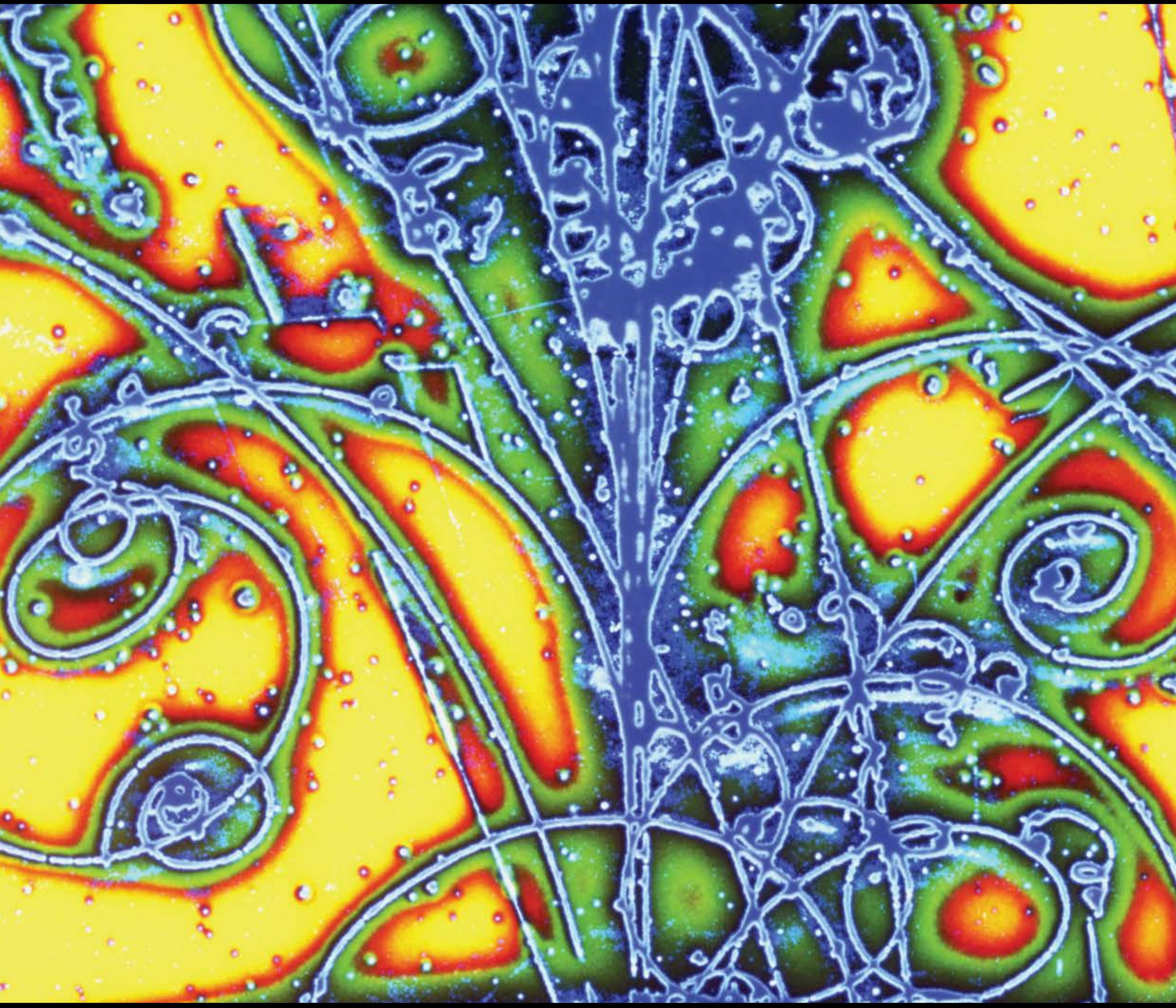
Advances in High Energy Physics

Classical and Quantum Gravity and Its Applications

Guest Editors: Seyed H. Hendi, Christian Corda, S. Habib Mazharimousavi, Davood Momeni, Masoud S. Rad, and Emmanuel N. Saridakis



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Contents

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Editorial

Classical and Quantum Gravity and Its Applications

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After discovery of accelerated expansion of the universe in 1998, understanding its theoretical reasons presents one of the fundamental open questions in physics. This is the so-called Dark Energy issue. Another important issue is the so-called Dark Matter problem, which started in the 30s of the last century. If we observe the Doppler shift of stars moving near the plane of our galaxy and calculate the velocities, we find a large amount of matter inside the galaxy which prevents the stars from escaping. That matter, which is supposed and unknown, generates a very large gravitational force that the luminous mass in the galaxy cannot explain. On one hand, if one wants to explain that large discrepancy, the sum of all the luminous components of the galaxy must be two or three times more massive. On the other hand, if one calculates the tangential velocity of stars in orbits around the galactic center like a function of distance from the center, one sees that stars which are far away from the galactic center move with the same velocity independent of their distance from the center. These strange issues generate a portion of the Dark Matter problem. In fact, either the luminous matter is not able to correctly describe the radial profile of our galaxy or the Newtonian theory of gravitation cannot describe dynamics far from the galactic center. Other issues of the general Dark Matter problem arise from the dynamical description of various self-gravitating astrophysical systems. Examples are stellar clusters, external galaxies, and clusters and groups of galaxies. In those cases, the problem is analogous. There is indeed more matter arising from dynamical analysis with respect to the total luminous matter. Identifying the cause of the Dark Matter and Dark Energy problems is a challenging problem in cosmology. Physicists are interested in considering Dark Matter and Dark Energy in a gravitational background and they proposed some candidates to explain them. Modifying general relativity opens a way to a large class of alternative theories of gravity ranging from higher dimensional physics to non-minimally coupled (scalar) fields. On the other hand, one of the interesting dreams of physicists is finding a consistent quantum theory of gravity. Although there are a lot of attempts to join gravity and quantum theories together, there is no complete description of the quantum gravity. The main idea of promoting general relativity to a quantum level scenario is one of the big challenges of our century. The fact that gravitational collapse is the dominant mechanism in formation of massive objects motivates one to study its various properties. It has been predicted that gravitational collapse of massive objects may lead to the formation of singularities. The recent observational evidences of Laser Interferometer Gravitational-Wave Observatory (LIGO) confirm not only the existence of gravitational waves but also the life of black holes. From the other side, an interesting topic in multidisciplinary branches of theoretical physics is to study relation between certain types of gravity models and quantum systems, called gauge/gravity duality or AdS/CFT. It is believed that this approach is able to explain all quantum phase transitions of systems using a unique and well-defined dictionary. For example, we can compute the entanglement entropy of a many-body-quantum system using the solutions of gravitational action in a higher dimensional asymptotically AdS space time. Furthermore, quantum information metric or fidelity and other condensed matter phenomena could be explained by this purely geometric approach using the classical black hole solutions. Different experimental data in labs supported this idea to treat strongly correlated quantum systems using the gravitational sector of a weakly coupled system.

