ be an NCS in $U = \{a, b\}$ given by Table 8.

Clearly, a and b are unstable elements of U and their complements are represented by Table 9.

Then, $\beta^c = \langle T_\beta^c, I_\beta^c, F_\beta^c, t_\beta^c, i_\beta^c, f_\beta^c \rangle$ is unstable since $a \in U_{\beta^c}$.

Theorem 4. *The P-union and P-intersection of two stable NCSs in U are stable cubic sets in U.*

Proof. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2} \rangle$ be two NCSs in U. Then,

$$\begin{aligned} S_\beta &= \left\{\mathring{u} \in U | l\left(\beta_\beta(\mathring{u})\right) \geq 0, r\left(\beta_{5\beta}(\mathring{u})\right) \geq 0\right\} = U, \\ S_{\beta_2} &= \left\{\mathring{u} \in U | l\left(\beta_{5\beta_2}(\mathring{u})\right) \geq 0, r\left(\beta_{5\beta_2}(\mathring{u})\right) \geq 0\right\} = U. \end{aligned} \tag{32}$$

It follows that

Table 8: Neutrosophic cubic set β of U.

U	$T_\beta(\mathring{u})$	$I_\beta(\mathring{u})$	$F_\beta(\mathring{u})$	$t_\beta(\mathring{u})$	$i_\beta(\mathring{u})$	$f_\beta(\mathring{u})$
a	[0.1, 0.5]	[0.3, 0.6]	[0.2, 0.4]	0.4	0.5	0.3
b	[0.6, 0.9]	[0.1, 0.9]	[0.1, 0.6]	0.7	0.6	0.5

Table 9: Complement of neutrosophic cubic set β of U provided in Table 8.

U	$T^c_\beta(\mathring{u})$	$I^c_\beta(\mathring{u})$	$F^c_\beta(\mathring{u})$	$t^c_\beta(\mathring{u})$	$i^c_\beta(\mathring{u})$	$f^c_\beta(\mathring{u})$
a	[0.5, 0.9]	[0.4, 0.7]	[0.6, 0.8]	0.6	0.5	0.7
b	[0.1, 0.4]	[0.1, 0.9]	[0.4, 0.9]	0.3	0.4	0.5

$$\begin{aligned}&\begin{pmatrix} t_\beta(\mathring{u}) - T^-_\beta(\mathring{u}) \geq 0, \\ T^+_\beta(\mathring{u}) - t_\beta(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} i_\beta(\mathring{u}) - I^-_\beta(\mathring{u}) \geq 0, \\ I^+_\beta(\mathring{u}) - i_\beta(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} f_\beta(\mathring{u}) - F^-_\beta(\mathring{u}) \geq 0, \\ F^+_\beta(\mathring{u}) - f_\beta(\mathring{u}) \geq 0 \end{pmatrix}, \quad \forall \mathring{u} \in U,\\ &\begin{pmatrix} t_{\beta_2}(\mathring{u}) - T^-_{\beta_2}(\mathring{u}) \geq 0, \\ T^+_{\beta_2}(\mathring{u}) - t_{\beta_2}(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} i_{\beta_2}(\mathring{u}) - I^-_{\beta_2}(\mathring{u}) \geq 0, \\ I^+_{\beta_2}(\mathring{u}) - i_{\beta_2}(\mathring{u}) \geq 0 \end{pmatrix}, \begin{pmatrix} f_{\beta_2}(\mathring{u}) - F^-_{\beta_2}(\mathring{u}) \geq 0, \\ F^+_{\beta_2}(\mathring{u}) - f_{\beta_2}(\mathring{u}) \geq 0 \end{pmatrix}, \quad \forall \mathring{u} \in U.\end{aligned} \tag{33}$$

Assume that $t_{\beta_1}(\mathring{u}) \geq t_{\beta_2}(\mathring{u}), i_{\beta_1}(\mathring{u}) \geq i_{\beta_2}(\mathring{u}), f_{\beta_1}(\mathring{u}) \geq f_{\beta_2}(\mathring{u})$ and consider the following cases:

(i) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(ii) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(iii) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(iv) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(v) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(vi) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \geq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(vii) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \leq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(viii) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \leq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(ix) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \leq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(x) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \leq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(xi) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \leq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

(xii) $\begin{matrix}(T^-_{\beta_1}(\mathring{u}) \geq T^-_{\beta_2}(\mathring{u}), T^+_{\beta_1}(\mathring{u}) \geq T^+_{\beta_2}(\mathring{u}))\\(I^-_{\beta_1}(\mathring{u}) \geq I^-_{\beta_2}(\mathring{u}), I^+_{\beta_1}(\mathring{u}) \geq I^+_{\beta_2}(\mathring{u}))\\(F^-_{\beta_1}(\mathring{u}) \geq F^-_{\beta_2}(\mathring{u}), F^+_{\beta_1}(\mathring{u}) \leq F^+_{\beta_2}(\mathring{u}))\end{matrix}$

The first case implies that

$$\begin{aligned}&\max\{(t_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}), f_{\beta_1}(\mathring{u})), (t_{\beta_2}(\mathring{u}), i_{\beta_2}(\mathring{u}), f_{\beta_2}(\mathring{u}))\}\\ &\quad = (t_{\beta_1}(\mathring{u}) \geq T^-_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}) \geq I^-_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}) \geq I^-_{\beta_1}(\mathring{u}))\\ &\quad = \max\{T^-_{\beta_1}(\mathring{u}), I^-_{\beta_1}(\mathring{u}), F^-_{\beta_1}(\mathring{u}), T^-_{\beta_2}(\mathring{u}), I^-_{\beta_2}(\mathring{u}), F^-_{\beta_2}(\mathring{u})\},\\ &\max\{(t_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}), f_{\beta_1}(\mathring{u})), (t_{\beta_2}(\mathring{u}, i_{\beta_2}(\mathring{u}), f_{\beta_2}(\mathring{u}))\}\\ &\quad = (t_{\beta_1}(\mathring{u}) \geq T^+_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}) \geq I^+_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}) \geq I^+_{\beta_1}(\mathring{u}))\\ &\quad = \max\{T^+_{\beta_1}(\mathring{u}), I^+_{\beta_1}(\mathring{u}), F^+_{\beta_1}(\mathring{u}), T^{+-}_{\beta_2}(\mathring{u}), I^+_{\beta_2}(\mathring{u}), F^+_{\beta_2}(\mathring{u})\}.\end{aligned} \tag{34}$$

It follows that

$$\begin{aligned}&(t_{\beta_1}(\mathring{u}) - T^-_{\beta_1}(\mathring{u}), i_{\beta_1}(\mathring{u}) - I^-_{\beta_1}(\mathring{u}), f_{\beta_1}(\mathring{u}) - F^-_{\beta_1}(\mathring{u})) \geq 0,\\ &(T_{\beta_1}(\mathring{u})^+ - t_{\beta_1}(\mathring{u}), I_{\beta_1}(\mathring{u})^+ - i_{\beta_1}(\mathring{u}), F_{\beta_1}(\mathring{u})^+ - f_{\beta_1}(\mathring{u})) \geq 0.\end{aligned} \tag{35}$$

The result of the remaining cases can be obtained in the same way. Therefore, $\beta_1 \cup_P \beta_2$ is a stable CS in U. By the same way, we also know that $\beta_1 \cup_P \beta_2$ is a stable CS in U.

Example 10 shows that the Ṙ-union and the Ṙ-intersection of two stable NCSs in U may not be a stable NCS in U.

Example 10. Let $\beta_1 = \langle T_{\beta_1}, I_{\beta_1}, F_{\beta_1}, t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\rangle$ be two NCSs in $U = \{a, b\}$ defined by Tables 10 and 11, respectively.

Then,

$$\beta_1 \cup_R \beta_2 = \left\{ \begin{array}{c} \langle a, [0.4, 0.5], [0.3, 0.9], [0.7, 0.9], 0.15, 0.35, 0.6\rangle, \\ \langle b, [0.6, 0.9], [0.8, 0.9], [0.5, 0.6], 0.6, 0.8, 0.25\rangle \end{array} \right\},$$
$$\beta_1 \cap_R \beta_2 = \left\{ \begin{array}{c} \langle a, [0.1, 0.3], [0.1, 0.4], [0.3, 0.7], 0.4, 0.8, 0.80\rangle, \\ \langle b, [0.6, 0.9], [0.1, 0.9], [0.2, 0.4], 0.7, 0.8, 0.56\rangle \end{array} \right\}. \tag{36}$$

Hence, we know that

$$\begin{aligned} E_{\beta_1 \cup_R \beta_2}(a) &= \langle(-0.25, 0.35), (0.05, 0.55), (-0.1, 0.3)\rangle, \\ E_{\beta_1 \cup \beta_2}(b) &= \langle(0, 0.3), (0, 0.1), (-0.25, 0.35)\rangle, \\ E_{\beta_1 \cap \beta_2}(a) &= \langle(0.3, -0.1), (0.7, -0.4), (0.5, -0.1)\rangle, \\ E_{\beta_1 \cap \beta_2}(b) &= \langle(0.1, 0.2), (0.7, 0.1), (0.36, -0.16)\rangle. \end{aligned} \tag{37}$$

Theorem 5. *Let $\beta_1 = \langle T_{\beta_1}, I_{\beta_1}, F_{\beta_1}, t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\rangle$ be two internal NCSs in U such that*

$$(\forall \mathring{u} \in U)\left(\max\left\{ \begin{array}{c} \left(T_{\beta_1}(\mathring{u})^-, I_{\beta_1}(\mathring{u})^-, F_{\beta_1}(\mathring{u})^-\right), \\ \left(T_{\beta_2}(\mathring{u})^-, I_{\beta_2}(\mathring{u})^-, F_{\beta_2}(\mathring{u})^-\right) \end{array} \right\} \leq \left(\left(t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\right) \wedge \left(t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\right)\right)(\mathring{u}) \right). \tag{38}$$

Then, the Ṙ-union of β_1 and β_2 is a stable NCS in U.

Proof. Let $\beta_1 = \langle T_{\beta_1}, I_{\beta_1}, F_{\beta_1}, t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\rangle$ be two internal NCSs in U. Then, $(T_{\beta_1}(\mathring{u})^- \leq t_{\beta_1}(\mathring{u}) \leq T_{\beta_1}(\mathring{u})^+)$, $(I_{\beta_1}(\mathring{u})^- \leq i_{\beta_1}(\mathring{u}) \leq I_{\beta_1}(\mathring{u})^+)$, and $(F_{\beta_1}(\mathring{u})^- \leq f_{\beta_1}(\mathring{u}) \leq F_{\beta_1}(\mathring{u})^+)$ and $(T_{\beta_2}(\mathring{u})^- \leq t_{\beta_2}(\mathring{u}) \leq T_{\beta_1}(\mathring{u})^+)$, $(I_{\beta_2}(\mathring{u})^- \leq i_{\beta_2}(\mathring{u}) \leq I_{\beta_2}(\mathring{u})^+)$, and $(F_{\beta_2}(\mathring{u})^- \leq f_{\beta_2}(\mathring{u}) \leq F_{\beta_R}(\mathring{u})^+)$, $\forall \mathring{u} \in U$. We know that

$$\begin{aligned} &\max\left\{\left(T_{\beta_1}(\mathring{u})^-, I^-_{\beta_1}(\mathring{u}) \leq F_{\beta_1}(\mathring{u})^-\right), \left(T_{\beta_1}(\mathring{u})^-, I_{\beta_1}(\mathring{u})^- \leq F_{\beta_1}(\mathring{u})^-\right)\right\} \\ &\leq \left(\left(t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\right) \wedge \left(t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\right)\right)(\mathring{u}) \\ &\leq \max\left\{\left(T_{\beta_1}(\mathring{u})^+, I_{\beta_1}(\mathring{u})^+ \leq F_{\beta_1}(\mathring{u})^+\right), \left(T_{\beta_1}(\mathring{u})^+, I_{\beta_1}(\mathring{u})^+ \leq F_{\beta_1}(\mathring{u})^+\right)\right\}, \end{aligned} \tag{39}$$

for all $\mathring{u} \in U$. Hence, the Ṙ-union of β_1 and β_2 is an internal NCS, and so it is stable by the fact that every internal NCS is stable.

Theorem 6. *Let $\beta_1 = \langle T_{\beta_1}, I_{\beta_1}, F_{\beta_1}, t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\rangle$ be two internal NCSs in U such that*

$$(\forall \mathring{u} \in U)\left(\max\left\{ \begin{array}{c} \left(T_{\beta_1}(\mathring{u})^+, I_{\beta_1}(\mathring{u})^+, F_{\beta_1}(\mathring{u})^+\right), \\ \left(T_{\beta_2}(\mathring{u})^+, I_{\beta_2}(\mathring{u})^+, F_{\beta_2}(\mathring{u})^+\right) \end{array} \right\} \leq \left(\left(t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\right) \vee \left(t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\right)\right)(\mathring{u}) \right). \tag{40}$$

Then, the Ṙ-intersection of β_1 and β_2 is a stable NCS in U.

Proof. Straightforward.

4. Neutro-Almost-Stable Neutrosophic Cubic Set

In this section, we introduce a new class of the stable neutrosophic cubic set, namely, the neutro-almost-stable neutrosophic cubic set.

Definition 8. Let $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ be an NCS with the evaluative set $E_\beta = \left\{(\mathring{u}, E_\beta(\mathring{u})) | \mathring{u} \in U\right\}$ in U. Then,

(1) The truth-stable degree of β in U is denoted by $\mathrm{Tru}(\mathrm{SD}_\beta)$ and is defined as

$$\mathrm{Tru}\left(\mathrm{SD}_\beta\right) = \left(\sum_{\mathring{u} \in U} l\left(E_{T\beta}(\mathring{u})\right), r\left(E_{T\beta}(\mathring{u})\right)\right). \tag{41}$$

(2) The indeterminacy-stable degree of β in U is denoted by $\mathrm{Ind}(\mathrm{SD}_\beta)$ and is defined as

$$\mathrm{Ind}\left(\mathrm{SD}_\beta\right) = \left(\sum_{\mathring{u} \in U} l\left(E_{I\beta}(\mathring{u})\right), r\left(E_{I\beta}(\mathring{u})\right)\right). \tag{42}$$

(3) The falsity-stable degree of β in U is denoted by $\mathrm{Fal}(\mathrm{SD}_\beta)$ and is defined as

Table 10: Neutrosophic cubic set β_1 of U.

U	$T_{\beta_1}(\mathring{u})$	$I_{\beta_1}(\mathring{u})$	$F_{\beta_1}(\mathring{u})$	$t_{\beta_1}(\mathring{u})$	$i_{\beta_1}(\mathring{u})$	$f_{\beta_1}(\mathring{u})$
a	[0.4, 0.5]	[0.3, 0.4]	[0.3, 0.7]	0.4	0.35	0.60
b	[0.3, 0.7]	[0.8, 0.9]	[0.5, 0.6]	0.60	0.8	0.56

Table 11: Neutrosophic cubic set β_2 of U.

U	$T_{\beta_2}(\mathring{u})$	$I_{\beta_2}(\mathring{u})$	$F_{\beta_2}(\mathring{u})$	$t_{\beta_2}(\mathring{u})$	$i_{\beta_2}(\mathring{u})$	$f_{\beta_2}(\mathring{u})$
a	[0.1, 0.3]	[0.1, 0.9]	[0.7, 0.9]	0.15	0.8	0.8
b	[0.6, 0.9]	[0.1, 0.9]	[0.2, 0.4]	0.7	0.8	0.25

$$\mathrm{Fal}(\mathrm{SD}_\beta) = \left(\sum_{\mathring{u}\in U} l(E_{F\beta}(\mathring{u})), r(E_{F\beta}(\mathring{u}))\right). \quad (43)$$

(4) The stable degree of β in U is denoted by SD_β and is defined as $\mathrm{SD}_\beta = (\mathrm{Tru}(\mathrm{SD}_\beta), \mathrm{Ind}(\mathrm{SD}_\beta), \mathrm{Fal}(\mathrm{SD}_\beta))$.

Definition 9. An NCS with the evaluative set $E_\beta = \{(\mathring{u}, E_\beta(\mathring{u}))|\mathring{u} \in U\}$ in U is said to be

(1) Almost truth-stable if $\mathrm{Tru}(\mathrm{SD}_\beta) \geq 0$

(2) Almost indeterminacy-stable if $\mathrm{Ind}(\mathrm{SD}_\beta) \geq 0$

(3) Almost falsity-stable if $\mathrm{Fal}(\mathrm{SD}_\beta) \geq 0$

(4) Almost stable if it is almost truth-stable, almost indeterminacy-stable, and almost falsity-stable, i.e., $\mathrm{Tru}(\mathrm{SD}_\beta) \geq 0, \mathrm{Ind}(\mathrm{SD}_\beta) \geq 0, \mathrm{Fal}(\mathrm{SD}_\beta) \geq 0$.

(5) Almost partially stable if it is almost partially truth-stable, almost partially indeterminacy-stable, and almost partially falsity-stable.

(6) Almost unstable if it is almost truth-unstable, almost indeterminacy-unstable, and almost falsity-unstable, i.e., $\mathrm{Tru}(\mathrm{SD}_\beta) < 0, \mathrm{Ind}(\mathrm{SD}_\beta) < 0, \mathrm{Fal}(\mathrm{SD}_\beta) < 0$.

Example 11. Let $\beta_1 = \langle T_{\beta_1}, I_{\beta_1}, F_{\beta_1}, t_{\beta_1}, i_{\beta_1}, f_{\beta_1}\rangle$ and $\beta_2 = \langle T_{\beta_2}, I_{\beta_2}, F_{\beta_2}, t_{\beta_2}, i_{\beta_2}, f_{\beta_2}\rangle$ be two NCSs in $U = \{a, b\}$ defined by Tables 12 and 13, respectively,with the evaluative set

$$E_{\beta_1} = \langle(a; \langle 0, 0.1\rangle, \langle 0.05, 0.05\rangle, \langle 0.3, 0.1\rangle), (b; \langle 0.3, 0.1\rangle, \langle 0, 0.1\rangle, \langle 0.06, 0.04\rangle)\rangle. \quad (44)$$

Then, $\mathrm{Tru}(\mathrm{SD}_{\beta_1}) = (0.3, 0.2) \geq 0$, $\mathrm{Ind}(\mathrm{SD}_{\beta_1}) = (0.05, 0.15) \geq 0, \mathrm{Fal}(\mathrm{SD}_{\beta_2}) = (0.36, 0.14) \geq 0$. Thus,

$$SD_{\beta_1} = (0.3, 0.2, 0.05, 0.15, 0.36, 0.14) \geq 0, \quad (45)$$

alsowith the evaluative set

$$E_{\beta_2} = \langle(a\langle 0.05, 0.15\rangle, \langle 0.7, 0.1\rangle, \langle 0.1, 0.1\rangle), (b\langle 0.1, 0.2\rangle, \langle 0.7, 0.1\rangle, \langle 0.05, 0.15\rangle)\rangle. \quad (46)$$

Then, $\mathrm{Tru}(\mathrm{SD}_{\beta_2}) = (0.15, 0.25) \geq 0$, $\mathrm{Ind}(\mathrm{SD}_{\beta_2}) = (0.14, 0.2) \geq 0, \mathrm{Fal}(\mathrm{SD})_{\beta_2} = (0.15, 0.25) \geq 0$. Thus,

$$\mathrm{SD}_{\beta_2} = (0.15, 0.35, 0.14, 0.2, 0.15, 0.25) \geq 0. \quad (47)$$

So, β_1 and β_2 both are almost stable NCSs.

Example 12. Let $\beta_3 = \langle T_{\beta_3}, I_{\beta_3}, F_{\beta_3}, t_{\beta_3}, i_{\beta_3}, f_{\beta_3}\rangle$ be an NCS in $U = \{a, b\}$ defined by Table 14.

The evaluative set is

$$E_{\beta_3} = \langle(a\langle -0.1, 0.2\rangle, \langle 0.3, -0.1\rangle, \langle -0.1, 0.3\rangle), (b\langle -0.1, 0.5\rangle, \langle -0.2, 0.3\rangle, \langle 0.2, -0.1\rangle)\rangle. \quad (48)$$

Then, $\mathrm{Tru}(\mathrm{SD}_{\beta_3}) = (-0.2, 0.7) < 0$. Thus, the NCS $\beta_3 = \langle T_{\beta_3}, I_{\beta_3}, F_{\beta_3}, t_{\beta_3}, i_{\beta_3}, f_{\beta_3}\rangle$ in U is not almost truth-stable as $\mathrm{Tru}(\mathrm{SD}_{\beta_3}) < 0$. Also, $\mathrm{Ind}(\mathrm{SD}_{\beta_3}) = (0.1, 0.2) \geq 0$. Thus, the NCS $\beta_3 = \langle T_{\beta_3}, I_{\beta_3}, F_{\beta_3}, t_{\beta_3}, i_{\beta_3}, f_{\beta_3}\rangle$ in U is almost indeterminacy-stable as $\mathrm{Ind}(\mathrm{SD}_{\beta_3}) \geq 0$. Similarly $\beta_3 = \langle T_{\beta_3}, I_{\beta_3}, F_{\beta_3}, t_{\beta_3}, i_{\beta_3}, f_{\beta_3}\rangle$ in U is almost falsity-stable as $\mathrm{Fal}(\mathrm{SD}_{\beta_3}) \geq 0$. So, finally, we can say that β_3 is an almost partially stable NCS.

Example 13. Let $\beta_4 = \langle T_{\beta_4}, I_{\beta_4}, F_{\beta_4}, t_{\beta_4}, i_{\beta_4}, f_{\beta_4}\rangle$ be an NCS in $U = \{a, b\}$ defined by Table 15

The evaluative set is

$$E_{\beta_4} = \langle(a; \langle 0.2, -0.1\rangle, \langle 0.3, -0.1\rangle, \langle -0.1, 0.3\rangle), (b; \langle -0.1, 0.5\rangle, \langle -0.2, 0.3\rangle, \langle 0.2, -0.1\rangle)\rangle. \quad (49)$$

Then, $\mathrm{Tru}(\mathrm{SD}_{\beta_4}) = (0.1, 0.4) \geq 0, \mathrm{Ind}(\mathrm{SD}_{\beta_4}) = (0.1, 0.2) \geq 0, \mathrm{Fal}(\mathrm{SD}_{\beta_4}) = (0.1, 0.2) \geq 0$. So, β_4 is an almost-stable NCS, but it is not a stable NCS, as from Definition 7; $S_\beta = \Phi, P_\beta = \Phi, U_\beta = \{a, b\}$.

Remark 5. From Examples 11, 12, and 13, we have the following results.

Theorem 7

(1) Every stable NCS $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta\rangle$ in U is an almost-stable NCS, but the converse is not true

(2) Every internal NCS is almost stable

(3) Every external NCS may or may not be stable

(4) The P-union and P-intersection of two stable NCSs are almost stable

(5) The complement of an almost-stable NC is also an almost-stable NCS

Proof. Straightforward.

Table 12: Neutrosophic cubic set β_1 of U.

U	$T_{\beta_1}(\mathring{u})$	$I_{\beta_1}(\mathring{u})$	$F_{\beta_1}(\mathring{u})$	$t_{\beta_1}(\mathring{u})$	$i_{\beta_1}(\mathring{u})$	$f_{\beta_1}(\mathring{u})$
a	[0.4, 0.5]	[0.3, 0.4]	[0.3, 0.7]	0.4	0.35	0.60
b	[0.3, 0.7]	[0.8, 0.9]	[0.5, 0.6]	0.60	0.8	0.56

Table 13: Neutrosophic cubic set β_2 of U.

U	$T_{\beta_2}(\mathring{u})$	$I_{\beta_2}(\mathring{u})$	$F_{\beta_2}(\mathring{u})$	$t_{\beta_2}(\mathring{u})$	$i_{\beta_2}(\mathring{u})$	$f_{\beta_2}(\mathring{u})$
a	[0.1, 0.3]	[0.1, 0.9]	[0.7, 0.9]	0.15	0.8	0.8
b	[0.6, 0.9]	[0.1, 0.9]	[0.2, 0.4]	0.7	0.8	0.25

Table 14: Neutrosophic cubic set β_3 of U.

U	$T_{\beta_3}(\mathring{u})$	$I_{\beta_3}(\mathring{u})$	$F_{\beta_3}(\mathring{u})$	$t_{\beta_3}(\mathring{u})$	$i_{\beta_3}(\mathring{u})$	$f_{\beta_3}(\mathring{u})$
a	[0.2, 0.3]	[0.3, 0.5]	[0.4, 0.6]	0.1	0.6	0.3
b	[0.3, 0.7]	[0.8, 0.9]	[0.5, 0.6]	0.2	0.6	0.7

Table 15: Neutrosophic cubic set β_4 of U.

U	$T_{\beta_4}(\mathring{u})$	$I_{\beta_4}(\mathring{u})$	$F_{\beta_4}(\mathring{u})$	$t_{\beta_4}(\mathring{u})$	$i_{\beta_4}(\mathring{u})$	$f_{\beta_4}(\mathring{u})$
a	[0.2, 0.3]	[0.3, 0.5]	[0.4, 0.6]	0.4	0.6	0.3
b	[0.3, 0.7]	[0.8, 0.9]	[0.5, 0.6]	0.2	0.6	0.7

5. Application in Decision Making

In this section, we shall define a new approach to multiple attribute group decision making wıth the help of stable neutrosophic cubic sets. We also provide a numerical example. Suppose $H = \{H_1, H_2, \ldots, H_m\}$. Each alternative H_i respects n criteria $G_j = \{G_1, G_2, \ldots, G_n\}$ which are expressed by a stable NCS $q_{i_j} = ((\tilde{q}_{\text{Tru}\,ij}, \tilde{q}_{\text{Ind}\,ij}, \tilde{q}_{\text{Fal}\,ij})(q_{\text{Tru}\,ij}, q_{\text{Ind}\,ij}, q_{\text{Fal}\,ij}))$, $(j = 1, 2, \ldots n, i = 1, 2, \ldots, m)$. The criteria $G_1, \ldots, G_k$ are benefit and criteria $G_{k+1}, \ldots, G_n$ are non-benefit criteria, and $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ is the weighted vector of the criteria, where, $\omega_i \varepsilon [0, 1]$ and $\sum \omega_i = 1$. So, the decision matrix is obtained as $D = (q_{ij})_{m\times n}$. The steps of the decision making based on stable NCSs are given as follows:

Step 1: we standardize the decision matrix.

Step 2: we construct the normalized decision matrix. Normalize score or data are as follows:

$$r_{i_j} = \frac{\mathring{u}_{i_j}}{\left(\sum \mathring{u}_{i_j}^2\right)}, \quad \text{for } i = 1, \ldots, m;\ j = 1, \ldots, n. \tag{50}$$

Step 3: we construct the weighted normalized decision matrix:

$$v_{i_j} = w_j \cdot r_{i_j}. \tag{51}$$

Step 4: we determine the ideal and negative ideal solutions. Ideal solution $A^* = \{v_1, \ldots, v_n\}$, where

$$v_j^* = \left\{\max\left(v_{i_j}\right), \text{ if } j \in J;\ \min\left(v_{i_j}\right), \text{ if } j \in J'\right\}. \tag{52}$$

Negative ideal solution is

$$A' = \{v_1', \ldots, v_n'\}, \tag{53}$$

where

$$v_j'\left\{\max\left(v_{i_j}\right), \quad \text{if } j \in J;\ \min\left(v_{i_j}\right), \text{ if } j \in J\prime\right\}. \tag{54}$$

Step 5: we calculate the separation measures for each alternative. Separation from the ideal alternatives is

$$S_i^* = \sqrt{\left[\sum\left(v_j^* - v_{i_j}\right)^2\right]}, \quad i = 1, \ldots, m. \tag{55}$$

Similarly, separation from negative ideal alternatives is

$$S_i' = \sqrt{\left[\sum\left(v_j' - v_{i_j}\right)^2\right]}, \quad i = 1, \ldots, m. \tag{56}$$

Step 6: we calculate the relative closeness to the ideal solution C_i^* where

$$C_i^* = \frac{S_i'}{(S_i^* + S_i')}, \quad 0 \le C_i^* \le 1. \tag{57}$$

We select the option with C_i^* closest to 1.

5.1. Numerical Application. At the end of December 2019 [22], in Wuhan, the China Health Commission reported a cluster of pneumonia cases of unknown etiology. The pathogen was identified as novel coronavirus 2019. Later, the World Health Organization named it Coronavirus Disease

2019 (COVID-19). After the discovery of COVID-19, it spread in more than 200 countries. COVID-19 has zoonotic basis, which was then spread through the human interaction to human population [23]. Common signs of COVID-19 infection are similar to those of common cold and include respiratory symptoms such as dry cough, fever, shortness of breath, and breathing difficulties. Initially its etiology was unknown. Later on, it was studied thoroughly and found that it has an incubation period of 14 days, during which some individuals show all the symptoms while others show mild symptoms. It is sensitive to know that someone have the disease due to the dual nature (same as common flu) of COVID-19 symptoms [24]. In this section, we use the TOPSIS method to rank the COVID-19 in four provinces of Pakistan. A numerical example which is solved using the TOPSIS method is presented to demonstrate the applicability and effectiveness of the proposed method.

5.2. Example. Let us consider the decision making problem. Suppose that there is a panel and they selected four possible alternatives (H_1, H_2, H_3, H_4) to find out the spreading of COVID-19 in provinces of Pakistan: $H_1$is KPK, $H_2$is Sindh, $H_3$is Punjab, and $H_4$is Balochistan. A group of doctors intends to choose one province be the most affected area from four provinces, to be further evaluated according to the four attributes, which are shown as G_1 effected people, G_2 recovered people, G_3 admitted people, and G_4 number of deaths. By this method, we can find out which province is more affected. Then, we must take some action to stop the cases in that province. The experts give them advice for quarantine. Also, they suggest them treatment and say that the treatment will be continued until the transmission of virus stops. By using the stable neutrosophic cubic information, the alternatives are evaluated by the decision maker and the results are presented in the decision matrix.

The decided steps of the TOPSIS method are presented as follows:

Step 1

(a) The decision makers take their analysis of each alternatives based on each criterion and the performance of each alternative H_i with respect to each criterion G_j (Tex translation failed).

$$D = \begin{array}{c c c c c} & G_1 & G_2 & G_3 & G_4 \\ H_1 & \left\{\begin{matrix}[0.1, 0.4],\\ [0.2, 0.6],\\ [0.1, 0.4],\\ (0.2, 0.5, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.3],\\ [0.1, 0.3],\\ [0.1, 0.3],\\ (0.2, 0.2, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.4],\\ [0.1, 0.4],\\ (0.2, 0.2, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.3],\\ [0.1, 0.4],\\ (0.2, 0.2, 0.2)\end{matrix}\right\} \\ H_2 & \left\{\begin{matrix}[0.1, 0.3],\\ [0.1, 0.4],\\ [0.2, 0.5],\\ (0.2, 0.2, 0.3)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.6],\\ [0.1, 0.4],\\ (0.3, 0.4, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.4],\\ [0.1, 0.4],\\ (0.2, 0.2, 0.3)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.3],\\ [0.2, 0.6],\\ [0.1, 0.4],\\ (0.2, 0.4, 0.3)\end{matrix}\right\} \\ H_3 & \left\{\begin{matrix}[0.2, 0.5],\\ [0.2, 0.5],\\ [0.1, 0.4],\\ (0.3, 0.3, 0.3)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.3],\\ [0.2, 0.6],\\ (0.3, 0.2, 0.4)\end{matrix}\right\} & \left\{\begin{matrix}[0.3, 0.6],\\ [0.3, 0.6],\\ [0.1, 0.5],\\ (0.4, 0.4, 0.3)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.4],\\ [0.2, 0.6],\\ (0.2, 0.3, 0.4)\end{matrix}\right\} \\ H_4 & \left\{\begin{matrix}[0.1, 0.4],\\ [0.1, 0.4],\\ [0.1, 0.4],\\ (0.2, 0.2, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.3, 0.6],\\ [0.1, 0.5],\\ (0.3, 0.4, 0.4)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.3],\\ [0.1, 0.5],\\ [0.1, 0.3],\\ (0.2, 0.3, 0.2)\end{matrix}\right\} & \left\{\begin{matrix}[0.1, 0.4],\\ [0.2, 0.4],\\ [0.3, 0.6],\\ (0.3, 0.3, 0.4)\end{matrix}\right\} \end{array} \tag{58}$$

(b) Then, the decision makers present their analysis in the form of a stable neutrosophic cubic set, according to Definitions 6 and 7 and Example 3:

$$D = \begin{array}{c} \\ H_1 \\ \\ H_2 \\ \\ H_3 \\ \\ H_4 \end{array} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \\ \left\{\begin{array}{c}(0.1, 0.2),\\ (0.3, 0.1),\\ (0.1, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.1),\\ (0.1, 0.1),\\ (0.1, 0.1)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.2),\\ (0.1, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.1),\\ (0.1, 0.2)\end{array}\right\} \\ \left\{\begin{array}{c}(0.1, 0.1),\\ (0.1, 0.2),\\ (0.1, 0.3)\end{array}\right\} & \left\{\begin{array}{c}(0.2, 0.1),\\ (0.3, 0.2),\\ (0.1, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.2),\\ (0.2, 0.1)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.1),\\ (0.2, 0.2),\\ (0.2, 0.1)\end{array}\right\} \\ \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.2),\\ (0.2, 0.1)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.1),\\ (0.2, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.2),\\ (0.2, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.2),\\ (0.2, 0.1),\\ (0.2, 0.2)\end{array}\right\} \\ \left\{\begin{array}{c}(0.1, 0.2),\\ (0.1, 0.2),\\ (0.1, 0.2)\end{array}\right\} & \left\{\begin{array}{c}(0.2, 0.1),\\ (0.1, 0.2),\\ (0.3, 0.1)\end{array}\right\} & \left\{\begin{array}{c}(0.1, 0.1),\\ (0.2, 0.2),\\ (0.1, 0.1)\end{array}\right\} & \left\{\begin{array}{c}(0.2, 0.1),\\ (0.1, 0.1),\\ (0.1, 0.2)\end{array}\right\} \end{array}. \quad (59)$$

Step 2. The normalized decision matrix is

$$\begin{array}{c} \\ H_1 \\ \\ H_2 \\ \\ H_3 \\ \\ H_4 \end{array} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \\ \left\{\begin{array}{c}(0.25, 0.29),\\ (0.5, 0.143),\\ (0.2, 0.25)\end{array}\right\} & \left\{\begin{array}{c}(0.17, 0.1),\\ (0.17, 0.17),\\ (0.143, 0.17)\end{array}\right\} & \left\{\begin{array}{c}(0.25, 0.29),\\ (0.2, 0.25),\\ (0.17, 0.33)\end{array}\right\} & \left\{\begin{array}{c}(0.20, 0.33),\\ (0.17, 0.20),\\ (0.17, 0.29)\end{array}\right\} \\ \left\{\begin{array}{c}(0.25, 0.143),\\ (0.17, 0.29),\\ (0.2, 0.38)\end{array}\right\} & \left\{\begin{array}{c}(0.33, 0.1),\\ (0.5, 0.33),\\ (0.143, 0.33)\end{array}\right\} & \left\{\begin{array}{c}(0.25, 0.29),\\ (0.2, 0.25),\\ (0.33, 0.17)\end{array}\right\} & \left\{\begin{array}{c}(0.20, 0.17),\\ (0.33, 0.4),\\ (0.33, 0.143)\end{array}\right\} \\ \left\{\begin{array}{c}(0.25, 0.29),\\ (0.17, 0.29),\\ (0.5, 0.125)\end{array}\right\} & \left\{\begin{array}{c}(0.17, 0.2),\\ (0.17, 0.17),\\ (0.29, 0.33)\end{array}\right\} & \left\{\begin{array}{c}(0.25, 0.29),\\ (0.2, 0.25),\\ (0.33, 0.33)\end{array}\right\} & \left\{\begin{array}{c}(0.20, 0.33),\\ (0.33, 0.20),\\ (0.33, 0.29)\end{array}\right\} \\ \left\{\begin{array}{c}(0.25, 0.29),\\ (0.17, 0.29),\\ (0.2, 0.25)\end{array}\right\} & \left\{\begin{array}{c}(0.33, 0.1),\\ (0.17, 0.33),\\ (0.43, 0.17)\end{array}\right\} & \left\{\begin{array}{c}(0.25, 0.143),\\ (0.4, 0.25),\\ (0.17, 0.17)\end{array}\right\} & \left\{\begin{array}{c}(0.40, 0.17),\\ (0.17, 0.20),\\ (0.17, 0.29)\end{array}\right\} \end{array}. \quad (60)$$

Step 3. The weighted normalized decision matrix where $w = (0.3, 0.1, 0.2, 0.4)$ is

$$\begin{array}{ccccc} & G_1 & G_2 & G_3 & G_4 \\ H_1 & \left\{\begin{array}{c}(0.075, 0.087),\\ (0.15, 0.043),\\ (0.06, 0.075)\end{array}\right\} & \left\{\begin{array}{c}(0.017, 0.01),\\ (0.017, 0.017),\\ (0.0143, 0.017)\end{array}\right\} & \left\{\begin{array}{c}(0.05, 0.06),\\ (0.04, 0.05),\\ (0.034, 0.066)\end{array}\right\} & \left\{\begin{array}{c}(0.08, 0.132),\\ (0.07, 0.08),\\ (0.07, 0.12)\end{array}\right\} \\ H_2 & \left\{\begin{array}{c}(0.075, 0.043),\\ (0.051, 0.087),\\ (0.06, 0.114)\end{array}\right\} & \left\{\begin{array}{c}(0.033, 0.01),\\ (0.05, 0.033),\\ (0.0143, 0.033)\end{array}\right\} & \left\{\begin{array}{c}(0.05, 0.06),\\ (0.04, 0.05),\\ (0.066, 0.034)\end{array}\right\} & \left\{\begin{array}{c}(0.08, 0.07),\\ (0.132, 0.16),\\ (0.132, 0.06)\end{array}\right\}. \\ H_3 & \left\{\begin{array}{c}(0.075, 0.087),\\ (0.051, 0.087),\\ (0.15, 0.038)\end{array}\right\} & \left\{\begin{array}{c}(0.017, 0.02),\\ (0.017, 0.017),\\ (0.029, 0.033)\end{array}\right\} & \left\{\begin{array}{c}(0.05, 0.06),\\ (0.04, 0.05),\\ (0.066, 0.066)\end{array}\right\} & \left\{\begin{array}{c}(0.08, 0.132),\\ (0.132, 0.08),\\ (0.132, 0.12)\end{array}\right\} \\ H_4 & \left\{\begin{array}{c}(0.075, 0.087),\\ (0.051, 0.087),\\ (0.06, 0.075)\end{array}\right\} & \left\{\begin{array}{c}(0.033, 0.01),\\ (0.017, 0.033),\\ (0.043, 0.017)\end{array}\right\} & \left\{\begin{array}{c}(0.05, 0.143),\\ (0.08, 0.05),\\ (0.034, 0.034)\end{array}\right\} & \left\{\begin{array}{c}(0.40, 0.07),\\ (0.07, 0.08),\\ (0.07, 0.12)\end{array}\right\} \end{array} \tag{61}$$

Step 4. Positive and negative ideal solution: the positive ideal solution $A^* = (a_1, a_2, a_3, a_4)$ contains the greatest numbers of the first, second, and third column and smallest numbers of the fourth column. The negative ideal solution $A\prime = (a_1', a_2', a_3', a_4')$ contains the smallest numbers of the first, second, and third column and greatest numbers of the fourth column.

$$\begin{array}{l} A^* \left\{\begin{array}{c}(0.075, 0.087),\\ (0.15, 0.087),\\ (0.15, 0.114)\end{array}\right\} \left\{\begin{array}{c}(0.033, 0.02),\\ (0.05, 0.033),\\ (0.029, 0.033)\end{array}\right\} \left\{\begin{array}{c}(0.05, 0.143),\\ (0.08, 0.05),\\ (0.066, 0.066)\end{array}\right\} \left\{\begin{array}{c}(0.08, 0.07),\\ (0.07, 0.07),\\ (0.07, 0.06)\end{array}\right\}, \\ A' \left\{\begin{array}{c}(0.075, 0.043),\\ (0.051, 0.043),\\ (0.06, 0.038)\end{array}\right\} \left\{\begin{array}{c}(0.017, 0.01),\\ (0.017, 0.017),\\ (0.0143, 0.017)\end{array}\right\} \left\{\begin{array}{c}(0.06, 0.05),\\ (0.04, 0.05),\\ (0.034, 0.034)\end{array}\right\} \left\{\begin{array}{c}(0.40, 0.132),\\ (0.132, 0.16),\\ (0.132, 0.12)\end{array}\right\}. \end{array} \tag{62}$$

Step 5. Separation measures for the positive and negative ideal solution are

$$\begin{aligned} a_1^* &= 0.3694, \\ a_2^* &= 0.2133, \\ a_3^* &= 0.0409, \\ a_4^* &= 0.1292, \\ a_1' &= 0.1308, \\ a_2' &= 0.1206, \\ a_3' &= 0.1236, \\ a_4' &= 0.0349. \end{aligned} \tag{63}$$

Step 6. Ranking order of the alternatives is shown by (Figures 1–4). Ranking of COVID-19 is obtained by completing the TOPSIS calculation.

$$\begin{aligned} H_1 &= 0.2615, \\ H_2 &= 0.3612, \\ H_3 &= 0.7514, \\ H_4 &= 0.2127, \\ H_3 &> H_2 > H_1 > H_4. \end{aligned} \tag{64}$$

Thus, we concluded that $H_3$is the most effected province of Pakistan till April 12, 2020. Here, we used stable neutrosophic cubic sets, but we may use other versions of stable neutrosophic cubic sets.

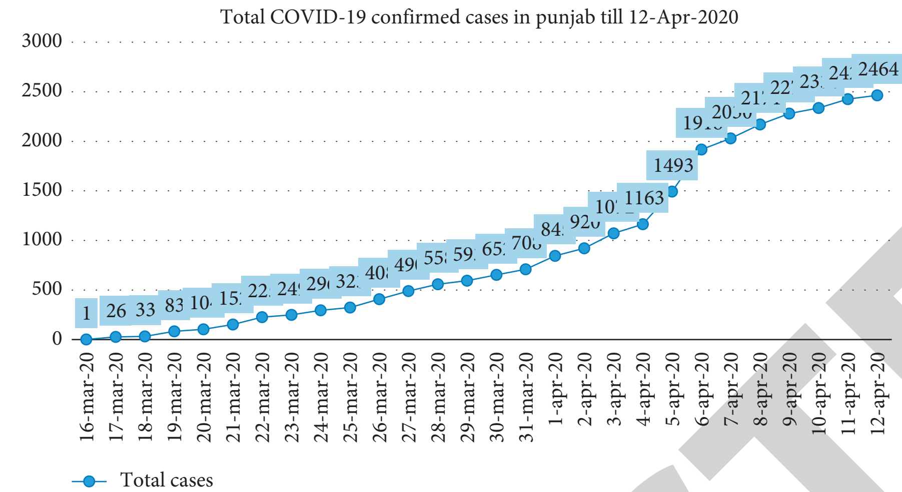

Figure 1: Total COVID-19 confirmed cases in Punjab till 12 Apr 2020.

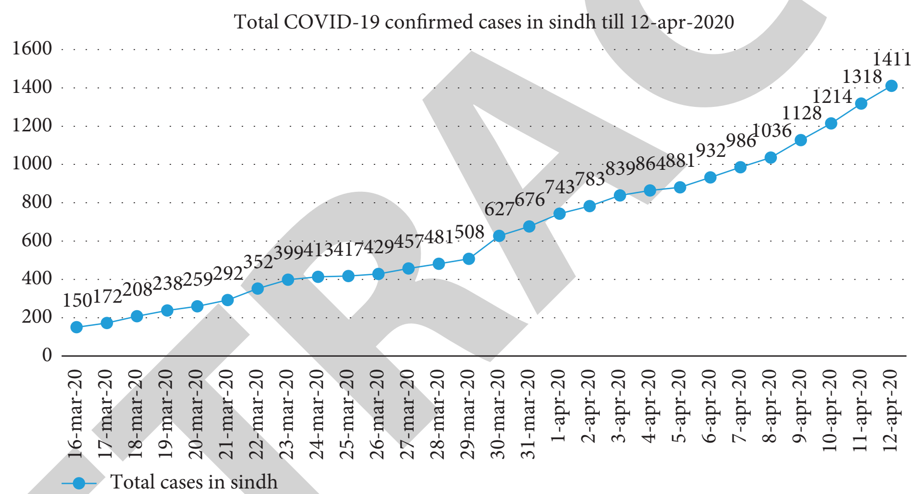

Figure 2: Total COVID-19 confirmed cases in Sindh till 12 Apr 2020.

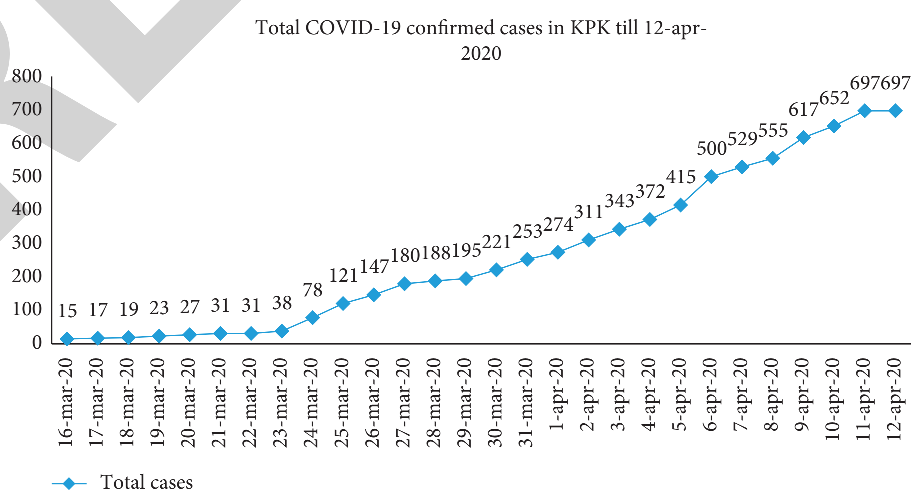

Figure 3: Total COVID-19 confirmed cases in KPK till 12 Apr 2020.

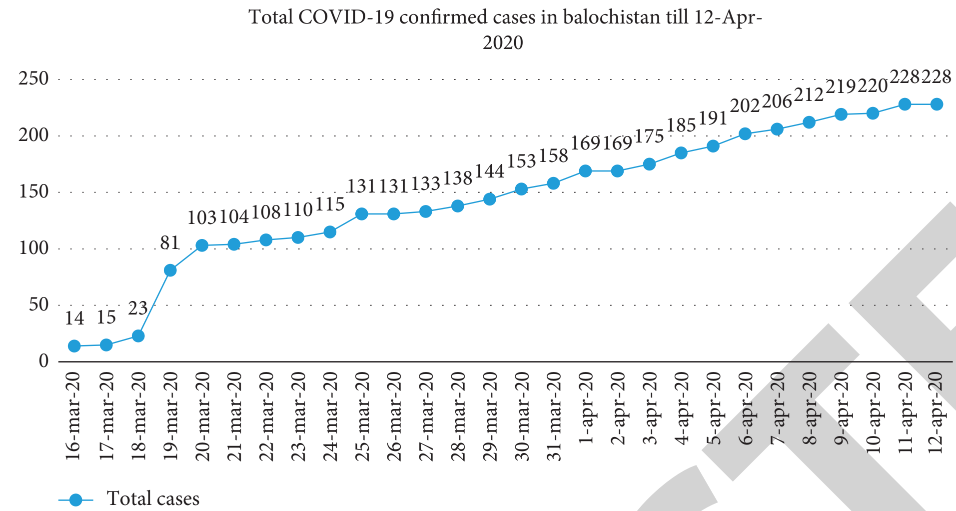

FIGURE 4: Total COVID-19 confirmed cases in Balochistan till 12 Apr 2020.

6. Conclusions

In this article, we work out with the idea of stable NCSs and internal and external stable NCSs. Also, we define their union, intersection, and complement with examples. After that, we demonstrate the application of the TOPSIS method to find out the ranking of COVID-19. For this purpose, we used a numerical example to find out the most affected area. We reached at the following key points:

Every stable NCS $\beta = \langle T_\beta, I_\beta, F_\beta, t_\beta, i_\beta, f_\beta \rangle$ in U is an almost-stable NCS, which is, of course, an NCS which turns into a cubic set with three different parts as truth, indeterminacy, and falsity, but the converse of this chain is not true always.

If we have an external NCS which is unstable such that

$$t(\mathring{u}) > [T^-(\mathring{u}), T^+(\mathring{u})], i(\mathring{u}) > [I^-(\mathring{u}), I^+(\mathring{u})], f(\mathring{u}) > [F^-(\mathring{u}), F^+(\mathring{u})], \quad (65)$$

then its right evaluative point becomes a neutrosophic bipolar fuzzy set.

If we have an external NCS which is unstable such that

$$t(\mathring{u}) < [T^-(\mathring{u}), T^+(\mathring{u})], i(\mathring{u}) < [I^-(\mathring{u}), I^+(\mathring{u})], f(\mathring{u}) < [F^-(\mathring{u}), F^+(\mathring{u})], \quad (66)$$

then its left evaluative point becomes a neutrosophic bipolar fuzzy set.

We used the idea of stable neutrosophic cubic sets in the application section, so results are within the range; otherwise, we may have results which lie outside the domain of neutrosophic cubic sets. This is the main advantage of stable neutrosophic cubic sets.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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Research Article

Linguistic Interval-Valued Intuitionistic Fuzzy Copula Heronian Mean Operators for Multiattribute Group Decision-Making

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As a generalization of the intuitionistic fuzzy number (IFN), the linguistic interval-valued intuitionistic fuzzy number (LIVIFN) is a flexible and superior tool to describe complex fuzzy uncertainty information. Heronian mean (HM) operator has the characteristic of considering the relationship between attributes. Extended copulas (ECs) and extended cocopulas (ECCs) are the promotion form of Archimedean *t*-norm and *t*-conorm (ATT). ECs and ECCs can generate versatile operational rules and can provide more choice for decision makers (DMs). Therefore, it is very necessary to take advantages of them. In this paper, ECs and ECCs, some specifics of ECs and ECCs, and score and accuracy functions of IVILFNs are gained first. Then, we propose the linguistic interval-valued intuitionistic fuzzy weighted copula Heronian mean (LIVIFWCHM) operator; also, some properties and five specific expressions of the LIVIFWCHM operator are discussed. Moreover, we also propose a new MAGDM approach based on the proposed LIVIFWCHM operator. Finally, a set of examples are used to demonstrate the effectiveness, generality, and flexibility of the proposed method.

1. Introduction

Decision-making problems (DMPs) exist in all aspects of people's life, ranging from the development of national politics, economy, and culture to the decision-making at the enterprise level. The multiattribute decision-making (MADM) problem is based on the analysis of multiple influencing factors or indicators in the decision-making. It is necessary to judge or evaluate the limited scheme set from multiple attributes, give the corresponding evaluation information or preference information, and then rank the limited alternatives. With the rapid development of society and the increasingly complex social environment, it is difficult for a decision maker to consider all aspects of the problem, so multiattribute group decision-making (MAGDM) came into being. MAGDM combines the characteristics of MADM and group decision-making. It mainly refers to that many members of the group evaluate several fixed attributes of several alternatives, give the order of alternatives, and obtain the best scheme.

In the classic MADM problems, decision makers (DMs) usually use accurate values to evaluate the attributes of alternatives. However, in the practical application process, due to the complexity and fuzziness of the problem, the evaluation values given by DMs are usually not accurate values but in the form of linguistic values, such as "good," "general," or "poor." Therefore, linguistic-based multiattribute decision-making has become the hot research content of MAGDM [1].

Since Zadeh put forward the concept of the linguistic variable in 1975 [2], the combination of the linguistic variable and other theories has been put forward constantly [3–5], such as linguistic hesitant fuzzy set (LHFS) [6, 7], linguistic neutrosophic set [8, 9], linguistic intuitionistic fuzzy set (LIFS) [10], and linguistic Pythagorean fuzzy set [11]. As an important extended linguistic fuzzy set, LIFS has drawn much more attention. Chen et al. [10] first introduced the concept of the LIFS by combining the linguistic term set (LTS) and the IFS in which the membership degree (MD) and nonmembership degree (NMD) are expressed by the

LTS. A LIFS A on a finite universe of discourse Z and CLTS $\overline{\mathcal{S}} = \{s_\alpha | s_0 \leq s_\alpha \leq s_t, \alpha \in [0, t]\}$ can be denoted as $A = \{(s_\alpha(z), s_\beta(z)) | z \in Z\}$, where $s_\alpha(z), s_\beta(z) \in \overline{\mathcal{S}}$ with the condition $\alpha + \beta \leq t$.

In order to better express fuzziness, Garg and Kumar [12] introduced the linguistic interval-valued IFS (LIVIFS). LIVIFS is a more general form of the LIFS. For example, selecting a suitable house from a certain number of alternatives is a typical decision problem. In this problem, factors affecting the house selection such as the direction of the house, unit price of area, area size, community environment, and credibility of the developer may be described by linguistic variables (LVs) such as "poor," "medium," and "good." Under this condition, IFS and IVIFS are no longer suitable. Furthermore, in the LIVIFS, the MD and NMD are expressed by the interval LVs (ILVs). When the upper and lower bounds of ILVs are equal, the LIVIFS degenerates to the LIFS. Based on the LIVIFS, scholars have proposed some aggregation operators (AOs), such as a prioritised weighted averaging operator by Kumar and Garg [13], a weighted Maclaurin symmetric mean operator (LIVIFWMSM) by Liu and Qin [14], and a weighted aggregation operator (LIVIFAWPMM) by Qin [15].

There are two limitations among these AOs. Firstly, the aforementioned AOs are only obtained by special t-norms (TNs) and t-conorms (TCs), i.e., algebraic TN and algebraic TC. Secondly, it is assumed that there is no relationship between attributes. In order to address some of the aforementioned limitations, some improved linguistic interval-valued intuitionistic fuzzy AOs are developed to solve some DMPs. For example, Xu et al. proposed the interval-valued intuitionistic fuzzy (IVIF) weighted power Muirhead mean (IVIFWPMM) operator [16]; a LIVIF-based Archimedean power Muirhead mean operator by Qin [15], which can tackle DMPs more generally and flexibly; entropic combined weighted averaging operator by Xian et al. [17]; LIVIFS-based Maclaurin symmetric mean (MSM) operator by Liu and Qin [14]; interval-valued intuitionistic 2-tuple linguistic setting and Bonferroni mean by Du and Yuan [18]; Pythagorean fuzzy linguistic (dual) Muirhead mean (PFLMM) operators and their weighted form by Liu [19]; Dombi Heronian mean operators by Wu et al. [20]; and Hamy mean operators by Wu et al. [21]. Besides, some DMP approaches have been built [22–32].

For the AOs above, although they can do well in their specific circumstances, only a small part of them can provide desirable generality and flexibility while taking into account the correlation between attributes. In [33], Liu and Chen proposed generalized AOs for the intuitionistic 2-tuple linguistic information with three kinds of additive generator. In [34], Tan and Chen proposed generalized AOs for the Archimedean intuitionistic fuzzy information with five kinds of additive generator.

Among various kinds of TNs and TCs, copulas and cocopulas are classical examples of TNs and TCs. Copula [35] can not only reflect the dependence among variables but also prevent information losing in the aggregation process. Copula is a method to deal with the correlation of random variables in statistics. The basic idea of the copula function is to simplify the problem by transforming the marginal variable into a uniformly distributed variable without looking at many different marginal distributions and then define the correlation as a joint distribution on the uniform distribution. As a tool for describing the dependence mechanism between variables, the copula function contains almost all the dependence information of random variables, especially when it is impossible to determine whether the traditional linear correlation coefficient can correctly measure the correlation between variables.

There are two distinguishing features of copulas: (1) copulas and cocopulas are flexible because DMs can select different types of copulas to define the operations under the fuzzy environment, and the results obtained from these operations are close; (2) copula function is flexible to capture the correlations among attributes in DMPs. Based on the two obvious characteristics, copulas have been applied to some DMPs. For example, Nelsen [35] applied copulas in the aggregation function. Tao et al. [36] extended copulas to the IFS and applied it to DMPs. Basically, there are two types of copula: Archimedean copula and Gaussian copula. In this paper, we only discuss Archimedean copula. In the light of Archimedean copula, Tao et al. [37] studied a new computational model for unbalanced LVs. Chen et al. [38] defined new AOs in the linguistic neutrosophic set based on the copula and applied them to solve DMPs. Xu et al. [39] also proposed fuzzy copula power AOs to solve MAGDM problems based on linguistic interval-valued intuitionistic information; however, the authors did not consider the correlation between attributes.

In order to solve the relationship between attributes, Bonferroni [40] proposed the Bonferroni mean (BM) operator firstly, and then Yager [41] further expanded the BM operator and enhanced its modeling capabilities. By replacing simple average operators (ordered weighted average operators and Choquet operators) with other forms of average operators, Yager proposed some more efficient AOs. However, the BM operator ignores the relationship between each attribute and itself and with the nature of computation redundancy. Heronian mean (HM) operator was first proposed by Beliakov [42]. Heronian mean (HM) operator and BM operator have similar structures, and both consider the correlation between attribute values. However, the HM operator has obvious advantages over the BM operator and can make up for the two shortcomings of the BM operator. On this basis, a series of extension models have been proposed, such as the intuitionistic fuzzy geometric HM (IFGHM) operator [43], IVIF Heronian mean (IVIFHM) operator [44], uncertain linguistic HM operators [45], partitioned HM operators [46], unbalanced linguistic generalized HM operator [32], normal intuitionistic fuzzy HM operator [47], and picture fuzzy Dombi HM operator [48]. However, the HM is not applied to aggregate the linguistic interval-valued intuitionistic fuzzy information (LIVIFI).

Although the existing AOs can provide the most commonly used way to aggregate the LIVIFS, they lack a unique way in practical applications. What is it the form of AOs on the basis of the copula function and LIVIFI? What are the differences between copula-based AOs and existing AOs?

Considering the HM operator has the ability to interrelate among the attributes, what is the form of the weighted HM operator based on the LIVIFS and copula function? So, the goal and motivation of the present work are to synthesize ECs (ECCs), HM operator, and LIVIFS and to develop a MAGDM approach with LIVIFI.

Accordingly, the main intentions and contributions of this work are summarized as follows:

(1) We propose a new version of copulas and cocopulas by extending the domain and the range of copulas and cocopulas from [0, 1] to $[0,t]\,(t>0)$, which is called extended copulas (ECs) and extended cocopulas (ECCs)
(2) We introduce several universal operational laws of LIVIFNs and discuss some special instances
(3) We develop the LIVIFWCHM operator, explore several characteristics, and give some particular cases
(4) In addition, we propose a novel decision approach for MAGDM with LIVIFI and investigate the efficacy and superiorities of the propounded approach

In order to achieve the above goals, the organizational structure of this paper is as follows. In Section 2, some basic concepts of the LIVIFS, copulas, and cocopulas and some properties of the LIVIFS based on ECs and ECCs are introduced. Furthermore, we redefine several novel operations for LIVIFNs and discuss some special cases. In Section 3, based on these operation rules, we derive the LIVIFWCHM operator as well as explore several properties and particular examples. In Section 4, a new method for MAGDM is proposed based on the LIVIFWCHM operator under LIVIFI. In Section 5, a set of examples are provided to investigate the efficacy and superiority of the propounded approach. The conclusion is obtained in Section 6.

2. Preliminaries

In this section, firstly, some basic concepts related to the LIVIFS, HM operator, and copulas and cocopulas are reviewed, which are the basis of the present work.

Definition 1 (see [12]). Let X be a finite universal set and $S_{[0,t]}$ be a continuous LTS. A LIVIFS A in X is defined as

$$A=\left(x,s_{\mu_A(x)},s_{\nu_A(x)}\right),\quad |x\in X, \tag{1}$$

where $s_{\mu A(x)}=[s_{\mu_A^L(x)},s_{\mu_A^U(x)}]$ and $s_{\nu A(x)}=[s_{\nu_A^L(x)},s_{\nu_A^U(x)}]$ are all subsets of $[s_0,s_t]$ and represent linguistic MD and NMD of x to A, respectively. For any $x\in X$, $s_{\mu_A^U(x)}+s_{\nu_A^U(x)}\le s_t$. The pair $([s_{\mu_A^L},s_{\mu_A^U}],[s_{\nu_A^L},s_{\nu_A^U}])$ is called the LIVIFN.

For convenience, we denote the LIVAIFN as $\alpha=([s_a,s_b],[s_c,s_d])$, where $s_a,s_b,s_c,s_d\in S_{[0,t]}$, and also, $[s_a,s_b]\in[s_0,s_t]$, $[s_c,s_d]\in[s_0,s_t]$, $b+d\le t$.

Definition 2 (see [12]). Let $\alpha=([s_a,s_b],[s_c,s_d])$ be a LIVAIFN; a score function and accuracy function of α are defined as

$$S(\alpha)=s_{(2t+a-c+b-d)/4}, \tag{2}$$

$$H(\alpha)=s_{(a+b+c+d)/2}. \tag{3}$$

Then, for any two different LIVAIFNs α_1 and α_2, we have the following:

(1) If $S(\alpha_1)<S(\alpha_2)$, then $\alpha_1<\alpha_2$
(2) If $S(\alpha_1)=S(\alpha_2)$ and $H(\alpha_1)=H(\alpha_2)$, then $\alpha_1<\alpha_2$

Definition 3 (see [33]). An extended t-norm $\mathcal{T}$ is a mapping from $[0,t]^2$ to $[0,t]$ if $\mathcal{T}$ fulfills the following: for all $c,d,e\in[0,t]$,

(i) $\mathcal{T}(c,t)=c$.
(ii) $\mathcal{T}(c,d)=\mathcal{T}(d,c)$.
(iii) $\mathcal{T}(c,\mathcal{T}(d,e))=\mathcal{T}(\mathcal{T}(c,d),e)$.

If T just satisfies (T1), then T is called a semicopula. With the help of extended TNs and extended TCs, we first introduce the concept of extended copulas (ECs) and extended cocopulas (ECCs) in order to handle some DMPs with LIFI.

Definition 4 (see [35]). A binary function $\mathbb{C}$: $[0,t]^2\longrightarrow[0,t]$ is called an EC if $\mathbb{C}$ fulfills the following conditions: for all $c,d,c',d'\in[0,t]$,

(i) $\mathbb{C}(c,d)+\mathbb{C}(c',d')\ge\mathbb{C}(c,d')+\mathbb{C}(c',d)$.
(ii) $\mathbb{C}(c,0)=\mathbb{C}(0,c)=0$.
(iii) $\mathbb{C}(c,t)=\mathbb{C}(t,c)=c$.

Definition 5 (see [35]). Let $\rangle$: $[0,t]\longrightarrow[0,+\infty)$ and ψ: $[0,+\infty)\longrightarrow[0,t]$. If ϱ,ψ satisfy the following conditions, for all $(c,d)\in[0,t]^2$,

(1) ϱ is continuous.
(2) ϱ is strictly decreasing.
(3) $\varrho(t)=0$.
(4)

$$\psi(c)=\begin{cases}\varrho^{-1}(c), & c\in[0,\varrho(0)],\\ 0, & c\in[\varrho(0),+\infty),\end{cases} \tag{4}$$
$$\mathbb{C}(c,d)=\psi(\varrho(c)+\varrho(d)),$$

the copula $\mathbb{C}$ is called ECs.

The generator ϱ of an EC is if a mapping from $[0,t]$ to $\mathbf{R}^+$ and ϱ^{-1} is the mapping from $\mathbf{R}^+$ to $[0,t]$ with $\varrho(0)=+\infty$ and $\varrho(t)=0$. According to Genest and Mackay [49], $\mathbb{C}$ can be rewritten as

$$\mathbb{C}(c,d)=\varrho^{-1}(\varrho(c)+\varrho(d)). \tag{5}$$

Definition 6. Let $\mathbb{C}$ be an EC, for all $(c,d)\in[0,t]^2$; then, ECCs are expressed as

$$\mathbb{C}^*(c,d)=t-\mathbb{C}(t-c,t-d). \tag{6}$$

Theorem 1. *For all $c_1, c_2, d_1, d_2 \in [0, t]$, if $c_i + d_i \leq t\,(i = 1, 2)$, then $0 \leq \mathbb{C}(c_1, c_2) + \mathbb{C}^*(d_1, d_2) \leq t$.*

Proof. It follows easily from the definitions of EC and ECC that $0 \leq \mathbb{C}(c_1, c_2) + \mathbb{C}^*(d_1, d_2)$. So, we just need to prove $\mathbb{C}(c_1, c_2) + \mathbb{C}^*(d_1, d_2) \leq t$.

It follows from the definitions of EC and ECC that

$$\begin{aligned}\mathbb{C}(c_1, c_2) + \mathbb{C}^*(d_1, d_2) &= \mathbb{C}(c_1, c_2) + (t - \mathbb{C}(t - d_1, t - d_2)) \\ &= \left(\varrho^{-1}(\varrho(c_1) + \varrho(c_2))\right) \\ &\quad + t - \left(\varrho^{-1}(\varrho(t - d_1) + \varrho(1 - d_2))\right).\end{aligned} \tag{7}$$

As ϱ is strictly decreasing and $c_i + d_i \leq t\,(i = 1, 2)$, it follows that

$$\varrho(c_1) + \varrho(c_2) \geq \varrho(t - d_1) + \varrho(t - d_1). \tag{8}$$

Therefore,

$$\varrho^{-1}(\varrho(c_1) + \varrho(c_2)) \leq \varrho^{-1}(\varrho(t - d_1) + \varrho(t - d_1)). \tag{9}$$

So, we have

$$\begin{aligned}\mathbb{C}(c_1, c_2) + \mathbb{C}^*(d_1, d_2) &= \left(\varrho^{-1}(\varrho(c_1) + \varrho(c_2))\right) \\ &\quad + t - \varrho^{-1}(\varrho(t - d_1) + \varrho(t - d_2)) \\ &\leq \left(\varrho^{-1}(\varrho(c_1) + \varrho(c_2))\right) \\ &\quad + t - \left(\varrho^{-1}(\varrho(c_1) + \varrho(c_2))\right) = t.\end{aligned} \tag{10}$$

According to Theorem 1, we know that the operation of ECs and ECCs is close. Table 1 shows five common Archimedean copulas, which can be considered for further consideration. In the following, we will give a new version of operational rules based on ECs and ECCs. □

Definition 7. Let $\alpha_1 = ([s_{a_1}, s_{b_1}], [s_{c_1}, s_{d_1}])$ and $\alpha_2 = ([s_{a_2}, s_{b_2}], [s_{c_2}, s_{d_2}])$ be two LIVIFNs; the novel operational rules of LIVIFNs are given as follows:

$$\begin{aligned}&(L1)\,\alpha_1 \oplus_{\mathbb{C}} \alpha_2 = \left(\left[s_{t-(\varrho^{-1}(\varrho(t-a_1)+\varrho(t-a_2)))}, s_{t-(\varrho^{-1}(\varrho(t-b_1)+\varrho(t-b_2)))}\right], \left[s_{\varrho^{-1}(\varrho(c_1)+\varrho(c_2))}, s_{\varrho^{-1}(\varrho(d_1)+\varrho(d_2))}\right]\right), \\ &(L2)\,\alpha_1 \otimes_{\mathbb{C}} \alpha_2 = \left(\left[s_{\varrho^{-1}(\varrho(a_1)+\varrho(a_2))}, s_{\varrho^{-1}(\varrho(b_1)+\varrho(b_2))}\right], \left[s_{t-(\varrho^{-1}(\varrho(t-c_1)+\varrho(t-c_2)))}, s_{t-(\varrho^{-1}(\varrho(t-d_1)+\varrho(t-d_2)))}\right]\right).\end{aligned} \tag{11}$$

It is easy to verify that $\oplus_{\mathbb{C}}$ and $\otimes_{\mathbb{C}}$ satisfy the associative law, that is, for all three LVIIFNs A, B, and C,

$$\begin{aligned}&(1)\,(A \oplus_{\mathbb{C}} B) \oplus_{\mathbb{C}} C = A \oplus_{\mathbb{C}} (B \oplus_{\mathbb{C}} C), \\ &(2)\,(A \otimes_{\mathbb{C}} B) \otimes_{\mathbb{C}} C = A \otimes_{\mathbb{C}} (B \otimes_{\mathbb{C}} C).\end{aligned} \tag{12}$$

Theorem 2. *Let $\alpha = ([s_a, s_b], [s_c, s_d])$ be a LIVIFN; for $n \in \mathbf{N}^*$, we have $n\alpha$ is still a LIVAIFN, and*

$$\begin{aligned}n\alpha = &\left(\left[s_{t-(\varrho^{-1}(n\varrho(t-a)))}, s_{t-(\varrho^{-1}(n\varrho(t-b)))}\right],\right. \\ &\left.\left[s_{\varrho^{-1}(n\varrho(c))}, s_{\varrho^{-1}(n\varrho(d))}\right]\right),\end{aligned} \tag{13}$$

where $n\alpha = \overbrace{\alpha \oplus_{\mathbb{C}} \alpha \oplus_{\mathbb{C}} \cdots \oplus_{\mathbb{C}} \alpha}^{n}$.

Proof. It is easy to obtain from Theorem 1 that $n\alpha$ is a LIVAIFN. Now, we only prove that equation (13) holds for $n \in \mathbf{N}^*$. When $n = 1$,

$$\begin{aligned}1\alpha &= \left(\left[s_{t-(\varrho^{-1}(\varrho(t-a)))}, s_{t-(\varrho^{-1}(\varrho(t-b)))}\right], \left[s_{\varrho^{-1}(\varrho(c))}, s_{\varrho^{-1}(\varrho(d))}\right]\right) \\ &= ([s_a, s_b], [s_c, s_d]) = \alpha.\end{aligned} \tag{14}$$

Presume equation (13) holds for $n = k$, i. e.,

$$k\alpha = \left(\left[s_{t-(\varrho^{-1}(k\varrho(t-a)))}, s_{t-(\varrho^{-1}(k\varrho(t-b)))}\right], \left[s_{\varrho^{-1}(k\varrho(c))}, s_{\varrho^{-1}(k\varrho(d))}\right]\right). \tag{15}$$

When $n = k + 1$, we have

$$\begin{aligned}(k+1)\alpha &= k\alpha \oplus_{\mathbb{C}} \alpha \\ &= \left(\left[s_{t-(\varrho^{-1}(k\varrho(t-a)))}, s_{t-(\varrho^{-1}(k\varrho(t-b)))}\right], \left[s_{\varrho^{-1}(k\varrho(c))}, s_{\varrho^{-1}(k\varrho(d))}\right]\right) \oplus_{\mathbb{C}} ([s_a, s_b], [s_c, s_d]) \\ &= \left(\left[s_{t-(\varrho^{-1}(\varrho(t-(t-(\varrho^{-1}(k\varrho(t-a)))))+\varrho(t-a))}, s_{t-(\varrho^{-1}(\varrho(t-(t-(\varrho^{-1}(k\varrho(t-b)))))+\varrho(t-b))}\right], \left[s_{\varrho^{-1}(k\varrho(c)+\varrho(c))}, s_{\varrho^{-1}(k\varrho(d)+\varrho(d))}\right]\right) \\ &= \left(\left[s_{t-(\varrho^{-1}((k\varrho(t-a))+\varrho(t-a)))}, s_{t-(\varrho^{-1}((k\varrho(t-b))+\varrho(t-b)))}\right], \left[s_{\varrho^{-1}(k\varrho(c)+\varrho(c))}, s_{\varrho^{-1}(k\varrho(d)+\varrho(d))}\right]\right) \\ &= \left(\left[s_{t-(\varrho^{-1}((k+1)\varrho(t-a)))}, s_{t-(\varrho^{-1}((k+1)\varrho(t-b)))}\right], \left[s_{\varrho^{-1}((k+1)\varrho(c))}, s_{\varrho^{-1}((k+1)\varrho(d))}\right]\right).\end{aligned} \tag{16}$$

Table 1: The influence of parameter θ on the rank of alternatives.

Type	Generator $\varrho(c)$	EC and ECC	Condition
Gumbel	$\varrho(c) = (-\ln(c/t))^{\theta}$	$\mathbb{C}(c,d) = \text{te}^{-((-\ln(c/t))^{\theta}+(-\ln(d/t))^{\theta})^{(1/\theta)}}$ $\mathbb{C}^{*}(c,d) = t - \text{te}^{-((-\ln(t-c/t))^{\theta}+(-\ln(t-d/t))^{\theta})^{(1/\theta)}}$	$\theta \geq 1$
Clayton	$\varrho(c) = (c/t)^{-\theta} - 1$	$\mathbb{C}(c,d) = t((c/t)^{-\theta} + (d/t)^{-\theta} - 1)^{-(1/\theta)}$ $\mathbb{C}^{*}(c,d) = t - t((t-c/t)^{-\theta} + (t-d/t)^{-\theta} - 1)^{-(1/\theta)}$	$\theta \neq 0$
Frank	$\varrho(c) = \ln(e^{-(\theta c/t)} - 1/e^{-\theta} - 1)$	$\mathbb{C}(c,d) = (-(t/\theta))\ln[((e^{-(\theta c/t)} - 1)(e^{-(\theta d/t)} - 1)/e^{-\theta} - 1) + 1]$ $\mathbb{C}^{*}(c,d) = t + (t/\theta)\ln[((e^{-(\theta(t-c)/t)} - 1)(e^{-(\theta(t-d)/t)} - 1)/e^{-\theta} - 1) + 1]$	$\theta \neq 0$
Ali-Mikhail-Haq	$\varrho(c) = \ln(t - \theta(t-c)/c)$	$\mathbb{C}(c,d) = (\text{tcd}/t^2 - \theta(t-c)(t-d))$ $\mathbb{C}^{*}(c,d) = t - (t(t-c)(t-d)/t^2 - \theta c d)$	$\theta \in [-1,1)$
Joe	$\varrho(c) = -\ln(1 - (1 - (c/t))^{\theta})$	$\mathbb{C}(c,d) = t - ((t^{\theta}((t-c)^{\theta} + (t-d)^{\theta}) - (t-c)^{\theta}(t-d)^{\theta})^{(1/\theta)}/t)$ $\mathbb{C}^{*}(c,d) = t(c^{\theta} + d^{\theta} - (\text{cd}/t)^{\theta})^{(1/\theta)}$	$\theta \geq 1$

So, equation (13) holds for all $n \in \mathbf{N}^{*}$.

