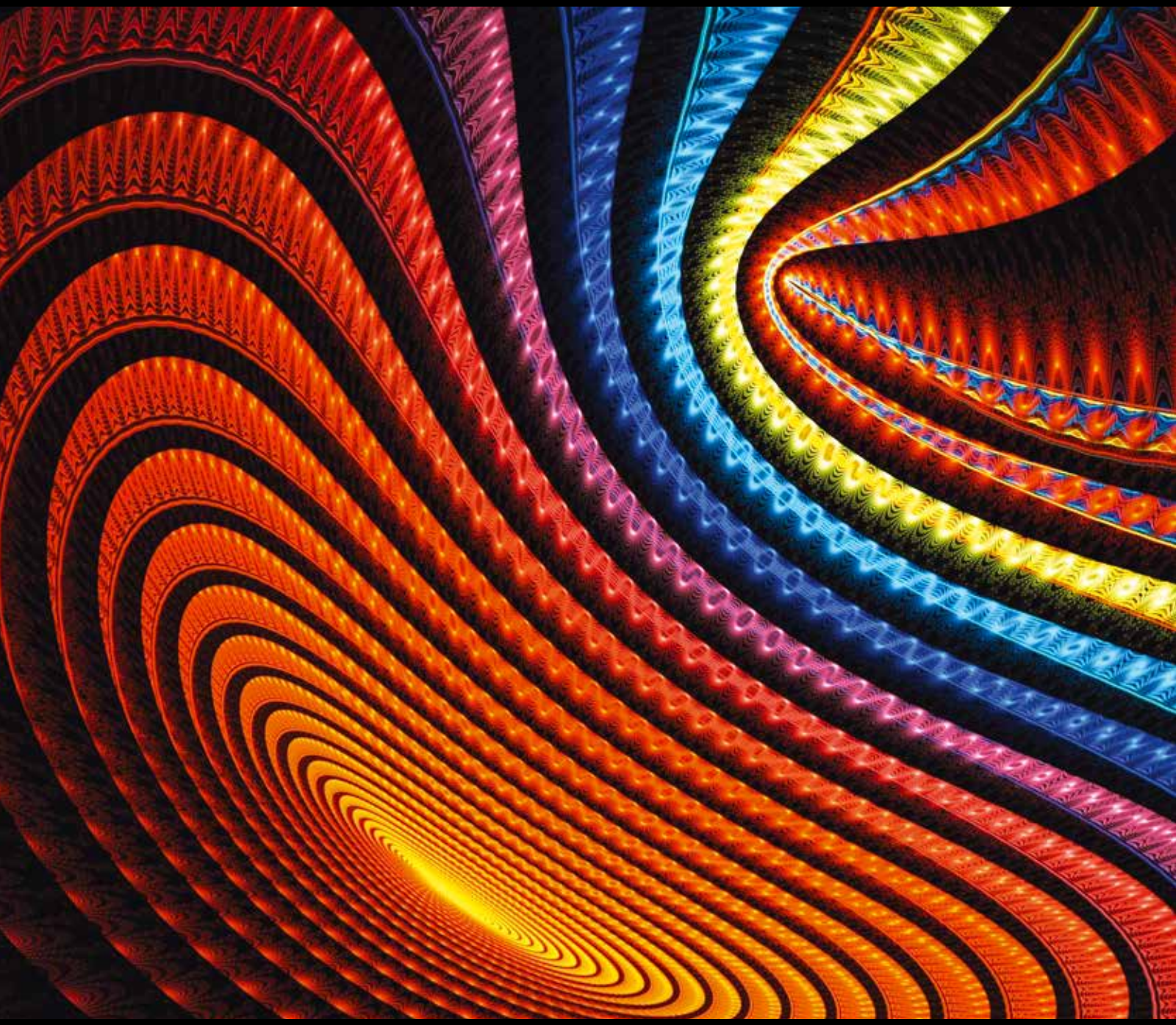
Abstract and Applied Analysis

Networked Systems with Incomplete Information

Guest Editors: Zidong Wang, Bo Shen, Hongli Dong, Xiao He, and Jun Hu



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Editorial Board

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Editorial

Networked Systems with Incomplete Information

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In this special issue, we have solicited submissions from electrical engineers, control engineers, computer scientists, and mathematicians. After a rigorous peer review process, 18 papers have been selected that provide overviews, solutions, or early promises, to manage, analyse, and interpret dynamical behaviours of networked systems. These papers have covered both the theoretical and practical aspects of networked system with incomplete information in the broad areas of dynamical systems, mathematics, statistics, operational research, and engineering.

In this special issue, there is a survey paper on the recent advances of control and filtering problems for Takagi-Sugeno (T-S) fuzzy systems with network-induced phenomena. Specifically, in the paper entitled "Analysis, Filtering, and Control for Takagi-Sugeno Fuzzy Models in Networked Systems" by S. Zhang et al., the focus is to provide a timely review on some recent advances on the T-S fuzzy control and filtering problems with various network-induced phenomena. Because of the advantages in dealing with various nonlinear systems, the fuzzy logic theory has great success in industry applications. Among various kinds of models for fuzzy systems, the T-S fuzzy model is quite popular due to its convenient, simple dynamic structure and the capability of approximating any smooth nonlinear function to any specified accuracy within any compact set. This survey discusses a variety of T-S fuzzy control and filtering issues with network-induced phenomena in great detail firstly. Four network-induced phenomena (communication delays, packet dropouts, signal quantization, and randomly occurring uncertainties (ROUs)) are introduced. Both theories and techniques for dealing with the controller or filter design are systematically reviewed. Then, some latest results on T-S fuzzy control/filtering problems (bilinear T-S fuzzy model, event-based fuzzy control, fuzzy filtering with multiple network-induced phenomena, l_2–l_∞ fuzzy filtering, H_-/H_∞ fault detection, H_∞ filtering with unknown membership functions, and nonfragile H_∞ fuzzy filtering) for networked systems are surveyed and some challenging issues for future research are raised. Finally, some conclusions are drawn and several possible related research directions are pointed out.

In the past decades, the stability analysis of the networked systems has attracted much research attention. In the work entitled "Uniform Stability Analysis of Fractional-Order BAM Neural Networks with Delays in the Leakage Terms" by X. Yang et al., the uniform stability analysis is studied for a class of fractional-order BAM neural networks with delays in the leakage terms. By introducing a novel norm, several delay-dependent sufficient conditions are obtained to ensure the uniform stability of the proposed system by using inequality technique and analysis method. Moreover, sufficient conditions are established to guarantee the existence, uniqueness, and uniform stability of the equilibrium point. Three simulation examples are given to demonstrate the effectiveness of the obtained results. It should be pointed out that it is possible to extend the main results of this paper to other complex systems and establish novel stability conditions

with less conservatism by using more up-to-date techniques. The μ-stability issue is discussed in "Global μ-Stability of Impulsive Complex-Valued Neural Networks with Leakage Delay and Mixed Delays" by X. Chen et al. for complex-valued neural networks (CVNNs) with leakage delay, discrete delay, and distributed delay under impulsive perturbations. The μ-stability is the concept for the purpose of unifying the exponential stability, power-rate stability, and log-stability of neural networks. CVNN is an extension of real-valued neural network which has been applied in physical systems dealing with electromagnetic, light, ultrasonic, and quantum waves. Based on the homeomorphism mapping principle of complex domain, a sufficient condition for the existence and uniqueness of the equilibrium point of the addressed CVNNs is proposed in terms of linear matrix inequality (LMI). By constructing appropriate Lyapunov-Krasovskii functionals and employing the free weighting matrix method, several delay-dependent criteria for checking the global μ-stability of the CVNNs are established in LMIs. As direct applications of these results, several criteria on the exponential stability, power-stability, and log-stability are obtained. In the paper entitled "A Switched Approach to Robust Stabilization of Multiple Coupled Networked Control Systems" by M. Yu et al., multiple coupled networked controlled systems (NCSs) with norm-bounded parameter uncertainties and multiple transmissions are considered. All the nodes in the proposed systems act over a limited bandwidth communication channel. The state information of every subsystem is split into different packets and there is only one packet of the subsystem that can be transmitted at a time. Based on the toking bus protocol, the nodes are arranged logically into a ring and transmit the corresponding packets in a prefixed circular order. Then, the proposed multiple NCSs can be modelled as periodic switched systems. Furthermore, the robust stabilization issue is dealt with by applying the switched system theory. State feedback controllers are constructed in terms of LMIs. A numerical example is given to show that the coupled NCSs considered can be effectively stabilized via the designed controller.

Control and fault estimation problems for stochastic systems have been of interest of many researchers during the past decades. In the paper entitled "Robust H_∞ Control for a Class of Discrete Time-Delay Stochastic Systems with Randomly Occurring Nonlinearities" by Y. Wang et al., the robust H_∞ problem is studied for a class of discrete time-delay stochastic systems with randomly occurring nonlinearities (RONs). It is assumed that all the system matrices contain the parameter uncertainties. The stochastic disturbances are both state- and control-dependent, and the RONs satisfy the sector boundedness conditions. The purpose of the problem proposed is to design a state feedback controller such that, for all admissible uncertainties, nonlinearities, and time-delays, the closed-loop system is robustly asymptotically stable in the mean square, and a prescribed H_∞ disturbance rejection attenuation level is also guaranteed. By using the Lyapunov stability theory and stochastic analysis tools, a LMI approach is developed to derive sufficient conditions ensuring the existence of the desired controllers, where the conditions are dependent on the lower and upper bounds of the time-varying delays. The explicit parameterization of the desired controller gains is also given. The problem of H_∞ control for network-based 2D systems with missing measurements is investigated in "H_∞ Control for Network-Based 2D Systems with Missing Measurements" by X. Bu et al. A state feedback controller is designed such that the closed-loop 2D stochastic system is mean-square asymptotic stability and has H_∞ disturbance attenuation performance. A sufficient condition is derived in terms of LMIs technique, and formulas can be given for the control law design. The result is also extended to more general cases where the system matrices contain uncertain parameters. Numerical examples are also provided to show the effectiveness of proposed approach. In the work entitled "Krein Space-Based H_∞ Fault Estimation for Discrete Time-Delay Systems" by X. Song and X. Yan, the finite-time H_∞ fault estimation issue is investigated for linear time-delay systems where the delay appears in both state output and measurement output. Firstly, the design of finite horizon H_∞ fault estimation is converted into a minimum problem of certain quadratic form. Then, a sufficient and necessary condition for the existence of the desired H_∞ fault estimator is derived by employing the Krein-space theory. A solution of the desired H_∞ fault estimator is obtained by recursively computing a partial difference Riccati equation which has the same dimension as the original systems. Therefore, solving a high dimension Riccati equation is avoided compared with the conventional augmented method. A numerical example is given to demonstrate the effectiveness of the approach.

Circulant type matrices have significant applications in network systems. In the work entitled "Equalities and Inequalities for Norms of Block Imaginary Circulant Operator Matrices" by X. Jiang and K. Hong, the block imaginary circulant operator matrices are studied. Firstly, by combining the special properties of block imaginary circulant operator matrix with unitarily invariant norm, several norm equalities are obtained. It should be pointed out that the norm in consideration is the weakly unitarily invariant norm. The usual operator norm and Schatten p-norm are included. Then, several pinching type inequalities are presented by the triangle inequality and the invariance property of unitarily invariant norms. Furthermore, some special cases and examples are considered. Circulant and left circulant matrices with Fermat and Mersenne numbers are considered in "Exact Inverse Matrices of Fermat and Mersenne Circulant Matrix" by Y. Zheng and S. Shon. Moreover, the exact determinants and the inverse matrices of Fermat and Mersenne left circulant matrix are given. The nonsingularity of these special matrices is discussed. In the paper entitled "Norms and Spread of the Fibonacci and Lucas RSFMLR Circulant Matrices" by W. Xu and Z. Jiang the norms and spread of Fibonacci row skew first-minus-last right (RSFMLR) circulant matrices are investigated as well as the Lucas RSFMLR circulant matrices. Firstly, these two kinds of special matrices are defined. Then, the lower and upper bounds for the spectral norms of these matrices are proposed as well as the upper bounds for the spread of these matrices. Afterwards, some corollaries related to norms of Hadamard and Kronecker products of these matrices are obtained, respectively. The determinants and

inverses of Tribonacci circulant type matrices are discussed in "Explicit Form of the Inverse Matrices of Tribonacci Circulant Type Matrices" by L. Liu and Z. Jian. The definition of Tribonacci circulant type matrices is given firstly. Then, the invertibility of Tribonacci circulant type matrices is studied. Based on constructing the transformation matrices, both the determinant and the inverse matrix are derived. Furthermore, by utilizing the relation between left circulant, g-circulant matrices, and circulant matrix, the invertibility of Tribonacci left circulant and Tribonacci g-circulant matrices is also studied. Finally, the determinants and inverse matrices of these matrices are presented, respectively. A future research direction is pointed out at last. In the paper entitled "Analysis of the Structured Perturbation for the BCSCB Linear System" by X. Tang and Z. Jian, the analysis problem associated with the BCSCB matrix is considered. The BCSCB matrix is an extension of the circulant matrix and skew circulant matrix. Firstly, the form of the BCSCB matrix is obtained based on the style spectral decomposition of the basic circulant matrix and the basic skew circulant matrix. Then, the structured perturbation analysis for BCSCB linear system is proposed, which includes the condition number and relative error of the BSCSB linear system. A new approach is presented to derive the minimal value of the perturbation bound, which is only related to the perturbation of the coefficient matrix and the vector. Simultaneously, the algorithm for the optimal backward perturbation bound is developed.

As is well known, the analysis of issues on network systems has important significance. In the paper entitled "On the Incidence Energy of Some Toroidal Lattices" by J.-B. Liu, et al., the closed-form formulae expressing the incidence energy of the 3.12.12 lattice and triangular kagomé lattice are derived as well as $S(m,n)$ lattice. The calculations of the energy of graphs become a popular topic of research. However, it is not an easy task to deal with the problem of the asymptotic incidence energy of various lattices with the free boundary. By utilizing the applications of analysis approach with the help of software calculation, the explicit asymptotic values of the incidence energy in these lattices are derived simultaneously. This developed method can be used widely to handle the asymptotic behaviour of other lattices and can obtain some useful results simultaneously. For the purpose of studying the distribution of evolving networks, a kind of evolving network is proposed in "Asymptotic Degree Distribution of a Kind of Asymmetric Evolving Network" by Z. Li et al., where the model is a combination of preferential attachment model and uniform model. The distribution of the number of vertices with given degree is studied as well as the asymptotic degree distribution. It is shown that the proportional degree sequence obeys power law, exponential distribution, and other forms according to the relation of the degree and parameter m. In the work entitled "Partial Synchronizability Characterized by Principal Quasi-Submatrices Corresponding to Clusters" by G. Zhang et al., a partial synchronization problem is studied in an oscillator network. In order to investigate the partial synchronization, the concept on a principal quasi-submatrix corresponding to the topology of a cluster is proposed. A novel criterion on partial synchronization is developed based on the analysis of principal quasi-submatrices corresponding to the clusters. The proposed criterion is not distinctly dependent on the intercluster couplings or the topology matrix of the whole network. If a network is composed of a large number of nodes, the enormous amount of calculation can be reduced by replacing the coupling matrix with several quasi-submatrices. Therefore, this criterion provides a novel index of partial synchronizability. It is shown that different types of partial synchronization occur in a star-global network when the coupling strength is increased. The proposed approach for partial synchronization might be applicable to the complex networks with networked induced phenomena. A backbone extraction heuristic with incomplete information (BEHwII) is investigated in "Extracting Backbones from Weighted Complex Networks with Incomplete Information" by L. Qian et al. The presence of the backbone is the signature or the abstraction of the nature of complex systems and can provide huge help for understanding them in more simplified forms. For the purpose of extracting backbones from large-scale weighted networks, a novel filter-based approach is presented which only needs incomplete information and then invokes the iteratively local search scheme for improving the efficiency. First, a strict filtering rule is designed to determine edges to be preserved or discarded. Then, a local search model is proposed to examine part of edges in an iterative way. Experimental results on four real-life networks demonstrate the advantage of BEHwII over the classic disparity filter method by either effectiveness or efficiency validity.

Recently, the application of networked systems has attracted a great deal of research interest. The data communication networks play an important role during the development of smart grid. Since the data communication network in smart gird is affected by plenty of decisive factors, different decision-making problems are presented according to the variable factors. In the paper entitled "Real-Time Pricing Decision Making for Retailer-Wholesaler in Smart Grid Based on Game Theory" by Y. Dai and Y. Gao, a novel game-theoretical decision-making scheme is investigated for electricity retailers and wholesaler in the smart grid with demand side management (DSM). The interaction between two retailers and their wholesaler is modelled by a two-stage dynamic game where the competition between two retailers is considered. According to the different action order between retailers and their wholesaler, two different game models are developed. The subgame perfect Nash equilibrium (SPE) for this game is determined through backward induction. It is shown that the wholesaler wants to decentralize certain management powers to the retailers through analysing the equilibrium revenues of the retailers for different situations. Imposing legal restrictions on the wholesaler's discretionary policy suggests that the time-inconsistency problem is mitigated. The packing problem of unit equilateral triangles is investigated in "A New Quasi-Human Algorithm for Solving the Packing Problem of Unit Equilateral Triangles" by R. Wang et al. The packing problem of unit equilateral triangles offers broad prospects in different fields including the network resource optimization. This problem is non-deterministic polynomial (NP) hard and has the feature of continuity. A novel quasi-human algorithm for solving this

problem is proposed according to the characteristic of the unit equilateral triangles and in the base of analysis of the general triangles packing problems. Time complexity analysis and the calculation results indicate that the proposed method is a polynomial time algorithm, which provides the possibility to solve the packing problem of arbitrary triangles.

Acknowledgments

This special issue is a timely reflection of the research progress in the area of networked systems with incomplete information. We would like to acknowledge all authors for their efforts in submitting high-quality papers. We are also very grateful to the reviewers for their thorough and on-time reviews of the papers.

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Research Article

Analysis of the Structured Perturbation for the BCSCB Linear System

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Circulant and block circulant type matrices are important tools in solving networked systems. In this paper, based on the style spectral decomposition of the basic circulant matrix and the basic skew circulant matrix, the block style spectral decomposition of the BCSCB matrix is obtained. And then, the structure perturbation is analysed, which includes the condition number and relative error of the BCSCB linear system. Then the optimal backward perturbation bound of the BCSCB linear system is discussed. Simultaneously, the algorithm for the optimal backward perturbation bound is given. Finally, a numerical example is provided to verify the effectiveness of the algorithm.

1. Introduction

It is an active objective that circulant and block circulant type matrices are applied to networks engineering. The stability region in the parameters space is extended by the breaking of a delayed ring neural network where the form of time-delay systems is $\dot{x} + Ax(t) + Bx(t-\tau) = 0$, where B is a circulant matrix, if the number of the neurons is sufficiently large in [1]. In [2], the question of when circulant quantum spin networks with nearest-neighbor couplings can give perfect state transfer is solved. The properties of linear diffusion algorithm are investigated both by a worst-case analysis and by a probabilistic analysis and are shown to depend on the spectral properties of the circulant matrix in [3]. A viable option for increasing the lifetime of the sensor network for a small loss in accuracy of the query results whose matrices are circulant is offered in [4]. In [5], the authors considered the kinetics of an autocatalytic reaction network in which replication and catalytic actions are separated by a translation step. They found that the behavior of such a system is closely related to second-order replicator equations, where the second-order replicator equations are circulant interaction matrices. In order to obtain the optimal routing in double loop networks, the problem of finding the shortest path in circulant graphs with an arbitrary number of jumps is studied in [6].

A block circulant with skew circulant blocks matrix with the first row $(c_{11}, \ldots, c_{1m}, c_{21}, \ldots, c_{2m}, \ldots, c_{n1}, \ldots, c_{nm})$ has the following form:

$$\mathbb{C} = \begin{pmatrix} C_1 & C_2 & \cdots & C_{n-1} & C_n \\ C_n & C_1 & C_2 & \cdots & C_{n-1} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ C_3 & \ddots & C_n & C_1 & C_2 \\ C_2 & C_3 & \cdots & C_n & C_1 \end{pmatrix}, \tag{1}$$

and for any $k = 1, 2, \ldots, n$,

$$C_k = \begin{pmatrix} c_{k1} & c_{k2} & \cdots & c_{k(m-1)} & c_{km} \\ -c_{km} & c_{k1} & c_{k2} & \cdots & c_{k(m-1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ -c_{k3} & \ddots & -c_{km} & c_{k1} & c_{k2} \\ -c_{k2} & -c_{k3} & \cdots & -c_{km} & c_{k1} \end{pmatrix}. \tag{2}$$

The matrix $\mathbb{C}$ is denoted by BCSCB$(c_{11}, \ldots, c_{1m}, \ldots, c_{n1}, \ldots, c_{nm})$.

Rigal and Gaches [7] considered a posteriori analysis of the compatibility of a computed solution to the uncertain data of a linear system by some new theorems generalizing a result of Oettli and Prager. In [8], the style spectral decomposition of the skew circulant matrix is given and the optimal backward perturbation analysis for the skew circulant linear system is discussed. Liu and Guo [9] obtained the bound of the optimal backward perturbation for a block circulant linear system. J.-G. Sun and Z. Sun [10] studied the optimal backward perturbation bounds for undetermined systems. In [11], the optimal backward perturbation analysis for the block skew circulant linear system with skew circulant blocks is given by Li et al.

2. The Block Style Spectral Decomposition of the BCSCB Matrix

Let matrix $\mathbb{C}$ be a BCSCB matrix as the form (1); then by using the properties of Kronecker products in [12], the $\mathbb{C}$ can be decomposed as

$$\mathbb{C} = \sum_{k=1}^{n} \left(\Pi^{k-1} \otimes C_k\right), \tag{3}$$

where Π is a square matrix of order n, and it has the following form:

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{4}$$

Based on (2.5) and (2.6) in [9], the style spectral decomposition of the matrix Π is

$$\Pi = Q\Pi_0 Q^T, \tag{5}$$

where Q is an orthogonal matrix.

When n is even,

$$\begin{gathered} \Pi_0 = \begin{pmatrix} \pi_1 & & & \\ & \pi_2 & & \\ & & \ddots & \\ & & & \pi_{n/2} \end{pmatrix}, \\ \pi_{n/2} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \pi_j = \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}, \\ \theta_j = \frac{2j}{n}\pi, \quad j = 1, 2, \ldots, \frac{n}{2} - 1. \end{gathered} \tag{6}$$

When n is odd,

$$\Pi_0 = \begin{pmatrix} \pi_1 & & & & \\ & \pi_2 & & & \\ & & \ddots & & \\ & & & \pi_{(n-1)/2} & \\ & & & & 1 \end{pmatrix},$$

$$\begin{gathered} \pi_j = \begin{pmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{pmatrix}, \qquad \theta_j = \frac{2j}{n}\pi, \\ j = 1, 2, \ldots, \frac{n-1}{2}. \end{gathered} \tag{7}$$

Taking (3) and (5) into consideration, the matrix $\mathbb{C}$ can be decomposed as

$$\begin{aligned} \mathbb{C} &= \sum_{k=1}^{n} \left(\Pi^{k-1} \otimes C_k\right) \\ &= \sum_{k=1}^{n} \left(Q\Pi_0^{k-1}Q^T\right) \otimes C_k \\ &= \sum_{k=1}^{n} \left(Q \otimes I_m\right)\left(\Pi_0^{k-1} \otimes C_k\right)\left(Q^T \otimes I_m\right) \\ &= \left(Q \otimes I_m\right) \sum_{k=1}^{n} \left(\Pi_0^{k-1} \otimes C_k\right)\left(Q^T \otimes I_m\right). \end{aligned} \tag{8}$$

$Q \otimes I_m$ is an orthogonal matrix obviously. So (8) is the block style spectral decomposition of the matrix $\mathbb{C}$.

3. Analysis of the Structured Perturbation

The structured perturbation analysis for BCSCB linear system is given in this section. We discuss the condition number and the relative error of the BCSCB linear system. The optimal backward perturbation bound of the BCSCB linear system is analysed. And, at the end of the section, we give the algorithm for the optimal backward perturbation bound.

3.1. Condition Number and Relative Error of BCSCB Linear System. Consider

$$\mathbb{C}x = b, \tag{9}$$

where $\mathbb{C}$ is defined in (1).

From (8) and the property of Kronecker products in [12], the matrix $\mathbb{C}$ can be expressed by using the elements in its first row as

$$\begin{aligned} \mathbb{C} &= \sum_{k=1}^{n} \left(\Pi^{k-1} \otimes C_k\right) \\ &= \sum_{k=1}^{n} \left[\Pi^{k-1} \otimes \left(\sum_{l=1}^{m} c_{kl}\Psi^{l-1}\right)\right] \\ &= \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\left(\Pi^{k-1} \otimes \Psi^{l-1}\right), \end{aligned} \tag{10}$$

where Ψ is a square matrix of order m, and it has the following form:

$$\Psi = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 1 \\ -1 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{11}$$

Based on (10) and (11) in [8], the style spectral decomposition of the matrix Ψ is

$$\Psi = J\Psi_0 J^T, \tag{12}$$

where J is an orthogonal matrix.

When m is even,

$$\Psi_0 = \begin{pmatrix} \psi_1 & & & \\ & \psi_2 & & \\ & & \ddots & \\ & & & \psi_{m/2} \end{pmatrix}. \tag{13}$$

When m is odd,

$$\Psi_0 = \begin{pmatrix} \psi_1 & & & & \\ & \psi_2 & & & \\ & & \ddots & & \\ & & & \psi_{(m-1)/2} & \\ & & & & -1 \end{pmatrix},$$

$$\psi_h = \begin{pmatrix} \cos\theta_h & \sin\theta_h \\ -\sin\theta_h & \cos\theta_h \end{pmatrix}, \qquad \theta_h = \frac{2h-1}{n}\pi, \tag{14}$$

$$h = \begin{cases} 1, 2, \ldots, \dfrac{m}{2}, & m \text{ is even.} \\ 1, 2, \ldots, \dfrac{m-1}{2}, & m \text{ is odd.} \end{cases}$$

Furthermore, (10) can be expressed as follows:

$$\mathbb{C} = \mathbb{Q}\left(\sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\Pi_0^{k-1} \otimes \Psi_0^{l-1}\right)\mathbb{Q}^T, \tag{15}$$

and here $\mathbb{Q} = (Q \otimes I_m)(I_n \otimes J)$, where I_n and I_m are identity matrices with orders n and m, respectively.

The problem will be discussed at two different situations.

(1) When n is even,

$$\sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\Pi_0^{k-1} \otimes \Psi_0^{l-1} = \begin{pmatrix} \Lambda_{11} & & & \\ & \ddots & & \\ & & \Lambda_{tt} & \\ & & & \Upsilon_1 \end{pmatrix},$$

$$\Lambda_{pp} = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\pi_p^{k-1} \otimes \Psi_0^{l-1}, \tag{16}$$

$$t = \frac{n}{2} - 1, \quad p = 1, 2, \ldots, t,$$

$$\Upsilon_1 = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\,\mathrm{diag}\left(-\Psi_0^{l-1}, \Psi_0^{l-1}\right).$$

(2) When n is odd,

$$\sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\Pi_0^{k-1} \otimes \Psi_0^{l-1} = \begin{pmatrix} \Lambda_{11} & & & \\ & \ddots & & \\ & & \Lambda_{tt} & \\ & & & \Upsilon_2 \end{pmatrix},$$

$$\Lambda_{pp} = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\pi_p^{k-1} \otimes \Psi_0^{l-1},$$

$$t = \frac{n-1}{2}, \quad p = 1, 2, \ldots, t,$$

$$\Upsilon_2 = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\Psi_0^{l-1}. \tag{17}$$

We denote by ω_i $(i = 1, 2, \ldots, n)$ the eigenvalues of matrix Π [9], and δ_j $(j = 1, 2, \ldots, m)$ are denoted as the eigenvalues of matrix Ψ [8], and then the eigenvalues of $\mathbb{C}$ are obtained (refer to [12, 13]). Consider

$$\lambda_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\omega_i^{k-1}\delta_j^{l-1}. \tag{18}$$

Lemma 1. *$\mathbb{C}$ is a nonsingular matrix if and only if $f(\omega_i, \delta_j) \neq 0$ $(i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m)$, where*

$$f\left(\omega_i, \delta_j\right) = \lambda_{ij} = \sum_{k=1}^{n}\sum_{l=1}^{m} c_{kl}\omega_i^{k-1}\delta_j^{l-1}. \tag{19}$$

Let

$$\sigma_{ij} = \left|f\left(\omega_i, \delta_j\right)\right|, \quad i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m,$$

$$\kappa = \frac{\max\left\{\sigma_{ij}\right\}}{\min\left\{\sigma_{ij}\right\}}. \tag{20}$$

Theorem 2. *If $\mathbb{C} = BCSCB(c_{11}, \ldots, c_{1m}, \ldots, c_{n1}, \ldots, c_{nm})$, then the singular values of the matrix $\mathbb{C}$ are $\sigma_{11}, \ldots, \sigma_{1m}, \sigma_{21}, \ldots, \sigma_{2m}, \ldots, \sigma_{n1}, \ldots, \sigma_{nm}$.*

Proof. Assume the conjugate transpose of $\mathbb{C}$ is

$$\mathbb{C}^* = \begin{pmatrix} C_1^* & C_n^* & \cdots & C_3^* & C_2^* \\ C_2^* & C_1^* & \cdots & \vdots & C_3^* \\ \vdots & C_2^* & \cdots & C_n^* & \vdots \\ C_{n-1}^* & \vdots & \ddots & C_1^* & C_n^* \\ C_n^* & C_{n-1}^* & \cdots & C_2^* & C_1^* \end{pmatrix}. \tag{21}$$

By a direct calculation, $\mathbb{C}$ is a normal matrix as $\mathbb{C}\mathbb{C}^* = \mathbb{C}^*\mathbb{C}$. Then matrix $\mathbb{C}$ is a unitarily diagonalizable matrix based on Theorem 2.5.4 in [14]. Then there exists a unitary matrix $\mathbb{U} \in M_{mn}$, such that

$$\mathbb{U}^*\mathbb{C}\mathbb{U} = \Lambda = \mathrm{diag}\left(\lambda_{11}, \ldots, \lambda_{1m}, \ldots, \lambda_{n1}, \ldots, \lambda_{nm}\right), \tag{22}$$

where λ_{ij} $(i = 1, 2, \ldots, n, j = 1, 2, \ldots, m)$ are the eigenvalues of matrix $\mathbb{C}$. Taking the conjugate transpose at both sides of (22), we get

$$\mathbb{U}^*\mathbb{C}^*\mathbb{U} = \Lambda = \mathrm{diag}\left(\overline{\lambda}_{11}, \ldots, \overline{\lambda}_{1m}, \ldots, \overline{\lambda}_{n1}, \ldots, \overline{\lambda}_{nm}\right); \tag{23}$$

then

$$\begin{aligned}\mathbb{U}^*(\mathbb{C}^*\mathbb{C})\mathbb{U} &= (\mathbb{U}^*\mathbb{C}^*\mathbb{U})(\mathbb{U}^*\mathbb{C}\mathbb{U}) \\ &= \operatorname{diag}\left(|\lambda_{11}|^2,\ldots,|\lambda_{1m}|^2,\ldots,|\lambda_{n1}|^2,\ldots,|\lambda_{nm}|^2\right).\end{aligned} \tag{24}$$

And $|\lambda_{ij}|^2$ are the eigenvalues of the matrix $\mathbb{C}^*\mathbb{C}$, for any $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. Therefore, the singular values of $\mathbb{C}$ are

$$\sigma_{ij}(\mathbb{C}) = \left[\lambda_{ij}(\mathbb{C}^*\mathbb{C})\right]^{1/2} = |\lambda_{ij}|. \tag{25}$$

Recall (19) and (20); the proof is completed. □

As the definition of the spectral norm of matrix $\mathbb{C}$ is

$$\|\mathbb{C}\|_2 = \max_{\substack{1\le i\le n\\1\le j\le m}}\left[\lambda_{ij}(\mathbb{C}^*\mathbb{C})\right]^{1/2}, \tag{26}$$

via Theorem 2, the following corollary is obtained.

Corollary 3. *Let $\mathbb{C} = BCSCB(c_{11},\ldots,c_{1m},\ldots,c_{n1},\ldots,c_{nm})$; then the spectrum norm of matrix $\mathbb{C}$ is*

$$\|\mathbb{C}\|_2 = \max_{\substack{1\le i\le n\\1\le j\le m}}\{\sigma_{ij}\}. \tag{27}$$

Let $\Delta\mathbb{C}$ be a perturbation of the coefficient matrix $\mathbb{C}$ and let Δb be a perturbation of the vector b, where $\Delta\mathbb{C} = \mathrm{BCSCB}(\varepsilon c_{11},\ldots,\varepsilon c_{1m},\ldots,\varepsilon c_{n1},\ldots,\varepsilon c_{nm})$ has the following form:

$$\Delta\mathbb{C} = \begin{pmatrix} \Delta C_1 & \cdots & \Delta C_{n-1} & \Delta C_n \\ \Delta C_n & \Delta C_1 & \cdots & \Delta C_{n-1} \\ \vdots & \ddots & \ddots & \vdots \\ \Delta C_2 & \cdots & \Delta C_n & \Delta C_1 \end{pmatrix}, \tag{28}$$

and for any $k = 1,2,\ldots,m$,

$$\Delta C_k = \begin{pmatrix} \varepsilon c_{k1} & \cdots & \varepsilon_{k(m-1)} & \varepsilon c_{km} \\ -\varepsilon c_{km} & \varepsilon c_{k1} & \cdots & \varepsilon c_{k(m-1)} \\ \vdots & \ddots & \ddots & \vdots \\ -\varepsilon c_{k2} & \cdots & -\varepsilon c_{km} & \varepsilon c_{k1} \end{pmatrix}. \tag{29}$$

Let

$$\begin{gathered}\widehat{\mathbb{C}} = \mathbb{C} + \Delta\mathbb{C}, \qquad \widehat{b} = b + \Delta b, \qquad \Delta b = \varepsilon b, \\ \widehat{f}(\omega_i,\delta_j) = \sum_{k=1}^{n}\sum_{l=1}^{m}(c_{kl} + \varepsilon c_{kl})\,\omega_i^{k-1}\delta_j^{l-1}.\end{gathered} \tag{30}$$

If

$$\sum_{k=1}^{n}\sum_{l=1}^{m}|\varepsilon c_{kl}| < \min_{\substack{1\le i\le n\\1\le j\le m}}\{\sigma_{ij}\}, \tag{31}$$

then

$$\begin{aligned}\left|\widehat{f}(\omega_i,\delta_j)\right| &= \left|\sum_{k=1}^{n}\sum_{l=1}^{m}(c_{kl} + \varepsilon c_{kl})\,\omega_i^{k-1}\delta_j^{l-1}\right| \\ &\ge \left|\sum_{k=1}^{n}\sum_{l=1}^{m}c_{kl}\omega_i^{k-1}\delta_j^{l-1}\right| \\ &\quad - \sum_{k=1}^{n}\sum_{l=1}^{m}|\varepsilon c_{kl}|\,|\omega_i|^{k-1}\,|\delta_j|^{l-1} \\ &\ge \min_{\substack{1\le i\le n\\1\le j\le m}}\{\sigma_{ij}\} - \sum_{k=1}^{n}\sum_{l=1}^{m}|\varepsilon c_{kl}| > 0;\end{aligned} \tag{32}$$

through Lemma 1, $\widehat{\mathbb{C}}$ is a nonsingular matrix. Let

$$\sigma_{\min} = \min_{\substack{1\le i\le n\\1\le j\le m}}\{\sigma_{ij}\}, \qquad \rho = \sum_{k=1}^{n}\sum_{l=1}^{m}|\varepsilon c_{kl}|. \tag{33}$$

By $\mathbb{C}x = b$, $\widehat{\mathbb{C}}\widehat{x} = \widehat{b}$, we obtain

$$\begin{aligned}\widehat{x} - x &= \widehat{\mathbb{C}}^{-1}\widehat{b} - \mathbb{C}^{-1}b \\ &= \widehat{\mathbb{C}}^{-1}(b + \varepsilon b) - \mathbb{C}^{-1}b \\ &= \widehat{\mathbb{C}}^{-1}\varepsilon b + \left(\widehat{\mathbb{C}}^{-1} - \mathbb{C}^{-1}\right)b \\ &= \widehat{\mathbb{C}}^{-1}\varepsilon b + \left(\widehat{\mathbb{C}}^{-1} - \mathbb{C}^{-1}\right)\mathbb{C}x \\ &= \widehat{\mathbb{C}}^{-1}\varepsilon b + \widehat{\mathbb{C}}^{-1}\left(\mathbb{C} - \widehat{\mathbb{C}}\right)x, \\ \|\widehat{x} - x\|_2 &\le \left\|\widehat{\mathbb{C}}^{-1}\right\|_2\|\varepsilon b\|_2 + \left\|\widehat{\mathbb{C}}^{-1}\right\|_2\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2\|x\|_2 \\ &\le \frac{\|\varepsilon b\|_2}{\sigma_{\min} - \rho} + \frac{\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2\|x\|_2}{\sigma_{\min} - \rho}, \\ \frac{\|\widehat{x} - x\|_2}{\|x\|_2} &\le \frac{\|\varepsilon b\|_2}{(\sigma_{\min} - \rho)\|x\|_2} + \frac{\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2}{\sigma_{\min} - \rho} \\ &= \frac{\|\mathbb{C}\|_2}{\sigma_{\min} - \rho}\left[\frac{\|\varepsilon b\|_2}{\|\mathbb{C}\|_2\|x\|_2} + \frac{\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2}{\|\mathbb{C}\|_2}\right] \\ &\le \frac{\|\mathbb{C}\|_2}{\sigma_{\min} - \rho}\left[\frac{\|\varepsilon b\|_2}{\|b\|_2} + \frac{\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2}{\|\mathbb{C}\|_2}\right],\end{aligned} \tag{34}$$

where

$$\|\mathbb{C}\|_2 = \max_{\substack{1\le i\le n\\1\le j\le m}}\{\sigma_{ij}\}. \tag{35}$$

$\widehat{\mathbb{C}} - \mathbb{C} = \Delta\mathbb{C}$ is a BCSCB matrix apparently, and $\|\mathbb{C} - \widehat{\mathbb{C}}\|_2 = |-1|\|\widehat{\mathbb{C}} - \mathbb{C}\|_2 = \|\widehat{\mathbb{C}} - \mathbb{C}\|_2$. So

$$\begin{aligned}\left\|\widehat{\mathbb{C}} - \mathbb{C}\right\|_2 &= \max_{\substack{1\le i\le n\\ 1\le j\le m}} \left|\sum_{k=1}^{n}\sum_{l=1}^{m} \varepsilon c_{kl}\omega_i^{k-1}\delta_j^{l-1}\right| \\ &\le \max_{\substack{1\le i\le n\\ 1\le j\le m}} \sum_{k=1}^{n}\sum_{l=1}^{m} |\varepsilon c_{kl}|\,|\omega_i|^{k-1}\left|\delta_j\right|^{l-1} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{m} |\varepsilon c_{kl}| = \rho.\end{aligned} \tag{36}$$

The following theorem can be obtained.

Theorem 4. *Let $\mathbb{C}$, $\widehat{\mathbb{C}}$, Δb, ρ, and $\sigma_{\min}$ be defined as above. If $\rho < \sigma_{\min}$, then*

$$\frac{\|\widehat{x} - x\|_2}{\|x\|_2} \le \frac{\sigma_{\max}}{\sigma_{\min} - \rho}\left(\frac{\|\varepsilon b\|_2}{\|b\|_2} + \frac{\rho}{\sigma_{\max}}\right), \tag{37}$$

where

$$\sigma_{\max} = \|\mathbb{C}\|_2. \tag{38}$$

Remark 5. The condition number κ of the BCSCB matrix can be easily computed with the basis of (37) and (38), the same as the bound of perturbation (37).

3.2. Optimal Backward Perturbation Bound of the BCSCB Linear System. In this part, a new method is given to obtain the minimal value of the perturbation bound, which is only related to the perturbation of the coefficient matrix and the vector. At the end of this part, the algorithm for the optimal backward perturbation bound is given.

Let $\widehat{x}$ be an approximate solution to $\mathbb{C}x = b$ and let

$$\begin{gathered}\Omega \equiv \{(\Delta\mathbb{C}, \Delta b) \mid (\mathbb{C} + \Delta\mathbb{C})\,\widehat{x} = b + \Delta b\}, \\ \xi(\widehat{x}) \equiv \inf_{(\Delta\mathbb{C},\Delta b)\in\Omega} \|\Delta\mathbb{C}, \Delta b\|, \\ (\mathbb{C} + \Delta\mathbb{C})\,\widehat{x} = b + \Delta b,\end{gathered} \tag{39}$$

which is equal to

$$(\Delta\mathbb{C}, \Delta b)\begin{pmatrix}\widehat{x}\\ -1\end{pmatrix} = b - \mathbb{C}\widehat{x}. \tag{40}$$

According to [7], we can get

$$\xi(\widehat{x}) = \frac{\|b - \mathbb{C}\widehat{x}\|_2}{\sqrt{1 + \|\widehat{x}\|_2^2}} \tag{41}$$

$$(\|\cdot\| \text{ is unitary invariant norm}).$$

Let $\widehat{x}$ be an approximate solution to $\mathbb{C}x = b$, where $\mathbb{C}$ is defined in (1):

$$\begin{gathered}\Omega \equiv \{(\Delta\mathbb{C}, \Delta b) \mid (\mathbb{C} + \Delta\mathbb{C})\,\widehat{x} = b + \Delta b, \\ \Delta\mathbb{C} \text{ is a BCSCB matrix}\}, \\ \xi(\widehat{x}) \equiv \inf_{(\Delta\mathbb{C},\Delta b\in\Omega)} \{\|\Delta\mathbb{C}, \Delta b\|_F\}.\end{gathered} \tag{42}$$

So $\Omega \ne \phi$ (as $\Delta\mathbb{C} = 0$ is a BCSCB matrix, $\Delta b = \widehat{\mathbb{C}}\widehat{x} - b$). Hence,

$$\xi^2(\widehat{x}) = \inf_{(\Delta\mathbb{C},\Delta b)\in\Omega} \left\{\|\Delta\mathbb{C}\|_F^2 + \|\Delta\mathbb{C}\widehat{x} + \mathbb{C}\widehat{x} - b\|_F^2\right\}. \tag{43}$$

Since

$$\begin{gathered}\|\Delta\mathbb{C}\|_F^2 = mn\sum_{k=1}^{n}\sum_{l=1}^{m}(\varepsilon c_{kl})^2, \\ \Delta\mathbb{C} = \mathbb{Q}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\Pi_0^{k-1}\otimes\Psi_0^{l-1}\right)\mathbb{Q}^T,\end{gathered} \tag{44}$$

the question will be analysed in two different conditions.

(1) When n is even,

$$\begin{aligned}&\|\Delta\mathbb{C}\widehat{x} + \mathbb{C}\widehat{x} - b\|_F^2 \\ &= \left\|\mathbb{Q}\begin{pmatrix}\varepsilon\Lambda_{11} & & & \\ & \ddots & & \\ & & \varepsilon\Lambda_{tt} & \\ & & & \varepsilon\Upsilon_1\end{pmatrix}\mathbb{Q}^T\widehat{x} + \mathbb{C}\widehat{x} - b\right\|_F^2 \\ &= \left\|\begin{pmatrix}\varepsilon\Lambda_{11} & & & \\ & \ddots & & \\ & & \varepsilon\Lambda_{tt} & \\ & & & \varepsilon\Upsilon_1\end{pmatrix}\begin{pmatrix}x_1^{(0)}\\ \vdots\\ x_t^{(0)}\\ x_{t+1}^{(0)}\end{pmatrix} - r_0\right\|_F^2 \\ &= \left\|\begin{pmatrix}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\pi_1^{k-1}\otimes\Psi_0^{l-1}\right)x_1^{(0)}\\ \vdots\\ \left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\pi_t^{k-1}\otimes\Psi_0^{l-1}\right)x_t^{(0)}\\ \left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\operatorname{diag}\left(-\Psi_0^{l-1},\Psi_0^{l-1}\right)\right)x_{t+1}^{(0)}\end{pmatrix} - r_0\right\|_F^2, \\ &= \left\|\Phi\left(\varepsilon c_{11},\ldots,\varepsilon c_{1m},\ldots,\varepsilon c_{n1},\ldots,\varepsilon c_{nm}\right)^T - r_0\right\|_F^2,\end{aligned} \tag{45}$$

where

$$\begin{gathered}r_0 = \mathbb{Q}^T(b - \mathbb{C}\widehat{x}), \\ \mathbb{Q}^T\widehat{x} = \left(x_1^{(0)} \;\cdots\; x_t^{(0)} \;\; x_{t+1}^{(0)}\right)^T, \\ \Phi = \left(\Phi_1, \Phi_2, \ldots, \Phi_{n-1}, \sum_{l=1}^{m}\operatorname{diag}\left(-\Psi_0^{l-1},\Psi_0^{l-1}\right)x_{t+1}^{(0)}\right), \\ \Phi_k = \begin{pmatrix}\phi_{1,k,1} & \cdots & \phi_{1,k,m}\\ \vdots & \ddots & \vdots\\ \phi_{t,k,1} & \cdots & \phi_{t,k,m}\end{pmatrix}, \\ \phi_{p,k,l} = \pi_p^{k-1}\otimes\Psi_0^{l-1}x_p^{(0)}, \\ t = \frac{n}{2} - 1, \quad p = 1, 2, \ldots, t, \\ k = 1, 2, \ldots, n-1, \quad l = 1, 2, \ldots, m.\end{gathered} \tag{46}$$

(2) When n is odd,

$$\begin{aligned}
&\|\Delta\mathbb{C}\hat{x}+\mathbb{C}\hat{x}-b\|_F^2 \\
&= \left\|\mathbb{Q}\begin{pmatrix}\varepsilon\Lambda_{11} & & & \\ & \ddots & & \\ & & \varepsilon\Lambda_{tt} & \\ & & & \varepsilon\Upsilon_2\end{pmatrix}\mathbb{Q}^T\hat{x}+\mathbb{C}\hat{x}-b\right\|_F^2 \\
&= \left\|\begin{pmatrix}\varepsilon\Lambda_{11} & & & \\ & \ddots & & \\ & & \varepsilon\Lambda_{tt} & \\ & & & \varepsilon\Upsilon_2\end{pmatrix}\begin{pmatrix}x_1^{(0)}\\ \vdots \\ x_t^{(0)} \\ x_{t+1}^{(0)}\end{pmatrix}-r_0\right\|_F^2 \\
&= \left\|\begin{pmatrix}\left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\pi_1^{k-1}\otimes\Psi_0^{l-1}\right)x_1^{(0)} \\ \vdots \\ \left(\sum_{k=1}^{n}\sum_{l=1}^{m}\varepsilon c_{kl}\pi_t^{k-1}\otimes\Psi_0^{l-1}\right)x_t^{(0)} \\ \left(\sum_{k=1}^{n}\sum_{l=1}^{n}\varepsilon c_{kl}\Psi_0^{l-1}\right)x_{t+1}^{(0)}\end{pmatrix}-r_0\right\|_F^2 \\
&= \left\|\Phi\left(\varepsilon c_{11},\ldots,\varepsilon c_{1m},\ldots,\varepsilon c_{n1},\ldots,\varepsilon c_{nm}\right)^T-r_0\right\|_F^2,
\end{aligned} \tag{47}$$

where

$$\begin{gathered}
r_0=\mathbb{Q}^T(b-\mathbb{C}\hat{x}), \\
\mathbb{Q}^T\hat{x}=\left(x_1^{(0)}\ \cdots\ x_t^{(0)}\ x_{t+1}^{(0)}\right)^T, \\
\Phi=\left(\Phi_1,\Phi_2,\ldots,\Phi_{n-1},\sum_{l=1}^{m}\Psi_0^{l-1}x_{t+1}^{(0)}\right), \\
\Phi_k=\begin{pmatrix}\phi_{1,k,1} & \cdots & \phi_{1,k,m} \\ \vdots & \ddots & \vdots \\ \phi_{t,k,1} & \cdots & \phi_{t,k,m}\end{pmatrix}, \\
\phi_{p,k,l}=\pi_p^{k-1}\otimes\Psi_0^{l-1}x_p^{(0)}, \\
t=\frac{n-1}{2},\quad p=1,2,\ldots,t, \\
k=1,2,\ldots,n-1,\quad l=1,2,\ldots,m.
\end{gathered} \tag{48}$$

Let

$$g\left(\varepsilon c_{11},\ldots,\varepsilon c_{nm}\right)=mn\sum_{k=1}^{n}\sum_{l=1}^{m}\left(\varepsilon c_{kl}\right)^2+\left\|\Phi\begin{pmatrix}\varepsilon c_{11}\\ \vdots \\ \varepsilon c_{nm}\end{pmatrix}-r_0\right\|_F^2, \tag{49}$$

and then

$$\frac{\partial g}{\partial\varepsilon c_{kl}}=0, \tag{50}$$

which is equal to

$$\begin{gathered}
\left(2mnI_{mn}+2\Phi^T\Phi\right)\begin{pmatrix}\varepsilon c_{11}\\ \vdots \\ \varepsilon c_{nm}\end{pmatrix}-2\Phi^Tr_0=0, \\
\frac{\partial^2 g}{\partial(\varepsilon c_{kl})^2}=2mnI_{mn}+2\Phi^T\Phi>0.
\end{gathered} \tag{51}$$

As g is a convex function of $(\varepsilon c_{11},\ldots,\varepsilon c_{nm})$, the point of the minimal value is

$$\begin{pmatrix}\varepsilon c_{11}\\ \vdots \\ \varepsilon c_{nm}\end{pmatrix}=\left(mnI_{mn}+\Phi^T\Phi\right)^{-1}\Phi^Tr_0. \tag{52}$$

Substituting it back into (49), we obtain the following.

Theorem 6. *Consider*

$$\begin{aligned}
\xi^2(x)&=mnr_0^T\Phi\left(mnI_{mn}+\Phi^T\Phi\right)^{-2}\Phi^Tr_0 \\
&=\left\|\left[\Phi\left(mnI_{mn}+\Phi^T\Phi\right)^{-1}\Phi^T-I_{mn}\right]r_0\right\|_F^2.
\end{aligned} \tag{53}$$

Let $\Phi=U\Sigma V^*$ be the singular value decomposition of Φ, where U and V are unitary matrices, $\Sigma=\operatorname{diag}(\sigma_1',\ldots,\sigma_{nm}')$, and $\sigma_j'\geq 0$ $(j=1,2,\ldots,nm)$; then

$$\begin{aligned}
\xi^2(x)&=mnr_0^TU\Sigma V^T\left(mnI_{mn}+\Sigma^2\right)^{-2}V\Sigma U^Tr_0 \\
&\quad+\left\|\left[U\Sigma V^T\left(mnI_{mn}+\Sigma^2\right)^{-1}V\Sigma U^T-I_{mn}\right]r_0\right\|_F^2 \\
&=mnr_1^T\Sigma\left(mnI_{mn}+\Sigma^2\right)^{-2}\Sigma r_1 \\
&\quad+\left\|\left[\Sigma\left(mnI_{mn}+\Sigma^2\right)^{-1}\Sigma-I_{mn}\right]r_0\right\|_F^2 \\
&=mnr_1^T\Sigma\left(mnI_{mn}+\Sigma^2\right)^{-2}\Sigma r_1 \\
&\quad+\left\|\left[\Sigma\left(mnI_{mn}+\Sigma^2\right)^{-1}\Sigma-I_{mn}\right]r_1\right\|_F^2 \\
&=mnr_1^T\Sigma\left(mnI_{mn}+\Sigma^2\right)^{-2}\Sigma r_1 \\
&\quad+m^2n^2r_1^T\left(mnI_{mn}+\Sigma^2\right)^{-2}r_1 \\
&=r_1^T\begin{pmatrix}\alpha_1 & & \\ & \ddots & \\ & & \alpha_{mn}\end{pmatrix}r_1,
\end{aligned} \tag{54}$$

where $r_1=U^Tr_0$, $\alpha_j=(mn\sigma_j'^2+m^2n^2)/(mn+\sigma_j'^2)^2=mn/(mn+\sigma_j'^2)$, $j=1,2,\ldots,mn$.

Remark 7. As $\sigma_j^2\leq\|\Phi\|_F^2=mn\|\hat{x}\|_2^2$, then $1+\|\hat{x}\|_2^2\geq 1+\sigma_j'^2/mn$ can be obtained; hence, $mn/(mn+\sigma_j'^2)\geq 1/(1+\|\hat{x}\|_2^2)$.

From what we analysed above, the following algorithm can be obtained.

Algorithm 8. We have the following steps.

Step 1. Form the style spectral decomposition of the matrixes Π and Ψ:

$$\Pi = Q\Pi_0 Q^T, \qquad \Psi = J\Psi_0 J^T. \tag{55}$$

Step 2. Form the block style spectral decomposition of the BCSCB matrix.

Step 3. Compute $r = b - \mathbb{C}\widehat{x}$.

Step 4. Compute $r_0 = \mathbb{Q}^T r$.

Step 5. Compute

$$\mathbb{Q}^T \widehat{x} = \begin{pmatrix} x_1^{(0)} \\ \vdots \\ x_t^{(0)} \\ x_{t+1}^{(0)} \end{pmatrix}. \tag{56}$$

Step 6. Form Φ.

Step 7. Compute the singular value decomposition of Φ.

Step 8. Compute $\xi^2(\widehat{x})$.

4. Numerical Example

In this section, a simple numerical example is given to verify the conclusion above. Suppose that $n = 2$, $m = 3$ in the following example.

If the coefficient matrix of the BCSCB linear system is $\mathbb{C} = \mathrm{BCSCB}(3, 5, 2, 7, 9, 8)$ and the constant vector $b = (11, 8, 10, -8, 9, 3)^T$, now, three perturbations are given as follows:

$$\begin{aligned} \Delta\mathbb{C}_1 &= 0.01\mathrm{BCSCB}\,(3, 5, 2, 7, 9, 8)\,, \\ \Delta b_1 &= 0.01\,(1, 3, 2, 3, 1, 5)^T\,, \\ \Delta\mathbb{C}_2 &= \mathrm{BCSCB}\,(0.01, 0.02, 0, 0.015, 0.033, 0.01)\,, \\ \Delta b_2 &= (0.012, 0.03, 0.02, 0.015, 0.01, 0.021)^T\,, \\ \Delta\mathbb{C}_3 &= \mathrm{BCSCB}\,(0.01, 0.02, 0.015, 0.01, 0, 0.01)\,, \\ \Delta b_3 &= (0.012, 0.035, 0.02, 0.01, 0.021, 0.015)^T\,. \end{aligned} \tag{57}$$

From the equation $\widehat{\mathbb{C}}\widehat{x} = \widehat{b}$, where $\widehat{\mathbb{C}}$, $\widehat{b}$ are defined as above, the approximate solution of $\mathbb{C}x = b$ can be obtained as follows:

$$x = \begin{pmatrix} -1.0556 \\ 0.3472 \\ -0.9306 \\ 0.4444 \\ -0.1528 \\ 1.5694 \end{pmatrix}, \qquad \widehat{x}_1 = \begin{pmatrix} -1.0556 \\ 0.3472 \\ -0.9306 \\ 0.4444 \\ -0.1528 \\ 1.5694 \end{pmatrix},$$

Table 1: The related date of the algorithm.

	ϵ	κ	$\xi_1(\widehat{x})$	$\xi_2(\widehat{x})$
Case 0	0	4.0000	0	0
Case 1	0	4.0000	$2.2112e^{-15}$	0.3796
Case 2	$1.9396e^{-4}$	4.0024	0.0028	0.1170
Case 3	0.0028	3.9971	0.0202	0.0899

$$\widehat{x}_2 = \begin{pmatrix} -1.0553 \\ 0.3472 \\ -0.9303 \\ 0.4444 \\ -0.1528 \\ 1.5694 \end{pmatrix}, \qquad \widehat{x}_3 = \begin{pmatrix} -1.0531 \\ 0.3439 \\ -0.9297 \\ 0.4418 \\ -0.1492 \\ 1.5699 \end{pmatrix}, \tag{58}$$

where x is the solution of $\mathbb{C}x = b$ and $\widehat{x}_i$, $i = 1, 2, 3$, is the solution of $(\mathbb{C} + \Delta\mathbb{C}_i)x = b + \Delta b_i$, $i = 1, 2, 3$, respectively.

Based on Algorithm 8, we obtain Table 1, where ϵ is the relative error of the BCSCB linear system, $\kappa = \max\{\sigma_{ij}\}/\min\{\sigma_{ij}\}$ is the condition number, $\xi_1(\widehat{x}) = \|b - \mathbb{C}\widehat{x}\|_2/\sqrt{1 + \|\widehat{x}\|_2^2}$, and $\xi_2(\widehat{x})$ can be obtained from the algorithm.

From the numerical example, the accuracy of the conclusion and the effectiveness of the algorithm are verified.

5. Conclusion

In this paper, we consider the problems associated with the BCSCB matrix. The BCSCB matrix is an extension of the circulant matrix and skew circulant matrix. We give the form of the BCSCB matrix and obtain its block style spectral decomposition. The algorithm of the optimal backward perturbation is given. Furthermore, by circulant matrices technology, we will develop solving problems in [15–17].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Equalities and Inequalities for Norms of Block Imaginary Circulant Operator Matrices

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Circulant, block circulant-type matrices and operator norms have become effective tools in solving networked systems. In this paper, the block imaginary circulant operator matrices are discussed. By utilizing the special structure of such matrices, several norm equalities and inequalities are presented. The norm τ in consideration is the weakly unitarily invariant norm, which satisfies $\tau(\mathscr{A}) = \tau(U\mathscr{A}V)$. The usual operator norm and Schatten p-norm are included. Furthermore, some special cases and examples are given.

1. Introduction

Circulant-type matrices have significant applications in network systems. For example, Noual et al. [1] presented some results on the dynamical behaviours of some specific non-monotone Boolean automata networks called XOR circulant networks. In [2], the authors proposed a special class of the feedback delay networks using circulant matrices. Based on the circulant adjacency matrices of the networks induced by these interior symmetries, Aguiar and Ruan [3] analyzed the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated with homogeneous networks. Jing and Jafarkhani [4] proposed distributed differential space-time codes that work for networks with any number of relays using circulant matrices. In [5], the authors showed a structure for the decoupling of circulant symmetric arrays of more than four elements.

The well-known circulant, block circulant-type matrices and operator norms have set up the strong basis with the work in [6–19].

In this paper, let $W(H)$ denote the imaginary circulant algebra of all bounded linear operators on a complex separable Hilbert space H. The direct sum of n copies of H is denoted by $H^{(n)} = \oplus_{n\,\text{copies}} H$. If A_{jk}, $j,k = 1,2,\ldots,n$, are operators in $W(H)$, then the operator matrix (or the partitioned operator) $\mathscr{A} = [A_{jk}]$ can be considered as an operator in $W(H^{(n)})$, which is defined by $\mathscr{A}x = (\sum_{k=1}^{n} A_{1k}x_k, \ldots, \sum_{k=1}^{n} A_{nk}x_k)^T$ for every vector $x = (x_1, \ldots, x_n)^T \in H^{(n)}$.

Recall that a norm τ on $W(H)$ is called weakly unitarily invariant if $\tau(\mathscr{A}) = \tau(U\mathscr{A}U^*)$ for all $\mathscr{A} \in W(H)$ and for all unitary operators $U \in W(H)$.

The Schatten p-norms $\|\cdot\|_p$, $1 \le p < \infty$, are important examples of unitarily invariant norms, which are defined on the Schatten p-classes.

If $V_1, V_2, \ldots, V_n$ are operators in $W(H)$, we write the direct sum $\oplus_{j=1}^{n} V_j$ for the $n \times n$ block-diagonal operator matrix $\begin{pmatrix} V_1 & & 0 \\ & \ddots & \\ 0 & & V_n \end{pmatrix}$, regarded as an operator on $H^{(n)}$. Thus, $\|\oplus_{j=1}^{n} V_j\| = \max\{\|V_j\| : j = 1,2,\ldots,n\}$ and $\|\oplus_{j=1}^{n} V_j\|_p = (\sum_{j=1}^{n} \|V_j\|_p^p)^{1/p}$ for $1 \le p < \infty$. In particular, $\|\oplus_{j=1}^{n} V\| = n^{1/p}\|V\|_p$ for $1 \le p < \infty$.

The pinching inequality asserts that if $\mathscr{A} = [A_{jk}]$, then

$$\tau\left(\oplus_{j=1}^{n} A_{jj}\right) \le \tau(A). \tag{1}$$

For the operator norm and the Schatten p-norms, the inequality (1) states that

$$\max\left\{\left\|A_{jj}\right\| : j = 1, 2, \ldots, n\right\} \le \|\mathscr{A}\|, \tag{2}$$

$$\left(\sum_{j=1}^{n}\left\|A_{jj}\right\|_p^p\right)^{1/p} \le \|\mathscr{A}\|_p \tag{3}$$

for $1 \le p < \infty$. It is known [18] that for $1 < p < \infty$, equality in (3) holds if and only if $\mathscr{A}$ is block-diagonal, that is, if and only if $A_{jk} = 0$, for $j \ne k$.

2. Equalities for the Norm of Imaginary Circulant Operator Matrices

In this section, we present block imaginary circulant operator matrix. By combining the special properties of block imaginary circulant operator matrix with unitarily invariant norm, we prove an equality in the following theorem.

If $A_1, A_2, \ldots, A_n$ are imaginary circulant operators in $W(H)$, the block imaginary circulant operator matrix $A = \mathrm{circ}_i(A_1, A_2, \ldots, A_n)$ is the $n \times n$ matrix whose first row has entries $A_1, A_2, \ldots, A_n$ and the other rows are obtained by successive cyclic permutations of these entries; that is,

$$\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_n \\ iA_n & A_1 & A_2 & \cdots & A_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ iA_3 & iA_4 & iA_5 & \cdots & A_2 \\ iA_2 & iA_3 & iA_4 & \cdots & A_1 \end{pmatrix}, \tag{4}$$

where $i = \sqrt{-1}$.

It is known that $\mathrm{circ}_i(A_1, A_2, \ldots, A_n) = T\mathrm{circ}(A_1, \kappa A_2, \ldots, \kappa^{n-1}A_n)T^*$, where

$$T = \begin{pmatrix} I & & & 0 \\ & \kappa I & & \\ & & \ddots & \\ 0 & & & \kappa^{n-1}I \end{pmatrix}, \quad \text{with } \kappa = e^{\pi i/2n}. \tag{5}$$

Thus, every imaginary circulant operator matrix is unitarily equivalent to a circulant operator matrix.

Theorem 1. *Let $A_1, A_2, \ldots, A_n$ be any operators in $W(H)$. Then, for every weakly unitarily invariant norm, one has*

$$\tau\left(\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right)\right) = \tau\left(\oplus_{k=0}^{n-1}\sum_{j=1}^{n}\left(\kappa\omega^k\right)^{j-1}A_j\right), \tag{6}$$

where $i = \sqrt{-1}$, $\kappa = e^{\pi i/2n}$, and $\omega = e^{2\pi i/n}$.

Proof. The n roots of $z^n = i$ are $\kappa, \kappa\omega, \kappa\omega^2, \ldots, \kappa\omega^{n-1}$.

Now, let $U = U_n \otimes I$, where

$$U_n = \frac{1}{\sqrt{n}} \times \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \kappa & \kappa\omega & \kappa\omega^2 & \cdots & \kappa\omega^{n-1} \\ \kappa^2 & (\kappa\omega)^2 & \left(\kappa\omega^2\right)^2 & \cdots & \left(\kappa\omega^{n-1}\right)^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \kappa^{n-1} & (\kappa\omega)^{n-1} & \left(\kappa\omega^2\right)^{n-1} & \cdots & \left(\kappa\omega^{n-1}\right)^{n-1} \end{pmatrix}_{n\times n}. \tag{7}$$

Then it is easy to prove that U is a unitary operator in $W(H)$ and

$$U^*\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right)U = \left(\oplus_{k=0}^{n-1}\sum_{j=1}^{n}\left(\kappa\omega^k\right)^{j-1}A_j\right). \tag{8}$$

By the invariance property of weakly unitarily invariant norms, we obtain

$$\tau\left(\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right)\right) = \tau\left(\oplus_{k=0}^{n-1}\sum_{j=1}^{n}\left(\kappa\omega^k\right)^{j-1}A_j\right). \tag{9}$$

□

Synthesizing the norm equality in Theorem 1 to the usual operator norm and to the Schatten p-norms, we obtain the corollary and remark as follows.

Corollary 2. *Let $A_1, A_2, \ldots, A_n$ be any operators in $W(H)$. Then one has*

$$\begin{aligned} &\left\|\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right)\right\| \\ &\quad = \max\left\{\left\|\sum_{j=1}^{n}\left(\kappa\omega^k\right)^{j-1}A_j\right\| : k = 0, 1, \ldots, n-1\right\}, \\ &\left\|\mathrm{circ}_i\left(A_1, A_2, \ldots, A_n\right)\right\|_p \\ &\quad = \left(\sum_{k=0}^{n-1}\left\|\sum_{j=1}^{n}\left(\kappa\omega^k\right)^{j-1}A_j\right\|_p^p\right)^{1/p} \end{aligned} \tag{10}$$

for $1 \le p < \infty$.

In particular, let $n = 2$; one has

$$\begin{aligned} &\left\|\mathrm{circ}_i\left(A_1, A_2\right)\right\| = \max\left(\left\|A_1 + e^{\pi i/4}A_2\right\|, \left\|A_1 - e^{\pi i/4}A_2\right\|\right), \\ &\left\|\mathrm{circ}_i\left(A_1, A_2\right)\right\|_p = \left(\left\|A_1 + e^{\pi i/4}A_2\right\|_p^p + \left\|A_1 - e^{\pi i/4}A_2\right\|_p^p\right)^{1/p} \end{aligned} \tag{11}$$

for $1 \le p < \infty$.

Remark 3. Here we give some special cases of Corollary 2.

(i) If $A \in W(H)$, then

$$\left\| \begin{pmatrix} \mathscr{A} & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \end{pmatrix}_{n\times n} \right\| = \frac{1-i}{1-\kappa} \|\mathscr{A}\|,$$

$$\left\| \begin{pmatrix} \mathscr{A} & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \end{pmatrix}_{n\times n} \right\|_p = \left[\sum_{k=0}^{n-1} \left(\frac{1-i}{1-\kappa\omega^k} \right)^p \right]^{1/p} \|\mathscr{A}\|_p \tag{12}$$

for $1 \le p < \infty$.

(ii) If $\mathscr{A} \in W(H)$, then

$$\left\| \begin{pmatrix} 0 & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & 0 & \mathscr{A} & \cdots & \mathscr{A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & 0 \end{pmatrix}_{n\times n} \right\| = \left| \frac{\kappa-i}{1-\kappa} \right| \|\mathscr{A}\|,$$

$$\left\| \begin{pmatrix} 0 & \mathscr{A} & \mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & 0 & \mathscr{A} & \cdots & \mathscr{A} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & \mathscr{A} \\ i\mathscr{A} & i\mathscr{A} & i\mathscr{A} & \cdots & 0 \end{pmatrix}_{n\times n} \right\| = \left[\sum_{k=0}^{n-1} \left| \frac{\kappa\omega^k - i}{1-\kappa\omega^k} \right|^p \right]^{1/p} \|\mathscr{A}\|_p \tag{13}$$

for $1 \le p < \infty$.

3. Pinching-Type Inequalities for Imaginary Circulant Operator Matrices

In this section, for imaginary circulant operator matrices, we obtain pinching-type inequalities by the triangle inequality and the invariance property of unitarily invariant norms.

Theorem 4. *Let $\mathscr{A} = [A_{jk}]$ be an operator matrices in $W(H^{(n)})$. Then, for every weakly unitarily invariant norm, one has*

$$\frac{1}{n}\tau\left(\oplus_{k=0}^{n-1} \sum_{j=1}^{n} \left(\kappa\omega^k\right)^{j-1} V_j \right) \le \tau(\mathscr{A}), \tag{14}$$

where

$$\begin{gathered} V_1 = \sum_{j=1}^{n} A_{jj}, \qquad V_2 = \sum_{j=2}^{n} A_{j-1,j} + e^{-i(\pi/2)} A_{n1}, \\ V_3 = \sum_{j=1}^{n-2} A_{j,j+2} + e^{-i(\pi/2)} \sum_{j=n-1}^{n} A_{j,j-(n-2)}, \ldots, \\ V_{n-1} = A_{1,n-1} + A_{2n} + e^{-i(\pi/2)} \sum_{j=2}^{n} A_{j-1,j}, \\ V_n = A_{1n} + e^{-i(\pi/2)} \sum_{j=2}^{n} A_{j,j-1}. \end{gathered} \tag{15}$$

Proof. Let $L_{k,n-k} = [l_{rs}]$ be the $n \times n$ operator with

$$l_{rs} = \begin{cases} I, & \text{if } s - r = k; \\ e^{i(\pi/2)} I, & \text{if } r - s = n - k; \\ 0, & \text{otherwise.} \end{cases} \tag{16}$$

We can prove easily that $L_{k,n-k} = [l_{rs}]$ is a unitary operator for all $k = 1, 2, 3, \ldots, n$ and

$$\sum_{k=1}^{n} L_{k,n-k} \mathscr{A} L_{k,n-k}^* = \operatorname{circ}_i (V_1, V_2, \ldots, V_n) = V, \tag{17}$$

where

$$\begin{gathered} V_1 = \sum_{j=1}^{n} A_{jj}, \qquad V_2 = \sum_{j=2}^{n} A_{j-1,j} + e^{-i(\pi/2)} A_{n1}, \\ V_3 = \sum_{j=1}^{n-2} A_{j,j+2} + e^{-i(\pi/2)} \sum_{j=n-1}^{n} A_{j,j-(n-2)}, \ldots, \\ V_{n-1} = A_{1,n-1} + A_{2n} + e^{-i(\pi/2)} \sum_{j=2}^{n} A_{j-1,j}, \\ V_n = A_{1n} + e^{-i(\pi/2)} \sum_{j=2}^{n} A_{j,j-1}. \end{gathered} \tag{18}$$

Now let

$$U = \frac{1}{\sqrt{n}} \times \begin{pmatrix} I & I & I & \cdots & I \\ \kappa I & \kappa\omega I & \kappa\omega^2 I & \cdots & \kappa\omega^{n-1} I \\ \kappa^2 I & (\kappa\omega)^2 I & \left(\kappa\omega^2\right)^2 I & \cdots & \left(\kappa\omega^{n-1}\right)^2 I \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \kappa^{n-1} I & (\kappa\omega)^{n-1} I & \left(\kappa\omega^2\right)^{n-1} I & \cdots & \left(\kappa\omega^{n-1}\right)^{n-1} I \end{pmatrix}_{n\times n}. \tag{19}$$

Then, from Theorem 1, we have

$$U^* V U = \oplus_{k=0}^{n-1} \sum_{j=1}^{n} \left(\kappa\omega^k\right)^{j-1} V_j. \tag{20}$$

Thus, by the invariance property of unitarily invariant norms and the triangle inequality, we get

$$\frac{1}{n}\tau\left(\oplus_{k=0}^{n-1}\sum_{j=1}^{n}\left(\kappa\omega^{k}\right)^{j-1}V_{j}\right)\leq\tau\left(\mathscr{A}\right). \tag{21}$$

□

Substituting the norm inequality (14) to the usual operator norm and to the Schatten p-norms, we obtain the following corollary.

Corollary 5. *Let $\mathscr{A} = [A_{jk}]$ be an operator matrix in $W(H^{(n)})$. Then*

$$\frac{1}{n}\max\left\{\left\|\sum_{j=1}^{n}\left(\kappa\omega^{k}\right)^{j-1}V_{j}\right\| : k=0,1,\ldots,n-1\right\}\leq\|\mathscr{A}\|,$$
$$\left(\frac{1}{n}\sum_{k=0}^{n-1}\left\|\sum_{j=1}^{n}\left(\kappa\omega^{k}\right)^{j-1}V_{j}\right\|_{p}^{p}\right)^{1/p}\leq\|\mathscr{A}\|_{p} \tag{22}$$

for $1\leq p<\infty$, where V_j is given in (15).

As a special case when $n=2$, Corollary 5 asserts that

$$\begin{aligned}&\frac{1}{2}\max\left(\left\|A_{11}+A_{22}+e^{\pi i/4}\left(A_{12}+e^{-(\pi/2)i}A_{21}\right)\right\|,\right.\\&\qquad\left.\left\|A_{11}+A_{22}-e^{\pi i/4}\left(A_{12}+e^{-(\pi/2)i}A_{21}\right)\right\|\right)\\&\quad\leq\left\|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\right\|,\\&\frac{1}{2}\left(\left\|A_{11}+A_{22}+e^{\pi i/4}\left(A_{12}+e^{-(\pi/2)i}A_{21}\right)\right\|_{p}^{p}\right.\\&\qquad\left.+\left\|A_{11}+A_{22}-e^{\pi i/4}\left(A_{12}+e^{-(\pi/2)i}A_{21}\right)\right\|_{p}^{p}\right)^{1/p}\\&\quad\leq\left\|\begin{pmatrix}A_{11}&A_{12}\\A_{21}&A_{22}\end{pmatrix}\right\|_{p}\end{aligned} \tag{23}$$

for $1\leq p<\infty$.

It should be mentioned here that the norm inequalities in Theorems 1 and 4 are sharp. This is demonstrated in the following proposition.

Proposition 6. *Let $A_1,A_2,\ldots,A_n$ be some operators in $W(H)$. If $\mathscr{A}=\mathrm{circ}_i(A_1,A_2,\ldots,A_n)$, then the inequality in Theorem 4 becomes an equality.*

Proof. Let $\mathscr{A}=\mathrm{circ}_i(A_1,A_2,\ldots,A_n)$. Then it follows from Theorem 1 that

$$\tau\left(\mathrm{circ}_i\left(A_1,A_2,\ldots,A_n\right)\right)=\tau\left(\oplus_{k=0}^{n-1}\sum_{j=1}^{n}\left(\kappa\omega^{k}\right)^{j-1}A_{j}\right). \tag{24}$$

Since $V_1=nA_1$, $V_2=nA_n,\ldots,V_{n-1}=nA_3$, and $V_n=nA_2$, it follows that

$$\frac{1}{n}\tau\left(\oplus_{k=0}^{n-1}D_{k}\right)=\tau\left(\mathrm{circ}_i\left(A_1,A_2,\ldots,A_n\right)\right), \tag{25}$$

where

$$\begin{aligned}D_0&=\sum_{j=1}^{n}\kappa^{j-1}V_j=n\left[A_1+\kappa A_2+\cdots+\kappa^{n-1}A_n\right],\\D_1&=\sum_{j=1}^{n}(\kappa\omega)^{j-1}V_j=n\left[A_1+(\kappa\omega)A_2+\cdots+(\kappa\omega)^{n-1}A_n\right],\\&\vdots\\D_{n-1}&=\sum_{j=1}^{n}\left(\kappa\omega^{n-1}\right)^{j-1}V_j\\&=n\left[A_1+\left(\kappa\omega^{n-1}\right)A_2+\cdots+\left(\kappa\omega^{n-1}\right)^{n-1}A_n\right].\end{aligned} \tag{26}$$

□

By Proposition 6, it is easy to see that equality holds in the inequality (14) if and only if $\mathscr{A}$ is imaginary circulant for $0<p<\infty$. Furthermore, we obtain the following proposition by using the general Clarkson inequalities which can be seen in Proposition 1 of [19].

Proposition 7. *Let $\mathscr{A}=[A_{jk}]$ be an operator matrix in $W(H^n)$, and let $1<p<\infty$. Then $\|\mathscr{A}\|_p=(1/n)\|\oplus_{k=0}^{n-1}\sum_{j=1}^{n}(\kappa\omega^k)^{j-1}V_j\|_p$ if and only if $\mathscr{A}$ is imaginary circulant.*

Proof. In view of Proposition 1, it if sufficient to prove the "only if" part. Let $L_{k,n-k}$ be as in the proof of Theorem 1. If $\|\mathscr{A}\|_p=(1/n)\|\oplus_{k=0}^{n-1}\sum_{j=1}^{n}(\kappa\omega^k)^{j-1}V_j\|_p$, then it follows from the proof of Theorem 1 that

$$\begin{aligned}&\left\|L_{1,n-1}\mathscr{A}L_{1,n-1}^{*}\right\|_{p}\\&\quad=\left\|L_{2,n-2}\mathscr{A}L_{2,n-2}^{*}\right\|_{p}=\cdots=\left\|L_{1,n-1}\mathscr{A}L_{1,n-1}^{*}\right\|_{p}=\|\mathscr{A}\|_{p},\\&\qquad\sum_{k=0}^{n-1}\left\|L_{k,n-k}\mathscr{A}L_{k,n-k}^{*}\right\|_{p}=n\|\mathscr{A}\|_{p}.\end{aligned} \tag{27}$$

Now invoking Clarkson inequalities for several operators, it follows that

$$L_{1,n-1}\mathscr{A}L_{1,n-1}^{*}=L_{2,n-2}\mathscr{A}L_{2,n-2}^{*}=\cdots=L_{1,n-1}\mathscr{A}L_{1,n-1}^{*}. \tag{28}$$

Consequently, $\mathscr{A}$ is imaginary circulant matrix. □

4. Conclusion

By utilizing the special structure of imaginary circulant matrices, we obtain several norm equalities and inequalities,

where the norm τ under consideration is the weakly unitarily invariant norm. Based on the existing problems in [20–23], we will exploit solving these problems by circulant matrices technique.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Review Article

Analysis, Filtering, and Control for Takagi-Sugeno Fuzzy Models in Networked Systems

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The fuzzy logic theory has been proven to be effective in dealing with various nonlinear systems and has a great success in industry applications. Among different kinds of models for fuzzy systems, the so-called Takagi-Sugeno (T-S) fuzzy model has been quite popular due to its convenient and simple dynamic structure as well as its capability of approximating any smooth nonlinear function to any specified accuracy within any compact set. In terms of such a model, the performance analysis and the design of controllers and filters play important roles in the research of fuzzy systems. In this paper, we aim to survey some recent advances on the T-S fuzzy control and filtering problems with various network-induced phenomena. The network-induced phenomena under consideration mainly include communication delays, packet dropouts, signal quantization, and randomly occurring uncertainties (ROUs). With such network-induced phenomena, the developments on T-S fuzzy control and filtering issues are reviewed in detail. In addition, some latest results on this topic are highlighted. In the end, conclusions are drawn and some possible future research directions are pointed out.

1. Introduction

Since its inception by Zadeh [1–3], the theory of fuzzy set and system has gone through substantial development and has been widely employed in various kinds of areas such as control engineering, signal processing, information processing, decision making, management, finance, medicine, motor industry, and robotics. As one of the most important research topics, the fuzzy logic control has been firstly proposed by Mamdani and Assilian [4, 5]. Note that, in this kind of control method, the fuzzy control algorithms including a set of heuristic control rules have been utilized to achieve the desired control performance. Fuzzy sets and fuzzy logic have been used, respectively, to represent linguistic terms and evaluate the rules. More than a decade later, one of the first fuzzy filters has been provided by Vicente and Aranguren [6] where the fuzzy topological concepts have been characterized by utilizing the convergence of the designed fuzzy filter.

According to the fuzzy rules and their generation methods, the research methodologies on fuzzy control can be roughly classified into six categories [7]: traditional fuzzy control, fuzzy proportional-integral-derivative (PID) control, neurofuzzy control, fuzzy sliding mode control, adaptive fuzzy control, and fuzzy control based on Takagi-Sugeno (T-S) model. It is worth mentioning that the T-S fuzzy model [8, 9] has attracted an ever-increasing research interest owing to (1) its convenient and simple dynamic structure and (2) its capability of using a set of fuzzy rules to approximate a global nonlinear system. Generally speaking, the T-S model consists of a set of local linear models which are smoothly connected by fuzzy membership functions. In this framework, the problem of T-S fuzzy control and filtering has long been

a fascinating focus of research attracting constant attention. It is not surprising that there has been a rich body of relevant literature published in the past two decades; see, for example, [10–19] for control problems and [20–26] for filtering issues.

In recent years, because of the ever-increasing popularity of communication networks, the study of networked systems has gradually become an active research area due to the advantages of using networked media in many aspects such as low cost, simple installation, reduced weight, and power requirements, as well as high reliability [27, 28]. It is well known that the signals are often transmitted through networks which may undergo unavoidable network-induced phenomena including communication delays, packet dropouts (also called missing measurements), signal quantization, and randomly occurring uncertainties (ROUs); see [29–37] and the references therein. These network-induced phenomena would bring in particular systems complexities (e.g., abrupt structural and parametric changes) which could seriously degrade the system performance if not adequately handled in practical engineering applications. Consequently, due to the merits of approximating nonlinear systems, the T-S fuzzy models in networked environments have been introduced to describe the nonlinear networked control systems (NCSs), and the corresponding control and filtering problems for such T-S fuzzy systems with aforementioned network-induced phenomena have attracted considerable attention by many researchers during the past few years [38–41].

In this paper, we focus mainly on the control and filtering problems for T-S fuzzy systems with network-induced phenomena and aim to provide a survey on some recent advances in this area. Firstly, a variety of T-S fuzzy control and filtering issues with network-induced phenomena are discussed in great detail. Both theories and techniques for dealing with the controller or filter design are systematically reviewed. Subsequently, latest results on T-S fuzzy control/filtering problems for networked systems are surveyed and some challenging issues for future research are raised. Finally, some conclusions are drawn and several possible related research directions are pointed out.

The rest of the paper is organized as follows. In Section 2, the control and filtering problems for the T-S fuzzy systems with network-induced phenomena are reviewed. Section 3 reviews the latest results on T-S fuzzy control, filtering, and fault detection problems for networked systems and some challenging issues are highlighted at the same time. The conclusions and future work are given in Section 4.

2. T-S Fuzzy Control and Filtering with Network-Induced Phenomena

The signal transmission via networked systems has become prevalent and, accordingly, the network-induced issues have drawn considerable research interest. In this section, we will recall the theoretical developments of T-S fuzzy control and filtering problems from the following four aspects: communication delays, packet dropouts, signal quantization, and ROUs.

2.1. T-S Fuzzy Control and Filtering with Communication Delays. It has been well recognized that communication delays, which could be one of the causes for poor performance or even instability of the closed loop, exist universally in practical systems. In the past decades, significant research efforts have been devoted to the control and filtering problems for T-S fuzzy systems with different types of time-delays including constant time-delays, time-varying delays, multiple time-delays, infinitely distributed time-delays, Markov jumping time-delays, and interval time-delays. For instance, in [42], some new results on stability properties (asymptotical stability and input-to-state stability) have been investigated for T-S fuzzy Hopfield neural networks with constant time-delay. Furthermore, in [33], the delay-dependent stabilizability condition has been integrated with the shifted-Chebyshev-series approach and the hybrid Taguchi-genetic algorithm. An effective control scheme has been proposed to handle the quadratic finite-horizon optimal parallel distributed compensation (PDC) control problem of the T-S fuzzy-model-based time-delay systems. In order to overcome the inherent difficulty of the nonlinear optimal control issue, the corresponding design has been turned into the feasibility problem for certain linear matrix inequalities (LMIs) in a suboptimal sense that can be easily solved by means of numerically efficient convex programming algorithms.

In the context of T-S fuzzy control, in addition to the communication delays, other factors contributing to the system complexities have also been studied. For example, the parameter uncertainties are unavoidable for modeling real-world engineering systems which would result in perturbations of the elements of a system matrix [43–45]. As such, in the past decade, considerable attention has been devoted to fuzzy systems with various time-delays and parameter uncertainties, and a large number of results have been reported by exploiting the LMI approach. According to the linear differential inclusion state-space representation, a novel design scheme of fuzzy controller has been proposed in [46] to stabilize the nonlinear multiple time-delay large-scale system. Moreover, the H_∞ performance index, which is closely related to the robustness of the closed-loop system, also has been employed to evaluate the design controller. In [47], the robust H_∞ control problem has been investigated for a class of discrete-time T-S fuzzy systems with the infinitely distributed time-delay which can be regarded as the discretization version of the infinite integral form in continuous-time case. It deserves attention that, in most of the literature mentioned above, the control problem has been considered for the case that the time-delay is not random. However, the time-delays may occur in a probabilistic method. So, stochastic time-delays over T-S fuzzy control systems also have been further researched. For instance, a T-S model has been employed to represent a networked control system with Markov jumping time-delays in [48], and the designed approach has addressed situations involving all possible network-induced delays. In [49], the probabilistic interval distribution of communication delay has been taken into account, and a robust networked controller for a class of T-S fuzzy systems has been designed, where the solvability of the networked controller design depends not only on the upper

and lower bounds of the delay but also on its probability distribution.

Similar to the control problem with communication delays, the problem of T-S fuzzy filtering also has been attracting considerable research interests and a lot of advanced methods have been proposed to handle the network-induced time-delays. For example, the fuzzy H_∞ filtering has been discussed in [50] for a class of nonlinear discrete-time systems with both multiple time-delays and unknown bounded disturbances. For the same kind of time-delays, in [51], a full-order H_∞ filter has been designed to guarantee that the filtering error dynamics are stochastically stable and the given H_∞ attenuation level is guaranteed. Additionally, it is worth mentioning that the results discussed above have been presented with delay-independent conditions. An interesting research problem is how to utilize the time-delay information to reduce the conservatism. Based on such an idea, the delay-dependent approach has been widely adopted in recent years for various time-varying delay T-S fuzzy systems. For instance, by utilizing the T-S fuzzy model, the H_∞ filtering problem has been addressed in [52] for a class of nonlinear systems where the nonlinearities have been assumed to satisfy global Lipschitz conditions. Furthermore, a novel delay-dependent piecewise Lyapunov-Krasovskii functional, which is dependent on both the upper bound of the delays and the delay interval, has been constructed in [30] to analyze the filtering error dynamics and then some sufficient conditions have been established in terms of LMIs.

Very recently, for the delay-dependent problems mentioned above, a novel technique has been provided by introducing some free-weighting matrices. The delay-dependent filter design for nonlinear systems with time-varying delay via T-S fuzzy model approach has been studied in [53], and the main technique used is the free-weighting matrix method combined with a matrix decoupling approach. The problem of delay-dependent robust H_∞ filtering design has been investigated for a class of uncertain discrete-time state-delayed T-S fuzzy systems in [54], where the state delay has been assumed to be time-varying and of an interval-like type, meaning that both the lower and upper bounds of the time-varying delay are available. Based on the delay-dependent piecewise Lyapunov-Krasovskii functional combined with an improved free-weighting matrix method, a delay-dependent H_∞ filter has been further designed for a class of discrete-time nonlinear interconnected systems with time-varying delays via the T-S fuzzy model in [55], and it guarantees both the delay-dependent stability and the prescribed H_∞ performance index.

2.2. T-S Fuzzy Control and Filtering with Packet Dropouts. It is a well-known fact that, in a networked environment, the measurement output is not consecutive but contains missing observations due to various causes such as network-induced packet loss and sensor temporal failure; see, for example, [56, 57]. Therefore, the control problem for T-S fuzzy systems with packet dropouts has recently attracted much attention. Assume that $\delta > 0$ represents a maximum allowable transfer interval. For no packet dropout case, a transmission of a packet takes place at time t_k and the control signal will reach the plant at the instant $t_k + \tau_k$, and then the next control signal must arrive within the time interval $(t_k, t_k+\delta]$. As such, the phenomenon of packet dropout in network transmission occurs when $\tau_k > \delta$. According to such a fact, a guaranteed cost networked control problem has been developed for T-S fuzzy systems with both network-induced delays and packet dropouts in network transmission [19]. Recently, in [58], the fault-tolerant control approach for linear controlled plant has been extended to the case of the nonlinear networked control systems in the presence of networked-induced delay and packet dropout as well as external disturbance. By using the Lyapunov-Krasovskii functional and introducing some slack matrices, the authors have established a less conservative robust H_∞ integrity design scheme for the T-S fuzzy systems with failures of both actuator and sensor.

Considering the unavoidable parameter uncertainties for modeling real-world engineering systems, some researchers have endeavoured to investigate fuzzy control systems with packet dropouts and parameter uncertainties. The problem of fuzzy controller design has been addressed for a class of nonlinear networked control systems approximated by uncertain networked T-S models in [59], where the time-varying network-induced delay and data packet dropout have been considered simultaneously. In [32], the robust H_∞ control problem has been investigated for a class of uncertain discrete-time fuzzy systems with both multiple probabilistic delays and multiple missing measurements. The measurement-missing phenomenon occurs in a random way and the missing probability for each sensor satisfies a certain probabilistic distribution in the interval $[0, 1]$. In [60], the problem of robust H_∞ output-feedback control has been studied for a class of network nonlinear systems with multiple packet dropouts, where stochastic variables satisfying the Bernoulli random binary distribution are adopted to characterize the data missing phenomenon. In [61], the stochastic stability has been researched for a class of discrete-time nonlinear control systems in the T-S form with uncertain network-induced time-delays and missing measurements by using a fuzzy decentralized control approach.

In comparison to the fruitful results for control problems of T-S fuzzy systems with packet dropouts, the corresponding filtering issues have also received considerable research attention. In [62], the network-based robust fault detection problem has been studied for a class of uncertain discrete-time T-S fuzzy systems with stochastic mixed time-delays and successive packet dropouts. A sequence of stochastic variables, which are mutually independent but obey the Bernoulli distribution, has been introduced to govern the random occurrences of the discrete time-delays, the distributed time-delays, and the successive packet dropouts. Moreover, if the network media are introduced to the filtering issues, the data packet dropout phenomenon, which occurs in a network environment, will naturally induce the intermittent measurements from the plant to the filter. Therefore, the problem of filter design with intermittent measurements is of significant importance and a significant amount of research effort has been made to analyze the T-S fuzzy systems with intermittent observations. The problem of H_∞ fuzzy filtering of nonlinear systems represented by a T-S fuzzy model

with intermittent measurements has been investigated in [63]. The measurements transmission from the plant to the filter is assumed to be imperfect and a stochastic variable satisfying the Bernoulli random binary distribution is utilized to model the phenomenon of the missing measurements, so the data packet dropout phenomenon happens intermittently. In [28], the authors have further discussed the problem of fault detection for T-S fuzzy systems with intermittent measurements. In most published papers, the measurement signal is usually assumed to be either completely missing or completely available and therefore the packet dropouts can be governed by some stochastic variables obeying the given Bernoulli distribution. However, such an assumption is quite restrictive in practice in case of fading measurements for an array of sensors. Recently, the stochastic variables with general probability distributions are adopted to characterize the data missing phenomenon in output channels. In [27], for this kind of general model, the problem of robust H_∞ state estimation has been investigated for a class of multichannel networked nonlinear systems with both multiple packet dropouts and norm-bounded uncertainties.

2.3. T-S Fuzzy Control and Filtering with Signal Quantization. In networked systems, due to the application of finite-precision arithmetic and the limited network bandwidth, quantization is an effective approach in order to reduce both the network burden and the energy consumption. However, the performance of networked systems will be inevitably subject to the effect of quantization error. As a result, it is necessary to research into (1) how the quantization phenomenon affects the system performance and (2) how to develop a suitable method to solve the addressed control and filtering problem with various network-induced phenomena involving signal quantization. It is worth noting that the problem of quantized control for nonnetworked systems has been reported as early as in 1990 [64].