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Review Article

Mimetic Gravity: A Review of Recent Developments and Applications to Cosmology and Astrophysics

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Mimetic gravity is a Weyl-symmetric extension of General Relativity, related to the latter by a singular disformal transformation, wherein the appearance of a dust-like perfect fluid can mimic cold dark matter at a cosmological level. Within this framework, it is possible to provide a unified geometrical explanation for dark matter, the late-time acceleration, and inflation, making it a very attractive theory. In this review, we summarize the main aspects of mimetic gravity, as well as extensions of the minimal formulation of the model. We devote particular focus to the reconstruction technique, which allows the realization of any desired expansionary history of the universe by an accurate choice of potential or other functions defined within the theory (as in the case of mimetic $f(R)$ gravity). We briefly discuss cosmological perturbation theory within mimetic gravity. As a case study within which we apply the concepts previously discussed, we study a mimetic Hořava-like theory, of which we explore solutions and cosmological perturbations in detail. Finally, we conclude the review by discussing static spherically symmetric solutions within mimetic gravity and apply our findings to the problem of galactic rotation curves. Our review provides an introduction to mimetic gravity, as well as a concise but self-contained summary of recent findings, progress, open questions, and outlooks on future research directions.

1. Introduction

The past decade has seen the astounding confirmation of the "dark universe" picture, wherein the energy budget of our universe is dominated by two dark components: dark matter and dark energy [1–50]. The race to determine the nature and origin of these components is in progress on both the observational and theoretical fronts. Theories of modified gravity appear quite promising in this respect, particularly given that gravity remains the least understood of the four fundamental forces. For an incomplete list of comprehensive reviews, as well as seminal works on the subject, we refer the reader to [51–68] and references therein.

A particularly interesting theory of modified gravity which has emerged in the past few years is mimetic gravity [69]. In mimetic gravity, as well as minimal modifications thereof, it is possible to describe the dark components of the universe as a purely geometrical effect, without the need of introducing additional matter fields. In the past three years, interest in this theory has grown rapidly, with over 90 papers following up on the original idea or at least touching on it in some way [70–161]. For this reason, we believe it is timely to present a review on the progress achieved thus far in the field of mimetic gravity. This review is not intended to provide a detailed pedagogical introduction to mimetic gravity but rather summarize the main findings and directions in current research on the theory, as well as providing useful directions to the reader, should she/he wish to deepen a given topic in mimetic gravity. For this reason, this review should not be seen as a complete introduction to mimetic gravity, nor should it substitute consultation of the original papers. No prior knowledge on the subject is assumed.

This review is structured as follows: in this section we will provide a historical and technical introduction to modified gravity, which shall justify our subsequent endeavour in mimetic gravity. In Section 2, we will introduce mimetic gravity. Given that understanding the reason behind the equations of motion of mimetic gravity differing from those of General Relativity is not obvious, a major goal of the section will be to provide a clear and concise explanation for this fact. Subsequently, in Section 3 we will explore a few solutions of mimetic gravity and correspondingly some extensions of the theory, as well as correspondences with related theories of modified gravity. Section 4 will provide a brief interlude focusing on perturbations in mimetic gravity. In Section 5 we will present a case study of a mimetic-like model, namely, mimetic covariant Hořava-like gravity, with a focus on its solutions and cosmological perturbations, and the need to extend the model beyond its basic formulation. Section 6 will be devoted to studying spherically symmetric solutions in mimetic gravity. In Section 7 we will touch upon the issue of rotation curves in mimetic gravity and how this issue is addressed. Finally, we will conclude in Section 8.

1.1. Why Modify General Relativity? General Relativity (GR henceforth), first formulated by Einstein in 1915 [162–166] (for a pedagogical review, see, e.g., [167]), is an extremely successful and predictive theory, and together with Quantum Field Theory forms one of the pillars of modern physics. The traditional picture of GR is a geometrical one, with the theory being one of space-time and its metric. A more modern view is free of geometrical concepts and sees GR as the unique theory of massless spin-2 particles.

Confirmations of GR abound (see, e.g., [168] for a complete review), ranging from gravitational lensing [169] to the precession of Mercury's orbit [165]. Shortly after its centennial in 2015 one of the pillars of GR, the existence of gravitational waves, was grandiosely confirmed by the detection of GW150914 [170] and GW151226 [171] by LIGO (see also [172]). Before we even embark on a review of mimetic gravity, then, it is worth reminding the reader why one should even consider questioning a theory as successful as GR. Aside from the philosophical perspective that questioning theories and exploring other approaches are a sensible route in science, provided of course that there is agreement with observations, hints persist in the literature that complicating the gravitational action may indeed have its merits. In fact, the reader should be reminded that as early as 1919 (four years after GR had been formulated), proposals started to be put forward as to how to extend this theory, notably, in the form of Weyl's scale independent theory [173] and Eddington's theory of connections [174]. These early attempts to modify GR were driven solely by scientific curiosity with no formal theoretical, or let alone experimental, motivation.

Nonetheless, theoretical motivation for modifying the gravitational action came quite soon. The underlying reason is that GR is nonrenormalizable and thus not quantizable in the way conventional Quantum Field Theories are quantized. However, it was proven that 1-loop renormalization requires the addition of higher order curvature terms to the Einstein-Hilbert action. In fact, it was later demonstrated that, while actions constructed from invariants quadratic in curvature are renormalizable [175], the addition of higher order time derivatives which follows from the addition of terms higher order in the curvature leads to the appearance of ghost degrees of freedom, which entail a loss of unitarity. More recent results show that when quantum corrections or string theory are taken into account, the effective low-energy gravitational action admits higher order curvature invariants (see, e.g., [176] for a general review). However, all these early attempts to modify GR had a common denominator in the fact that terms which modified the gravitational action were only considered to be relevant in proximity of the Planck scale, thus not affecting the late universe.

With the emergence of the "dark universe" picture in recent years, the limits of GR have been fully exposed, and further motivations to modify this theory have emerged. A series of experiments and surveys, including but not limited to CMB experiments, galaxy redshift surveys, cluster surveys, supernovae surveys, lensing experiments, and quasar surveys, have depicted a peculiar picture of our universe [1–50]. This scenario suggests that our näive picture of the world we live in being described by the Standard Model of Particle Physics (SM) supplemented by General Relativity is, at best, incomplete. The concordance cosmology model suggests a scenario where only ~4% of the energy budget of the universe consists of baryonic matter, whereas ~24% consists of nonbaryonic dark matter (DM), and the remaining ~76% consists of dark energy (DE). Of the last two extra components, dark matter is (presumably) the one with properties most similar to ordinary matter. It shares the clustering properties of ordinary matter, but not its couplings to SM gauge bosons (e.g., electromagnetic ones), and is believed to be responsible for structure formation during the matter domination era of the cosmological history. As ordinary matter, DM satisfies the strong energy condition. Dark energy, instead, is more peculiar still, given that it does not share the clustering properties of ordinary matter or DM, as it violates the strong energy condition. It is believed to be responsible for the late-time speed-up of the universe, which has been inferred from a variety of cosmological and astrophysical observations, ranging from type Ia-supernovae to the CMB. Whereas evidence for DE is somewhat indirect and exclusively of cosmological origin, clues as to the existence of DM are instead present on a wide variety of scales, from cosmological to astrophysical (galactic and subgalactic) ones. For technical reviews on DM, see, for instance, [177–181]; for similar reviews on DE, see, for instance, [182–186].

The late-time acceleration of our universe, however, is most likely not the only period of accelerated expansion that our universe has experienced. A period of accelerated (exponential) expansion during the very early universe, prior to the conventional radiation and matter domination epochs, is required to solve the horizon, flatness, and monopole problems. This period of accelerated expansion is known as inflation (see, e.g., [187–196] for pioneering work), and a vast class of models attempting to reproduce such period exists in the literature (for an incomplete list, see, e.g., [197–206] and references therein, see also, e.g., [207–231] for more recent inflationary model-building which is extremely closely relevant to mimetic gravity and variations of it). For reviews on inflation, see, for instance, [232–236]. Inflation also purports to be the mechanism generating primordial inhomogeneities which are quantum in origin [237, 238], providing the seeds which grow under gravitational instability to form the large-scale structure of the universe. The fact that the universe presumably undergoes acceleration at both early and late times or, equivalently, at high and low curvature is very puzzling and might be hinting to a more profound structure.