Similarly, the following theorem can be obtained easily. □

Theorem 3. *Let $\alpha = ([s_a, s_b], [s_c, s_d])$ be a LIVIFN; for all $n \in \mathbf{N}^{*}$, we have α^{n} is still a LIVIFN, and*

$$
\begin{aligned}
(L3)\lambda\alpha &= \left(\left[s_{t-(\varrho^{-1}(\lambda\varrho(t-a)))}, s_{t-(\varrho^{-1}(\lambda\varrho(t-b)))}\right], \left[s_{\varrho^{-1}(\lambda\varrho(c))}, s_{\varrho^{-1}(\lambda\varrho(d))}\right]\right),\\
(L4)\alpha^{\lambda} &= \left(\left[s_{\varrho^{-1}(\lambda\varrho(a))}, s_{\varrho^{-1}(\lambda\varrho(b))}\right], \left[s_{t-(\varrho^{-1}(\lambda\varrho(t-c)))}, s_{t-(\varrho^{-1}(\lambda\varrho(t-d)))}\right]\right).
\end{aligned}
\tag{17}
$$

It is easy to verify that the operational laws hold, for all three LIVIFNs $\alpha, \alpha_1, \alpha_2$ and $\lambda, \lambda_1, \lambda_2 > 0$:

$$
\begin{aligned}
&(3)\ \lambda_1\alpha \oplus_{\mathbb{C}} \lambda_2\alpha = (\lambda_1 + \lambda_2)\alpha,\\
&(4)\ \alpha_1^{\lambda} \otimes_{\mathbb{C}} \alpha_2^{\lambda} = (\alpha_1 \otimes_{\mathbb{C}} \alpha_2)^{\lambda},\\
&(5)\ \alpha_1^{\lambda} \otimes_{\mathbb{C}} \alpha_2^{\lambda} = \alpha^{\lambda_1+\lambda_2}.
\end{aligned}
\tag{18}
$$

Definition 8 (see [43]). Let $I = [0,t]$, $p, q \geq 0$, $H^{p,q}\colon I^{n} \longrightarrow I$ if $H^{p,q}$ satisfies

$$
H^{p,q}(x_1, \ldots, x_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n}\sum_{j=1}^{i} x_i^{p} x_j^{q}\right)^{(1/p+q)}. \tag{19}
$$

3. LIVIF Heronian Mean Operator Based on ECs and ECCs

Under this part, we proposed the LIVIFWCHM operator through the novel operational laws. The particular cases of the propounded operator are explored, and several desired properties are proved in detail.

Definition 9. Let $\alpha_i = ([s_{a_i}, s_{b_i}], [s_{c_i}, s_{d_i}])$ be a collection of LIVIFNs and $p, q > 0$; then, the linguistic interval-valued intuitionistic fuzzy weight copula Heronian mean (LIVIFWCHM) operator is expressed as

$$
\text{LIVIFWCHM}^{p,q}(\alpha_1, \ldots, \alpha_n) = \left(\frac{2}{n(n+1)} \sum_{i=1}^{n}\sum_{j=1}^{i} (\omega_i\alpha_i)^{p} \otimes (\omega_j\alpha_j)^{q}\right)^{(1/p+q)}, \tag{20}
$$

where ω_i is the weight vector (WV) of α_i, $\omega_i \geq 0$, and $\sum_{k=1}^{n}\omega_i = 1$.

Theorem 4. *Let $\alpha_i = ([s_{a_i}, s_{b_i}], [s_{c_i}, s_{d_i}])$ be a collection of LIVIFNs and $p, q > 0$; then, the aggregated result form is still LIVIFNs and has*

$$
\begin{aligned}
&\mathrm{LIVIFWCHM}^{p,q}(\alpha_1,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]),\\
&a=\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)\right)\right),\\
&b=\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)\right)\right)\right)\right),\\
&c=t-\left(\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)\right)\right)\right)\right)\right),\\
&d=t-\left(\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)\right)\right)\right)\right)\right).
\end{aligned}
\tag{21}
$$

Proof. Since

$$
\omega_i\alpha_i=\left(\left[s_{t-\left(\varrho^{-1}\left(\omega_i\varrho(t-a)\right)\right)},s_{t-\left(\varrho^{-1}\left(\omega_i\varrho(t-b)\right)\right)}\right],\left[s_{\varrho^{-1}\left(\omega_i\varrho(c)\right)},s_{\varrho^{-1}\left(\omega_i\varrho(d)\right)}\right]\right),
\tag{22}
$$

$$
\left(\omega_i\alpha_i\right)^p=\left(\left[s_{\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)\right)},s_{\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)\right)}\right],\left[s_{t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)\right)},s_{t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)\right)}\right]\right),
\tag{23}
$$

and similarly, we have

$$
\left(\omega_j\alpha_j\right)^q=\left(\left[s_{\varrho^{-1}\left(q\varrho\left(t-\left(\varrho^{-1}\left(n\varpi_j\varrho\left(t-a_j\right)\right)\right)\right)\right)},s_{\varrho^{-1}\left(q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)}\right],\left[s_{t-\varrho^{-1}\left(q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)},s_{t-\varrho^{-1}\left(q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)}\right]\right).
\tag{24}
$$

Then,

$$
\begin{aligned}
&\left(\omega_i\alpha_i\right)^p\otimes\left(\omega_j\alpha_j\right)^q\\
&=\begin{pmatrix}
\left[s_{\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)},s_{\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)}\right],\\
\left[s_{t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)},s_{t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)}\right]
\end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{i}(\omega_i\alpha_i)^p\otimes(\omega_j\alpha_j)^q\\
&=\left(\begin{array}{l}
\left[s_{t-\varrho^{-1}\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{t-\varrho^{-1}\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)\right)\right)}\right],\\
\left[s_{\varrho^{-1}\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{\varrho^{-1}\left(\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)\right)\right)}\right]
\end{array}\right)\\
&\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}(\omega_i\alpha_i)^p\otimes(\omega_j\alpha_j)^q\\
&=\left(\begin{array}{l}
\left[s_{t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)+q\varrho\left(\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)\right)\right)}\right],\\
\left[s_{\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)\right)\right)}\right]
\end{array}\right),\\
&\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}(\omega_i\alpha_i)^p\otimes(\omega_j\alpha_j)^q\right)^{(1/(p+q))}\\
&=\left(\begin{array}{l}
\left[s_{\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)\right)\right)\right)\right)}\right],\\
\left[s_{t-\left(\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(c_j\right)\right)\right)\right)\right)\right)\right)\right)\right)},\right.\\
\left.s_{t-\left(\varrho^{-1}\left(\frac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\Sigma_{i=1}^{n}\Sigma_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)+q\varrho\left(t-\varrho^{-1}\left(\omega_j\varrho\left(d_j\right)\right)\right)\right)\right)\right)\right)\right)\right)}\right]
\end{array}\right).
\end{aligned}
\tag{25}
$$

□

Theorem 5 (monotonicity). *Let* $\alpha_i = ([s_{a_i}, s_{b_i}], [s_{c_i}, s_{d_i}])$ *and* $\beta_i = ([s_{\tau_i}, s_{\theta_i}], [s_{\eta_i}, s_{\nu_i}])$ *be a collection of LIVIFNs; if* $a_i \leq \tau_i, b_i \leq \theta_i, c_i \geq \eta_i, d_i \geq \nu_i$ *for all i, then*

$$\mathrm{LIVIFWCHM}^{p,q}(\alpha_1, \ldots, \alpha_n) \leq \mathrm{LIVIFWCHM}^{p,q}(\beta_1, \ldots, \beta_n). \tag{26}$$

Proof. On the one hand, since $a_i \leq \tau_i, b_i \leq \theta_i, c_i \geq \eta_i, d_i \geq \nu_i$ for all i, we have $t - a_i \geq t - \tau_i$ and $t - b_i \geq t - \theta_i$. As ϱ and ϱ^{-1} are monotonicity decreasing, $\varrho(t - a_i) \leq \varrho(t - \tau_i)$ and $\varrho(t - b_i) \leq \varrho(t - \theta_i)$; furthermore,

$$\left(\varrho^{-1}(w_i\varrho(t - a_i))\right) \geq \left(\varrho^{-1}(w_i\varrho(t - \tau_i))\right), \tag{27}$$

and so,

$$\begin{gathered}
t-\left(\varrho^{-1}(w_i\varrho(t-a_i))\right) \leq t-\left(\varrho^{-1}(w_i\varrho(t-\tau_i))\right)\\
p\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-a_i))\right)\right) \geq p\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-\tau_i))\right)\right),\\
\begin{pmatrix}\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-a_i))\right)\right)\right.\\ \left.+q\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-a_i))\right)\right)\right)\end{pmatrix} \leq \begin{pmatrix}\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-\tau_i))\right)\right)\right.\\ \left.+q\varrho\left(t-\left(\varrho^{-1}(w_i\varrho(t-\tau_i))\right)\right)\right)\end{pmatrix},\\
\begin{pmatrix}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-a_i))\right)\right)\right)\right.\\ \left.\left(+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\end{pmatrix} \leq \begin{pmatrix}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-\tau_i))\right)\right)\right)\right.\\ \left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\tau_j\right)\right)\right)\right)\right)\end{pmatrix},\\
\begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-a_i))\right)\right)\right)\right.\\ \left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\end{pmatrix} \leq \begin{pmatrix}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-(\omega_i\varrho(t-\tau_i))\right)\right)\right.\\ \left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\tau_j\right)\right)\right)\right)\right)\end{pmatrix},\\
\begin{pmatrix}\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-a_i))\right)\right)\right)\right.\right.\\ \left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\end{pmatrix} \geq \begin{pmatrix}\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-\tau_i))\right)\right)\right)\right.\right.\\ \left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\tau_j\right)\right)\right)\right)\right)\right)\end{pmatrix},\\
\begin{pmatrix}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-a_i))\right)\right)\right)\right.\right.\right.\\ \left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)\end{pmatrix} \geq \begin{pmatrix}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-\tau_i))\right)\right)\right)\right.\right.\right.\\ \left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\tau_j\right)\right)\right)\right)\right)\right)\right)\end{pmatrix}.\\
\begin{pmatrix}\varrho^{-1}\left(\dfrac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-a_i))\right)\right)\right)\right.\right.\right.\right.\\ \left.\left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-a_j\right)\right)\right)\right)\right)\right)\right)\right)\end{pmatrix} \leq \begin{pmatrix}\varrho^{-1}\left(\dfrac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-\tau_i))\right)\right)\right)\right.\right.\right.\right.\\ \left.\left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\tau_j\right)\right)\right)\right)\right)\right)\right)\right)\end{pmatrix}.
\end{gathered} \tag{28}$$

Similarly, we have

$$\begin{aligned}
&\begin{pmatrix}\varrho^{-1}\left(\dfrac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-b_i))\right)\right)\right)\right.\right.\right.\right.\\ \left.\left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-b_j\right)\right)\right)\right)\right)\right)\right)\right)\end{pmatrix}\\
&\leq \begin{pmatrix}\varrho^{-1}\left(\dfrac{1}{(p+q)}\varrho\left(t-\varrho^{-1}\left(\dfrac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}(\omega_i\varrho(t-\theta_i))\right)\right)\right)\right.\right.\right.\right.\\ \left.\left.\left.\left.+q\varrho\left(t-\left(\varrho^{-1}\left(\omega_j\varrho\left(t-\theta_j\right)\right)\right)\right)\right)\right)\right)\right)\end{pmatrix}.
\end{aligned} \tag{29}$$

On the other hand, as $c_i \geq \eta_i$ and $d_i \geq \nu_i$, we have $\varrho(c_i) \leq \varrho(\eta_i)$, $\varrho^{-1}(w_i\varrho(c_i)) \geq \varrho^{-1}(w_i\varrho(\eta_i))$, and $p\varrho(t-\varrho^{-1}(\omega_i\varrho(c_i)) \geq p\varrho(t-\varrho^{-1}(\omega_i\varrho(\eta_i))$.

$$\begin{aligned}
&\begin{pmatrix} \varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(c_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(c_j)\big)\big)\big) \end{pmatrix} \leq \begin{pmatrix} \varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(\eta_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(\eta_j)\big)\big)\big) \end{pmatrix},\\
&\begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(c_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(c_j)\big)\big)\big)\big) \end{pmatrix} \leq \begin{pmatrix} \sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(\eta_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(\eta_j)\big)\big)\big)\big) \end{pmatrix},\\
&\begin{pmatrix} \varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(c_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(c_j)\big)\big)\big)\big)\Big)\Big) \end{pmatrix} \geq \begin{pmatrix} \varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(\eta_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(\eta_j)\big)\big)\big)\big)\Big)\Big) \end{pmatrix},\\
&\begin{pmatrix} t-\Big(\varrho^{-1}\Big(\frac{1}{(p+q)}\varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(c_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(c_j)\big)\big)\big)\big)\Big)\Big)\Big)\Big) \end{pmatrix} \geq \begin{pmatrix} t-\Big(\varrho^{-1}\Big(\frac{1}{(p+q)}\varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(\eta_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(\eta_j)\big)\big)\big)\big)\Big)\Big)\Big)\Big) \end{pmatrix}.
\end{aligned} \tag{30}$$

Similarly, we have

$$\begin{aligned}
&\begin{pmatrix} t-\Big(\varrho^{-1}\Big(\frac{1}{(p+q)}\varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(d_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(d_j)\big)\big)\big)\big)\Big)\Big)\Big)\Big) \end{pmatrix}\\
&\geq \begin{pmatrix} t-\Big(\varrho^{-1}\Big(\frac{1}{(p+q)}\varrho\Big(t-\varrho^{-1}\Big(\frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\varrho\big(t-\varrho^{-1}\big(p\varrho\big(t-\varrho^{-1}(\omega_i\varrho(\nu_i))\big) \\ +q\varrho\big(t-\varrho^{-1}\big(\omega_j\varrho(\nu_j)\big)\big)\big)\big)\Big)\Big)\Big)\Big) \end{pmatrix}.
\end{aligned} \tag{31}$$

Therefore, $\text{LIVIFWCHM}^{p,q}(\alpha_1,\ldots,\alpha_n) \leq \text{LIVIFWCHM}^{p,q}(\beta_1,\ldots,\beta_n)$. □

Theorem 6 (boundness). *Let $\alpha_i = ([s_{a_i}, s_{b_i}], [s_{c_i}, s_{d_i}])$ be a collection of LIVIFNs; if $\alpha^+, ([\max(s_{a_i}), \max(s_{b_i})], [\min(s_{c_i}), \min(s_{d_i})])$ and $\alpha^-, ([\min(s_{a_i}), \min(s_{b_i})], [\max(s_{c_i}), \max(s_{d_i})])$ for all i, then*

$$\text{LIVIFWCHM}^{p,q}(\alpha^-,\ldots,\alpha^-) \leq \text{LIVIFWCHM}^{p,q}(\beta_1,\ldots,\beta_n) \leq \text{LIVIFWCHM}^{p,q}(\alpha^+,\ldots,\alpha^+). \tag{32}$$

Proof. According to Theorem 5, the conclusion is obvious, so we omitted it here.

According to formula (20) and Theorem 4, it is easy to know that operators do not satisfy idempotency.

Now, we can discuss some special cases of the $\text{LIVIFWCHM}^{p,q}$ operator with respect to the parameters p and q.

(1) When $q \longrightarrow 0$, the formula reduces to

$$\text{LIVIFWCHM}^{p,q}(\alpha_1,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]),$$

$$a=\varrho^{-1}\left(\frac{1}{p}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left((n+1-i)\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$a=\varrho^{-1}\left(\frac{1}{p}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left((n+1-i)\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$c=t-\left(\varrho^{-1}\left(\frac{1}{p}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left((n+1-i)\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$c=t-\left(\varrho^{-1}\left(\frac{1}{p}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left((n+1-i)\varrho\left(t-\varrho^{-1}\left(p\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right). \quad (33)$$

(2) When $p\longrightarrow 0$, the formula reduces to

$$\text{LIVIFWCHM}^{p,q}(\alpha_1,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]),$$

$$a=\varrho^{-1}\left(\frac{1}{q}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left(i\varrho\left(t-\varrho^{-1}\left(q\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-a_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$a=\varrho^{-1}\left(\frac{1}{q}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left(i\varrho\left(t-\varrho^{-1}\left(q\varrho\left(t-\left(\varrho^{-1}\left(\omega_i\varrho\left(t-b_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$c=t-\left(\varrho^{-1}\left(\frac{1}{q}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left(i\varrho\left(t-\varrho^{-1}\left(q\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(c_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right),$$

$$c=t-\left(\varrho^{-1}\left(\frac{1}{q}\varrho\left(t-\varrho^{-1}\left(\frac{2}{n(n+1)}\sum_{i=1}^{n}\varrho^{-1}\left(i\varrho\left(t-\varrho^{-1}\left(q\varrho\left(t-\varrho^{-1}\left(\omega_i\varrho\left(d_i\right)\right)\right)\right)\right)\right)\right)\right)\right)\right). \quad (34)$$

(3) When $p=q=(1/2)$, the formula reduces to an interval-valued intuitionistic fuzzy basic Heronian operator.

(4) When $p=q=1$, the formula reduces to an interval-valued intuitionistic fuzzy basic line Heronian mean operator.

Some different types of $\text{LIVIFWCHM}^{p,q}$ are as follows:

Case 1: Gumbel type: when $\varrho(c)=(-\ln(c/t))^{\theta}$, $\varrho^{-1}(c)=te^{-c(1/\theta)}$, and $\theta\geq 1$, we have

$$G-\text{LIVIFWCHM}^{p,q}(\alpha_1,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]), \quad (35)$$

where

$$
\begin{aligned}
a &= \mathrm{te}^{-\left(\frac{1}{p+q}\left(-\ln\left(1-e^{-a_4^{(1/\theta)}}\right)\right)^{\theta}\right)^{(1/\theta)}},\\
b &= \mathrm{te}^{-\left(\frac{1}{p+q}\left(-\ln\left(1-e^{-b_4^{(1/\theta)}}\right)\right)^{\theta}\right)^{(1/\theta)}},\\
c &= t-\mathrm{te}^{-\left(\frac{1}{p+q}\left(-\ln\left(1-e^{-c_4^{(1/\theta)}}\right)\right)^{\theta}\right)^{(1/\theta)}},\\
c &= t-\mathrm{te}^{-\left(\frac{1}{p+q}\left(-\ln\left(1-e^{-d_4^{(1/\theta)}}\right)\right)^{\theta}\right)^{(1/\theta)}}\\
a_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\ln\left(1-e^{-a_4'^{(1/\theta)}}\right)\right)^{\theta},\\
b_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\ln\left(1-e^{-b_4'^{(1/\theta)}}\right)\right)^{\theta},\\
c_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\ln\left(1-e^{-c_4'^{(1/\theta)}}\right)\right)^{\theta},\\
d_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(-\ln\left(1-e^{-d_4'^{(1/\theta)}}\right)\right)^{\theta},\\
a_4' &= p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(t-a_i/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(t-a_j/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta},\\
b_4' &= p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(t-b_i/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(t-b_j/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta},\\
c_4' &= p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(c_i/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(c_j/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta},\\
d_4' &= p\left(-\ln\left(1-e^{-\left(w_i\left(-\ln\left(d_i/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta}+q\left(-\ln\left(1-e^{-\left(w_j\left(-\ln\left(d_j/t\right)\right)^{\theta}\right)^{(1/\theta)}}\right)\right)^{\theta}.
\end{aligned}
\tag{36}
$$

Case 2: Clayton type: when $\varrho(c)=(c/t)^{-\theta}-1$, where $\varrho^{-1}(c)=t(c+1)^{-(1/\theta)}$, $\theta\geq -1$, and $\theta\neq 0$, we have

$$C-\mathrm{LIVIFWCHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]), \tag{37}$$

where

$$
\begin{aligned}
a &= t\left(\frac{1}{p+q}\left(\left(\frac{t-t\left(a_4+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)},\\
b &= t\left(\frac{1}{p+q}\left(\left(\frac{t-t\left(b_4+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)},\\
c &= t-t\left(\frac{1}{p+q}\left(\left(\frac{t-t\left(c_4+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)},\\
d &= t-t\left(\frac{1}{p+q}\left(\left(\frac{t-t\left(d_4+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)},\\
a_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{t-t\left(a_4'+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1,\\
b_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{t-t\left(b_4'+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1,\\
c_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{t-t\left(c_4'+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1,\\
d_4 &= \frac{2}{n(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(\frac{t-t\left(d_4'+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-1,\\
a_4' &= p\left(\frac{t-t\left(w_i\left(\left(t-a_i/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}+q\left(\frac{t-t\left(w_j\left(\left(t-a_j/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-2,\\
b_4' &= p\left(\frac{t-t\left(w_i\left(\left(t-b_i/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}+q\left(\frac{t-t\left(w_j\left(\left(t-b_j/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-2,\\
c_4' &= p\left(\frac{t-t\left(w_i\left(\left(c_i/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}+q\left(\frac{t-t\left(w_j\left(\left(c_j/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-2,\\
d_4' &= p\left(\frac{t-t\left(w_i\left(\left(d_i/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}+q\left(\frac{t-t\left(w_j\left(\left(d_j/t\right)^{-\theta}-1\right)+1\right)^{-(1/\theta)}}{t}\right)^{-\theta}-2.
\end{aligned}
\tag{38}
$$

Case 3: Frank type: when $\varrho(c) = \ln(e^{-(\theta c/t)} - 1/e^{-\theta} - 1)$, $\varrho^{-1}(c) = (-(t/\theta))\ln(e^{c}(e^{-\theta} - 1) + 1)$, and $\theta \neq 0$, we have

$$F - \text{LIVIFWCHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s_a, s_b], [s_c, s_d]), \tag{39}$$

where

$$\begin{aligned}
a &= -\frac{t}{\theta}\ln\left(\left(\left(\frac{e^{-(\theta a_4/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(1/p+q)}\left(e^{-\theta} - 1\right) + 1\right), b\\
c &= t + \frac{t}{\theta}\ln\left(\left(\left(\frac{e^{-(\theta a_4/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(1/p+q)}\left(e^{-\theta} - 1\right) + 1\right), d\\
a_4 &= t + \frac{t}{\theta}\ln\left(\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(\frac{e^{-(\theta a'_{4ij}/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(2/n(n+1))}\left(e^{-\theta} - 1\right) + 1\right),\\
b_4 &= t + \frac{t}{\theta}\ln\left(\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(\frac{e^{-(\theta b'_{4ij}/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(2/n(n+1))}\left(e^{-\theta} - 1\right) + 1\right),\\
c_4 &= t + \frac{t}{\theta}\ln\left(\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(\frac{e^{-(\theta c'_{4ij}/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(2/n(n+1))}\left(e^{-\theta} - 1\right) + 1\right),\\
d_4 &= t + \frac{t}{\theta}\ln\left(\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(\frac{e^{-(\theta d'_{4ij}/t)} - 1}{e^{-\theta} - 1}\right)\right)^{(2/n(n+1))}\left(e^{-\theta} - 1\right) + 1\right),\\
a'_{4ij} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta a'_{4i}/t)} - 1}{e^{-\theta} - 1}\right)^{p}\left(\frac{e^{-(\theta a'_{4j}/t)} - 1}{e^{-\theta} - 1}\right)^{q}\left(e^{-\theta} - 1\right) + 1\right),\\
b'_{4ij} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta b'_{4i}/t)} - 1}{e^{-\theta} - 1}\right)^{p}\left(\frac{e^{-(\theta b'_{4j}/t)} - 1}{e^{-\theta} - 1}\right)^{q}\left(e^{-\theta} - 1\right) + 1\right),\\
a'_{4i} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(t-a_i)/t)} - 1}{e^{-\theta} - 1}\right)^{w_i}\left(e^{-\theta} - 1\right) + 1\right), a'_{4j} = t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(t-a_j)/t)} - 1}{e^{-\theta} - 1}\right)^{w_j}\left(e^{-\theta} - 1\right) + 1\right),\\
b'_{4i} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(t-b_i)/t)} - 1}{e^{-\theta} - 1}\right)^{w_i}\left(e^{-\theta} - 1\right) + 1\right), b'_{4j} = t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(t-b_j)/t)} - 1}{e^{-\theta} - 1}\right)^{w_j}\left(e^{-\theta} - 1\right) + 1\right),\\
c'_{4ij} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta c'_{4i}/t)} - 1}{e^{-\theta} - 1}\right)^{p}\left(\frac{e^{-(\theta c'_{4j}/t)} - 1}{e^{-\theta} - 1}\right)^{q}\left(e^{-\theta} - 1\right) + 1\right),\\
d'_{4ij} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta d'_{4i}/t)} - 1}{e^{-\theta} - 1}\right)^{p}\left(\frac{e^{-\left(\theta d'_{4j}/t\right)} - 1}{e^{-\theta} - 1}\right)^{q}\left(e^{-\theta} - 1\right) + 1\right),\\
c'_{4i} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(c_i)/t)} - 1}{e^{-\theta} - 1}\right)^{w_i}\left(e^{-\theta} - 1\right) + 1\right), c'_{4j} = t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(c_j)/t)} - 1}{e^{-\theta} - 1}\right)^{w_j}\left(e^{-\theta} - 1\right) + 1\right),\\
d'_{4i} &= t + \frac{t}{\theta}\ln\left(\left(\frac{e^{-(\theta(d_i)/t)} - 1}{e^{-\theta} - 1}\right)^{w_i}\left(e^{-\theta} - 1\right) + 1\right), d'_{4j} = t + \frac{t}{\theta}\ln\left(\left(\frac{e^{--(\theta(d_j)/t)} - 1}{e^{-\theta} - 1}\right)^{w_j}\left(e^{-\theta} - 1\right) + 1\right).
\end{aligned} \tag{40}$$

Case 4: Ali-Mikhail-Haq type: when $\varrho(c)=\ln(t-\theta(t-c)/c)$, $\varrho^{-1}(c)=(t(1-\theta)/e^{c}-\theta)$, and $\theta\in[-1,1)$, we have

$$A-\mathrm{LIVIFWCHM}^{p,q}(\alpha_1,\alpha_2,\ldots,\alpha_n)=([s_a,s_b],[s_c,s_d]), \tag{41}$$

where

$$\begin{aligned}
a&=\frac{t(1-\theta)}{(t-\theta a_4/t-a_4)^{(1/p+q)}-\theta}, b\\
c&=t-\frac{t(1-\theta)}{(t-\theta c_4/t-c_4)^{(1/p+q)}-\theta}, d\\
a_4&=\frac{t(1-\theta)}{\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(t-\theta a'_{4ij}/t-a'_{4ij}\right)\right)^{(2/n(n+1))}-\theta},\\
b_4&=\frac{t(1-\theta)}{\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(t-\theta b'_{4ij}/t-b'_{4ij}\right)\right)^{(2/n(n+1))}-\theta},\\
c_4&=\frac{t(1-\theta)}{\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(t-\theta c'_{4ij}/t-a'_{4ij}\right)\right)^{(2/n(n+1))}-\theta},\\
d_4&=\frac{t(1-\theta)}{\left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(t-\theta d'_{4ij}/t-d'_{4ij}\right)\right)^{(2/n(n+1))}-\theta},\\
a'_{4ij}&=\frac{t(1-\theta)}{(t-\theta a'_{4i}/t-a'_{4i})^{p}\left(t-\theta a'_{4j}/t-a'_{4j}\right)^{q}-\theta},\\
b'_{4ij}&=\frac{t(1-\theta)}{(t-\theta b'_{4i}/t-a'_{4i})^{p}\left(t-\theta b'_{4j}/t-a'_{4j}\right)^{q}-\theta},\\
c'_{4ij}&=\frac{t(1-\theta)}{(t-\theta c'_{4i}/t-c'_{4i})^{p}\left(t-\theta c'_{4j}/t-c'_{4j}\right)^{q}-\theta},\\
d'_{4ij}&=\frac{t(1-\theta)}{(t-\theta d'_{4i}/t-c'_{4i})^{p}\left(t-\theta d'_{4j}/t-d'_{4j}\right)^{q}-\theta},\\
a'_{4i}&=\frac{t(1-\theta)}{(t-\theta a_i/t-a_i)^{w_i}-\theta},\\
a'_{4j}&=\frac{t(1-\theta)}{\left(t-\theta a_j/t-a_j\right)^{w_j}-\theta},\\
b'_{4i}&=\frac{t(1-\theta)}{(t-\theta b_i/t-b_i)^{w_i}-\theta},\\
b'_{4j}&=\frac{t(1-\theta)}{\left(t-\theta b_j/t-b_j\right)^{w_j}-\theta},\\
c'_{4i}&=\frac{t(1-\theta)}{(t-\theta(t-c_i)/t-c_i)^{w_i}-\theta},\\
c'_{4j}&=\frac{t(1-\theta)}{\left(t-\theta\left(t-c_j\right)/t-c_j\right)^{w_j}-\theta},\\
d'_{4i}&=\frac{t(1-\theta)}{(t-\theta(t-d_i)/t-d_i)^{w_i}-\theta},\\
d'_{4j}&=\frac{t(1-\theta)}{\left(t-\theta\left(t-d_j\right)/t-d_j\right)^{w_j}-\theta}.
\end{aligned} \tag{42}$$

Case 5: Joe type: when $\varrho(c) = -\ln(1 - (1 - (c/t))^{\theta})$ and $\varrho^{-1}(c) = t - t(1 - e^{-c})^{(1/\theta)}$, where $\theta \geq 1$, we have

$$J - \mathrm{LIVIFWCHM}^{p,q}(\alpha_1, \alpha_2, \ldots, \alpha_n) = ([s_a, s_b], [s_c, s_d]), \tag{43}$$

where

$$\begin{aligned}
a &= t - t\left(1 - \left(1 - \left(1 - (1 - a_4)^{(1/\theta)}\right)^{\theta}\right)^{(1/p+q)}\right)^{(1/\theta)},\\
b &= t - t\left(1 - \left(1 - \left(1 - (1 - b_4)^{(1/\theta)}\right)^{\theta}\right)^{(1/p+q)}\right)^{(1/\theta)},\\
c &= t\left(1 - \left(1 - \left(1 - (1 - c_4)^{(1/\theta)}\right)^{\theta}\right)^{(1/p+q)}\right)^{(1/\theta)},\\
d &= t\left(1 - \left(1 - \left(1 - (1 - d_4)^{(1/\theta)}\right)^{\theta}\right)^{(1/p+q)}\right)^{(1/\theta)},\\
a_4 &= \left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(1 - \left(1 - (1 - a_4')^{(1/\theta)}\right)^{\theta}\right)\right)^{(2/n(n+1))},\\
b_4 &= \left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(1 - \left(1 - (1 - b_4')^{(1/\theta)}\right)^{\theta}\right)\right)^{(2/n(n+1))},\\
c_4 &= \left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(1 - \left(1 - (1 - c_4')^{(1/\theta)}\right)^{\theta}\right)\right)^{(2/n(n+1))},\\
d_4 &= \left(\prod_{i=1}^{n}\prod_{j=1}^{i}\left(1 - \left(1 - (1 - d_4')^{(1/\theta)}\right)^{\theta}\right)\right)^{(2/n(n+1))},\\
a_4' &= \left(1 - \left(1 - \left(1 - \left(1 - (a_i/t)^{\theta}\right)^{w_i}\right)^{(1/\theta)}\right)^{\theta}\right)^{p}\left(1 - \left(1 - \left(1 - \left(1 - (a_j/t)^{\theta}\right)^{w_j}\right)^{(1/\theta)}\right)^{\theta}\right)^{q},\\
b_4' &= \left(1 - \left(1 - \left(1 - \left(1 - (b_i/t)^{\theta}\right)^{w_i}\right)^{(1/\theta)}\right)^{\theta}\right)^{p}\left(1 - \left(1 - \left(1 - \left(1 - (b_j/t)^{\theta}\right)^{w_j}\right)^{(1/\theta)}\right)^{\theta}\right)^{q},\\
c_4' &= \left(1 - \left(1 - \left(1 - \left(1 - (1 - (c_i/t))^{\theta}\right)^{w_i}\right)^{(1/\theta)}\right)^{\theta}\right)^{p}\left(1 - \left(1 - \left(1 - \left(1 - (1 - (c_i/t))^{\theta}\right)^{w_j}\right)^{(1/\theta)}\right)^{\theta}\right)^{q},\\
d_4' &= \left(1 - \left(1 - \left(1 - \left(1 - (1 - (d_i/t))^{\theta}\right)^{w_i}\right)^{(1/\theta)}\right)^{\theta}\right)^{p}\left(1 - \left(1 - \left(1 - \left(1 - (1 - (d_i/t))^{\theta}\right)^{w_j}\right)^{(1/\theta)}\right)^{\theta}\right)^{q}.
\end{aligned} \tag{44}$$

□

4. LIMADM Approach

In this part, we will give an approach for MAGDM. In general, a MAGDM problem consists of the following parts: (1) alternative set: $\Xi = \{\Psi_1, \ldots, \Psi_m\}$; (2) attribute (criteria) set: $A = \{a_1, \ldots, a_n\}$; (3) WV of attribute $W = (w_1, \ldots, w_n)^T$ satisfies $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$; and (4) $D = \{D_1, D_2, \ldots, D_p\}$ is the set of DMs.

DMs evaluate the attribute value of alternative Ψ_i under the attribute a_j which can be expressed by LIVIFNs: $\gamma_{ij}^{k} = ([s_{a_{ij}^k}, s_{b_{ij}^k}], [s_{c_{ij}^k}, s_{d_{ij}^k}])$. Then, an algorithm and process of MAGDM will be designed and given as follows:

Step 1: a revised decision matrix $\widehat{R}^{k} = (\widehat{\gamma}_{ij}^{k})_{m\times n}$ is obtained by normalizing the original decision matrix R in terms of the following equation:

$$\widehat{\gamma}_{ij}^{k} = \begin{cases} s\left(\left[s_{a_{ij}^k}, s_{b_{ij}^k}\right], \left[s_{c_{ij}^k}, s_{d_{ij}^k}\right]\right), & \text{for benefit type} \\ s\left(\left[s_{c_{ij}^k}, s_{d_{ij}^k}\right], \left[s_{a_{ij}^k}, s_{b_{ij}^k}\right]\right), & \text{for cost type.} \end{cases} \tag{45}$$

Step 2: all attribute values $\widehat{\gamma}_{ij}^{k}\,(i = 1, \ldots, m; j = 1, \ldots, n; k = 1, \ldots, p;)$ are aggregated to a comprehensive value Z_i^k by the $G-\text{LIVIFWCHM}^{p,q}$ operator.

Step 3: the supports are calculated:

$$\text{Sup}\left(Z_i^k, Z_i^t\right) = 1 - \left|K_F\left(Z_i^k\right) - K_F\left(Z_i^t\right)\right|, \tag{46}$$

$$K_F\left(Z_i^k\right) = \frac{\sqrt{\sum_{m=a}^{d}\left(m_i^k\right)^2 + \left(a_i^k + c_i^k\right)^2 + \left(b_i^k + d_i^k\right)^2}}{4}. \tag{47}$$

Step 4: $T(Z_i^k)$ and weights w_i^k are calculated:

$$T\left(Z_i^k\right) = \sum_{t=1,k\neq t}^{p} \text{Sup}\left(Z_i^k, Z_i^t\right), \tag{48}$$

$$w_i^k = \frac{\lambda_k\left(1 + T\left(Z_i^k\right)\right)}{\sum_{k=1}^{p}\lambda_k\left(1 + T\left(Z_i^k\right)\right)}, \tag{49}$$

where $w_i^k \geq 0$, and $\sum_{k=1}^{p} w_i^k = 1$.

Step 5: the $G-\text{LIVIFWCHM}^{p,q}$ operator is used to obtain the collective preference values Z_i.

Step 6: the alternatives are ranked, and the desirable one is selected by equation (2).

The flowchart is shown in Figure 1.

5. Case Analysis

This example is from [12]. In the selection of companies for investment in the rural areas, there are four companies Ψ_1, Ψ_2, Ψ_3, and Ψ_4 as candidates. The following four attributes $(c_1, \ldots, c_4)$ should be considered: c_1: project cost; c_2: technical capability; c_3: financial status; and c_4: company background.

The experts use LVs $\mathcal{S}=\{s_0$=extremelypoor, s_1=verypoor, s_2=poor, s_3=slightlypoor, s_4=fair, s_5=slightlygood, s_6=good, s_7=verygood, s_8=extremelygood$\}$ to evaluate the companies in terms of LIVIFNs. The decision matrix can be found in Table 2.

5.1. Determining the Optimal Company

Example 1. In this section, we use $G-\text{LIVIFWCHM}^{p,q}$ operators to solve this MAGDM problem, where $q = 1$ and $\theta = 1$.

Let $\lambda = (0.243, 0.514, 0.243)$ be the WV of the three experts and $w = (0.4, 0.25, 0.2, 0.15)$ be the WV of the attributes.

Step 1: since all attributes are of the same type, the normalization procedure is omitted.

Step 2: the proposed $G-\text{LIVIFWCHM}^{p,q}$ operator is employed to aggregate the evaluation values of each attribute into the integrated matrices Z_i^k, as shown in Table 3.

Step 3: the supports $S_i^{kt} = \text{Sup}(Z_i^k, Z_i^t)$ are obtained according to equation (46): $S_1^{12} = S_1^{21} = 0.9715$, $S_1^{13} = S_1^{31} = 0.9663$, $S_1^{23} = S_1^{32} = 0.9948$, $S_2^{12} = S_2^{21} = 0.9602$, $S_2^{13} = S_2^{31} = 0.9537$, $S_2^{23} = S_2^{32} = 0.9936$, $S_3^{12} = S_3^{21} = 0.9625$, $S_3^{13} = S_3^{31} = 0.9774$, $S_3^{23} = S_3^{32} = 0.9851$, $S_4^{12} = S_4^{21} = 0.9657$, $S_4^{13} = S_4^{31} = 0.9640$, and $S_4^{23} = S_4^{32} = 0.9982$.

Step 4: the supports $T_i^k = T(Z_i^k)$ and the weights w_i^k are obtained according to equations (47) and (48):

$$\begin{aligned}
T_1^1 &= \sum_{t=1,t\neq 1}^{3} \text{Sup}\left(Z_1^1, Z_1^t\right) = \text{Sup}\left(Z_1^1, Z_1^2\right) + \text{Sup}\left(Z_1^1, Z_1^3\right) \\
&= 1.9378, \\
T_1^2 &= 1.9663, \\
T_1^3 &= 1.9610, \\
T_2^1 &= 1.9139, \\
T_2^2 &= 1.9537 \\
T_2^3 &= 1.9473, \\
T_3^1 &= 1.9399, \\
T_3^2 &= 1.9477, \\
T_3^3 &= 1.9625, \\
T_4^1 &= 1.9297, \\
T_4^2 &= 1.9640, \\
T_4^3 &= 1.9622 \\
\omega_1^1 &= \frac{\lambda_1\left(1 + T_1^1\right)}{\sum_{k=1}^{3}\lambda_k\left(1 + T_1^k\right)}
\end{aligned} \tag{50}$$

Similarly, we have

$$\begin{aligned}
\omega_1^2 &= 0.5154, \\
\omega_1^3 &= 0.2432, \\
\omega_2^1 &= 0.2406, \\
\omega_2^2 &= 0.5160, \\
\omega_2^3 &= 0.2434, \\
\omega_3^1 &= 0.2422, \\
\omega_3^2 &= 0.5137, \\
\omega_3^2 &= 0.2441, \\
\omega_4^1 &= 0.2409, \\
\omega_4^2 &= 0.5155, \\
\omega_3^2 &= 0.2436.
\end{aligned} \tag{51}$$

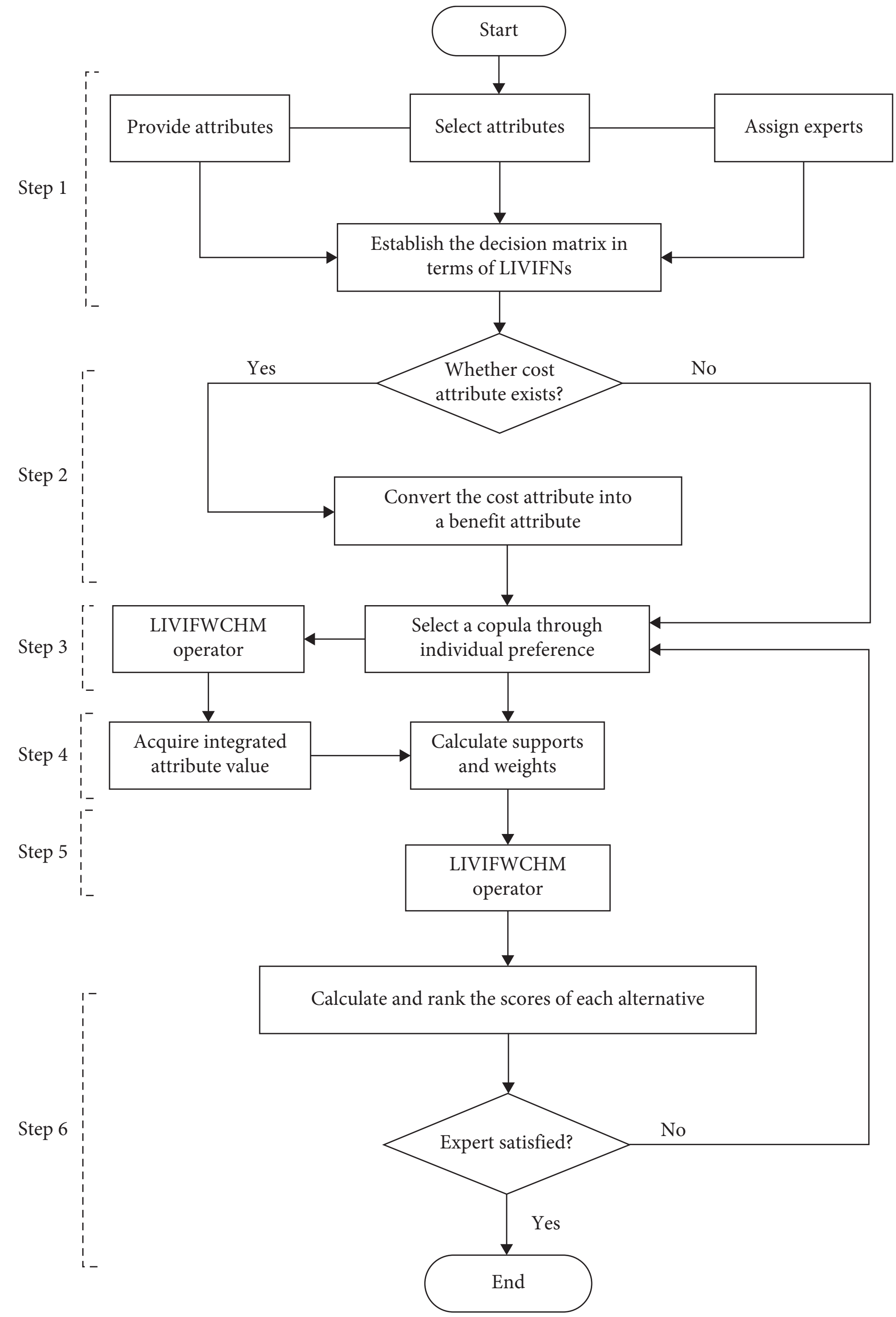

Figure 1: Flowchart of MAGDM.

Step 5: the collective preference values Z_i are obtained according to the $G-\text{LIVIFWCHM}^{p,q}$ operator:

$$\begin{aligned} Z_1 &= \left(\left[s_{3.3828}, s_{5.5831}\right], \left[s_{6.7725}, s_{7.1948}\right]\right),\\ Z_2 &= \left(\left[s_{2.2704}, s_{4.4453}\right], \left[s_{6.9295}, s_{7.3946}\right]\right),\\ Z_3 &= \left(\left[s_{3.3408}, s_{5.5389}\right], \left[s_{6.8444}, s_{7.2586}\right]\right),\\ Z_4 &= \left(\left[s_{3.3786}, s_{4.4738}\right], \left[s_{7.0047}, s_{7.2643}\right]\right). \end{aligned} \tag{52}$$

Step 6: computing the score values of every alternative on the basis of Definition 2, we have

$$\begin{aligned} S\left(\Psi_1\right) &= 0.7497,\\ S\left(\Psi_2\right) &= 0.5979,\\ S\left(\Psi_3\right) &= 0.6942\\ S\left(\Psi_4\right) &= 0.6458. \end{aligned} \tag{53}$$

Table 2: Decision matrix $R^k\,(k = 1, 2, 3)$.

		c_1	c_2	c_3	c_4
R^1	Ψ_1	$([s_5, s_6], [s_1, s_2])$	$([s_4, s_6], [s_1, s_1])$	$([s_4, s_5], [s_2, s_3])$	$([s_6, s_7], [s_1, s_1])$
	Ψ_2	$([s_3, s_5], [s_2, s_3])$	$([s_5, s_6], [s_1, s_2])$	$([s_2, s_4], [s_3, s_4])$	$([s_3, s_4], [s_2, s_3])$
	Ψ_3	$([s_5, s_6], [s_1, s_2])$	$([s_5, s_6], [s_1, s_2])$	$([s_3, s_5], [s_2, s_3])$	$([s_3, s_5], [s_1, s_3])$
	Ψ_4	$([s_4, s_5], [s_2, s_3])$	$([s_1, s_3], [s_3, s_4])$	$([s_3, s_5], [s_1, s_3])$	$([s_6, s_7], [s_1, s_1])$
R^2	Ψ_1	$([s_2, s_4], [s_1, s_3])$	$([s_4, s_5], [s_1, s_2])$	$([s_4, s_5], [s_1, s_3])$	$([s_3, s_6], [s_1, s_2])$
	Ψ_2	$([s_3, s_5], [s_1, s_3])$	$([s_1, s_2], [s_1, s_4])$	$([s_2, s_3], [s_3, s_4])$	$([s_3, s_5], [s_1, s_3])$
	Ψ_3	$([s_3, s_4], [s_1, s_2])$	$([s_3, s_6], [s_1, s_2])$	$([s_2, s_5], [s_2, s_3])$	$([s_3, s_4], [s_2, s_3])$
	Ψ_4	$([s_4, s_5], [s_1, s_2])$	$([s_3, s_3], [s_3, s_5])$	$([s_3, s_3], [s_2, s_3])$	$([s_4, s_6], [s_1, s_1])$
R^3	Ψ_1	$([s_2, s_4], [s_1, s_2])$	$([s_2, s_3], [s_1, s_4])$	$([s_3, s_5], [s_2, s_3])$	$([s_5, s_7], [s_1, s_1])$
	Ψ_2	$([s_1, s_4], [s_2, s_3])$	$([s_4, s_5], [s_1, s_2])$	$([s_2, s_4], [s_1, s_3])$	$([s_3, s_4], [s_2, s_4])$
	Ψ_3	$([s_2, s_3], [s_1, s_5])$	$([s_3, s_5], [s_1, s_2])$	$([s_3, s_5], [s_1, s_3])$	$([s_3, s_5], [s_2, s_3])$
	Ψ_4	$([s_3, s_4], [s_2, s_3])$	$([s_1, s_2], [s_3, s_4])$	$([s_3, s_5], [s_1, s_2])$	$([s_5, s_6], [s_1, s_1])$

Table 3: Integrated decision matrix Z_i^k.

	Z^1	Z^2	Z^3
Z_1	$([s_{1.6304}, s_{2.3639}], [s_{4.9640}, s_{5.4021}])$	$([s_{0.9388}, s_{1.6541}], [s_{4.7882}, s_{6.0148}])$	$([s_{0.8144}, s_{1.6166}], [s_{4.9640}, s_{5.8864}])$
Z_2	$([s_{1.0577}, s_{1.7387}], [s_{5.5472}, s_{6.1952}])$	$([s_{0.6849}, s_{1.3032}], [s_{5.0710}, s_{6.4657}])$	$([s_{0.6894}, s_{1.3951}], [s_{5.2463}, s_{6.1729}])$
Z_3	$([s_{1.4590}, s_{2.1252}], [s_{4.9640}, s_{5.8748}])$	$([s_{0.8296}, s_{1.6500}], [s_{5.1007}, s_{5.8748}])$	$([s_{0.7574}, s_{1.4120}], [s_{4.9224}, s_{6.4236}])$
Z_3	$([s_{1.1480}, s_{1.7950}], [s_{5.4708}, s_{6.1195}])$	$([s_{1.1087}, s_{1.4725}], [s_{5.3195}, s_{5.9689}])$	$([s_{0.8829}, s_{1.3765}], [s_{5.4708}, s_{5.9967}])$

The rank of alternatives is $\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$, and so, Ψ_1 is the best alternative.

The ordering results of alternatives use other ECs proposed in the present work which are listed in Table 4.

5.2. Sensitivity Analysis. The following two aspects reflect the flexibility of this method: firstly, DMs can select different types of ECs and ECCs with parameter θ; secondly, the HM operator contains two important parameters, which can reflect the correlation between attributes. Therefore, different ranking results may be obtained according to different parameters.

In the following, the influence of parameters p, q, and θ on the results will be analyzed. Without loss of generality, the following analysis adopts the $G - \text{LIVIFWCHM}^{p,q}$ operator. Firstly, we assign different values to θ with fixed p and q, and the results are listed in Table 5. In addition, we explore the effect of parameters p and q on the ultimate ranking results which can be found in Table 6 and Figures 2–9.

From Table 5, we can find out that when p and q are fixed, θ has little influence on the sorting result. From Figures 2–9 and Table 6, it is easy to derive the following conclusions: (1) the scores and ranking order will be different with respect to different parameters p and q. (2) The optimal candidate will change when θ is small, and the absolute value of p minus q is large. (3) The best and worst alternatives are always the same when $\theta \geq 2$, and $q = 1$, or $q = 1$.

In application, the larger the value of p or q, the more prominent the interaction between attributes, and if one of the parameters is zero, the relationship between attributes is not considered. For the actual applications, we can choose a simple integral number for p and q to simplify the process.

5.3. Comparative Analysis. In the following, the proposed approach will be analyzed and compared with other existing methods.

Example 2. This example is to select a new management information system. There are four alternatives $A_i\,(i = 1, 2, 3, 4)$ to be considered. Following this, there are four attributes C_1, C_2, C_3, and C_4 to be evaluated by three DMs using LIVIFI, where C_1: the costs; C_2: the reliability of software development from outsourcing enterprise; C_3: the contribution to the enterprise performance; and C_4: the effort to transition to a new system from the old systems.

For the decision matrices, see Tables III–V in [50]. The score values and rankings of alternatives are separately displayed in Table 7. From it, we can draw a conclusion that the orders are almost the same, and the optimal selections are all A_4, so we can see the proposed approach is workable and efficient.

Example 3. This is a MADM problem which is from [14]. In the example, a company wants to establish a new subsidiary on four potential sites $\Psi = \{\Psi_1, \Psi_2, \Psi_3, \Psi_4\}$, and there are five attributes $C = \{c_1, c_2, c_3, c_4, c_5\}$ that affect decision-making, and the weight of attributes is $\omega = \{0.2, 0.25, 0.15, 0.18, 0.22\}$. The evaluation value is expressed by the IVLIFN which is shown in Table 8. The proposed method used the LIVIFWCHM operator (see equation (21)), and also, the same score function proposed by Garg and Kumar [12] (equation (2)) was used for easy comparison. The comparison results with the recent existing work are listed in Table 9.

In the comparison, the WA operator in [12] and PWA operator in [13] were chosen which ignored the interaction

Table 4: The ordering results of alternatives using other different copulas.

Type of copulas	Parameters	Score index of $\Psi_i\,(i=1,2,3,4)$	Ranking order
Clayton	$\theta = 1$	1.3856, 0.9479, 1.1720, 1.0664	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
Frank	$\theta = 1$	0.9278, 0.7177, 0.8458, 0.7839	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
Ali-Mikhail-Haq	$\theta = -1$	1.6916, 1.6132, 1.6703, 1.6391	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
Joe	$\theta = 1$	0.7704, 0.6149, 0.7154, 0.6653	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$

Table 5: The influence of parameter θ on the rank of alternatives $(p = q = 1)$.

θ	Score index of $\Psi_i\,(i=1,2,3,4)$	Ranking order
$\theta = 2$	2.3003, 1.8367, 2.0952, 2.0216	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
$\theta = 3$	3.2580, 2.6363, 2.9632, 2.9331	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
$\theta = 5$	4.2744, 3.5199, 3.8836, 3.9829	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
$\theta = 10$	5.3432, 4.4847, 4.8243, 5.1902	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$

Table 6: The influence of parameter q on the rank of alternatives $(\theta = 1)$.

p	q	Score index of $\Psi_i\,(i=1,2,3,4)$	Ranking order
$p=0$	$q=1$	0.8477, 0.6880, 0.8302, 0.6917	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
	$q=2$	0.9082, 0.7696, 0.9108, 0.7984	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
	$q=5$	1.0758, 0.9895, 1.1331, 1.0987	$\Psi_3 \succ \Psi_4 \succ \Psi_1 \succ \Psi_2$
$p=1$	$q=0$	0.6644, 0.5043, 0.5727, 0.5897	$\Psi_1 \succ \Psi_4 \succ \Psi_3 \succ \Psi_2$
	$q=1$	0.7497, 0.5979, 0.6942, 0.6458	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
	$q=2$	0.8285, 0.6897, 0.7972, 0.7386	$\Psi_3 \succ \Psi_1 \succ \Psi_4 \succ \Psi_2$
	$q=5$	1.0126, 0.9099, 1.0362, 1.0089	$\Psi_3 \succ \Psi_1 \succ \Psi_4 \succ \Psi_2$
$p=2$	$q=0$	0.7219, 0.5713, 0.6485, 0.6551	$\Psi_1 \succ \Psi_4 \succ \Psi_3 \succ \Psi_2$
	$q=1$	0.7755, 0.6379, 0.7321, 0.6983	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
	$q=2$	0.8367, 0.7129, 0.8155, 0.7740	$\Psi_1 \succ \Psi_3 \succ \Psi_4 \succ \Psi_2$
	$q=5$	0.9978, 0.9034, 1.0237, 1.0031	$\Psi_3 \succ \Psi_4 \succ \Psi_1 \succ \Psi_2$

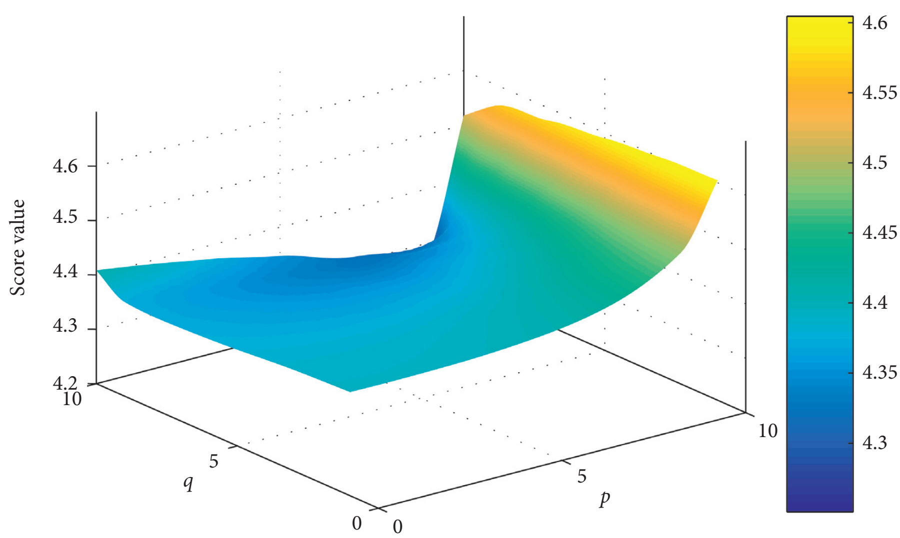

Figure 2: Scores of A_1 when $p, q \in [0, 10]\,(\theta = 5)$.

between attributes. However, the YWMSM operator in [14] and WPMM operator in [15] considered the interaction between attributes. As can be seen from Table 9, the ranking order of the proposed method is exactly the same with Liu and Qin [14] and Qin [15]. Therefore, it can be concluded that the method proposed in this paper is feasible and effective for dealing with MAGDM problems based on LIVIFNs. In addition, the ranking results of the WA operator and PWA operator are different from those of other methods. The reason is that the former methods assume that all attributes are independent. Furthermore, compared with the YWMSM operator in [14] and WPMM operator in [15], the proposed method is based on ECs and ECCs, which have 5 different generator functions, so it can provide DMs more options. Therefore, this method is more flexible.

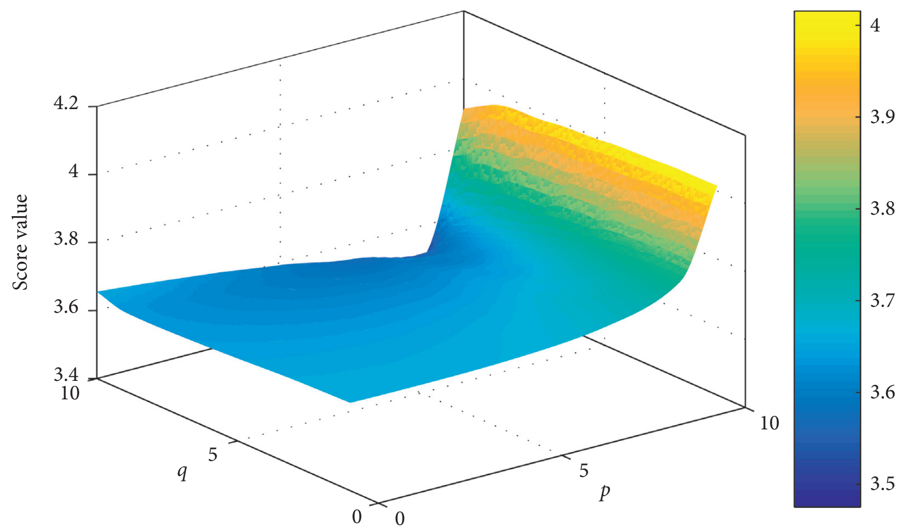

Figure 3: Scores of A_2 when $p, q \in [0, 10]$ ($\theta = 5$).

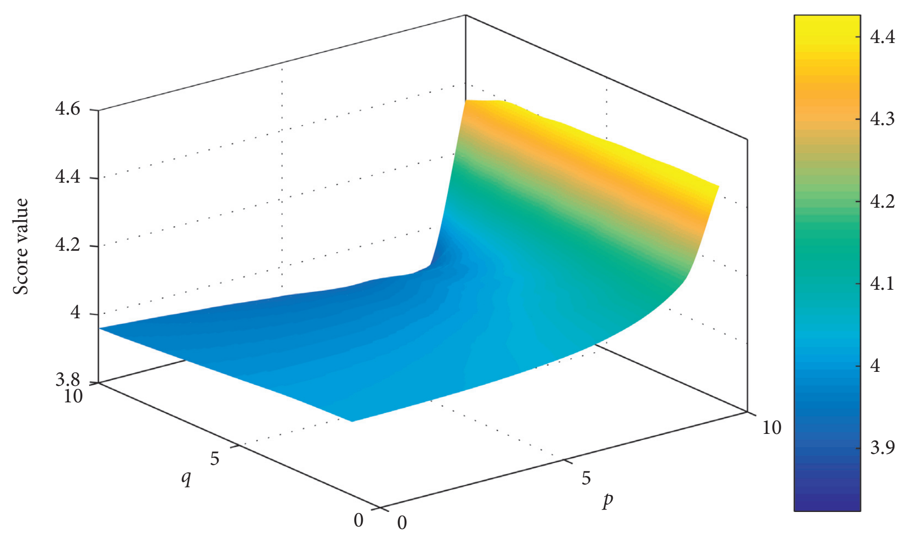

Figure 4: Scores of A_3 when $p, q \in [0, 10]$ ($\theta = 5$).

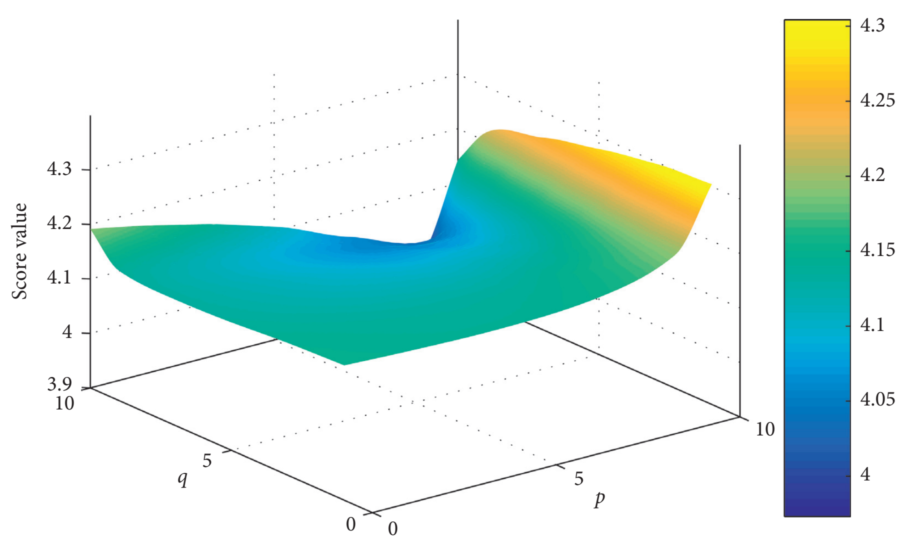

Figure 5: Scores of A_4 when $p, q \in [0, 10]$ ($\theta = 5$).

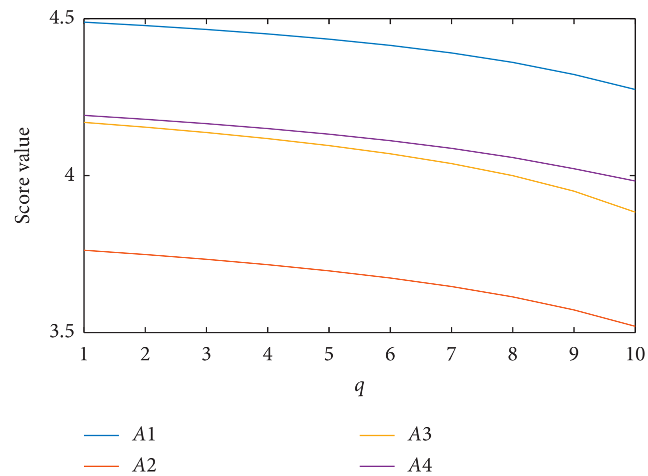

FIGURE 6: Scores of A_i $(i = 1, 2, 3, 4)$ when $p = 1$ and $q \in [0, 10]$ $(\theta = 5)$.

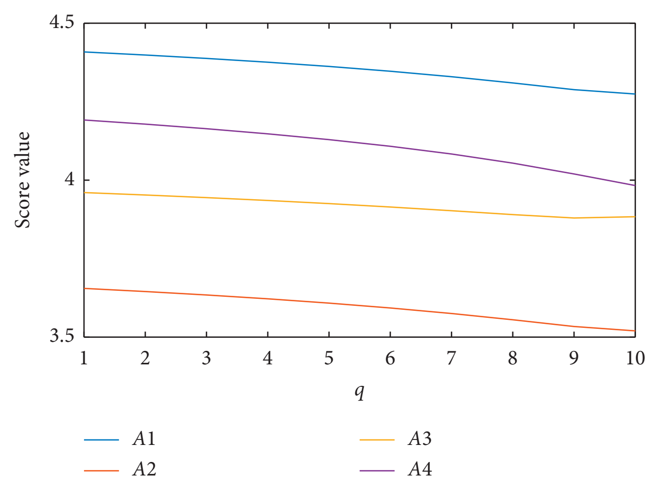

FIGURE 7: Scores of A_i $(i = 1, 2, 3, 4)$ when $q = 1$ and $p \in [0, 10]$ $(\theta = 5)$.

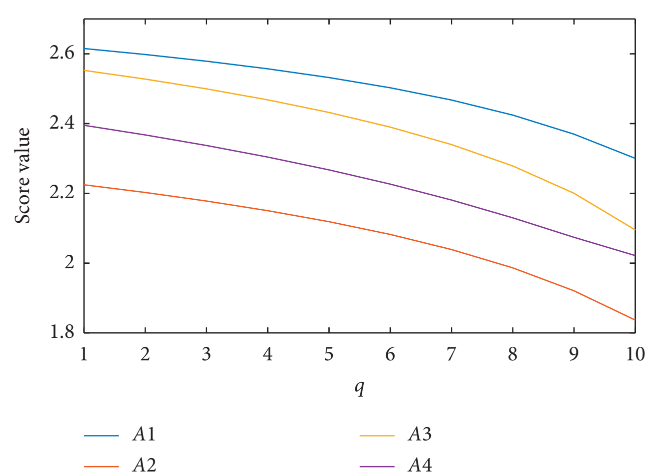

FIGURE 8: Scores of A_i $(i = 1, 2, 3, 4)$ when $p = 1$ and $q \in [0, 10]$ $(\theta = 2)$.

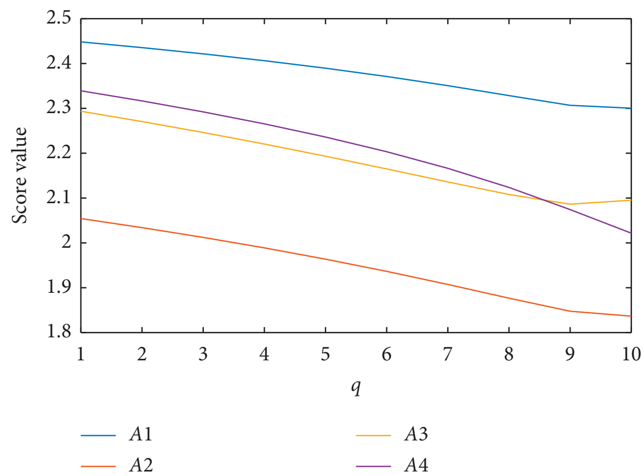

Figure 9: Scores of $A_i\,(i=1,2,3,4)$ when $q=1$ and $p\in[0,10]$ $(\theta=2)$.

Table 7: Comparison with existing approaches for Example 2.

Methods	Parameters	Score index of $A_i\,(i=1,2,3,4)$	Ranking order
He et al. [51]	$\lambda=2$	0.0167, 0.0931, 0.0808, 0.2676	$A_4 \succ A_3 \succ A_2 \succ A_1$
Yu and Wu [44]	$p=2, q=2$	0.0740, 0.1078, 0.2868, 0.3025	$A_4 \succ A_3 \succ A_2 \succ A_1$
Liu [50]	$p=2, q=2$	0.0719, 0.1090, 0.2846, 0.3022	$A_4 \succ A_3 \succ A_2 \succ A_1$
Gumbel type	$\theta=1, p=q=1$	0.0565, 0.0598, 0.0721, 0.0777	$A_4 \succ A_3 \succ A_2 \succ A_1$
Clayton type	$\theta=1, p=q=1$	0.0712, 0.0837, 0.0811, 0.1026	$A_4 \succ A_2 \succ A_3 \succ A_1$
Frank type	$\theta=1, p=q=1$	0.0656, 0.0705, 0.0850, 0.0912	$A_4 \succ A_3 \succ A_2 \succ A_1$
Ali-Mikhail-Haq type	$\theta=-1, p=q=1$	0.1884, 0.1904, 0.1969, 0.1981	$A_4 \succ A_3 \succ A_2 \succ A_1$
Joe type	$\theta=1, p=q=1$	0.0590, 0.0622, 0.0770, 0.0821	$A_4 \succ A_3 \succ A_2 \succ A_1$

Table 8: Decision matrix R.

	c_1	c_2	c_3	c_4	c_5
Ψ_1	$([s_6,s_7],[s_1,s_1])$	$([s_5,s_6],[s_1,s_2])$	$([s_4,s_5],[s_1,s_3])$	$(s_6,[s_7]),[s_1,s_1]$	$([s_5,s_6],[s_1,s_2])$
Ψ_2	$([s_4,s_5],[s_1,s_2])$	$([s_5,s_7],[s_1,s_1])$	$([s_5,s_6],[s_1,s_2])$	$([s_5,s_6],[s_1,s_2])$	$([s_6,s_7],[s_1,s_1])$
Ψ_3	$([s_5,s_6],[s_1,s_2])$	$([s_4,s_5],[s_2,s_3])$	$([s_6,s_7],[s_1,s_1])$	$([s_5,s_6],[s_1,s_2])$	$([s_3,s_4],[s_3,s_4])$
Ψ_4	$([s_4,s_5],[s_2,s_3])$	$([s_6,s_7],[s_1,s_1])$	$([s_4,s_5],[s_2,s_3])$	$([s_4,s_6],[s_1,s_2])$	$([s_3,s_4],[s_3,s_4])$

Table 9: Comparison with existing approaches for Example 3.

Methods	Used AOs	Parameters	Score index of $\Psi\,(i=1,2,3,4)$	Ranking order
Garg and Kumar [12]	WA	None	6.2621 6.2709 5.6111 5.5823	$\Psi_2 \succ \Psi_1 \succ \Psi_3 \succ \Psi_4$
Garg and Kumar [13]	PWA	None	6.1407 6.1620 5.3326 5.2541	$\Psi_2 \succ \Psi_1 \succ \Psi_3 \succ \Psi_4$
Liu and Qin [14]	YWMSM	$k=3$	6.1034 6.0611 5.5196 5.1819	$\Psi_1 \succ \Psi_2 \succ \Psi_3 \succ \Psi_4$
Qin [15]	WPMM	Q=(1, 2, 3, 0, 0)	6.1584 6.1157 5.5637 5.3056	$\Psi_1 \succ \Psi_2 \succ \Psi_3 \succ \Psi_4$
The proposed method	WCHM	$\theta=1$ and $p=q=2$	6.1101 6.0791 5.3637 5.2688	$\Psi_1 \succ \Psi_2 \succ \Psi_3 \succ \Psi_4$

In the following, the proposed approach will be analyzed and compared with other existing method approaches:

(1) Chen et al.'s LIFWA operator [10], Zhang's LIFWA operator [52], and Liu and Wang's ILIFWA operator [53] are all based on the LIFS. In our proposed method, when $t=1$, $\theta=1$, $p=1$, and $q=0$ and only $s_{(a+b)/2}$ and $s_{(c+d)/2}$ are considered, LIVIFWCHM^{p,q} reduces to LIFWCHM^{p,q}. In those methods mentioned above, the operational rules are based on algebraic TN and algebraic TC, which are special forms of EC and ECC. So, our method can also be applied to intuitionistic fuzzy DMPs. Therefore, our proposed method is effective and feasible. Furthermore, the proposed approach will provide more choice for the decision maker in real DMPs.

(2) Compared with Tao et al.'s method [36], if $t=1$, $p=1$, and $q=0$ and only $s_{(a+b)/2}$ and $s_{(c+d)/2}$ are considered, LIFWCHM^{p,q} reduces to IFCAA$_\omega$. Therefore, compared with IFCAA$_\omega$ [36], the proposed method is the generalization of Tao et al.

Table 10: The characteristic comparison of different AOs.

AOs	Capture correlation among attributes	Generalization	Flexibility	Deal with LIVIFNs
Chen et al. [10]	No	No	No	No
Zhang [52]	No	Yes	No	No
Liu and Wang [53]	Yes	No	No	No
Garg and Kumar [12]	No	Yes	No	Yes
Tao et al. [36]	No	Yes	Yes	No
Liu and Qin [14]	Yes	Yes	No	Yes
Qin [15]	Yes	Yes	No	Yes
The proposed method	Yes	Yes	Yes	Yes

(3) Compared to the method of Garg and Kumar and Liu and Qin [12, 14], the proposed method is more universal and flexible. Five aggregation functions can be regarded through assigning diverse copulas to them; also, parameters θ, p, and q can be selected according to decision maker's attitude.

(4) In [15] Qin combined the MM operator and the PA operator under the ATT operations. Compared with our method, Qin considered the interconnection of diverse attributes, but the designed method is based on ATT which is the special form of ECs and ECCs.

A detailed comparative analysis for the aforementioned approaches is displayed in Table 10.

6. Conclusions

In this paper, we propose a LIVIFWCHM operator to deal with MAGDM problems under LIVIFI. We establish a new version of copulas and cocopulas and several universal operational laws of LIVIFNs and study some special instances of them based on dissimilar copulas. Then, we give the generalized expression of the LIVIFWCHM operator and explore several characteristics and five specific expressions of the LIVIFWCHM operator. On this basis, we bring forward an approach to solve MAGDM problems based on the LIVIFN. Then, a detailed numerical example has been given to show how it works, and a set of experiments have been carried out to verify the efficacy and superiority of the propounded approach. The results also show that the proposed method is more general and flexible and can consider the correlation between attributes. In future, we shall focus, especially, on the correlation between attributes and incomplete attribute information, as well as the large-scale decision-making algorithm based on linguistic assessment theory and methodology.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Review Article

Decision-Making Approach with Fuzzy Type-2 Soft Graphs

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Molodtsov's theory of soft sets is free from the parameterizations insufficiency of fuzzy set theory. Type-2 soft set as an extension of a soft set has an essential mathematical structure to deal with parametrizations and their primary relationship. Fuzzy type-2 soft models play a key role to study the partial membership and uncertainty of objects along with underlying and primary set of parameters. In this research article, we introduce the concept of fuzzy type-2 soft set by integrating fuzzy set theory and type-2 soft set theory. We also introduce the notions of fuzzy type-2 soft graphs, regular fuzzy type-2 soft graphs, irregular fuzzy type-2 soft graphs, fuzzy type-2 soft trees, and fuzzy type-2 soft cycles. We construct some operations such as union, intersection, AND, and OR on fuzzy type-2 soft graphs and discuss these concepts with numerical examples. The fuzzy type-2 soft graph is an efficient model for dealing with uncertainty occurring in vertex-neighbors structure and is applicable in computational analysis, applied intelligence, and decision-making problems. We study the importance of fuzzy type-2 soft graphs in chemical digestion and national engineering services.

1. Introduction

Fuzzy set theory has its remarkable origin to the work of Zadeh [1] in 1965 to interact with vagueness and imprecision between absolute true and absolute false. The range of the values in a fuzzy set lies in [0, 1]. This remarkable discovery of fuzzy set theory paved a different way for dealing with uncertainties in various domains of science and technology.

Graph theory is moving quickly into the mainstream of mathematics, primarily due to its applications in engineering, communication networks, computer science, and artificial intelligence. In 1973, Kauffmann [2] introduced the notion of fuzzy graph, which is based on Zadeh's fuzzy relation [3]. Another elaborated definition of fuzzy graph was introduced by Rosenfeld [4]. Bhattacharya [5] subsequently gave some helpful results on fuzzy graphs and some operations on fuzzy graph theory were explored by Mordeson and Nair [6]. Many researchers studied fuzzy graphs in recent decades [7–9].

However, the theory of fuzzy sets has inadequacy to deal with parametrization tool. Soft set theory proposed by Molodtsov [10] has the ability to cope with this difficulty and is defined as a pair (ξ, M), where ξ is a mapping given by $\xi\colon M \longrightarrow P(E)$. Soft sets have been generalized to numerous directions beginning with Maji et al. [11, 12] who introduced fuzzy soft sets and Ahmad and Kharal [13] discussed some properties of fuzzy soft sets. In algebraic structures, soft sets and their hybrid models based on fuzzy soft sets, generalized fuzzy soft sets, rough soft sets, and soft rough sets have been implemented effectively [14–19]. Sarwar [20] elaborated the notion of rough graph and discussed decision-making approaches based on rough numbers and rough graphs. Akram and Nawaz [21] introduced the concepts of fuzzy soft graphs (named as fuzzy type-1 soft graph), vertex-induced soft graphs, and edge-induced soft graphs and also discussed some operations on soft graphs. Akram and Zafar [22] introduced various hybrid models based on fuzzy sets, soft sets, and rough sets. Further, Akram in cooperation with other researchers [23–26] discussed various applications and extensions of graph theory to study different types of uncertainties in real-world problems. Nowadays, researchers are actively working on interval type-2 fuzzy arc lengths [27], trapezoidal interval type-2 fuzzy soft sets [28], total uniformity of graph under fuzzy soft information [29], fuzzy soft cycles [30], and fuzzy soft β−coverings.