So far, some preliminary results have been available in the literature; see [36, 65, 66], for example. The problem of quantized output-feedback networked control for the continuous-time T-S fuzzy system with impulsive effects has been investigated in [66] where both the observed state and the control signals have been quantized before they were sent to the controller and the actuator. In [36], the problem of robust control has been concerned for uncertain discrete-time T-S fuzzy networked control systems with state quantization, network-induced delays, and packet dropouts. Based on the same model, by employing the fuzzy Lyapunov-Krasovskii functional, a less conservative delay-dependent stability condition has been derived in order to guarantee the desired stability and H_∞ performance. For a class of continuous-time T-S fuzzy affine dynamic systems with quantized measurements, a suitable observer-based dynamic output-feedback controller has been designed by using the common/piecewise quadratic Lyapunov functions combined with both the S-procedure and the matrix inequality convexification technique in [65].

Unfortunately, the filtering problem for T-S fuzzy system with signal quantization has gained very little research attention despite its practical importance. In [67], the problem of generalized H_2 filtering has been studied for a class of discrete-time T-S fuzzy systems with measurement quantization and packet loss. The quantized measurements are transmitted to the filter via an imperfect communication channel and the quantization errors are treated as sector bound uncertainties.

2.4. T-S Fuzzy Control and Filtering with ROUs. In the networked world nowadays, the parameter uncertainty serves as one of the important complexities for system modeling. The parameter uncertainties may be subject to random changes in environmental circumstances, for example, network-induced random failures and repairs of components, changing subsystem interconnections, and sudden environmental disturbances; see [34], for more details. It is worth noting that a stochastic variable obeying the given Bernoulli distribution has been utilized to characterize such a phenomenon. Such a description is more suitable for reflecting parameter variations in a random nature, particularly in the network transmission. In [34], the concept of ROUs has been firstly introduced to reflect the uncertain parameter variations and the robust sliding mode control problem has been considered for discrete time stochastic systems. By using the Lyapunov stability theory, the stability and the reachability have been investigated in detail.

Inspired by the work in [34], ROUs have been investigated for T-S fuzzy systems in recent years. For instance, the problem of robust passive control for networked fuzzy systems has been considered in [68], where ROUs, variable sampling intervals, and constant network-induced delay have been taken into account, simultaneously. In this paper, a discontinuous Lyapunov functional has been introduced for the closed-loop T-S fuzzy systems, which takes full advantage of the sawtooth structure of the time-varying interval delay induced by sample-and-hold and signal transmission. Furthermore, in [69], the H_∞ fuzzy filtering problem has been discussed for a class of discrete-time T-S fuzzy systems with ROUs and randomly occurring interval time-varying delays as well as channel fadings. However, up until now, ROUs have not yet received adequate research attention.

3. Latest Progress

Very recently, the control and filtering problems for the networked T-S fuzzy systems have been intensively studied and some elegant results have been reported. In this section, we highlight some of the newest works with respect to this topic.

3.1. Bilinear T-S Fuzzy Model. In [70], the fuzzy bilinear state feedback controller based on the T-S fuzzy bilinear model has been addressed for direct current-direct current (DC-DC) converters. Via Taylor series expansion, the DC-DC converters can be approximated by the bilinear model and one of this extended system's state variables is the error between the output voltage and the reference output voltage. A fuzzy bilinear state feedback controller has been designed to track the reference output voltage and it ensures that the closed-loop system is globally asymptotically stable.

3.2. Event-Based Fuzzy Control. In [71], a discrete-time event-based communication protocol and a parallel distribution compensation controller have been appropriately codesigned so as to trade off the communication bandwidth utilization and the stability of the controlled continuous-time T-S fuzzy system. Compared with a time-triggered periodic communication scheme, the advantage of event-triggered communication scheme will not only greatly lighten the network loads but also save power of sensors. Moreover, by employing the networked T-S fuzzy model and the discrete event-triggered communication scheme, a stability criterion and a stabilization criterion about the networked T-S fuzzy system are derived in terms of matrix inequalities, and the maximum allowable delay and the feedback gain can be obtained simultaneously through solving an optimization problem. Recently, the event-triggered distributed filtering is a promising research topic.

3.3. Fuzzy Control with Multiple Network-Induced Phenomena. In [72], the H_∞ output-feedback control problem has been addressed for a class of discrete-time fuzzy systems with randomly occurring infinite distributed delays and channel fadings. The stochastic Rice fading model has been employed to simultaneously describe the phenomena of random time-delays and channel fadings via setting different values of the channel coefficients. An H_∞ output-feedback fuzzy controller has been designed such that the closed-loop T-S fuzzy control system is exponentially mean-square stable and the disturbance rejection attenuation is constrained to a given level by means of the H_∞-performance index. Subsequently, sufficient conditions have been obtained for the existence of desired output-feedback controllers ensuring both the exponential mean-square stability and the prescribed H_∞ performance. However, the phenomena of channel fadings have not yet been thoroughly studied for the T-S fuzzy system and this deserves deep investigation.

3.4. l_2-l_∞ Fuzzy Filtering. In [73], the design problem of l_2-l_∞ filters has been investigated for a class of discrete-time T-S fuzzy systems with time-varying delay. The full-order and reduced-order filters that guarantee the filtering error system to be asymptotically stable with a prescribed H_∞ performance have been designed. By employing an input-output approach and a two-term approximation method, which is applied to approximate the time-varying delay, sufficient conditions have been proposed for the designed filtering error system. In this paper, the main attention has been focused on the reduction of conservativeness, such that the filter synthesis problems have feasible solutions in the full-order and reduced-order cases. The time-varying delay in the original system has been transformed into the uncertainties so as to use the small-scale gain theorem, and then the original system has been transformed into a comparison system including two subsystems which are a constant time-delay forward subsystem and a delayed "uncertainty" feedback one.

3.5. H_-/H_∞ Fault Detection. Reference [74] has been concerned with the design problem of the H_-/H_∞ fault detection filter under relax conditions for the T-S fuzzy system affected by sensor faults and unknown bounded disturbances. This method has employed the technique of descriptor systems through considering sensor faults as an auxiliary state variable and a robust fault detection filter design which has the best robustness to disturbances and sensitivity to faults has been formulated. Based on nonquadratic Lyapunov functions, this design has approached the fault detection filtering problem by means of the minimization of the H_∞ norm and the maximization of the H_- index to obtain the observer gains and the residual weighting matrix. Hence, the problems of fault detection filter with randomly occurring network-induced phenomena would be another interesting topic.

3.6. H_∞ Filtering with Unknown Membership Functions. In [75], the H_∞ filter has been designed for T-S fuzzy systems with unknown or partially unknown membership functions, which refer to the ones with unknown parameters or unknown perturbations or unknown variables. In this case, the designed filter over fuzzy systems, which is based on parallel distributed compensator strategy, is infeasible. In order to tackle this difficulty, a switching mechanism, which depends on the lower and upper bounds of the unknown membership functions, has been employed and further modified to construct the H_∞ filter with varying gains. Nevertheless, in these approaches, the system states are required to be available, and thus, it is rather difficult to be used to design the H_∞ filter.

3.7. Nonfragile H_∞ Fuzzy Filtering. In [76], the design problem of nonfragile H_∞ filter has been studied for continuous-time T-S fuzzy systems. The proposed filter has been assumed to have two types of multiplicative gain variations. At first, two relaxed H_∞ filtering analysis conditions have been proposed based on useful LMIs. Next, some results have been utilized to obtain sufficient conditions for designing a nonfragile H_∞ filter guaranteeing a H_∞ performance of the fuzzy filtering error system. It is clear that the designed methods not only suit a standard form of the fuzzy filter but also give more relaxed design conditions through comparing them with the existing results. As a result, the analysis and design problems for the nonfragile H_∞ filtering with randomly occurring gain variations still remain as challenging research topics.

4. Conclusions and Future Work

Throughout the paper, we have reviewed some recent advances on the T-S fuzzy control and filtering problems for networked systems with the network-induced phenomena. We have discussed, in great detail, various control and filtering problems for network-induced phenomena consisting of communication delays, packet dropouts, signal quantization, and ROUs. In addition, we have surveyed some latest results on control and filtering problems for T-S fuzzy systems with network-induced phenomena and pointed out some challenging issues.

Finally, based on the literature review, we provide some related directions for the future research work as follows.

(i) The T-S fuzzy control and filtering issues for networked systems are still active research topics. In practical engineering, there still exist many more complex yet important network-induced phenomena such as channel fadings and randomly occurring gain variations, which have not yet been thoroughly studied yet. Therefore, a trend for future research is to investigate the T-S fuzzy control and filtering with more network-induced phenomena [77–81].

(ii) The problems of fault detection for T-S fuzzy systems in the presence of network-induced phenomena are of engineering significance, especially when the system is time-varying. Hence, it would be interesting to investigate the problems of fault detection filtering for time-varying T-S fuzzy systems with randomly occurring network-induced phenomena over a finite time horizon.

(iii) Note that the network-induced phenomena usually occur in a random way which makes the dynamics under consideration stochastic. In particular, the mathematical expectation and variance of random variables are two time-varying positive scalar sequences that take values on two intervals (rather than two distinctive points as in the current studies). Therefore, the control and filtering problems for T-S fuzzy systems with the time-varying occurrence probability are of significant engineering importance.

(iv) Taking the energy efficiency into consideration, the T-S fuzzy control and filtering algorithms with event-triggered mechanism are of significant engineering importance. Future works may involve the study on how to construct an exact mathematical description of event-based T-S fuzzy systems.

(v) Some other future research directions are to further investigate self-adaptive fuzzy control, nonfragile H_∞ filtering, and multiobjective H_2/H_∞ filtering problems for T-S fuzzy systems with randomly occurring network-induced phenomena.

(vi) Applications of the existing theories and methodologies to some practical engineering problems (e.g., consumer electronics, medical services, pilotless aircrafts, and industrial robots) would be another topic for future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Exact Inverse Matrices of Fermat and Mersenne Circulant Matrix

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The well known circulant matrices are applied to solve networked systems. In this paper, circulant and left circulant matrices with the Fermat and Mersenne numbers are considered. The nonsingularity of these special matrices is discussed. Meanwhile, the exact determinants and inverse matrices of these special matrices are presented.

1. Introduction

Circulant matrices are an important tool in solving networked systems. In [1], the authors investigated the storage of binary cycles in Hopfield-type and other neural networks involving circulant matrix. In [2], the authors considered a special class of the feedback delay network using circulant matrices. Distributed differential space-time codes that work for networks with any number of relays using circulant matrices were proposed by Jing and Jafarkhani in [3]. Bašić [4] solved the question for when circulant quantum spin networks with nearest-neighbor couplings can give perfect state transfer. Wang et al. considered two-way transmission model ensured that circular convolution between two frequency selective channels in [5]. Li et al. [6] presented a low-complexity binary framewise network coding encoder design based on circulant matrix.

Circulant matrices have been applied to various disciplines including image processing, communications, signal processing, and encoding. Circulant type matrices have established the substantial basis with the work in [7–12] and so on.

Lately, some authors gave the explicit determinant and inverse of the circulant and skew-circulant involving famous numbers. For example, Yao and Jiang [13] presented the determinants, inverses, norm, and spread of skew circulant type matrices involving any continuous Lucas numbers. Jiang et al. [14] considered circulant type matrices with the k-Fibonacci and k-Lucas numbers and presented the explicit determinant and inverse matrix by constructing the transformation matrices. Dazheng [15] got the determinant of the Fibonacci-Lucas quasi-cyclic matrices. Determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers were obtained by Bozkurt and Tam in [16].

For any integer $m \geq 0$, let $F_m = 2^{2^m} + 1$ be the mth Fermat number. It is well known that F_m is prime for $m \leq 4$, but there is no other m for which F_m is known to prime. The Mersenne and Fermat sequences are defined by the following recurrence relations [17, 18], respectively:

$$\begin{aligned} \mathbb{M}_{n+1} &= 3\mathbb{M}_n - 2\mathbb{M}_{n-1} \\ \mathbb{F}_{n+1} &= 3\mathbb{F}_n - 2\mathbb{F}_{n-1} \end{aligned} \tag{1}$$

with the initial condition $\mathbb{M}_0 = 0$, $\mathbb{M}_1 = 1$, $\mathbb{F}_0 = 2$, $\mathbb{F}_1 = 3$, for $n \geq 1$.

Let α and β be the roots of the characteristic equation $x^2 - 3x + 2 = 0$; then the Binet formulas of the sequences $\{\mathbb{M}_{k+n}\}$ and $\{\mathbb{F}_{k+n}\}$ have the form

$$\begin{aligned} \mathbb{M}_{k+n} &= \frac{\alpha^{k+n} - \beta^{k+n}}{\alpha - \beta}, \\ \mathbb{F}_{k+n} &= \alpha^{k+n} + \beta^{k+n}. \end{aligned} \tag{2}$$

Lemma 1. *Let $\mathbb{M}_{k+n}$ be the $(k+n)$th Mersenne number and let $\mathbb{F}_{k+n}$ be the $(k+n)$th Fermat number; then*

(1) $\mathbb{M}_{n+1} - \mathbb{M}_n = 2^n$,
$\mathbb{M}_{n+1} - 2\mathbb{M}_n = 1$,
$\mathbb{M}_n = 2^n - 1$,
$\mathbb{M}_n^2 - \mathbb{M}_{n+1}\mathbb{M}_{n-1} = 2^{n-1}$.

(2) $\mathbb{F}_{n+1} - \mathbb{F}_n = 2^n$,
$\mathbb{F}_{n+1} - 2\mathbb{F}_n = -1$,
$\mathbb{F}_n = 2^n + 1$,
$\mathbb{F}_n^2 - \mathbb{F}_{n+1}\mathbb{F}_{n-1} = -2^{n-1}$.

We define a Fermat circulant matrix which is an $n \times n$ matrix with the following form:

$$\operatorname{Circ}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n}) = \begin{bmatrix} \mathbb{F}_{k+1} & \mathbb{F}_{k+2} & \cdots & \mathbb{F}_{k+n} \\ \mathbb{F}_{k+n} & \mathbb{F}_{k+1} & \cdots & \mathbb{F}_{k+n-1} \\ \vdots & \vdots & & \vdots \\ \mathbb{F}_{k+2} & \mathbb{F}_{k+3} & \cdots & \mathbb{F}_{k+1} \end{bmatrix}. \tag{3}$$

A Mersenne circulant matrix which is an $n \times n$ matrix is defined with the following form:

$$\operatorname{Circ}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n}) = \begin{bmatrix} \mathbb{M}_{k+1} & \mathbb{M}_{k+2} & \cdots & \mathbb{M}_{k+n} \\ \mathbb{M}_{k+n} & \mathbb{M}_{k+1} & \cdots & \mathbb{M}_{k+n-1} \\ \vdots & \vdots & & \vdots \\ \mathbb{M}_{k+2} & \mathbb{M}_{k+3} & \cdots & \mathbb{M}_{k+1} \end{bmatrix}. \tag{4}$$

Besides, a Fermat left circulant matrix is given by

$$\operatorname{LCirc}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n}) = \begin{bmatrix} \mathbb{F}_{k+1} & \mathbb{F}_{k+2} & \cdots & \mathbb{F}_{k+n} \\ \mathbb{F}_{k+2} & \mathbb{F}_{k+3} & \cdots & \mathbb{F}_{k+1} \\ \vdots & \vdots & & \vdots \\ \mathbb{F}_{k+n} & \mathbb{F}_{k+1} & \cdots & \mathbb{F}_{k+n-1} \end{bmatrix}. \tag{5}$$

A Mersenne left circulant matrix is given by

$$\operatorname{LCirc}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n}) = \begin{bmatrix} \mathbb{M}_{k+1} & \mathbb{M}_{k+2} & \cdots & \mathbb{M}_{k+n} \\ \mathbb{M}_{k+2} & \mathbb{M}_{k+3} & \cdots & \mathbb{M}_{k+1} \\ \vdots & \vdots & & \vdots \\ \mathbb{M}_{k+n} & \mathbb{M}_{k+1} & \cdots & \mathbb{M}_{k+n-1} \end{bmatrix}. \tag{6}$$

The main content of this paper is to obtain the results for the exact determinants and inverses of Fermat and Mersenne circulant matrix. In this paper, let k be a nonnegative integer, $A_{k,n} = \operatorname{Circ}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$, and $B_{k,n} = \operatorname{Circ}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$.

2. Determinant and Inverse of Fermat Circulant Matrix

In this section, let $A_{k,n} = \operatorname{Circ}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ be a Fermat circulant matrix. Firstly, we obtain the exact form determinant of the matrix $A_{k,n}$. Afterwards, we find the exact form inverse of the matrix $A_{k,n}$.

Theorem 2. *Let $A_{k,n} = \operatorname{Circ}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ be a Fermat circulant matrix. Then one has*

$$\det A_{k,n} = \mathbb{F}_{k+1} \cdot \left[\sum_{j=1}^{n-2} \left(\mathbb{F}_{j+k+2} - \tau_k \mathbb{F}_{j+k+1} \right) \cdot y^{n-j-1} + \mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} \right] \cdot (-f)^{n-2}, \tag{7}$$

where $y = -e/f$, $e = 2(\mathbb{F}_k - \mathbb{F}_{k+n})$, $f = \mathbb{F}_{k+n+1} - \mathbb{F}_{k+1}$, $\tau_k = \mathbb{F}_{k+2}/\mathbb{F}_{k+1}$, and $\mathbb{F}_{k+n}$ is the $(k+n)$th Fermat number. Moreover, $A_{k,n}$ is singular if and only if $(1-\alpha\kappa_l)(1-\beta\kappa_l) \neq 0$ and $\mathbb{F}_{k+1} - 2\kappa_l\mathbb{F}_k - \mathbb{F}_{k+n+1} + 2\kappa_l\mathbb{F}_{k+n} = 0$, for $k \in N$, $n \in N_+$, where $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$, $l = 1, 2, \ldots, n$.

Proof. It is clear that $\det A_{0,n} = \mathbb{F}_1 \cdot [\sum_{j=1}^{n-2}(\mathbb{F}_{j+2} - \tau_0\mathbb{F}_{j+1})[2(\mathbb{F}_n - \mathbb{F}_0)/(\mathbb{F}_{n+1} - \mathbb{F}_1)]^{n-j-1} + \mathbb{F}_1 - \tau_0\mathbb{F}_n] \cdot [\mathbb{F}_1 - \mathbb{F}_{n+1}]^{n-2}$ satisfies (7). In the following, let

$$\Sigma = \begin{pmatrix} 1 & & & & & & & \\ -\tau_k & & & & & & & 1 \\ 2 & & & & & & 1 & -3 \\ 0 & & & 0 & & 1 & -3 & 2 \\ \vdots & & & & \ddots & \ddots & \ddots & \\ 0 & & & 1 & \ddots & \ddots & & \\ 0 & & 1 & -3 & \ddots & & 0 & \\ 0 & 1 & -3 & 2 & & & & \end{pmatrix}, \tag{8}$$

$$\Omega_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & y^{n-2} & 0 & \cdots & 0 & 0 \\ 0 & y^{n-3} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & y & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices; we have

$$\Sigma A_{k,n} \Omega_1 = \begin{pmatrix} \mathbb{F}_{k+1} & h'_{k,n} & -\mathbb{F}_{k+n} & -\mathbb{F}_{k+n-1} & \cdots & -\mathbb{F}_{k+3} \\ 0 & h_{k,n} & a_3 & a_4 & \cdots & a_n \\ 0 & 0 & e & f & & 0 \\ 0 & 0 & 0 & e & & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & f \\ 0 & 0 & 0 & 0 & & e \end{pmatrix}, \tag{9}$$

where

$$\begin{aligned}
\tau_k &= \frac{\mathbb{F}_{k+2}}{\mathbb{F}_{k+1}}, \\
y &= -\frac{e}{f}, \\
a_3 &= \tau_k \mathbb{F}_{k+n} - \mathbb{F}_{k+1}, \\
a_j &= \tau_k \mathbb{F}_{k+n+3-j} - \mathbb{F}_{k+n+4-j} \\
&\quad (j = 4, 5, \ldots, n), \\
h'_{k,n} &= \sum_{t=1}^{n-1} \mathbb{F}_{t+k+1} \left[\frac{2(\mathbb{F}_{k+n} - \mathbb{F}_k)}{\mathbb{F}_{k+n+1} - \mathbb{F}_{k+1}} \right]^{n-t-1}, \\
h_{k,n} &= \sum_{t=1}^{n-2} (\mathbb{F}_{t+k+2} - \tau_k \mathbb{F}_{t+k+1}) y^{n-t-1} \\
&\quad + \mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n}.
\end{aligned} \tag{10}$$

We obtain

$$\begin{aligned}
&\det \Sigma \det A_{k,n} \det \Omega_1 \\
&= \mathbb{F}_{k+1} \cdot \left[\sum_{t=1}^{n-2} (\mathbb{F}_{t+k+2} - \tau_k \mathbb{F}_{t+k+1}) y^{n-t-1} \right. \\
&\quad \left. + \mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} \right] \cdot e^{n-2},
\end{aligned} \tag{11}$$

while

$$\begin{aligned}
\det \Sigma &= (-1)^{(n-1)(n-2)/2}, \\
\det \Omega_1 &= (-1)^{(n-1)(n-2)/2} \left[\frac{2(\mathbb{F}_k - \mathbb{F}_{k+n})}{\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}} \right]^{n-2}.
\end{aligned} \tag{12}$$

We have

$$\begin{aligned}
\det A_{k,n} &= \mathbb{F}_{k+1} \\
&\quad \cdot \left[\sum_{t=1}^{n-2} (\mathbb{F}_{t+k+2} - \tau_k \mathbb{F}_{t+k+1}) \cdot y^{n-t-1} \right. \\
&\quad \left. + \mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} \right] \cdot (-f)^{n-2}.
\end{aligned} \tag{13}$$

Next, we discuss the singularity of the matrix $A_{k,n}$.

The roots of polynomial $g(x) = x^n - 1$ are κ_l ($l = 1, 2, \ldots, n$), where $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$. We have

$$\begin{aligned}
f(\kappa_l) &= \mathbb{F}_{k+1} + \mathbb{F}_{k+2}\kappa_l + \cdots + \mathbb{F}_{k+n} (\kappa_l)^{n-1} \\
&= \frac{\mathbb{F}_{k+1} - 2\kappa_l \mathbb{F}_k - \mathbb{F}_{k+n+1} + 2\kappa_l \mathbb{F}_{k+n}}{(1 - \alpha\kappa_l)(1 - \beta\kappa_l)}.
\end{aligned} \tag{14}$$

By Lemma 1 in [14], the matrix $A_{k,n}$ is nonsingular if and only if $f(\kappa_l) \neq 0$; that is, when $(1 - \alpha\kappa_l)(1 - \beta\kappa_l) \neq 0$, $A_{k,n}$ is nonsingular if and only if $\mathbb{F}_{k+1} - 2\kappa_l \mathbb{F}_k - \mathbb{F}_{k+n+1} + 2\kappa_l \mathbb{F}_{k+n} \neq 0$; when $(1 - \alpha\kappa_l)(1 - \beta\kappa_l) = 0$, we obtain $\kappa_l = 1/\alpha$ or $\kappa_l = 1/\beta$.

Let $\kappa_l = 1/\alpha$; then the eigenvalue of $A_{k,n}$ is

$$f(\kappa_l) = \frac{n\alpha^{k+n} - \beta^{k+1}\mathbb{F}_n}{\alpha^{n-1}(\alpha - \beta)} \neq 0, \tag{15}$$

for $\alpha = 2$, $\beta = 1$, $k \in N$, $n \in N_+$, $l = 1, 2, \ldots, n$, so $A_{k,n}$ is nonsingular. The arguments for $\kappa_l = 1/\beta$ are similar. Thus, the proof is completed. □

Lemma 3. *Let the matrix $\mathfrak{M} = [m'_{i,l}]_{i,l=1}^{n-2}$ be of the form*

$$m_{i,l} = \begin{cases} 2(\mathbb{F}_k - \mathbb{F}_{k+n}) = e, & i = l, \\ \mathbb{F}_{k+n+1} - \mathbb{F}_{k+1} = f, & l = i + 1, \\ 0, & \text{otherwise}. \end{cases} \tag{16}$$

Then the inverse $\mathfrak{M}^{-1} = [m'_{i,l}]_{i,l=1}^{n-2}$ of the matrix $\mathfrak{M}$ is equal to

$$m'_{i,l} = \begin{cases} \dfrac{(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1})^{l-i}}{[2(\mathbb{F}_k - \mathbb{F}_{k+n})]^{l-i+1}} = \dfrac{(-f)^{l-i}}{e^{l-i+1}}, & l \geq i, \\ 0, & l < i. \end{cases} \tag{17}$$

Proof. Let $e_{i,l} = \sum_{k=1}^{n-2} m_{i,k} m'_{k,l}$. Distinctly, $c_{i,l} = 0$ for $l < i$. In the case $i = l$, we obtain

$$\begin{aligned}
e_{i,i} &= m_{i,i} m'_{i,i} \\
&= (\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}) \cdot \frac{1}{(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1})} \\
&= 1.
\end{aligned} \tag{18}$$

For $l \geq i + 1$, we get

$$\begin{aligned}
e_{i,l} &= \sum_{k=1}^{n-2} m_{i,k} m'_{k,l} \\
&= m_{i,i} m'_{i,l} + m_{i,i+1} m'_{i+1,l} \\
&= e \cdot \frac{(-f)^{l-i}}{e^{l-i+1}} + f \cdot \frac{(-f)^{l-i-1}}{e^{l-i}} \\
&= 0.
\end{aligned} \tag{19}$$

We check on $\mathfrak{M}\mathfrak{M}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2) \times (n-2)$ identity matrix. Similarly, we can verify $\mathfrak{M}^{-1}\mathfrak{M} = I_{n-2}$. Thus, the proof is completed. □

Theorem 4. *Let $A_{k,n} = \mathrm{Circ}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ be a Fermat circulant matrix. Then one acquires $A_{k,n}^{-1} = \mathrm{Circ}(v_1, v_2, \ldots, v_n)$, where*

$$\begin{aligned}
v_1 &= \frac{1}{h_{k,n}} + \left(\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}\right) \\
&\quad \cdot \frac{-\mathbb{F}_{k+n+1} + 3\mathbb{F}_{k+n} + \mathbb{F}_{k+1} - 3\mathbb{F}_k}{2h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)^2} \\
&\quad + \frac{\left(\mathbb{F}_{k+n} - \tau_k \mathbb{F}_{k+n-1}\right)}{h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)}, \\
v_2 &= \frac{-2^k - h_{k,n}\mathbb{F}_{k+1}}{\mathbb{F}_{k+1} h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)}, \\
v_3 &= \frac{\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}}{h_{k,n}} \\
&\quad \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-3}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-2}} + \frac{1}{h_{k,n}} \\
&\quad \cdot \sum_{i=4}^{n} \left(\mathbb{F}_{k+n+4-i} - \tau_k \mathbb{F}_{k+n+3-i}\right) \\
&\quad \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-i}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-i+1}}, \\
v_4 &= \frac{\mathbb{F}_{k+n+2} - \mathbb{F}_{k+2}}{h_{k,n}} \\
&\quad \times \left[\left(\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}\right)\right. \\
&\quad \times \frac{\left(\mathbb{F}_{k+1} - \mathbb{M}_{k+n+1}\right)^{n-4}}{\left[2\left(\mathbb{M}_k - \mathbb{M}_{k+n}\right)\right]^{n-2}} \\
&\quad + \sum_{i=4}^{n} \left(\mathbb{F}_{k+n+4-i} - \tau_k \mathbb{F}_{k+n+3-i}\right) \\
&\quad \left. \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-i-1}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-i+1}}\right], \\
v_s &= 0 \quad (s = 5, 6, \ldots, n).
\end{aligned} \tag{20}$$

Proof. Let

$$\Omega_2 = \begin{pmatrix} 1 & -\dfrac{h'_{k,n}}{\mathbb{F}_{k+1}} & x'_3 & x'_4 & \cdots & x'_n \\ 0 & 1 & y'_3 & y'_4 & \cdots & y'_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{21}$$

where

$$\begin{aligned}
\tau_k &= \frac{\mathbb{F}_{k+2}}{\mathbb{F}_{k+1}}, \\
x'_3 &= \frac{\mathbb{F}_{k+n}}{\mathbb{F}_{k+1}} + \frac{h'_{k,n}}{h_{k,n}} \cdot \frac{\left(\tau_k \mathbb{F}_{k+n} - \mathbb{F}_{k+1}\right)}{\mathbb{F}_{k+1}}, \\
y'_3 &= \frac{\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n}}{h_{k,n}}, \\
x'_i &= \frac{\mathbb{F}_{k+n+3-i}}{\mathbb{F}_{k+1}} + \frac{h'_{k,n}}{h_{k,n}} \\
&\quad \cdot \frac{\tau_k \mathbb{F}_{k+n+3-i} - \mathbb{F}_{k+n+4-i}}{\mathbb{F}_{k+1}} \quad (i = 4, \ldots, n), \\
y'_i &= \frac{\mathbb{F}_{k+n+4-i} - \tau_k \mathbb{F}_{k+n+3-i}}{h_{k,n}} \quad (i = 4, \ldots, n), \\
h'_{k,n} &= \sum_{i=1}^{n-1} \mathbb{F}_{i+k+1} \left[\frac{2\left(\mathbb{F}_{k+n} - \mathbb{F}_k\right)}{\mathbb{F}_{k+n+1} - \mathbb{F}_{k+1}}\right]^{n-i-1}, \\
h_{k,n} &= \sum_{i=1}^{n-2} \left(\mathbb{F}_{i+k+2} - \tau_k \mathbb{F}_{i+k+1}\right) y^{n-i-1} \\
&\quad + \mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n}.
\end{aligned} \tag{22}$$

We have

$$\Sigma A_{k,n} \Omega_1 \Omega_2 = \mathscr{D}_2 \oplus \mathfrak{M}, \tag{23}$$

where $\mathscr{D}_2 = \mathrm{diag}(\mathbb{F}_{k+1}, h_{k,n})$ is a diagonal matrix, and $\mathscr{D}_2 \oplus \mathfrak{M}$ is the direct sum of $\mathscr{D}_2$ and $\mathfrak{M}$. If we denote $\Omega = \Omega_1 \Omega_2$, then we obtain

$$A_{k,n}^{-1} = \Omega\left(\mathscr{D}_2^{-1} \oplus \mathfrak{M}^{-1}\right)\Sigma. \tag{24}$$

Let $A_{k,n}^{-1} = \mathrm{Circ}(v_1, v_2, \ldots, v_n)$. Since the last row elements of the matrix Ω are $0, 1, y'_3 - 1, y'_4, \ldots, y'_{n-1}, y'_n$, according to Lemma 3, then the last row elements of $A_{k,n}^{-1}$ are given by the following equations:

$$\begin{aligned}
v_2 &= -\frac{\tau_k}{h_{k,n}} + \frac{y'_3 - 1}{\mathbb{F}_k - \mathbb{F}_{k+n}}, \\
v_3 &= \left(y'_3 - 1\right)\frac{(-f)^{n-3}}{e^{n-2}} + \sum_{i=4}^{n} y'_i \cdot \frac{(-f)^{n-i}}{e^{n-i+1}}, \\
v_4 &= \left(y'_3 - 1\right)\left[\frac{(-f)^{n-4}}{(-f)^{n-3}} - \frac{3(-f)^{n-3}}{e^{n-2}}\right] \\
&\quad + \sum_{i=4}^{n} y'_i \cdot \left[\frac{(-f)^{n-i-1}}{e^{n-i}} - \frac{3(-f)^{n-i}}{e^{n-i+1}}\right] \\
&\quad \left(t < 0, (-f)^t = 0\right),
\end{aligned}$$

$$v_s = \left(y_3' - 1\right)\left[\frac{(-f)^{n-s}}{e^{n-s+1}} - \frac{3(-f)^{n-s+1}}{e^{n-s+2}} + \frac{2(-f)^{n-s+2}}{e^{n-s+3}}\right] + \sum_{i=4}^{n-s+5} y_i' \cdot \left[\frac{(-f)^{n-i-s+3}}{e^{n-i-s+4}} - \frac{3(-f)^{n-i-s+4}}{e^{n-i-s+5}} + \frac{2(-f)^{n-i-s+5}}{e^{n-i-s+6}}\right]$$

$$\left(s = 5, 6, \ldots, n; t < 0, (-f)^t = 0\right),$$

$$v_1 = \frac{1}{h_{k,n}} + \frac{-2f - 3e}{e^2}(y_3 - 1) + \frac{2}{e} y_4, \tag{25}$$

where $f = \mathbb{F}_{k+n+1} - \mathbb{F}_{k+1}$, $e = 2(\mathbb{F}_k - \mathbb{F}_{k+n})$, according to Lemma 1; then we have

(i) $e + f = 0$,

(ii) $e + 2f = 2^{k+n+1} - 2^{k+1}$.

Hence, we obtain

$$\begin{aligned}
v_1 &= \frac{1}{h_{k,n}} + \left(\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}\right) \cdot \frac{-\mathbb{F}_{k+n+1} + 3\mathbb{F}_{k+n} + \mathbb{F}_{k+1} - 3\mathbb{F}_k}{2h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)^2} + \frac{\left(\mathbb{F}_{k+n} - \tau_k \mathbb{F}_{k+n-1}\right)}{h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)}, \\
v_2 &= \frac{-2^k - h_{k,n}\mathbb{F}_{k+1}}{\mathbb{F}_{k+1} h_{k,n}\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)}, \\
v_3 &= \frac{\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}}{h_{k,n}} \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-3}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-2}} + \frac{1}{h_{k,n}} \cdot \sum_{i=4}^{n}\left(\mathbb{F}_{k+n+4-i} - \tau_k \mathbb{F}_{k+n+3-i}\right) \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-i}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-i+1}}, \\
v_4 &= \frac{\mathbb{F}_{k+n+2} - \mathbb{F}_{k+2}}{h_{k,n}} \times \left[\left(\mathbb{F}_{k+1} - \tau_k \mathbb{F}_{k+n} - h_{k,n}\right) \cdot \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-4}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-2}} + \sum_{i=4}^{n}\left(\mathbb{F}_{k+n+4-i} - \tau_k \mathbb{F}_{k+n+3-i}\right) \times \frac{\left(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1}\right)^{n-i-1}}{\left[2\left(\mathbb{F}_k - \mathbb{F}_{k+n}\right)\right]^{n-i+1}}\right], \\
v_s &= 0 \quad (s = 5, 6, \ldots, n).
\end{aligned} \tag{26}$$

Thus, the proof is completed. □

3. Determinant and Inverse of Mersenne Circulant Matrix

In this section, let $B_{k,n} = \mathrm{Circ}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ be a Mersenne circulant matrix. Firstly, we obtain the determinant of the matrix $B_{k,n}$. Afterwards, we seek out the inverse of the matrix $B_{k,n}$.

Theorem 5. *Let $B_{k,n} = \mathrm{Circ}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ be a Mersenne circulant matrix. Then one obtains*

$$\det B_{k,n} = \mathbb{M}_{k+1} \cdot \left[\sum_{k=1}^{n-2}\left(\mathbb{M}_{k+k+2} - \mu_k \mathbb{M}_{k+k+1}\right) x^{n-k-1} + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}\right] \cdot (-d)^{n-2}, \tag{27}$$

where $x = -c/d$, $c = 2(\mathbb{M}_k - \mathbb{M}_{k+n})$, $d = \mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}$, $\mu_k = \mathbb{M}_{k+2}/\mathbb{M}_{k+1}$, and $\mathbb{M}_{k+n}$ is the $(k+n)$th Mersenne number. Furthermore, $B_{k,n}$ is singular if and only if $(1-\alpha\kappa_l)(1-\beta\kappa_l) \neq 0$ and $\mathbb{M}_{k+1} - 2\kappa_l \mathbb{M}_k - \mathbb{M}_{k+n+1} + 2\kappa_l \mathbb{M}_{k+n} = 0$, for $k \in N$, $n \in N_+$, where $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$, $l = 1, 2, \ldots, n$.

Proof. Obviously,

$$\det A_{0,n} = \mathbb{M}_1 \cdot \left[\sum_{i=1}^{n-2}\left(\mathbb{M}_{i+2} - \mu_0 \mathbb{M}_{i+1}\right) \cdot \left[\frac{2\left(\mathbb{M}_n - \mathbb{M}_0\right)}{\mathbb{M}_{n+1} - \mathbb{M}_1}\right]^{n-k-1} + \mathbb{M}_1 - \mu_0 \mathbb{M}_n\right] \cdot \left[2\left(\mathbb{M}_0 - \mathbb{M}_n\right)\right]^{n-2} \tag{28}$$

satisfies (27). In the following, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & & \\ -\mu_k & & & & & & & 1 \\ 2 & & & & & & 1 & -3 \\ 0 & & & 0 & & 1 & -3 & 2 \\ \vdots & & & & \iddots & \iddots & \iddots & \\ 0 & & & 1 & \iddots & \iddots & & \\ 0 & & 1 & -3 & \iddots & & 0 & \\ 0 & 1 & -3 & 2 & & & & \end{pmatrix}, \tag{29}$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x^{n-2} & 0 & \cdots & 0 & 0 \\ 0 & x^{n-3} & 0 & \cdots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices; then we have

$$\Gamma B_{k,n} \Pi_1 = \begin{pmatrix} \mathbb{M}_{k+1} & f'_{k,n} & -\mathbb{M}_{k+n} & \cdots & -\mathbb{M}_{k+3} \\ 0 & f_{k,n} & h_3 & \cdots & h_n \\ 0 & 0 & c & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & d \\ 0 & 0 & 0 & \cdots & c \end{pmatrix}, \tag{30}$$

where

$$\begin{gathered} c = 2\left(\mathbb{M}_k - \mathbb{M}_{k+n}\right), \\ d = \mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}, \\ x = -\frac{c}{d}, \qquad \mu_k = \frac{\mathbb{M}_{k+2}}{\mathbb{M}_{k+1}}, \\ h_3 = \mu_k \mathbb{M}_{k+n} - \mathbb{M}_{k+1}, \\ h_j = \left(\mu_k \mathbb{M}_{k+n+3-j} - \mathbb{M}_{k+n+4-j}\right) \\ \left(j = 4, 5, \ldots, n\right), \\ f'_{k,n} = \sum_{i=1}^{n-1} \mathbb{M}_{i+k+1} \left[\frac{2\left(\mathbb{M}_{k+n} - \mathbb{M}_k\right)}{\mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}}\right]^{n-i-1}, \\ f_{k,n} = \sum_{i=1}^{n-2} \left(\mathbb{M}_{i+k+2} - \mu_k \mathbb{M}_{i+k+1}\right), x^{n-i-1} \\ + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}. \end{gathered} \tag{31}$$

We get

$$\begin{aligned} &\det \Gamma \det B_{k,n} \det \Pi_1 \\ &= \mathbb{M}_{k+1} \cdot \left[\sum_{i=1}^{n-2} \left(\mathbb{M}_{i+k+2} - \mu_k \mathbb{M}_{i+k+1}\right) x^{n-i-1} \right. \\ &\qquad \left. + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}\right] \cdot c^{n-2}; \end{aligned} \tag{32}$$

besides

$$\begin{gathered} \det \Gamma = (-1)^{(n-1)(n-2)/2}, \\ \det \Pi_1 = (-1)^{(n-1)(n-2)/2} \left[\frac{2\left(\mathbb{M}_{k+n} - \mathbb{M}_k\right)}{\mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}}\right]^{n-2}. \end{gathered} \tag{33}$$

We have

$$\begin{aligned} \det B_{k,n} = \mathbb{M}_{k+1} \cdot &\left[\sum_{i=1}^{n-2} \left(\mathbb{M}_{i+k+2} - \mu_k \mathbb{M}_{i+k+1}\right) x^{n-i-1} \right. \\ &\left. + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}\right] \cdot (-d)^{n-2}. \end{aligned} \tag{34}$$

Now, we discuss the singularity of the matrix $B_{k,n}$.

The roots of polynomial $g(x) = x^n - 1$ are κ_l $(l = 1, 2, \ldots, n)$, where $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$. So we have

$$\begin{aligned} f(\kappa_l) &= \mathbb{M}_{k+1} + \mathbb{M}_{k+2}\kappa_l + \cdots + \mathbb{M}_{k+n}(\kappa_l)^{n-1} \\ &= \frac{\mathbb{M}_{k+1} - 2\kappa_l \mathbb{M}_k - \mathbb{M}_{k+n+1} + 2\kappa_l \mathbb{M}_{k+n}}{(1 - \alpha\kappa_l)(1 - \beta\kappa_l)}. \end{aligned} \tag{35}$$

By Lemma 1 in [14], the matrix $B_{k,n}$ is nonsingular if and only if $f(\kappa_l) \neq 0$. That is when $(1 - \alpha\kappa_l)(1 - \beta\kappa_l) \neq 0$, $B_{k,n}$ is nonsingular if and only if $\mathbb{M}_{k+1} - 2\kappa_l \mathbb{M}_k - \mathbb{M}_{k+n+1} + 2\kappa_l \mathbb{M}_{k+n} \neq 0$, for $k \in N, n \in N_+, l = 1, 2, \ldots, n$. When $(1 - \alpha\kappa_l)(1 - \beta\kappa_l) = 0$, we obtain $\kappa_l = 1/\alpha$ or $\kappa_l = 1/\beta$. Let $\kappa_l = 1/\alpha$; then the eigenvalue of $B_{k,n}$ is

$$f(\kappa_l) = \frac{n\alpha^{k+n} - \beta^{k+1}\mathbb{M}_n}{\alpha^{n-1}(\alpha - \beta)} \neq 0, \tag{36}$$

for $\alpha = 2$, $\beta = 1$, $k \in N$, $n \in N_+$, $l = 1, 2, \ldots, n$, so $B_{k,n}$ is nonsingular. The arguments for $\kappa_l = 1/\beta$ are similar. Thus, the proof is completed. □

Lemma 6. *Let the matrix $\mathfrak{G} = [g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{i,j} = \begin{cases} 2\left(\mathbb{M}_k - \mathbb{M}_{k+n}\right) = c, & i = j, \\ \mathbb{M}_{k+n+1} - \mathbb{M}_{k+1} = d, & j = i + 1, \\ 0, & \textit{otherwise}. \end{cases} \tag{37}$$

Then the inverse $\mathfrak{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-2}$ of the matrix $\mathfrak{G}$ is equal to

$$g'_{i,j} = \begin{cases} \dfrac{\left(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1}\right)^{j-i}}{\left[2\left(\mathbb{M}_k - \mathbb{M}_{k+n}\right)\right]^{j-i+1}} = \dfrac{(-d)^{j-i}}{c^{j-i+1}}, & j \geq i, \\ 0, & j < i. \end{cases} \tag{38}$$

Proof. Let $c_{i,j} = \sum_{k=1}^{n-2} g_{i,k} g'_{k,j}$. Distinctly, $c_{i,j} = 0$ for $j < i$. When $i = j$, we obtain

$$c_{i,i} = g_{i,i} g'_{i,i} = -d \cdot \frac{1}{-d} = 1. \tag{39}$$

For $j \geq i+1$, we obtain

$$\begin{aligned} c_{i,j} &= \sum_{k=1}^{n-2} g_{i,k} g'_{k,j} = g_{i,i} g'_{i,j} + g_{i,i+1} g'_{i+1,j} \\ &= c \cdot \frac{(-d)^{j-i}}{c^{j-i+1}} + d \cdot \frac{(-d)^{j-i-1}}{c^{j-i}} = 0. \end{aligned} \tag{40}$$

We verify $\mathfrak{G}\mathfrak{G}^{-1} = I_{n-2}$, where I_{n-2} is $(n-2)\times(n-2)$ identity matrix. Similarly, we check on $\mathfrak{G}^{-1}\mathfrak{G} = I_{n-2}$. Thus, the proof is completed. □

Theorem 7. *Let $B_{k,n} = \operatorname{Circ}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ be a Mersenne circulant matrix. Then one acquires*

$$B_{k,n}^{-1} = \operatorname{Circ}(u_1, u_2, \ldots, u_n), \tag{41}$$

where

$$\begin{aligned} u_1 &= \frac{1}{f_{k,n}} + (\mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n} - f_{k,n}) \\ &\quad \cdot \frac{-\mathbb{M}_{k+n+1} + 3\mathbb{M}_{k+n} + \mathbb{M}_{k+1} - 3\mathbb{M}_k}{2 f_{k,n} (\mathbb{M}_k - \mathbb{M}_{k+n})^2} \\ &\quad + \frac{(\mathbb{M}_{k+n} - \mu_k \mathbb{M}_{k+n-1})}{f_{k,n}(\mathbb{M}_k - \mathbb{M}_{k+n})}, \\ u_2 &= \frac{2^k - f_{k,n}\mathbb{M}_{k+1}}{\mathbb{M}_{k+1} f_{k,n} (\mathbb{M}_k - \mathbb{M}_{k+n})}, \\ u_3 &= \frac{\mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n} - f_{k,n}}{f_{k,n}} \\ &\quad \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-3}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-2}} + \frac{1}{f_{k,n}} \\ &\quad \cdot \sum_{i=4}^{n} (\mathbb{M}_{k+n+4-i} - \mu_k \mathbb{M}_{k+n+3-i}) \\ &\quad \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-i}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-i+1}}, \\ u_4 &= \frac{\mathbb{M}_{k+n+2} - \mathbb{M}_{k+2}}{f_{k,n}} \\ &\quad \times \left[(\mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n} - f_{k,n}) \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-4}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-2}} \right. \\ &\quad + \sum_{i=4}^{n} (\mathbb{M}_{k+n+4-i} - \mu_k \mathbb{M}_{k+n+3-i}) \\ &\quad \left. \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-i-1}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-i+1}} \right], \\ u_s &= 0 \quad (s = 5, 6, \ldots, n), \end{aligned} \tag{42}$$

where

$$\begin{aligned} \mu_k &= \frac{\mathbb{M}_{k+2}}{\mathbb{M}_{k+1}}, \\ f_{k,n} &= \sum_{i=1}^{n-2} (\mathbb{M}_{i+k+2} - \mu_k \mathbb{M}_{i+k+1}) x^{n-i-1} \\ &\quad + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}. \end{aligned} \tag{43}$$

Proof. Let

$$\Pi_2 = \begin{pmatrix} 1 & -\dfrac{f'_{k,n}}{\mathbb{M}_{k+1}} & x_3 & x_4 & \cdots & x_n \\ 0 & 1 & y_3 & y_4 & \cdots & y_n \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{44}$$

where

$$\begin{aligned} \mu_k &= \frac{\mathbb{M}_{k+2}}{\mathbb{M}_{k+1}}, \\ x_3 &= \frac{\mathbb{M}_{k+n}}{\mathbb{M}_{k+1}} + \frac{f'_{k,n}}{f_{k,n}} \cdot \frac{(\mu_k \mathbb{M}_{k+n} - \mathbb{M}_{k+1})}{\mathbb{M}_{k+1}}, \\ y_3 &= \frac{\mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}}{f_{k,n}}, \\ x_i &= \frac{\mathbb{M}_{k+n+3-i}}{\mathbb{M}_{k+1}} + \frac{f'_{k,n}}{f_{k,n}} \\ &\quad \cdot \frac{\mu_k \mathbb{M}_{k+n+3-i} - \mathbb{M}_{k+n+4-i}}{\mathbb{M}_{k+1}} \\ &\quad (i = 4, \ldots, n), \\ y_i &= \frac{\mathbb{M}_{k+n+4-i} - \mu_k \mathbb{M}_{k+n+3-i}}{f_{k,n}} \\ &\quad (i = 4, \ldots, n), \\ f'_{k,n} &= \sum_{i=1}^{n-1} \mathbb{M}_{i+k+1} \left[\frac{2(\mathbb{M}_{k+n} - \mathbb{M}_k)}{\mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}} \right]^{n-i-1}, \\ f_{k,n} &= \sum_{i=1}^{n-2} (\mathbb{M}_{i+k+2} - \mu_k \mathbb{M}_{i+k+1}) x^{n-i-1} \\ &\quad + \mathbb{M}_{k+1} - \mu_k \mathbb{M}_{k+n}. \end{aligned} \tag{45}$$

We have

$$\Gamma B_{k,n} \Pi_1 \Pi_2 = \mathscr{D}_1 \oplus \mathfrak{G}, \tag{46}$$

where $\mathscr{D}_1 = \operatorname{diag}(\mathbb{M}_{k+1}, f_{k,n})$ is a diagonal matrix, and $\mathscr{D}_1 \oplus \mathfrak{G}$ is the direct sum of $\mathscr{D}_1$ and $\mathfrak{G}$. If we denote $\Pi = \Pi_1 \Pi_2$, then we obtain

$$B_{k,n}^{-1} = \Pi \left(\mathscr{D}_1^{-1} \oplus \mathfrak{G}^{-1} \right) \Gamma. \tag{47}$$

Let $B_{k,n}^{-1} = \mathrm{Circ}(u_1, u_2, \ldots, u_n)$. Since the last row elements of the matrix Π are $0, 1, y_3 - 1, y_4, \ldots, y_{n-1}, y_n$, according to Lemma 6, then the last row elements of $B_{k,n}^{-1}$ are given by the following equations:

$$\begin{aligned}
u_2 &= -\frac{\mu_k}{f_{k,n}} + \frac{2(y_3-1)}{c},\\
u_3 &= (y_3-1)\cdot\frac{(-d)^{n-3}}{c^{n-2}} + \sum_{i=4}^{n} y_i \cdot \frac{(-d)^{n-i}}{c^{n-i+1}},\\
u_4 &= (y_3-1)\cdot\left[\frac{(-d)^{n-4}}{c^{n-3}} - \frac{3(-d)^{n-3}}{c^{n-2}}\right]\\
&\quad + \sum_{i=4}^{n} y_i \cdot \left[\frac{(-d)^{n-i-1}}{c^{n-i}} - \frac{3(-d)^{n-i}}{c^{n-i+1}}\right]\\
&= (y_3-1)\cdot\frac{(-d)^{n-4}}{c^{n-2}}(c+3d)\\
&\quad + \sum_{i=4}^{n} y_i \cdot \frac{(-d)^{n-i-1}}{c^{n-i+1}}(c+3d) \quad \left(t<0, (-d)^t = 0\right),\\
u_s &= (y_3-1)\\
&\quad \cdot\left[\frac{(-d)^{n-s}}{c^{n-s+1}} - \frac{3(-d)^{n-s+1}}{c^{n-s+2}} + \frac{2(-d)^{n-s+2}}{c^{n-s+3}}\right]\\
&\quad + \sum_{i=4}^{n-s+5} y_i \cdot \left[\frac{(-d)^{n-i-s+3}}{c^{n-i-s+4}}\right.\\
&\qquad \left. - \frac{3(-d)^{n-i-s+4}}{c^{n-i-s+5}} + \frac{2(-d)^{n-i-s+5}}{c^{n-i-s+6}}\right]\\
&= \left[(y_3-1)\cdot\frac{(-d)^{n-s}}{c^{n-s+3}} + \sum_{i=4}^{n-s+5} y_i \cdot \frac{(-d)^{n-i-s+3}}{c^{n-i-s+6}}\right]\\
&\quad \times (c+2d)(c+d) \quad \left(s=5,6,\ldots,n; t<0, (-d)^t = 0\right),\\
u_1 &= \frac{1}{f_{k,n}} + \frac{-2d-3c}{c^2}(y_3-1) + \frac{2}{c}y_4,
\end{aligned} \tag{48}$$

where $d = \mathbb{M}_{k+n+1} - \mathbb{M}_{k+1}, c = 2(\mathbb{M}_k - \mathbb{M}_{k+n})$, according to Lemma 1; then we have

(i) $c + d = 0$,

(ii) $c + 2d = 2^{k+n+1} - 2^{k+1}$.

We get

$$\begin{aligned}
u_1 &= \frac{1}{f_{k,n}} + (\mathbb{M}_{k+1} - \mu_k\mathbb{M}_{k+n} - f_{k,n})\\
&\quad \cdot \frac{-\mathbb{M}_{k+n+1} + 3\mathbb{M}_{k+n} + \mathbb{M}_{k+1} - 3\mathbb{M}_k}{2f_{k,n}(\mathbb{M}_k - \mathbb{M}_{k+n})^2}\\
&\quad + \frac{(\mathbb{M}_{k+n} - \mu_k\mathbb{M}_{k+n-1})}{f_{k,n}(\mathbb{M}_k - \mathbb{M}_{k+n})},\\
u_2 &= \frac{2^k - f_{k,n}\mathbb{M}_{k+1}}{\mathbb{M}_{k+1}f_{k,n}(\mathbb{M}_k - \mathbb{M}_{k+n})},\\
u_3 &= \frac{\mathbb{M}_{k+1} - \mu_k\mathbb{M}_{k+n} - f_{k,n}}{f_{k,n}}\\
&\quad \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-3}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-2}} + \frac{1}{f_{k,n}}\\
&\quad \cdot \sum_{i=4}^{n}(\mathbb{M}_{k+n+4-i} - \mu_k\mathbb{M}_{k+n+3-i})\\
&\qquad \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-i}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-i+1}},\\
u_4 &= \frac{\mathbb{M}_{k+n+2} - \mathbb{M}_{k+2}}{f_{k,n}}\\
&\quad \times \left[(\mathbb{M}_{k+1} - \mu_k\mathbb{M}_{k+n} - f_{k,n})\frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-4}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-2}}\right.\\
&\qquad + \sum_{i=4}^{n}(\mathbb{M}_{k+n+4-i} - \mu_k\mathbb{M}_{k+n+3-i})\\
&\qquad \left. \cdot \frac{(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1})^{n-i-1}}{[2(\mathbb{M}_k - \mathbb{M}_{k+n})]^{n-i+1}}\right],\\
u_s &= 0 \quad (s = 5, 6, \ldots, n).
\end{aligned} \tag{49}$$

Thus, the proof is completed. □

4. Determinants and Inverses of Fermat and Mersenne Left Circulant Matrix

In this section, let $A'_{k,n} = \mathrm{LCirc}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ and $B'_{k,n} = \mathrm{LCirc}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ be Mersenne and Fermat left circulant matrices, respectively. By using the obtained conclusions, we give a determinant formula for the matrix $A'_{k,n}$ and $B'_{k,n}$. In addition, the inverse matrices of $A'_{k,n}$ and $B'_{k,n}$ are derived.

According to Lemma 2 in [14] and Theorems 2, 4, 5, and 7, we can obtain the following theorems.

Theorem 8. *Let $A'_{k,n} = \mathrm{LCirc}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ be a Fermat left circulant matrix; then one has*

$$\begin{aligned}
\det A'_{k,n} &= (-1)^{(n-1)(n-2)/2}\cdot\mathbb{F}_{k+1}\\
&\quad \cdot\left[\sum_{j=1}^{n-2}\left(\mathbb{F}_{j+k+2} - \tau_k\mathbb{F}_{j+k+1}\right)p^{n-j-1} + \mathbb{F}_{k+1} - \tau_k\mathbb{F}_{k+n}\right]\\
&\quad \cdot(\mathbb{F}_{k+1} - \mathbb{F}_{k+n+1})^{n-2},
\end{aligned} \tag{50}$$

where $\tau_k = \mathbb{F}_{k+2}/\mathbb{F}_{k+1}$, $p = 2(\mathbb{F}_{k+n} - \mathbb{F}_k)/(\mathbb{F}_{k+n+1} - \mathbb{F}_{k+1})$, and $\mathbb{F}_{k+n}$ is the $(k+n)$th Fermat number. Moreover, $A'_{k,n}$ is singular if and

only if $(1-\alpha\kappa_l)(1-\beta\kappa_l) \neq 0$ *and* $\mathbb{F}_{k+1}-2\kappa_l\mathbb{F}_k-\mathbb{F}_{k+n+1}+2\kappa_l\mathbb{F}_{k+n} = 0$, *for* $k \in N$, $n \in N_+$, *where* $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$, $l = 1, 2, \ldots, n$.

Theorem 9. *Let* $A'_{k,n} = \mathrm{LCirc}(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n})$ *be a Fermat left circulant matrix; then*

$$\begin{aligned}\left(A'_{k,n}\right)^{-1} &= \mathrm{Circ}^{-1}\left(\mathbb{F}_{k+1}, \mathbb{F}_{k+2}, \ldots, \mathbb{F}_{k+n}\right)\cdot\Delta \\ &= \mathrm{Circ}\left(v_1, v_2, \ldots, v_n\right)\cdot\Delta \\ &= \mathrm{LCirc}\left(v_1, v_n, \ldots, v_2\right),\end{aligned} \tag{51}$$

where $v_1, v_2, \ldots, v_n$ *were given by Theorem 4 and* $\Delta = \mathrm{LCirc}(1, 0, \ldots, 0)$ *was given by Lemma 2 in [14].*

Theorem 10. *Let* $B'_{k,n} = \mathrm{LCirc}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ *be a Mersenne left circulant matrix; then one has*

$$\begin{aligned}\det B'_{k,n} = {} & (-1)^{(n-1)(n-2)/2}\cdot\mathbb{M}_{k+1} \\ & \cdot\Bigg[\mathbb{M}_{k+1} - \mu_k\mathbb{M}_{k+n} \\ & + \sum_{j=1}^{n-2}\left(\mathbb{M}_{j+k+2} - \mu_k\mathbb{M}_{j+k+1}\right)z^{n-j-1}\Bigg] \\ & \cdot\left(\mathbb{M}_{k+1} - \mathbb{M}_{k+n+1}\right)^{n-2},\end{aligned} \tag{52}$$

where $\mu_k = \mathbb{M}_{k+2}/\mathbb{M}_{k+1}$, $z = 2(\mathbb{M}_{k+n} - \mathbb{M}_k)/(\mathbb{M}_{k+n+1} - \mathbb{M}_{k+1})$, *and* $\mathbb{M}_{k+n}$ *is the* $(k+n)$*th Mersenne number. Furthermore,* $B'_{k,n}$ *is singular if and only if* $(1-\alpha\kappa_l)(1-\beta\kappa_l) \neq 0$ *and* $\mathbb{M}_{k+1} - 2\kappa_l\mathbb{M}_k - \mathbb{M}_{k+n+1} + 2\kappa_l\mathbb{M}_{k+n} = 0$, *for* $k \in N$, $n \in N_+$, *where* $\kappa_l = \cos(2l\pi/n) + i\sin(2l\pi/n)$, $l = 1, 2, \ldots, n$.

Theorem 11. *Let* $B'_{k,n} = \mathrm{LCirc}(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n})$ *be a Mersenne left circulant matrix; then one has*

$$\begin{aligned}\left(B'_{k,n}\right)^{-1} &= \mathrm{Circ}^{-1}\left(\mathbb{M}_{k+1}, \mathbb{M}_{k+2}, \ldots, \mathbb{M}_{k+n}\right)\cdot\Delta \\ &= \mathrm{Circ}\left(u_1, u_2, \ldots, u_n\right)\cdot\Delta \\ &= \mathrm{LCirc}\left(u_1, u_n, \ldots, u_2\right),\end{aligned} \tag{53}$$

where $u_1, u_2, \ldots, u_n$ *were given by Theorem 7 and* $\Delta = \mathrm{LCirc}(1, 0, \ldots, 0)$ *was given by Lemma 2 in [14].*

5. Conclusion

In this paper, we present the exact determinants and the inverse matrices of Fermat and Mersenne circulant matrix, respectively. Furthermore, we give the exact determinants and the inverse matrices of Fermat and Mersenne left circulant matrix. Meanwhile, the nonsingularity of these special matrices is discussed. On the basis of circulant matrices technology, we will develop solving the problems in [19–22].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Extracting Backbones from Weighted Complex Networks with Incomplete Information

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The backbone is the natural abstraction of a complex network, which can help people understand a networked system in a more simplified form. Traditional backbone extraction methods tend to include many outliers into the backbone. What is more, they often suffer from the computational inefficiency—the exhaustive search of all nodes or edges is often prohibitively expensive. In this paper, we propose a backbone extraction heuristic with incomplete information (BEHwII) to find the backbone in a complex weighted network. First, a strict filtering rule is carefully designed to determine edges to be preserved or discarded. Second, we present a local search model to examine part of edges in an iterative way, which only relies on the local/incomplete knowledge rather than the global view of the network. Experimental results on four real-life networks demonstrate the advantage of BEHwII over the classic disparity filter method by either effectiveness or efficiency validity.

1. Introduction

Complex networks have become an important approach for understanding systems involving interacting objects [1]. Thus, networked systems have permeated a wide spectrum of domains, ranging from the biology and the automatic control to the computer science [2, 3]. With networked systems being increasingly large, to understand and reveal the underlying phenomena taking place in such systems are facing considerable challenges. The presence of the backbone is a signature or an abstraction of the nature of complex systems and can provide huge help for understanding them in more simplified forms [4]. For example, detecting the backbones in criminal networks can better target suspects [5]. Also, urban planners attempt to examine the topologies of public transport systems by analyzing their backbones [6].

Recent years have witnessed an increasing interest in extracting backbones in large-scale weighted networks of various kinds [4, 7–9]. As many networks are evolving into large scale and the weight distributions are spanning several orders of magnitude, extracting backbones from them has become a critical task for research and applications of various purposes. In general, the backbone should be thought of as a set of nodes and edges that interconnect various pieces of network, providing a path for the exchange of information between different subnetworks [10]. Thus, a promising way for backbone extraction is to map the original network into a smaller network, in which the numbers of nodes and edges should be small enough to be amenable to analysis and visualization.

In the literature, the existing methods can be roughly divided into two categories, one based on the coarse graining and the other is filter-based. The methods based on the coarse graining [4, 7, 11–14] clump nodes sharing common attributes together in the same group/community and then consider the whole group as one single unit in the new networks. However, there is often no clear statement on whether properties of the initial network should be preserved in the network of clusters [15].

The filter-based methods [8, 9, 16–18] typically employ a bottom-up strategy to extract the backbone. They often start by defining a statistical property of a node or an edge, and then this property is used as a criterion to determine nodes/edges to be preserved or discarded. In this case,

the observation scale is fixed and the representation that the network symbolizes is not changed. Instead, those elements, nodes, and edges, which carry relevant information about the network structure, are kept while the rest are discarded. However, the filter-based methods may include a multitude of outliers, which should not be included into the backbone naturally. What is more, they often suffer from the computational inefficiency: the exhaustive search of all nodes or edges is often prohibitively expensive.

In this work, we attempt to design a novel filter-based method for extracting backbones from large-scale weighted networks. Unlike the exhaustive search adopted by the existing methods, the proposed approach only needs incomplete information and then invokes the iteratively local search scheme for improving the efficiency. So, this novel method is called backbone extraction heuristic with incomplete information (BEHwII). In particular, although α_{ij} proposed in [8] is employed as the filtering criterion, BEHwII imposes max instead of min to enhance the filtering rule, so that the case of extracting too many outliers into the backbone can be avoided. Our method is naturally a heuristic, since it does not examine all edges in the network. Alternatively, BEHwII greedily selects an optimal edge in one iteration and adds this edge into the backbone if the predefined max filtering rule is satisfied. Extensive experiments on various real-world networks demonstrate the superiority of BEHwII over the global filtering method in terms of effectiveness and efficiency.

The remainder of this paper is organized as follows. In Section 2, we introduce preliminaries and motivation of this work. In Section 3, we discuss the local search mechanism and then present the algorithmic details of BEHwII. Experimental results will be given in Section 4. We present the related work in Section 5 and finally conclude this paper in Section 6.

2. Preliminaries and Motivation

Since the proposed method for backbone extraction is a filter-based model in essence, we begin by providing the preliminary knowledge about the filter-based model. Thus, we analyze some drawbacks of existing filter-based methods, which leads to a better understanding of the motivation of this paper.

The filter-based models typically employ a bottom-up strategy to extract the backbone. They often start by defining a statistical property of a node or an edge, and then this property is used as a criterion to determine nodes/edges to be preserved or discarded. As a result, preserved nodes and their links, or preserved edges and their endpoints, composed the backbone of the network. Therefore, the key step in filter-based methods is how to define a reasonable filtering property for nodes/edges. For instance, k-core is a well-known filtering property that is used to construct a hierarchical topological filter in [16]. However, many simple filtering properties (e.g., k-core) are not suitable for weighted networks. Meanwhile, the real-world weighted networks are usually with strong disorder heavy-tailed distributions of weights [19]. That is, the probability distribution $P(w)$ that any given link carries a weight w is broadly distributed, spanning several orders of magnitude. This feature exerts nontrivial challenges to define the filtering property for weighted networks, due in large part to the lack of a characteristic scale. Serrano et al. [8] addressed this challenge by introducing the disparity filter based on the null hypothesis; that is, the normalized weights that correspond to the connections of a certain node of degree k are produced by a random assignment from a uniform distribution. Given a node i and its associated link with weight w_{ij}, the normalized weight p_{ij} is defined as

$$p_{ij} = \frac{w_{ij}}{\sum_l w_{il}}. \tag{1}$$

Under the null hypothesis, a *null model* is then presented, in which $k-1$ points are distributed with uniform probability in the interval $[0, 1]$. As a result, k subintervals are generated, of which lengths represent the expected values for the k normalized weights p_{ij} according to the null hypothesis. The probability density function for one of these variables taking a particular value x is

$$\rho(x)\,dx = (k-1)(1-x)^{k-2}\,dx. \tag{2}$$

Based on (2), given an edge, the probability α_{ij} indicating its normalized weight p_{ij} is compatible with the null model and can be defined as

$$\begin{aligned}\alpha_{ij} &= 1 - (k_i - 1)\int_0^{p_{ij}} (1-x)^{k_i-2}\,dx \\ &= \left(1 - p_{ij}\right)^{k_i-1},\end{aligned} \tag{3}$$

where k_i is the degree of node i. Thus, α_{ij} is adopted as the filtering criterion in [8] for weighted networks. Given a significance level α, the edges that carry weights which can be considered not compatible with a random distribution can be filtered out with a certain statistical significance. That is, edges with $\alpha_{ij} < \alpha$ should be kept, since they reject the null hypothesis.

The criterion α_{ij} gave birth to an effective filter-based method for backbone extraction [8]. However, two drawbacks have attracted our attention. One of the biggest limitations is that it may include a multitude of outliers, which should not be included into the backbone naturally. In what follows, we try to explore its cause and give a modified scheme.

For node i with degree k_i, the level of local heterogeneity in the weights can be calculated as

$$\gamma(k_i) = k_i \sum_j p_{ij}^2. \tag{4}$$

Thus, under perfect homogeneity, when all the links share the same amount of the strength of the node, $\gamma(k_i)$ equals 1 independently of k_i, while in the case of perfect heterogeneity, when just one of the links carries the whole strength of the node, $\gamma(k_i)$ is equal to k_i. With predefined null model, the join probability distribution for two intervals can be defined as

$$\begin{aligned}\rho(x, y)\,dx\,dy = (k-1)(k-2)(1-x-y)^{k-3}\,\Theta \\ \times (1-x-y)\,dx\,dy,\end{aligned} \tag{5}$$

where $\Theta(\cdot)$ is the Heaviside step function, which can be used to calculate the statistics of $\gamma_{\text{null}}(k_i)$ for the null model. The average $\mu(\gamma_{\text{null}}(k_i))$ and the standard deviation $\sigma^2(\gamma_{\text{null}}(k_i))$ are estimated to be

$$\mu\left(\gamma_{\text{null}}\left(k_i\right)\right) = \frac{2k_i}{k_i+1},$$
$$\sigma^2\left(\gamma_{\text{null}}\left(k_i\right)\right) = k_i^2\left(\frac{4k_i+20}{\left(k_i+1\right)\left(k_i+2\right)\left(k_i+3\right)} - \frac{4}{\left(k_i+1\right)^2}\right). \tag{6}$$

In real networks, the observed level of local heterogeneity, denoted by $\gamma_{\text{ob}}(k_i)$, can be compared against the null model expectations. Namely, the observed values are compatible with the null hypotheses when they lie between the perfect homogeneity and $\mu(\gamma_{\text{null}}(k_i)) + a \cdot \sigma(\gamma_{\text{null}}(k_i))$. And the local heterogeneity will be recognized only if $\gamma_{\text{ob}}(k_i)$ obeys

$$\gamma_{\text{ob}}\left(k_i\right) > \mu\left(\gamma_{\text{null}}\left(k_i\right)\right) + a \cdot \sigma\left(\gamma_{\text{null}}\left(k_i\right)\right). \tag{7}$$

The parameter a is a constant determining the confidence interval for the evaluation of the null hypothesis. The larger it is, the more restrictive the null model becomes and the more disordered weights should be for local heterogeneity to be detected. A typical value of a in analogy to Gaussian statistics could be set as 2. In Figure 1, we show two regions (local heterogeneity and local compatibility) associated with different k_i. Obviously, small nodes in terms of degree (e.g., $k_i < 5$) are more likely to fall into the local compatible region, which implies that those nodes with small degree should not be preserved in the backbone.

In [8], the multiscale backbone is obtained by preserving all the links which beat the significant level α for at least one of the two nodes at the ends of the link while discounting the rest. Notice that α_{ij} is not symmetrical; that is, $\alpha_{ij} \neq \alpha_{ji}$, if $k_i \neq k_j$. In the case of a node i with degree $k_i < 5$ connected to a node j with degree $k_j \gg 5$, we might have $\alpha_{ji} < \alpha < \alpha_{ij}$. Then this link will be preserved as it holds $\min(\alpha_{ij}, \alpha_{ji}) < \alpha$. However, as discussed above, node i is likely to fall into the local compatible region, which should be kept away from the backbone. Considering that an intermediate power law degree distribution is usually observed in real systems, the disparity filter in [8] may include a multitude of outliers. To avoid including many outliers into the backbone, one can impose max instead of min to enhance the filtering rule, so that a connection is preserved whenever its intensity is significant for both nodes involved.

Secondly, most of the existing filter-based methods [8, 9, 16, 17] suffer from the computational inefficiency, the exhaustive search of all nodes or edges in a network. For example, the filtering method based on α_{ij} is heavily dependent on the number of links. As many social networking sites are evolving into superlarge scales, for example, containing millions even billions of nodes and edges, the computation will be terrible!

According to the above analysis, this paper proposes a local method for extracting backbones from weighted networks. In particular, we try to answer the following two questions:

(i) Q1: how to carefully design a filtering criterion to avoid including many outliers into the backbone?

(ii) Q2: how to reduce the computational complexity of the backbone extraction algorithm?

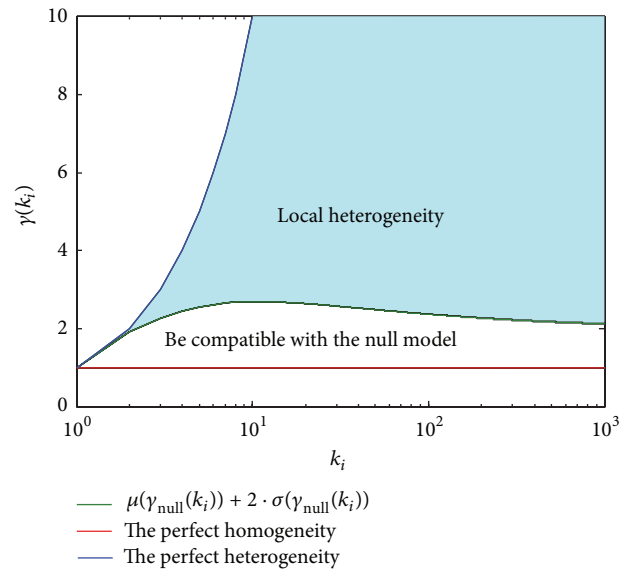


Figure 1: $\gamma(k_i)$ compared against the null model expectations.

3. Backbone Extraction Heuristic with Incomplete Information (BEHwII)

Let $\mathcal{G} = (V, E, W)$ be a given weighted graph, where V is the set of nodes ($|V| = n$), E is the set of edges ($|E| = m$) that connect the nodes in V, and W is the weight of every edge in E. Backbone extraction is formulated as finding a subset of graph $\mathcal{G}' = (V', E')$, that is, the backbone, where $|E'| \ll |E|$ and $\forall e_{ij} \in E'$, $\alpha_{ij} < \alpha$. This implies that the backbone should also significantly reduce the number of edges, while preserving most essential connections.

In this section, we propose a backbone extraction heuristic with incomplete information (BEHwII for short). First, we introduce the basic idea of BEHwII, covering the local search mechanism. Second, we present algorithmic details including the complexity analysis for BEHwII.

3.1. Local Search Model. In this paper, we employ the filtering criterion α_{ij} proposed in [8]. However, one major drawback lies in that it is probable to include too many outliers into the backbone as stated in Section 2. To explore its cause, we argue that this drawback originates from the looseness of the filtering rule, that is, $\min(\alpha_{ij}, \alpha_{ji}) < \alpha$. Therefore, BEHwII attempts to impose max instead of min to enhance the filtering rule, so that a connection is preserved whenever its intensity is significant for both nodes involved. In BEHwII, an edge e_{ij} is *preserved* in the backbone, if

$$\alpha_{ij}^* = \max\left(\alpha_{ij}, \alpha_{ji}\right) < \alpha, \tag{8}$$

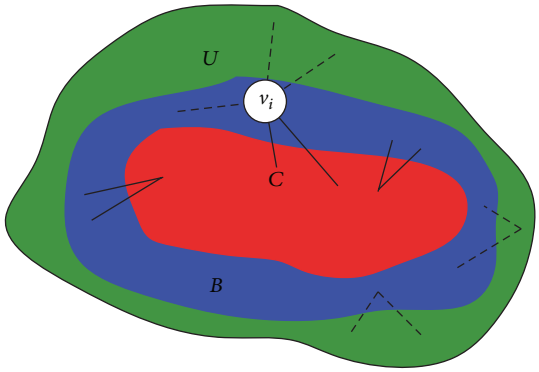


Figure 2: Illustration for the local search.

where α_{ij} is the probability derived by comparing the normalized weight p_{ij} with the null model, as shown in (3). With the filtering rule, BEHwII aims to extract a certain percentage (denoted by $\%m_t$) of edges satisfying (8) as the backbone.

A straightforward way for backbone extraction is to apply the exhaustive search, that is, to examine all of the edges one by one, and add the edge to the backbone as (8) satisfied. Obviously, this exhaustive search suffers from the computational inefficiency, especially when the network becomes much larger. Here, we introduce a local search model to solve this problem. We divide the explored graph into three regions: the known local area $\mathscr{C}$, the boundary area $\mathscr{B}$, and a larger unknown area $\mathscr{U}$, as illustrated in Figure 2. Initially, we randomly select a node v_s as the start node and add v_s to $\mathscr{C}$. Then, all neighbors of nodes in $\mathscr{C}$ (e.g., v_s) are added to $\mathscr{B}$. The local search model selects an optimal edge e_{ij} with minimum α^*_{ij} from $\mathscr{C} \cup \mathscr{B}$ and adds it into the backbone if it holds (8). Areas $\mathscr{C}$ and $\mathscr{B}$ are expanded accordingly. Another edge will be selected and checked, until a certain number of edges are included into the backbone.

Remark 1. The local search model is a streaming and iterative scheme in essence [20]. An iterative process is invoked to examine each node along with its neighbors and performs a computation, of which the result is associated with the processed node. Such scheme is a very promising technique of scaling the existing method. Moreover, the local search model is independent of the "global knowledge"; that is, it only needs to fetch part of the node adjacency lists into main-memory. Due to the small-world effect, our model is validated to be slightly dependent on the initial node selection, of which the experimental results will be given in Section 4.1.

3.2. Algorithmic Details. In this section, we introduce how to use BEHwII to extract the backbone starting from any randomly selected node. BEHwII initially places the randomly selected source node v_s into the known local area ($\mathscr{C} \leftarrow \{v_s\}$) and adds its neighbors into $\mathscr{B}$. Two data structures used in BEHwII are described as follows:

(i) *Min-heap* H, which stores the edge information, including e_{ij} and $\max(\alpha_{ij}, \alpha_{ji})$, in $\mathscr{C} \cup \mathscr{B}$, so that every update process will take $O(\log |H|)$ time;

(ii) *List* L^E, which stores the edges of the backbone, and every insert process will take $O(1)$ time.

```
procedure BEHwII(v_s, α, %m_t)
    𝒞 ← {v_s};
    ℬ ← {v_i | v_i ∈ N_s};
    H ← {⟨e_si, α*_si⟩}, where α*_si = max(α_si, α_is);
    while |L^E| ≤ %m_t * m do
        Get the minimal α*_ij from H;
        if α*_ij < α then
            L^E ← e_ij;
        end if
        H ← H/e_ij;
        if ∃i' ∈ {i, j}, v_i' ∉ 𝒞 then
            𝒞 ← 𝒞 ⋃ {v_i'};
            ℬ ← ℬ ⋃ {v_j' | v_j' ∈ N_i'};
            H ← H ⋃ {⟨e_i'j', α*_i'j'⟩};
        end if
        if |𝒞| ≥ n then
            break;
        end if
    end while
    return L^E;
end procedure
```

Algorithm 1: BEHwII algorithm.

We describe the BEHwII Algorithm step by step roughly as follows.

Step 1. Find the edge e_{ij} with the minimal value of $\max(\alpha_{ij}, \alpha_{ji})$ in $\mathscr{C} \cup \mathscr{B}$ and add it into L^E if it satisfies (8).

Step 2. If any endpoints on the considered edge e_{ij} are not included in $\mathscr{C}$ ($\exists i' \in \{i, j\}, v_{i'} \notin \mathscr{C}$), remove $v_{i'}$ from $\mathscr{B}$ to $\mathscr{C}$; otherwise, delete edge e_{ij} and turn to Step 1.

Step 3. Delete edge e_{ij} and remove additional nodes ($v_{j'} \mid v_{j'} \in N_{i'}, v_{j'} \in \mathscr{U}$) from $\mathscr{U}$ to $\mathscr{B}$.

The above process continues until it has agglomerated a certain percentage of edges, or it has discovered the entire enclosing component, whichever happens first. Note that if e_{ij} with the minimal value of $\max(\alpha_{ij}, \alpha_{ji})$ in Step 1 does not satisfy (8), we still check its endpoints and add corresponding edges into $\mathscr{C} \cup \mathscr{B}$. Here, the nodes between e_{ij} can be seen as the excessive nodes to continue the search process. See Algorithm 1 for more exact pseudocode.

Computational Complexity. The main computational cost of the above algorithm originates from the number of examined edges M. For each examined edge e_{ij}, BEHwII needs to calculate the value of $\max(\alpha_{ij}, \alpha_{ji})$ on it and update the min-heap H. Because $\max(\alpha_{ij}, \alpha_{ji})$ depends on the degrees of nodes v_i and v_j and on the normalized weights p_{ij} and p_{ji}, thus, it takes $O(k_i + k_j)$ time to calculate $\max(\alpha_{ij}, \alpha_{ji})$ on each examined edge. The updating (inserting or deleting) cost of H for each examined edge is $O(\log |H|)$. In general, the running

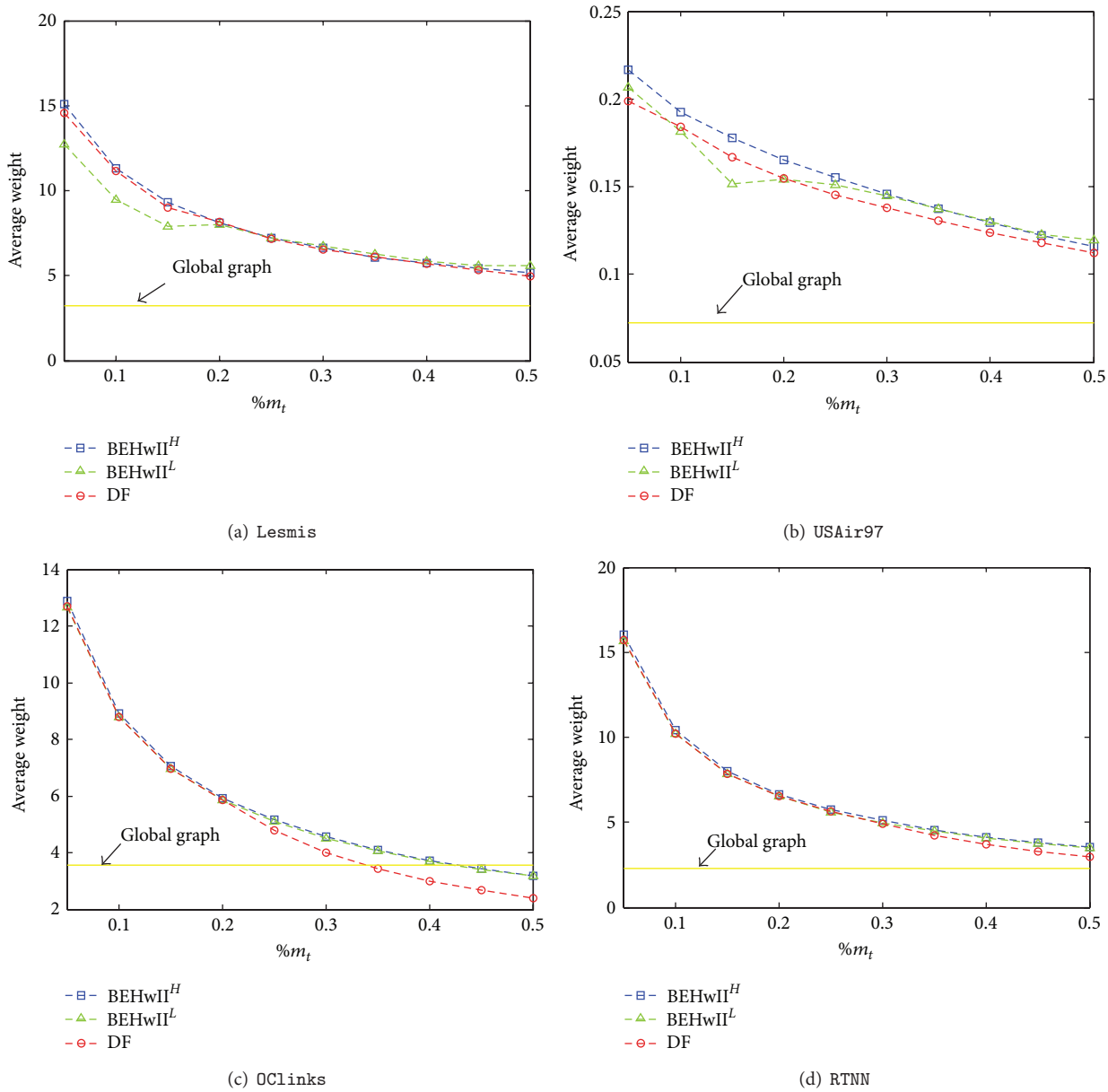


Figure 3: Comparison in terms of the average weight.

Table 1: Real-world networks for experiments.

Network	$\lvert V\rvert$	$\lvert E\rvert$	$\langle k\rangle$	$\langle w\rangle$
Lesmis	77	254	6.60	3.22
USAir97	322	2,126	12.80	0.07
OClinks	1,899	20,296	14.60	2.95
RTNN	13,308	148,035	22.25	2.29

time for the algorithm is $O(M(2\langle k\rangle + \log|H|))$, where $\langle k\rangle$ is the average degree of the graph.

4. Experimental Results

Four real-world undirected and weighted networks, `Lesmis`, `USAir97`, `OClinks`, and `RTNN`, are used for experiments. Some characteristics of these networks are shown in Table 1, where $|V|$ and $|E|$ indicate the numbers of nodes and edges, respectively, in the network, $\langle k\rangle$ indicates the average degree, and $\langle w\rangle$ indicates the average weight. `Lesmis` [21] is the network of coappearances of characters in Victor Hugo's novel, where nodes represent characters and edges connect any pair of characters that appear in the same chapter of the book. `USAir97` [22] gathers 2126 flight information between 332 US airports, where the weight represents the normalized distance among two airports. `OClinks` [23] is a network created from an online community, where nodes represent students at the University of California and edges are established between two students if one or more messages have been sent from one to the other. `RTNN` [24] is also a coappearance network including all words/terms in online

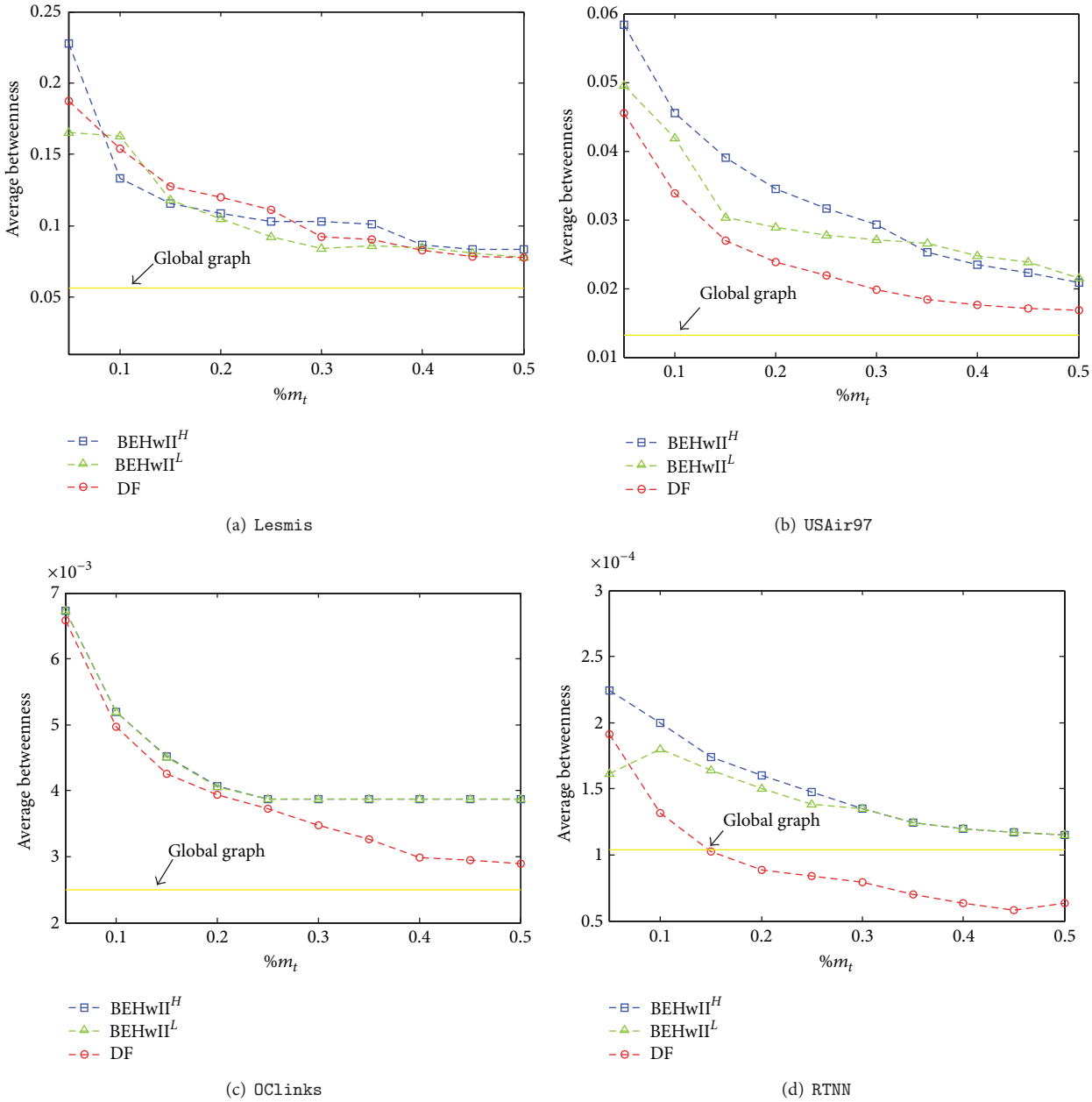
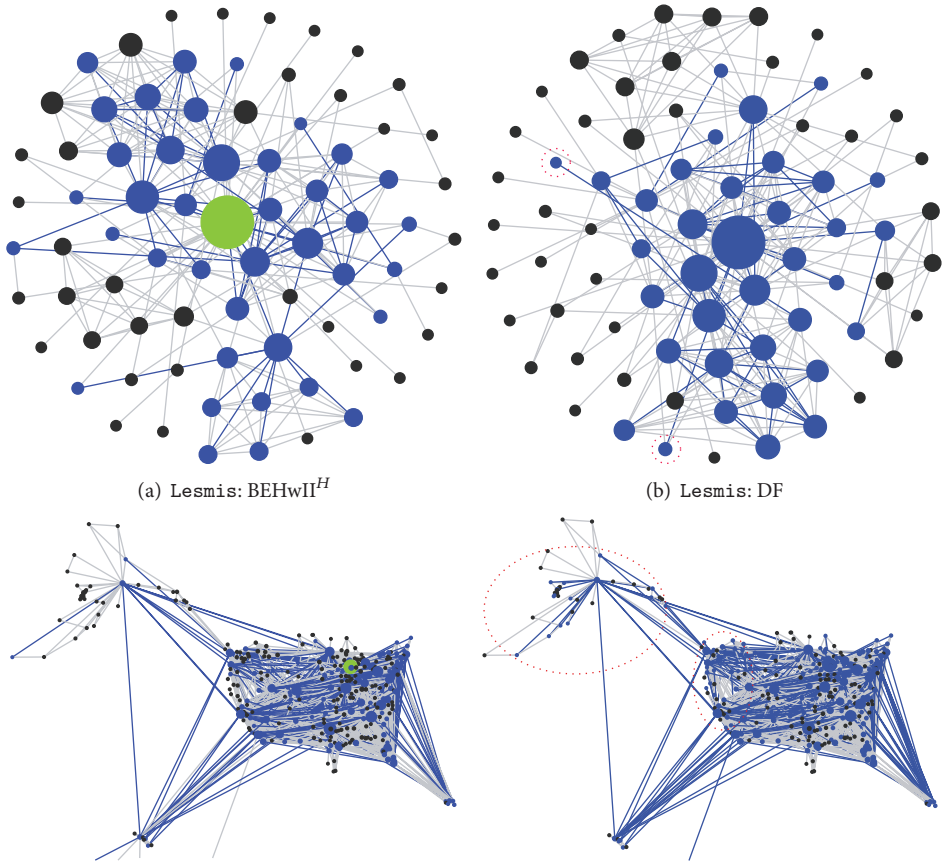


Figure 4: Comparison in terms of the average betweenness.

stories about the September 11 attack, where each node represents a word and each tie means that the two words appear in the same story.

4.1. Comparison Results. In this subsection, we compare BEHwII with the disparity filter (DF for short) proposed by Serrano et al. [8] in performance and scalability. BEHwII is a local search based algorithm, which can start from any randomly selected source node. To investigate the impact of the parameter v_s, we fix $\alpha = 0.5$ and take $v_s = v_h, v_l$, respectively, where v_h is a high-connected node and v_l is a low-connected one. Both v_h and v_l are randomly selected from the original network. For convenience, we denote BEHwII starting from v_h by BEHwII^H; then BEHwII^L represents BEHwII starting from v_l. For a given extraction goal (the percent edges kept in the backbone), the effectiveness of BEHwII^H, BEHwII^L, and DF can be validated by measuring the average weight and node betweenness of the extracted backbones, while the efficiencies can be measured by the number of examined edges and the overall running time.