It thus appears that concordance cosmology requires at least three extra (possibly dark) cosmological components: one or more dark matter components, some form of dark energy, and one or more inflaton fields. There is no shortage of ideas as to what might be the nature of each of these components. Nonetheless, adding these three or more components opens another set of questions, which include but are not limited to the compatibility with the current SM and the consistency of formulation. On the other hand, gravity is the least understood of the four fundamental interactions and the most relevant one on cosmological and astrophysical scales. If so, it could be that our understanding of gravity on these scales is inadequate or incomplete, and modifying our theory of gravitation could indeed be the answer to the dark components of the universe. One could argue that this solution is indeed more economical and possibly the one to pursue in the spirit of Occam's razor. In other words, modifications to Einstein's theory of General Relativity might provide a consistent description of early and late-time acceleration and of the dark matter which appears to pervade the universe. Modified theories of gravity not only can provide a solution to the "dark universe riddle" but also possess a number of alluring features such as unification of the various epochs of acceleration and deceleration (matter domination) of the universe's evolution, transition from nonphantom to phantom phase being transient (and thus without Big Rip), solution to the coincidence problem, and also interesting connections to string theory.

Having presented some motivation to modify our theory of gravitation, we now proceed to briefly discuss systematic ways by means of which this purpose can be achieved.

1.2. How to Modify General Relativity? Essentially all attempts to modify General Relativity are guided by Lovelock's theorem [239]. Lovelock's theorem states that the only possible second-order Euler-Lagrange expression obtainable in 4D space from a scalar Lagrangian density of the form $\mathscr{L} = \mathscr{L}(g_{\mu\nu})$, where $g_{\mu\nu}$ is the metric tensor, is given by the following:

$$E^{\mu\nu} = \beta\sqrt{-g}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right) + \kappa\sqrt{-g}g^{\mu\nu}, \tag{1}$$

where β and κ are constants and $R^{\mu\nu}$ and R are the Ricci tensor and scalar, respectively. It follows that constructing metric theories of gravity whose equations differ from those of GR requires at least one of the following to be satisfied:

(i) Presence of other fields apart from or in lieu of the metric tensor

(ii) Work in a number of dimensions different from 4

(iii) Accept metric derivatives of degree higher than 2 in the field equations

(iv) Giving up locality or Lorentz invariance

Therefore, we can imagine broadly classifying the plethora of modified gravity theories according to which of the above assumptions is broken.

1.2.1. New Degrees of Freedom. Relaxing the first assumption leads to what is probably the largest class of modified gravity theories. Theories corresponding to the addition of *scalar* degrees of freedom include quintessence (e.g., [240–242]) and coupled quintessence (e.g., [243]) theories, the Chern-Simons theory (e.g., [244]), Cuscuton cosmology (e.g., [245–247]), Chaplygin gases (e.g., [248, 249]), torsion theories such as $f(T)$ theories (e.g., [250–253], see also, e.g., [254–264] for recent work) or the Einstein-Cartan-Sciama-Kibble theory (e.g., [265–268]), scalar-tensor theories (e.g., [269]) such as the Brans-Dicke theory [270], ghost condensates (e.g., [271]), galileons (e.g., [272, 273]), KGB [274], Horndeski's theory [275], and many others.

One can instead choose to add *vector* degrees of freedom, as in the case of the Einstein-aether theory (e.g., [276–280]). The addition of a vector field leads to the introduction of a preferred direction in space-time, which entails breaking Lorentz invariance.

Theories where *tensor* degrees of freedom are added include Eddington-Born-Infeld gravity (e.g., [281–284]) and bimetric MOND (e.g., [285, 286]) among others. TeVeS (tensor vector scalar gravity, [287]) is instead a theory which features the addition of all three types of degrees of freedom together.

Broadly speaking, mimetic gravity belongs to the class of theories of modified gravity where an additional scalar degree of freedom is added. Caution is needed with this identification though because, as we shall see later, mimetic gravity does not possess a proper scalar degree of freedom, but rather a constrained one.

1.2.2. Extra Dimensions. Relaxing the second assumption instead brings us to consider models with extra dimensions, the prototype of which is constituted by Kaluza-Klein models (e.g., [288–290]). Models of modified gravity with extra dimensions abound when considering string theory, including Randall-Sundrum I [291] and II [292] models, Einstein-Dilaton-Gauss-Bonnet gravity (e.g., [293, 294]), cascading gravity (e.g., [295–298]), and Dvali-Gabadadze-Porrati gravity (e.g., [299, 300]). Another interesting theory that falls within the extra dimension category is represented by 2T gravity [301].

1.2.3. Higher Order. The most famous and studied example of a theory falling within this category is undoubtedly represented by $f(R)$ gravity ([302], see also, e.g., [60, 65–67, 303–309] or [310] for black holes phenomenology). In fact, unification of inflation and late-time acceleration was proposed in the context of $f(R)$ gravity in [303, 305, 311–313]. Belonging to the same family are also variations of the former such as $f(R_{\mu\nu}R^{\mu\nu})$, $f(\Box R)$, $f(R,T)$, $f(R,T,R^{\mu\nu}T_{\mu\nu})$ gravity (see, e.g., [132, 314–323]), but also Gauss-Bonnet (see, e.g., [324–334]) and conformal gravity (see, e.g., [335–337]). Another well-known theory which lies within this family is represented by Hořava-Lifshitz gravity ([338], see also [339–363], which also violates Lorentz invariance explicitly), and correspondingly the vast category of Hořava-like theories, including those which break Lorentz invariance dynamically (e.g., [356, 361, 362, 364–372]).

1.2.4. Nonlocal. If we choose to relax the assumption of locality (we have already seen cases where the assumption of Lorentz invariance is relaxed above), we can consider nonlocal gravity models whose action contains the inverse of differential operators of curvature invariants, such as $f(R/\Box)$ and $f(R_{\mu\nu}\Box^{-1}R^{\mu\nu})$ gravity (e.g., [373–377]). Some degravitation scenarios belong to this family as well (e.g., [378, 379]).

Broadly speaking, mimetic gravity belongs to the class of theories of modified gravity where an additional scalar degree of freedom is added. Caution is needed with this identification though because mimetic gravity does not possess a proper scalar degree of freedom. Instead, the would-be scalar degree of freedom is constrained by a Lagrange multiplier, which kills all higher derivatives. As such, the mimetic field cannot have oscillating solutions and the sound speed satisfies $c_s = 0$, confirming that there is no propagation of scalar degrees of freedom (however, this is true only in the original mimetic model but does not necessarily hold in extensions thereof). Furthermore, the same Lagrange multiplier term introduced a preferred foliation of space-time, which breaks Lorentz invariance (although this is preserved at the level of the action). These aspects of mimetic gravity will be discussed in more detail in the following section.