All these existing models have the same restriction that one cannot freely select the parameters. That is, if a correspondence or association occurs between parameters, then none of these models can solve the problems completely. Chatterjee et al. [31] proposed the concept of type-2 soft sets to deal with the correspondence between parameters, which is a generalization of Molodtsov's soft sets (called type-1 soft sets). Type-2 soft sets reparameterize the already parameterized crisp sets and thus have more freedom and effectiveness in dealing with imprecision as compared to type-1 soft sets. Hayat et al. [32–34] introduced vertex-neighbors-based type-2 soft sets, type-2 soft graphs, and irregular type-2 soft graphs and presented certain types of type-2 soft graphs.

The motives of this study are as follows:

(1) Soft sets and their hybrid models are used to deal with uncertainty based on parametrization tool. The correspondence, association, or relation occurring among parameters cannot be discussed with existing approaches. Type-2 soft models tackle this difficulty and present a mathematical approach to reparameterize the existing soft models. To deal with partial membership of objects, the main focus of this study is to introduce a hybrid model by combining fuzzy set theory with type-2 soft sets.

(2) Graph theory is an essential approach to study relations among objects using a figure consisting of vertices and lines joining these vertices. But there is an information loss in graphical models whether the objects are fully related or partially related, that is, uncertain and parameterized relations among objects. To handle this information loss, there is a need to represent the graphical models under fuzzy type-2 soft environment.

The main contribution of this study is as follows:

(1) The present study introduces the mathematical approaches of vertex-neighbors-based type-2 soft set and vertex-neighbors-based type-2 soft graphs under fuzzy environment. The notions of fuzzy type-2 soft graphs, regular fuzzy type-2 soft graphs, irregular fuzzy type-2 soft graphs, fuzzy type-2 soft trees, and fuzzy type-2 soft cycles are discussed with certain operations and numerical examples.

(2) The importance of presented concepts is studied with an application in chemical digestion and national engineering services.

2. Preliminaries

The term crisp graph on a nonvoid universe (named as set of vertices) J is defined as a pair $G = (J, K)$, where $K \subseteq J \times J$ is named as set of edges. Crisp graph (J, K) is a special case of the fuzzy graph with each vertex and edge of (J, K) having degree of membership 1. A soft graph corresponding to a crisp graph G is a parameterized family of subgraphs of G. A soft graph on a nonempty set J is a 3−tuple (J, K, A) such that, for each $e \in A$, $(J(e), K(e))$ is a graph, where $J(e) \subseteq J$ and $K(e) \subseteq J(e) \times J(e)$.

Definition 1 (see [31]). Let (E, M) be a soft universe and let $\eta(E)$ be the set of all T1SSs over (E, M). Then a mapping $W\colon B \longrightarrow \eta(E)$, $B \subseteq M$ is called a type-2 soft set (T2SS) over (E, M) and it is denoted by $[W^*, B]$. For all $\delta \in B, W^*(\delta)$ is a T1SS $(W_{(\delta)}, F_{(\delta)})$ such that $W^*(\delta) = (W_{(\delta)}, F_{(\delta)})$, where $W_{(\delta)}\colon F_{(\delta)} \longrightarrow P(E)$ and $F_{(\delta)} \subset M$. We refer to the parameter set B as the "primary set of parameters" although the collection of parameters denoted by $\cup F_{(\delta)}$ is called "underlying set of parameters."

Definition 2 (see [32]). Suppose that $G = (J, K)$ is a simple graph. Suppose that $B \subset J$ and $\Gamma(J)$ is the set of all T1SSs over J. Suppose that $[\xi^*, B]$ is a T2SS over J. Then a mapping $\xi^*\colon B \longrightarrow \Gamma(J)$ is said to be a T2SS over J and is denoted as $[\xi^*, B]$. For every vertex $x \in B$, $[\xi^*, B]$ is a T1SS, where $\xi^*(x) = (\xi_{(x)}, \mathcal{NB}_x)$ and $\xi_{(x)}\colon \mathcal{NB}_x \longrightarrow P(J)$ can be explained as $\xi_x(u) = \{v \in J | uRv\}\ \forall u \in \mathcal{NB}_x \subseteq J$, This T2SS is said to be a vertex-neighbors type-2 soft set (VN-T2SS) over J.

Definition 3 (see [32]). Suppose that $G = (J, K)$ is a simple graph. Suppose that $B \subset J$ and $\Gamma(K)$ is the set of all T1SSs over K. Suppose that $[\xi^*, B]$ is a VN-T2SS over J. Then a mapping $\psi^*\colon B \longrightarrow \Gamma(K)$ is said to be a T2SS over K and is denoted as $[\psi^*, B]$. For every vertex $x \in B$, $[\psi^*, B]$ is a T1SS, where $\psi^*(x) = (\psi_{(x)}, \mathcal{N}B_x)$ and $\psi_{(x)}\colon \mathcal{NB}_x \longrightarrow P(K)$ can be explained as $\psi_x(u) = \{vw \in K | v, w \subseteq \xi_x(u)\}\quad \forall x \in \mathcal{NB}_x \subseteq J$. This T2SS is said to be a VN-T2SS over K.

We present the notations that are used in this research article in Table 1.

3. Fuzzy Type-2 Soft Graphs

We refer to Maji's [11] fuzzy soft set as fuzzy type-1 soft set (FT1SS). Consider B as a set of parameters that have a random nature (characterization of object, some functions, numeric values, etc.). Consider E as a universal set and the class of all FT1SSs over E will be indicated by $P(E)$. Recently, researchers have shown attraction to the application of fuzzy soft sets in science, advance technology, and decision problems. Fuzzy type-2 soft sets are considered as a generalized form of fuzzy soft set. Consider E as a universal set and M as the set of parameters. Fuzzy type-2 soft set is defined as follows.

Definition 4. Let (E, M) be a fuzzy soft universe and let $P(E)$ be the collection of all FT1SSs over (E, M). Then a mapping $S^*\colon B \longrightarrow P(E)$, $B \subset M$, is called a fuzzy type-2 soft set (FT2SS) over (E, M) and it is denoted by $[S^*, B]$. In this case, corresponding to each parameter $e \in B$, $S^*(e)$ is

Table 1: List of abbreviations.

Abbreviation	Description
T1SS	Type-1 soft set
T2SS	Type-2 soft set
VN-T2SS	Vertex-neighbors type-2 soft set
FT1SS	Fuzzy type-1 soft set
FT2SS	Fuzzy type-2 soft set
VN-FT2SS	Vertex-neighbors fuzzy type-2 soft set
FT1SG	Fuzzy type-1 soft graph
FT2SG	Fuzzy type-2 soft graph
FT1ST	Fuzzy type-1 soft tree
FT2ST	Fuzzy type-2 soft tree
FT2SST	Fuzzy type-1 soft subtree
FT2SST	Fuzzy type-2 soft subtree
FT1SC	Fuzzy type-1 soft cycle
FT2SC	Fuzzy type-2 soft cycle

FT1SS. Thus, for each $e \in B$, there exists a FT1SS (S_e, L_e) such that $S^*(e) = (S_e, L_e)$, where $S_e\colon L_e \longrightarrow P(E)$ and $L_e \subset M$. In this case, we refer to the parameter set B as the "primary set of parameters," while the set of parameters $\cup L_e$ is known as the "underlying set of parameters."

Definition 5. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph. The set of neighbors of an element $(j, \mu(j))$ is denoted by $\mathcal{N}\mathcal{B}_j$ and defined by $\mathcal{N}\mathcal{B}_j = ((i, \mu(i) | ij \in \mathcal{K}))$. Then $\mathcal{N}\mathcal{B}_B = \cup_{j\in B}\mathcal{N}\mathcal{B}_j$.

Definition 6. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph. Suppose that $B \subset \mathcal{J}$ and $\Gamma(\mathcal{J})$ is the set of all FT1SSs over $\mathcal{J}$. Suppose that $[\xi, B]$ is a FT2SS over $\mathcal{J}$. Then a mapping $\xi\colon B \longrightarrow \Gamma(\mathcal{J})$ is said to be a FT2SS over $\mathcal{J}$ and is denoted as $[\xi, B]$. For every vertex $j \in B$, $\xi(j) = (\xi_{(j)}, \mathcal{N}\mathcal{B}_j)$ is a FT1SS and $\xi_{(j)}\colon \mathcal{N}\mathcal{B}_j \longrightarrow P(\mathcal{J})$ can be explained as $\xi_j(u) = \{v \in \mathcal{J} | uRv\}\ \forall u \in \mathcal{N}\mathcal{B}_j \subseteq \mathcal{J}$. This FT2SS is said to be a vertex-neighbors fuzzy type-2 soft set (VN-FT2SS) over $\mathcal{J}$.

Definition 7. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph. Suppose that $B \subset \mathcal{J}$ and $\Gamma(K)$ is the set of all T1SSs over $\mathcal{K}$. Suppose that $[\psi, B]$ is a FT2SS over $\mathcal{K}$. Then a mapping $\psi\colon B \longrightarrow \Gamma(K)$ is said to be a FT2SS over $\mathcal{K}$ and is denoted as $[\psi, B]$. For every vertex $j \in B$, $\psi(j) = (\psi_{(j)}, \mathcal{N}\mathcal{B}_j)$ is a FT1SS and $\psi_{(j)}\colon \mathcal{N}\mathcal{B}_j \longrightarrow P(\mathcal{K})$ can be explained as $\psi_j(u) = \left\{uv \in \mathcal{K} | \{u, v\} \subseteq \xi_j(u)\right\}\ \forall u \in \mathcal{N}\mathcal{B}_j \subseteq \mathcal{J}$. This FT2SS is said to be a VN-FT2SS over $\mathcal{K}$.

$\xi(j) = (\xi_{(j)}, \mathcal{N}\mathcal{B}_j)$ and $\psi(j) = (\psi_{(j)}, \mathcal{N}\mathcal{B}_j)\ \forall j \in B$ are FT1SS over $\mathcal{J}$ and $\mathcal{K}$, respectively. If $(\xi_j(u), \psi_j(u))\ \forall u \in \mathcal{N}\mathcal{B}_j$ represent a fuzzy graph in fuzzy type-2 soft graph $\mathbb{G}$, then $(\xi(j), \psi(j))\ \forall j \in B$ is called FT1SG.

Definition 8. A 5-tuple $\mathbb{G} = (\mathcal{G}, \xi, \psi, B, \mathcal{N}\mathcal{B}_b)$ is called a fuzzy type-2 soft graph (FT2SG) if it satisfies the following conditions:

(a) $\mathcal{G} = (\mathcal{J}, \mathcal{K} \subseteq \mathcal{J} \times \mathcal{J})$ is a fuzzy graph.

(b) B is a nonempty set of parameters.

(c) $[\xi, B]$ is a VN-FT2SS over $\mathcal{J}$.

(d) $[\psi, B]$ is a VN-FT2SS over $\mathcal{K}$.

(e) FT1SS corresponding to $(\xi(j), \psi(j))\ \forall j \in B$ represents a VN-fuzzy type-1 soft graph (FT1SG).

A FT2SG can also be defined by $\mathbb{G} = <\xi, \psi, B> = \{Z(j) | j \in B\}$, where $Z(j) = (Z_j, \mathcal{N}\mathcal{B}_j)$ such that $Z_j(u) = (\xi_j(u), \psi_j(u))$ for all $u \in \mathcal{N}\mathcal{B}_j$.

Example 1. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 1. Let $B = \{(e_3, 0.5)\}$, $\mathcal{N}\mathcal{B}_{e_3} = \{(e_1, 0.8), (e_4, 0.4)\}$. Suppose that $[\xi, B]$ and $[\psi, B]$ are two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{N}\mathcal{B}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{N}\mathcal{B}_j\right), \quad \text{for all } j \in B. \end{aligned} \tag{1}$$

Define $\xi_{e_3}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow S^{\infty}(P) = 0.3\}$ and $\psi_{e_3}(u) = \left\{vw \in \mathcal{K} | \{v, w\} \subseteq \xi_{e_3}(u)\right\}\ \forall u \in \mathcal{N}\mathcal{B}_{e_3} \subseteq \mathcal{J}$. Then FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned} \xi_{e_3} &= \{\{(e_1, 0.8), \{(e_4, 0.4)\}\}, \{(e_4, 0.4), \{(e_1, 0.8), (e_2, 0.7), (e_3, 0.5), (e_5, 0.9)\}\}\}, \\ \psi_{e_3} &= \{\{(e_1, 0.8), \varnothing\}, \{(e_4, 0.4), \{(e_1e_3, 0.4), (e_1e_2, 0.6)\}\}\}. \end{aligned} \tag{2}$$

Fuzzy type-2 soft graph $\mathbb{G} = (Z(e_3))$ is shown in Figure 2.

Definition 9. Let $\mathbb{G} = (\mathcal{G}, \xi, \psi, B, \mathcal{N}\mathcal{B}_B)$ be a fuzzy type-2 soft graph; the complement of $\mathbb{G}$ is denoted by $\mathbb{G}^c$ and defined by $\mathbb{G}^c = (Z^c(z_1), Z^c(z_2), \ldots, Z^c(z_n))$ for all $z_1, z_2, \ldots, z_n \in B$, where $Z^c(z_i) = (\xi^c(z_i), \psi^c(z_i))$ is the complement of FT1SG corresponding to $Z(z_i) = (\xi(z_i), \psi(z_i))$ for all $z_i \in B, i = 1, 2, \ldots, n$.

Example 2. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 3.

Let $B = \{(e_1, 0.7), (e_5, 0.5)\}$, $\mathcal{N}\mathcal{B}_{e_1} = \{(e_2, 0.8), (e_3, 0.5), (e_6, 0.6)\}$, and $\mathcal{N}\mathcal{B}_{e_5} = \{(e_4, 0.8), (e_6, 0.6)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{N}\mathcal{B}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{N}\mathcal{B}_j\right), \quad \text{for all } j \in B. \end{aligned} \tag{3}$$

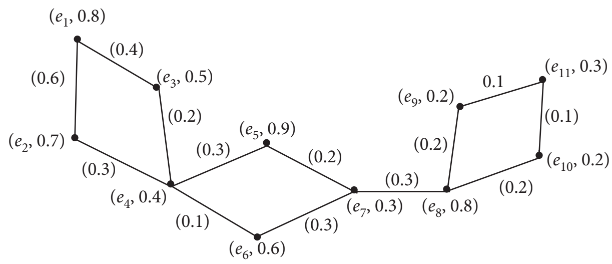

Figure 1: Fuzzy graph $\mathscr{G} = (\mathscr{J}, \mathscr{K})$.

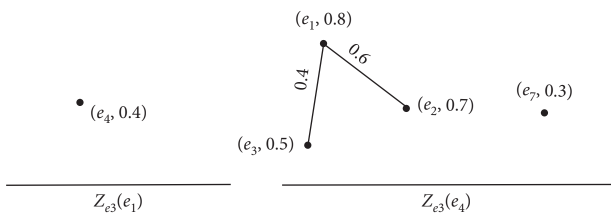

Figure 2: Fuzzy type-2 soft graph $\mathbb{G} = (Z(e_3))$.

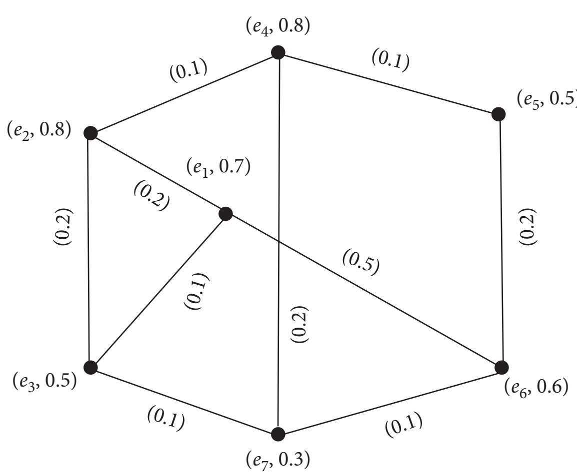

Figure 3: Fuzzy graph $\mathscr{G} = (\mathscr{J}, \mathscr{K})$.

Define

$$
\begin{aligned}
&\xi_{e_1}(u) = \{v \in \mathscr{J} \mid u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.2\},\\
&\psi_{e_1}(u) = \left\{vw \in \mathscr{K} \mid \{v,w\} \subseteq \xi_{e_1}(u)\right\}, \quad \text{for all } u \in \mathscr{N}\mathscr{B}_{e_1} \subseteq \mathscr{J},\\
&\xi_{e_5}(u) = \{v \in \mathscr{J} \mid u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.2\},\\
&\psi_{e_5}(u) = \left\{vw \in \mathscr{K} \mid \{v,w\} \subseteq \xi_{e_5}(u)\right\}, \quad \text{for all } u \in \mathscr{N}\mathscr{B}_{e_5} \subseteq \mathscr{J}.
\end{aligned} \tag{4}
$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$
\begin{aligned}
&\xi_{e_1} = \left\{\begin{array}{c}
\{(e_2,0.8),\{(e_1,0.7),(e_3,0.5),(e_4,0.8),(e_5,0.5)\}\},\\
\{(e_3,0.5),\{(e_1,0.7),(e_2,0.8),(e_6,0.6),(e_7,0.3)\}\},\\
\{(e_6,0.6),\{(e_3,0.5),(e_5,0.5),(e_7,0.3)\}\}
\end{array}\right\},\\
&\psi_{e_1} = \left\{\begin{array}{c}
\{(e_2,0.8),\{(e_1e_3,0.1),(e_4e_5,0.1)\}\},\\
\{(e_3,0.5),\{(e_1e_2,0.2),(e_1e_6,0.5),(e_6e_7,0.1)\}\},\\
\{(e_6,0.6),\{(e_3e_7,0.1)\}\}
\end{array}\right\},\\
&\xi_{e_5} = \{\{(e_4,0.8),\{(e_2,0.8),(e_5,0.5),(e_7,0.3)\}\},\{(e_6,0.6),\\
&\quad \{(e_7,0.3),(e_3,0.5),(e_5,0.5)\}\}\},\\
&\psi_{e_5} = \{\{(e_6,0.6),\{(e_3e_7,0.1)\}\},\{(e_4,0.8),\varnothing\}\}.
\end{aligned} \tag{5}
$$

Then $[\xi^c, B]$ and $[\psi^c, B]$ are defined as follows:

$$
\begin{aligned}
&\xi^c_{e_1} = \left\{\begin{array}{c}
\{(e_2,0.8),\{(e_4,0.8),(e_3,0.5),(e_1,0.7),(e_5,0.5)\}\},\\
\{(e_3,0.5),\{(e_1,0.7),(e_2,0.8),(e_6,0.6)(e_7,0.3)\}\},\\
\{(e_6,0.6),\{(e_5,0.5),(e_3,0.5),(e_7,0.3)\}\}
\end{array}\right\},\\
&\psi^c_{e_1} = \left\{\begin{array}{c}
\{(e_2,0.8),\{(e_1e_4,0.7),(e_1e_3,0.4),(e_1e_5,0.5)(e_4e_3,0.5),(e_3e_5,0.5),(e_4e_5,0.4)\}\},\\
\{(e_3,0.5),\{(e_1e_2,0.5),(e_1e_7,0.3),(e_1e_6,0.1),(e_2e_6,0.6),(e_2e_7,0.3),(e_6e_7,0.2)\}\},\\
\{(e_6,0.6),\{(e_5e_7,0.3),(e_5e_3,0.5),(e_7e_3,0.2)\}\}
\end{array}\right\},\\
&\xi^c_{e_5} = \{\{(e_4,0.8),\{(e_2,0.8),(e_5,0.5),(e_7,0.3)\}\},\{(e_6,0.6),\{(e_7,0.3),(e_3,0.5),(e_5,0.5)\}\}\},\\
&\psi^c_{e_5} = \{\{(e_4,0.8),\{(e_2e_7,0.3),(e_5e_7,0.3),(e_2e_5,0.5)\}\},\{(e_6,0.6),\{(e_7e_3,0.2),(e_5e_3,0.5),(e_5e_7,0.3)\}\}\}.
\end{aligned} \tag{6}
$$

The complement of $\mathbb{G} = (Z_{e_1}, Z_{e_5})$ is a FT2SG $\mathbb{G}^c = (Z^c_{e_1}, Z^c_{e_5})$ such that $Z^c_{e_1} = (\xi^c_{e_1}, \psi^c_{e_1})$ is the complement of FT1SG corresponding to $Z(e_1) = (\xi_{e_1}, \psi_{e_1})$ and $Z^c_{e_5} = (\xi^c_{e_5}, \psi^c_{e_5})$ is the complement of FT1SG corresponding to $Z(e_5) = (\xi_{e_5}, \psi_{e_5})$ as shown in Figure 4.

Definition 10. Let $\mathbb{G}$ be a FT2SG; $\mathbb{G}$ is said to be a regular FT2SG if FT1SG corresponding to $Z(\chi)$ is a regular FT1SG for all $\chi \in B$.

Proposition 1. *If $\mathbb{G}$ is a regular FT2SG, then $\mathbb{G}^c$ is a regular FT2SG.*

Proof. Let $\mathbb{G}$ be a regular FT2SG. Suppose that $(Z_\sigma, \mathcal{NB}_\sigma)$ is a FT1SG corresponding to $Z(\sigma)$ for all $\sigma \in B$; then $Z_\sigma(j)$ for all $j \in \mathcal{NB}_\sigma$ must be a regular fuzzy graph. As we know that complement of a regular graph is regular, $Z^c_\sigma(j)\,\forall j \in \mathcal{NB}_\sigma$ is also a regular fuzzy graph. It provides FT1SG corresponding to a $Z^c(\sigma)$ for all $\sigma \in B$ being regular FT1SG. Thus, $\mathbb{G}^c$ is a regular FT2SG of $\mathbb{G}$. □

Definition 11. Let $\mathbb{G}$ be a FT2SG; $\mathbb{G}$ is said to be an irregular FT2SG if FT1SG corresponding to $Z(\chi)$ is an irregular FT1SG for all $\chi \in B$.

Example 3. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 5. Let $B = \{(e_4, 0.4), (e_5, 0.5)\}$, $\mathcal{NB}_{e_4} = \{(e_1, 0.1), (e_3, 0.3), (e_5, 0.5)\}$, and $\mathcal{NB}_{e_5} = \{(e_4, 0.4), (e_6, 0.9)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{NB}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{NB}_j\right), \quad \text{for all } j \in B. \end{aligned} \tag{7}$$

Define

$$\begin{aligned} \xi_{e_4}(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \le 0.2\}, \\ \psi_{e_4}(u) &= \left\{vw \in \mathcal{K} | v, w \subseteq \xi_{e_4}(u)\right\}, \quad \text{for all } u \in \mathcal{NB}_{e_4} \subseteq \mathcal{J}, \\ \xi_{e_5}(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \le 0.2\}, \\ \psi_{e_5}(u) &= \left\{vw \in \mathcal{K} | v, w \subseteq \xi_{e_5}(u)\right\}, \quad \text{for all } u \in \mathcal{NB}_{e_5} \subseteq \mathcal{J}. \end{aligned} \tag{8}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned} \xi_{e_4} &= \left\{ \begin{array}{c} \{(e_1, 0.1), \{(e_2, 0.2), (e_3, 0.3), (e_4, 0.4), (e_5, 0.5)\}\}, \\ \{(e_3, 0.3), \{(e_1, 0.1), (e_2, 0.2), (e_4, 0.4)\}\}, \\ \{(e_5, 0.5), \{(e_1, 0.1), (e_4, 0.4), (e_6, 0.9)\}\} \end{array} \right\}, \\ \psi_{e_4} &= \left\{ \begin{array}{c} \{(e_1, 0.1), \{(e_2e_3, 0.1), (e_3e_4, 0.2), (e_4e_5, 0.1)\}\}, \{(e_3, 0.3), \{(e_1e_4, 0.1), (e_1e_2, 0.1)\}\}, \\ \{(e_5, 0.5), \{(e_1e_4, 0.1)\}\} \end{array} \right\}, \\ \xi_{e_5} &= \{(e_4, 0.4), \{(e_1, 0.1), (e_2, 0.2), (e_3, 0.3), (e_5, 0.5), (e_6, 0.9)\}, \{(e_6, 0.9), \{(e_4, 0.4), (e_5, 0.5)\}\}\}, \\ \psi_{e_5} &= \{\{(e_4, 0.4), \{(e_1e_2, 0.1), (e_2e_3, 0.1), (e_5e_6, 0.1), (e_6e_3, 0.3)\}\}, \{(e_6, 0.9), \{(e_4e_5, 0.1)\}\}\}. \end{aligned} \tag{9}$$

Then $\mathbb{G} = (Z(e_4), Z(e_5))$ is an irregular FT2SG as shown in Figure 6.

Proposition 2. *If $\mathcal{G}$ is a regular fuzzy graph, then every FT2SG of $\mathcal{G}$ is not necessarily a regular FT2SG.*

Definition 12. Let $\mathbb{G}$ be a FT2SG; $\mathbb{G}$ is called a neighborly irregular FT2SG if FT1SGs corresponding to $Z(\chi)$ are neighborly irregular FT1SG for all $\chi \in B$.

Example 4. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 7. Let $B = \{(e_3, 0.9), (e_5, 0.4)\}$, $\mathcal{NB}_{e_3} = \{(e_1, 0.2), (e_2, 0.4), (e_4, 0.6)\}$, and $\mathcal{NB}_{e_5} = \{(e_4, 0.6), (e_6, 0.1), (e_7, 0.3)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{NB}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{NB}_j\right), \quad \text{for all } j \in B. \end{aligned} \tag{10}$$

Define

$$\begin{aligned} \xi_{e_3}(u) &= (v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \le 0.3), \\ \psi_{e_3}(u) &= \left\{vw \in \mathcal{K}\, |\{v, w\} \subseteq \xi_{e_3}(u)\right\}, \quad \text{for all } u \in \mathcal{NB}_{e_3} \subseteq \mathcal{J}, \\ \xi_{e_5}(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \le 0.4\}, \\ \psi_{e_5}(u) &= \left\{vw \in \mathcal{K}\, |\{v, w\} \subseteq \xi_{e_5}(u)\right\}, \quad \text{for all } u \in \mathcal{NB}_{e_5} \subseteq \mathcal{J}. \end{aligned} \tag{11}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

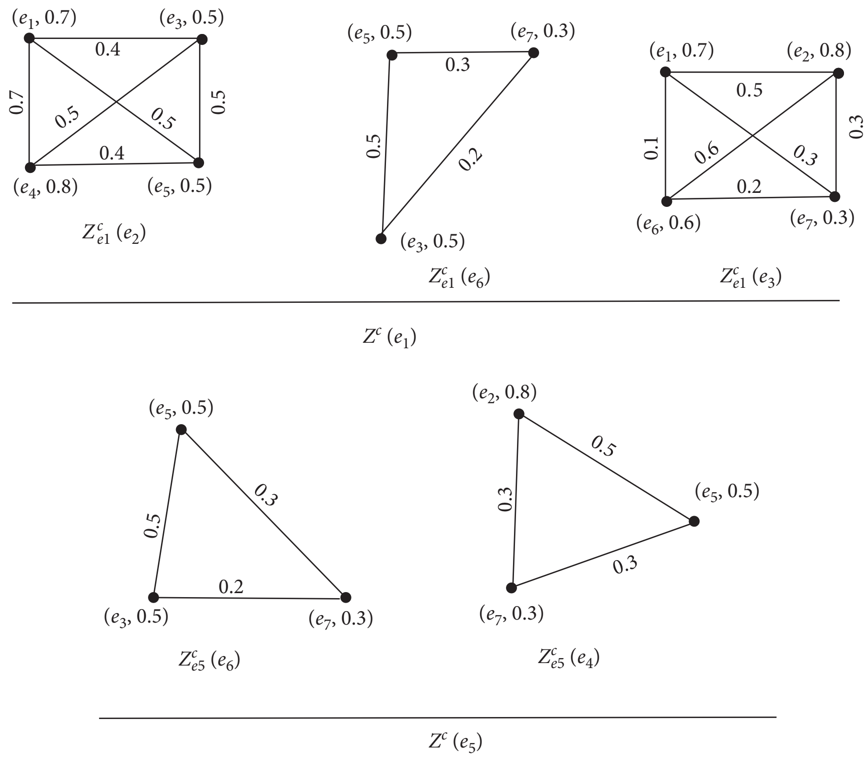

FIGURE 4: $\mathbb{G}^c = (Z^c(e_1), Z^c(e_5))$.

$$
\begin{aligned}
\xi_{e_3} &= \left\{ \begin{array}{c} \{(e_1, 0.2), \{(e_2, 0.4), (e_3, 0.9), (e_4, 0.6)\}\}, \{(e_2, 0.4), \{(e_1, 0.2), (e_3, 0.9), (e_4, 0.6)\}\}, \\ \{(e_4, 0.6), \{(e_1, 0.2), (e_2, 0.4), (e_3, 0.9), (e_5, 0.4)\}\} \end{array} \right\}, \\
\psi_{e_3} &= \left\{ \begin{array}{c} \{(e_1, 0.2), \{(e_2e_3, 0.3), (e_3e_4, 0.4)\}\}, \\ \{(e_2, 0.4), (e_1e_3, 0.2), (e_3e_4, 0.4), (e_1e_4, 0.1)\}, \\ \{(e_4, 0.6), \{(e_1e_2, 0.1), (e_3e_2, 0.3), (e_1e_3, 0.2)\}\} \end{array} \right\}, \\
\xi_{e_5} &= \left\{ \begin{array}{c} \{(e_4, 0.6), (e_1, 0.2), (e_2, 0.4), (e_3, 0.9), (e_5, 0.4)\}, \\ \{(e_6, 0.1), (\{e_5, 0.4\}, (e_8, 0.9), (e_7, 0.3))\}, \\ \{(e_7, 0.3), \{(e_6, 0.1), (e_8, 0.9), (e_5, 0.4)\}\} \end{array} \right\}, \\
\psi_{e_5} &= \left\{ \begin{array}{c} \{(e_4, 0.6), \{(e_1e_2, 0.1), (e_2e_3, 0.3), (e_1e_3, 0.2)\}\}, \\ \{(e_6, 0.1), \{(e_5e_7, 0.2), (e_7e_8, 0.1)\}\}, \\ \{(e_7, 0.3), \{(e_6e_5, 0.2), (e_6e_8, 0.1)\}\} \end{array} \right\}.
\end{aligned} \tag{12}
$$

Then $\mathbb{G} = (Z(e_3), Z(e_5))$ is a neighborly irregular FT2SG as shown in Figure 8.

Definition 13. Let $\mathbb{G}$ be a FT2SG and $Z(\chi) = (Z_\chi, \mathcal{NB}_\chi)$ is a FT1SG for all $\chi \in B$. An edge uv in $\mathbb{G}$ is said to be a FT2S bridge if its removal disconnects $Z_\chi(u)$ for all $u \in \mathcal{NB}_\chi$.

Definition 14. Let $\mathbb{G}$ be a FT2SG and $Z(\chi) = (Z_\chi, \mathcal{NB}_\chi)$ is a FT1SG for all $\chi \in B$. A vertex z in $\mathbb{G}$ is said to be a FT2S cut-vertex if its removal disconnects $Z_\chi(u)$ for all $u \in \mathcal{NB}_\chi$.

Definition 15. Let $\mathbb{G} = (\mathcal{G}, \xi, \psi, B, \mathcal{NB}_B)$ be a FT2SG; $\mathbb{G}$ is called a fuzzy type-2 soft tree (FT2ST) if FT1SGs corresponding to $Z(\chi)$ are FT1STs for all $\chi \in B$.

Example 5. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 9. Let $B = \{(c, 0.6), (f, 0.2)\}, \mathcal{NB}_c = \{(b, 0.2), (d, 0.4)\}$, and $\mathcal{NB}_f = \{(e, 0.5), (g, 0.8)\}$.

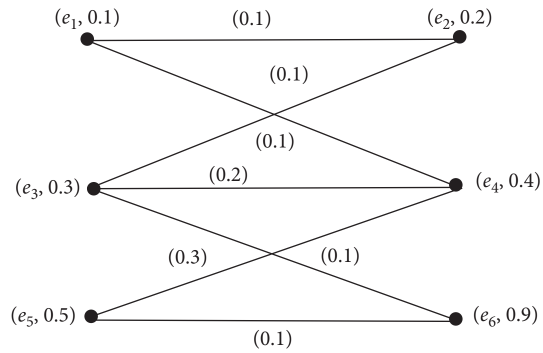

Figure 5: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

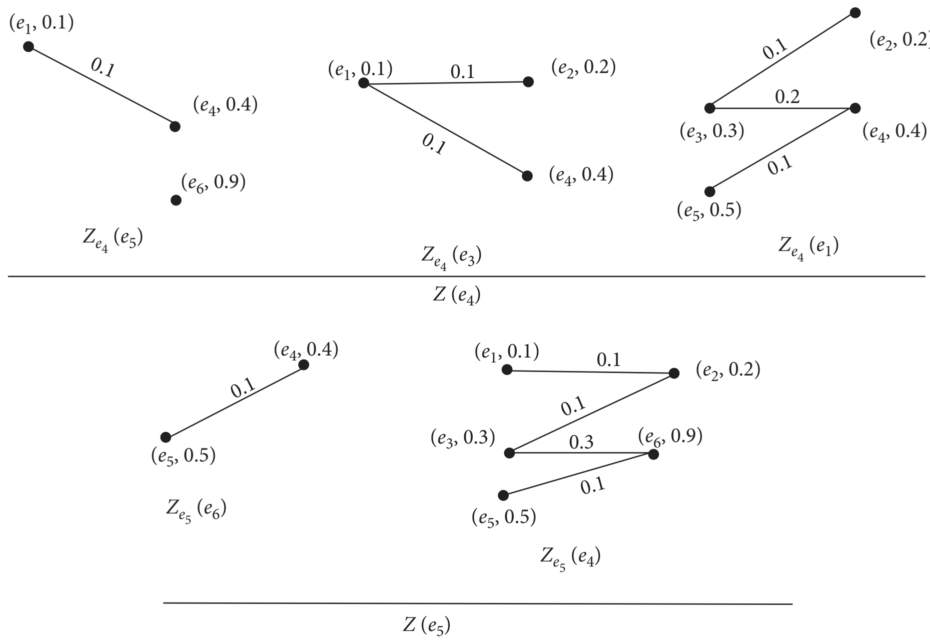

Figure 6: $\mathbb{G} = (Z(e_4), Z(e_5))$.

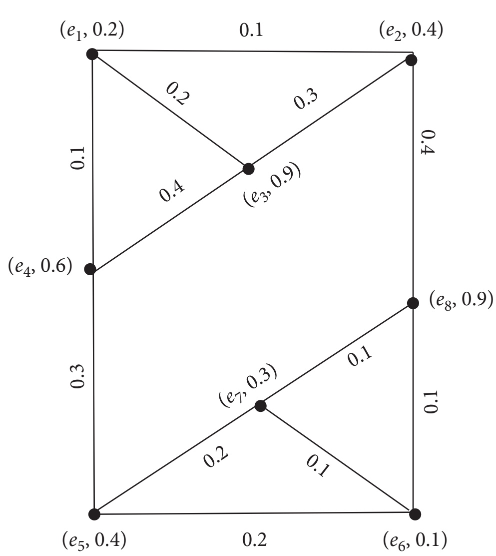

Figure 7: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{N}\mathcal{B}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{N}\mathcal{B}_j\right), \quad \text{for all } j \in B. \end{aligned} \tag{13}$$

Define

$$\begin{aligned} \xi_c(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) = \mathrm{rad}(G)\}, \\ \psi_c(u) &= \{vw \in \mathcal{K} | v, w \subseteq \xi_c(u)\}, \quad \text{for all } u \in \mathcal{N}\mathcal{B}_c \subseteq \mathcal{J}, \\ \xi_f(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) = \mathrm{rad}(G)\}, \\ \psi_f(u) &= \left\{vw \in \mathcal{K} | v, w \subseteq \xi_f(u)\right\}, \quad \text{for all } u \in \mathcal{N}\mathcal{B}_f \subseteq \mathcal{J}. \end{aligned} \tag{14}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

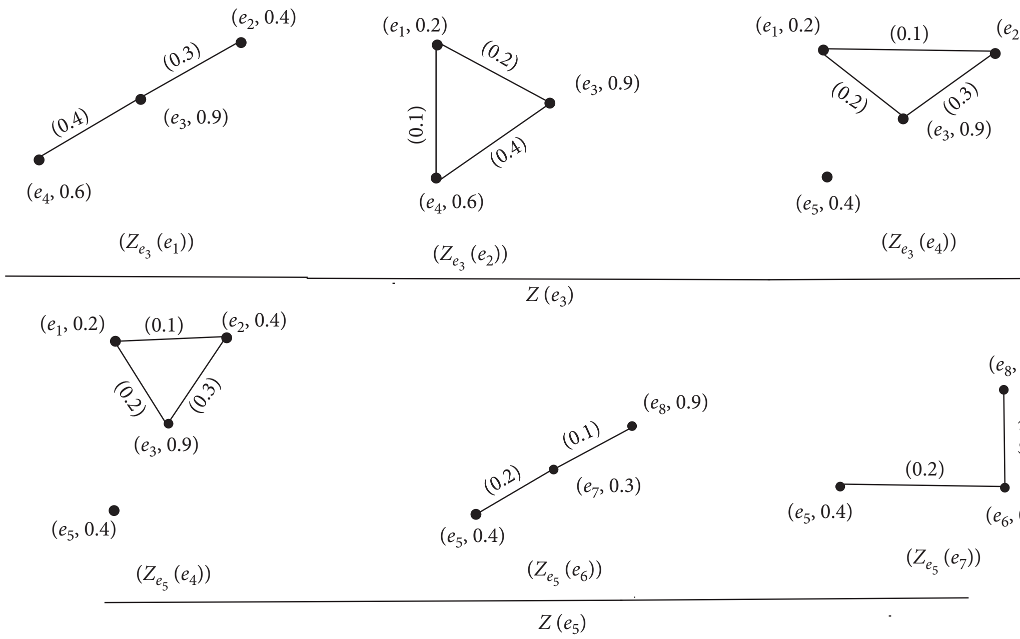

Figure 8: $\mathbb{G} = (Z(e_3), Z(e_5))$.

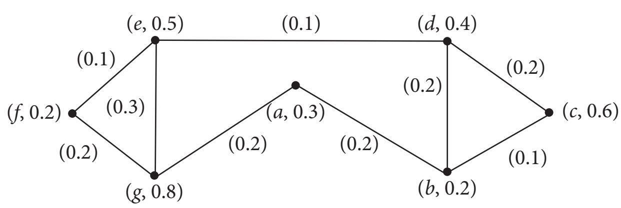

Figure 9: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

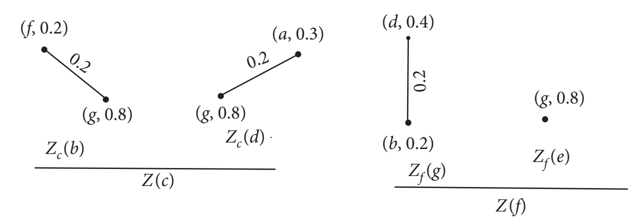

Figure 10: $\mathbb{G} = (Z(c), Z(f))$.

$$\begin{aligned}
\xi_c &= \{\{(b,0.2),\{(g,0.8),(f,0.2)\}\},\{(d,0.4),\{(a,0.3),(g,0.8)\}\}\},\\
\psi_c &= \{\{(b,0.2),\{(gf,0.2)\}\},\{(d,0.4),\{(ag,0.2)\}\}\},\\
\xi_f &= \{\{(e,0.5),\varnothing\},\{(g,0.8),\{(b,0.2),(d,0.4)\}\}\},\\
\psi_f &= \{\{(e,0.5),\varnothing\},\{(g,0.8),\{(bd,0.2)\}\}\}.
\end{aligned} \tag{15}$$

Then $\mathbb{G} = (Z(c), Z(f))$ is a FT2ST as shown in Figure 10. It can also be defined as VN-type-2 soft tree.

Theorem 1. *Let $\mathbb{G}$ be a FT2SG and $Z(\chi) = (Z_\chi, \mathcal{NB}_\chi)$ is a FT1SG for all $\chi \in B$. If $Z_\chi(j)\,\forall j \in \mathcal{NB}_\chi$ is a FT1SG with $n \geq 3$ vertices, then $\mathbb{G}$ will not be a complete FT2SG.*

Proof. Let $\mathbb{G}$ be a FT2SG and $Z(\chi) = (Z_\chi, \mathcal{NB}_\chi)$ is a FT1SG for all $\chi \in B$. On the contrary, assume that $\mathbb{G}$ is a complete FT2SG; then each $Z_\chi(j)\,\forall j \in \mathcal{NB}_\chi$ will also be complete. Let v, w be arbitrary nodes of $Z_\chi(j)$ joined by a line vw. Since $Z_\chi(j)$ having $n \geq 3$ vertices of $\mathcal{G}$ is a FT1SG, then a minimum one vertex η which is connected to v by an edge $v\eta$ and to w by an edge $w\eta$ as $Z_\chi(j)$ be a complete fuzzy graph. Then there is a cycle $vw\eta v$. Therefore, $Z_\chi(j)\,\forall j \in \mathcal{NB}_\chi$ cannot be a FT1ST, which is opposite to the fact that $Z_\chi(j)$ is a connected FT1SG of FT2SG. So, $\mathbb{G}$ is not a complete FT2SG. □

Definition 16. Let $\mathbb{G}$ be a FT2SG and $Z(\chi) = (Z_\chi, \mathcal{NB}_\chi)$ is a FT1SG for all $\chi \in B$. Then $\mathbb{G}$ is called a fuzzy type-2 soft forest if $Z_\chi(j)$ consists of several disjoined fuzzy trees $\forall j \in \mathcal{NB}_\chi$.

Definition 17. Let $\mathbb{G}$ be a FT2SG; $\mathbb{G}$ is said to be a FT2SC if FT1SG corresponding to $Z(\chi)$ is a fuzzy type-1 soft cycle, for all $\chi \in B$.

Example 6. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 11, where

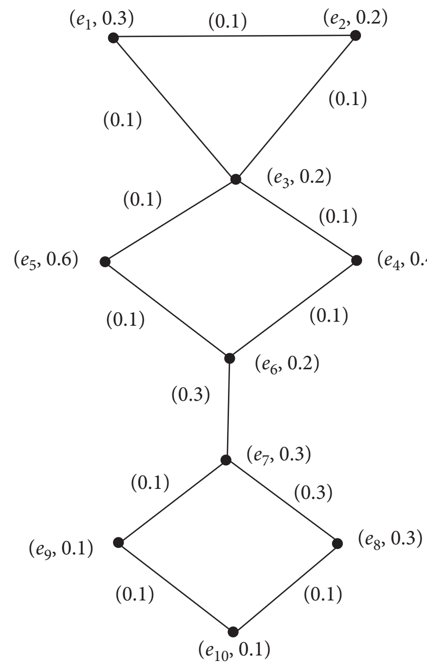

FIGURE 11: Fuzzy graph $\mathscr{G} = (\mathscr{J}, \mathscr{K})$.

$$\begin{aligned}
&\mathscr{J} = \{(e_1, 0.3), (e_2, 0.2), (e_3, 0.2), (e_4, 0.4), (e_5, 0.6), (e_6, 0.2), (e_7, 0.3), (e_8, 0.3), (e_9, 0.1), (e_{10}, 0.1)\},\\
&\mathscr{K} = \left\{ \begin{array}{c} (e_1e_2, 0.1), (e_2e_3, 0.1), (e_3e_1, 0.1), (e_3e_4, 0.1), (e_5e_3, 0.1), \\ (e_5e_6, 0.1), (e_6e_4, 0.1), (e_7e_6, 0.3), (e_7e_9, 0.1), (e_7e_8, 0.3), (e_9e_{10}, 0.1), (e_8e_{10}, 0.1) \end{array} \right\}.
\end{aligned} \tag{16}$$

Let $B = \{(e_8, 0.3), (e_9, 0.1)\}$, $\mathscr{NB}_{e_9} = \{(e_7, 0.3), (e_{10}, 0.1)\}$, and $\mathscr{NB}_{e_8} = \{(e_7, 0.3), (e_{10}, 0.1)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathscr{J}$ and $\mathscr{K}$, respectively. We have

$$\begin{aligned}
&\xi(j) = \left(\xi_j, \mathscr{NB}_j\right),\\
&\psi(j) = \left(\psi_j, \mathscr{NB}_j\right), \quad \text{for all } j \in B.
\end{aligned} \tag{17}$$

Define

$$\begin{aligned}
&\xi_{e_9}(u) = \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow 0.5 \le \mathrm{d}(u, v) \le 0.7\},\\
&\psi_{e_9}(u) = \left\{vw \in \mathscr{K} | \{v, w\} \subseteq \xi_{e_9}(u)\right\}, \quad \text{for all } u \in \mathscr{NB}_{e_9} \subseteq \mathscr{J},\\
&\xi_{e_8}(u) = \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow 0.5 \le \mathrm{d}(u, v) \le 0.7\},\\
&\psi_{e_8}(u) = \left\{vw \in \mathscr{K} | \{v, w\} \subseteq \xi_{e_8}(u)\right\}, \quad \text{for all } u \in \mathscr{NB}_{e_8} \subseteq \mathscr{J}.
\end{aligned} \tag{18}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned}
&\xi_{e_9} = \{\{(e_7, 0.3), \{(e_1, 0.3), (e_2, 0.2), (e_3, 0.2)\}\}, \{(e_{10}, 0.1), \{(e_3, 0.2), (e_4, 0.4), (e_5, 0.6), (e_6, 0.2)\}\}\},\\
&\psi_{e_9} = \left\{ \begin{array}{c} \{(e_7, 0.3), \{(e_3e_2, 0.1), (e_1e_3, 0.1), (e_1e_2, 0.1)\}\}, \\ \{(e_{10}, 0.1), \{(e_4e_3, 0.1), (e_3e_5, 0.1), (e_6e_5, 0.1), (e_6e_4, 0.1)\}\} \end{array} \right\},\\
&\xi_{e_8} = \{\{(e_7, 0.3), \{(e_1, 0.3), (e_2, 0.2), (e_3, 0.2)\}\}, \{(e_{10}, 0.1), \{(e_3, 0.2), (e_4, 0.4), (e_5, 0.6), (e_6, 0.2)\}\}\},\\
&\psi_{e_8} = \left\{ \begin{array}{c} \{(e_7, 0.3), \{(e_3e_2, 0.1), (e_1e_3, 0.1), (e_1e_2, 0.1)\}\}, \\ \{(e_{10}, 0.1), \{(e_4e_3, 0.1), (e_3e_5, 0.1), (e_6e_5, 0.1), (e_6e_4, 0.1)\}\} \end{array} \right\}.
\end{aligned} \tag{19}$$

We can check that $\mathbb{G} = (Z(e_9), Z(e_8))$ is a FT2SC as shown in Figure 12. It is also defined as a fuzzy VN-type-2 soft cycle.

Example 7. Let $\mathscr{G} = (\mathscr{J}, \mathscr{K})$ be a fuzzy graph as shown in Figure 13. Let $B = \{(a, 0.4), (b, 0.2)\} \subset \mathscr{J}$, $\mathscr{NB}_a = \{(b, 0.2), (c, 0.3), (d, 0.3)\}$, and $\mathscr{NB}_b = \{(a, 0.4), (c, 0.3), (d, 0.3)\}$. Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathscr{J}$ and $\mathscr{K}$, respectively. We have

$$\begin{aligned}
&\xi(j) = \left(\xi_j, \mathscr{NB}_j\right),\\
&\psi(j) = \left(\psi_j, \mathscr{NB}_j\right), \quad \text{for all } j \in B.
\end{aligned} \tag{20}$$

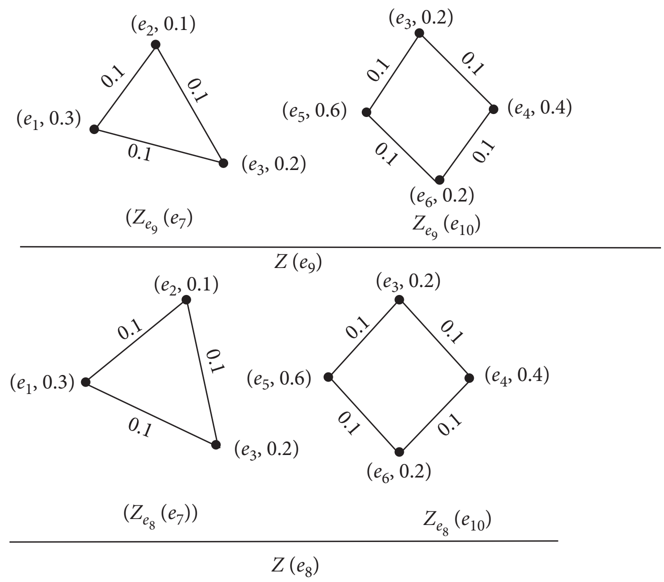

Figure 12: $\mathbb{G} = (Z(e_9), Z(e_8))$.

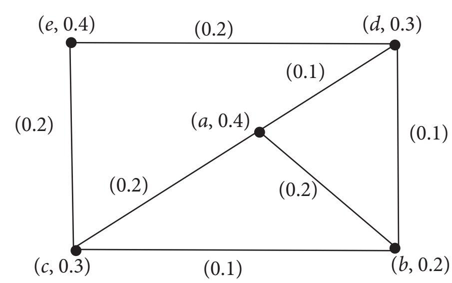

Figure 13: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

Define $\xi_a(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.2\}$, $\psi_a(u) = \{vw \in \mathcal{K} | v,\ w \subseteq \xi_a(u)\}\ \forall u \in \mathcal{N}\mathcal{B}_a \subseteq \mathcal{J}$, $\xi_b(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.2\}$, $\psi_b(u) = \{vw \in \mathcal{K} | v, w \subseteq \xi_b(u)\} \forall u \in \mathcal{N}\mathcal{B}_b \subseteq \mathcal{J}$.

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned}
\xi_a &= \left\{ \begin{array}{c} \{(b, 0.2), \{(a, 0.4), (c, 0.3), (d, 0.3)\}\}, \{(c, 0.3), \{(a, 0.4), (b, 0.2), (d, 0.3), (e, 0.4)\}\}, \\ \{(d, 0.3), \{(a, 0.4), (b, 0.2), (c, 0.3), (e, 0.4)\}\} \end{array} \right\}, \\
\psi_a &= \left\{ \begin{array}{c} \{(b, 0.2), \{(ad, 0.1), (ac, 0.2)\}\}, \{(c, 0.3), \{(ab, 0.2), (ad, 0.1), (bd, 0.1), (de, 0.2)\}\}, \\ \{(d, 0.3), \{(ab, 0.2), (ac, 0.2), (bc, 0.1), (ce, 0.2)\}\} \end{array} \right\}, \\
\xi_b &= \left\{ \begin{array}{c} \{(a, 0.4), \{(c, 0.3), (b, 0.2), (d, 0.3)\}\}, \{(c, 0.3), \{(a, 0.4), (b, 0.2), (e, 0.4), (d, 0.3)\}\}, \\ \{(d, 0.3), \{(a, 0.4), (b, 0.2), (c, 0.3), (e, 0.4)\}\} \end{array} \right\}, \\
\psi_b &= \left\{ \begin{array}{c} \{(a, 0.4), \{(bc, 0.1), (bd, 0.1)\}\}, \{(c, 0.3), \{(ad, 0.1), (ed, 0.2), (ab, 0.2), (bd, 0.1)\}\}, \\ \{(d, 0.3), \{(ec, 0.2), (ab, 0.2), (ac, 0.2), (bc, 0.1)\}\} \end{array} \right\}.
\end{aligned} \tag{21}$$

$Z(a) = (\xi(a), \psi(a))$ and $Z(b) = (\xi(b), \psi(b))$ are FT1SGs as shown in Figure 14. We can see that $Z_a(b) = (\xi_a(b), \psi_a(b))$, $Z_a(c) = (\xi_a(c), \psi_a(c))$, $Z_a(d) = (\xi_a(d), \psi_a(d))$, $Z_b(c) = (\xi_b(c), \psi_b(c))$ and $Z_b(d) = (\xi_b(d), \psi_b(d))$ are all not trees. Hence $\mathbb{G} = (Z(a), Z(b))$ is not a FT2ST and $\mathbb{G}$ is also not a FT2SC.

Proposition 3. *Every fuzzy type-2 soft cycle is a regular fuzzy type-2 soft cycle.*

Proof. Let $\mathbb{G}$ be a FT2SC. Let $(Z_\chi, \mathcal{N}\mathcal{B}_\chi)$ be a T1FSC corresponding to $Z(\chi)$ for every $\chi \in B$. Then, $Z_\chi(j)$ is a cycle $\forall j \in \mathcal{N}\mathcal{B}_\chi$. We know that cycle is a path that is closed and

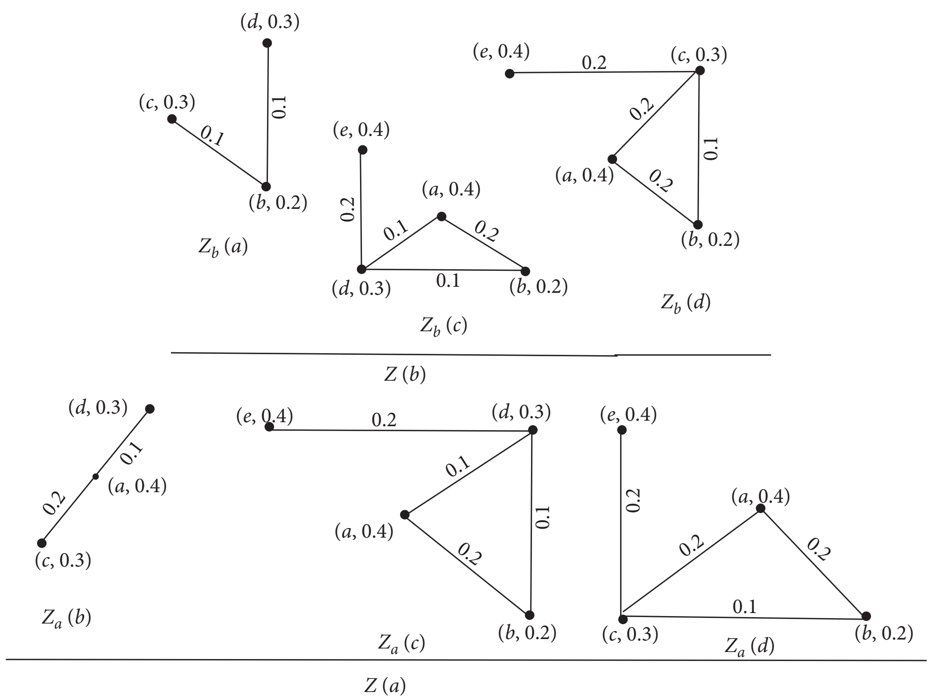

Figure 14: $\mathbb{G} = (Z(b), Z(a))$.

every vertex of cycle is of degree 2; this signifies that $Z_\chi(j)$ is a regular fuzzy graph for all $j \in \mathcal{NB}_\chi$. Therefore, $(Z_\chi, \mathcal{NB}_\chi)$ is a regular FT1SG, for all $\chi \in B$. Hence $\mathbb{G}$ is a regular FT2SG. □

Definition 18. Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. $\mathbb{G}_2$ is a fuzzy type-2 soft subtree (FT2SST) of $\mathbb{G}_1$ if

(i) $B_2 \subseteq B_1$

(ii) For each $j \in B_2$, FT1ST corresponding to $Z_{2(j)} = (\xi_{2(j)}, \psi_{2(j)})$ is a fuzzy type-1 soft subtree (FT1SST) of FT1ST corresponding to $Z_{1(j)} = (\xi_{1(j)}, \psi_{1(j)})$

Example 8. Let $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ be a fuzzy graph as shown in Figure 15, where

$$\begin{aligned}\mathcal{J} &= \{(e_1, 0.3), (e_2, 0.2), (e_3, 0.2), (e_4, 0.4), (e_5, 0.6), (e_6, 0.2), (e_7, 0.3)\},\\ \mathcal{K} &= \{(e_1e_2, 0.1), (e_2e_3, 0.1), (e_3e_4, 0.2), (e_5e_4, 0.1), (e_5e_6, 0.1), (e_7e_6, 0.2), (e_7e_1, 0.1)\}.\end{aligned} \tag{22}$$

Let $B = \{(e_2, 0.2), (e_4, 0.4)\}$, $B^* = \{(e_2, 0.2), (e_4, 0.4)\}$, $\mathcal{NB}_{e_2} = \{(e_1, 0.3), (e_3, 0.2)\}$, $\mathcal{NB}_{e_4} = \{(e_3, 0.2), (e_5, 0.6)\}$. Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\xi(j) = \left(\xi_j, \mathcal{NB}_j\right) \psi(j) = \left(\psi_j, \mathcal{NB}_j\right), \quad \text{for all } j \in B. \tag{23}$$

Define $\xi_{e_2}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.2\}$, $\psi_{e_2}(u) = \left\{vw \in \mathcal{K} | v, w \subseteq \xi_{e_2}(u)\right\}$ $\forall u \in \mathcal{NB}_{e_2} \subseteq \mathcal{J}$ and $\xi_{e_4}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) = 0.2\}$, $\psi_{e_4}(u) = \left\{vw \in \mathcal{K} | v, w \subseteq \xi_{e_4}(u)\right\}$ $\forall u \in \mathcal{NB}_{e_4} \subseteq \mathcal{J}$.

Then FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned}\xi_{e_2} &= \{\{(e_1, 0.3), \{(e_2, 0.2), (e_3, 0.2), (e_7, 0.3)\}\}, \{(e_3, 0.2), \{(e_1, 0.3), (e_2, 0.2), (e_4, 0.4)\}\}\},\\ \psi_{e_2} &= \{\{(e_1, 0.3), \{(e_2e_3, 0.1)\}\}, \{(e_3, 0.2), \{(e_1e_2, 0.1)\}\}\},\\ \xi_{e_4} &= \{\{(e_3, 0.2), \{(e_1, 0.3), (e_4, 0.4)\}\}, \{(e_5, 0.6), \varnothing\}\},\\ \psi_{e_4} &= \{\{(e_3, 0.2), \varnothing\}, \{(e_5, 0.6), \varnothing\}\}.\end{aligned} \tag{24}$$

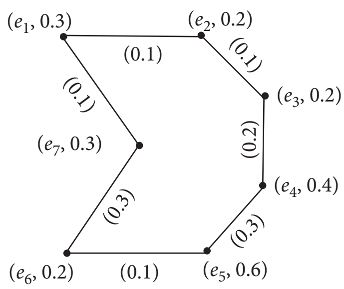

FIGURE 15: $\mathcal{G} = (\mathcal{J}, \mathcal{K})$ is a fuzzy graph.

$\mathbb{G} = (Z(e_2), Z(e_4))$ is a FT2ST as shown in Figure 16.

Let $[\xi', N]$ and $[\psi', N]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi'(j) &= \left(\xi_j, \mathcal{N}\mathcal{B}_j\right), \\ \psi'(j) &= \left(\psi_j, \mathcal{N}\mathcal{B}_j\right), \quad \text{for all } j \in N. \end{aligned} \tag{25}$$

Define $\xi'_{e_2}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.3\}$, $\psi'_{e_2}(u) = \left\{vw \in \mathcal{K} | v, w \subseteq \xi'_{e_2}(u)\right\}$ $\forall u \in \mathcal{N}\mathcal{B}_{e_2} \subseteq \mathcal{J}$ and $\xi'_{e_4}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.3\}$, $\psi'_{e_4}(u) = \left\{vw \in \mathcal{K} | v, w \subseteq \xi'_{e_4}(u)\right\}$ $\forall u \in \mathcal{N}\mathcal{B}_{e_4} \subseteq \mathcal{J}$.

FT2SSs $[\xi', N]$ and $[\psi', N]$ are defined as follows:

$$\begin{aligned} \xi'_{e_2} &= \{\{(e_1, 0.3), \{(e_2, 0.2), (e_3, 0.2), (e_7, 0.3)\}\}, \{(e_3, 0.2), \{(e_1, 0.3), (e_2, 0.2), (e_4, 0.4), (e_7, 0.3)\}\}\}, \\ \psi'_{e_2} &= \{\{(e_1, 0.3), \{(e_2 e_3, 0.1)\}\}, \{(e_3, 0.2), \{(e_1 e_7, 0.1), (e_1 e_2, 0.1)\}\}\}, \\ \xi'_{e_4} &= \{\{(e_3, 0.2), \{(e_1, 0.3), (e_2, 0.2), (e_4, 0.4), (e_7, 0.3)\}\}, \{(e_5, 0.6), \{(e_4, 0.4), (e_6, 0.2)\}\}\}, \\ \psi'_{e_4} &= \{\{(e_3, 0.2), \{(e_2 e_1, 0.1)(e_1 e_7, 0.1)\}\}, \{(e_5, 0.6), \varnothing\}\}. \end{aligned} \tag{26}$$

$\mathbb{G}' = (Z'(e_2), Z'(e_4))$ is a FT2SST of $\mathbb{G}$ as shown in Figure 17. We can see that $B \subset B^*$ and $Z(e_2) \subseteq Z'(e_2), Z(e_4) \subseteq Z'(e_4)$. Hence, $\mathbb{G}$ is a FT2SST of $\mathbb{G}'$.

Theorem 2. *Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. Then $\mathbb{G}_2$ is said to be a FT2SST of $\mathbb{G}_1$ if and only if $\xi_2 \subseteq \xi_1$ and $\psi_2 \subseteq \psi_1$.*

Proof. Let $\mathbb{G}_2$ be a FT2SST of $\mathbb{G}_1$. Then, by using the definition of FT2SST,

(i) $B_2 \subseteq B_1$

(ii) For all $j \in B_2$, FT1ST corresponding to $Z_2(j) = (\xi_2(j), \psi_2(j))$ is a FT1SST of FT1ST corresponding to $Z_1(j) = (\xi_1(j), \psi_1(j))$

Since FT1ST corresponding to $Z_2(j)$ is a FT1SST of FT1ST corresponding to $Z_1(j)$ for all $j \in B_2$, we have $\xi_2 \subseteq \xi_1$ and $\psi_2 \subseteq \psi_1$ $\forall j \in B_2$. Conversely, we have $\xi_2(j) \subseteq \xi_1(j)$ and $\psi_2(j) \subseteq \psi_1(j) \forall j \in B_2$. As $\mathbb{G}_1$ is a fuzzy type-2 soft tree, fuzzy type-1 soft set corresponding to $Z_1(j)$ forms a FT1ST of $\mathbb{G}_2$ for all $j \in B_1$. Also, $\mathbb{G}_2$ is a fuzzy type-2 soft tree, and fuzzy type-1 soft set corresponding to $Z_2(j)$ forms a FT1ST of $\mathbb{G}_1$ for all $j \in B_2$. This implies that FT1ST corresponding to $Z_2(j)$ is a FT1SST of FT1ST corresponding to $Z_1(j)$ for all $j \in B_2$. Hence, $\mathbb{G}_2$ is a FT2SST of $\mathbb{G}_1$. □

Definition 19. Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. The union of $\mathbb{G}_1$ and $\mathbb{G}_2$ is denoted by $\mathbb{G}_1 \cup \mathbb{G}_2 = \mathbb{G} = <\xi, \psi, P>$, where $P = B_1 \cup B_2$, such that

$$\begin{aligned} \xi(v) &= \begin{cases} \xi_1(v), & \text{if } v \in B_1 - B_2, \\ \xi_2(v), & \text{if } v \in B_2 - B_1, \\ \xi_1(v) \cup \xi_2(v), & \text{if } v \in B_1 \cap B_2, \end{cases} \\ \psi(v) &= \begin{cases} \psi_1(v), & \text{if } v \in B_1 - B_2, \\ \psi_2(v), & \text{if } v \in B_2 - B_1, \\ \psi_1(v) \cup \psi_2(v), & \text{if } v \in B_2 \cap B_1, \end{cases} \end{aligned} \tag{27}$$

where $(\xi_1(v) \cup \xi_2(v), \psi_1(v) \cup \psi_2(v))$ for all $v \in B_1 \cap B_2$ relates to the fuzzy type-1 soft union between the relevant FT1STs corresponding to $(\xi_1(v), \psi_1(v))$ and $(\xi_2(v), \psi_2(v))$, respectively. It can be written as $\mathbb{G}_1 \cup \mathbb{G}_2 = \{Z(v) = (\xi(v), \psi(v)) | v \in P\}$.

Theorem 3. *Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs with $B_1 \cap B_2 = \varnothing$. Then $\mathbb{G}_1 \cup \mathbb{G}_2$ is a FT2ST.*

Proof. Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. The union of $\mathbb{G}_1$ and $\mathbb{G}_2$ is denoted by $\mathbb{G}_1 \cup \mathbb{G}_2 = \mathbb{G} = <\xi, \psi, P>$, where $P = B_1 \cup B_2$ is defined $\forall v \in P$:

$$\begin{aligned} \xi(v) &= \begin{cases} \xi_1(v), & \text{if } v \in B_1 - B_2, \\ \xi_2(v), & \text{if } v \in B_2 - B_1, \\ \xi_1(v) \cup \psi_2(v), & \text{if } v \in B_1 \cap B_2, \end{cases} \\ \psi(v) &= \begin{cases} \psi_1(v), & \text{if } v \in B_1 - B_2, \\ \psi_2(v), & \text{if } v \in B_2 - B_1, \\ \psi_1(v) \cup \psi_2(v), & \text{if } v \in B_2 \cap B_1, \end{cases} \end{aligned} \tag{28}$$

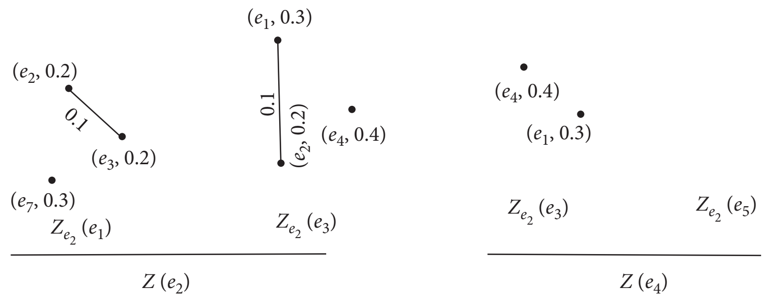

FIGURE 16: $\mathbb{G} = (Z(e_2), Z(e_4))$.

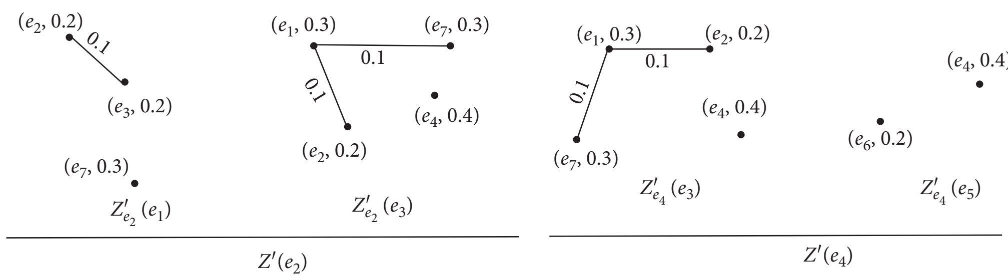

FIGURE 17: $\mathbb{G}' = (Z_{e_4}', Z_{e_2}')$.

where $\psi_1(v) \cup \psi_2(v)$ for all $v \in B_2 \cap B_1$ relates to the fuzzy type-1 soft extended union among the relevant FT1STs corresponding to $\psi_1(v)$ and $\psi_2(v)$, respectively, and $\xi_1(v) \cup \xi_2(v)$ for all $v \in B_1 \cap B_2$ relates to the fuzzy type-1 soft union between the relevant FT1STs corresponding to $\xi_1(v)$ and $\xi_2(v)$, respectively. Since $\mathbb{G}_1$ is a FT2ST, FT1ST corresponding to $(\xi_1(j)$ and $\psi_1(j))$ is a FT2ST for all $j \in B_1 - B_2$.

Since $\mathbb{G}_2$ is a FT2ST, FT1ST corresponding to $(\xi_2(j)$ and $\psi_2(j))$ is a FT2ST for all $j \in B_2 - B_1$. It is given that $B_1 \cap B_2 = \varnothing$. Thus, $\mathbb{G}_1 \cup \mathbb{G}_2 = \mathbb{G} = \ <\xi, \psi, B_1 \cap B_2>$ is a FT2ST. □

Definition 20. Let $\mathbb{G}_1 = <\xi_1, \psi_1, B_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. The intersection of $\mathbb{G}_1$ and $\mathbb{G}_2$ is denoted by $\mathbb{G}_1 \cap \mathbb{G}_2 = \mathbb{G} = <\xi, \psi, P>$, where $P = B_1 \cup B_2$ such that

$$\xi(v) = \begin{cases} \xi_1(v), & \text{if } v \in B_1 - B_2, \\ \xi_2(v), & \text{if } v \in B_2 - B_1, \\ \xi_1(v) \cap \xi_2(v), & \text{if } v \in B_1 \cap B_2, \end{cases}$$
$$\psi(v) = \begin{cases} \psi_1(v), & \text{if } v \in B_1 - B_2, \\ \psi_2(v), & \text{if } v \in B_2 - B_1, \\ \psi_1(v) \cap \psi_2(v), & \text{if } v \in B_2 \cap B_1, \end{cases} \quad (29)$$

where $(\xi_1(v) \cap \xi_2(v), \psi_1(v) \cap \psi_2(v))$ for all $v \in B_1 \cap B_2$ relates to the fuzzy type-1 soft intersection between the relevant FT1STs corresponding to $(\xi_1(v), \psi_1(v))$ and $(\xi_2(v), \psi_2(v))$, respectively.

It can be written as $\mathbb{G}_1 \cap \mathbb{G}_2 = \{Z(v) = \ (\xi(v), \psi(v)) \,|\, v \in P\}$.

Example 9. Let $\mathcal{G}$ be a fuzzy graph as shown in Figure 18. Let $B = \{(a, 0.9), (b, 0.1)\}$, $B^* = \{(a, 0.9), (e, 0.2)\}$. It can be written as $\mathcal{NB}_a = \{(v, 0.7), (b, 0.1)\}$, $\mathcal{NB}_b = \{(a, 0.9), (c, 0.1)\}$, $\mathcal{NB}_e = \{(d, 0.3), (f, 0.5)\}$. Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathcal{NB}_j\right), \\ \psi(j) &= \left(\psi_j, \mathcal{NB}_j\right), \quad \text{for all } j \in B. \end{aligned} \quad (30)$$

Define $\xi_a(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.3\}$, $\psi_a(u) = \{vw \in \mathcal{K} | \{v, w\} \subseteq \xi_a(u)\}\ \forall u \in \mathcal{NB}_a$ and $\xi_b(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.3\}$, $\psi_b(u) = \{vw \in \mathcal{K} | \{v, w\} \subseteq \xi_b(u)\}\ \forall z \in \mathcal{NB}_b$.

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned} \xi_a &= \{\{(b, 0.1), \{(a, 0.9), (c, 0.1), (d, 0.3)\}\}, \{(v, 0.7), \{(d, 0.3), (g, 0.1), (f, 0.5), (h, 0.2)\}\}\}, \\ \psi_a &= \{\{(b, 0.1), \{(cd, 0.1)\}\}, \{(v, 0.7), \{(gh, 0.1), (gf, 0.1)\}\}\}, \\ \xi_b &= \{\{(c, 0.1), \{(a, 0.9), (b, 0.1), (d, 0.3), (e, 0.2)\}\}, \{(a, 0.9), \{(b, 0.1), (c, 0.1), (d, 0.3)\}\}\}, \\ \psi_b &= \{\{(c, 0.1), \{(ab, 0.1), (ed, 0.2)\}\}, \{(a, 0.9), \{(bc, 0.1), (cd, 0.1)\}\}\}. \end{aligned} \quad (31)$$

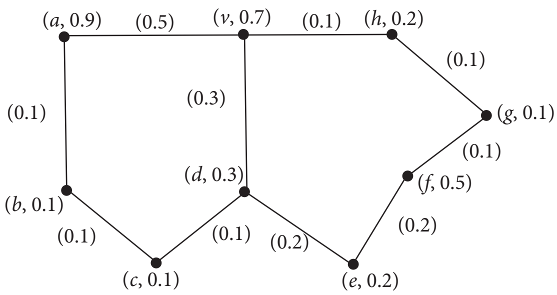

Figure 18: Fuzzy graph $\mathscr{G} = (\mathscr{J}, \mathscr{K})$.

Let $[\xi', B^*]$ and $[\psi', B^*]$ be two FT2SSs over $\mathscr{J}$ and $\mathscr{K}$, respectively. We have

$$\begin{aligned} \xi'(j) &= \left(\xi_j, \mathscr{N}\mathscr{B}_j\right), \\ \psi'(j) &= \left(\psi_j, \mathscr{N}\mathscr{B}_j\right), \quad \text{for all } j \in B^*, \\ \text{define } \xi'_a(u) &= \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.2\}, \\ \psi'_a(u) &= \{vw \in \mathscr{K} | \{v,w\} \subseteq \xi'_a(u)\}, \quad \forall u \in \mathscr{N}\mathscr{B}_a \subseteq \mathscr{J}, \\ \xi'_e(u) &= \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.3\}, \\ \psi'_e(u) &= \{vw \in \mathscr{K} | \{v,w\} \subseteq \xi'_e(u)\}, \quad \forall u \in \mathscr{N}\mathscr{B}_e \subseteq \mathscr{J}. \end{aligned} \tag{32}$$

The FT2SSs $[\xi', B^*]$ and $[\psi', B^*]$ are defined as follows:

$$\begin{aligned} \xi'_a &= \{\{(b,0.1), \{(a,0.9), (c,0.1), (d,0.3)\}\}, \{(v,0.7), \{(g,0.1), (h,0.2)\}\}\}, \\ \psi'_a &= \{\{(b,0.1), \{(cd,0.1)\}\}, \{(v,0.7), \{(gh,0.1)\}\}\}, \\ \xi'_e &= \left\{ \begin{array}{c} \{(d,0.3), \{(a,0.9), (b,0.1), (c,0.1), (e,0.2), (v,0.7)\}\}, \\ \{(f,0.5), \{(e,0.2), (g,0.1), (h,0.2), (v,0.7)\}\} \end{array} \right\}, \\ \psi'_e &= \{\{(d,0.3), \{(bc,0.1), (av,0.5), (ba,0.1)\}\}, \{(f,0.5), \{(vh,0.1), (hg,0.1)\}\}\}. \end{aligned} \tag{33}$$

Then $\mathbb{G} = (Z(a), Z(b))$ and $\mathscr{G}' = (Z'(e), Z'(a))$ are FT2STs as shown in Figure 19. By the definition of intersection of FT2STs, $\xi(a) = \xi(a) \cap \xi'(a)$ and $\psi(a) = \psi(a) \cap \psi'(a)$ where $a \in B^* \cap B$.

Therefore, $\mathbb{G}_1 \cap \mathbb{G}_2 = \mathbb{G} = <\xi, \psi, B \cup B^*>$ is a FT2ST as shown in Figure 20.