Effectiveness. Figure 3 shows the average weight of the extracted backbones when the original graphs are extracted by BEHwII^H, BEHwII^L, and DF, respectively. Note that as the only parameter for DF is α, for a given network, the fraction of extracted edges $\%m_t$ is a monotonically increasing function



(a) Lesmis: BEHwII^H (b) Lesmis: DF

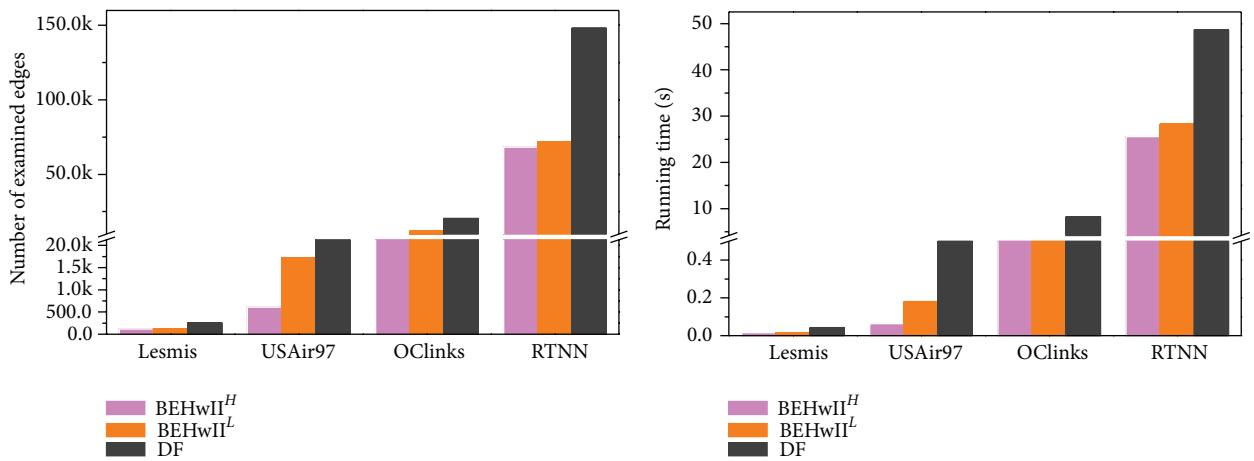
(c) USAir97: BEHwII^H (d) USAir97: DF

Figure 5: Comparison in terms of network visualizations.

of α. For convenient comparison, both DF and BEHwII use the same parameter $\%m_t$, which is gradually increased so that the number of extracted edges grows accordingly. Two observations are noteworthy from Figure 3. First, compared with DF, BEHwII^H shows slight improvements in terms of the average weight, no matter what $\%m_t$ is input. BEHwII^L does not perform well when $\%m_t$ is set to be too small. For instance, BEHwII^L obtains the $\%m_t = 0.1$ backbone with the average weight lower than 10 on the Lesmis network, but, after using BEHwII^H and DF to extract backbones, the average weight increases significantly. Another important observation is that BEHwII^H and BEHwII^L will trend consistently as $\%m_t$ grows to a certain level. As can be seen from Figures 3(a) and 3(b), when the fraction of edges grows to around 0.25, the backbones extracted by BEHwII^H and BEHwII^L will have the same value of average weight. As BEHwII^L adds local optimum edge into the backbone, even if it starts from a low-connected source node, it can sniff several high-connected nodes within limited steps. Therefore, BEHwII^L will evolve to a BEHwII^H after a certain percentage of edges have been discovered.

We then extensively explore the average node betweenness in the backbones extracted from Lesmis, USAir97, OClinks, and RTNN. Node betweenness centrality is the fraction of all shortest paths in the network that contain a given node, which reflects the connectedness of the node. Figure 4 shows the average betweenness of extracted nodes for different fractions of edges $\%m_t$ in the backbones. We can clearly find out that both BEHwII^H and BEHwII^H outperform DF in all of the test graphs. This implies that the edges extracted by BEHwII always lie between two high-connected nodes. As for DF, the filtering rule is so loose that some outliers (nodes with degree equal 1) will be included in the backbones, which will drop the connectedness of extracted backbone.

We then take a direct look at the extracted backbones. The Lesmis and USAir97 networks are used here as two examples. We set $\%m_t = 0.25$ and $\alpha = 0.5$ for BEHwII^H. In the case of Lesmis, the extracted backbone obtained by BEHwII^L is shown in Figure 5(a). The source node is colored with green, the nodes and edges colored with blue are those kept in the backbones, the size of the node expresses its strength ($\sum_l w_{il}$), and the thickness of the edge represents the weight on it. Interestingly, the backbone obtained by BEHwII^H preserves almost all high-connectivity nodes and essential connections. We then employ DF directly on this network and obtain a backbone as shown in Figure 5(b). The clique-like pattern on the top is missed, and, what is more, two outliers (highlighted by dashed circles) are kept.



(a) Lesmis (b) USAir97

Figure 6: Comparison on efficiency.

As for the USAir97 network, nodes are placed in the plane according to their actual coordinates on the earth. The backbone extracted by BEHwII^H, as shown in Figure 5(c), almost covers all the geographic regions of USA. In addition, the hierarchy of the transportation system is fully highlighted, including not just the most high flux connections but also small weight edges that are statistically significant because they represent relevant signal at the small scales. However, the backbone extracted by DF includes many small airports in Alaska and the west coast of USA (highlighted in dashed ellipses).

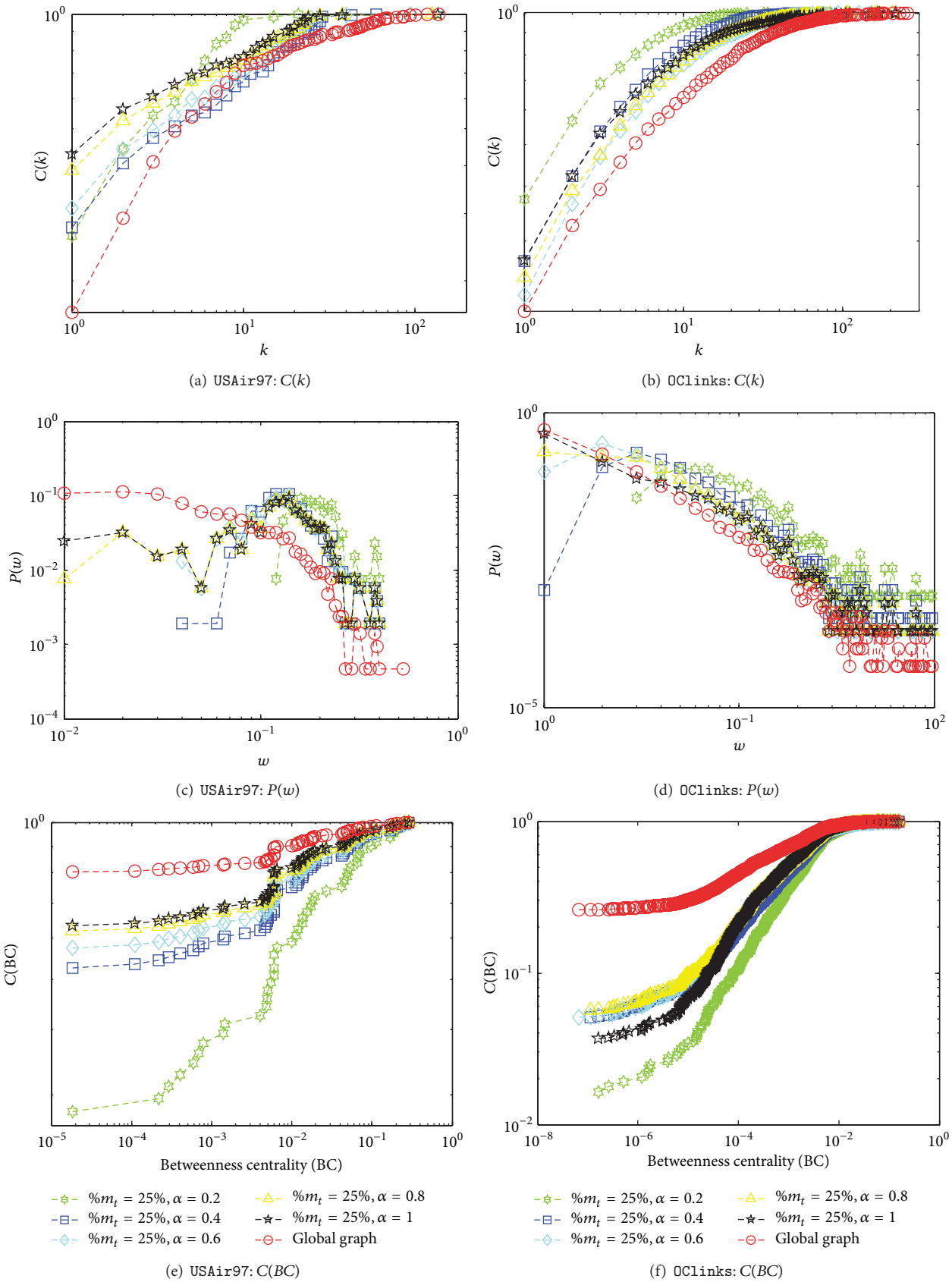
The Efficiency. Figure 6 compares the efficiencies of BEHwII and DF, given the extraction goal $\%m_t = 0.25$. The numbers of examined edges by BEHwII^H, BEHwII^L, and DF for the four test networks are shown in Figure 6(a). Apparently, BEHwII^H and BEHwII^L examine fewer edges than DF does. The latter will examine all nodes and edges in the network. Figure 6(b) verifies our analysis in Section 3.2; that is, the running time of BEHwII originates from the number of examined edges. It is interesting to find that the running time of BEHwII^H and BEHwII^L remains nearly constant in relative large dense graphs (e.g., OClinks and RTNN), that is, because those two networks have the "small world" effect [23, 24], in which most nodes can be reached from each other by a small number of hops or steps. In this context, both BEHwII^H and BEHwII^L can rapidly sniff those high-connected nodes; therefore their overall running times are almost consistent.

4.2. Inside BEHwII. Here, we take a further step to explore several factors that affect the performance of BEHwII. We select BEHwII starting from a high-connected source node, that is, BEHwII^H, for experiments. Two inside factors have been investigated: the significant level α and the inside filtering rule.

The Significant Level α. It is particularly interesting to analyze the behavior of the topological properties of the backbones extracted by BEHwII^H at increasing levels of the significant level α. Figures 7(a) and 7(b) show the evolution of the cumulative degree distribution, $C(k) = \sum_{k' \leq k} P(k')$, with different values of α for USAir97 and OClinks, respectively. The backbones extracted by BEHwII^H have the cumulative degree distributions similar to the original networks. Smaller values of α have flat startups, indicating that the extracted backbones contain fewer low-degree nodes. The evolution of the weight distribution ($P(w)$) with different values of α is shown in Figures 7(c) and 7(d), from which we observe that the original USAir97 and OClinks networks are both heavy tailed. Interestingly, almost all scales are kept during the search process until BEHwII^H becomes too restrictive, in which case BEHwII^H applies a very small value of α. A restrictive BEHwII^H cuts $P(w)$ off below w_c, which may discard the region of small weights. Finally, we analyze the cumulative node betweenness centrality distributions of extracted backbones. It is worth mentioning that the node betweenness centrality in the backbone is given as that in the original network. Figures 7(e) and 7(f) give the evolution of the cumulative betweenness centrality distribution with different α. For both test graphs, $C(BC)$ starts from a very low value if BEHwII^H applies a very small value of α, which implies that those low-connected nodes will not be included in the backbones.

Therefore, we can conclude that values of α in the range $[0.4, 0.8]$ are optimal, in the sense that backbones extracted by BEHwII^H in this region have a large proportion of high-connective nodes and essential connections, and the stable stationary degree/weight distributions, compared with the original network. It is important to stress that BEHwII^H also includes the connections with the largest weight present in the network. This is because the heavy tail of the $P(w)$ distribution is mainly determined by relevant large-scale weight. This is clearly illustrated in Figures 7(c) and 7(d).

The Inside Filtering Rule. We further explore the critical factor that contributes to the success of BEHwII^H. As discussed



(a) USAir97: $C(k)$

(b) OClinks: $C(k)$

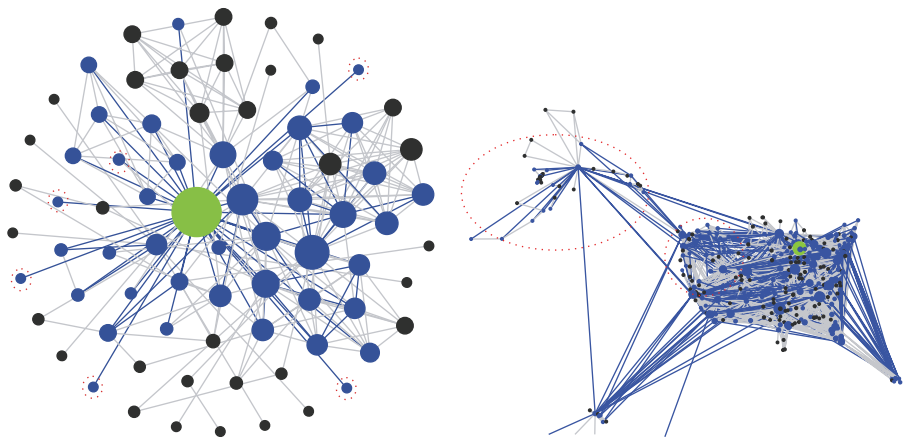
(c) USAir97: $P(w)$

(d) OClinks: $P(w)$

(e) USAir97: $C(BC)$

(f) OClinks: $C(BC)$

FIGURE 7: Impact of the threshold α.



(a) Lesmis (b) USAir97

Figure 8: Backbones extracted by BEHwII^{H*}.

in Section 3.1, BEHwII^{H} uses a strict filtering rule to absorb edges. Here, we relax the previous inside filtering rule, by imposing min instead of max, so that a connection is preserved whenever its intensity is significant for one of the nodes involved. In this loose BEHwII^{H} (denoted by BEHwII^{H*}), an edge e_{ij} is *preserved* in the backbone, if $\min(\alpha_{ij}, \alpha_{ji}) < \alpha$. We visualize the backbones of Lesmis and USAir97 extracted by BEHwII^{H*} in Figure 8. For each test network, we set $\alpha = 0.5$ and $\%m_t = 0.25$. In the case of the Lesmis network, six outliers (highlighted by dashed circles) are extracted by BEHwII^{H*}, and it also fails to discover many essential connections. Obviously, its performance is worse than BEHwII^{H} by comparing Figures 8(a) and 5(a). BEHwII^{H*} has made progress in the case of USAir97, as most regions of USA have been covered in the extracted backbone as shown in Figure 8(b). However, it still includes many small airports in Alaska and the west coast of USA (highlighted in dashed ellipses) as DF does.

5. Related Work

In the literature, the existing backbone extraction methods can fall into two categories: the coarse graining based methods and the filter-based methods. The methods based on the coarse graining clump nodes sharing common attributes together in the same group/community and then consider the whole group as one single unit in the new networks. Some methods along this line include the box-covering technique [4], fractal skeleton [7], and traditional community detection techniques such as the Kernighan-Lin algorithm [11], latent space models [12], stochastic block models [13], and modularity optimization [14]. The differences between these methods ultimately come down to the precise definition of a community. However, there is often no clear statement on whether properties of the initial network are preserved in the network of groups.

The filter-based methods typically employ a bottom-up strategy to extract the backbone. They often start by defining a statistical property of a node or an edge, and then this property is used as a criterion to determine nodes/edges to be preserved or discarded. In this case, the observation scale is fixed and the representation that the network symbolizes is not changed. Instead, those elements, nodes, and edges, which carry relevant information about the network structure, are kept while the rest are discarded. An example of a well-known hierarchical topological filter is the k-core decomposition [16], with a filtering rule that acts on the connectivity of the nodes. In the case of weighted networks, two basic reduction techniques refer to the extraction of the minimum spanning tree [17] and the application of a global threshold [18] on the edge-weights, so that just those that beat the threshold are preserved, as real-world weighted networks that are usually with strong disorder heavy-tailed distributions of weight, which exerts nontrivial challenges to define the filtering property. Serrano et al. [8] addressed this challenge by introducing the disparity filter based on the null hypothesis.

In summary, although backbone extraction based on the coarse graining and filter models are extensively studied, they all need the knowledge of the entire network. Further study is still needed on finding a nice balance between the good performance and high efficiency. Our work attempts to fill this void by conducting backbone extraction based on an efficient BEHwII method.

6. Conclusion

In this work, we propose a backbone extraction heuristic with incomplete information (BEHwII) to find the backbone in a complex weighted network. First, a strict filtering rule is carefully designed to determine edges to be preserved or discarded. Second, we present a local search model to examine part of edges in an iterative way, which only relies on the local/incomplete knowledge rather than the global view of the network. Experimental results on four real-life networks demonstrate the advantage of BEHwII over the classic disparity filter method by either effectiveness or efficiency validity.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Explicit Form of the Inverse Matrices of Tribonacci Circulant Type Matrices

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It is a hot topic that circulant type matrices are applied to networks engineering. The determinants and inverses of Tribonacci circulant type matrices are discussed in the paper. Firstly, Tribonacci circulant type matrices are defined. In addition, we show the invertibility of Tribonacci circulant matrix and present the determinant and the inverse matrix based on constructing the transformation matrices. By utilizing the relation between left circulant, g-circulant matrices and circulant matrix, the invertibility of Tribonacci left circulant and Tribonacci g-circulant matrices is also discussed. Finally, the determinants and inverse matrices of these matrices are given, respectively.

1. Introduction

Circulant type matrices have important applications in various networks engineering. Exploiting the circulant structure of the channel matrices, Eghbali et al. [1] analysed the realistic near fast fading scenarios with circulant frequency selective channels. The optimum sampling in the one- and two-dimensional (1D and 2D) wireless sensor networks (WSNs) with spatial temporally correlated data was studied with circulant matrices in [2]. The repeat space theory (RST) was extended to apply to carbon nanotubes and related molecular networks, where the corresponding matrices are pseudocirculant in [3]. Preconditioners obtained by circulant approximations of stochastic automata networks were considered in [4]. In [5], circulant mutation whose differential equations obtained neither are of repliator-type nor can they be transformed straightway into a linear equation was introduced into autocatalytic reaction networks. Jing and Jafarkhani [6] proposed distributed differential space-time codes that work for networks with any number of relays using circulant matrices. Wang and Cheng [7] studied the existence of doubly periodic travelling waves in cellular networks involving the discontinuous Heaviside step function by circulant matrix. Pais et al. [8] proved conditions for the existence of stable limit cycles arising from multiple distinct Hopf bifurcations of the dynamics in the case of circulant fitness matrices.

Circulant type matrices have been put on the firm basis with the work in [9, 10] and so on. Furthermore, the g-circulant matrices are focused on by many researchers; for the details please refer to [11–13] and the references therein.

Lately, some scholars gave the explicit determinant and inverse of the circulant and skew-circulant involving famous numbers. Jiang et al. [14] discussed the invertibility of circulant type matrices with the sum and product of Fibonacci and Lucas numbers and presented the determinants and the inverses of these matrices. Jiang et al. [15] considered circulant type matrices with the k-Fibonacci and k-Lucas numbers and presented the explicit determinant and inverse matrix by constructing the transformation matrices. Jiang and Hong [16] gave exact form determinants of the RSFPLR circulant matrices and the RSLPFL circulant matrices involving Perrin, Padovan, Tribonacci, and the generalized Lucas number by the inverse factorization of polynomial. Bozkurt and Tam gave determinants and inverses of circulant matrices with Jacobsthal and Jacobsthal-Lucas numbers in [17]. Cambini presented an explicit form of the inverse of a particular circulant matrix in [18]. Shen et al. considered circulant matrices

with Fibonacci and Lucas numbers and presented their explicit determinants and inverses in [19].

The Tribonacci sequences are defined by the following recurrence relations [20–22], respectively:

$$T_n = T_{n-1} + T_{n-2} + T_{n-3}, \tag{1}$$

where $T_0 = 0, T_1 = 1, T_2 = 1, n \geq 3$.

The first few values of the sequences are given by the following table:

$$\begin{array}{c|cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ \hline T_n & 0 & 1 & 1 & 2 & 4 & 7 & 13 & 24 & 44 & \cdots \end{array}. \tag{2}$$

Let t_1, t_2, and t_3 be the roots of the characteristic equation $x^3 - x^2 - x - 1 = 0$ and then we have

$$\begin{gathered} t_1 + t_2 + t_3 = 1, \\ t_1t_2 + t_1t_3 + t_2t_3 = -1, \\ t_1t_2t_3 = 1. \end{gathered} \tag{3}$$

Hence the Binet formulas of the sequences $\{T_n\}$ have the form

$$T_n = b_1t_1^n + b_2t_2^n + b_3t_3^n, \tag{4}$$

where b_i is the ith root of the polynomial $44y^3 - 2y - 1$ for $i = 1, 2, 3$.

In this paper, we consider circulant type matrices, including the circulant and left circulant and g-circulant matrices. If we suppose T_n is the nth Tribonacci number, then we define a Tribonacci circulant matrix which is an $n \times n$ matrix with the following form:

$$\begin{aligned} &Circ\left(T_1, T_2, \cdots, T_n\right) \\ &\quad = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ T_n & T_1 & \cdots & T_{n-1} \\ \vdots & \vdots & & \vdots \\ T_2 & T_3 & \cdots & T_1 \end{bmatrix}. \end{aligned} \tag{5}$$

Besides, a Tribonacci left circulant matrix is given by

$$\begin{aligned} &LCirc\left(T_1, T_2, \cdots, T_n\right) \\ &\quad = \begin{bmatrix} T_1 & T_2 & \cdots & T_n \\ T_2 & T_3 & \cdots & T_1 \\ \vdots & \vdots & & \vdots \\ T_n & T_1 & \cdots & T_{n-1} \end{bmatrix}, \end{aligned} \tag{6}$$

where each row is a cyclic shift of the row above to the left.

A Tribonacci g-circulant matrix is an $n \times n$ matrix with the following form:

$$A_{g,n} = \begin{pmatrix} T_1 & T_2 & \cdots & T_n \\ T_{n-g+1} & T_{n-g+2} & \cdots & T_{n-g} \\ T_{n-2g+1} & T_{n-2g+2} & \cdots & T_{n-2g} \\ \vdots & \vdots & \ddots & \vdots \\ T_{g+1} & T_{g+2} & \cdots & T_g \end{pmatrix}, \tag{7}$$

where g is a nonnegative integer and each of the subscripts is understood to be reduced modulo n. The first row of $A_{g,n}$ is $(T_1, T_2, \ldots, T_n)$, and its $(j+1)$th row is obtained by giving its jth row a right circular shift by g positions (equivalently, g mod n positions). Note that $g = 1$ or $g = n + 1$ yields standard Tribonacci circulant matrix. If $g = n - 1$, then we obtain Tribonacci left circulant matrix.

Lemma 1. *The $n \times n$ tridiagonal matrix is given by*

$$A_n = \begin{pmatrix} \tau_2 & \tau_1 & & & & & 0 \\ \tau_3 & \tau_2 & \tau_1 & & & & \\ & \tau_3 & \tau_2 & \tau_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tau_3 & \tau_2 & \tau_1 & \\ & & & & \tau_3 & \tau_2 & \tau_1 \\ 0 & & & & & \tau_3 & \tau_2 \end{pmatrix}; \tag{8}$$

then

$$\det A_n = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1} - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1}}{\sqrt{\tau_2^2 - 4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ (n+1)\left(\dfrac{\tau_2}{2}\right)^n, \quad \tau_2^2 = 4\tau_1\tau_3. \end{cases} \tag{9}$$

Proof. $\det A_n = \tau_2 \cdot \det A_{n-1} - \tau_1\tau_3 \cdot \det A_{n-2}$; let $x + y = \tau_2$, $xy = \tau_1\tau_3$ and then let x, y be the roots of the equation $x^2 - \tau_2 x + \tau_1\tau_3 = 0$.

We have

$$\begin{aligned} \det A_n &= y^n + xy^{n-1} + \cdots + x^{n-1}y + x^n \\ &= \begin{cases} \dfrac{x^{n+1} - y^{n+1}}{x - y}, & x \neq y, \\ (n+1)\,x^n, & x = y, \end{cases} \end{aligned} \tag{10}$$

where $x = (\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3})/2$ and $y = (\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3})/2$.

Hence,

$$\det A_n = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1} - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^{n+1}}{\sqrt{\tau_2^2 - 4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ (n+1)\left(\dfrac{\tau_2}{2}\right)^n, \quad \tau_2^2 = 4\tau_1\tau_3. \end{cases} \tag{11}$$

□

Lemma 2. *Let*

$$B_n = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_{n-2} & a_{n-1} & a_n \\ \tau_2 & \tau_1 & & & & & \\ \tau_3 & \tau_2 & \tau_1 & & & & \\ & \tau_3 & \tau_2 & \tau_1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & \tau_3 & \tau_2 & \tau_1 & \\ & & & & \tau_3 & \tau_2 & \tau_1 \end{pmatrix} \tag{12}$$

be an $n \times n$ matrix; one has

$$\det B_n = \sum_{i=1}^{n} (-1)^{1+i} \tau_1^{n-i} a_i \cdot \det A_{i-1}, \tag{13}$$

where

$$\det A_{i-1} = \begin{cases} \dfrac{\left(\left(\tau_2 + \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^i - \left(\left(\tau_2 - \sqrt{\tau_2^2 - 4\tau_1\tau_3}\right)/2\right)^i}{\sqrt{\tau_2^2 - 4\tau_1\tau_3}}, & \tau_2^2 \neq 4\tau_1\tau_3, \\ i\left(\dfrac{\tau_2}{2}\right)^{i-1}, \quad \tau_2^2 = 4\tau_1\tau_3. \end{cases} \tag{14}$$

Specifically, $\det A_0 = 1$.

Proof. Accoding to the last column determinant expansion and Lemma 1, we obtain

$$\begin{aligned} \det B_n &= \tau_1 \cdot \det B_{n-1} + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \tau_1 \left(\tau_1 \cdot \det B_{n-2} + (-1)^n a_{n-1} \cdot \det A_{n-2}\right) \\ &\quad + (-1)^{n+1} a_n \cdot \det A_{n-1} \\ &= \tau_1^{n-1} \cdot \det B_1 + (-1)^{1+2} \tau_1^{n-2} a_2 \cdot \det A_1 \\ &\quad + (-1)^{1+3} \tau_1^{n-3} a_3 \cdot \det A_2 \\ &\quad + \cdots + (-1)^{1+n} a_n \cdot \det A_{n-1} \\ &= \sum_{i=1}^{n} (-1)^{1+i} \tau_1^{n-i} a_i \cdot \det A_{i-1}. \end{aligned} \tag{15}$$

□

2. Determinant and Inverse of Tibonacci Circulant Matrix

In this section, let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci circulant matrix. Firstly, we give the determinant of the matrix D_n. Afterwards, we discuss the invertibility of the matrix D_n, and we find the inverse of the matrix D_n.

Theorem 3. *Let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci circulant matrix; then we have*

$$\begin{aligned} \det D_n = &\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ &- \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2, \end{aligned} \tag{16}$$

where T_n is the nth Tribonacci number, and

$$\begin{aligned} \Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\ \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}, \\ \det A_{i-1} &= \begin{cases} \dfrac{\left(\left(b + \sqrt{b^2 - 4ac}\right)/2\right)^i - \left(\left(b - \sqrt{b^2 - 4ac}\right)/2\right)^i}{\sqrt{b^2 - 4ac}}, & b^2 \neq 4ac, \\ i\left(\dfrac{b}{2}\right)^{i-1}, \quad b^2 = 4ac, \end{cases} \\ a &= T_1 - T_{n+1}, \\ b &= -T_n - T_{n-1}, \\ c &= -T_n. \end{aligned} \tag{17}$$

Proof. Obviously, $\det D_1 = 1$ satisfies (16). In the case where $n > 1$, let

$$\Gamma_1 = \begin{pmatrix} 1 & & & & & & 0 \\ -1 & & & & & & 1 \\ -1 & & & & & 1 & -1 \\ -1 & & & & 1 & -1 & -1 \\ 0 & & & 1 & -1 & -1 & -1 \\ \vdots & & \iddots & \iddots & \iddots & \iddots & \\ 0 & 1 & -1 & -1 & -1 & & \\ 0 & 1 & -1 & -1 & -1 & & \end{pmatrix}, \tag{18}$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \Delta^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & \Delta^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & \Delta & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices and then we have

$$\Gamma_1 D_n \Pi_1 = \begin{pmatrix} T_1 & p_1 & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 \\ 0 & p_2 & \varphi_3 & \varphi_4 & \cdots & \varphi_{n-1} & \varphi_n \\ 0 & p_3 & \phi & T_{n-3} & \cdots & T_2 & T_1 \\ 0 & 0 & b & a & & & \\ 0 & 0 & c & b & a & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & c & b & a \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned} p_1 &= \sum_{i=1}^{n-1} T_{i+1} \Delta^{n-(i+1)}, \\ p_2 &= T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right), \\ p_3 &= -T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}, \\ \varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n), \\ \phi &= T_1 - T_n - T_{n-1}, \\ \Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \\ a &= T_1 - T_{n+1}, \qquad b = -T_n - T_{n-1}, \qquad c = -T_n. \end{aligned} \quad (20)$$

We obtain

$$\begin{aligned} &\det \Gamma_1 \det D_n \det \Pi_1 \\ &\quad = T_1 \cdot \left(p_2 \delta_1 - p_3 \delta_2\right) \\ &\quad = \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ &\qquad - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2, \end{aligned} \quad (21)$$

where

$$\begin{aligned} \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \cdot T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}. \end{aligned} \quad (22)$$

Let

$$\mathscr{B}_n = \begin{pmatrix} \phi & T_{n-3} & \cdots & T_3 & T_2 & T_1 \\ b & a & & & & \\ c & b & a & & & \\ & \ddots & \ddots & \ddots & & \\ & & c & b & a & \\ & & & c & b & a \end{pmatrix},$$
$$\mathscr{C}_n = \begin{pmatrix} \varphi_3 & \varphi_4 & \cdots & \varphi_{n-2} & \varphi_{n-1} & \varphi_n \\ b & a & & & & \\ c & b & a & & & \\ & \ddots & \ddots & \ddots & & \\ & & c & b & a & \\ & & & c & b & a \end{pmatrix} \quad (23)$$

be two $(n-2) \times (n-2)$ matrices, and $\delta_1 = \det \mathscr{B}_n$, and $\delta_2 = \det \mathscr{C}_n$.

Aoccording to Lemma 2, thus

$$\begin{aligned} \delta_1 &= \left(T_1 - T_n - T_{n-1}\right)\left(T_1 - T_{n+1}\right)^{n-3} \\ &\quad + \sum_{i=2}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \cdot T_{n-i-1} \cdot \det A_{i-1}, \\ \delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} \left(T_1 - T_{n+1}\right)^{n-i-2} \left(T_{n-i+1} - T_{n-i}\right) \cdot \det A_{i-1}. \end{aligned} \quad (24)$$

While

$$\det \Gamma_1 = \det \Pi_1 = (-1)^{(n-1)(n-2)/2}, \quad (25)$$

we have

$$\begin{aligned} \det D_n &= \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right] \delta_1 \\ &\quad - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right] \delta_2. \end{aligned} \quad (26)$$

□

Theorem 4. *Let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci circulant matrix; if $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then D_n is an invertible matrix.*

Proof. When $n = 2$ in Theorem 3, then we have $\det D_2 = 0$. Hence, D_2 is not invertible.

In the case where $n > 2$, since $T_n = b_1 t_1^n + b_2 t_2^n + b_3 t_3^n$, where b_i is the ith root of the polynomial $44y^3 - 2y - 1$, we have

$$\begin{aligned} f\left(\omega^k\right) &= \sum_{j=1}^{n} T_j \left(\omega^k\right)^{j-1} \\ &= \sum_{j=1}^{n} \left(b_1 t_1^j + b_2 t_2^j + b_3 t_3^j\right) \left(\omega^k\right)^{j-1} \end{aligned}$$

$$= \frac{b_1 t_1 (1 - t_1^n)}{1 - t_1 \omega^k} + \frac{b_2 t_2 (1 - t_2^n)}{1 - t_2 \omega^k} + \frac{b_3 t_3 (1 - t_3^n)}{1 - t_3 \omega^k}$$
$$= \frac{1 - T_{n+1} + (-T_n - T_{n-1}) \omega^k - T_n \omega^{2k}}{1 - \omega^k - \omega^{2k} - \omega^{3k}}$$
$$(k = 1, 2, \ldots, n-1). \tag{27}$$

If there exists ω^l $(l = 1, 2, \ldots, n-1)$ such that $f(\omega^l) = 0$, we obtain $1 - T_{n+1} + (-T_n - T_{n-1})\omega^k - T_n\omega^{2k} = 0$ for $1 - \omega^k - \omega^{2k} - \omega^{3k} \neq 0$. Thus, $\omega^l = (T_n + T_{n-1} \pm \sqrt{(-T_n - T_{n-1})^2 + 4T_n(T_1 - T_{n+1})})/-2T_n$.

Let $a = T_1 - T_{n+1}$, $b = -T_n - T_{n-1}$, and $c = -T_n$; if $b^2 - 4ac \geq 0$, we have ω^l is a real number.

While

$$\omega^l = \exp\left(\frac{2l\pi i}{n}\right) = \cos\left(\frac{2l\pi}{n}\right) + i \sin\left(\frac{2l\pi}{n}\right), \tag{28}$$

$\sin(2l\pi/n) = 0$. We have $\omega^l = -1$ for $0 < (2l\pi/n) < 2\pi$, but $1 - \omega^k - \omega^{2k} - \omega^{3k} = 0$. We obtain $f(\omega^k) \neq 0$ for any ω^k $(k = 1, 2, \ldots, n-1)$, while $f(1) = \sum_{j=1}^{n} T_j = -(1/2)(1 - T_n - T_{n+2}) \neq 0$.

If $b^2 - 4ac < 0$, ω^l is an imaginary number.

If

$$\cos\left(\frac{2l\pi}{n}\right) = \frac{T_n + T_{n-1}}{-2T_n}$$
$$\sin\left(\frac{2l\pi}{n}\right) = \frac{\sqrt{-4T_n(T_1 - T_{n+1}) - (-T_n - T_{n-1})^2}}{-2T_n} \tag{29}$$

or

$$\cos\left(\frac{2l\pi}{n}\right) = \frac{T_n + T_{n-1}}{-2T_n},$$
$$\sin\left(\frac{2l\pi}{n}\right) = -\frac{\sqrt{-4T_n(T_1 - T_{n+1}) - (-T_n - T_{n-1})^2}}{-2T_n}, \tag{30}$$

we obtain $n = 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$, such that $f(\omega^l) = 0$. If $1 - \omega^k - \omega^{2k} - \omega^{3k} = 0$, we have $\omega^k = -1$ and if n is an even number, then $f(\omega^k) = \sum_{j=1}^{n} T_j(\omega^k)^{j-1} = T_1 - T_2 + \cdots - T_n < 0$. By Lemma 1 in [15], the proof is completed. □

Lemma 5. *Let $\Phi = \left(\begin{smallmatrix} a & V \\ U & A \end{smallmatrix}\right)$ be an $n \times n$ matrix; then*

$$\Phi^{-1} = \begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} VA^{-1} \\ -\frac{1}{\ell} A^{-1}U & A^{-1} + \frac{1}{\ell} A^{-1}UVA^{-1} \end{pmatrix}, \tag{31}$$

where $\ell = a - VA^{-1}U$, V is a row vector, and U is a column vector.

Proof. Consider

$$\begin{pmatrix} a & V \\ U & A \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} VA^{-1} \\ -\frac{1}{\ell} A^{-1}U & A^{-1} + \frac{1}{\ell} A^{-1}UVA^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} = I_n,$$
$$\begin{pmatrix} \frac{1}{\ell} & -\frac{1}{\ell} VA^{-1} \\ -\frac{1}{\ell} A^{-1}U & A^{-1} + \frac{1}{\ell} A^{-1}UVA^{-1} \end{pmatrix} \cdot \begin{pmatrix} a & V \\ U & A \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & I_{n-1} \end{pmatrix} = I_n. \tag{32}$$

□

Lemma 6. *Let the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-3}$ be of the form*

$$g_{i,j} = \begin{cases} T_1 - T_{n+1}, & i = j, \\ -T_n - T_{n-1}, & i = j+1, \\ -T_n, & i = j+2, \\ 0, & \text{otherwise}; \end{cases} \tag{33}$$

then the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-3}$ of the matrix $\mathcal{G}$ is equal to

$$g'_{i,j} = \begin{cases} \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha}\right), & i \geq j, \\ 0, & i < j, \end{cases} \tag{34}$$

where

$$\alpha = -\frac{(-T_n - T_{n-1}) + \sqrt{(-T_n - T_{n-1})^2 + 4T_n(T_1 - T_{n+1})}}{2(T_1 - T_{n+1})},$$
$$\beta = -\frac{(-T_n - T_{n-1}) - \sqrt{(-T_n - T_{n-1})^2 + 4T_n(T_1 - T_{n+1})}}{2(T_1 - T_{n+1})}. \tag{35}$$

Proof. Let $c_{i,j} = \sum_{k=1}^{n-3} g_{i,k} g'_{k,j}$. Obviously, $c_{i,j} = 0$ for $i < j$. In the case where $i = j$, we obtain

$$c_{i,i} = g_{i,i} g'_{i,i} = (T_1 - T_{n+1}) \cdot \frac{1}{T_1 - T_{n+1}} = 1. \tag{36}$$

For $i \geq j+1$, we obtain

$$\begin{aligned} c_{i,j} &= \sum_{k=1}^{n-3} g_{i,k} g'_{k,j} \\ &= g_{i,i-2} g'_{i-2,j} + g_{i,i-1} g'_{i-1,j} + g_{i,i} g'_{i,j} \\ &= -T_n \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j-1} - \alpha^{i-j-1}}{\beta - \alpha} \right) \\ &\quad + \left(-T_n - T_{n-1}\right) \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j} - \alpha^{i-j}}{\beta - \alpha} \right) \\ &\quad + \left(T_1 - T_{n+1}\right) \cdot \frac{1}{T_1 - T_{n+1}} \left(\frac{\beta^{i-j+1} - \alpha^{i-j+1}}{\beta - \alpha} \right) \\ &= 0. \end{aligned} \tag{37}$$

Hence, we verify $\mathcal{G}\mathcal{G}^{-1} = I_{n-3}$, where I_{n-3} is $(n-3) \times (n-3)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-3}$. Thus, the proof is completed. □

Theorem 7. *Let $D_n = Circ(T_1, T_2, \ldots, T_n)$ be an invertible Tribonacci circulant matrix; then one has*

$$\begin{aligned} D_n^{-1} = Circ\Bigg(& x'_2 + \left(-1 - \frac{p_3}{p_2}\right) x'_3 - x'_4 - x'_5, \\ & - x'_2 + \left(-1 + \frac{p_3}{p_2}\right) x'_3 - x'_4, x'_n, x'_{n-1} - x'_n, \\ & x'_{n-2} - x'_{n-1} - x'_n, \ldots, x'_3 - x'_4 - x'_5 - x'_6 \Bigg), \end{aligned} \tag{38}$$

where

$$\begin{aligned} p_2 &= T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right), \\ p_3 &= -T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}, \\ x'_1 &= 0, \\ x'_2 &= \frac{1}{p_2}, \\ x'_3 &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right) \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ x'_4 &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3} \\ &\quad - \frac{\left(T_1 - T_{n+1}\right) \sum_{i=1}^{n-3} g_{i1}\rho_{i+3} \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ &\vdots \\ x'_k &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1} \\ &\quad - \frac{\left(T_1 - T_{n+1}\right) \sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1} \sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ &\qquad (k \geq 4). \end{aligned} \tag{39}$$

Proof. Let

$$\Gamma_2 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & \\ 0 & -\frac{p_3}{p_2} & 1 & 0 & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}; \tag{40}$$

thus

$$\Gamma_2 \Gamma_1 D_n \Pi_1 = \begin{pmatrix} T_1 & p_1 & T_{n-1} & T_{n-2} & \cdots & T_3 & T_2 \\ 0 & p_2 & \varphi_3 & \varphi_4 & \cdots & \varphi_{n-1} & \varphi_n \\ 0 & 0 & \rho_3 & \rho_4 & \cdots & \rho_{n-1} & \rho_n \\ 0 & 0 & b & a & & & \\ 0 & 0 & c & b & a & & \\ & & & \ddots & \ddots & \ddots & \\ & & & & c & b & a \end{pmatrix}, \tag{41}$$

where

$$\begin{aligned} \rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2}\left(T_n - T_{n-1}\right), \\ \rho_i &= T_{n-i+1} - \frac{p_3}{p_2}\left(T_{n-i+3} - T_{n-i+2}\right) \\ &\qquad (i = 4, 5, \cdots, n), \\ \varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n). \end{aligned} \tag{42}$$

According to Lemma 5, let

$$F = \begin{pmatrix} \rho_3 & V \\ U & \mathcal{G} \end{pmatrix} \tag{43}$$

be an $(n-2) \times (n-2)$ matrix and we obtain

$$F^{-1} = \begin{pmatrix} \frac{1}{\ell} & -\frac{V\mathcal{G}^{-1}}{\ell} \\ -\frac{\mathcal{G}^{-1}U}{\ell} & \mathcal{G}^{-1} + \frac{UV\mathcal{G}^{-1}}{\ell} \end{pmatrix}, \tag{44}$$

where

$$\begin{aligned}
\rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2}\left(T_n - T_{n-1}\right),\\
U &= \left(-T_n - T_{n-1}, -T_n, 0, \ldots, 0\right)^T,\\
V &= \left(\rho_4, \rho_5, \ldots, \rho_n\right),\\
\ell &= \rho_3 - b\sum_{i=1}^{n-3} g_{i1}\rho_{i+3} - c\sum_{i=1}^{n-4}\rho_{i+4}.
\end{aligned} \tag{45}$$

Let

$$\Pi_2 = \begin{pmatrix} 1 & -p_1 & -T_{n-1} + \dfrac{p_1\varphi_3}{p_2} & \cdots & -T_2 + \dfrac{p_1\varphi_n}{p_2} \\ 0 & 1 & \dfrac{-\varphi_3}{p_2} & \cdots & \dfrac{-\varphi_n}{p_2} \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{46}$$

where

$$\begin{aligned}
p_1 &= \sum_{i=1}^{n-1} T_{i+1}\Delta^{n-(i+1)},\\
p_2 &= T_1 - T_n + \sum_{i=1}^{n-2}\Delta^i\left(T_{n-i+1} - T_{n-i}\right),\\
\varphi_i &= T_{n+3-i} - T_{n+2-i}, \quad (i = 3, \ldots, n),\\
\Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\\
a &= T_1 - T_{n+1}, \quad b = -T_n - T_{n-1}, \quad c = -T_n.
\end{aligned} \tag{47}$$

We have

$$\Gamma_2\Gamma_1 D_n \Pi_1 \Pi_2 = \mathscr{D}_1 \oplus F, \tag{48}$$

where $\mathscr{D}_1 = \operatorname{diag}(T_1, p_2)$ is a diagonal matrix and $\mathscr{D}_1 \oplus F$ is the direct sum of $\mathscr{D}_1$ and F. If we denote $\Gamma = \Gamma_2\Gamma_1$ and $\Pi = \Pi_1\Pi_2$, we obtain

$$D_n^{-1} = \Pi\left(\mathscr{D}_1^{-1} \oplus F^{-1}\right)\Gamma. \tag{49}$$

Since the last row elements of the matrix Π are $0, 1, (T_{n-1} - T_n)/p_2, (T_{n-2} - T_{n-1})/p_2, \ldots, (T_3 - T_4)/p_2, (T_2 - T_3)/p_2$, then the elements of last row of $\Pi(\mathscr{D}_1^{-1} \oplus F^{-1})$ are given by the following equations:

$$\begin{aligned}
x_1' &= 0,\\
x_2' &= \frac{1}{p_2},\\
x_3' &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right)\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
x_4' &= -\frac{\gamma_3\sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3}\\
&\quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-3} g_{i1}\rho_{i+3}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\vdots\\
x_k' &= -\frac{\gamma_3\sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1}\\
&\quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\qquad (k \geq 4),
\end{aligned} \tag{50}$$

where

$$\begin{aligned}
\gamma_i &= \frac{T_{n-i+2} - T_{n-i+3}}{p_2}, \quad (i = 3, 4 \ldots, n),\\
\rho_3 &= T_1 - T_n - T_{n-1} - \frac{p_3}{p_2}\left(T_n - T_{n-1}\right),\\
\rho_i &= T_{n-i+1} - \frac{p_3}{p_2}\left(T_{n-i+3} - T_{n-i+2}\right)\\
&\qquad (i = 4, 5, \ldots, n).
\end{aligned} \tag{51}$$

By Lemma 6, if $D_n^{-1} = Circ(x_1, x_2, \ldots, x_n)$, then its last row elements are given by the following equations:

$$\begin{aligned}
x_2 &= -x_2' + \left(-1 + \frac{p_3}{p_2}\right)x_3' - x_4',\\
x_3 &= x_n',\\
x_4 &= x_{n-1}' - x_n',\\
x_5 &= x_{n-2}' - x_{n-1}' - x_n',\\
&\vdots\\
x_k &= x_{n-k+3}' - x_{n-k+4}' - x_{n-k+5}' - x_{n-k+6}'\\
&\qquad (5 < k \leq n),\\
x_1 &= x_2' + \left(-1 - \frac{p_3}{p_2}\right)x_3' - x_4' - x_5'.
\end{aligned} \tag{52}$$

Hence, the proof is compeleted. □

3. Determinant and Inverse of Tibonacci Left Circulant Matrix

In this section, let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tibonacci left circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix D'_n. Afterwards, we discuss the invertibility of the matrix D'_n. The inverse of the matrix D'_n is also presented. According to Lemma 2 in [15] and Theorems 3, 4, and 7, we can obtain the following theorems.

Theorem 8. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix; then one has*

$$\begin{aligned}\det D'_n = {} & (-1)^{(n-1)(n-2)/2} \\ & \times \left[\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right]\delta_1 \right. \\ & - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) \right. \\ & \left.\left. + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right]\delta_2\right],\end{aligned} \tag{53}$$

where T_n is the nth Tribonacci number.

Theorem 9. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix. If $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then D'_n is an invertible matrix.*

Theorem 10. *Let $D'_n = LCirc(T_1, T_2, \ldots, T_n)$ be a Tribonacci left circulant matrix. If D_n is an invertible matrix, then we have*

$$\begin{aligned}{D'_n}^{-1} = Circ\Bigg(& x'_2 + \left(-1 - \frac{p_3}{p_2}\right)x'_3 - x'_4 - x'_5, \\ & x'_3 - x'_4 - x'_5 - x'_6, \ldots, x'_{n-2} - x'_{n-1} - x'_n, \\ & x'_{n-1} - x'_n, x'_n, -x'_2 + \left(-1 + \frac{p_3}{p_2}\right)x'_3 - x'_4\Bigg),\end{aligned} \tag{54}$$

where

$$\begin{aligned}x'_1 &= 0, \\ x'_2 &= \frac{1}{p_2}, \\ x'_3 &= \frac{\gamma_3}{\ell} + \frac{\left(T_1 - T_{n+1}\right)\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ x'_4 &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3} \\ & \quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-3} g_{i1}\rho_{i+3}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ & \vdots \\ x'_k &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1} \\ & \quad - \frac{\left(T_1 - T_{n+1}\right)\sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell}, \\ & \quad (k \geq 4).\end{aligned} \tag{55}$$

4. Determinant and Inverse of Tibonacci g-Circulant Matrix

In this section, let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tibonacci g-circulant matrix. By using the obtained conclusions, we give a determinant formula for the matrix $D_{g,n}$. Afterwards, we discuss the invertibility of the matrix $D_{g,n}$. The inverse of the matrix $D_{g,n}$ is also presented. From Lemmas 3 and 4 in [15] and Theorems 3, 4, and 7, we deduce the following results.

Theorem 11. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$; then one has*

$$\begin{aligned}\det D_{g,n} = {} & \det \mathbb{Q}_g \cdot \left[\left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i \left(T_{n-i+1} - T_{n-i}\right)\right]\delta_1 \right. \\ & - \left[-T_n + \Delta\left(T_1 - T_n - T_{n-1}\right) \right. \\ & \left.\left. + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right]\delta_2\right],\end{aligned} \tag{56}$$

where T_n is the nth Tribonacci number.

Theorem 12. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, \ldots, T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$. If $n \neq 2$ and $n \neq 2k\pi(\arctan(\pm\sqrt{4ac - b^2}/-b))^{-1}$ $(k = 1, 2, \ldots, n-1)$, then $D_{g,n}$ is an invertible matrix.*

Theorem 13. *Let $D_{g,n} = g\text{-}Circ(T_1, T_2, ..., T_n)$ be a Tribonacci g-circulant matrix and $(n, g) = 1$. If D_n is an invertible matrix, then one has*

$$\begin{aligned}D_{g,n}^{-1} = \Bigg[& Circ\Bigg(x'_2 + \left(-1 - \frac{p_3}{p_2}\right)x'_3 - x'_4 - x'_5, \\ & -x'_2 + \left(-1 + \frac{p_3}{p_2}\right)x'_3 - x'_4, x'_n, x'_{n-1} - x'_n, x'_{n-2} \\ & - x'_{n-1} - x'_n, \ldots, x'_3 - x'_4 - x'_5 - x'_6\Bigg)\Bigg]\mathbb{Q}_g^T,\end{aligned} \tag{57}$$

where

$$\begin{aligned}
x_1' &= 0,\\
x_2' &= \frac{1}{p_2},\\
x_3' &= \frac{\gamma_3}{\ell} + \frac{(T_1 - T_{n+1})\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
x_4' &= -\frac{\gamma_3 \sum_{i=2}^{n-3} g_{i1}\rho_{i+3}}{\ell} + \sum_{i=1}^{n-3} g_{i1}\gamma_{i+3}\\
&\quad - \frac{(T_1 - T_{n+1})\sum_{i=1}^{n-3} g_{i1}\rho_{i+3}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\vdots\\
x_k' &= -\frac{\gamma_3 \sum_{i=2}^{n-k+1} g_{i1}\rho_{i+k-1}}{\ell} + \sum_{i=1}^{n-k+1} g_{i1}\gamma_{i+k-1}\\
&\quad - \frac{(T_1 - T_{n+1})\sum_{i=1}^{n-k+1} g_{i1}\rho_{i+k-1}\sum_{i=2}^{n-2} g_{i1}\gamma_{i+2}}{\ell},\\
&\qquad (k \geq 4).
\end{aligned} \tag{58}$$

5. Conclusion

The related problem of Tribonacci circualnt type matrices is studied in this paper. We not only discuss nonsingularity of Tibonacci circulant type matrices, but also give the explicit determinant and inverse of Tribonacci circulant matrix, Tribonacci left circualant matrix, and Tribonacci g-circualant matrix. Furthermore, according to Theorem 11 in [23] and the result in Theorem 3 in the paper, identities can be easily obtained:

$$\begin{aligned}
&\frac{(1 - T_{n+1})^n - (c_1^n + d_1^n) + (-T_n)^n}{\mathbb{L}_{-n} - \mathbb{L}_n}\\
&\quad = \left[T_1 - T_n + \sum_{i=1}^{n-2} \Delta^i (T_{n-i+1} - T_{n-i})\right]\delta_1\\
&\qquad - \left[-T_n + \Delta(T_1 - T_n - T_{n-1}) + \sum_{i=1}^{n-3} \Delta^{i+1} T_{n-i-2}\right]\delta_2,
\end{aligned} \tag{59}$$

where T_n is the nth Tribonacci number, $\mathbb{L}_n$ is the nth generalized Lucas number, and

$$\begin{aligned}
c_1 &= \frac{(T_{n+2} - T_{n+1}) + \mu_1}{2},\\
d_1 &= \frac{(T_{n+2} - T_{n+1}) - \mu_1}{2},\\
\mu_1 &= \sqrt{(T_{n+2} - T_{n+1})^2 - 4T_n(T_{n+1} - 1)},\\
\Delta &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},\\
\delta_1 &= (T_1 - T_n - T_{n-1})(T_1 - T_{n+1})^{n-3}\\
&\quad + \sum_{i=2}^{n-2} (-1)^{1+i} (T_1 - T_{n+1})^{n-i-2} \cdot T_{n-i-1} \det A_{i-1},\\
\delta_2 &= \sum_{i=1}^{n-2} (-1)^{1+i} (T_1 - T_{n+1})^{n-i-2} (T_{n-i+1} - T_{n-i}) \det A_{i-1},\\
\det A_{i-1} &= \begin{cases} \dfrac{\left(\left(b + \sqrt{b^2 - 4ac}\right)/2\right)^i - \left(\left(b - \sqrt{b^2 - 4ac}\right)/2\right)^i}{\sqrt{b^2 - 4ac}}, & b^2 \neq 4ac,\\ i\left(\dfrac{b}{2}\right)^{i-1}, \quad b^2 = 4ac. \end{cases}\\
a &= T_1 - T_{n+1}, \qquad b = -T_n - T_{n-1}, \qquad c = -T_n.
\end{aligned} \tag{60}$$

In addition, we will develop solving the problem in [24–26] by circulant matrices technology.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Norms and Spread of the Fibonacci and Lucas RSFMLR Circulant Matrices

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Circulant type matrices have played an important role in networks engineering. In this paper, firstly, some bounds for the norms and spread of Fibonacci row skew first-minus-last right (RSFMLR) circulant matrices and Lucas row skew first-minus-last right (RSFMLR) circulant matrices are given. Furthermore, the spectral norm of Hadamard product of a Fibonacci RSFMLR circulant matrix and a Lucas RSFMLR circulant matrix is obtained. Finally, the Frobenius norm of Kronecker product of a Fibonacci RSFMLR circulant matrix and a Lucas RSFMLR circulant matrix is presented.

1. Introduction

Circulant type matrices have been put on the firm basis with the work in [1–4] and so on. Circulant type matrices have significant applications in networks systems. In [5], some preliminary results on the dynamical behaviours of some specific nonmonotone Boolean automata networks which are called xor circulant networks were showed. In [6], the authors proposed a special class of the feedback delay network using circulant matrices. In [7], the impact of interior symmetries on the multiplicity of the eigenvalues of the Jacobian matrix at a fully synchronous equilibrium for the coupled cell systems associated with homogeneous networks was analyzed by Aguiar and Ruan, which was based on the circulant adjacency matrices of the networks induced by these interior symmetries. Exploiting the circulant structure of the channel matrices, the realistic near fast fading scenarios with circulant frequency selective channels were analysed by Eghbali et al. in [8]. The existence of doubly periodic travelling waves in cellular networks involving the discontinuous Heaviside step function by circulant matrix was studied by Wang and Cheng in [9].

The Fibonacci and Lucas sequences F_n and L_n are defined by the recurrence relations [10, 11]:

$$F_0 = 0, \quad F_1 = 1, \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2, \tag{1}$$

$$L_0 = 2, \quad L_1 = 1, \quad L_n = L_{n-1} + L_{n-2} \quad \text{for } n \geq 2. \tag{2}$$

If we start from $n = 0$, then Fibonacci and Lucas sequences are given by

$$\begin{array}{cccccccccc} n & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \cdots \\ F_n & 0 & 1 & 1 & 2 & 3 & 5 & 8 & 13 & \cdots \\ L_n & 2 & 1 & 3 & 4 & 7 & 11 & 18 & 29 & \cdots \end{array} \tag{3}$$

In [10], their Binet forms are given by

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}\right], \\ L_n &= \left(\frac{1+\sqrt{5}}{2}\right)^n + \cos(\pi n)\left(\frac{1+\sqrt{5}}{2}\right)^{-n}. \end{aligned} \tag{4}$$

The following sum formulations for the Fibonacci and Lucas numbers are well known [11]:

$$\sum_{s=1}^{n-1} F_s^2 = F_n F_{n-1}, \tag{5}$$

$$\sum_{s=1}^{n-1} L_s^2 = L_n L_{n-1} - 2, \tag{6}$$

$$\begin{aligned} \sum_{s=1}^{n} F_s F_{s-1} &= \begin{cases} F_n^2, & n \text{ even}, \\ F_n^2 - 1, & n \text{ odd}, \end{cases} \\ \sum_{s=1}^{n} L_s L_{s-1} &= \begin{cases} L_n^2 - 4, & n \text{ even}, \\ L_n^2 + 1, & n \text{ odd}. \end{cases} \end{aligned} \tag{7}$$

Lately, some authors studied the problems of the norms of some special matrices [11–21]. The author [11] found upper and lower bounds for the spectral norms of Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{i-j} \equiv L_{i-j}$. In [13], the authors obtain upper and lower bounds for the spectral norms of matrices $A = C_r(F_{k,0}, F_{k,1}, \ldots, F_{k,n-1})$ and $B = C_r(L_{k,0}, L_{k,1}, \ldots, L_{k,n-1})$, where $\{F_{k,n}\}_{n\in N}$ and $\{L_{k,n}\}_{n\in N}$ are k-Fibonacci and k-Lucas sequences, respectively, and they also give the bounds for the spectral norms of Kronecker and Hadamard products of these special matrices, respectively [14]. Solak and Bozkurt [16] have found out upper and lower bounds for the spectral norms of Cauchy-Toeplitz and Cauchy-Hankel matrices. Solak [18–20] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ circulant matrices, where $a_{ij} \equiv F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} \equiv L_{(\mathrm{mod}(j-i,n))}$; then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms.

In this paper, we define two kinds of special matrices as follows.

A Fibonacci row skew first-minus-last right (RSFMLR) circulant matrix is defined as a square matrix of the form

$$\begin{pmatrix} F_0 & F_1 & \cdots & F_{n-1} \\ -F_{n-1} & F_0 - F_{n-1} & F_1 & F_{n-2} \\ \vdots & -F_{n-1} - F_{n-2} & \ddots & \vdots \\ -F_2 & \ddots & \ddots & F_1 \\ -F_1 & -F_2 - F_1 & \cdots & F_0 - F_{n-1} \end{pmatrix}. \tag{8}$$

A Lucas row skew first-minus-last right (RSFMLR) circulant matrix is defined as a square matrix of the form

$$\begin{pmatrix} L_0 & L_1 & \cdots & L_{n-1} \\ -L_{n-1} & L_0 - L_{n-1} & L_1 & L_{n-2} \\ \vdots & -L_{n-1} - L_{n-2} & \ddots & \vdots \\ -L_2 & \ddots & \ddots & L_1 \\ -L_1 & -L_2 - L_1 & \cdots & L_0 - L_{n-1} \end{pmatrix}. \tag{9}$$

Obviously, the RSFMLR circulant matrix is determined by its first row, and RSFMLR circulant matrix is a x^n+x+1 circulant matrix [22].

We define $\Theta_{(-1,-1)}$ as the basic RSFMLR circulant matrix; that is,

$$\begin{aligned} \Theta_{(-1,-1)} &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -1 & -1 & 0 & \cdots & 0 \end{pmatrix}_{n\times n} \\ &= \mathrm{RSFMLRcircfr}\,(0, 1, 0, \ldots, 0)\,. \end{aligned} \tag{10}$$

It is easily verified that $g(x) = x^n + x + 1$ has no repeated roots in its splitting field and $g(x) = x^n + x + 1$ is both the minimal polynomial and the characteristic polynomial of the matrix $\Theta_{(-1,-1)}$. In addition, $\Theta_{(-1,-1)}$ is nonderogatory and satisfies $\Theta^j_{(-1,-1)} = \mathrm{RFMLRcircfr}(\underbrace{0,\ldots,0}_{j}, 1, \underbrace{0,\ldots,0}_{n-j-1})$ and $\Theta^n_{(-1,-1)} = -I_n + \Theta_{(-1,-1)}$.

As we all know, letting $A = \mathrm{RSFMLRcircfr}(a_0, a_1, \ldots, a_{n-1})$ be a RSFMLR circulant matrix with the first row $(a_0, a_1, \ldots, a_{n-1})$, it is clear that

$$\begin{aligned} A &= \mathrm{RSFMLRcircfr}\,(a_0, a_1, \ldots, a_{n-1}) \\ &= \sum_{i=0}^{n-1} a_i \Theta^i_{(-1,-1)}. \end{aligned} \tag{11}$$

Thus, A is a RSFMLR circulant matrix if and only if $A = f(\Theta_{(-1,-1)})$ for some polynomial $f(x)$. The polynomial $f(x) = \sum_{i=0}^{n-1} a_i x^i$ will be called the representer of the RSFMLR circulant matrix A. By (11), it is clear that A is a RSFMLR circulant matrix if and only if A commutes with $\Theta_{(-1,-1)}$; that is, $A\Theta_{(-1,-1)} = \Theta_{(-1,-1)}A$.

In addition to the algebraic properties that can be easily derived from the representation (11), we mention that RSFMLR circulant matrices have very nice structure. The product of two RSFMLR circulant matrices is a RSFMLR circulant matrix and A^{-1} is a RSFMLR circulant matrix too.

Let $A = (a_{ij})$ be an $n \times n$ matrix. The Euclidean (or Frobenius) norm, the spectral norm, the maximum column sum matrix norm, and the maximum row sum matrix norm of the matrix A are, respectively [11],

$$\|A\|_F = \left(\sum_{i,j=1}^{n} \left|a_{ij}\right|^2\right)^{1/2}, \tag{12}$$

$$\|A\|_2 = \left(\max_{1\le i\le n} \lambda_i(A^* A)\right)^{1/2}, \tag{13}$$

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{n} \left|a_{ij}\right|, \tag{14}$$

$$\|A\|_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} \left|a_{ij}\right|, \tag{15}$$

where A^* denotes the conjugate transpose of A. The following inequality holds:

$$\frac{1}{\sqrt{n}} \|A\|_F \le \|A\|_2 \le \|A\|_F\,. \tag{16}$$

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be $n \times n$ matrices. The Hadamard product of A and B is defined by $A \circ B = [a_{ij}b_{ij}]$. If $\|\cdot\|$ is any norm on $n \times m$ matrices, then [18, 23]

$$\|A \circ B\| \le \|A\| \cdot \|B\|\,. \tag{17}$$

Kronecker product of A and B is given to be [18]

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1m}B \\ \vdots & & \vdots \\ a_{n1}B & \cdots & a_{nm}B \end{bmatrix}. \tag{18}$$

Then [18]

$$\|A \otimes B\|_F = \|A\|_F \|B\|_F\,. \tag{19}$$

Let $A = (a_{ij})$ be an $n \times n$ matrix with eigenvalues λ_i, $i = 1, 2, \ldots, n$. The spread of A is defined as [24, 25]

$$s(A) = \max_{i,j} \left|\lambda_i - \lambda_j\right|. \tag{20}$$

An upper bound for the spread due to Mirsky [24] states that

$$s(A) \le \sqrt{2\|A\|_F^2 - \frac{2}{n}|\operatorname{tr} A|^2}, \tag{21}$$

where $\|A\|_F$ denotes the Frobenius norm of A and $\operatorname{tr} A$ is the trace of A.

2. Norms and Spread of Fibonacci RSFMLR Circulant Matrices

Theorem 1. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ denote Fibonacci numbers given by (1); then two kinds of norms of A are given by*

$$\|A\|_1 = \|A\|_\infty = 2\left(F_{n+1} - 1\right). \tag{22}$$

Proof. The matrix A is of the form (8), by (14), (15); then we have

$$\begin{aligned}\|A\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}\left|a_{ij}\right| \\ &= \sum_{i=1}^{n-1} F_i + F_{n+1} - F_2, \\ \|A\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}\left|a_{ij}\right| \\ &= \sum_{i=1}^{n-1} F_i + F_{n+1} - F_2.\end{aligned} \tag{23}$$

Since the Fibonacci sequences F_n are defined by the recurrence relations (1), then we obtain

$$F_{n-1} = F_n - F_{n-2} \quad n \ge 2. \tag{24}$$

To sum up, we can get

$$\sum_{s=1}^{n-1} F_s = F_{n+1} - F_2. \tag{25}$$

Then

$$\|A\|_1 = \|A\|_\infty = 2\left(F_{n+1} - 1\right), \tag{26}$$

which completes the proof. □

Theorem 2. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ denote Fibonacci numbers given by (1); then*

$$\begin{gathered}\sqrt{\frac{\Gamma}{n}} \le \|A\|_2, \\ \|A\|_2 \le 2\left(F_{n+1} - 1\right),\end{gathered} \tag{27}$$

where

$$\begin{aligned}\Gamma = {} & F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ & + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{28}$$

Proof. Since $F_{n+2} = F_{n+1} + F_n$ and $F(0) = 0$ given by (1), the matrix A is of the form

$$\begin{pmatrix} F_0 & F_1 & F_{n-2} & F_{n-1} \\ -F_{n-1} & -F_{n-1} & \cdots & F_{n-2} \\ \vdots & -F_{n-1} - F_{n-2} & \ddots & \vdots \\ -F_2 & \vdots & \ddots & F_1 \\ -F_1 & -F_3 & -F_{n-1} - F_{n-2} & -F_{n-1} \end{pmatrix}. \tag{29}$$

We know that $(1/\sqrt{n})\|A\|_F \le \|A\|_2 \le \|A\|_F$ from equivalent norms. By (5), we can get

$$\begin{aligned}\|A\|_F^2 &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{i=1}^{n-1} iF_i^2 + 2\sum_{i=1}^{n-2} iF_iF_{i+1} \\ &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{k=1}^{n-1}\sum_{i=n-k}^{n-1}F_i^2 + 2\sum_{k=1}^{n-2}\sum_{i=n-k-1}^{n-2}F_iF_{i+1} \\ &= n\sum_{i=0}^{n-1}F_i^2 + \sum_{k=1}^{n-1}\left(\sum_{i=0}^{n-1}F_i^2 - \sum_{i=0}^{n-k-1}F_i^2\right) \\ &\quad + 2\sum_{k=1}^{n-2}\left(\sum_{i=0}^{n-2}F_iF_{i+1} - \sum_{i=0}^{n-k-2}F_iF_{i+1}\right) \\ &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{30}$$

Then

$$\frac{1}{\sqrt{n}}\|A\|_F = \sqrt{\frac{\Gamma}{n}}, \tag{31}$$

where

$$\begin{aligned}\Gamma = {} & F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ & + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2.\end{aligned} \tag{32}$$

We have

$$\sqrt{\frac{\Gamma}{n}} \le \|A\|_2. \tag{33}$$

On the other hand, suppose that

$$M_1 = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

$$M_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{34}$$

$$M_3 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Then

$$A = \sum_{i=0}^{n-1} F_i M_1^i - \sum_{i=1}^{n-2} F_{n-i-1} M_2^i + F_{n-1} M_3. \tag{35}$$

We can get

$$\begin{aligned} \|A\|_2 &= \left\| \sum_{i=0}^{n-1} F_i M_2^i + \sum_{i=1}^{n-2} F_{n-i-1} M_2^i + F_{n-1} M_3 \right\|_2 \\ &\le \sum_{i=0}^{n-1} F_i \|M_1\|_2^i - \sum_{i=1}^{n-2} F_{n-i-1} \|M_2\|_2^i + F_{n-1} \|M_3\|_2 . \end{aligned} \tag{36}$$

Furthermore,

$$M_1^H M_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$M_2^H M_2 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{37}$$

$$M_3^H M_3 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

We obtain

$$\|M_1\|_2 = \|M_2\|_2 = \|M_3\|_2 = 1. \tag{38}$$

The other result is obtained as follows:

$$\begin{aligned} \|A\|_2 &\le \sum_{i=1}^{n-1} F_i \|M_1\|_2^i + \sum_{i=1}^{n-2} F_{n-i-1} \|M_2\|_2^i + F_{n-1} \|M_3\|_2 \\ &= 2 \sum_{i=0}^{n-1} F_i = 2 \left(F_{n+1} - 1 \right), \end{aligned} \tag{39}$$

which completes the proof. □

Theorem 3. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix, where $\{F_i\}_{0 \le i \le n-1}$ denote Fibonacci numbers given by (1); then the bound for the spread of A is*

$$s(A) \le \sqrt{\tau_1(n) - \frac{2}{n} \tau_2(n)}, \tag{40}$$

where

$$\begin{aligned} \tau_1(n) &= 2 \left(F_1^2 + (2n-1) F_{n-1}^2 + (n-1) F_{n-3} F_{n-2} \right. \\ &\quad \left. + (2n-4) F_{n-1} F_{n-2} + (2n-3) F_{n-2}^2 \right), \\ \tau_2(n) &= \left[(n-1) F_{n-1} \right]^2 . \end{aligned} \tag{41}$$

Proof. The trace of A is $\operatorname{tr} A = nF_0 + (n-1)F_{n-1}$. By Theorem 2 and inequation (21), we have

$$s(A) \le \sqrt{2 \|A\|_F^2 - \frac{2}{n} \operatorname{tr} A^2}, \tag{42}$$

where

$$\begin{aligned} \|A\|_F^2 &= F_1^2 + (3n-4) F_{n-1}^2 + (n-1) F_{n-3} F_{n-2} \\ &\quad + (2n-4) F_{n-1} F_{n-2} + (2n-3) F_{n-2}^2, \\ \operatorname{tr} A &= nF_0 - (n-1) F_{n-1}. \end{aligned} \tag{43}$$

We can get

$$s(A) \le \sqrt{\tau_1(n) - \frac{2}{n} \tau_2(n)}, \tag{44}$$

where

$$\begin{aligned} \tau_1(n) &= 2 \left(F_1^2 + (2n-1) F_{n-1}^2 + (n-1) F_{n-3} F_{n-2} \right. \\ &\quad \left. + (2n-4) F_{n-1} F_{n-2} + (2n-3) F_{n-2}^2 \right), \\ \tau_2(n) &= \left[(n-1) F_{n-1} \right]^2 , \end{aligned} \tag{45}$$

which completes the proof. □

3. Norms and Spread of Lucas RSFMLR Circulant Matrices

Theorem 4. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0 \le i \le n-1}$ denote Lucas numbers given by (2); then two kinds of norms of B are given by*

$$\|B\|_1 = \|B\|_\infty = 2 \left(L_{n+1} - 3 \right) + 2. \tag{46}$$

Proof. The matrix B is of the form (9), by (14), (15); then we get

$$\begin{aligned}\|B\|_1 &= \max_{1\le j\le n}\sum_{i=1}^{n}\left|b_{ij}\right| = \sum_{i=1}^{n-1}L_i + L_0 + L_{n+1} - L_2,\\ \|B\|_\infty &= \max_{1\le i\le n}\sum_{j=1}^{n}\left|b_{ij}\right| = \sum_{i=1}^{n-1}L_i + L_0 + L_{n+1} - L_2.\end{aligned} \tag{47}$$

Since the Lucas sequences L_n are defined by the recurrence relations (2), then we obtain

$$L_{n-1} = L_n - L_{n-2} \quad n \ge 2. \tag{48}$$

To sum up, we can get

$$\sum_{s=1}^{n-1}L_s = L_{n+1} - L_2. \tag{49}$$

Then

$$\|B\|_1 = \|B\|_\infty = 2\left(L_{n+1} - 3\right) + 2, \tag{50}$$

which completes the proof. □

Theorem 5. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0\le i\le n-1}$ denote Lucas numbers given by (2); then*

$$\begin{aligned}\sqrt{\frac{\Pi}{n}} &\le \|B\|_2,\\ \|B\|_2 &\le 2\left(L_{n+1} - 2\right),\end{aligned} \tag{51}$$

where

$$\begin{aligned}\Pi = {} & L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ & + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ & + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{52}$$

Proof. Since $L_{n+2} = L_{n+1} + L_n$ and $L(0) = 2$, the matrix B is of the form

$$\begin{pmatrix} L_0 & L_1 & L_{n-2} & L_{n-1}\\ -L_{n-1} & L_0 - L_{n-1} & \cdots & L_{n-2}\\ \vdots & -L_{n-1} - L_{n-2} & \ddots & \vdots\\ -L_2 & \vdots & \ddots & L_1\\ -L_1 & -L_3 & -L_{n-1} - L_{n-2} & L_0 - L_{n-1}\end{pmatrix}. \tag{53}$$

We know that $(1/\sqrt{n})\|B\|_F \le \|B\|_2 \le \|B\|_F$ from equivalent norms. By (6), we can get

$$\begin{aligned}\|B\|_F^2 &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{i=1}^{n-1}iL_i^2 + 2\sum_{i=1}^{n-2}iL_iL_{i+1} - 4(n-1)L_{n-1}\\ &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{k=1}^{n-1}\sum_{i=n-k}^{n-1}L_i^2 + 2\sum_{k=1}^{n-2}\sum_{i=n-k-1}^{n-2}L_iL_{i+1}\\ &\quad - 4(n-1)L_{n-1}\\ &= n\sum_{i=0}^{n-1}L_i^2 + \sum_{k=1}^{n-1}\left(\sum_{i=0}^{n-1}L_i^2 - \sum_{i=0}^{n-k-1}L_i^2\right)\\ &\quad + 2\sum_{k=1}^{n-2}\left(\sum_{i=0}^{n-2}L_iL_{i+1} - \sum_{i=0}^{n-k-2}L_iL_{i+1}\right)\\ &\quad - 4(n-1)L_{n-1}\\ &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{54}$$

Then

$$\frac{1}{\sqrt{n}}\|B\|_F = \sqrt{\frac{\Pi}{n}}. \tag{55}$$

We have

$$\sqrt{\frac{\Pi}{n}} \le \|B\|_2, \tag{56}$$

where

$$\begin{aligned}\Pi = {} & L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2}\\ & + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1}\\ & + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{57}$$

On the other hand, supposing that

$$\begin{aligned}M_1 &= \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0\\ 0 & 0 & 1 & \ddots & 0 & 0\\ 0 & 0 & 0 & \ddots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 1\\ -1 & 0 & 0 & \cdots & 0 & 0\end{pmatrix},\\ M_2 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & 1 & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \ddots & 0 & 0\\ 0 & 0 & 0 & \cdots & 1 & 0\end{pmatrix},\\ M_3 &= \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0\\ 0 & -1 & 0 & \cdots & 0 & 0\\ 0 & 0 & -1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 0\\ 0 & 0 & 0 & \cdots & 0 & -1\end{pmatrix},\end{aligned} \tag{58}$$

then

$$B = \sum_{i=0}^{n-1}L_iM_1^i - \sum_{i=1}^{n-2}L_{n-i-1}M_2^i + L_{n-1}M_3. \tag{59}$$

We obtain

$$\begin{aligned}\|B\|_2 &= \left\|\sum_{i=0}^{n-1} L_i M_2^i + \sum_{i=1}^{n-2} L_{n-i-1} M_2^i + L_{n-1} M_3\right\|_2 \\ &\le \sum_{i=0}^{n-1} L_i \|M_1\|_2^i - \sum_{i=1}^{n-2} L_{n-i-1} \|M_2\|_2^i + L_{n-1} \|M_3\|_2 .\end{aligned} \tag{60}$$

We have

$$\begin{aligned} M_1^H M_1 &= \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \\ M_2^H M_2 &= \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \\ M_3^H M_3 &= \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \end{aligned} \tag{61}$$

We get

$$\|M_1\|_2 = \|M_2\|_2 = \|M_3\|_2 = 1. \tag{62}$$

The other result is obtained as follows:

$$\begin{aligned}\|B\|_2 &\le \sum_{i=1}^{n-1} L_i \|M_1\|_2^i + \sum_{i=1}^{n-2} L_{n-i-1} \|M_2\|_2^i + L_{n-1} \|M_3\|_2 \\ &= 2\sum_{i=0}^{n-1} L_i = 2\left(L_{n+1} - 2\right),\end{aligned} \tag{63}$$

which completes the proof. □

Theorem 6. *Let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{L_i\}_{0\le i\le n-1}$ denote Lucas numbers given by (2); then*

$$s(B) \le \sqrt{\kappa_1(n) - \frac{2}{n}\kappa_2(n)}, \tag{64}$$

where

$$\begin{aligned}\kappa_1(n) &= 2\Big(L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2\Big), \\ \kappa_2(n) &= \left[nL_0 - (n-1)L_{n-1}\right]^2 .\end{aligned} \tag{65}$$

Proof. The trace of B is $\operatorname{tr} B = nL_0 + (n-1)L_{n-1}$. By Theorem 5 and by inequation (21), we have

$$s(B) \le \sqrt{2\|B\|_F^2 - \frac{2}{n}\operatorname{tr} B^2}, \tag{66}$$

where

$$\begin{aligned}\|B\|_F^2 &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2, \\ \operatorname{tr} B &= nL_0 - (n-1)L_{n-1}.\end{aligned} \tag{67}$$

We obtain

$$s(B) \le \sqrt{\kappa_1(n) - \frac{2}{n}\kappa_2(n)}, \tag{68}$$

where

$$\begin{aligned}\kappa_1(n) &= 2\Big(L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2\Big), \\ \kappa_2(n) &= \left[nL_0 - (n-1)L_{n-1}\right]^2 ,\end{aligned} \tag{69}$$

which completes the proof. □

Corollary 7. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ and $\{L_i\}_{0\le i\le n-1}$ denote Fibonacci numbers and Lucas numbers, respectively; then the spectral norm of Hadamard product of A and B satisfies the following inequality:*

$$\|A \circ B\|_2 \le 4\left(F_{n+1} - 1\right) \times \left(L_{n+1} - 2\right). \tag{70}$$

Proof. The proof is trivial by Theorems 2 and 5; we obtain

$$\|A\|_2 \le 2\left(F_{n+1} - 1\right), \qquad \|B\|_2 \le 2\left(L_{n+1} - 2\right). \tag{71}$$

By inequation (17), we have

$$\|A \circ B\|_2 \le 4\left(F_{n+1} - 1\right) \times \left(L_{n+1} - 2\right), \tag{72}$$

which completes the proof. □

Corollary 8. *Let $A = RSFMLRcircfr(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = RSFMLRcircfr(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix, where $\{F_i\}_{0\le i\le n-1}$ and $\{L_i\}_{0\le i\le n-1}$ denote Fibonacci numbers and Lucas numbers, respectively; then the Frobenius norm of Kronecker product of A and B is*

$$\|A \otimes B\|_F = \sqrt{\Gamma} \times \sqrt{\Pi}, \tag{73}$$

where

$$\begin{aligned}\Gamma &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \Pi &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{74}$$

Proof. Since the proof is trivial by Theorems 2 and 5, we obtain

$$\begin{aligned}\|A\|_F^2 &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \|B\|_F^2 &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2.\end{aligned} \tag{75}$$

By (19), then

$$\|A \otimes B\|_F = \sqrt{\Gamma} \times \sqrt{\Pi}, \tag{76}$$

where

$$\begin{aligned}\Gamma &= F_1^2 + (3n-4)F_{n-1}^2 + (n-1)F_{n-3}F_{n-2} \\ &\quad + (2n-4)F_{n-1}F_{n-2} + (2n-3)F_{n-2}^2, \\ \Pi &= L_1^2 + (3n-4)L_{n-1}^2 + (n-1)L_{n-3}L_{n-2} \\ &\quad + (2n-3)L_{n-2}^2 - (4n-4)L_{n-1} \\ &\quad + (2n-4)L_{n-1}L_{n-2} + 2n + 2,\end{aligned} \tag{77}$$

which completes the proof. □

4. Conclusion

In this study, we define matrices of the following forms: let $A = \mathrm{RSFMLRcircfr}(F_0, F_1, \ldots, F_{n-1})$ be a Fibonacci RSFMLR circulant matrix and let $B = \mathrm{RSFMLRcircfr}(L_0, L_1, \ldots, L_{n-1})$ be a Lucas RSFMLR circulant matrix. Firstly, we get lower and upper bounds for the spectral norms of these matrices. Upper bounds for the spread of the matrix A and the matrix B are given. Afterwards, we obtain some corollaries related to norms of Hadamard and Kronecker products of these matrices. Based on the existing problems in [26–28], we will explore solving these problems by circulant matrices technology.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

On the Incidence Energy of Some Toroidal Lattices

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The incidence energy IE(G), defined as the sum of the singular values of the incidence matrix of G, is a much studied quantity with well known applications in chemical physics. In this paper, we derived the closed-form formulae expressing the incidence energy of the 3.12.12 lattice, triangular kagomé lattice, and $S(m,n)$ lattice, respectively. Simultaneously, the explicit asymptotic values of the incidence energy in these lattices are obtained by utilizing the applications of analysis method with the help of software calculation.

1. Introduction

A general problem of interest in physics, chemistry, and mathematics is the calculations of the energy of graphs [1–3], which has now become a popular topic of research; however, almost all of literature deal with the energy of the finite graphs. Yan and Zhang [4] first considered the asymptotic energy of the infinite lattice graphs; they obtained the asymptotic formulae for energies of various lattices. Historically in lattice statistics, the hexagonal lattice, 3.12.12 lattice, triangular kagomé lattice, and $3^3 \cdot 4^2$ lattice have attracted the most attention [4–9]. Ising spins and XXZ/Ising spins on the TKL(m,n) have been studied in [10, 11].

Let G be a simple graph with n vertices, let $A(G)$ be the adjacency matrix, and let $D(G)$ be the diagonal matrix of vertex degrees of G, respectively. The Laplacian eigenvalues of G are $L(G) = D(G) - A(G)$ and the signless Laplacian matrix is $Q(G) = D(G) + A(G)$. The characteristic polynomial $P_G(x) = \det(xI_n - A(G))$ (resp., $L_G(x) = \det(xI_n - L(G))$, $Q_G(x) = \det(xI_n - Q(G))$) of $A(G)$ (resp., $L(G), Q(G)$) is called the $A(G)$ (resp., $L(G), Q(G)$) characteristic polynomial or $A(G)$ (resp., $L(G), Q(G)$) polynomial of G and is denoted by $A_G(x)$ (resp., $L_G(x), Q_G(x)$). The spectrum of $A(G)$ (resp., $L(G), Q(G)$) which consists of the $A(G)$ (resp., $L(G), Q(G)$) eigenvalues is also called the $A(G)$ (resp., $L(G), Q(G)$)) spectrum of G, respectively. It is well known that $A(G)$, $L(G)$, and $Q(G)$ are symmetric and positive semidefinite; then we denote the eigenvalues of $A(G)$, $L(G)$, and $Q(G)$ by $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G) = 0$, and $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G) \geq 0$, respectively. Details on its theory can be found in recent papers [12–14] and the references cited therein.

The famous graph energy $E(G)$ for a simple graph G, introduced by Gutman [1], is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$. The quantity can be used to estimate the total π-electron energy in conjugated hydrocarbons. As an analogue of $E(G)$, the incidence energy IE(G), is a novel topological index, inspired by Nikiforov idea [2], Jooyandeh et al. [15] introduced the concept IE(G) of a graph G as $\mathrm{IE}(G) = \sum_{i=1}^{n} \sqrt{q_i}$, which is the sum of the singular values of the incidence matrix $B(G)$. The index has attracted extensive attention due to its wide applications in physics, chemistry, graph theory, and so forth; for more work on IE(G), the readers are referred to papers [15–18].

In [4, 19] the energy $E(G)$ and Kirchhoff index Kf(G) of toroidal lattices were studied. It is an interesting problem to study the incidence energy of some lattices with toroidal boundary condition. Motivated by results above, we consider the problem of computations of the IE(G) of the 3.12.12 lattice, triangular kagomé lattice, and $S(m,n)$ lattice with toroidal condition in this paper.

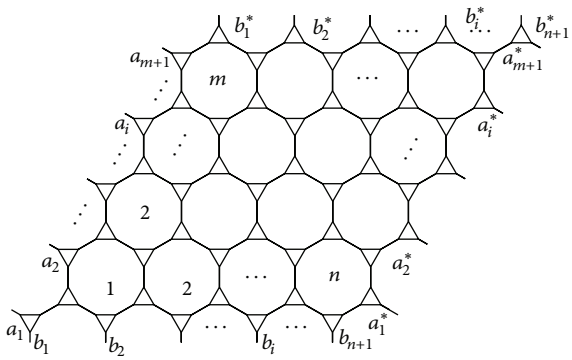

FIGURE 1: The $J^t(m,n)$ lattice with toroidal boundary condition [5].

2. Main Results

2.1. The 3.12.12 Lattice. The 3.12.12 lattice with toroidal boundary condition by physicists [5], denoted by $J^t(m,n)$, is illustrated in Figure 1.

Recently, the adjacency spectrum of 3.12.12 lattice has been proposed in [5] as follows.

Theorem 1 (see [5]). *Let $J^t(m,n)$ be the 3.12.12 lattice with toroidal boundary condition. Then the adjacency spectrum is*

$$\begin{aligned}&\operatorname{Spec}_A\left(J^t(m,n)\right)\\&=\left\{\underbrace{-2,-2,\ldots,-2}_{(m+1)(n+1)},\underbrace{0,0,\ldots,0}_{(m+1)(n+1)}\right\}\\&\quad\cup\left\{\frac{1\pm\sqrt{13\pm4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}{2}\right\},\end{aligned}\tag{1}$$

where $\alpha_i = 2\pi i/(m+1)$, $\beta_j = 2\pi j/(n+1)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$.

The following result is an important relationship between $\operatorname{Spec}_A(G)$ and $\operatorname{Spec}_Q(G)$.

Consider that if G is an r-regular graph of order n, then

$$D(G) = rI_n. \tag{2}$$

Consequently,

$$Q(G) = D(G) + A(G) = D(G) + rI_n. \tag{3}$$

One can conclude that

$$Q_G(x) = P_G(x-r). \tag{4}$$

Define the mapping $\varphi(\lambda_i) = \lambda_i + r$ maps the eigenvalues of $A(G)$ to the eigenvalues of $Q(G)$ and can be considered as an isomorphism of the A-spectrum to the corresponding the Q-spectrum for regular graphs.

Suppose that G is an r-regular graph with n vertices and $\operatorname{Spec}_A(G) = \{\lambda_1,\lambda_2,\ldots,\lambda_n\}$. Then

$$\operatorname{Spec}_Q(G) = \{r+\lambda_1, r+\lambda_2,\ldots,r+\lambda_n\}. \tag{5}$$

Note that $J^t(m,n)$ is the line graph of the subdivision of $H^t(n,m)$ which is a 3-regular graph with $2(m+1)(n+1)$ vertices, and $J^t(m,n)$ has $6(m+1)(n+1)$ vertices. Hence, we get the following theorem.

Theorem 2. *Let $J^t(m,n)$ be the 3.12.12 lattice with toroidal boundary condition and $\alpha_i = 2\pi i/(m+1)$, $\beta_j = 2\pi j/(n+1)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$. Then the signless Laplacian spectrum is*

$$\begin{aligned}&\operatorname{Spec}_Q\left(J^t(m,n)\right)\\&=\left\{\underbrace{1,1,\ldots,1}_{(m+1)(n+1)},\underbrace{3,3,\ldots,3}_{(m+1)(n+1)}\right\}\\&\quad\cup\left\{\frac{7\pm\sqrt{13\pm4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}{2}\right\}.\end{aligned}\tag{6}$$

By the definition of the incidence energy, we can easily get the incidence energy of $J^t(m,n)$.

Theorem 3. *Let $\alpha_i = 2\pi i/(m+1)$, $\beta_j = 2\pi j/(n+1)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$. Then the incidence energy of $J^t(m,n)$ can be expressed as*

$$\begin{aligned}&\operatorname{IE}\left(J(m,n)\right)\\&=\left(\sqrt{3}+1\right)(m+1)(n+1)\\&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}.\end{aligned}\tag{7}$$

From theorem above, we consider that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{\mathrm{IE}\left(J^t(m,n)\right)}{6(m+1)(n+1)}\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{12(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{12(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{12(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{12(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{7+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\frac{\sqrt{3}+1}{6}.
\end{aligned}\tag{8}$$

Consequently, one can easily arrive to the asymptotic value of incidence energy

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{IE\left(J^t(m,n)\right)}{6(m+1)(n+1)}\\
&=\frac{1}{12}\int_0^1\int_0^1\sqrt{7-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy\\
&\quad+\frac{1}{12}\int_0^1\int_0^1\sqrt{7-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy\\
&\quad+\frac{1}{12}\int_0^1\int_0^1\sqrt{7+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy\\
&\quad+\frac{1}{12}\int_0^1\int_0^1\sqrt{7+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy\\
&\quad+\frac{\sqrt{3}+1}{6}\approx 1.3040.
\end{aligned}\tag{9}$$

The numerical integration value in last line is calculated with MATLAB software calculation.

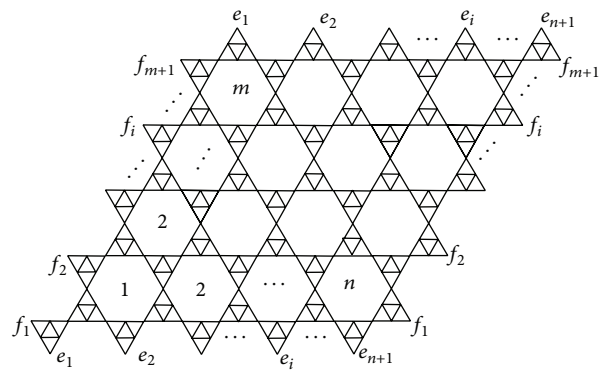

Figure 2: The $\mathrm{TKL}^t(m,n)$ lattice with toroidal boundary condition [5].

Hence $J^t(m,n)$ has the asymptotic incidence energy $\mathrm{IE}(J^t(m,n))\approx 7.8240(m+1)(n+1)$.

2.2. The Triangular Kagomé Lattice. The triangular kagomé lattice [5] with toroidal boundary condition, denoted by $\mathrm{TKL}^t(m,n)$, is depicted in Figure 2.

In order to obtain the $\mathrm{IE}(G)$ of toroidal boundary condition, we recall the spectrum and the Laplacian spectrum of $\mathrm{TKL}^t(m,n)$.

Theorem 4 (see [5]). *The spectrum and the Laplacian spectrum of* $\mathrm{TKL}^t(m,n)$ *are*

$$\begin{aligned}
&\mathrm{Spec}_A\left(\mathrm{TKL}^t(m,n)\right)\\
&=\left\{\underbrace{-2,-2,\ldots,-2}_{3(m+1)(n+1)},\underbrace{-1,-1,\ldots,-1}_{(m+1)(n+1)},\underbrace{1,1,\ldots,1}_{(m+1)(n+1)}\right\}\\
&\quad\cup\left\{\frac{3\pm\sqrt{13\pm4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}{2}\right\},\\
&\mathrm{Spec}_L\left(\mathrm{TKL}^t(m,n)\right)\\
&=\left\{\underbrace{6,6,\ldots,6}_{3(m+1)(n+1)},\underbrace{3,3,\ldots,3}_{(m+1)(n+1)},\underbrace{5,5,\ldots,5}_{(m+1)(n+1)}\right\}\\
&\quad\cup\left\{\frac{5\pm\sqrt{13\pm4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}{2}\right\},
\end{aligned}\tag{10}$$

where $\alpha_i=2\pi i/(m+1)$, $\beta_j=2\pi j/(n+1)$, $i=0,1,\ldots,m$, $j=0,1,\ldots,n$.