2. Mimetic Gravity

The expression "mimetic dark matter" was first coined in a 2013 paper by Chamseddine and Mukhanov [69] although, as we shall see later, the foundation for mimetic theories had actually been developed a few years earlier in three independent papers [380–382]. In [69], the proposed idea is to isolate the conformal degree of freedom of gravity by introducing a parametrization of the physical metric $g_{\mu\nu}$ in terms of an auxiliary metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ, dubbed mimetic field, as follows:

$$g_{\mu\nu} = -\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi. \tag{2}$$

From (2) it is clear that, in such a way, the physical metric is invariant under conformal transformations of the auxiliary metric of the type $\tilde{g}_{\mu\nu} \rightarrow \Omega(t,\mathbf{x})^2\tilde{g}_{\mu\nu}$, $\Omega(t,\mathbf{x})$ being a function of the space-time coordinates. It is also clear that, as a consistency condition, the mimetic field must satisfy the following constraint:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi = -1. \tag{3}$$

Thus, the gravitational action, taking into account the reparametrization given by (2) now takes the form:

$$I = \frac{1}{2}\int_{\mathscr{M}} d^4x\sqrt{-g\left(\tilde{g}_{\mu\nu},\phi\right)}\left[R\left(\tilde{g}_{\mu\nu},\phi\right) + 2\mathscr{L}_m\right], \tag{4}$$

where $\mathscr{M}$ is the space-time manifold, $R \equiv R(g_{\mu\nu},\phi)$ is the Ricci scalar, $\mathscr{L}_m$ is the matter Lagrangian, and $g \equiv g(\tilde{g}_{\mu\nu},\phi)$ is the determinant of the physical metric.

By varying the action with respect to the physical metric one obtains the equations for the gravitational field. However, this process must be done with care, for the (variation of the) physical metric can be written in terms of the (variation of the) auxiliary metric and the (variation of the) mimetic field. Taking this dependency into account, variation of the action with respect to the physical metric yields [69]

$$G_{\mu\nu} - T_{\mu\nu} + (G-T)\,\partial_\mu\phi\partial_\nu\phi = 0, \tag{5}$$

where $G_{\mu\nu} = R_{\mu\nu} - g_{\mu\nu}R/2$ is the Einstein tensor, with $R_{\mu\nu}$ being the Ricci tensor, while $G\ (= -R)$ and T are the trace of Einstein's tensor and the stress-energy tensor of matter, respectively. Notice that the auxiliary metric does not enter the gravitational field equation explicitly, but only through the physical metric, whereas the mimetic field enters explicitly. In fact, the mimetic field contributes to the right-hand side of Einstein's equation through an additional stress-energy tensor component:

$$\tilde{T}_{\mu\nu} = -(G-T)\,\partial_\mu\phi\partial_\nu\phi. \tag{6}$$

We note that both energy-momentum tensors, $T_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$, are covariantly conserved, that is, $\nabla^\mu T_{\mu\nu} = \nabla^\mu\tilde{T}_{\mu\nu} = 0$ (with ∇^μ the covariant derivative), whereas the continuity equation for $\tilde{T}_{\mu\nu}$ with the mimetic constraint (3) leads to

$$\begin{aligned}\nabla^\kappa\left((G-T)\,\partial_\kappa\phi\right) &\equiv \frac{1}{\sqrt{-g}}\partial_\kappa\left(\sqrt{-g}\,(G-T)\,g^{\kappa\sigma}\partial_\sigma\phi\right)\\ &= 0.\end{aligned} \tag{7}$$

Finally, the trace of (5) is found to be

$$(G-T)\left(1 + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right) = 0. \tag{8}$$

It is clear that the above is automatically satisfied if one takes into account (3) even if $G \neq T$. Thus, the theory admits nontrivial solutions and the conformal degree of freedom becomes dynamical even in the absence of matter ($T = 0$ but $G \neq 0$) [69].

Let us examine the structure of the mimetic stress-energy tensor. Recall that the stress-energy tensor of a perfect fluid whose energy density is ρ and pressure p is given by

$$\begin{aligned}T_{\mu\nu} &= (\rho + p)\,u_\mu u_\nu + pg_{\mu\nu},\\ u_\mu u^\mu &= -1.\end{aligned} \tag{9}$$

Notice that the mimetic stress-energy tensor in (6) assumes the same form of the one of a perfect fluid with pressure $p = 0$ and energy density $\rho = -(G-T)$, while the gradient of the mimetic field, $\partial_\mu\phi$, plays the role of 4-velocity. Thus, the mimetic fluid is pressureless, suggesting it could play the role of dust in a cosmological setting. To confirm whether the mimetic fluid can indeed play the role of dust, it is necessary to investigate cosmological solutions. In fact, this is easy to do on a Friedmann-Lemaître-Robertson-Walker (FLRW) setting, with a metric of the form:

$$ds^2 = -dt^2 + a(t)^2 \, d\mathbf{x}^2, \tag{10}$$

where $a \equiv a(t)$ is the scale factor. If we take the hypersurfaces of constant time to be equal to those of constant ϕ, we immediately see that the constraint equation (3) is automatically satisfied if the mimetic field is identified with time up to an integration constant (which we arbitrarily set to 0). Thus, the mimetic field plays the role of "clock" on an FLRW background. It is then easy to show that (7) implies that $(G - T)$, which corresponds to the energy density of the mimetic stress-energy tensor, decays with the scale factor of the FLRW universe as $(G - T) \propto 1/a^3$. Recall that the energy density of a component with equation of constant state parameter w evolves as $a^{-3(w+1)}$ in an FLRW universe, and hence the evolution in the energy density of the mimetic field corresponds to $w = 0$, namely, the equation of state for dust. In other words, the conformal degree of freedom of gravity can mimic the behaviour of dark matter at a cosmological level, hence the name "mimetic dark matter" [69].

Lagrange Multiplier Formulation. Before further discussing some fundamental aspects of mimetic dark matter (or mimetic gravity henceforth), such as the reason behind the different solutions from GR despite the seemingly innocuous parametrization given by (2), let us discuss an alternative but equivalent formulation of mimetic gravity. The equations of motion obtained from the action written in terms of the auxiliary metric $\tilde{g}$ are equivalent to those one would conventionally obtain from the action expressed in terms of the physical metric with the imposition of an additional constraint on the mimetic field. This suggests that the mimetic constraint given by (3) can actually be implemented at the level of the action by using a Lagrange multiplier. That is, the action for mimetic gravity (4) can be written as

$$I = \frac{1}{2} \int_{\mathscr{M}} d^4x \sqrt{-g} \left[R + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) + 2\mathscr{L}_m \right]. \tag{11}$$

Variation of the Lagrangian with respect to the Lagrange multiplier field λ leads to (3), while variation with respect to the physical metric $g_{\mu\nu}$ yields

$$G_{\mu\nu} - T_{\mu\nu} + \lambda \partial_\mu \phi \partial_\nu \phi = 0, \tag{12}$$

whose trace, when one takes into account (3), is given by

$$\lambda = (G - T). \tag{13}$$

Thus, one recovers (5) again. In this review, we will always make use of the action given by (11) to introduce the mimetic field.

A remark is in order here. Actions such as (11) had actually been introduced three years before the term "mimetic dark matter" was first coined. Three independent papers in 2010, by Lim et al. [380], Gao et al. [381], and Capozziello et al. [382], respectively, presented models with two additional scalar fields, one of which playing the role of Lagrange multiplier enforcing a constraint on the derivative of the other (see also [383] for recent work on the role of Lagrange multiplier constrained terms in cosmology). In fact, it was shown that these types of models can produce a unified theory describing dark energy and dark matter, because the term inside the Lagrange multiplier can always be arranged in such a way to reproduce the conventional expansion history of ΛCDM. Thus, it is fair to state that mimetic theories had actually seen birth prior to the 2013 paper by Chamseddine and Mukhanov.