Definition 21. Let $\mathbb{G}_1 = <\xi_1, \psi_1, M_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. The AND operation of $\mathbb{G}_1$ and $\mathbb{G}_2$ is denoted by $\mathbb{G}_1 \wedge \mathbb{G}_2 = \mathbb{G} = <\xi, \psi, B_1 \times B_2>$ such that $\xi(\chi, \eta) = \xi_1(\chi) \wedge \xi_2(\eta)$, $\psi(\chi, \eta) = \psi_1(\chi) \wedge \psi_2(\eta)$ for all $(\chi, \eta) \in B_1 \times B_2$. $(\xi(\chi, \eta), \psi(\chi, \eta))$ for all $(\chi, \eta) \in B_1 \times B_2$ is the fuzzy type-1 soft AND operation between the relevant FT1SGs corresponding to $(\xi_1(\chi), \psi_1(\chi))$ and $(\xi_2(\eta), \psi_2(\eta))$, respectively.

Example 10. Let $\mathscr{G} = (\mathscr{J}, \mathscr{K})$ be the fuzzy graph as shown in Figure 21, where

$$\begin{aligned} \mathscr{J} &= \{(e_1, 0.3), (e_2, 0.1), (e_3, 0.2), (e_4, 0.4), (e_5, 0.6), \\ &\quad (e_6, 0.2), (e_7, 0.3), (e_8, 0.3)\}, \\ \mathscr{K} &= \{(e_1e_2, 0.1), (e_2e_3, 0.1), (e_3e_4, 0.2), (e_5e_4, 0.3), \\ &\quad (e_5e_6, 0.1), (e_7e_6, 0.2), (e_7e_8, 0.2), (e_8e_1, 0.1)\}. \end{aligned} \tag{34}$$

Let $B = \{(e_3, 0.2), (e_4, 0.4)\}, B^* = \{(e_7, 0.3)\}$, $\mathscr{N}\mathscr{B}_{e_3} = \{(e_2, 0.1), (e_4, 0.4)\}, \mathscr{N}\mathscr{B}_{e_4} = \{(e_3, 0.2), (e_5, 0.6)\}$, $\mathscr{N}\mathscr{B}_{e_7} = \{(e_6, 0.2), (e_8, 0.3)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathscr{J}$ and $\mathscr{K}$, respectively. We have

$$\begin{aligned} \xi(j) &= \left(\xi_j, \mathscr{N}\mathscr{B}_j\right), \\ \psi(j) &= \left(\psi_j, \mathscr{N}\mathscr{B}_j\right), \quad \text{for all } J \in B, \\ \text{define } \xi_{e_3}(u) &= \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.3\}, \\ \psi_{e_3}(u) &= \left\{vw \in \mathscr{K} | \{v,w\} \subseteq \xi_{e_3}(u)\right\}, \quad \forall u \in \mathscr{N}\mathscr{B}_{e_3}, \\ \xi_{e_4}(u) &= \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow \mathrm{d}(u,v) \le 0.4\}, \\ \psi_{e_4}(u) &= \left\{vw \in \mathscr{K} | \{v,w\} \subseteq \xi_{e_4}(u)\right\}, \quad \forall z \in \mathscr{N}\mathscr{B}_{e_3}. \end{aligned} \tag{35}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned} \xi_{e_3} &= \{\{(e_4, 0.4), \{(e_2, 0.1), (e_3, 0.2), (e_5, 0.6)\}\}, \{(e_2, 0.1), \{(e_1, 0.3), (e_3, 0.2), (e_4, 0.4), (e_8, 0.3)\}\}\}, \\ \psi_{e_3} &= \{\{(e_4, 0.4), \{(e_2e_3, 0.1)\}\}, \{(e_2, 0.1), \{(e_3e_4, 0.2), (e_1e_8, 0.1)\}\}\}, \\ \xi_{e_4} &= \{\{(e_5, 0.6), \{(e_4, 0.4), (e_6, 0.2), (e_7, 0.3)\}\}, \{(e_3, 0.2), \{(e_1, 0.3), (e_2, 0.1), (e_4, 0.4), (e_8, 0.3)\}\}\}, \\ \psi_{e_4} &= \{\{(e_3, 0.2), \{(e_8e_1, 0.1), (e_1e_2, 0.1)\}\}, \{(e_5, 0.6), \{(e_6e_7, 0.2)\}\}\}. \end{aligned} \tag{36}$$

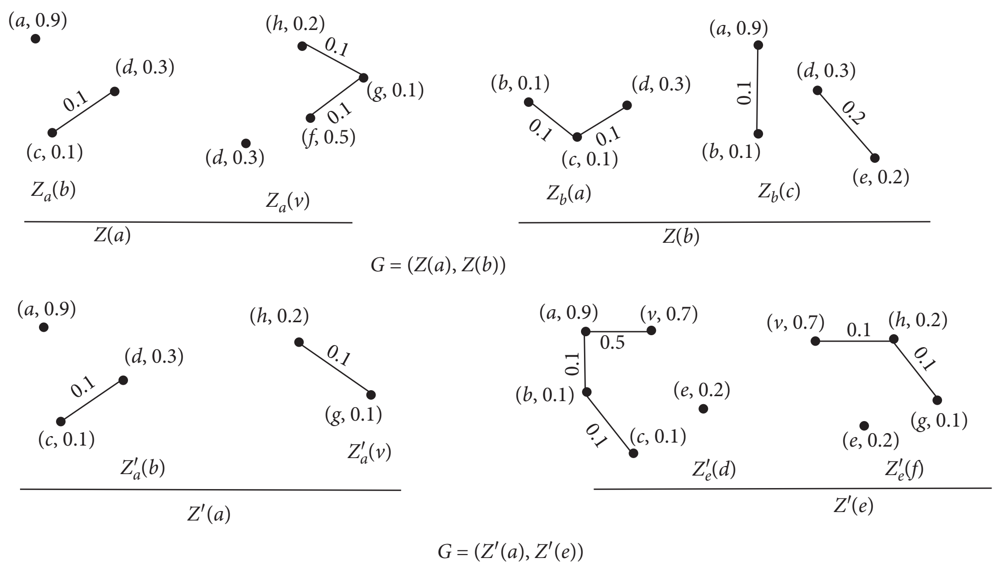

Figure 19: $\mathbb{G} = (Z(a), Z(b))$ and $\mathbb{G}' = (Z'(e), Z'(a))$.

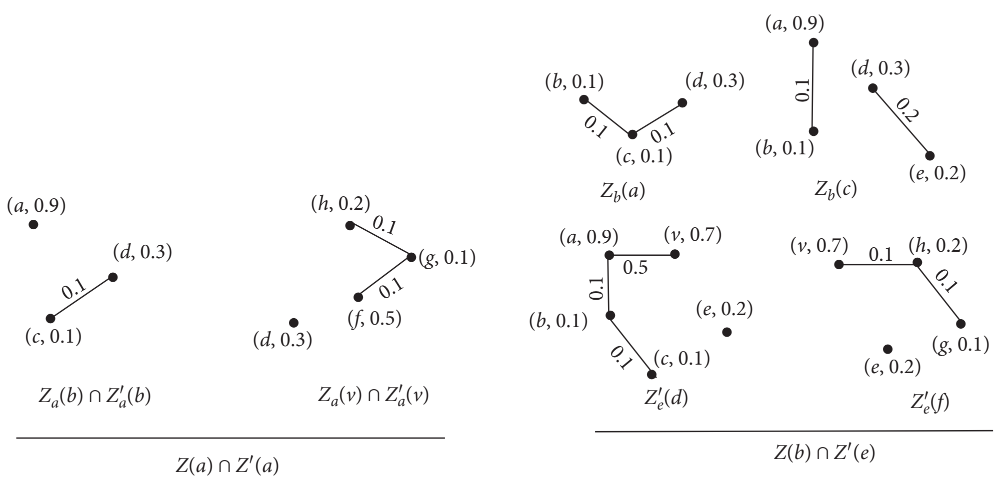

Figure 20: Intersection of $\mathbb{G}$ and $\mathbb{G}'$ is $\mathbb{G} \cap \mathbb{G}'$.

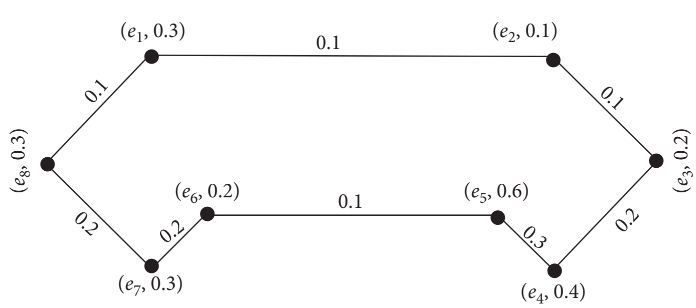

Figure 21: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

Then $\mathbb{G} = (Z(e_3), Z(e_4))$ is a FT2ST. Let $[\xi', B^*]$ and $[\psi', B^*]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned} \xi'(j) &= (\xi_j, \mathcal{NB}_j), \\ \psi'(j) &= (\psi_j, \mathcal{NB}_j), \quad \text{for all } j \in B^*. \end{aligned} \tag{37}$$

Define $\xi'_{e_7}(u) = \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \le 0.4\}$, $\psi'_{e_7}(u) = \{vw \in \mathcal{K} | \{v, w\} \subseteq \xi'_{e_7}(u)\} \ \forall u \in \mathcal{NB}_{e_7} \subseteq \mathcal{J}$,

$$\begin{aligned} \xi'_{e_7} &= \left\{ \begin{array}{c} \{(e_8, 0.3), \{(e_1, 0.3), (e_2, 0.1), (e_3, 0.2), (e_6, 0.2), (e_7, 0.3)\}\}, \\ \{(e_6, 0.2), \{(e_4, 0.4), (e_5, 0.6), (e_7, 0.3), (e_8, 0.3)\}\} \end{array} \right\}, \\ \psi'_{e_7} &= \left\{ \begin{array}{c} \{(e_8, 0.3), \{(e_1e_2, 0.1), (e_2e_3, 0.1), (e_7e_6, 0.2)\}\}, \\ \{(e_6, 0.2), \{(e_4e_5, 0.3), (e_7e_8, 0.2)\}\} \end{array} \right\}. \end{aligned} \tag{38}$$

$\mathbb{G}' = Z'(e_7)$ is a FT2ST. The AND operation of $\mathbb{G}$ and $\mathbb{G}'$ is defined as follows:

$$\begin{aligned} \xi(e_3, e_7) &= \xi_{e_3} \wedge \xi'_{e_7} \left\{ \begin{array}{c} \{((e_4, 0.4), (e_8, 0.3)), \{(e_3, 0.2), (e_2, 0.1)\}\}, \\ \{((e_4, 0.4), (e_6, 0.2)), \{(e_5, 0.6)\}\}, \\ \{(e_2, 0.1), (e_6, 0.2), \{(e_4, 0.4), (e_8, 0.3)\}\}, \\ \{(e_8, 0.3), (e_2, 0.1), \{(e_1, 0.3), (e_3, 0.2)\}\} \end{array} \right\}, \\ \psi(e_3, e_7) &= \psi_{e_3} \wedge \psi'_{e_7} \left\{ \begin{array}{c} \{((e_4, 0.4), (e_8, 0.3)), \{(e_2e_3, 0.1)\}\}, \{((e_4, 0.4), (e_6, 0.2)), \varnothing\}, \\ \{((e_2, 0.1), (e_8, 0.3)), \varnothing\}, \{((e_2, 0.1), (e_6, 0.2)), \varnothing\} \end{array} \right\}, \\ \xi(e_4, e_7) &= \xi_{e_4} \wedge \xi'_{e_7} \left\{ \begin{array}{c} \{((e_5, 0.6), (e_8, 0.3)), \{(e_6, 0.2), (e_7, 0.3)\}\}, \\ \{((e_5, 0.6), (e_6, 0.2)), \{(e_7, 0.3), (e_4, 0.4)\}\}, \\ \{((e_3, 0.2), (e_8, 0.3)), \{(e_1, 0.3), (e_2, 0.1)\}\}, \\ \{((e_3, 0.2), (e_6, 0.2)), \{(e_8, 0.3), (e_4, 0.4)\}\} \end{array} \right\}, \\ \psi(e_4, e_7) &= \psi_{e_4} \wedge \psi'_{e_7} \left\{ \begin{array}{c} \{((e_5, 0.6), (e_8, 0.3)), \{(e_6e_7, 0.2)\}\}, \{((e_5, 0.6), (e_6, 0.2)), \varnothing\}, \\ \{\{((e_3, 0.2), (e_8, 0.3)), \{(e_2e_1, 0.1)\}\}, \{(e_3, 0.2), (e_6, 0.2), \varnothing\}\} \end{array} \right\}. \end{aligned} \tag{39}$$

The AND operation of $\mathbb{G}$ and $\mathbb{G}'$ is shown in Figure 22.

Definition 22. Let $\mathbb{G}_1 = <\xi_1, \psi_1, M_1>$ and $\mathbb{G}_2 = <\xi_2, \psi_2, B_2>$ be two FT2STs. The OR operation of $\mathbb{G}_1$ and $\mathbb{G}_2$ is denoted by $\mathbb{G}_1 \vee \mathbb{G}_2 = \mathbb{G} = <\xi, \ \psi, B_1 \times B_2>$ such that $\xi(\chi, \eta) = \xi_1(\chi) \vee \xi_2(\eta)$, $\psi(\chi, \eta) = \psi_1(\chi) \vee \psi_2(\eta)$ for all $(\chi, \eta) \in B_1 \times B_2$. $(\xi(\chi, \eta), \psi(\chi, \eta))$ for all $(\chi, \eta) \in B_1 \times B_2$ is the fuzzy type-1 soft OR operation between the relevant FT1SGs corresponding to $(\xi_1(\chi), \psi_1(\chi))$ and $(\xi_2(\eta), \psi_2(\eta))$, respectively.

4. Applications of Fuzzy Type-2 Soft Graphs

In this section, we apply the concept of fuzzy type-2 soft graphs to decision-making problems in chemical digestion and national engineering services. The selection of a suitable object problem can be considered as a decision-making problem, in which final identification of object is decided on a given set of information. A detailed description of the algorithm for the selection of most suitable object based on available set of parameters is given in Algorithm 1 and the flow chart shown in Figure 23; purposed algorithm can be used to find out the best correspondence relationship between the neighboring objects in the decision-making problem. This method can be applied in various domains for multicriteria selection of objects.

4.1. Determination of Dominant Food Components in Chemical Digestion. We present an application of FT2SG in chemical digestion and discuss how to apply FT2SG in

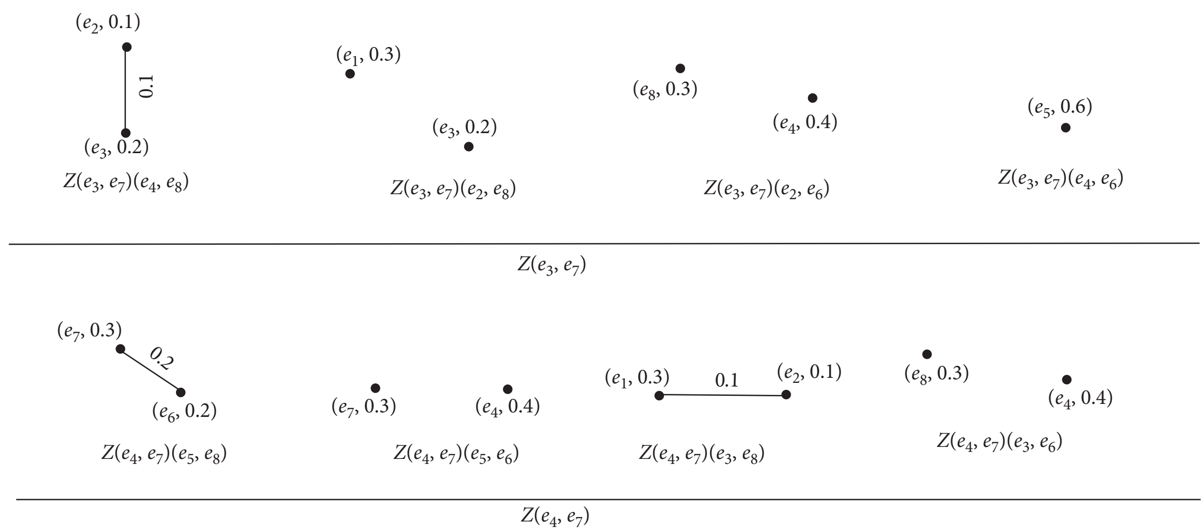

FIGURE 22: AND operation of $\mathbb{G}$ and $\mathbb{G}'$ is $\mathbb{G}\wedge\mathbb{G}' = (Z(e_3, e_7), Z(e_4, e_7))$.

(1) Input the fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.
(2) Input the choice parameter set $B = \{e_1, e_2, \ldots, e_n\}$ for suitable selection of the object.
(3) Input the VN-FT2SS $[\xi, B]$ over $\mathcal{J}$ and VN-FT2SS $[\psi, B]$ over $\mathcal{K}$.
(4) Construct a FT2SG $\mathbb{G} = <\xi, \psi, B> = \{Z(j) | j \in B\}$, where $Z(j) = (Z_{(j)}, \mathcal{NB}_j)$ such that $Z_j(u) = (\xi_j(u), \psi_j(u)), \forall u \in \mathcal{NB}_j$.
(5) Construct the resultant VN-fuzzy graph by taking the intersection of vertex-neighbors fuzzy graphs $Z^*(j) = \bigcap_u Z_j(u), \forall u \in \mathcal{NB}_j$.
(6) Tabular representation of resultant VN-fuzzy graph $Z^*(j) \forall j \in B$ with the choice values C_i^j.
(7) The decision is S_i if $S_i = \bigvee_i^n (\bigwedge_j C_i^j)$.
(8) If i has more than one value, then any one of S_i may be chosen.

ALGORITHM 1: Algorithm for the selection of most suitable objects.

chemical digestion of spinach. Spinach is generally composed of carbohydrates, protein, lipids, minerals, vitamins, and nucleic acids. We mainly focused on the digestion of carbohydrates, proteins, lipids, and nucleic acids, which is carried out by a variety of salivary enzymes and the enzymes present in other parts of digestive system; that is, amylase, pepsin, and trypsin are released as a result of involuntary signal generated by our body to digest the food. When 25 g of spinach is taken, it contains carbohydrates (0.9g), protein (0.7g), lipids (0.1g), nucleic acids (0.3g), involuntary signal (0.3), pepsin (0.2), amylase (0.2), and trypsin (0.1), represented as vertices donated by $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$, respectively. "Chemical digestion" is the enzyme-mediated, hydrolysis method that converts large macronutrients into smaller molecules.

(i) Carbohydrate mostly comprises amylose and glycogen. Long carbohydrates chains are broken down into disaccharides which are decomposed by amylase enzyme.
(ii) Proteins are usually broken down into amino acids by peptidase enzyme as well as trypsin and chymotrypsin.
(iii) Lipids are hydrolyzed by pancreatic lipase enzyme.
(iv) Nucleic acids, that is, DNA and RNA, are hydrolyzed by pancreatic nuclease.
(v) Involuntary signal is generated by the brain in order to carry out chemical digestion in the digestive system.

Protein digestion occurs in stomach and duodenum by the action of three primary enzymes.

(i) Pepsin, disguised by abdomen
(ii) Trypsin, disguised through pancreas
(iii) Amylase, disguised through saliva and pancreas

Note that the values of pepsin, trypsin, amylase, and involuntary signal are supposed as we cannot calculate the amounts of these products released as a result of con-

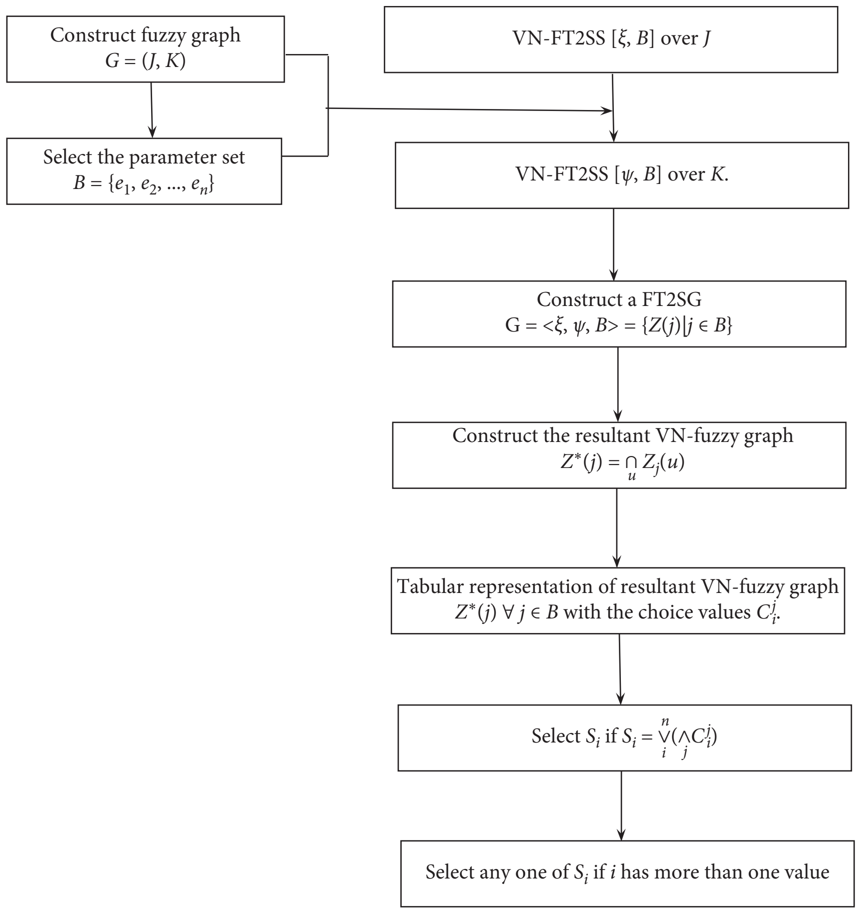

Figure 23: Flow chart for suitable selection of objects.

sumption of little amount of food through previous literature findings.

Consider

$$\begin{aligned}\mathcal{J} &= \{(e_1, 0.9), (e_2, 0.7), (e_3, 0.1), (e_4, 0.3), (e_5, 0.3), (e_6, 0.2), (e_7, 0.2), (e_8, 0.1)\},\\ \mathcal{K} &= \left\{\begin{array}{c}(e_1e_2, 0.3), (e_1e_4, 0.3), (e_3e_2, 0.1), (e_3e_4, 0.1), (e_5e_4, 0.2),\\ (e_5e_6, 0.2), (e_5e_8, 0.1), (e_8e_7, 0.1), (e_6e_7, 0.1)\end{array}\right\}.\end{aligned} \tag{40}$$

In fuzzy graph $(\mathcal{J}, \mathcal{K})$ as shown in Figure 24, edges represent the amount of energy utilized by the body in order to carry out the digestion process. Let $B = \{(e_1, 0.9), (e_2, 0.7)\}$ represent the amounts of carbohydrates and protein released when 25 g of spinach is consumed. We have $\mathcal{NB}_{e_1} = \{(e_2, 0.7), (e_4, 0.3)\}$, $\mathcal{NB}_{e_2} = \{(e_3, 0.1), (e_1, 0.9)\}$.

Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathcal{J}$ and $\mathcal{K}$, respectively. We have

$$\begin{aligned}\xi(j) &= (\xi_j, \mathcal{NB}_j),\\ \psi(j) &= (\psi_j, \mathcal{NB}_j), \quad \text{for all } j \in B,\\ \text{define } \xi_{e_1}(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.5\},\\ \psi_{e_1}(u) &= \{vw \in \mathcal{K} | v, w \subseteq \xi_{e_1}(u)\},\\ \xi_{e_2}(u) &= \{v \in \mathcal{J} | u\mathcal{R}v \Longleftrightarrow \mathrm{d}(u, v) \leq 0.5\},\\ \psi_{e_2}(u) &= \{vw \in \mathcal{K} | v, w \subseteq \xi_{e_2}(u)\}.\end{aligned} \tag{41}$$

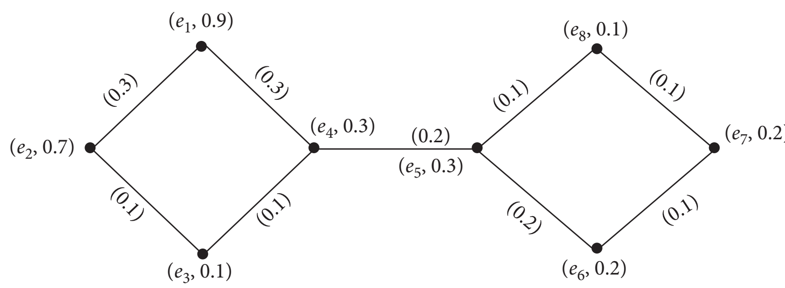

Figure 24: $\mathscr{G} = (\mathscr{J}, \mathscr{K})$.

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

$$\begin{aligned}
\xi_{e_1} &= \left\{ \begin{array}{c} \{(e_2, 0.7), \{(e_1, 0.9), (e_3, 0.1), (e_4, 0.3), (e_5, 0.3)\}\}, \\ \{(e_4, 0.3), \{(e_1, 0.9), (e_2, 0.7), (e_3, 0.1), (e_5, 0.3), (e_6, 0.2), (e_7, 0.2), (e_8, 0.1)\}\} \end{array} \right\}, \\
\psi_{e_1} &= \left\{ \begin{array}{c} \{(e_2, 0.7), \{(e_3e_4, 0.1), (e_1e_4, 0.3), (e_5e_4, 0.2)\}\}, \\ \{(e_4, 0.5), (e_1e_2, 0.3), (e_3e_2, 0.1), (e_5e_6, 0.2), (e_5e_8, 0.1), (e_6e_7, 0.1), (e_7e_8, 0.1)\} \end{array} \right\}, \\
\xi_{e_2} &= \left\{ \begin{array}{c} \{(e_1, 0.9), \{(e_2, 0.7), (e_3, 0.1), (e_4, 0.3), (e_5, 0.3)\}\}, \\ \{(e_3, 0.1), \{(e_1, 0.9), (e_2, 0.7), (e_4, 0.3), (e_5, 0.3), (e_6, 0.2), (e_8, 0.1)\}\} \end{array} \right\}, \\
\psi_{e_2} &= \left\{ \begin{array}{c} \{(e_1, 0.9), \{(e_5e_4, 0.2), (e_3e_4, 0.2), (e_2e_3, 0.1)\}\}, \\ \{(e_3, 0.4), \{(e_1e_2, 0.3), (e_1e_4, 0.3), (e_4e_5, 0.2), (e_5e_6, 0.2), (e_5e_8, 0.1)\}\} \end{array} \right\}.
\end{aligned} \tag{42}$$

The fuzzy type-2 soft graph $\mathbb{G}$ is shown in Figure 25.

The tabular representations of resultant vertex-neighbors fuzzy graphs $Z^*(e_j)$ shown in Figure 26 corresponding to the parameter e_j, $j = 1, 2$ with the choice values $C_i^j = \sum_k S_{ik}$ for all i, k are given in Tables 2 and 3.

The decision value is $S_i = \vee_i^7 (\wedge_j C_i^j) = \vee_{i=1}^7 \{0.6 \wedge 0.6, 0.4 \wedge 0.4, 0.2 \wedge 0.2, 0.6 \wedge 0.5, 0.3 \wedge 0.2, 0.2 \wedge 0, 0.3 \wedge 0.2\} = 0.6$ from the choice value C_i^j of fuzzy type-2 soft graphs for $j = 1, 2$. The prominent food components are e_1 as carbohydrates and e_2 as lipids as carbohydrates are consumed as sugar and lipids are consumed as fats. Clearly, the dominant food components are e_1 or e_4.

4.2. Water Supply for National Engineering Services. We present the application of fuzzy type-2 soft graph in the National Engineering Services Pakistan (NESPAK). The National Engineering Services Pakistan is a Pakistani multinational state-owned corporation that provides construction, management, and consulting services globally. Every government project has something to do with NESPAK at some time of its planning or implementation. In fuzzy graph $\mathscr{G} = (\mathscr{J}, \mathscr{K})$ as shown in Figure 27, vertices represent some important projects.

$$\mathscr{J} = \left\{ \begin{array}{c} (a = \text{Water Supply}, 0.9), (b = \text{Sewerage}, 0.7), (c = \text{Drainage}, 0.6), \\ (\text{d} = \text{Solid Waste Management}, 0.4), (e = \text{Plumbing}, 0.3), (f = \text{Industrial Wastes}, 0.2) \end{array} \right\}. \tag{43}$$

NESPAK provides engineering services for these projects, the membership value of a vertex showing the working capability of the relevant project and values of edges represents the strength of the relationship between different projects to complete the tasks.

Now, we take two important projects Plumbing and Solid Waste Management named as $(e, 0.3)$, $(d, 0.4)$, respectively, and $B = \{(e, 0.3), (d, 0.4)\} \subset \mathscr{J}$. The vertex-neighbors of these selected projects are $\mathscr{N}\mathscr{B}_e = \{(b, 0.7), (c, 0.6), (d, 0.4), (f, 0.2)\}$ and $\mathscr{N}\mathscr{B}_d = \{(a, 0.9), (c, 0.6), (e, 0.3), (f, 0.2)\}$. Let $[\xi, B]$ and $[\psi, B]$ be two FT2SSs over $\mathscr{J}$ and $\mathscr{K}$, respectively. We have

$$\begin{aligned}
&\xi(j) = \left(\xi_j, \mathscr{N}\mathscr{B}_j\right), \\
&\psi(j) = \left(\psi_j, \mathscr{N}\mathscr{B}_j\right), \quad \text{for all } j \in B, \\
&\text{define } \xi_e(u) = \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow \text{d}(u, v) \le 0.2\}, \\
&\psi_e(u) = \{vw \in \mathscr{K} | v, w \subseteq \xi_e(u)\}, \quad \forall u \in \mathscr{N}\mathscr{B}_e \subseteq \mathscr{J}, \\
&\xi_d(u) = \{v \in \mathscr{J} | u\mathscr{R}v \Longleftrightarrow 0.1 \le \text{d}(u, v) \le 0.2\}, \\
&\psi_d(u) = \{vw \in \mathscr{K} | v, w \subseteq \xi_d(u)\}, \quad \forall u \in \mathscr{N}\mathscr{B}_d \subseteq \mathscr{J}.
\end{aligned} \tag{44}$$

The FT2SSs $[\xi, B]$ and $[\psi, B]$ are defined as follows:

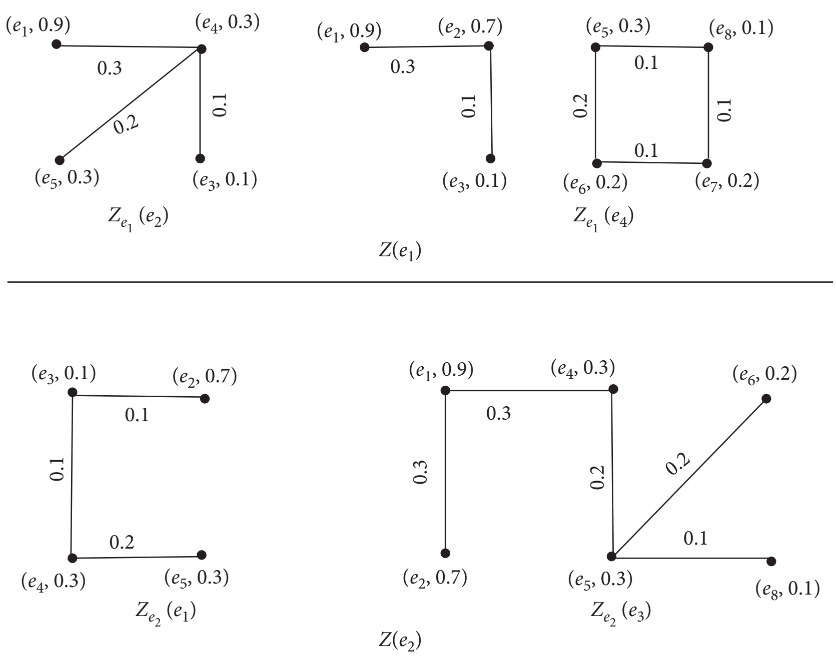

Figure 25: Fuzzy type-2 soft graph $\mathbb{G} = (Z(e_1), Z(e_2))$.

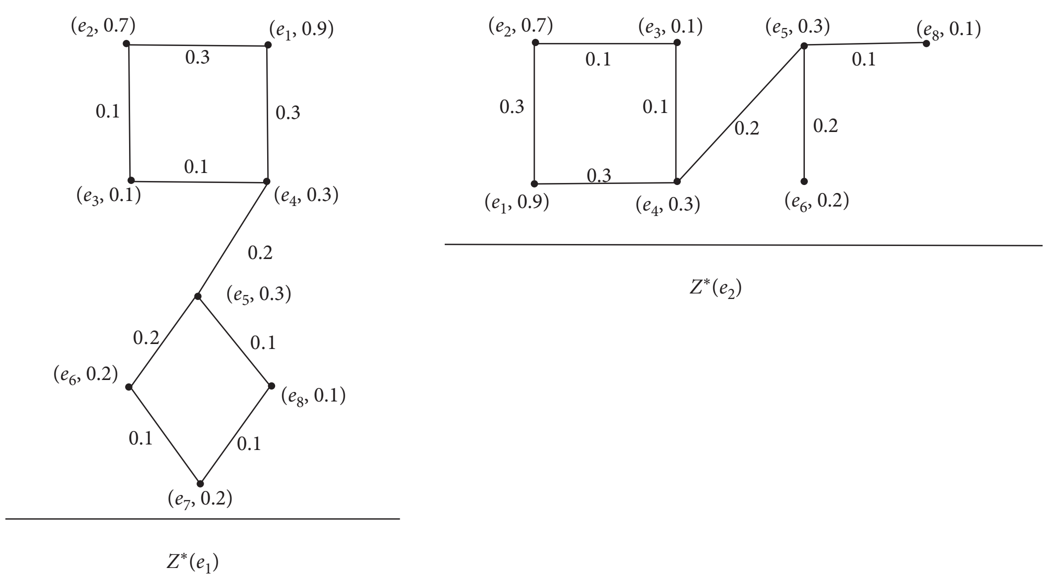

Figure 26: Resultant VN-fuzzy graphs $Z^*(e_1)$ and $Z^*(e_2)$.

Table 2: The tabular representation of $Z^*(e_1)$ with choice values.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	C_i^1
e_1	0	0.3	0	0.3	0	0	0	0	0.6
e_2	0.3	0	0.1	0	0	0	0	0	0.4
e_3	0	0.1	0	0.1	0	0	0	0	0.2
e_4	0.3	0	0.1	0	0.2	0	0	0	0.6
e_5	0	0	0	0.2	0	0.2	0	0.1	0.5
e_6	0	0	0	0	0.2	0	0.1	0	0.3
e_7	0	0	0	0	0	0.1	0	0.1	0.2
e_8	0	0	0	0	0.1	0	0.1	0	0.2

Table 3: The tabular representation of $Z^*(e_2)$ with choice values C_i^2.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	C_i^2
e_1	0	0.3	0	0.3	0	0	0	0.6
e_2	0.3	0	0.1	0	0	0	0	0.4
e_3	0	0.1	0	0.1	0	0	0	0.2
e_4	0.3	0	0.1	0	0.2	0	0	0.6
e_5	0	0	0	0.2	0	0.2	0.1	0.5
e_6	0	0	0	0	0.2	0	0	0.2
e_8	0	0	0	0	0.1	0	0	0.1

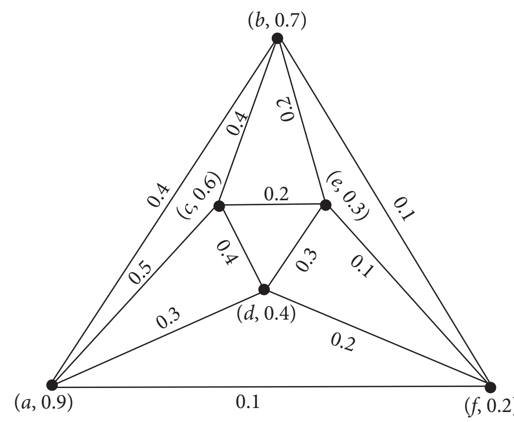

Figure 27: Fuzzy graph $\mathcal{G} = (\mathcal{J}, \mathcal{K})$.

$$\begin{aligned}
\xi_e &= \left\{ \begin{array}{c} \{(f,0.2),\{(a,0.9),(b,0.7),(d,0.4),(e,0.3)\}\},\{(d,0.4),\{(f,0.2)\}\},\{(c,0.6),\{(e,0.3)\}\}, \\ \{(b,0.7),\{(e,0.3),(f,0.2)\}\} \end{array} \right\}, \\
\psi_e &= \left\{ \begin{array}{c} \{(f,0.2),\{(ad,0.3),(ab,0.4),(de,0.3),(eb,0.2)\}\}, \\ \{(d,0.4),\varnothing\},\{(c,0.6),\varnothing\},\{(b,0.7),\{(fe,0.1)\}\} \end{array} \right\}, \\
\xi_d &= \left\{ \begin{array}{c} \{(e,0.3),\{(b,0.7),(c,0.6),(f,0.2)\}\},\{(a,0.9),\{(f,0.2)\}\}, \\ \{(f,0.2),\{(a,0.9),(b,0.7),(d,0.4),(e,0.3)\}\},\{(c,0.6),\{(e,0.3)\}\} \end{array} \right\}, \\
\psi_d &= \left\{ \begin{array}{c} \{(e,0.3),\{(bc,0.4),(fb,0.1)\}\},\{(a,0.9),\varnothing\}, \\ \{(c,0.6),\varnothing\},\{(f,0.2),\{(ad,0.3),(ab,0.4),(de,0.3),(eb,0.2)\}\} \end{array} \right\}.
\end{aligned} \tag{45}$$

FT1SGs corresponding to $Z(e) = (\xi(e), \psi(e))$ and $Z(d) = (\xi(d), \psi(d))$, respectively, are shown in fuzzy type-2 soft graph 28. (Figure 28)

The tabular representations of resultant vertex-neighbors fuzzy graphs $Z^*(e)$ and $Z^*(d)$ shown in Figure 29 with the choice values $C_i^j = \sum_k S_{ik}$ for all i, k are given in Tables 4 and 5.

The decision value is $S_i = \vee_i^6 (\wedge_j C_i^j) = \vee_{i=1}^6 \{0.7 \wedge 0.7, 0.6 \wedge 1.1, 0 \wedge 0.4, 0.6 \wedge 0.6, 0.6 \wedge 0.5, 0.1 \wedge 0.1\} = 0.7$, from the choice value C_i^j of fuzzy type-2 soft graphs for $j = 1, 2$. The optimal project is "a = water supply." So, NESPAK provides the best engineering services to the project of "water supply."

Advantages of the Proposed Method.

The advantages of the proposed method based on FT2SGs are as follows:

(1) The method can be effectively used to handle uncertainty and vagueness with correspondence, assertion, and relations among parameters.

(2) The proposed method incorporates parametrization tool with fuzzy information to effectively handle more uncertain conditions and errors in given data.

(3) The presented method considers vertex-neighbors coordination tool along with reparametrization to study the interrelationship and ambiguity among objects.

5. Comparison Analysis

In this section, we discuss the comparison of fuzzy type-2 soft graphs with fuzzy soft graphs and type-2 soft graphs.

5.1. Comparison with Fuzzy Soft Graphs. Fuzzy soft graph [21] is a parameterized family of fuzzy graphs, and it is an extension of a soft graph. The fuzzy type-2 soft graph is a parameterized family of VN-fuzzy soft graphs and an extension of type-2 soft graph. Fuzzy type-2 soft graphs show

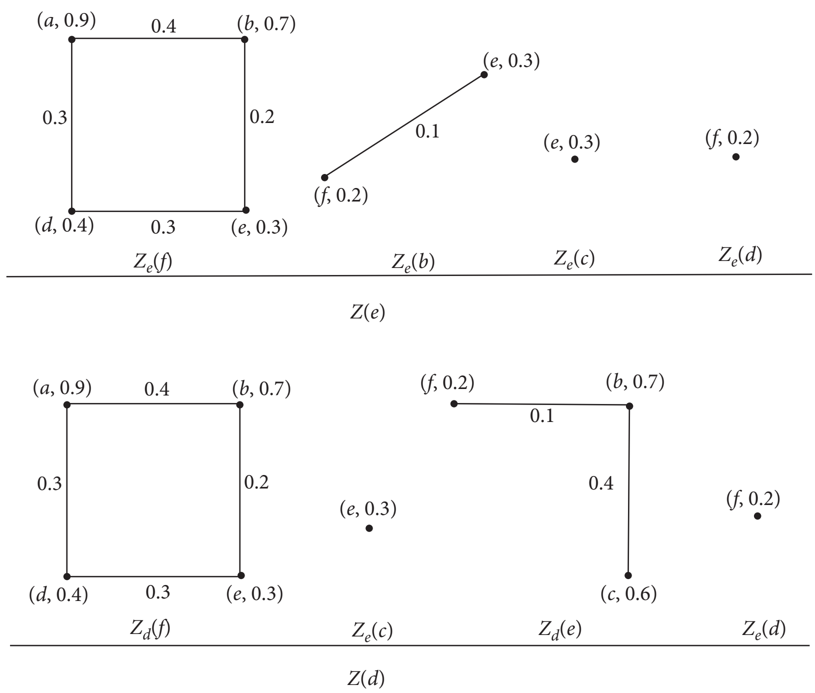

Figure 28: Fuzzy type-2 soft graph for national engineering services.

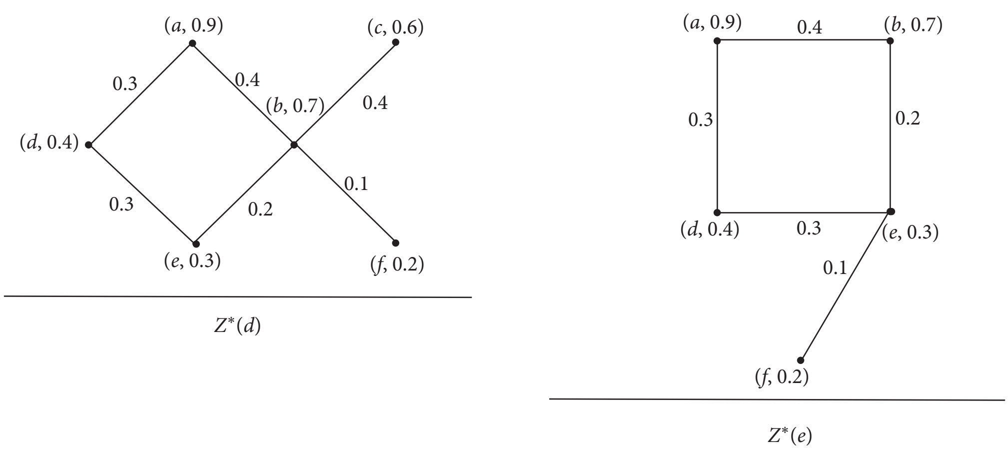

Figure 29: Resultant VN-fuzzy graphs $Z^*(e)$ and $Z^*(d)$.

vertex-neighbors coordination relation among objects in a parameterized VN-fuzzy graph. The proposed models take the set of parameters from a given fuzzy vertex set and, corresponding to each selected parameter, there exists a VN-fuzzy soft graph. As fuzzy soft graph is a parameterized family of fuzzy graphs and, corresponding to each parameter, there exists a fuzzy graph. For handling vagueness and ambiguity in decision-making problems, different fuzzy models were introduced. Fuzzy type-2 soft graph shows vertex-neighbors correspondence among objects as well as relations of parameters, while fuzzy soft graphs cannot study these correspondences and thus cannot give accurate and effective results. The decision-making problem discussed in Section 4.1 can be discussed using fuzzy soft graphs.

We consider a fuzzy soft graph $G = (\Phi, \Psi, M)$, where (Φ, M) is a fuzzy soft set over $V = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8\}$ which describes the membership of the objects based upon the given parameters e_1 and e_2; (Ψ, M) is a fuzzy soft set over $E = \{e_1e_2, e_1e_4, e_2e_3, e_3e_4, e_4e_5, e_5e_6, e_5e_6, e_5e_8, e_6e_7, e_8e_7\} \subseteq V \times V$ describing the membership between two objects corresponding to the given parameters e_1 and e_2. A fuzzy soft graph $G = \{H(e_1), H(e_2)\}$ is given in Tables 6 and 7.

The fuzzy graphs $H(e_1)$ and $H(e_2)$ of fuzzy soft graph $G = \{H(e_1), H(e_2)\}$ corresponding to the parameters "carbohydrates" and "protein" are shown in Figure 30.

The fuzzy graphs $H(e_1)$ and $H(e_2)$ and the choice values $C_i^k = \sum_j S_{ij}$ for all $i, j, k = 1, 2$ are given in Tables 8 and 9, respectively.

The decision value is $S_i = \vee_i^8 (\wedge_k C_i^k) = \vee_{i=1}^8 \{0.5 \wedge 0.6, 0.6 \wedge 0.4, 0.4 \wedge 0.3, 0.4 \wedge 0.5, 0.4 \wedge 0.3, 0.4 \wedge 0.1, 0.1 \wedge 0.2, 0.3 \wedge 0.2\} = 0.5$ from the choice value C_i^k of fuzzy graph $H(e_k)$ for $k = 1, 2$. Clearly, the dominant object is e_1 or e_4. The

Table 4: The tabular representation of $Z^*(e)$ with choice values.

	a	b	d	e	f	C_i^1
a	0	0.4	0.3	0	0	0.7
b	0.4	0	0	0.2	0	0.6
d	0.3	0	0	0.3	0	0.6
e	0	0.2	0.3	0	0.1	0.6
f	0	0	0	0.1	0	0.1

Table 5: The tabular representation of $Z^*(d)$ with choice values.

	a	b	c	d	e	f	C_i^2
a	0	0.4	0	0.3	0	0	0.7
b	0.4	0	0.4	0	0.2	0.1	1.1
c	0	0.4	0	0	0	0	0.4
d	0.3	0	0	0	0.3	0	0.6
e	0	0.2	0	0.3	0	0	0.5
f	0	0.1	0	0	0	0	0.1

Table 6: Tabular representation of a fuzzy soft vertex set.

Φ	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
e_1	0.8	0.7	0.2	0.4	0.3	0.3	0.2	0.2
e_2	0.8	0.5	0.5	0.4	0.4	0.5	0.6	0.5

Table 7: Tabular representation of a fuzzy soft edge set.

Ψ	e_1e_2	e_1e_6	e_1e_4	e_1e_7	e_2e_3	e_2e_4	e_2e_8	e_3e_4	e_3e_7	e_4e_5	e_5e_6	e_5e_8	e_7e_8
e_1	0.3	0.2	0.0	0.0	0.2	0.1	0.0	0.1	0.1	0.2	0.1	0.1	0.1
e_2	0.3	0.0	0.2	0.1	0.0	0.0	0.1	0.2	0.1	0.2	0.1	0.1	0.0

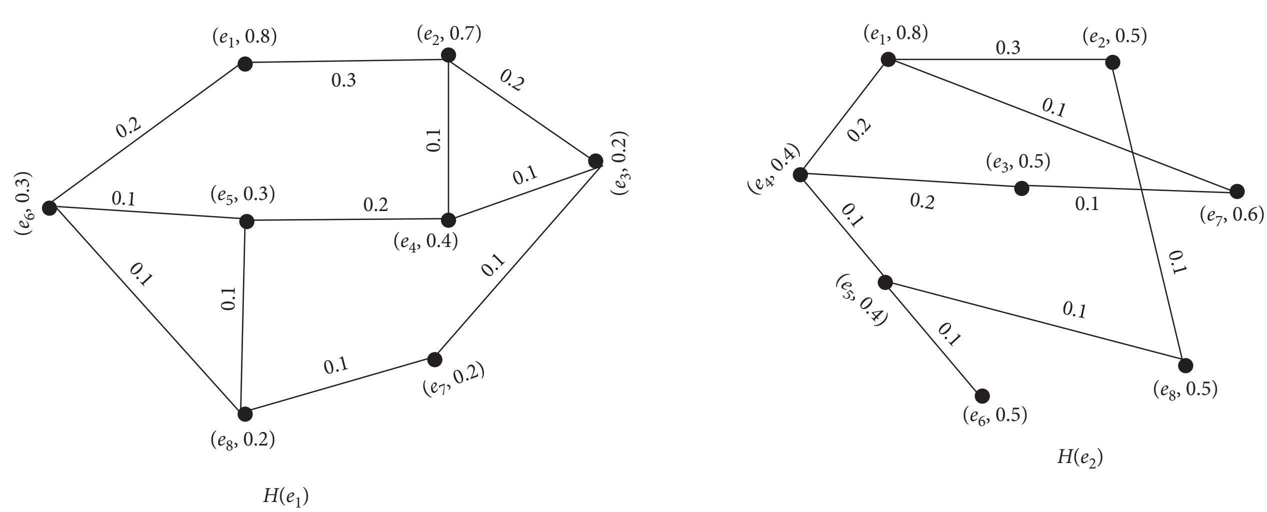

Figure 30: Fuzzy soft graph $G = \{H(e_1), H(e_2)\}$.

suitable object determined by fuzzy soft graph as above and fuzzy type-2 soft graph in Section 4.1 is dependent on information determined by selected set of parameters and fuzzy values in VN-fuzzy graphs, respectively. As the coordination among objects varies, the solution changes accordingly. So, in this case, when the objects show close vertex-neighbors coordination according to observed data, fuzzy type-2 soft graph model can be used and in the case when fuzzy relations are given along with different parameters, fuzzy soft graph model can be used.

Table 8: Tabular representation of $H(e_1)$ with choice values.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	C_i^1
e_1	0.0	0.3	0.0	0.0	0.0	0.2	0.0	0.0	0.5
e_2	0.3	0.0	0.2	0.1	0.0	0.0	0.0	0.0	0.6
e_3	0.0	0.2	0.0	0.1	0.0	0.0	0.1	0.0	0.4
e_4	0.0	0.1	0.1	0.0	0.2	0.0	0.0	0.0	0.4
e_5	0.0	0.0	0.0	0.2	0.0	0.1	0.0	0.1	0.4
e_6	0.2	0.0	0.0	0.0	0.1	0.0	0.0	0.1	0.4
e_7	0.0	0.0	0.1	0.0	0.0	0.0	0.0	0.1	0.1
e_8	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.0	0.3

Table 9: Tabular representation of $H(e_2)$ with choice values.

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8	C_i^2
e_1	0.0	0.3	0.0	0.2	0.0	0.0	0.1	0.0	0.6
e_2	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.4
e_3	0.0	0.0	0.0	0.2	0.0	0.0	0.1	0.0	0.3
e_4	0.2	0.0	0.2	0.0	0.1	0.0	0.0	0.0	0.5
e_5	0.0	0.0	0.0	0.1	0.0	0.1	0.0	0.1	0.3
e_6	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.1
e_7	0.1	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.2
e_8	0.0	0.1	0.0	0.0	0.1	0.0	0.0	0.0	0.2

5.2. Comparison with Type-2 Soft Graphs. In structure of a graph, the vertex-neighbors correspondence has an important role. The type-2 soft graph [32] is based on the correspondence of initial parameters (vertex soft set) and underlying parameters. The type-2 soft graph is an efficient model for dealing with uncertainty occurring in vertex-neighbors' structure and is applicable in computational analysis, applied intelligence, and decision-making problems. The theory of fuzzy sets has played an important role to form useful models for handling partial membership of objects. To overcome the parameterized limitations of fuzzy set, the theory of fuzzy type-2 soft set was introduced. Fuzzy type-2 soft graph model is a more efficient model as compared to type-2 soft graph model to represent the parametric uncertainty in graphical networks. It is observed that, for the selection of dominant food components in chemical digestion using given type-2 soft information, we are not able to identify any object (dominating component). In this case, the simple type-2 soft information provides no solution. To determine the solution of the problem, it is necessary to have fuzzy information or define a fuzzy relation in order to attain a suitable approximation approach for selecting at least one object. So, fuzzy type-2 soft graph is more reliable in such decision-making problems.

6. Conclusions and Future Directions

Molodtsov's soft set theory is an effective and rational approach to understand uncertainties in terms of parameters. Type-2 soft sets have been introduced by adding the primary relations among parameters in soft sets. We have introduced the notions of fuzzy type-2 soft sets and fuzzy type-2 soft graphs to study the partial membership and uncertainty of objects along with underlying and primary set of parameters. We have discussed certain properties of fuzzy type-2 soft graphs, regular fuzzy type-2 soft graphs, irregular fuzzy type-2 soft graphs, fuzzy type-2 soft trees, and fuzzy type-2 soft cycles. We have discussed different methods of construction of fuzzy type-2 soft graphs with certain operations and elaborated these concepts with numerical examples. We have studied the importance of fuzzy type-2 soft graphs in chemical digestion and national engineering services. The present study can be extended to various directions including (1) Pythagorean fuzzy type-2 soft graphs, (2) spherical fuzzy type-2 soft graphs, and (3) picture fuzzy type-2 soft trees.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

The ILHWLAD-MCDM Framework for the Evaluation of Concrete Materials under an Intuitionistic Linguistic Fuzzy Environment

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Concrete, one of the sources of energy consumption and carbon emissions, is widely used in the construction industry. The selection of concrete materials raises the question of energy sustainability and turns it into a complex multicriteria decision-making (MCDM) issue. To address this, we present an MCDM framework based on the intuitionistic linguistic hybrid weighted logarithmic averaging distance (ILHWLAD). To begin with, the intuitionistic linguistic numbers are used to deal with the uncertainty and fuzziness of the decision-making process. In addition, in view of the significance and the ordered position of the input arguments, an intuitionistic linguistic hybrid weighted logarithmic averaging distance (ILHWLAD) operator is defined. We, then, initiate the criteria system and present the MCDM framework based on the ILHWLAD to select the finest concrete. A case study involving four alternative materials, namely, autoclaved aerated concrete (AAC), hollow concrete blocks (HCB), expanded polystyrene (EPS), and lime hemp concrete (LHC), is presented to verify the scientificity of the framework.

1. Introduction

The rapid advance of urbanization has meant that land available for construction is becoming increasingly scarce. However, the number of individuals who crowd into the city to improve their living standards is growing day by day [1]. This has led to an explosion in demand for housing because the existing stock does not meet citizens' needs. Building materials have become an issue of crucial consideration, and because high-rise buildings accommodate more individuals, safety is critical [2]. Concrete, a mixture of paste and aggregate or rocks, is one of the most widely used building materials. Different kinds are used for building external filled walls, frame structured filled walls, non-load-bearing walls, load-bearing walls, roofs, floors, insulation systems, and so on [3–5]. However, as a result, energy consumption is increasing drastically and so is environmental pollution [6]. The United Nations Environment Program (UNEP) reports that the finest concrete materials account for more than 36% of global energy consumption and up to 40% of energy-related carbon dioxide emissions [7]. Therefore, the choice of suitable concrete for construction is of great significance for energy conservation and emissions reduction, manufacturing, composition, innovation, and so on.

Concrete gives buildings strength and durability. The concrete itself must have great compressive strength. Concrete also affects the thermal performance of buildings. Climatic conditions can change enormously [8]; concrete materials consume heat during the summer season, and this is stored and released in the night in winter, which reduces the effect of external temperature variations. In addition, the residential comfort of the building is not adversely affected because concrete is able to balance variations in outdoor humidity and to avoid excessive variations in humidity within the building. Because concrete is widely used in the construction industry, economy and accessibility are especially important in the selection process of the finest types.

It is difficult to use specific data in the analysis of the performance and functionality of specific concrete materials. For example, the energy of concrete material varies according to levels of technology, production conditions, and so on [9]. The intuitionistic linguistic number (ILN) is an effective way to resolve these issues, and it has been frequently used in different fields [10], e.g., energy performance contracting [11], strategy decisions [12], the selection of offshore wind farms [13], and assessment of green building insulation materials [14].

Compared to the value of language evaluation, ILN increases the degree of membership and nonmembership, which reflects the nonmembership degree of the language evaluation value and the magnitude of hesitation of decision makers [15]. We can utilize ILN to characterize the grades of each criterion for concrete materials because the information is hard to measure with specific values.

The ultimate goal of this paper is to select the finest concrete material. Its contribution to the field can be summarized as follows:

(1) The intuitionistic linguistic weighted logarithmic average distance (ILWLAD) operator and the intuitionistic linguistic ordered weighted logarithmic average distance (ILOWLAD) operator are projected by combining the weighted logarithmic and the ordered weighted logarithmic average methods. Furthermore, to address the defects of ILWLAD and ILOWLAD, we introduce a new intuitionistic linguistic hybrid weighted logarithmic average distance (ILHWLAD) operator for better handling of the data.

(2) As mentioned previously, the selection of concrete materials is a comprehensive decision-making problem. To improve the scientificity of the decision-making process, a multidimensional examination of concrete materials is required. Thus, a six-criterion evaluation system for concrete material selection has been developed, and four concrete materials have been graded using ILNs.

(3) The HWLAD-MCDM framework of concrete material selection is presented [16,17], and the ILHWLAD operator is used to select the most suitable material by using the fuzzy concept. To illustrate the rationality of the ILHWLAD operator, we compare the results obtained by using the ILHWLAD operator with those obtained using the ILOWLAD operator, ILOWAD operator, and ILWLAD operator.

The remainder of this paper is arranged as follows: Section 2 introduces the evaluation criteria of concrete materials and briefly reviews the related concepts. Section 3 presents the new ILHWLAD operator and introduces the HWLAD-MCDM framework used to select the most suitable and finest concrete material. In Section 4, we apply the framework to the four alternative concrete materials and display the results. In addition, some comparisons are made and discussed to explain the rationale behind the ILHWALD operator. The conclusions, limitations, and recommendations for further applications are presented in Section 5.

2. Materials and Methods

2.1. Criteria for Concrete Materials Selection. Constructing the evaluation criteria system is a key step in the concrete materials selection process. In accordance with the available literature, four main aspects and six criteria were selected (see Table 1) [18, 23].

(1) Embodied energy (A1): the embodied energy of the concrete materials refers to the exact total energy consumed in the entire process of concrete material production. It includes production, processing, transportation, and construction, that is, is the sum of direct and indirect energy consumption. The lower the energy content, the lower the energy consumption of the concrete material and the more the energy that is saved.

(2) Embodied carbon (A2): the concrete material in production, processing, and other processes expel carbon dioxide into the atmosphere. Carbon dioxide contributes to air pollution and provokes the global greenhouse effect, which is the cause of global climate change. Hence, the less the carbon contained in the concrete material, the lower the carbon dioxide expelled.

(3) Purchase cost (A3): the finest concrete materials are frequently used in the construction business. The purchase cost of concrete changes greatly, and a reduction contributes to economies in construction costs. Therefore, the purchase cost can be regard as an economic criterion for measurement.

(4) Thermal performance (A4): the outdoor temperature of buildings is generally high in the summer. Good thermal performance enables indoor temperatures to reach a lower level than the day's peak temperature, thus improving the comfort of indoor living. In contrast, outdoor temperatures are generally lower in winter and reach their lowest point at the night. Therefore, the indoor temperature is capable of reaching a higher level at the lowest external night-time temperature as a result of the concrete's outstanding thermal performance. In this paper, the highest indoor temperature of the building in summer and the lowest indoor temperature in winter at night-time are combined to measure the thermal performance of the concrete materials.

(5) Ability to balance outdoor humidity fluctuations (A5): a criterion that measures the ability of certain concrete materials to balance outdoor humidity fluctuations is the variation range of indoor humidity within a day. The smaller the range, the better the ability to balance outdoor humidity fluctuations. We combine the indoor humidity variation of the building in the summer and in the winter to measure the ability of the concrete materials to balance fluctuations in outdoor humidity.

(6) Compressive strength (A6): the compressive strength of concrete refers to the strength limit

Table 1: Criteria for concrete material selection.

Aspects	Criteria	Abbreviation	Reference
Energy	Embodied energy	A_1	[8, 18, 19]
Sustainability	Embodied carbon	A_2	[8, 20–22]
Economy	Purchase cost	A_3	[23, 24]
	Thermal performance	A_4	[25–27]
Comfort	Ability to balance outdoor humidity fluctuations	A_5	[28, 29]
Safety	Compressive strength	A_6	[30–33]

applied by external forces, which is obtained by a test of a cube specimen with a side length of 150 mm under the strength of C60. As has been noted, it is one of the factors that influence the stability of a building. The greater the compressive strength, the greater the maximum pressure the material can withstand.

In terms of energy sustainability, embodied energy and embodied carbon are the two main criteria that reflect the energy consumed and carbon dioxide emitted [19,20]. When taking the comfort of the building's interior into account, the thermal performance of the concrete materials and the ability to balance outdoor humidity fluctuations are especially significant [34]. For the economy of the concrete materials, the purchase cost is the critical criterion by which to estimate economic possibilities, given concrete's widespread use in the construction industry. Finally, the criterion of its compressive strength has to be considered because this is what makes buildings strong [30].

2.2. Weighting Method of Criteria. In this section, we briefly review some concepts related to the linguistic approach and the intuitionistic linguistic set (ILS) and present a programming model for calculating the weights of the criteria.

Definition 1. The linguistic approach is an approximate technique that expresses qualitative aspects as linguistic values through the use of linguistic terms [35]. For ease of calculation, let $K = \{k_a | a = 1, 2, \ldots, t\}$ be an ordered linguistic term set, where t is the positive odd value and k_a represents a possible value for a linguistic variable.

For instance, taking $t = 7$, a set K could be expressed as follows: $K = \{k_1, k_2, k_3, k_4, k_5, k_6, k_7\} = \{$*extremely bad*, *quite bad*, *bad*, *medium*, *good*, *quite good*, *extremely good*$\}$.

Any label k_α should satisfy the following operational laws:

(1) $\mathrm{Neg}(k_i) = k_{t-i}$

(2) $k_i \geq k_j \Longleftrightarrow i \geq j$

(3) $\max(k_i, k_j) = k_i$, if $\geq i \geq j$

(4) $\min(k_i, k_j) = k_i$, if $\geq i \leq j$

Considering two linguistic terms $k_\alpha, k_\beta \in K$, and $\mu > 0$, the operations are defined as follows:

(1) $k_\alpha \oplus k_\beta = k_{\alpha+\beta}$

(2) $\mu k_\alpha = k_{\mu\alpha}$

Definition 2. Let X be a nonempty set. An ILS A in X is, then, expressed as

$$A = \left\{ \left\langle x \left[k_{\theta(x)}, (\mu_A(x), \nu_A(x)) \right] \right\rangle | x \in X \right\}, \tag{1}$$

where $k_\theta \in \overline{K}$ and the $\mu A(x)$ and $\nu_A(x)$ indicate the membership degree and nonmembership degree of the element $x \in X$ to the set A, respectively. Hence, we have $\mu A(x), \nu_A(x) \in [0, 1]$ and $0 \leq \mu A(x) + \nu_A(x) \leq 1$, for all $x \in X$. For convenience, the ILN is generally denoted as $\langle k_{\theta(x)}, (\mu A(x), \nu_A(x)) \rangle$.

More specifically, for each ILS A in X, we have

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{2}$$

where $\pi_A(a)$ is called the hesitation degree of x to linguistic variable $k_\theta(x)$.

After the notion of the ILN has been presented, we then calculate the weights of criteria. The weight calculation methods commonly used to determine the weights when the information is completely unknown include the analytic hierarchy process (AHP) [36, 37] and entropy method [38, 39]. Here, the criterion information is partially known, so the programming model is preferred to calculate the weights. The steps are as follows:

Step 1: establish a matrix with ILNs $(u \times v)$, which includes u different experts and v different criteria

Step 2: calculate the positive ideal solution (PIS) and negative ideal solution (NIS) for each criterion by using the following equations:

$$A_j^+ = \max_{j\in\text{benefit}}\left(H\left(A_j\right)\right)\text{or}\min_{j\in\text{cost}}\left(H\left(A_j\right)\right), \tag{3}$$

$$A_j^- = \max_{j\in\text{cost}}\left(H\left(A_j\right)\right)\text{or}\min_{j\in\text{benefit}}\left(H\left(A_j\right)\right), \tag{4}$$

where $H(A_i) = (\theta/t - 1) \times (\mu + \nu)$, A_j^+ represents the value of PIS for the j-th criterion, and A_j^- represents the value of NIS for the j-th criterion

Step 3: determine an objective function by using the PIS and NIS values:

$$\min T = \sum_{j=1}^{n} w_j \sum_{i=1}^{m} \left(d\left(A_{ij}, A_j^+\right) - d\left(A_{ij}, A_j^-\right) \right), \tag{5}$$

where $0 \le w_j \le 1$ and $\sum_j w_j = 1 \cdot d(\cdot)$ represents the distance between two ILNs

For example, for the two ILNs A_1 and A_2, we have

$$d(A_1, A_2) = \frac{1}{2(t-1)} \times \Big(\big| (1 + \mu(A_1) - \nu(A_1))\theta(A_1) - (1 + \mu(A_2) - \nu(A_2))\theta(A_1) \big| \Big). \tag{6}$$

2.3. The ILOWAD and the ILWALD Operators. In this section, some related operators are briefly reviewed, including the OWAD [40], the WLAD [41], the OWLAD [42], and the ILOWAD [43] measures.

Definition 3. Let A and B be two intuitionistic linguistic sets; the normalized hamming distance between A and B is given by the mathematical form:

$$d(A, B) = \frac{1}{n} \sum_{i=1}^{n} d_{\mathrm{ILN}}\big(\tilde{a}_i, \tilde{b}_i\big) = \sum_{i=1}^{n} \frac{1}{2n(t-1)} \times \Big(\big| (1 + \mu(a_i) - \nu(a_i))\theta(a_i) - (1 + \mu(b_i) - \nu(b_i))\theta(b_i) \big| \Big), \tag{7}$$

where $\tilde{a}_i = \langle k_{\theta(ai)}, (\mu(a_i), \nu(a_i)) \rangle$ and $\tilde{b}_i = \langle k_{\theta(bi)}, (\mu(b_i), \nu(b_i)) \rangle$ are the i-th ILN of A and B, respectively, and $\theta(\cdot)$ represents the i-th linguistic value of A or B.

Definition 4. Let $A = \{a_1, a_2, \ldots, a_n\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ be two crisp sets and $d_i = |a_i - b_i|$ the distance between a_i and b_i. The OWAD measure is, then, defined as

$$\mathrm{OWAD}(A, B) = \mathrm{OWAD}(d_1, d_2, \ldots, d_n) = \sum_{j=1}^{n} w_j d_{\sigma(j)}, \tag{8}$$

where $d_{\sigma(j)}\,(j = 1, 2, \ldots, n)$ is the j-th largest value of $d_j\,(j = 1, 2, \ldots, n)$. $w = \{w_j | \sum_{j=1}^{n} w_j = 1, 0 \le w_j \le 1\}$ is the associated weighting vector of *OWAD*.

Definition 5. The WLAD operator of dimension n is a mapping WLAD: $R^n \times R^n \longrightarrow R$ has a relative weighting vector $W = \{w_1, w_2, \ldots, w_n\}$, with $\sum_{i=1}^{n} w_i = 1$ and $w_i \in [0, 1]$.

$$\mathrm{WLAD}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \exp\left\{ \sum_{i=1}^{n} w_i \ln(d_i) \right\}, \tag{9}$$

where $d_i = |x_i - y_i|$ represents the individual distance between x_i and y_i.

Definition 6. Let $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ be two crisp sets and $d_i = |x_i - y_i|$ be the distance between xi and yi. The OWLAD operator of dimension n is a mapping $OWLAD$: $R^n \times R^n \longrightarrow R$ that has a relative weighting vector $W = \{\omega_1, \omega_2, \ldots, \omega_n\}$, with $\sum_{i=1}^{n} wi = 1$ and $w_j \in [0, 1]$, satisfying

$$\mathrm{OWLAD}(x_1, y_1, x_2, y_2, \ldots, x_n, y_n) = \exp\left\{ \sum_{j=1}^{n} w_j \ln\big(\mathrm{d}_{\sigma(j)}\big) \right\}, \tag{10}$$

where $d_{\sigma(j)}$ has the same meaning and value range as Definition 4.

Definition 7. The ILOWAD operator of dimension n is a mapping ILOWAD: $\Omega^n \times \Omega^n \longrightarrow \Omega$ that is defined by an associated weighting vector W. Hence, the sum of weights is equal to 1 and $w_j \in [0, 1]$. Then, we have

$$\mathrm{ILOWAD}\big(\big(\tilde{a}_1, \tilde{b}_1\big), \ldots, \big(\tilde{a}_n, \tilde{b}_n\big)\big) = \sum_{j=1}^{n} w_j d_{\sigma(j)}, \tag{11}$$

where $d_{\sigma(j)}$ is the j-th largest value among the intuitionistic linguistic distance $d_{\mathrm{ILN}}(\tilde{a}_i, \tilde{b}_i)$ and $\tilde{a}_i = \langle k_{\theta(ai)}, (\mu(a_i), \nu(a_i)) \rangle$ and $\tilde{b}_i = \langle k_{\theta(bi)}, (\mu(b_i), \nu(b_i)) \rangle$ are the i-th ILN of A and B, respectively.

3. Proposed Method

3.1. Intuitionistic Linguistic Weighted Logarithmic Distance Measures. In this section, we present the intuitionistic linguistic weighted logarithmic average distance (*ILWLAD*) operator, the *ILOWLAD* operator, and the *ILHWLAD* operator.

Definition 8. The ILWLAD operator of dimension n is a mapping $ILWALD$: $\mathrm{R}^n \times \mathrm{R}^n \longrightarrow R$. This operator can be formulated as:

$$ILWLAD\big(\big(\tilde{a}_1, \tilde{b}_1\big), \ldots, \big(\tilde{a}_n, \tilde{b}_n\big)\big) = \exp\left\{ \sum_{i=1}^{n} w_i \mathrm{In}(d_i) \right\}, \tag{12}$$

where $W = \{\omega_1, \omega_2, \ldots, \omega_n\}$ is the relative weighting vector of *ILWLAD*, $\sum_{i=1}^{n} wi = 1$, $w_i \in [0, 1]$.

Definition 9. The ILOWLAD operator maps the parameter vector of dimension n to a real number, which has a relative weighting vector $W = \{\omega_1, \omega_2, \ldots, \omega_n\}$, with $\sum_{i=1}^{n} wi = 1$ and $w_i \in [0, 1]$. Hence, we define this operator as follows:

$$\text{ILOWLAD}\left(\left(\tilde{a_1}, \tilde{b_1}\right), \ldots, \left(\tilde{a_n}, \tilde{b_n}\right)\right) = \exp\left\{\sum_{j=1}^{n} w_j \ln\left(d_{\sigma(j)}\right)\right\}, \quad (13)$$

where $d_{\sigma(j)}$ represents the j-th largest value of all intuitionistic linguistic distances $d_{\text{ILN}}(\tilde{a_1}, \tilde{b_1})$.

In the aggregation process, the ILWLAD measure examines the importance of criteria and the ILOWLAD measure examines the importance of the ordered deviation. However, the ILWLAD is unable to perform the aggregation function in order, while the ILOWLAD fails to integrate the criteria in a way that the ILWLAD can. To compensate for this disadvantage, we present the ILHWLAD measure.

Definition 10. Let $\tilde{a_i} = \langle k_{\theta(ai)}, (\mu(a_i), v(a_i))\rangle (i = 1, 2, \ldots, n)$ and $\tilde{b_i} = \langle k_{\theta(bi)}, (\mu(b_i), v(b_i))\rangle$ $(i = 1, 2, \ldots, n)$ two sets of ILN. An ILHWLAD operator of dimension n is a mapping ILHWLAD: $R^n \times R^n \longrightarrow R$. The ILHWLAD measure is given as follows:

$$\text{ILHWLAD}\left(\left(\tilde{a_1}, \tilde{b_1}\right), \cdots, \left(\tilde{a_n}, \tilde{b_n}\right)\right) = \exp\left\{\sum_{j=1}^{n} w_j \ln\left(\overline{D}_{\sigma(j)}\right)\right\}, \quad (14)$$

where $\overline{D}_{\sigma(j)}$ represents the j-th largest value among $\overline{D}_j$, which is defined as $\overline{D}_j = (nw_j D_j)$, $(j = 1, 2, \ldots, n.)$ $w_j = (w_1, w_2, \ldots, w_n)$ is the weight vector corresponding to $\overline{D}_{\sigma(j)}$, and w_j is the associated weight of the unordered value D_j, satisfying $\sum_{i=1}^{n} wi = 1$ and $w_i \in [0, 1]$. N is used as a balancing factor to compensate for the double weighting. Moreover, we can explore a wide range of special cases of the ILHWLAD operator utilizing the similar methods proven in [16, 44].

3.2. The MCDM Framework Based on the ILHWLAD Measure. The selection of concrete materials is a multicriteria decision-making problem. Based on the *ILHWLAD* operator, this section uses a weight programming model to calculate weight, and an MCDM framework is used to select the finest concrete material. The specific steps are shown in Figure 1.

An MCDM problem includes j different alternatives, denoted as $C_1, C_2, \ldots, C_j$, and a total of t experts are invited to evaluate the alternatives under k finite criteria $E_1, E_2, \ldots, E_k$. The process can be summarized into the following steps; Table 2:

Step 1: each expert $e_q (q = 1, 2, \ldots, t)$ (the corresponding weight is τ_q, which meets $\tau_q \geq 0$ and $\sum_{q=1}^{t} \tau_q = 1$) measures his or her performance using criteria from the ILNs. Afterwards, the individual decision matrix $R^q = (r_{ij}^{(q)})_{m \leftrightarrow n}$ is obtained, where $r_{ij}(q)$ is the evaluation of the alternative C_i by q-th experts with regard to criterion E_j.

Step 2: calculate the collective decision matrix $R^q = (r_{ij}^{q})_{m \times n}$ to aggregate individual evaluations, where $r_{ij} = \sum_{q=1}^{t} \tau_q r_{ij}^{(q)}$.

Step 3: establish the ideal alternative by setting the ideal performances for each criterion (see Table 2).

Step 4: use the programming model presented to obtain the weights of criteria according to equation (5). Then, the weights of operators can be determined by experts.

Step 5: the distances between the alternative $C_i (I = 1, 2, \ldots, j)$ and the ideal alternative I are computed by utilizing the ILHWLAD measure:

$$\begin{aligned}&\text{ILHWLAD}\left(\left(\tilde{a_1}, \tilde{b_1}\right), \cdots, \left(\tilde{a_n}, \tilde{b_n}\right)\right)\\&= \exp\left\{\sum_{j=1}^{n} w_j \ln\left(\overline{D}_{\sigma(j)}\right)\right\}.\end{aligned} \quad (15)$$

Step 6: based on the value of the distances obtained in the previous steps, we can order the alternatives and select the finest one.

4. Case Study

4.1. Description of Concretes. The concrete materials used in the construction industry are mainly autoclaved aerated concrete (AAC) [10, 45], hollow concrete blocks (HCB) [46, 47], expanded polystyrene (EPS) [48, 49], and lime hemp concrete (LHC) [50].