Note that the triangular kagomé lattice is the line graph of the 3.12.12 lattice and $\mathrm{TKL}^t(m,n)$ is a 4-regular graph with $9(m+1)(n+1)$ vertices.

Consequently, we can easily get the signless Laplacian spectrum of $\mathrm{TKL}^t(m,n)$:

$$\begin{aligned}
&\mathrm{Spec}_Q\left(\mathrm{TKL}^t(m,n)\right)\\
&= \left\{\underbrace{2,2,\ldots,2}_{3(m+1)(n+1)},\underbrace{3,3,\ldots,3}_{(m+1)(n+1)},\underbrace{5,5,\ldots,5}_{(m+1)(n+1)}\right\}\\
&\quad\cup\left\{\frac{11\pm\sqrt{13\pm 4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}{2}\right\},
\end{aligned}\tag{11}$$

where $\alpha_i = 2\pi i/(m+1)$, $\beta_j = 2\pi j/(n+1)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$.

Theorem 5. *Let $\alpha_i = 2\pi i/(m+1)$, $\beta_j = 2\pi j/(n+1)$, $i = 0,1,\ldots,m$, $j = 0,1,\ldots,n$. Then the incidence energy of $\mathrm{TKL}^t(m,n)$ can be expressed as*

$$\begin{aligned}
&\mathrm{IE}\left(\mathrm{TKL}^t(m,n)\right)\\
&= 3\sqrt{2}(m+1)(n+1)+\left(\sqrt{3}+\sqrt{5}\right)(m+1)(n+1)\\
&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\frac{1}{2}\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}.
\end{aligned}\tag{12}$$

Hence,

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{\mathrm{IE}\left(\mathrm{TKL}^t(m,n)\right)}{9(m+1)(n+1)}\\
&= \lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{18(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{18(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{18(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{18(m+1)(n+1)}\\
&\quad\times\sum_{i=0}^{m}\sum_{j=0}^{n}\sqrt{11+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\\
&\quad+\frac{\sqrt{3}+\sqrt{5}}{9}+\frac{\sqrt{2}}{3}\\
&= \frac{1}{18}\int_0^1\int_0^1\sqrt{11-\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy
\end{aligned}$$

$$+\frac{1}{18}\int_0^1\int_0^1\sqrt{11-\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy$$
$$+\frac{1}{18}\int_0^1\int_0^1\sqrt{11+\sqrt{13-4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy$$
$$+\frac{1}{18}\int_0^1\int_0^1\sqrt{11+\sqrt{13+4\sqrt{3+2\cos\alpha_i+2\cos\beta_j+2\cos\left(\alpha_i+\beta_j\right)}}}\,dx\,dy$$
$$+\frac{\sqrt{3}+\sqrt{5}}{9}+\frac{\sqrt{2}}{3}\approx 1.6390. \tag{13}$$

The above numerical integration value implies that $\mathrm{TKL}^t(m,n)$ has the asymptotic incidence energy $\mathrm{IE}(\mathrm{TKL}^t(m,n))\approx 14.7510(m+1)(n+1)$.

Remark 6. In comparison to [5], the authors have derived the formulae of the number of spanning trees, the energy, and the Kirchhoff index of the triangular kagomé lattice with toroidal boundary condition in [5], while we have handled the $\mathrm{IE}(G)$ of the 3.12.12 lattice, triangular kagomé lattice, which enriches and extends the earlier results by Liu and Yan [5].

2.3. The $S^t(m,n)$ Lattice. The $S^t(m,n)$ lattice [20] with toroidal boundary condition, denoted by $S^t(m,n)$, can be constructed by starting with an $m\times n$ square lattice and adding two diagonal edges to each square, which are illustrated in Figure 3.

The eigenvalues of $A(S^t(m,n))$ have been obtained in [20].

Lemma 7. *The eigenvalues of $A(S^t(m,n))$ are*

$$2\cos\frac{2\pi i}{m}+2\cos\frac{2\pi j}{n}+4\cos\frac{2\pi i}{m}\cos\frac{2\pi j}{n},\qquad i=0,1,\ldots,m-1;\quad j=0,1,\ldots,n-1. \tag{14}$$

Notice that $S^t(m,n)$ is a 8-regular graph. Let $Q(S^t(m,n))$ be the signless Laplacian matrix of $S^t(m,n)$, and then the signless Laplacian eigenvalues of $S^t(m,n)$ are

$$8+2\cos\frac{2\pi i}{m}+2\cos\frac{2\pi j}{n}+4\cos\frac{2\pi i}{m}\cos\frac{2\pi j}{n},\qquad i=0,1,\ldots,m-1;\ j=0,1,\ldots,n-1. \tag{15}$$

Based on Lemma 7 and the definition of the incidence energy, it is easy to deduce the following.

Theorem 8. *Let $\alpha_i=2\pi i/m$, $\beta_j=2\pi j/n$, $i=0,1,\ldots,m-1$, $j=0,1,\ldots,n-1$, and then the incidence energy of $S^t(m,n)$ can be expressed as*

$$\mathrm{IE}\left(S^t(m,n)\right)=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sqrt{8+2\cos\frac{2\pi i}{m}+2\cos\frac{2\pi j}{n}+4\cos\frac{2\pi i}{m}\cos\frac{2\pi j}{n}}. \tag{16}$$

Similarly, one can readily derive that

$$\begin{aligned}
&\lim_{m\to\infty}\lim_{n\to\infty}\frac{\mathrm{IE}\left(S^t(m,n)\right)}{mn}\\
&=\lim_{m\to\infty}\lim_{n\to\infty}\frac{1}{mn}\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}\sqrt{8+2\cos\frac{2\pi i}{m}+2\cos\frac{2\pi j}{n}+4\cos\frac{2\pi i}{m}\cos\frac{2\pi j}{n}}\\
&=\int_0^1\int_0^1\sqrt{8+2\cos 2\pi x+2\cos 2\pi y+4\cos 2\pi x\cos 2\pi y}\,dx\,dy\\
&=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sqrt{8+2\cos x+2\cos y+4\cos x\cos y}\,dx\,dy\\
&\approx 2.7883.
\end{aligned} \tag{17}$$

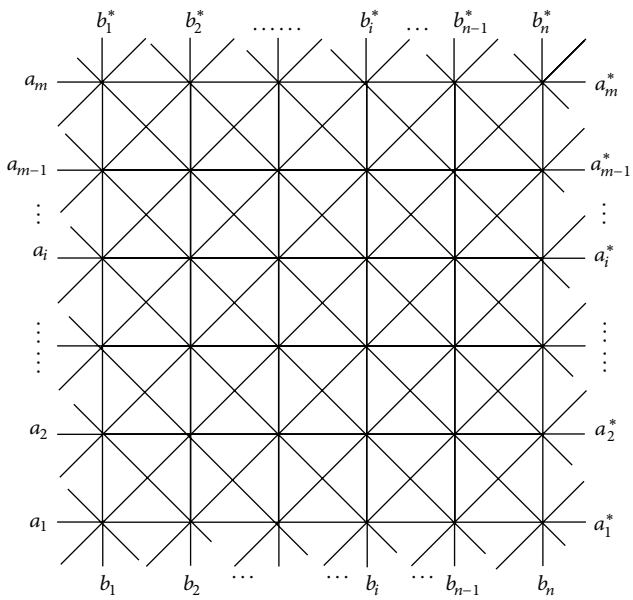

FIGURE 3: The $S^t(m,n)$ lattice with toroidal boundary condition [20].

The above numerical integration value implies that $S^t(m,n)$ has the asymptotic incidence energy $\mathrm{IE}(S^t(m,n)) \approx 2.7883mn$. Summing up, we complete the proof.

3. Remarking Conclusions

In this paper, we deduced the formulae and asymptotic formulae expressing the incidence energy of the 3.12.12 lattice, triangular kagomé lattice, and $S(m,n)$ lattice with toroidal boundary condition, respectively.

It is well known that dealing with the problem of the asymptotic incidence energy of various lattices with the free boundary is not an easy task; however, we can convert the more difficult problems to relatively simple ones via the applications of analysis approach with the help of calculational software. In fact, our approach can be used widely to handle the asymptotic behavior of other lattices and can obtain some useful results simultaneously.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Robust H_∞ Control for a Class of Discrete Time-Delay Stochastic Systems with Randomly Occurring Nonlinearities

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In this paper, we consider the robust H_∞ control problem for a class of discrete time-delay stochastic systems with randomly occurring nonlinearities. The parameter uncertainties enter all the system matrices; the stochastic disturbances are both state and control dependent, and the randomly occurring nonlinearities obey the sector boundedness conditions. The purpose of the problem addressed is to design a state feedback controller such that, for all admissible uncertainties, nonlinearities, and time delays, the closed-loop system is robustly asymptotically stable in the mean square, and a prescribed H_∞ disturbance rejection attenuation level is also guaranteed. By using the Lyapunov stability theory and stochastic analysis tools, a linear matrix inequality (LMI) approach is developed to derive sufficient conditions ensuring the existence of the desired controllers, where the conditions are dependent on the lower and upper bounds of the time-varying delays. The explicit parameterization of the desired controller gains is also given. Finally, a numerical example is exploited to show the usefulness of the results obtained.

1. Introduction

Various engineering systems, such as electrical networks, turbojet engines, microwave oscillators, nuclear reactors, and hydraulic systems, have the characteristics of time delay in signal transmissions. The existence of time delay is often a source of instability and poor performance. So far, the stability analysis and robust control for dynamic time-delay systems have attracted a number of researchers over the past years; see [1–4] and the references therein.

Since stochastic modeling has had extensive applications in control and communication problems, the stability analysis problem for linear stochastic time-delay systems has been studied by many authors. For example, in [5], the stability analysis problem for uncertain stochastic fuzzy systems with time delays has been considered. In [6], an LMI approach has been developed to cope with the robust H_∞ control problem for linear uncertain stochastic systems with state delay. In [7], the robust energy-to-peak filtering problem has been dealt with for uncertain stochastic time-delay systems. In [8], the robust integral sliding mode control problem has been studied for uncertain stochastic systems with time-varying delays, and the H_∞ performance has been analyzed in [9] for continuous-time stochastic systems with polytopic uncertainties.

On the other hand, it is well known that nonlinearity is ubiquitous and all pervading in the physical world and therefore nonlinear control has been an ever hot topic in the past few decades. It is worth mentioning that, among different descriptions of the nonlinearities, the so-called *sector nonlinearity* [10] has gained much attention for *deterministic* systems, and both the control analysis and model reduction problems have been investigated; see [11–13]. Recently, the control problem of nonlinear stochastic

systems has stirred renewed research interests [14], and a variety of nonlinear stochastic systems have been investigated by different approaches, such as the minimax dynamic game approach [15], the input-to-state stabilization method [16, 17], the infinite-horizon risk-sensitive scheme [18], and the Lyapunov-based recursive design method [19]. Most recently, in [20–27], an H_∞-type theory has been developed for a large class of continuous- and discrete-time nonlinear stochastic systems. Notice that it is also quite common to describe nonlinearities as additive nonlinear disturbances. Such nonlinear disturbances may occur in a probabilistic way. For example, in a particular moment for networked control systems, the transmission channel for a large amount of packets may encounter severe network-induced congestions due to the bandwidth limitations, and the resulting phenomenon could be reflected by certain randomly occurring nonlinearities where the occurrence probability can be estimated via statistical tests. Recently some initial work has been reported on the dynamics for the systems with randomly occurring nonlinearities; see [27, 28] and the references therein. As far as we know, to date, however, the robust H_∞ control problem for stochastic systems with time delays and randomly occurring nonlinearities has not been fully investigated, especially for discrete-time cases, which motivates us to shorten such a gap in the present investigation.

The main contribution of this paper can be summarized as follows. (1) A general framework is established to cope with robust H_∞ control problem for a class of uncertain discrete-time stochastic systems involving randomly occurring sector nonlinearities and time-varying delays. (2) An effective LMI approach is proposed to design the state feedback controllers such that, for all the randomly occurring nonlinearities and time-delays, the overall uncertain closed-loop system is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We first establish the sufficient conditions for the uncertain nonlinear stochastic time-delay systems to be stable in the mean square and then derive the explicit expression of the desired controller gains. (3) Sector nonlinearity technique is utilized to reduce the conservativeness for the main results.

Notation. Throughout this paper, $\mathbb{N}$ and $\mathbb{N}^+$ stand for the natural numbers and the nonnegative integer set, respectively; $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ denote, respectively, the set of real numbers, the n dimensional Euclidean space, and the set of all $n\times m$ real matrices. The superscript T represents the transpose for a matrix; the notation $X \geq Y$ (resp., $X > Y$), where X and Y are symmetric matrices, means that $X - Y$ is positive semidefinite (resp., positive definite). $|\cdot|$ denotes the standard Euclidean norm. If A is a square matrix, denote by $\lambda_{\max}(A)$ (resp., $\lambda_{\min}(A)$) the largest (resp., the smallest) eigenvalue of A. Matrices, if not explicitly specified, are assumed to have compatible dimensions. Sometimes, the arguments of a function will be omitted in the analysis when no confusion can arise. In an underlying probability space $(\Omega, \mathscr{F}, \mathscr{P})$, $\mathbb{E}[\cdot]$ denotes the mean for a random variable, and Prob$\{\cdot\}$ means the occurrence probability of the event.

2. Problem Formulation

Consider, on a probability space $(\Omega, \mathscr{F}, \mathscr{P})$, the following uncertain discrete nonlinear stochastic system with time delays of the form:

$$\begin{aligned}
(\Sigma):\ & x(k+1) \\
&= A(k)x(k) + A_d(k)x(k-d(k)) \\
&\quad + \vartheta_k f(x(k)) + \zeta_k f_d(x(k-d(k))) + B_1(k)u(k) \\
&\quad + D_1(k)v(k) \\
&\quad + \big[G(k)x(k) + G_d(k)x(k-d(k)) \\
&\quad + \vartheta_k f(x(k)) + \zeta_k f_d(x(k-d(k))) \\
&\quad + B_2(k)u(k) + D_2(k)v(k)\big]w(k), \\
&\qquad y(k) = Cx(k) + Bu(k), \\
&x(j) = \phi(j), \quad j = -d_M, -d_M+1, \ldots, -1, 0,
\end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state vector; $u(k) \in \mathbb{R}^m$ is the control input; $y(k) \in \mathbb{R}^q$ is the controlled output; $w(k) \in \mathbb{R}$ is a white process noise with

$$\mathbb{E}[w(k)] = 0, \qquad \mathbb{E}\left[w^2(k)\right] = 1, \qquad \mathbb{E}\{w(i)w(j)\} = 0 \quad (i \neq j), \tag{2}$$

ϑ_k and ζ_k are mutually independent Bernoulli-distributed white sequences and also assumed to be independent of $w(k)$.

Let distribution laws of ϑ_k and ζ_k be given by

$$\begin{aligned}
\text{Prob}\{\vartheta_k = 1\} &= \mathbb{E}\{\vartheta_k\} = \beta_1, \\
\text{Prob}\{\vartheta_k = 0\} &= 1 - \mathbb{E}\{\vartheta_k\} = 1 - \beta_1, \\
\text{Prob}\{\zeta_k = 1\} &= \mathbb{E}\{\zeta_k\} = \beta_2, \\
\text{Prob}\{\zeta_k = 0\} &= 1 - \mathbb{E}\{\zeta_k\} = 1 - \beta_2.
\end{aligned} \tag{3}$$

It is well known that $\mathbb{E}\{\vartheta_k\} = \beta_1$, $\mathbb{E}\{\zeta_k\} = \beta_2$, and the variances of ϑ_k and ζ_k are, respectively, $\sigma_1 = \beta_1(1-\beta_1)$ and $\sigma_2 = \beta_2(1-\beta_2)$. Notice that ϑ_k and ζ_k characterize the random occurrence of nonlinear functions.

For the exogenous disturbance signal $v(k) \in \mathbb{R}^p$, it is assumed that $v(\cdot) \in l_{e_2}([0,\infty);\mathbb{R}^p)$, where $l_{e_2}([0,\infty);\mathbb{R}^p)$ is the space of nonanticipatory square-summable stochastic process $f(\cdot) = (f(k))_{k\in\mathbb{N}}$ with respect to $(\mathscr{F}_k)_{k\in\mathbb{N}}$ with the following norm:

$$\|f\|_{e_2} = \left\{\mathbb{E}\sum_{k=0}^{\infty}|f(k)|^2\right\}^{1/2} = \left\{\sum_{k=0}^{\infty}\mathbb{E}|f(k)|^2\right\}^{1/2}. \tag{4}$$

In the system (Σ), the positive integer $d(k)$ denotes the time-varying delay satisfying

$$d_m \le d(k) \le d_M, \quad k \in \mathbb{N}^+, \tag{5}$$

where d_m and d_M are known positive integers. The $\phi(j)$ $(j = -d_M, -d_M+1, \ldots, -1, 0)$ are the initial conditions independent of the process $\{w(\cdot)\}$.

In (Σ), C and B are known real constant matrices. The matrices $A(k)$, $A_d(k)$, $B_1(k)$, $D_1(k)$, $G(k)$, $G_d(k)$, $B_2(k)$, and $D_2(k)$ are time-varying matrices of the following form:

$$\begin{aligned} A(k) &= A + \Delta A(k), & A_d(k) &= A_d + \Delta A_d(k), \\ B_1(k) &= B_1 + \Delta B_1(k), & D_1(k) &= D_1 + \Delta D_1(k), \\ G(k) &= G + \Delta G(k), & G_d(k) &= G_d + \Delta G_d(k), \\ B_2(k) &= B_2 + \Delta B_2(k), & D_2(k) &= D_2 + \Delta D_2(k). \end{aligned} \tag{6}$$

Here, A, A_d, B_1, D_1, G, G_d, B_2, and D_2 are known real constant matrices; $\Delta A(k)$, $\Delta A_d(k)$, $\Delta B_1(k)$, $\Delta D_1(k)$, $\Delta G(k)$, $\Delta G_d(k)$, $\Delta B_2(k)$, and $\Delta D_2(k)$ are unknown matrices representing time-varying parameter uncertainties, which are assumed to satisfy the following admissible condition:

$$\begin{aligned} &\begin{bmatrix} \Delta A(k) & \Delta A_d(k) & \Delta B_1(k) & \Delta D_1(k) \\ \Delta G(k) & \Delta G_d(k) & \Delta B_2(k) & \Delta D_2(k) \end{bmatrix} \\ &\quad = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} F(k) \begin{bmatrix} N_1 & N_2 & N_3 & N_4 \end{bmatrix}, \end{aligned} \tag{7}$$

where M_i $(i = 1, 2)$ and N_i $(i = 1, 2, 3, 4)$ are known real constant matrices, and $F(k)$ is the unknown time-varying matrix-valued function subject to the following condition:

$$F^T(k) F(k) \le I, \quad \forall k \in \mathbb{N}^+. \tag{8}$$

Furthermore, the vector-valued functions f and f_d satisfy the sector nonlinearity condition:

$$[f(x) - L_1 x]^T [f(x) - L_2 x] \le 0, \quad \forall x \in \mathbb{R}^n, \tag{9}$$

$$[f_d(x) - U_1 x]^T [f_d(x) - U_2 x] \le 0, \quad \forall x \in \mathbb{R}^n, \tag{10}$$

where $L_1, L_2, U_1, U_2 \in \mathbb{R}^{n\times n}$ are known real constant matrices.

Remark 1. It is customary that the nonlinear functions f, f_d are said to belong to sectors $[L_1, L_2]$, $[U_1, U_2]$, respectively [10]. The nonlinear descriptions in (9)-(10) are quite general descriptions that include the usual Lipschitz conditions as a special case. Note that both the control analysis and model reduction problems for systems with sector nonlinearities have been intensively studied; see for example [11–13].

Substituting the state feedback $u(k) = Kx(k)$ to system (Σ) gives the following closed-loop system:

$$\begin{aligned} (\Sigma_c) : x(k+1) &= A_K(k) x(k) + A_d(k) x(k - d(k)) + \vartheta_k f(x(k)) \\ &\quad + \zeta_k f_d(x(k - d(k))) + D_1(k) v(k) \\ &\quad + [G_K(k) x(k) + G_d(k) x(k - d(k)) + \vartheta_k f(x(k)) \\ &\quad + \zeta_k f_d(x(k - d(k))) + D_2(k) v(k)] w(k), \\ y(k) &= C_K x(k), \\ x(k) &= \phi(k), \quad k = -d_M, -d_M+1, \ldots, -1, 0, \end{aligned} \tag{11}$$

where

$$\begin{aligned} A_K(k) &= A(k) + B_1(k) K, \\ G_K(k) &= G(k) + B_2(k) K, \qquad C_K = C + BK. \end{aligned} \tag{12}$$

In this paper, we aim to solve the robust H_∞ control problem for the uncertain discrete nonlinear stochastic systems (Σ) with time-varying delays. More specifically, a state feedback controller of the form $u(k) = Kx(k)$ is to be designed such that the following two requirements are met simultaneously.

(1) The closed-loop system (Σ_c) with $v(k) = 0$ is mean-square asymptotically stable for all admissible uncertainties.

(2) Under zero initial condition, the closed-loop system satisfies $\|y\|_{e_2} \le \gamma \|v\|_{e_2}$ for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n\times m})$.

3. Main Results

In this section, we begin with the robust stabilization problem for the closed-loop system (Σ_c). A sufficient condition is derived in the form of an LMI in order to guarantee the robustly mean-square asymptotic stability for the closed-loop system (Σ_c) with $v(k) = 0$.

Theorem 2. *Let K be a given constant feedback gain matrix. The closed-loop system (Σ_c) with $v(t) = 0$ is robustly asymptotically stable in the mean square if there exist two positive definite matrices X, Q and two positive constant scalars ε_1, ε_2 such that the following LMI holds:*

$$\Psi < 0, \tag{13}$$

where

$$\Psi = \begin{bmatrix} \Upsilon & 0 & -X\check{L}_2 & 0 & \Theta_1 & \Theta_2 & X & 0 & XN_1^T & Y^T N_3^T \\ * & -Q & 0 & -X\check{U}_2 & XA_d^T & XG_d^T & 0 & X & XN_2^T & 0 \\ * & * & (2\sigma_1 - 1) I & 0 & \beta_1 I & \beta_1 I & 0 & 0 & 0 & 0 \\ * & * & * & (2\sigma_2 - 1) I & \beta_2 I & \beta_2 1 I & 0 & 0 & 0 & 0 \\ * & * & * & * & \Xi_0 & \Gamma_0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & \Pi_0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \check{L}_1^{-1} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \check{U}_1^{-1} & 0 & 0 \\ * & * & * & * & * & * & * & * & -\varepsilon_1 I & 0 \\ * & * & * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix}, \tag{14}$$

$$\breve{L}_1 = \frac{\left(L_1^T L_2 + L_2^T L_1\right)}{2}; \qquad \breve{L}_2 = -\frac{\left(L_1^T + L_2^T\right)}{2}; \tag{15}$$

$$\breve{U}_1 = \frac{\left(U_1^T U_2 + U_2^T U_1\right)}{2}; \qquad \breve{U}_2 = -\frac{\left(U_1^T + U_2^T\right)}{2}; \tag{16}$$

$$Y = KX; \tag{17}$$

$$\Upsilon = -X + \left(d_M - d_m + 1\right) Q; \tag{18}$$

$$\Theta_1 = XA^T + Y^T B_1^T, \qquad \Theta_2 = XG^T + Y^T B_2^T, \tag{19}$$

$$\Xi_0 = -X + \left(\varepsilon_1 + \varepsilon_2\right) M_1 M_1^T, \tag{20}$$

$$\Gamma_0 = \left(\varepsilon_1 + \varepsilon_2\right) M_1 M_2^T, \tag{21}$$

$$\Pi_0 = -X + \left(\varepsilon_1 + \varepsilon_2\right) M_2 M_2^T. \tag{22}$$

Proof. For the stability analysis of the system (Σ_c), we construct the following Lyapunov-Krasovskii functional:

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{23}$$

where

$$\begin{aligned} V_1(k) &= x^T(k) P x(k); \\ V_2(k) &= \sum_{i=k-d(k)}^{k-1} x^T(i) \widehat{Q} x(i); \\ V_3(k) &= \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} x^T(i) \widehat{Q} x(i), \end{aligned} \tag{24}$$

with $P = X^{-1}$ and $\widehat{Q} = X^{-1} Q X^{-1}$.

By calculating the difference of $V(k)$ along the system (Σ_c) with $v(k) = 0$ and taking the mathematical expectation, we have

$$\mathbb{E}\{\Delta V(k)\} = \mathbb{E}\{\Delta V_1(k)\} + \mathbb{E}\{\Delta V_2(k)\} + \mathbb{E}\{\Delta V_3(k)\}, \tag{25}$$

where

$$\begin{aligned} &\mathbb{E}\{\Delta V_1(k)\} \\ &\quad = \mathbb{E}\{V_1(k+1) - V_1(k)\} \\ &\quad = \mathbb{E}\Big\{\mathscr{F}_0^T(k) P \mathscr{F}_0(k) + \mathscr{G}_0^T(k) P \mathscr{G}_0(k) \\ &\qquad + 2\sigma_1 f^T(x(k)) f(x(k)) \\ &\qquad + 2\sigma_2 f_d^T(x(k-d(k))) f_d(x(k-d(k))) \\ &\qquad - x^T(k) P x(k)\Big\}, \end{aligned} \tag{26}$$

with

$$\begin{aligned} \mathscr{F}_0(k) &= A_K(k) x(k) + A_d(k) x(k-d(k)) \\ &\quad + \beta_1 f(x(k)) + \beta_2 f_d(x(k-d(k))), \\ \mathscr{G}_0(k) &= G_K(k) x(k) + G_d(k) x(k-d(k)) + \beta_1 f(x(k)) \\ &\quad + \beta_2 f_d(x(k-d(k))), \end{aligned}$$

$$\begin{aligned} &\mathbb{E}\{\Delta V_2(k)\} \\ &\quad = \mathbb{E}\{V_2(k+1) - V_2(k)\} \\ &\quad = \mathbb{E}\left\{\sum_{i=k+1-d(k+1)}^{k} x^T(i) \widehat{Q} x(i) - \sum_{i=k-d(k)}^{k-1} x^T(i) \widehat{Q} x(i)\right\} \\ &\quad = \mathbb{E}\Bigg\{x^T(k) \widehat{Q} x(k) - x^T(k-d(k)) \widehat{Q} x(k-d(k)) \\ &\qquad + \sum_{i=k-d(k+1)+1}^{k-1} x^T(i) \widehat{Q} x(i) \\ &\qquad - \sum_{i=k-d(k)+1}^{k-1} x^T(i) \widehat{Q} x(i)\Bigg\} \\ &\quad = \mathbb{E}\Bigg\{x^T(k) \widehat{Q} x(k) - x^T(k-d(k)) \widehat{Q} x(k-d(k)) \\ &\qquad + \sum_{i=k-d_m+1}^{k-1} x^T(i) \widehat{Q} x(i) \\ &\qquad + \sum_{i=k-d(k)+1}^{k-d_m} x^T(i) \widehat{Q} x(i) \\ &\qquad - \sum_{i=k-d(k)+1}^{k-1} x^T(i) \widehat{Q} x(i)\Bigg\} \\ &\quad \le \mathbb{E}\Bigg\{x^T(k) \widehat{Q} x(k) - x^T(k-d(k)) \widehat{Q} x(k-d(k)) \\ &\qquad + \sum_{i=k-d_M+1}^{k-d_m} x^T(i) \widehat{Q} x(i)\Bigg\}, \\ &\mathbb{E}\{\Delta V_3(k)\} \\ &\quad = \mathbb{E}\{V_3(k+1) - V_3(k)\} \\ &\quad = \mathbb{E}\Bigg\{\sum_{j=k-d_M+2}^{k-d_m+1} \sum_{i=j}^{k} x^T(i) \widehat{Q} x(i) \\ &\qquad - \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} x^T(i) \widehat{Q} x(i)\Bigg\} \\ &\quad = \mathbb{E}\Bigg\{\sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j+1}^{k} x^T(i) \widehat{Q} x(i) \\ &\qquad - \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} x^T(i) \widehat{Q} x(i)\Bigg\} \\ &\quad = \mathbb{E}\Bigg\{\sum_{j=k-d_M+1}^{k-d_m} \left(x^T(k) \widehat{Q} x(k) - x^T(j) \widehat{Q} x(j)\right)\Bigg\} \\ &\quad = \mathbb{E}\Bigg\{\left(d_M - d_m\right) x^T(k) \widehat{Q} x(k) \\ &\qquad - \sum_{i=k-d_M+1}^{k-d_m} x^T(i) \widehat{Q} x(i)\Bigg\}. \end{aligned} \tag{27}$$

Substituting (26)-(27) into (25) leads to

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad \le \mathbb{E}\Big\{\mathscr{F}_0^T(k)P\mathscr{F}_0(k)+\mathscr{G}_0^T(k)P\mathscr{G}_0(k) \\
&\qquad + x^T(k)\left[-P+\left(d_M-d_m+1\right)\widehat{Q}\right]x(k) \\
&\qquad - x^T(k-d(k))\widehat{Q}x(k-d(k)) \\
&\qquad + 2\sigma_1 f^T(x(k))f(x(k)) \\
&\qquad + 2\sigma_2 f_d^T(x(k-d(k)))f_d(x(k-d(k)))\Big\} \\
&\quad = \mathbb{E}\Big\{\xi_0^T(k)\Psi_1(k)\xi_0(k)+\xi_0^T(k)\widehat{F}_0^T(k)P\widehat{F}_0(k)\xi_0(k) \\
&\qquad + \xi_0^T(k)\widehat{G}_0^T(k)P\widehat{G}_0(k)\xi_0(k)\Big\},
\end{aligned} \tag{28}$$

where

$$\begin{aligned}
&\xi_0(k) \\
&\quad = \left[x^T(k)\ \ x^T(k-d(k))\ \ f^T(x(k))\ \ f_d^T(x(k-d(k)))\right]^T, \\
&\quad \widehat{F}_0(k) = \left[A_K(k)\ \ A_d(k-d(k))\ \ \beta_1 I\ \ \beta_2 I\right], \\
&\quad \widehat{G}_0(k) = \left[G_K(k)\ \ G_d(k-d(k))\ \ \beta_1 I\ \ \beta_2 I\right], \\
&\quad \Psi_1(k) = \operatorname{diag}\left(-P+\left(d_M-d_m+1\right)\widehat{Q}, -\widehat{Q}, 2\sigma_1 I, 2\sigma_1 I\right).
\end{aligned} \tag{29}$$

Notice that (9) is equivalent to

$$\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} \check{L}_1 & \check{L}_2 \\ \check{L}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix} \le 0, \tag{30}$$

where $\check{L}_1$, $\check{L}_2$ are defined in (15) and, similarly, it follows from (10) that

$$\begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \check{U}_1 & \check{U}_2 \\ \check{U}_2^T & I \end{bmatrix} \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix} \le 0, \tag{31}$$

where $\check{U}_1$, $\check{U}_2$ are defined in (16).

It follows from (28), (30), and (31) that

$$\begin{aligned}
&\mathbb{E}\{\Delta V(k)\} \\
&\quad \le \mathbb{E}\Big\{\xi_0^T(k)\Psi_1(k)\xi_0(k)+\xi_0^T(k)\widehat{F}_0^T(k)P\widehat{F}_0(k)\xi_0(k) \\
&\qquad + \xi_0^T(k)\widehat{G}_0^T(k)P\widehat{G}_0(k)\xi_0(k)\Big\} \\
&\qquad - \mathbb{E}\left\{\begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}^T \begin{bmatrix} \check{L}_1 & \check{L}_2 \\ \check{L}_2^T & I \end{bmatrix} \begin{bmatrix} x(k) \\ f(x(k)) \end{bmatrix}\right. \\
&\qquad + \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix}^T \begin{bmatrix} \check{U}_1 & \check{U}_2 \\ \check{U}_2^T & I \end{bmatrix} \\
&\qquad \left.\times \begin{bmatrix} x(k-d(k)) \\ f_d(x(k-d(k))) \end{bmatrix}\right\} \\
&\quad = \mathbb{E}\Big\{\xi_0^T(k)\Big[\Psi_2(k)+\widehat{F}_0^T(k)P\widehat{F}_0(k) \\
&\qquad + \widehat{G}_0^T(k)P\widehat{G}_0(k)\Big]\xi_0(k)\Big\},
\end{aligned} \tag{32}$$

where

$$\Psi_2(k) = \begin{bmatrix} -P+\left(d_M-d_m+1\right)\widehat{Q}-\check{L}_1 & 0 & \check{L}_2 & 0 \\ 0 & -\widehat{Q}-\check{U}_1 & 0 & \check{U}_2 \\ \check{L}_2^T & 0 & \left(2\sigma_1-1\right)I & 0 \\ 0 & \check{U}_2^T & 0 & \left(2\sigma_2-1\right)I \end{bmatrix}. \tag{33}$$

From Lyapunov stability theory, in order to ensure the stability of the closed-loop system (Σ_c) with $v(k)=0$, we just need to show

$$\Psi_2(k)+\widehat{F}_0^T(k)P\widehat{F}_0(k)+\widehat{G}_0^T(k)P\widehat{G}_0(k) < 0. \tag{34}$$

By *Schur complement*, (34) is equivalent to

$$\Psi_3(k) < 0, \tag{35}$$

where

$$\Psi_3(k) = \begin{bmatrix} -P+\left(d_M-d_m+1\right)\widehat{Q}-\check{L}_1 & 0 & -\check{L}_2 & 0 & A_K^T(k) & G_K^T(k) \\ 0 & -\widehat{Q}-\check{U}_1 & 0 & -\check{U}_2 & A_d^T(k) & G_d^T(k) \\ -\check{L}_2^T & 0 & \left(2\sigma_1-1\right)I & 0 & \beta_1 I & \beta_1 I \\ 0 & -\check{U}_2^T & 0 & \left(2\sigma_2-1\right)I & \beta_2 I & \beta_2 I \\ A_K(k) & A_d(k) & \beta_1 I & \beta_2 I & -P^{-1} & 0 \\ G_K(k) & G_d(k) & \beta_1 I & \beta_2 I & 0 & -P^{-1} \end{bmatrix}. \tag{36}$$

Therefore, it remains to show $\Psi_3(k) < 0$. Let $\widehat{X} = \mathrm{diag}(X, X, I, I, I, I)$, and

$$\Psi_4(k) = \widehat{X}^T \Psi_3(k) \widehat{X} = \begin{bmatrix} \Upsilon - X\breve{L}_1 X & 0 & -X\breve{L}_2 & 0 & XA_K^T(k) & XG_K^T(k) \\ 0 & -Q - X\breve{U}_1 X & 0 & -X\breve{U}_2 & XA_d^T(k) & XG_d^T(k) \\ -\breve{L}_2^T X & 0 & (2\sigma_1 - 1) I & 0 & \beta_1 I & \beta_1 I \\ 0 & -\breve{U}_2^T X & 0 & (2\sigma_2 - 1) I & \beta_2 I & \beta_2 I \\ A_K(k) X & A_d(k) X & \beta_1 I & \beta_2 I & -X & 0 \\ G_K(k) X & G_d(k) X & \beta_1 I & \beta_2 I & 0 & -X \end{bmatrix}, \tag{37}$$

where Υ is defined in (18).

Obviously, $\Psi_3(k) < 0$ is equivalent to $\Psi_4(k) < 0$. Now, we can rewrite $\Psi_4(k)$ as follows:

$$\Psi_4(k) = \Psi_5(k) + \widetilde{X}_0^T \begin{bmatrix} -\breve{L}_1 & 0 \\ 0 & -\breve{U}_1 \end{bmatrix} \widetilde{X}_0, \tag{38}$$

where

$$\widetilde{X}_0 = \begin{bmatrix} X & 0 & 0 & 0 & 0 & 0 \\ 0 & X & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\Psi_5(k) = \begin{bmatrix} \Upsilon & 0 & -X\breve{L}_2 & 0 & \Theta_1 & \Theta_2 \\ 0 & -Q & 0 & -X\breve{U}_2 & XA_d^T(k) & XG_d^T(k) \\ -\breve{L}_2^T X & 0 & (2\sigma_1 - 1) I & 0 & \beta_1 I & \beta_1 I \\ 0 & -\breve{U}_2^T X & 0 & (2\sigma_2 - 1) I & \beta_2 I & \beta_2 I \\ \Theta_1^T & A_d(k) X & \beta_1 I & \beta_2 I & -X & 0 \\ \Theta_2^T & G_d(k) X & \beta_1 I & \beta_2 I & 0 & -X \end{bmatrix}, \tag{39}$$

where Θ_1 and Θ_2 are defined in (19).

According to *Schur complement*, $\Phi_4(k) < 0$ is equivalent to

$$\Psi_6(k) < 0, \tag{40}$$

where

$$\Psi_6(k) = \begin{bmatrix} \Upsilon & 0 & -X\breve{L}_2 & 0 & \Theta_1 & \Theta_2 & X & 0 \\ 0 & -Q & 0 & -X\breve{U}_2 & XA_d^T(k) & XG_d^T(k) & 0 & X \\ -\breve{L}_2^T X & 0 & (2\sigma_1 - 1) I & 0 & \beta_1 I & \beta_1 I & 0 & 0 \\ 0 & -\breve{U}_2^T X & 0 & (2\sigma_2 - 1) I & \beta_2 I & \beta_2 I & 0 & 0 \\ \Theta_1^T & A_d(k) X & \beta_1 I & \beta_2 I & -X & 0 & 0 & 0 \\ \Theta_2^T & G_d(k) X & \beta_1 I & \beta_2 I & 0 & -X & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & \breve{L}_1^{-1} & 0 \\ 0 & X & 0 & 0 & 0, & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}. \tag{41}$$

Noticing (7), we can rearrange $\Phi_6(k)$ as follows:

$$\Psi_6(k) = \Psi_6 + \Delta\Psi_6(k), \tag{42}$$

where

$$\Psi_6 = \begin{bmatrix} \Upsilon & 0 & -X\breve{L}_2 & 0 & \Theta_1 & \Theta_2 & X & 0 \\ 0 & -Q & 0 & -X\breve{U}_2 & XA_d^T & XG_d^T & 0 & X \\ -\breve{L}_2^T X & 0 & (2\sigma_1 - 1) I & 0 & \beta_1 I & \beta_1 2I & 0 & 0 \\ 0 & -\breve{U}_2^T X & 0 & (2\sigma_2 - 1) I & \beta_2 I & \beta_2 I & 0 & 0 \\ \Theta_1^T & A_d X & \beta_1 I & \beta_2 I & -X & 0 & 0 & 0 \\ \Theta_2^T & G_d X & \beta_1 I & \beta_2 I & 0 & -X & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & \breve{L}_1^{-1} & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}, \tag{43}$$

$$\begin{aligned}\Delta\Psi_6(k) &= \widehat{M}F(k)\widehat{N}_1 + \widehat{N}_1^T F^T(k)\widehat{M}^T \\ &\quad + \widehat{M}F(k)\widehat{N}_2 + \widehat{N}_2^T F^T(k)\widehat{M}^T,\end{aligned} \tag{44}$$

with

$$\begin{aligned}\widehat{M} &= \begin{bmatrix}0 & 0 & 0 & 0 & M_1^T & M_2^T & 0 & 0\end{bmatrix}^T, \\ \widehat{N}_1 &= \begin{bmatrix}N_1X & N_2X & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}, \\ \widehat{N}_2 &= \begin{bmatrix}N_3Y & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}.\end{aligned} \tag{45}$$

From (44), it follows

$$\Delta\Psi_6(k) \le \varepsilon_1\widehat{M}\widehat{M}^T + \varepsilon_2\widehat{M}\widehat{M}^T + \varepsilon_1^{-1}\widehat{N}_1^T\widehat{N}_1 + \varepsilon_2^{-1}\widehat{N}_2^T\widehat{N}_2, \tag{46}$$

and then we obtain from (42)–(46)

$$\Psi_6(k) \le \Psi_7(k) + \varepsilon_1^{-1}\widehat{N}_1^T\widehat{N}_1 + \varepsilon_2^{-1}\widehat{N}_2^T\widehat{N}_2, \tag{47}$$

where

$$\Psi_7(k) = \begin{bmatrix} \Upsilon & 0 & -X\breve{L}_2 & 0 & \Theta_1 & \Theta_2 & X & 0 \\ 0 & -Q & 0 & -X\breve{U}_2 & XA_d^T & XG_d^T & 0 & X \\ -\breve{L}_2^T X & 0 & (2\sigma_1 - 1)I & 0 & \beta_1 I & \beta_1 I & 0 & 0 \\ 0 & -\breve{U}_2^T X & 0 & (2\sigma_2 - 1)I & \beta_2 I & \beta_2 I & 0 & 0 \\ \Theta_1^T & A_d X & \beta_1 I & \beta_2 I & \Xi_0 & \Gamma_0 & 0 & 0 \\ \Theta_2^T & G_d X & \beta_1 I & \beta_2 I & \Gamma_0^T & \Pi_0 & 0 & 0 \\ X & 0 & 0 & 0 & 0 & 0 & \breve{L}_1^{-1} & 0 \\ 0 & X & 0 & 0 & 0 & 0 & 0 & \breve{U}_1^{-1} \end{bmatrix}, \tag{48}$$

and Ξ_0, Γ_0, and Π_0 are defined in (18)–(22).

Using *Schur complement* once again, it is not difficult to see that $\Psi < 0$ (i.e., the condition (13)) is equivalent to that the right-hand side of (47) is negative definite; that is, $\Psi_6(k) < 0$. Thus, we arrive at $\Psi_3(k) < 0$, which completes the proof of the theorem. □

Next we will analyze the H_∞ performance of the closed-loop system (Σ_c).

Theorem 3. *Let K be a given constant feedback gain matrix and γ a given positive constant. Then, the closed-loop system (Σ_c) is robustly mean-square asymptotically stable for $v(k) = 0$ and satisfies $\|y\|_{e_2} \le \gamma\|v\|_{e_2}$ under the zero initial condition for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n\times m})$ if there exist two positive definite matrices X, Q and three scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ such that the following LMI holds*

$$\Phi_0 < 0, \tag{49}$$

where

$$\Phi_0 = \begin{bmatrix} \Upsilon & 0 & -X\breve{L}_2 & 0 & 0 & \Theta_1 & \Theta_2 & X & 0 & \Theta_3 & XN_1^T & Y^TN_3 & 0 \\ * & -Q & 0 & -X\breve{U}_2 & 0 & XA_d^T & XG_d^T & 0 & X & 0 & XN_2^T & 0 & 0 \\ * & * & (2\sigma_1 - 1)I & 0 & 0 & \beta_1 I & \beta_1 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & (2\sigma_2 - 1)I & 0 & \beta_2 I & \beta_2 I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & D_1^T & D_2^T & 0 & 0 & 0 & 0 & 0 & N_4^T \\ * & * & * & * & * & \Xi & \Gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & \Pi & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & \breve{L}_1^{-1} & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \breve{U}_1^{-1} & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & -\varepsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_3 I \end{bmatrix}, \tag{50}$$

$$\begin{aligned}\Theta_3 &= XC^T + Y^T B^T,\\ \Xi &= -X + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) M_1 M_1^T,\\ \Gamma &= (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) M_1 M_2^T,\\ \Pi &= -X + (\varepsilon_1 + \varepsilon_2 + \varepsilon_3) M_2 M_2^T,\end{aligned} \tag{51}$$

and $\breve{L}_1$, $\breve{L}_2$, $\breve{U}_1$, $\breve{U}_2$, Y, Υ, Θ_1, *and* Θ_2 *are defined as in Theorem 2.*

Proof. It is not difficult to verify that $\Phi_0 < 0$ implies $\Psi < 0$ (i.e., (13)). According to Theorem 2, the closed-loop system (Σ_c) is robustly asymptotically stable in the mean square.

Next, we shall deal with the H_∞ performance of the closed-loop system. As in Theorem 2, define Lyapunov-Krasovskii functional candidate for the system (Σ_c) as follows:

$$V(k) = V_1(k) + V_2(k) + V_3(k), \tag{52}$$

where

$$V_1(k) = x^T(k) P x(k), \qquad V_2(k) = \sum_{i=k-d(k)}^{k-1} x^T(i) \widehat{Q} x(i),$$
$$V_3(k) = \sum_{j=k-d_M+1}^{k-d_m} \sum_{i=j}^{k-1} x^T(i) \widehat{Q} x(i). \tag{53}$$

And we also introduce

$$J(n) = \mathbb{E}\sum_{k=0}^{n} \left[y^T(k) y(k) - \gamma^2 v^T(k) v(k) \right], \tag{54}$$

where n is nonnegative integer. Under the zero initial condition, one has

$$\begin{aligned}J(n) = \mathbb{E}\sum_{k=0}^{n} &\left[y^T(k) y(k) - \gamma^2 v^T(k) v(k) + \Delta V(k) \right]\\ &- \mathbb{E}V(n+1).\end{aligned} \tag{55}$$

Along the similar lines to the proof of Theorem 2, we can derive that $J(n) \le 0$. Letting $n \to \infty$, we obtain $\|y\|_{e_2} \le \gamma \|v\|_{e_2}$, which concludes the proof of the theorem. □

Finally, let us consider the design of H_∞ control for the system (Σ). Based on Theorem 3, we have the following result.

Theorem 4. *Let $\gamma > 0$ be a given positive constant. Then, for the nonlinear stochastic system (Σ), there exists a state feedback controller such that the closed-loop system (Σ_c) is robustly mean-square asymptotically stable for $v(k) = 0$ and satisfies $\|y\|_{e_2} \le \gamma \|v\|_{e_2}$ under the zero initial condition for any nonzero $v(\cdot) \in l_{e_2}([0, +\infty); \mathbb{R}^{n\times m})$ if there exist two positive definite matrices X and Q, a matrix Y, and three scalars $\varepsilon_1 > 0$, $\varepsilon_2 > 0$, $\varepsilon_3 > 0$ such that the following LMI holds:*

$$\Phi < 0, \tag{56}$$

where

$$\Phi_0 = \begin{bmatrix}
\Upsilon & 0 & -X\breve{L}_2 & 0 & 0 & \Theta_1 & \Theta_2 & X & 0 & \Theta_3 & XN_1^T & Y^T N_3 & 0\\
* & -Q & 0 & -X\breve{U}_2 & 0 & XA_d^T & XG_d^T & 0 & X & 0 & XN_2^T & 0 & 0\\
* & * & (2\sigma_1 - 1) I & 0 & 0 & \beta_1 I & \beta_1 I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & (2\sigma_2 - 1) I & 0 & \beta_2 I & \beta_2 I & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\gamma^2 I & D_1^T & D_2^T & 0 & 0 & 0 & 0 & 0 & N_4^T\\
* & * & * & * & * & \Xi & \Gamma & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & \Pi & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & \breve{L}_1^{-1} & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & \breve{U}_1^{-1} & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\varepsilon_1 I & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -\varepsilon_2 I & 0\\
* & * & * & * & * & * & * & * & * & * & * & * & -\varepsilon_3 I
\end{bmatrix}, \tag{57}$$

where $\breve{L}_1$, $\breve{L}_2$, $\breve{U}_1$, $\breve{U}_2$, Υ, Ξ, Γ, Π, Θ_1, Θ_2, *and* Θ_3 *are defined as in Theorems 2 and 3. In this case, the state feedback gain matrix can be designed as*

$$K = YX^{-1}. \tag{58}$$

Remark 5. In Theorem 4, the robust H_∞ controller design problem is reduced to the solvability of an LMI, which could be easily checked by utilizing the LMI Matlab toolbox. It is also worth pointing out that, in LMI framework, the sector nonlinearity condition is superior to commonly used Lipschitz condition in reducing the possible conservativeness.

4. Numerical Example

In this section, a numerical example is presented to demonstrate the usefulness of the developed method on the design of robust H_∞ control for the discrete uncertain nonlinear stochastic systems with time-varying delays.

Consider the system (Σ) with the following parameters:

$$A = \begin{bmatrix} 0.4 & 0.1 & 0 \\ 0 & -0.3 & 0.1 \\ 0.1 & 0 & -0.2 \end{bmatrix}, \qquad A_d = \begin{bmatrix} 0.2 & -0.1 & 0.1 \\ 0.1 & -0.2 & 0 \\ -0 & -0.2 & -0.1 \end{bmatrix},$$

$$E = H = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0.2 & 0.1 \end{bmatrix}, \qquad E_d = H_d = \begin{bmatrix} 0.1 & 0 & 0.1 \\ 0.1 & 0.2 & 0 \\ 0.1 & 0 & 0.1 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.1 & 0.1 & 0 \\ -0 & 0.2 & 0.1 \\ -0.1 & -0.2 & 0.1 \end{bmatrix}, \qquad G_d = \begin{bmatrix} -0.1 & 0.2 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.1 & 0.2 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.2 & -0 & 0.1 \\ -0.2 & -0.1 & 0.1 \\ -0 & 0.2 & -0.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1 & 0.1 \\ -0.1 & 0 \\ 0 & -0.1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 0.1 \\ 0.2 & 1 \\ 0 & -0.1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \\ 0 & -0.1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} -0.2 & 0 \\ -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \qquad D_2 = \begin{bmatrix} -0.2 & 0.1 \\ 0.1 & 0.2 \\ 0 & 0.3 \end{bmatrix},$$

$$L_1 = U_1 = \begin{bmatrix} 0.2 & 0.1 & 0 \\ 0.1 & 0.3 & 0 \\ -0.1 & 0.1 & 0.3 \end{bmatrix},$$

$$L_2 = U_2 = \begin{bmatrix} -0.2 & 0.1 & 0 \\ 0.1 & -0.3 & -0.1 \\ -0.1 & 0 & -0.3 \end{bmatrix}, \qquad M_1 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.1 \end{bmatrix}, \qquad N_1 = N_2 = N_4 = N_5 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}^T,$$

$$N_3 = N_6 = \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix}^T, \qquad d_m = 2,$$

$$d_M = 3, \qquad \beta_1 = 0.5, \qquad \beta_2 = 0.5. \tag{59}$$

The H_∞ performance level is taken as $\gamma = 0.9$. With the above parameters and by using the Matlab LMI Toolbox, we solve the LMI (56) and obtain

$$X = \begin{bmatrix} 5.7714 & 0.4304 & 2.2321 \\ 0.4304 & 4.1813 & 0.8207 \\ 2.2321 & 0.8207 & 13.0932 \end{bmatrix},$$

$$Q = \begin{bmatrix} 1.4729 & -0.1464 & 1.5125 \\ -0.1464 & 1.1697 & 0.7317 \\ 1.5125 & 0.7317 & 4.1582 \end{bmatrix},$$

$$Y = \begin{bmatrix} 0.1037 & -1.5046 & 0.0842 \\ -0.7253 & -0.8760 & -2.1560 \end{bmatrix}, \tag{60}$$

$$\varepsilon_1 = 10.0151, \qquad \varepsilon_2 = 4.8030,$$

$$\varepsilon_3 = 8.1230.$$

Therefore, the state feedback gain matrix can be designed as

$$K = YX^{-1} = \begin{bmatrix} 0.0364 & -0.3681 & 0.0233 \\ -0.0569 & -0.1754 & -0.1440 \end{bmatrix}. \tag{61}$$

5. Conclusions

In this paper, we have studied the robust H_∞ control problem for a class of uncertain discrete-time stochastic systems involving randomly occurring nonlinearities and time-varying delays. An effective linear matrix inequality (LMI) approach has been proposed to design the state feedback controllers such that the overall uncertain closed-loop system is robustly asymptotically stable in the mean square and a prescribed H_∞ disturbance rejection attenuation level is guaranteed. We have first investigated the sufficient conditions for the uncertain stochastic time-delay systems under consideration to be stable in the mean square and then derived the explicit expression of the desired controller gains. A numerical example has been provided to show the usefulness and effectiveness of the proposed design method. It should be pointed out that the main results of this paper can be extended to the case where the parameter uncertainties are of the polytopic type [9, 29] and the case where system measurements suffer from quantization effects.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Krein Space-Based H_∞ Fault Estimation for Discrete Time-Delay Systems

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This paper investigates the finite-time H_∞ fault estimation problem for linear time-delay systems, where the delay appears in both state and measurement equations. Firstly, the design of finite horizon H_∞ fault estimation is converted into a minimum problem of certain quadratic form. Then we introduce a stochastic system in Krein space, and a sufficient and necessary condition for the minimum is derived by applying innovation analysis approach and projection theory. Finally, a solution to the H_∞ fault estimation is obtained by recursively computing a partial difference Riccati equation, which has the same dimension as the original system. Compared with the conventional augmented approach, the solving of a high dimension Riccati equation is avoided.

1. Introduction

Krein-space theory has proven to be an effective tool in dealing with the indefinite quadratic control/filtering problems. Some recent researches on H_∞ filtering have led to an interesting connection with Kalman filtering in Krein space [1, 2]. Comparing with the linear estimation approaches in Hilbert space, the Krein-space theory can lead to not only less conservative results but also computationally attractive algorithms. It has been shown in [1] that a finite horizon linear estimation problem can be cast into a problem of calculating the minimum point of a certain quadratic form. By applying linear estimation in Krein space, one can calculate recursively the minimum point via Riccati equation. In [2] the authors consider the H_∞ prediction problem for time-varying continuous-time systems with delayed measurements in the finite horizon case. The necessary and sufficient condition for the existence of an H_∞ predictor is obtained by applying a reorganized innovation approach in Krein space.

On the other hand, fault estimation is one of the most important issues. The paper [3] designs a fuzzy fault detection filter for T-S fuzzy systems with intermittent measurements, and all the results are formulated in the form of linear matrix inequalities. In [4], a sufficient condition for the existence of a fault filter is exploited in terms of certain linear matrix inequality. Reference [5] is concerned with the robust fault detection problem for a class of discrete-time networked systems with distributed sensors. The existence of the desired fault detection filter can be determined from the feasibility of a set of linear matrix inequalities. The paper [6] addresses the fault detection problem for discrete-time Markovian jump systems. The characterization of the gains of the desired fault detection filters is derived in terms of the solution to a convex optimization problem that can be easily solved by using the semidefinite program method. As for the fault estimation problem, the Krein-space approach has received much attention so far [7–12]. A Krein-space approach is presented in [7] to H_∞ fault estimation for LDTV system, where the augmented approach [13] is also used. Different from [7], a more further result is obtained by a Krein-space approach and nonaugmented approach for the same problem in [8]. Recently, by applying Krein-space approach and reorganized innovation approach, [9] considers the finite-horizon H_∞ fault estimation for linear discrete time-varying systems with delayed measurements [9]. Finite-horizon H_∞ fault estimation for uncertain linear discrete time-varying systems with known inputs is considered in [10].

Recently, we note that time-delay systems have received much attention [14–19]. For optimal estimation problem, when the delay appears in state, the reorganized innovation approach is not suitable for estimation problem. Motivated by this point, we consider the finite-horizon problem H_∞ fault estimation for linear discrete systems with time delay, where the delay appears in both state and measurement, which contain [9] as a special case. To the best of our knowledge, this problem has not yet been investigated, and this constitutes the primary motivation for our research. On the other hand, we desire to obtain the necessary and sufficient condition for the existence of an H_∞ fault estimator. A natural idea is to use Krein space to deal with the finite-horizon H_∞ fault estimation for linear time-delay systems, and this gives rise to another motivation of our work. The main contributions of the paper are highlighted as follows. (i) The necessary and sufficient condition will be derived for fault estimation problem with time delay. (ii) Compared with the augmentation approach [13], our result on estimation is given based on a partial difference Riccati equation, and hence the solving of an high dimension Riccati equation is avoided.

The organization of this paper is as follows. The problem statement is given in Section 2. Section 3 presents the fault estimator design in terms of a partial difference Riccati equation. A numerical example is given to demonstrate the effectiveness of the approach in Section 4, and the paper is concluded in Section 5.

Notation. Throughout this paper, a real symmetric matrix $P > 0$ (≥ 0) denotes P being a positive definite (or positive semidefinite) matrix. I denotes an identity matrix of appropriate dimension. The superscripts "-1" and "T" represent the inverse and transpose of a matrix. $\mathscr{R}^n$ denotes the n-dimensional Euclidean space. $\mathscr{R}^{n\times m}$ is the set of all $n\times m$ real matrices. $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ii} = 1$. For stochastic vectors α and β, inner product $\langle \alpha, \beta\rangle$ equals the covariance matrix of α and β. $\theta(k) \in l_2[0, N]$ means $\sum_{k=0}^{N} \theta^T(k)\theta(k) < \infty$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problems Statement

Consider the following linear systems with time delay:

$$x(k+1) = \sum_{i=0}^{d} A_i x(k-h_i) + B_d d(k) + B_f f(k), \quad (1)$$

$$y(k) = \sum_{i=0}^{d} C_i x(k-l_i) + D_f f(k) + v(k), \quad (2)$$

where $x(k) \in \mathscr{R}^n$, $d(k) \in \mathscr{R}^p$, and $f(k) \in \mathscr{R}^r$ are the state, the driving disturbance, and the fault to be estimated, respectively. Also, $y(k) \in \mathscr{R}^m$ and $v(k) \in \mathscr{R}^m$ are measurements and noises, respectively. Without loss of generality, the delays are assumed to be of an increasing order: $0 = h_0 < h_1 < \cdots < h_d$, $0 = l_0 < l_1 < \cdots < l_d$. Moreover, it is assumed that $d(k)$, $f(k)$, and $v(k)$ belong to $l_2[0, N]$. For simplicity of presentation, we assume that A_i, B_d, B_f, C_i, and D_f are constant matrices even though the later development and results can be easily adapted to the time-varying case.

Problem. Given the observation $\{y(0), \ldots, y(N)\}$, seek a fault estimator $r(k)$ $(k = 0, \ldots, N)$ such that

$$\sup_{(x_0,w)\neq 0} \frac{\sum_{k=0}^{N} \|r(k) - f(k)\|^2}{(x_0 - \hat{x}_0)^T \Pi_0^{-1} (x_0 - \hat{x}_0) + \sum_{k=0}^{N} \|w(k)\|^2} < \gamma^2, \quad (3)$$

where γ is a given positive scalar, Π_0 is a positive definite matrix, and

$$w(k) = \left[d^T(k) \quad f^T(k) \quad v^T(k)\right]^T. \quad (4)$$

Without loss of generality, the initial state estimator $\hat{x}_0$ is assumed to be zero. The value of $x(-k)$ is assumed to be zero, where $1 \leq k \leq \tau$, $\tau = \max(h_d, l_d)$, $E\{x(-i)x^T(-j)\} = 0$.

Define the fault estimation error between $r(k)$ and $f(k)$ as

$$v_f(k) = r(k) - f(k), \quad (5)$$

and introduce the following quadratic form:

$$J_N = x_0^T \Pi_0^{-1} x_0 + \sum_{k=0}^{N} \|w(k)\|^2 - \gamma^{-2} \sum_{k=0}^{N} \left\|v_f(k)\right\|^2. \quad (6)$$

Obviously, the H_∞ performance (3) is satisfied if and only if $J_N > 0$ for all $(x_0, w(k)) \neq 0$.

3. Main Result

We consider constructing an equivalent Krein-space problem to the minimum for J_N. To do so we need to introduce the following stochastic systems in a Krein space:

$$\mathbf{x}(k+1) = \sum_{i=0}^{d} A_i \mathbf{x}(k-h_i) + B_d \mathbf{d}(k) + B_f \mathbf{f}(k), \quad (7)$$

$$\mathbf{y}(k) = \sum_{i=0}^{d} C_i \mathbf{x}(k-l_i) + D_f \mathbf{f}(k) + \mathbf{v}(k), \quad (8)$$

$$\mathbf{r}(k) = \mathbf{f}(k) + \mathbf{v}_f(k), \quad (9)$$

with

$$\left\langle \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{d}(k) \\ \mathbf{f}(k) \\ \mathbf{v}(k) \\ \mathbf{v}_f(k) \end{bmatrix}, \begin{bmatrix} \mathbf{x}(0) \\ \mathbf{d}(j) \\ \mathbf{f}(j) \\ \mathbf{v}(j) \\ \mathbf{v}_f(j) \end{bmatrix} \right\rangle = \operatorname{diag}\left(\Pi_0, I\delta_{kj}, I\delta_{kj}, I\delta_{kj}, -\gamma^2 I\delta_{kj}\right). \quad (10)$$

Let $\mathbf{y}_r(k) = \left[\mathbf{y}^T(k) \quad \mathbf{r}^T(k)\right]^T$; then the linear space generated by the measurements in the Krein space up to time N can be written as

$$\mathscr{L}\{\mathbf{y}_r(k), 0 \leq k \leq N\}. \quad (11)$$

It is readily known that $\mathbf{y}_r(k)$ satisfies

$$\mathbf{y}_r(k) = \sum_{i=0}^{d} C_{ri}\mathbf{x}(k-l_i) + D_{fr}\mathbf{f}(k) + \mathbf{v}_r(k), \tag{12}$$

$$C_{ri} = \begin{bmatrix} C_i \\ 0 \end{bmatrix}, \qquad D_{fr} = \begin{bmatrix} D_f \\ I \end{bmatrix}, \qquad \mathbf{v}_r(k) = \begin{bmatrix} \mathbf{v}(k) \\ \mathbf{v}_f(k) \end{bmatrix}. \tag{13}$$

In the sequel, we denote the Krein-space projection of $\mathbf{m}(k)$ onto $\mathscr{L}\{\{\mathbf{y}_r(j)\}_{j=0}^{s}\}$ by $\widehat{\mathbf{m}}(k \mid s)$. Construct the innovations

$$\widetilde{\mathbf{y}}_r(k) = \mathbf{y}_r(k) - \widehat{\mathbf{y}}_r(k \mid k-1). \tag{14}$$

Defining $\widetilde{\mathbf{x}}(k \mid j) = \mathbf{x}(k) - \widehat{\mathbf{x}}(k \mid j)$, we further have

$$\widetilde{\mathbf{y}}_r(k) = \sum_{i=0}^{d} C_{ri}\widetilde{\mathbf{x}}(k-l_i \mid k-1) + D_{fr}\mathbf{f}(k) + \mathbf{v}_r(k). \tag{15}$$

Furthermore, we define the cross-covariance matrices of the state estimation error:

$$P(i,j,k) = E\left\{(\mathbf{x}(i) - \widehat{\mathbf{x}}(i \mid k))\left(\mathbf{x}(j) - \widehat{\mathbf{x}}(j \mid k)\right)^T\right\}. \tag{16}$$

By employing the Krein-space theory, a necessary and sufficient condition for the existence of the desired H_∞ fault estimator is given in the following.

Lemma 1. *Consider the stochastic systems* (7)–(9). *For a given* $\gamma > 0$, *a fault estimator* $\mathbf{r}(k)$ *achieving the performance* (3) *exists if and only if*

$$\Theta(k) = \sum_{i=0}^{d}\sum_{j=0}^{d} C_i P\left(k-l_i, k-l_j, k-1\right) C_j^T + I + D_f D_f^T > 0,$$
$$\Xi(k) = \left(1-\gamma^2\right) I - D_f^T \Theta^{-1}(k) D_f < 0. \tag{17}$$

Furthermore, if the above conditions are satisfied, the desired H_∞ *fault estimator is given by*

$$\mathbf{r}(k) = D_f^T \Theta^{-1}(k)\widetilde{\mathbf{y}}(k), \tag{18}$$

where $\widetilde{\mathbf{y}}(k) = \mathbf{y}(k) - \widehat{\mathbf{y}}(k \mid k-1)$, *and the minimum of the quadratic form* J_N *is*

$$\min J_N = \sum_{k=0}^{N} \widetilde{\mathbf{y}}^T(k)\Theta^{-1}(k)\widetilde{\mathbf{y}}(k). \tag{19}$$

Proof. It can be seen from (12) and (15) that

$$\begin{aligned} R_{\widetilde{\mathbf{y}}_r(k)} &= \langle \widetilde{\mathbf{y}}_r(k), \widetilde{\mathbf{y}}_r(k)\rangle \\ &= \sum_{i=0}^{d}\sum_{j=0}^{d} C_{ri} P\left(k-l_i, k-l_j, k-1\right) C_{ri}^T + D_{fr} D_{fr}^T - \gamma^2 I \\ &= \begin{bmatrix} \Theta(k) & D_f \\ D_f^T & \left(1-\gamma^2\right) I \end{bmatrix}. \end{aligned} \tag{20}$$

Equation (20) can be further written as

$$M(k) R_{\widetilde{\mathbf{y}}_r(k)} M^T(k) = \Lambda(k), \tag{21}$$

where

$$M(k) = \begin{bmatrix} I & 0 \\ -D_f^T \Theta^{-1}(k) & I \end{bmatrix}, \qquad \Lambda(k) = \begin{bmatrix} \Theta(k) & 0 \\ 0 & \Xi(k) \end{bmatrix}. \tag{22}$$

We can draw the conclusion from (17) and (21) that $R_{\widetilde{\mathbf{y}}_r(k)}$ and $R_{\mathbf{v}_r(k)}$ have the same inertia. Therefore following the same line as in [1], the minimum value of J_N can be obtained as follows:

$$\begin{aligned} \min J_N &= \sum_{k=0}^{N} \widetilde{\mathbf{y}}_r^T(k) R_{\widetilde{\mathbf{y}}_r(k)}^{-1} \widetilde{\mathbf{y}}_r(k) \\ &= \sum_{k=0}^{N} \left[\widetilde{\mathbf{y}}^T(k) \quad \mathbf{r}^T(k) - \widetilde{\mathbf{y}}^T(k)\Theta^{-1}(k) D_f\right] \\ &\quad \times \begin{bmatrix} \Theta^{-1}(k) & 0 \\ 0 & \Xi^{-1}(k) \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{y}}(k) \\ \mathbf{r}(k) - D_f^T \Theta^{-1}(k)\widetilde{\mathbf{y}}(k) \end{bmatrix}. \end{aligned} \tag{23}$$

Thus the rest of the proof is clear. □

In the following, we are devoted to the estimator design in terms of the solution to a partial difference Riccati equation.

Lemma 2. *The state estimate is recursively calculated as*

$$\begin{aligned} \widehat{\mathbf{x}}(k+1 \mid k) &= \sum_{i=0}^{d} A_i \widehat{\mathbf{x}}(k-h_i \mid k), \\ \widehat{\mathbf{x}}(k-j \mid k) &= \widehat{\mathbf{x}}(k-j \mid k-1) + K_j(k)\widetilde{\mathbf{y}}_r(k) \\ &= \widehat{\mathbf{x}}(k-j \mid k-1) + \sum_{i=0}^{d} P(k-j, k-h_i, k-1) C_i^T \\ &\quad \Theta^{-1}(k)\widetilde{\mathbf{y}}(k), \quad j = 0, \ldots, l, \end{aligned} \tag{24}$$

where the initial values are $\widehat{\mathbf{x}}(-j \mid -1) = 0$ *and* $K_j(k)$ *can be calculated by*

$$\begin{aligned} K_j(k) &= \sum_{i=0}^{d} P(k-j, k-h_i, k-1) C_{ri}^T R_{\widetilde{\mathbf{y}}_r(k)}^{-1}, \\ R_{\widetilde{\mathbf{y}}_r(k)} &= \begin{bmatrix} \Theta(k) & D_f \\ D_f^T & \left(1-\gamma^2\right) I \end{bmatrix}, \end{aligned} \tag{25}$$

while $P(.,.,.)$ is calculated by the partial difference Riccati equation as

$$\begin{aligned} &P(k-i,k-j,k) \\ &\quad = P(k-i,k-j,k-1) \\ &\qquad - K_i(k) R_{\tilde{\mathbf{y}}_r(k)} K_j^T(k), \quad 0 \le i \le j \le \tau, \end{aligned} \tag{26}$$

$$P(k+1,k-j,k) = \sum_{i=0}^{d} A_i P(k-h_i,k-j,k), \quad 0 \le j \le \tau, \tag{27}$$

$$\begin{aligned} &P(k+1,k+1,k) \\ &\quad = \sum_{i=0}^{d}\sum_{j=0}^{d} A_i P(k-h_i,k-h_j,k) A_j^T \\ &\qquad + B_d B_d^T + B_f B_f^T, \end{aligned} \tag{28}$$

$$P(k-i,k-j,k) = P^T(k-j,k-i,k), \tag{29}$$

$$P(-i,-j,-1) = P_0(-i,-j), \quad 0 \le i \le j, \ 0 \le j \le \tau. \tag{30}$$

Proof. Applying projection theory, we have

$$\hat{\mathbf{x}}(k+1 \mid k) = \sum_{i=0}^{d} A_i \hat{\mathbf{x}}(k-h_i \mid k), \tag{31}$$

$$\hat{\mathbf{x}}(k-j \mid k) = \hat{\mathbf{x}}(k-j \mid k-1) + K_j(k)\tilde{\mathbf{y}}_r(k), \tag{32}$$

where $K_j(k)$ is given as

$$K_j(k) = E\left\{\mathbf{x}(k-j)\tilde{\mathbf{y}}_r^T(k)\right\} R_{\tilde{\mathbf{y}}_r(k)}^{-1}. \tag{33}$$

Noting that $\mathbf{x}(k-j) = \hat{\mathbf{x}}(k-j \mid k-1) + \tilde{\mathbf{x}}(k-j \mid k-1)$, then based on (15) one has

$$\begin{aligned} &K_j(k) \\ &\quad = E\left\{\tilde{\mathbf{x}}(k-j \mid k-1)\tilde{\mathbf{y}}_r^T(k)\right\} R_{\tilde{\mathbf{y}}_r(k)}^{-1} \\ &\quad = E\left\{\tilde{\mathbf{x}}(k-j \mid k-1)\left[\sum_{i=0}^{d} C_{ri}\tilde{\mathbf{x}}(k-l_i \mid k-1)\right.\right. \\ &\qquad\qquad \left.\left. + D_{fr}\mathbf{f}(k) + \mathbf{v}_r(k)\right]^T\right\} R_{\tilde{\mathbf{y}}_r(k)}^{-1} \\ &\quad = \sum_{i=0}^{d} P(k-j,k-l_i,k-1) C_{ri}^T R_{\tilde{\mathbf{y}}_r(k)}^{-1}. \end{aligned} \tag{34}$$

Then one has

$$\begin{aligned} &\hat{\mathbf{x}}(k-j \mid k) \\ &\quad = \hat{\mathbf{x}}(k-j \mid k-1) + K_j(k)\tilde{\mathbf{y}}_r(k) \\ &\quad = \hat{\mathbf{x}}(k-j \mid k-1) \\ &\qquad + \sum_{i=0}^{d} P(k-j,k-l_i,k-1) C_{ri}^T R_{\tilde{\mathbf{y}}_r(k)}^{-1} \tilde{\mathbf{y}}_r(k) \\ &\quad = \hat{\mathbf{x}}(k-j \mid k-1) + \sum_{i=0}^{d} P(k-j,k-l_i,k-1) \\ &\qquad \times \left[C_i^T \ \ 0\right] M^T(k)\Lambda^{-1}(k) M(k)\tilde{\mathbf{y}}_r(k) \\ &\quad = \hat{\mathbf{x}}(k-j \mid k-1) \\ &\qquad + \sum_{i=0}^{d} P(k-j,k-l_i,k-1) C_i^T \Theta^{-1}(k)\tilde{\mathbf{y}}(k). \end{aligned} \tag{35}$$

Next it follows from (7) and (32) that

$$\begin{aligned} \tilde{\mathbf{x}}(k-j \mid k) &= \tilde{\mathbf{x}}(k-j \mid k-1) - K_j(k)\tilde{\mathbf{y}}_r(k), \\ \tilde{\mathbf{x}}(k-i \mid k) &= \tilde{\mathbf{x}}(k-i \mid k-1) - K_i(k)\tilde{\mathbf{y}}_r(k). \end{aligned} \tag{36}$$

Based on (36), we have the estimation error covariance matrices

$$\begin{aligned} &P(k-i,k-j,k) \\ &\quad = E\left\{\tilde{\mathbf{x}}(k-i \mid k)\tilde{\mathbf{x}}^T(k-j \mid k)\right\} \\ &\quad = E\left\{\tilde{\mathbf{x}}(k-i \mid k-1)\tilde{\mathbf{x}}^T(k-j \mid k-1)\right\} \\ &\qquad + E\left\{K_i(k)\tilde{\mathbf{y}}_r(k)\tilde{\mathbf{y}}_r^T(k)K_j^T(k)\right\} \\ &\qquad - E\left\{\tilde{\mathbf{x}}(k-i \mid k-1)\tilde{\mathbf{y}}_r^T(k)\right\} K_j^T(k) \\ &\qquad - K_i(k) E\left\{\tilde{\mathbf{y}}_r(k)\tilde{\mathbf{x}}^T(k-j \mid k-1)\right\} \\ &\quad = P(k-i,k-j,k-1) \\ &\qquad - K_i(k) R_{\tilde{\mathbf{y}}_r(k)} K_j^T(k), \end{aligned} \tag{37}$$

which is (26). Combining (7) and (31), we obtain

$$\tilde{\mathbf{x}}(k+1 \mid k) = \sum_{i=0}^{d} A_i \tilde{\mathbf{x}}(k-h_i \mid k) + B_d\mathbf{d}(k) + B_f\mathbf{f}(k). \tag{38}$$

Furthermore, one has

$$\begin{aligned}
&P(k+1,k-j,k)\\
&\quad= E\left\{\tilde{\mathbf{x}}(k+1\mid k)\tilde{\mathbf{x}}^T(k-j\mid k)\right\}\\
&\quad= E\left\{\left[\sum_{i=0}^{d}A_i\tilde{\mathbf{x}}(k-h_i\mid k)+B_d\mathbf{d}(k)\right.\right.\\
&\qquad\left.\left.+B_f\mathbf{f}(k)\right]\tilde{\mathbf{x}}^T(k-j\mid k)\right\}\\
&\quad= \sum_{i=0}^{d}A_iP(k-h_i,k-j,k),\\
&P(k+1,k+1,k)\\
&\quad= E\left\{\tilde{\mathbf{x}}(k+1\mid k)\tilde{\mathbf{x}}^T(k+1\mid k)\right\}\\
&\quad= E\left\{\left[\sum_{i=0}^{d}A_i\tilde{\mathbf{x}}(k-h_i\mid k)+B_d\mathbf{d}(k)\ +B_f\mathbf{f}(k)\right]\right.\\
&\qquad\times\left[\sum_{i=0}^{d}A_i\tilde{\mathbf{x}}(k-h_i\mid k)\right.\\
&\qquad\left.\left.+B_d\mathbf{d}(k)+B_f\mathbf{f}(k)\right]^T\right\}\\
&\quad= \sum_{i=0}^{d}\sum_{j=0}^{d}A_iP\left(k-h_i,k-h_j,k\right)A_j^T+B_dB_d^T+B_fB_f^T.
\end{aligned}\tag{39}$$

Finally (29) is straightforward by virtue of the definition of (16). Thus the proof is completed here. □

Theorem 3. *Consider the system* (1)-(2). *For a given $\gamma > 0$, a fault estimator $r(k)$ that achieves the performance index* (3) *exists if and only if $\Theta(k) > 0$ and $\Xi(k) < 0$, where $\Theta(k)$ and $\Xi(k)$ are defined in Lemma 1. In this case, one possible finite-time H_∞ fault estimator is given by*

$$\begin{aligned}
r(k)&= D_f^T\Theta^{-1}(k)\tilde{y}(k)\\
&= D_f^T\Theta^{-1}(k)\left[y(k)-\sum_{i=0}^{d}C_i\hat{x}(k-l_i\mid k-1)\right],
\end{aligned}\tag{40}$$

where

$$\begin{aligned}
&\hat{x}(k+1\mid k)=\sum_{i=0}^{d}A_i\hat{x}(k-h_i\mid k),\\
&\hat{x}(k-j\mid k)\\
&\quad=\hat{x}(k-j\mid k-1)\\
&\qquad+\sum_{i=0}^{d}P(k-j,k-l_i,k-1)C_i^T\Theta^{-1}(k)\tilde{y}(k),\\
&\qquad j=0,\ldots,l.
\end{aligned}\tag{41}$$

Proof. According to [1], we can see that the fault estimation problem addressed for the deterministic systems (1) with (2) and (5) is partially equivalent to that for the stochastic systems (7) with (8) and (9) in a Krein space, and therefore the proof is readly and we omitted here. □

Remark 4. In fact, the problem mentioned in this paper can be converted into the problem in [7, 8] by applying augmented approach. However, due to the existence of time delay, we need to solve a high dimension Riccati equation. Here the solutions to the fault estimator can be obtained by solving partial difference Riccati equations (26)–(28), which have the same dimension as the original system (1). Therefore solving a high dimension Riccati equation is avoided. Here we present a simple explanation. Because the multiplications and divisions cost much more in computation than additions, hence we only use the number of multiplications and divisions as the operation count. Denote C_{aug} and C_{new} as the numbers of multiplications and divisions for augmentation method and our proposed approach in one step, respectively. According to [18], one can see that the order of h_d in C_{aug} is 3, while the order of h_d in C_{new} is 2. Therefore if h_d is large enough, $C_{\text{aug}} \gg C_{\text{new}}$.

Remark 5. Recently, by applying Krein-space approach and reorganized innovation approach, [9] has considered the finite-horizon H_∞ fault estimation for linear discrete time-varying systems with two-channel single measurement delay. In this paper, we have investigated the finite-horizon problem H_∞ fault estimation for linear discrete systems with time delay, where the delay appears in both state and measurement, which contain [9] as a special case.

4. Numerical Example

Consider the linear discrete-time system:

$$\begin{aligned}
&x(k+1)\\
&\quad=\begin{bmatrix}0.3&0.5\\0&0.4\end{bmatrix}x(k)+\begin{bmatrix}0.2&0.1\\-0.05&0.2\end{bmatrix}x(k-1)\\
&\qquad+\begin{bmatrix}0.4&0.1\\-0.5&0.3\end{bmatrix}x(k-2)\\
&\qquad+\begin{bmatrix}0.5\\0.4\end{bmatrix}d(k)+\begin{bmatrix}1.2\\1.8\end{bmatrix}f(k),\\
&y(k)=\begin{bmatrix}-0.5&0.5\end{bmatrix}x(k)\\
&\qquad+\begin{bmatrix}0.5&0\end{bmatrix}x(k-1)+\begin{bmatrix}0.7&-0.3\end{bmatrix}x(k-2)\\
&\qquad+2.5f(k)+v(k).
\end{aligned}\tag{42}$$

The finite time horizon concerned here is $[0, 100]$. The driving disturbance and measurement noise are selected as $d(k) = 0.4\cos(k)$, $v(k) = 0.6\sin(k)$. The fault to be estimated is assumed to be

$$f(k)=\begin{cases}1, & 10\le k\ge 25,\ 50\le k\ge 70,\\ 0, & \text{others}.\end{cases}\tag{43}$$

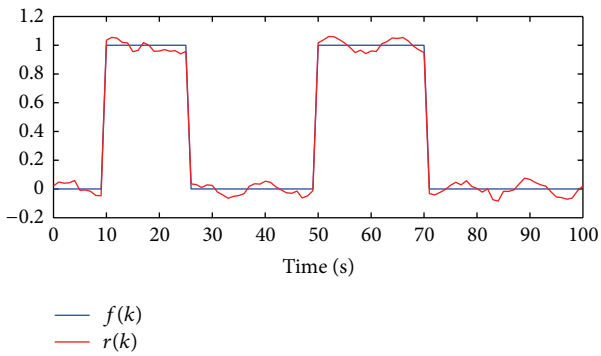

Figure 1: Fault and its estimation.

Set initial values as $x_0 = [1 \quad -0.5]^T$, $P(i, j, -1) = 0$, $-2 \le i \le -1$, $P(i, j, -1) = 0$, $-2 \le j \le -1$, $P(0, 0, -1) = I$. By using the result given in Theorem 3, the desired fault estimator is designed with $\gamma = 0.85$. Figure 1 shows the fault and its estimate, which confirm that the designed estimator performs very well.

5. Conclusion

The finite-time H_∞ fault estimation problem for linear time-delay systems has been investigated. The design of finite horizon H_∞ fault estimation has been converted into a minimum problem of certain quadratic form. Then an stochastic system in Krein space has been proposed, and a sufficient and necessary condition for the minimum has been derived by applying innovation analysis approach and projection theory. Finally a solution to the H_∞ fault estimation has been obtained by recursively computing a partial difference Riccati equation. Compared with the conventional augmented approach, the presented approach lessens the computational demand when the delay is large. In the further study, we will consider the H_∞ fault estimation problem for linear time-delay systems with multiplicative noise.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A New Quasi-Human Algorithm for Solving the Packing Problem of Unit Equilateral Triangles

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The packing problem of unit equilateral triangles not only has the theoretical significance but also offers broad prospects in material processing and network resource optimization. Because this problem is nondeterministic polynomial (NP) hard and has the feature of continuity, it is necessary to limit the placements of unit equilateral triangles before optimizing and obtaining approximate solution (e.g., the unit equilateral triangles are not allowed to be rotated). This paper adopts a new quasi-human strategy to study the packing problem of unit equilateral triangles. Some new concepts are put forward such as side-clinging action, and an approximation algorithm for solving the addressed problem is designed. Time complexity analysis and the calculation results indicate that the proposed method is a polynomial time algorithm, which provides the possibility to solve the packing problem of arbitrary triangles.

1. Introduction

The solution of NP hard problem has both popularity and intractability, which is of great value in philosophy of science and real life. Packing problem in two-dimension plane is a typical NP hard issue, which is about how to use the two-dimensional space efficiently. To date, research results show that a complete axiomatic approach is not currently viable. Packing problem of unit equilateral triangles is actually a special case of the two-dimension packing problem. Therefore, researches on the packing problem of unit equilateral triangles are theoretically significant to look for an efficient approximate algorithm for a NP hard issue, especially for a general packing problem. Moreover, researches on the rational distribution of a set of unit equilateral triangles, which do not overlap mutually in a limited region, are extremely useful in practical application. The packing problem discussed in this paper is one of the key issues in the field of CAD/CAM, the task of which is to design a high-performance algorithm to improve the quality of a layout scheme for the purposes of saving raw materials, shortening the construction period, reducing the costs, increasing productivity, and so forth.

A classification approach for packing problem has been brought forward by Dyckhoff in 1990 (see, e.g., [1]), and an improved classification method has been proposed by Wäscher et al. in 2007 (see, e.g., [2]). Wäscher et al. have classified the packing problem into four cases, namely, one-dimensional (1D), two-dimensional (2D), three-dimensional (3D), and high-dimensional (n-D) spaces. 1D packing problem only considers one factor, such as weight, volume, or length. 2D packing problem always considers two factors. Some common issues include geographic division of parking lots, trim of packing materials, and leathers. 3D packing problem considers three factors, usually the length, width, and height. For example, three dimensions should not go beyond the set bounds in ship or car loading process. n-D packing problem is always the optimal operation of rectangle packing problem in space and time dimension. Reference [3] can be deemed as a world's earlier result about n-D

packing problem, where the layout feature of rods with n-1 unit sides has been analyzed, but a specific algorithm has not been described. Subsequently, Fekete et al. have published a series of articles to discuss the general n-D packing problem (see, e.g., [4–6]). According to graph theory, a branch-and-bound framework and a tree searching algorithm have been provided based on all kinds of legitimate classifications. Typical examples regarding 1D and 2D spaces are presented, but the situations of 4D or more are not discussed. Moreover, the famous OR-Library and PackLib[2] provide only the 2D and 3D packing instances; see [7, 8], respectively.

The most common packing problems in daily life include cloth cutting and steel processing. Nowadays, the academia has conducted large number of valuable studies on the 2D and 3D packing problems; see, for example, [9, 10]. 2D packing problem mainly includes circles packing problem, rectangles packing problem, and triangles packing problem. Based on the population control (PERM) strategy and corner-occupying approach, a new hybrid algorithm is proposed to solve the problem of packing equal or unequal circles into a larger circle container in [11]. In [12], a novel computational approach is designed to place n identical nonoverlapping disks into a unit square, by which the radii are maximized. And based on the conjecture, a stochastic search algorithm that displays excellent numerical performance is developed. By elaborately simulating the movement of the smooth elastic disks in the container in the physical world, a heuristic quasiphysical strategy is provided in [13] for solving disks packing problem. Subsequently, based on the simulated annealing, that is, imitating the displacements of the objects under different temperature, the calculation speed is also improved. In [14], a coarse-to-fine quasiphysical optimization method is presented for solving the circle packing problem with equilibrium constraints, where the dense packing of n circular disks satisfying the equilibrium constraints is considered. 3D packing problem is confined primarily to a cuboid packing problem, which mainly includes the block arrangement method [15], spatial representation technique [16], genetic algorithm [17], dynamic space decomposition approach [18], and sequence triplet method [19].

In the past decade, the rectangles packing problem has been widely studied by researchers at home and abroad; for instance, a population heuristic is proposed in [20]. An effective deterministic heuristic, namely, the Less Flexibility First strategy, is studied in [21]. A hybrid heuristic algorithm is provided in [22], which is based on the divide-and-conquer and greedy strategies. Unfortunately, to the best of authors' knowledge, so far, the triangles packing problem and the convex polygons packing problem have not yet been discussed well. In [23], a preliminary study has been conducted on the general triangles packing problem, but further improvement and some indepth studies are required.

Therefore, the primary purpose of this paper is to study the packing problem of unit equilateral triangles according to the characteristic of the unit equilateral triangles and in the base of analysis of the general triangles packing problems. The main contribution of this paper can be listed as follows.

(i) A mathematical description of the packing problem of unit equilateral triangles is proposed, and the characteristic and advantage of which are colloquially stated.

(ii) A reasonable quasi-human strategy is formed according to the natural law of the "like attracts like."

(iii) A new algorithm for solving the packing problem of unit equilateral triangles is presented based on the proposed quasi-human strategy.

2. Problem Formulation

Packing problem of unit equilateral triangles: given one square container whose side is L, if N unit equilateral triangles can be put into the given container, then develop a concrete algorithm accordingly or else provide an opposite answer. Actually, the problem can be formally described as follows. For any given positive integer N and one square container with every side of L, let S_{ij} be the overlap area of the ith triangle ($\triangle_i$) and the jth triangle ($\triangle_j$). Does there exist the following array of $6N$ real numbers? Consider

$$A = \{x_{11}, y_{11}, x_{12}, y_{12}, x_{13}, y_{13}, x_{21}, y_{21}, x_{22}, y_{22}, x_{23}, y_{23}, \ldots, x_{N1}, y_{N1}, x_{N2}, y_{N2}, x_{N3}, y_{N3}\} \tag{1}$$

such that

$$\begin{aligned}
&\text{(i)} \quad (x_{i1} - x_{i2})^2 + (y_{i1} - y_{i2})^2 \\
&\qquad = (x_{i2} - x_{i3})^2 + (y_{i2} - y_{i3})^2 \\
&\qquad = (x_{i3} - x_{i1})^2 + (y_{i3} - y_{i1})^2 \\
&\qquad = 1; \qquad 0 \le xi1, xi2, xi3, \ \ yi1, yi2, yi3 \le L; \\
&\qquad\qquad \forall i \in 1, 2, \ldots, N, \\
&\text{(ii)} \quad \sum_{i<j} S_{ij} = 0; \quad \forall i, j \in 1, 2, \ldots, N.
\end{aligned} \tag{2}$$

If there exists one which meets (i) and (ii), then provide the corresponding solution A.

Remark 1. It is important to note that if the given L is sufficiently large, it is very easy to find the solution constrained by (i) and (ii) to the packing problem of unit equilateral triangles. However, it is not an easy task to put the unit equilateral triangles into the square container quickly, when the given L is relatively small. On the other hand, the packing problem of unit equilateral triangles can be described as follows. Given N unit equilateral triangles, if these triangles can be put into a square container without overlap, then how long should the sides of the square container be at least?

Since figures in a plane can be translated and rotated continuously, there always is an infinite number of ways of layout scheme. Therefore, it is always to limit the placements of the filler when solving the packing problem quickly. The restriction policy in [1] only allows the polygons to translate

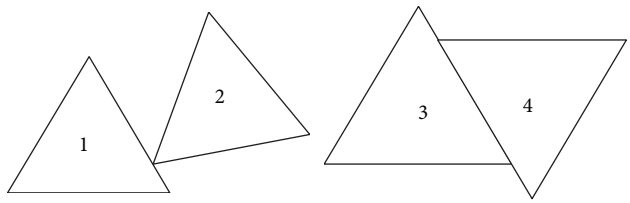

Figure 1: Tangent triangles.

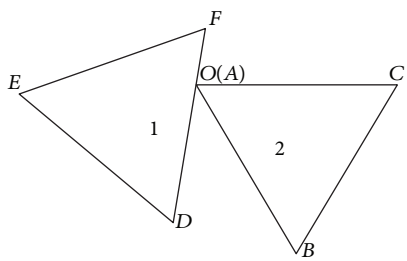

Figure 2: Angle region.

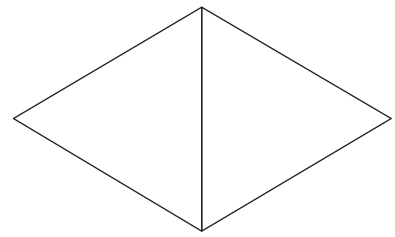

Figure 3: Zero angle regions.

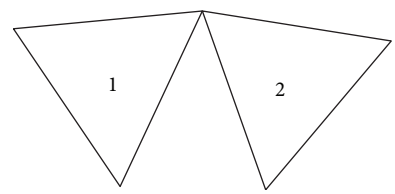

Figure 4: One angle region.

but not to rotate; thus, the polygons packing problem can be converted into a NP complete problem.

Definition 2 (tangency). For two unit equilateral triangles in a same plane, if they intersect each other, but the overlapping area is zero, then the two triangles are said to be tangent. As shown in Figure 1, $\triangle_1$ and $\triangle_2$ are tangent to with each other, and $\triangle_3$ and $\triangle_4$ are also tangent.

Definition 3 (angle region). Given two tangent unit equilateral triangles, for any two sides which are taken from the two triangles, respectively, if all items below are matched:

(1) there is one and only one point of intersection between the two sides, which is denoted by O;

(2) the angle (let O be the vertex) formed by the two sides is positive, but less than π;

(3) in the angle of (2), there is on side of the two unit equilateral triangles, except these in (1),

then the angle in (2) is said to be angle region of the two tangent triangles. Its size is called the angle of the angle region. The initial side of the angle in (2) is called the initial side of the angle region, and the terminal side of the angle in (2) is the terminal side of the angle region. The intersection in (1) is called the vertex of the angle region.

As shown in Figure 2, $\triangle_{ABC}$ is tangent to $\triangle_{DEF}$ at A; $\angle_{BAD}$ is an angle region of $\triangle_{ABC}$ and $\triangle_{DEF}$. Size of the angle region is denoted by $\angle_{BAD}$. AB and AD are the initial and terminal sides of the angle region, respectively.

To judge whether there is an angle region formed by two tangent unit equilateral triangles, it is just to take out two sides with public vertex from each unit equilateral triangle and judge whether the angle formed by the two sides is positive and less than π and verify that there is no other side of the two unit equilateral triangles with the angle. The computational procedure can be realized according to vertex coordinate of the triangle.