2.1. Understanding Mimetic Gravity. Before we can make further progress in exploring solutions in mimetic gravity, generalizing the theory, or studying connections to other theories, we need to touch on two very important points: first, why the seemingly innocuous parametrization given by (5) has led to a completely new class of solutions not contemplated by GR, and second, whether the theory is stable or not. As we shall see, the first point can ultimately be explained in terms of singular disformal transformations.

It might appear puzzling at first that, only by rearranging parts of the metric, one is faced with a different model altogether. A first explanation appeared in [70], which explained this property in terms of variation of the action taking place over a restricted class of functions. This is the case in mimetic gravity, precisely because the consistency equation (3) enforces an additional condition for any admissible variation of the action, in particular demanding that

$$\int_{t_{\text{in}}}^{t_{\text{fin}}} dt \sqrt{\Omega(x)} = \left(t_{\text{fin}} - t_{\text{in}} \right), \quad \Omega(x) \equiv \tilde{g}^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi. \tag{14}$$

Thus, for a spatially homogeneous mimetic field $\phi = t$, $\dot{\phi} = \sqrt{\Omega}$. Varying over a restricted class of functions now provides less conditions for the stationarity of the action and hence more freedom in the dynamics. This is a well-known property relevant when one makes derivative substitutions $x \equiv f(\dot{y})$ into an action $I(x, t)$: the class of functions over which the variation is admissible does not only comprise those for which the variation of x is vanishing at the boundary but also those where the integral of the variation of x is zero. This extra restriction leads to less conditions for stationarity of the action and the appearance of additional dynamics with respect to the original case.

Another explanation was presented in [71], which identified mimetic gravity as a conformal (Weyl-symmetric) extension of GR. The first important point to notice is that the parametrization $g^{\mu\nu} = g^{\mu\nu}(\tilde{g}^{\mu\nu}, \phi)$ is noninvertible even for fixed ϕ, owing to the fact that the map $g \to \tilde{g}, \phi$ is a map from ten variables to eleven. With this parametrization, the theory is manifestly conformally invariant, that is, invariant with respect to transformations of the auxiliary metric (and correspondingly the action I) of the form:

$$\begin{gathered} \partial_\alpha \tilde{g}_{\mu\nu}(x) = \alpha(x) \, g_{\mu\nu}(x), \\ \partial_\alpha I \left[g_{\mu\nu} \left(\tilde{g}_{\mu\nu}, \phi \right) \right] = 0, \end{gathered} \tag{15}$$

where $\alpha(x)$ is a function of the space-time coordinates. Two immediate corollaries of the theory's conformal invariance are its yielding of identically traceless equations of motion for the gravitational field and requiring conformal gauge fixing. In fact, recall that the equation of motion for the gravitational field (5) is automatically traceless if the consistency condition given by (3) is satisfied. Therefore, we can identify (3) with the conformal gauge condition in the locally gauge-invariant theory with action $I[g_{\mu\nu}(\tilde{g}_{\mu\nu},\phi)]$. Mimetic gravity can thus be seen as a conformal extension of GR which is Weyl-invariant in terms of the auxiliary metric $\tilde{g}_{\mu\nu}$. However, this theory is quite different from the off-shell conformal extensions of GR proposed in [384] (which preserves GR on-shell but modifies its effective action off-shell); here the gravitational action is already modified at a classical level, by adding an extra degree of freedom provided by a collisionless perfect fluid. This additional degree of freedom, which can mimic collisionless cold dark matter for cosmological purposes, arises from gauging out local Weyl invariance.

2.1.1. Singular Disformal Transformations. As we anticipated above, mimetic gravity and the appearance of the extra degree of freedom which can mimic cosmological dark matter are rooted into the role played by singular disformal transformations. As was shown by Bekenstein [385], because GR enjoys invariance under diffeomorphisms, one is free to parametrize a metric $g_{\mu\nu}$ in terms of a fiducial metric $\tilde{g}_{\mu\nu}$ and a scalar field ϕ. The map between the two is defined as a "disformal transformation," or a "disformation," and takes the following form:

$$g_{\mu\nu}=\mathscr{A}(\phi,X)\,\tilde{g}_{\mu\nu}+\mathscr{B}(\phi,X)\,\partial_\mu\phi\partial_\nu\phi, \qquad (16)$$

where $X\equiv\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$. The functions $\mathscr{A}$ and $\mathscr{B}$ are referred to as conformal factor and disformal factor, respectively. In general the functions $\mathscr{A}(\phi,X)$, $\mathscr{B}(\phi,X)$ are arbitrary, with $\mathscr{A}\neq 0$. It is easy to show that, provided the transformation is invertible, the equations of motion for the theory (obtained by variation of the action with respect to $\tilde{g}_{\mu\nu}$ and ϕ) reduce to those of obtained by varying with respect to the metric $g_{\mu\nu}$ [79].

To make progress, it is useful to contract the two equations of motion with $\tilde{g}_{\mu\nu}$ and $\partial_\mu\phi\partial_\nu\phi$. Although we will not perform the steps explicitly, it is easy to show that this leads to the following two equations of motion:

$$\begin{gathered}\Omega\left(\mathscr{A}-X\frac{\partial\mathscr{A}}{\partial X}\right)-\omega X\frac{\partial\mathscr{B}}{\partial X}=0,\\ \Omega X^2\frac{\partial\mathscr{A}}{\partial X}-\omega\left(\mathscr{A}-X^2\frac{\partial\mathscr{B}}{\partial X}\right)=0,\end{gathered} \qquad (17)$$

where the two quantities Ω and ω are defined as

$$\begin{gathered}\Omega\equiv\left(G^{\mu\nu}-T^{\mu\nu}\right)\tilde{g}_{\mu\nu},\\ \omega\equiv\left(G^{\mu\nu}-T^{\mu\nu}\right)\partial_\mu\phi\partial_\nu\phi.\end{gathered} \qquad (18)$$

The determinant of system (17) is given by

$$\det=X^2\mathscr{A}\frac{\partial}{\partial X}\left(\mathscr{B}+\frac{\mathscr{A}}{X}\right). \qquad (19)$$

If the above is nonzero, it is trivial to obtain that the resulting set of equations consists of Einstein's equation ($G_{\mu\nu}=T_{\mu\nu}$) and a second empty equation. Therefore, the theory does not feature new solutions with respect to GR [79].

The situation is quite different if the determinant in (19) is zero, which corresponds to the physical case when the disformal transformation given by (16) is noninvertible or singular. In this case, being the function $\mathscr{A}\neq 0$, this immediately determines $\mathscr{B}$, which is of the form:

$$\mathscr{B}(X,\phi)=-\frac{\mathscr{A}(X,\phi)}{X}+\mathscr{E}(\phi), \qquad (20)$$

where $\mathscr{E}(\phi)\neq 0$ is an arbitrary function. We will not show the steps explicitly, which are instead discussed in detail in Section IV of [79], but it is not hard to obtain the equations of motion and notice that they differ from those of GR, due to the presence of an extra term on the right-hand side of Einstein's equation (i.e., an additional contribution to the stress-energy tensor). Therefore, when the disformal transformation is singular, one is faced with the appearance of extra degrees of freedom which result in equations of motion differing from those of GR [79].

The parametrization (5) defining mimetic gravity can be identified with a singular disformal transformation, with $\mathscr{A}=X$ and $\mathscr{B}=0$ in (16), and correspondingly $\mathscr{E}=1$ in (20). In general, when the relation defined by (20) exists between the conformal factor $\mathscr{A}$ and the disformal factor $\mathscr{B}$, the resulting disformal transformation is singular and, as a result, the system possesses additional degrees of freedom, explaining the origin of the extra degree of freedom in mimetic gravity which mimics a dust component. This aspect has been at the center of a number of studies recently; see, for instance, [79, 90, 96, 102, 104, 107, 128, 140, 142, 386–389]. Moreover, [104] has shown that the two approaches towards mimetic gravity (and further extensions to be discussed later, such as mimetic Horndeski theories), namely, Lagrange multiplier (11) and singular disformal transformation (16), are in fact equivalent.