LHC (C_1) is a new composite material of lime and hemp, which maintains an excellent thermal and moisture processing performance. Because of its lower embodied energy (EE) and embodied carbon (EC), the energy consumption and emission of carbon dioxide are compact during its manufacture. Its low mechanical property means that it is widely used in roofs, walls, slabs, and insulation.

AAC (C_2) forms numerous small air holes in the interior during the production process, so it possesses good heat and sound insulation functions. Moreover, it is relatively light, and the density is about 1/3 of clay brick. It is generally used in the outside filled walls of buildings and non-load-bearing internal partitions.

HCB (C_3) is of low density and possesses a good thermal performance, which are advantageous for masonry. It is commonly used in industrial and civil buildings, particularly in bearing walls and frame structure fill walls of multistorey buildings. It is also frequently adopted to construct fences, flower beds, bridges, and so on.

EPS (C_4) is a lightweight polymer with good thermal insulation. It is commonly used in the heat insulation system of external walls, roofs, and floors of buildings.

4.2. Decision Procedure. In this section, we use the framework to deal with selection problems under IL environments. Four possible concretes $C_i (I = 1, 2, 3, 4)$ are evaluated from the following criteria: embodied energy (E1); embodied carbon (E2); purchase cost (E3); thermal performance (E4); ability to balance outdoor humidity

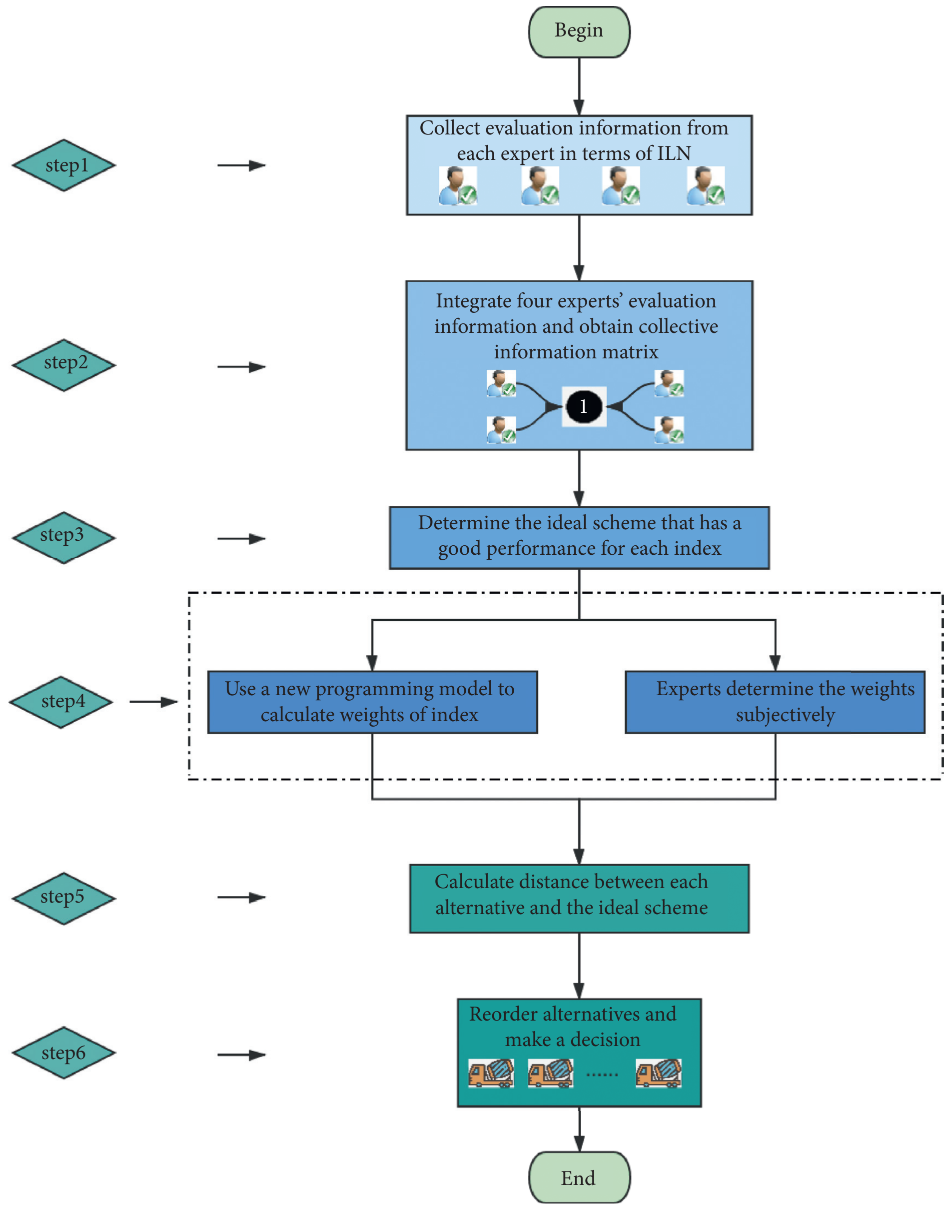

Figure 1: The procedure of the MCDM framework based on ILHWLAD for concrete materials.

Table 2: Ideal alternative.

	E_1	E_2	...	E_k
I	I_1	I_2	...	I_k

fluctuation (E5); and compressive strength (E6). Four experts (expert's weight $\tau = (0.25, 0.3, 0.2, 0.25)$) utilize IL information to evaluate four candidate concretes under six criteria, where the linguistic term set is assumed to be $K = (k_1, k_2, k_3, k_4, k_5, k_6, k_7)$; Tables 3–9 and Table 10.

Step 1: let each expert express evaluation of four concrete materials under given criteria through ILNs. The intuitionistic linguistic individual decision matrixes are shown in Tables 3–6.

Step 2: on the basis of the individual decision matrices and weights of the experts, we can obtain the collective decision matrix; see Table 7.

Step 3: having acquired the relevant information on the four concrete materials, the experts construct the ideal

Table 3: Intuitionistic linguistic matrix-expert 1.

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	$\langle k_6, (0.5, 0.4)\rangle$	$\langle k_6, (0.3, 0.4)\rangle$	$\langle k_3, (0.6, 0.3)\rangle$	$\langle k_6, (0.2, 0.6)\rangle$	$\langle k_6, (0.4, 0.4)\rangle$	$\langle k_6, (0.7, 0.4)\rangle$
C_2	$\langle k_4, (0.3, 0.6)\rangle$	$\langle k_5, (0.5, 0.4)\rangle$	$\langle k_6, (0.7, 0.2)\rangle$	$\langle k_6, (0.5, 0.5)\rangle$	$\langle k_5, (0.5, 0.5)\rangle$	$\langle k_4, (0.2, 0.8)\rangle$
C_3	$\langle k_5, (0.2, 0.7)\rangle$	$\langle k_4, (0.6, 0.2)\rangle$	$\langle k_4, (0.6, 0.3)\rangle$	$\langle k_4, (0.9, 0.1)\rangle$	$\langle k_4, (0.4, 0.4)\rangle$	$\langle k_7, (0.1, 0.9)\rangle$
C_4	$\langle k_3, (0.7, 0.2)\rangle$	$\langle k_3, (0.2, 0.8)\rangle$	$\langle k_5, (0.1, 0.9)\rangle$	$\langle k_3, (0.3, 0.6)\rangle$	$\langle k_2, (0.4, 0.4)\rangle$	$\langle k_2, (0.3, 0.6)\rangle$

Table 4: Intuitionistic linguistic matrix-expert 2.

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	$\langle k_7, (0.7, 0.2)\rangle$	$\langle k_7, (0.8, 0)\rangle$	$\langle k_3, (0.6, 0.3)\rangle$	$\langle k_6, (0.5, 0.5)\rangle$	$\langle k_7, (0.3, 0.6)\rangle$	$\langle k_2, (0.4, 0.5)\rangle$
C_2	$\langle k_5, (0.2, 0.7)\rangle$	$\langle k_5, (0.4, 0.6)\rangle$	$\langle k_6, (0.7, 0.2)\rangle$	$\langle k_6, (0.6, 0.4)\rangle$	$\langle k_5, (0.6, 0.3)\rangle$	$\langle k_3, (0.9, 0)\rangle$
C_3	$\langle k_5, (0.4, 0.5)\rangle$	$\langle k_4, (0.7, 0.3)\rangle$	$\langle k_5, (0.5, 0.3)\rangle$	$\langle k_4, (0.2, 0.7)\rangle$	$\langle k_5, (0.3, 0.6)\rangle$	$\langle k_7, (0.4, 0.5)\rangle$
C_4	$\langle k_3, (0.2, 0.8)\rangle$	$\langle k_4, (0.7, 0.2)\rangle$	$\langle k_5, (0.4, 0.5)\rangle$	$\langle k_3, (0.5, 0.4)\rangle$	$\langle k_2, (0.1, 0.8)\rangle$	$\langle k_2, (0.8, 0.1)\rangle$

Table 5: Intuitionistic linguistic matrix-expert 3.

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	$\langle k_7, (0.3, 0.6)\rangle$	$\langle k_6, (0.4, 0.5)\rangle$	$\langle k_3, (0.3, 0.6)\rangle$	$\langle k_7, (0.8, 0.1)\rangle$	$\langle k_6, (0.8, 0.2)\rangle$	$\langle k_3, (0.7, 0.2)\rangle$
C_2	$\langle k_4, (0.3, 0.7)\rangle$	$\langle k_5, (0.5, 0.4)\rangle$	$\langle k_7, (0.8, 0.1)\rangle$	$\langle k_7, (0.3, 0.6)\rangle$	$\langle k_5, (0.4, 0.5)\rangle$	$\langle k_4, (0.4, 0.6)\rangle$
C_3	$\langle k_6, (0.8, 0.2)\rangle$	$\langle k_5, (0.4, 0.6)\rangle$	$\langle k_5, (0.5, 0.4)\rangle$	$\langle k_5, (0.2, 0.7)\rangle$	$\langle k_4, (0.8, 0.2)\rangle$	$\langle k_7, (0.7, 0.2)\rangle$
C_4	$\langle k_3, (0.4, 0.5)\rangle$	$\langle k_3, (0.7, 0.2)\rangle$	$\langle k_6, (0.5, 0.4)\rangle$	$\langle k_3, (0.6, 0.3)\rangle$	$\langle k_3, (0.6, 0.4)\rangle$	$\langle k_3, (0.6, 0.2)\rangle$

Table 6: Intuitionistic linguistic matrix-expert 4.

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	$\langle k_7, (0.4, 0.5)\rangle$	$\langle k_6, (0.6, 0.3)\rangle$	$\langle k_3, (0.7, 0.2)\rangle$	$\langle k_6, (0.7, 0.2)\rangle$	$\langle k_6, (0.2, 0.7)\rangle$	$\langle k_3, (0.4, 0.4)\rangle$
C_2	$\langle k_4, (0.3, 0.4)\rangle$	$\langle k_4, (0.5, 0.4)\rangle$	$\langle k_6, (0.6, 0.2)\rangle$	$\langle k_6, (0.5, 0.5)\rangle$	$\langle k_4, (0.2, 0.8)\rangle$	$\langle k_4, (0.8, 0.1)\rangle$
C_3	$\langle k_6, (0.6, 0.3)\rangle$	$\langle k_4, (0.7, 0.2)\rangle$	$\langle k_4, (0.6, 0.3)\rangle$	$\langle k_5, (0.3, 0.6)\rangle$	$\langle k_4, (0.1, 0.9)\rangle$	$\langle k_7, (0.4, 0.4)\rangle$
C_4	$\langle k_4, (0.6, 0.2)\rangle$	$\langle k_3, (0.5, 0.5)\rangle$	$\langle k_6, (0.1, 0.9)\rangle$	$\langle k_4, (0.4, 0.5)\rangle$	$\langle k_2, (0.6, 0.3)\rangle$	$\langle k_3, (0.3, 0.5)\rangle$

Table 7: Collective IL decision matrix.

	E_1	E_2	E_3	E_4	E_5	E_6
C_1	$\langle k_{6.46}, (0.52, 0.37)\rangle$	$\langle k_{6.35}, (0.60, 0)\rangle$	$\langle k_{5.27}, (0.58, 0.31)\rangle$	$\langle k_{6.32}, (0.59, 0.30)\rangle$	$\langle k_{6.35}, (0.46, 0.45)\rangle$	$\langle k_{5.11}, (0.56, 0.37)\rangle$
C_2	$\langle k_{5.77}, (0.27, 0.59)\rangle$	$\langle k_{5.91}, (0.47, 0.45)\rangle$	$\langle k_{6.32}, (0.70, 0.17)\rangle$	$\langle k_{6.32}, (0.50, 0.49)\rangle$	$\langle k_{5.91}, (0.46, 0.48)\rangle$	$\langle k_{5.54}, (0.71, 0)\rangle$
C_3	$\langle k_{6.11}, (0.53, 0.40)\rangle$	$\langle k_{5.72}, (0.63, 0.28)\rangle$	$\langle k_{5.83}, (0.55, 0.32)\rangle$	$\langle k_{5.89}, (0.54, 0.41)\rangle$	$\langle k_{5.77}, (0.44, 0.48)\rangle$	$\langle k_{6.52}, (0.42, 0.46)\rangle$
C_4	$\langle k_{5.37}, (0.50, 0.36)\rangle$	$\langle k_{5.39}, (0.56, 0.36)\rangle$	$\langle k_{6.11}, (0.32, 0.64)\rangle$	$\langle k_{5.37}, (0.46, 0.44)\rangle$	$\langle k_{4.86}, (0.44, 0.46)\rangle$	$\langle k_{4.98}, (0.57, 0.33)\rangle$

Table 8: Ideal concrete.

	E_1	E_2	E_3	E_4	E_5	E_6
I_1	$\langle k_7, (0.9, 0.1)\rangle$	$\langle k_7, (0.9, 0)\rangle$	$\langle k_7, (0.8, 0.2)\rangle$	$\langle k_6, (0.9, 0.1)\rangle$	$\langle k_7, (0.8, 0.1)\rangle$	$\langle k_7, (0.9, 0.1)\rangle$

Table 9: Positive ideal solution (PIS) and negative ideal solution (NIS) for each criterion.

	E_1	E_2	E_3	E_4	E_5	E_6
A_j^+	$\langle k_{6.46}, (0.52, 0.37)\rangle$	$\langle k_{5.91}, (0.47, 0.45)\rangle$	$\langle k_{6.11}, (0.32, 0.64)\rangle$	$\langle k_{6.32}, (0.50, 0.49)\rangle$	$\langle k_{6.35}, (0.46, 0.45)\rangle$	$\langle k_{6.52}, (0.42, 0.46)\rangle$
A_j^-	$\langle k_{5.37}, (0.50, 0.36)\rangle$	$\langle k_{6.35}, (0.60, 0)\rangle$	$\langle k_{5.27}, (0.58, 0.31)\rangle$	$\langle k_{5.37}, (0.46, 0.44)\rangle$	$\langle k_{4.86}, (0.44, 0.46)\rangle$	$\langle k_{5.54}, (0.71, 0)\rangle$

Table 10: The total distance of each criterion.

	E_1	E_2	E_3	E_4	E_5	E_6
$\sum_{i=1}^{m}(d(A_{ij}, A_j^+) - d(A_{ij}, A_j^-))$	0.080	-0.320	0.426	-0.161	-0.024	-0.533

Table 11: The ranking orders of different measures.

Measures	Ranking orders
ILHWLAD	$C_1 \succ C_2 \succ C_3 \succ C_4$
ILOWLAD	$C_1 \succ C_2 \succ C_3 \succ C_4$
ILOWAD	$C_1 \succ C_3 \succ C_2 \succ C_4$
ILWLAD	$C_2 \succ C_1 \succ C_3 \succ C_4$

concrete with good performance in each criterion; see Table 8.

Step 4: according to Table 7 and equations (2)–(4), the PIS and NIS of each criterion are determined; see Table 9.

The distance matrix is calculated in accordance with equation (6); see Table 10.

In many real-world situations, the information about criteria weights is incomplete, so we should first determine the criteria weights in Step 4 if the known weight information on the criteria is set as in the following set: $H = \{\omega_2 + \omega_4 + \omega_5 + \omega_6 \leq 0.6, \omega_1 \leq 0.15, \omega_3 \leq 0.25, \omega_2 \leq 0.2, \omega_6 \leq 0.15, \omega_4 + \omega_6 \leq 0.3\}$. The objective function and the constraints are determined by using equation (5). The objective function is given by

$$Z = 0.080\omega_1 - 0.320\omega_2 + 0.426\omega_3 - 0.161\omega_4 - 0.024\omega_5 - 0.533\omega. \quad (16)$$

Finally, the weighting vector of the criteria is obtained as $\omega = (0.15, 0.2, 0.25, 0.15, 0.1, 0.15)^T$ using the Python programming language. In the meantime, the weighting vectors of the *ILHWLAD* are set to be $w = (0.2, 0.15, 0.25, 0.1, 0.15, 0.15)^T$.

Step 5: in accordance with equation (15) and the available information, the distances between the alternatives and the ideal concrete are computed by using the ILHWLAD as follows:

(1) *ILHWLAD* $(C_1, I) = 0.3651$
(2) *ILHWLAD* $(C_2, I) = 0.3717$
(3) *ILHWLAD* $(C_3, I) = 0.4319$
(4) *ILHWLAD* $(C_4, I) = 0.5338$

Step 6: the smaller the value of the *ILHWLAD* (C_i, I), the closer the C_i to the ideal concrete. Therefore, we can rearrange the order of $C_i: C_1 \succ C_2 \succ C_3 \succ C_4$

Hence, the best alternative is C_1.

4.3. Comparisons and Discussion. In this section, to verify the superiority and rationality of the ILHWLAD method, and the results of the ILOWLAD, the ILOWAD, and the ILWLAD measures are compared with those of the ILHWLAD measures in the selection of the concrete materials. According to equation (13), the distances between the alternatives and the ideal concrete are calculated by using the ILOWLAD operator as follows:

(1) *ILOWLAD* $(C_1, I) = 0.3786$
(2) *ILOWLAD* $(C_2, I) = 0.3991$
(3) *ILOWLAD* $(C_3, I) = 0.4445$
(4) *ILOWLAD* $(C_4, I) = 0.5476$

By the *ILOWAD* measure, we have

(1) *ILOWAD* $(C_1, I) = 0.3963$
(2) *ILOWAD* $(C_2, I) = 0.4579$
(3) *ILOWAD* $(C_3, I) = 0.4516$
(4) *ILOWAD* $(C_4, I) = 0.5499$

The results obtained by the *ILWLAD* measure are

(1) *ILWLAD* $(C_1, \mathrm{I}) = 0.3563$
(2) *ILWLAD* $(C_2, I) = 0.3387$
(3) *ILWLAD* $(C_3, \mathrm{I}) = 0.4237$
(4) *ILWLAD* $(C_4, I) = 0.5455$

Thus, the final ranking of the four alternatives according to the *ILOWLAD*, the *ILOWAD*, and the *ILWLAD* measures are $C_1 \succ C_2 \succ C_3 \succ C_4$, $C_1 \succ C_3 \succ C_2 \succ C_4$, and $C_2 \succ C_1 \succ C_3 \succ C_4$, respectively. The final results are shown in Table 11.

The ranking results obtained by the abovementioned four methods are contrasting. As Table 11 shows, LHC is the finest concrete material measured by the ILHWLAD, ILOWLAD, and ILOWAD measures. LHC performs very well in terms of thermal performance, embodied energy, and embodied carbon; it is a high-quality environmentally friendly insulating material. A building constructed with LHC not only reduces carbon dioxide emissions but also cuts down energy consumption. However, according to the measurement results of the

ILWLAD operator, AAC is the finest concrete material. EPS performs poorly under all four measures, mainly because, according to the evaluation criteria system designed for this paper, its compressive strength is low, it consumes more energy, and produces more carbon dioxide in the manufacturing process than the other three materials.

The reasons for the inconsistent results can be summarized as follows:

First, ILOWLAD and ILOWAD take the ordering mechanism of the parameters into account and pay more attention to the importance of ordered deviation. But, ILWLAD considers the importance of the criteria. Under this measure, AAC is superior to the other three concrete materials in the economic aspect. Hence, the distance between AAC and the ideal solution is lowest in terms of the purchase cost criterion. However, during the aggregation process of the ILOWLAD and the ILOWAD operators, the higher weights are coordinated by those larger intuitionistic linguistic distance values, in which LHC performs best. Hence, the distance between it and the ideal solution is the smallest.

Second, unlike the ILOWAD operator, the ILOWLAD operator performs logarithmic transformation of distance. If the evaluation of an alternative is closer to the ideal solution under a certain criterion, the advantage of logarithmic transformation will be clearer. In this case, AAC is the closest to the ideal solution under the purchase cost criterion, which increases the gap between AAC and HCB, so that the second and third place results obtained by the two measures are different.

As has been noted, ILWLAD prioritizes the criteria, whereas the ILOWLAD measure only accounts for the importance of the ordered deviation. Therefore, ILWLAD and ILOWLAD take into account different stages in the assembly process. ILHWLAD makes up for this deficiency by taking input arguments and the ordered position into account simultaneously. Hence, it is the modest among the four operators.

5. Conclusions

Concrete plays a vital role in the construction industry; it helps to determine the strength, thermal performance, and relative humidity of buildings. At the same time, it is one of the major sources of energy consumption and carbon emissions, though certain types of concrete contribute to energy sustainability more than others. In this paper, we proposed an HWLAD-MCDM framework for the selection of the finest concrete materials under an intuitionistic linguistic fuzzy environment. The four alternative concrete materials LHC, AAC, HCB, and EPS were evaluated using six criteria (embodied energy, embodied carbon, purchase cost, thermal performance, ability to balance outdoor humidity fluctuations, and compressive strength), which encompass energy sustainability, economic performance, comfort, and safety. The OWLAD, WLAD, ILOWLAD, and ILWLAD operators were used in combination with an ILS. We, then, presented the new ILHWLAD operator, which addresses the limitations of the latter two. In addition, instead of using the traditional AHP and entropy method, we used a programming model to calculate the weight under incomplete information. The MCDM method based on the ILHWLAD operator was proposed for the selection of the finest concrete material, and the results are compared with those obtained by ILOWLAD, ILOWAD, and ILWLAD.

The MCDM framework based on the ILHWLAD operator allows a new way of making decisions, and its field of application is not limited to the selection of concrete materials. In further research, both methodological developments and new areas of application should be considered. With regard to the former, the expert weighting scheme could be employed in a more objective manner. In addition, we should try to expand further the new operator and implement it in more complex areas.

Data Availability

All data generated and analyzed during this study are included in the published paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Decision-Making Framework for an Effective Sanitizer to Reduce COVID-19 under Fermatean Fuzzy Environment

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The purpose of this article is to develop some general aggregation operators (AOs) based on Einstein's norm operations, to cumulate the Fermatean fuzzy data in decision-making environments. A Fermatean fuzzy set (FFS), possessing the more flexible structure than the intuitionistic fuzzy set (IFS) and Pythagorean fuzzy set (PFS), is a competent tool to handle vague information in the decision-making process by the means of membership degree (MD) and nonmembership degree (NMD). Our target is to empower the AOs using the theoretical basis of Einstein norms for the FFS to establish some advantageous operators, namely, Fermatean fuzzy Einstein weighted averaging (FFEWA), Fermatean fuzzy Einstein ordered weighted averaging (FFEOWA), generalized Fermatean fuzzy Einstein weighted averaging (GFFEWA), and generalized Fermatean fuzzy Einstein ordered weighted averaging (GFFEOWA) operators. Some properties and important results of the proposed operators are highlighted. As an addition to the MADM strategies, an approach, based on the proposed operators, is presented to deal with Fermatean fuzzy data in MADM problems. Moreover, multiattribute decision-making (MADM) problem for the selection of an effective sanitizer to reduce coronavirus is presented to show the capability and proficiency of this new idea. The results are compared with the Fermatean fuzzy TOPSIS method to exhibit the potency of the proposed model.

1. Introduction

In decision sciences, it is an important aspect to find the ranking order of the alternatives corresponding to different attributes according to the preferences of the decision-making experts. Therefore, selection of various attributes of the alternatives is a very complex task. These decisions cannot be interpreted by the exact data so the need of a powerful model was raised to handle the ambiguous data. For that issue, Zadeh [1] initiated the innovative idea of fuzzy set (FS) which served as the backbone of the FS theory. FS permits the experts to describe their satisfaction level (membership degree) regarding performance of a member within the unit interval. Although, the FSs provide the grounds to the uncertain assessments but they were not adequate enough to describe the NMD. To overcome the limitations of FS, Atanassov [2] introduced a more dominant model, namely, IFS which has both MD μ and NMD ν with condition $\mu + \nu \leq 1$. The theory of IFS was felt to be inept and insufficient to represent the inexact data as there are a lot of problems where the sum of MD and NMD is exceeded by 1. To reduce such type of complications, Yager [3] delivered the idea of PFS with condition $\mu^2 + \nu^2 \leq 1$. However, PFS has also some limitations if MD of an element is 0.8 and NMD is 0.7, then sum of square of these values is greater than 1. Then, Yager [4] developed the theory of q-rung orthopair fuzzy set (q-ROFS) with condition $\mu^q + \nu^q \leq 1$. Recently, Senapati and Yager [5] gave the concept of FFS as a generalization of IFS and PFS.

The worthwhile theory of AOs is widely applied to decision-making scenarios for the sake of data aggregation and to identify the best alternative from the possible choices. Xu [6] gave the idea of intuitionistic fuzzy (IF) AOs. The concept of generalized AOs for IFS was developed by Zhao et al. [7]. Rahman et al. [8]

introduced Pythagorean fuzzy (PF) AOs. Zhao and Wei [9] studied the Einstein hybrid AOs under IF environment. The idea of IF aggregation using Einstein operations was discussed by Wang and Liu [10]. The induced interval-valued IF Einstein AOs were developed by Cai and Han [11]. Garg [12] studied the generalized PF Einstein weighted arithmetic AOs. Garg [13] also proposed the generalized PF Einstein weighted geometric AOs. The Pythagorean Dombi fuzzy AOs with applications were discussed by Akram et al. [14]. Shahzadi et al. [15] proposed the decision-making approach using PF Yager AOs. Liu and Wang [16] expressed q-rung orthopair fuzzy (q-ROF) weighted AOs. Wei et al. [17] studied weighted Heronian mean AOs under q-ROF information. q-ROF power Maclaurin AOs were developed by Liu et al. [18]. Jana et al. [19] studied Dombi AOs for q-ROFS. Liu and Liu [20] proposed q-ROF Bonferroni mean operators. Joshi and Gegov [21] studied the confidence levels q-ROF aggregation operators. Akram and Shahzadi [22] developed the hybrid decision-making model under q-ROF Yager AOs. Liu et al. [23] extended the concept of prioritized weighted AOs for complex q-ROFS. Senapati and Yager [24] studied subtraction, division, and Fermatean arithmetic mean operations over FFS. The idea of Fermatean fuzzy (FF) weighted averaging/geometric operators was also given by Senapati and Yager [25]. For more information and applications, the readers can refer to [26–59].

The motivations of this article are described as follows:

(1) The judgement of a perfect alternative in an FF environment is a laborious MADM problem. The prevalent model, possessing the more space than the IF model and PF model, vigorously elaborates the imprecise decisions for the selection of best alternative.
(2) As Einstein AOs are the simplest and quite creative approach for dealing with DM affairs, basically, this article directs Einstein AOs in FF surroundings to face complex issues.
(3) The outcomes based on conclusion are quite accurate under Einstein AOs when it is put on to the reality-based MADM problems in FF data.
(4) The proposed operators are keen to provide the optimal solution not only for FF environment but also to work efficiently for IF and PF environment.

The contributions of this article are described as follows:

(1) The feasibility of FFNs is merged with the aggregation skills of Einstein norms to establish more powerful, multiskilled, and practical AOs which can be deployed to aggregate FF data and to get more accuracy in decision-making scenarios
(2) The dominant properties as well as the notable results of the proposed operators are highlighted
(3) An algorithm is studied to handle complex realistic problems with FF data
(4) A MADM problem for the selection of an effective sanitizer to reduce coronavirus is discussed by using proposed operators
(5) A validity test is discussed for the approval and authenticity of proposed theory
(6) At the end, the benefits and characteristics of the proposed work are discussed by comparison analysis

The remaining paper is as follows. In Section 2, we recall the concept of FFS and related score functions. Section 3 provides Einstein operational laws for FFNs. In Sections 4 and 5, we study the FFEWA and FFEOWA operators, respectively, and related properties to them. In Sections 6 and 7, we present the idea of GFFEWA and GFFEOWA operators, respectively. In Section 8, we propose an algorithm for our new model and discuss a MADM problem for the selection of a good sanitizer to reduce the coronavirus. Section 9 provides the validity criteria to prove the consistency of the proposed work. Section 10 gives the comparison analysis of proposed theory with the FF TOPSIS method. In Section 11, we have concluded the results related to the proposed model.

The list of acronyms in research paper is given in Table 1.

2. Preliminaries

In this section, we recall some basic definitions including IFS, PFS, FFS, and score functions related to FFS.

Definition 1. (see [2]). An IFS I on nonempty set $\mathscr{V}$ is given by

$$I = \{\langle x, \mu_I(x), \nu_I(x)\rangle\}, \tag{1}$$

where $\mu_I\colon \mathscr{V} \longrightarrow [0,1]$ and $\nu_I\colon \mathscr{V} \longrightarrow [0,1]$ specify MD and NMD of an element $x \in \mathscr{V}$, respectively. $\varpi_I(x) = 1 - \mu_I(x) - \nu_I(x)$ is indeterminacy degree (InD) of an element $x \in \mathscr{V}$.

Definition 2. (see [3]). A PFS P on nonempty set $\mathscr{V}$ is given by

$$P = \{\langle x, \mu_P(x), \nu_P(x)\rangle\}, \tag{2}$$

where $\mu_P\colon \mathscr{V} \longrightarrow [0,1]$ and $\nu_P\colon \mathscr{V} \longrightarrow [0,1]$ specify MD and NMD of an element, respectively. $\varpi_P(x) = \sqrt{1 - (\mu_P(x))^2 - (\nu_P(x))^2}$ is InD.

Definition 3. (see [5]). An FFS $\mathscr{R}$ on nonempty set $\mathscr{V}$ is given by

$$\mathscr{R} = \{\langle x, \mu_{\mathscr{R}}(x), \nu_{\mathscr{R}}(x)\rangle\}, \tag{3}$$

where $\mu_{\mathscr{R}}\colon \mathscr{V} \longrightarrow [0,1]$, $\nu_{\mathscr{R}}\colon \mathscr{V} \longrightarrow [0,1]$, and $\varpi_{\mathscr{R}}(x) = \sqrt[3]{1 - (\mu_{\mathscr{R}}(x))^3 - (\nu_{\mathscr{R}}(x))^3}$ specify MD, NMD, and InD, respectively. FFNs are components of the FFS.

Definition 4. (see [5]). The score function and accuracy function for FFN $\mathscr{R} = (\mu_{\mathscr{R}}, \nu_{\mathscr{R}})$ are represented by

$$\begin{aligned} S(\mathscr{R}) &= \mu_{\mathscr{R}}^3 - \nu_{\mathscr{R}}^3, \quad \text{where } S(\mathscr{R}) \in [-1,1],\\ \mathscr{A}(\mathscr{R}) &= \mu_{\mathscr{R}}^3 + \nu_{\mathscr{R}}^3, \quad \text{where } \mathscr{A}(\mathscr{R}) \in [0,1]. \end{aligned} \tag{4}$$

Table 1: List of acronyms.

Acronyms	Description
FS	Fuzzy set
IFS	Intuitionistic fuzzy set
PFS	Pythagorean fuzzy set
q-ROFS	q-rung orthopair fuzzy set
FFS	Fermatean fuzzy set
FFN	Fermatean fuzzy number
AOs	Aggregation operators
MADM	Multiattribute decision-making
FF TOPSIS	Fermatean fuzzy TOPSIS

Definition 5. (see [5]). Consider two FFNs $\mathscr{R}_1 = \langle \mu_{\mathscr{R}_1}, \nu_{\mathscr{R}_1} \rangle$ and $\mathscr{R}_2 = \langle \mu_{\mathscr{R}_2}, \nu_{\mathscr{R}_2} \rangle$. Then,

(1) If $S(\mathscr{R}_1) < S(\mathscr{R}_2)$, then $\mathscr{R}_1 \prec \mathscr{R}_2$.

(2) If $S(\mathscr{R}_1) > S(\mathscr{R}_2)$, then $\mathscr{R}_1 \succ \mathscr{R}_2$.

(3) If $S(\mathscr{R}_1) = S(\mathscr{R}_2)$, then

(a) If $\mathscr{A}(\mathscr{R}_1) < \mathscr{A}(\mathscr{R}_2)$, then $\mathscr{R}_1 \prec \mathscr{R}_2$.
(b) If $\mathscr{A}(\mathscr{R}_1) > \mathscr{A}(\mathscr{R}_2)$, then $\mathscr{R}_1 \succ \mathscr{R}_2$.
(c) If $\mathscr{A}(\mathscr{R}_1) = \mathscr{A}(\mathscr{R}_2)$, then $\mathscr{R}_1 \sim \mathscr{R}_2$.

3. Einstein Operational Law of FFNs

In this section, we present concepts of the Einstein t-norm and t-conorm operations for FFNs and some of their properties. The Einstein operations on FFNs are defined as follows.

Definition 6. Let $\mathscr{R} = \langle \mu, \nu \rangle$, $\mathscr{R}_1 = \langle \mu_1, \nu_1 \rangle$, and $\mathscr{R}_2 = \langle \mu_2, \nu_2 \rangle$ be FFNs and $\lambda > 0$; then,

(i) $\overline{\mathscr{R}} = \langle \nu_{\mathscr{R}}, \mu_{\mathscr{R}} \rangle$

(ii) $\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2 = \langle \min\{\mu_1, \mu_2\}, \max\{\nu_1, \nu_2\} \rangle$

(iii) $\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2 = \langle \max\{\mu_1, \mu_2\}, \min\{\nu_1, \nu_2\} \rangle$

(iv) $\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2 = \left\langle \sqrt[3]{(\mu_1^3 + \mu_2^3)/(1 + \mu_1^3 \cdot {}_\varepsilon \mu_2^3)}, (\nu_1 \cdot {}_\varepsilon \nu_2 / \sqrt[3]{1 + (1 - \nu_1^3) \cdot \varepsilon (1 - \nu_2^3)}) \right\rangle$

(v) $\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2 = \left\langle (\mu_1 \cdot {}_\varepsilon \mu_2 / \sqrt[3]{1 + (1 - \mu_1^3) \cdot \varepsilon (1 - \mu_2^3)}), \sqrt[3]{(\nu_1^3 + \nu_2^3)/(1 + \nu_1^3 \cdot {}_\varepsilon \nu_2^3)} \right\rangle$

(vi) $\lambda \cdot_\varepsilon \mathscr{R} = \left\langle \sqrt[3]{((1+\mu^3)^\lambda - (1-\mu^3)^\lambda)/((1+\mu^3)^\lambda + (1-\mu^3)^\lambda)}, (\sqrt[3]{2}\nu^\lambda / \sqrt[3]{(2-\nu^3)^\lambda + (\nu^3)^\lambda}) \right\rangle$

(vii) $\mathscr{R}^\lambda = \left\langle (\sqrt[3]{2}\mu^\lambda / \sqrt[3]{(2-\mu^3)^\lambda + (\mu^3)^\lambda}), \sqrt[3]{((1+\nu^3)^\lambda - (1-\nu^3)^\lambda)/((1+\nu^3)^\lambda + (1-\nu^3)^\lambda)} \right\rangle$

Theorem 1. *Let $\mathscr{R} = \langle \mu_{\mathscr{R}}, \nu_{\mathscr{R}} \rangle$, $\mathscr{R}_1 = \langle \mu_1, \nu_1 \rangle$, and $\mathscr{R}_2 = \langle \mu_2, \nu_2 \rangle$ be three FFNs; then, $\mathscr{R}_3 = \mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2$ and $\mathscr{R}_4 = \lambda \cdot {}_\varepsilon \mathscr{R}$ are also FFNs.*

Proof. Since $\lambda > 0$ and $\mathscr{R}$ is an FFN, therefore, $0 \le \mu_{\mathscr{R}}(x) \le 1$, $0 \le \nu_{\mathscr{R}}(x) \le 1$, and $0 \le (\mu_{\mathscr{R}}(x))^3 + (\nu_{\mathscr{R}}(x))^3 \le 1$; then, $1 - (\mu_{\mathscr{R}}(x))^3 \ge (\nu_{\mathscr{R}}(x))^3 \ge 0$, $1 - (\nu_{\mathscr{R}}(x))^3 \ge (\mu_{\mathscr{R}}(x))^3 \ge 0$, and $(1 - (\mu_{\mathscr{R}}(x))^3)^\lambda \ge (\nu_{\mathscr{R}}(x))^3$; then,

$$
\begin{aligned}
&\sqrt[3]{\frac{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda - \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}} \\
&\le \sqrt[3]{\frac{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda - \left((\nu_{\mathscr{R}}(x))^3\right)^\lambda}{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left((\nu_{\mathscr{R}}(x))^3\right)^\lambda}}, \\
&\frac{\sqrt[3]{2}(\nu_{\mathscr{R}}(x))^\lambda}{\sqrt[3]{\left(2 - (\nu_{\mathscr{R}}(x))^3\right)^\lambda + \left(\nu_{\mathscr{R}}(x)^3\right)^\lambda}} \\
&\le \frac{\sqrt[3]{2}(\nu_{\mathscr{R}}(x))^\lambda}{\sqrt[3]{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left((\nu_{\mathscr{R}}(x))^3\right)^\lambda}}.
\end{aligned}
\tag{5}
$$

Thus,

$$
\left(\sqrt[3]{\frac{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda - \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}}\right)^3 + \left(\frac{\sqrt[3]{2}(\nu_{\mathscr{R}}(x))^\lambda}{\sqrt[3]{\left(2 - (\nu_{\mathscr{R}}(x))^3\right)^\lambda + \left((\nu_{\mathscr{R}}(x))^3\right)^\lambda}}\right)^3 \le 1.
\tag{6}
$$

Furthermore,

$$
\left(\sqrt[3]{\frac{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda - \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}}\right)^3 + \left(\frac{\sqrt[3]{2}(\nu_{\mathscr{R}}(x))^\lambda}{\sqrt[3]{\left(2 - (\nu_{\mathscr{R}}(x))^3\right)^\lambda + \left(\nu_{\mathscr{R}}(x)^3\right)^\lambda}}\right)^3 = 0,
\tag{7}
$$

iff $\mu_{\mathscr{R}}(x) = \nu_{\mathscr{R}}(x) = 0$ and

$$
\left(\sqrt[3]{\frac{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda - \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}{\left(1 + (\mu_{\mathscr{R}}(x))^3\right)^\lambda + \left(1 - (\mu_{\mathscr{R}}(x))^3\right)^\lambda}}\right)^3 + \left(\frac{\sqrt[3]{2}(\nu_{\mathscr{R}}(x))^\lambda}{\sqrt[3]{\left(2 - (\nu_{\mathscr{R}}(x))^3\right)^\lambda + \left(\nu_{\mathscr{R}}(x)^3\right)^\lambda}}\right)^3 = 1,
\tag{8}
$$

iff $(\mu_{\mathscr{R}}(x))^3 + (\nu_{\mathscr{R}}(x))^3 = 1$.

Thus, $\mathscr{R}_4 = \lambda \cdot {}_\varepsilon \mathscr{R}$ is an FFN for $\lambda > 0$. □

Theorem 2. *Let $\lambda, \lambda_1, \lambda_2 \ge 0$; then,*

(i) $\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2 = \mathscr{R}_2 \oplus_\varepsilon \mathscr{R}_1$

(ii) $\mathscr{R}_1 \otimes_\varepsilon \mathscr{R}_2 = \mathscr{R}_1 \otimes_\varepsilon \mathscr{R}_2$

(iii) $\lambda \cdot_\varepsilon (\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2) = \lambda._\varepsilon \mathscr{R}_1 \oplus_\varepsilon \lambda \cdot_\varepsilon \mathscr{R}_2$

(iv) $(\mathscr{R}_1 \otimes_{\varepsilon} \mathscr{R}_2)^{\lambda} = \mathscr{R}_1^{\lambda} \otimes_{\varepsilon} \mathscr{R}_2^{\lambda}$

(v) $\lambda_1 \cdot_{\varepsilon} \mathscr{R} \oplus_{\varepsilon} \lambda_2 \cdot_{\varepsilon} \mathscr{R} = (\lambda_1 + \lambda_2) \cdot_{\varepsilon} \mathscr{R}$

(vi) $\mathscr{R}^{\lambda_1} \otimes_{\varepsilon} \mathscr{R}^{\lambda_2} = \mathscr{R}^{(\lambda_1+\lambda_2)}$

Proof

(i)

$$\begin{aligned}\mathscr{R}_1 \oplus_{\varepsilon} \mathscr{R}_2 &= \left\langle \sqrt[3]{\frac{\mu_1^3 + \mu_2^3}{1 + \mu_1^3 \cdot_{\varepsilon} \mu_2^3}}, \frac{\nu_1 \cdot_{\varepsilon} \nu_2}{\sqrt[3]{1 + \left(1 - \nu_1^3\right) \cdot \varepsilon \left(1 - \nu_2^3\right)}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\mu_2^3 + \mu_1^3}{1 + \mu_2^3 \cdot_{\varepsilon} \mu_1^3}}, \frac{\nu_2 \cdot_{\varepsilon} \nu_1}{\sqrt[3]{1 + \left(1 - \nu_2^3\right) \cdot \varepsilon \left(1 - \nu_1^3\right)}} \right\rangle \\ &= \mathscr{R}_2 \oplus_{\varepsilon} \mathscr{R}_1.\end{aligned} \tag{9}$$

(ii)

$$\mathscr{R}_1 \oplus_{\varepsilon} \mathscr{R}_2 = \left\langle \sqrt[3]{\frac{\mu_1^3 + \mu_2^3}{1 + \mu_1^3 \cdot_{\varepsilon} \mu_2^3}}, \frac{\nu_1 \cdot_{\varepsilon} \nu_2}{\sqrt[3]{1 + \left(1 - \nu_1^3\right) \cdot \varepsilon \left(1 - \nu_2^3\right)}} \right\rangle \tag{10}$$

is equivalent to

$$\mathscr{R}_1 \oplus_{\varepsilon} \mathscr{R}_2 = \left\langle \sqrt[3]{\frac{\left(1 + \mu_1^3\right) \cdot_{\varepsilon} \left(1 + \mu_2^3\right) - \left(1 - \mu_1^3\right) \cdot_{\varepsilon} \left(1 - \mu_2^3\right)}{\left(1 + \mu_1^3\right) \cdot_{\varepsilon} \left(1 + \mu_2^3\right) + \left(1 - \mu_1^3\right) \cdot_{\varepsilon} \left(1 - \mu_2^3\right)}}, \frac{\sqrt[3]{2}\nu_1 \cdot_{\varepsilon} \nu_2}{\sqrt[3]{\left(2 - \nu_1^3\right) \cdot \varepsilon \left(2 - \nu_2^3\right) + \nu_1^3 \cdot_{\varepsilon} \nu_2^3}} \right\rangle. \tag{11}$$

Take $a = (1 + \mu_1^3) \cdot_{\varepsilon} (1 + \mu_2^3)$, $b = (1 - \mu_1^3) \cdot_{\varepsilon} (1 - \mu_2^3)$, $c = \nu_1^3 \cdot_{\varepsilon} \nu_2^3$, and $d = (2 - \nu_1^3) \cdot \varepsilon (2 - \nu_2^3)$; then,

$$\mathscr{R}_1 \oplus_{\varepsilon} \mathscr{R}_2 = \left\langle \sqrt[3]{\frac{a - b}{a + b}}, \frac{\sqrt[3]{2c}}{\sqrt[3]{d + c}} \right\rangle. \tag{12}$$

By the Einstein FF law,

$$\begin{aligned}\lambda \cdot_{\varepsilon} \left(\mathscr{R}_1 \oplus_{\varepsilon} \mathscr{R}_2\right) &= \lambda \cdot_{\varepsilon} \left\langle \sqrt[3]{\frac{a - b}{a + b}}, \frac{\sqrt[3]{2c}}{\sqrt[3]{d + c}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{(1 + ((a - b)/(a + b)))^{\lambda} - (1 - ((a - b)/(a + b)))^{\lambda}}{(1 + ((a - b)/(a + b)))^{\lambda} + (1 - ((a - b)/(a + b)))^{\lambda}}}, \frac{\sqrt[3]{2} \cdot \left(\sqrt[3]{2c}/\sqrt[3]{d + c}\right)^{\lambda}}{\sqrt[3]{(2 - (2c/(d + c)))^{\lambda} + (2c/(d + c))^{\lambda}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{a^{\lambda} - b^{\lambda}}{a^{\lambda} + b^{\lambda}}}, \frac{\sqrt[3]{2c^{\lambda}}}{\sqrt[3]{d^{\lambda} + c^{\lambda}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\left(1 + \mu_1^3\right)^{\lambda} \cdot_{\varepsilon} \left(1 + \mu_2^3\right)^{\lambda} - \left(1 - \mu_1^3\right)^{\lambda} \cdot_{\varepsilon} \left(1 - \mu_2^3\right)^{\lambda}}{\left(1 + \mu_1^3\right)^{\lambda} \cdot_{\varepsilon} \left(1 + \mu_2^3\right)^{\lambda} + \left(1 - \mu_1^3\right)^{\lambda} \cdot_{\varepsilon} \left(1 - \mu_2^3\right)^{\lambda}}}, \frac{\sqrt[3]{2}\nu_1^{\lambda} \cdot_{\varepsilon} \nu_2^{\lambda}}{\sqrt[3]{\left(2 - \nu_1^3\right)^{\lambda} \cdot \varepsilon \left(2 - \nu_2^3\right)^{\lambda} + \left(\nu_1^3\right)^{\lambda} \cdot_{\varepsilon} \left(\nu_2^3\right)^{\lambda}}} \right\rangle.\end{aligned} \tag{13}$$

On the other hand,

$$\begin{aligned}\lambda \cdot_{\varepsilon} \mathscr{R}_1 &= \left\langle \sqrt[3]{\frac{\left(1 + \mu_1^3\right)^{\lambda} - \left(1 - \mu_1^3\right)^{\lambda}}{\left(1 + \mu_1^3\right)^{\lambda} + \left(1 - \mu_1^3\right)^{\lambda}}}, \frac{\sqrt[3]{2}\nu_1^{\lambda}}{\sqrt[3]{\left(2 - \nu_1^3\right)^{\lambda} + \left(\nu_1^3\right)^{\lambda}}} \right\rangle = \left\langle \sqrt[3]{\frac{a_1 - b_1}{a_1 + b_1}}, \frac{\sqrt[3]{2c_1}}{\sqrt[3]{d_1 + c_1}} \right\rangle, \\ \lambda \cdot_{\varepsilon} \mathscr{R}_2 &= \left\langle \sqrt[3]{\frac{\left(1 + \mu_2^3\right)^{\lambda} - \left(1 - \mu_2^3\right)^{\lambda}}{\left(1 + \mu_2^3\right)^{\lambda} + \left(1 - \mu_2^3\right)^{\lambda}}}, \frac{\sqrt[3]{2}\nu_2^{\lambda}}{\sqrt[3]{\left(2 - \nu_2^3\right)^{\lambda} + \left(\nu_2^3\right)^{\lambda}}} \right\rangle = \left\langle \sqrt[3]{\frac{a_2 - b_2}{a_2 + b_2}}, \frac{\sqrt[3]{2c_2}}{\sqrt[3]{d_2 + c_2}} \right\rangle,\end{aligned} \tag{14}$$

where $a_1 = (1+\mu_1^3)^\lambda$, $b_1 = (1-\mu_1^3)^\lambda$, $c_1 = (\nu_1^3)^\lambda$, $d_1 = (2-\nu_1^3)^\lambda$, $a_2 = (1+\mu_2^3)^\lambda$, $b_2 = (1-\mu_2^3)^\lambda$, $c_2 = (\nu_2^3)^\lambda$, and $d_2 = (2-\nu_2^3)^\lambda$; therefore,

$$\begin{aligned}(\lambda \cdot_\varepsilon \mathscr{R}_1)\oplus_\varepsilon(\lambda \cdot_\varepsilon \mathscr{R}_2) &= \left\langle \sqrt[3]{\frac{a_1-b_1}{a_1+b_1}}, \frac{\sqrt[3]{2c_1}}{\sqrt[3]{d_1+c_1}} \right\rangle \oplus_\varepsilon \left\langle \sqrt[3]{\frac{a_2-b_2}{a_2+b_2}}, \frac{\sqrt[3]{2c_2}}{\sqrt[3]{d_2+c_2}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{((a_1-b_1)/(a_1+b_1)) + ((a_2-b_2)/(a_2+b_2))}{1+((a_1-b_1)/(a_1+b_1))\cdot_\varepsilon((a_2-b_2)/(a_2+b_2))}}, \frac{2^{2/3}\sqrt[3]{c_1\cdot_\varepsilon c_2/(d_1+c_1)\cdot_\varepsilon(d_2+c_2)}}{\sqrt[3]{1+(1-(2c_1/(d_1+c_1))).{}_\varepsilon(1-(2c_2/(d_2+c_2)))}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{a_1\cdot_\varepsilon a_2 - b_1\cdot_\varepsilon b_2}{a_1\cdot_\varepsilon a_2 + b_1\cdot_\varepsilon b_2}}, \frac{\sqrt[3]{2c_1\cdot_\varepsilon c_2}}{\sqrt[3]{d_1\cdot_\varepsilon d_2 + c_1\cdot_\varepsilon c_2}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{(1+\mu_1^3)^\lambda\cdot_\varepsilon(1+\mu_2^3)^\lambda - (1-\mu_1^3)^\lambda\cdot_\varepsilon(1-\mu_2^3)^\lambda}{(1+\mu_1^3)^\lambda\cdot_\varepsilon(1+\mu_2^3)^\lambda + (1-\mu_1^3)^\lambda\cdot_\varepsilon(1-\mu_2^3)^\lambda}}, \frac{\sqrt[3]{2}\nu_1^\lambda\cdot_\varepsilon\nu_2^\lambda}{\sqrt[3]{(2-\nu_1^3)^\lambda\cdot\varepsilon(2-\nu_2^3)^\lambda + (\nu_1^3)^\lambda\cdot_\varepsilon(\nu_2^3)^\lambda}} \right\rangle.\end{aligned} \tag{15}$$

Hence, $\lambda\cdot_\varepsilon(\mathscr{R}_1\oplus_\varepsilon\mathscr{R}_2) = \lambda\cdot_\varepsilon\mathscr{R}_1\oplus_\varepsilon\lambda\cdot_\varepsilon\mathscr{R}_2$.

(v) For $\lambda_1, \lambda_2 > 0$,

$$\begin{aligned}\lambda_1\cdot_\varepsilon\mathscr{R} &= \left\langle \sqrt[3]{\frac{(1+\mu^3)^{\lambda_1} - (1-\mu^3)^{\lambda_1}}{(1+\mu^3)^{\lambda_1} + (1-\mu^3)^{\lambda_1}}}, \frac{\sqrt[3]{2}\nu^{\lambda_1}}{\sqrt[3]{(2-\nu^3)^{\lambda_1} + (\nu^3)^{\lambda_1}}} \right\rangle = \left\langle \sqrt[3]{\frac{a_1-b_1}{a_1+b_1}}, \frac{\sqrt[3]{2c_1}}{\sqrt[3]{d_1+c_1}} \right\rangle, \\ \lambda_2\cdot_\varepsilon\mathscr{R} &= \left\langle \sqrt[3]{\frac{(1+\mu^3)^{\lambda_2} - (1-\mu^3)^{\lambda_2}}{(1+\mu^3)^{\lambda_2} + (1-\mu^3)^{\lambda_2}}}, \frac{\sqrt[3]{2}\nu^{\lambda_2}}{\sqrt[3]{(2-\nu^3)^{\lambda_2} + (\nu^3)^{\lambda_2}}} \right\rangle = \left\langle \sqrt[3]{\frac{a_2-b_2}{a_2+b_2}}, \frac{\sqrt[3]{2c_2}}{\sqrt[3]{d_2+c_2}} \right\rangle,\end{aligned} \tag{16}$$

where $a_{\mathscr{J}} = (1+\mu^3)^{\lambda_{\mathscr{J}}}$, $b_{\mathscr{J}} = (1-\mu^3)^{\lambda_{\mathscr{J}}}$, $c_{\mathscr{J}} = (\nu^3)^{\lambda_{\mathscr{J}}}$, and $d_{\mathscr{J}} = (2-\nu^3)^{\lambda_{\mathscr{J}}}$, for $\mathscr{J} = 1, 2$.

$$\begin{aligned}(\lambda_1\cdot_\varepsilon\mathscr{R})\oplus_\varepsilon(\lambda_2\cdot_\varepsilon\mathscr{R}) &= \left\langle \sqrt[3]{\frac{a_1-b_1}{a_1+b_1}}, \frac{\sqrt[3]{2c_1}}{\sqrt[3]{d_1+c_1}} \right\rangle \oplus_\varepsilon \left\langle \sqrt[3]{\frac{a_2-b_2}{a_2+b_2}}, \frac{\sqrt[3]{2c_2}}{\sqrt[3]{d_2+c_2}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{((a_1-b_1)/(a_1+b_1)) + ((a_2-b_2)/(a_2+b_2))}{1+((a_1-b_1)/(a_1+b_1))\cdot_\varepsilon((a_2-b_2)/(a_2+b_2))}}, \frac{2^{2/3}\sqrt[3]{c_1\cdot_\varepsilon c_2/(d_1+c_1)\cdot_\varepsilon(d_2+c_2)}}{\sqrt[3]{1+(1-(2c_1/(d_1+c_1)))\cdot_\varepsilon(1-(2c_2/(d_2+c_2)))}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{a_1\cdot_\varepsilon a_2 - b_1\cdot_\varepsilon b_2}{a_1\cdot_\varepsilon a_2 + b_1\cdot_\varepsilon b_2}}, \frac{\sqrt[3]{2c_1\cdot_\varepsilon c_2}}{\sqrt[3]{d_1\cdot_\varepsilon d_2 + c_1\cdot_\varepsilon c_2}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{(1+\mu^3)^{\lambda_1+\lambda_2} - (1-\mu^3)^{\lambda_1+\lambda_2}}{(1+\mu^3)^{\lambda_1+\lambda_2} + (1-\mu^3)^{\lambda_1+\lambda_2}}}, \frac{\sqrt[3]{2}\nu^{\lambda_1+\lambda_2}}{\sqrt[3]{(2-\nu^3)^{\lambda_1+\lambda_2} + (\nu_1^3)^{\lambda_1+\lambda_2}}} \right\rangle \\ &= (\lambda_1+\lambda_2)\cdot_\varepsilon\mathscr{R}.\end{aligned} \tag{17}$$

Hence, $\lambda_1 \cdot_\varepsilon \mathscr{R} \oplus_\varepsilon \lambda_2 \cdot_\varepsilon \mathscr{R} = (\lambda_1 + \lambda_2) \cdot_\varepsilon \mathscr{R}$.
Similarly, others can be verified. □

Theorem 3. *Let $\mathscr{R}_1 = \langle \mu_1, \nu_1 \rangle$ and $\mathscr{R}_2 = \langle \mu_2, \nu_2 \rangle$ be FFNs; then,*

(i) $\mathscr{R}_1^c \wedge_\varepsilon \mathscr{R}_2^c = (\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2)^c$
(ii) $\mathscr{R}_1^c \vee_\varepsilon \mathscr{R}_2^c = (\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2)^c$
(iii) $\mathscr{R}_1^c \oplus_\varepsilon \mathscr{R}_2^c = (\mathscr{R}_1 \otimes_\varepsilon \mathscr{R}_2)^c$
(iv) $\mathscr{R}_1^c \otimes_\varepsilon \mathscr{R}_2^c = (\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2)^c$
(v) $(\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2) \oplus_\varepsilon (\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2) = \mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2$
(vi) $(\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2) \oplus_\varepsilon (\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2) = \mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2$

Proof. It is obvious. □

Theorem 4. *Let $\mathscr{R}_1 = \langle \mu_1, \nu_1 \rangle$, $\mathscr{R}_2 = \langle \mu_2, \nu_2 \rangle$, and $\mathscr{R}_3 = \langle \mu_3, \nu_3 \rangle$ be three FFNs; then,*

(i) $(\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2) \wedge_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_3) \vee_\varepsilon (\mathscr{R}_2 \wedge_\varepsilon \mathscr{R}_3)$
(ii) $(\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2) \vee_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_3) \wedge_\varepsilon (\mathscr{R}_2 \vee_\varepsilon \mathscr{R}_3)$
(iii) $(\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2) \oplus_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_3) \vee_\varepsilon (\mathscr{R}_2 \oplus_\varepsilon \mathscr{R}_3)$
(iv) $(\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2) \oplus_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_3) \wedge_\varepsilon (\mathscr{R}_2 \oplus_\varepsilon \mathscr{R}_3)$
(v) $(\mathscr{R}_1 \vee_\varepsilon \mathscr{R}_2) \otimes_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \otimes_\varepsilon \mathscr{R}_3) \vee_\varepsilon (\mathscr{R}_2 \otimes_\varepsilon \mathscr{R}_3)$
(vi) $(\mathscr{R}_1 \wedge_\varepsilon \mathscr{R}_2) \otimes_\varepsilon \mathscr{R}_3 = (\mathscr{R}_1 \otimes_\varepsilon \mathscr{R}_3) \wedge_\varepsilon (\mathscr{R}_2 \otimes_\varepsilon \mathscr{R}_3)$

Proof. The proof is trivial, so we omit it. □

4. Fermatean Fuzzy Einstein Weighted Averaging Operators

The Einstein weighted averaging operators under FF environment are defined here.

Definition 7. Let $\mathscr{R}_{\mathscr{j}} = \langle \mu_{\mathscr{j}}, \nu_{\mathscr{j}} \rangle$ $(\mathscr{j} = 1, 2, \ldots, \mathscr{s})$ be a collection of FFNs and $w_{\mathscr{j}}$ be the weight vector (WV) of $\mathscr{R}_{\mathscr{j}}$ with $w_{\mathscr{j}} > 0$ and $\sum_{\mathscr{j}=1}^{\mathscr{s}} w_{\mathscr{j}} = 1$; then, FFEWA operator is a mapping $\mathscr{Q}^{\mathscr{s}} \longrightarrow \mathscr{Q}$ such that

$$\text{FFEWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_{\mathscr{s}}) = w_1 \cdot_\varepsilon \mathscr{R}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \mathscr{R}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon w_{\mathscr{s}} \cdot_\varepsilon \mathscr{R}_{\mathscr{s}}. \quad (18)$$

If $w_{\mathscr{j}} = (1/\mathscr{s})$, $\forall \mathscr{j}$, then FFEWA operator becomes FFWA operator:

$$\text{FFA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_{\mathscr{s}}) = \frac{1}{\mathscr{s}} (\mathscr{R}_1 \oplus_\varepsilon \mathscr{R}_2 \oplus_\varepsilon \cdots \oplus_\varepsilon \mathscr{R}_{\mathscr{s}}). \quad (19)$$

Theorem 5. *Let $\mathscr{R}_{\mathscr{j}} = (\mu_{\mathscr{j}}, \nu_{\mathscr{j}})$ be FFNs; then, the aggregated value by using equation (18) is*

$$\text{FFEWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_{\mathscr{s}}) = \left\langle \sqrt[3]{\frac{\prod_{\mathscr{j}=1}^{\mathscr{s}} \left(1 + \mu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}} - \prod_{\mathscr{j}=1}^{\mathscr{s}} \left(1 + \mu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}}}{\prod_{\mathscr{j}=1}^{\mathscr{s}} \left(1 + \mu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}} + \prod_{\mathscr{j}=1}^{\mathscr{s}} \left(1 + \mu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}}}}, \frac{\sqrt[3]{2} \prod_{\mathscr{j}=1}^{\mathscr{s}} \nu_{\mathscr{j}}^{w_{\mathscr{j}}}}{\sqrt[3]{\prod_{\mathscr{j}=1}^{\mathscr{s}} \left(2 - \nu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}} + \prod_{\mathscr{j}=1}^{\mathscr{s}} \left(\nu_{\mathscr{j}}^3\right)^{w_{\mathscr{j}}}}} \right\rangle. \quad (20)$$

Proof. Use the mathematical induction to prove equation (20).

When $\mathscr{s} = 2$,

$$\text{FFEWA}(\mathscr{R}_1, \mathscr{R}_2) = w_1 \cdot_\varepsilon \mathscr{R}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \mathscr{R}_2. \quad (21)$$

By Theorem 1, both $w_1 \cdot_\varepsilon \mathscr{R}_1$ and $w_2 \cdot_\varepsilon \mathscr{R}_2$ are FFNs and value of $w_1 \cdot_\varepsilon \mathscr{R}_1 \oplus_\varepsilon w_2 \cdot_\varepsilon \mathscr{R}_2$ is an FFN. By using (vi) in Definition 6,

$$\begin{aligned} w_1 \cdot_\varepsilon \mathscr{R}_1 &= \left\langle \sqrt[3]{\frac{\left(1 + \mu_1^3\right)^{w_1} - \left(1 - \mu_1^3\right)^{w_1}}{\left(1 + \mu_1^3\right)^{w_1} + \left(1 - \mu_1^3\right)^{w_1}}}, \frac{\sqrt[3]{2} \nu_1^{w_1}}{\sqrt[3]{\left(2 - \nu_1^3\right)^{w_1} + \left(\nu_1^3\right)^{w_1}}} \right\rangle, \\ w_2 \cdot_\varepsilon \mathscr{R}_2 &= \left\langle \sqrt[3]{\frac{\left(1 + \mu_2^3\right)^{w_2} - \left(1 - \mu_2^3\right)^{w_2}}{\left(1 + \mu_2^3\right)^{w_2} + \left(1 - \mu_2^3\right)^{w_2}}}, \frac{\sqrt[3]{2} \nu_2^{w_2}}{\sqrt[3]{\left(2 - \nu_2^3\right)^{w_2} + \left(\nu_2^3\right)^{w_2}}} \right\rangle. \end{aligned} \quad (22)$$

Then,

$$\mathrm{FFEWA}(\mathscr{R}_1,\mathscr{R}_2) = w_1 \cdot_{\varepsilon} \mathscr{R}_1 \oplus_{\varepsilon} w_2 \cdot_{\varepsilon} \mathscr{R}_2$$

$$= \left\langle \sqrt[3]{\frac{\left(\left(\left(1+\mu_1^3\right)^{w_1} - \left(1-\mu_1^3\right)^{w_1}\right)/\left(\left(1+\mu_1^3\right)^{w_1} + \left(1-\mu_1^3\right)^{w_1}\right)\right) + \left(\left(\left(1+\mu_2^3\right)^{w_2} - \left(1-\mu_2^3\right)^{w_2}\right)/\left(\left(1+\mu_2^3\right)^{w_2} + \left(1-\mu_2^3\right)^{w_2}\right)\right)}{1 + \left(\left(\left(1+\mu_1^3\right)^{w_1} - \left(1-\mu_1^3\right)^{w_1}\right)/\left(\left(1+\mu_1^3\right)^{w_1} + \left(1-\mu_1^3\right)^{w_1}\right)\right) \cdot_{\varepsilon} \left(\left(\left(1+\mu_2^3\right)^{w_2} - \left(1-\mu_2^3\right)^{w_2}\right)/\left(\left(1+\mu_2^3\right)^{w_2} + \left(1-\mu_2^3\right)^{w_2}\right)\right)}}, \right.$$

$$\left. \frac{\left(\sqrt[3]{2}\nu_1^{w_1}/\sqrt[3]{\left(2-\nu_1^3\right)^{w_1} + \left(\nu_1^3\right)^{w_1}}\right) \cdot_{\varepsilon} \left(\sqrt[3]{2}\nu_2^{w_2}/\sqrt[3]{\left(2-\nu_2^3\right)^{w_2} + \left(\nu_2^3\right)^{w_2}}\right)}{\sqrt[3]{1 + \left(1 - \left(2\nu_1^{3w_1}/\left(\left(2-\nu_1^3\right)^{w_1} + \left(\nu_1^3\right)^{w_1}\right)\right)\right) \cdot_{\varepsilon} \left(1 - \left(2\nu_2^{3w_2}/\left(\left(2-\nu_2^3\right)^{w_2} + \left(\nu_2^3\right)^{w_2}\right)\right)\right)}} \right\rangle$$

$$= \left\langle \sqrt[3]{\frac{\left(1+\mu_1^3\right)^{w_1} \cdot_{\varepsilon} \left(1+\mu_2^3\right)^{w_2} - \left(1-\mu_1^3\right)^{w_1} \cdot_{\varepsilon} \left(1-\mu_2^3\right)^{w_2}}{\left(1+\mu_1^3\right)^{w_1} \cdot_{\varepsilon} \left(1+\mu_2^3\right)^{w_2} + \left(1-\mu_1^3\right)^{w_1} \cdot_{\varepsilon} \left(1-\mu_2^3\right)^{w_2}}}, \frac{\sqrt[3]{2}\nu_1^{w_1} \cdot_{\varepsilon} \nu_2^{w_2}}{\sqrt[3]{\left(2-\nu_1^3\right)^{w_1} \cdot \varepsilon\left(2-\nu_2^3\right)^{w_2} + \left(\nu_1^3\right)^{w_1} \cdot_{\varepsilon} \left(\nu_2^3\right)^{w_2}}} \right\rangle. \tag{23}$$

Thus, equation (20) is true when $s = 2$.

Suppose result is true for $s = k$:

$$\mathrm{FFEWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_k) = \left\langle \sqrt[3]{\frac{\prod_{j=1}^{k}\left(1+\mu_j^3\right)^{w_j} - \prod_{j=1}^{k}\left(1-\mu_j^3\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\mu_j^3\right)^{w_j} + \prod_{j=1}^{k}\left(1-\mu_j^3\right)^{w_j}}}, \frac{\sqrt[3]{2}\prod_{j=1}^{k}\nu_j^{w_j}}{\sqrt[3]{\prod_{j=1}^{k}\left(2-\nu_j^3\right)^{w_j} + \prod_{j=1}^{k}\left(\nu_j^3\right)^{w_j}}} \right\rangle. \tag{24}$$

Now, for $s = k+1$,

$$\begin{aligned}\mathrm{FFEWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{k+1}) &= \left\langle \sqrt[3]{\frac{\prod_{j=1}^{k}\left(1+\mu_j^3\right)^{w_j} - \prod_{j=1}^{k}\left(1-\mu_j^3\right)^{w_j}}{\prod_{j=1}^{k}\left(1+\mu_j^3\right)^{w_j} + \prod_{j=1}^{k}\left(1-\mu_j^3\right)^{w_j}}}, \frac{\sqrt[3]{2}\prod_{j=1}^{k}\nu_j^{w_j}}{\sqrt[3]{\prod_{j=1}^{k}\left(2-\nu_j^3\right)^{w_j} + \prod_{j=1}^{k}\left(\nu_j^3\right)^{w_j}}} \right\rangle \\ &\oplus_{\varepsilon} \left\langle \sqrt[3]{\frac{\left(1+\mu_{k+1}^3\right)^{w_{k+1}} - \left(1-\mu_{k+1}^3\right)^{w_{k+1}}}{\left(1+\mu_{k+1}^3\right)^{w_{k+1}} + \left(1-\mu_{k+1}^3\right)^{w_{k+1}}}}, \frac{\sqrt[3]{2}\nu_{k+1}^{w_{k+1}}}{\sqrt[3]{\left(2-\nu_{k+1}^3\right)^{w_{k+1}} + \left(\nu_{k+1}^3\right)^{w_{k+1}}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\prod_{j=1}^{k+1}\left(1+\mu_j^3\right)^{w_j} - \prod_{j=1}^{k+1}\left(1-\mu_j^3\right)^{w_j}}{\prod_{j=1}^{k+1}\left(1+\mu_j^3\right)^{w_j} + \prod_{j=1}^{k+1}\left(1-\mu_j^3\right)^{w_j}}}, \frac{\sqrt[3]{2}\prod_{j=1}^{k+1}\nu_j^{w_j}}{\sqrt[3]{\prod_{j=1}^{k+1}\left(2-\nu_j^3\right)^{w_j} + \prod_{j=1}^{k+1}\left(\nu_j^3\right)^{w_j}}} \right\rangle.\end{aligned} \tag{25}$$

Thus, the result is true for $s = k+1$. Hence, equation (20) holds, $\forall s$. □

Lemma 1. *Let* $\mathscr{R}_j = \langle \mu_j, \nu_j \rangle, w_j > 0,$ *and* $\sum_{j=1}^{s} w_j = 1$; *then,*

$$\prod_{j=1}^{s} R_j^{w_j} \leq \sum_{j=1}^{s} w_j R_j, \tag{26}$$

where equality holds iff $R_1 = R_2 = \cdots = R_s$.

Theorem 6. *If* $\mathscr{R}_j = \langle \mu_j, \nu_j \rangle$ *are FFNs, then FFEWA* $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_s)$ *is also an FFN.*

Proof. Since $\mathscr{R}_j = \langle \mu_j, \nu_j \rangle$ are FFNs, so $0 \leq \mu_j, \nu_j \leq 1$ and $0 \leq \mu_j^3 + \nu_j^3 \leq 1$. Therefore,

$$\frac{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}=1-\frac{2\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}\leq 1-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}\leq 1. \tag{27}$$

Also, $(1+\mu_{\jmath}^{3})\geq(1-\mu_{\jmath}^{3})\Rightarrow\prod_{\jmath=1}^{\beth}(1+\mu_{\jmath}^{3})-\prod_{\jmath=1}^{\beth}(1-\mu_{\jmath}^{3})\geq 0$. Therefore,

$$\frac{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}\geq 0. \tag{28}$$

Thus, $0\leq\mu_{\mathrm{FFEWA}}\leq 1$.

Moreover,

$$\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}\leq\frac{2\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}\leq\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}\leq 1. \tag{29}$$

Also,

$$\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}\geq 0\Longleftrightarrow\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}\geq 0. \tag{30}$$

Thus, $0\leq\nu_{\mathrm{FFEWA}}\leq 1$. Moreover,

$$\begin{aligned}\mu_{\mathrm{FFEWA}}^{3}+\nu_{\mathrm{FFEWA}}^{3}&=\frac{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}+\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}\\&\leq 1-\frac{2\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}+\frac{2\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}\\&=1.\end{aligned} \tag{31}$$

Hence, FFEWA $\in[0,1]$. Therefore, FFEWA $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\beth})\in$ FFN. □

Corollary 1. *The FFEWA and FFWA operators have the relationship:*

$$\mathrm{FFEWA}\left(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\beth}\right)\leq\mathrm{FFWA}\left(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\beth}\right). \tag{32}$$

Proof. Let FFEWA $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\beth})=(\mu_{\mathscr{R}}^{\beta},\nu_{\mathscr{R}}^{\beta})=\mathscr{R}^{\beta}$ and FFWA $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\beth})=(\mu_{\mathscr{R}},\nu_{\mathscr{R}})=\mathscr{R}$. Since $\prod_{\jmath=1}^{\beth}(1+\mu_{\jmath}^{3})^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}(1-\mu_{\jmath}^{3})^{w_{\jmath}}\leq\sum_{\jmath=1}^{\beth}(1+\mu_{\jmath}^{3})^{w_{\jmath}}+\sum_{\jmath=1}^{\beth}(1-\mu_{\jmath}^{3})^{w_{\jmath}}=2$, then from equation (27), we obtain

$$\sqrt[3]{\frac{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}}\leq\sqrt[3]{1-\prod_{\jmath=1}^{\beth}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}\Longleftrightarrow\mu_{\mathscr{R}}^{\beta}\leq\mu_{\mathscr{R}}, \tag{33}$$

equality holds iff $\mu_1=\mu_2=\cdots=\mu_{\beth}$.

Also,

$$\begin{aligned}\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}&\geq\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\sum_{\jmath=1}^{\beth}w_{\jmath}\left(2-\nu_{\jmath}^{3}\right)+\sum_{\jmath=1}^{\beth}w_{\jmath}\nu_{\jmath}^{3}}\\&\geq\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}\Rightarrow\sqrt[3]{\frac{2\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\beth}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\beth}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}}\geq\prod_{\jmath=1}^{\beth}\nu_{\jmath}^{w_{\jmath}}\Rightarrow\nu_{\mathscr{R}}^{\beta}\leq\nu_{\mathscr{R}},\end{aligned} \tag{34}$$

equality holds iff $\nu_1 = \nu_2 = \cdots = \nu_s$.