3. Classification of an Angle Region and Side-Clinging Action

When a unit equilateral triangle is tangent to another unit equilateral triangle, zero, one, or two angle regions may be formed, which is shown in Figures 3, 4, and 5(a)–5(c), respectively.

Based on the long-term production practice of the human society, in order to put some fillers into a given container as much as possible, one always chooses some appropriate fillers which matches the size of the free space in the container first. Then, put these fillers into some relatively stable positions. This so-called stable position cannot be rotated or slide freely.

For a unit equilateral triangle, if it can be rotated clockwise or counterclockwise around one of its vertexes, then the unit equilateral triangle is said to be freely rotated or it is said to be restrictively rotated. Let one side of the unit equilateral triangle cling to a side of another; namely, the length of the overlapping part of the two unit equilateral triangles is greater than zero. If the triangle can move along the clinging side in one direction at most, then the unit equilateral triangle is said to be restrictively sliding or it is said to be freely sliding. One point should be noted that unit equilateral triangles cannot intersect others and must always be within the square container.

If a side of the unit equilateral triangle $\triangle_1$ clings to a side of others, then the triangle can only slide along the clinging side, and the position of $\triangle_1$ is said to be a slidable position. If $\triangle_1$ slides along the clinging side to cling to a third unit equilateral triangle $\triangle_3$, where $\triangle_1$ still clings to $\triangle_2$, then the position of $\triangle_1$ is stable. If a unit equilateral triangle can slide or be rotated freely, then it is said to be impending. Obviously, the position of this unit equilateral triangle at the moment is extremely unstable, as $\triangle_1$ is shown in Figure 6. If a unit equilateral triangle is restrictively sliding but freely

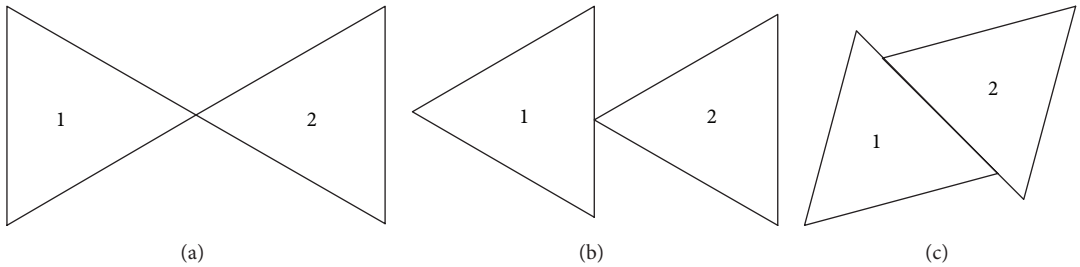

Figure 5: Two angle regions.

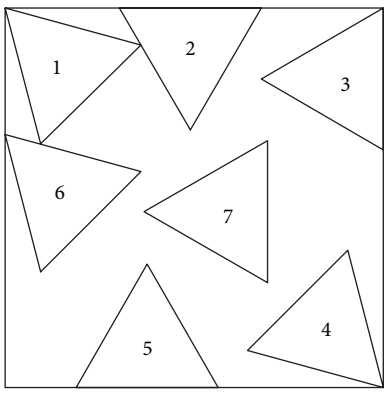

Figure 6: Stability of unit equilateral triangles.

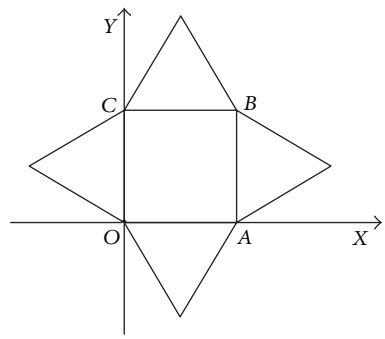

Figure 7: The square container.

rotated, then the position of which is actually unstable, as $\triangle_2$ is shown in Figure 6. Moreover, if a unit equilateral triangle is restrictively rotated but freely sliding, the position of which is still unstable, as $\triangle_3$ (which can move left and right), $\triangle_4$, and $\triangle_5$ (which do not overlap with side of any triangle) are shown in Figure 6. Generally speaking, the placement of a unit equilateral triangle is stable, only if it is restrictively rotated and restrictively sliding, as $\triangle_6$ and $\triangle_7$ are shown in Figure 6.

In general, the more stable the position of a unit equilateral triangle is, the closer it will cling to other triangles and therefore the higher the space utilization will be.

Definition 4 (side-clinging action). If one side of a unit equilateral triangle clings to an initial or terminal side of an angle region (the overlapping part is greater than zero) and another side of the unit equilateral triangle is tangent to terminal or initial side of the angle region, then the process of putting the triangle into the angle region is said to be a side-clinging action.

One can find a side-clinging action in this way. Let the side a of a unit equilateral triangle cling to the initial side BA or the terminal side DA of the angle region $\angle_{BAD}$ and push the side a along $\overrightarrow{BA}$ or $\overrightarrow{DA}$ until it cannot move (the triangle is tangent to initial or terminal side of the angle region). If the side a also clings to the initial side BA or terminal side DA of $\angle_{BAD}$, then fix the unit equilateral triangle to the position where it is. Therefore, the problem of putting a unit equilateral triangle into an angle region formed by two given tangent unit equilateral triangles is converted into the positional relationship between a side of the unit equilateral triangle and a line segment; thus, the search space is limited to just a few points from continuous Euclidean space.

Definition 5 (side-clinging degree). If one side of a unit equilateral triangle clings to one side of an angle region (the length is a_1), set the length of overlapping part as b_1, then the side-clinging degree of the side-clinging action is $b_1/\max(1, a_1)$. Specifically, if the other side of the unit equilateral triangle clings to another side (the length is a_2) of the angle region, set the length of overlapping part as b_2, then the side-clinging degree of the side-clinging action is $b_1/\max(1, a_1) + b_2/\max(1, a_2)$.

Remark 6. Actually, side-clinging degree can be used to measure how well a unit equilateral triangle fits the position which is proposed to be filled. The bigger the side-clinging degree is, the better the triangle will fit.

4. The Side-Clinging Algorithm

As shown in Figure 7, the square container $OABC$ can be deemed as a plane figure formed by four equilateral triangles.

Definition 7 (pattern). A pattern refers to a kind of ordered pair $\langle P, R\rangle$ at some points, where P is the set of the four triangles constituting the square container and unit equilateral triangles which has been put in the container (each element

in the set is denoted by three vertex coordinates of a triangle) and R is the set of angle regions formed by triangles in P.

At the first moment, let $N > 0$; and there are four triangles constituting the square container in P; R is the set of angle regions formed by four triangles constituting the square container. Then, initial pattern at this moment is denoted by $\langle P_0, R_0 \rangle$. Putting the kth $(0 < k \leq N)$ triangle into the square container after the initial pattern, then the corresponding pattern is called the kth pattern, which is denoted by $\langle P_k, R_k \rangle$. In the kth pattern, let a triangle perform a side-clinging action; if the triangle is still in the square container and does not intersect any triangle in P_k, then the side-clinging action is considered to be an appropriate side-clinging action.

Side-Clinging Strategy. Set the side-clinging degree of two appropriate side-clinging actions (a_1 and a_2) as q_1 and q_2, respectively. If $q_1 > q_2$, then priority of the side-clinging action a_1 is higher than a_2.

Based on the side-clinging strategy, the quasi-human algorithm for solving a given packing problem of unit equilateral triangles can be presented as follows.

Given an initial pattern $\langle P_0, R_0 \rangle$, where P_0 is the set of triangles constituting a given square container, R_0 is the set of angle regions formed by four triangles constituting a given square container. Arrange angle regions and appropriate side-clinging actions in the pattern according to time sequences.

Step 1. Set $t = 0, S = S_t$, and the number of the unit equilateral triangles to be put into the container as $N - t$.

Step 2. In the pattern S_t, if $N - t = 0$, then print the information indicating that all the triangles have been put into the square container and stop; if $N - t \neq 0$ and there is no appropriate side-clinging action to be performed, then print that it is failure to obtain the solution and stop; if $N - t \neq 0$ and there are some appropriate side-clinging actions to be performed in S_t, then go to Step 3.

Step 3. Perform the highest-priority side-clinging action according to the side-clinging strategy. If there is only one appropriate side-clinging action based on the side-clinging strategy, then put a unit equilateral triangle into the container according to the side-clinging action or according to the first corner-occupying action.

Step 4. Set $t \leq t + 1$ and add the three-vertex coordinates of the triangle in Step 3 to P_t. Accordingly, R_t represents the set of angle regions formed by triangles in P_t; then, go to Step 2.

5. Time Complexity Analysis of the Side-Clinging Algorithm

In general, one can assume that there are $n-1$ unit equilateral triangles that have been put into the container at the pattern $\langle P, R \rangle$; namely, there are $n + 3$ triangles in P, and it will take $f(n)$ time units to find and perform the highest-priority side-clinging action with respect to the nth unit equilateral triangle. Sequence of the implementation is as follows.

(1) Time Complexity of Forming Angle Regions by $n + 3$ Triangles in P. Given triangles, if it will take a time units to judge whether the two sides taken from each triangle can form an angle region, then it will take $C(3,1) \times C(3,1) \times a = 9a$ time units at most to find the angle region formed by the two triangles. Besides, it is easy to know that it will take $C(n + 3, 2) \times 9a$ time units at most to find all the possible angle regions formed by $n+3$ triangles in P. According to the definition of angle region, two triangles can form two angle regions at most. Therefore, there will be up to $2 \times C(n + 3, 2)$ angle regions in R.

Remark 8. In fact, it is impossible that a unit equilateral triangle is tangent to each triangle in the square container. Generally speaking, the number of angle regions formed by $n+3$ triangles is far less than $2 \times C(n+3, 2)$. And this is exactly the essential reason for the fast property of the proposed algorithm.

(2) Time Complexity of All the Side-Clinging Actions for Putting the nth Unit Equilateral Triangle into All the Angle Regions in R. Assume that it will take b time units to put a unit equilateral triangle into an angle region, with one side of the unit equilateral triangle clinging to one side of the angle region. Because there are only two different side-clinging actions to be performed while putting a unit equilateral triangle into a given angle region, it will take $2b$ time units at most. One can easily find that it will take up to $C(n+3, 2) \times 4b$ time units to put a unit equilateral triangle into all angle regions in R.

(3) Time Complexity of Judging Whether Every Side-Clinging Action Is an Appropriate Side-Clinging Action. If it will take c_1 time units to judge whether two triangles can intersect each other and c_2 time units to judge whether a unit equilateral triangle is in the square container, then it will take $C(n+3, 2) \times 4((n-1) \times c_1 + c_2)$ time units to find all appropriate side-clinging actions.

(4) Time Complexity of Finding the Highest-Priority Side-Clinging Action from All the Appropriate Side-Clinging Actions. If it will take d time units to compare the priority of two side-clinging actions, then it will take a maximum of $C(n + 3, 2) \times 4d$ time units to find the highest-priority side-clinging action.

Based on the above analysis, one can know that the maximum time units to put the nth unit equilateral triangle into a square container should be $C(n + 3, 2) \times 9a + C(n + 3, 2) \times 4b + C(n+3, 2) \times 4((n-1) \times c_1 + c_2) + C(n+3, 2) \times 4d = C(n + 3, 2)(9a + 4b + 4((n - 1) \times c_1 + c_2) + 4d)$, where a, b, c_1, c_2, and d are constants. Thus, the time complexity of putting the nth triangle is $O(n^3)$, and the time complexity of putting n triangles will be no more than $1 + 2^3 + 3^3 + \cdots + n^3 = (n(n + 1)/2)^2$, which implies that the time complexity of the designed algorithm is $O(n^4)$. If there is no reasonable side-clinging action to perform, then stop. Thus, the order of magnitude of the total computing time is not more than $O(n^4)$.

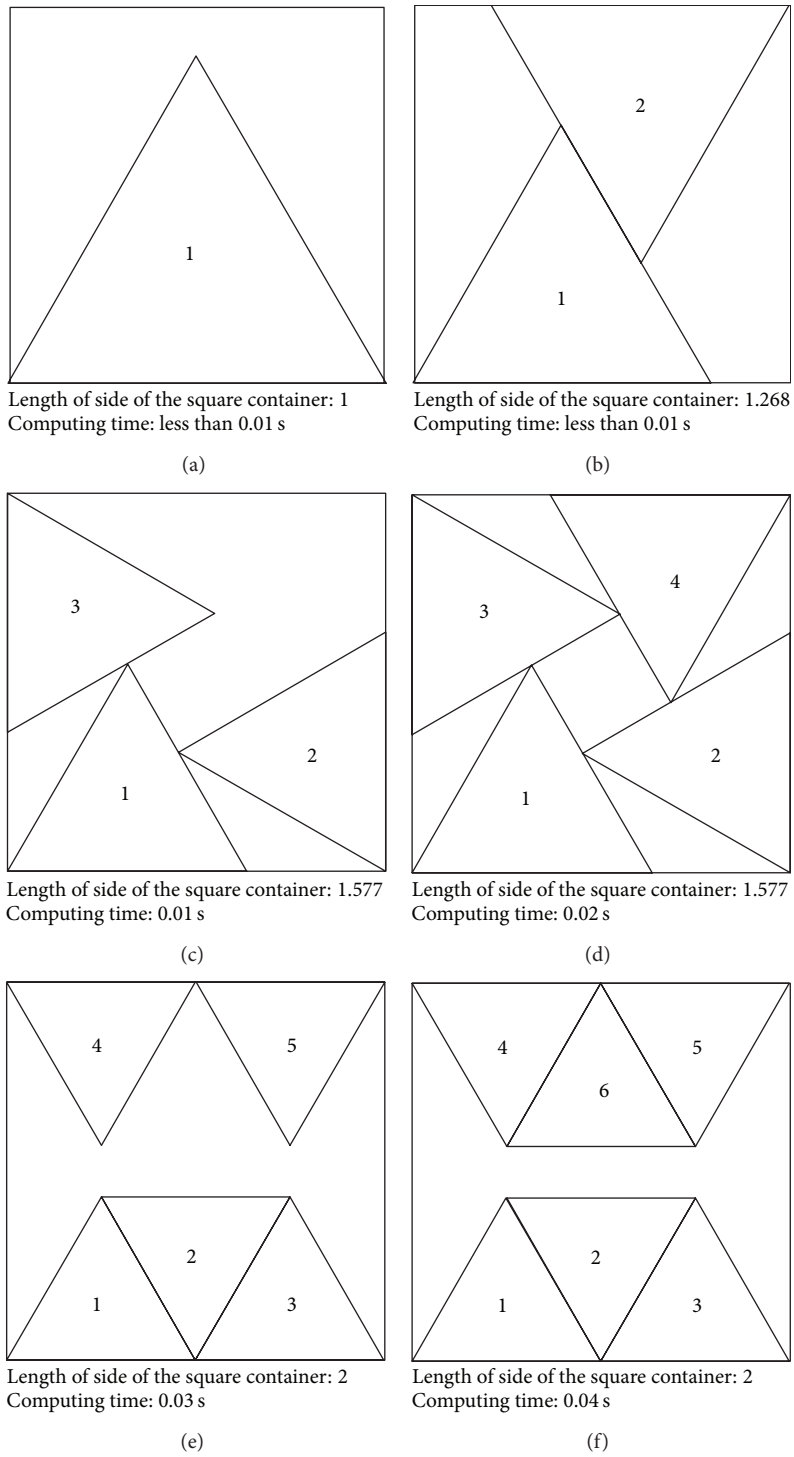

Figure 8: Outputs of simulation examples.

In the process of solving the packing problem of unit equilateral triangles, we find that some similar packing problems can be solved by experience accumulated in long-term practice. For example, the triangles packing problem is the same as with bricklayers: how to place irregular stones to make the most of the limited space? Based on past experience, they always occupy corners first, then sides and centers. In fact, this idea just provides an arrangement sequence. Inspired by which, we only consider the properties of side-clinging actions for unit equilateral triangles, which can guarantee the compactness of placement method. According to the proposed side-clinging algorithm, an approximation algorithm for solving packing problem of general polygon can be developed in the future.

6. Illustrative Examples

This program is developed by using Visual C++6.0. The simulation example is tested on a computer with Pentium 2.8 GHz processor and 1 G of RAM.

Inputs:

Length of the side of the square container (±0.1);

Number of unit equilateral triangles to be put (±0.000001).

Outputs:

Utilization (0.1%);

Computing time (±0.01 sec);

Sequence of putting unit equilateral triangles into a square container.

The simulation results are shown in Figure 8, from which we can know that the computing speed of the suggested quasi-human algorithm is polynomial time. However, the size of the square container for placing three unit equilateral triangles is the same as the case of four in some individual examples, indicating that space utilization of the container has yet to be improved. In future research, we will explore how to adjust and improve the quasi-human algorithm to increase the space utilization.

7. Conclusions and Future Work

In this note, a novel quasi-human algorithm for solving the packing problem of unit equilateral triangles has been proposed. We have categorized angle regions of two tangent triangles based on their position relations. Some concepts and terms have been defined, such as tangency, angle region, and side-clinging action, and the position stability of triangles has been analyzed. According to the side-clinging strategy, an effective quasi-human algorithm has been developed to solve the packing problem of unit equilateral triangles. On the basis of the simulation results, we can find that there is always only one angle region between two successive unit equilateral triangles, and the mount of angle regions can be relatively stable. Thus, search scope of available space for the subsequent triangles can be reduced, which is the root cause of the lower complexity of the suggested quasi-human algorithm.

Although the computational results are already very satisfactory, there are still many possible ways that may further improve the proposed quasi-human algorithm, such as the sound fuzzy method; see [24, 25] and references therein. As is well known, networked systems (NSs), complex networks, sensor networks, and multiagent systems are some important systems in daily life, and researches on which have become increasingly active in recent years, primarily due to their wide applications in many fields; see, for example, [26–30]. Naturally, how to apply our quasi-human algorithm to optimize resources in different systems is still a thoughtful issue. For another, disturbance (such as white noise and periodic narrowband noise) and incomplete information (such as data-packet dropouts and missing measurements) inevitably exist in many kinds of systems and networks; see, for example, [31–35], which will undoubtedly influence the feasibility of the suggested algorithm. Therefore, accuracy and robustness of the designed quasi-human algorithm in complicated background are a promising and valuable research direction in the future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Partial Synchronizability Characterized by Principal Quasi-Submatrices Corresponding to Clusters

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A partial synchronization problem in an oscillator network is considered. The concept on a principal quasi-submatrix corresponding to the topology of a cluster is proposed for the first time to study partial synchronization. It is shown that partial synchronization can be realized under the condition depending on the principal quasi-submatrix, but not distinctly depending on the intercluster couplings. Obviously, the dimension of any principal quasi-submatrix is usually far less than the one of the network topology matrix. Therefore, our criterion provides us a novel index of partial synchronizability, which reduces the network size when the network is composed of a great mount of nodes. Numerical simulations are carried out to confirm the validity of the method.

1. Introduction

Since the pioneering work of Pecora and Carroll [1], extensive researches on chaos synchronization have been carried out due to their potential applications in various disciplines such as physics, engineering, and biology [2–5]. For example, wireless sensor networks are widely researched due to the important applications in the real scenarios. One of the biggest challenges in investigating wireless sensor networks is how to obtain higher synchronization accuracy with minimal overhead [6]. And many other network-induced phenomena have also been discussed extensively in engineering [7].

Up to date, there have been many synchronization types being proposed and discussed, including complete synchronization [8, 9], partial synchronization [10], and inner and outer synchronization [11]. Generally speaking, there exists an intimate relationship between the phenomena of synchronization and invariant manifolds of coupled systems [9–12]. Thus, when we carry out researches on partial synchronization of coupled systems, it is always supposed that the corresponding full or partial synchronization manifolds are invariant manifolds. The established tools for the stability of invariant synchronization manifolds mainly consist of local linearization method and Lyapunov function method. One of the prominent results of the former is the master stability function method [9]. The method has obtained two significant factors determining the local stability of the synchronous state, that is, the maximum Lyapunov exponent of the node dynamics and the eigenvalues of the topology matrix. One of the prominent results of the latter is the study of the global attractiveness of the synchronization manifold. A crucial requirement for the method is the condition of the node dynamics satisfying $f \in \mathrm{QUAD}(\Delta, P, \Omega)$ [10]. In some sense, the condition means that the system can synchronize when the coupling is made sufficiently large. Recently, some new conditions have been obtained for synchronization of networks without Lipschitz condition or QUAD condition [12].

The type of synchronization concerned in this paper is partial synchronization. It means that the coupled oscillators split into subgroups called clusters, and all the oscillators in the same cluster behave in the same fashion. Many relative

studies have been carried out [13–18]. By using pinning control strategy, partial synchronization of coupled stochastic delayed neural networks was discussed in [14]. Similar control strategy is also proposed to select controlled communities by analyzing the information of each community such as in-degrees and out-degrees [15]. Afterwards, some simple intermittent pinning controls and centralized adaptive intermittent controls are proposed [16]. However, a suitable control law must be presented in order to use pinning control strategy. Some other researches focused on partial synchronization induced by the coupling configuration. An arbitrarily selected partial synchronization manifold was constructed for a network with cooperative and competitive couplings [17]. Recently, a sufficient and necessary condition for partial synchronization manifolds being invariant manifolds was obtained in networks coupled linearly and symmetrically [10]. More significantly, some sufficient conditions for the global attractiveness of the partial synchronization manifold were derived by decomposing the whole space into a direct sum of the synchronization manifold and the transverse space. The results are meaningful and interesting. Based on the method in [10], partial synchronization bifurcations [19] were analyzed for a globally coupled network with a parameter, which topology is not complex. Nevertheless, all the eigenvalues of the topology matrix are essential for that method, which needs a large quantity of computation when the network size is very large.

In this paper, a novel criterion on partial synchronization is proposed through the analysis of principal quasi-submatrices corresponding to the clusters. Previous researches have obtained several criteria on partial synchronization [10]. However, these criteria depend heavily on the topology matrix of the whole network. For many complex networks in real world, it is tedious to obtain the eigenvalues and eigenvectors of the topology matrix of the whole network. Therefore, this paper aims to propose a novel criterion, which is not distinctly dependent on the intercluster couplings and the topology matrix of the whole network. Instead, it is sufficient for partial synchronization to verify that the conditions on the intracluster couplings are satisfied. Therefore, it will be advantageous to discuss partial synchronization in the complex network with a large number of nodes because the dimension of any principal quasi-submatrix is usually far less than the one of the network topology matrix. At first, the tedious work of solving the eigenvectors corresponding to the second-largest eigenvalue is avoided. Secondly, partial synchronization is studied by the inner topologies of the individual clusters. Obviously, the network size reduction provides convenience for the study of partial synchronization in networks with great mounts of oscillators. In order to confirm its effectiveness, some numerical simulations are carried out to study partial synchronization in a star-global network. The numerical simulations are in good agreement with the theoretical analysis.

The present paper is built up as follows. Some concerned concepts and a lemma for the invariance of the partial synchronization manifold are introduced in Section 2. Principal quasi-submatrices corresponding to the clusters are proposed, and our main results are then established in Section 3. Numerical examples are presented to confirm the effectiveness of the results in Section 4. Finally, a brief discussion of the obtained results is given in Section 5.

2. Preliminaries

In this section, we introduce some basic concepts of invariant synchronization manifolds and a related lemma, which are required throughout the paper.

In past years, many studies [9, 10, 12] of synchronization phenomena focused on oscillator networks coupled linearly and symmetrically. The system can be described by the following ordinary differential equations:

$$\dot{x}_i = f(x_i, t) + \varepsilon \sum_{j=1}^{m} a_{ij} \Gamma x_j, \quad i = 1, \ldots, m, \tag{1}$$

where $x_i = (x_i^1, \ldots, x_i^n)^\top$ is the n-dimensional state variable of the ith oscillator, $m > 1$ is the network size, $t \in [0, +\infty)$ is a continuous time, $f : R^n \times [0, +\infty) \to R^n$ is a continuous map, $\varepsilon > 0$ is the coupling strength, and $\Gamma = \mathrm{diag}\{\gamma_1, \ldots, \gamma_n\}$ is a nonnegative matrix determining the interaction of variables. The coupling weight matrix $A = (a_{ij})_{m\times m}$ is assumed to satisfy that $a_{ij} = a_{ji} \geq 0$, for $i \neq j$, and $\sum_{j=1}^{m} a_{ij} = s$ for $i = 1, \ldots, m$.

In order to study partial synchronization of the system (1), the set of nodes $\{1, \ldots, m\}$ is divided into d nonempty subsets (clusters). Let $G = \{G^1, \ldots, G^d\}$ be the partition, and denote $K = (K_1, K_2, \ldots, K_d)$, where $K_k \geq 1$ is the cardinal number of the cluster G^k, $k = 1, \ldots, d$. Without loss of generality, suppose that $G^1 = \{1, \ldots, K_1\}, \ldots, G^d = \{\sum_{p=1}^{d-1} K_p + 1, \ldots, m\}$. Based on the partition G, we rewrite the coupling matrix A as a block matrix,

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1d} \\ A_{21} & A_{22} & \cdots & A_{2d} \\ \cdots & \cdots & \cdots & \cdots \\ A_{d1} & A_{d2} & \cdots & A_{dd} \end{bmatrix}, \tag{2}$$

where $A_{kl} \in R^{K_k \times K_l}, k, l = 1, \ldots, d$.

We will discuss sufficient conditions for the K_k nodes in the cluster G^k to synchronize with each other, $k = 1, \ldots, d$. Before that, the concepts of partial synchronization manifolds and transverse spaces introduced in the following [10]:

$$\mathbb{M}_K = \left\{ (x_1^\top, \ldots, x_m^\top)^\top \mid \left(x^\top_{\sum_{p=0}^{k-1} K_p + 1}, \ldots, x^\top_{\sum_{p=0}^{k} K_p} \right)^\top \in \mathbb{M}_K^k, \right.$$
$$\left. k = 1, \ldots, d \right\}, \tag{3}$$

where $K_0 = 0$,

$$\mathbb{M}_K^k = \left\{ \left(x^\top_{\sum_{p=0}^{k-1} K_p + 1}, \ldots, x^\top_{\sum_{p=0}^{k} K_p} \right)^\top \right.$$
$$\left. \in R^{nK_k} \mid x_{\sum_{p=0}^{k-1} K_p + 1} = \cdots = x_{\sum_{p=0}^{k} K_p} \right\}, \tag{4}$$

are called a partial synchronization manifold. We also call

$$\mathbb{L}_K = \left\{ \left(x_1^\top, \ldots, x_m^\top\right)^\top \mid \left(x^\top_{\sum_{p=0}^{k-1} K_p+1}, \ldots, x^\top_{\sum_{p=0}^{k} K_p}\right)^\top \in \mathbb{L}_K^k, \; k = 1, \ldots, d \right\}, \tag{5}$$

where

$$\mathbb{L}_K^k = \left\{ \left(x^\top_{\sum_{p=0}^{k-1} K_p+1}, \ldots, x^\top_{\sum_{p=0}^{k} K_p}\right)^\top \in R^{nK_k} \mid \sum_{i=1}^{K_k} x_{\sum_{p=0}^{k-1} K_p+i} = 0 \right\}, \tag{6}$$

a transverse space for $\mathbb{M}_K$.

In case $n = 1$, the four sets mentioned above are denoted by M_K, M_K^k, L_K, and L_K^k, respectively. In case $d = 1$, the synchronization manifold $\mathbb{M}_K$ is called a full synchronization manifold. For simplicity, we denote the full synchronization manifold $\mathbb{M}_K$ by $\mathbb{M}$, and the corresponding transverse space $\mathbb{L}_K$ by $\mathbb{L}$. Sometimes, the manifold $\mathbb{M}_K$ is also denoted by $\mathbb{M}_K(G)$ to emphasize its partition G.

Definition 1. The synchronization manifold $\mathbb{M}_K$ is said to be globally attractive for the system (1), or the system (1) is said to realize partial synchronization with the partition $G = \{G^1, G^2, \ldots, G^d\}$ if, for any initial condition $(x_1^\top(0), x_2^\top(0), \ldots, x_m^\top(0))^\top \in R^{mn}$,

$$\lim_{t \to +\infty} \sum_{k=1}^{d} \sum_{i \in G^k} \left\| x_i(t) - x_{\sum_{p=0}^{k-1} K_p+1}(t) \right\| = 0, \tag{7}$$

where $\|\cdot\|$ denotes 2-norm of vectors. In case of $d = 1$, the system (1) is said to realize full synchronization, if the full synchronization manifold $\mathbb{M}$ is globally attractive.

Synchronization manifolds are always supposed to be invariant in order to discuss the attractiveness. Therefore, it is necessary to recall the definition of an invariant manifold [20] for the ordinary differential equations,

$$\dot{x} = X(x, t), \quad x \in R^N, \; X : R^N \times [0, +\infty) \longrightarrow R^N. \tag{8}$$

Denote $\mathscr{M}$ as the manifold of codimension p defined by a vector equation $H(x) = 0$, $H = (h_1, h_2, \ldots, h_p)$, $1 \le p \le N-1$. $\mathscr{M}$ is called an invariant manifold of the system (8) if

$$\left(\operatorname{grad} H \cdot X\right)|_{H=0} \equiv 0, \tag{9}$$

which implies that the vector field (8) is tangent to $\mathscr{M}$. For more details of the existence of invariant manifolds, one can refer to the papers by Golubitsky and coworkers [21, 22]. The following lemma gives a sufficient and necessary condition for a partial synchronization manifold being an invariant manifold.

Lemma 2 (see [10]). *Let $K = (K_1, K_2, \ldots, K_d)$. The synchronization manifold $\mathbb{M}_K$ is an invariant manifold of the system* (1) *if and only if the coupling matrix A has the form* (2) *with every A_{ij} being equal row sum.*

Remark 3. Based on Lemma 2, we can find all invariant synchronization manifolds for a given coupling matrix. As we know, each diagonal block reveals the intracluster information communication, and each nondiagonal block represents the information communication among different clusters. By the condition that every A_{ij} is equal row sum, we mean that every node in the same cluster receive an equal amount of information communication from every other cluster.

3. Main Results

Noticing the sufficient and necessary condition in Lemma 2, we assume that every submatrix $A_{kl} \in R^{K_k \times K_l}$ is an equal row sum matrix with row sum s_{kl}, $k, l = 1, \ldots, d$. Since A^s is an equal row sum matrix, that is, $\sum_{j=1}^{m} a_{ij} = \sum_{k=1}^{d} s_{\hat{i}k} = s$, where the denotation $\hat{i}$ represents k , for all $i \in G^k$, we define the matrices $\overline{A}_{kk} = (\overline{a}_{ij})_{K_k \times K_k}$ as

$$\overline{A}_{kk} = A_{kk} + (s - s_{kk}) E_{K_k} = A_{kk} + \sum_{l=1, l \ne k}^{d} s_{kl} E_{K_k}, \tag{10}$$

where $E_{K_k} \in R^{K_k \times K_k}$ is the identity matrix, $k = 1, \ldots, d$. It is easy to conclude from (10) that $\sum_{j \in G^{\hat{i}}} \overline{a}_{ij} = s$ and

$$\overline{a}_{ij} = \begin{cases} a_{ij}, & i \ne j; \\ a_{ij} + \displaystyle\sum_{l=1, l \ne \hat{i}}^{d} s_{\hat{i}l}, & i = j. \end{cases} \tag{11}$$

Noticing that A_{kk} is a principal submatrix of A, we call the matrix $\overline{A}_{kk}$ a principal quasi-submatrix corresponding to the cluster G^k, $k = 1, \ldots, d$.

Now, we are now in a position to carry out the following theorem with the help of Lyapunov function method.

Theorem 4. *Let $K = (K_1, K_2, \ldots, K_d)$, $P = \operatorname{diag}(p_1, \ldots, p_n)$ be a positive-definite diagonal matrix, and let $\Delta = \operatorname{diag}(\delta_1, \ldots, \delta_n)$ be a diagonal matrix. Suppose $\delta_j \le 0$ if $j \notin J$, where $J = \{j : 1 \le j \le n, \gamma_j \ne 0\}$. Then under the following three conditions.*

(i) *Every submatrix A_{kl} in the block matrix* (2) *has equal row sum s_{kl}, and every principal submatrix A_{kk} is irreducible.*

(ii) *There exists a constant $\epsilon > 0$ such that, for any $u, v \in R^n$ and all $t \ge 0$,*

$$(u - v)^\top P \left\{ \left[f(u, t) - f(v, t) \right] - \Delta (u - v) \right\} \le -\epsilon (u - v)^\top (u - v). \tag{12}$$

(iii) *For all $j = 1, \ldots, n$, $k = 1, \ldots, d$, the matrices $\varepsilon \gamma_j \overline{A}_{kk} + \delta_j E_{K_k}$ are negative semidefinite in the transverse space L_K^k, that is,*

$$z^\top \left(\varepsilon \gamma_j \overline{A}_{kk} + \delta_j E_{K_k} \right) z \le 0, \quad z \in L_K^k, \tag{13}$$

or, in particular,

$$\varepsilon\gamma_j\lambda_k^{(2)} + \delta_j \le 0, \tag{14}$$

where $\lambda_k^{(2)}$ *is the second-largest eigenvalue of* $\overline{A}_{kk}$.

The synchronization manifold $\mathbb{M}_K$ *is globally attractive for the system* (1).

For convenience of the proof, the following notations are introduced.

$$\begin{gathered}
\overline{x}_k(t) = \frac{1}{K_k}\sum_{i\in G^k} x_i(t), \qquad \overline{x}(t) = \left[\overline{x}_{\hat{1}}^\top(t), \ldots, \overline{x}_{\hat{m}}^\top(t)\right]^\top, \\
\delta x_i(t) = x_i(t) - \overline{x}_{\hat{i}}(t), \\
\delta x(t) = \left[\delta x_1^\top(t), \ldots, \delta x_m^\top(t)\right]^\top, \\
\delta\widetilde{x}_k^s(t) = \left[\delta x^s_{\sum_{p=0}^{k-1} K_p+1}(t), \ldots, \delta x^s_{\sum_{p=0}^{k} K_p}(t)\right]^\top, \\
\delta\widetilde{x}_k(t) = \left[\delta\widetilde{x}_k^{1\top}(t), \ldots, \widetilde{x}_k^{n\top}(t)\right]^\top,
\end{gathered} \tag{15}$$

where $k = 1, \ldots, d$, $i = 1, \ldots, m$, $s = 1, \ldots, n$.

Then any vector $x = (x_1^\top, \ldots, x_m^\top)^\top \in R^{mn}$ can be decomposed into

$$x = \overline{x} \oplus \delta x, \quad \overline{x} \in \mathbb{M}_K, \ \delta x \in \mathbb{L}_K. \tag{16}$$

Therefore, the attractiveness of the invariant synchronization manifold $\mathbb{M}_K$ is equivalent to $\delta x \to 0$ when $t \to +\infty$; that is, the dynamical flow in the transverse subspace $\mathbb{L}_K$ converges to zero.

Proof. Noticing the condition (i), one gets

$$\begin{aligned}
\sum_{j=1}^m a_{ij}\Gamma x_j(t) &= \sum_{l=1}^d \sum_{j\in G^l} a_{ij}\Gamma\left[\delta x_j(t) + \overline{x}_l(t)\right] \\
&= \sum_{l=1}^m a_{ij}\Gamma\delta x_j(t) + \sum_{l=1}^d s_{\hat{i}l}\Gamma\overline{x}_l(t).
\end{aligned} \tag{17}$$

Therefore,

$$\begin{aligned}
\frac{d\delta x_i(t)}{dt} &= \frac{dx_i(t)}{dt} - \frac{1}{K_{\hat{i}}}\sum_{p\in G^{\hat{i}}}\frac{dx_p(t)}{dt} \\
&= f\left(x_i(t), t\right) - f\left(\overline{x}_{\hat{i}}(t), t\right) + \varepsilon\sum_{j=1}^m a_{ij}\Gamma\delta x_j(t) + J_{\hat{i}},
\end{aligned} \tag{18}$$

where

$$\begin{aligned}
J_{\hat{i}} = {} & f\left(\overline{x}_{\hat{i}}(t), t\right) - \frac{1}{K_{\hat{i}}}\sum_{p\in G^{\hat{i}}}\left[f\left(x_p(t), t\right) + \varepsilon\sum_{q=1}^m a_{pq}\Gamma x_q(t)\right] \\
& + \varepsilon\sum_{l=1}^d s_{\hat{i}l}\Gamma\overline{x}_l(t).
\end{aligned} \tag{19}$$

In order to utilize the QUAD(Δ, P, R^n) condition, a Lyapunov function is defined as follows:

$$V(\delta x(t)) = \frac{1}{2}\sum_{j=1}^m \delta x_i^\top(t) P\delta x_i(t). \tag{20}$$

One can conclude from $\sum_{i\in G^k}\delta x_i(t) = 0$ that

$$\sum_{i=1}^m \delta x_i^\top(t) PJ_{\hat{i}} = \sum_{k=1}^d\left[\sum_{i\in G^k}\delta x_i^\top(t)\right] PJ_{\hat{i}} = 0, \tag{21}$$

and then,

$$\begin{aligned}
&\frac{dV(\delta x_i(t))}{dt} \\
&= \sum_{i=1}^m \delta x_i^\top(t) P\left[f\left(x_i(t), t\right) - f\left(\overline{x}_{\hat{i}}(t), t\right) + \varepsilon\sum_{j=1}^m a_{ij}\Gamma\delta x_j(t)\right] \\
&\le -\epsilon\sum_{i=1}^m \delta x_i^\top(t)\delta x_i(t) \\
&\quad + \sum_{i=1}^m \delta x_i^\top(t) P\left[\varepsilon\sum_{j=1}^m a_{ij}\Gamma\delta x_j(t) + \Delta\delta x_i(t)\right].
\end{aligned} \tag{22}$$

Denote the second term in the right hand of (22) as S for convenience; then,

$$\begin{aligned}
S &= \sum_{k=1}^d\sum_{i\in G^k}\delta x_i^\top(t) P\left[\varepsilon\sum_{l=1}^d\sum_{j\in G^l} a_{ij}\Gamma\delta x_j(t) + \Delta\delta x_i(t)\right] \\
&= \sum_{k=1}^d\sum_{i\in G^k}\delta x_i^\top(t) P\left[\varepsilon\sum_{j\in G^k} a_{ij}\Gamma\delta x_j(t) + \Delta\delta x_i(t)\right. \\
&\qquad \left. + \varepsilon\sum_{l=1, l\ne k}^d\sum_{j\in G^l} a_{ij}\Gamma\delta x_j(t)\right].
\end{aligned} \tag{23}$$

Since $i, j \in G^k$ holds for the first term in the right hand above, replacing a_{ij} with $\overline{a}_{ij}$ according to (11) gives rise to that

$$\begin{aligned}
S &= \sum_{k=1}^d\sum_{i\in G^k}\delta x_i^\top(t) P\left[\varepsilon\sum_{j\in G^k}\overline{a}_{ij}\Gamma\delta x_j(t) + \Delta\delta x_i(t)\right] \\
&\quad + \varepsilon\sum_{k=1}^d\sum_{i\in G^k}\delta x_i^\top(t) P\sum_{l=1, l\ne\hat{i}}^d\left[-s_{\hat{i}l}\Gamma\delta x_i(t) + \sum_{j\in G^l} a_{ij}\Gamma\delta x_j(t)\right] \\
&= S_1 + S_2.
\end{aligned} \tag{24}$$

As a result of the condition (13), one obtains that

$$\begin{aligned} S_1 &= \sum_{k=1}^{d}\sum_{i\in G^k}\delta x_i^\top(t)\,P\left[\varepsilon\sum_{j\in G^k}\overline{a}_{ij}\Gamma\delta x_j(t)+\Delta\delta x_i(t)\right] \\ &= \sum_{k=1}^{d}\sum_{s=1}^{n}p_s\delta\widetilde{x}_k^{s\top}(t)\left(\varepsilon\gamma_s\overline{A}_{kk}+\delta_s E_{K_k}\right)\delta\widetilde{x}_k^s(t)\le 0. \end{aligned} \tag{25}$$

Since $s_{\hat{i}l}=\sum_{j\in G^l}a_{ij}$, the sum S_2 can be decomposed into

$$\begin{aligned} S_2 &= \varepsilon\sum_{k=1}^{d}\sum_{i\in G^k}\delta x_i^\top(t)\,P\sum_{l=1,l\neq k}^{d}\sum_{j\in G^l}a_{ij}\Gamma\left(\delta x_j(t)-\delta x_i(t)\right) \\ &= \varepsilon\sum_{k=1}^{d-1}\sum_{l=k+1}^{d}\sum_{i\in G^k}\sum_{j\in G^l}a_{ij}\delta x_i^\top(t)\,P\Gamma\left(\delta x_j(t)-\delta x_i(t)\right) \\ &\quad+\varepsilon\sum_{l=1}^{d-1}\sum_{k=l+1}^{d}\sum_{i\in G^k}\sum_{j\in G^l}a_{ij}\delta x_i^\top(t)\,P\Gamma\left(\delta x_j(t)-\delta x_i(t)\right). \end{aligned} \tag{26}$$

Renaming in the second term k by l, i by j, and vice versa [23] and utilizing the symmetry of a_{ij}, or $A_{lk}^\top=A_{kl}$, one gets

$$\begin{aligned} S_2 &= \varepsilon\sum_{k=1}^{d-1}\sum_{l=k+1}^{d}\sum_{i\in G^k}\sum_{j\in G^l}a_{ij}\delta x_i^\top(t)\,P\Gamma\left(\delta x_j(t)-\delta x_i(t)\right) \\ &\quad+\varepsilon\sum_{k=1}^{d-1}\sum_{l=k+1}^{d}\sum_{j\in G^l}\sum_{i\in G^k}a_{ji}\delta x_j^\top(t)\,P\Gamma\left(\delta x_i(t)-\delta x_j(t)\right) \\ &= -\varepsilon\sum_{k=1}^{d-1}\sum_{l=k+1}^{\mathrm{d}}\sum_{i\in G^k}\sum_{j\in G^l}a_{ij}\left(\delta x_j(t)-\delta x_i(t)\right)^\top P\Gamma \\ &\qquad\qquad\times\left(\delta x_j(t)-\delta x_i(t)\right) \\ &\le 0. \end{aligned} \tag{27}$$

Therefore, one obtains that $S\le 0$ and

$$\frac{dV(\delta x_i(t))}{dt}\le-\epsilon\sum_{i=1}^{m}\delta x_i^\top(t)\,\delta x_i(t)\le-2\epsilon\frac{V(\delta x_i(t))}{\max_i p_i}, \tag{28}$$

which implies that the partial synchronization manifold $\mathbb{M}_K$ is globally attractive for the system (1).

The remainder of the proof is to show that condition (14) is also sufficient for $S_1\le 0$.

It is well known that a symmetric matrix $\overline{A}_{kk}$ has the decomposition $\overline{A}_{kk}=U_k\Lambda_kU_k^\top$, where $\Lambda_k=\mathrm{diag}\{\lambda_k^{(1)},\ldots,\lambda_k^{(K_k)}\}$ is a real diagonal matrix and $U_k\in R^{K_k\times K_k}$ is an unitary matrix, that is, $U_kU_k^\top=E_{K_k}$. The diagonal elements of Λ_k are the eigenvalues of $\overline{A}_{kk}$ satisfying $s=\lambda_k^{(1)}>\lambda_k^{(2)}\ge\cdots\ge\lambda_k^{(K_k)}$. The ith column of U_k is the eigenvector of $\overline{A}_{kk}$ corresponding to the eigenvalue $\lambda_k^{(i)}$, $i=1,\ldots,K_k$. By changes of variables $\delta\widetilde{x}_k^s(t)=U\eta_k^s(t)$, the quadratic form (25) can be diagonalized as follows:

$$\begin{aligned} S_1 &= \sum_{k=1}^{d}\sum_{s=1}^{n}p_s\delta\widetilde{x}_k^{s\top}(t)\left(\varepsilon\gamma_s\overline{A}_{kk}+\delta_sE_{K_k}\right)\delta\widetilde{x}_k^s(t) \\ &= \sum_{k=1}^{d}\sum_{s=1}^{n}p_s\eta_k^{s\top}(t)\left(\varepsilon\gamma_s\Lambda_k+\delta_sE_{K_k}\right)\eta_k^s(t). \end{aligned} \tag{29}$$

Noticing that the first column of U_k is $[1,\ldots,1]^\top$, one can conclude from $\sum_{i\in G^k}\delta x_i(t)=0$ and $\eta_k^s(t)=U^\top\delta\widetilde{x}_k^s(t)$ that $\eta_k^1(t)=0$. Therefore, condition (14) is sufficient for $S_1\le 0$.

The proof is completed. □

Compared with the previous results [10], the conditions in Theorem 4 are not dependent on the intercluster couplings, which are eliminated in the proof through a set of mathematical skills in inequalities (25) and (27). The obtained results greatly reduce the network sizes theoretically. However, another question arises naturally: how to implement the proposed condition in reality? The following remarks might answer these questions.

Remark 5. In case that the network size is not very large and the coupling matrix is given, it is easy to verify all the conditions in Theorem 4. In case that the network consists of great mounts of oscillators, it should still be full of challenges to implement though our result reduces the network size greatly. And it might be hard to implement.

In order to make clear the implications of Theorem 4, several remarks on the conditions are given as follows.

Remark 6. (1) Many chaotic oscillators have been proved to satisfy condition (ii), such as Chua circuits [24] and standard Hopfield neural networks [25]. However, many other systems are not the case such as a lattice of x-coupled Rössler systems, in which the stability of synchronization regime is lost with the increasing of coupling [26].

(2) Providing that the trajectories of the uncoupled systems $x_i=f(x_i,t)$, $i=1,\ldots,m$ are eventually dissipative, that is, the trajectories will be in the absorbing domain $\mathscr{B}$ eventually, it has been proved that each trajectory of the coupled system (1) is also eventually dissipative and will be in the absorbing domain $\underbrace{\mathscr{B}\times\cdots\times\mathscr{B}}_{m}$ [17] eventually. Therefore, condition (ii) holds when the time t is large enough. For example, the uncoupled Lorenz system

$$\begin{aligned} \dot{u} &= \sigma(v-u), \quad \sigma=10, \\ \dot{v} &= ru-v-uw, \quad r=28, \\ \dot{w} &= -bw+uv, \quad b=\frac{8}{3}, \end{aligned} \tag{30}$$

is eventually dissipative, where

$$\mathscr{B}=\left\{(u)^2+(v)^2+(w-\sigma-r)^2<\frac{b^2(\sigma+r)^2}{4(b-1)}=B\right\}. \tag{31}$$

It has been proved that x-coupled [27] or y-coupled Lorenz systems [28] satisfy condition (ii) when the time t is large enough.

(3) If there exists a $k_0 \in \{1, 2, \ldots, d\}$ such that $K_{k_0} = 1$, which implies that the subset G^{k_0} contains only one element, which does not synchronize with any other node. Without loss of generality, suppose that $K_k = 1$ for $k > \overline{d}$. In this case, the corresponding principal quasi-submatrix $\overline{A}_{kk} = s$ and the constant s can be regarded as the single eigenvalue of $\overline{A}_{kk}$ since $\overline{A}_{kk}\vec{v} = s\vec{v}$ for any $\vec{v} \in R$. Therefore, the second-largest eigenvalue $\lambda_k^{(2)}$ does not exist. But notice the definition of partial synchronization in Section 2, synchronization in the cluster G^k always occurs, and conditions (13) and (14) should be regarded to hold for any $\varepsilon > 0$, $k > \overline{d}$.

(4) Since A_{kk} is irreducible and diffusive, the largest eigenvalue of A_{kk} is zero, which is simple. Therefore, the largest eigenvalue of $\overline{A}_{kk}$ is s, and it is simple also. In case $s = 0$, it can be seen that $\lambda_k^{(2)} < 0$ and condition (14) is equivalent to

$$\varepsilon \geq \frac{\max_{j\in J}\left\{0, \delta_j/\gamma_j\right\}}{\min_{1\leq k\leq \overline{d}}\left|\lambda_k^{(2)}\right|}. \tag{32}$$

Condition (14) provides us a novel index of partial synchronizability.

4. Numerical Examples

Consider the system (1) with $2m$ y-coupled Lorenz systems

$$\dot{x}_i = f(x_i, t) + \varepsilon\sum_{j=1}^{2m} a_{ij}(\theta)\Gamma x_j, \quad i = 1, \ldots, 2m, \tag{33}$$

where $f(x_i, t), x_i = (u_i, v_i, w_i)^{\top}$ defined by the system (30), $\Gamma = \mathrm{diag}(0, 1, 0)$. Let $P = \mathrm{diag}(B/2(b-\epsilon)\sigma + \epsilon/\sigma, 1, 1)$, and $\Delta = \mathrm{diag}(0, 2b-\epsilon-1+[B-2(b-\epsilon)(\sigma-\epsilon)]^2/2B(b-\epsilon), 0)$, and the f in (27) has been proved to satisfy condition (ii) when the time t is large enough. Through translating time, the network system (27) satisfies condition (ii) in Theorem 4.

4.1. A Star-Global Network. Design the following topology matrix of the system (33) as

$$A(\theta) = \begin{bmatrix} A_{11}(\theta) & \theta E_m \\ \theta E_m & A_{22}(\theta) \end{bmatrix}_{2m\times 2m}, \tag{34}$$

where $\theta \geq 0$ is the coupling weight parameter of couplings between the two clusters, and the identity matrix E_m implies that the ith $(1 \leq i \leq m)$ oscillator in the first cluster is coupled with the $(m+i)$th oscillator in the second cluster. As a special case, the submatrices $A_{11}(\theta)$ and $A_{22}(\theta)$ are taken as follows:

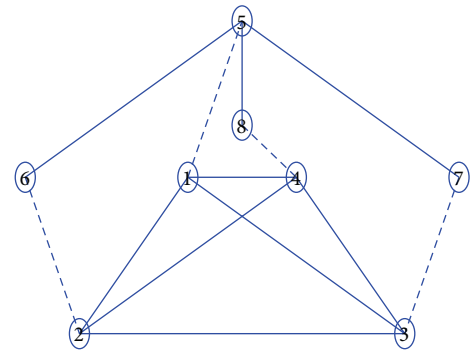


Figure 1: Topology structure of a star-global network composed of 8 oscillators. The solid lines represent the inner couplings of each clusters; the dashed lines represent the outer couplings between different clusters.

$$A_{11}(\theta) = \begin{bmatrix} 1-m-\theta & 1 & \cdots & 1 \\ 1 & 1-m-\theta & \cdots & 1 \\ & \cdots & \cdots & \\ 1 & 1 & \cdots & 1-m-\theta \end{bmatrix},$$
$$A_{22}(\theta) = \begin{bmatrix} 1-m-\theta & 1 & 1 & \cdots & 1 \\ 1 & -1-\theta & 0 & \cdots & 0 \\ & \cdots & \cdots & & \\ 1 & 0 & 0 & \cdots & -1-\theta \end{bmatrix}. \tag{35}$$

Obviously, $A_{11}(\theta)$ and $A_{22}(\theta)$ are the topology matrices of a globally coupled network and a star-coupled network, respectively. Therefore, we call the network a star-global (coupled) network.

4.2. Numerical Simulations. As an example, the topology structure of a star-global network with $m = 4$ is shown in Figure 1. Due to the specific topological structure of the network, the following partitions satisfy condition (i),

$$\begin{aligned} G_1 &= \{1, 2, 3, 4, 5, 6, 7, 8\}, \\ G_2 &= \{1, 2, 3, 4; 5, 6, 7, 8\}, \\ G_3 &= \{1, 5; 2, 3, 4, 6, 7, 8\}. \end{aligned} \tag{36}$$

Remark 7. According to the criterion in the previous researches [10], we should firstly find the eigenvalues and the corresponding eigenvectors of the 8×8 matrix $A(\theta)$. As we know, it should be of a vast amount of calculations. However, based on Theorem 4, it is enough for partial synchronization with the partition $G_2(G_3)$ if we can find the eigenvalues of two 4×4 matrices (four 2×2 matrices). Obviously, Theorem 4 reduces the calculations greatly.

The principal quasi-submatrices corresponding to the clusters in the partitions G_1 or G_2, are $A(\theta)$ or $A_{11}(\theta)|_{\theta=0}$ and $A_{22}(\theta)|_{\theta=0}$, respectively. And denote the ones in the partition

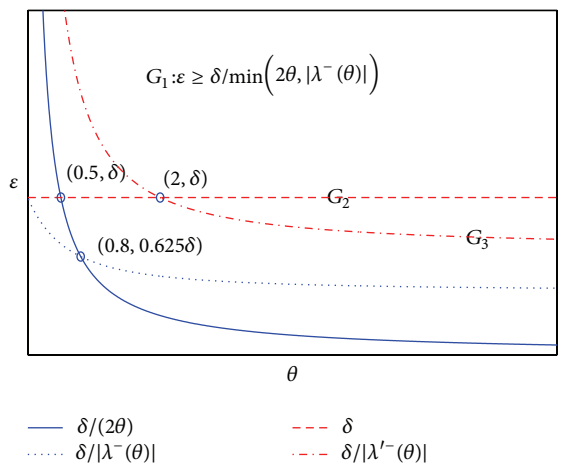


FIGURE 2: Threshold of the coupling strength ε and the coupling weight θ for partial synchronization with different partitions.

G_3 as $\overline{A}_{11}(\theta)$ and $\overline{A}_{22}(\theta)$. Further analysis gives rise to the eigenvalues sets of the quasi-submatrices,

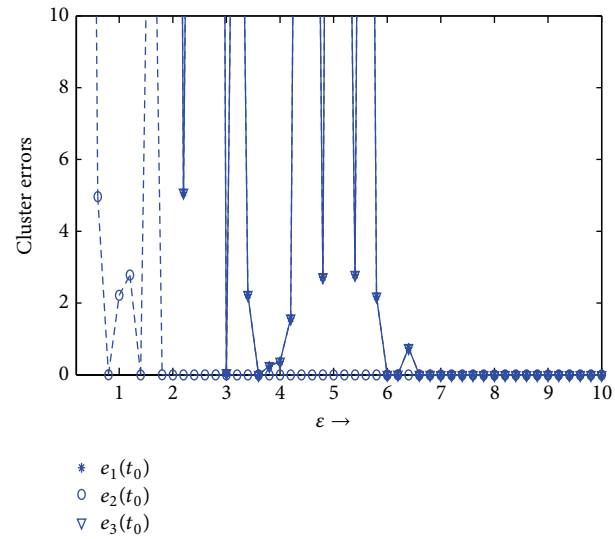
$$\begin{aligned} \sigma(A(\theta)) &= \{0, -4, -2\theta, -4-2\theta, \lambda^{\pm}, \lambda^{\pm}\}, \\ \sigma(A_{11}(\theta)|_{\theta=0}) &= \{0, -4, -4, -4\}, \\ \sigma(A_{22}(\theta)|_{\theta=0}) &= \{0, -1, -1, -4\}, \\ \sigma(\overline{A}_{11}(\theta)) &= \{0, -2\theta\}, \\ \sigma(\overline{A}_{22}(\theta)) &= \{0, -2\theta, \lambda'^{\pm}, \lambda'^{\pm}\}, \end{aligned} \tag{37}$$

where $\lambda^{\pm} = -(2\theta + 5 \pm \sqrt{9+4\theta^2})/2$, $\lambda'^{\pm} = -(2\theta + 3 \pm \sqrt{9+4\theta^2})/2$. Therefore, partial synchronization with the partition G_2 occurs if $\varepsilon \geq \delta$; partial synchronization with the partition G_3 occurs if $\varepsilon \geq \delta/|\lambda'^-(\theta)|$; and full synchronization with the partition G_1 occurs if $\varepsilon \geq \max\{\delta/2\theta, \delta/|\lambda^-(\theta)|\}$. These are seen much more clearly in Figure 2.

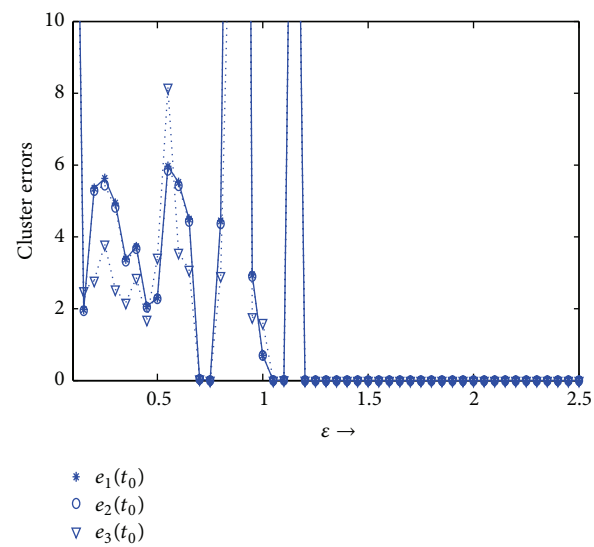
By fixing $\theta \in (0, 0.5]$ and increasing ε gradually, partial synchronization with the partition G_2 will firstly occur; then, the one with G_3 and full synchronization occurs at the same time. Figure 2 also implies that the threshold for partial synchronization with the partition G_3 is sufficient for full synchronization.

In order to validate the effectiveness of Figure 2 numerically, the following average cluster errors of the system (33) are defined to measure partial synchronization with partitions G_1, G_2, and G_3, respectively,

$$\begin{aligned} e_1 &= \frac{1}{8}\sum_{i=1}^{8} \|x_i(t_0) - x_1(t_0)\|, \\ e_2 &= \frac{1}{8}\sum_{i=1}^{4} \|x_i(t_0) - x_1(t_0)\| + \frac{1}{8}\sum_{i=5}^{8} \|x_i(t_0) - x_5(t_0)\|, \\ e_3 &= \frac{1}{8}\|x_5(t_0) - x_1(t_0)\| \\ &\quad + \frac{1}{8}\sum_{i=2, i\neq 5}^{8} \|x_i(t_0) - x_2(t_0)\|. \end{aligned} \tag{38}$$



(a) $\theta = 0.2$



(b) $\theta = 5$

FIGURE 3: Dependence of the average cluster errors on the coupling strength ε for the system (33) with the topology matrix (34). The errors $e_1(t_0), e_2(t_0)$, and $e_3(t_0)$ measure the attractiveness of the manifolds $\mathbb{M}_K(G_1)$, $\mathbb{M}_K(G_2)$, and $\mathbb{M}_K(G_3)$, respectively.

Choose the initial conditions ($u_i(0)$, $v_i(0)$, and $w_i(0)$) randomly on $[-1,1] \times [-1,1] \times [-1,1]$, and pick $t_0 = 500$. Fix the coupling weight θ at 0.2, and 5; the dependence of the cluster errors on ε is shown, respectively, in Figure 3. As can be seen, there is a good agreement between Figures 3 and 2.

5. Conclusions

In summary, this paper introduced a novel index of partial synchronizability of a network. It is shown that partial synchronization can be ensured by the conditions merely on the quasi-submatrices corresponding to the clusters. If a network is composed of a great mount of nodes, the enormous amount of calculation can be reduced by replacing the

coupling matrix with several quasi-submatrices. Numerically, different types of partial synchronization occur in a star-global network when the coupling strength is increased, the order of which is forecasted accurately by our result. It should be a meaningful and effective method to study partial synchronization with different partitions.

In the past decades, the networked control systems have attracted much attention due to their applications covering a wide range of industries. And the network-induced phenomena under consideration in engineering have been discussed widely, including missing measurements [29], fading measurements [30], and probabilistic sensor delays [31]. Therefore, the obtained approach for partial synchronization in this paper might be applicable to the complex networks with networked induced phenomena. The related studies should be one of the future research topics.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

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Research Article

Asymptotic Degree Distribution of a Kind of Asymmetric Evolving Network

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We propose a kind of evolving network which shows tree structure. The model is a combination of preferential attachment model and uniform model. We show that the proportional degree sequence $\{p_k\}_{k>1}$ obeys power law, exponential distribution, and other forms according to the relation of k and parameter m.

1. Introduction

In recent ten years, there has been much interest in understanding the properties of real large-scale complex networks which describe a wide range of systems in nature and society. Examples of such networks appear in communications, biology, social science, economics, and so forth [1]. In pursuit of such understanding, mathematicians and physicists usually use random graphs to model all these real-life networks. In the investigation of various complex networks, the degree distribution is always the main concern because it characterizes the fundamental topological properties of complex networks which show importance in network control, estimation, and sensor [2–8]. Several models were introduced to explain the properties. Bollobás [9] proposed a model with n vertices and m edges. In this model, the degree distribution is approximately Poisson distribution. Later, Barabási and Albert [10] proposed the following model: at each time step, add a new vertex v and a fixed number r of edges originating at v and directed towards vertices chosen at random with probability proportional to their degrees. Based on simulation and heuristic approximation, they predicted that the degree distribution behaves like d^{-3} for all $r \geq 1$. The result was confirmed by Barabási et al. [11, 12]. In order to generate power laws with arbitrary exponents, Dorogovtsev et al. [13] and Drine et al. [14] introduced the following natural generalization of the above model: the destination of the r new edges added at each time step is chosen with probability proportional to the degree plus an initial attractiveness αr; they gave a nonrigorous argument that the degree distribution p_d behaves like $d^{-2-\alpha}$ for large d.

In some real networks, experiments show that the distribution obeys neither power law nor exponential. To explain the phenomenon, we propose a model as follows: starting with a single vertex, at each time step, a new vertex is added and linked to one of the existing vertices, which is chosen according the following rule: at time $m, 2m, 3m, \ldots$, where m is integer, we choose one of the existing vertices with probability proportional to the degree; that is, we have probability k/s_n, where k is the degree of the vertex chosen and s_n is the total degree of vertices; at another time step, we choose one of the existing vertices with equal probability. Related models were also proposed by Krapivsky and Redner [15] and Li [16] to describe the organization of growing networks. In this paper, we will focus on the distribution of evolving network and the distribution of the number of vertices with given degree will be considered in Section 2. In Section 3, we will consider the asymptotic degree distribution.

2. The Number of Vertices with Given Degrees

Let $D_n(k)$ denote the number of vertices with degree k at time n. We will consider the case $k = 1, 2$ in this section and the case $k > 2$ will be considered in the next section. As $k = 1$, we obtain the following result.

Lemma 1. *In the evolving network, the expectation of the number of degree 1 satisfies*

$$ED_n(1) = \frac{2m}{4m-1}n. \tag{1}$$

Proof. Herein after, $=$ denotes asymptotic equivalence as $n \to \infty$. From the way the network is formed, we can see that, for $n > 1$, the number of vertices of degree 1 does not change if we attach a new vertex v_n to a vertex with degree 1 and increases by 1 if we attach v_n to vertices of degree larger than 1 after joining the vertex v_n. Assuming n is multiple of m, that is, $n = km$, where k is integer number, and taking expectation of $D_n(1)$, we obtain

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \left(1 - \frac{1}{2(km-1)}\right)\mathrm{ED}_{km-1}(1) + 1 \\
&= \left(1 - \frac{1}{2(km-1)}\right) \\
&\quad \times \left[\mathrm{ED}_{km-2}(1)\left(1 - \frac{1}{km-2}\right) + 1\right] + 1 \\
&= \left(1 - \frac{1}{2(km-1)}\right)\left(1 - \frac{1}{km-2}\right)\mathrm{ED}_{km-2}(1) \\
&\quad + \left(1 - \frac{1}{2(km-1)}\right) + 1.
\end{aligned} \tag{2}$$

The first equation shows that when we add a new vertex and link it to one of existing vertices with preferential attachment, the number of vertex increases by 1, while the second equation comes from the uniform attachment. Continuing the iteration and noticing the boundary condition $\mathrm{ED}_1(1) = 0$, we have

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \sum_{j=1}^{km}\prod_{i=0}^{[j/m]}\left(1 - \frac{1}{2[(k-i)m-1]}\right) \\
&\quad \times \prod_{v=2}^{j}\left(1 - \frac{1}{km-v}\right) \\
&\quad \times \left(\prod_{s=0}^{[j/m]}\left(1 - \frac{1}{(k-i)m-1}\right)\right)^{-1}.
\end{aligned} \tag{3}$$

Considering the term

$$\begin{aligned}
&\prod_{i=0}^{[j/m]}\left(1 - \frac{1}{2[(k-i)m-1]}\right)\prod_{v=2}^{j}\left(1 - \frac{1}{km-v}\right) \\
&\quad \times \left(\prod_{s=0}^{[j/m]}\left(1 - \frac{1}{(k-i)m-1}\right)\right)^{-1},
\end{aligned} \tag{4}$$

we have

$$\begin{aligned}
&e^{\ln\prod_{i=0}^{[j/m]}(1-(1/2[(k-i)m-1]))\prod_{v=2}^{j}(1-(1/(km-v)))-\ln\prod_{s=0}^{[j/m]}(1-(1/((k-i)m-1)))} \\
&\quad = e^{-\sum_{i=0}^{[j/m]}(1/2[(k-i)m-1])-\sum_{v=2}^{j}(1/(km-v))+\sum_{i=0}^{[j/m]}(1/((k-i)m-1))} \\
&\quad = e^{-(1/(2k-1))\int_{t=0}^{[j/m]}(1/(1-(mt/(2k-1))))dt-(1/km)\int_{v=2}^{j}(1/(1-(v/km)))dv+(1/(km-1))\int_{t=0}^{[j/m]}(1/(1-(mt/(km-1))))dt} \\
&\quad = \left(1 - \frac{j}{km}\right)^{(1-(1/2m))}.
\end{aligned} \tag{5}$$

We obtain that

$$\begin{aligned}
\mathrm{ED}_{km}(1) &= \sum_{j=1}^{km}\left(1 - \frac{j}{km}\right)^{(1-(1/2m))} \\
&= \frac{2m}{4m-1}\cdot km
\end{aligned} \tag{6}$$

when n is not a multiple of m, assuming $n = k'm + s$, $1 < s < m$, where k' is an integer number; we also obtain

$$\begin{aligned}
&\mathrm{ED}_{k'm+s}(1) \\
&\quad = \mathrm{ED}_{k'm+s-1}(1)\left(1 - \frac{1}{k'm+s-1}\right) + 1 \\
&\quad = \cdots
\end{aligned}$$

$$
\begin{aligned}
&= \left(1 - \frac{1}{k'm + s - 1}\right) \\
&\quad \times \left(1 - \frac{1}{k'm + s - 2}\right) \cdots \left(1 - \frac{1}{k'm}\right) \mathrm{ED}_{k'm}^{m}(1) \\
&\quad + \left(1 - \frac{1}{k'm + s - 1}\right) \\
&\quad \times \left(1 - \frac{1}{k'm + s - 2}\right) \cdots \left(1 - \frac{1}{k'm + 1}\right) \\
&\quad + \cdots + \left(1 - \frac{1}{k'm + s - 1}\right)\left(1 - \frac{1}{k'm + s - 2}\right) \\
&\quad + \left(1 - \frac{1}{k'm + s - 1}\right).
\end{aligned} \tag{7}
$$

When n is large enough, we can see that

$$\mathrm{ED}_{k'm}(1) = \mathrm{ED}_{k'm+s}(1)(1 + o(1)). \tag{8}$$

As a result, we have

$$\mathrm{ED}_n(1) = \frac{2m}{4m - 1} n. \tag{9}$$

□

Now we discuss the number of degree 2 in the network; we have the following.

Lemma 2. *For $n > 2$,*

$$ED_n(2) = \frac{2m - 1}{8m - 2} n. \tag{10}$$

Proof. We prove the case that n is a multiple of m and assume $n = km$, where k is an integer number; considering the expectation of $D_n(2)$, we have

$$
\begin{aligned}
\mathrm{ED}_{km}(2) &= \left(1 - \frac{2}{2(km - 1)}\right) \mathrm{ED}_{km-1}(2) + \frac{\mathrm{ED}_{km-1}(1)}{2(km - 1)} \\
&= \left(1 - \frac{1}{km - 1}\right) \left[\mathrm{ED}_{km-2}(2)\left(1 - \frac{1}{km - 2}\right) \right. \\
&\qquad \left. + \frac{\mathrm{ED}_{km-2}(1)}{km - 2}\right] + \frac{\mathrm{ED}_{km-1}(1)}{2(km - 1)}.
\end{aligned} \tag{11}
$$

Noticing the boundary condition $\mathrm{ED}_2(2) = 0$ and Lemma 1, we have

$$
\begin{aligned}
\mathrm{ED}_{km}(2) = &\left(\sum_{j=1}^{km} \prod_{i=1}^{j} \left(1 - \frac{1}{km - i}\right) \right. \\
&\left. - \frac{1}{2} \sum_{j=1}^{k} \prod_{v=1}^{jm} \left(1 - \frac{1}{km - v}\right)\right) \frac{2m}{4m - 1}.
\end{aligned} \tag{12}
$$

By the estimation $\ln(1 + x) = x$ and the fact that

$$
\begin{aligned}
\prod_{i=1}^{j} \left(1 - \frac{1}{km - i}\right) &= 1 - \frac{j}{km}, \\
\prod_{v=0}^{jm} \left(1 - \frac{1}{km - v}\right) &= 1 - \frac{j}{k},
\end{aligned} \tag{13}
$$

we obtain that

$$
\begin{aligned}
\mathrm{ED}_{km}(2) &= \frac{2m}{4m - 1} \left[\sum_{j=1}^{km} \left(1 - \frac{j}{km}\right) - \frac{1}{2} \sum_{j=0}^{k} \left(1 - \frac{j}{k}\right)\right] \\
&= km \frac{2m}{4m - 1} \left[\int_0^1 (1 - x)\, dx - \frac{1}{2} \int_0^1 (1 - x)\, dx\right] \\
&= \frac{2m - 1}{8m - 2} \cdot km.
\end{aligned} \tag{14}
$$

The case n, which is not a multiple of m, is the same as Lemma 1, just a little tedious. □

3. Asymptotic Degree Distribution of Network

Let

$$p_k(n) = \frac{D_n(k)}{n} \tag{15}$$

denote the proportion of vertices with degree k at time n. Considering the expectation of $D_n(k)$, we have the following theorem.

Theorem 3. *For arbitrary $k > 1$ and n, the expectation of the number of degree k satisfies*

$$ED_n(k) = \frac{2m}{4m - 1} \prod_{i=2}^{k} \frac{2m + i - 3}{4m + j - 2} n. \tag{16}$$

Proof. The case $k = 1, 2$ is just the result of Lemmas 1 and 2. Assume the result is true for k; that is,

$$\mathrm{ED}_n(k) = \frac{2m}{4m - 1} \prod_{i=2}^{k} \frac{2m + i - 3}{4m + j - 2} n. \tag{17}$$

We will prove the result is true for $k+1$. We just prove the case n is a multiple of m; that is, $n = lm$, where l is integer number. From the network constructed, we have

$$\begin{aligned}
&\mathrm{ED}_{ml}(k+1)\\
&\quad=\left(1-\frac{k+1}{2(ml-1)}\right)\mathrm{ED}_{ml-1}(k+1)\\
&\qquad+\frac{k}{2(ml-1)}\mathrm{ED}_{ml-1}(k)\\
&\quad=\left(1-\frac{k+1}{2(ml-1)}\right)\\
&\qquad\times\left[\mathrm{ED}_{ml-2}(k+1)\left(1-\frac{1}{ml-2}\right)+\frac{\mathrm{ED}_{ml-2}(k)}{ml-2}\right]\\
&\qquad+\frac{k}{2(ml-1)}\mathrm{ED}_{ml-1}(k).
\end{aligned}\tag{18}$$

Continuing the iteration and noticing the boundary condition $\mathrm{ED}_s(k+1)=0,\ s<k$, we obtain that

$$\begin{aligned}
&\mathrm{ED}_{ml}(k+1)\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\\
&\qquad\times\sum_{j=1}^{lm}\prod_{i=0}^{[j/m]}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{[j/m]}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\sum_{j=1}^{l}\prod_{i=0}^{j}\left(1-\frac{k+1}{2(l-i)m-1}\right)\\
&\qquad\times\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\times\left(\prod_{i=0}^{j}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}.
\end{aligned}\tag{19}$$

Noticing the fact that

$$\begin{aligned}
&\prod_{i=0}^{[j/m]}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{j}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{[j/m]}\left(1-\frac{1}{(l-i)m-1}\right)\right)^{-1}\\
&\quad=\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)},
\end{aligned}\tag{20}$$

$$\begin{aligned}
&\prod_{i=0}^{j}\left(1-\frac{k+1}{2(l-i)m-1}\right)\prod_{v=1}^{jm}\left(1-\frac{1}{lm-v}\right)\\
&\qquad\times\left(\prod_{i=0}^{j}\left(1-\frac{1}{(l-i)m-1}\right)\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)}\right)^{-1}\\
&\quad=\left(1-\frac{j}{l}\right)^{(((k-1)/2m)+1)}.
\end{aligned}\tag{21}$$

We obtain

$$\begin{aligned}
&\mathrm{ED}_{lm}(k+1)\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\\
&\qquad\times\sum_{j=1}^{lm}\left(1-\frac{j}{lm}\right)^{(((k-1)/2m)+1)}+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\sum_{j=1}^{l}\left(1-\frac{j}{l}\right)^{(((k-1)/2m)+1)}\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\int_0^1(1-x)^{(((k-1)/2m)+1)}dx\cdot lm\\
&\qquad+\frac{k-2}{2}\frac{2m}{4m-1}\\
&\qquad\times\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\int_0^1(1-x)^{(((k-1)/2m)+1)}dx\cdot l\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k}\frac{2m+i-3}{4m+j-2}\left(1+\frac{k-2}{2m}\right)\\
&\qquad\cdot\frac{2m}{4m+k-1}\cdot lm\\
&\quad=\frac{2m}{4m-1}\prod_{i=2}^{k+1}\frac{2m+i-3}{4m+j-2}\cdot lm.
\end{aligned}\tag{22}$$

The result is true for $k+1$. □

From Theorem 3, we can see that $\lim_{n\to\infty}(\mathrm{ED}_n(k)/n)$ exists; we denote it by p_k. Now we consider the relation of p_k and $p_k(n)$; we introduce the following lemma.

Lemma 4. *There exists a bound constant $C(k)$ such that for arbitrary $a>0$,*

$$P\left(\left|D_n(k)-ED_n(k)\right|\geq a\right)\leq 2e^{-a^2/2C(k)^2n}.\tag{23}$$

Proof. Let $\mathfrak{F}_n=\sigma(D_1(1),\ldots,D_k(1),D_k(2),\ldots D_k(k),\ldots,D_n(1),\ldots D_n(k)\ldots,D_n(n))$ denote the *σ-algbra*. For $m=0,1,\ldots,n$, we define

$$M_m=E\left(D_k(n)\mid\mathfrak{F}_m\right).\tag{24}$$

By the tower property of conditional expectation and the fact that the σ-*algbra* $\mathfrak{F}_n$ can be deduced from $\mathfrak{F}_{n+1}$, we obtain that, for $m < n$,

$$\begin{aligned} E\left(M_{m+1} \mid \mathfrak{F}_m\right) &= E\left[E\left(D_n(k) \mid \mathfrak{F}_{m+1}\right) \mid \mathfrak{F}_m\right] \\ &= E\left(D_n(k) \mid \mathfrak{F}_m\right) \\ &= M_m. \end{aligned} \tag{25}$$

Noticing the fact that

$$E\left[\left|M_m\right|\right] = EM_m = \mathrm{E}D_n(k) < n < \infty, \tag{26}$$

we have $\{M_m\}_{m=0}^n$ as a martingale sequence. According to the definition of the σ-*algbra*, we know the $\mathfrak{F}_0$ has no information of the network and $\mathfrak{F}_n$ has the whole information, so we have

$$\begin{aligned} M_0 &= E\left[D_n(k) \mid \mathfrak{F}_0\right] = \mathrm{E}D_n(k), \\ M_n &= E\left[D_n(k) \mid \mathfrak{F}_n\right] = D_n(k). \end{aligned} \tag{27}$$

Therefore, we have

$$D_n(k) - \mathrm{E}D_n(k) = M_n - M_0 = \sum_{j=0}^{n}\left(M_{j+1} - M_j\right). \tag{28}$$

Now we prove that there exists a bound constant $C(k)$, such that $|M_{j+1} - M_j| \le C(k)$. We will prove the result by induction. For the case $k = 1$, we have

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \left|E\left(D_n(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_n(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \left|E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right)\right. \\ &\qquad - E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_j\right) \\ &\qquad \left. + E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \left|E\left(E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{n-1}\right) \mid \mathfrak{F}_{j+1}\right)\right. \\ &\qquad - E\left(E\left(D_n(1) - D_{n-1}(1) \mid \mathfrak{F}_{n-1}\right) \mid \mathfrak{F}_j\right) \\ &\qquad \left. + E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \left(1 - \frac{1}{n-1}\right) \\ &\qquad \times \left|E\left(D_{n-1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(1) \mid \mathfrak{F}_j\right)\right|. \end{aligned} \tag{29}$$

Continuing the iteration and noticing the fact that $E(D_m(1) \mid \mathfrak{F}_{j+1}) - E(D_m(1) \mid \mathfrak{F}_j) = 0$, for $m < j$, we obtain that

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ &\qquad \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \\ &\qquad \cdot \left|E\left(D_{j+1}(1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{j+1}(1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ &\qquad \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \\ &\qquad \cdot \left|\left(D_{j+1}(1) - D_j(1)\right) - E\left(D_{j+1}(1) - D_j(1) \mid \mathfrak{F}_j\right)\right|. \end{aligned} \tag{30}$$

Obviously,

$$\begin{gathered} \left|D_{j+1}(1) - D_j(1)\right| \le 1, \\ \prod_{i=j}^{n-1}\left(1 - \frac{1}{i}\right)\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{2(sm-1)}\right) \\ \times \left(\prod_{s=1}^{[n/m]}\left(1 - \frac{1}{sm-1}\right)\right)^{-1} \le 1, \end{gathered} \tag{31}$$

so we have

$$\left|M_{j+1} - M_j\right| \le 2. \tag{32}$$

Assume the result is true for k; that is, there exists a bound constant $C(k)$, such that

$$\left|M_{j+1} - M_j\right| \le C(k). \tag{33}$$

For $k + 1$, by the definition of M_{j+1}, we have

$$\begin{aligned} &\left|M_{j+1} - M_j\right| \\ &\quad = \left|E\left(D_n(k+1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_n(k+1) \mid \mathfrak{F}_j\right)\right| \\ &\quad = \left(1 - \frac{1}{n-1}\right) \\ &\qquad \times \left|E\left(D_n(k+1) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(k+1) \mid \mathfrak{F}_j\right)\right| \\ &\qquad + \frac{1}{n-1}\left[E\left(D_{n-1}(k) \mid \mathfrak{F}_{j+1}\right) - E\left(D_{n-1}(k) \mid \mathfrak{F}_j\right)\right]. \end{aligned} \tag{34}$$

Continuing the iteration and using the assumption for k, we obtain that

$$\begin{aligned}
&\left|M_{j+1}-M_j\right| \\
&\quad\le \prod_{v=j+1}^{n-1}\left(1-\frac{1}{v}\right)\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{k+1}{2(im-1)}\right) \\
&\quad\times\left(\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{1}{(im-1)}\right)\right)^{-1} \\
&\quad\cdot\left|E\left(D_{j+1}(k+1)\mid\mathfrak{F}_{j+1}\right)-E\left(D_{j+1}(k+1)\mid\mathfrak{F}_j\right)\right| \\
&\quad+\sum_{v=1}^{n-j-1}\prod_{s=1}^{v}\left(1-\frac{1}{n-s}\right)\frac{1}{n-v-1}C(k) \\
&\quad+\sum_{v=0}^{[(n-j-1)/m]}\prod_{s=1}^{n-([n/m]-v)m}\left(1-\frac{1}{n-s}\right) \\
&\quad\cdot\prod_{s=0}^{v}\left(1-\frac{k+1}{2(([n/m]-s)m-1)}\right) \\
&\quad\times\left(\prod_{s=0}^{j}\left(1-\frac{1}{([n/m]-s)m-1}C(k)\right)\right)^{-1}.
\end{aligned}\tag{35}$$

Noticing the fact that $1-((k+1)/2j)<1-(1/j)$ and

$$\begin{aligned}
&\prod_{v=j+1}^{n-1}\left(1-\frac{1}{v}\right)\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{k+1}{2(im-1)}\right) \\
&\quad\times\left(\prod_{i=[(j+1)/m]}^{[n/m]}\left(1-\frac{1}{(im-1)}\right)\right)^{-1}\le 1,
\end{aligned}\tag{36}$$

we obtain that

$$\begin{aligned}
\left|M_{j+1}-M_j\right| &\le 2+\frac{n-j-1}{n-1}C(k)+\frac{[(n-j-1)/m]}{n-1}C(k) \\
&\le 2+C(k)\left(1+\frac{1}{m}\right).
\end{aligned}\tag{37}$$

We just let $C(k+1)=2+C(k)(1+(1/m))$ and the result for $k+1$ is proved. By Asume-Hoeffding's inequality, we have the following for arbitrary $a>0$:

$$P\left(\left|D_n(k)-\mathrm{ED}_n(k)\right|\ge a\right)\le 2e^{-a^2/2C(k)^2n}.\tag{38}$$

□

Theorem 5. *For a fixed k, one has*

$$\lim_{n\to\infty}p_k(n)=p_k\quad a.e.\tag{39}$$

Proof. By the Borel-Cantelli Lemma, we need to prove the following for arbitrary ε:

$$\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right)<\infty.\tag{40}$$

We have

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right) \\
&\quad=\sum_{n=1}^{\infty}P\left(\left|\frac{D_n(k)}{n}-\frac{\mathrm{ED}_n(k)}{n}+\frac{\mathrm{ED}_n(k)}{n}-p_k\right|>\varepsilon\right) \\
&\quad\le\sum_{n=1}^{\infty}P\left(\left|\frac{D_n(k)}{n}-\frac{\mathrm{ED}_n(k)}{n}\right|\ge\frac{\varepsilon}{2}\right) \\
&\qquad+\sum_{n=1}^{\infty}P\left(\left|\frac{\mathrm{ED}_n(k)}{n}-p_k\right|\ge\frac{\varepsilon}{2}\right).
\end{aligned}\tag{41}$$

Noticing that $\lim_{n\to\infty}\mathrm{ED}_k(n)/n=p_k$ and using Lemma 4, we obtain that there exists N, such that

$$\begin{aligned}
&\sum_{n=1}^{\infty}P\left(\left|p_k(n)-p_k\right|>\varepsilon\right) \\
&\quad\le\sum_{n=1}^{\infty}P\left(\left|D_n(k)-\mathrm{ED}_n(k)\right|\ge\frac{\varepsilon}{2}n\right)+N \\
&\quad\le\sum_{n=1}^{\infty}2e^{-(\varepsilon^2/4)n}+N \\
&\quad<\infty.
\end{aligned}\tag{42}$$

□

Remark 6. As a result, we can see that the distribution p_k obeys the following rule.

When $m\ll k$, $p_k\propto k^{-(2m+1)}$, the degree distribution obeys power law; when $m\gg k$, $p_k\propto 2^{-k}$, the degree distribution obeys exponential distribution; otherwise, $p_k=(2m/(4m-1))\prod_{j=2}^{k}((2m+j-3)/(4m+j-2))$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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Research Article

Uniform Stability Analysis of Fractional-Order BAM Neural Networks with Delays in the Leakage Terms

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A class of fractional-order BAM neural networks with delays in the leakage terms is considered. By using inequality technique and analysis method, several delay-dependent sufficient conditions are established to ensure the uniform stability of such networks. Moreover, the sufficient conditions guaranteeing the existence, uniqueness, and stability of the equilibrium point are also obtained. In addition, three simulation examples are given to demonstrate the effectiveness of the obtained results.

1. Introduction

The bidirectional associative memory (BAM) neural networks models, first proposed and studied by Kosko [1], have been widely applied within various engineering and scientific fields such as pattern recognition, signal and image processing, artificial intelligence, and combinatorial optimization [2]. In such applications, it is of prime importance to ensure that the designed neural networks are stable [3].

In hardware implementation of a neural network using analog electronic circuits, time delay will be inevitable and occur in the signal transmission among the neurons [4], which will affect the stability of the neural system and may lead to some complex dynamic behaviors such as oscillation, divergence, chaos, and instability or other poor performances of the neural networks [5]. In this case, the time delay may substantially affect the performance of the recurrent neural networks. Therefore, the study of stability for delayed neural networks is of both theoretical and practical importance. In the past few decades, a considerable number of sufficient conditions on the existence, uniqueness, and stability of equilibrium point for delayed BAM neural networks were reported under some assumptions; for example, see [2–17] and references therein.

In recent years, since the theory and application of fractional differential equations gradually developed [18–20], efforts have been made to study the complex dynamics of fractional-order neural networks. In [21], the authors firstly introduced a new class of cellular neural networks with fractional order. The peculiarity of the new cellular neural networks model consisted in replacing the traditional first order cell with a noninteger order one. The introduction of fractional-order cells, with a suitable choice of the coupling parameters, led to the onset of chaos in a two-cell system of a total order of less than three. A theoretical approach, based on the interaction between equilibrium points and limit cycles, was used to discover chaotic motions in fractional cellular neural networks. In [22], the authors investigated the existence of chaos by using the harmonic balance theory. A circuit realization of the proposed fractional two-cell chaotic cellular neural networks was reported and the corresponding strange attractor was also shown. In [23], the authors presented

an algorithm of numerical solution for fractional differential equations and investigated chaos control and synchronization in a fractional neuron network system. In [24], the authors proposed a fractional-order Hopfield neural network and investigated its stability by using energy function. In [25], a new type of stability and synchronization, α-exponential stability and α-synchronization, was investigated for a class of fractional-order neural networks. Several criteria were derived for such kind of stability of the addressed networks by handling a new fractional-order differential inequality. In [26], chaos and hyperchaos for fractional-order cellular neural networks were investigated by means of numerical simulations. The existence of chaotic and hyperchaotic attractors was verified with the related Lyapunov exponent spectrum, bifurcation diagram, and phase portraits. In [27], the authors investigated stability, multistability, bifurcations, and chaos for fractional-order Hopfield neural networks. In [28], the synchronization problem was studied for a class of fractional-order chaotic neural networks. By using the Mittag-Leffler function, M-matrix and linear feedback control, a sufficient condition was developed ensuring the synchronization of such neural models with the Caputo fractional derivatives. In [29], a class of fractional-order neural networks with delay was considered; a sufficient condition was established for the uniform stability of such networks. Moreover, the existence, uniqueness, and stability of its equilibrium point were also proved. In [30], the authors introduced memristor-based fractional-order neural networks. The conditions on the global Mittag-Leffler stability and synchronization were established by using Lyapunov method for these networks. In [31], the authors investigated the finite-time stability for Caputo fractional neural networks with distributed delay and established a delay-dependent stability criterion by using the theory of fractional calculus and generalized Gronwall-Bellman inequality approach. In [32], the global projective synchronization for fractional-order neural networks was investigated. Based on the preparation and some analysis techniques, several criteria were obtained to realize projective synchronization of fractional-order neural networks via combining open loop control and adaptive control.

Recently, some authors considered the uniform stability of delayed neural networks; for example, see [33–36] and references therein. In [33], the local uniform stability of competitive neural networks with different time-scales under vanishing perturbations was investigated; several stability conditions were established based on Gershgorin's Theorem. In [34], the authors considered the uniform asymptotic stability and global asymptotic stability of the equilibrium point for time-delays Hopfield neural networks. Several criteria of the system were derived by using the Lyapunov functional method and the linear matrix inequality approach for estimating the upper bound of the derivative of Lyapunov functional. In [35], the authors showed the uniform stability and existence and uniqueness of the equilibrium point of the fractional-order complex-valued neural networks with time delays firstly. In [36], the authors discuss the existence and global uniform asymptotic stability of almost periodic solutions for cellular neural networks. By utilizing the theory of the almost periodic differential equation and the Lyapunov functionals method, some sufficient conditions were obtained to ensure the existence and global uniform asymptotic stability. To the best of our knowledge, however, there are few results on the uniform stability analysis of fractional-order BAM neural networks.

Motivated by the above discussions, the objective of this paper is to study the uniform stability analysis of fractional-order BAM neural networks with delays in the leakage terms. In order to demonstrate the stability of our proposed model, a novel norm which can be found in [29, 35] will be introduced, and several delay-dependent sufficient conditions ensuring the uniform stability of our model will be established. Incidentally, when it comes to the proof of the existence, uniqueness, and stability of the equilibrium point of the proposed model, we will utilize the common norm for convenience.

The rest of the paper is structured as follows. In Section 2, we will present the proposed model and recall some necessary definitions and lemmas. In Section 3, a sufficient criterion ensuring the uniform stability of such neural networks is presented and the existence and uniqueness of the equilibrium of the model is also demonstrated. Three numerical examples are presented to manifest the effectiveness of our theoretical results in Section 4. Finally, the paper is concluded in Section 5.

2. Model Description and Preliminaries

In this paper, we consider the following fractional-order BAM neural networks with delays in the leakage terms:

$$\begin{aligned} D^{\alpha} x_i(t) &= -a_i x_i(t-\sigma) + \sum_{j=1}^{m} c_{ij} F_j\left(y_j(t)\right) \\ &\quad + \sum_{j=1}^{m} p_{ij} V_j\left(y_j(t-\tau)\right) + I_i, \quad i = 1, 2, \ldots, n, \\ D^{\alpha} y_j(t) &= -b_j y_j(t-\sigma) + \sum_{i=1}^{n} d_{ji} G_i\left(x_i(t)\right) \\ &\quad + \sum_{i=1}^{n} q_{ji} U_i\left(x_i(t-\tau)\right) + J_j, \quad j = 1, 2, \ldots, m, \end{aligned} \tag{1}$$

or in the vector form

$$\begin{aligned} D^{\alpha} x(t) &= -Ax(t-\sigma) + CF\left(y(t)\right) + PV\left(y(t-\tau)\right) + I, \\ D^{\alpha} y(t) &= -By(t-\sigma) + DG\left(x(t)\right) + QU\left(x(t-\tau)\right) + J, \end{aligned} \tag{2}$$

where D^{α} denotes Caputo fractional derivative of order α, $0 < \alpha < 1$; $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$, $y(t) = (y_1(t), y_2(t), \ldots, y_m(t))^T \in \mathbb{R}^m$, $x_i(t)$, and $y_j(t)$ are the state of the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y at time t, respectively; $F_j(y)$ and $V_j(y)$ denote the activation functions of the jth neuron from the neural field F_Y and $G_i(x)$ and $U_i(x)$ denote the activation functions of the ith neuron from the neural field F_X; I_i and

J_j are constants, which denote the external inputs on the ith neuron from F_X and the jth neuron from F_Y, respectively; the positive constants a_i and b_j denote the rates with which the ith neuron from the neural field F_X and the jth neuron from the neural field F_Y will reset their potential to the resting state in isolation when disconnected from the networks and external inputs, respectively; the constants c_{ij}, p_{ij}, d_{ji}, and q_{ji} denote the connection strengths; the nonnegative constants σ and τ denote the leakage delay and the transmission delay, respectively; $A = \operatorname{diag}(a_1, a_2, \ldots, a_n)$ and $B = \operatorname{diag}(b_1, b_2, \ldots, b_m)$ are diagonal matrices, C, D, P, and Q are the connection weight matrices; and $I = (I_1, I_2, \ldots, I_n)^T$ and $J = (J_1, J_2, \ldots, J_m)^T$ are the external inputs.