2.1.2. Mimetic Gravity from the Brans-Dicke Theory. There actually exists a third route to mimetic gravity, apart from disformal transformations and Lagrange multiplier, whose starting point is the singular Brans-Dicke theory. Namely, by starting from the action (4) (neglecting matter terms) and by performing the conformal transformation given by (5), we end up with the action [129]:

$$I(\tilde{g},\phi)=\int d^4x\sqrt{-\tilde{g}}\left(XR(\tilde{g})+\frac{3}{2}\frac{\tilde{g}^{\mu\nu}\partial_\mu X\partial_\nu X}{X}\right), \qquad (21)$$

where we have defined $X\equiv\tilde{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi/2$. One immediately sees that (21) corresponds to the singular/conformal Brans-Dicke action [270] with density parameter $\omega=-3/2$. Thus, we conclude that a third way of obtaining mimetic gravity is by substituting the kinetic term in lieu of the scalar field in the singular Brans-Dicke action [129]. In case matter fields are included, the substitution has to be performed on the matter part of the Lagrangian as well, which means that matter will not be coupled to $\tilde{g}_{\mu\nu}$ but to $2X\tilde{g}_{\mu\nu}$ [129].

2.2. Stability. Is mimetic gravity stable? In other words, does its spectrum contemplate the presence of states with negative norm, or fields whose kinetic term has the wrong sign (corresponding to negative energy states), which could possibly destabilize the theory? This is an important question which has yet to find a definitive answer. Recall that the original mimetic theories formulated in 2010 were found to suffer from a tachyonic instability [390]; therefore, the question of whether mimetic gravity is stable is totally pertinent.

If we formulate the theory of mimetic gravity using the physical metric $g_{\mu\nu}$, we inevitably incur into the risk of the appearance of higher derivatives of the mimetic field, which could entail the emergence of ghosts. Addressing this question requires performing a Hamiltonian analysis of the theory, identifying all constraints and counting the local degrees of freedom. A preliminary analysis of this problem was conducted in [71], which concluded that the theory is stable if the energy density of the mimetic field $\epsilon = T - G = T + R$ is positive. This condition is, of course, easy to understand physically. Moreover, it indicates a preference for de Sitter-type backgrounds with a positive cosmological constant, since in that case both contributions to the energy density, given by curvature and trace of the matter stress-energy tensor, are positive. Therefore, it is presumed that mimetic is gravity is stable provided the time evolution of the system preserves the positivity of the energy density stored in the mimetic degree of freedom. The work of [71] also identified another possible instability issue, namely, caustic singularities (which are not dangerous at the quantum level, unlike ghost instabilities). These are presumably due to the pressureless nature of the mimetic field and can possibly be circumvented if one modifies the theory with higher derivative terms, as we will discuss later.

The analysis of [71] imposed the conformal gauge condition (3) prior to proceeding to the canonical formulation. A proper analysis should instead take place in full generality and has been conducted in [75]. This analysis finds that the Hamiltonian constraint depends linearly on the momentum, which in most cases signals that the Hamiltonian density of the theory is unbounded from below, a classical sign of instability. As anticipated, this occurs frequently for higher derivative theories, which are prone to the Ostrogradski instability. The work in [75] concluded, as [71], that mimetic gravity is stable as long as the energy density in the dust degree of freedom in the theory remains positive. However, this is not always consistent with the dynamics of the theory, given that for some initial configurations, the energy density could start its evolution with a positive value but then end up with a negative value, which would cause instability. In fact, [75] finds that the requirement that the theory be stable correspond to the requirement that initial configurations do not cross the surface for which the momentum conjugate to the mimetic field, p_ϕ, satisfies $p_\phi = 0$.

A possible solution to these instability issues was presented in [71] and studied in [75]. The idea is to modify the parametrization (5) by making use, instead of the gradient of a scalar field, of a dynamical vector (Proca) field:

$$g_{\mu\nu} = -\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}u_\alpha u_\beta. \tag{22}$$

The Proca field is made dynamical by adding a Maxwell kinetic term F^2 to the action, where F is the field-strength tensor of the vector field. It is beyond the scope of our review to provide details of the analysis, conducted in [75] which finds that the Hamiltonian in the Proca mimetic model shows no sign of instability. Furthermore, [75] proposes an interesting extension of mimetic gravity to a mimetic tensor vector scalar model, by generalizing (2) to

$$g_{\mu\nu} = -f(\phi)\tilde{g}_{\mu\nu}\tilde{g}^{\alpha\beta}u_\alpha u_\beta, \tag{23}$$

where now both the scalar and vector degrees of freedom contribute to mimetic matter. It is furthermore demonstrated that the theory is free of ghosts [75].

Another recent work confirmed in all generality that the original mimetic gravity theory suffers from ghost instability [140], in the following way. It is immediate to show that mimetic gravity is invariant under the local symmetry defined by

$$\begin{aligned} \delta\phi &= 0, \\ \delta g_{\mu\nu} &= \epsilon\left(\frac{\partial\mathscr{A}}{\partial X}g_{\mu\nu} + \frac{\partial\mathscr{B}}{\partial X}\partial_\mu\phi\partial_\nu\phi\right), \end{aligned} \tag{24}$$

where as usual $\mathscr{A}$ and $\mathscr{B}$ correspond to the conformal and disformal factors. Being $\mathscr{A} = X$ and $\mathscr{B} = 0$ for mimetic gravity, (24) corresponds as expected to invariance of the physical metric under conformal transformations of the auxiliary metric. In the Hamiltonian description, this symmetry is associated with a first class constraint. In fact, one can show that the primary constraint, which corresponds to the generator of infinitesimal conformal transformations, is first class, with its Poisson commuting with the Hamiltonian and momentum constraints. This leaves no place for a secondary constraint which could eliminate the Ostrogradski ghost. Thus, this result confirms indeed that the original mimetic gravity proposal suffers from a ghost instability.

3. Solutions and Extensions of Mimetic Gravity

Having discussed the underlying physical foundation of mimetic gravity, and its stability, we can now proceed to study solutions and extensions of this theory.

3.1. Potential for Mimetic Gravity. Recall that, in a cosmological setting, the mimetic field plays the role of "clock." Therefore, one can imagine making the mimetic field dynamical by adding a potential for such field to the action. A field-dependent potential corresponds to a time-dependent potential which, by virtue of the Friedmann equation, corresponds to a time-varying Hubble parameter (and correspondingly scale factor). Therefore, by adding an appropriate potential for the mimetic field, one can in principle reconstruct any desired expansion history of the universe. This is the idea behind the minimal extension of mimetic gravity first proposed in [73]. The action of mimetic gravity (in the Lagrange multiplier formulation) is thus extended to include a potential for the mimetic field:

$$I = \frac{1}{2}\int_{\mathcal{M}} d^4x\sqrt{-g} \cdot \left[R + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1\right) - V(\phi) + 2\mathcal{L}_m\right]. \tag{25}$$

The equations of motion of the theory are then given by

$$G_{\mu\nu} - 2\lambda\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}V(\phi) = T_{\mu\nu}, \tag{26}$$

which, by taking the trace, can be used to determine the Lagrange multiplier:

$$\lambda = \frac{1}{2}(G - T - 4V). \tag{27}$$

When plugged into (26), (27) yields

$$G_{\mu\nu} = (G - T - 4V)\partial_\mu\phi\partial_\nu\phi + g_{\mu\nu}V(\phi) + T_{\mu\nu}. \tag{28}$$

Of course, variation with respect to the Lagrange multiplier as usual yields the constraint (3). Thus, when a potential is added to the action, the mimetic field contributes a pressure and energy density of $p = -V$ and $\rho = G - T - 3V$, respectively [73]. One further equation of motion can be derived by varying the action with respect to the mimetic field, which gives

$$\nabla^\nu\left[(G - T - 4V)\partial_\nu\phi\right] = -\frac{\partial V}{\partial\phi}, \tag{29}$$

when taking into account the expression for the Lagrange multiplier (27).