Thus,

$$\mathcal{S}\left(\mathcal{R}^{\beta}\right) = \left(\mu^{\beta}_{\mathcal{R}}\right)^3 - \left(\nu^{\beta}_{\mathcal{R}}\right)^3 \leq \left(\mu_{\mathcal{R}}\right)^3 - \left(\nu_{\mathcal{R}}\right)^3 = \mathcal{S}(\mathcal{R}). \quad (35)$$

If $\mathcal{S}(\mathcal{R}^{\beta}) < \mathcal{S}(\mathcal{R})$, then

$$\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) < \text{FFWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right). \quad (36)$$

If $\mathcal{S}(\mathcal{R}^{\beta}) = \mathcal{S}(\mathcal{R})$, that is, $(\mu^{\beta}_{\mathcal{R}})^3 - (\nu^{\beta}_{\mathcal{R}})^3 = (\mu_{\mathcal{R}})^3 - (\nu_{\mathcal{R}})^3$, then by condition $\mu^{\beta}_{\mathcal{R}} \leq \mu_{\mathcal{R}}$ and $\nu^{\beta}_{\mathcal{R}} \geq \nu_{\mathcal{R}}$; thus, the accuracy function $\mathcal{A}(\mathcal{R}^{\beta}) = (\mu^{\beta}_{\mathcal{R}})^3 - (\nu^{\beta}_{\mathcal{R}})^3 = (\mu_{\mathcal{R}})^3 - (\nu_{\mathcal{R}})^3 = \mathcal{A}(\mathcal{R})$. Thus,

$$\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) = \text{FFWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right). \quad (37)$$

Hence,

$$\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) \leq \text{FFWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right), \quad (38)$$

equality holds iff $\mathcal{R}_1 = \mathcal{R}_2 = \cdots = \mathcal{R}_s$. □

Example 1. Let $\mathcal{R}_1 = (0.8, 0.5)$, $\mathcal{R}_2 = (0.9, 0.4)$, $\mathcal{R}_3 = (0.6, 0.7)$, and $\mathcal{R}_4 = (0.8, 0.7)$ be four FFNs and $w = (0.4, 0.2, 0.2, 0.2)^T$; then,

$$\begin{aligned}\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\right) &= \left\langle \sqrt[3]{\frac{\prod_{j=1}^{4}\left(1+\mu_j^3\right)^{w_j} - \prod_{j=1}^{4}\left(1-\mu_j^3\right)^{w_j}}{\prod_{j=1}^{4}\left(1+\mu_j^3\right)^{w_j} + \prod_{j=1}^{4}\left(1-\mu_j^3\right)^{w_j}}}, \frac{\sqrt[3]{2}\prod_{j=1}^{4}\nu_j^{w_j}}{\sqrt[3]{\prod_{j=1}^{4}\left(2-\nu_j^3\right)^{w_j} + \prod_{j=1}^{4}\left(\nu_j^3\right)^{w_j}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{1.49 - 0.48}{1.49 + 0.48}}, \frac{\sqrt[3]{2} \times 0.55}{\sqrt[3]{1.80 + 0.16}} \right\rangle = \langle 0.80, 0.55 \rangle.\end{aligned} \quad (39)$$

Now,

$$\begin{aligned}&\text{FFWA}\left(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\right) \\ &= \left\langle \sqrt[3]{1 - \prod_{j=1}^{4}\left(1-\mu_j^3\right)^{w_j}}, \prod_{j=1}^{4}\left(\nu_j\right)^{w_j} \right\rangle = \langle 0.80, 0.55 \rangle, \Rightarrow \text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\right) \leq \text{FFWA}\left(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4\right).\end{aligned} \quad (40)$$

Proposition 1. *Let $\mathcal{R}_j = \langle \mu_j, \nu_j \rangle$ be FFNs and w_j be the WV of $\mathcal{R}_j$, such that $w_j \in [0, 1]$ and $\sum_{j=1}^{s} w_j = 1$.*

(i) Idempotency: if $\mathcal{R}_j = \mathcal{R}_o = \langle \mu_o, \nu_o \rangle$ for all j, then

$$\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) = \mathcal{R}_o. \quad (41)$$

(ii) Boundedness: let $\mathcal{R}^- = (\min_j(\mu_j), \max_j(\nu_j))$ and $\mathcal{R}^+ = (\max_j(\mu_j), \min_j(\nu_j))$; then,

$$\mathcal{R}^- \leq \text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) \leq \mathcal{R}^+. \quad (42)$$

(iii) Monotonicity: when $\mathcal{R}_j \leq \mathcal{P}_j$, $\forall j$, then

$$\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) \leq \text{FFEWA}\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_s\right). \quad (43)$$

Proof. (i) As $\mathcal{R}_j = \langle \mu_o, \nu_o \rangle$ are FFNs, $\forall j$, then

$$\begin{aligned}\text{FFEWA}\left(\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_s\right) &= \left\langle \sqrt[3]{\frac{\prod_{j=1}^{s}\left(1+\mu_o^3\right)^{w_j} - \prod_{j=1}^{s}\left(1-\mu_o^3\right)^{w_j}}{\prod_{j=1}^{s}\left(1+\mu_o^3\right)^{w_j} + \prod_{j=1}^{s}\left(1-\mu_o^3\right)^{w_j}}}, \frac{\sqrt[3]{2}\prod_{j=1}^{s}\nu_o^{w_j}}{\sqrt[3]{\prod_{j=1}^{s}\left(2-\nu_o^3\right)^{w_j} + \prod_{j=1}^{s}\left(\nu_o^3\right)^{w_j}}} \right\rangle \\ &= \left\langle \sqrt[3]{\frac{\left(1+\mu_o^3\right)^{\sum_{j=1}^{s} w_j} - \left(1-\mu_o^3\right)^{\sum_{j=1}^{s} w_j}}{\left(1+\mu_o^3\right)^{\sum_{j=1}^{s} w_j} + \left(1-\mu_o^3\right)^{\sum_{j=1}^{s} w_j}}}, \frac{\sqrt[3]{2}\nu_o^{\sum_{j=1}^{s} w_j}}{\sqrt[3]{\left(2-\nu_o^3\right)^{\sum_{j=1}^{s} w_j} + \left(\nu_o^3\right)^{\sum_{j=1}^{s} w_j}}} \right\rangle = \langle \mu_o, \nu_o \rangle.\end{aligned} \quad (44)$$

(ii) Consider $f(x) = ((1-x)/(1+x))$, $x \in [0,1]$, then $f\prime(x) = -(2/(1+x)^2) < 0$, so $f(x)$ is a decreasing function (DF). As $\mu^3_{\jmath,\min} \le \mu^3_{\jmath} \le \mu^3_{\jmath,\max}$, $\forall \jmath = 1,2,\ldots,\mathfrak{s}$, then $f(\mu^3_{\jmath,\max}) \le f(\mu^3_{\jmath}) \le f(\mu^3_{\jmath,\min})$, $\forall \jmath$, that is, $((1-\mu^3_{\jmath,\max})/(1+\mu^3_{\jmath,\max})) \le ((1-\mu^3_{\jmath})/(1+\mu^3_{\jmath})) \le ((1-\mu^3_{\jmath,\min})/(1+\mu^3_{\jmath,\min}))$, for all $\jmath$. Let $w_{\jmath} \in [0,1]$ and $\sum_{\jmath=1}^{\mathfrak{s}} w_{\jmath} = 1$, and we have

$$\left(\frac{1-\mu^3_{\jmath,\max}}{1+\mu^3_{\jmath,\max}}\right)^{w_{\jmath}} \le \left(\frac{1-\mu^3_{\jmath}}{1+\mu^3_{\jmath}}\right)^{w_{\jmath}} \le \left(\frac{1-\mu^3_{\jmath,\min}}{1+\mu^3_{\jmath,\min}}\right)^{w_{\jmath}},$$

$$\prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath,\max}}{1+\mu^3_{\jmath,\max}}\right)^{w_{\jmath}} \le \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath}}{1+\mu^3_{\jmath}}\right)^{w_{\jmath}} \le \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath,\min}}{1+\mu^3_{\jmath,\min}}\right)^{w_{\jmath}}$$

$$\Longleftrightarrow \left(\frac{1-\mu^3_{\jmath,\max}}{1+\mu^3_{\jmath,\max}}\right)^{\sum_{\jmath=1}^{\mathfrak{s}} w_{\jmath}} \le \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath}}{1+\mu^3_{\jmath}}\right)^{w_{\jmath}} \le \left(\frac{1-\mu^3_{\jmath,\min}}{1+\mu^3_{\jmath,\min}}\right)^{\sum_{\jmath=1}^{\mathfrak{s}} w_{\jmath}}$$

$$\Longleftrightarrow \left(\frac{1-\mu^3_{\jmath,\max}}{1+\mu^3_{\jmath,\max}}\right) \le \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath}}{1+\mu^3_{\jmath}}\right)^{w_{\jmath}} \le \left(\frac{1-\mu^3_{\jmath,\min}}{1+\mu^3_{\jmath,\min}}\right)$$

$$\Longleftrightarrow \left(\frac{2}{1+\mu^3_{\jmath,\max}}\right) \le 1 + \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{1-\mu^3_{\jmath}}{1+\mu^3_{\jmath}}\right)^{w_{\jmath}} \le \left(\frac{2}{1+\mu^3_{\jmath,\min}}\right) \tag{45}$$

$$\Longleftrightarrow \left(\frac{1+\mu^3_{\jmath,\min}}{2}\right) \le \frac{1}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(1-\mu^3_{\jmath}\right)/\left(1+\mu^3_{\jmath}\right)\right)^{w_{\jmath}}} \le \left(\frac{1+\mu^3_{\jmath,\max}}{2}\right)$$

$$\Longleftrightarrow \left(1+\mu^3_{\jmath,\min}\right) \le \frac{2}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(1-\mu^3_{\jmath}\right)/\left(1+\mu^3_{\jmath}\right)\right)^{w_{\jmath}}} \le \left(1+\mu^3_{\jmath,\max}\right)$$

$$\Longleftrightarrow \mu^3_{\jmath,\min} \le \frac{2}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(1-\mu^3_{\jmath}\right)/\left(1+\mu^3_{\jmath}\right)\right)^{w_{\jmath}}} - 1 \le \mu^3_{\jmath,\max}$$

$$\Longleftrightarrow \mu^3_{\jmath,\min} \le \frac{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu^3_{\jmath}\right)^{w_{\jmath}} - \prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu^3_{\jmath}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu^3_{\jmath}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu^3_{\jmath}\right)^{w_{\jmath}}} \le \mu^3_{\jmath,\max}.$$

Thus,

$$\mu_{\jmath,\min} \le \sqrt[3]{\frac{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu^3_{\jmath}\right)^{w_{\jmath}} - \prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu^3_{\jmath}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu^3_{\jmath}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu^3_{\jmath}\right)^{w_{\jmath}}}} \le \mu_{\jmath,\max}. \tag{46}$$

Consider $g(y) = ((2-y)/y)$, $y \in (0,1]$, then $g'(y) = -(2/y^2)$, i.e., $g(y)$ is a DF on $(0,1]$. Since $\nu^3_{\jmath,\min} \le \nu^3_{\jmath} \le \nu^3_{\jmath,\max}$, $\forall \jmath$, then $g(\nu^3_{\jmath,\max}) \le g(\nu^3_{\jmath}) \le g(\nu^3_{\jmath,\min})$, $\forall \jmath$, that is, $((2-\nu^3_{\jmath,\max})/\nu^3_{\jmath,\max}) \le ((2-\nu^3_{\jmath})/\nu^3_{\jmath}) \le ((2-\nu^3_{\jmath,\min})/\nu^3_{\jmath,\min})$. Then,

$$\begin{aligned}
&\left(\frac{2-\nu_{\jmath,\max}^{3}}{\nu_{\jmath,\max}^{3}}\right)^{w_{\jmath}} \leq \left(\frac{2-\nu_{\jmath}^{3}}{\nu_{\jmath}^{3}}\right)^{w_{\jmath}} \leq \left(\frac{2-\nu_{\jmath,\min}^{3}}{\nu_{\jmath,\min}^{3}}\right)^{w_{\jmath}},\\
&\prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath,\max}^{3}}{\nu_{\jmath,\max}^{3}}\right)^{w_{\jmath}} \leq \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath}^{3}}{\nu_{\jmath}^{3}}\right)^{w_{\jmath}} \leq \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath,\min}^{3}}{\nu_{\jmath,\min}^{3}}\right)^{w_{\jmath}}\\
&\Rightarrow\left(\frac{2-\nu_{\jmath,\max}^{3}}{\nu_{\jmath,\max}^{3}}\right)^{\sum_{\jmath=1}^{\mathfrak{s}} w_{\jmath}} \leq \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath}^{3}}{\nu_{\jmath}^{3}}\right)^{w_{\jmath}} \leq \left(\frac{2-\nu_{\jmath,\min}^{3}}{\nu_{\jmath,\min}^{3}}\right)^{\sum_{\jmath=1}^{\mathfrak{s}} w_{\jmath}}\\
&\Rightarrow\left(\frac{2-\nu_{\jmath,\max}^{3}}{\nu_{\jmath,\max}^{3}}\right) \leq \prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath}^{3}}{\nu_{\jmath}^{3}}\right)^{w_{\jmath}} \leq \left(\frac{2-\nu_{\jmath,\min}^{3}}{\nu_{\jmath,\min}^{3}}\right)\\
&\Rightarrow\left(\frac{2}{\nu_{\jmath,\max}^{3}}\right) \leq 1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\frac{2-\nu_{\jmath}^{3}}{\nu_{\jmath}^{3}}\right)^{w_{\jmath}} \leq \left(\frac{2}{\nu_{\jmath,\min}^{3}}\right)\\
&\Rightarrow\left(\frac{\nu_{\jmath,\min}^{3}}{2}\right) \leq \frac{1}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(2-\nu_{\jmath}^{3}\right)/\nu_{\jmath}^{3}\right)^{w_{\jmath}}} \leq \left(\frac{\nu_{\jmath,\max}^{3}}{2}\right)\\
&\Rightarrow\left(\nu_{\jmath,\min}^{3}\right) \leq \frac{2}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(2-\nu_{\jmath}^{3}\right)/\nu_{\jmath}^{3}\right)^{w_{\jmath}}} \leq \left(\nu_{\jmath,\max}^{3}\right)\\
&\Rightarrow\nu_{\jmath,\min}^{3} \leq \frac{2}{1+\prod_{\jmath=1}^{\mathfrak{s}}\left(\left(2-\nu_{\jmath}^{3}\right)/\nu_{\jmath}^{3}\right)^{w_{\jmath}}} \leq \nu_{\jmath,\max}^{3}\\
&\Rightarrow\nu_{\jmath,\min}^{3} \leq \frac{2\prod_{\jmath=1}^{\mathfrak{s}}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\mathfrak{s}}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\mathfrak{s}}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}} \leq \nu_{\jmath,\max}^{3}\\
&\Rightarrow\nu_{\jmath,\min} \leq \frac{\sqrt[3]{2}\prod_{\jmath=1}^{\mathfrak{s}}\nu_{\jmath}^{w_{\jmath}}}{\sqrt[3]{\prod_{\jmath=1}^{\mathfrak{s}}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\mathfrak{s}}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}} \leq \nu_{\jmath,\max}.
\end{aligned} \tag{47}$$

Let FFEWA $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\mathfrak{s}})=\mathscr{R}=\langle\mu_{\mathscr{R}},\nu_{\mathscr{R}}\rangle$; then, from equations (46) and (47),

$$\begin{aligned}
&\mu_{\min}\leq\mu_{\mathscr{R}}\leq\mu_{\max},\\
&\nu_{\min}\leq\nu_{\mathscr{R}}\leq\nu_{\max},
\end{aligned} \tag{48}$$

where $\mu_{\min}=\min_{\jmath}\{\mu_{\jmath}\}$, $\mu_{\max}=\max_{\jmath}\{\mu_{\jmath}\}$, $\nu_{\min}=\min_{\jmath}\{\nu_{\jmath}\}$, and $\nu_{\max}=\max_{\jmath}\{\nu_{\jmath}\}$. So, $\mathcal{S}(\mathscr{R})=\mu_{\mathscr{R}}^3-\nu_{\mathscr{R}}^3\leq\mu_{\max}^3-\nu_{\min}^3=\mathcal{S}(\mathscr{R}^+)$ and $\mathcal{S}(\mathscr{R})=\mu_{\mathscr{R}}^3-\nu_{\mathscr{R}}^3\geq\mu_{\min}^3-\nu_{\max}^3=\mathcal{S}(\mathscr{R}^-)$. As $S(\mathscr{R})<S(\mathscr{R}^+)$ and $S(\mathscr{R})>S(\mathscr{R}^-)$, so

$$\mathscr{R}^-\leq\mathrm{FFEWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\mathfrak{s}})\leq\mathscr{R}^+. \tag{49}$$

(iii) It is similar to (ii), so we omit it. □

5. Fermatean Fuzzy Einstein Ordered Weighted Averaging Operators

Definition 8. Let $\mathscr{R}_{\jmath}=\langle\mu_{\jmath},\nu_{\jmath}\rangle$ be a family of FFNs and $w_{\jmath}$ be the WV of $\mathscr{R}_{\jmath}$ with $w_{\jmath}>0$ and $\sum_{\jmath=1}^{\mathfrak{s}}w_{\jmath}=1$; then, FFEOWA operator is a mapping $\mathcal{Q}^{\mathfrak{s}}\longrightarrow\mathcal{Q}$ such that

$$\begin{aligned}
&\mathrm{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\mathfrak{s}})\\
&\quad=w_1\cdot_{\varepsilon}\mathscr{R}_{\varrho(1)}\oplus_{\varepsilon}w_2\cdot_{\varepsilon}\mathscr{R}_{\varrho(2)}\oplus_{\varepsilon}\cdots\oplus_{\varepsilon}w_{\mathfrak{s}}\cdot_{\varepsilon}\mathscr{R}_{\varrho(\mathfrak{s})},
\end{aligned} \tag{50}$$

where $(\varrho(1),\varrho(2),\ldots,\varrho(\mathfrak{s}))$ is the permutation of $(\jmath=1,2,\ldots,\mathfrak{s})$ such that $\mathscr{R}_{\varrho(\jmath-1)}\geq\mathscr{R}_{\varrho(\jmath)}$, $\forall\jmath=1,2,\ldots,\mathfrak{s}$.

Theorem 7. *Let $\mathscr{R}_{\jmath}=(\mu_{\jmath},\nu_{\jmath})$ be FFNs; then, the aggregated value by using FFEOWA is an FFN and*

$$\begin{aligned}
&\mathrm{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_{\mathfrak{s}})\\
&=\left\langle\sqrt[3]{\frac{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}-\prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{\mathfrak{s}}\left(1+\mu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\mathfrak{s}}\left(1-\mu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}}},\right.\\
&\qquad\left.\frac{\sqrt[3]{2}\prod_{\jmath=1}^{\mathfrak{s}}\nu_{\varrho(\jmath)}^{w_{\jmath}}}{\sqrt[3]{\prod_{\jmath=1}^{\mathfrak{s}}\left(2-\nu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}+\prod_{\jmath=1}^{\mathfrak{s}}\left(\nu_{\varrho(\jmath)}^3\right)^{w_{\jmath}}}}\right\rangle.
\end{aligned} \tag{51}$$

Proof. It is similar to Theorem 4.

We give some properties without their proofs. □

Corollary 2. *The FFEOWA and FFOWA operators have the relation:*

$$\text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)\leq \text{FFOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath). \tag{52}$$

Example 2. Let $\mathscr{R}_1=(0.6,0.7)$, $\mathscr{R}_2=(0.8,0.7)$, $\mathscr{R}_3=(0.6,0.9)$, and $\mathscr{R}_4=(0.9,0.4)$ be four FFNs and $w=(0.3,0.3,0.2,0.2)^T$ as

$$\begin{aligned}\mathcal{S}(\mathscr{R}_1)&=(0.6)^3-(0.7)^3=-0.13,\\ \mathcal{S}(\mathscr{R}_2)&=(0.8)^3-(0.7)^3=0.17,\\ \mathcal{S}(\mathscr{R}_3)&=(0.6)^3-(0.9)^3=-0.51,\\ \mathcal{S}(\mathscr{R}_4)&=(0.9)^3-(0.4)^3=0.67.\end{aligned} \tag{53}$$

Since $\mathcal{S}(\mathscr{R}_4)>\mathcal{S}(\mathscr{R}_2)>\mathcal{S}(\mathscr{R}_1)>\mathcal{S}(\mathscr{R}_3)$, therefore

$$\begin{aligned}\mathscr{R}_{\varrho(1)}&=\mathscr{R}_4=(0.9,0.4),\\ \mathscr{R}_{\varrho(2)}&=\mathscr{R}_2=(0.8,0.7),\\ \mathscr{R}_{\varrho(3)}&=\mathscr{R}_1=(0.6,0.7),\\ \mathscr{R}_{\varrho(4)}&=\mathscr{R}_3=(0.6,0.9).\end{aligned} \tag{54}$$

Thus, by applying the FFEOWA operator, we obtain

$$\begin{aligned}\text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3,\mathscr{R}_4)&=\left\langle \sqrt[3]{\frac{\prod_{\jmath=1}^4\left(1+\mu^3_{\varrho(\jmath)}\right)^{w_\jmath}-\prod_{\jmath=1}^4\left(1-\mu^3_{\varrho(\jmath)}\right)^{w_\jmath}}{\prod_{\jmath=1}^4\left(1+\mu^3_{\varrho(\jmath)}\right)^{w_\jmath}+\prod_{\jmath=1}^4\left(1-\mu^3_{\varrho(\jmath)}\right)^{w_\jmath}}}, \frac{\sqrt[3]{2}\prod_{\jmath=1}^4 \nu^{w_\jmath}_{\varrho(\jmath)}}{\sqrt[3]{\prod_{\jmath=1}^4\left(2-\nu^3_{\varrho(\jmath)}\right)^{w_\jmath}+\prod_{\jmath=1}^4\left(\nu^3_{\varrho(\jmath)}\right)^{w_\jmath}}}\right\rangle\\ &=\left\langle\sqrt[3]{\frac{1.44-0.49}{1.44+0.49}},\frac{\sqrt[3]{2}\times 0.66}{\sqrt[3]{1.62+0.66}}\right\rangle=\langle 0.79,0.55\rangle.\end{aligned} \tag{55}$$

Now,

$$\begin{aligned}&\text{FFOWA}(\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3,\mathscr{R}_4)\\ &=\left\langle\sqrt[3]{1-\prod_{\jmath=1}^4\left(1-\mu^3_{\varrho(\jmath)}\right)^{w_\jmath}},\prod_{\jmath=1}^4\left(\nu_{\varrho(\jmath)}\right)^{w_\jmath}\right\rangle=\langle 0.80,0.66\rangle,\\ &\Rightarrow \text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3,\mathscr{R}_4)<\text{FFOWA}(\mathscr{R}_1,\mathscr{R}_2,\mathscr{R}_3,\mathscr{R}_4).\end{aligned} \tag{56}$$

Proposition 1. *Let $\mathscr{R}_\jmath=\langle\mu_\jmath,\nu_\jmath\rangle$ be FFNs and $w_\jmath$ be the WV of $\mathscr{R}_\jmath$, such that $w_\jmath\in[0,1]$ and $\sum_{\jmath=1}^{\jmath} w_\jmath=1$.*

(i) *Idempotency: if $\mathscr{R}_\jmath=\mathscr{R}_o=\langle\mu_o,\nu_o\rangle$, $\forall\jmath$, then*

$$\text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)=\mathscr{R}_o. \tag{57}$$

(ii) *Boundedness: let $\mathscr{R}^-=(\min_\jmath(\mu_\jmath),\max_\jmath(\nu_\jmath))$ and $\mathscr{R}^+=(\max_\jmath(\mu_\jmath),\min_\jmath(\nu_\jmath))$; then,*

$$\mathscr{R}^-\leq\text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)\leq\mathscr{R}^+. \tag{58}$$

(iii) *Monotonicity: when $\mathscr{R}_\jmath\leq\mathscr{P}_\jmath$, $\forall\jmath$, then*

$$\text{FFEOWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)\leq\text{FFEOWA}(\mathscr{P}_1,\mathscr{P}_2,\ldots,\mathscr{P}_\jmath). \tag{59}$$

6. Generalized Fermatean Fuzzy Einstein Weighted Averaging Operators

Definition 9. Let $\mathscr{R}_\jmath=\langle\mu_\jmath,\nu_\jmath\rangle$ be a collection of FFNs and $w_\jmath$ be the WV of $\mathscr{R}_\jmath$ with $w_\jmath>0$ and $\sum_{\jmath=1}^{\jmath}w_\jmath=1$; then, GFFEWA operator is a mapping $\mathcal{Q}^\jmath\longrightarrow\mathcal{Q}$ such that

$$\text{GFFEWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)=\left(\oplus_{\jmath=1}^{\jmath}\left(w_\jmath\cdot_\varepsilon\mathscr{R}^\lambda_\jmath\right)\right)^{1/\lambda}, \tag{60}$$

where $\lambda>0$.

Particularly,

(i) If $\lambda=1$, then GFFEWA becomes FFEWA

(ii) If $w=((1/\jmath),(1/\jmath),\ldots,(1/\jmath))^T$, then GFFEWA $(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_\jmath)=((1/\jmath)\cdot{}_\varepsilon\oplus_{\jmath=1}^{\jmath}\mathscr{R}^\lambda_\jmath)^{1/\lambda}$

Theorem 8. *Let $\mathscr{R}_\jmath=\langle\mu_\jmath,\nu_\jmath\rangle$ be FFNs and $w_\jmath$ be the WV of $\mathscr{R}_\jmath$ with $w_\jmath>0$ and $\sum_{\jmath=1}^{\jmath}w_\jmath=1$; then, the aggregated value by applying the GFFEWA operator is an FFN and*

$$\operatorname{GFFEWA}(\mathscr{R}_1,\mathscr{R}_2,\ldots,\mathscr{R}_s)$$
$$=\left\langle\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda\right\}^{w_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}},\right.$$
$$\left.\cdot\sqrt[3]{\frac{\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}}\right\rangle. \tag{61}$$

Proof. Since

$$\mathscr{R}_j^\lambda=\left\langle\frac{\sqrt[3]{2}\mu_j^\lambda}{\sqrt[3]{(2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda}},\sqrt[3]{\frac{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda}{(1+\nu_j^3)^\lambda+(1-\nu_j^3)^\lambda}}\right\rangle,$$
$$\Rightarrow\oplus_{j=1}^{s}w_j\cdot{}_{\varepsilon}\mathscr{R}_j^\lambda=\left\langle\sqrt[3]{\frac{\prod_{j=1}^{s}\left(1+\left(2\mu_j^{3\lambda}/\left((2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda\right)\right)\right)^{w_j}-\prod_{j=1}^{s}\left(1-\left(2\mu_j^{3\lambda}/\left((2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda\right)\right)\right)^{w_j}}{\prod_{j=1}^{s}\left(1+\left(2\mu_j^{3\lambda}/\left((2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda\right)\right)\right)^{w_j}+\prod_{j=1}^{s}\left(1-\left(2\mu_j^{3\lambda}/\left((2-\mu_j^3)^\lambda+(\mu_j^3)^\lambda\right)\right)\right)^{w_j}}},\right.$$
$$\left.\frac{\sqrt[3]{2}\prod_{j=1}^{s}\left(\sqrt[3]{\left((1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right)/\left((1+\nu_j^3)^\lambda+(1-\nu_j^3)^\lambda\right)}\right)^{w_j}}{\sqrt[3]{\prod_{j=1}^{s}\left(2-\left(\left((1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right)/\left((1+\nu_j^3)^\lambda+(1-\nu_j^3)^\lambda\right)\right)\right)^{w_j}+\prod_{j=1}^{s}\left(\left((1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right)/\left((1+\nu_j^3)^\lambda+(1-\nu_j^3)^\lambda\right)\right)^{w_j}}}\right\rangle$$
$$=\left\langle\sqrt[3]{\frac{\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}}{\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}}},\right.$$
$$\left.\frac{\sqrt[3]{2}\prod_{j=1}^{s}\left(\sqrt[3]{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda}\right)^{w_j}}{\sqrt[3]{\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}}}\right\rangle. \tag{62}$$

Therefore,

$$\left(\oplus_{j=1}^{s}w_j\cdot{}_{\varepsilon}\mathscr{R}_j^\lambda\right)^{1/\lambda}=\left\langle\frac{\sqrt[3]{2\left(\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)/\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)\right)^{1/\lambda}}}{\sqrt[3]{\left(2-\left(\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)/\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)\right)\right)^{1/\lambda}+\left(\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)/\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)\right)^{1/\lambda}}},\right.$$
$$\left.\sqrt[3]{\frac{\left(1+\left(2\prod_{j}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}/\left(\prod_{j}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+\prod_{j}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)\right)\right)^{1/\lambda}-\left(1-\left(2\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}/\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)\right)\right)^{1/\lambda}}{\left(1+\left(2\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}/\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)\right)\right)^{1/\lambda}+\left(1-\left(2\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}/\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)\right)\right)^{1/\lambda}}}\right\rangle$$
$$=\left\langle\frac{\sqrt[3]{2}\left\{\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda+3(\mu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(2-\mu_j^3)^\lambda-(\mu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}},\right.$$
$$\left.\sqrt[3]{\frac{\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}-\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}{\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}+3\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}+\left(\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda+3(1-\nu_j^3)^\lambda\right\}^{w_j}-\prod_{j=1}^{s}\left\{(1+\nu_j^3)^\lambda-(1-\nu_j^3)^\lambda\right\}^{w_j}\right)^{1/\lambda}}}\right\rangle. \tag{63}$$

When $\lambda = 1$, then

$$\mathrm{GFFEWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s) = \left\langle \sqrt[3]{\frac{\prod_{\jmath=1}^{s}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}} \prod_{\jmath=1}^{s}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{s}\left(1+\mu_{\jmath}^{3}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{s}\left(1-\mu_{\jmath}^{3}\right)^{w_{\jmath}}, \sqrt[3]{2}\prod_{\jmath=1}^{s}\nu_{\jmath}^{w_{\jmath}} / \sqrt[3]{\prod_{\jmath=1}^{s}\left(2-\nu_{\jmath}^{3}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{s}\left(\nu_{\jmath}^{3}\right)^{w_{\jmath}}}}} \right\rangle. \tag{64}$$

7. Generalized Fermatean Fuzzy Einstein Ordered Weighted Averaging Operators

Definition 10. Let $\mathscr{R}_{\jmath} = \langle \mu_{\jmath}, \nu_{\jmath} \rangle$ be a collection of FFNs and $w_{\jmath}$ be the WV of $\mathscr{R}_{\jmath}$ with $w_{\jmath} > 0$ and $\sum_{\jmath=1}^{s} w_{\jmath} = 1$; then, the GFFEOWA operator is a mapping $\mathcal{Q}^{s} \longrightarrow \mathcal{Q}$ such that

$$\mathrm{GFFEOWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s) = \left(\oplus_{\jmath=1}^{s}\left(w_{\jmath} \cdot_{\varepsilon} \mathscr{R}_{\varrho(\jmath)}^{\lambda}\right)\right)^{1/\lambda}, \tag{65}$$

where $\lambda > 0$.

Theorem 9. *Let $\mathscr{R}_{\jmath} = \langle \mu_{\jmath}, \nu_{\jmath} \rangle$ be FFNs and $w_{\jmath}$ be the WV of $\mathscr{R}_{\jmath}$ with $w_{\jmath} > 0$ and $\sum_{\jmath=1}^{s} w_{\jmath} = 1$; then, the aggregated value by applying the GFFEOWA operator is an FFN and*

$$\mathrm{GFFEOWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s) = \left\langle \frac{\sqrt[3]{2}\left\{\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} + 3\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda} + \left(\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left\{\left(2-\mu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(\mu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda}}}, \right.$$
$$\left. \sqrt[3]{\frac{\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} + 3\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda} - \left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda}}{\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} + 3\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda} + \left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} + 3\left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left\{\left(1+\nu_{\varrho(\jmath)}^{3}\right)^{\lambda} - \left(1-\nu_{\varrho(\jmath)}^{3}\right)^{\lambda}\right\}^{w_{\jmath}}\right)^{1/\lambda}}} \right\rangle. \tag{66}$$

Proof. It is similar to Theorem 6, and we can prove it. □

When $\lambda = 1$, then

$$\mathrm{GFFEOWA}(\mathscr{R}_1, \mathscr{R}_2, \ldots, \mathscr{R}_s) = \left\langle \sqrt[3]{\frac{\prod_{\jmath=1}^{s}\left(1+\mu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}} - \prod_{\jmath=1}^{s}\left(1-\mu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}}}{\prod_{\jmath=1}^{s}\left(1+\mu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{s}\left(1-\mu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}}}}, \frac{\sqrt[3]{2}\prod_{\jmath=1}^{s}\nu_{\varrho(\jmath)}^{w_{\jmath}}}{\sqrt[3]{\prod_{\jmath=1}^{s}\left(2-\nu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}} + \prod_{\jmath=1}^{s}\left(\nu_{\varrho(\jmath)}^{3}\right)^{w_{\jmath}}}} \right\rangle. \tag{67}$$

8. MADM Problem Using FF Information

To handle an MADM problem under FF environment, let $\mathscr{K} = \{\mathscr{K}_1, \mathscr{K}_2, \ldots, \mathscr{K}_m\}$ be a set of possible alternatives and $\mathscr{J} = \{\mathscr{J}_1, \mathscr{J}_2, \ldots, \mathscr{J}_s\}$ be a set of possible attributes chosen by the decision maker. Let $w = (w_1, w_2, \ldots, w_s)^T$ be the WV with $w_{\jmath} > 0$ and $\sum_{\jmath=1}^{s} w_{\jmath} = 1$. Suppose that $\widetilde{\mathscr{E}} = (\mu_{l\jmath}, \nu_{l\jmath})_{m \times s}$ is the FF decision matrix (FFDM), where $\mu_{l\jmath}$ and $\nu_{l\jmath}$ are the MD and NMD of the alternative $\mathscr{K}_l$ for the attribute $\mathscr{J}_{\jmath}$, respectively, where $0 \le \mu_{l\jmath}^{3} + \nu_{l\jmath}^{3} \le 1$.

The following Algorithm 1 is used to solve the MADM problem with FFN based on using the GFFEWA operator.

(1) **Input:** selection of suitable alternatives and attributes.
(2) Use the FFDM and GFFEWA operator:

ALGORITHM 1

$$\mathscr{B}_l = \mathrm{GFFEWA}\left(\mathscr{K}_{l1}, \mathscr{K}_{l2}, \ldots, \mathscr{K}_{ls}\right)$$
$$= \left\langle \frac{\sqrt[3]{2}\left\{\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}+3\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}-\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}-\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right\}^{1/3\lambda}}{\sqrt[3]{\left(\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}+3\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}+3\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}-\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}+\left(\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}+3\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}-\prod_{\jmath=1}^{s}\left\{\left(2-\mu_{l\jmath}^3\right)^{\lambda}-\left(\mu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}}}, \right.$$
$$\left. \sqrt[3]{\frac{\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}+3\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}+3\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}-\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}-\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}+3\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}-\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}-\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}}{\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}+3\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}+3\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}-\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}+\left(\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}+3\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}-\prod_{\jmath=1}^{s}\left\{\left(1+\nu_{l\jmath}^3\right)^{\lambda}-\left(1-\nu_{l\jmath}^3\right)^{\lambda}\right\}^{w_\jmath}\right)^{1/\lambda}}} \right\rangle, \tag{68}$$

for overall preference values $\mathscr{B}_l\,(l = 1, 2, \ldots, m)$ of the alternatives $\mathscr{K}_l$.

(3) Use the score function $\mathscr{S}(\mathscr{B}_l)\,(l = 1, 2, \ldots, m)$ for the ranking of alternatives. If score values are equal, then compute the accuracy functions $\mathscr{A}(\mathscr{B}_l)$ and rank according to these values.

Output: the alternative containing maximum score value will be the decision.

8.1. Selection of an Effective Sanitizer to Reduce Coronavirus. Hand sanitizer is a liquid or gel mostly used to reduce infectious agents on the hands. Alcohol-based hand sanitizers are preferred for hand washing in most healthcare settings. The Centers for Disease Control and Prevention (CDC) advise the people to wash hands with soap and water to restrain the spread of infections and decrease the endanger of getting sick. In shortage of soap and water, CDC suggests people to use an alcohol-based (at least 60 percent) hand sanitizer. According to the World Health Organization (WHO), in this pandemic situation of coronavirus, good hygiene and physical distancing are the best ways to protect ourself and everyone around us from coronavirus. This virus spreads by a person who has the disease and also spread by touching a sick person. We cannot isolate ourselves entirely to prudent from coronavirus. So, good hand hygiene can be the final barrier between us and the disease. WHO recommends alcohol-based hand sanitizers to remove the novel coronavirus. Alcohol-based hand sanitizer works to prevent the proteins of microbes—including bacteria and some viruses—from functioning normally. Hand sanitizers must contain ethanol, isopropanol, n-propanol, or a combination of these alcohols. All are effective against viruses such as the novel coronavirus.

Demand of a hand sanitizer is increased in such critical situation of COVID-19. Due to increasing demand, it is difficult to get good and effective hand sanitizers in local markets. Increasing demand has also led to low quality hand sanitizers entering the market. The main motive of this application is to select an effective sanitizer to mitigate transmission of coronavirus by applying the GFFYWA operator. Let $\mathscr{K} = \{\mathscr{K}_1, \mathscr{K}_2, K_3, \mathscr{K}_4\}$ be a set of sanitizers. Let $\mathscr{J} = \{\mathscr{J}_1, \mathscr{J}_2, \mathscr{J}_3\}$ be a set of three attributes for the evaluation of an effective sanitizer, where

$$\begin{aligned} &\mathscr{J}_1\text{: represents quantity of ethanol,} \\ &\mathscr{J}_2\text{: represents quantity of glycerol,} \\ &\mathscr{J}_3\text{: represents quantity of hydrogenperoxide.} \end{aligned} \tag{69}$$

(1) The FFDM is shown in Table 2.
(2) The weights assigned by the decision maker are

$$\begin{aligned} w_1 &= 0.60, \\ w_2 &= 0.25, \\ w_3 &= 0.15, \\ \sum_{\jmath=1}^{3} w_\jmath &= 1. \end{aligned} \tag{70}$$

We use the GFFEWA operator for the selection of an effective sanitizer.

Step 1. For performance values $\mathscr{B}_l$ of sanitizers, use the GFFEWA operator for $\lambda = 1$:

$$\begin{aligned} \mathscr{B}_1 &= (0.64, 0.48), \\ \mathscr{B}_2 &= (0.46, 0.28), \\ \mathscr{B}_3 &= (0.72, 0.31), \\ \mathscr{B}_4 &= (0.77, 0.34). \end{aligned} \tag{71}$$

Step 2. Calculate the scores $\mathscr{S}(\mathscr{B}_l)$ of FFNs $\mathscr{B}_l$ and rank the sanitizers:

Table 2: FFDM.

$\tilde{\mathscr{E}}$	$\mathscr{J}_1$	$\mathscr{J}_2$	$\mathscr{J}_3$
$\mathscr{K}_1$	(0.6, 0.7)	(0.6, 0.2)	(0.8, 0.4)
$\mathscr{K}_2$	(0.4, 0.2)	(0.6, 0.4)	(0.3, 0.6)
$\mathscr{K}_3$	(0.8, 0.3)	(0.5, 0.3)	(0.6, 0.4)
$\mathscr{K}_4$	(0.7, 0.5)	(0.8, 0.1)	(0.9, 0.6)

$$\begin{aligned}\mathcal{S}(\mathscr{B}_1) &= 0.15,\\ \mathcal{S}(\mathscr{B}_2) &= 0.08,\\ \mathcal{S}(\mathscr{B}_3) &= 0.34,\\ \mathcal{S}(\mathscr{B}_4) &= 0.42.\end{aligned} \tag{72}$$

The ranking of sanitizers is

$$\mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_1 > \mathscr{K}_2. \tag{73}$$

Step 3. Therefore, $\mathscr{K}_4$ is the best sanitizer.

9. Validity Test

For the validity and authenticity of MADM methods, Wang and Triantaphyllou [53] developed testing criteria, given as follows:

(i) Criterion 1: a MADM technique is valid if the most desirable alternative remains same on changes a nonoptimal alternative with some other poor or weak alternative, without changing the respective decision criteria

(ii) Criterion 2: the transitive property should be followed by a valid MADM technique

(iii) Criterion 3: the ranking result of alternatives should not change on splitting the problem into the smaller subproblems and by applying the same MADM technique on subproblems

Now, we discuss the validity of our proposed MADM technique by testing the above criteria.

(1) Validity test by criterion 1: if we replace the decision values of a nonoptimal alternative $\mathscr{K}_2$ by $\tilde{\mathscr{K}}_2$, then the new DM is given in Table 3.

By applying the GFFEWA operator for $\lambda = 1$ and score function, the score values of alternatives are

$$\begin{aligned}\mathcal{S}(\mathscr{B}_1) &= 0.15,\\ \mathcal{S}(\tilde{\mathscr{B}}_2) = -0.03, \mathcal{S}(\mathscr{B}_3) &= 0.34, \mathcal{S}(\mathscr{B}_4) = 0.42.\end{aligned} \tag{74}$$

The ranking of sanitizers is $\mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_1 > \tilde{\mathscr{K}}_2$, which is the same as the original ranking order, and the best sanitizer is $\mathscr{K}_4$. Thus, our presented MADM model fulfills the test criterion 1.

(2) Validity test by criteria 2 and 3: for the validity of proposed algorithm, using criteria 2 and 3, we split the problem into the smaller subproblems $\{\mathscr{K}_1, \mathscr{K}_2, \mathscr{K}_4\}, \{\mathscr{K}_1, \mathscr{K}_3, \mathscr{K}_4\}$, and $\{\mathscr{K}_2, \mathscr{K}_3, \mathscr{K}_4\}$.

Table 3: Reconstructed FFDM.

$\tilde{\mathscr{E}}$	$\mathscr{J}_1$	$\mathscr{J}_2$	$\mathscr{J}_3$
$\mathscr{K}_1$	(0.6, 0.7)	(0.6, 0.2)	(0.8, 0.4)
$\tilde{\mathscr{K}}_2$	(0.3, 0.4)	(0.5, 0.4)	(0.2, 0.8)
$\mathscr{K}_3$	(0.8, 0.3)	(0.5, 0.3)	(0.6, 0.4)
$\mathscr{K}_4$	(0.7, 0.5)	(0.8, 0.1)	(0.9, 0.6)

Table 4: Distance of alternatives from FFPIS and FFNIS.

$D(\mathscr{K}_l, \mathcal{S}^+)$	$D(\mathscr{K}_l, \mathcal{S}^-)$
0.30	0.08
0.46	0.29
0.18	0.21
0.09	0.32

By utilizing the proposed technique, the ranking orders of alternatives in these subproblems are $\mathscr{K}_4 > \mathscr{K}_1 > \mathscr{K}_2, \mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_1$, and $\mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_2$, respectively. The combined ranking of alternatives is $\mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_1 > \mathscr{K}_2$, which is the same as that of the original ranking. Hence, the proposed MADM technique is authentic and proficient under criteria 2 and 3.

10. Comparison Analysis

Here, we discuss the comparison of proposed theory with the FF TOPSIS method [5]. The steps to find out the best alternative by the FF TOPSIS method are

(1) Table 2 represents the FF decision matrix in which each entry corresponds to an FFN.

(2) The FF positive ideal solution (FFPIS) $\mathcal{S}^+$ and FF negative ideal solution (FFNIS) $\mathcal{S}^-$ are

$$\begin{aligned}\mathcal{S}^+ &= \{(0.8, 0.3), (0.8, 0.1), (0.9, 0.6)\},\\ \mathcal{S}^- &= \{(0.6, 0.7), (0.5, 0.3), (0.3, 0.6)\}.\end{aligned} \tag{75}$$

(3) The distance between the alternative $\mathscr{K}_l$ and FFPIS $\mathcal{S}^+$ together with the FFNIS $\mathcal{S}^-$ are given in Table 4.

(4) The revised closeness degree of each alternative is given as

$$\begin{aligned}\xi(\mathscr{K}_1) &= -3.05,\\ \xi(\mathscr{K}_2) &= -4.11,\\ \xi(\mathscr{K}_3) &= -1.28,\\ \xi(\mathscr{K}_4) &= 0.10.\end{aligned} \tag{76}$$

(5) We get the following ranking list by arranging the alternatives in the decreasing order with respect to $\xi(\mathscr{K}_l)$:

$$\mathscr{K}_4 > \mathscr{K}_3 > \mathscr{K}_1 > \mathscr{K}_2. \tag{77}$$

(6) $\mathscr{K}_4$ is the best alternative.

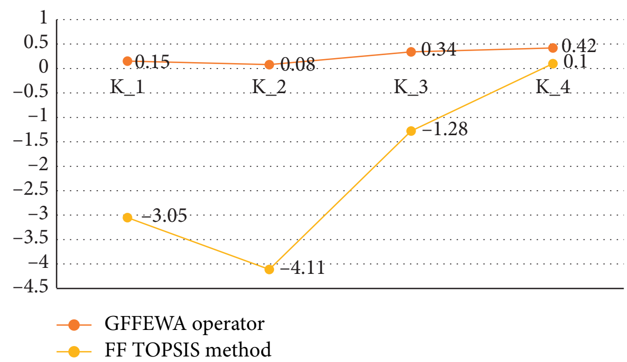

Figure 1: Comparison of the GFFEWA operator and FF TOPSIS method.

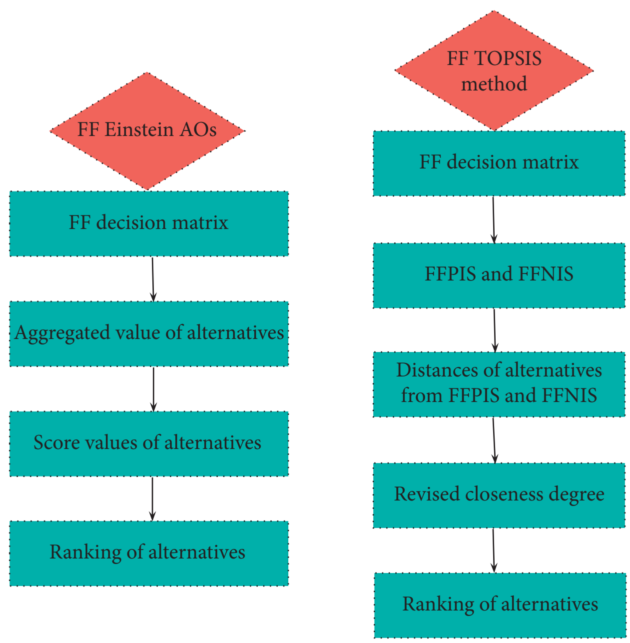

Figure 2: Flow chart for MADM problem using FF Einstein AOs and FF TOPSIS method.

From the outcomes of proposed operators and FF TOPSIS method, as shown in Figure 1, we conclude that ranking lists obtained from both compared methods are the same and the best alternative from both approaches is $\mathscr{K}_4$. The FF TOPSIS method is a good approach to solve DM problems but there are many hindrances which can be solved by using our proposed theory. The FF Einstein AOs are more flexible and easy approach. A best alternative can be obtained by a short process. The results from proposed theory are more accurate and closest to original results.

The steps to solve any MADM problem by FF Einstein AOs and FF TOPSIS method are shown in Figure 2.

10.1. Advantages and Limitations of Proposed Model. The proposed model is superior than the IF and PF models because it contains the space of IF and PF models. The cubic sum of membership and nonmembership degrees is bounded by 1 in the proposed model. The MADM approaches discussed in [10, 12, 13, 15] failed to handle the proposed application because $0.9 + 0.6 > 1$ and $0.9^2 + 0.6^2 > 1$ but proposed approach covers all such situations. The results are more precise and accurate by using the proposed model. However, there are some limitations of this model. It cannot be applied in the situations where we take the parameters for the evaluation of anything. It means this theory lacks parametrization property.

11. Conclusions

An FFS is an extension of IFS and PFS which has more flexible structure to solve decision-making problems owing to the condition $\mu^3 + \nu^3 \leq 1$. Moreover, Einstein's t-norm and t-conorm have more generalized structure that operates efficiently to integrate the intricate information. The limitations of existing operators and beneficial characteristics of Einstein AOs motivated us to endeavor for the development of a fruitful combination of Einstein AOs with FFNs.

A major contribution of the study is the development new tremendous AOs, called, FFEWA, FFEOWA, GFFEWA, and GFFEOWA operators. Some captivating properties of these operators have been discussed. Another achievement of this study is the establishment of a MADM technique on the basis of the proposed operators to manifest the application of the proposed operators. A MADM problem for the selection of an effective sanitizer to reduce the coronavirus has been presented to demonstrate the potency of the proposed strategy.

The validity test has been discussed to unfold the consistency of proposed work. A comparison analysis of our proposed theory with the FF TOPSIS method has been presented to exhibit the dominance of our proposed operators over the FF TOPSIS method. In short, this article builds up a tool that has the rich properties of Einstein AOs and flexibility of the FF model. In future, our aim is to develop some worthwhile AOs using the theoretical foundations of Einstein norms for the Fermatean fuzzy soft set.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the research article.

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Research Article

A Novel Approach towards Bipolar Soft Sets and Their Applications

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The notion of bipolar soft sets has already been defined, but in this article, the notion of bipolar soft sets has been redefined, called T-bipolar soft sets. It is shown that the new approach is more close to the concept of bipolarity as compared to the previous ones, and further it is discussed that so far in the study of soft sets and their generalizations, the concept introduced in this manuscript has never been discussed earlier. We have also discussed the operational laws of T-bipolar soft sets and their basic properties. In the end, we have deliberated the algebraic structures associated with T-bipolar soft sets and the applications of T-bipolar soft sets in decision-making problems.

1. Introduction

To handle the uncertainty has always been a problem for the researchers and decision makers as it appears in almost every field of real life and all sciences including basic sciences, management sciences, social sciences, and information sciences. Many efforts have been made to cope with this concern. The first compact attempt in this direction was made by Zadeh [1] when he familiarized the notion of fuzzy sets in 1965. In 1982, Pawlak [2] familiarized the notion of rough sets. Although these theories have their own advantages and these theories proved their effectiveness, the theory of soft sets by Molodtsov [3] in 1999 did shovel work as it generalizes both the theories. Maji et al. [4] furnished some operations to soft sets. Later on, Ali et al. [5] piercing out some inadequacies in the operations defined in [4] bequeathed some new operations to soft sets like extended union, restricted union, restricted intersection, and the restricted difference of two soft sets. In [6], Ali et al. deliberated some algebraic structures associated with the new defined operations on soft sets. Aktaş and Çağman [7] evidenced that soft sets generalize both fuzzy sets and rough sets and they are pragmatic soft sets in group theory. After the remarkable start of the era of soft sets, many researchers put their share in the progress of the theory of soft sets, for example, Acar et al. [8] presented the notion of soft rings, Sezer and Atagün [9] originated soft vector spaces, Ali et al. [10] represented graphs based on neighbourhoods and soft sets, Shabir and Naz [11] opened the notion of soft topological spaces, Sezer et al. [12] worked on soft intersection semigroups, Ali et al. [13] initiated the notion of lattice ordered soft sets, and Cagman [14] initiated a new approach in soft set theory.

The applications of soft sets in decision making were initiated by Maji et al. [15] in 2002. Since then many other authors contributed in this direction, for example, Cagman and Enginoglu [16, 17] and Kong et al. [18] did copious work in the applications of soft sets in decision making. For more studies and applications of soft sets, one may study [19–23].

The notion of fuzzy soft sets was introduced by Maji et al. [24]. Deng and Wang [25] espoused object parameter methodology for predicting unknown data in incomplete fuzzy soft sets. Naz and Shabir [26] instigated the study of algebraic structures associated with fuzzy soft sets. Roy and Maji [27] toiled on fuzzy soft set theoretic approach to decision-making problems. For more applications of fuzzy soft sets in decision making and other fields, one may study [28–34].

The notion of bipolar-valued fuzzy sets was instigated by Lee [35] in 2000. Abdullah et al. [36] commenced the perception of bipolar fuzzy soft sets and applied this perception in a decision-making problem. In 2013, Shabir and Naz [37] instigated the idea of bipolar soft sets, and then keeping this concept in view, Naz and Shabir [38] familiarized the idea of fuzzy bipolar soft sets and studied their algebraic structures and their applications. In 2014, Karaaslan and Karatas [39] espoused a different methodology to introduce bipolar soft sets, and later on, Karaaslan et al. [40] toiled on bipolar soft groups. For additional work and applications of the impression of bipolarity in soft sets and allied topics, one may study [41–45].

If we sum up all the above debate, then we noticed that keeping in view the association between fuzzy sets and soft sets and keeping in view the significance of bipolar-valued fuzzy sets, two attempts have been made to define bipolar soft sets: one by Shabir and Naz and the other by Karaaslan and Karatas. But if we notice, then we come to know that in both approaches, the conception of bipolar soft sets has some shortcomings, which we will discuss in our upcoming sections of the article (see Remark 1). So keeping this downside of the defined bipolar soft sets, in this article, we have embraced a new approach to define bipolar soft set and we named it T-bipolar soft set. Rest of the article is organized as follows:

(1) In Section 2 of the article, we have given some basic definitions to make the article self-contained and to justify redefining the notion of bipolar soft set.
(2) In Section 3 of the article, the notion of T-bipolar soft sets is familiarized, its basic operational laws are given, and related results are conferred.
(3) In Section 4, some algebraic structures are discussed associated with new defined T-BSSs.
(4) In Section 5, some applications of T-BSSs towards decision making are discussed.
(5) In Section 6, conclusion of the work presented is drawn and some future directions are discussed.

2. Preliminaries

In this section of the article, we will provide and deliberate some basic definitions of fuzzy sets, intuitionistic fuzzy sets, bipolar-valued fuzzy sets, soft sets, double framed soft sets, and bipolar-valued soft sets to make the article self-contained and also to justify the need to define T-bipolar soft sets. We will also debate the motivation to define T-bipolar soft sets.

Definition 1 (see [1]). Let A be a nonempty set. Then, a fuzzy set in A is characterized by a membership function $f\colon A \longrightarrow [0,1]$.

Definition 2 (see [3]). Let A be a nonempty set of parameters and U be an initial universe. Then, a soft set (F, A) over U is characterized by a set valued function $F\colon A \longrightarrow P(U)$.

Definition 3 (see [46]). Let A be a nonempty set. Then, an intuitionistic fuzzy set in A is characterized by two functions $f\colon A \longrightarrow [0,1]$ and $g\colon A \longrightarrow [0,1]$, where f is called a membership function and g is called nonmembership function. The condition that the sum of the values of f and g must belong to $[0,1]$ is the part of the definition of intuitionistic fuzzy set.

Definition 4 (see [47]). Let A be a nonempty set of parameters and U be an initial universe. Then, a double framed soft set over U is characterized by two set valued functions $\mathrm{F}\colon A \longrightarrow P(U)$ and $\mathrm{G}\colon A \longrightarrow P(U)$.

Definition 5 (see [35]). Let A be a nonempty set. Then, a bipolar-valued fuzzy set in A is characterized by two functions $f\colon A \longrightarrow [0,1]$ and $g\colon A \longrightarrow [-1,0]$, where for some $x \in A$, $f(x)$ denotes the satisfaction degree of the element x to the property corresponding to the bipolar-valued fuzzy set, which we denote by $\langle f, g, A\rangle$, and further $g(x)$ denotes the satisfaction degree of x to some implicit counterproperty of the bipolar-valued fuzzy set $\langle f, g, A\rangle$.

Definition 6 (see [37]). Let A be a nonempty set of parameters, $A = \{x\colon x \in A\}$ denotes the NOT set of A, and let U be an initial universe. Then, a bipolar soft set, denoted by (F, G, A), over U is characterized by two set valued functions $F\colon A \longrightarrow P(U)$ and $G\colon A \longrightarrow P(U)$ such that for all $x \in A, F(x) \cap G(x) = \varnothing$ (empty set).

Definition 7 (see [39]). Let A be a parameter set and A_1 and A_2 be two nonempty subsets of A such that $(A_1 \cup A_2 = A)$ and $A_1 \cap A_2 = \varnothing$. Then, the triplet (F, G, A) is thought to be a bipolar soft set over U, where F and G are set valued mappings given by $\mathrm{F}\colon A_1 \longrightarrow P(U)$ and $\mathrm{G}\colon A_2 \longrightarrow P(U)$ such that $F(x) \cap G(f(x)) = \varnothing$, where $f\colon A_1 \longrightarrow A_2$ is a bijective function.

Remark 1. From above definitions, we note that

(1) The definitions of fuzzy sets and that of soft sets have same characteristics in the sense that
  (i) Both are characterized by a single function
  (ii) Both have a single set as domain set
  (iii) Both have a single set, which is a lattice in either case, as codomain set
(2) The definitions of intuitionistic fuzzy sets and that of double framed soft sets have same characteristics in the sense that
  (i) Both are characterized by two functions
  (ii) Both have a single set as domain set for both the functions
  (iii) Both have a single set, which is a lattice in either case, as codomain set for both the functions
(3) But this is not the case for bipolar-valued fuzzy sets as compared to the definitions of bipolar soft sets defined in [37, 39]

All this dialogue demonstrates that the space to define bipolar soft set has not yet been filled. As every definition in mathematics has its own importance and it does not mean that the already existing definitions of bipolar soft sets are of no use and the proposed definition of bipolar soft set will nullify the existing definitions, but the purpose to redefine the notion of bipolar soft set is, one to elaborate the notion of bipolarity in soft sets more affectively and the other is, as there is, up to the best of our knowledge, no such type of situation is discussed in soft sets earlier.

Now, we deliberate some basic definitions connected to soft sets. In this section, from now onwards, E will denote a set of parameters, $A, B, C \ldots \subseteq E$, and U will denote an initial universe. Further, the set of all soft sets over U will be denoted by $(SS)_{(U)}$.

Definition 8 (see [14]). Let (F_1, A) and $(F_2, B) \in (\mathrm{SS})_{(U)}$. Then, (F_1, A) is called a soft subset of (F_2, B) if

(i) $A \subseteq B$

(ii) For all $a \in A, F_1(a) \subseteq F_2(a)$

Then, we write $(F_1, A) \subseteq (F_2, B)$. (F_1, A) and (F_2, B) are said to be soft equal if and only if $(F_1, A) \subseteq (F_2, B)$ and $(F_2, B) \subseteq (F_1, A)$. Then, we write $(F_1, A) = (F_2, B)$.

Definition 9 (see [14]). Let $(F, A) \in (\mathrm{SS})_{(U)}$. Then,

(i) Complement of (F, A) is designated and specified by $(F, A)^c = (F^c, A)$ where $F^c(a) = U - F(a)$, for all $a \in A$

(ii) (F, A) is said to be null if and only if for all $a \in A, F(a) = \varnothing$

(iii) (F, A) is said to be absolute if and only if for all $a \in A, F(a) = U$

Definition 10 (see [4]). Let (F_1, A) and $(F_2, B) \in (\mathrm{SS})_{(U)}$. Then,

(i) "AND" product of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \wedge (F_2, B) = (F_3, A \times B)$ where $F_3(a, b) = F_1(a) \cap F_2(b)$ for all $(a, b) \in A \times B$

(ii) "OR" product of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \vee (F_2, B) = (F_3, A \times B)$ where $F_3(a, b) = F_1(a) \cup F_2(b)$ for all $(a, b) \in A \times B$

Definition 11 (see [4]). Let (F_1, A) and $(F_2, B) \in (\mathrm{SS})_{(U)}$. Then, "union" (which we may also call extended union) of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \cup_E (F_2, B) = (H, A \cup B)$, where

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ F_1(e) \cup F_2(e), & \text{if } e \in A \cap B. \end{cases} \quad (1)$$

Definition 12 (see [5]). Let (F_1, A) and $((F_2, B) \in (\mathrm{SS})_{(U)})$. Then, "extended intersection" of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \cap_E (F_2, B) = (H, A \cup B)$, where

$$H(e) = \begin{cases} F_1(e), & \text{if } e \in A - B, \\ F_2(e), & \text{if } e \in B - A, \\ F_1(e) \cap F_2(e), & \text{if } e \in A \cap B. \end{cases} \quad (2)$$

Definition 13 (see [5]). Let (F_1, A) and $(F_2, B) \in (\mathrm{SS})_{(U)}$ such that $(A \cap B)$ is nonempty. Then,

(i) "Restricted union" of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \cup_R (F_2, B) = (H, A \cap B)$, where $H(e) = F_1(e) \cup F_2(e)$

(ii) "Restricted intersection" of (F_1, A) and (F_2, B) is designated and demarcated by $(F_1, A) \cap_R (F_2, B) = (H, A \cap B)$, where $H(e) = F_1(e) \cap F_2(e)$

3. T-Bipolar Soft Set

In this section, we will familiarize the perception of T-bipolar soft set (T-BSS), we will delineate binary operations for T-BSSs, and we will also deliberate some basic properties and some results concomitant with these concepts. First, we contemplate the succeeding example.

Example 1. Let us consider the case, where a researcher Dr. Shabir wants to submit his four research articles a_1 (on homological algebra), a_2 (on fuzzy sets), a_3 (on soft sets), and a_4 (on rough sets) in some research journals. For the purpose, he has to propose some potential referees and also he has the option to oppose some referees. Keeping in view all the aspects, he prepared a set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ of some proposed referees as well as a set $Y = \{y_1, y_2, y_3, y_4, y_5\}$ of some referees to oppose. Hence, in this case, he has under consideration the set $U = \{x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4, y_5\}$ of all referees. For each of his article, he selects some referees from X to propose and selects some referees from Y to oppose. Keeping in view all the aspects for the article

(i) a_1 (on homological algebra), he decided to propose x_2, x_3, x_4 and he opposed y_1, y_2

(ii) a_2 (on fuzzy sets), he decided to propose x_1, x_3, x_6 and he opposed y_2, y_4

(iii) a_3 (on soft sets), he decided to propose x_4, x_5 and he opposed y_3

(iv) a_4 (on rough sets), he decided to propose x_1, x_5, x_6 and he opposed y_1, y_4, y_5

Note that all this information can be modeled mathematically as follows: let $A = \{a_1, a_2, a_3, a_4\}$, $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, and $Y = \{y_1, y_2, y_3, y_4, y_5\}$. Define $(F: A \longrightarrow P(X))$ as $F(a_1) = \{x_2, x_3, x_4\}$, $F(a_2) = \{x_1, x_3, x_6\}$, $F(a_3) = \{x_4, x_5\}$, $F(a_4) = \{x_1, x_5, x_6\}$, and $(G: A \longrightarrow P(Y))$ as $G(a_1) = \{y_1, y_2\}$, $G(a_2) = \{y_2, y_4\}$, $G(a_3) = \{y_3\}$, $G(a_4) = \{y_1, y_4, y_5\}$.

Here, it can be perceived that both the functions F and G have common domain A, and codomains of F and G have nothing in common except the empty set ϕ. Further we notice that same is the case in bipolar-valued fuzzy sets. Hence, we have the following definition.

Definition 14. Let E be a set of parameters, $A \subseteq E$, and U be an initial universe, $X \subset U$ and $Y = U - X$. Then, a triplet F, G, A is said to be a T-BSS over U, where F and G are set valued mappings given by $F\colon A \longrightarrow P(X)$ and $G\colon A \longrightarrow P(Y)$. In this case, we write $(F, G, A) = \{\langle a, F(a), G(a)\colon F(a) \in P(X) \text{ and } G(a) \in P(Y)\rangle\}$ or simply $(F, G, A) = \{\langle a, F(a), G(a)\rangle\}$. The collection of all T-BSSs over U is denoted by $(T - \mathrm{BSS})_{(U)}$.

Remark 2. Let $A = \{a_1, a_2, a_3, \ldots, a_l\} \subseteq E$, $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$, and (F, G, A) be corresponding T-BSS. Then, we can represent (F, G, A) as follows (Table 1).

$$\zeta_{ijk} = (\mu_j, \nu_k) = \begin{cases} (0,0) & \text{if } x_j \notin F(a_i) \text{ and } y_k \notin F(a_i), \\ (1,0) & \text{if } x_j \in F(a_i) \text{ and } y_k \notin F(a_i), \\ (0,1) & \text{if } x_j \notin F(a_i) \text{ and } y_k \in F(a_i), \\ (1,1) & \text{if } x_j \in F(a_i) \text{ and } y_k \in F(a_i), \end{cases}$$
$$\zeta^*_{ijk} = \mu_j,$$
$$\zeta^\circ_{ijk} = \nu_k. \tag{3}$$

Example 2. A university wants to appoint a permanent faculty member from the set $A = \{a_1, a_2, a_3, a_4, a_5\}$ of visiting faculty members. For the purpose, the university authorities constitute two panels $X = \{x_1, x_2, x_3, x_4\}$ and $Y = \{y_1, y_2, y_3\}$ of experts, where panel X consists of members from outside the university and panel Y consists of members from inside the university. Further each member of the panel X will decide about each candidate by considering his/her experience, number of research publications, number of conferences attended, etc., while each member of the panel Y will decide about each candidate by considering his/her regularity and punctuality, attitude towards other faculty members, and behavior with students during class. Now the university authorities decided the selection criteria that towards each candidate each member of the panel X will have to select a candidate by keeping in view his/her positive points while each member of the panel Y has to reject a candidate by keeping in view his/her negative points. According to the decisions taken by the members of the panels X, the experts x_1 and x_3 are in favor to select the candidate a_1 while x_2 and x_4 decided to remain neutral for the candidate a_1. Similarly the decisions taken by the members of the panels Y, the member y_3 is not in favor to select the candidate a_1 while y_1 and y_2 decided to remain neutral for the candidate a_1. Hence, for the candidate a_1, the situation can be modeled as $\langle a_1, \{x_1, x_3\}, \{y_3\}\rangle$. Now keeping under consideration the decisions taken by all the members from the panels X and Y, the result can be modeled mathematically as given in the following T-BSS: $((F, G, A) = \{\langle a_1, \{x_1, x_3\}, \{y_3\}\rangle, \langle a_2, \{x_1, x_3, x_4\}, \{y_1\}\rangle, \langle a_3, \{x_1, x_4\}, \{y_1, y_3\}\rangle, \langle a_4, \{x_2, x_3, x_4\}, \{y_2, y_3\}\rangle, \langle a_5, \{x_2, x_4\}, \{y_2\}\rangle\})$.

Tabular form of the (F, G, A) is given as follows (Table 2).

Definition 15. Let $(F_1, G_1, A) \in (T - BSS)_{(U)}$. Then, (F_1, G_1, A) is said to be T-bipolar soft subset of (F_2, G_2, B) if

(i) $A \subseteq B$

(ii) For all $a \in A$, $F_1(a) \subseteq F_2(a)$ and $G_2(a) \subseteq G_1(a)$.

Then, we write $(F_1, G_1, A) \subseteq (F_2, G_2, B)$. (F_1, G_1, A) and (F_2, G_2, B) are said to be equal if and only if $(F_1, G_1, A) \subseteq (F_2, G_2, B)$ and $(F_2, G_2, B) \subseteq (F_1, G_1, A)$. Then, we write $(F_1, G_1, A) = (F_2, G_2, B)$.

Definition 16. Let $(F, G, A) \in (\mathrm{T - BSS})_{(U)}$. Then,

(i) Complement of (F, G, A) is denoted and given by $(F, G, A)^c = (F^c, G^c, A) = \{\langle a, F^c(a) = X - F(a), \ G^c(a) = Y - G(a)\rangle\}$.

(ii) (F, G, A) is said to be null if and only if for all $a \in A$, $F(a) = \varnothing$ and $G(a) = Y$. In our study, it will further be designated by ϕ, that is, $\phi = \{\langle a, \varnothing, Y\rangle\}$.

(iii) (F, G, A) is said to be absolute if and only if for all $a \in A$, $F(a) = X$ and $G(a) = \varnothing$. In our study, it will further be designated by $\mathscr{A}$ that is, $\mathscr{A} = \{\langle a, X, \varnothing\rangle\}$.

Definition 17. Let $(F_1, G_1, A), (F_2, G_2, B) \in (\mathrm{T - BSS})_{(U)}$. Then,

(i) "AND" product of (F_1, G_1, A) and (F_2, G_2, B) is designated and demarcated by

$$(F_1, G_1, A) \wedge (F_2, G_2, B) = \{<(a,b), F_1(a) \cap F_2(b), G_1(a) \cup G_2(b)>\colon (a,b) \in A \times B\}. \tag{4}$$

(ii) "OR" product of (F_1, G_1, A) and (F_2, G_2, B) is designated and demarcated by

$$(F_1, G_1, A) \vee (F_2, G_2, B) = \{<(a,b), F_1(a) \cup F_2(b), G_1(a) \cap G_2(b)>\colon (a,b) \in A \times B\}. \tag{5}$$

Proposition 1. *Let* (F_1, G_1, A), (F_2, G_2, B) $(F_2, G_2, B) \in (T - BSS)_{(U)}$. *Then,*

(i) $[(F_1, G_1, A) \wedge (F_2, G_2, B)]^c = [(F_1, G_1, A)]^c \vee [(F_2, G_2, B)]^c$

TABLE 1: Tabular form of a T-bipolar soft set.

(F,G,A)	(x_1,y_1)	(x_1,y_2)	$\ldots$	(x_1,y_n)	(x_2,y_1)	(x_2,y_2)	$\ldots$	(x_2,y_n)	$\ldots$	(x_m,y_1)	(x_m,y_2)	$\ldots$	(x_m,y_n)
a_1	ζ_{111}	ζ_{112}	$\ldots$	ζ_{11n}	ζ_{121}	ζ_{122}	$\ldots$	ζ_{12n}	$\ldots$	ζ_{1m1}	ζ_{1m2}	$\ldots$	ζ_{1mn}
a_2	ζ_{211}	ζ_{212}	$\ldots$	ζ_{21n}	ζ_{221}	ζ_{222}	$\ldots$	ζ_{22n}	$\ldots$	ζ_{2m1}	ζ_{2m2}	$\ldots$	ζ_{2mn}
a_3	ζ_{311}	ζ_{312}	$\ldots$	ζ_{31n}	ζ_{321}	ζ_{322}	$\ldots$	ζ_{32n}	$\ldots$	ζ_{3m1}	ζ_{3m2}	$\ldots$	ζ_{3mn}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
a_l	ζ_{l11}	ζ_{l12}	$\ldots$	ζ_{l1n}	ζ_{l21}	ζ_{l22}	$\ldots$	ζ_{l2n}	$\ldots$	ζ_{lm1}	ζ_{lm2}	$\ldots$	ζ_{lmn}

TABLE 2: Tabular form of the T-bipolar soft set (F,G,A).

(F,G,A)	(x_1,y_1)	(x_1,y_2)	(x_1,y_3)	(x_2,y_1)	(x_2,y_2)	(x_2,y_3)	(x_3,y_1)	(x_3,y_2)	(x_3,y_3)	(x_4,y_1)	(x_4,y_2)	(x_4,y_3)
a_1	(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 0)	(0, 1)
a_2	(1, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 0)	(0, 0)	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)
a_3	(1, 1)	(1, 0)	(1, 1)	(0, 1)	(0, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 1)	(1, 1)	(1, 0)	(1, 1)
a_4	(0, 0)	(0, 1)	(0, 1)	(1, 0)	(1, 1)	(1, 1)	(1, 0)	(1, 1)	(1, 1)	(1, 0)	(1, 1)	(1, 1)
a_5	(0, 0)	(0, 1)	(0, 0)	(1, 0)	(1, 1)	(1, 0)	(0, 0)	(0, 1)	(0, 0)	(1, 0)	(1, 1)	(1, 0)

(ii) $[(F_1,G_1,A)\vee(F_2,G_2,B)]^c = [(F_1,G_1,A)]^c \wedge [(F_2,G_2,B)]^c$

Proof

(i) $[(F_1,G_1,A)\wedge(F_2,G_2,B)]^c = \{\langle(a,b),F_1(a)\cap F_2(b), G_1(a)\cup G_2(b)\rangle\}^c = \{\langle(a,b),X-(F_1(a)\cap F_2(b)),\ Y-(G_1(a)\cup G_2(b))\rangle\} = \{\langle(a,b),(X-F_1(a))\cup(X-F_2(b)),(Y-G_1(a))\cap(Y-G_2(b))\rangle\} = \{\langle<(a,b), F_1^c(a)\cup F_2^c(b), G_1^c(a)\cap G_2^c(b)\rangle\} = [(F_1,G_1,A)]^c\vee[(F_2,G_2,B)]^c$.

(ii) Similar to part (i). □

Definition 18. Let $(F_1,G_1,A),\ (F_2,G_2,B)\in(T-BSS)_{(U)}$. Then, "extended union" of (F_1,G_1,A) and (F_2,G_2,B) is designated and demarcated by $((F_1,G_1,A)\cup_E(F_2,G_2,B) = (H,K,A\cup B))$, where

$$
\begin{aligned}
H(e) &= \begin{cases} F_1(e), & \text{if } e\in A-B,\\ F_2(e), & \text{if } e\in B-A,\\ F_1(e)\cup F_2(e), & \text{if } e\in A\cap B,\end{cases}\\
K(e) &= \begin{cases} G_1(e), & \text{if } e\in A-B,\\ G_2(e), & \text{if } e\in B-A,\\ G_1(e)\cap G_2(e), & \text{if } e\in A\cap B.\end{cases}
\end{aligned}
\tag{6}
$$

Definition 19. Let $(F_1,G_1,A),\ (F_2,G_2,B)\in(\mathrm{T-BSS})_{(U)}$ Then, "extended intersection" of (F_1,G_1,A) and (F_2,G_2,B) is designated and demarcated by

$$(F_1,G_1,A)\cap_E(F_2,G_2,B) = (H,K,A\cup B), \tag{7}$$

where

$$
\begin{aligned}
H(e) &= \begin{cases} F_1(e), & \text{if } e\in A-B,\\ F_2(e), & \text{if } e\in B-A,\\ F_1(e)\cap F_2(e), & \text{if } e\in A\cap B,\end{cases}\\
K(e) &= \begin{cases} G_1(e), & \text{if } e\in A-B,\\ G_2(e), & \text{if } e\in B-A,\\ G_1(e)\cup G_2(e), & \text{if } e\in A\cap B.\end{cases}
\end{aligned}
\tag{8}
$$

Proposition 2. *For any T-BSSs (F_1,G_1,A), (F_2,G_2,B), and (F_3,G_3,C),*

(i) $(F_1,G_1,A)\cap_E\phi=\phi$, $(F_1,G_1,A)\cup_E\phi=(F_1,G_1,A)$, $(F_1,G_1,A)\cap_E\mathscr{A}=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_E\mathscr{A}=\mathscr{A}$

(ii) $(F_1,G_1,A)\cap_E(F_1,G_1,A)=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_E(F_1,G_1,A)=(F_1,G_1,A)$

(iii) $(F_1,G_1,A)\cap_E(F_2,G_2,B)=(F_2,G_2,B)\cap_E(F_1,G_1,A)$, $(F_1,G_1,A)\cup_E(F_2,G_2,B)=(F_2,G_2,B)\cup_E(F_1,G_1,A)$

(iv) $(F_1,G_1,A)\cup_E[(F_2,G_2,B)\cup_E(F_3,G_3,C)]=[(F_1,G_1,A)\cup_E(F_2,G_2,B)]\cup_E(F_3,G_3,C)$

(v) $(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cup_E(F_1,G_1,A)]=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_E[(F_2,G_2,B)\cap_E(F_1,G_1,A)]=(F_1,G_1,A)$

(vi) $[(F_1,G_1,A)^c]^c=(F_1,G_1,A)$, $(F_1,G_1,A)\cap_E[(F_1,G_1,A)]^c=\phi$, $(F_1,G_1,A)\cup_E[(F_1,G_1,A)]^c=\mathscr{A}$

(vii) $[(F_1,G_1,A)\cap_E(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cup_E[(F_2,G_2,B)]^c$, $[(F_1,G_1,A)\cup_E(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cap_E[(F_2,G_2,B)]^c$

Proof. We prove (iv) and (vii); rest are straightforward.

(iv) When $x\in A$, $x\notin B$, and $x\in C$, then

$$\begin{aligned}&(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]\\&=(F_1,G_1,A)\cap_E(F_3,G_3,C)\\&=\{\langle\{<x\in A\cap C\colon F_1(x)\cap F_3(x),\ G_1(x)\cup G_3(x)>\}\rangle\},\\&\cdot[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cap_E(F_3,G_3,C)\\&=(F_1,G_1,A)\cap_E(F_3,G_3,C)\\&=\{\langle x\in A\cap C\colon F_1(x)\cap F_3(x),\ G_1(x)\cup G_3(x)\rangle\}.\end{aligned}\tag{9}$$

When $x\in A$, $x\in B$, and $x\notin C$, then

$$\begin{aligned}&(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]\\&=(F_1,G_1,A)\cap_E(F_2,G_2,B)\\&=\{\langle x\in A\cap B\colon F_1(x)\cap F_2(x),\ G_1(x)\cup G_2(x)\rangle\},\\&\cdot[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cap_E(F_3,G_3,C)\\&=(F_1,G_1,A)\cap_E(F_2,G_2,B)\\&=\{\langle x\in A\cap B\colon F_1(x)\cap F_2(x),\ G_1(x)\cup G_2(x)\rangle\}.\end{aligned}\tag{10}$$

When $x\notin A$, $x\in B$, and $x\in C$, then

$$\begin{aligned}&(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]\\&=(F_2,G_2,B)\cap_E(F_3,G_3,C)\\&=\{\langle x\in B\cap C\colon F_2(x)\cap F_3(x),\ G_2(x)\cup G_3(x)\rangle\},\\&\cdot[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cap_E(F_3,G_3,C)\\&=(F_2,G_2,B)\cap_E(F_3,G_3,C)\\&=\{\langle x\in B\cap C\colon F_2(x)\cap F_3(x),\ G_2(x)\cup G_3(x)\rangle\}.\end{aligned}\tag{11}$$

When $x\notin A$, $x\in B$, and $x\in C$, then result is obvious. Hence, it concludes that $(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]=[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cap_E(F_3,G_3,C)$. Similarly $(F_1,G_1,A)\cup_E[(F_2,G_2,B)\cup_E(F_3,G_3,C)]=[(F_1,G_1,A)\cup_E(F_2,G_2,B)]\cup_E(F_3,G_3,C)$.