The initial conditions associated with system (1) are of the form

$$\begin{aligned} x_i(s) &= \psi_i(s), \quad s \in [-\gamma, 0], \ i = 1, 2, \ldots, n, \\ y_j(s) &= \phi_j(s), \quad s \in [-\gamma, 0], \ j = 1, 2, \ldots, m, \end{aligned} \tag{3}$$

where $\gamma = \max\{\sigma, \tau\}$, and it is usually assumed that $\psi_i(s)$, $\phi_j(s) \in C([-\gamma, 0], 0)$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$, with the norm given by $\|\psi(t)\| = \sum_{i=1}^{n} \sup_{t\in[-\gamma,0]}\{e^{-t}|\psi_i(t)|\}$ and $\|\phi(t)\| = \sum_{j=1}^{m} \sup_{t\in[-\gamma,0]}\{e^{-t}|\phi_i(t)|\}$.

Throughout this paper, we make the following assumption.

(H) The activation functions $F_j(\cdot)$, $Q_i(\cdot)$, $V_j(\cdot)$, and $U_i(\cdot)$ are Lipschitz continuous; that is, there exist constants $F_j > 0$, $G_i > 0$, $V_j > 0$, and $U_i > 0$ such that

$$\begin{aligned} \left|F_j(u) - F_j(v)\right| &\le F_j |u - v|, \\ \left|G_i(u) - G_i(v)\right| &\le G_i |u - v|, \\ \left|V_j(u) - V_j(v)\right| &\le V_j |u - v|, \\ \left|U_i(u) - U_i(v)\right| &\le U_i |u - v|, \end{aligned} \tag{4}$$

for any $u, v \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and $j = 1, 2, \ldots, m$.

To prove our results, the following definitions and lemma are necessary.

Definition 1 (see [18]). The Riemann-Liouville fractional integral with fractional-order $\alpha > 0$ of function $f(t)$ is defined as follows:

$${}_{t0}I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t - \tau)^{\alpha-1} f(\tau)\, d\tau, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function and $\Gamma(\tau) = \int_0^{\infty} t^{\tau-1} e^{-t} dt$.

Definition 2 (see [18]). The Caputo fractional derivative of fractional-order α of function $f(t)$ is defined as follows:

$$\begin{aligned} {}_{t0}^{C}D_t^{\alpha} f(t) &= {}_{t0}I_t^{n-\alpha} \frac{d^n}{dt^n} f(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t - \tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \end{aligned} \tag{6}$$

where n is the first integer greater than α; that is, $n-1 < \alpha < n$.

Particularly, when $0 < \alpha < 1$,

$$D^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t - \tau)^{-\alpha} f'(\tau)\, d\tau. \tag{7}$$

Definition 3. The solution of system (1) is said to be uniformly stable if for any $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that, for any two solutions $(x(t), y(t))^T$, $(\overline{x}(t), \overline{y}(t))^T$, of system (1) with initial functions $(\psi(t), \phi(t))^T$, $(\overline{\psi}(t), \overline{\phi}(t))^T$, respectively, it holds that

$$\|\overline{x}(t) - x(t)\| < \varepsilon, \qquad \left\|\overline{y}(t) - y(t)\right\| < \varepsilon, \tag{8}$$

for all $t \ge 0$, whenever

$$\left\|\overline{\psi}(s) - \psi(s)\right\| < \delta, \quad \left\|\overline{\phi}(s) - \phi(s)\right\| < \delta, \quad s \in [-\gamma, 0], \tag{9}$$

where

$$\begin{aligned} \left\|\overline{\psi}(s) - \psi(s)\right\| &= \sum_{i=1}^{n} \sup_{s\in[-\gamma,0]} \left\{e^{-s} \left|\overline{\psi}_i(s) - \psi_i(s)\right|\right\}, \\ \left\|\overline{\phi}(s) - \phi(s)\right\| &= \sum_{j=1}^{m} \sup_{s\in[-\gamma,0]} \left\{e^{-s} \left|\overline{\phi}_j(s) - \phi_j(s)\right|\right\}, \\ \|\overline{x}(t) - x(t)\| &= \sum_{i=1}^{n} \sup_{t} \left\{e^{-t} \left|\overline{x}_i(t) - x_i(t)\right|\right\}, \\ \left\|\overline{y}(t) - y(t)\right\| &= \sum_{j=1}^{m} \sup_{t} \left\{e^{-t} \left|\overline{y}_j(t) - y_j(t)\right|\right\}. \end{aligned} \tag{10}$$

Lemma 4 (see [20]). *Let n be a positive integer such that $n-1 < \alpha < n$; if $y(t) \in C^{n-1}[a, b]$, then*

$$I^{\alpha} D^{\alpha} y(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t - a)^k. \tag{11}$$

In particular, if $0 < \alpha \le 1$ and $y(t) \in C[a, b]$, then

$$I^{\alpha} D^{\alpha} y(t) = y(t) - y(a). \tag{12}$$

Remark 5. It is noted that when the leakage delay $\sigma = 0$, the system (1) becomes the following fractional-order BAM neural networks with delay

$$\begin{aligned} D^{\alpha} x_i(t) &= -a_i x_i(t) + \sum_{j=1}^{m} c_{ij} F_j\left(y_j(t)\right) \\ &\quad + \sum_{j=1}^{m} p_{ij} V_j\left(y_j(t - \tau)\right) + I_i, \\ D^{\alpha} y_j(t) &= -b_j y_j(t) + \sum_{i=1}^{n} d_{ji} G_i\left(x_i(t)\right) \\ &\quad + \sum_{i=1}^{n} q_{ji} U_i\left(x_i(t - \tau)\right) + J_j, \end{aligned} \tag{13}$$

with initial conditions

$$x_i(s) = \psi_i(s), \quad s \in [-\tau, 0], \qquad y_j(s) = \phi_j(s), \quad s \in [-\tau, 0], \tag{14}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

3. Main Results

Theorem 6. *Under assumption (H), the system* (1) *is uniformly stable, if* $S_1 > 0$, $T_1 > 0$, *and* $S_1T_1 > S_2T_2$ *hold, where*

$$S_1 = 1 - \max_{1\le i\le n}\{a_i\} e^{-\sigma}, \qquad T_1 = 1 - \max_{1\le j\le m}\{b_j\} e^{-\sigma},$$
$$S_2 = \sum_{i=1}^{n}\max_{1\le j\le m}\{|c_{ij}| F_j\} + \sum_{i=1}^{n}\max_{1\le j\le m}\{|p_{ij}| V_j\} e^{-\tau},$$
$$T_2 = \sum_{j=1}^{m}\max_{1\le i\le n}\{|d_{ji}| G_i\} + \sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}| U_i\} e^{-\tau}. \tag{15}$$

Proof. Assume that $(x(t), y(t))^T = (x_1(t), \ldots, x_n(t), y_1(t), \ldots, y_m(t))^T$ and $(\overline{x}(t), \overline{y}(t))^T = (\overline{x}_1(t), \ldots, \overline{x}_n(t), \overline{y}_1(t), \ldots, \overline{y}_m(t))^T$ are any two solutions of system (1) with the initial conditions (3), then

$$\begin{aligned}
D^\alpha(\overline{x}_i(t) - x_i(t)) &= -a_i(\overline{x}_i(t-\sigma) - x_i(t-\sigma)) \\
&\quad + \sum_{j=1}^{m} c_{ij}\left(F_j(\overline{y}_j(t)) - F_j(y_j(t))\right) \\
&\quad + \sum_{j=1}^{m} p_{ij}\left(V_j(\overline{y}_j(t-\tau)) - V_j(y_j(t-\tau))\right), \\
&\qquad i = 1, 2, \ldots, n, \\
D^\alpha(\overline{y}_j(t) - y_j(t)) &= -b_j(\overline{y}_j(t-\sigma) - y_j(t-\sigma)) \\
&\quad + \sum_{i=1}^{n} d_{ji}(G_i(\overline{x}_i(t)) - G_i(x_i(t))) \\
&\quad + \sum_{i=1}^{n} q_{ji}(U_i(\overline{x}_i(t-\tau)) - U_i(x_i(t-\tau))), \\
&\qquad j = 1, 2, \ldots, m.
\end{aligned} \tag{16}$$

From Lemma 4, we can obtain

$$\begin{aligned}
\overline{x}_i(t) - x_i(t) &= \overline{\psi}_i(0) - \psi_i(0) \\
&\quad + I^\alpha\Big[-a_i(\overline{x}_i(t-\sigma) - x_i(t-\sigma)) \\
&\quad + \sum_{j=1}^{m} c_{ij}\left(F_j(\overline{y}_j(t)) - F_j(y_j(t))\right) \\
&\quad + \sum_{j=1}^{m} p_{ij}\left(V_j(\overline{y}_j(t-\tau)) - V_j(y_j(t-\tau))\right)\Big], \\
&\qquad i = 1, 2, \ldots, n, \\
\overline{y}_j(t) - y_j(t) &= \overline{\phi}_j(0) - \phi_j(0) \\
&\quad + I^\alpha\Big[-b_j(\overline{y}_j(t-\sigma) - y_j(t-\sigma)) \\
&\quad + \sum_{i=1}^{n} d_{ji}(G_i(\overline{x}_i(t)) - G_i(x_i(t))) \\
&\quad + \sum_{i=1}^{n} q_{ji}(U_i(\overline{x}_i(t-\tau)) - U_i(x_i(t-\tau)))\Big], \\
&\qquad j = 1, 2, \ldots, m.
\end{aligned} \tag{17}$$

Further, we have that

$$\begin{aligned}
e^{-t}|\overline{x}_i(t) - x_i(t)| &\le e^{-t}|\overline{\psi}_i(0) - \psi_i(0)| + \frac{1}{\Gamma(\alpha)} e^{-t} \\
&\quad \times \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Big[a_i|\overline{x}_i(s-\sigma) - x_i(s-\sigma)| \\
&\quad + \sum_{j=1}^{m} |c_{ij}| \left|F_j(\overline{y}_j(s)) - F_j(y_j(s))\right| \\
&\quad + \sum_{j=1}^{m} |p_{ij}| \left|V_j(\overline{y}_j(s-\tau)) - V_j(y_j(s-\tau))\right|\Big] ds,
\end{aligned} \tag{18}$$

$$\begin{aligned}
e^{-t}|\overline{y}_j(t) - y_j(t)| &\le e^{-t}|\overline{\phi}_j(0) - \phi_j(0)| + \frac{1}{\Gamma(\alpha)} e^{-t} \\
&\quad \times \int_0^t (t-s)^{\alpha-1} \\
&\quad \times \Big[b_j|\overline{y}_j(s-\sigma) - y_j(s-\sigma)| \\
&\quad + \sum_{i=1}^{n} |d_{ji}| \left|G_i(\overline{x}_i(s)) - G_i(x_i(s))\right| \\
&\quad + \sum_{i=1}^{n} |q_{ji}| \left|U_i(\overline{x}_i(s-\tau)) - U_i(x_i(s-\tau))\right|\Big] ds,
\end{aligned} \tag{19}$$

for $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$.

It follows from assumption (H) and inequality (18) that

$$
\begin{aligned}
&e^{-t}\left|\overline{x}_i(t)-x_i(t)\right| \\
&\quad\le e^{-t}\left|\overline{\psi}_i(0)-\psi_i(0)\right|+a_i\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s+\sigma)}e^{-(s-\sigma)} \\
&\qquad\quad\times\left|\overline{x}_i(s-\sigma)-x_i(s-\sigma)\right|ds \\
&\qquad+\sum_{j=1}^m\left|c_{ij}\right|\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s)}e^{-s} \\
&\qquad\quad\times\left|F_j\left(\overline{y}_j(s)\right)-F_j\left(y_j(s)\right)\right|ds \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s+\tau)}e^{-(s-\tau)} \\
&\qquad\quad\times\left|V_j\left(\overline{y}_j(s-\tau)\right)-V_j\left(y_j(s-\tau)\right)\right|ds \\
&\quad\le e^{-t}\left|\overline{\psi}_i(0)-\psi_i(0)\right|+a_i\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s+\sigma)}e^{-(s-\sigma)} \\
&\qquad\quad\times\left|\overline{x}_i(s-\sigma)-x_i(s-\sigma)\right|ds \\
&\qquad+\sum_{j=1}^m\left|c_{ij}\right|F_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s)}e^{-s}\left|\overline{y}_j(s)-y_j(s)\right|ds \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|V_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s+\tau)}e^{-(s-\tau)} \\
&\qquad\quad\times\left|\overline{y}_j(s-\tau)-y_j(s-\tau)\right|ds \\
&\quad= e^{-t}\left|\overline{\psi}_i(0)-\psi_i(0)\right|+a_i\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^{\sigma} (t-s)^{\alpha-1}e^{-(t-s+\sigma)}e^{-(s-\sigma)} \\
&\qquad\quad\times\left|\overline{\psi}_i(s-\sigma)-\psi_i(s-\sigma)\right|ds \\
&\qquad+a_i\frac{1}{\Gamma(\alpha)}\int_{\sigma}^t (t-s)^{\alpha-1}e^{-(t-s+\sigma)}e^{-(s-\sigma)} \\
&\qquad\quad\times\left|\overline{x}_i(s-\sigma)-x_i(s-\sigma)\right|ds \\
&\qquad+\sum_{j=1}^m\left|c_{ij}\right|F_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s)}e^{-s}\left|\overline{y}_j(s)-y_j(s)\right|ds \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|V_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^{\tau} (t-s)^{\alpha-1}e^{-(t-s+\tau)}e^{-(s-\tau)} \\
&\qquad\quad\times\left|\overline{\phi}_j(s-\tau)-\phi_j(s-\tau)\right|ds \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|V_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_{\tau}^t (t-s)^{\alpha-1}e^{-(t-s+\tau)}e^{-(s-\tau)} \\
&\qquad\quad\times\left|\overline{y}_j(s-\tau)-y_j(s-\tau)\right|ds \\
&\quad= e^{-t}\left|\overline{\psi}_i(0)-\psi_i(0)\right|+a_i\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_{-\sigma}^0 (t-\gamma-\sigma)^{\alpha-1}e^{-(t-\gamma)}e^{-\gamma} \\
&\qquad\quad\times\left|\overline{\psi}_i(\gamma)-\psi_i(\gamma)\right|d\gamma \\
&\qquad+a_i\frac{1}{\Gamma(\alpha)}\int_0^{t-\sigma} (t-\gamma-\sigma)^{\alpha-1}e^{-(t-\gamma)}e^{-\gamma} \\
&\qquad\quad\times\left|\overline{x}_i(\gamma)-x_i(\gamma)\right|d\gamma \\
&\qquad+\sum_{j=1}^m\left|c_{ij}\right|F_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^t (t-s)^{\alpha-1}e^{-(t-s)}e^{-s}\left|\overline{y}_j(s)-y_j(s)\right|ds \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|V_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_{-\tau}^0 (t-\beta-\tau)^{\alpha-1}e^{-(t-\beta)}e^{-\beta}\left|\overline{\phi}_j(\beta)-\phi_j(\beta)\right|d\beta \\
&\qquad+\sum_{j=1}^m\left|p_{ij}\right|V_j\frac{1}{\Gamma(\alpha)} \\
&\qquad\times\int_0^{t-\tau} (t-\beta-\tau)^{\alpha-1}e^{-(t-\beta)}e^{-\beta}\left|\overline{y}_j(\beta)-y_j(\beta)\right|d\beta \\
&\quad\le \sup_{t\in[-\gamma,0]}\left\{e^{-t}\left|\overline{\psi}_i(t)-\psi_i(t)\right|\right\} \\
&\qquad+a_i\sup_{t\in[-\gamma,0]}\left\{e^{-t}\left|\overline{\psi}_i(t)-\psi_i(t)\right|\right\}e^{-\sigma}
\end{aligned}
$$

$$
\begin{aligned}
&\times \frac{1}{\Gamma(\alpha)} \int_{t-\sigma}^{t} \theta^{\alpha-1} e^{-\theta} d\theta \\
&+ a_i \sup_t \left\{ e^{-t} \left| \overline{x}_i(t) - x_i(t) \right| \right\} e^{-\sigma} \\
&\times \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\sigma} \theta^{\alpha-1} e^{-\theta} d\theta \\
&+ \max_{1\le j\le m} \left\{ \left| c_{ij} \right| F_j \right\} \sum_{j=1}^{m} \sup_t \left\{ e^{-t} \left| \overline{y}_j(t) - y_j(t) \right| \right\} \\
&\times \frac{1}{\Gamma(\alpha)} \int_{0}^{t} u^{\alpha-1} e^{-u} du \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} \sum_{j=1}^{m} \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\phi}_j(t) - \phi_j(t) \right| \right\} e^{-\tau} \\
&\times \frac{1}{\Gamma(\alpha)} \int_{t-\tau}^{t} v^{\alpha-1} e^{-v} dv \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} \sum_{j=1}^{m} \sup_t \left\{ e^{-t} \left| \overline{y}_j(t) - y_j(t) \right| \right\} e^{-\tau} \\
&\times \frac{1}{\Gamma(\alpha)} \int_{0}^{t-\tau} v^{\alpha-1} e^{-v} dv \\
\le{}& \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\psi}_i(t) - \psi_i(t) \right| \right\} \\
&+ a_i \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\psi}_i(t) - \psi_i(t) \right| \right\} e^{-\sigma} \\
&+ a_i \sup_t \left\{ e^{-t} \left| \overline{x}_i(t) - x_i(t) \right| \right\} e^{-\sigma} \\
&+ \max_{1\le j\le m} \left\{ \left| c_{ij} \right| F_j \right\} \sum_{j=1}^{m} \sup_t \left\{ e^{-t} \left| \overline{y}_j(t) - y_j(t) \right| \right\} \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} \sum_{j=1}^{m} \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\phi}_j(t) - \phi_j(t) \right| \right\} e^{-\tau} \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} \sum_{j=1}^{m} \sup_t \left\{ e^{-t} \left| \overline{y}_j(t) - y_j(t) \right| \right\} e^{-\tau} \\
={}& \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\psi}_i(t) - \psi_i(t) \right| \right\} \\
&+ a_i e^{-\sigma} \sup_{t\in[-\gamma,0]} \left\{ e^{-t} \left| \overline{\psi}_i(t) - \psi_i(t) \right| \right\} \\
&+ a_i e^{-\sigma} \sup_t \left\{ e^{-t} \left| \overline{x}_i(t) - x_i(t) \right| \right\} \\
&+ \max_{1\le j\le m} \left\{ \left| c_{ij} \right| F_j \right\} \left\| \overline{y}(t) - y(t) \right\| \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau} \left\| \overline{\phi}(t) - \phi(t) \right\| \\
&+ \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau} \left\| \overline{y}(t) - y(t) \right\|.
\end{aligned}
\tag{20}
$$

From (20), we can get

$$
\begin{aligned}
\left\| \overline{x}(t) - x(t) \right\| &= \sum_{i=1}^{n} \sup_t \left\{ e^{-t} \left| \overline{x}_i(t) - x_i(t) \right| \right\} \\
&\le \left\| \overline{\psi}(t) - \psi(t) \right\| + \max_{1\le i\le n} \{a_i\} e^{-\sigma} \left\| \overline{\psi}(t) - \psi(t) \right\| \\
&\quad + \max_{1\le i\le n} \{a_i\} e^{-\sigma} \left\| \overline{x}(t) - x(t) \right\| \\
&\quad + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| c_{ij} \right| F_j \right\} \left\| \overline{y}(t) - y(t) \right\| \\
&\quad + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau} \left\| \overline{\phi}(t) - \phi(t) \right\| \\
&\quad + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau} \left\| \overline{y}(t) - y(t) \right\|,
\end{aligned}
\tag{21}
$$

which implies

$$
\begin{aligned}
&\left\| \overline{x}(t) - x(t) \right\| \\
&\quad \le \frac{\sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| c_{ij} \right| F_j \right\} + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau}}{1 - \max_{1\le i\le n} \{a_i\} e^{-\sigma}} \\
&\qquad \times \left\| \overline{y}(t) - y(t) \right\| + \frac{1 + \max_{1\le i\le n} \{a_i\} e^{-\sigma}}{1 - \max_{1\le i\le n} \{a_i\} e^{-\sigma}} \\
&\qquad \times \left\| \overline{\psi}(t) - \psi(t) \right\| + \frac{\sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left| p_{ij} \right| V_j \right\} e^{-\tau}}{1 - \max_{1\le i\le n} \{a_i\} e^{-\sigma}} \\
&\qquad \times \left\| \overline{\phi}(t) - \phi(t) \right\| \\
&\quad = \frac{S_2}{S_1} \left\| \overline{y}(t) - y(t) \right\| + \frac{S_3}{S_1} \left\| \overline{\psi}(t) - \psi(t) \right\| \\
&\qquad + \frac{S_4}{S_1} \left\| \overline{\phi}(t) - \phi(t) \right\|,
\end{aligned}
\tag{22}
$$

where $S_3 = 1 + \max_{1\le i\le n}\{a_i\}e^{-\sigma}$, $S_4 = \sum_{i=1}^{n} \max_{1\le j\le m} \{|p_{ij}|V_j\}e^{-\tau}$.

Similarly, it follows from assumption (H) and inequality (19) that

$$
\begin{aligned}
&\left\| \overline{y}(t) - y(t) \right\| \\
&\quad \le \frac{\sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left| d_{ji} \right| G_i \right\} + \sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left| q_{ji} \right| U_i \right\} e^{-\tau}}{1 - \max_{1\le j\le m} \{b_j\} e^{-\sigma}} \\
&\qquad \times \left\| \overline{x}(t) - x(t) \right\| + \frac{1 + \max_{1\le j\le m} \{b_j\} e^{-\sigma}}{1 - \max_{1\le j\le m} \{b_j\} e^{-\sigma}}
\end{aligned}
$$

$$\times \left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{\sum_{j=1}^{m}\max_{1\le i\le n}\left\{\left|q_{ji}\right|U_i\right\}e^{-\tau}}{1-\max_{1\le j\le m}\left\{b_j\right\}e^{-\sigma}} \times \left\|\overline{\psi}(t)-\psi(t)\right\|$$
$$= \frac{T_2}{T_1}\left\|\overline{x}(t)-x(t)\right\| + \frac{T_3}{T_1}\left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{T_4}{T_1}\left\|\overline{\psi}(t)-\psi(t)\right\|, \tag{23}$$

where $T_3 = 1+\max_{1\le j\le m}\{b_j\}e^{-\sigma}$, $T_4 = \sum_{j=1}^{m}\max_{1\le i\le n}\{|q_{ji}|U_i\}e^{-\tau}$.

Substituting (23) into (22), we can obtain

$$\begin{aligned}\left\|\overline{x}(t)-x(t)\right\| &\le \frac{S_2}{S_1}\left[\frac{T_2}{T_1}\left\|\overline{x}(t)-x(t)\right\| + \frac{T_3}{T_1}\left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{T_4}{T_1}\left\|\overline{\psi}(t)-\psi(t)\right\|\right] \\ &\quad + \frac{S_3}{S_1}\left\|\overline{\psi}(t)-\psi(t)\right\| + \frac{S_4}{S_1}\left\|\overline{\phi}(t)-\phi(t)\right\| \\ &= \frac{S_2T_2}{S_1T_1}\left\|\overline{x}(t)-x(t)\right\| + \frac{S_2T_4+S_3T_1}{S_1T_1}\left\|\overline{\psi}(t)-\psi(t)\right\| + \frac{S_2T_3+S_4T_1}{S_1T_1}\left\|\overline{\phi}(t)-\phi(t)\right\|.\end{aligned} \tag{24}$$

By using condition $S_1T_1 > S_2T_2$, (24) implies

$$\left\|\overline{x}(t)-x(t)\right\| \le \frac{S_2T_4+S_3T_1}{S_1T_1-S_2T_2}\left\|\overline{\psi}(t)-\psi(t)\right\| + \frac{S_2T_3+S_4T_1}{S_1T_1-S_2T_2}\left\|\overline{\phi}(t)-\phi(t)\right\|. \tag{25}$$

And, substituting (22) into (23), we can get

$$\begin{aligned}\left\|\overline{y}(t)-y(t)\right\| &\le \frac{T_2}{T_1}\left[\frac{S_2}{S_1}\left\|\overline{y}(t)-y(t)\right\| + \frac{S_3}{S_1}\left\|\overline{\psi}(t)-\psi(t)\right\| + \frac{S_4}{S_1}\left\|\overline{\phi}(t)-\phi(t)\right\|\right] \\ &\quad + \frac{T_3}{T_1}\left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{T_4}{T_1}\left\|\overline{\psi}(t)-\psi(t)\right\| \\ &= \frac{T_2S_2}{T_1S_1}\left\|\overline{y}(t)-y(t)\right\| + \frac{T_2S_4+T_3S_1}{T_1S_1}\left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{T_2S_3+T_4S_1}{T_1S_1}\left\|\overline{\psi}(t)-\psi(t)\right\|.\end{aligned} \tag{26}$$

By using condition $S_1T_1 > S_2T_2$, (26) implies

$$\left\|\overline{y}(t)-y(t)\right\| \le \frac{T_2S_4+T_3S_1}{T_1S_1-T_2S_2}\left\|\overline{\phi}(t)-\phi(t)\right\| + \frac{T_2S_3+T_4S_1}{T_1S_1-T_2S_2}\left\|\overline{\psi}(t)-\psi(t)\right\|. \tag{27}$$

If we take

$$\left\|\overline{\psi}(t)-\psi(t)\right\| \le \frac{\varepsilon_1}{(S_2T_4+S_3T_1)/(S_1T_1-S_2T_2)} = \frac{\varepsilon_1}{2\delta_1}, \qquad \left\|\overline{\phi}(t)-\phi(t)\right\| \le \frac{\varepsilon_1}{(S_2T_3+S_4T_1)/(S_1T_1-S_2T_2)} = \frac{\varepsilon_1}{2\delta_2}, \tag{28}$$

where

$$\delta_1 = \frac{S_2T_4+S_3T_1}{S_1T_1-S_2T_2}, \qquad \delta_2 = \frac{S_2T_3+S_4T_1}{S_1T_1-S_2T_2}. \tag{29}$$

From (25), we can obtain

$$\left\|\overline{x}(t)-x(t)\right\| \le \varepsilon_1. \tag{30}$$

If we take

$$\left\|\overline{\phi}(t)-\phi(t)\right\| \le \frac{\varepsilon_2}{(T_2S_4+T_3S_1)/(T_1S_1-T_2S_2)} = \frac{\varepsilon_2}{2\delta_3}, \qquad \left\|\overline{\psi}(t)-\psi(t)\right\| \le \frac{\varepsilon_2}{(T_2S_3+T_4S_1)/(T_1S_1-T_2S_2)} = \frac{\varepsilon_2}{2\delta_4}, \tag{31}$$

where

$$\delta_3 = \frac{T_2S_4+T_3S_1}{T_1S_1-T_2S_2}, \qquad \delta_4 = \frac{T_2S_3+T_4S_1}{T_1S_1-T_2S_2}, \tag{32}$$

from (27), we can get

$$\left\|\overline{y}(t)-y(t)\right\| \le \varepsilon_2. \tag{33}$$

From (30) and (33), we say that, for any $\varepsilon = \max\{\varepsilon_1, \varepsilon_2\} > 0$, then there exists a constant $\delta = \varepsilon/\max\{\delta_5, \delta_6\} > 0$, $\delta_5 = \max\{\delta_1, \delta_4\}$, or $\delta_6 = \max\{\delta_2, \delta_3\}$ such that $\|\overline{x}(t)-x(t)\| < \varepsilon$, $\|\overline{y}(t)-y(t)\| < \varepsilon$, when $\|\overline{\psi}(t)-\psi(t)\| < \delta$, $\|\overline{\phi}(t)-\phi(t)\| < \delta$, which means that the solution of system (1) is uniformly stable. The proof is completed. □

Theorem 7. *Under assumption (H) and the conditions of Theorem 6, the system* (1) *has a unique equilibrium point, which is uniformly stable if* $W_1 < \min_{1\le j\le m}\{b_j\}$ *and* $W_2 < \min_{1\le i\le n}\{a_i\}$ *hold, where*

$$W_1 = \sum_{i=1}^{n}\max_{1\le j\le m}\left\{\left|c_{ij}\right|F_j\right\} + \sum_{i=1}^{n}\max_{1\le j\le m}\left\{\left|p_{ij}\right|V_j\right\},$$
$$W_2 = \sum_{j=1}^{m}\max_{1\le i\le n}\left\{\left|d_{ji}\right|G_i\right\} + \sum_{j=1}^{m}\max_{1\le i\le n}\left\{\left|q_{ji}\right|U_i\right\}. \tag{34}$$

Proof. Let $a_i x_i^* = u_i^*$, $b_j y_j^* = v_j^*$, and construct a mapping $\Upsilon(u,v) : \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ defined by

$$\Upsilon(u,v) = \left(\Upsilon_1(u,v), \Upsilon_2(u,v), \ldots, \Upsilon_n(u,v), \Upsilon_{n+1}(u,v), \Upsilon_{n+2}(u,v), \ldots, \Upsilon_{n+m}(u,v)\right)^T, \tag{35}$$

where

$$\Upsilon_i(u,v) = \sum_{j=1}^{m} c_{ij}F_j\left(\frac{v_j}{b_j}\right) + \sum_{j=1}^{m} p_{ij}V_j\left(\frac{v_j}{b_j}\right) + I_i,$$
$$\Upsilon_{n+j}(u,v) = \sum_{i=1}^{n} d_{ji}G_i\left(\frac{u_i}{a_i}\right) + \sum_{i=1}^{n} q_{ji}U_i\left(\frac{u_i}{a_i}\right) + J_j, \tag{36}$$

for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$.

Now, we will show that $\Upsilon(u,v)$ is a contraction mapping on $\mathbb{R}^{n+m}$. In fact, for any two different points $(u,v) = (u_1,\ldots,u_n,v_1,\ldots,v_m)^T$ and $(\overline{u},\overline{v}) = (\overline{u}_1,\ldots,\overline{u}_n,\overline{v}_1,\ldots,\overline{v}_m)^T$, we have

$$\begin{aligned}
&\|\Upsilon(\overline{u},\overline{v}) - \Upsilon(u,v)\| \\
&= \sum_{i=1}^{n}\left|\sum_{j=1}^{m} c_{ij}\left(F_j\left(\frac{\overline{v}_j}{b_j}\right) - F_j\left(\frac{v_j}{b_j}\right)\right) + \sum_{j=1}^{m} p_{ij}\left(V_j\left(\frac{\overline{v}_j}{b_j}\right) - V_j\left(\frac{v_j}{b_j}\right)\right)\right| \\
&\quad + \sum_{j=1}^{m}\left|\sum_{i=1}^{n} d_{ji}\left(G_i\left(\frac{\overline{u}_i}{a_i}\right) - G_i\left(\frac{u_i}{a_i}\right)\right) + \sum_{i=1}^{n} q_{ji}\left(U_i\left(\frac{\overline{u}_i}{a_i}\right) - U_i\left(\frac{u_i}{a_i}\right)\right)\right| \\
&\le \sum_{i=1}^{n}\left[\sum_{j=1}^{m}\left|c_{ij}\right|F_j\frac{\left|\overline{v}_j - v_j\right|}{b_j} + \sum_{j=1}^{m}\left|p_{ij}\right|V_j\frac{\left|\overline{v}_j - v_j\right|}{b_j}\right] \\
&\quad + \sum_{j=1}^{m}\left[\sum_{i=1}^{n}\left|d_{ij}\right|G_i\frac{\left|\overline{u}_i - u_i\right|}{a_i} + \sum_{i=1}^{n}\left|q_{ij}\right|U_i\frac{\left|\overline{u}_i - u_i\right|}{a_i}\right] \\
&\le \left(\frac{\sum_{i=1}^{n}\max_{1\le j\le m}\left\{\left|c_{ij}\right|F_j\right\} + \sum_{i=1}^{n}\max_{1\le j\le m}\left\{\left|p_{ij}\right|V_j\right\}}{\min_{1\le j\le m}\left\{b_j\right\}}\right) \times \sum_{j=1}^{m}\left|\overline{v}_j - v_j\right| \\
&\quad + \left(\frac{\sum_{j=1}^{m}\max_{1\le i\le n}\left\{\left|d_{ji}\right|G_i\right\} + \sum_{j=1}^{m}\max_{1\le i\le n}\left\{\left|q_{ji}\right|U_i\right\}}{\min_{1\le i\le n}\left\{a_i\right\}}\right) \times \sum_{i=1}^{n}\left|\overline{u}_i - u_i\right| \\
&= \frac{W_1}{\min_{1\le j\le m}\left\{b_j\right\}}\sum_{j=1}^{m}\left|\overline{v}_j - v_j\right| + \frac{W_2}{\min_{1\le i\le n}\left\{a_i\right\}}\sum_{i=1}^{n}\left|\overline{u}_i - u_i\right|.
\end{aligned} \tag{37}$$

By using conditions $W_1 < \min_{1\le j\le m}\{b_j\}$ and $W_2 < \min_{1\le i\le n}\{a_i\}$, (37) implies

$$\|\Upsilon(\overline{u},\overline{v}) - \Upsilon(u,v)\| < \sum_{j=1}^{m}\left|\overline{v}_j - v_j\right| + \sum_{i=1}^{n}\left|\overline{u}_i - u_i\right| = \|(\overline{u},\overline{v}) - (u,v)\|, \tag{38}$$

which implies that $\Upsilon(u,v)$ is a contraction mapping on $\mathbb{R}^{n+m}$. Hence, there exists a unique fixed point $(u^*,v^*) = (u_1^*,\ldots,u_n^*,v_1^*,\ldots,v_m^*)^T$ such that $\Upsilon(u^*,v^*) = (u^*,v^*)$; that is

$$u_i^* = \sum_{j=1}^{m} c_{ij}F_j\left(\frac{v_j}{b_j}\right) + \sum_{j=1}^{m} p_{ij}V_j\left(\frac{v_j}{b_j}\right) + I_i,$$
$$v_j^* = \sum_{i=1}^{n} d_{ji}G_i\left(\frac{u_i}{a_i}\right) + \sum_{i=1}^{n} q_{ji}U_i\left(\frac{u_i}{a_i}\right) + J_j, \tag{39}$$

for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$. That is

$$-a_i x_i^* + \sum_{j=1}^{m} c_{ij}F_j\left(y_j^*\right) + \sum_{j=1}^{m} p_{ij}V_j\left(y_j^*\right) + I_i = 0,$$
$$-b_j y_j^* + \sum_{i=1}^{n} d_{ji}G_i\left(x_i^*\right) + \sum_{i=1}^{n} q_{ji}U_i\left(x_i^*\right) + J_j = 0, \tag{40}$$

for $i = 1,2,\ldots,n$, $j = 1,2,\ldots,m$, which implies that (x^*,y^*) is an equilibrium point of system (1). Moreover, it follows from Theorem 6 that (x^*,y^*) is uniformly stable. The proof is completed. □

Corollary 8. *Under assumption (H), the system* (13) *is uniformly stable, if* $\widetilde{S}_1 > 0$, $\widetilde{T}_1 > 0$, *and* $\widetilde{S}_1\widetilde{T}_1 > \widetilde{S}_2\widetilde{T}_2$ *hold, where*

$$\widetilde{S}_1 = 1 - \max_{1\le i\le n}\{a_i\}, \qquad \widetilde{T}_1 = 1 - \max_{1\le j\le m}\{b_j\},$$

$$\widetilde{S}_2 = \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left|c_{ij}\right| F_j \right\} + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left|p_{ij}\right| V_j \right\} e^{-\tau},$$

$$\widetilde{T}_2 = \sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left|d_{ji}\right| G_i \right\} + \sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left|q_{ji}\right| U_i \right\} e^{-\tau}. \tag{41}$$

Proof. Similar to the proof of Theorem 6, we can obtain the above Corollary 8; thus, we omit it. □

Corollary 9. *Under assumption (H) and the conditions of Corollary 8, the system* (13) *has a unique equilibrium point, which is uniformly stable if* $\widetilde{W}_1 < \min_{1\le j\le m}\{b_j\}$ *and* $\widetilde{W}_2 < \min_{1\le i\le n}\{a_i\}$ *hold, where*

$$\widetilde{W}_1 = \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left|c_{ij}\right| F_j \right\} + \sum_{i=1}^{n} \max_{1\le j\le m} \left\{ \left|p_{ij}\right| V_j \right\},$$
$$\widetilde{W}_2 = \sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left|d_{ji}\right| G_i \right\} + \sum_{j=1}^{m} \max_{1\le i\le n} \left\{ \left|q_{ji}\right| U_i \right\}. \tag{42}$$

Proof. Similar to the proof of Theorem 7, we can obtain the above Corollary 9; thus, we omit it.

□

Remark 10. In [25], the authors investigated α-stability and α-synchronization of fractional-order neural networks without delays. In [28], the authors introduced a class of fractional-order chaotic neural networks without delays and discussed the synchronization of such networks. In [29], the authors took the constant delay into account and discussed the dynamic analysis of a class of fractional-order neural networks with constant delay. In [35], the authors investigated the uniform stability of fractional-order complex-valued neural networks with constant delay. Different from the previous works, here, we have viewed the stability analysis of fractional-order BAM neural networks with delays in the leakage terms.

Remark 11. In [29, 35], several delay-independent stability conditions were given for fractional-order neural networks with constant delay. In this paper, a delay-dependent stability condition was provided. It is known that delay-dependent conditions are usually less conservative than delay-independent ones, especially in the case when the delay size is small [10]. In addition, the positive constants c_i $(i = 0, 1, \ldots, n)$ in the model of [29] was required satisfying $0 < c_i < 1$ $(i = 0, 1, \ldots, n)$. However, the obtained results in this paper show that when the leakage delay $\sigma > 0$, the positive constants $a_i > 1$ $(i = 0, 1, \ldots, n)$ and $b_j > 1$ $(j = 0, 1, \ldots, m)$ could be possible, and the simulation examples in the next section verify the validity of our results.

4. Examples

Example 1. Consider the following fractional-order BAM neural networks with delays in the leakage terms:

$$\begin{aligned} D^{\alpha} x_i(t) &= -a_i x_i(t-\sigma) + \sum_{j=1}^{2} c_{ij} F_j\left(y_j(t)\right) \\ &\quad + \sum_{j=1}^{2} p_{ij} V_j\left(y_j(t-\tau)\right) + I_i, \quad i = 1, 2, \\ D^{\alpha} y_j(t) &= -b_j y_j(t-\sigma) + \sum_{i=1}^{2} d_{ji} G_i\left(x_i(t)\right) \\ &\quad + \sum_{i=1}^{2} q_{ji} U_i\left(x_i(t-\tau)\right) + J_j, \quad j = 1, 2, \end{aligned} \tag{43}$$

where $\alpha = 0.95$, $\sigma = 0.25$, $\tau = 0.50$, $A = \mathrm{diag}(0.55, 0.60)$, $B = \mathrm{diag}(0.50, 0.50)$,

$$\begin{aligned} C &= \begin{bmatrix} 1.22 & 0.80 \\ 0.65 & -0.45 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.55 & 0.43 \\ -0.32 & 0.42 \end{bmatrix}, \\ P &= \begin{bmatrix} 0.38 & -0.32 \\ 0.45 & 0.80 \end{bmatrix}, \\ Q &= \begin{bmatrix} -0.62 & 0.34 \\ 0.45 & 1.10 \end{bmatrix}, \qquad I = [0.84, 1.22]^T, \\ J &= [-0.48, 0.75]^T, \\ G_1(x) &= G_2(x) = U_1(x) = U_2(x) \\ &= \frac{1}{10}\left(|x+1| + |x-1|\right), \\ F_1(y) &= F_2(y) = V_1(y) = V_2(y) \\ &= \frac{1}{20}\left(|y+1| + |y-1|\right). \end{aligned} \tag{44}$$

By calculation, $F_1 = F_2 = V_1 = V_2 = 0.1$, $G_1 = G_2 = U_1 = U_2 = 0.2$, $S_1 = 1 - 0.60e^{-0.25} = 0.5327$, $T_1 = 1 - 0.5e^{-0.25} = 0.6106$, $S_2 = 1.22 \times 0.1 + 0.65 \times 0.1 + (0.38 \times 0.1 + 0.80 \times 0.1)e^{-0.5} = 0.2586$, and $T_2 = 0.55 \times 0.2 + 0.42 \times 0.2 + (0.62 \times 0.2 + 1.10 \times 0.2)e^{-0.5} = 0.4026$, which satisfy $S_1T_1 > S_2T_2$; according to Theorem 6, when we select the appropriate initial values, the system (43) could realize uniform stability. Furthermore, we have $W_1 = 1.22 \times 0.1 + 0.65 \times 0.1 + 0.38 \times 0.1 + 0.80 \times 0.1 = 0.3050 < \min_{1\le j\le 2}\{b_j\} = 0.50$, $W_2 = 0.55 \times 0.2 + 0.42 \times 0.2 + 0.62 \times 0.2 + 1.10 \times 0.2 = 0.5380 < \min_{1\le i\le 2}\{a_i\} = 0.55$; by utilizing Theorem 7, we can obtain that the system (43) has an unique equilibrium point which is uniformly stable.

In order to check the validity of Theorems 6 and 7, the following five cases are given: case 1 with the initial values $(x_1, x_2, y_1, y_2)^T = (-2.5, 4.0, 3.5, -5.0)^T$, case 2 with the initial values $(x_1, x_2, y_1, y_2)^T = (6.5, -3.0, -7.0, 4.5)^T$, case 3 with the initial values $(x_1, x_2, y_1, y_2)^T = (3.0, 8.0, -1.5, -2.0)^T$, case 4 with the initial values $(x_1, x_2, y_1, y_2)^T =$

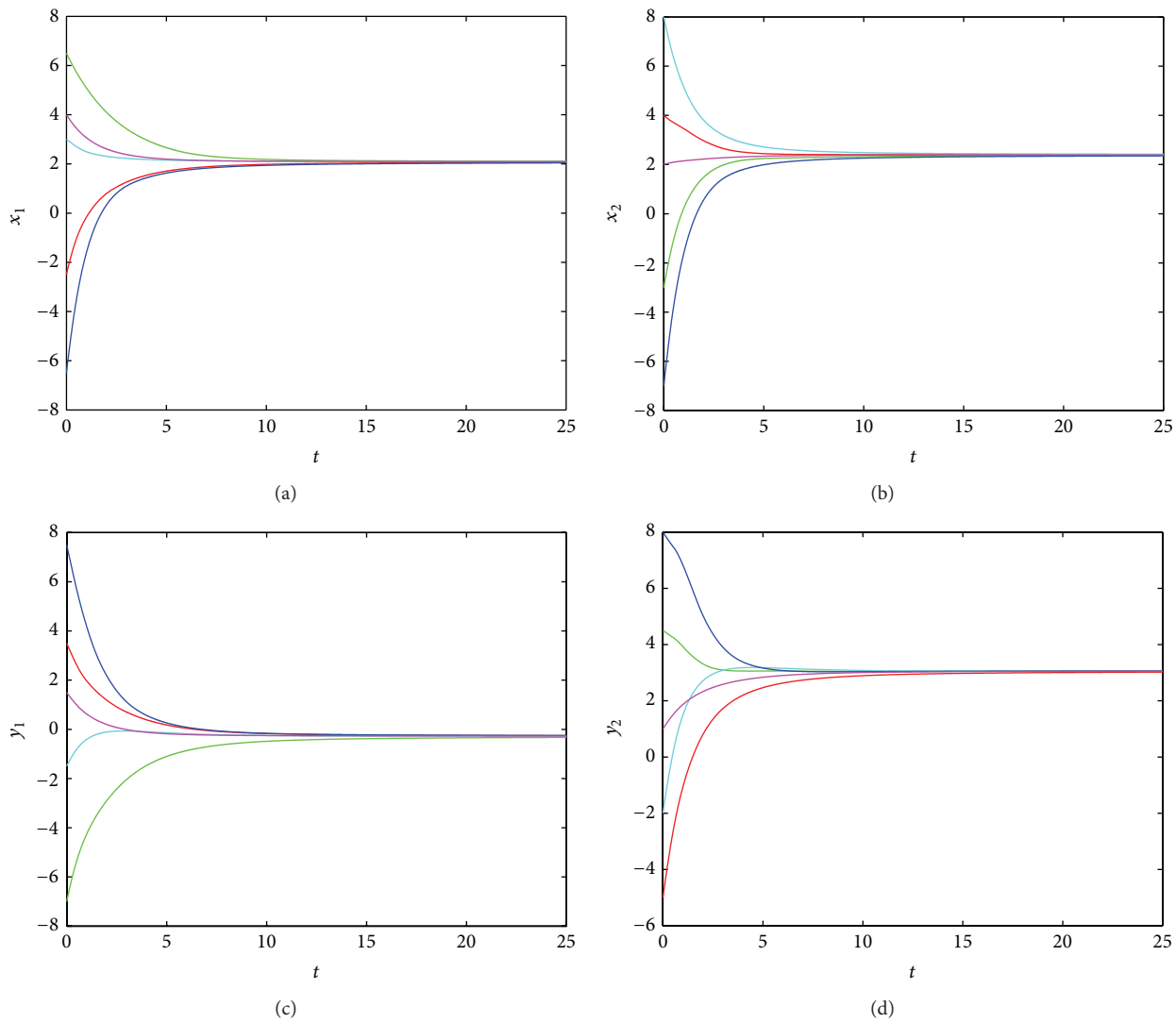

FIGURE 1: Transient states of the fractional-order BAM neural networks in Example 1 with $\alpha = 0.95$, $\sigma = 0.25$, and $\tau = 0.50$.

$(-6.5, -7.0, 7.5, 8.0)^T$, and case 5 with the initial values $(x_1, x_2, y_1, y_2)^T = (4.0, 2.0, 1.5, 1.0)^T$. The time responses of state variables are shown in Figure 1.

When $\sigma = 0$, consider the following three cases: case 1 with the initial values $(x_1, x_2, y_1, y_2)^T = (3.0, 8.0, -1.5, -2.0)^T$, case 2 with the initial values $(x_1, x_2, y_1, y_2)^T = (-6.5, -7.0, 7.5, 8.0)^T$, and case 3 with the initial values $(x_1, x_2, y_1, y_2)^T = (4.0, 2.0, 1.5, 1.0)^T$. The time responses of state variables are shown in Figure 2 with the leakage delay $\sigma = 0$.

Example 2. Consider the following fractional-order BAM neural networks with delays in the leakage terms:

$$D^\alpha x_i(t) = -a_i x_i(t-\sigma) + \sum_{j=1}^{2} c_{ij} F_j\left(y_j(t)\right) + \sum_{j=1}^{2} p_{ij} V_j\left(y_j(t-\tau)\right) + I_i, \quad i = 1, 2,$$

$$D^\alpha y_j(t) = -b_j y_j(t-\sigma) + \sum_{i=1}^{2} d_{ji} G_i\left(x_i(t)\right) + \sum_{i=1}^{2} q_{ji} U_i\left(x_i(t-\tau)\right) + J_j, \quad j = 1, 2, \tag{45}$$

where $\alpha = 0.98$, $\sigma = 0.20$, $\tau = 0.40$, $A = \mathrm{diag}(0.50, 0.65)$, $B = \mathrm{diag}(0.55, 0.42)$,

$$C = \begin{bmatrix} 0.55 & -0.75 \\ 0.85 & 0.66 \end{bmatrix}, \qquad D = \begin{bmatrix} 0.76 & 0.42 \\ 0.39 & -0.68 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.22 & 0.46 \\ -0.75 & 0.82 \end{bmatrix},$$

$$Q = \begin{bmatrix} -0.24 & 0.65 \\ 0.37 & 0.55 \end{bmatrix}, \qquad I = [0.50, -0.45]^T,$$

$$J = [-0.20, 0.30]^T,$$

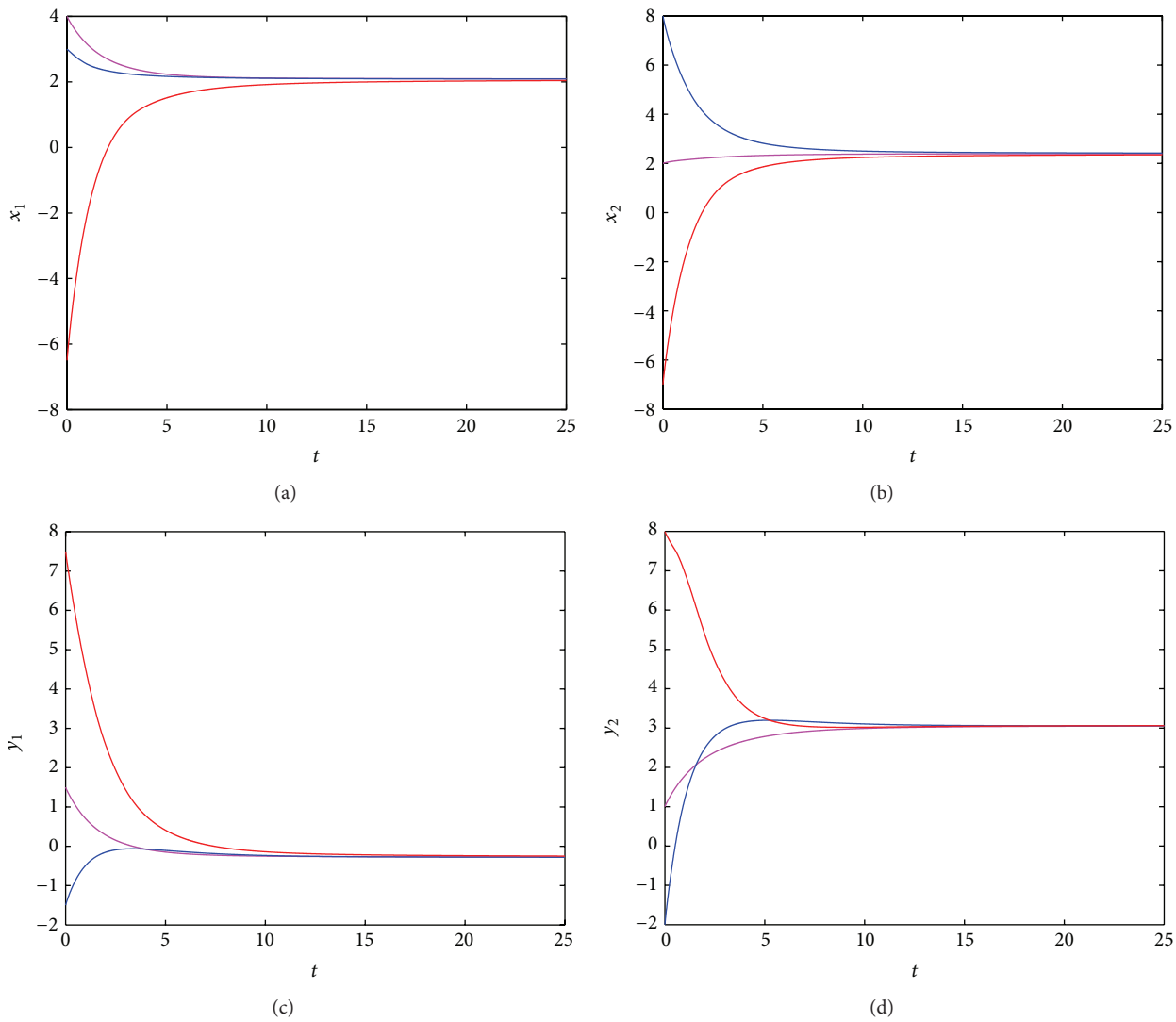

FIGURE 2: Transient states of the fractional-order BAM neural networks in Example 1 with $\alpha = 0.95$, $\sigma = 0$, and $\tau = 0.50$.

$$G_1(x) = G_2(x) = \frac{1}{10}(|x+1| + |x-1|),$$
$$U_1(x) = U_2(x) = \frac{1}{20}(|x+1| + |x-1|),$$
$$F_1(y) = F_2(y) = \frac{1}{15}(|y+1| + |y-1|),$$
$$V_1(y) = V_2(y) = \frac{1}{30}(|y+1| + |y-1|). \tag{46}$$

By calculation, $F_1 = F_2 = 2/15$, $V_1 = V_2 = 1/15$, $G_1 = G_2 = 1/5$, $U_1 = U_2 = 1/10$, $S_1 = 1 - 0.65e^{-0.2} = 0.4678$, $T_1 = 1 - 0.55e^{-0.2} = 0.5497$, $S_2 = 0.75 \times 2/15 + 0.85 \times 2/15 + (0.46\times1/15+0.82\times1/15)e^{-0.4} = 0.2665$, and $T_2 = 0.76 \times 1/5 + 0.68 \times 1/5 + (0.65 \times 1/10 + 0.55 \times 1/10)e^{-0.4} = 0.3684$, which satisfy $S_1T_1 > S_2T_2$; according to Theorem 6, when we select the appropriate initial values, the system (45) could realize uniform stability. Furthermore, we have $W_1 = 0.75 \times 2/15 + 0.85\times2/15+0.46\times1/15+0.82\times1/15 = 0.2987 < \min_{1\le j\le 2}\{b_j\} = 0.42$, $W_2 = 0.76 \times 1/5 + 0.68 \times 1/5 + 0.65 \times 1/10 + 0.55 \times 1/10 = 0.4080 < \min_{1\le i\le 2}\{a_i\} = 0.50$; by utilizing Theorem 7, we can obtain that the system (45) has an unique equilibrium point which is uniformly stable.

In order to check the validity of Theorems 6 and 7, the following five cases are given: case 1 with the initial values $(x_1, x_2, y_1, y_2)^T = (-2.5, 4.0, 3.5, -5.0)^T$, case 2 with the initial values $(x_1, x_2, y_1, y_2)^T = (6.5, -3.0, -7.0, 4.5)^T$, case 3 with the initial values $(x_1, x_2, y_1, y_2)^T = (3.0, 8.0, -1.5, -2.0)^T$, case 4 with the initial values $(x_1, x_2, y_1, y_2)^T = (-6.5, -7.0, 7.5, 8.0)^T$, and case 5 with the initial values $(x_1, x_2, y_1, y_2)^T = (4.0, 2.0, 1.5, 1.0)^T$. The time responses of state variables are shown in Figure 3.

When $\sigma = 0$, consider the following three cases: case 1 with the initial values $(x_1, x_2, y_1, y_2)^T = (-2.5, 4.0, 3.5, -5.0)^T$, case 2 with the initial values $(x_1, x_2, y_1, y_2)^T = (3.0, 8.0, -1.5, -2.0)^T$, and case 3 with

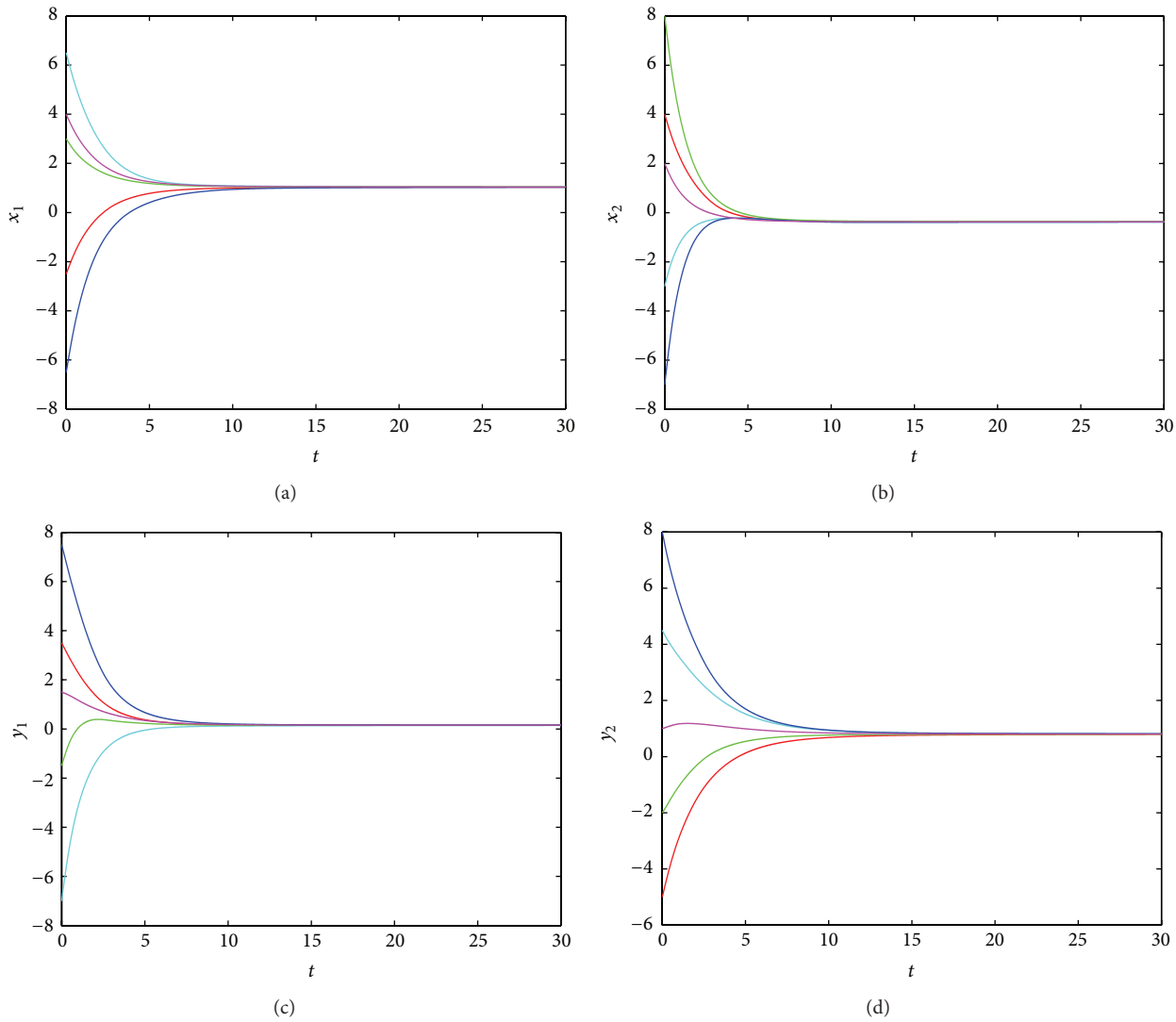

FIGURE 3: Transient states of the fractional-order BAM neural networks in Example 2 with $\alpha = 0.98$, $\sigma = 0.20$, and $\tau = 0.40$.

the initial values $(x_1, x_2, y_1, y_2)^T = (-6.5, -7.0, 7.5, 8.0)^T$. The time responses of state variables are shown in Figure 4 with the leakage delay $\sigma = 0$.

Example 3. Consider the following fractional-order BAM neural networks with delays in the leakage terms:

$$\begin{aligned} D^{\alpha} x_i(t) &= -a_i x_i(t-\sigma) + \sum_{j=1}^{2} c_{ij} F_j\left(y_j(t)\right) \\ &\quad + \sum_{j=1}^{2} p_{ij} V_j\left(y_j(t-\tau)\right) + I_i, \quad i = 1, 2, \\ D^{\alpha} y_j(t) &= -b_j y_j(t-\sigma) + \sum_{i=1}^{2} d_{ji} G_i\left(x_i(t)\right) \\ &\quad + \sum_{i=1}^{2} q_{ji} U_i\left(x_i(t-\tau)\right) + J_j, \quad j = 1, 2, \end{aligned} \tag{47}$$

where $\alpha = 0.95$, $\sigma = 0.50$, $\tau = 0.50$, $A = \mathrm{diag}(1.30, 1.40)$, $B = \mathrm{diag}(1.20, 1.15)$,

$$C = \begin{bmatrix} 0.45 & 0.25 \\ 0.36 & -0.27 \end{bmatrix}, \qquad D = \begin{bmatrix} -0.45 & 0.43 \\ 0.29 & 0.35 \end{bmatrix},$$

$$P = \begin{bmatrix} 0.38 & 0.33 \\ -0.45 & 0.52 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.62 & 0.34 \\ -0.36 & -0.48 \end{bmatrix}, \qquad I = [0.84, 1.22]^T,$$

$$J = [-0.48, 0.75]^T,$$

$$\begin{aligned} G_1(x) = G_2(x) = U_1(x) = U_2(x) \\ = \frac{1}{10}\left(|x+1| + |x-1|\right), \end{aligned}$$

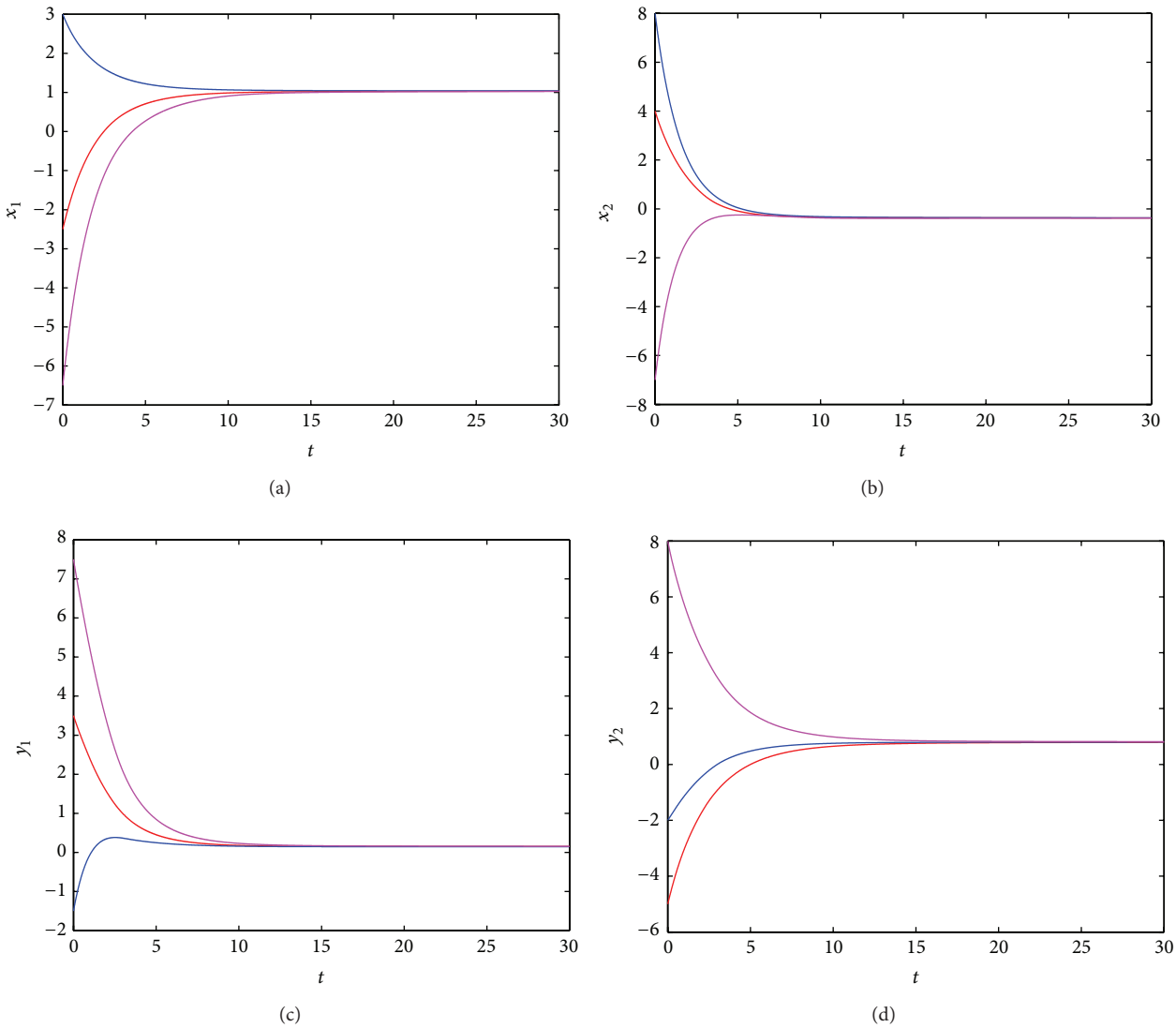

Figure 4: Transient states of the fractional-order BAM neural networks in Example 2 with $\alpha = 0.98$, $\sigma = 0$, and $\tau = 0.40$.

$$F_1(y) = F_2(y) = V_1(y) = V_2(y) = \frac{1}{20}\left(|y+1| + |y-1|\right). \tag{48}$$

By calculation, $F_1 = F_2 = V_1 = V_2 = 0.1$, $G_1 = G_2 = U_1 = U_2 = 0.2$, $S_1 = 1-1.4e^{-0.5} = 0.1509$, $T_1 = 1-1.2e^{-0.5} = 0.2722$, $S_2 = 0.45 \times 0.1 + 0.36 \times 0.1 + (0.38 \times 0.1 + 0.52 \times 0.1)e^{-0.5} = 0.1356$, and $T_2 = 0.45 \times 0.2 + 0.35 \times 0.2 + (0.62 \times 0.2 + 0.48 \times 0.2)e^{-0.5} = 0.3954$, which satisfy $S_1T_1 > S_2T_2$; according to Theorem 6, when we select the appropriate initial values, the system (47) could realize uniform stability. Furthermore, we have $W_1 = 0.45 \times 0.1 + 0.36 \times 0.1 + 0.38 \times 0.1 + 0.52 \times 0.1 = 0.171 < \min_{1\le j\le 2}\{b_j\} = 1.15$, $W_2 = 0.45 \times 0.2 + 0.35 \times 0.2 + 0.62 \times 0.2 + 0.48 \times 0.2 = 0.38 < \min_{1\le i\le 2}\{a_i\} = 1.30$; by utilizing Theorem 7, we can obtain that the system (47) has an unique equilibrium point which is uniformly stable.

In order to check the validity of Theorems 6 and 7, the following five cases are given: case 1 with the initial values $(x_1, x_2, y_1, y_2)^T = (-2.5, 4.0, 3.5, -5.0)^T$, case 2 with the initial values $(x_1, x_2, y_1, y_2)^T = (6.5, -3.0, -7.0, 4.5)^T$, case 3 with the initial values $(x_1, x_2, y_1, y_2)^T = (3.0, 8.0, -1.5, -2.0)^T$, case 4 with the initial values $(x_1, x_2, y_1, y_2)^T = (-6.5, -7.0, 7.5, 8.0)^T$, and case 5 with the initial values $(x_1, x_2, y_1, y_2)^T = (4.0, 2.0, 1.5, 1.0)^T$. The time responses of state variables are shown in Figure 5.

5. Conclusions

In this paper, the uniform stability for a class of fractional-order BAM neural networks with leakage delays has been discussed. Several sufficient conditions ensuring the uniform stability of such systems have been derived based on inequality technique and analysis method. Meanwhile, the existence,

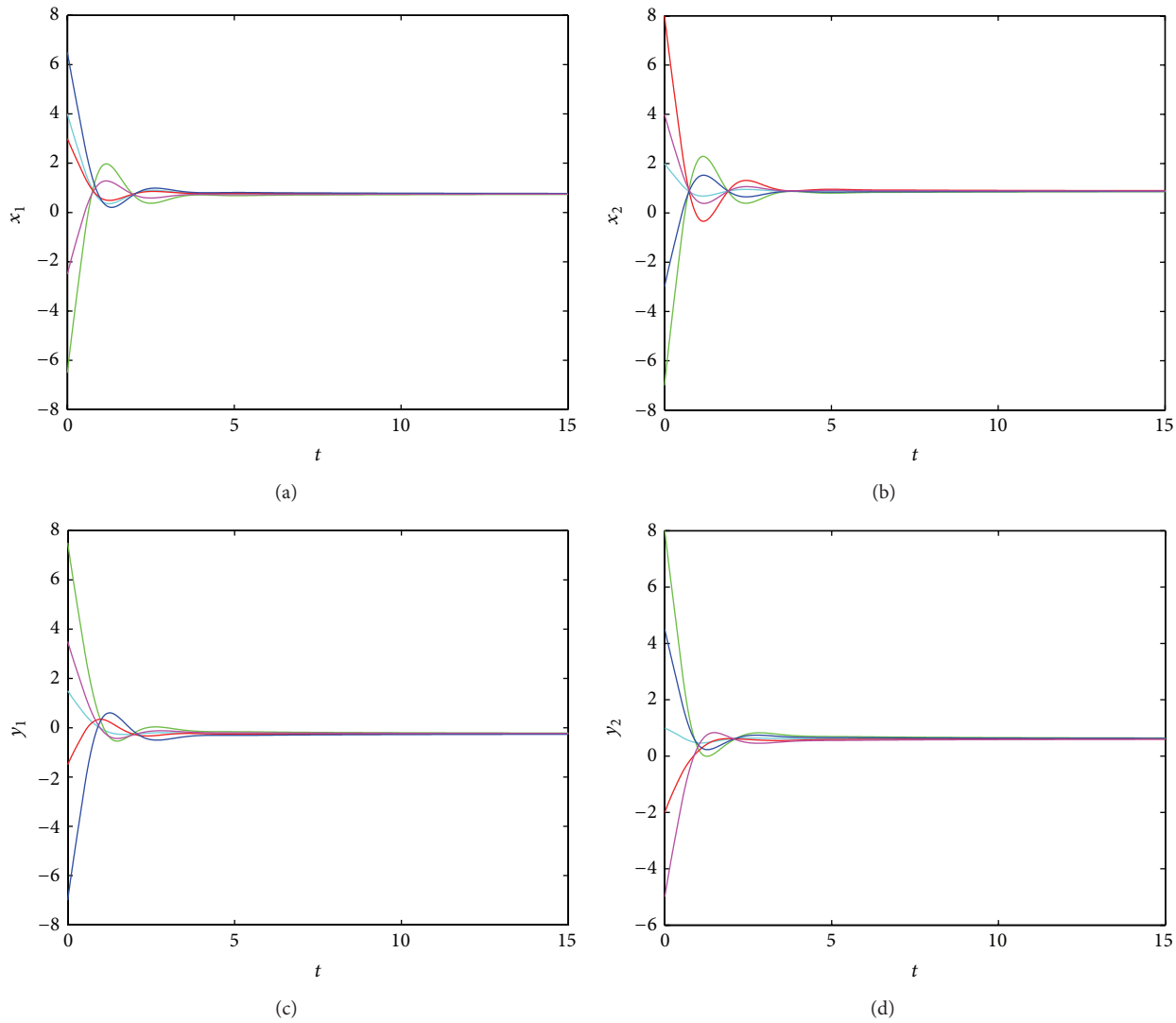

Figure 5: Transient states of the fractional-order BAM neural networks in Example 3 with $\alpha = 0.95$, $\sigma = 0.50$, and $\tau = 0.50$.

uniqueness, and uniform stability of the equilibrium point have been investigated. Finally, three simulation examples have been provided to demonstrate the effectiveness of the obtained results.

We would like to point out that it is possible to extend our main results to other complex systems [37–41] and establish novel stability conditions with less conservatism by using more up-to-date techniques in [42–46]. The corresponding results will appear in the near future.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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Research Article

Real-Time Pricing Decision Making for Retailer-Wholesaler in Smart Grid Based on Game Theory

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Real-time pricing DSM (demand side management) is widely used to dynamically change or shift the electricity consumption in the smart grid. In this paper, a game decision making scheme is proposed in the smart grid with DSM. The interaction between two retailers and their wholesaler is modeled as a two-stage game model. Considering the competition between two retailers, two different game models are developed in terms of the different action order between retailers and their wholesaler. Through analyzing the equilibrium revenues of the retailers for different situations we find that although the wholesaler expects to decentralize certain management powers to the retailers, it has retained the right to change the rules of the game and frequently reneged on the promises. More specifically, the law should ensure that any change of the revenue-sharing formula must go through certain legal procedures. Imposing legal restrictions on the wholesaler's discretionary policy suggests that the time-inconsistency problem is mitigated. Numerical simulation shows the effectiveness of proposed scheme.

1. Introduction

Today much more electricity grids have operated more than half a century and tend to be outdated even in some developed countries. Modernising electricity grids can increase the efficiency of electricity production and promote the use of grid assets and meanwhile makes the whole power network more reliable and secure so as to decrease carbon emission. The concept of smart grid has, especially, been arousing significant attention of much more researchers. The data communication networks play an important role during the development of smart grid. However, data communication network in smart grid is affected by many decisive factors such as different load and congestion level, changing customer demand, power generation, and different prices. These variable factors lead to different decision making problems. To solve these problems, demand side management (DSM) is a fine choice for residential customers to reduce the peak load and decrease the demand ability and costs in much more cases. DSM has been practiced since the early 1980s [1–3]. DSM is designed to affect the consumption of the customer electricity through implementing and monitoring practical activities. Usually, DSM can make users flatten the demand curve or shift the energy use to off-peak hours [4]. It is especially urgent to improve efficiency of the customers both in quantity and quality in power grid [5]. There is a significant scope for DSM to contribute to increasing the efficiency and use of system assets, for example, peak clipping, valley filling, load shifting, and flexible load shape [6]. Real-time pricing is one of the most effective DSM tools that can encourage users to consume electricity wisely. The reason is that the electricity is a very short-term commodity and economically nonstorable; that is, it has to be consumed the moment it is produced, where markets constantly experience short-term changes as capacity fluctuations from surplus to scarcity due to the hourly and daily fluctuation in demand. Considering the enhancement of the current power transmission and distribution systems with communication facilities and information technologies, real-time and adaptive pricing attract more attention. Adaptive pricing and peak load pricing have been practiced for many years [7–10]. In peak load pricing, the operating cycle is divided into several periods and instinct

price is determined for each period. The prices are then announced ahead of time at the beginning of the operation cycle [9]. If real-time pricing is implemented, the price would become elastic on the demand side opposed to fixed price tariff. In real-time pricing, random events and the reaction of the customers to the previous prices will affect the price in the future [7]; then real-time pricing is conducive to realizing the linkage between demand side price and wholesale market clearing price as a kind of ideal dynamic electricity price mechanism. The market risk at proper points between power suppliers and users makes the retail price truly reflect real-time electricity costs change, really achieve paretooptimal efficiency of the market, and realize the optimal allocation of power resources and maximization of total social benefit.

The spreading technologies and services in smart grid imply that game theory and some other more up-to-date techniques [11–15] will naturally become a prominent tool in the design and analysis of real-time pricing in smart grids. In recent years, several efforts have been carried out to design intelligent systems for managing the energy consumption in real-time pricing. Mohsenian-Rad and Leon-Garcia utilize the smart grid and smart meters to provide an efficient power dispatching scheme for studying a single user's reaction [16]; Saad et al. use game theory to study the various decision making problems in the smart grid based on the DSM [17]. The Stackelberg game model is used to study the pricing problem of hierarchical decision problem in [18–20] and is adopted to study the real-time pricing based on the demand side in [21–24]. However, the proposed framework might not be suitable for all circumstances in practical electricity market. The reason is that the parallel structure in each layer and the order problem of the hierarchical structure are excluded from the object of study.

The Stackelberg game and Cournot game are unified into our model in our paper. The Stackelberg game is a game model in economics in which the leader firm moves first and then the follower firms move sequentially. Firms may engage in Stackelberg game if one has some sort of advantage enabling it to move first. More generally, the leader must have one commitment in power market. The Cournot game is an economic model used to describe an industry structure in which firms compete on quantity. The Stackelberg and Cournot models are similar in both competitions on quantity. However, a crucial advantage is given to the leader in Stackelberg game. The assumption of perfect information is also needed in the Stackelberg game; the follower must observe the strategy of the leader; otherwise the game reduces to Cournot game. Inspired by [25], we propose a real-time pricing method based on DSM using optimization technique and game theory. The novelty of this approach is threefold. Firstly, we jointly consider the optimization of consumers' revenues, integrating them into the retailers' problem. Secondly, in our model we not only consider the Stackelberg game between the retailers and wholesaler but also consider the Cournot game between the retailers to study the real-time pricing problem of bilevel decision. Last but not least, we consider the Stackelberg game between the retailers and wholesaler in different order of the hierarchical structure.

The remainder of this paper is organized as follows. In Section 2 we introduce the system model and the problem formulations, and we propose an efficient game model for electricity consumption scheduling between the wholesaler and retailers based on two different situations with elaborate mathematical analysis. In Section 3 the solution of model is proposed, and then a numerical analysis and simulation are done in Section 4. In Section 5, we evaluated the performance of the proposed model and a summary is provided.

2. System Model and Problem Formulation

We consider a smart grid with more than one retailer from which customers purchase electricity in electricity market liberalization. We aim to maximize the customers' utilities with minimum payment and increase the retailers' profit, so as to reduce peak to average load demand ratio through considering real-time varying prices. The retailers can compete or cooperate with each other in the electricity market to obtain the highest individual or combined revenue by varying price for the customers.

2.1. Electricity Demand Models

Utility Functions of Customers. Each customer is equipped with a smart meter in our model. The retailers set the real-time retail electricity prices and information for the customers via LAN. As far as customers are concerned, the energy scheduler in the smart meter can compute and distribute optimal energy consumption according to the prices for the upcoming time. Certainly customers always prefer to take lower prices to consume more electricity till reaching maximal consumption level if possible. Similar to [17], the utility function of each customer is taken as

$$U(p,d) = Xd - \frac{\alpha}{2}d^2 - \xi pd, \tag{1}$$

where X is varying parameter at different times of the day and among different customers, d denotes the customer's electricity demand, α is a parameter that is pre-determined, ξ indicates the price elasticity of electricity demand, and p is the price provided by the retailer.

Since real-time pricing DSM is an effective tool to direct and affect the electricity consumption behavior of customers, each customer adjusts electricity consumption level to maximize utility according to real-time prices which are offered by the retailers. Each customer's electricity consumption can be calculated on the base of utility function. The electricity demand function of each customer $D(p)$ can be obtained by maximizing the following utility function:

$$D(p) = \frac{X - \xi p}{\alpha}. \tag{2}$$

Retailers' Electricity Demand Function. Assuming that there are M wholesalers and N electricity retailers in the electricity market, the retailers procure electricity from the wholesalers and provide the electricity for customers. Different retailers offered different price for customers they serve. The lower

price attracts more customers and the customers can turn to other retailers in that area. So the electricity demand of retailer i can be shown as follows:

$$D_i(p) = D_i - \xi_i p_i + \sum_{n=1, n\neq i}^{N} v_{i,n} p_n, \tag{3}$$

where $p = (p_1, p_2, \ldots, p_n)$ denotes the price vector offered by the retailers, D_i denotes varying parameter with different retailers and time, ξ_i denotes the price elasticity of electricity demand of retailer i, p_i denotes the price offered by retailer i, and $v_{i,n}$ $(0 \leq v_{i,n} \leq 1)$ represents the proportion of electricity demand flowing from retailer n to retailer i for a given price from retailer n.

2.2. Revenue of Electricity Retailers and Wholesaler. Solving the equilibrium of game based on multiple wholesalers and multiple retailers will yield nonsmooth problems. It is a hard work for the traditional game algorithm and needs to use the corresponding nonsmooth optimization algorithm to solve them; in this paper we only consider the case that there are one wholesaler and two retailers in the electricity market for simplicity. The wholesaler wholesales electricity to retailers and chooses the percentages of retailers' revenue to be submitted to him, where the retailers make the decision on how much revenue to procure in their jurisdictions. The revenue function of retailer i can be expressed as

$$y_i(p) = D_i(p) p_i, \quad i = 1, 2. \tag{4}$$

The budget revenue function of wholesaler can be expressed as

$$R_w = y_1(p) x_1 + y_2(p) x_2, \tag{5}$$

where x_i $(i = 1, 2)$ denotes the share of revenue submitted to the wholesaler from the retailer i. The budget revenue function of retailer i can be expressed as

$$L_i = (1 - x_i) y_i(p), \quad i = 1, 2, \tag{6}$$

where the net income of the retailer i after deducting the cost from budget revenue is, namely,

$$R_i = (1 - x_i) y_i - C_i(y_i), \quad i = 1, 2, \tag{7}$$

where $C_i(y_i)$ denotes cost; assume marginal costs rise with the increase of income, specially; we set $C_i(y_i) = a_i y_i^2$. In general, $a_1 \neq a_2$; we can regard a_i as the parameter representing regional development level. The higher the level of development is, the smaller the a_i is.

The wholesaler aims to minimize the income gap between retailers after ensuring its fundamental spending needs. So the wholesaler's preferences can be expressed with a logarithmic function defined on the retailer' budget revenue as follows:

$$U = \ln(1 - x_1) y_1 + \ln(1 - x_2) y_2. \tag{8}$$

The retailers can be noncooperative or cooperative with each other; that is, each retailer maximizes its individual utility disregarding the benefit of the other retailer, or the retailers maximize the sum of their utility in the game.

2.3. Game Model Formulation

2.3.1. The Wholesaler Can Abide by the Commitment. The wholesaler and the two retailers play a two-stage game. The timing of this game is as follows.

Let the wholesaler move first. The retailers move and will have incentive to obtain more revenues.

The game proceeds as follows:

(1) Stage 1: the wholesaler announces x_1 and x_2;

(2) Stage 2: retailer 1 and retailer 2 choose p_1 and p_2 simultaneously after observing x_1 and x_2.

The above assumption says that the wholesaler plays a Stackelberg game with the retailers and the wholesaler moves first (as the leader). Under such regime, the retailers obtain their revenues through serving electricity customers after knowing the wholesaler's offer of x_i, while the wholesaler takes into account the retailers' reaction to x_i and finds the optimal x_i. In terms of retailers, they play a Cournot game in which both players do not have the information about other player's move and maximize the sum of their utility.

Definition 1. The equilibrium is defined as follows.

(1) The wholesaler acts optimally given the retailers' reaction functions. (2) The retailer i optimizes his revenue given the wholesaler's announced x_i and retailer's revenue.

The wholesaler's aim is to solve the problem,

$$\begin{aligned} \max_{x_1, x_2} \quad & U = \ln(1 - x_1) y_1 + \ln(1 - x_2) y_2 \\ \text{s.t.} \quad & x_1 y_1 + x_2 y_2 \geq E, \end{aligned} \tag{9}$$

where E is spending needs which is assumed as small enough so that $x_i \leq 1$, $i = 1, 2$, while the aim of the retailers is to maximize the net income after deducting the cost from budget revenue; namely,

$$\max_{y_i} R_i = (1 - x_i) y_i - C_i(y_i), \quad i = 1, 2. \tag{10}$$

2.3.2. The Wholesaler Cannot Abide by the Commitment. The wholesaler and the two retailers play a two-stage game, but the order of game is changed. The wholesaler can modify arbitrarily the payout rate despite the commitment in advance; the action sequence of this game is that retailers move firstly.

(1) Stage 1: retailer 1 and retailer 2 choose p_1 and p_2;

(2) Stage 2: after observing p_1 and p_2, the wholesaler chooses x_1 and x_2.

The above assumption states that the wholesaler plays a Stackelberg game with the retailers and the retailers move first. The retailers are the leaders and the wholesaler is the follower in the game. This assumption embodies the true feature of current Chinese economic system. The wholesaler does not precommit to a fixed revenue sharing method and the retailers often take this reality into account when determining their revenues; in this ease the two retailers are still noncooperative and play a Cournot game between them as they move simultaneously.

Definition 2. The equilibrium is defined as follows.

(1) The wholesaler responds optimally given each retailer's collected revenues.

(2) The retailer i optimizes its collected revenue given the wholesaler's reaction function and retailer j's choice of revenue collection ($i \neq j$, $i, j = 1, 2$).

3. Solution of the Game Model

3.1. The Wholesaler Can Abide by the Commitment. To find the equilibrium of the game when the wholesaler can abide by the commitment, we use backward induction. In other words, we first solve for the retailers' reaction function. Believing that the wholesaler will commit constant x_1, retailer 1 chooses p_1 to maximize

$$R_1 = (1 - x_1) y_1 - a_1 y_1^2, \tag{11}$$

where x_1 is fixed. This yields the optimal revenue y_1 under the commitment regime satisfying $dR_1/dy_1 = 0$; the solution is

$$y_1^* = \frac{1 - x_1}{2a_1}. \tag{12}$$

For retailer 2,

$$R_2 = (1 - x_2) y_2 - a_2 y_2^2, \tag{13}$$

where x_1 is fixed. This leads to the optimal revenue y_2 under the commitment regime satisfying $dR_1/dy_1 = 0$; its optimal revenue is

$$y_2^* = \frac{1 - x_2}{2a_2}. \tag{14}$$

By introducing (3) and (4) to (12) and (14), respectively, we have

$$\begin{aligned} (D_1 - \xi_1 p_1 + v_{1,2} p_2) p_1 &= \frac{1 - x_1}{2a_1}, \\ (D_2 - \xi_2 p_2 + v_{2,1} p_1) p_2 &= \frac{1 - x_2}{2a_2}. \end{aligned} \tag{15}$$

By (15), we obtain

$$\begin{aligned} p_1^* &= \frac{(((1 - x_2)/2a_2 p_2^*) + \xi_2 p_2^* - D_2)}{v_{2,1}}, \\ p_2^* &= \frac{(((1 - x_1)/2a_1 p_1^*) + \xi_1 p_1^* - D_1)}{v_{1,2}}. \end{aligned} \tag{16}$$

Now return to the wholesaler's problem. The wholesaler chooses x_1 and x_2 to aim for

$$\begin{aligned} &\max_{x_1, x_2} \quad U = \ln(1 - x_1) y_1^* + \ln(1 - x_2) y_2^*, \\ &\text{s.t.} \quad x_1 y_1^* + x_2 y_2^* \geq E, \end{aligned} \tag{17}$$

where y_1^* and y_2^* are given by (12) and (14). Substituting (12) and (14) into (17), we can construct the Lagrange function

$$\begin{aligned} L = {} & \frac{\ln(1 - x_1)^2}{2a_1} + \frac{\ln(1 - x_2)^2}{2a_2} \\ & + \lambda \left[\frac{x_1(1 - x_1)}{2a_1} + \frac{x_2(1 - x_2)}{2a_2} - E \right]. \end{aligned} \tag{18}$$

The first-order optimality conditions are

$$\begin{aligned} \frac{\partial L}{\partial x_1} &= -\frac{2}{1 - x_1} + \lambda \frac{1 - 2x_1}{2a_1} = 0; \\ \frac{\partial L}{\partial x_2} &= -\frac{2}{1 - x_2} + \lambda \frac{1 - 2x_2}{2a_1} = 0; \\ \frac{\partial L}{\partial \lambda} &= \frac{x_1(1 - x_1)}{2a_1} + \frac{x_2(1 - x_2)}{2a_2} - E = 0. \end{aligned} \tag{19}$$

Eliminating λ from (19) gives $(1/a_1)(1 - x_1)(1 - 2x_1) = (1/a_2)(1-x_2)(1-2x_2)$, and we also can see $x_1 < 1/2$, $x_2 < 1/2$ from (19); therefore, we have

$$y_i^{c*} > \frac{1}{4a_i} \quad (i = 1, 2). \tag{20}$$

On the other hand, the budget constraint can be rewritten as

$$\frac{x_1(1 - x_1)}{2a_1} + \frac{x_2(1 - x_2)}{2a_2} = E. \tag{21}$$

Equation (21) defines the optimal point x_i^* ($i = 1, 2$) under commitment. But solving the two equations yields a third-order polynomial equation. We have the following proposition through the previous derivation.

Proposition 3. *The equilibrium of the game when the wholesaler can abide by the commitment satisfies the following conditions:*

$$\begin{aligned} p_2^* &= \frac{(((1 - x_1^*)/2a_1 p_1^*) + \xi_1 p_1^* - D_1)}{v_{1,2}}, \\ p_1^* &= \frac{(((1 - x_2^*)/2a_2 p_2^*) + \xi_2 p_2^* - D_2)}{v_{2,1}}, \\ \frac{1}{a_1}(1 - x_1^*)(1 - 2x_1^*) &= \frac{1}{a_2}(1 - x_2^*)(1 - 2x_2^*), \\ \frac{x_1^*(1 - x_1^*)}{2a_1} &+ \frac{x_2^*(1 - x_2^*)}{2a_2} = E. \end{aligned} \tag{22}$$

3.2. The Wholesaler Cannot Abide by the Commitment. To find the equilibrium, we use backward induction. The wholesaler's aim is

$$\begin{aligned} &\max_{x_1, x_2} \quad U = \ln(1 - x_1) y_1 + \ln(1 - x_2) y_2 \\ &\text{s.t.} \quad x_1 y_1 + x_2 y_2 \geq E. \end{aligned} \tag{23}$$

The first-order condition of problem (23) is equivalent to the following equation:

$$(1 - x_1) y_1 = (1 - x_2) y_2. \tag{24}$$

Namely, the wholesaler will equalize the budget revenues between the two retailers, given the optimal levels of x_1 and x_2 as functions of y_1, y_2, and E; the reaction function of the wholesaler is

$$\begin{aligned} x_1(y_1, y_2) &= \frac{1}{2} - \frac{y_2 - E}{2y_1}, \\ x_2(y_1, y_2) &= \frac{1}{2} - \frac{y_1 - E}{2y_2}. \end{aligned} \tag{25}$$

The above reaction function means payout rate of a retailer increases with the rising of its relative income and decreases with the falling of the other retailers' falling of the other retailer's relative income. Because the retailers know the reaction function of the wholesaler, in the first stage of game, the problems of the retailers are to maximize their individual utility

$$R_i = (1 - x_i(y_1, y_2)) y_i - a_i y_i^2, \quad i = 1, 2. \tag{26}$$

The first-order condition determines that the optimal y_i is

$$\frac{\partial R_i}{\partial y_i} = \frac{1}{2} + \frac{\partial y_j}{\partial y_i} - 2a_i y_i = 0, \quad i \neq j,\ i, j = 1, 2. \tag{27}$$

Through the Cournot game assumption $\partial y_i / \partial y_j = 0$, $i \neq j$, $i, j = 1, 2$, we can solve the Nash equilibrium as follows:

$$y_1^{\mathrm{nc}*} = \frac{1}{4a_1}, \qquad y_2^{\mathrm{nc}*} = \frac{1}{4a_2}. \tag{28}$$

The wholesaler substitutes (28) into (25) and obtains equilibrium

$$\begin{aligned} x_1^* &= \frac{1}{2} - \frac{a_1}{2a_2} + 2a_1 E, \qquad x_2^* = \frac{1}{2} - \frac{a_2}{2a_1} + 2a_2 E, \\ (D_1 - \xi_1 p_1 + v_{1,2} p_2) p_1 &= \frac{1}{4a_1}, \\ (D_2 - \xi_2 p_2 + v_{2,1} p_1) p_2 &= \frac{1}{4a_2}. \end{aligned} \tag{29}$$

By solving the two equations, we obtain the equilibrium price

$$\begin{aligned} p_1^* &= \frac{((1/4a_2 p_2^*) + \xi_2 p_2^* - D_2)}{v_{2,1}}, \\ p_2^* &= \frac{((1/4a_1 p_1^*) + \xi_1 p_1^* - D_1)}{v_{1,2}}. \end{aligned} \tag{30}$$

Therefore, we have the following proposition through the previous derivation.