To study cosmological solutions, it is useful to consider a flat FLRW background (10), since therein the mimetic field can be identified with time. In this case, it is not hard to show that (29) reduces to [73]

$$\frac{1}{a^3}\frac{d}{dt}\left[a^3(\rho - V)\right] = -\frac{dV}{dt}, \tag{30}$$

which can be integrated to give

$$\rho = \frac{3}{a^3}\int da\, a^2 V, \tag{31}$$

whereas the pressure remains $p = -V$. The Friedmann equation can instead be manipulated to the form:

$$2\dot{H} + 3H^2 = V(t), \tag{32}$$

where as usual the Hubble parameter is defined as $H \equiv \dot{a}/a$. Further progress can be made by performing the substitution $y \equiv a^{3/2}$, which yields the following equation [73]:

$$\ddot{y} - \frac{3}{4}V(t)\,y = 0. \tag{33}$$

It should be noticed that the equations of motion simplify greatly because of the identification of the mimetic field with time on an FLRW background. In this way, the pressure becomes a known function of time and y satisfies a linear differential equation. We now proceed to study a few interesting potentials and the corresponding solutions.

3.1.1. $V \propto 1/\phi^2$. Let us consider the following potential [73]:

$$V(\phi) = \frac{\alpha}{\phi^2} = \frac{\alpha}{t^2}. \tag{34}$$

Solving the corresponding (33) and substituting for the scale factor, $a \equiv y^{2/3}$, yield the following solution for $\alpha \geq -1/3$:

$$a(t) = t^{(1/3)(1+\sqrt{1+3\alpha})}\left(1 + \beta t^{-\sqrt{1+3\alpha}}\right)^{2/3}, \tag{35}$$

where β is an integration constant. For $\alpha < -1/3$ the solution describes an oscillating flat universe with amplitude of oscillations which grows with time; however, the solution presents singularities and for this reason we will not write it down explicitly [73]. We can furthermore determine the equation of state parameter (EoS) $w \equiv p/\rho$ if we recall that the energy density is given by (31) whereas the pressure reads $p = -V$. Explicit calculation gives the following:

$$w = -3\alpha\left(1 + \sqrt{1+3\alpha}\,\frac{1 - \beta t^{-\sqrt{1+3\alpha}}}{1 + \beta t^{-\sqrt{1+3\alpha}}}\right)^{-2}. \tag{36}$$

It is interesting to note that, for $\alpha \gg 1$, the EoS approaches $w = -1$, that is, a cosmological constant, at late times.

We can consider the case where mimetic matter is a subdominant energy component in the universe, which is instead dominated by another form of matter with EoS $\widetilde{w}$. The scale factor in this situation evolves as

$$a \propto t^{2/3(1+\widetilde{w})}, \tag{37}$$

and hence (31) can be used to deduce the evolution of the energy density of mimetic matter, which decays as

$$\rho = -\frac{\alpha}{\widetilde{w}t^2}. \tag{38}$$

Given that the pressure of mimetic matter reads $p = -V$, we immediately see that the EoS for mimetic matter is $w = \widetilde{w}$, demonstrating that mimetic matter, when subdominant, can imitate the EoS of the dominant energy component [73]. A comment is in order here. Mimetic matter can only be subdominant if $\alpha/w \ll 1$. If this condition is not satisfied, mimetic matter will only start imitating the dominant matter component at late times, while acting as a cosmological constant at earlier times.

3.1.2. Power-Law Potential. We can consider an arbitrary power-law potential:

$$V(\phi) = \alpha\phi^n = \alpha t^n, \tag{39}$$

for which the solution of (33) can be written in terms of the Bessel functions [73]:

$$y = t^{1/2} Z_{1/(n+2)}\left(\frac{\sqrt{-3\alpha}}{n+2}t^{(n+2)/2}\right). \tag{40}$$

For $n < -2$ (with $n = -2$ corresponding to the case we have studied previously) the limiting behaviour of the scale factor is that of a dust-dominated universe, with EoS $w = 0$. For $n > -2$ and $\alpha < 0$ (which corresponds to a positive pressure), the corresponding solution is a singular oscillating universe. For $n > -2$ and $\alpha > 0$ instead, the pressure is negative and hence we expect accelerating solutions [73]. In fact, $n = 0$ corresponds to a cosmological constant as expected (the potential is simply a constant), whereas $n = 2$ gives an inflationary expansion solution with scale factor:

$$a \propto t^{-1/3} e^{\sqrt{\alpha/12} t^2}, \tag{41}$$

which resembles that of chaotic inflation sourced by a quadratic potential [73].

3.1.3. Inflation in Mimetic Gravity. One can always reconstruct the appropriate potential for the mimetic field which can provide an inflationary solution. The method is very simple: choose a desired expansion history of the universe [encoded in the Hubble parameter H or equivalently in the scale factor $a(t)$], find the corresponding y-parameter through $a = y^{2/3}$, and then invert (33) to find the corresponding potential which can provide the desired expansion:

$$V(\phi) = V(t) = \frac{4\ddot{y}}{3y}. \tag{42}$$

As an example, we can consider the following potential [73]:

$$V(\phi) = \frac{\alpha\phi^2}{e^{\phi} + 1}, \tag{43}$$

whose corresponding solution for the scale factor is exponential ($a \propto e^{-t^2}$) for large negative times and corresponding to EoS $w = 0$ (i.e., dust, $a \propto t^{2/3}$) at late times. Thus, we see that with this potential mimetic gravity can provide us with an inflationary solution with graceful exit to the matter dominated era [73].

Another interesting possibility is given by an exponential potential [83]:

$$V(\phi) = \alpha e^{-\kappa\phi} = \alpha e^{-\kappa t}. \tag{44}$$

In this case, the scale factor can be expressed in terms of Bessel functions of the first and second kinds [83]:

$$a(t) = \left[\beta J_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{-kt/2}\right) + \gamma Y_0\left(\frac{\sqrt{-3\alpha}}{\kappa} e^{-kt/2}\right)\right]^{2/3}, \tag{45}$$

where β and γ are integration constants and J_0 and Y_0 are the modified Bessel functions of order zero, of the first and second kinds, respectively. At late times, the behaviour of the scale factor is that of a matter dominated universe, that is, with EoS $w = 0$ ($a \propto t^{2/3}$). At early times the behaviour of the scale factor depends on the sign of α, in particular providing us with an inflationary solution for $\alpha > 0$, whereas the solution is a bouncing nonsingular universe for $\alpha < 0$. A similar behaviour can be obtained if one chooses the potential:

$$V(\phi) = \frac{\alpha\phi^{2n}}{e^{\kappa\phi} + 1} = \frac{\alpha t^{2n}}{e^{\kappa t} + 1}, \tag{46}$$

that is, a solution with inflation at early times and matter domination at late times. More complicated potentials which can reproduce qualitatively similar behaviours were studied in [103].