(vii) When $(x\in A)$ and $(x\notin B)$, then

$$\begin{aligned}&[(F_1,G_1,A)\cap_E(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c,\\&[(F_1,G_1,A)]^c\cup_E[(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c.\end{aligned}\tag{12}$$

When $(x\notin A)$ and $(x\in B)$, then

$$\begin{aligned}&[(F_1,G_1,A)\cap_E(F_2,G_2,B)]^c=[(F_2,G_2,B)]^c,\\&[(F_1,G_1,A)]^c\cup_E[(F_2,G_2,B)]^c=[(F_2,G_2,B)]^c.\end{aligned}\tag{13}$$

When $x\in A$ and $x\in B$, then the result is a trivial case. Hence, in either case $([(F_1,G_1,A)\cap_E(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cup_E[(F_2,G_2,B)]^c)$ Similarly, $([(F_1,G_1,A)\cup_E(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cap_E[(F_2,G_2,B)]^c)$. □

Remark 3. For any arbitrary (F_1,G_1,A) (F_2,G_2,B), $(F_3,G_3,C)\in(T-\mathrm{BSS})_{(U)}$, it is not necessary that

(i) $(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cup_E(F_3,G_3,C)]=[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cup_E[(F_1,G_1,A)\cap_E(F_3,G_3,C)]$

(ii) $(F_1,G_1,A)\cup_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]=[(F_1,G_1,A)\cup_E(F_2,G_2,B)]\cap_E[(F_1,G_1,A)\cup_E(F_3,G_3,C)]$

Example 3. Let $E=\{e_1,e_2,e_3,e_4,e_5\}$, $A=\{e_1,e_2,e_3\}$, $B=\{e_3,e_4\}$, $C=\{e_4,e_5\}$, $U=\{x_1,x_2,x_3,x_4,x_5\}$, $X=\{x_1,x_2,x_3\}$, and $Y=\{x_4,x_5\}$.

Now let

$$\begin{aligned}&(F_1,G_1,A)=\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\\&\qquad\cdot\langle e_3,\{x_1,x_3\},\{x_4\}\rangle\},\\&(F_2,G_2,B)=\{\langle e_3,\{x_2,x_3\},\{x_5\}\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle\},\\&(F_3,G_3,C)=\{\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}.\end{aligned}\tag{14}$$

Now

$$\begin{aligned}&(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cup_E(F_3,G_3,C)]\\&=(F_1,G_1,A)\cap_E\{\langle e_3,\{x_2,x_3\},\{x_5\}\rangle,\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\\&\quad\cdot\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\},\\&=\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\\&\langle e_3,\{x_3\},\{x_4,x_5\}\rangle,\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}.\end{aligned}\tag{15}$$

Next

$$\begin{aligned}&[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cup_E[(F_1,G_1,A)\cap_E(F_3,G_3,C)]\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\\&\quad\cdot\langle e_3,\{x_3\},\{x_4,x_5\}\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle\}\cup_E\\&\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\langle e_3,\{x_1,x_3\},\{x_4\}\rangle,\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}\\&=\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\langle e_3,\{x_1,x_3\},\{x_4\}\rangle,\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\},\\&\Longrightarrow(F_1,G_1,A)\cap_E[(F_2,G_2,B)\cup_E(F_3,G_3,C)]\neq[(F_1,G_1,A)\cap_E(F_2,G_2,B)]\cup_E[(F_1,G_1,A)\cap_E(F_3,G_3,C)].\end{aligned}\tag{16}$$

Now

$$\begin{aligned}(F_1,G_1,A)\cup_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)] &= (F_1,G_1,A)\cup_E\{\langle e_3,\{x_2,x_3\},\{x_5\}\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}\\ &=\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\langle e_3,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}.\end{aligned} \tag{17}$$

Next

$$\begin{aligned}&[(F_1,G_1,A)\cup_E(F_2,G_2,B)]\cap_E[(F_1,G_1,A)\cup_E(F_3,G_3,C)] = \{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\\ &\quad\cdot\langle e_3,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle\}\cap_E\\ &\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\langle e_3,\{x_1,x_3\},\{x_4\}\rangle,\langle e_4,\{x_1,x_2,x_3\},\varnothing\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}\\ &=\{\langle e_1,\{x_1,x_2\},\{x_4\}\rangle,\langle e_2,\{x_1\},\{x_4,x_5\}\rangle,\langle e_3,\{x_1,x_3\},\{x_4\}\rangle,\langle e_4,\{x_1,x_2\},\{x_4,x_5\}\rangle,\langle e_5,\{x_1,x_2\},\{x_5\}\rangle\}\\ &\Longrightarrow (F_1,G_1,A)\cup_E[(F_2,G_2,B)\cap_E(F_3,G_3,C)]\neq[(F_1,G_1,A)\cup_E(F_2,G_2,B)]\cap_E[(F_1,G_1,A)\cup_E(F_3,G_3,C)].\end{aligned} \tag{18}$$

Definition 20. Let (F_1,G_1,A) $(F_2,G_2,B)\in(T-\mathrm{BSS})_{(U)}$ with $A\cap B\neq\varnothing$. Then,

(i) "Restricted union" of (F_1,G_1,A) and (F_2,G_2,B) is designated and demarcated by

$$(F_1,G_1,A)\cup_R(F_2,G_2,B)=\{\langle c,F_1(c)\cup F_2(c),G_1(c)\cap G_2(c)\rangle : c\in A\cap B\}. \tag{19}$$

(ii) "Restricted intersection" of (F_1,G_1,A) and (F_2,G_2,B) is designated and demarcated by

$$(F_1,G_1,A)\cap_R(F_2,G_2,B)=\{\langle c,F_1(c)\cap F_2(c),G_1(c)\cup G_2(c)\rangle : c\in A\cap B\}. \tag{20}$$

Proposition 3. *For any T-BSSs (F_1,G_1,A), (F_2,G_2,B), and (F_3,G_3,C),*

(i) $(F_1,G_1,A)\cap_R\phi=\phi$, $(F_1,G_1,A)\cup_R\phi=(F_1,G_1,A)$, $(F_1,G_1,A)\cap_R\mathscr{A}=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R\mathscr{A}=\mathscr{A}$

(ii) $(F_1,G_1,A)\cap_R(F_1,G_1,A)=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R(F_1,G_1,A)=(F_1,G_1,A)$

(iii) $(F_1,G_1,A)\cap_R(F_2,G_2,B)=(F_2,G_2,B)\cap_R(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R(F_2,G_2,B)=(F_2,G_2,B)\cup_R(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R(F_2,G_2,B)=(F_2,G_2,B)\cup_R(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R(F_2,G_2,B)=(F_2,G_2,B)\cup_R(F_1,G_1,A)$

(iv) $(F_1,G_1,A)\cap_R[(F_2,G_2,B)\cap_R(F_3,G_3,C)]=[(F_1,G_1,A)\cap_R(F_2,G_2,B)]\cap_R(F_3,G_3,C)$, $(F_1,G_1,A)\cup_R[(F_2,G_2,B)\cup_R(F_3,G_3,C)]=[(F_1,G_1,A)\cup_R(F_2,G_2,B)]\cup_R(F_3,G_3,C)$

(v) $(F_1,G_1,A)\cap_R[(F_2,G_2,B)\cup_R(F_3,G_3,C)]=[(F_1,G_1,A)\cap_R(F_2,G_2,B)]\cup_R[(F_1,G_1,A)\cap_R(F_3,G_3,C)]$, $(F_1,G_1,A)\cup_R[(F_2,G_2,B)\cap_R(F_3,G_3,C)]=[(F_1,G_1,A)\cup_R(F_2,G_2,B)]\cap_R[(F_1,G_1,A)\cup_R(F_3,G_3,C)]$

(vi) $(F_1,G_1,A)\cap_R[(F_2,G_2,B)\cup_R(F_1,G_1,A)]=(F_1,G_1,A)$, $(F_1,G_1,A)\cup_R[(F_2,G_2,B)\cap_R(F_1,G_1,A)]=(F_1,G_1,A)$

(vii) $[(F_1,G_1,A)^c]^c=(F_1,G_1,A)$, $(F_1,G_1,A)\cap_R[(F_1,G_1,A)]^c=\phi$, $(F_1,G_1,A)\cup_R[(F_1,G_1,A)]^c=\mathscr{A}$

(viii) $[(F_1,G_1,A)\cap_R(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cup_R[(F_2,G_2,B)]^c$, $[(F_1,G_1,A)\cup_R(F_2,G_2,B)]^c=[(F_1,G_1,A)]^c\cap_R[(F_2,G_2,B)]^c$

Proof. Straightforward. □

4. Algebraic Structures Associated with T-BSSs

In this section, we will discuss some algebraic structures associated with T-BSSs. Recall that $(\mathrm{T}-\mathrm{BSS})_{(U)}$ denotes the collection of all T-BSSs over U. Now in this section, $(\mathrm{T}-\mathrm{BSS})^{A}_{(U)}$ denotes the collection of all T-BSSs over U with domain A.

Proposition 4. *For any $\Delta \in \{\cap_E, \cup_E, \cap_R, \cup_R\}$, $((T-BSS)_{(U)}, \Delta)$ is a commutative semigroup whose every element is idempotent.*

Proof. The proof is straightforward by using the Proposition 2 and Proposition 3. □

Proposition 5. $((T-BSS)_{(U)}, \cap_R, \cup_R)$ *is a commutative semiring.*

Proof. The proof follows from the definitions of restricted intersection of T-BSSs, restricted union of T-BSSs, and parts (iv) and (v) of Proposition 3. □

Remark 4. It follows from Proposition 2 and Remark 3 that $((\mathrm{T}-\mathrm{BSS})_{(U)}, \cap_E, \cup_E)$ is not a semiring.

Proposition 6. $((T-BSS)^{A}_{(U)}, \cap_E, \cup_E)$ *is a commutative semiring.*

Proof. This is straightforward as extended intersection of T-BSSs and extended union of T-BSSs satisfy the distributive laws, which usually do not hold (Remark 3), if all the T-BSSs have same domain A. □

Proposition 7. $(T-BSS)_{(U)}, \cap_E, \cup_E, {}^c, (\phi, \mathscr{A})$ *is a bounded lattice.*

Proof. The result follows from conditions (i)–(v) of Proposition 2. □

Proposition 8. $(T-BSS)_{(U)}, \cap_R, \cup_R, {}^c, (\phi, \mathscr{A})$ *is a bounded distributive lattice.*

Proof. The result follows from Proposition 3. □

5. Applications of T-BSSs in Decision Making

In this section, we will discuss some decision-making problems by using T-BSSs. We will discuss decision-making problems in the absence of weights, in the presence of weights selected randomly, and in the presence of weights taken as discussed in [48].

Definition 21. Let $A = \{a_1, a_2, a_3, \ldots, a_l\} \subseteq E$, $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$, and (F, G, A) be corresponding T-BSS. Then, score of $(a_i,\ 1 \le i \le l)$ is denoted and defined as $(S_i = \overline{\sigma_i} - \underline{\sigma_i})$, where $(\overline{\sigma_i} = \sum_{j,k} \zeta^{*}_{\mathrm{ijk}})$ and $(\underline{\sigma_i} = \sum_{j,k} \zeta^{\circ}_{\mathrm{ijk}})$.

Definition 22. Let $A = \{a_1, a_2, a_3, \ldots, a_l\} \subseteq E$, $X = \{x_1, x_2, x_3, \ldots, x_m\}$, $Y = \{y_1, y_2, y_3, \ldots, y_n\}$, and (F, G, A) be corresponding T-BSS. Then, $(a_i,\ 1 \le i \le l)$ is said to be optimal if and only if $(S_i > S_{i'})$, for all $(i' \ne i)$.

Example 4. Consider Example 2 with $(F, G, A) = \{\langle a_1, \{x_1, x_3\}, \{y_3\}\rangle, \langle a_2, \{x_1, x_2, x_4\}, \{y_1\}\rangle, \langle a_3, \{x_1, x_4\}, \{y_1, y_3\}\rangle, \langle a_4, \{x_2, x_3, x_4\}, \{y_2, y_3\}\rangle, \langle a_5, \{x_2, x_4\}, \{y_2\}\rangle\}$, with tabular form as in Table 3.

Then, the score values are given in Table 4.

Then, according to the Algorithm 1, the candidate "a_2" will be selected.

Remark 5. Sometimes in decision making, some decision makers have less importance as compared to other decision makers, for example, to decide about admission policy of a school, a meeting was called in which four persons participated who were the school owner, school principal, school vice principal, and accountant of the school. Now here it is clear that all the decision makers have not the same weightage. So, in decision making, the weightage of a decision maker also matters a lot. So, now we establish an algorithm to handle a decision-making problem in the presence of weights.

Definition 23. Let $A = \{a_1, a_2, a_3, \ldots, a_l\} \subseteq E$, $X = \{x_1, x_2, x_3, \ldots, x_m\}$, and $Y = \{y_1, y_2, y_3, \ldots, y_n\}$ such that each x_j has weight w_j and each y_k has weight w'_k with $\sum_j w_j = 1 \& \sum_k w'_k = 1$. Then, for all i, $(S_i = \overline{\sigma_i} - \underline{\sigma_i})$, where $\overline{\sigma_i} = \sum_{j,k} w_j \zeta^{*}_{ijk}$ and $\underline{\sigma_i} = \sum_{j,k} w'_k \zeta^{\circ}_{ijk}$.

Remark 6. The above stated algorithm (Algorithm 1) also works in the present case.

Example 5. Consider Example 4, with Table 5 representing weight values and Table 6 representing score values.

Then, according to the new criteria, the candidate "a_1" will be selected.

Remark 7. According to Xu [48], the weight vector $w = (w_1, w_2, w_3, \ldots, w_n)^T$ can also be calculated as

$$w_i = \frac{e^{-\left[\left(\left(j-\mu_n\right)^2/2\sigma_n^2\right)\right]}}{\sum_{j=1}^{n} e^{-\left[\left(\left(j-\mu_n\right)^2/2\sigma_n^2\right)\right]}}, \tag{21}$$

where

$$\mu_n = \frac{1+n}{2}, \qquad \sigma_n = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(i-\mu_n\right)^2}. \tag{22}$$

In this case, Example 5 takes the following form (Tables 7 and 8).

Then, in this case, the candidate "a_2" will be selected.

Table 3: Tabular expression of the T-bipolar soft set (F, G, A).

(F, G, A)	(x_1, y_1)	(x_1, y_2)	(x_1, y_3)	(x_2, y_1)	(x_2, y_2)	(x_2, y_3)	(x_3, y_1)	(x_3, y_2)	(x_3, y_3)	(x_4, y_1)	(x_4, y_2)	(x_4, y_3)
a_1	(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 0)	(0, 1)	(1, 0)	(1, 0)	(1, 1)	(0, 0)	(0, 0)	(0, 1)
a_2	(1, 1)	(1, 0)	(1, 0)	(1, 1)	(1, 0)	(1, 0)	(0, 1)	(0, 0)	(0, 0)	(1, 1)	(1, 0)	(1, 0)
a_3	(1, 1)	(1, 0)	(1, 1)	(0, 1)	(0, 0)	(0, 1)	(0, 1)	(0, 0)	(0, 1)	(1, 1)	(1, 0)	(1, 1)
a_4	(0, 0)	(0, 1)	(0, 1)	(1, 0)	(1, 1)	(1, 1)	(1, 0)	(1, 1)	(1, 1)	(1, 0)	(1, 1)	(1, 1)
a_5	(0, 0)	(0, 1)	(0, 0)	(1, 0)	(1, 1)	(1, 0)	(0, 0)	(0, 1)	(0, 0)	(1, 0)	(1, 1)	(1, 0)

Table 4: Scores of a_1, a_2, a_3, a_4, a_5.

(F, G, A)	$\overline{\sigma_i}$	$\underline{\sigma_i}$	S_i
a_1	6	4	2
a_2	9	4	5
a_3	6	8	−2
a_4	9	8	1
a_5	6	4	2

(1 Here we state an algorithm for finding an optimal value for a given data.
Step 1. Write given T-BSS in tabular form.
Step 2. Calculate $S_1, S_2, S_3, \ldots, S_l$.
Step 3. Put $\max_i S_i = S_p$.
Step 4. S_p is optimal value.

Algorithm 1: Finding an optimal value for a given data.

Table 5: Weight values.

x_j	Weight of x_j	y_k	Weight of y_k
x_1	0.3	y_1	0.4
x_2	0.2	y_2	0.4
x_3	0.4	y_3	0.2
x_4	0.1	—	—

Table 6: Scores of a_1, a_2, a_3, a_4, a_5.

(F, G, A)	$\overline{\sigma_i}$	$\underline{\sigma_i}$	S_i
a_1	2.1	0.8	1.3
a_2	1.8	1.6	0.2
a_3	1.2	2.4	−1.2
a_4	2.1	2.4	−0.3
a_5	0.9	1.6	−0.7

Table 7: Weight values.

x_j	Weight of x_j	y_k	Weight of y_k
x_1	0.1550	y_1	0.2429
x_2	0.3450	y_2	0.5142
x_3	0.3450	y_3	0.2429
x_4	0.1550	—	—

Table 8: Scores of a_1, a_2, a_3, a_4, a_5.

(F, G, A)	$\overline{\sigma_i}$	$\underline{\sigma_i}$	S_i
a_1	1.5	0.9716	0.5284
a_2	1.965	0.9716	0.9934
a_3	0.93	1.9432	−1.0132
a_4	2.535	3.0284	−0.4934
a_5	1.5	2.0568	−0.5568

6. Conclusion and Future Prospective

Keeping in view the shortcoming in predefined notions of BSSs, in this article, we have defined and discussed the notion of T-BSS. Then, rendering to new definition, we have defined different binary operations for T-BSSs and then we conferred some results associated with these binary operations. We evidenced the existence of bounded lattices and De Morgan algebras interrelated with these binary operations. We also established some algorithms to solve decision-making problems and then solved the problems from daily life by using these algorithms. In future, this work can be extended to its applications in algebraic structures and in rough set theory.

Data Availability

The data used in this article are artificial and hypothetical, and anyone can use these data before prior permission by just citing this article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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Research Article

Integrated Weighted Distance Measure for Single-Valued Neutrosophic Linguistic Sets and Its Application in Supplier Selection

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The purpose of this study is to propose an integrated distance-based methodology for multiple attribute group decision making (MAGDM) within single-valued neutrosophic linguistic (SVNL) environments. A new SVNL distance measure, namely the SVNL integrated weighted distance (SVNLIWD) measure, is first developed for achieving the aim. The remarkable feature of the SVNLIWD is that it integrates both merits of ordered weighting and average weighting into aggregating SVNL distances; therefore, it can account for both the importance of aggregated deviations as well as ordered positions. Thus, it can highlight the decision makers' subjective risk attitudes and combine the importance of objective decision information. Some distinctive characteristics and special forms of the presented distance framework are then specifically studied. Moreover, a MAGDM model on the basis of the proposed SVNLIWD form is formulated. Finally, an illustrative numerical case regarding selecting low-carbon supplier is used to test the performance of the designed method.

1. Introduction

With the increasing vagueness and uncertainties of objects in multiple attribute group decision making (MAGDM) problems, people may find it more and more difficult to express accurate evaluation on the attributes during decision process. Therefore, it has become a hot issue in decision making areas to research a scientific and reasonable tool for handling such vague and uncertain information. Linguistic term sets [1, 2], intuitionistic fuzzy sets (FSs) [3], hesitant FSs [4], single-valued neutrosophic sets (SVNSs) [5], Pythagorean FSs [6], and spherical FSs [7] emerge at a historic moment in recent years, which have been widely used to express uncertainties or vagueness in various complex decision making situations. The emergence of these methods greatly reduces the pressure of decision makers' depiction of the fuzziness of evaluation objects in the process of decision making.

Generally speaking, due to the complexity of people's judgement and the fuzziness of objective things, people tend to use language terms instead of actual values or fuzzy values. However, the use of linguistic variables usually means that the truth degree of a linguistic term is 1, while the degrees of indeterminacy and falsity cannot be described. This defect hinders its application in decision making problems. To improve this limitation, a new powerful fuzzy tool introduced by Ye [8], called the single-valued neutrosophic linguistic set (SVNLS), has attracted growing concerns from worldwide authors. The key feature of the SVNLS is that it takes advantage of both the linguistic terms and SVNSs, and thus, it can successfully describe the uncertain information comprehensively and reasonably. In addition, it can eliminate the limitations of intuitionistic linguistic set [9] and the Pythagorean linguistic set [10] as it has three membership (i.e., truth, indeterminacy, and falsity) elements, which makes it more suitable to handle a higher degree of imprecise evaluations.

From the latest research trends, it can be seen that the SVNLS is widely used to deal with MAGDM problems in uncertain and complex environments. Guo and Sun [11] gave a SVNL decision making using prospect theory. Zhao et al. [12] developed some induced Choquet integral weighted operators for SVNLS and explored their application in MAGDM. Ye [8] extended the classic TOPSIS to handle SVNL information and investigated its application in selecting investment context. Ye [13] introduced several neutrosophic linguistic aggregation methods and used them to select the flexible operating system supplier. Wang et al. [14] studied the usefulness of Maclaurin symmetric mean technique in aggregating SVNL preferences. Chen et al. [15] presented a new aggregated SVNL distance framework by utilizing the ordered weight technique. Based on the results obtained by Chen et al. [15], Cao et al. [16] introduced a combined SVNL distance measure. Kazimieras et al. [17] constructed a WASPAS model to solve SVNL MAGDM problems. Garg and Nancy [18] introduced some prioritized weighted methods to aggregate SVNL information with priority among the attributes.

In MAGDM problems, it is often necessary to measure the deviations between the alternatives and certain ideal schemes, wherein the construction of the distance measure plays a decisive role. Until now, the weighted distance (WD) and the ordered weighted averaging (OWAD) measures [19] are two most widely used tools for reflecting deviations in practical application. In general, the WD measure can account for the importance of the attributes, while the OWAD measure is helpful to highlight decision makers' risk attitude through the weight designing schemes in the aggregation process. At present, numerous OWAD's extensions and their corresponding usefulness in MAGDM problems have shown an increasing trend in recent research, such as the induced OWAD [20, 21], probabilistic OWAD [22], continuous OWAD [23], intuitionistic fuzzy OWAD [24], hesitant fuzzy OWAD [25, 26], intuitionistic fuzzy weighted induced OWAD [27], and Pythagorean OWAD measures [28, 29]. In particular, Chen et al. [15] defined the single-valued neutrosophic linguistic OWAD (SVNLOWAD) measure and explored its extension with the TOPSIS model for handling MAGDM with SVNL information.

Following the previous literature analysis, one can see that the SVNLS is regarded as a popularized tool, while the OWAD measure is of great strategic significance measurement tool and has shown its advantages in actual use. Therefore, it is a very interesting topic to study the theoretical development and application of OWAD framework in the SVNL context. For doing so, this paper tries to further explore the usefulness of the OWAD in solving SVNL decision making problems. To achieve this aim, we first develop a new distance measure for SVNLSs, named the SVNL integrated weighted distance (SVNLIWD) measure, which is a useful extension of the existing SVNLOWAD measure. Moreover, the SVNLIWD measure can overcome the defects of the SVNLOWAD measure as it unifies the superiority of the weighted distance and ordered weighted distance. Several properties and main families of the proposed distance measures are then explored. A MAGDM framework based on the SVNLIWD measure is constructed and its application is verified.

The remainder of this research is carried out as follows: Section 2 reviews some concepts of SVNLS and the OWAD measure. Section 3 proposes the SVNLIWD measure and explores some of its properties and families. Section 4 mainly describes the usefulness of the proposed SVNLIWD in MAGDM field. In Section 5, feasibility and effectiveness of the presented method are discussed through comparing with existing methods. Finally, Section 6 makes a systematic summary of this paper.

2. Preliminaries

Some important concepts concerning the definitions of the SVNLS, the OWAD, and the SVNLOWAD measures are briefly reviewed in this section.

2.1. Single-Valued Neutrosophic Set (SVNS). To improve the computational efficiency of the neutrosophic set [30], Ye [5] gave the definition of SVNS.

Definition 1 (see [5]). A single-valued neutrosophic set (SVNS) Z in finite set X is denoted by a mathematical form as follows:

$$Z = \left\{ \langle x, T_Z(x), I_Z(x), F_Z(x) \rangle \,\middle|\, , \quad x \in X \right\}, \tag{1}$$

where $T_Z(x)$, $I_Z(x)$, and $F_Z(x)$, respectively, denote the truth, the indeterminacy, and the falsity-membership functions, and they must satisfy the following conditions:

$$\begin{aligned} &0 \le T_Z(x), I_Z(x), F_Z(x) \le 1, \\ &0 \le T_Z(x) + I_Z(x) + F_Z(x) \le 3. \end{aligned} \tag{2}$$

The triplet $(T_Z(x), I_Z(x), F_Z(x))$ is named SVN number (SVNN) and simply described as $Z = (T_Z, I_Z, F_Z)$. Let $y = (T_y, I_y, F_y)$ and $z = (T_z, I_z, F_z)$ be two SVNNs; some mathematical operational rules are given as follows [30]:

(1) $y \oplus z = (T_y + T_z - T_y * T_z, I_y * T_z, F_y * F_z)$
(2) $\lambda y = (1 - (1 - T_y)^\lambda, (I_y)^\lambda, (F_y)^\lambda), \lambda > 0$
(3) $y^\lambda = ((T_y)^\lambda, 1 - (1 - I_y)^\lambda, 1 - (1 - F_y)^\lambda), \lambda > 0$

2.2. Linguistic Set. A linguistic term set S is generally defined as a finitely ordered discrete set $S = \{s_\alpha \mid \alpha = 1, \ldots, l\}$, where l is an odd number and s_α is a possible linguistic term. Let $l = 7$; then, S shall be specified $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\} = \{$*extremely poor, very poor, poor, fair, good, very good, extremely good*$\}$. Let s_i and s_j be two linguistic terms in S, and they should meet the following rules [1]:

(1) $s_i \le s_j \Longleftrightarrow i \le j$
(2) $\mathrm{Neg}(s_i) = s_{-i}$

In practical application, discrete set S shall be extended into a continuous set $\overline{S} = \{s_\alpha \mid \alpha \in R\}$ for minimizing information loss. In this case, for $s_\alpha, s_\beta \in \overline{S}$, they shall meet the following operational laws [31]:

(1) $s_\alpha \oplus s_\beta = s_{\alpha+\beta}$
(2) $\mu s_\alpha = s_{\mu\alpha}$, $\mu \geq 0$
(3) $(s_\alpha / s_\beta) = s_{(\alpha/\beta)}$

2.3. Single-Valued Neutrosophic Linguistic Set (SVNLS)

Definition 2 (see [8]). The mathematical form of a SVNLS in X is described as in

$$P = \left\{ \langle x, \left[s_{\theta(x)}, \left(T_P(x), I_P(x), F_P(x)\right)\right] \rangle \,\middle|\, , \quad x \in X \right\}, \tag{3}$$

where $s_{\theta(x)} \in \overline{S}$, while $T_p(x)$, $I_p(x)$, and $F_p(x)$ have the following constraints:

$$\begin{aligned} &0 \leq T_p(x), I_p(x), F_p(x) \leq 1, \\ &0 \leq T_p(x) + I_p(x) + F_p(x) \leq 3. \end{aligned} \tag{4}$$

For a SVNLS P in X, the SVNL number (SVNLN) $\langle s_{\theta(x)}, (T_P(x), I_P(x), F_P(x))\rangle$ is simply formulated as $x = \langle s_{\theta(x)}, (T_x, I_x, F_x)\rangle$ for the convenience of application. Let $x_i = \langle s_{\theta(x_i)}, (T_{x_i}, I_{x_i}, F_{x_i})\rangle$ $(i = 1, 2)$ be two SVNLNs; then, the following are considered:

(1) $x_1 \oplus x_2 = \langle s_{\theta(x_1)+\theta(x_2)}, (T_{x_1} + T_{x_2} - T_{x_1} * T_{x_2}, I_{x_1} * I_{x_2}, F_{x_1} * F_{x_2})\rangle$
(2) $\lambda x_1 = \langle s_{\lambda\theta(x_1)}, (1 - (1 - T_{x_1})^\lambda, (I_{x_1})^\lambda, (F_{x_1})^\lambda)\rangle, \lambda > 0$
(3) $x_1^\lambda = \langle s_{\theta^\lambda(x_1)}, ((T_{x_1})^\lambda, 1 - (1 - I_{x_1})^\lambda, 1 - (1 - F_{x_1})^\lambda)\rangle, \lambda > 0$

Definition 3 (see [8]). Let $\lambda > 0$; then, the distance measure between SVNLNs $x_i = \langle s_{\theta(x_i)}, (T_{x_i}, I_{x_i}, F_{x_i})\rangle$ $(i = 1, 2)$ is defined as follows:

$$d_{\text{SVNL}}(x_1, x_2) = \left[\left|\theta(x_1)T_{x_1} - \theta(x_2)T_{x_2}\right|^\lambda + \left|\theta(x_1)I_{x_1} - \theta(x_2)I_{x_2}\right|^\lambda + \left|\theta(x_1)F_{x_1} - \theta(x_2)F_{x_2}\right|^\lambda \right]^{(1/\lambda)}. \tag{5}$$

On the basis of Definition 3, the SVNL weighted distance (SVNLWD) measure is formulated in equation (6) if we consider different importance for the individual deviation:

$$\text{SVNLWD}\left((x_1, x_1'), \ldots, (x_n, x_n')\right) = \sum_{j=1}^{n} w_j d_{\text{SVNL}}\left(x_j, x_j'\right), \tag{6}$$

where the relative weight vector W satisfies $w_j \in [0, 1]$ and $\sum_{j=1}^n w_j = 1$.

2.4. OWAD Measure. The OWAD measure introduced by Merigó and Gil-Lafuente [19] is used to characterize individual distances on the basis of the ordered weighted averaging method [32]. Let $\eta = \{\eta_1, \eta_2, \ldots, \eta_n\}$ and $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be two crisp sets and $d_i = |\eta_i - \gamma_i|$ be the distance between the crisp numbers η_i and γ_i; then, we can define the OWAD measure as follows.

Definition 4 (see [19]). An OWAD measure with the weighting vector $W = \left\{w_j \mid \sum_{i=1}^n w_j = 1, \quad 0 \leq w_j \leq 1\right\}$ is defined as

$$\text{OWAD}(A, B) = \text{OWAD}\left(d_1, \ldots, d_n\right) = \sum_{j=1}^{n} w_j d_{\sigma(j)}, \tag{7}$$

where $d_{\sigma(j)} (j = 1, \ldots, n)$ is the reorder values of $d_j (j = 1, \ldots, n)$, such that $d_{\sigma(1)} \geq \ldots \geq d_{\sigma(n)}$.

The OWAD measure is generally effective for crisp sets. In order to adapt the OWAD measure to deal with SVNL information, Chen et al. [15] developed the SVNLOWAD measure.

Definition 5 (see [15]). Let $d_{\text{SVNL}}(x_j, x_j')$ be the deviation between two SVNLNs x_j, x_j' $(j = 1, \ldots, n)$ defined in equation (5); then, SVNLOWAD measure is defined as

$$\text{SVNLOWAD}\left((x_1, x_1'), \ldots, (x_n, x_n')\right) = \sum_{j=1}^{n} w_j d_{\text{SVNL}}\left(x_{\sigma(j)}, x_{(j)}'\right), \tag{8}$$

where $d_{\text{SVNL}}(x_{\sigma(j)}, x_{\sigma(j)}')$ $(j = 1, \ldots, n)$ is the reorder values of $d_{\text{SVNL}}(x_j, x_j')$ $(j = 1, \ldots, n)$ such that $d_{\text{SVNL}}(x_{\sigma(1)}, x_{\sigma(1)}') \geq \ldots \geq d_{\text{SVNL}}(x_{\sigma(n)}, x_{\sigma(n)}')$. $w = (w_1, \ldots, w^n)^T$ is the associated weighting vector of the SVNLOWAD measure, satisfying $\sum_{j=1}^n w_j = 1$ and $w_j \in [0, 1]$.

Chen et al. [15] explored some characteristics of the SVNLOWAD measure, such as commutativity, boundedness, idempotency, and monotonicity. Moreover, they verified its desired performance in solving SVNL MAGDM problems by constructing a new TOPSIS model. However, the SVNLOWAD measure has some shortcomings; that is, it can only integrate the decision makers' special interests but fails to account for the weights of attributes in aggregation outcomes, which goes against its further application. So we shall present a new SVNL distance measure in the next section.

3. SVNL Integrated Weighted Distance (SVNLIWD) Measure

The SVNL integrated weighted distance (SVNLIWD) is a new extension of SVNL distance that unifies both the merits of the SVNLOWAD and the the SVNLWD measures. Therefore, it can highlight the decision makers' attitudes through the ordered weighted arguments and combine the importance of attributes' weights in decision making. Moreover, it enables decision makers the chance to flexibly change the weight ratio of the SVNLWD and the SVNLOWAD according to the demands for the specific problem or actual preferences.

Definition 6. Let $d_{\mathrm{SVNL}}(x_j, x_j')$ be the distance between two SVNLNs $x_j, x_j'(j = 1, \ldots, n)$ described as in equation (5); if

$$\mathrm{SVNLIWD}((x_1, x_1'), \ldots, (x_n, x_n')) = \sum_{j=1}^{n} \overline{w}_j d_{\mathrm{SVNL}}\left(x_{\sigma(j)}, x_{(j)}'\right), \tag{9}$$

then the SVNLIWD is called the SVNL integrated weighted distance measure, where $d_{\mathrm{SVNL}}(x_{\sigma(j)}, x_{\sigma(j)}')$ $(j = 1, \ldots, n)$ is the reorder values of $d_{\mathrm{SVNL}}(x_j, x_j')$ $(j = 1, \ldots, n)$ such that $d_{\mathrm{SVNL}}(x_{\sigma(1)}, x_{\sigma(1)}') \geq \ldots \geq d_{\mathrm{SVNL}}(x_{\sigma(n)}, x_{\sigma(n)}')$. The integrated weight $\overline{w}_j$ is determined by two weight values: one is the weight w_j for the OWA satisfying $\sum_{j=1}^{n} w_j = 1$ and $w_j \in [0, 1]$, and the other is the weight ω_j for weighted average with $\sum_{j=1}^{n} \omega_j = 1$ and $\omega_j \in [0, 1]$. The unified weight $\overline{w}_j(j = 1, \ldots, n)$ is defined as

$$\overline{w}_j = \frac{w_j^{\theta} \omega_{\sigma(j)}^{1-\theta}}{\sum_{j=1}^{n}\left(w_j^{\theta} \omega_{\sigma(j)}^{1-\theta}\right)}, \tag{10}$$

with $\theta \in [0, 1]$ and $\omega_{\sigma(j)}$ is the reordered element of the weight ω_j.

Following the Definition 6, one can see that the SVNLIWD is generalized to the SVNLWD and SVNLOWAD measures when $\theta = 0$ and $\theta = 1$, respectively. Thus, the SVNLIWD measure is a generalized model that unifies the SVNLWD, SVNLOWAD, and many other existing distance measures. A mathematical example is utilized to illustrate the computational process of the SVNLIWD measure.

Example 1. Let $X = (x_1, x_2, x_3, x_4, x_5) = (\langle s_3, (0.6, 0.3, 0.1)\rangle, \langle s_5, (0.2, 0.5, 0.5)\rangle, \langle s_6, (0.7, 0.1, 0.1)\rangle, \langle s_1, (0.6, 0.1, 0.6)\rangle, \langle s_4, (0.3, 0.1, 0.9)\rangle)$ and $X' = (x_1', x_2', x_3', x_4', x_5') = (\langle s_5, (0.2, 0.9, 0)\rangle, \langle s_4, (0.5, 0.7, 0.2)\rangle, \langle s_5, (0.4, 0.4, 0.5)\rangle, \langle s_3, (0.5, 0.7, 0.2)\rangle, \langle s_3, (0.4, 0.2, 0.6)\rangle)$ be two SVNLSs defined in set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$. The weighting vector of SVNLIWD measure is supposed to be $w = (0.3, 0.15, 0.25, 0.2, 0.1)^T$. Then, the computational process through the SVNLUWD can be performed as follows:

(1) Calculate distances $d_{\mathrm{SVNL}}(x_i, x_i')$ $(i = 1, 2, \ldots, 5)$ according to equation (5) (let $\lambda = 1$):

$$\begin{aligned}
d_{\mathrm{SVNL}}(x_1, x_1') &= |3 \times 0.6 - 5 \times 0.2| + |3 \times 0.3 - 5 \times 0.9| + |3 \times 0.1 - 5 \times 0| = 4.7,\\
d_{\mathrm{SVNL}}(x_2, x_2') &= |5 \times 0.2 - 4 \times 0.5| + |5 \times 0.5 - 4 \times 0.7| + |5 \times 0.5 - 4 \times 0.2| = 3,\\
d_{\mathrm{SVNL}}(x_3, x_3') &= |6 \times 0.7 - 5 \times 0.4| + |6 \times 0.1 - 5 \times 0.4| + |6 \times 0.1 - 5 \times 0.5| = 5.5,\\
d_{\mathrm{SVNL}}(x_4, x_4') &= |1 \times 0.6 - 3 \times 0.5| + |1 \times 0.1 - 3 \times 0.7| + |1 \times 0.6 - 3 \times 0.2| = 2.9,\\
d_{\mathrm{SVNL}}(x_5, x_5') &= |4 \times 0.3 - 3 \times 0.4| + |4 \times 0.1 - 3 \times 0.2| + |4 \times 0.9 - 3 \times 0.6| = 2.
\end{aligned} \tag{11}$$

(2) Sort the $d(x_i, x_i')$ $(i = 1, 2, \ldots, 5)$ in nonincreasing order:

$$\begin{aligned}
d_{\mathrm{SVNL}}\left(x_{\sigma(1)}, x_{\sigma(1)}'\right) &= d_{\mathrm{SVNL}}(x_3, x_3') = 5.5,\\
d_{\mathrm{SVNL}}\left(x_{\sigma(2)}, x_{\sigma(2)}'\right) &= d_{\mathrm{SVNL}}(x_1, x_1') = 4.7,\\
d_{\mathrm{SVNL}}\left(x_{\sigma(3)}, x_{\sigma(3)}'\right) &= d_{\mathrm{SVNL}}(x_2, x_2') = 3,\\
d_{\mathrm{SVNL}}\left(x_{\sigma(4)}, x_{\sigma(4)}'\right) &= d_{\mathrm{SVNL}}(x_4, x_4') = 2.9,\\
d_{\mathrm{SVNL}}\left(x_{\sigma(5)}, x_{\sigma(5)}'\right) &= d_{\mathrm{SVNL}}(x_5, x_5') = 2.
\end{aligned} \tag{12}$$

(3) Let $\omega = (0.15, 0.2, 0.1, 0.35, 0.2)^T$ and $\theta = 0.5$; compute the integrated weights $\overline{w}_j$ according to equation (10):

$$\overline{w}_1 = \frac{w_1^{0.5}\omega_{\sigma(1)}^{1-0.5}}{\sum_{j=1}^{n}\left(w_j^{0.5}\omega_{\sigma(j)}^{1-0.5}\right)} = \frac{0.3^{0.5}\times 0.1^{0.5}}{\left(0.3^{0.5}\times 0.1^{0.5} + 0.15^{0.5}\times 0.15^{0.5} + 0.25^{0.5}\times 0.2^{0.5} + 0.2^{0.5}\times 0.35^{0.5} + 0.1^{0.5}\times 0.2^{0.5}\right)} = 0.1791. \quad (13)$$

Similarly, we can obtain

$$\begin{aligned}\overline{w}_2 &= 0.1901,\\ \overline{w}_3 &= 0.2183,\\ \widehat{w}_4 &= 0.2366,\\ \overline{w}_5 &= 0.1757.\end{aligned} \quad (14)$$

(4) Utilize the SVNLIWD given in equation (9) to compute the distance measure between X and X':

$$\begin{aligned}\text{SVNLIWD}(X, X') &= 0.1791\times 5.5 + 0.1901\times 4.7\\ &\quad + 0.2183\times 3 + 0.2366\times 2.9\\ &\quad + 0.1757\times 2 = 3.5719.\end{aligned} \quad (15)$$

If we use the SVNLOWAD and the SVNLWD to perform the aggregation process, we have

$$\begin{aligned}\text{SVNLOWAD}(X, X') &= 0.3\times 5.5 + 0.15\times 4.7 + 0.25\times 3\\ &\quad + 0.2\times 2.9 + 0.1\times 2 = 3.885,\\ \text{SVNLWD}(X, X') &= 0.15\times 4.7 + 0.2\times 3 + 0.1\times 5.5\\ &\quad + 0.35\times 2.9 + 0.2\times 2 = 3.27.\end{aligned} \quad (16)$$

Apparently, we obtain different results from three methods. In fact, the SVNLWD model only considers the importance of the individual deviations, while the SVNLOWAD focuses on the weights of the ordered deviations. The SVNLIWD measure unifies the features of both the SVNLOWAD and the SVNLWD measures, and thus, it can not only highlight the ordered weights of positions but also incorporate deviations' importance.

Moreover, some particular SVNL weighted distance measures can be obtained if we sign different weighted schemes for the SVNLIWD measure:

(i) If $w_1 = 1$ and $w_j = 0$ for $j \in [2, n]$, then we obtain the max-SVNLIWD measure

(ii) If $w_n = 1$ and $w_j = 0$ for $j \in [1, n-1]$, then the min-SVNLIWD measure is constructed

(iii) The step-SVNLIWD measure is formed by signing $w_1 = \ldots = w_{k-1} = 0$, $w_k = 1$, and $w_{k+1} = \ldots = w_n = 0$

(iv) Other special cases of the SVNLIWD can be created by using the similar methods provided in references [15, 33–36]

The SVNLIWD measure is monotonic, bounded, idempotent, and commutative, which can be demonstrated by following theorems.

Theorem 1 (monotonicity). *If $d_{SVNL}(y_i, y_i') \geq d_{SVNL}(x_i, x_i')$ for all i, then the following feature holds:*

$$\begin{aligned}\text{SVNLIWD}((y_1, y_1'), \ldots, (y_n, y_n')) \geq \text{SVNLIWD}((x_1, x_1'),\\ \ldots, (x_n, x_n')).\end{aligned} \quad (17)$$

Theorem 2 (boundedness). *Let $d_{\min} = \min_i\{d_{SVNL}(x_i, x_i')\}$ and $d_{\max} = \max_i\{d_{SVNL}(x_i, x_i')\}$; then,*

$$d_{\min} \leq \text{SVNLIWD}((x_1, x_1'), \ldots, (x_n, x_n')) \leq d_{\max}. \quad (18)$$

Theorem 3 (idempotency). *If $d_{SVNL}(x_i, x_i') = D$ for all i, then*

$$\text{SVNLIWD}((x_1, x_1'), \ldots, (x_n, x_n')) = D. \quad (19)$$

Theorem 4 (commutativity). *This property can also be rendered from the following equation:*

$$\begin{aligned}\text{SVNLIWD}((x_1, x_1'), \ldots, (x_n, x_n')) = \text{SVNLIWD}((x_1', x_1),\\ \ldots, (x_n', x_n)).\end{aligned} \quad (20)$$

It is noted that the proof of these theorems are omitted as they are straightforward.

In addition, we can utilize the generalized mean method [37] to achieve a more generalization for SVNL distance measure, obtaining the SVNL generalized integrated weighted distance (SVNLGIWD) measure:

$$\text{SVNLGIWD}\left((x_1, x_1'), \ldots, (x_n, x_n')\right) = \left\{ \sum_{j=1}^{n} \overline{w}_j \left(d_{\text{SVNL}}\left(x_{\sigma(j)}, x'_{(j)}\right)\right)^p \right\}^{(1/p)}, \quad (21)$$

where p is a parameter that meets $p \in (-\infty, +\infty) - \{0\}$. Several representative cases of the SVNLGIWD measure can be determined based on the variation of parameter p; for example, the SVNLIWD is formed when $p = 1$, the SVNL integrated weighted quadratic distance (SVNLIWQD) is obtained if $p = 2$, and the SVNL integrated weighted harmonic distance (SVNLIWHD) is rendered if $p = -1$. Many other cases of the SVNLGIWD measure can be analyzed by using the similar method provided in references [37–43].

4. Application of SVNLIWD in MAGDM Problems

As a more representative distance measurement method, the SVNLIWD can be broadly used in different areas, such as social management, pattern recognition, decision making, data analysis, medical diagnosis, and financial investment. Subsequently, an application of the SVNLIWD measure in MAGDM is presented within SVNL environments. Let $A = \{A_1, A_2, \ldots, A_n\}$ be a set of finite attributes and $B = \{B_1, B_2, \ldots, B_m\}$ be the set of schemes; then, the decision procedure is summarized as follows.

Step 1. Each expert e_t $(t = 1, 2, \ldots, k)$ (the weight is δ_t with $\delta_t \geq 0$ and $\sum_{t=1}^{k} \delta_t = 1$) expresses his or her evaluation on each attribute of the assessed objects in the form of SVNLNs, thus forming the individual SVNL decision matrix $X^t = (x_{ij}^{(t)})_{m\times n}$.

Step 2. Apply the SVNL weighted average (SVNLWA) operator [8] to aggregate all individual evaluations into a group decision matrix:

$$X = \left(x_{ij}\right)_{m\times n} = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}, \quad (22)$$

where the SVNLN $x_{ij} = \sum_{t=1}^{k} \delta_t x_{ij}^{(t)}$.

Step 3. Determine the ideal gradation of each attribute to construct the ideal solution shown in Table 1.

Step 4. Calculate the deviations between the alternative B_i $(i = 1, 2, \ldots, m)$ and the ideal alternative I by utilizing the SVNLIWD measure.

Step 5. Rank all alternatives and select the best one(s) according to the distances rendered from the previous step.

Step 6. End.

5. Application in Low-Carbon Supplier Selection

The green and low-carbon economic development mode has received more and more attention from the governments and enterprises all over the world. Choosing a suitable low-carbon supplier has become an important issue for the development of enterprises. As a result, many supplier selection methods have been proposed in the existing literature [44, 45]. In this section, a mathematical case of selecting a low-carbon supplier introduced by Chen et al. [15] is used to verify the usefulness of the proposed method. A company invites three experts to evaluate four potential low-carbon suppliers B_i $(i = 1, 2, 3, 4)$ from the following aspects: low-carbon technology (A_1), cost (A_2), risk factor (A_3), and capacity (A_4). The SVNL decision matrices expressed by the experts regarding these four attributes within set $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ are given in Tables 2–4.

The weights of the experts are supposed to be $\delta_1 = 0.30$, $\delta_2 = 0.37$, and $\delta_3 = 0.33$, respectively. The group SVNL decision matrix is then formed by aggregating the three individual opinions, which are listed in Table 5.

According to the actual performance level of these alternative companies, the experts determine the ideal scheme listed in Table 6.

Let the weighting vectors of the SVNLIWD measure and the attributes be $w = (0.15, 0.3, 0.3, 0.25)^T$ and $\omega = (0.2, 0.3, 0.3, 0.2)^T$, respectively. According to the available information, let the parameter $\theta = 0.5$; then, the SVNLIWD can be used to compute the deviations between the alternative B_i $(i = 1, 2, 3, 4)$ and the ideal supplier I:

$$\begin{aligned} \text{SVNLIWD}(I, B_1) &= 5.0563, \\ \text{SVNLIWD}(I, B_2) &= 5.7334, \\ \text{SVNLIWD}(I, B_3) &= 6.5700, \\ \text{SVNLIWD}(I, B_4) &= 6.5798. \end{aligned} \quad (23)$$

The smaller the value of $\text{SVNLIWD}(I, B_i)$ is, the closer the alternative B_i is to the ideal scheme and the better scheme B_i is. Thus, the ranking of all alternatives yields

$$B_1 \succ B_2 \succ B_3 \succ B_4. \quad (24)$$

The results show that B_1 is the most desirable alternative as it possesses the smallest distance from the ideal scheme.

To more effectively show the superiority of the SVNLIWD measure, we also utilize the SVNLOWAD and the SVNLWD measures to calculate the subsequent distances of each alternative to the ideal supplier. For the SVNLOWAD measure, we have

Table 1: Ideal solution.

	A_1	A_2	$\cdots$	A_n
I	$\widetilde{y}_1$	$\widetilde{y}_2$	$\cdots$	$\widetilde{y}_n$

Table 2: SVNL decision matrix X^1.

	A_1	A_2	A_3	A_4
B_1	$\langle s_5^{(3)}, (0.7, 0.2, 0.1)\rangle$	$\langle s_4^{(3)}, (0.5, 0.2, 0.2)\rangle$	$\langle s_3^{(3)}, (0.4, 0.1, 0.1)\rangle$	$\langle s_4^{(3)}, (0.6, 0.1, 0.2)\rangle$
B_2	$\langle s_6^{(3)}, (0.4, 0.6, 0.2)\rangle$	$\langle s_4^{(3)}, (0.7, 0.2, 0.2)\rangle$	$\langle s_5^{(3)}, (0.7, 0.2, 0.1)\rangle$	$\langle s_5^{(3)}, (0.5, 0.2, 0.3)\rangle$
B_3	$\langle s_4^{(3)}, (0.3, 0.6, 0.2)\rangle$	$\langle s_5^{(3)}, (0.6, 0.1, 0.3)\rangle$	$\langle s_4^{(3)}, (0.6, 0.2, 0.1)\rangle$	$\langle s_6^{(3)}, (0.5, 0.1, 0.3)\rangle$
B_4	$\langle s_4^{(3)}, (0.5, 0.2, 0.3)\rangle$	$\langle s_6^{(3)}, (0.6, 0.2, 0.4)\rangle$	$\langle s_5^{(3)}, (0.2, 0.1, 0.6)\rangle$	$\langle s_6^{(3)}, (0.5, 0.2, 0.3)\rangle$

Table 3: SVNL decision matrix X^2.

	A_1	A_2	A_3	A_4
B_1	$\langle s_6^{(1)}, (0.6, 0.1, 0.2)\rangle$	$\langle s_4^{(1)}, (0.6, 0.1, 0.2)\rangle$	$\langle s_3^{(1)}, (0.3, 0.1, 0.2)\rangle$	$\langle s_5^{(1)}, (0.7, 0.0, 0.1)\rangle$
B_2	$\langle s_3^{(1)}, (0.6, 0.2, 0.4)\rangle$	$\langle s_5^{(1)}, (0.6, 0.1, 0.2)\rangle$	$\langle s_4^{(1)}, (0.5, 0.2, 0.2)\rangle$	$\langle s_6^{(1)}, (0.6, 0.1, 0.2)\rangle$
B_3	$\langle s_5^{(1)}, (0.3, 0.5, 0.2)\rangle$	$\langle s_4^{(1)}, (0.5, 0.2, 0.3)\rangle$	$\langle s_3^{(1)}, (0.5, 0.3, 0.1)\rangle$	$\langle s_4^{(1)}, (0.3, 0.2, 0.3)\rangle$
B_4	$\langle s_4^{(1)}, (0.5, 0.3, 0.3)\rangle$	$\langle s_5^{(1)}, (0.4, 0.2, 0.3)\rangle$	$\langle s_3^{(1)}, (0.3, 0.2, 0.5)\rangle$	$\langle s_5^{(1)}, (0.4, 0.2, 0.3)\rangle$

Table 4: SVNL decision matrix X^3.

	A_1	A_2	A_3	A_4
B_1	$\langle s_6^{(2)}, (0.6, 0.3, 0.3)\rangle$	$\langle s_5^{(2)}, (0.7, 0.2, 0.3)\rangle$	$\langle s_4^{(2)}, (0.4, 0.2, 0.2)\rangle$	$\langle s_4^{(2)}, (0.8, 0.1, 0.2)\rangle$
B_2	$\langle s_4^{(2)}, (0.5, 0.4, 0.2)\rangle$	$\langle s_6^{(2)}, (0.7, 0.2, 0.3)\rangle$	$\langle s_5^{(2)}, (0.6, 0.2, 0.2)\rangle$	$\langle s_6^{(2)}, (0.7, 0.2, 0.3)\rangle$
B_3	$\langle s_5^{(2)}, (0.4, 0.4, 0.1)\rangle$	$\langle s_6^{(2)}, (0.6, 0.3, 0.4)\rangle$	$\langle s_4^{(2)}, (0.6, 0.1, 0.3)\rangle$	$\langle s_6^{(2)}, (0.4, 0.2, 0.4)\rangle$
B_4	$\langle s_3^{(2)}, (0.7, 0.1, 0.1)\rangle$	$\langle s_6^{(2)}, (0.5, 0.1, 0.2)\rangle$	$\langle s_5^{(2)}, (0.3, 0.1, 0.6)\rangle$	$\langle s_5^{(2)}, (0.4, 0.3, 0.4)\rangle$

Table 5: Group SVNL decision matrix R.

	A_1	A_2	A_3	A_4
C_1	$\langle s_{5.70}, (0.633, 0.180, 0.186)\rangle$	$\langle s_{4.33}, (0.611, 0.155, 0.229)\rangle$	$\langle s_{3.67}, (0.365, 0.128, 0.163)\rangle$	$\langle s_{4.37}, (0.714, 0.000, 0.155)\rangle$
C_2	$\langle s_{4.23}, (0.514, 0.350, 0.258)\rangle$	$\langle s_{4.70}, (0.666, 0.155, 0.229)\rangle$	$\langle s_{2.37}, (0.602, 0.200, 0.162)\rangle$	$\langle s_{5.70}, (0.611, 0.155, 0.258)\rangle$
C_3	$\langle s_{4.70}, (0.335, 0.491, 0.159)\rangle$	$\langle s_{4.96}, (0.566, 0.186, 0.330)\rangle$	$\langle s_{3.37}, (0.566, 0.185, 0.144)\rangle$	$\langle s_{5.26}, (0.399, 0.163, 0.330)\rangle$
C_4	$\langle s_{3.67}, (0.578, 0.185, 0.209)\rangle$	$\langle s_{5.63}, (0.450, 0.159, 0.286)\rangle$	$\langle s_{2.37}, (0.271, 0.129, 0.561)\rangle$	$\langle s_{5.30}, (0.432, 0.229, 0.330)\rangle$

Table 6: Ideal scheme.

	A_1	A_2	A_3	A_4
I	$\langle s_7, (0.9, 0.1, 0)\rangle$	$\langle s_7, (1, 0, 0.1)\rangle$	$\langle s_7, (0.9, 0, 0.1)\rangle$	$\langle s_6, (0.9, 0, 0)\rangle$

$$\begin{aligned}
\text{SVNLOWAD}(I, B_1) &= 5.0171,\\
\text{SVNLOWAD}(I, B_2) &= 5.6742,\\
\text{SVNLOWAD}(I, B_3) &= 6.5613,\\
\text{SVNLOWAD}(I, B_4) &= 6.6086.
\end{aligned} \tag{25}$$

And for the SVNLWD measure, we have

$$\begin{aligned}
\text{SVNLWD}(I, B_1) &= 5.1268,\\
\text{SVNLWD}(I, B_2) &= 5.8078,\\
\text{SVNLWD}(I, B_3) &= 6.6038,\\
\text{SVNLWD}(I, B_4) &= 6.5743.
\end{aligned} \tag{26}$$

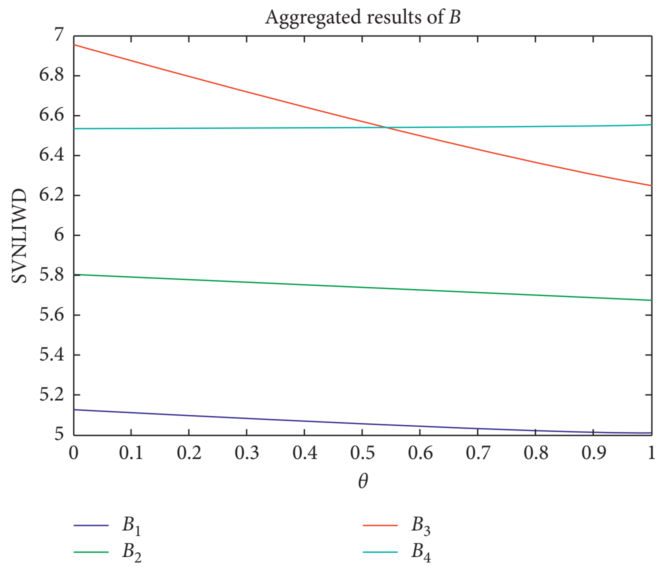

Figure 1: The performance of alternatives rendered by the SVNLIWD measure with θ.

From the results, one can find that B_1 is the best choice for both the SVNLOWAD and SVNLWD measures, which is consistent with the result obtained by the SVNLIWD measure. However, from the comparisons with the SVNLWD and SVNLOWAD measures discussed in the previous example, one can see that the SVNLIWD measure can not only overcome the SVNLWD's disadvantage of just considering the importance of attributes but also make up for the SVNLOWAD's defects of only reflecting expert's risk preference but fails to integrate attributes' weights; therefore, it can yield a more reasonable result. Furthermore, the SVNLIWD-based MAGDM method will not be affected by the parameter θ change, which can be verified by Figure 1.

Following the results from Figure 1, the best alternative is A_1 for all $\theta \in [0, 1]$. It shows that the variation of parameter θ will not affect the final integration results; that is, the MAGDM approach based on the SVNLIWD will not be affected by the parameter variation. Thus, the proposed method has certain stability and robustness.

6. Conclusions

This paper introduces a new integrated aggregation distance method for handling single-valued neutrosophic linguistic MAGDM problems. Thus, we obtain the SVNL integrated weighted (SVNLIWD) measure. Given that the presented distance measure generalizes both advantages of the arithmetic weight and ordered weight approaches during aggregating process, the importance for separate attributes and attitudes towards ordered deviations is taken into account. Moreover, the SVNLIWD measure generalizes a wide type of SVNL distance measures, such as the SVNLWD and the SVNLOWAD measures. Therefore, it provides a much wider model to solve complex situations in a more efficient and flexible way, which further illustrates the promotion of the previous methods. The application of the proposed model is taken to deal with the supplier selection problem, which demonstrates that the presented methodology can consider capricious decision makers' preferences as well as the different importance of attributes during the decision process. Finally, we verify that the presented SVNLIWD-based MAGDM method will not be affected by the parameter variation. Therefore, this method has certain stability and robustness and can achieve more accurate results.

In future work, both extensions of mathematical formula and application in different areas will be considered. Various variables can be considered in the SVNLIWD for future analysis, such as the induced variables, heavy aggregation, and q-rung orthopair fuzzy set [46]. Also, the method of entropy will be considered to account for the weighting schemes.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Research Article

Extensions of Dombi Aggregation Operators for Decision Making under *m*-Polar Fuzzy Information

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An m-polar fuzzy set is a powerful mathematical model to analyze multipolar, multiattribute, and multi-index data. The m-polar fuzzy sets have appeared as a useful tool to portray uncertainty in multiattribute decision making. The purpose of this article is to analyze the aggregation operators under the m-polar fuzzy environment with the help of Dombi norm operations. In this article, we develop some averaging and geometric aggregation operators using Dombi t-norm and t-conorm to handle uncertainty in m-polar fuzzy (mF, henceforth) information, which are mF Dombi weighted averaging (mFDWA) operator, mF Dombi ordered weighted averaging (mFDOWA) operator, mF Dombi hybrid averaging (mFDHA) operator, mF Dombi weighted geometric (mFDWG) operator, mF Dombi weighted ordered geometric operator, and mF Dombi hybrid geometric (mFDHG) operator. We investigate properties, namely, idempotency, monotonicity, and boundedness, for the proposed operators. Moreover, we give an algorithm to solve multicriteria decision-making issues which involve mF information with mFDWA and mFDWG operators. To prove the validity and feasibility of the proposed model, we solve two numerical examples with our proposed models and give comparison with mF-ELECTRE-I approach (Akram et al. 2019) and mF Hamacher aggregation operators (Waseem et al. 2019). Finally, we check the effectiveness of the developed operators by a validity test.

1. Introduction

Multicriteria decision making (MCDM) is performing a vital role in different areas, including social, physical, medical, and environmental sciences. MCDM methods are not only used to determine a suitable object but also used to rank the objects in an appointed problem. To solve different uncertain problems for decision making, Atanassov [1] presented the concept of intuitionistic fuzzy set (IFS) which considers both membership and nonmembership parts, an extension of fuzzy set [2] in which simple membership part is characterized.

Aggregation operators (AOs) perform an important role in order to combine data into a single form and solve MCDM problems. For example, Yager [3] introduced weighted AOs. Xu [4] proposed some new AOs under IFSs. Xu and Yager [5] developed certain new geometric AOs and solved some real-world MCDM problems. From the inspection of an object, it can be easily seen that there exist two properties of the object which are opposite to each other. With this perspective, Zhang [6] presented the idea of bipolar fuzzy set (BFS). BFSs provide generalized structure as compared to fuzzy sets [2] whose memberships belong to $[-1, 0] \times [0, 1]$. Bipolarity plays an important role in different research areas and provides more flexibility as compared to the fuzzy methods. In the last decades, a lot of researchers, attracted by this efficient concept, applied it to aggregate bipolar information using different t-norms and their corresponding conorms, including Hamacher and Dombi t-norms and their corresponding conorms. For example, Wei et al. [7] developed some bipolar fuzzy Hamacher weighted averaging and geometric AOs. By combining Hamacher operations and prioritized AOs, Gao et al. [8] proposed dual hesitant bipolar fuzzy Hamacher prioritized weighted AOs and applied the proposed methodologies to an MCDM problem. Liu [9] utilized interval-valued intuitionistic fuzzy numbers with Hamacher AOs and developed multicriteria methods for group decision making. Jana et al. [10] applied weighted, ordered weighted, and hybrid average and geometric AOs for the aggregation of bipolar fuzzy information using Dombi t-conorm and t-norm. They also proposed bipolar fuzzy Dombi prioritized AOs in [11]. He [12] developed hesitant fuzzy Dombi AOs and investigated typhoon disaster assessment using proposed theory. Xu and Wei [13] introduced different dual hesitant bipolar fuzzy AOs to solve MCDM problems. Xu [14] proposed intuitionistic fuzzy power AOs for multiattribute group decision making. Xiao [15] constructed induced interval-valued intuitionistic fuzzy Hamacher AOs and discussed their application to MCDM. Chen and Ye [16] discussed MCDM problem under Dombi operations in single-valued neutrosophic situation. Garg [17] presented some generalized interactive AOs under Einstein operations in Pythagorean fuzzy environment and discussed a decision-making issue. Akram et al. [18] proposed different Pythagorean Dombi fuzzy AOs and studied their applications in MCDM. Shahzadi et al. [19] introduced Pythagorean fuzzy Yager AOs for decision making. Peng and Yang [20] investigated different basic properties of interval-valued Pythagorean fuzzy AOs. Wang et al. [21] introduced some new types of q-rung orthopair fuzzy Hamy mean AOs to handle MCDM situations. Arora and Garg [22] proposed robust AOs with an intuitionistic fuzzy soft environment. Wang and Li [23] developed Pythagorean fuzzy interaction power Bonferroni mean AOs and discussed their applications to MCDM. Chiclana et al. [24] introduced some ordered weighted geometric operators and solved a decision-making problem. Liang et al. [25] developed Pythagorean fuzzy Bonferroni mean AOs.

Nowadays, experts believe that multipolarity performs a vital role in many practical situations. Due to the presence of multipolar data in different daily life problems of science and technology, Chen et al. [26] initiated the notion of mF set theory as generalization of fuzzy and bipolar fuzzy sets. Waseem et al. [27] studied recently MCDM problems based on mF Hamacher AOs. Khameneh and Kilicman [28] proposed mF soft weighted AOs and applied these AOs in decision making. In view of the fact that mF sets have an efficient strength to handle vague data which arise in several real-life problems, in this paper, we generalize Dombi AOs to aggregate the mF information. The study of AOs under Dombi operations is very popular. Thus, an efficient research topic is how to aggregate mF numbers with Dombi operations. To tackle this dilemma, in this article, we present some mF Dombi AOs on the ground of classical geometric, arithmetic, and Dombi operations. For more information and terminologies on AOs, the readers are referred to [29–45].

An mF model is more general than the fuzzy sets and BFSs due to the wider range of applicability over different complex problems. The mF sets can handle much more details about an element and can explain uncertainties concurrently more precisely than the other existing methods, like fuzzy set and BFS. The motivation of developed AOs is summarized as below.

(1) A very difficult MCDM problem is the estimation of the supreme option in an mF environment due to the involvement of several imprecise factors. Assessment of information in different MCDM techniques is simply depicted through fuzzy and bipolar fuzzy numbers which may not consider all the data in a real-world problem.

(2) As a general theory, mF numbers describe efficient execution in the assessment process about uncertain, imprecise, and vague multipolar information. Thus, mF theory provides an excellent approach for the assessment of objects under multinary data.

(3) In view of the fact that Dombi AOs are simple but provide a pioneering tool for solving MCDM problems when combined with other powerful mathematical tools, this article aims to develop Dombi AOs in an mF environment to handle complex problems.

(4) An mF model is different from the mathematical tools like fuzzy sets and BFSs because the fuzzy set and BFS can only handle one-dimensional data and two-dimensional data, respectively, which may prompt a loss in data. Nevertheless, in many daily life problems, we handle the situations having higher dimension to sort out all the attributes and their subcharacteristics.

(5) The Dombi AOs employed in the construction of mF Dombi AOs are more suitable than all other aggregation approaches to tackle the MCDM situations as developed AOs have ability to consider all the information within the aggregation procedure.

(6) Dombi AOs make the optimal outcomes more accurate and definite when utilized in practical MCDM problems under mF environment.

(7) The proposed mF Dombi operators handle the drawbacks of existing AOs, including bipolar fuzzy Dombi AOs [10].

Therefore, some mF Dombi AOs are developed to choose the best option in different decision-making situations. The developed operators have some advantages over other approaches which are given as follows:

(1) Our proposed methods explain the problems more accurately which involve multiple attributes because they consider mF numbers.

(2) The developed AOs are more precise and efficient with single attribute.

(3) To solve practical problems by using Dombi AOs with mF numbers is very significant.

The rest of this article is structured as follows. Section 2 recalls some fundamental definitions and operations of the mF numbers (mFNs). Section 3 presents mFDWA, mFDOWA, mFDHA, mFDWG, mFDOWG, and mFDHG operators. Section 4 develops a methodology of these AOs to model mF MCDM problems. Section 5 discusses two applications: first for the selection of best agricultural land and second for the selection of best commercial bank. Section 6 provides comparative analysis of developed approaches with mF ELECTRE-I model [46] and mF Hamacher AOs [27]. Section 7 discusses the conclusions and future directions.

2. Preliminaries

Definition 1 (see [26]). An mF set on a universal set U is a mapping $\zeta\colon U \longrightarrow [0,1]^m$. The membership of every object is described by $\zeta(u) = (p_1\circ\zeta(u), p_2\circ\zeta(u), \ldots, p_m\circ\zeta(u))$ where $p_r\circ\zeta\colon [0,1]^m \longrightarrow [0,1]$ is the r-th projection mapping.

Let $\widehat{\zeta} = (p_1\circ\zeta, \ldots, p_m\circ\zeta)$ be an mFN, where $p_r\circ\zeta \in [0,1]$, $\forall r = 1,2,\ldots,m$. We define the score and accuracy functions of $\widehat{\zeta}$, respectively, as follows.

Definition 2 (see [27]). For an mFN $\widehat{\zeta} = (p_1\circ\zeta, \ldots, p_m\circ\zeta)$, we define a score function S as follows:

$$S(\widehat{\zeta}) = \frac{1}{m}\left(\sum_{r=1}^{m}(p_r\circ\zeta)\right), \quad S(\widehat{\zeta}) \in [0,1]. \tag{1}$$

Definition 3 (see [27]). For an mFN $\widehat{\zeta} = (p_1\circ\zeta, \ldots, p_m\circ\zeta)$, an accuracy function H is defined as

$$H(\widehat{\zeta}) = \frac{1}{m}\left(\sum_{r=1}^{m}(-1)^{r+1}(p_r\circ\zeta - 1)\right), \quad H(\widehat{\zeta}) \in [-1,1]. \tag{2}$$

From Definitions 2 and 3, it can be readily seen that for any mFN $\widehat{\zeta}$, $S(\widehat{\zeta}) \in [0,1]$ and $H(\widehat{\zeta}) \in [-1,1]$. Notice that $H(\widehat{\zeta})$ represents the accuracy degree of $\widehat{\zeta}$. Thus, a higher value of $H(\widehat{\zeta})$ represents a higher accuracy degree for mFN $\widehat{\zeta}$.

Using Definitions 2 and 3, we now give the following ordered relation criteria for any two mFNs.

Definition 4 (see [27]). Let $\widehat{\zeta}_1 = (p_1\circ\zeta_1, \ldots, p_m\circ\zeta_1)$ and $\widehat{\zeta}_2 = (p_1\circ\zeta_2, \ldots, p_m\circ\zeta_2)$ be two mFNs. Then,

(1) $\widehat{\zeta}_1 < \widehat{\zeta}_2$, if $S(\widehat{\zeta}_1) < S(\widehat{\zeta}_2)$.

(2) $\widehat{\zeta}_1 > \widehat{\zeta}_2$, if $S(\widehat{\zeta}_1) > S(\widehat{\zeta}_2)$.

(3) $\widehat{\zeta}_1 = \widehat{\zeta}_2$, if $S(\widehat{\zeta}_1) = S(\widehat{\zeta}_2)$ and $H(\widehat{\zeta}_1) = H(\widehat{\zeta}_2)$.

(4) $\widehat{\zeta}_1 < \widehat{\zeta}_2$, if $S(\widehat{\zeta}_1) = S(\widehat{\zeta}_2)$, but $H(\widehat{\zeta}_1) < H(\widehat{\zeta}_2)$.

(5) $\widehat{\zeta}_1 > \widehat{\zeta}_2$, if $S(\widehat{\zeta}_1) = S(\widehat{\zeta}_2)$, but $H(\widehat{\zeta}_1) > H(\widehat{\zeta}_2)$.

Some basic operations for mFNs are given by [27]

(1) $\widehat{\zeta}_1 \boxplus \widehat{\zeta}_2 = (p_1\circ\zeta_1 + p_1\circ\zeta_2 - p_1\circ\zeta_1.p_1\circ\zeta_2, \ldots, p_m\circ\zeta_1 + p_m\circ\zeta_2 - p_m\circ\zeta_1.p_m\circ\zeta_2)$.

(2) $\widehat{\zeta}_1 \boxtimes \widehat{\zeta}_2 = (p_1\circ\zeta_1.p_1\circ\zeta_2, \ldots, p_m\circ\zeta_1.p_m\circ\zeta_2)$.

(3) $\beta\widehat{\zeta} = (1-(1-p_1\circ\zeta)^\beta), \ldots, 1-(1-p_m\circ\zeta)^\beta)$, $\beta > 0$.

(4) $(\widehat{\zeta})^\beta = ((p_1\circ\zeta)^\beta, \ldots, (p_m\circ\zeta)^\beta)$, $\beta > 0$.

(5) $\widehat{\zeta}^c = (1-p_1\circ\zeta, \ldots, 1-p_m\circ\zeta)$.

(6) $\widehat{\zeta}_1 \subseteq \widehat{\zeta}_2$, if and only if $p_1\circ\zeta_1 \le p_1\circ\zeta_2, \ldots, p_m\circ\zeta_1 \le p_m\circ\zeta_2$.

(7) $\widehat{\zeta}_1 \cup \widehat{\zeta}_2 = (\max(p_1\circ\zeta_1, p_1\circ\zeta_2), \ldots, \max(p_m\circ\zeta_1, p_m\circ\zeta_2))$.

(8) $\widehat{\zeta}_1 \cap \widehat{\zeta}_2 = (\min(p_1\circ\zeta_1, p_1\circ\zeta_2), \ldots, \min(p_m\circ\zeta_1, p_m\circ\zeta_2))$.

Theorem 1 (see [27]). *For two mFNs $\widehat{\zeta}_1 = (p_1\circ\zeta_1, \ldots, p_m\circ\zeta_1(u))$ and $\widehat{\zeta}_2 = (p_1\circ\zeta_2, \ldots, p_m\circ\zeta_2)$ with $\beta, \beta_1, \beta_2 > 0$, we have*

(1) $\widehat{\zeta}_1 \boxplus \widehat{\zeta}_2 = \widehat{\zeta}_2 \boxplus \widehat{\zeta}_1$.

(2) $\widehat{\zeta}_1 \boxtimes \widehat{\zeta}_2 = \widehat{\zeta}_2 \boxtimes \widehat{\zeta}_1$.

(3) $\beta(\widehat{\zeta}_1 \boxplus \widehat{\zeta}_2) = \beta(\widehat{\zeta}_1) \boxplus \beta(\widehat{\zeta}_2)$.

(4) $(\widehat{\zeta}_1 \boxtimes \widehat{\zeta}_2)^\beta = (\widehat{\zeta}_1)^\beta \boxplus (\widehat{\zeta}_2)^\beta$.

(5) $\beta_1\widehat{\zeta}_1 \boxplus \beta_2\widehat{\zeta}_1 = (\beta_1+\beta_2)\widehat{\zeta}_1$.

(6) $(\widehat{\zeta}_1)^{\beta_1} \boxtimes (\widehat{\zeta}_2)^{\beta_2} = (\widehat{\zeta}_1)^{\beta_1+\beta_2}$.

(7) $((\widehat{\zeta}_1)^{\beta_1})^{\beta_2} = (\widehat{\zeta}_1)^{\beta_1\beta_2}$.

Dombi [47] proposed operations, namely, Dombi sum $\oplus$ and Dombi product $\otimes$, which are, respectively, t-conorm and t-norm given by

$$D^*(a,b) = a\oplus b = 1 - \frac{1}{1+\left\{(a/1-a)^k + (b/1-b)^k\right\}^{1/k}},$$

$$D(a,b) = a\otimes b = \frac{1}{1+\left\{(1-a/a)^k + (1-b/b)^k\right\}^{1/k}}, \tag{3}$$

where $k \ge 1$ and $a,b \in [0,1]$.