Proposition 4. *The equilibrium of the game when the wholesaler cannot abide by the commitment satisfies the following conditions:*

$$\begin{aligned} p_1^* &= \frac{((1/4a_2 p_2^*) + \xi_2 p_2^* - D_2)}{v_{2,1}}, \\ p_2^* &= \frac{((1/4a_1 p_1^*) + \xi_1 p_1^* - D_1)}{v_{1,2}}, \\ x_1^* &= \frac{1}{2} - \frac{a_1}{2a_2} + 2a_1 E, \\ x_2^* &= \frac{1}{2} - \frac{a_2}{2a_1} + 2a_2 E. \end{aligned} \tag{31}$$

4. Model Analysis and Simulation

We observe the change trend when we assume that $a_1 = 0.04$, $a_2 = 0.01$, and $E = 20, v_{1,2} = v_{2,1} = 0.3, \xi_1 = \xi_2 = 0.5$. We vary the number of D_1 when D_2 changes from 0 to 5 to study how they affect the equilibrium price.

Firstly, we simulate the situation that the wholesaler can abide by the commitment. From (22) we have

$$\begin{aligned} p_2^* &= \frac{(((1 - x_1^*)/0.056 p_1^*) + 0.5 p_1^* - D_1)}{0.3}, \\ p_1^* &= \frac{(((1 - x_2^*)/0.014 p_2^*) + 0.5 p_2^* - D_2)}{0.3}, \\ x_1^* &= 0.2, \quad x_2^* = 0.4. \end{aligned} \tag{32}$$

In Figure 1, when D_1 keeps constant and D_2 grows to the low regional development level the increase of electricity demand results in short supply which causes prices to rise for retailer 2. But the change trends of two equilibrium price vary greatly. Retailer 1 maintains price stability in electricity demand stable circumstances, which can be explained by the low level of development. Retailer 1 can only expand electricity price instead of expanding production.

Next we simulate the situation that the wholesaler cannot abide by the commitment. From (31) we have

$$\begin{aligned} p_1^* &= \frac{((1/0.028 p_2^*) + 0.5 p_2^* - D_2)}{0.3}, \\ p_2^* &= \frac{((1/0.112 p_1^*) + 0.5 p_1^* - D_1)}{0.3}, \\ x_1^* &= 0.1, \\ x_2^* &= 0.775. \end{aligned} \tag{33}$$

In Figure 2, the change trends of equilibrium price stay similar to Figure 1; p_1^* increases much faster than it does in Figure 1 which shows the effect of actions in a different order.

In addition to the above situation, we also can verify the key differences of the retailers' revenue between the two cases through comparing the two kinds of equilibrium. From the above section it can be seen that each retailer ignores

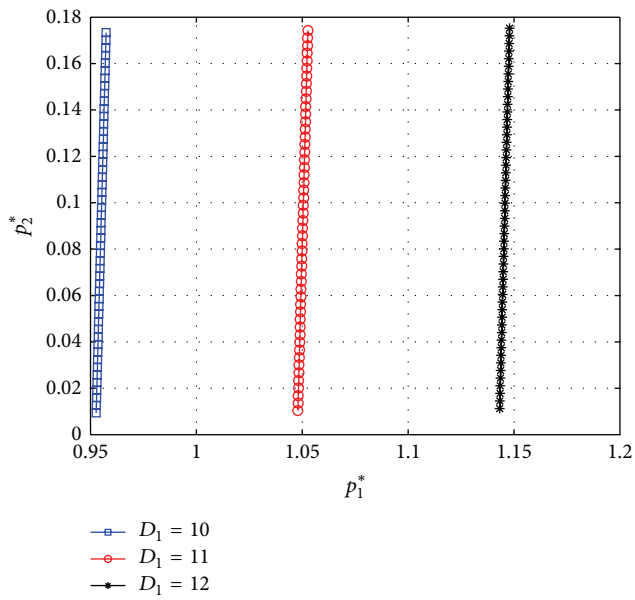

Figure 1: Equilibrium price chart affected by D_1, D_2 when wholesaler abides by the commitment.

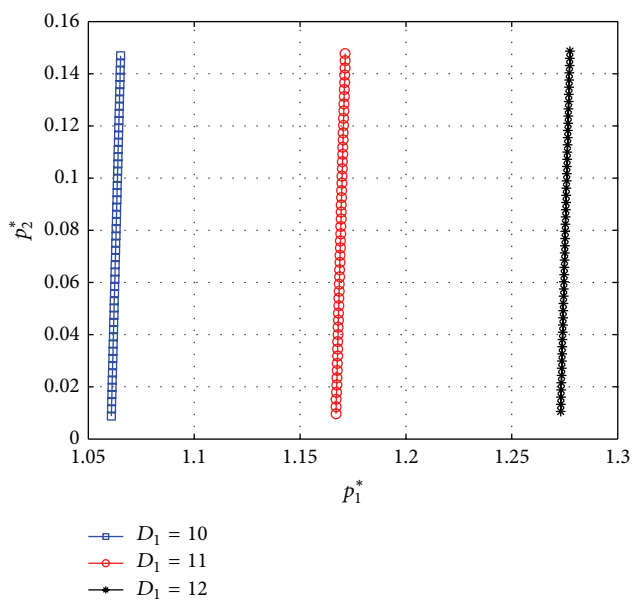

Figure 2: Equilibrium price change trend chart affected by D_1, D_2 when the wholesaler cannot abide by the commitment.

the externality problem (one retailer's revenue collection can affect the other retailers' share of revenue submitted to the wholesaler). Moreover, a revenue distortion is involved in the no-commitment case as retailers are tempted to reduce their efforts in order to avoid high rate. However, the externality of a retailer's revenue disappears under the commitment regime since the wholesaler precommits to fixed rates, which increases the retailers' revenue, but the predetermined rates can create distortions on the retailer's revenue. So we should consider two factors when comparing the two cases. On one hand, the absence of the externality can contribute to higher revenue in the commitment case compared with the no-commitment case; on the other hand, the two cases involve a distortion that reduces retailers' revenue which yields higher revenue levels in the commitment case than the no-commitment case. From (20) and (28) we know $y_1^{nc*} < y_1^{c*}$, $y_2^{nc*} < y_2^{c*}$ at different equilibrium price. The commitment case provides a way to partially overcome the retailers' incentive problem in the no-commitment case. Firstly, the institution is not adequate for restricting the wholesaler from reneging on preannounced declaring; that is, there is no legal restriction on the wholesaler's contract revision. Secondly, the wholesaler still does not want to commit after knowing that the commitment case can yield higher level of revenue than the no-commitment case. The problem is that the wholesaler's policy is time-inconsistent. Suppose the wholesaler preannounces the optimal rates, but the wholesaler wants to change the preannounced rates after having observed the realized revenues; then the optimal policy is time-inconsistent. In view of this, the promise made by the wholesaler becomes incredible to the retailers.

5. Conclusion

In this paper, we propose a novel game-theoretical decision making scheme for electricity retailers and wholesaler in the smart grid with DSM. The interaction between two retailers and their wholesaler has been modeled as a two-stage dynamic game, in which the competition between two retailers is considered. Two different game models are constructed in terms of the different action order between retailers and their wholesaler. Backward induction is used to determine the SPE of the game.

Through analyzing the equilibrium revenues of the retailers in different situations we find that the wholesaler wants to decentralize certain management powers to the retailers. As he has retained the right to change the rules of the game, he frequently reneged on the promises when he thought "necessary." Imposing legal restrictions on the wholesaler's discretionary policy suggests that the time-inconsistency problem is mitigated. Numerical simulation shows the effectiveness of proposed conclusion and effect of parameters on the equilibrium price.

More wholesalers and retailers will be researched in the direction as a possible future extension for the electricity market. The main results of this paper will be extended to other complex systems [26–30].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

A Switched Approach to Robust Stabilization of Multiple Coupled Networked Control Systems

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This paper proposes a switched approach to robust stabilization of a collection of coupled networked controlled systems (NCSs) with node devices acting over a limited communication channel. We suppose that the state information of every subsystem is split into different packets and only one packet of the subsystem can be transmitted at a time. Multiple NCSs with norm-bounded parameter uncertainties and multiple transmissions are modeled as a periodic switched system in this paper. State feedback controllers can be constructed in terms of linear matrix inequalities. A numerical example is given to show that a collection of uncertain NCSs with the problem of limited communication can be effectively stabilized via the designed controller.

1. Introduction

Networked control systems (NCSs) are feedback control systems with network communication channels used for the communications between spatially distributed system components like sensors, actuators, and controllers. In recent years, the studies of NCSs have received increasing attention in control theory [1–4]. The use of the communication channels can reduce the costs of cables and power, simplify the installation and maintenance of the whole system, and increase the reliability. The insertion of communication network in the feedback control loop complicates the analysis and design of an NCS because many ideal assumptions made in the traditional control theory cannot be applied to NCSs directly. The traditional control theory requires that all the feedback information be obtained by the controller. Meanwhile one major problem must be solved, the limited bandwidth of the communication network. Sometimes, data packets containing full plant measurements may not be permitted to be transmitted because of limited bandwidth and this may deteriorate the system performance and destabilize the system. Examples include fleets of unmanned autonomous vehicles, wide area power system, remote planetary exploration with multiple coordinated robots, and the control of microactuator arrays. In such systems, simultaneous communication with all subsystems may not be possible because of physical or performance constraints. So how to control the NCSs via a limited communication channel is a big problem. Potential application can be found in [5–7].

The problem of stabilization with finite communication bandwidth has received much attention; see, for example, [2, 8–11]. In NCSs, communication capacity depends on the topology of the network. In many cases, the communication among the devices of the network is through one communication channel as shown in Figure 1, where P_i is the plant and C_i is the corresponding controller ($i = 1, \ldots, N$). The problem of stabilization with finite communication bandwidth was introduced by [12, 13]. The switched system approach was introduced to study NCSs with multiple-packet transmission in [14, 15] and further pursued by [16, 17]. The merit of the switched approach is that the controllers can make full use of the previous information to stabilize NCSs when the current state measurements are available from the network. By switched approach, the stabilization

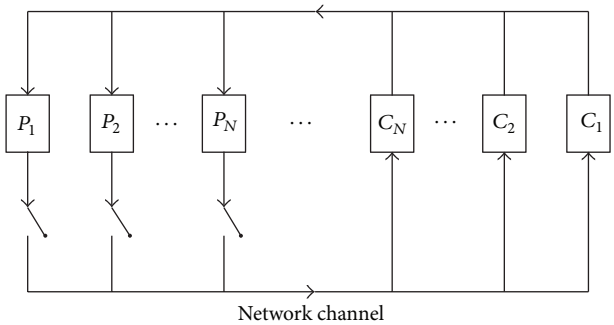

Figure 1: A shared network.

of NCSs with multiple-packet transmissions over token-passing bus networks was considered in Yu et al. [14, 15], where sufficient conditions for stabilization and design of controllers have been obtained through modeling NCSs as periodically switched systems. Further results for single NCS were presented in Yu et al. [18–21]. Zhang and Hristu-Varsakelis [22] also studied stabilization problem of NCSs by modeling these systems as switched systems. Reference [23] studied the stability for a class of systems with time varying delay subject to controller failure, where the delay system with failed controller was modelled as a class of switched delay systems. Unfortunately, all the above literatures focused on studying NCSs without uncertainty.

Uncertainty is ubiquitous in control systems and it inevitably exists in system model due to the complexity of the system itself, exogenous disturbance, measurement errors, and so on. Qiu et al. [24] dealt with mode and parameter-dependent robust mixed H_2/H_∞ filtering design for a class of discrete-time switched polytopic linear systems. Furthermore, the problem of delay-dependent robust energy-to-peak filtering design for a class of discrete-time switched linear systems with a time-varying state delay and polytopic uncertainties was revisited in [25]. Recently, [26] studied exponential H_∞ filtering for discrete-time switched state-delay systems under asynchronous switching. Reference [27] considered the problem of robust stabilization of linear uncertain discrete-time systems via limited capacity communication channels. In [28], a class of networked control systems was investigated where the plant had time-varying norm-bounded parameter uncertainties. In [29], sufficient conditions were given to ensure the stability of uncertain NCSs using switched approach. Reference [30] studied the stability of NCSs that were subject to time-varying transmission intervals, time-varying transmission delays, and communication constraints. Reference [31] addressed the problem of stabilizing uncertain nonlinear plants over a shared limited bandwidth packet-switching network. However, all the above literatures consider performance or stabilization of single NCS, which was not coupled with any other system.

Yu et al. [32] investigated stabilization of a collection of linear systems with limited information. Reference [16] further modeled multiple NCSs as a periodic switched system, with only one subsystem able to access the network to transmit all of its present state information at a time. But they did not deal with the problem of multiple packet transmission. Ding et al. [33] studied multiple networked control systems with multiple transmissions, but they did not consider uncertainties and they did not consider the two-side channel transmission. In [34, 35], Dai et al. developed a scheduling strategy for a collection of discrete-time NCSs subjected to communication constraints, which were modelled as a switched delay system. The systems they considered were not coupled.

To the best of our knowledge, there is no result on the robust stabilization of a collection of coupled NCSs with multiple transmissions. Motivated by the references above, this paper will model multiple coupled NCSs with norm-bounded parameter uncertainties as switched system and then we can apply the theory of the switched system to NCSs [36, 37]. We consider the case that all the nodes act over a limited bandwidth communication channel, state information of every subsystem is split into different packets and only one packet can be transmitted at a time. Applying the toking bus protocol, the nodes are arranged logically into a ring and transmit their packets in a predetermined order. For multiple-packet transmitted NCSs with one-side channel and with both S/C and C/A network channel, we model such multiple coupled uncertain NCSs as a periodic switched system. We consider the setup with a clock-driven sensor, and both the controller and the actuator are event driven. The controller is installed to use the old state measurement if there is no new data updating. Then robust stability of the NCSs with periodic transmission is considered. State feedback controllers can be constructed in terms of linear matrix inequalities (LMIs).

The remainder of this paper is structured as follows. Section 2 models multiple NCSs with multipacket transmission and norm-bounded parameter uncertainties. Section 3 develops stabilization and stability results for state feedback case. Section 4 presents a numerical simulation to illustrate the efficiency and feasibility of our proposed approach. Section 5 concludes this paper.

Notation. We use standard notations throughout this paper. Denote by A^T the transpose of a matrix A. $A > 0$ ($A < 0$) means that A is positive definite (negative definite). I is the identifying matrix of appropriate dimension. R^n and R^{nm} denote, respectively, the n dimensional Euclidean space and the set of all nm real matrices. In symmetric block matrices, the symbol $*$ is used as an ellipsis for terms induced by symmetry. In token bus, the computers are connected so that the signal travels around the network from one computer to another in a logical ring. A single electronic token moves around the ring from one computer to the next. If a computer does not have information to transmit, it simply passes the token on to the next workstation.

2. System Modelling

We consider a finite collection of linear systems that are coupled together through their dynamics and feedback. We consider the case that the state information is transmitted

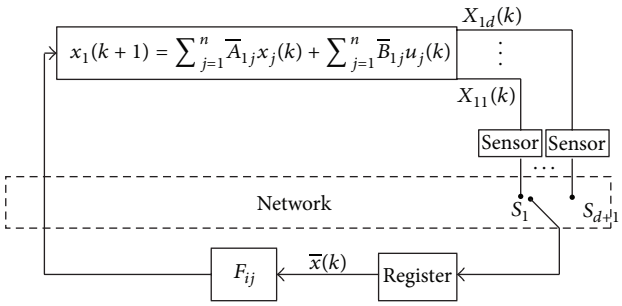

Figure 2: The structure of an NCS with S/C channel.

over a limited bandwidth communication channel. State information of every subsystem is split into different packets and only one packet can be transmitted at a time. We first consider multiple-packet transmission NCSs with only S/C channel, illustrated in Figure 2. The n coupled NCSs with norm-bounded parameter uncertainties are described by

$$x_i(k+1) = \sum_{j=1}^{n} \left(A_{ij} + \Delta A_{ij}\right) x_j(k) + \sum_{j=1}^{n} \left(B_{ij} + \Delta B_{ij}\right) u_j(k),$$
$$u_i(k) = \sum_{j=1}^{n} F_{ij} \overline{x}_j, \quad i, j = 1, 2, \ldots, n, \ k = 1, 2, \ldots, \tag{1}$$

where $x_i(k) \in \mathbf{R}^n$ and $u_i(k) \in \mathbf{R}^p$ are the plant state and the plant input, respectively. $\overline{x}_i(k)$ is the content of the register. F_{ij} is the feedback gain to be designed. A_{ij}, B_{ij} are known real constant matrices with appropriate dimensions. ΔA_{ij}, ΔB_{ij} characterize the uncertainties in the system and satisfy the following assumption:

$$\left[\Delta A_{ij} \quad \Delta B_{ij}\right] = E_{ij} \Gamma_j(k) \left[G_j \quad H_j\right], \tag{2}$$

where E_{ij}, G_j, and H_j are known real constant matrices of appropriate dimensions, and $\Gamma_j(k)$ are unknown matrix functions with Lebesgue-measurable elements which satisfies

$$\Gamma_j(k)^T \Gamma_j(k) \le I, \quad \forall k. \tag{3}$$

Define $\overline{A}_{ij} = A_{ij} + \Delta A_{ij}$, $\overline{B}_{ij} = B_{ij} + \Delta B_{ij}$; then the networked control systems with norm-bounded parameter uncertainties are described as

$$x_i(k+1) = \sum_{j=1}^{n} \overline{A}_{ij} x_j(k) + \sum_{j=1}^{n} \overline{B}_{ij} u_j(k),$$
$$u_j(k) = \sum_{j=1}^{n} F_{ij} \overline{x}_j, \quad i, j = 1, 2, \ldots, N, \ k = 1, 2, \ldots. \tag{4}$$

Suppose the state is split into d packets

$$x_i(k) = \left[X_{i1}^T(k), \ldots, X_{id}^T(k)\right]^T, \tag{5}$$

where $X_{it}(k) = [x_{ir_{t-1}+1}(k) \cdots x_{ir_t}(k)]^T$ and $1 \le t \le d$, $0 = r_0 < r_1 < \cdots < r_d = n$. The controller is installed to use the old state measurement if there is no new data updating. That is,

$$\overline{x}_i(k) = \left[\overline{X}_{i1}^T(k), \ldots, \overline{X}_{id}^T(k)\right]^T, \tag{6}$$

where

$$\overline{X}_{it}(k) = \begin{cases} X_{it}(k) & \text{if the packet containing} \\ & X_{it}(k) \text{ is transmitted;} \\ \overline{X}_{it}(k-1) & \text{otherwise.} \end{cases} \tag{7}$$

2.1. An Example of Two Subsystems. For simplicity, we first consider an example of two subsystems and the plant states of each subsystem are split into two packets. Consider

$$x_1(k+1) = \overline{A}_{11} x_1(k) + \overline{A}_{12} x_2(k) + \overline{B}_{11} u_1(k) + \overline{B}_{12} u_2(k),$$
$$x_2(k+1) = \overline{A}_{21} x_1(k) + \overline{A}_{22} x_2(k) + \overline{B}_{21} u_1(k) + \overline{B}_{22} u_2(k), \tag{8}$$

where $x_1(k) = [X_{11}^T(k), X_{12}^T(k)]^T$, $x_2(k) = [X_{21}^T(k), X_{22}^T(k)]^T$.

In the standard token-passing bus protocol, the token is repeatedly transmitted in a fixed order in the network, that is, $[1, 1, 2, 2]$ in turn. Here, we regard different packets as different nodes for the network protocol, and this means the packets would be transmitted in a periodic manner via the network channel. With the given four-step communication cycle, the system equations will evolve according to

$$\begin{aligned} x_1(k+1) &= \overline{A}_{11} x_1(k) + \overline{A}_{12} x_2(k) \\ &\quad + \overline{B}_{11} F_{11} \left[X_{11}^T(k), \overline{X}_{12}^T(k-1)\right]^T \\ &\quad + \overline{B}_{12} F_{12} \left[\overline{X}_{21}^T(k-1), \overline{X}_{22}^T(k-1)\right]^T, \\ x_1(k+2) &= \overline{A}_{11} x_1(k+1) + \overline{A}_{12} x_2(k+1) \\ &\quad + \overline{B}_{11} F_{11} \left[X_{11}^T(k), X_{12}^T(k+1)\right]^T \\ &\quad + \overline{B}_{12} F_{12} \left[\overline{X}_{21}^T(k-1), \overline{X}_{22}^T(k-1)\right]^T, \\ x_1(k+3) &= \overline{A}_{11} x_1(k+2) + \overline{A}_{12} x_2(k+2) \\ &\quad + \overline{B}_{11} F_{11} \left[X_{11}^T(k), X_{12}^T(k+1)\right]^T \end{aligned}$$

$$
\begin{aligned}
&+\overline{B}_{12}F_{12}\left[X_{21}^{T}(k+2),\overline{X}_{22}^{T}(k-1)\right]^{T},\\
x_1(k+4) &= \overline{A}_{11}x_1(k+3)+\overline{A}_{12}x_2(k+3)\\
&+\overline{B}_{11}F_{11}\left[X_{11}^{T}(k),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{12}F_{12}\left[X_{21}^{T}(k+2),X_{22}^{T}(k+3)\right]^{T},\\
x_1(k+5) &= \overline{A}_{11}x_1(k+4)+\overline{A}_{12}x_2(k+4)\\
&+\overline{B}_{11}F_{11}\left[X_{11}^{T}(k+4),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{12}F_{12}\left[X_{21}^{T}(k+2),X_{22}^{T}(k+3)\right]^{T},\\
&\vdots\\
x_2(k+1) &= \overline{A}_{21}x_1(k)+\overline{A}_{22}x_2(k)\\
&+\overline{B}_{21}F_{21}\left[X_{11}^{T}(k),\overline{X}_{12}^{T}(k-1)\right]^{T}\\
&+\overline{B}_{22}F_{22}\left[\overline{X}_{21}^{T}(k-1),\overline{X}_{22}^{T}(k-1)\right]^{T},\\
x_2(k+2) &= \overline{A}_{21}x_1(k+1)+\overline{A}_{22}x_2(k+1)\\
&+\overline{B}_{21}F_{21}\left[X_{11}^{T}(k),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{22}F_{22}\left[\overline{X}_{21}^{T}(k-1),\overline{X}_{22}^{T}(k-1)\right]^{T},\\
x_2(k+3) &= \overline{A}_{21}x_1(k+2)+\overline{A}_{22}x_2(k+2)\\
&+\overline{B}_{21}F_{21}\left[X_{11}^{T}(k),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{22}F_{22}\left[X_{21}^{T}(k+2),\overline{X}_{22}^{T}(k-1)\right]^{T},\\
x_2(k+4) &= \overline{A}_{21}x_1(k+3)+\overline{A}_{22}x_2(k+3)\\
&+\overline{B}_{21}F_{21}\left[X_{11}^{T}(k),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{22}F_{22}\left[X_{21}^{T}(k+2),X_{22}^{T}(k+3)\right]^{T},\\
x_2(k+5) &= \overline{A}_{21}x_1(k+4)+\overline{A}_{22}x_2(k+4)\\
&+\overline{B}_{21}F_{21}\left[X_{11}^{T}(k+4),X_{12}^{T}(k+1)\right]^{T}\\
&+\overline{B}_{22}F_{22}\left[X_{21}^{T}(k+2),X_{22}^{T}(k+3)\right]^{T}.
\end{aligned}
\tag{9}
$$

In this case, the two subsystems x_1 and x_2 are coupled and their dynamics have a periodicity of four steps. Similar to [10], define the buffered states as

$$
\begin{aligned}
\widehat{x}_1(k) &= \begin{bmatrix} x_1(k-3)\\ x_1(k-2)\\ x_1(k-1)\\ x_1(k)\end{bmatrix},\\
\widehat{x}_2(k) &= \begin{bmatrix} x_2(k-3)\\ x_2(k-2)\\ x_2(k-1)\\ x_2(k)\end{bmatrix}, \qquad \widetilde{x} = \begin{bmatrix}\widehat{x}_1\\ \widehat{x}_2\end{bmatrix}.
\end{aligned}
\tag{10}
$$

Then the corresponding drift dynamics are

$$
\begin{aligned}
\widehat{x}_1(k+1) &= \begin{bmatrix} 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ \overline{B}_{11}F_{11}D_{10} & 0 & 0 & \overline{A}_{11}+\overline{B}_{11}F_{11}D_{11}\end{bmatrix}\widehat{x}_1(k)\\
&\quad+\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \overline{B}_{12}F_{12}D_{21} & \overline{B}_{12}F_{12}D_{20} & \overline{A}_{12}\end{bmatrix}\widehat{x}_2(k),\\
\widehat{x}_2(k+1) &= \begin{bmatrix} 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\\ \overline{B}_{21}F_{21}D_{10} & 0 & 0 & \overline{A}_{21}+\overline{B}_{21}F_{21}D_{11}\end{bmatrix}\widehat{x}_1(k)\\
&\quad+\begin{bmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & \overline{B}_{22}F_{22}D_{21} & \overline{B}_{22}F_{22}D_{20} & \overline{A}_{22}\end{bmatrix}\widehat{x}_2(k),
\end{aligned}
\tag{11}
$$

where

$$
\begin{aligned}
D_{11} &= \begin{bmatrix} I & 0\\ 0 & 0\end{bmatrix}, \qquad D_{10} = \begin{bmatrix} 0 & I\\ 0 & 0\end{bmatrix},\\
D_{21} &= \begin{bmatrix} 0 & 0\\ I & 0\end{bmatrix}, \qquad D_{20} = \begin{bmatrix} 0 & 0\\ 0 & I\end{bmatrix}.
\end{aligned}
\tag{12}
$$

The combined drift dynamics at each of the four steps of the communication sequence are

$$
\widetilde{x}(k+1) = M_q\widetilde{x}(k), \tag{13}
$$

where

$$M_1 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ \overline{B}_{11}F_{11}D_{10} & 0 & 0 & \overline{A}_{11}+\overline{B}_{11}F_{11}D_{11} & 0 & \overline{B}_{12}F_{12}D_{21} & \overline{B}_{12}F_{12}D_{20} & \overline{A}_{12} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ \overline{B}_{21}F_{21}D_{10} & 0 & 0 & \overline{A}_{21}+\overline{B}_{21}F_{21}D_{11} & 0 & \overline{B}_{22}F_{22}D_{21} & \overline{B}_{22}F_{22}D_{20} & \overline{A}_{22} \end{bmatrix},$$

$$M_2 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & \overline{B}_{11}F_{11}D_{11} & \overline{A}_{11}+\overline{B}_{11}F_{11}D_{10} & \overline{B}_{12}F_{12}D_{21} & \overline{B}_{12}F_{12}D_{20} & 0 & \overline{A}_{12} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & \overline{B}_{21}F_{21}D_{10} & \overline{A}_{21}+\overline{B}_{21}F_{21}D_{11} & \overline{B}_{22}F_{22}D_{21} & \overline{B}_{22}F_{22}D_{20} & 0 & \overline{A}_{22} \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & \overline{B}_{11}F_{11}D_{11} & \overline{B}_{11}F_{11}D_{10} & \overline{A}_{11} & \overline{B}_{12}F_{12}D_{20} & 0 & 0 & \overline{A}_{12}+\overline{B}_{12}F_{12}D_{21} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & \overline{B}_{21}F_{21}D_{11} & \overline{B}_{21}F_{21}D_{10} & \overline{A}_{21} & \overline{B}_{22}F_{22}D_{20} & 0 & 0 & \overline{A}_{22}+\overline{B}_{22}F_{22}D_{21} \end{bmatrix},$$

$$M_4 = \begin{bmatrix} 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \\ \overline{B}_{11}F_{11}D_{11} & \overline{B}_{11}F_{11}D_{10} & 0 & \overline{A}_{11} & 0 & 0 & \overline{B}_{12}F_{12}D_{21} & \overline{A}_{12}+\overline{B}_{12}F_{12}D_{20} \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I \\ \overline{B}_{21}F_{21}D_{11} & \overline{B}_{21}F_{21}D_{10} & 0 & \overline{A}_{21} & 0 & 0 & \overline{B}_{22}F_{22}D_{21} & \overline{A}_{22}+\overline{B}_{22}F_{22}D_{20} \end{bmatrix}. \tag{14}$$

Remark 1. For data packets transmitted in both C/A and S/C channels periodically, similar model can be obtained.

As shown in Figure 3, $\overline{x}(k)$ is the received state information by the controller over S/C channel, $u(k)$, which will be sent to the system over C/A channel, is the output of the controller, and $\overline{u}(k)$ is the real system input. We assume that plant output and controller output are divided into d $(d \geq 1)$ packets to be transmitted. The plant input is presented as (5) and the controller output is described as

$$u_i(k) = \left[\overline{U}_{i1}^T(k), \ldots, \overline{U}_{id}^T(k)\right]^T, \tag{15}$$

where

$$U_{it}(k) = \left[u_{ir_{t-1}+1}(k) \ \cdots \ u_{ir_t}(k)\right]^T, \tag{16}$$

where $1 \leq t \leq d$, $0 = r_0 < r_1 < \cdots < r_d = n$.

Consequently, denote

$$u_i(k) = \left[U_{i1}^T(k), \ldots, U_{id}^T(k)\right]^T, \tag{17}$$

where

$$U_{it}(k) = \begin{cases} U_{it}(k), & \text{if the packet containing} \\ & U_{it}(k) \text{ is transmitted;} \\ \overline{U}_{it}(k-1), & \text{otherwise.} \end{cases} \tag{18}$$

Then

$$u_i(k) = \sum_{j=1}^{n} F_{ij}\left[\overline{X}_{i1}^T(k) \cdots \overline{X}_{id}^T(k)\right]^T, \tag{19}$$

where $\overline{x}_i(k)$ is defined in (6).

In the case that both C/A and S/C channels are standard token-passing bus networks, the packets in the two channels

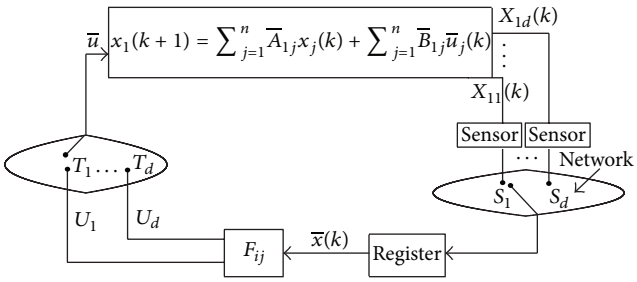

Figure 3: The structure of an NCS with both S/C and C/A channel.

would be transmitted synchronously in a periodic manner. We still study the following two subsystems and there is only one packet that can be transmitted every time:

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) + B_{11}\overline{u}_1(k) + B_{12}\overline{u}_2(k), \\ x_2(k+1) &= A_{21}x_1(k) + A_{22}x_2(k) + B_{21}\overline{u}_1(k) + B_{22}\overline{u}_2(k), \end{aligned} \tag{20}$$

where

$$\begin{aligned} x_1(k) &= \left[X_{11}^T(k), X_{12}^T(k)\right]^T, \\ x_2(k) &= \left[X_{21}^T(k), X_{22}^T(k)\right]^T, \\ u_1(k) &= \left[U_{11}^T(k), U_{12}^T(k)\right]^T, \\ u_2(k) &= \left[U_{21}^T(k), U_{22}^T(k)\right]^T. \end{aligned} \tag{21}$$

Suppose the packets are transmitted over the communication channel with a periodic communication sequence $[1, 1, 2, 2]$ in turn, the system equations will evolve according to

$$\begin{aligned} x_1(k+1) &= A_{11}x_1(k) + A_{12}x_2(k) \\ &\quad + B_{11}\left[U_{11}^T(k), \overline{U}_{12}^T(k-1)\right]^T \\ &\quad + B_{12}\left[\overline{U}_{21}^T(k-1), \overline{U}_{22}^T(k-1)\right]^T \\ &= A_{11}x_1(k) + A_{12}x_2(k) \\ &\quad + B_{11}\left[F_{11}X_{11}^T(k), F_{11}\overline{X}_{12}^T(k-1)\right]^T \\ &\quad + B_{12}\left[F_{12}\overline{X}_{21}^T(k-1), F_{12}\overline{X}_{22}^T(k-1)\right]^T, \\ x_1(k+2) &= A_{11}x_1(k+1) + A_{12}x_2(k+1) \\ &\quad + B_{11}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[\overline{U}_{21}^T(k-1), \overline{U}_{22}^T(k-1)\right]^T \\ &= A_{11}x_1(k+1) + A_{12}x_2(k+1) \\ &\quad + B_{11}\left[F_{11}X_{11}^T(k), F_{11}X_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[F_{12}\overline{X}_{21}^T(k-1), F_{12}\overline{X}_{22}^T(k-1)\right]^T, \\ x_1(k+3) &= A_{11}x_1(k+2) + A_{12}x_2(k+2) \\ &\quad + B_{11}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[U_{21}^T(k+2), \overline{U}_{22}^T(k-1)\right]^T \\ &= A_{11}x_1(k+2) + A_{12}x_2(k+2) \\ &\quad + B_{11}\left[F_{11}X_{11}^T(k), F_{11}X_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[F_{12}X_{21}^T(k+2), F_{12}\overline{X}_{22}^T(k-1)\right]^T, \\ x_1(k+4) &= A_{11}x_1(k+3) + A_{12}x_2(k+3) \\ &\quad + B_{11}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[U_{21}^T(k+2), U_{22}^T(k+3)\right]^T \\ &= A_{11}x_1(k+3) + A_{12}x_2(k+3) \\ &\quad + B_{11}\left[F_{11}X_{11}^T(k), F_{11}X_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[F_{12}X_{21}^T(k+2), F_{12}X_{22}^T(k+3)\right]^T, \\ x_1(k+5) &= A_{11}x_1(k+4) + A_{12}x_2(k+4) \\ &\quad + B_{11}\left[U_{11}^T(k+4), U_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[U_{21}^T(k+2), U_{22}^T(k+3)\right]^T \\ &= A_{11}x_1(k+4) + A_{12}x_2(k+4) \\ &\quad + B_{11}\left[F_{11}X_{11}^T(k+4), F_{11}X_{12}^T(k+1)\right]^T \\ &\quad + B_{12}\left[F_{12}X_{21}^T(k+2), F_{12}X_{22}^T(k+3)\right]^T, \\ &\vdots \\ x_2(k+1) &= A_{21}x_1(k+1) + A_{22}x_2(k+1) \\ &\quad + B_{21}\left[U_{11}^T(k), \overline{U}_{12}^T(k+1)\right]^T \\ &\quad + B_{22}\left[\overline{U}_{21}^T(k-1), \overline{U}_{22}^T(k-1)\right]^T, \end{aligned}$$

$$
\begin{aligned}
&= A_{21}x_1(k) + A_{22}x_2(k) \\
&\quad + B_{21}\left[F_{21}X_{11}^T(k), F_{21}\overline{X}_{12}^T(k-1)\right]^T \\
&\quad + B_{22}\left[F_{22}\overline{X}_{21}^T(k-1), F_{22}\overline{X}_{22}^T(k-1)\right]^T, \\
x_2(k+2) &= A_{21}x_1(k+1) + A_{22}x_2(k+1) \\
&\quad + B_{21}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[\overline{U}_{21}^T(k-1), \overline{U}_{22}^T(k-1)\right]^T \\
&= A_{21}x_1(k+1) + A_{22}x_2(k+1) \\
&\quad + B_{21}\left[F_{21}X_{11}^T(k), F_{21}X_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[F_{22}\overline{X}_{21}^T(k-1), F_{22}\overline{X}_{22}^T(k-1)\right]^T, \\
x_2(k+3) &= A_{21}x_1(k+2) + A_{22}x_2(k+2) \\
&\quad + B_{21}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[U_{21}^T(k+2), \overline{U}_{22}^T(k-1)\right]^T \\
&= A_{21}x_1(k+2) + A_{22}x_2(k+2) \\
&\quad + B_{21}\left[F_{21}X_{11}^T(k), F_{21}X_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[F_{22}X_{21}^T(k+2), F_{22}\overline{X}_{22}^T(k-1)\right]^T, \\
x_2(k+4) &= A_{21}x_1(k+3) + A_{22}x_2(k+3) \\
&\quad + B_{21}\left[U_{11}^T(k), U_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[U_{21}^T(k+2), U_{22}^T(k+3)\right]^T \\
&= A_{21}x_1(k+3) + A_{22}x_2(k+3) \\
&\quad + B_{21}\left[F_{21}X_{11}^T(k), F_{21}X_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[F_{22}X_{21}^T(k+2), F_{22}X_{22}^T(k+3)\right]^T, \\
x_2(k+5) &= A_{21}x_1(k+4) + A_{22}x_2(k+4) \\
&\quad + B_{21}\left[U_{11}^T(k+4), U_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[U_{21}^T(k+2), U_{22}^T(k+3)\right]^T \\
&= A_{21}x_1(k+4) + A_{22}x_2(k+4) \\
&\quad + B_{21}\left[F_{21}X_{11}^T(k+4), F_{21}X_{12}^T(k+1)\right]^T \\
&\quad + B_{22}\left[F_{22}X_{21}^T(k+2), F_{22}X_{22}^T(k+3)\right]^T.
\end{aligned}
\tag{22}
$$

Using the definition of the buffered states (10), we still obtain the evolution equation (13).

2.2. The General Case. Similarly, if the states of each subsystem are split into d packets and the packets are transmitted in a periodic pattern, we can obtain the general case.

Suppose $\widetilde{x} = [\widehat{x}_1^T \cdots \widehat{x}_n^T]^T$, $\widehat{x}_i$ contains past state values for each step in the communication sequence and $\widehat{x}_i = [x_i^T(k-p+1) \cdots x_i^T(k)]^T$ and p is the length of communication sequence. To describe the general case, we define an integer matrix, T, that contains an update sequence for each subsystem's packet. The matrix T will have n rows (one for each subsystem) and p columns. The (i, j)th entry of T will denote the number of steps from the jth step of the communication sequence to the next communication with the ith subsystem. Define $\widetilde{A}_{ij} = J_p \otimes I + \sum_{j=1}^{n}(E_{n,n} \otimes A_{ij})$, $\Delta\widetilde{A}_{ij} = \sum_{j=1}^{n} E_{n,n} \otimes \Delta A_{ij}$, where J_p is the $p \times p$ Jordan matrix composed of all zeros except for ones on the superdiagonal; $E_{i,j}$ is a matrix of zeros with a one in the (i, j)th position. Define

$$
D_j q = \begin{bmatrix} 0 & & & & \\ & \ddots & & 0 & \\ & & I_{(r_j - r_{j-1})q} & & \\ & 0 & & \ddots & \\ & & & & 0 \end{bmatrix}, \tag{23}
$$

$\widetilde{F}_{ijq} = E_{n,T_{jq}} \otimes B_{ij}F_{ij}D_{jq}$, and $\Delta\widetilde{F}_{ijq} = E_{n,T_{jq}} \otimes \Delta B_{ij}F_{ij}D_{jq}$, which is related to the feedback gains for subsystem i at the qth step in the communication sequence. Now, at each sequence step, $q = 1, \ldots, p$, we can write the evolution of all subsystems as

$$
\widetilde{x}(k+1) = M_q\widetilde{x}(k), \quad k \in \{np + q\}, \tag{24}
$$

where

$$
\begin{aligned}
M_q = &\begin{bmatrix} \widetilde{A}_{11} & \cdots & \widetilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \widetilde{A}_{n1} & \cdots & \widetilde{A}_{nn} \end{bmatrix} + \begin{bmatrix} \Delta\widetilde{A}_{11} & \cdots & \Delta\widetilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ \Delta\widetilde{A}_{n1} & \cdots & \Delta\widetilde{A}_{nn} \end{bmatrix} \\
&+ \begin{bmatrix} \widetilde{F}_{11q} & \cdots & \widetilde{F}_{1nq} \\ \vdots & \ddots & \vdots \\ \widetilde{F}_{n1q} & \cdots & \widetilde{F}_{nnq} \end{bmatrix} + \begin{bmatrix} \Delta\widetilde{F}_{11q} & \cdots & \Delta\widetilde{F}_{1nq} \\ \vdots & \ddots & \vdots \\ \Delta\widetilde{F}_{n1q} & \cdots & \Delta\widetilde{F}_{nnq} \end{bmatrix}.
\end{aligned}
\tag{25}
$$

Remark 2. This paper models uncertain coupled NCSs with multiple packet transmission as switched system. The transmission pattern of the NCSs is closely related to the switching law of the switched system. This switched model not only enables us to design the controller for the NCSs, but also makes it more convenient to deal with the uncertain part.

3. Stability Analysis and Stabilization Result

We consider the case that in which all nodes transmitted in a toking-bus. With the toking bus protocol applied, the nodes

are arranged logically into a ring and transmit their packets in a predetermined order. It can be seen that system (24) is a switched linear system switching among the following subsystems:

$$\{M_1, \ldots, M_p\} \tag{26}$$

in a periodic manner. Clearly, the original system (1) is stable if the switched system (24) is stable.

The following result gives a sufficient condition on the stability of the NCS (1) with packets in different network channels transmitted in a periodic manner.

Lemma 3 (see [19]). *Let the states of multiple NCSs* (1) *be split into multiple data packets and suppose the transmission of these data packets is in a periodic manner. Then NCS* (1) *is uniformly asymptotically stable if all the eigenvalues of* Ψ *are contained within the unit circle; that is,* $|\lambda_i(\Psi)| < 1$ *for* $i = 1, 2, \ldots, n$, *where* $\Psi = \prod_{i=1}^{i=p} M_i$.

The following lemma will play a key rule to design the feedback gain for NCS (1).

Lemma 4 (see [38]). *Let the matrices* U, W, *and* $\Phi = \Phi^*$ *be given. Then the following statements are equivalent.*

(i) *There exists a matrix* V *satisfying*

$$UVW + (UVW)^* + \Phi < 0. \tag{27}$$

(ii) *The following two conditions hold:*

$$\begin{aligned} N_u \Phi N_u^* < 0 \quad &or \quad UU^* > 0, \\ N_w^* \Phi N_w < 0 \quad &or \quad W^* W > 0, \end{aligned} \tag{28}$$

where N_u *and* N_w^* *are, respectively, orthogonal complements of* N *and* W^*; *that is,*

$$N_u U = 0, \qquad N_w^* W^* = 0. \tag{29}$$

The following lemma will be used to deal with the uncertain part of the NCSs.

Lemma 5 (see [17]). M, N, *and* Λ *are real matrices with proper dimensions which satisfy* $\Lambda^T \Lambda \le I$; *then for any positive scalar* ϵ, *we can get the following inequality:*

$$M\Lambda N + N^T \Lambda^T M^T \le \varepsilon M M^T + \varepsilon^{-1} N^T N. \tag{30}$$

We give the stabilization result in the following.

Theorem 6. *If there exist a positive definite matrix* P, *matrices* K_{ij}, Y_{ij}, *and positive scalars* ϵ, ε_i *satisfying*

$$PB_{ij} = B_{ij} K_{ij}, \tag{31}$$

and the following LMI

$$\begin{bmatrix} \Omega & \overline{\overline{M}} & \overline{\overline{N}}^T & \overline{\overline{M_1}} & \overline{\overline{N_1}}^T & \cdots & \overline{\overline{M_n}} & \overline{\overline{N_n}}^T \\ \overline{\overline{M}}^T & -\Omega_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \overline{\overline{N}} & 0 & -\Omega_0 & 0 & 0 & \cdots & 0 & 0 \\ \overline{\overline{M_1}}^T & 0 & 0 & -\Omega_1 & 0 & \cdots & 0 & 0 \\ \overline{\overline{N_1}} & 0 & 0 & 0 & -\Omega_1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \overline{\overline{M_n}}^T & 0 & 0 & 0 & 0 & \cdots & -\Omega_n & 0 \\ \overline{\overline{N_n}} & 0 & 0 & 0 & 0 & \cdots & 0 & -\Omega_n \end{bmatrix} < 0, \tag{32}$$

where

$$\Omega = \begin{bmatrix} -\overline{P} & \bigwedge_p^T & 0 & \cdots & 0 & 0 \\ \bigwedge_p & -2\overline{P} & \bigwedge_{p-1}^T & \cdots & 0 & 0 \\ 0 & \bigwedge_{p-1} & -2\overline{P} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\overline{P} & \bigwedge_1^T \\ 0 & 0 & 0 & \cdots & \bigwedge_1 & -\overline{P} \end{bmatrix},$$

$$\overline{P} = \begin{bmatrix} P & & 0 \\ & \ddots & \\ 0 & & P \end{bmatrix},$$

$$\bigwedge_q = \begin{bmatrix} P\widetilde{A}_{11} & \cdots & P\widetilde{A}_{1n} \\ \vdots & \ddots & \vdots \\ P\widetilde{A}_{n1} & \cdots & P\widetilde{A}_{nn} \end{bmatrix} + \begin{bmatrix} E_{nT_{1t_q}} \otimes B_{11} Y_{11} D_{1t_q} & \cdots & E_{nT_{nt_q}} \otimes B_{1n} Y_{1n} D_{nt_q} \\ \vdots & \ddots & \vdots \\ E_{nT_{1t_q}} \otimes B_{n1} Y_{n1} D_{1t_q} & \cdots & E_{nT_{nt_q}} \otimes B_{nn} Y_{nn} D_{nt_q} \end{bmatrix},$$

$$\overline{\overline{M}} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & M & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M \end{bmatrix},$$

$$M = \begin{bmatrix} PE_{n,n} \otimes \overline{E}_{11} & \cdots & PE_{n,n} \otimes \overline{E}_{1n} \\ \vdots & \ddots & \vdots \\ PE_{n,n} \otimes \overline{E}_{n1} & \cdots & PE_{n,n} \otimes \overline{E}_{nn} \end{bmatrix},$$

$$\overline{\overline{N}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \epsilon N & 0 & 0 & \cdots & 0 & 0 \\ 0 & \epsilon N & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \epsilon N & 0 \end{bmatrix},$$

$$N = \begin{bmatrix} E_{n,n} \otimes \overline{G}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{n,n} \otimes \overline{G}_n \end{bmatrix},$$

$$\overline{\overline{M_i}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_i M_{ip} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \varepsilon_i M_{i(p-1)} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon_i M_{i2} & 0 \\ 0 & 0 & 0 & \cdots & 0 & \varepsilon_i M_{i1} \end{bmatrix},$$

$$M_{iq} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ E_{nT_{1t_q}} \otimes \overline{E}_{i1} & \cdots & E_{nT_{nt_q}} \otimes \overline{E}_{in} \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix},$$

$$\overline{\overline{N_i}} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ N_{ip} & 0 & 0 & \cdots & 0 & 0 \\ 0 & N_{i(p-1)} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & N_{i1} & 0 \end{bmatrix},$$

$$N_{iq} = \begin{bmatrix} E_{T_{1t_q}T_{1t_q}} \otimes \overline{H}_1 Y_{i1} D_{1t_q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{T_{nt_q}T_{nt_q}} \otimes \overline{H}_n Y_{in} D_{nt_q} \end{bmatrix},$$

$$\overline{E}_{ij} = [E_{ij} \quad 0], \qquad \overline{G}_j = \begin{bmatrix} G_j \\ 0 \end{bmatrix}, \qquad \overline{H}_j = \begin{bmatrix} H_j \\ 0 \end{bmatrix},$$

$$\Omega_0 = \operatorname{diag}\{\epsilon I \ \cdots \ \epsilon I\}, \qquad \Omega_j = \operatorname{diag}\{\epsilon_i I \ \cdots \ \epsilon_i I\}, \tag{33}$$

where $q = 1, \ldots, p$, $j = 1, 2, \ldots, n$, and $i = 1, 2, \ldots, n$, then multiple NCSs (1) can be robustly stabilized with the state feedback gain

$$F_{ij} = K_{ij}^{-1} Y_{ij}. \tag{34}$$

Proof. From Theorem 6, the stabilization problem is to compute the feedback gain F_{ij} such that Ψ is Schur-stable. In the Lyapunov framework, the Schur-stability of matrix Ψ can be guaranteed by the existence of a symmetric positive definite matrix $\overline{P}$ such that the following inequality holds:

$$-\overline{P} + \Psi^T \overline{P} \Psi < 0. \tag{35}$$

Condition (35) can be written as

$$\begin{bmatrix} I & M_p^T \end{bmatrix} \begin{bmatrix} -\overline{P} & 0 \\ 0 & \Pi_{p-1}^T \overline{P} \Pi_{p-1} \end{bmatrix} \begin{bmatrix} I \\ M_p \end{bmatrix} < 0, \tag{36}$$

where

$$\Pi_{p-1} = M_1 M_2 \cdots M_{p-1}. \tag{37}$$

We define $N_u = [I \quad M_p^T]$, $V = \overline{P}$, and $W = [0 \quad I]$. Using Lemma 4, (36) is equivalent to the existence of symmetric positive definite matrix $\overline{P}$ such that the following inequality holds:

$$\begin{aligned} &\begin{bmatrix} -\overline{P} & 0 \\ 0 & \Pi_{p-1}^T \overline{P} \Pi_{p-1} \end{bmatrix} + \begin{bmatrix} M_p^T \\ -I \end{bmatrix} \overline{P} \begin{bmatrix} 0 & I \end{bmatrix} \\ &+ \begin{bmatrix} 0 \\ I \end{bmatrix} \overline{P} \begin{bmatrix} M_p & -I \end{bmatrix} < 0. \end{aligned} \tag{38}$$

Rearranging it, we obtain

$$\begin{bmatrix} -\overline{P} & M_p^T \overline{P} \\ \overline{P} M_p & -2\overline{P} + \Pi_{p-1}^T \overline{P} \Pi_{p-1} \end{bmatrix} < 0, \tag{39}$$

which can be written as

$$\begin{aligned} &\begin{bmatrix} I & 0 & 0 \\ 0 & I & M_{p-1}^T \end{bmatrix} \begin{bmatrix} -\overline{P} & M_p^T \overline{P} & 0 \\ \overline{P} M_p & -2\overline{P} & 0 \\ 0 & 0 & \Pi_{p-2}^T \overline{P} \Pi_{p-2} \end{bmatrix} \\ &\times \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & M_{p-1} \end{bmatrix} < 0, \end{aligned} \tag{40}$$

where

$$\Pi_{p-2} = M_1 M_2 \cdots M_{p-2}. \tag{41}$$

Repeating this procedure, we can show that (36) can be guaranteed by the following inequality:

$$\begin{bmatrix} -\overline{P} & M_p^T\overline{P} & 0 & \cdots & 0 & 0 \\ \overline{P}M_p & -2\overline{P} & M_{p-1}^T\overline{P} & \cdots & 0 & 0 \\ 0 & \overline{P}M_{p-1} & -2\overline{P} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\overline{P} & M_1^T\overline{P} \\ 0 & 0 & 0 & \cdots & \overline{P}M_1 & -\overline{P} \end{bmatrix} < 0. \tag{42}$$

Let $Y_{ij} = K_{ij}F_{ij}$. Using the definition of M_q $(i = 1, \ldots, p)$, together with (31), we know (42) is equivalent to

$$\begin{bmatrix} -\overline{P} & \bigwedge_p^T + \Delta\bigwedge_p^T & 0 & \cdots & 0 & 0 \\ \bigwedge_p + \Delta\bigwedge_p & -2\overline{P} & \bigwedge_{p-1}^T + \Delta\bigwedge_{p-1}^T & \cdots & 0 & 0 \\ 0 & \bigwedge_{p-1} + \Delta\bigwedge_{p-1} & -2\overline{P} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\overline{P} & \bigwedge_1^T + \Delta\bigwedge_1^T \\ 0 & 0 & 0 & \cdots & \bigwedge_1 + \Delta\bigwedge_1 & -\overline{P} \end{bmatrix} < 0, \tag{43}$$

which can be written as

$$\begin{bmatrix} -\overline{P} & \bigwedge_p^T & 0 & \cdots & 0 & 0 \\ \bigwedge_p & -2\overline{P} & \bigwedge_{p-1}^T & \cdots & 0 & 0 \\ 0 & \bigwedge_{p-1} & -2\overline{P} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -2\overline{P} & \bigwedge_1^T \\ 0 & 0 & 0 & \cdots & \bigwedge_1 & -\overline{P} \end{bmatrix} + \begin{bmatrix} 0 & \Delta\bigwedge_p^T & 0 & \cdots & 0 & 0 \\ \Delta\bigwedge_p & 0 & \Delta\bigwedge_{p-1}^T & \cdots & 0 & 0 \\ 0 & \Delta\bigwedge_{p-1} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \Delta\bigwedge_1^T \\ 0 & 0 & 0 & \cdots & \Delta\bigwedge_1 & 0 \end{bmatrix} < 0, \tag{44}$$

where

$$\begin{aligned} \Delta\bigwedge_q = {} & \begin{bmatrix} PE_{n,n} \otimes \overline{E}_{11} & \cdots & PE_{n,n} \otimes \overline{E}_{1n} \\ \vdots & \ddots & \vdots \\ PE_{n,n} \otimes \overline{E}_{n1} & \cdots & PE_{n,n} \otimes \overline{E}_{nn} \end{bmatrix} \begin{bmatrix} E_{n,n} \otimes \overline{\Gamma}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{n,n} \otimes \overline{\Gamma}_n \end{bmatrix} \begin{bmatrix} E_{n,n} \otimes \overline{G}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{n,n} \otimes \overline{G}_n \end{bmatrix} \\ & + \sum_{i=1}^{n} \begin{bmatrix} 0 & & \\ E_{nT_{1t_q}} \otimes \overline{E}_{i1} & \cdots & E_{nT_{nt_q}} \otimes \overline{E}_{in} \\ 0 & & \end{bmatrix} \begin{bmatrix} E_{T_{1t_q}T_{1t_q}} \otimes \overline{\Gamma}_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{T_{nt_q}T_{nt_q}} \otimes \overline{\Gamma}_n \end{bmatrix} \\ & \times \begin{bmatrix} E_{T_{1t_q}T_{1t_q}} \otimes \overline{H}_1 Y_{i1} D_{1t_q} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & E_{T_{nt_q}T_{nt_q}} \otimes \overline{H}_n Y_{in} D_{nt_q} \end{bmatrix}. \end{aligned} \tag{45}$$

Define $\overline{\overline{M}} = \overline{M}$, $\overline{\overline{N}} = \epsilon\overline{N}$, $\overline{\overline{M_i}} = \varepsilon_i\overline{M_i}$, $\overline{\overline{N_i}} = \overline{N_i}$, $q = 1,\ldots,p$. Then LMIs (44) can be rewritten as

$$\begin{aligned}\Omega &+ \overline{M}\,\overline{\overline{\Gamma}}_1(k)\,\overline{N} + \overline{N}^T\overline{\overline{\Gamma}}_1(k)^T\overline{M}^T \\ &+ \sum_{i=1}^{n}\overline{M_i}\,\overline{\overline{\Gamma}}_i(k)\,\overline{N_i} + \overline{N_i}^T\overline{\overline{\Gamma}}_i(k)^T\overline{M_i}^T < 0,\end{aligned} \tag{46}$$

where

$$\begin{aligned}&\overline{\overline{\Gamma}}_1(k) = \operatorname{diag}\left\{\Gamma',\Gamma',\ldots,\Gamma'\right\},\\ &\Gamma' = \begin{bmatrix} E_{n,n}\otimes\overline{\Gamma}_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & E_{n,n}\otimes\overline{\Gamma}_n\end{bmatrix},\\ &\overline{\Gamma}_j = \operatorname{diag}\{\Gamma(k),\Gamma(k),\ldots,\Gamma(k)\},\\ &\overline{\overline{\Gamma}}_2(k) = \operatorname{diag}\left\{0,\Gamma'_p(k),\Gamma'_{p-1}(k),\ldots,\Gamma'_1(k)\right\},\\ &\Gamma'_q = \begin{bmatrix} E_{T_{1t_q}T_{1t_q}}\otimes\overline{\Gamma}_1 & \cdots & 0\\ \vdots & \ddots & \vdots\\ 0 & \cdots & E_{T_{nt_q}T_{nt_q}}\otimes\overline{\Gamma}_n\end{bmatrix}.\end{aligned} \tag{47}$$

It can be deduced from Lemma 4 that LMI (46) is satisfied if there exist positive scalars ε, ε_i satisfying

$$\Omega + \varepsilon\overline{M}\,\overline{M}^T + \varepsilon^{-1}\overline{N}^T\overline{N} + \sum_{i=1}^{n}\varepsilon_i\overline{M_i}\,\overline{M_i}^T + \sum_{i=1}^{n}\varepsilon_i^{-1}\overline{N_i}^T\overline{N_i} < 0. \tag{48}$$

By Lemma 3, it is known inequality (48) is equivalent to inequality (32), and NCS (1) can be robustly stabilized with the state feedback gain

$$F_{ij} = K_{ij}^{-1}Y_{ij}. \tag{49}$$

□

Remark 7. It is noted that special structured matrices D_{jq} have been introduced into the closed-loop system to deal with multiple packet transmission and this makes the feedback controller design difficult. To solve this problem, equation constraints have been introduced in Theorem 6 and this brings the conservatism.

4. A Simulation Example

In this section, a numerical example is given to demonstrate the effectiveness of our method. We consider the case in which the state of the system is split into two parts and the plant has time-varying norm-bounded parameter uncertainties. The coupled system is given by

$$\begin{aligned}x_1(k+1) &= \left(A_{11} + E_{11}\Gamma_1(k)G_1\right)x_1(t)\\ &\quad + \left(A_{12} + E_{12}\Gamma_2(k)G_2\right)x_2(t)\\ &\quad + \left(B_{11} + E_{11}\Gamma_1(t)H_1\right)u_1(t)\\ &\quad + \left(B_{12} + E_{12}\Gamma_2(t)H_2\right)u_2(t),\\ x_2(k+1) &= \left(A_{21} + E_{21}\Gamma_1(k)G_1\right)x_1(t)\\ &\quad + \left(A_{22} + E_{22}\Gamma_2(k)G_2\right)x_2(t)\\ &\quad + \left(B_{21} + E_{21}\Gamma_1(t)H_1\right)u_1(t)\\ &\quad + \left(B_{22} + E_{22}\Gamma_2(t)H_2\right)u_2(t),\end{aligned} \tag{50}$$

where

$$\begin{aligned}&A_{11} = \begin{bmatrix}0.3 & 0\\ 0.4 & 0.4\end{bmatrix}, \qquad A_{12} = \begin{bmatrix}0.2 & 0.01\\ 0 & 0.2\end{bmatrix},\\ &A_{21} = \begin{bmatrix}0.05 & 0\\ 0.3 & 0.3\end{bmatrix}, \qquad A_{22} = \begin{bmatrix}0 & 0.05\\ 0.3 & 0.3\end{bmatrix},\\ &B_{11} = \begin{bmatrix}0.3\\ 0.6\end{bmatrix}, \qquad B_{12} = \begin{bmatrix}0.2\\ 0.5\end{bmatrix},\\ &B_{21} = \begin{bmatrix}0.3\\ 1\end{bmatrix}, \qquad B_{22} = \begin{bmatrix}0.08\\ 0.02\end{bmatrix},\\ &E_{11} = \begin{bmatrix}0.01\\ -0.001\end{bmatrix}, \qquad E_{12} = \begin{bmatrix}0.02\\ -0.002\end{bmatrix},\\ &E_{21} = \begin{bmatrix}0.002\\ -0.002\end{bmatrix}, \qquad E_{22} = \begin{bmatrix}0.001\\ -0.001\end{bmatrix},\\ &G_1 = \begin{bmatrix}0.001 & 0.001\end{bmatrix}, \qquad G_2 = \begin{bmatrix}0.002 & 0.002\end{bmatrix},\\ &H_1 = 0.001, \qquad H_2 = 0.002,\\ &\Gamma_1(k) = \sin(20k), \qquad \Gamma_2(k) = \sin(50k).\end{aligned} \tag{51}$$

We will show that the NCSs can be stabilized with only a quarter of the state information transmitted, even including time-varying norm-bounded parameter uncertainties. Suppose the controller pays equal attention to each of the two subsystems and the communication sequence is [1, 1, 2, 2]. By solving the LMI in Theorem 6 with LMI toolbox [39], we have

$$\begin{aligned}&P = \begin{bmatrix}18.0282 & -0.2108\\ -0.2108 & 2.1266\end{bmatrix},\\ &\epsilon = 15.6610, \qquad \varepsilon_1 = 15.6556, \qquad \varepsilon_2 = 15.6592,\\ &F_{11} = \begin{bmatrix}-0.1137 & 0\end{bmatrix}, \qquad F_{12} = \begin{bmatrix}0 & -0.2026\end{bmatrix},\\ &F_{21} = \begin{bmatrix}-0.027 & 0\end{bmatrix}, \qquad F_{22} = \begin{bmatrix}0 & -0.5755\end{bmatrix}.\end{aligned} \tag{52}$$

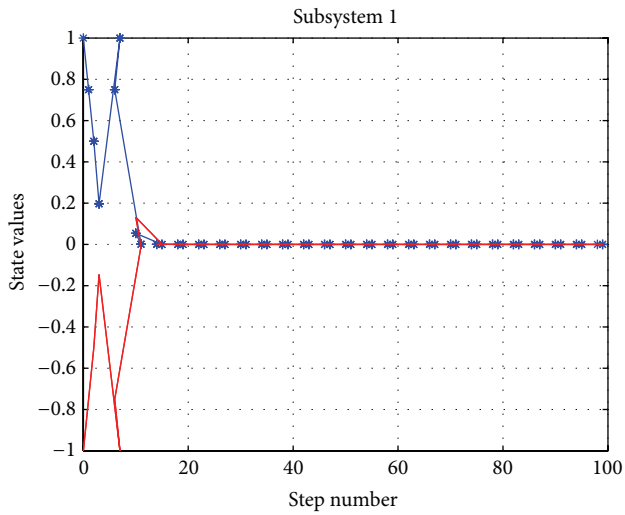

FIGURE 4: The state trajectories of subsystem 1.

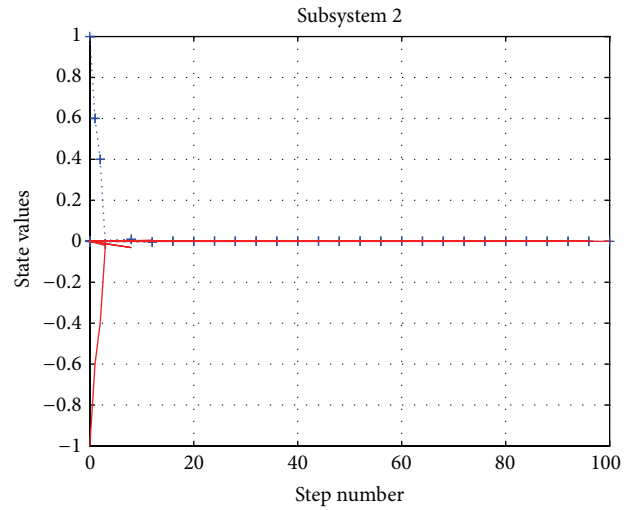

FIGURE 5: The state trajectories of subsystem 2.

With the initial condition $x_1(1) = [1\ \ -1]^T$, $x_1(2) = [0.75\ \ -0.75]^T$, $x_1(3) = [0.5\ \ -0.5]^T$, $x_1(4) = [0.125\ \ -0.125]^T$, $x_2(1) = [1\ \ -1]^T$, $x_2(2) = [0.6\ \ -0.6]^T$, $x_2(3) = [0.4\ \ -0.4]^T$, $x_2(4) = [0.1\ \ -0.1]^T$, the state trajectories of two coupled second-order NCSs with multiple packet transmission are shown in Figures 4 and 5. From which we can see that multiple uncertain NCSs (50) with only a quarter of state information transmitted every step can be effectively stabilized with the designed feedback controller. This is a remarkable result, which shows the efficiency of our proposed method. This example illustrates that the switched approach proposed in this paper leads to useful results, because it only requires plant state measurements to be transmitted sparsely. This reduces network traffic without sacrificing stability.

5. Conclusion

In this paper, we dealt with robust stabilization of multiple coupled uncertain NCSs with multipacket transmitted over a shared channel. For NCSs acted over a toking-bus, multiple NCSs with multipacket were modeled as a periodically switched system, whose stability guaranteed that of the original system. Sufficient conditions on stability and stabilization of the NCSs with norm-bounded parameter uncertainties were derived. The results obtained here suggest that data packet can be transmitted sparsely to save network bandwidth while preserving the stability of the NCS. This is of practical interest in the application of NCSs.

Future work focuses on studying the case where the communication sequence is stochastic, on one step time delay, and on the real life application.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Global μ-Stability of Impulsive Complex-Valued Neural Networks with Leakage Delay and Mixed Delays

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The impulsive complex-valued neural networks with three kinds of time delays including leakage delay, discrete delay, and distributed delay are considered. Based on the homeomorphism mapping principle of complex domain, a sufficient condition for the existence and uniqueness of the equilibrium point of the addressed complex-valued neural networks is proposed in terms of linear matrix inequality (LMI). By constructing appropriate Lyapunov-Krasovskii functionals, and employing the free weighting matrix method, several delay-dependent criteria for checking the global μ-stability of the complex-valued neural networks are established in LMIs. As direct applications of these results, several criteria on the exponential stability, power-stability, and log-stability are obtained. Two examples with simulations are provided to demonstrate the effectiveness of the proposed criteria.

1. Introduction

The nonlinear systems are ubiquitous in the real world [1–5]. As one of the most important nonlinear systems, the real-valued neural networks have been extensively studied and developed due to their extensive applications in pattern recognition, associative memory, signal processing, image processing, combinatorial optimization, and other areas [6]. In implementation of neural networks, however, time delays are unavoidably encountered [7]. It has been found that the existence of time delays may lead to instability and oscillation in a neural network [8]. Therefore, dynamics analysis of neural networks with time delays has received much attention. In [6–10], the exponential stability and asymptotic stability of delayed neural networks were investigated; some sufficient conditions for checking stability were given. In [11, 12], authors investigated the synchronization of chaotic neural networks with delay and obtained several criteria for checking the synchronization. In [13], the passivity of uncertain neural networks with both leakage delay and time-varying delay was considered. In [14], authors investigated the state estimation for neural networks with leakage delay and time-varying delays.

However, besides delay effect, impulsive effects are also likely to exist in neural networks [15]. For instance, in implementation of electronic networks, the state of the networks is subject to instantaneous perturbations and experiences abrupt change at certain instants, which may be caused by switching phenomenon, frequency change. or other sudden noise; that is, it exhibits impulsive effects [16–18]. Therefore, it is necessary to consider both impulsive effect and delay effect on dynamical behaviors of neural networks. Some results on impulsive effect have been gained for delayed neural networks; for example, see [15, 16, 19] and the references therein.

Recently, the power-rate global stability of the equilibrium is proposed in [20]. Moreover, in [21], the authors proposed a new concept of global μ-stability to unify the exponential stability, power-rate stability, and log-stability of neural networks. In [22], the authors investigated the global robust μ-stability in the mean square for a class of stochastic neural networks. In [23], the delayed neural systems with

impulsion were considered, and the μ-stability criteria were derived by using Lyapunov-Krasovskii functional method. In [24], the multiple μ-stability of delayed neural networks was investigated, and several criteria for the coexistence of equilibrium points and their local μ-stability were derived.

As an extension of real-valued neural networks, complex-valued neural networks with complex-valued state, output, connection weight, and activation function become strongly desired because of their practical applications in physical systems dealing with electromagnetic, light, ultrasonic, and quantum waves [25, 26]. It has been shown that such applications strongly depend on the stability of CVNNs [27]. Therefore, stability analysis of CVNNs has received much attention and various stability conditions have been obtained [28–33]. In [28], authors considered a discrete-time CVNNs and obtained several sufficient conditions for checking global exponential stability of a unique equilibrium. In [29, 30], the discrete-time CVNNs with linear threshold neurons were investigated, and some conditions for the boundedness, global attractivity, and complete stability as well as global exponential stability of the considered neural networks were also derived. In [31], the continuous-time CVNNs with delays were considered, and the boundedness, complete stability, and exponential stability were investigated. In [32, 33], the global stability was investigated for CVNNs on time scales, which is useful to unify the continuous-time and discrete-time CVNNs under the same framework. In [34], the authors investigated the μ-stability of delayed CVNNs and obtained several sufficient conditions to ensure the global μ-stability. To the best of our knowledge, there are no results on μ-stability of impulsive CVNNs with the three kinds of time delays including leakage delay, discrete delay, and distributed delay in the literature, and it remains as an open topic for further investigation.

Motivated by the above discussions, in this paper, we will deal with the problem of μ-stability for CVNNs with leakage delay, discrete delay, and distributed delay under impulsive perturbations. Based on the homeomorphism mapping principle of complex domain, a LMI condition for the existence and uniqueness of the equilibrium point of the addressed CVNNs is proposed. Several delay-dependent criteria for checking the global μ-stability of the CVNNs are obtained by constructing appropriate Lyapunov-Krasovskii functionals and employing the free weighting matrix method. The obtained results can also be applied to several special cases and we can get the exponential stability, power-stability, and log-stability of the CVNNs, correspondingly. Finally, two illustrative examples are provided to show the effectiveness of the proposed criteria.

Notations. The notations are quite standard. Throughout this paper, let $\mathbb{Z}^+$ denote the set of positive integers. Let i denote the imaginary unit; that is, $i = \sqrt{-1}$. $\mathbb{C}^n$, $\mathbb{R}^{m\times n}$, and $\mathbb{C}^{m\times n}$ denote, respectively, the set of n-dimensional complex vectors, $m \times n$ real matrices and complex matrices. The subscripts T and $*$ denote matrix transposition and matrix conjugate transpose, respectively. For complex vector $z \in \mathbb{C}^n$, let $|z| = (|z_1|, |z_2|, \ldots, |z_n|)^T$ be the module of the vector z, and let $\|z\| = \sqrt{\sum_{k=1}^{n} |z_k|^2}$ be the norm of the vector z. I denotes the identity matrix with appropriate dimensions. The notation $X \geq Y$ (resp., $X > Y$) means that $X - Y$ is positive semidefinite (resp., positive definite). $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ are defined as the largest and the smallest eigenvalue of positive definite matrix P, respectively. For any $t \geq 0$, z_t is defined by $z_t = z(t+s)$, $z_{t^-} = z(t^- + s)$, $s \in [-\delta, 0]$. In addition, the notation $\star$ always denotes the conjugate transpose of block in a Hermitian matrix.

2. Problems Formulation and Preliminaries

Consider the following complex-valued neural networks with leakage delay and mixed delays under impulsive perturbations by a nonlinear differential equation of the form

$$\begin{aligned} \dot{z}(t) &= -Dz(t-\delta) + Af(z(t)) + Bf(z(t-\tau_1)) \\ &\quad + C\int_{t-\tau_2}^{t} f(z(s))\,ds + h, \quad t \neq t_k,\ t > 0, \\ \Delta z(t_k) &= z(t_k) - z(t_k^-) = J_k\left(z(t_k^-), z_{t_k^-}\right), \quad k \in \mathbb{Z}^+, \end{aligned} \tag{1}$$

where the impulse times t_k satisfy $0 = t_0 < t_1 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = +\infty$; $z(t) = (z_1(t), z_2(t), \ldots, z_n(t))^T \in \mathbb{C}^n$ is the state vector of the neural networks; $D = \operatorname{diag}\{d_1, d_2, \ldots, d_n\} \in \mathbb{R}^{n\times n}$ is the self-feedback connection weight matrix with $d_j > 0$ $(j = 1, 2, \ldots, n)$; A, B, and $C \in \mathbb{C}^{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix, and distributively delayed connection weight matrix; $f(z(t)) = (f_1(z_1(t)), f_2(z_2(t)), \ldots, f_n(z_n(t)))^T \in \mathbb{C}^n$ represents the neuron activation function; $h = (h_1, h_2, \ldots, h_n)^T \in \mathbb{C}^n$ is the external input vector; $\delta \geq 0$, $\tau_1 \geq 0$, and $\tau_2 \geq 0$ are the leakage time delay, the discrete time delay, and the distributed time delay, respectively; J_k is the impulsive function.

In the analysis of complex-valued neural networks, it is usually assumed that the activation functions are differentiable [31]. However, in this paper, we adopt the following assumption on the activation functions in which the differentiability is not required:

(H1) the neuron activation functions f_j are continuous and satisfy

$$\left\|f_j(z_1) - f_j(z_2)\right\| \leq \gamma_j \|z_1 - z_2\|, \tag{2}$$

for any $z_1, z_2 \in \mathbb{C}$, $j = 1, 2, \ldots, n$, where γ_j is a constant. Moreover, we define $\Gamma = \operatorname{diag}\{\gamma_1^2, \gamma_2^2, \ldots, \gamma_n^2\}$.

Remark 1. Note that the assumptions on activation functions are weaker than those generally used in the literature. Namely, the boundedness and differentiability of the activation functions f_j are not required in this paper.

Next we introduce some definitions and lemmas to be used in the stability analysis.

Definition 2. Let $\hat{z}$ be an equilibrium point of system (1). Suppose that $z(t)$ is an arbitrary solution of system (1); $\mu(t)$ is a positive continuous function and satisfies $\mu(t) \to \infty$ as $t \to \infty$. If there is a positive constant M such that

$$\|z(t) - \hat{z}\| \le \frac{M}{\mu(t)}, \quad t \ge 0, \tag{3}$$

then the equilibrium point $\hat{z}$ is said to be μ-stable.

In particular, if taking $\mu(t)$ in Definition 2 to exponential function, power function, and logarithmic function, we can get the definitions of exponential stability, power-stability, and log-stability, correspondingly.

Definition 3. Let $\hat{z}$ be an equilibrium point of system (1). Suppose that $z(t)$ is an arbitrary solution of system (1). If there are two positive constants ε and M such that

$$\|z(t) - \hat{z}\| \le \frac{M}{e^{\varepsilon t}}, \quad t \ge 0, \tag{4}$$

then the equilibrium point $\hat{z}$ is said to be exponentially stable.

Definition 4. Let $\hat{z}$ be an equilibrium point of system (1). Suppose that $z(t)$ is an arbitrary solution of system (1). If there are two positive constants ε and M such that

$$\|z(t) - \hat{z}\| \le \frac{M}{t^{\varepsilon}}, \quad t \ge 0, \tag{5}$$

then the equilibrium point $\hat{z}$ is said to be power-stable.

Definition 5. Let $\hat{z}$ be an equilibrium point of system (1). Suppose that $z(t)$ is an arbitrary solution of system (1). If there are two positive constants ε and M such that

$$\|z(t) - \hat{z}\| \le \frac{M}{\ln(\varepsilon t + 1)}, \quad t > 0, \tag{6}$$

then the equilibrium point $\hat{z}$ is said to be log-stable.

Lemma 6. *If $H(z) : \mathbb{C}^n \to \mathbb{C}^n$ is a continuous map and satisfies the following conditions:*

(i) *$H(z)$ is injective on $\mathbb{C}^n$,*

(ii) $\lim_{\|z\| \to \infty} \|H(z)\| = \infty$,

then $H(z)$ is a homeomorphism of $\mathbb{C}^n$ onto itself.

Proof. Let $z = x + iy$ and $\alpha = (x^T, y^T)^T$, where $x, y \in \mathbb{R}^n$. Define a homeomorphism $\mathscr{I} : \mathbb{C}^n \to \mathbb{R}^{2n}$ by

$$\mathscr{I}(z) = \alpha. \tag{7}$$

Obviously, $\mathscr{I}$ is invertible. Let $\mathscr{L} = \mathscr{I} \circ H \circ \mathscr{I}^{-1}$. Then $\mathscr{L}$ is injective on $\mathbb{R}^{2n}$, since H and $\mathscr{I}$ are injective. In addition, $\|\mathscr{L}(\alpha)\| \to \infty$ when $\|\alpha\| \to \infty$, since $\lim_{\|z\| \to \infty} \|\mathscr{I}(z)\| = \infty$ and $\lim_{\|z\| \to \infty} \|H(z)\| = \infty$. Therefore, $\mathscr{L}$ is a homeomorphism of $\mathbb{R}^{2n}$ onto itself. Then $H = \mathscr{I}^{-1} \circ \mathscr{L} \circ \mathscr{I}$ is a homeomorphism of $\mathbb{C}^n$ onto itself. The proof iscompleted. □

Lemma 7. *For any $a, b \in \mathbb{C}^n$, if $P \in \mathbb{C}^{n\times n}$ is a positive definite Hermitian matrix, then $a^* b + b^* a \le a^* P a + b^* P^{-1} b$.*

Proof. Since P is a positive definite Hermitian matrix, there exists an invertible matrix $Q \in \mathbb{C}^{n\times n}$, such that $P = Q^* Q$. For any $a, b \in \mathbb{C}^n$, it follows from Cauchy inequality that

$$\begin{aligned}
&a^* b + b^* a \\
&= (Qa)^* \left[(Q^*)^{-1} b\right] + \left[(Q^*)^{-1} b\right]^* (Qa) \\
&= 2\,\mathrm{Re}\left\{(Qa)^* \left[(Q^*)^{-1} b\right]\right\} \le 2 \left\|(Qa)^* \left[(Q^*)^{-1} b\right]\right\| \\
&\le 2\|Qa\| \left\|(Q^*)^{-1} b\right\| \le (Qa)^* (Qa) + \left[(Q^*)^{-1} b\right]^* \left[(Q^*)^{-1} b\right] \\
&= a^* P a + b^* P^{-1} b.
\end{aligned} \tag{8}$$

The proof is completed. □

Lemma 8 (see [33]). *For any constant matrix $W \in \mathbb{C}^{n\times n}$ and $W > 0$, a scalar function $\omega : [a, b] \to \mathbb{C}^n$ with scalars $a < b$ such that the integration concerned are well defined, then*

$$\begin{aligned}
&(b - a) \int_a^b \omega^*(s) W \omega(s)\, \mathrm{d}s \\
&\quad \le \left(\int_a^b \omega(s)\, \mathrm{d}s\right)^* W \left(\int_a^b \omega(s)\, \mathrm{d}s\right).
\end{aligned} \tag{9}$$

Lemma 9. *A given Hermitian matrix*

$$S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} < 0, \tag{10}$$

where $S_{11}^* = S_{11}$, $S_{12}^* = S_{21}$ and $S_{22}^* = S_{22}$, is equivalent to any one of the following conditions:

(1) *$S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{21} < 0$,*

(2) *$S_{11} < 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} < 0$.*

Proof. (1) Note that

$$\begin{aligned}
S &= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \\
&= \begin{pmatrix} I & S_{12} S_{22}^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} S_{11} - S_{12} S_{22}^{-1} S_{21} & 0 \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} I & S_{12} S_{22}^{-1} \\ 0 & I \end{pmatrix}^*.
\end{aligned} \tag{11}$$

Therefore, $S < 0 \Leftrightarrow S_{22} < 0$ and $S_{11} - S_{12} S_{22}^{-1} S_{21} < 0$.

(2) Note that

$$\begin{aligned}
S &= \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \\
&= \begin{pmatrix} I & 0 \\ S_{21} S_{11}^{-1} & I \end{pmatrix} \begin{pmatrix} S_{11} & 0 \\ 0 & S_{22} - S_{21} S_{11}^{-1} S_{12} \end{pmatrix} \begin{pmatrix} I & 0 \\ S_{21} S_{11}^{-1} & I \end{pmatrix}^*.
\end{aligned} \tag{12}$$

Therefore, $S < 0 \Leftrightarrow S_{11} < 0$ and $S_{22} - S_{21} S_{11}^{-1} S_{12} < 0$. The proof iscompleted. □

3. Existence and Uniqueness of Equilibrium Point

Now we study the existence and uniqueness of the equilibrium point of system (1). As usual, we denote an equilibrium point of the system (1) by the constant complex vector $\hat{z} \in \mathbb{C}^n$, where $\hat{z}$ satisfies

$$\begin{gathered} -D\hat{z} + Af(\hat{z}) + Bf(\hat{z}) + \tau_2 Cf(\hat{z}) + h = 0, \\ J_k(\hat{z}, \hat{z}) = 0, \quad k \in \mathbb{Z}^+. \end{gathered} \tag{13}$$

In this paper, it is assumed that the constant complex vector $\hat{z}$ satisfies $J_k(\hat{z}, \hat{z}) = 0$ for $k \in \mathbb{Z}^+$, if $\hat{z}$ satisfies $-D\hat{z} + Af(\hat{z}) + Bf(\hat{z}) + \tau_2 Cf(\hat{z}) + h = 0$. Hence, to prove the existence and uniqueness of a solution of (13), it suffices to show that the following map $\mathscr{H} : \mathbb{C}^n \to \mathbb{C}^n$ has a unique zero point:

$$\mathscr{H}(z) = -Dz + (A + B + \tau_2 C) f(z) + h. \tag{14}$$

In the following, we will give some conditions for checking that $\mathscr{H}$ is a homeomorphism on $\mathbb{C}^n$, that is, for assuring the existence and uniqueness of the equilibrium point of system (1).

Theorem 10. *Under condition (H1), the system* (1) *has a unique equilibrium point, if there exist two complex matrices U_1, U_2 and a real positive diagonal matrix R, such that the following LMI holds:*

$$\begin{pmatrix} \Pi_{11} & \Pi_{12} \\ \star & \Pi_{22} \end{pmatrix} < 0, \tag{15}$$

where $\Pi_{11} = \Gamma^* R\Gamma - U_1 D - DU_1^*$, $\Pi_{12} = U_1(A+B+\tau_2 C) - DU_2^*$, $\Pi_{22} = U_2(A + B + \tau_2 C) + (A^* + B^* + \tau_2 C^*)U_2^* - R$.

Proof. In the following, we will prove that $\mathscr{H}(z)$ is a homeomorphism of $\mathbb{C}^n$ onto itself.

First, we prove that $\mathscr{H}(z)$ is an injective map on $\mathbb{C}^n$. Suppose that there exist $z, z' \in \mathbb{C}^n$ with $z \neq z'$, such that $\mathscr{H}(z) = \mathscr{H}(z')$. Then

$$-D(z - z') + (A + B + \tau_2 C)(f(z) - f(z')) = 0. \tag{16}$$

Multiply both sides above by $(z - z')^* U_1 + (f(z) - f(z'))^* U_2$; we get

$$\begin{aligned} 0 = & -(z - z')^* U_1 D (z - z') \\ & + (z - z')^* U_1 (A + B + \tau_2 C)(f(z) - f(z')) \\ & - (f(z) - f(z'))^* U_2 D (z - z') \\ & + (f(z) - f(z'))^* U_2 (A + B + \tau_2 C)(f(z) - f(z')). \end{aligned} \tag{17}$$

Taking the conjugate transpose of (17), we get that

$$\begin{aligned} 0 = & -(z - z')^* DU_1^* (z - z') \\ & + (f(z) - f(z'))^* (A^* + B^* + \tau_2 C^*) U_1^* (z - z') \\ & - (z - z')^* DU_2^* (f(z) - f(z')) + (f(z) - f(z'))^* \\ & \times (A^* + B^* + \tau_2 C^*) U_2^* (f(z) - f(z')). \end{aligned} \tag{18}$$

Let $\Sigma_1 = R - U_2(A + B + \tau_2 C) - (A^* + B^* + \tau_2 C^*)U_2^*$, $\Sigma_2 = -U_1 D - DU_1^* + [U_1(A + B + \tau_2 C) - DU_2^*]\Sigma_1^{-1}[(A^* + B^* + \tau_2 C^*)U_1^* - U_2 D]$. Then Σ_1 is a positive definite Hermitian matrix from Lemma 9 and LMI (15). Summing (17) and (18), applying Lemma 7, we have

$$\begin{aligned} 0 = & -(z - z')^* (U_1 D + DU_1^*)(z - z') \\ & + (z - z')^* [U_1 (A + B + \tau_2 C) - DU_2^*](f(z) - f(z')) \\ & + (f(z) - f(z'))^* \\ & \times [(A^* + B^* + \tau_2 C^*) U_1^* - U_2 D](z - z') \\ & + (f(z) - f(z'))^* \\ & \times [U_2 (A + B + \tau_2 C) + (A^* + B^* + \tau_2 C^*) U_2^*] \\ & \times (f(z) - f(z')) \\ \leq & -(z - z')^* (U_1 D + DU_1^*)(z - z') \\ & + (z - z')^* [U_1 (A + B + \tau_2 C) - DU_2^*] \\ & \times \Sigma_1^{-1} [(A^* + B^* + \tau_2 C^*) U_1^* - U_2 D](z - z') \\ & + (f(z) - f(z'))^* \Sigma_1 (f(z) - f(z')) \\ & + (f(z) - f(z'))^* \\ & \times [U_2 (A + B + \tau_2 C) + (A^* + B^* + \tau_2 C^*) U_2^*] \\ & \times (f(z) - f(z')) \\ = & (z - z')^* \Sigma_2 (z - z') \\ & + (f(z) - f(z'))^* R (f(z) - f(z')). \end{aligned} \tag{19}$$

Since R is a positive diagonal matrix, from condition (H1), we can get

$$\begin{aligned} & (f(z) - f(z'))^* R (f(z) - f(z')) \\ & \leq (z - z')^* \Gamma^* R \Gamma (z - z'). \end{aligned} \tag{20}$$

It follows from (19) and (20) that

$$0 \leq (z - z')^* \Sigma_3 (z - z'), \tag{21}$$

where $\Sigma_3 = \Gamma^* R\Gamma + \Sigma_2 = \Gamma^* R\Gamma - U_1 D - DU_1^* + [U_1(A + B + \tau_2 C) - DU_2^*]\Sigma_1^{-1}[(A^* + B^* + \tau_2 C^*)U_1^* - U_2 D]$. From Lemma 9 and LMI (15), we can know $\Sigma_3 < 0$. Then $z - z' = 0$ from (21). Therefore, $\mathscr{H}(z)$ is an injective map on $\mathbb{C}^n$.

Second, we prove $\|\mathscr{H}(z)\| \to \infty$ as $\|z\| \to \infty$. Let $\widetilde{\mathscr{H}}(z) = \mathscr{H}(z) - \mathscr{H}(0)$. By Lemma 7, we have

$$\begin{aligned}
&\left[z^* U_1 + (f(z) - f(0))^* U_2\right] \widetilde{\mathscr{H}}(z) \\
&\quad + \widetilde{\mathscr{H}}(z)^* \left[z^* U_1 + (f(z) - f(0))^* U_2\right]^* \\
&= -z^* (U_1 D + DU_1^*) z \\
&\quad + z^* [U_1 (A + B + \tau_2 C) - DU_2^*] (f(z) - f(0)) \\
&\quad + (f(z) - f(0))^* [(A^* + B^* + \tau_2 C^*) U_1^* - U_2 D] z \\
&\quad + (f(z) - f(0))^* \\
&\quad \times [U_2 (A + B + \tau_2 C) + (A^* + B^* + \tau_2 C^*) U_2^*] \\
&\quad \times (f(z) - f(0)) \\
&\le -z^* (U_1 D + DU_1^*) z \\
&\quad + z^* [U_1 (A + B + \tau_2 C) - DU_2^*] \\
&\quad \times \Sigma_1^{-1} [(A^* + B^* + \tau_2 C^*) U_1^* - U_2 D] z \\
&\quad + (f(z) - f(0))^* \Sigma_1 (f(z) - f(0)) \\
&\quad + (f(z) - f(0))^* \\
&\quad \times [U_2 (A + B + \tau_2 C) + (A^* + B^* + \tau_2 C^*) U_2^*] \\
&\quad \times (f(z) - f(0)) \\
&= z^* \Sigma_2 z + (f(z) - f(0))^* R (f(z) - f(0)).
\end{aligned} \tag{22}$$

It follows from condition (H1) that

$$(f(z) - f(0))^* R (f(z) - f(0)) \le z^* \Gamma^* R \Gamma z. \tag{23}$$

Then from (22), (23), and $\Sigma_3 = \Gamma^* R\Gamma + \Sigma_2 < 0$, we obtain

$$\begin{aligned}
&\left[z^* U_1 + (f(z) - f(0))^* U_2\right] \widetilde{\mathscr{H}}(z) \\
&\quad + \widetilde{\mathscr{H}}(z)^* \left[z^* U_1 + (f(z) - f(0))^* U_2\right]^* \\
&\quad \le z^* \Sigma_3 z \le -\rho_{\min}(-\Sigma_3) \|z\|^2,
\end{aligned} \tag{24}$$

which can imply that

$$\begin{aligned}
\lambda_{\min}(-\Sigma_3) \|z\|^2 &\le 2 \left\| \left[z^* U_1 + (f(z) - f(0))^* U_2\right] \right\| \left\| \widetilde{\mathscr{H}}(z) \right\| \\
&\le 2 (\|U_1\| + \|\Gamma^* U_2\|) \|z\| \left\| \widetilde{\mathscr{H}}(z) \right\|.
\end{aligned} \tag{25}$$

When $z \ne 0$, we have

$$\left\| \widetilde{\mathscr{H}}(z) \right\| \ge \frac{\lambda_{\min}(-\Sigma_3) \|z\|}{2 (\|U_1\| + \|\Gamma^* U_2\|)}. \tag{26}$$

Therefore, $\|\widetilde{\mathscr{H}}(z)\| \to \infty$ as $\|z\| \to \infty$ which implies $\|\mathscr{H}(z)\| \to \infty$ as $\|z\| \to \infty$. From Lemma 6, we know that $\mathscr{H}(z)$ is a homeomorphism of $\mathbb{C}^n$. Thus, the system (1) has a unique equilibrium point. □

Remark 11. It should be noted that Theorem 10 is independent of leakage time delay and initial conditions. So, the time delays in the leakage terms do not affect the existence and uniqueness of the equilibrium point.

4. Global μ-Stability Results

In the preceding section, we have shown the existence and uniqueness of the equilibrium point for system (1). In this section, we will further investigate the global μ-stability of the unique equilibrium point. For this purpose, the impulsive function J_k which is viewed as a perturbation of the equilibrium point $\widehat{z}$ of system (1) without impulses is defined by

$$J_k\left(z(t_k^-), z_{t_k^-}\right) = E_k \left[z(t_k^-) - \widehat{z} - D \int_{t_k - \delta}^{t_k} (z(s) - \widehat{z})\, \mathrm{d}s \right], \tag{27}$$

where $k \in \mathbb{Z}^+$, $E_k \in \mathbb{C}^{n \times n}$. It is obvious that $J_k(\widehat{z}, \widehat{z}) = 0$.

Remark 12. The type of impulse such as (27) describes the fact that the instantaneous perturbations are not only related to the state of neurons at impulse times t_k but also related to the state of neurons in recent history, which reflects more realistic dynamics [19].

Theorem 13. *Under the conditions of Theorem 10, the equilibrium point of system* (1) *is globally μ-stable, if there exist six positive definite Hermitian matrices P_1, P_2, P_3,P_4, P_5, and P_6, two real positive diagonal matrices R_1 and R_2, three complex matrices Q_1, Q_2, and Q_3, a positive differential function $\mu(t)$, and three constants $\alpha \ge 0$, $\beta > 0$, and $T > 0$ such that*

$$\begin{gathered}
0 \le \frac{\dot{\mu}(t)}{\mu(t)} \le \alpha, \\
\frac{\min\{\mu(t - \delta), \mu(t - \tau_1), \mu(t - \tau_2)\}}{\mu(t)} \ge \beta, \\
\forall t \in [T, \infty),
\end{gathered} \tag{28}$$

and the following LMIs hold:

$$\begin{pmatrix} \Omega_{11} & \Omega_{12} & P_1 D & \frac{\beta^2}{\tau_1} P_5 & \Omega_{15} & 0 & 0 & 0 \\ \star & \Omega_{22} & Q_1^* D & Q_3 & -P_1 D & -Q_1^* A & -Q_1^* B & -Q_1^* C \\ \star & \star & \Omega_{33} & DQ_3 & -DP_1 D & -Q_2^* A & -Q_2^* B & -Q_2^* C \\ \star & \star & \star & \Omega_{44} & 0 & -Q_3^* A & -Q_3^* B & -Q_3^* C \\ \star & \star & \star & \star & -\frac{\beta^2}{\delta} P_3 & 0 & 0 & 0 \\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0 \\ \star & \star & \star & \star & \star & \star & -R_2 & 0 \\ \star & \star & \star & \star & \star & \star & \star & -\frac{\beta^2}{\tau_2} P_6 \end{pmatrix} < 0, \tag{29}$$

$$\begin{pmatrix} P_1 & (I + E_k) P_1 \\ \star & P_1 \end{pmatrix} > 0, \tag{30}$$

where $\Omega_{11} = 2\alpha P_1 - DP_1 - P_1 D + 2\alpha P_2 + \delta P_3 + P_4 - (\beta^2/\tau_1) P_5 + R_1 \Gamma$, $\Omega_{12} = P_1 + P_2$, $\Omega_{15} = DP_1 D - 2\alpha P_1 D$, $\Omega_{22} = Q_1^* + Q_1 + \tau_1 P_5$, $\Omega_{33} = Q_2^* D + DQ_2$, $\Omega_{44} = -\beta^2 P_4 - (\beta^2/\tau_1) P_5 + R_2 \Gamma$, *and* $\Omega_{66} = \tau_2 P_6 - R_1$.

Proof. Under the condition of Theorem 10, system (1) has a unique equilibrium point $\hat{z}$. Then we shift the equilibrium point of (1) to the origin by the translation $\tilde{z}(t) = z(t) - \hat{z}$ and obtain

$$\begin{aligned} \dot{\tilde{z}}(t) &= -D\tilde{z}(t-\delta) + Ag(\tilde{z}(t)) + Bg(\tilde{z}(t-\tau_1)) \\ &\quad + C\int_{t-\tau_2}^{t} g(\tilde{z}(s))\,\mathrm{d}s, \quad t \neq t_k,\ t > 0, \\ \Delta\tilde{z}(t_k) &= \tilde{z}(t_k) - \tilde{z}(t_k^-) \\ &= E_k\left[\tilde{z}(t_k^-) - D\int_{t_k-\delta}^{t_k} \tilde{z}(s)\,\mathrm{d}s\right], \quad k \in \mathbb{Z}^+. \end{aligned} \tag{31}$$

Consider the following Lyapunov-Krasovskii functional candidate:

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{32}$$

where

$$\begin{aligned} V_1(t) &= \mu^2(t)\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right]^* \\ &\quad \times P_1\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right], \\ V_2(t) &= \mu^2(t)\tilde{z}^*(t) P_2 \tilde{z}(t) \\ &\quad + \int_0^{\delta}\int_{t-u}^{t} \mu^2(s)\tilde{z}^*(s) P_3 \tilde{z}(s)\,\mathrm{d}s\,\mathrm{d}u, \\ V_3(t) &= \int_{t-\tau_1}^{t} \mu^2(s)\tilde{z}^*(s) P_4 \tilde{z}(s)\,\mathrm{d}s \\ &\quad + \int_0^{\tau_1}\int_{t-u}^{t} \mu^2(s)\dot{\tilde{z}}^*(s) P_5 \dot{\tilde{z}}(s)\,\mathrm{d}s\,\mathrm{d}u, \\ V_4(t) &= \int_0^{\tau_2}\int_{t-u}^{t} \mu^2(s) g^*(\tilde{z}(s)) P_6 g(\tilde{z}(s))\,\mathrm{d}s\,\mathrm{d}u. \end{aligned} \tag{33}$$

Calculating the upper right derivative of V along the solution of (31), applying Lemma 8, we get

$$\begin{aligned} &D^+V_1(t) \\ &= 2\mu(t)\dot{\mu}(t)\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right]^* \\ &\quad \times P_1\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right] \\ &\quad + \mu^2(t)\left[\dot{\tilde{z}}(t) - D\tilde{z}(t) + D\tilde{z}(t-\delta)\right]^* \\ &\quad \times P_1\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right] \\ &\quad + \mu^2(t)\left[\tilde{z}(t) - D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right]^* \\ &\quad \times P_1\left[\dot{\tilde{z}}(t) - D\tilde{z}(t) + D\tilde{z}(t-\delta)\right] \\ &\le \mu^2(t)\left[2\alpha\tilde{z}^*(t) P_1 \tilde{z}(t) - 2\alpha\tilde{z}^*(t) P_1 D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s\right. \\ &\quad - 2\alpha\int_{t-\delta}^{t} \tilde{z}^*(s)\,\mathrm{d}s DP_1 \tilde{z}(t)\,\mathrm{d}s \\ &\quad + 2\alpha\int_{t-\delta}^{t} \tilde{z}^*(s)\,\mathrm{d}s DP_1 D\int_{t-\delta}^{t} \tilde{z}(s)\,\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&+\dot{\tilde{z}}^{*}(t)P_1\tilde{z}(t)-\dot{\tilde{z}}^{*}(t)P_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\\
&-\tilde{z}^{*}(t)DP_1\tilde{z}(t)+\tilde{z}^{*}(t)DP_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\\
&+\tilde{z}^{*}(t-\delta)DP_1\tilde{z}(t)\\
&-\tilde{z}^{*}(t-\delta)DP_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s+\tilde{z}^{*}(t)P_1\dot{\tilde{z}}(t)\\
&-\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1\dot{\tilde{z}}(t)\\
&-\tilde{z}^{*}(t)P_1D\tilde{z}(t)+\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1D\tilde{z}(t)\\
&+\tilde{z}^{*}(t)P_1D\tilde{z}(t-\delta)\\
&-\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1D\tilde{z}(t-\delta)\Big]\\
=\mu^2(t)\Big[&\tilde{z}^{*}(t)\left(2\alpha P_1-DP_1-P_1D\right)\tilde{z}(t)\\
&+\tilde{z}^{*}(t)\left(DP_1D-2\alpha P_1D\right)\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\\
&+\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}s\left(DP_1D-2\alpha DP_1\right)\tilde{z}(t)\\
&+2\alpha\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\\
&+\dot{\tilde{z}}^{*}(t)P_1\tilde{z}(t)+\tilde{z}^{*}(t)P_1\dot{\tilde{z}}(t)\\
&-\dot{\tilde{z}}^{*}(t)P_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s-\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1\dot{\tilde{z}}(t)\\
&+\tilde{z}^{*}(t-\delta)DP_1\tilde{z}(t)+\tilde{z}^{*}(t)P_1D\tilde{z}(t-\delta)\\
&-\tilde{z}^{*}(t-\delta)DP_1D\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\\
&-\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sDP_1D\tilde{z}(t-\delta)\Big],
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
&D^{+}V_2(t)\\
&=2\mu(t)\dot{\mu}(t)\tilde{z}^{*}(t)P_2\tilde{z}(t)+\mu^2(t)\dot{\tilde{z}}^{*}(t)P_2\tilde{z}(t)\\
&\quad+\mu^2(t)\tilde{z}^{*}(t)P_2\dot{\tilde{z}}(t)\\
&\quad+\delta\mu^2(t)\tilde{z}^{*}(t)P_3\tilde{z}(t)\\
&\quad-\int_{0}^{\delta}\mu^2(t-u)\tilde{z}^{*}(t-u)P_3\tilde{z}(t-u)\,\mathrm{d}u\\
&\le 2\alpha\mu^2(t)\tilde{z}^{*}(t)P_2\tilde{z}(t)\\
&\quad+\mu^2(t)\dot{\tilde{z}}^{*}(t)P_2\tilde{z}(t)+\mu^2(t)\tilde{z}^{*}(t)P_2\dot{\tilde{z}}(t)\\
&\quad+\delta\mu^2(t)\tilde{z}^{*}(t)P_3\tilde{z}(t)\\
&\quad-\int_{t-\delta}^{t}\mu^2(s)\tilde{z}^{*}(s)P_3\tilde{z}(s)\,\mathrm{d}s\\
&\le 2\alpha\mu^2(t)\tilde{z}^{*}(t)P_2\tilde{z}(t)+\mu^2(t)\dot{\tilde{z}}^{*}(t)P_2\tilde{z}(t)\\
&\quad+\mu^2(t)\tilde{z}^{*}(t)P_2\dot{\tilde{z}}(t)\\
&\quad+\delta\mu^2(t)\tilde{z}^{*}(t)P_3\tilde{z}(t)\\
&\quad-\beta^2\mu^2(t)\int_{t-\delta}^{t}\tilde{z}^{*}(s)P_3\tilde{z}(s)\,\mathrm{d}s\\
&\le\mu^2(t)\Big[\tilde{z}^{*}(t)\left(2\alpha P_2+\delta P_3\right)\tilde{z}(t)+\tilde{z}^{*}(t)P_2\dot{\tilde{z}}(t)\\
&\qquad+\dot{\tilde{z}}^{*}(t)P_2\tilde{z}(t)\\
&\qquad-\frac{\beta^2}{\delta}\int_{t-\delta}^{t}\tilde{z}^{*}(s)\,\mathrm{d}sP_3\int_{t-\delta}^{t}\tilde{z}(s)\,\mathrm{d}s\Big],
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
&D^{+}V_3(t)\\
&=\mu^2(t)\tilde{z}^{*}(t)P_4\tilde{z}(t)\\
&\quad-\mu^2(t-\tau_1)\tilde{z}^{*}(t-\tau_1)P_4\tilde{z}(t-\tau_1)\\
&\quad+\tau_1\mu^2(t)\dot{\tilde{z}}^{*}(t)P_5\dot{\tilde{z}}(t)\\
&\quad-\int_{0}^{\tau_1}\mu^2(t-u)\dot{\tilde{z}}^{*}(t-u)P_5\dot{\tilde{z}}(t-u)\,\mathrm{d}u\\
&=\mu^2(t)\tilde{z}^{*}(t)P_4\tilde{z}(t)\\
&\quad-\mu^2(t-\tau_1)\tilde{z}^{*}(t-\tau_1)P_4\tilde{z}(t-\tau_1)\\
&\quad+\tau_1\mu^2(t)\dot{\tilde{z}}^{*}(t)P_5\dot{\tilde{z}}(t)\\
&\quad-\int_{t-\tau_1}^{t}\mu^2(s)\dot{\tilde{z}}^{*}(s)P_5\dot{\tilde{z}}(s)\,\mathrm{d}s\\
&\le\mu^2(t)\Big[\tilde{z}^{*}(t)P_4\tilde{z}(t)-\beta^2\tilde{z}^{*}(t-\tau_1)P_4\tilde{z}(t-\tau_1)\\
&\qquad+\tau_1\dot{\tilde{z}}^{*}(t)P_5\dot{\tilde{z}}(t)\\
&\qquad-\frac{\beta^2}{\tau_1}\int_{t-\tau_1}^{t}\dot{\tilde{z}}^{*}(s)\,\mathrm{d}sP_5\int_{t-\tau_1}^{t}\dot{\tilde{z}}(s)\,\mathrm{d}s\Big]\\
&=\mu^2(t)\Big[\tilde{z}^{*}(t)P_4\tilde{z}(t)-\beta^2\tilde{z}^{*}(t-\tau_1)P_4\tilde{z}(t-\tau_1)\\
&\qquad+\tau_1\dot{\tilde{z}}^{*}(t)P_5\dot{\tilde{z}}(t)-\frac{\beta^2}{\tau_1}\left(\tilde{z}^{*}(t)-\tilde{z}^{*}(t-\tau_1)\right)\\
&\qquad\times P_5\left(\tilde{z}(t)-\tilde{z}(t-\tau_1)\right)\Big],
\end{aligned}
\tag{36}
$$

$$\begin{aligned}
&D^{+}V_4(t)\\
&= \tau_2\mu^2(t)\, g^*(\tilde{z}(t))\, P_6 g(\tilde{z}(t))\\
&\quad - \int_0^{\tau_2} \mu^2(t-u)\, g^*(\tilde{z}(t-u))\, P_6 g(\tilde{z}(t-u))\, \mathrm{d}u\\
&= \tau_2\mu^2(t)\, g^*(\tilde{z}(t))\, P_6 g(\tilde{z}(t))\\
&\quad - \int_{t-\tau_2}^{t} \mu^2(s)\, g^*(\tilde{z}(s))\, P_6 g(\tilde{z}(s))\, \mathrm{d}s\\
&\le \tau_2\mu^2(t)\, g^*(\tilde{z}(t))\, P_6 g(\tilde{z}(t))\\
&\quad - \beta^2\mu^2(t) \int_{t-\tau_2}^{t} g^*(\tilde{z}(s))\, P_6 g(\tilde{z}(s))\, \mathrm{d}s\\
&\le \mu^2(t)\left[\tau_2 g^*(\tilde{z}(t))\, P_6 g(\tilde{z}(t))\right.\\
&\qquad \left. - \frac{\beta^2}{\tau_2}\int_{t-\tau_2}^{t} g^*(\tilde{z}(s))\, \mathrm{d}s P_6 \int_{t-\tau_2}^{t} g(\tilde{z}(s))\, \mathrm{d}s\right].
\end{aligned} \tag{37}$$

Combining (34), (35), (36), and (37), we can deduce that

$$\begin{aligned}
&D^{+}V(t)\\
&\le \mu^2(t)\left[\tilde{z}^*(t)\left(2\alpha P_1 - DP_1 - P_1 D + 2\alpha P_2\right.\right.\\
&\qquad \left. + \delta P_3 + P_4 - \frac{\beta^2}{\tau_1}P_5\right)\tilde{z}(t)\\
&\qquad + \tilde{z}^*(t)\left(P_1 + P_2\right)\dot{\tilde{z}}(t)\\
&\qquad + \dot{\tilde{z}}^*(t)\left(P_1 + P_2\right)\tilde{z}(t)\\
&\qquad + \tilde{z}^*(t)\, P_1 D\tilde{z}(t-\delta)\\
&\qquad + \tilde{z}^*(t-\delta)\, DP_1\tilde{z}(t)\\
&\qquad + \frac{\beta^2}{\tau_1}\tilde{z}^*(t)\, P_5\tilde{z}(t-\tau_1)\\
&\qquad + \frac{\beta^2}{\tau_1}\tilde{z}^*(t-\tau_1)\, P_5\tilde{z}(t)\\
&\qquad + \tilde{z}^*(t)\left(DP_1 D - 2\alpha P_1 D\right)\int_{t-\delta}^{t}\tilde{z}(s)\, \mathrm{d}s\\
&\qquad + \int_{t-\delta}^{t}\tilde{z}^*(s)\, \mathrm{d}s\left(DP_1 D - 2\alpha DP_1\right)\tilde{z}(t)\\
&\qquad - \dot{\tilde{z}}^*(t)\, P_1 D\int_{t-\delta}^{t}\tilde{z}(s)\, \mathrm{d}s\\
&\qquad - \int_{t-\delta}^{t}\tilde{z}^*(s)\, \mathrm{d}s DP_1\dot{\tilde{z}}(t)\\
&\qquad - \tilde{z}^*(t-\delta)\, DP_1 D\int_{t-\delta}^{t}\tilde{z}(s)\, \mathrm{d}s\\
&\qquad - \int_{t-\delta}^{t}\tilde{z}^*(s)\, \mathrm{d}s DP_1 D\tilde{z}(t-\delta)\\
&\qquad - \frac{\beta^2}{\delta}\int_{t-\delta}^{t}\tilde{z}^*(s)\, \mathrm{d}s P_3\int_{t-\delta}^{t}\tilde{z}(s)\, \mathrm{d}s\\
&\qquad - \tilde{z}^*(t-\tau_1)\left(\beta^2 P_4 + \frac{\beta^2}{\tau_1}P_5\right)\tilde{z}(t-\tau_1)\\
&\qquad + \tau_2 g^*(\tilde{z}(t))\, P_6 g(\tilde{z}(t))\\
&\qquad \left. - \frac{\beta^2}{\tau_2}\int_{t-\tau_2}^{t} g^*(\tilde{z}(s))\, \mathrm{d}s P_6\int_{t-\tau_2}^{t} g(\tilde{z}(s))\, \mathrm{d}s\right].
\end{aligned} \tag{38}$$

From assumption (H1), we have

$$\left|g_j\left(\tilde{z}_j(t)\right)\right| \le \gamma_j\left|\tilde{z}_j(t)\right| \tag{39}$$

for $j = 1, 2, \ldots, n$. Let $R_1 = \mathrm{diag}(r_1, r_2, \ldots, r_n) > 0$. It follows from (39) that

$$r_j\mu^2(t)\, g_j^*\left(\tilde{z}_j(t)\right) g_j\left(\tilde{z}_j(t)\right) - r_j\mu^2(t)\, \gamma_j^2\tilde{z}_j^*(t)\, \tilde{z}_j(t) \le 0 \tag{40}$$

for $j = 1, 2, \ldots, n$. Hence

$$\mu^2(t)\, g^*(\tilde{z}(t))\, R_1 g(\tilde{z}(t)) - \mu^2(t)\, \tilde{z}^*(t)\, R_1\Gamma\tilde{z}(t) \le 0. \tag{41}$$

Also, we can get that

$$\begin{aligned}
&\mu^2(t)\, g^*\left(\tilde{z}(t-\tau_1)\right) R_2 g\left(\tilde{z}(t-\tau_1)\right)\\
&\qquad - \mu^2(t)\, \tilde{z}^*(t-\tau_1)\, R_2\Gamma\tilde{z}(t-\tau_1) \le 0.
\end{aligned} \tag{42}$$

From (31), we have that

$$\begin{aligned}
0 &= \mu^2(t)\left[Q_1\dot{\tilde{z}}(t) + Q_2\tilde{z}(t-\delta) + Q_3\tilde{z}(t-\tau_1)\right]^*\\
&\quad \times\left[\dot{\tilde{z}}(t) + D\tilde{z}(t-\delta) - Ag(\tilde{z}(t)) - Bg\left(\tilde{z}(t-\tau_1)\right)\right.\\
&\qquad \left. - C\int_{t-\tau_2}^{t} g(\tilde{z}(s))\, \mathrm{d}s\right]
\end{aligned}$$

$$+\mu^2(t)\left[\dot{\tilde{z}}(t)+D\tilde{z}(t-\delta)-Ag(\tilde{z}(t))\right.$$
$$\left.-Bg(\tilde{z}(t-\tau_1))-C\int_{t-\tau_2}^{t}g(\tilde{z}(s))\,\mathrm{d}s\right]^*$$
$$\times\left[Q_1\dot{\tilde{z}}(t)+Q_2\tilde{z}(t-\delta)+Q_3\tilde{z}(t-\tau_1)\right]. \tag{43}$$

It follows from (38), (41), (42), and (43) that

$$D^+V(t)\le\mu^2(t)\,w^*(t)\,\Omega w(t), \tag{44}$$

where $w(t)=(\tilde{z}^*(t),\dot{\tilde{z}}^*(t),\tilde{z}^*(t-\delta),\tilde{z}(t-\tau_1),\int_{t-\delta}^{t}\tilde{z}^*(s)\mathrm{d}s,g^*(\tilde{z}(t)),g^*(\tilde{z}(t-\tau_1)),\int_{t-\tau_2}^{t}g^*(\tilde{z}(s))\mathrm{d}s)^*$ and

$$\Omega=\begin{pmatrix}\Omega_{11} & \Omega_{12} & P_1D & \frac{\beta^2}{\tau_1}P_5 & \Omega_{15} & 0 & 0 & 0\\ \star & \Omega_{22} & Q_1^*D & Q_3 & -P_1D & -Q_1^*A & -Q_1^*B & -Q_1^*C\\ \star & \star & \Omega_{33} & DQ_3 & -DP_1D & -Q_2^*A & -Q_2^*B & -Q_2^*C\\ \star & \star & \star & \Omega_{44} & 0 & -Q_3^*A & -Q_3^*B & -Q_3^*C\\ \star & \star & \star & \star & -\frac{\beta^2}{\delta}P_3 & 0 & 0 & 0\\ \star & \star & \star & \star & \star & \Omega_{66} & 0 & 0\\ \star & \star & \star & \star & \star & \star & -R_2 & 0\\ \star & \star & \star & \star & \star & \star & \star & -\frac{\beta^2}{\tau_2}P_6\end{pmatrix} \tag{45}$$

with $\Omega_{11}=2\alpha P_1-DP_1-P_1D+2\alpha P_2+\delta P_3+P_4-(\beta^2/\tau_1)P_5+R_1\Gamma$, $\Omega_{12}=P_1+P_2$, $\Omega_{15}=DP_1D-2\alpha P_1D$, $\Omega_{22}=Q_1^*+Q_1+\tau_1P_5$, $\Omega_{33}=Q_2^*D+DQ_2$, $\Omega_{44}=-\beta^2P_4-(\beta^2/\tau_1)P_5+R_2\Gamma$, $\Omega_{66}=\tau_2P_6-R_1$. Thus, we get from (29) and (44) that

$$D^+V(t)\le 0,\quad t\in[t_{k-1},t_k)\cap[T,\infty),\ k\in\mathbb{Z}^+. \tag{46}$$

In addition, we note that

$$\begin{aligned}&\begin{pmatrix}P_1 & (I+E_k)P_1\\ \star & P_1\end{pmatrix}>0\\ &\Longleftrightarrow\begin{pmatrix}I & 0\\ 0 & P_1^{-1}\end{pmatrix}\begin{pmatrix}P_1 & (I+E_k)P_1\\ \star & P_1\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & P_1^{-1}\end{pmatrix}>0\\ &\Longleftrightarrow\begin{pmatrix}P_1 & (I+E_k)\\ \star & P_1^{-1}\end{pmatrix}>0\\ &\Longleftrightarrow P_1-(I+E_k)^*P_1(I+E_k)>0\end{aligned} \tag{47}$$

in which the last equivalent relation is obtained by Lemma 9. Thus, it yields

$$\begin{aligned}V_1(t_k)&=\mu^2(t_k)\left[\tilde{z}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]^*\\&\quad\times P_1\left[\tilde{z}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]\\&=\mu^2(t_k^-)\left[\tilde{z}(t_k^-)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]^*\\&\quad\times(I+E_k)^*P_1(I+E_k)\\&\quad\times\left[\tilde{z}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]\\&\le\mu^2(t_k^-)\left[\tilde{z}(t_k^-)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]^*\\&\quad\times P_1\left[\tilde{z}(t_k)-D\int_{t_k-\delta}^{t_k}\tilde{z}(s)\,\mathrm{d}s\right]\\&=V_1(t_k^-).\end{aligned} \tag{48}$$

Hence, we can deduce that

$$V(t_k)\le V(t_k^-),\quad k\in\mathbb{Z}^+. \tag{49}$$

From (46) and (49), we know that V is monotonically nonincreasing for $t\in[T,\infty)$, which implies that

$$V(t)\le V(T),\quad t\ge T. \tag{50}$$

From the definition of $V(t)$ in (32), we obtain that

$$\mu^2(t)\,\lambda_{\min}(P_2)\,\|\tilde{z}(t)\|^2\le V(t)\le V_0<\infty,\quad t\ge 0, \tag{51}$$

where $V_0=\max_{0\le s\le T}V(s)$. It implies that

$$\|\tilde{z}(t)\|\le\frac{M}{\mu(t)},\quad t\ge 0, \tag{52}$$

where $M=\sqrt{V_0/\lambda_{\min}(P_2)}$. The proof is completed. □

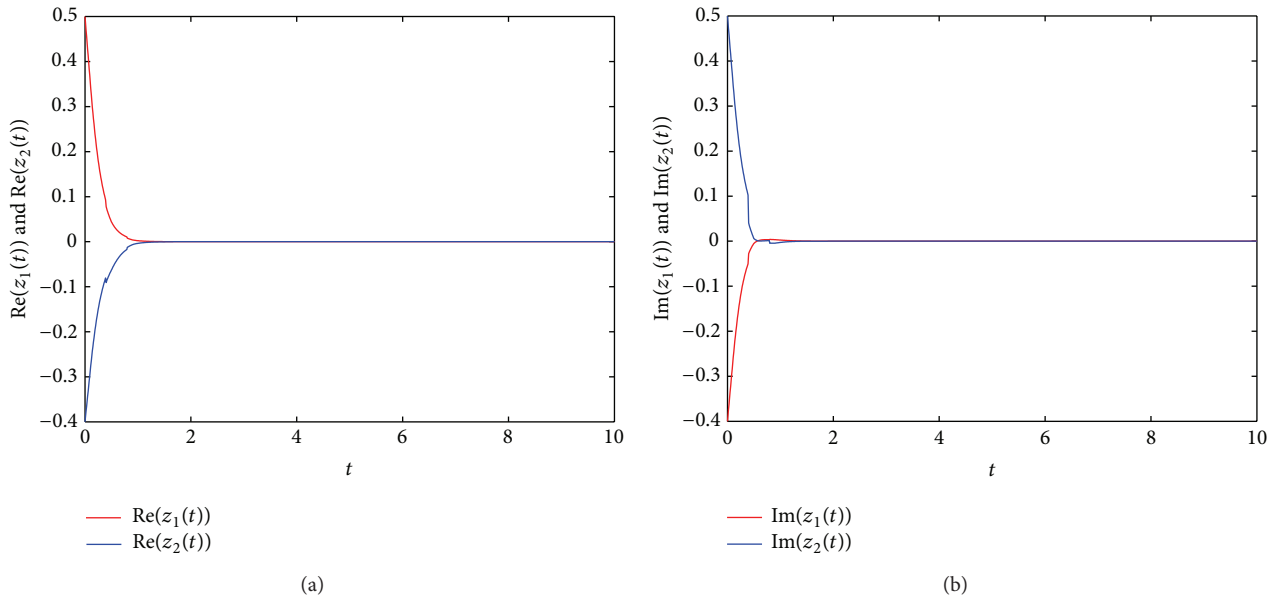

Figure 1: State trajectories of system (54).

Corollary 14. *Under the conditions of Theorem 10, the equilibrium point of system* (1) *is globally exponentially stable, if there exist six positive definite Hermitian matrices P_1, P_2, P_3, P_4, P_5, and P_6, two real positive diagonal matrices R_1 and R_2, three complex matrices Q_1, Q_2, and Q_3, and a positive constant ε such that conditions* (29) *and* (30) *in Theorem 13 are satisfied, where $\alpha = \varepsilon$, $\beta = e^{-\varepsilon \max\{\delta,\tau_1,\tau_2\}}$.*

Proof. Let $\mu(t) = e^{\varepsilon t}$ $(t \geq 0)$; then $\dot{\mu}(t)/\mu(t) = \varepsilon$, $\min\{\mu(t-\delta), \mu(t-\tau_1), \mu(t-\tau_2)\}/\mu(t) = \min\{e^{-\varepsilon\delta}, e^{-\varepsilon\tau_1}, e^{-\varepsilon\tau_2}\} = e^{-\varepsilon \max\{\delta,\tau_1,\tau_2\}}$. Take $\alpha = \varepsilon$, $\beta = e^{-\varepsilon \max\{\delta,\tau_1,\tau_2\}}$, and $T = 0$. Then it is obvious that condition (28) in Theorem 13 is satisfied. The proof is completed. □

Corollary 15. *Under the conditions of Theorem 10, the equilibrium point of system* (1) *is globally power-stable, if there exist six positive definite Hermitian matrices P_1, P_2, P_3, P_4, P_5, and P_6, two real positive diagonal matrices R_1 and R_2, three complex matrices Q_1, Q_2, and Q_3, and a positive constant ε such that conditions* (29) *and* (30) *in Theorem 13 are satisfied, where $\alpha = \varepsilon/2$, $\beta = 2^{-\varepsilon}$.*

Proof. Let $\mu(t) = t^{\varepsilon}$. For all $t \geq 2\max\{1, \delta, \tau_1, \tau_2\}$, we have $\dot{\mu}(t)/\mu(t) = \varepsilon/t \leq \varepsilon/2$, $\min\{\mu(t-\delta), \mu(t-\tau_1), \mu(t-\tau_2)\}/\mu(t) = (1 - (\max\{\delta, \tau_1, \tau_2\}/t))^{\varepsilon} \geq 2^{-\varepsilon}$. Take $\alpha = \varepsilon/2$, $\beta = 2^{-\varepsilon}$, and $T = 2\max\{1, \delta, \tau_1, \tau_2\}$. Then it is obvious that condition (28) in Theorem 13 is satisfied. The proof is completed. □

Corollary 16. *Under the conditions of Theorem 10, the equilibrium point of system* (1) *is globally log-stable, if there exist six positive definite Hermitian matrices P_1, P_2, P_3,P_4, P_5, and P_6, two real positive diagonal matrices R_1 and R_2, three complex matrices Q_1, Q_2, and Q_3, and a positive constant ε such that conditions* (29) *and* (30) *in Theorem 13 are satisfied, where $\alpha = \varepsilon/e$, $\beta = 1/\ln(e + \varepsilon\max\{\delta, \tau_1, \tau_2\})$.*

Proof. Let $\mu(t) = \ln(\varepsilon t + 1)$. For all $t \geq ((e-1)/\varepsilon) + \max\{\delta, \tau_1, \tau_2\}$, we have $\dot{\mu}(t)/\mu(t) = \varepsilon/(\varepsilon t + 1)\ln(\varepsilon t + 1) \leq \varepsilon/e$, $\min\{\mu(t-\delta), \mu(t-\tau_1), \mu(t-\tau_2)\}/\mu(t) = \ln[\varepsilon(t - \max\{\delta, \tau_1, \tau_2\}) + 1]/\ln(\varepsilon t + 1) \geq 1/\ln(e + \varepsilon\max\{\delta, \tau_1, \tau_2\})$. Take $\alpha = \varepsilon/e$, $\beta = 1/\ln(e + \varepsilon\max\{\delta, \tau_1, \tau_2\})$, and $T = ((e-1)/\varepsilon) + \max\{\delta, \tau_1, \tau_2\}$. Then it is obvious that condition (28) in Theorem 13 is satisfied. The proof is completed. □

Remark 17. It is noted that LMIs (15), (29), and (30) are complex-valued, which cannot be directly handled via MATLAB LMI Toolbox. However, the authors in [33] give the result that a complex Hermitian matrix L satisfies $L < 0$ if and only if

$$\begin{pmatrix} \operatorname{Re}(L) & \operatorname{Im}(L) \\ -\operatorname{Im}(L) & \operatorname{Re}(L) \end{pmatrix} < 0. \tag{53}$$

Therefore, applying the result, the complex-valued LMIs (15), (29), and (30) can be turned into real-valued LMIs, which can be checked numerically using LMI toolbox in MATLAB.

5. Numerical Examples

The following two illustrative examples will demonstrate the effectiveness and superiority of our results.

Example 1. Consider the following two-neuron CVNNs with leakage, discrete, and distributed delays, described by

$$\dot{z}(t) = -Dz(t-\delta) + Af(z(t)) + Bf(z(t-\tau_1)) + C\int_{t-\tau_2}^{t} f(z(s))\,ds + h, \quad t \neq t_k,\ t > 0,$$

$$\begin{aligned}\Delta z(t_k) &= z(t_k) - z(t_k^-)\\ &= E_k\left[z(t_k^-) - D\int_{t_k-\delta}^{t_k} z(s)\,\mathrm{d}s\right], \quad k \in \mathbb{Z}^+,\end{aligned} \tag{54}$$

where $f_1(u) = f_2(u) = (1/20)(|u+1| - |u-1|)$, $\delta = 0.1$, $\tau_1 = 0.2$, $\tau_2 = 0.2$, $h = (0,0)^T$, $t_k = 0.4k$, $k \in \mathbb{Z}^+$, and the parameter matrices D, A, B, C, and E_k are given as follows:

$$\begin{aligned}D &= \begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}, \qquad A = \begin{pmatrix}0.5+i & -0.2-0.3i\\ 0.2+0.5i & 1-0.5i\end{pmatrix},\\ B &= \begin{pmatrix}0.4+0.3i & -0.1-i\\ -1.2+i & -0.4-0.3i\end{pmatrix},\\ C &= \begin{pmatrix}0.1-0.5i & 0.6+0.4i\\ 0.2-0.5i & 0.2\end{pmatrix},\\ E_k &= \begin{pmatrix}-0.3+0.2i & 0\\ 0.1-0.6i & -0.2+0.2i\end{pmatrix}, \quad k \in \mathbb{Z}^+.\end{aligned} \tag{55}$$

It can be verified that the activation functions f_1 and f_2 satisfy condition (H1), and $\Gamma = \mathrm{diag}\{0.01, 0.01\}$. Then LMI in Theorem 10 has the following feasible solution via the MATLAB LMI toolbox:

$$\begin{aligned}U_1 &= \begin{pmatrix}19.0818+6.7535i & -9.2220+2.2044i\\ 8.0728+0.4398i & 19.0186+9.3571i\end{pmatrix},\\ U_2 &= \begin{pmatrix}4.7472-0.2821i & -6.9954-6.3515i\\ 1.8858+0.8953i & 4.5109+1.3211i\end{pmatrix},\\ R &= \begin{pmatrix}161.0117 & 0\\ 0 & 129.7777\end{pmatrix}.\end{aligned} \tag{56}$$

Also there exists a constant $\varepsilon = 0.1$, and by employing MATLAB LMI Toolbox, we can find the solutions to the LMIs in Corollary 14 as follows:

$$\begin{aligned}P_1 &= \begin{pmatrix}12.4935 & -0.6208-3.5072i\\ -0.6208+3.5072i & 38.9645\end{pmatrix},\\ P_2 &= \begin{pmatrix}16.5671 & -1.5742-1.2335i\\ -1.5742+1.2335i & 27.6891\end{pmatrix},\\ P_3 &= \begin{pmatrix}43.4484 & -1.3614-5.3324i\\ -1.3614+5.3324i & 87.1860\end{pmatrix},\\ P_4 &= \begin{pmatrix}12.8707 & -0.3721-2.5769i\\ -0.3721+2.5769i & 30.8832\end{pmatrix},\\ P_5 &= \begin{pmatrix}18.0542 & -0.3996+0.2524i\\ -0.3996-0.2524i & 16.1543\end{pmatrix},\\ P_6 &= \begin{pmatrix}49.8004 & -0.2065+3.8010i\\ -0.2065-3.8010i & 48.9421\end{pmatrix},\\ Q_1 &= \begin{pmatrix}-33.7168+0.0480i & 3.0570-2.0568i\\ 2.2459+1.1090i & -39.3508+0.0752i\end{pmatrix},\\ Q_2 &= \begin{pmatrix}-66.6664-1.5581i & 6.8518-15.0791i\\ 1.2207+10.1224i & -73.9430+1.7006i\end{pmatrix},\\ Q_3 &= \begin{pmatrix}-4.2664+0.1047i & 0.4653-0.9119i\\ 0.3881+1.2719i & 0.9145-0.0078i\end{pmatrix},\\ R_1 &= \begin{pmatrix}260.7956 & 0\\ 0 & 276.7577\end{pmatrix},\\ R_2 &= \begin{pmatrix}293.9133 & 0\\ 0 & 256.1810\end{pmatrix}.\end{aligned} \tag{57}$$

Hence, from Theorem 10 and Corollary 14, the unique equilibrium $(0,0)^T$ of system (54) is globally exponentially stable. Figure 1 depicts the real and imaginary parts of states of the considered system (54), where the initial condition is $z_1(t) = 0.5 - 0.4i$, $z_2(t) = -0.4 + 0.5i$.

Example 2. Consider the following two-neuron CVNNs with leakage, discrete, and distributed delays, described by

$$\begin{aligned}\dot{z}(t) &= -Dz(t-\delta) + Af(z(t)) + Bf(z(t-\tau_1))\\ &\quad + C\int_{t-\tau_2}^{t} f(z(s))\,\mathrm{d}s + h, \quad t \neq t_k,\ t > 0,\\ \Delta z(t_k) &= z(t_k) - z(t_k^-)\\ &= E_k\left[z(t_k^-) - D\int_{t_k-\delta}^{t_k} z(s)\,\mathrm{d}s - \gamma\right], \quad k \in \mathbb{Z}^+,\end{aligned} \tag{58}$$

where $f_1(u) = f_2(u) = (1/4)[\max\{0, \mathrm{Re}(u)\} + i\max\{0, \mathrm{Im}(u)\}]$, $\delta = 0.2$, $\tau_1 = 0.6$, $\tau_2 = 1$, $h = (1+i, -1+i)^T$, $\gamma = (0.0295+0.0623i, -0.0604+0.0397i)^T$, $t_k = 0.8k$, $k \in \mathbb{Z}^+$, and the parameter matrices D, A, B, C, and E_k are given as follows:

$$\begin{aligned}D &= \begin{pmatrix}4 & 0\\ 0 & 4\end{pmatrix}, \qquad A = \begin{pmatrix}-1+3i & -1+2i\\ -2-2i & 1+0.5i\end{pmatrix},\\ B &= \begin{pmatrix}0.8+i & i\\ -1+0.5i & 1+0.5i\end{pmatrix}, \qquad C = \begin{pmatrix}1+i & 1-2i\\ -1+i & 0.5+i\end{pmatrix},\\ E_k &= \begin{pmatrix}-1 & 0.5i\\ 0.5i & -0.5\end{pmatrix}, \quad k \in \mathbb{Z}^+.\end{aligned} \tag{59}$$

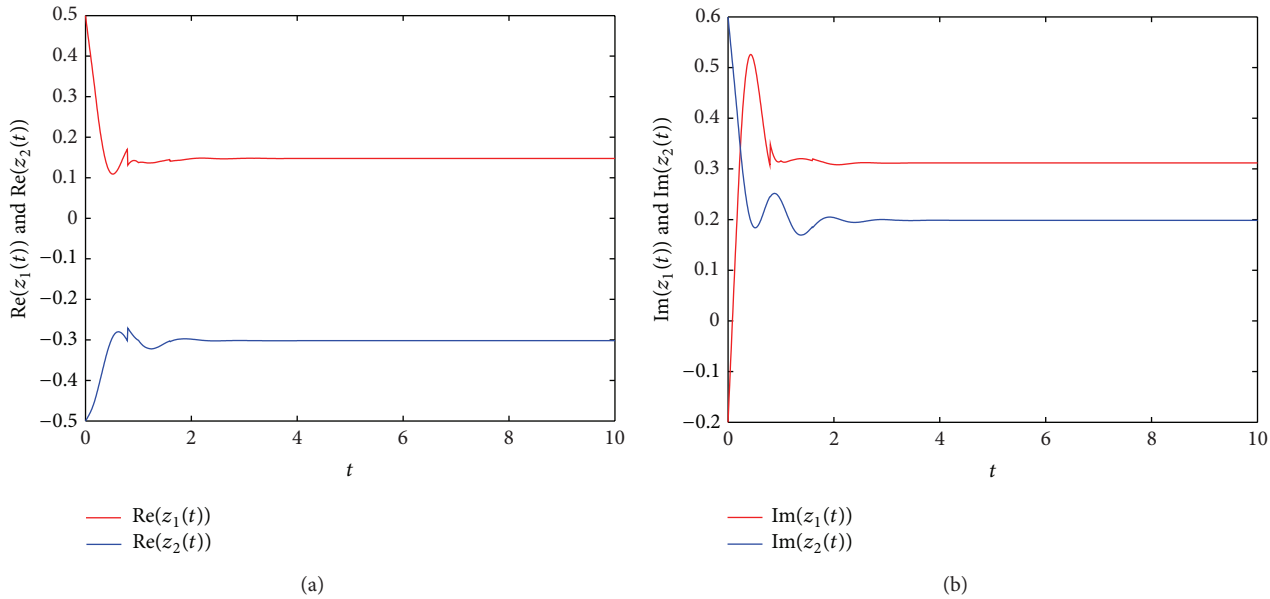

Figure 2: State trajectories of system (58) with $\delta = 0.2$.

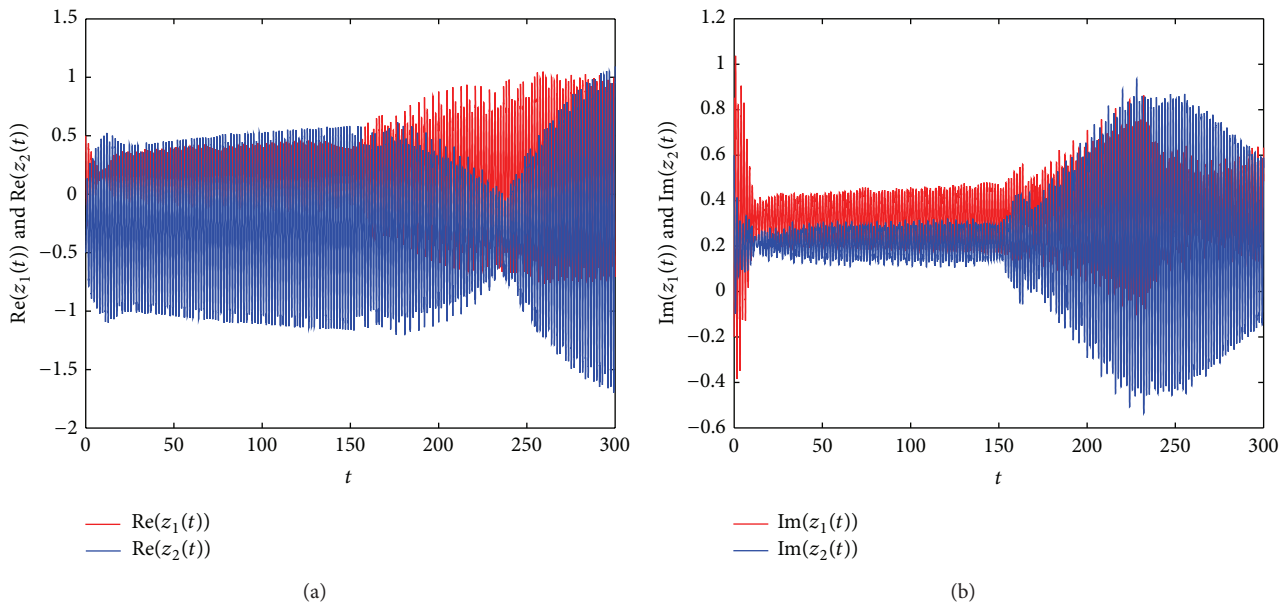

Figure 3: State trajectories of system (58) with $\delta = 0.37$.

It can be verified that the activation functions f_1 and f_2 satisfy condition (H1), and $\Gamma = \mathrm{diag}\{1/16, 1/16\}$. Then LMI in Theorem 10 has the following feasible solution via the MATLAB LMI toolbox:

$$U_1 = \begin{pmatrix} 2.9099 - 5.3413i & 2.7078 + 7.2154i \\ -2.5180 + 7.0990i & 2.6469 + 0.8628i \end{pmatrix},$$

$$U_2 = \begin{pmatrix} 5.4524 + 4.8583i & -11.7495 + 2.6580i \\ -0.5273 - 7.2479i & -0.8154 - 1.1665i \end{pmatrix}, \quad (60)$$

$$R = \begin{pmatrix} 79.4022 & 0 \\ 0 & 37.6964 \end{pmatrix}.$$

Also there exists a constant $\varepsilon = 0.1$, and by employing MATLAB LMI Toolbox, we can find the solutions to LMIs in Corollary 15 as follows:

$$P_1 = \begin{pmatrix} 1.6537 & 0.0741 + 0.0987i \\ 0.0741 - 0.0987i & 2.5236 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0.1984 & 0.0640 - 0.0118i \\ 0.0640 + 0.0118i & 0.2985 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 21.5659 & 0.7498 + 1.4712i \\ 0.7498 - 1.4712i & 32.7091 \end{pmatrix},$$

$$P_4 = \begin{pmatrix} 0.6852 & 0.0304 + 0.0669i \\ 0.0304 - 0.0669i & 1.1632 \end{pmatrix},$$
$$P_5 = \begin{pmatrix} 0.2419 & 0.0941 - 0.1415i \\ 0.0941 + 0.1415i & 0.5347 \end{pmatrix},$$
$$P_6 = \begin{pmatrix} 7.7723 & -0.2071 - 0.7161i \\ -0.2071 + 0.7161i & 10.5317 \end{pmatrix},$$
$$Q_1 = \begin{pmatrix} -0.7898 + 0.0341i & -0.1269 - 0.5099i \\ -0.1170 + 0.5513i & -1.2707 - 0.0430i \end{pmatrix},$$
$$Q_2 = \begin{pmatrix} -2.5965 + 0.0654i & -0.5192 - 2.1874i \\ -0.5396 + 2.4027i & -4.2140 - 0.0583i \end{pmatrix},$$
$$Q_3 = \begin{pmatrix} -0.0376 + 0.0030i & -0.0029 - 0.0779i \\ -0.0046 + 0.0854i & -0.0596 - 0.0047i \end{pmatrix},$$
$$R_1 = \begin{pmatrix} 19.9880 & 0 \\ 0 & 32.1768 \end{pmatrix},$$
$$R_2 = \begin{pmatrix} 6.8268 & 0 \\ 0 & 14.7991 \end{pmatrix}. \tag{61}$$

Hence, from Theorem 10 and Corollary 15, system (58) has a unique equilibrium, which is globally exponentially power-stable. Figure 2 depicts the real and imaginary parts of states of the considered system (58), where the initial condition is $z_1(t) = 0.5 - 0.2i, z_2(t) = -0.5 + 0.6i$.

Remark 18. If we take leakage delay $\delta \geq 0.37$ in system (58), one may check that LMIs in Corollary 15 do not have a feasible solution via MATLAB. In other words, our results cannot guarantee the stability of system (58) with $\delta \geq 0.37$. From the simulations, it is easy to check that the unique equilibrium point of system (58) with $\delta = 0.37$ is not stable (see Figure 3); this implies that the effect of leakage delay on the dynamics of CVNNs cannot be ignored.

6. Conclusion

In this paper, the μ-stability of impulsive CVNNs with leakage delay, discrete delay, and distributed delay has been investigated. Several sufficient conditions to ensure the existence, uniqueness, and global μ-stability of the equilibrium point of the considered neural networks have been established in LMIs by applying the homeomorphism mapping principle of complex domain, constructing appropriate Lyapunov-Krasovskii functionals, and employing the free weighting matrix method. As direct applications of these results, several criteria on the exponential stability, power-stability and log-stability have been obtained. Two numerical examples are given to illustrate the effectiveness of the proposed theoretical results.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

H_∞ Control for Network-Based 2D Systems with Missing Measurements

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The problem of H_∞ control for network-based 2D systems with missing measurements is considered. A stochastic variable satisfying the Bernoulli random binary distribution is utilized to characterize the missing measurements. Our attention is focused on the design of a state feedback controller such that the closed-loop 2D stochastic system is mean-square asymptotic stability and has an H_∞ disturbance attenuation performance. A sufficient condition is established by means of linear matrix inequalities (LMIs) technique, and formulas can be given for the control law design. The result is also extended to more general cases where the system matrices contain uncertain parameters. Numerical examples are also given to illustrate the effectiveness of proposed approach.

1. Introduction

Two-dimensional (2D) systems have attracted considerable research interest over the past few decades due to their wide applications in the areas such as multidimensional digital filtering, linear image processing, signal processing, process control, and iterative learning control [1–5]. Thus the stability and stabilization of 2D systems have attracted a lot of interests; see, for example, [6–13] and the references therein. H_∞ optimization is a powerful tool that can be used to design a robust controller or filter, which has been proved to be one of the most important strategies. Recently, such problems on 2D systems have stirred a great deal of research attention. For example, several versions of 2D bounded real lemma have been established in [2, 6, 14]. The problem of H_∞ control for 2D systems with state delays has been considered in [15]. The problem of H_∞ filtering for 2D systems has been studied extensively in [16]. An H_∞ controller is designed for a class of 2D nonlinear discrete systems with sector nonlinearity in [17, 18].

Notice that all the above-mentioned results are based on an implicit assumption that the communication between the physical plant and controller is perfect; that is, the signals transmitted from the plant will arrive at the controller simultaneously and perfectly. However, in many practical situations, there may be a nonzero probability that all the signals can be measured during their transmission. In other words, the systems may have missing measurements. Moreover, networked systems are becoming more and more popular for the reason that they have several advantages over traditional systems, such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability [19–22]. If network is introduced to controller design, the data packet dropout phenomenon, which appears in a typical network environment, will naturally induce missing measurements from the plant to the controller. In 1D system, the problems of stability, stabilization, filtering, or state estimation for networked control systems have been widely researched [23–30]. However, in the network-based 2D system case, only few results have been available. For instance, the problem of robust H_∞ filtering for 2D systems described by the Fornasini-Marchesini (FM) second model with missing measurements is considered in [31]. To the best of the authors' knowledge, the H_∞ control for 2D systems represented by the Roesser model which is structurally quite distinct from FM second model has not been addressed in the literature so far.

Motivated by the aforementioned observations, this paper considers the problem of H_∞ control for network-based 2D systems presented by the Roesser model with missing measurements. We also notice that some dynamical processes such as gas absorption, water stream heating, and

air drying can be described by a 2D Roesser model. In practical, these systems are often implemented by distribute control systems (DCSs) or field-bus control systems (FCSs), where control loops that are closed over a communication network. Hence, the considered topic in this paper is of practical significance. Compared with existing results, this paper proposes a state feedback controller design method for 2D systems in the framework of networked control systems. Meanwhile, the introduction of the random missing measurements renders the 2D system to be stochastic instead of a deterministic one. To analyze the stability, we introduce the stochastic mean-square asymptotically stable and stochastic H_∞ disturbance attenuation level. The controller is also designed under the framework of 2D stochastic systems.

The remainder of this paper is organized as follows. In Section 2, the mathematical description and design objectives of this paper are presented. In Section 3, a sufficient condition of mean-square asymptotic stability with H_∞ performance for such 2D stochastic systems is derived by means of LMI technique, and then formulas can be given for the control law design. In Section 4, the design result is extended to the 2D systems with uncertain parameters. Numerical examples are given in Section 5 and conclusions are drawn in Section 6.

Notation 1. The superscript "T" denotes the matrix transposition, R^n denotes the n-dimensional Euclidean space, I denotes the identity matrix, 0 denotes the zero vector or matrix with the required dimensions, and diag$\{\cdot\}$ denotes the standard (block) diagonal matrix whose off-diagonal elements are zero. In symmetric block matrices, an asterisk $*$ is used to denote the term that is induced by symmetry. The notation $\|\cdot\|$ refers to the Euclidean vector norm and $\lambda_{\min}(\cdot), \lambda_{\max}(\cdot)$ denote the minimum and the maximum eigenvalue of the corresponding matrix, respectively. $E\{x\}, E\{x \mid y\}$ mean the expectation of x and the expectation of x conditional on y. Matrices, if the dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. Problem Formulation

Consider the following 2D discrete system given by

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = A \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + Bu(i,j) + B_1 w(i,j), \quad (1)$$

where $x^h(i,j) \in R^{n_1}$, $x^v(i,j) \in R^{n_2}$, and $u(i,j) \in R^m$ represent the horizontal state, vertical state, and control input, respectively. $w(i,j) \in R^p$ denotes the noise input, which belongs to ℓ_2. A, B, B_1 are real matrices with appropriate dimension.

We make the following assumption on the boundary condition.

Assumption 1. The boundary condition is assumed to satisfy

$$\lim_{N\to\infty} E\left\{\sum_{k=0}^{N}\left(|x_{k,0}|^2 + |x_{0,k}|^2\right)\right\} < \infty. \quad (2)$$

Now, we consider the following state feedback control law:

$$u(i,j) = G\tilde{x}(i,j), \quad (3)$$

where $\tilde{x}(i,j)$ is the measurement of state signals, G is appropriately dimensioned controller matrix to be determined. When the feedback control is implemented via a networked control system, the data $x^h(i,j), x^v(i,j)$ are transferred as two separate packets from the remote plant to the controller. In this process, the data may be missed due to the network transmission failure, resulting in what we call missing measurement. It is assumed that the data packet dropout can be described by a stochastic variable; that is,

$$\tilde{x}(i,j) = \alpha_{i,j} x(i,j), \quad (4)$$

where $x(i,j) = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$, the stochastic parameters $\{\alpha_{i,j}\}$ is a Bernoulli distributed white sequence taking the values of 0 and 1 with

$$\begin{aligned} \text{Prob}\left\{\alpha_{i,j} = 1\right\} &= E\left\{\alpha_{i,j}\right\} = \alpha, \\ \text{Prob}\left\{\alpha_{i,j} = 0\right\} &= 1 - E\left\{\alpha_{i,j}\right\} = 1 - \alpha, \end{aligned} \quad (5)$$

and $0 \le \alpha \le 1$ is a known constant.

Notice that the introduction of the stochastic variable $\{\alpha_{i,j}\}$ renders the 2D system to be stochastic instead of a deterministic one. Before proceeding further, we need to introduce the following definition of stochastic stability for the 2D system, which will be essential for our derivation.

Definition 2 (see [31]). The 2D system (1) is said to be mean-square asymptotically stable if under the zero input and for every bounded initial condition $x^h(i,0)$, $x^v(0,j)$, the following is satisfied

$$\lim_{i+j\to\infty} E\left\{\|x(i,j)\|^2\right\} = 0. \quad (6)$$

Definition 3. Given a scalar $\gamma > 0$, the 2D system (1) is said to have an H_∞ disturbance attenuation level γ, if it is mean-square asymptotically stable and under zero initial conditions, $\|x\|_E < \gamma\|w\|_2$ is satisfied for any external disturbance $w(i,j) \in \ell_2$, where

$$\begin{aligned} \|x\|_E &= \sqrt{E\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|x(i,j)\|^2\right\}}, \\ \|w\|_2 &= \sqrt{E\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\|w(i,j)\|^2\right\}}. \end{aligned} \quad (7)$$

To this end, the design objective of this paper can be described as follows.

Problem Statement. For any initial condition satisfying Assumption (1) and missing measurements described as (5), design a state feedback law (3) such that the closed-loop 2D system is mean-square asymptotically stable and has an H_∞ disturbance attenuation level γ.

3. Main Results

In this section, we assume that the system matrices A, B, B_1 and controller gain matrix G are known, and then we study the condition under which the closed-loop 2D system is mean-square asymptotically stable with a guaranteed H_∞ performance. Then, a feasible controller gain matrix can be given based on the condition.

From system (1), (3), and (4), we can obtain the following closed-loop system:

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(A + \alpha_{i,j}BG\right)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_1 w(i,j). \quad (8)$$

Define $\tilde{\alpha}_{i,j} = \alpha_{i,j} - \alpha$; it is obvious that

$$E\left\{\tilde{\alpha}_{i,j}\right\} = 0, \qquad E\left\{\tilde{\alpha}_{i,j}\tilde{\alpha}_{i,j}\right\} = \alpha(1-\alpha). \quad (9)$$

Then, the 2D closed-loop system can be rewritten as

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(A + \alpha BG + \tilde{\alpha}_{i,j}BG\right)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + B_1 w(i,j). \quad (10)$$

Theorem 4. *The 2D closed-loop system* (10) *is mean-square asymptotically stable with a given H_∞ disturbance attenuation level γ, if there exists a positive define symmetric matrix $P = \mathrm{diag}\{P_h, P_v\} > 0$, satisfying*

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix} < 0, \quad (11)$$

where

$$\Xi_{11} = (A+\alpha BG)^T P (A+\alpha BG) + \theta^2 (BG)^T PBG + I - P,$$
$$\Xi_{12} = (A+\alpha BG)^T PB_1, \qquad \Xi_{22} = B_1^T PB_1 - \gamma^2 I,$$
$$\theta^2 = \alpha(1-\alpha). \quad (12)$$

Proof. We first prove the mean-square asymptotically stability of 2D system (10) with zero disturbance $w(i,j) = 0$. In this case, the system becomes

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = \left(A + \alpha BG + \tilde{\alpha}_{i,j}BG\right)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}. \quad (13)$$

Define

$$W_1 = E\left\{\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}^T P \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} \mid \tilde{x}\right\},$$
$$W_2 = \tilde{x}^T P \tilde{x}, \quad (14)$$

where $\tilde{x} = \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}$.

Consider the following index

$$J \triangleq W_1 - W_2. \quad (15)$$

Substituting (13) into the above index, we can obtain

$$\begin{aligned} J &= E\left\{\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}^T P \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} \mid \tilde{x}\right\} - \tilde{x}^T P \tilde{x} \\ &= E\left\{\begin{bmatrix} \left(A + \alpha BG + \tilde{\alpha}_{i,j}BG\right)\tilde{x} \\ \left(A + \alpha BG + \tilde{\alpha}_{i,j}BG\right)\tilde{x} \end{bmatrix}^T P \mid \tilde{x}\right\} - \tilde{x}^T P \tilde{x} \\ &= \tilde{x}^T\left[(A+\alpha BG)^T P (A+\alpha BG) + \theta^2 (BG)^T PBG - P\right]\tilde{x} \\ &= \tilde{x}^T \Psi \tilde{x}, \end{aligned} \quad (16)$$

where

$$\Psi = (A+\alpha BG)^T P (A+\alpha BG) + \theta^2 (BG)^T PBG - P. \quad (17)$$

From (11), it is easy to see that $\Psi < 0$. Hence, for all $\tilde{x} \neq 0$, we have

$$\frac{W_1 - W_2}{W_2} = -\frac{\tilde{x}^T(-\Psi)\tilde{x}}{\tilde{x}^T P \tilde{x}} \le -\frac{\lambda_{\min}(-\Psi)}{\lambda_{\max}(P)} = \delta - 1, \quad (18)$$

where $\delta = 1 - (\lambda_{\min}(-\Psi)/\lambda_{\max}(P))$.

Notice that $(\lambda_{\min}(-\Psi)/\lambda_{\max}(P)) > 0$; we have $\delta < 1$. From (18), it is also easy to see that

$$\delta \ge \frac{W_1}{W_2} > 0. \quad (19)$$

Hence, $\delta \in (0,1)$ and it is independent of $\tilde{x}$. Thus, we obtain $W_1 \le \delta W_2$, and taking expectation of both sides yields

$$\begin{aligned} &E\left\{\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}^T P \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix}\right\} \\ &\quad \le \delta E\left\{\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}^T P \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix}\right\}. \end{aligned} \quad (20)$$

That is,

$$\begin{aligned} &E\left\{x^v(k+1,0)^T P_v x^v(k+1,0)\right\} \\ &\quad = E\left\{x^v(k+1,0)^T P_v x^v(k+1,0)\right\} \\ &E\left\{x^h(k+1,0)^T P_h x^h(k+1,0) + x^v(k,1)^T P_v x^v(k,1)\right\} \\ &\quad \le \delta E\left\{x^h(k,0)^T P_h x^h(k,0) + x^v(k,0)^T P_v x^v(k,0)\right\} \\ &E\left\{x^h(k,1)^T P_h x^h(k,1) + x^v(k-1,2)^T P_v x^v(k-1,2)\right\} \\ &\quad \le \delta E\left\{x^h(k-1,1)^T P_h x^h(k-1,1)\right. \\ &\qquad \left. + x^v(k-1,1)^T P_v x^v(k-1,1)\right\} \end{aligned}$$

$$\begin{aligned}
&E\left\{x^h(k-1,2)^T P_h x^h(k-1,2)\right.\\
&\quad\left.+x^v(k-2,3)^T P_v x^v(k-2,3)\right\}\\
&\quad\le \delta E\left\{x^h(k-2,2)^T P_h x^h(k-2,2)\right.\\
&\qquad\left.+x^v(k-2,2)^T P_v x^v(k-2,2)\right\}\\
&\quad\vdots\\
&E\left\{x^h(1,k)^T P_h x^h(1,k)+x^v(0,k+1)^T P_v x^v(0,k+1)\right\}\\
&\quad\le \delta E\left\{x^h(0,k)^T P_h x^h(0,k)+x^v(0,k)^T P_v x^v(0,k)\right\}\\
&E\left\{x^h(0,k+1)^T P_h x^h(0,k+1)\right\}\\
&\quad=\left\{x^h(0,k+1)^T P_h x^h(0,k+1)\right\}.
\end{aligned} \tag{21}$$

Adding both sides of the inequality system (21) yields

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}\left[x^h(k+1-j,j)^T P_h x^h(k+1-j,j)\right.\right.\\
&\qquad\left.\left.+x^v(k+1-j,j)^T P_v x^v(k+1-j,j)\right]\right\}\\
&\quad\le \delta E\left\{\sum_{j=0}^{k}\left[x^h(k-j,j)^T P_h x^h(k-j,j)\right.\right.\\
&\qquad\left.\left.+x^v(k-j,j)^T P_v x^v(k-j,j)\right]\right\}\\
&\qquad+E\left\{x^v(k+1,0)^T P_v x^v(k+1,0)\right\}\\
&\qquad+E\left\{x^h(k+1,0)^T P_h x^h(k+1,0)\right\}.
\end{aligned} \tag{22}$$

Using this relationship iteratively, we can obtain

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}\left[x^h(k+1-j,j)^T P_h x^h(k+1-j,j)\right.\right.\\
&\qquad\left.\left.+x^v(k+1-j,j)^T P_v x^v(k+1-j,j)\right]\right\}\\
&\quad\le \delta^{k+1}E\left\{x^h(0,0)^T P_h x^h(0,0)+x^v(0,0)^T P_v x^v(0,0)\right\}\\
&\qquad+E\left\{\sum_{j=0}^{k}\delta^j\left[x^v(k+1-j,0)^T P_v x^v(k+1-j,0)\right.\right.\\
&\qquad\left.\left.+x^h(0,k+1-j)^T P_h x^h(0,k+1-j)\right]\right\}\\
&\quad=E\left\{\sum_{j=0}^{k+1}\delta^j\left[x^v(k+1-j,0)^T P_v x^v(k+1-j,0)\right.\right.\\
&\qquad\left.\left.+x^h(0,k+1-j)^T P_h x^h(0,k+1-j)\right]\right\},
\end{aligned} \tag{23}$$

which implies

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}\left\|x(k+1-j,j)\right\|^2\right\}\\
&\quad\le \kappa\sum_{j=0}^{k+1}\delta^j E\left\{\left\|x^v(k+1-j,0)\right\|^2\right.\\
&\qquad\left.+\left\|x^h(0,k+1-j)\right\|^2\right\},
\end{aligned} \tag{24}$$

where

$$\kappa := \frac{\lambda_{\max}(P)}{\lambda_{\min}(P)}. \tag{25}$$

Now, denote $\chi_k := \sum_{j=0}^{k}\|x(k-j,j)\|^2$; then, upon the inequality (24) we have

$$\begin{aligned}
E\{\chi_0\} &\le \kappa E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\\
E\{\chi_1\} &\le \kappa\left[\delta E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\right.\\
&\qquad\left.+E\left\{\left\|x^v(1,0)\right\|^2+\left\|x^h(0,1)\right\|^2\right\}\right]\\
E\{\chi_2\} &\le \kappa\left[\delta^2 E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\right.\\
&\qquad+\delta E\left\{\left\|x^v(1,0)\right\|^2+\left\|x^h(0,1)\right\|^2\right\}\\
&\qquad\left.+E\left\{\left\|x^v(2,0)\right\|^2+\left\|x^h(0,2)\right\|^2\right\}\right]\\
&\vdots\\
E\{\chi_N\} &\le \kappa\left[\delta^N E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\right.\\
&\qquad+\delta^{N-1}E\left\{\left\|x^v(1,0)\right\|^2+\left\|x^h(0,1)\right\|^2\right\}\\
&\qquad\left.+\cdots+E\left\{\left\|x^v(N,0)\right\|^2+\left\|x^h(0,N)\right\|^2\right\}\right].
\end{aligned} \tag{26}$$

Adding both sides of the inequalities yields

$$\begin{aligned}
&\sum_{k=0}^{N}E\{\chi_k\}\\
&\quad\le \kappa\left(1+\delta+\cdots+\delta^N\right)E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\\
&\qquad+\kappa\left(1+\delta+\cdots+\delta^{N-1}\right)E\left\{\left\|x^v(1,0)\right\|^2+\left\|x^h(0,1)\right\|^2\right\}\\
&\qquad+\cdots+\kappa E\left\{\left\|x^v(N,0)\right\|^2+\left\|x^h(0,N)\right\|^2\right\}
\end{aligned}$$

$$\begin{aligned}
&\le \kappa\left(1+\delta+\cdots+\delta^N\right)E\left\{\left\|x^v(0,0)\right\|^2+\left\|x^h(0,0)\right\|^2\right\}\\
&\quad+\kappa\left(1+\delta+\cdots+\delta^N\right)E\left\{\left\|x^v(1,0)\right\|^2+\left\|x^h(0,1)\right\|^2\right\}\\
&\quad+\cdots+\kappa\left(1+\delta+\cdots+\delta^N\right)E\left\{\left\|x^v(N,0)\right\|^2\right.\\
&\qquad\qquad\left.+\left\|x^h(0,N)\right\|^2\right\}\\
&=\kappa\times\frac{1-\delta^N}{1-\delta}E\left\{\sum_{i=0}^{N}\left[\left\|x^v(k,0)\right\|^2+\left\|x^h(0,k)\right\|^2\right]\right\}.
\end{aligned}\tag{27}$$

Then, under Assumption 1, the right side of this inequality is bounded, which means that $\lim_{k\to\infty}E\{\chi_k\}=0$; that is, $\lim_{i+j\to\infty}E\{\|x(i,j)\|^2\}=0$. Then the 2D stochastic system (10) is mean-square asymptotically stable.

Now, the H_∞ performance for the 2D stochastic system (10) will be established. To this end, assume zero initial boundary conditions; that is, $x^h(0,i)=0, x^v(i,0)=0$ for all i. In this case, the index J becomes

$$\begin{aligned}
J&=E\left\{\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}^T P\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}\mid\tilde{x}\right\}-\tilde{x}^TP\tilde{x}\\
&=E\left\{\begin{matrix}\left[\left(A+\alpha BG+\tilde{\alpha}_{i,j}BG\right)\tilde{x}+B_1w(i,j)\right]^TP\\\left[\left(A+\alpha BG+\tilde{\alpha}_{i,j}BG\right)\tilde{x}+B_1w(i,j)\right]\mid\tilde{x}\end{matrix}\right\}-\tilde{x}^TP\tilde{x}.
\end{aligned}\tag{28}$$

Another index is introduced as

$$\begin{aligned}
\Pi&\triangleq J+\tilde{x}^T\tilde{x}-\gamma^2w^Tw\\
&=E\left\{\begin{matrix}\left[\left(A+\alpha BG+\tilde{\alpha}_{i,j}BG\right)\tilde{x}+B_1w(i,j)\right]^TP\\\left[\left(A+\alpha BG+\tilde{\alpha}_{i,j}BG\right)\tilde{x}+B_1w(i,j)\right]^T\mid\tilde{x}\end{matrix}\right\}\\
&\quad-\tilde{x}^TP\tilde{x}+\tilde{x}^T\tilde{x}-\gamma^2w^Tw\\
&=\zeta^T\Xi\zeta,
\end{aligned}\tag{29}$$

where $\zeta=\left[\tilde{x}^T\quad\tilde{w}^T\right]^T$.

From condition (11), we have $\Pi<0$ for any $\zeta\neq0$; that is,

$$\begin{aligned}
&E\left\{\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}^T P\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}\mid\tilde{x}\right\}\\
&\quad<\tilde{x}^TP\tilde{x}-\tilde{x}^T\tilde{x}+\gamma^2\tilde{w}^T\tilde{w}.
\end{aligned}\tag{30}$$

Taking the expectation of both sides yields

$$\begin{aligned}
&E\left\{\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}^T P\begin{bmatrix}x^h(i+1,j)\\x^v(i,j+1)\end{bmatrix}\mid\tilde{x}\right\}\\
&\quad<E\left\{\tilde{x}^TP\tilde{x}-\tilde{x}^T\tilde{x}\right\}+\gamma^2\tilde{w}^T\tilde{w}.
\end{aligned}\tag{31}$$

Due to the relationship (31), it can be established that

$$\begin{aligned}
&E\left\{x^v(k+1,0)^TP_2x^v(k+1,0)\right\}\\
&\quad=E\left\{x^v(k+1,0)^TP_2x^v(k+1,0)\right\}\\
&E\left\{x^h(k+1,0)^TP_1x^h(k+1,0)+x^v(k,1)^TP_2x^v(k,1)\right\}\\
&\quad\le E\left\{x^h(k,0)^TP_1x^h(k,0)+x^v(k,0)^TP_2x^v(k,0)\right\}\\
&\qquad-E\left\{\tilde{x}(k,0)^T\tilde{x}(k,0)\right\}+\gamma^2w(k,0)^Tw(k,0)\\
&E\left\{x^h(k,1)^TP_1x^h(k,1)+x^v(k-1,2)^TP_2x^v(k-1,2)\right\}\\
&\quad\le E\left\{x^h(k-1,1)^TP_1x^h(k-1,1)\right.\\
&\qquad\left.+x^v(k-1,1)^TP_2x^v(k-1,1)\right\}\\
&\qquad-E\left\{\tilde{x}(k-1,1)^T\tilde{x}(k-1,1)\right\}\\
&\qquad+\gamma^2w(k-1,1)^Tw(k-1,1)\\
&E\left\{x^h(k-1,2)^TP_1x^h(k-1,2)\right.\\
&\quad\left.+x^v(k-2,3)^TP_2x^v(k-2,3)\right\}\\
&\quad\le E\left\{x^h(k-2,2)^TP_1x^h(k-2,2)\right.\\
&\qquad\left.+x^v(k-2,2)^TP_2x^v(k-2,2)\right\}\\
&\qquad-E\left\{\tilde{x}(k-2,2)^T\tilde{x}(k-2,2)\right\}\\
&\qquad+\gamma^2w(k-2,2)^Tw(k-2,2)\\
&\quad\vdots\\
&E\left\{x^h(1,k)^TP_1x^h(1,k)+x^v(0,k+1)^TP_2x^v(0,k+1)\right\}\\
&\quad\le E\left\{x^h(0,k)^TP_1x^h(0,k)+x^v(0,k)^TP_2x^v(0,k)\right\}\\
&\qquad-E\left\{\tilde{x}(0,k)^T\tilde{x}(0,k)\right\}+\gamma^2w(0,k)^Tw(0,k)\\
&E\left\{x^h(0,k+1)^TP_1x^h(0,k+1)\right\}\\
&\quad=\left\{x^h(0,k+1)^TP_1x^h(0,k+1)\right\}.
\end{aligned}\tag{32}$$

Adding both sides of the inequality system, we have

$$\begin{aligned}
&E\left\{\sum_{j=0}^{k+1}\left[x^h(k+1-j,j)^TP_1x^h(k+1-j,j)\right.\right.\\
&\qquad\left.\left.+x^v(k+1-j,j)^TP_2x^v(k+1-j,j)\right]\right\}\\
&\quad<\left\{\sum_{j=0}^{k}\left[x^h(k-j,j)^TP_1x^h(k-j,j)\right.\right.\\
&\qquad\left.\left.+x^v(k-j,j)^TP_2x^v(k-j,j)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&+ E\left\{x^v(k+1,0)^T P_2 x^v(k+1,0)\right\} \\
&+ E\left\{x^h(0,k+1)^T P_1 x^h(0,k+1)^T\right\} \\
&- E\left\{\sum_{j=0}^{k}\left[\tilde{x}(k-j,j)^T \tilde{x}(k-j,j)\right]\right\} \\
&+ \gamma^2 \sum_{j=0}^{i} w(k-j,j)^T w(k-j,j).
\end{aligned} \tag{33}
$$

Summing up both sides of this inequality from $k = 0$ to $k = N$, we have

$$
\begin{aligned}
&E\left\{\sum_{k=0}^{N}\sum_{j=0}^{k}\tilde{x}(k-j,j)^T \tilde{x}(k-j,j)\right\} \\
&\quad < \gamma^2 \sum_{k=0}^{N}\sum_{j=0}^{k} w(k-j,j)^T w(k-j,j) \\
&\quad + \sum_{k=0}^{N}\left(E\left\{x^v(k+1,0)^T P_2 x^v(k+1,0)\right\}\right. \\
&\qquad \left. + E\left\{x^h(0,k+1)^T P_1 x^h(0,k+1)^T\right\}\right) \\
&\quad - E\left\{\sum_{j=0}^{N}\left[x^h(N+1-j,j)^T P_1 x^h(N+1-j,j)\right.\right. \\
&\qquad \left.\left. + x^v(N+1-j,j)^T P_2 x^v(N+1-j,j)\right]\right\} \\
&\quad + E\left\{x^v(0,0)^T P_2 x^v(0,0)\right\} + E\left\{x^h(0,0)^T P_1 x^h(0,0)^T\right\};
\end{aligned} \tag{34}
$$

that is,

$$
\begin{aligned}
&E\left\{\sum_{k=0}^{\infty}\sum_{j=0}^{k}\tilde{x}(k-j,j)^T \tilde{x}(k-j,j)\right\} \\
&\quad < \gamma^2 \sum_{k=0}^{\infty}\sum_{j=0}^{k} w(k-j,j)^T w(k-j,j) \\
&\quad + \sum_{k=0}^{\infty}\left(E\left\{x^v(k,0)^T P_2 x^v(k,0)\right\}\right. \\
&\qquad \left. + E\left\{x^h(0,k)^T P_1 x^h(0,k)^T\right\}\right).
\end{aligned} \tag{35}
$$

Considering the zero initial boundary conditions, (35) means

$$
\|\tilde{x}\|_E < \gamma^2 \|w\|_2. \tag{36}
$$

This completes the proof. □

Remark 5. Theorem 4 provides a sufficient condition of the mean-square asymptotic stability and H_∞ disturbance attenuation level γ for 2D systems with missing measurements. If the communication links existing between the plant and the controller are perfect, that is, there is no packet dropout during their transmission, then $\alpha = 1$ and $\theta = 0$. In this case, the condition in Theorem 4 becomes the condition obtained in [6] for 2D deterministic system. From this point of view, Theorem 4 can be seen as an extension of [6] to 2D systems with missing measurement.

Theorem 4 gives a mean-square asymptotic stability condition with H_∞ disturbance attenuation level γ for system (10) where the controller gain matrix G is known. However, our eventual purpose is to determine a suitable G by system matrices A, B, B_1 and parameter α.

The following well-known lemma is needed in the proof of our main result.

Lemma 6 (Schur complement). *Assume W, L, V are given matrices with appropriate dimensions, where W and V are positive definite symmetric matrices. Then*

$$
L^T V L - W < 0, \tag{37}
$$

if and only if

$$
\begin{bmatrix} -W & L^T \\ L & -V^{-1} \end{bmatrix} < 0, \tag{38}
$$

or

$$
\begin{bmatrix} -V^{-1} & L \\ L^T & -W \end{bmatrix} < 0. \tag{39}
$$

Based on the above lemma, we can give our main result.

Theorem 7. *For the 2D closed-loop system* (10)*, if there exist a positive define symmetric matrix Q and a matrix M satisfying*

$$
\Omega = \begin{bmatrix} -Q & 0 & QA^T + \alpha M^T B^T & \theta M^T B^T & Q \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & -Q & 0 & 0 \\ & * & & -Q & 0 \\ & & & & -I \end{bmatrix} < 0, \tag{40}
$$

then the 2D closed-loop system (10) *is mean-square asymptotically stability and has an H_∞ disturbance attenuation level γ. In this case, a suitable state feedback control law can be given as $G = MQ^{-1}$.*

Proof. The condition in Theorem 4 can be rewritten as

$$
\Theta^T \begin{bmatrix} P & & \\ & P & \\ & & I \end{bmatrix} \Theta + \begin{bmatrix} -P & \\ & -\gamma^2 I \end{bmatrix} < 0, \tag{41}
$$

where

$$\Theta^T = \begin{bmatrix} (A+\alpha BG)^T & \theta(BG)^T & I \\ B_1^T & 0 & 0 \end{bmatrix}. \tag{42}$$

By applying Lemma 6, condition (41) is equivalent to the following LMI condition:

$$\begin{bmatrix} -P & 0 & (A+\alpha BG)^T & \theta(BG)^T & I \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & -P^{-1} & 0 & 0 \\ & * & & -P^{-1} & 0 \\ & & & & -I \end{bmatrix} < 0. \tag{43}$$

Define $Q = P^{-1}$, and pro- and postmultiplying $\mathrm{diag}(Q, I, I, I, I, I)$ for the above condition give

$$\begin{bmatrix} -Q & 0 & QA^T + \alpha(GQ)^T B^T & \theta(GQ)^T B^T & Q \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & -Q & 0 & 0 \\ & * & & -Q & 0 \\ & & & & -I \end{bmatrix} < 0. \tag{44}$$

Set $GQ = M$ to obtain the LMI of (40) and the proof is complete. □

Remark 8. Theorem 7 provides an LMI condition for the mean-square asymptotic stability and H_∞ disturbance attenuation level γ of 2D stochastic system, which can be solved by LMI Toolbox. Then by (40), we also can give a suitable state feedback control law. It is noted that LMI (40) only contains few elements; hence, the computation is not big. However, when the system matrices get bigger, the solution of LMI (40) may be time-consuming due to complicated computation. In practical system, the system dimension is not very big (often less than 5). Hence, the computation complexity is acceptable.

Remark 9. For a fixed γ, the feasibility of (40) is a suboptimal H_∞ controller. When $\gamma$is not fixed, the minimization of γ that satisfies (40) can be searched. Hence, an optimal H_∞ controller can be obtained by solving the following optimization problem:

$$\begin{aligned} &\min_{Q,M} \quad \gamma^2 \\ &\text{s.t} \quad (40). \end{aligned} \tag{45}$$

4. Robust H_∞ Control for Uncertain 2D Systems

In this section, we extend the above design to the case of robust H_∞ control. Consider the following 2D system with uncertain parameter perturbations

$$\begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} = (A+\Delta A)\begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + (B+\Delta B)\,u(i,j) + (B_1+\Delta B_1)\,w(i,j), \tag{46}$$

where $\Delta A, \Delta B, \Delta B_1$ denote admissible uncertain perturbations of matrices A, B, and B_1, which can be represented as

$$\Delta A = E\Sigma F_1, \qquad \Delta B = E\Sigma F_2, \qquad \Delta B_1 = E\Sigma F_3, \tag{47}$$

where E, F_1, F_2, F_3 are known real constant matrices characterizing the structures of uncertain perturbations and Σ is an uncertain perturbation of the system that satisfies $\Sigma^T\Sigma \le I$.

Lemma 10. *Assume X, Y are matrices or vectors with appropriate dimensions. For any scalar $\varepsilon > 0$ and all matrices Δ with appropriate dimensions satisfying $\Delta\Delta^T \le I$, the following inequality holds:*

$$X\Delta Y + Y^T\Delta^T X^T \le \varepsilon XX^T + \varepsilon^{-1}Y^T Y. \tag{48}$$

The main result of this section is given in the following theorem.

Theorem 11. *For the uncertain 2D system* (46) *used the state feedback control law* (3) *with missing measurement, if there exist a positive define symmetric matrix Q, a matrix M , and scalars $\varepsilon > 0$ satisfying*

$$\begin{bmatrix} -Q & 0 & \Omega_1 & \theta M^T B^T & Q & \Omega_2 & \theta M^T {F_2}^T \\ & -\gamma^2 I & B_1^T & 0 & 0 & F_3^T & 0 \\ & & \varepsilon EE^T - Q & 0 & 0 & 0 & 0 \\ & & & \varepsilon EE^T - Q & 0 & 0 & 0 \\ & & & & -I & 0 & 0 \\ & * & & & & -\varepsilon I & 0 \\ & & & & & & -\varepsilon I \end{bmatrix} < 0, \tag{49}$$

where

$$\Omega_1 = QA^T + \alpha M^T B^T, \qquad \Omega_2 = Q{F_1}^T + \alpha M^T {F_2}^T, \tag{50}$$

then the uncertain 2D system is mean-square asymptotically stability and has an H_∞ disturbance attenuation level γ. In this case, a suitable state feedback control law can be given as $G = MQ^{-1}$.

Proof. By applying Theorem 7, the uncertain 2D system with state feedback control law (3) is mean-square asymptotically stable and has an H_∞ disturbance attenuation level γ, and there exist a positive define symmetric matrix Q, a matrix M satisfying

$$\Omega = \begin{bmatrix} -Q & 0 & Q(A+\Delta A)^T + \alpha M^T (B+\Delta B)^T & \theta M^T (B+\Delta B)^T & Q \\ & -\gamma^2 I & (B_1+\Delta B_1)^T & 0 & 0 \\ & & -Q & 0 & 0 \\ & * & & -Q & 0 \\ & & & & -I \end{bmatrix} < 0. \tag{51}$$

That is,

$$\begin{aligned} &\begin{bmatrix} -Q & 0 & \Omega_1 & \theta M^T B^T & Q \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & -Q & 0 & 0 \\ & * & & -Q & 0 \\ & & & & -I \end{bmatrix} \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix} \Sigma \begin{bmatrix} \Omega_2 \\ F_3^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} \Omega_2 \\ F_3^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma^T \begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix}^T \\ &\quad + \begin{bmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \end{bmatrix} \Sigma \begin{bmatrix} \theta M^T F_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T + \begin{bmatrix} \theta M^T F_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Sigma^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \end{bmatrix}^T \\ &\quad < 0. \end{aligned} \tag{52}$$

Therefore, using Lemma 10, there exists a scalar $\varepsilon > 0$ such that

$$\begin{aligned} &\begin{bmatrix} -Q & 0 & \Omega_1 & \theta M^T B^T & Q \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & -Q & 0 & 0 \\ & * & & -Q & 0 \\ & & & & -I \end{bmatrix} \\ &\quad + \varepsilon \begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ E \\ 0 \\ 0 \end{bmatrix}^T + \varepsilon^{-1} \begin{bmatrix} \Omega_2 \\ F_3^T \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \Omega_2 \\ F_3^T \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\quad + \varepsilon \begin{bmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 0 \\ 0 \\ E \\ 0 \end{bmatrix} + \varepsilon^{-1} \begin{bmatrix} \theta M^T F_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} \theta M^T F_2^T \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\quad < 0. \end{aligned} \tag{53}$$

Note that the above condition is equivalent to

$$\begin{aligned} &\begin{bmatrix} -Q & 0 & \Omega_1 & \theta M^T B^T & Q \\ & -\gamma^2 I & B_1^T & 0 & 0 \\ & & \varepsilon E E^T - Q & 0 & 0 \\ & * & & \varepsilon E E^T - Q & 0 \\ & & & & -I \end{bmatrix} \\ &\quad + \varepsilon^{-1} \begin{bmatrix} \Omega_2 & \theta M^T F_2{}^T \\ F_3^T & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} (\Omega_2)^T & F_3 & 0 & 0 & 0 \\ \theta F_2 M & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\quad < 0. \end{aligned} \tag{54}$$

By the Schur complement, LMI (54) implies LMI in Theorem 11 holds.

This completes the proof. □

In addition, by solving the following optimization problem

$$\begin{aligned} &\min_{Q,M,\varepsilon} \quad \gamma^2 \\ &\text{s.t} \quad (49). \end{aligned} \tag{55}$$

we can obtain a robust optimal H_∞ controller for the uncertain 2D stochastic systems (46).

Remark 12. This paper considers the problem of H_∞ stabilization for a class of 2D systems with missing measurements. Here, we describe the missing measurements as a Bernoulli random binary distribution, which renders the 2D systems to be stochastic ones. Under this framework, we give the definition of stochastic mean-square stability and H_∞ performance, and then a state feedback control design approach is addressed. The proposed design approach is systematic for 2D stochastic system. The results in this paper can be extended to solve other problems such as network-induced delay and H_∞ filter design.

5. Illustrative Examples

In this section, two numerical examples are used to illustrate the effectiveness of the proposed results.

Example 1. Let us consider 2D system (1) with the following parameters:

$$A = \begin{bmatrix} 0.5 & 0.3 \\ 0.1 & 0.7 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.1 & 0.05 \\ 0.2 & 0 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{56}$$

It is assumed that measurements transmitted between the plant and the controller are imperfect; that is, the state signal may be lost during their transmission. Suppose $\alpha = 0.8$; that is, in the communication link, the probability of the data packet missing is 20%. By applying Theorem 7 and solving the optimization problem (45), the minimum H_∞ disturbance attenuation level is $\gamma_{\text{opt}} = 0.32$. Meanwhile, we can obtain

$$Q = \begin{bmatrix} 0.8663 & -0.0405 \\ -0.0405 & 0.8554 \end{bmatrix},$$
$$M = \begin{bmatrix} -0.4429 & -3.7218 \\ -3.7218 & 2.1220 \end{bmatrix}. \tag{57}$$

Hence, a feasible state feedback control law can be selected as

$$G = MQ^{-1} = \begin{bmatrix} -0.7162 & -4.3849 \\ -4.1895 & -2.2824 \end{bmatrix}. \tag{58}$$

Assume the disturbance is

$$w(i, j) = \begin{bmatrix} \dfrac{1}{10ij} \\ \dfrac{1}{10ij} \end{bmatrix}. \tag{59}$$

Simulation results are shown in Figures 1 and 2, where the state response of $x^h(i, j)$ is plotted in Figure 1 and $x^v(i, j)$ is plotted in Figure 2. It can be seen from Figures 1 and 2 that the closed-loop 2D system is asymptotically stable. Hence, even though the 2D system is affected by external disturbances and has significant data dropout in the output measurements, the proposed design approach can guarantee the stability and disturbance attenuation ability for the 2D system.

Example 2. Let us consider uncertain 2D system (46) with the following parameters:

$$A = \begin{bmatrix} 0.8 & 0.021 \\ 0.01 & 0.9 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.03 & 0 \\ 0.1 & 0.1 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \qquad E = \begin{bmatrix} 0.01 & 0.01 \\ 0 & 0.02 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.02 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0.01 & 0.01 \\ 0.03 & 0 \end{bmatrix},$$
$$F_3 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.02 \end{bmatrix}. \tag{60}$$

It is also assumed that the probability of the data packet missing is 0.2; that is, $\alpha = 0.8$. In this case, by applying Theorem 11 and solving the optimization problem (55), the minimum H_∞ disturbance attenuation level is $\gamma_{\text{opt}} = 0.35$. Meanwhile, we can obtain

$$Q = \begin{bmatrix} 0.5125 & -0.2522 \\ -0.2522 & 0.6073 \end{bmatrix},$$
$$M = \begin{bmatrix} -10.4172 & 10.0783 \\ 10.0783 & -16.9824 \end{bmatrix}, \tag{61}$$
$$\varepsilon = 14.4592.$$

Hence, a feasible state feedback control law can be selected as

$$G = MQ^{-1} = \begin{bmatrix} -15.2830 & 10.2485 \\ 7.4205 & -24.8822 \end{bmatrix}. \tag{62}$$

The disturbance $w(i, j)$ is also given as Example 1. Simulation results are shown in Figures 3 and 4, where the state response of $x^h(i, j)$ is plotted in Figure 3 and $x^v(i, j)$ is plotted in Figure 4. It is observed that the closed-loop uncertain 2D system is also asymptotically stable. Hence, the proposed design approach is also effective for uncertain 2D system.

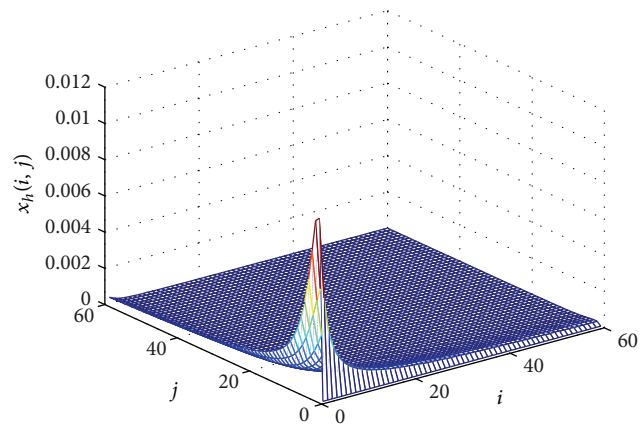

Figure 1: State response of $x^h(i, j)$ for Example 1.

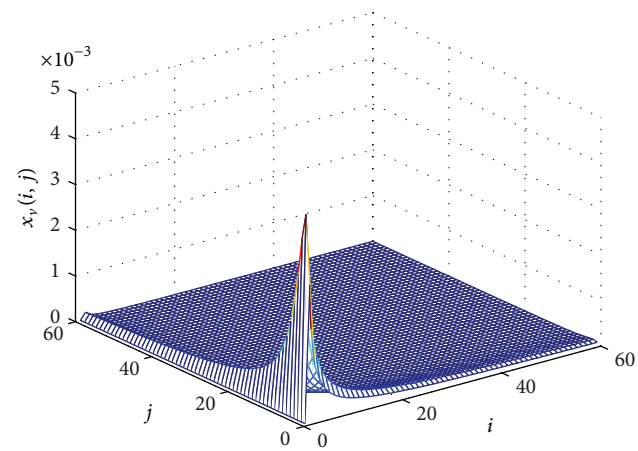

Figure 2: State response of $x^v(i, j)$ for Example 1.

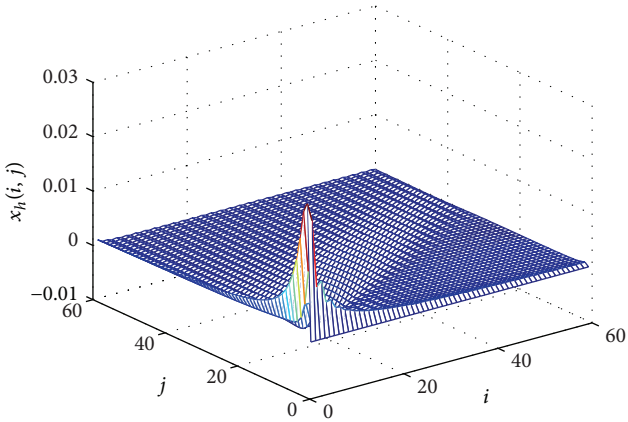

Figure 3: State response of $x^h(i, j)$ for Example 2.

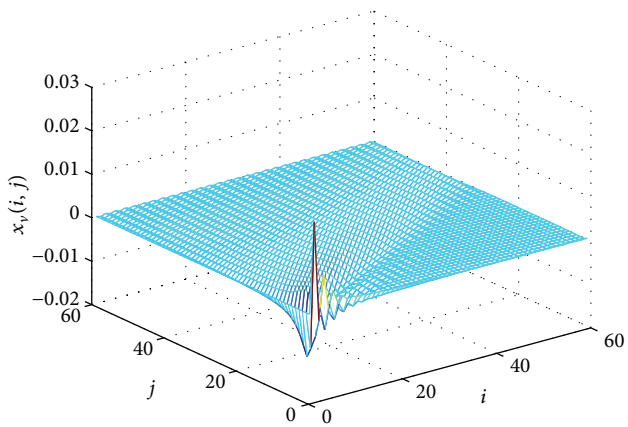

Figure 4: State response of $x^v(i, j)$ for Example 2.

6. Conclusions

In this paper, we have investigated the problem of H_∞ stabilization for a class of 2D systems described with missing measurements. A sufficient condition has been developed in terms of LMIs, which guarantees mean-square asymptotic stability and H_∞ disturbance attenuation level for closed-loop 2D system. The result is also extended to more general cases where the system matrices contain uncertain parameters. Numerical examples have been provided to illustrate the effectiveness of proposed approach. In our future work, the control and filtering problems for networked-based 2D system with packet dropouts and network-induced delay will be discussed.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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