3. mF Dombi AOs

In this section, we first give Dombi operations for mFNs via Dombi t-conorm and Dombi t-norm and then we present mF Dombi arithmetic and geometric AOs. Let $\widehat{\zeta}_1 = (p_1\circ\zeta_1, \ldots, p_m\circ\zeta_1)$, $\widehat{\zeta}_2 = (p_1\circ\zeta_2, \ldots, p_m\circ\zeta_2)$ and $\widehat{\zeta} = (p_1\circ\zeta, \ldots, p_m\circ\zeta)$ be mFNs. We give some fundamental Dombi operations of mFNs as follows:

$$\begin{aligned}
\widehat{\zeta}_1 \oplus \widehat{\zeta}_2 &= \left(1 - \frac{1}{1 + \left\{\left(p_1 \circ \zeta_1/1 - p_1 \circ \zeta_1\right)^k + \left(p_1 \circ \zeta_2/1 - p_1 \circ \zeta_2\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\left(p_m \circ \zeta_1/1 - p_m \circ \zeta_1\right)^k + \left(p_m \circ \zeta_2/1 - p_m \circ \zeta_2\right)^k\right\}^{1/k}}\right),\\
\widehat{\zeta}_1 \otimes \widehat{\zeta}_2 &= \left(\frac{1}{1 + \left\{\left(1 - p_1 \circ \zeta_1/p_1 \circ \zeta_1\right)^k + \left(1 - p_1 \circ \zeta_2/p_1 \circ \zeta_2\right)^k\right\}^{1/k}}, \ldots, \frac{1}{1 + \left\{\left(1 - p_m \circ \zeta_1/p_m \circ \zeta_1\right)^k + \left(1 - p_m \circ \zeta_2/p_m \circ \zeta_2\right)^k\right\}^{1/k}}\right),\\
\beta\widehat{\zeta} &= \left(1 - \frac{1}{1 + \left\{\beta\left(p_1 \circ \zeta/1 - p_1 \circ \zeta\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\beta\left(p_m \circ \zeta/1 - p_m \circ \zeta\right)^k\right\}^{1/k}}\right),\\
(\widehat{\zeta})^\beta &= \left(1 - \frac{1}{1 + \left\{\beta\left(1 - p_1 \circ \zeta/p_1 \circ \zeta\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\beta\left(1 - p_m \circ \zeta/p_m \circ \zeta\right)^k\right\}^{1/k}}\right),
\end{aligned} \tag{4}$$

where $k > 0$.

3.1. mF Dombi Arithmetic AOs. We present mF Dombi arithmetic AOs as follows.

Definition 5. For a collection of mFNs $\widehat{\zeta}_j = (p_1 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$ where $j = 1, 2, \ldots, n$, a mapping from $\widehat{\zeta}^n$ to $\widehat{\zeta}$ is called an mFDWA operator, which is given by

$$m\text{FDWA}_\Theta\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) = \oplus_{j=1}^n \left(\Theta_j \widehat{\zeta}_j\right), \tag{5}$$

where $\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n)^T$ denotes the weights of $\widehat{\zeta}_j$, $\forall j = 1, \ldots, n$ and $\Theta_j > 0$ with $\sum_{j=1}^n \Theta_j = 1$.

We give the following theorem, which is used to apply the Dombi operations on mFNs.

Theorem 2. *For a collection of mFNs $\widehat{\zeta}_j = (p_1 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$ where $j = 1, 2, \ldots, n$, an accumulated value of these mFNs using the mFDWA operators is defined as*

$$\begin{aligned}
m\text{FDWA}_\Theta\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) &= \oplus_{j=1}^n \left(\Theta_j \widehat{\zeta}_j\right),\\
&= \left(1 - \frac{1}{1 + \left\{\sum_{j=1}^n \Theta_j \left(p_1 \circ \zeta_j/1 - p_1 \circ \zeta_j\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\sum_{j=1}^n \Theta_j \left(p_m \circ \zeta_j/1 - p_m \circ \zeta_j\right)^k\right\}^{1/k}}\right).
\end{aligned} \tag{6}$$

Proof. We utilize the induction approach to show it.

Case 1. For $n = 1$, by equation (6), we obtain

$$\begin{aligned}
m\text{FDWA}_\Theta\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) &= \Theta_1 \widehat{\zeta}_1 = \widehat{\zeta}_1, \quad (\text{since } \Theta_1 = 1)\\
&= \left(1 - \frac{1}{1 + \left\{\left(p_1 \circ \zeta_1/1 - p_1 \circ \zeta_1\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\left(p_m \circ \zeta_1/1 - p_m \circ \zeta_1\right)^k\right\}^{1/k}}\right).
\end{aligned} \tag{7}$$

Hence, equation (6) satisfies when $n = 1$.

Case 2. Now, we presume that equation (6) satisfies for $n = t$; here t is an arbitrary natural number; then,

$$\begin{aligned}
m\text{FDWA}_\Theta\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_t\right) &= \oplus_{j=1}^t \left(\Theta_j \widehat{\zeta}_j\right),\\
&= \left(1 - \frac{1}{1 + \left\{\sum_{j=1}^t \Theta_j \left(p_1 \circ \zeta_j/1 - p_1 \circ \zeta_j\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\sum_{j=1}^t \Theta_j \left(p_m \circ \zeta_j/1 - p_m \circ \zeta_j\right)^k\right\}^{1/k}}\right).
\end{aligned} \tag{8}$$

For $n = t + 1$,

$$
\begin{aligned}
m\mathrm{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_t,\widehat{\zeta}_{t+1}\right) &= \oplus_{j=1}^{t}\left(\Theta_j\widehat{\zeta}_j\right)\oplus\left(\Theta_{t+1}\widehat{\zeta}_{t+1}\right),\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{t}\Theta_j\left(p_1\circ\zeta_j/1-p_1\circ\zeta_j\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{t}\Theta_j\left(p_m\circ\zeta_j/1-p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right)\\
&\oplus\left(1-\frac{1}{1+\left\{\Theta_{t+1}\left(p_1\circ\zeta_{t+1}/1-p_1\circ\zeta_{t+1}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\Theta_{t+1}\left(p_m\circ\zeta_{t+1}/1-p_m\circ\zeta_{t+1}\right)^k\right\}^{1/k}}\right)\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{t+1}\Theta_j\left(p_1\circ\zeta_j/1-p_1\circ\zeta_j\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{t+1}\Theta_j\left(p_m\circ\zeta_j/1-p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right).
\end{aligned}
\tag{9}
$$

Therefore, equation (6) satisfies for $n = t + 1$. Hence, we deduce that equation (6) satisfies for every natural number n. □

Example 1. Let $\widehat{\zeta}_1 = (0.4, 0.3, 0.8), \widehat{\zeta}_2 = (0.3, 0.5, 0.1), \widehat{\zeta}_3 = (0.7, 0.2, 0.4)$, and $\widehat{\zeta}_4 = (0.5, 0.4, 0.6)$ be 3FNs and $\Theta = (0.2, 0.3, 0.1, 0.4)^T$ be weights related to these 3FNs. Then, for $k = 3$,

$$
\begin{aligned}
m\mathrm{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\widehat{\zeta}_3\right) &= \oplus_{j=1}^{3}\left(\Theta_j\widehat{\zeta}_j\right)\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(p_1\circ\zeta_j/1-p_1\circ\zeta_j\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(p_m\circ\zeta_j/1-p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right)\\
&= \left(1-\frac{1}{1+\left(0.2\times(0.4/1-0.4)^3+0.3\times(0.3/1-0.3)^3+0.1\times(0.7/1-0.7)^3+0.4\times(0.5/1-0.5)^3\right)^{1/3}},\right.\\
&\quad 1-\frac{1}{1+\left(0.2\times(0.3/1-0.3)^3+0.3\times(0.5/1-0.5)^3+0.1\times(0.2/1-0.2)^3+0.4\times(0.4/1-0.4)^3\right)^{1/3}},\\
&\quad \left.1-\frac{1}{1+\left(0.2\times(0.8/1-0.8)^3+0.3\times(0.1/1-0.1)^3+0.1\times(0.4/1-0.4)^3+0.4\times(0.6/1-0.6)^3\right)^{1/3}}\right),\\
&= (0.5467, 0.4312, 0.7076).
\end{aligned}
\tag{10}
$$

We now explore some useful laws of mFDWA operators as follows.

Theorem 3 (idempotent law). *Let $\widehat{\zeta}_j = (p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ be a family of "n" mFNs, which are equal, i.e., $\widehat{\zeta}_j = \widehat{\zeta}$; then,*

$$
m\mathrm{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right) = \widehat{\zeta}. \tag{11}
$$

Proof. Since $\widehat{\zeta}_j = (p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j) = \widehat{\zeta}$, where $j = 1,\ldots,n$, then by equation (6),

$$
\begin{aligned}
m\text{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right) &= \oplus_{j=1}^{n}\left(\Theta_j\widehat{\zeta}_j\right),\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(p_1\circ\zeta_j/1-p_1\circ\zeta_j\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(p_m\circ\zeta_j/1-p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right),\\
&= \left(1-\frac{1}{1+\left\{\left(p_1\circ\zeta/1-p_1\circ\zeta\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\left(p_m\circ\zeta/1-p_m\circ\zeta\right)^k\right\}^{1/k}}\right),\\
&= \left(p_1\circ\zeta,p_2\circ\zeta,\ldots,p_m\circ\zeta\right),\quad \text{for } k=1\\
&= \widehat{\zeta}.
\end{aligned}
\tag{12}
$$

Hence, $m\text{FDWA}_{\Theta}(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n)=\widehat{\zeta}$ holds if $\widehat{\zeta}_j=\widehat{\zeta}$, for all "$j$" varies from 1 to n. □

Theorem 4 (bounded law). *Let $\widehat{\zeta}_j=(p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ be a collection of "n" mFNs, $\widehat{\zeta}^{-}=\cap_{j=1}^{n}(\zeta_j)$, and $\widehat{\zeta}^{+}=\cup_{j=1}^{n}(\zeta_j)$; then,*

$$
\widehat{\zeta}^{-}\leq m\text{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)\leq\widehat{\zeta}^{+}. \tag{13}
$$

Theorem 5 (monotonic law). *For two collections of mFNs $\widehat{\zeta}_j$ and $\widehat{\zeta}'_j$, $j=1,2,\ldots,n$, if $\widehat{\zeta}_j\leq\widehat{\zeta}'_j$,*

$$
m\text{FDWA}_{\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)\leq m\text{FDWA}_{\Theta}\left(\widehat{\zeta}'_1,\widehat{\zeta}'_2,\ldots,\widehat{\zeta}'_n\right). \tag{14}
$$

Now, we present mFDOWA operator.

Definition 6. For a collection of mFNs $\widehat{\zeta}_j=(p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$, $j=1,2,\ldots,n$, an mFDOWA operator is a function mFDOWA: $\widehat{\zeta}^{n}\longrightarrow\widehat{\zeta}$, which is given by

$$
m\text{FDOWA}_{w}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)=\oplus_{j=1}^{n}\left(w_j\widehat{\zeta}_{\sigma(j)}\right), \tag{15}
$$

where $w=(w_1,w_2,\ldots,w_n)^T$ denotes the weights and $w_j\in(0,1]$ with $\sum_{j=1}^{n}w_j=1$. $\sigma(j)$, $(j=1,2,\ldots,n)$ represents the permutation, for which $\widehat{\zeta}_{\sigma(j-1)}\geq\widehat{\zeta}_{\sigma(j)}$.

Theorem 6. *For a collection of mFNs $\widehat{\zeta}_j=(p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ where $j=1,2,\ldots,n$, an accumulated value of these mFNs using the mFDOWA operators is defined as*

$$
\begin{aligned}
m\text{FDOWA}_{w}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right) &= \oplus_{j=1}^{n}\left(w_j\widehat{\zeta}_{\sigma(j)}\right)\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_1\circ\zeta_{\sigma(j)}/1-p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_m\circ\zeta_{\sigma(j)}/1-p_m\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right).
\end{aligned}
\tag{16}
$$

Example 2. Let $\widehat{\zeta}_1=(0.4,0.5,0.3,0.8)$, $\widehat{\zeta}_2=(0.3,0.4,0.1,0.7)$, and $\widehat{\zeta}_3=(0.8,0.7,0.6,0.4)$ be 4FNs with weights $w=(0.3,0.1,0.6)^T$. Then, for $k=3$, we compute the score values as

$$
\begin{aligned}
S\left(\widehat{\zeta}_1\right) &= \frac{0.4+0.5+0.3+0.8}{4}=0.5,\\
S\left(\widehat{\zeta}_2\right) &= \frac{0.3+0.4+0.1+0.7}{4}=0.375,\\
S\left(\widehat{\zeta}_3\right) &= \frac{0.8+0.7+0.6+0.4}{4}=0.625.
\end{aligned}
\tag{17}
$$

Since $S(\widehat{\zeta}_3)>S(\widehat{\zeta}_1)>S(\widehat{\zeta}_2)$,

$$
\begin{aligned}
\widehat{\zeta}_{\sigma(1)} &= \widehat{\zeta}_3=(0.8,0.7,0.6,0.4),\\
\widehat{\zeta}_{\sigma(2)} &= \widehat{\zeta}_1=(0.4,0.5,0.3,0.8),\\
\widehat{\zeta}_{\sigma(3)} &= \widehat{\zeta}_2=(0.3,0.4,0.1,0.7).
\end{aligned}
\tag{18}
$$

Then, from Definition 6,

$$mFDOWA_w\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\widehat{\zeta}_3\right)=\oplus_{j=1}^{3}\left(w_j\widehat{\zeta}_{\sigma(j)}\right),$$

$$=\left(1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_1\circ\zeta_{\sigma(j)}/1-p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_m\circ\zeta_{\sigma(j)}/1-p_m\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right),$$

$$=\left(1-\frac{1}{1+\left(0.3\times(0.8/1-0.8)^3+0.1\times(0.4/1-0.4)^3+0.6\times(0.3/1-0.3)^3\right)^{1/3}},\right.$$

$$1-\frac{1}{1+\left(0.3\times(0.7/1-0.7)^3+0.1\times(0.5/1-0.5)^3+0.6\times(0.4/1-0.4)^3\right)^{1/3}},$$

$$1-\frac{1}{1+\left(0.3\times(0.6/1-0.6)^3+0.1\times(0.3/1-0.3)^3+0.6\times(0.1/1-0.1)^3\right)^{1/3}},$$

$$\left.1-\frac{1}{1+\left(0.3\times(0.4/1-0.4)^3+0.1\times(0.8/1-0.8)^3+0.6\times(0.7/1-0.7)^3\right)^{1/3}}\right),$$

$$=(0.7284,0.6152,0.5017,0.7073). \quad (19)$$

Remark 1. Note that mFDOWA operators satisfy properties, namely, idempotency, boundedness, and monotonicity, as described in Theorems 3, 4, and 5.

Theorem 7 (commutative law). *For any two collections of mFNs $\widehat{\zeta}_j$ and $\widehat{\zeta}'_j j=1,2,\ldots,n$, we get*

$$mFDOWA_w\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)=mFDOW\zeta_w\left(\widehat{\zeta}'_1,\widehat{\zeta}'_2,\ldots,\widehat{\zeta}'_n\right), \quad (20)$$

where $\widehat{\zeta}'_j$ is any permutation of $\widehat{\zeta}_j$.

We see that mFDWA and mFDOWA operators aggregate weighted mFNs and their ordering, respectively. Now, we develop a novel operator called mFDHA operator, which obtains the properties of both mFDWA and mFDOWA operators.

Definition 7. For a family of mFNs $\widehat{\zeta}_j=(p_1\circ\zeta_j,p_2\circ\zeta_j,\ldots,p_m\circ\zeta_j)$, $j=1,2,\ldots,n$, an mFDHA operator is defined as

$$mFDHA_{w,\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)=\oplus_{j=1}^{n}\left(w_j\widetilde{\widehat{\zeta}}_{\sigma(j)}\right), \quad (21)$$

where $w=(w_1,w_2,\ldots,w_n)^T$ denotes the weights corresponding to the mFNs $\widehat{\zeta}_j$ with the conditions $w_j\in(0,1]$, $\sum_{j=1}^{n}w_j=1$, $\widehat{\zeta}_{\sigma(j)}$ is the jth biggest mFN, $\widetilde{\widehat{\zeta}}_{\sigma(j)}=(n\Theta_j)\widehat{\zeta}_j$, $(j=1,2,\ldots,n)$, and $\Theta=(\Theta_1,\Theta_2,\ldots,\Theta_n)$ is a vector having weights, with $\Theta_j\in(0,1]$, $\sum_{j=1}^{n}\Theta_j=1$.

Notice that when $w=((1/n),(1/n),\ldots,(1/n))$, mFDHA operator converts into mFDWA operator. If $\Theta=((1/n),(1/n),\ldots,(1/n))$, then mFDHA operator becomes mFDOWA operator. Thus, mFDHA operator is a generalization for both operators, mFDWA and mFDOWA, which describes the degrees and ordering of mFNs.

The following theorem can be readily showed by same steps as in Theorem 2.

Theorem 8. *For a collection of mFNs $\widehat{\zeta}_j=(p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ where $j=1,2,\ldots,n$, an accumulated score of these mFNs using the mFDHA operators is defined as*

$$mFDHA_{w,\Theta}\left(\widehat{\zeta}_1,\widehat{\zeta}_2,\ldots,\widehat{\zeta}_n\right)=\oplus_{j=1}^{n}\left(w_j\widetilde{\widehat{\zeta}}_{\sigma(j)}\right),$$

$$=\left(1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_1\circ\zeta_{\sigma(j)}/1-p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n}w_j\left(p_m\circ\zeta_{\sigma(j)}/1-p_m\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right). \quad (22)$$

Example 3. Let $\widehat{\zeta}_1 = (0.7, 0.3, 0.5)$, $\widehat{\zeta}_2 = (0.2, 0.5, 0.7)$, $\widehat{\zeta}_3 = (0.8, 0.2, 0.1)$, and $\widehat{\zeta}_4 = (0.6, 0.7, 0.9)$ be 3FNs with $w = (0.2, 0.3, 0.1, 0.4)^T$, a weight vector corresponding to given 3FNs, and a vector $\Theta = (0.3, 0.1, 0.4, 0.2)^T$ having weights. Then, by Definition 7, for $k = 3$,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_1 &= \left(1 - \frac{1}{1 + \left\{n\Theta_1 \left(p_1 \circ \zeta_1/1 - p_1 \circ \zeta_1\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{n\Theta_1 \left(p_3 \circ \zeta_1/1 - p_3 \circ \zeta_1\right)^k\right\}^{1/k}}\right),\\ &= \left(1 - \frac{1}{1 + \left(4 \times 0.3 \times (0.7/1 - 0.7)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.3 \times (0.3/1 - 0.3)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.3 \times (0.5/1 - 0.5)^3\right)^{1/3}}\right),\\ &= (0.7126, 0.3129, 0.5152).\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_2 &= \left(1 - \frac{1}{1 + \left(4 \times 0.1 \times (0.2/1 - 0.2)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.1 \times (0.5/1 - 0.5)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.1 \times (0.7/1 - 0.7)^3\right)^{1/3}}\right),\\ &= (0.1555, 0.4242, 0.6322),\\ \widetilde{\widetilde{\zeta}}_3 &= \left(1 - \frac{1}{1 + \left(4 \times 0.4 \times (0.8/1 - 0.8)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.4 \times (0.2/1 - 0.2)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.4 \times (01/1 - 0.1)^3\right)^{1/3}}\right),\\ &= (0.8239, 0.2262, 0.1150),\\ \widetilde{\widetilde{\zeta}}_4 &= \left(1 - \frac{1}{1 + \left(4 \times 0.2 \times (0.6/1 - 0.6)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (0.7/1 - 0.7)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (0.9/1 - 0.9)^3\right)^{1/3}}\right),\\ &= (0.5820, 0.6842, 0.8931).\end{aligned} \tag{24}$$

Then, scores of mFNs for $k = 3$ are calculated as

$$\begin{aligned}S\left(\widetilde{\widetilde{\zeta}}_1\right) &= \frac{0.7126 + 0.3129 + 0.5152}{3} = 0.5136,\\ S\left(\widetilde{\widetilde{\zeta}}_2\right) &= \frac{0.1555 + 0.4242 + 0.6322}{3} = 0.4040,\\ S\left(\widetilde{\widetilde{\zeta}}_3\right) &= \frac{0.8239 + 0.2262 + 0.1150}{3} = 0.3884,\\ S\left(\widetilde{\widetilde{\zeta}}_4\right) &= \frac{0.5820 + 0.6842 + 0.8931}{3} = 0.7198.\end{aligned} \tag{25}$$

Since $S(\widetilde{\widetilde{\zeta}}_4) > S(\widetilde{\widetilde{\zeta}}_1) > S(\widetilde{\widetilde{\zeta}}_2) > S(\widetilde{\widetilde{\zeta}}_3)$,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_{\sigma(1)} &= \widehat{\zeta}_4 = (0.5820, 0.6842, 0.8931),\\ \widetilde{\widetilde{\zeta}}_{\sigma(2)} &= \widehat{\zeta}_1 = (0.7126, 0.3129, 0.5152),\\ \widetilde{\widetilde{\zeta}}_{\sigma(3)} &= \widehat{\zeta}_2 = (0.1555, 0.4242, 0.6322),\\ \widetilde{\widetilde{\zeta}}_{\sigma(4)} &= \widehat{\zeta}_3 = (0.8239, 0.2262, 0.1150).\end{aligned} \tag{26}$$

Then, from Theorem 8,

$$mFDHA_{w,\Theta}(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3,\hat{\zeta}_4) = \oplus_{j=1}^{4}(w_j\hat{\zeta}_{\sigma(j)})$$
$$= \left(1 - \frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(p_1\circ\zeta_{\sigma(j)}/1-p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(p_3\circ\zeta_{\sigma(j)}/1-p_3\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right),$$
$$= \left(1 - \frac{1}{1+\left(0.2\times(0.5820/1-0.5820)^3+0.3\times(0.7126/1-0.7126)^3+0.1\times(0.1555/1-0.1555)^3+0.4\times(0.8239/1-0.8239)^3\right)^{1/3}},\right.$$
$$1-\frac{1}{1+\left(0.2\times(0.6842/1-0.6842)^3+0.3\times(0.3129/1-0.3129)^3+0.1\times(0.4242/1-0.4242)^3+0.4\times(0.2262/1-0.2262)^3\right)^{1/3}},$$
$$\left.1-\frac{1}{1+\left(0.2\times(0.8931/1-0.8931)^3+0.3\times(0.5152/1-0.5152)^3+0.1\times(0.6322/1-0.6322)^3+0.4\times(0.1150/1-0.1150)^3\right)^{1/3}}\right),$$
$$= (0.7819, 0.5620, 0.8304). \tag{27}$$

3.2. mF Dombi Geometric AOs. We now propose different types of Dombi geometric AOs with mFNs, namely, mFDWG operator, mFDOWG operator, and mFDHG operator.

Definition 8. For a family of mFNs $\hat{\zeta}_j = (p_1\circ\zeta_j, p_2\circ\zeta_j, \ldots, p_m\circ\zeta_j)$, $j = 1,2,\ldots,n$, a mapping mFDWG: $\hat{\zeta}^n \longrightarrow \hat{\zeta}$ is called mFDWG operator, which is given by

$$mFDWG_{\Theta}(\hat{\zeta}_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_n) = \otimes_{j=1}^{n}(\hat{\zeta}_j)^{\Theta_j}, \tag{28}$$

where $\Theta = (\Theta_1,\Theta_2,\ldots,\Theta_n)^T$ represents the weights, with $\sum_{j=1}^{n}\Theta_j = 1, \Theta_j \in (0,1]$.

Theorem 9. *For a collection of mFNs $\hat{\zeta}_j = (p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ where $j = 1,2,\ldots,n$, an accumulated score of these mFNs using the mFDWG operators is defined by*

$$mFDWG_{\Theta}(\hat{\zeta}_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_n) = \otimes_{j=1}^{n}(\hat{\zeta}_j)^{\Theta_j},$$
$$\left(1 - \frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(1-p_1\circ\zeta_j/p_1\circ\zeta_j\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1+\left\{\sum_{\Theta_j}^{n}\Theta_j\left(1-p_m\circ\zeta_j/p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right). \tag{29}$$

Proof. Its proof is identical to Theorem 2. □

Example 4. Let $\hat{\zeta}_1 = (0.2, 0.6, 0.3)$, $\hat{\zeta}_2 = (0.9, 1.0, 0.7)$, $\hat{\zeta}_3 = (0.1, 0.8, 0.4)$, and $\hat{\zeta}_4 = (0.4, 0.7, 0.3)$ be 3FNs with weights $\Theta = (0.1, 0.5, 0.3, 0.1)^T$. Then, for $k = 3$,

$$mFDWG_{\Theta}(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3) = \otimes_{j=1}^{3}(\hat{\zeta}_j)^{\Theta_j},$$
$$= \left(1 - \frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(1-p_1\circ\zeta_j/p_1\circ\zeta_j\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1+\left\{\sum_{j=1}^{n}\Theta_j\left(1-p_m\circ\zeta_j/p_m\circ\zeta_j\right)^k\right\}^{1/k}}\right),$$
$$= \left(1 - \frac{1}{1+\left(0.1\times(1-0.2/0.2)^3+0.5\times(1-0.9/0.9)^3+0.3\times(1-0.1/0.1)^3+0.1\times(1-0.4/0.4)^3\right)^{1/3}},\right.$$
$$1-\frac{1}{1+\left(0.1\times(1-0.6/0.6)^3+0.5\times(1-0.1/0.1)^3+0.3\times(1-0.8/0.8)^3+0.1\times(1-0.7/0.7)^3\right)^{1/3}},$$
$$\left.1-\frac{1}{1+\left(0.1\times(1-0.3/0.3)^3+0.5\times(1-0.7/0.7)^3+0.3\times(1-0.4/0.4)^3+0.1\times(1-0.3/0.3)^3\right)^{1/3}}\right),$$
$$= (0.8589, 0.2582, 0.6052). \tag{30}$$

It can be readily shown that the mFDWG operator holds the notions given below.

Theorem 10 (idempotent law). *Let $\widehat{\zeta}_j = (p_1 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$ be a family of "n" mFNs, which are equal, i.e., $\widehat{\zeta}_j = \widehat{\zeta}$; then,*

$$mFDWG_{\Theta}\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) = \widehat{\zeta}. \tag{31}$$

Theorem 11 (bounded law). *Let $\widehat{\zeta}_j = (p_1 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$ be a collection of "n" mFNs, $\widehat{\zeta}^- = \cap_{j=1}^{n} (\zeta_j)$, and $\widehat{\zeta}^+ = \cup_{j=1}^{n} (\zeta_j)$; then,*

$$\widehat{\zeta}^- \leq mFDWG_{\Theta}\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) \leq \widehat{\zeta}^+. \tag{32}$$

Theorem 12 (monotonic law). *For two collections of mFNs $\widehat{\zeta}_j$ and $\widehat{\zeta}'_j$, $(j = 1, 2, \ldots, n)$, if $\widehat{\zeta}_j \leq \widehat{\zeta}'_j$, then*

$$mFDWG_{\Theta}\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) \leq mFDWG_{\Theta}\left(\widehat{\zeta}'_1, \widehat{\zeta}'_2, \ldots, \widehat{\zeta}'_n\right). \tag{33}$$

Now, we develop mFDOWG operators.

Definition 9. For a family of mFNs $\widehat{\zeta}_j = (p_1 \circ \zeta_j, p_2 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$, $j = 1, 2, \ldots, n$, an mFDOWG operator is a mapping mFDOWG: $\widehat{\zeta}^n \longrightarrow \widehat{\zeta}$, which is given as

$$mFDOWG_w\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) = \otimes_{j=1}^{n}\left(w_j \widehat{\zeta}_{\sigma(j)}\right), \tag{34}$$

where $w = (w_1, w_2, \ldots, w_n)^T$ is the weight vector and $w_j \in (0, 1]$ with $\sum_{j=1}^{n} w_j = 1$. $\sigma(j)$, $j = 1, 2, \ldots, n$ represents the permutation, such that $\widehat{\zeta}_{\sigma(j-1)} \geq \widehat{\zeta}_{\sigma(j)}$.

Theorem 13. *For a collection of mFNs $\widehat{\zeta}_j = (p_1 \circ \zeta_j, \ldots, p_m \circ \zeta_j)$ where $j = 1, 2, \ldots, n$, an accumulated score of these mFNs using an mFDOWG operator is defined by*

$$\begin{aligned} mFDOWG_w\left(\widehat{\zeta}_1, \widehat{\zeta}_2, \ldots, \widehat{\zeta}_n\right) &= \otimes_{j=1}^{n}\left(\widehat{\zeta}_{\sigma(j)}\right)^{w_j}, \\ &= \left(1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j\left(1 - p_1 \circ \zeta_{\sigma(j)}/p_1 \circ \zeta_{\sigma(j)}\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{\sum_{j=1}^{n} w_j\left(1 - p_m \circ \zeta_{\sigma(j)}/p_m \circ \zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right). \end{aligned} \tag{35}$$

Example 5. Let $\widehat{\zeta}_1 = (0.2, 0.4, 0.7)$, $\widehat{\zeta}_2 = (0.3, 0.6, 0.1)$, $\widehat{\zeta}_3 = (0.8, 0.3, 0.5)$, and $\widehat{\zeta}_4 = (0.6, 0.4, 0.7)$ be 3FNs and $w = (0.3, 0.1, 0.2, 0.4)^T$ be a weight vector. Then, score values of mFNs for $k = 3$ are calculated as

$$\begin{aligned} S\left(\widehat{\zeta}_1\right) &= \frac{0.2 + 0.4 + 0.7}{3} = 0.4333, \\ S\left(\widehat{\zeta}_2\right) &= \frac{0.3 + 0.6 + 0.1}{3} = 0.3333, \\ S\left(\widehat{\zeta}_3\right) &= \frac{0.8 + 0.3 + 0.5}{3} = 0.5333, \\ S\left(\widehat{\zeta}_4\right) &= \frac{0.6 + 0.4 + 0.7}{3} = 0.5667. \end{aligned} \tag{36}$$

Since $S(\widehat{\zeta}_4) > S(\widehat{\zeta}_3) > S(\widehat{\zeta}_1) > S(\widehat{\zeta}_2)$,

$$\begin{aligned} \widehat{\zeta}_{\sigma(1)} &= \widehat{\zeta}_3 = (0.6, 0.4, 0.7), \\ \widehat{\zeta}_{\sigma(2)} &= \widehat{\zeta}_3 = (0.8, 0.3, 0.5), \\ \widehat{\zeta}_{\sigma(3)} &= \widehat{\zeta}_1 = (0.2, 0.4, 0.7), \\ \widehat{\zeta}_{\sigma(4)} &= \widehat{\zeta}_2 = (0.3, 0.6, 0.1). \end{aligned} \tag{37}$$

Then, from Definition 9,

$$\begin{aligned}
m\text{FDOWG}_w\left(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3,\hat{\zeta}_4\right) &= \otimes_{j=1}^{4}\left(\hat{\zeta}_{\sigma(j)}\right)^{w_j},\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(1-p_1\circ\zeta_{\sigma(j)}/p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(1-p_3\circ\zeta_{\sigma(j)}/p_3\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right),\\
&= \left(1-\frac{1}{1+\left(0.3\times(1-0.6/0.6)^3+0.1\times(1-0.8/0.8)^3+0.2\times(1-0.2/0.2)^3+0.4\times(1-0.3/0.3)^3\right)^{1/3}},\right.\\
&\quad 1-\frac{1}{1+\left(0.3\times(1-0.4/0.4)^3+0.1\times(1-0.3/0.3)^3+0.2\times(1-0.4/0.4)^3+0.4\times(1-0.6/0.6)^3\right)^{1/3}},\\
&\quad \left. 1-\frac{1}{1+\left(0.3\times(1-0.7/0.7)^3+0.1\times(1-0.5/0.5)^3+0.2\times(1-0.7/0.7)^3+0.4\times(1-0.1/0.1)^3\right)^{1/3}}\right),\\
&= (0.7237, 0.5926, 0.8690).
\end{aligned} \tag{38}$$

Remark 2. Note that mFDOWG operators satisfy properties, namely, idempotency, boundedness, and monotonicity, as described in Theorems 10, 11, and 12.

Theorem 14 (commutative law). *For two arbitrary collections of mFNs $\hat{\zeta}_j$ and $\hat{\zeta}'_j\,(j=1,2,\ldots,n)$, if $\hat{\zeta}_j\le\hat{\zeta}'_j$, then*

$$m\text{FDOWG}_w\left(\hat{\zeta}_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_n\right) = m\text{FDOWG}_w\left(\hat{\zeta}'_1,\hat{\zeta}'_2,\ldots,\hat{\zeta}'_n\right), \tag{39}$$

where $\hat{\zeta}'_j$ is any permutation of $\hat{\zeta}_j$.

In Definitions 5 and 6, we see that mFDWG and mFDOWG operators aggregate weighted mFNs and their ordering, respectively. Now, we develop a new operator called mFDHG operator, which contains the properties of both mFDWG and mFDOWG operators.

Definition 10. For a family of mFNs $\hat{\zeta}_j=(p_1\circ\zeta_j,p_2\circ\zeta_j,\ldots,p_m\circ\zeta_j)$, $j=1,2,\ldots,n$, an mFDHG operator is defined by

$$m\text{FDHG}_{w,\Theta}\left(\hat{\zeta}_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_n\right) = \otimes_{j=1}^{n}\left(\tilde{\hat{\zeta}}_{\sigma(j)}\right)^{w_j}, \tag{40}$$

where $w=(w_1,w_2,\ldots,w_n)^T$ denotes the weights associated to the mFNs $\hat{\zeta}_j$, $j=1,2,\ldots,n$, $w_j\in(0,1]$, $\sum_{j=1}^{n} w_j=1$, $\tilde{\hat{\zeta}}_{\sigma(j)}$ is the j-th largest mFN, $\tilde{\hat{\zeta}}_{\sigma(j)}=(n\Theta_j)\hat{\zeta}_j$, $(j=1,2,\ldots,n)$, and $\Theta=(\Theta_1,\Theta_2,\ldots,\Theta_n)$ is a vector having weights, with $\Theta_j\in(0,1]$, $\sum_{j=1}^{n}\Theta_j=1$.

Notice that when $w=((1/n),(1/n),\ldots,(1/n))^T$, mFDHG operator becomes mFDWG operator. When $\Theta=((1/n),(1/n),\ldots,(1/n))^T$, then mFDHG operator converts into mFDOWG operator. Thus, mFDHG operator is a generalization of mFDWG and mFDOWG operators.

With the induction technique, one can readily show the next theorem.

Theorem 15. *For a collection of mFNs $\hat{\zeta}_j=(p_1\circ\zeta_j,\ldots,p_m\circ\zeta_j)$ where $j=1,2,\ldots,n$, an accumulated score of these mFNs using an mFDHG operator is defined as*

$$\begin{aligned}
m\text{FDHG}_{w,\Theta}\left(\hat{\zeta}_1,\hat{\zeta}_2,\ldots,\hat{\zeta}_n\right) &= \otimes_{j=1}^{n}\left(\tilde{\hat{\zeta}}_{\sigma(j)}\right)^{w_j},\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(1-p_1\circ\zeta_{\sigma(j)}/p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{n} w_j\left(1-p_m\circ\zeta_{\sigma(j)}/p_m\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right).
\end{aligned} \tag{41}$$

Example 6. Let $\widehat{\zeta}_1 = (0.4, 0.6, 0.3)$, $\widehat{\zeta}_2 = (0.3, 0.2, 0.9)$, $\widehat{\zeta}_3 = (0.6, 0.3, 0.5)$, and $\widehat{\zeta}_4 = (0.3, 0.5, 0.7)$ be 3FNs and $w = (0.4, 0.1, 0.3, 0.2)^T$ be an associated weight vector and a vector $\Theta = (0.5, 0.2, 0.1, 0.2)^T$ be weights. By Definition 10, for $k = 3$,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_1 &= \left(1 - \frac{1}{1 + \left\{n\Theta_1 \left(1 - p_1 \circ \zeta_1 / p_1 \circ \zeta_1\right)^k\right\}^{1/k}}, \ldots, 1 - \frac{1}{1 + \left\{n\Theta_1 \left(1 - p_3 \circ \zeta_1 / p_3 \circ \zeta_1\right)^k\right\}^{1/k}}\right),\\ &= \left(1 - \frac{1}{1 + \left(4 \times 0.5 \times (1 - 0.4/0.4)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.5 \times (1 - 0.6/0.6)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.5 \times (1 - 0.3/0.3)^3\right)^{1/3}}\right),\\ &= (0.6540, 0.4565, 0.7462).\end{aligned} \tag{42}$$

Similarly,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_2 &= \left(1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.3/0.3)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.2/0.2)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.9/0.9)^3\right)^{1/3}}\right),\\ &= (0.6842, 0.7878, 0.0935),\\ \widetilde{\widetilde{\zeta}}_3 &= \left(1 - \frac{1}{1 + \left(4 \times 0.1 \times (1 - 0.6/0.6)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.1 \times (1 - 0.3/0.3)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.1 \times (1 - 0.5/0.5)^3\right)^{1/3}}\right),\\ &= (0.3294, 0.6322, 0.4242),\\ \widetilde{\widetilde{\zeta}}_4 &= \left(1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.3/0.3)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.5/0.5)^3\right)^{1/3}}, 1 - \frac{1}{1 + \left(4 \times 0.2 \times (1 - 0.7/0.7)^3\right)^{1/3}}\right),\\ &= (0.6842, 0.4814, 0.2846).\end{aligned} \tag{43}$$

Then, score values of mFNs for $k = 3$ are given as follows:

$$\begin{aligned}S\left(\widetilde{\widetilde{\zeta}}_1\right) &= \frac{0.6540 + 0.4565 + 0.7462}{3} = 0.6189,\\ S\left(\widetilde{\widetilde{\zeta}}_2\right) &= \frac{0.6842 + 0.7878 + 0.0935}{3} = 0.5218,\\ S\left(\widetilde{\widetilde{\zeta}}_3\right) &= \frac{0.3294 + 0.6322 + 0.4242}{3} = 0.4620,\\ S\left(\widetilde{\widetilde{\zeta}}_4\right) &= \frac{0.6842 + 0.4814 + 0.2846}{3} = 0.4834.\end{aligned} \tag{44}$$

Since $S(\widetilde{\widetilde{\zeta}}_1) > S(\widetilde{\widetilde{\zeta}}_2) > S(\widetilde{\widetilde{\zeta}}_4) > S(\widetilde{\widetilde{\zeta}}_3)$,

$$\begin{aligned}\widetilde{\widetilde{\zeta}}_{\sigma(1)} &= \widehat{\zeta}_1 = (0.6540, 0.4565, 0.7462),\\ \widetilde{\widetilde{\zeta}}_{\sigma(2)} &= \widehat{\zeta}_2 = (0.6842, 0.7878, 0.0935),\\ \widetilde{\widetilde{\zeta}}_{\sigma(3)} &= \widehat{\zeta}_4 = (0.6842, 0.4814, 0.2846),\\ \widetilde{\widetilde{\zeta}}_{\sigma(4)} &= \widehat{\zeta}_3 = (0.3294, 0.6322, 0.4242).\end{aligned} \tag{45}$$

Then, from Definition 9,

$$
\begin{aligned}
mFDH\zeta_{w,\Theta}\left(\hat{\zeta}_1,\hat{\zeta}_2,\hat{\zeta}_3,\hat{\zeta}_4\right) &= \otimes_{j=1}^{4}\left(\hat{\zeta}_{\sigma(j)}\right)^{w_j},\\
&= \left(1-\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(1-p_1\circ\zeta_{\sigma(j)}/p_1\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}},\ldots,1-\frac{1}{1+\left\{\sum_{j=1}^{4} w_j\left(1-p_3\circ\zeta_{\sigma(j)}/p_3\circ\zeta_{\sigma(j)}\right)^k\right\}^{1/k}}\right),\\
&= \left(1-\frac{1}{1+\left(0.4\times(1-0.6540/0.6540)^3+0.1\times(1-0.6842/0.6842)^3+0.3\times(1-0.6842/0.6842)^3+0.2\times(1-0.3294/0.3294)^3\right)^{1/3}},\right.\\
&\quad 1-\frac{1}{1+\left(0.4\times(1-0.4565/0.4565)^3+0.1\times(1-0.7878/0.7878)^3+0.3\times(1-0.4814/0.4814)^3+0.2\times(1-0.6322/0.6322)^3\right)^{1/3}},\\
&\quad \left.1-\frac{1}{1+\left(0.4\times(1-0.7462/0.7462)^3+0.1\times(1-0.0935/0.0935)^3+0.3\times(1-0.2846/0.2846)^3+0.2\times(1-0.4242/0.4242)^3\right)^{1/3}}\right),\\
&= (0.5482, 0.5073, 0.8210).
\end{aligned}
\tag{46}
$$

4. Mathematical Method for MCDM with mF Data

To solve MCDM problems containing mF data, we apply mF Dombi AOs. The following notions are utilized to tackle the MCDM situations having mF information. Suppose that $\{Y_1, Y_2, \ldots, Y_k\}$ is a universal set and $\{S_1, S_2, \ldots, S_n\}$ is the universe of attributes. Assume $\Theta = \{\Theta_1, \Theta_2, \ldots, \Theta_n\}$ is a weight vector with $\sum_{j=1}^{n}\Theta_j = 1$, $\Theta_j \in (0,1]$, for all $j = 1, \ldots, n$. Consider $\hat{S} = (\hat{s}_{ij})_{k\times n} = (p_1\circ\zeta_{ij}, p_2\circ\zeta_{ij}, \ldots, p_m\circ\zeta_{ij})_{k\times n}$ is an mF decision matrix, which represents the membership values evaluated by the experts.

We construct an algorithmic method to handle MCDM problems by an mFDWA (or mFDWG) operator.

5. Applications

5.1. Agriculture Land Selection. Agriculture is an essential part of Pakistan's economic system. This area directly supports the population of the country and accounts for 26% of gross domestic product (GDP). The leading agricultural crops include sugarcane, wheat, rice, cotton, vegetables, and fruits. A business man wants to invest in agriculture sector and is searching for an appropriate land. The options in his brain are $Y_1, Y_2, \ldots, Y_5$. He consults an expert to get his suggestion about the alternatives based on the following desired parameters:

S_1 denotes the "Location"

S_2 denotes the "Climate"

S_3 denotes the "Fertility"

S_4 denotes the "Price"

Each parameter has been characterized into three parts to construct a 3FN.

(i) "Location" includes near to market, near to water channel, and transport availability.

(ii) "Climate" includes temperature, pollution level, and humidity level.

(iii) "Fertility" includes soil PH, level of nutrients, and water retention capacity of land.

(iv) "Price" includes low, medium, and high.

The 3F decision matrix is shown in Table 1.

According to the businessman, the expert assigns weights to parameters as follows:

$$
\begin{aligned}
\Theta_1 &= 0.35,\\
\Theta_2 &= 0.25,\\
\Theta_3 &= 0.30,\\
\Theta_4 &= 0.10.
\end{aligned}
\tag{47}
$$

Clearly, $\sum_{j=1}^{3}\Theta_j = 1$. To compute the most suitable land regarding agriculture, we use the two operators, namely, mFDWA and mFDWG, respectively:

(1) For $k = 3$, by applying the mFDWA operator, we calculate the values $\hat{s}_i$ of the lands $Y_i, i = 1, 2, \ldots, 5$ regarding agriculture.

$$
\begin{aligned}
\hat{s}_1 &= (0.8107, 0.6224, 0.6109),\\
\hat{s}_2 &= (0.8662, 0.6679, 0.7297),\\
\hat{s}_3 &= (0.5443, 0.7418, 0.6515),\\
\hat{s}_4 &= (0.8192, 0.7347, 0.5366),\\
\hat{s}_5 &= (0.8663, 0.6334, 0.6327).
\end{aligned}
\tag{48}
$$

(2) Find the score values $S(\hat{s}_i)$ of 3FNs $\hat{s}_i$, $(i = 1, 2, \ldots, 5)$ of the lands Y_i:

$$
\begin{aligned}
S(\hat{s}_1) &= 0.6814,\\
S(\hat{s}_2) &= 0.7546,\\
S(\hat{s}_3) &= 0.6459,\\
S(\hat{s}_4) &= 0.6968,\\
S(\hat{s}_5) &= 0.7108.
\end{aligned}
\tag{49}
$$

(3) Rank the lands using scores $S(s_i)$, $(i = 1, 2, \ldots, 5)$ obtained from the preference values in the form of 3FNs: $Y_2 > Y_5 > Y_4 > Y_1 > Y_3$.

Table 1: 3F decision matrix.

	S_1	S_2	S_3	S_4
Y_1	(0.7, 0.6, 0.3)	(0.5, 0.7, 0.2)	(0.6, 0.3, 0.7)	(0.9, 0.5, 0.4)
Y_2	(0.9, 0.6, 0.5)	(0.8, 0.5, 0.3)	(0.4, 0.5, 0.8)	(0.5, 0.8, 0.5)
Y_3	(0.4, 0.7, 0.3)	(0.5, 0.4, 0.3)	(0.6, 0.8, 0.4)	(0.6, 0.3, 0.8)
Y_4	(0.5, 0.6, 0.3)	(0.8, 0.6, 0.5)	(0.7, 0.8, 0.2)	(0.9, 0.3, 0.7)
Y_5	(0.9, 0.7, 0.6)	(0.8, 0.4, 0.3)	(0.6, 0.5, 0.7)	(0.4, 0.6, 0.5)

(4) Y_2 has a high score value, so it is the best land for agriculture.

In a similar way, apply an mFDWG operator to find an appropriate land.

(1) Take $k = 3$. Apply an mFDWG operator to determine the values $\hat{s}_i$ of the lands Y_i.

$$\begin{aligned} \hat{s}_1 &= (0.4171, 0.6142, 0.7334), \\ \hat{s}_2 &= (0.5092, 0.4648, 0.6058), \\ \hat{s}_3 &= (0.5364, 0.5633, 0.6723), \\ \hat{s}_4 &= (0.4196, 0.5311, 0.7423), \\ \hat{s}_5 &= (0.4303, 0.5153, 0.6007). \end{aligned} \tag{50}$$

(2) Determine the score values $S(\hat{s}_i)$ of 3FNs $\hat{s}_i$ of the lands Y_i:

$$\begin{aligned} S(\hat{s}_1) &= 0.5882, \\ S(\hat{s}_2) &= 0.5266, \\ S(\hat{s}_3) &= 0.5907, \\ S(\hat{s}_4) &= 0.5643, \\ S(\hat{s}_5) &= 0.5154. \end{aligned} \tag{51}$$

(3) Rank the lands using scores $S(\hat{s}_i)$, $(i = 1, 2, \ldots, 5)$ obtained from the preference values in the form of 3FNs: $Y_3 > Y_1 > Y_4 > Y_2 > Y_5$.

(4) Y_3 has high score, so it is the best land for agriculture.

5.2. Performance Evaluation of Commercial Banks. Commercial bank is one of the largest essential economic institutions. It can pull in money related streams, offering credit and different monetary administrations. These activities vitally affect national monetary improvements. Hence, commercial banks ought to be assessed by the modern and reliable procedures to rank commercial banks in the financial framework. This research establishes a MCDM model that uses mFDWA, mFDWG, and mF ELECTRE-I methods under a set of criteria and rank commercial banks. The board of specialists will assess each bank under chosen criteria. After a primer evaluation, six banks $\{B_1, B_2, B_3, B_4, B_5, B_6\}$ are assessed and ranked to pick the best bank. The banks are evaluated on the basis of four parameters.

S_1 denotes the "Net Income"

S_2 denotes the "Customer Service"

S_3 denotes the "Nonfinancial Performance"

S_4 denotes the "Potential Attractiveness"

Each parameter has been characterized into four parts to form a 4FN.

(i) "Net Income" includes total equity, operating income, total assets, and net interest income.

(ii) "Customer Service" includes accessibility for customers, the evaluation of Internet page, the number of new services, and the number of new products.

(iii) "Nonfinancial Performance" includes support from main stake holders, bank management, employee stability, and ownership structure.

(iv) "Potential Attractiveness" includes location, involving environment, strategic dimension, and external and internal characteristics.

The 4F decision matrix is represented by Table 2.

The expert assigns weights to parameters as follows:

$$\begin{aligned} \Theta_1 &= 0.28, \\ \Theta_2 &= 0.34, \\ \Theta_3 &= 0.22, \\ \Theta_4 &= 0.16. \end{aligned} \tag{52}$$

Clearly, $\sum_{j=1}^{4} \Theta_j = 1$. To select the most efficient bank, we use the two operators, namely, mFDWA and mFDWG, respectively:

(1) For $k = 3$, utilize the mFDWA operator to compute the values $\hat{s}_i$ for the banks B_i, $i = 1, 2, \ldots, 6$.

$$\begin{aligned} \hat{s}_1 &= (0.6188, 0.5392, 0.6306, 0.6723), \\ \hat{s}_2 &= (0.8617, 0.8639, 0.7895, 0.6875), \\ \hat{s}_3 &= (0.5139, 0.7319, 0.5492, 0.5097), \\ \hat{s}_4 &= (0.7385, 0.6188, 0.5699, 0.5492), \\ \hat{s}_5 &= (0.6551, 0.6323, 0.5227, 0.6112), \\ \hat{s}_6 &= (0.7689, 0.7302, 0.6562, 0.6228). \end{aligned} \tag{53}$$

(2) Calculate the score values $S(\hat{s}_i)$ of 4FNs $\hat{s}_i$, $(i = 1, 2, \ldots, 6)$ for the banks B_i.

$$\begin{aligned} S(\hat{s}_1) &= 0.6152, \\ S(\hat{s}_2) &= 0.8007, \\ S(\hat{s}_3) &= 0.5762, \\ S(\hat{s}_4) &= 0.6191, \\ S(\hat{s}_5) &= 0.6053, \\ S(\hat{s}_6) &= 0.6945. \end{aligned} \tag{54}$$

(3) Now, rank the banks using scores $S(s_i)$, $(i = 1, 2, \ldots, 6)$ obtained from the preference values in the form of 4F numbers: $B_2 > B_6 > B_4 > B_1 > B_5 > B_3$.

(4) B_2 has a high score value, so it is the best bank.

Table 2: 4F decision matrix.

	S_1	S_2	S_3	S_4
B_1	(0.7, 0.6, 0.7, 0.7)	(0.5, 0.7, 0.6, 0.7)	(0.5, 0.3, 0.5, 0.5)	(0.5, 0.4, 0.4, 0.6)
B_2	(0.9, 0.7, 0.8, 0.7)	(0.8, 0.9, 0.8, 0.7)	(0.8, 0.7, 0.7, 0.6)	(0.7, 0.7, 0.8, 0.7)
B_3	(0.4, 0.8, 0.5, 0.6)	(0.5, 0.6, 0.6, 0.4)	(0.5, 0.6, 0.5 0.4)	(0.6, 0.6, 0.5, 0.3)
B_4	(0.8, 0.7, 0.6, 0.5)	(0.7, 0.5, 0.4, 0.6)	(0.5, 0.5, 0.6, 0.5)	(0.4, 0.5, 0.6, 0.5)
B_5	(0.7, 0.7, 0.6, 0.7)	(0.5, 0.6, 0.4, 0.4)	(0.7, 0.5, 0.5, 0.4)	(0.5, 0.5, 0.4, 0.5)
B_6	(0.8, 0.8, 0.7, 0.6)	(0.7, 0.6, 0.6, 0.5)	(0.8, 0.5, 0.5, 0.4)	(0.6, 0.6, 0.7, 0.6)

In a similar way, apply the mFDWG operator to determine the most efficient bank.

(1) Take $k = 3$. We employ the mFDWG operator to compute the values $\widehat{s}_i$ for the banks B_i.

$$\begin{aligned}
\widehat{s}_1 &= (0.4752, 0.6016, 0.4896, 0.4052),\\
\widehat{s}_2 &= (0.2182, 0.2723, 0.2361, 0.3343),\\
\widehat{s}_3 &= (0.5366, 0.3756, 0.4772, 0.6136),\\
\widehat{s}_4 &= (0.4006, 0.5206, 0.5246, 0.4772),\\
\widehat{s}_5 &= (0.4487, 0.4429, 0.4952, 0.5604),\\
\widehat{s}_6 &= (0.3029, 0.4185, 0.4146, 0.5118).
\end{aligned} \tag{55}$$

(2) Find the score values $S(\widehat{s}_i)$ of 4FNs $\widehat{s}_i$ for the banks B_i.

$$\begin{aligned}
S(\widehat{s}_1) &= 0.4929,\\
S(\widehat{s}_2) &= 0.2652,\\
S(\widehat{s}_3) &= 0.5008,\\
S(\widehat{s}_4) &= 0.4808,\\
S(\widehat{s}_5) &= 0.4868,\\
S(\widehat{s}_6) &= 0.4269.
\end{aligned} \tag{56}$$

(3) Rank the banks with scores $S(\widehat{s}_i)$, $(i = 1, 2, \ldots, 6)$ obtained from the preference values in the form of 4F numbers: $B_3 > B_1 > B_5 > B_4 > B_6 > B_2$.

(4) B_3 has high score, so it is the best bank.

The methodology utilized in the applications to find the best alternative is shown in Figure 1.

5.3. mF-ELECTRE-I Method. In this section, we apply mF-ELECTRE-I approach [27] to the problem (performance evaluation of commercial banks, Section 5.2) (Tables 3–5).

(1) Table 3 describes the 4F decision matrix.

(2) Tables 4 and 5, respectively, describe the 4F concordance and discordance values.

(3) The 4F concordance matrix is calculated by

$$F = \begin{pmatrix} - & 0 & 0.84 & 0.62 & 0.62 & 0.34\\ 1 & - & 1 & 1 & 1 & 1\\ 0.16 & 0 & - & 0.16 & 0.34 & 0\\ 0.38 & 0 & 1 & - & 0.56 & 0.22\\ 0.66 & 0 & 0.66 & 0.44 & - & 0\\ 0.66 & 0 & 1 & 0.78 & 1 & - \end{pmatrix}. \tag{57}$$

(4) The 4F concordance level $\overline{f} = 0.498$.

(5) The 4F discordance matrix is computed by

$$G = \begin{pmatrix} - & 1 & 0.522 & 0.289 & 0.7484 & 1\\ 0 & - & 0 & 0 & 0 & 0\\ 1 & 1 & - & 1 & 1 & 1\\ 1 & 1 & 0.413 & - & 1 & 0.8303\\ 1 & 1 & 0.7010 & 1 & - & 1\\ 0.95 & 0 & 0 & 1 & 0 & - \end{pmatrix}. \tag{58}$$

(6) The 4F discordance level $\overline{g} = 0.65$.

(7) The 4F concordance and discordance matrices are given by

$$\begin{aligned}
H &= \begin{pmatrix} - & 0 & 0 & 1 & 1 & 0\\ 1 & - & 1 & 1 & 1 & 1\\ 0 & 0 & - & 0 & 0 & 0\\ 0 & 0 & 1 & - & 1 & 0\\ 1 & 0 & 1 & 0 & - & 0\\ 1 & 0 & 1 & 1 & 1 & - \end{pmatrix},\\
L &= \begin{pmatrix} - & 0 & 1 & 1 & 0 & 0\\ 0 & - & 1 & 1 & 1 & 1\\ 0 & 0 & - & 0 & 0 & 0\\ 0 & 0 & 1 & - & 0 & 0\\ 0 & 0 & 0 & 0 & - & 0\\ 0 & 1 & 1 & 0 & 1 & - \end{pmatrix}.
\end{aligned} \tag{59}$$

(8) The 4F aggregated dominance matrix is constructed as

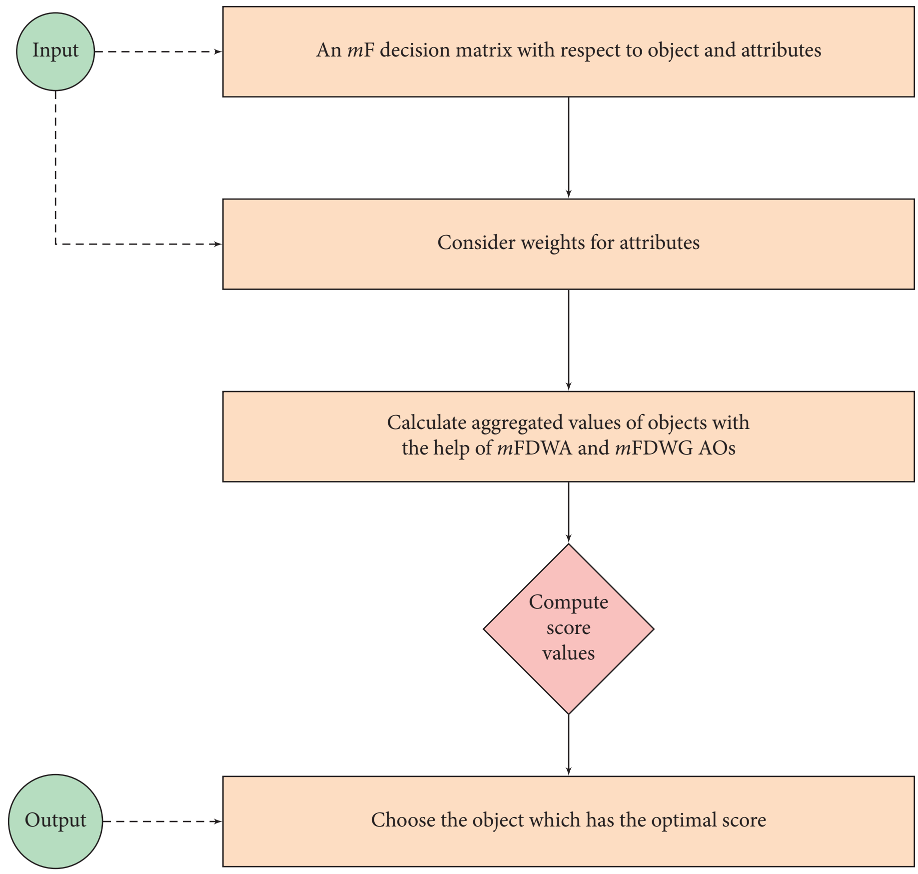

Figure 1: Flowchart of selecting the best option.

Table 3: 4F weighted decision matrix.

	S_1	S_2	S_3	S_4
B_1	(0.196, 0.168, 0.196, 0.196)	(0.17, 0.238, 0.204, 0.238)	(0.11, 0.066, 0.11, 0.11)	(0.08, 0.064, 0.064, 0.096)
B_2	(0.252, 0.196, 0.224, 0.196)	(0.272, 0.306, 0.272, 0.238)	(0.176, 0.154, 0.154, 0.132)	(0.112, 0.112, 0.128, 0.112)
B_3	(0.112, 0.224, 0.14, 0.168)	(0.17, 0.204, 0.204, 0.136)	(0.11, 0.132, 0.11, 0.038)	(0.096, 0.096, 0.08, 0.048)
B_4	(0.224, 0.196, 0.168, 0.14)	(0.238, 0.17, 0.136, 0.204)	(0.11, 0.11, 0.132, 0.11)	(0.064, 0.08, 0.096, 0.08)
B_5	(0.196, 0.196, 0.168, 0.196)	(0.17, 0.204, 0.136, 0.136)	(0.154, 0.11, 0.11, 0.038)	(0.08, 0.08, 0.096, 0.08)
B_6	(0.224, 0.224, 0.196, 0.168)	(0.238, 0.204, 0.204, 0.17)	(0.176, 0.11, 0.11, 0.038)	(0.096, 0.096, 0.112, 0.096)

Table 4: 4F concordance set.

j	1	2	3	4	5	6
F_{1j}	—	{}	{1, 2, 3}	{1, 2}	{1, 2}	{2}
F_{2j}	{1, 2, 3, 4}	—	{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3, 4}	{1, 2, 3, 4}
F_{3j}	{4}	{}	—	{4}	{2}	{}
F_{4j}	{3, 4}	{}	{1, 2, 3, 4}	—	{2, 3}	{3}
F_{5j}	{1, 3, 4}	{}	{1, 3, 4}	{1, 4}	—	{}
F_{6j}	{1, 3, 4}	{}	{1, 2, 3, 4}	{1, 2, 4}	{1, 2, 3, 4}	—

Table 5: 4F discordance set.

j	1	2	3	4	5	6
G_{1j}	—	{1, 2, 3, 4}	{4}	{3, 4}	{1, 3, 4}	{1, 3, 4}
G_{2j}	{}	—	{}	{}	{}	{}
G_{3j}	{1, 2, 3}	{1, 2, 3, 4}	—	{1, 2, 3, 4}	{1, 3, 4}	{1, 2, 3, 4}
G_{4j}	{1, 2}	{1, 2, 3, 4}	{4}	—	{1, 4}	{1, 2, 4}
G_{5j}	{1, 2}	{1, 2, 3, 4}	{2}	{2, 3}	—	{1, 2, 3, 4}
G_{6j}	{2}	{}	{}	{3}	{}	—

$$M = \begin{pmatrix} - & 0 & 0 & 1 & 0 & 0 \\ 1 & - & 1 & 1 & 1 & 1 \\ 0 & 0 & - & 0 & 0 & 0 \\ 0 & 0 & 1 & - & 0 & 0 \\ 0 & 0 & 0 & 0 & - & 0 \\ 0 & 0 & 1 & 0 & 1 & - \end{pmatrix}. \quad (60)$$

(9) Figure 2 shows the preference relations between the banks.

From Figure 2, it is clear that B_2 is the best option (Tables 6 and 7).

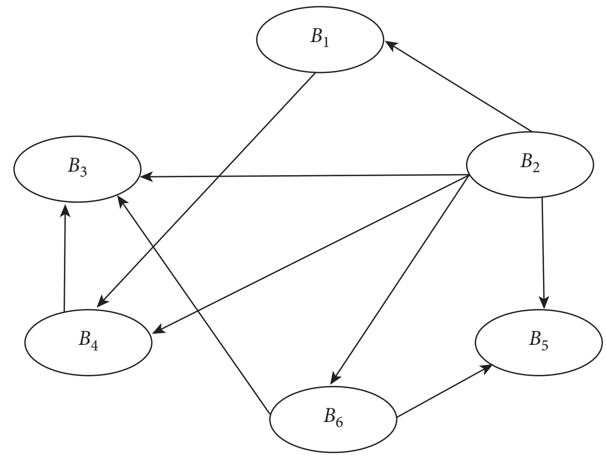

Figure 2: Outranking relation of banks.

6. Comparison Analysis and Discussion

This section gives a comparison of the developed Dombi AOs with *m*F Hamacher AOs [27] and an *m*F-ELECTRE-I model [46] to show their feasibility and practicality.

(1) We compare the results of developed *m*F Dombi AOs with *m*F Hamacher AOs [27]. The results computed by applying both operators in first application are explained by Table 6 and Figure 3. In a similar way, the results computed using both operators in second application are explained by Table 7 and Figure 4. Clearly, the results of *m*F Hamacher weighted average (*m*FHWA) and *m*F Hamacher weighted geometric (*m*FHWG) operators are different from our newly constructed *m*FDWA and *m*FDWG operators. The results of *m*FHWA and *m*FHWG operators are the same. Therefore, our developed AOs are more generalized and versatile than some existing models to handle *m*F MCDM problems.

(2) From the second application, it can be observed that the final rankings by applying the *m*FDWA and *m*FDWG operators are $B_2 > B_6 > B_4 > B_1 > B_5 > B_3$ and $B_3 > B_1 > B_5 > B_4 > B_6 > B_2$, respectively. However, the final score values are not the same. When *m*F-ELECTRE-I method is applied, the best option is B_2. Clearly, the optimal decision using *m*F-ELECTRE-I method and *m*FDWA operator is B_2.

(3) When a number of *m*FNs are aggregated with the help of *m*FDWA and *m*FDWG operators, different computations will increase rapidly. But our developed AOs can explain the assessed data more flexibly for decision making. The developed method ranks every objects in a given problem in comparison with *m*F-ELECTRE-I approach [46].

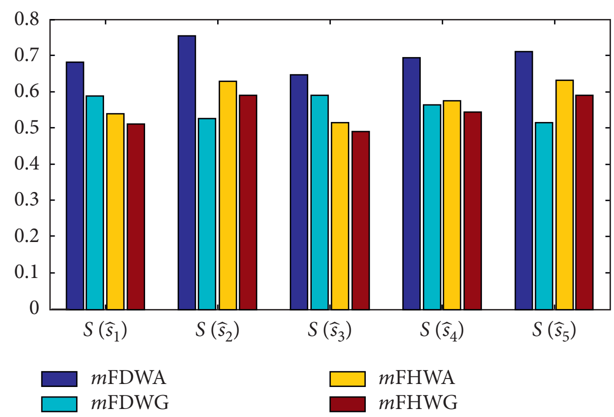

Figure 3: Comparison of first application in Section 5.1.

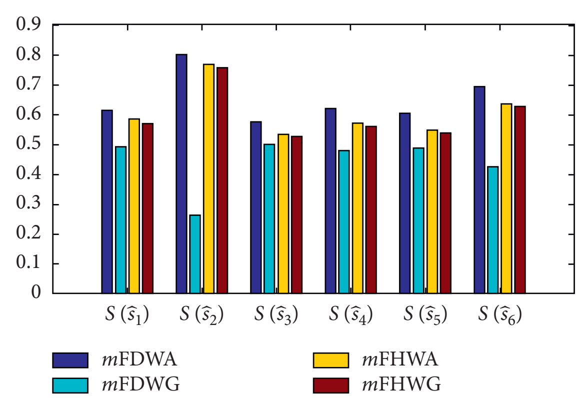

Figure 4: Comparison of second application in Section 5.2.

6.1. Effectiveness Test. To examine the validity of the provided algorithm, we verify it with test criteria developed by Wang and Triantaphyllou [21] as follows (Tables 8 and 9):

(i) Test criterion I: if we change the membership grades of nonoptimal object with worse membership values without effecting criteria, then the optimal object should not change.

(ii) Test criterion II: MCDM approach should satisfy transitive property.

(iii) Test criterion III: when a designated problem is resolved into different small issues and the similar

TABLE 6: Comparison of mF Dombi AOs with mF Hamacher AOs [27] in agriculture land selection.

AO	$S(\hat{s}_1)$	$S(\hat{s}_2)$	$S(\hat{s}_3)$	$S(\hat{s}_4)$	$S(\hat{s}_5)$	Ranking order
Proposed mFDWA	0.6814	0.7546	0.6459	0.6968	0.7108	$Y_2 > Y_5 > Y_4 > Y_1 > Y_3$
Proposed mFDWG	0.5882	0.5266	0.5907	0.5643	0.5154	$Y_3 > Y_1 > Y_4 > Y_2 > Y_5$
mFHWA [27]	0.5403	0.6287	0.5151	0.5725	0.6327	$Y_5 > Y_2 > Y_4 > Y_1 > Y_3$
mFHWG [27]	0.5084	0.5881	0.4908	0.5445	0.5909	$Y_5 > Y_2 > Y_4 > Y_1 > Y_3$

TABLE 7: Comparison of mF Dombi AOs with mF Hamacher AOs [27] in bank selection.

AO	$S(\hat{s}_1)$	$S(\hat{s}_2)$	$S(\hat{s}_3)$	$S(\hat{s}_4)$	$S(\hat{s}_5)$	$S(\hat{s}_6)$	Ranking order
Proposed mFDWA	0.6152	0.8007	0.5762	0.6191	0.6053	0.6945	$B_2 > B_6 > B_4 > B_1 > B_5 > B_3$
Proposed mFDWG	0.4929	0.2652	0.5008	0.4808	0.4868	0.4269	$B_3 > B_1 > B_5 > B_4 > B_6 > B_2$
mFHWA [27]	0.5851	0.7681	0.5342	0.5727	0.5490	0.6355	$B_2 > B_6 > B_1 > B_4 > B_5 > B_3$
mFHWG [27]	0.5711	0.7592	0.5267	0.5611	0.5381	0.6262	$B_2 > B_6 > B_1 > B_4 > B_5 > B_3$

(1) **Input:**
$\hat{S}$, an mF decision matrix having k objects and n attributes.
$\Theta = (\Theta_1, \Theta_2, \ldots, \Theta_n)$, the vector having weights.
(2) Apply the mFDWA operators to aggregate the data in mF decision matrix $\hat{S}$ and calculate the preference values $\hat{s}_i$, where "i" varies from 1 to k for the mFNs ζ_i.
$\hat{s}_i = m\text{FDWA}_{\Theta}(\hat{\zeta}_{i1}, \hat{\zeta}_{i2}, \ldots, \hat{\zeta}_{in}) = \oplus_{j=1}^{n}(\Theta_j \hat{\zeta}_{ij})$
$= (1 - (1/1 + \{\sum_{j=1}^{n} \Theta_j (p_1 \circ \zeta_{ij}/1 - p_1 \circ \zeta_{ij})^k\}^{1/k}), \ldots, 1 - (1/1 + \{\sum_{j=1}^{n} \Theta_j (p_m \circ \zeta_{ij}/1 - p_m \circ \zeta_{ij})^k\}^{1/k}))$.
When we use mFDWG operators,
$\hat{s}_i = m\text{FDWG}_{\Theta}(\hat{\zeta}_{i1}, \hat{\zeta}_{i2}, \ldots, \hat{\zeta}_{in}) = \otimes_{j=1}^{n}(\hat{\zeta}_{ij})^{\Theta_j}$,
$= (1 - (1/1 + \{\sum_{j=1}^{n} \Theta_j (1 - p_1 \circ \zeta_{ij}/p_1 \circ \zeta_{ij})^k\}^{1/k}), \ldots, 1 - (1/\{\sum_{j=1}^{n} \Theta_j (1 - p_m \circ \zeta_{ij}/p_m \circ \zeta_{ij})^k\}))$.
(3) Compute the score values $S(\hat{s}_i)$, $i = 1, 2, \ldots, k$.
(4) Rank the objects u_i, $i = 1, 2, \ldots, k$ with respect to their scores $S(\hat{s}_i)$. When the scores of two objects are equal, we apply the accuracy function to find the order of alternatives.
Output: the object containing maximum score value in last step will be the decision.

ALGORITHM 1: Computing maximum score value.

MCDM technique has been utilized, then the rank order of the objects should be similar to the original ranking order. We have checked these test criteria on developed MCDM approach under mF Dombi AOs as below.

(1) Effectiveness test by criterion I: using this test, if we change only the membership grades of alternative Y_3 with $Y_3' = (0.3, 0.6, 0.2), (0.4, 0.3, 0.2), (0.5, 0.7, 0.3), (0.5, 0.1, 0.6)$ in Table 1 (that is, 3F decision matrix), then the new 3F decision matrix is shown in Table 8. By using mFDWA operator, the score values of the alternative are $S(\hat{s}_1) = 0.6814$, $S(\hat{s}_2) = 0.7546$, $S(\hat{s}_3') = 0.6459$, $S(\hat{s}_4) = 0.6968$, $S(\hat{s}_5) = 0.7108$. Clearly, $S(\hat{s}_2) > S(\hat{s}_5) > S(\hat{s}_4) > S(\hat{s}_1) > S(\hat{s}_3')$; consequently, the ranking of the objects is $Y_2 > Y_5 > Y_4 > Y_1 > Y_3'$. Thus, Y_2 is the best alternative. According to above information, the presented AOs have been employed, and the decision alternative is Y_2 which is similar to the original optimal object. Similarly, if we change the membership grades of alternative Y_4 with $Y_4' = (0.4, 0.5, 0.2), (0.7, 0.5, 0.5), (0.8, 0.7, 0.1), (0.8, 0.2, 0.5)$ in Table 1 (that is, 3F decision matrix), then the new 3F decision matrix is shown in Table 9. By applying the mFDWA operator, the ranking order of the alternatives is $Y_2 > Y_5 > Y_1 > Y_3 > Y_4'$. Thus, the optimal alternative is Y_2 which is same as that of the original ranking. Therefore, the proposed algorithm is feasible under test criterion I.

(2) Effectiveness test by criteria II and III: based upon these test criteria, if we dissolved the designated problem (Application 1) into the subissues $\{Y_1, Y_2, Y_3\}, \{Y_2, Y_3, Y_4\}, \{Y_3, Y_4, Y_5\}$, and $\{Y_4, Y_5, Y_1\}$ and employed the procedure steps of Algorithm 1, then we obtain the ranking of these smaller issues as $Y_2 > Y_1 > Y_3$, $Y_2 > Y_4 > Y_3$, $Y_5 > Y_4 > Y_3$, and $Y_5 > Y_4 > Y_1$, respectively. Hence, by uniting above criteria II and III, we obtain the overall ranking order of the alternatives as $Y_2 > Y_5 > Y_4 > Y_1 > Y_3$, which is exactly same as the original ranking order. Therefore, the developed algorithm is feasible under test criteria II and III.

Table 8: 3F decision matrix.

	S_1	S_2	S_3	S_4
Y_1	(0.7, 0.6, 0.3)	(0.5, 0.7, 0.2)	(0.6, 0.3, 0.7)	(0.9, 0.5, 0.4)
Y_2	(0.9, 0.6, 0.5)	(0.8, 0.5, 0.3)	(0.4, 0.5, 0.8)	(0.5, 0.8, 0.5)
Y_3'	(0.3, 0.6, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.7, 0.3)	(0.5, 0.1, 0.6)
Y_4	(0.5, 0.6, 0.3)	(0.8, 0.6, 0.5)	(0.7, 0.8, 0.2)	(0.9, 0.3, 0.7)
Y_5	(0.9, 0.7, 0.6)	(0.8, 0.4, 0.3)	(0.6, 0.5, 0.7)	(0.4, 0.6, 0.5)

Table 9: 3F decision matrix.

	S_1	S_2	S_3	S_4
Y_1	(0.7, 0.6, 0.3)	(0.5, 0.7, 0.2)	(0.6, 0.3, 0.7)	(0.9, 0.5, 0.4)
Y_2	(0.9, 0.6, 0.5)	(0.8, 0.5, 0.3)	(0.4, 0.5, 0.8)	(0.5, 0.8, 0.5)
Y_3	(0.3, 0.6, 0.2)	(0.4, 0.3, 0.2)	(0.5, 0.7, 0.3)	(0.5, 0.1, 0.6)
Y_4'	(0.4, 0.5, 0.2)	(0.7, 0.5, 0.5)	(0.8, 0.7, 0.1)	(0.8, 0.2, 0.5)
Y_5	(0.9, 0.7, 0.6)	(0.8, 0.4, 0.3)	(0.6, 0.5, 0.7)	(0.4, 0.6, 0.5)

7. Conclusions and Future Directions

Aggregation operators are mathematical functions and essential tools of unifying several inputs into single valuable output. Due to the existence of multipolar data and multiple attributes in many real-world problems, classical MCDM methods are not useful to tackle complicated decision-making situations. To overcome the difficulties of existing models, we have combined *m*FNs with Dombi AOs.

In this article, we have discussed MCDM issues based on *m*F information. Motivated by Dombi operations, we have proposed certain *m*F Dombi AOs, namely, *m*FDWA, *m*FDOWA, *m*FDHA, *m*FDWG, *m*FDOWG, and *m*FDHG operators. We have investigated different features of these operators. We have employed these AOs to enlarge the applicability scope of MCDM. We have given real-life applications for the selection of best agricultural land and for the selection of best bank regarding performance. At the end, we have provided a comparison of developed AOs with *m*F-ELECTRE-I method [46] and *m*F Hamacher AOs [27] and have authenticated the proposed strategy by effectiveness tests to check its validity. In the comparison, we have seen that the optimal alternative is the same by applying *m*F-ELECTRE-I method [46], *m*F Hamacher AOs [27], and proposed *m*FDWA operator. However, it is different in case of *m*FDWG operator. In the future, we plan to extend our work to (i) *m*F Dombi prioritized AOs, (ii) *m*F soft Dombi AOs, (iii) *q*-rung orthopair fuzzy Dombi AOs, and (iv) *q*-rung orthopair fuzzy soft Dombi AOs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of the research article.

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