3.1.4. Bouncing Universes in Mimetic Gravity. As we have already seen in previous cases, one can easily construct bouncing solutions in mimetic gravity. Let us work through one further example here. Consider a potential of the form:

$$V(\phi) = \frac{4}{3}\frac{1}{(1+\phi^2)^2} = \frac{4}{3}\frac{1}{(1+t^2)^2}. \tag{47}$$

As usual, the scale factor can be determined by solving (33), which yields [73]

$$a(t) = \left[\sqrt{t^2+1}\,(1 + \beta \arctan t)\right]^{2/3}. \tag{48}$$

If we set the integration constant β to 0, the corresponding energy density and pressure (one again, refer to (31) and recall that $p = -V$) are given by

$$\rho = \frac{4}{3}\frac{t^2}{(1+t^2)^2}, \qquad p = -\frac{4}{3}\frac{1}{(1+t^2)^2}. \tag{49}$$

At very early times (large negative t) the EoS approaches $w \to 0$; the universe is dominated by dust and contracts. At a certain time corresponding to $|t| \sim 1$, the energy density drops suddenly to zero, after which the universe begins expanding. During the first instants of the expansion (within one Planckian time), the energy density of the universe is Planckian but subsequently drops as the expansion proceeds as a conventional expansion in a dust-dominated universe. The interesting feature of this potential is that the EoS crosses the phantom divide without singularity. This remains true even in the general case where the integration constant β is nonzero, provided $|\beta| < 2/\pi$ [73].

In the case we have just examined, the bounce occurs at the Planck scale, and hence the classical analysis we provided might not be valid as quantum gravity effects would be playing an important role. However, a minimal modification allows lowering the scale of the bounce and correspondingly increases the duration of the bounce (which now lasts more than a Planckian time). The corresponding potential which can provide this behaviour is given by [73]

$$V(\phi) = \frac{4}{3}\frac{\alpha}{(\phi_0^2+\phi^2)^2} = \frac{4}{3}\frac{\alpha}{(t_0^2+t^2)^2}. \tag{50}$$

Although we will not show the solution explicitly, in this case the scale of the bounce is reduced to α/t_0^2 and the duration of the bounce is now t_0 [73].

3.2. Mimetic $F(R)$ Gravity. The next step which was performed by Nojiri and Odintsov is to generalize mimetic gravity to mimetic $F(R)$ gravity [81]. In this theory we expect to have two additional degrees of freedom compared to GR: the constrained (nonpropagating) scalar degree of freedom of mimetic gravity and the additional scalar degree of freedom arising from the $F(R)$ term. The action of the theory is given by [81]

$$I = \int d^4x\sqrt{-g\left(\tilde{g}_{\mu\nu},\phi\right)}\left[F\left(R\left(\tilde{g}_{\mu\nu},\phi\right)\right)+\mathscr{L}_m\right], \tag{51}$$

where as usual the relation between the physical and auxiliary metric and the mimetic field is given by (2), and the constraint equation (3) has to be satisfied. Because of this, the action of mimetic $F(R)$ gravity can be equally written employing a Lagrange multiplier field analogously to (11) [81]:

$$\begin{aligned} I = \int d^4x\sqrt{-g}\,&\left[F\left(R\left(g_{\mu\nu}\right)\right)-V\left(\phi\right)\right.\\ &\left.+\lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1\right)+\mathscr{L}_m\right], \end{aligned}\tag{52}$$

where the Lagrange multiplier enforces the constraint on the gradient of the mimetic field and in addition we have added a potential for the mimetic field.

The equations of motion of the theory are slightly more complicated than that of conventional mimetic gravity. Varying with respect to the metric gives the gravitational field equations [81]:

$$\begin{aligned} 0 = {}&\frac{1}{2}g_{\mu\nu}F\left(R\right)-R_{\mu\nu}F'\left(R\right)+\nabla_\mu\nabla_\nu F'\left(R\right)\\ &-g_{\mu\nu}\Box F'\left(R\right)+\frac{1}{2}g_{\mu\nu}\left[-V\left(\phi\right)+\lambda\left(g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right)\right]\\ &-\lambda\partial_\mu\phi\partial_\nu\phi+\frac{1}{2}T_{\mu\nu}. \end{aligned}\tag{53}$$

Variation with respect to the mimetic field instead yields the following equation [81]:

$$2\nabla^\mu\left(\lambda\partial_\mu\phi\right)+\frac{dV}{d\phi}=0. \tag{54}$$

As usual, by construction, variation with respect to the Lagrange multiplier gives the mimetic constraint:

$$g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi=-1. \tag{55}$$

As we have mentioned previously, in mimetic $F(R)$ gravity one has two additional degrees of freedom. Therefore, by appropriately tuning either or both the potential for the mimetic field, or the form of the $F(R)$ function, one can reconstruct basically any desired expansion history of the universe. On the other hand, the interpretation of the cosmological role played by mimetic dark matter remains the same as in mimetic gravity [81].

Let us proceed to study some of the properties of mimetic $F(R)$ gravity in a cosmological setting. As usual, we consider a flat FLRW universe, and we model the matter contribution as that of a perfect fluid with energy density ρ and pressure p. Assuming that the mimetic field depends only on time, (53), (54), and (55) can be expressed as follows [81]:

$$\begin{aligned} 0 = {}&-F\left(R\right)+6\left(\dot{H}+H^2\right)F'\left(R\right)-6H\frac{dF'\left(R\right)}{dt}\\ &-\lambda\left(\dot{\phi}^2+1\right)+V\left(\phi\right)+\rho, \end{aligned}\tag{56}$$

$$\begin{aligned} 0 = {}&F\left(R\right)-2\left(\dot{H}+3H\right)+2\frac{d^2F'\left(R\right)}{dt^2}+4H\frac{dF'\left(R\right)}{dt}\\ &-\lambda\left(\dot{\phi}^2-1\right)-V\left(\phi\right)+p, \end{aligned}\tag{57}$$

$$0 = 2\frac{d}{dt}\left(\lambda\dot{\phi}\right)+6H\lambda\dot{\phi}-\frac{dV}{d\phi}, \tag{58}$$

$$0 = \dot{\phi}^2-1. \tag{59}$$

The last equation shows that up to an integration constant, which we can set to zero, the mimetic field can be identified with time just as in ordinary mimetic gravity. Thus, (58) can be expressed as follows:

$$\begin{aligned} 0 = {}&F\left(R\right)-2\left(\dot{H}+3H\right)+2\frac{d^2F'\left(R\right)}{dt^2}+4H\frac{dF'\left(R\right)}{dt}\\ &-V\left(t\right)+p, \end{aligned}\tag{60}$$

which, if we assume that the contribution of ordinary matter is negligible ($\rho = p = 0$), reduces to

$$\begin{aligned} V\left(t\right) = {}&F\left(R\right)-2\left(\dot{H}+3H\right)+2\frac{d^2F'\left(R\right)}{dt^2}\\ &+4H\frac{dF'\left(R\right)}{dt}. \end{aligned}\tag{61}$$

On the other hand, (57) can be solved for λ as follows:

$$\lambda\left(t\right) = -\frac{1}{2}F\left(R\right)+3\left(\dot{H}+H^2\right)F'\left(R\right)-3H\frac{dF'\left(R\right)}{dt}, \tag{62}$$

which shows that (59) is automatically satisfied.

The above equations put on a quantitative footing the statement we previously made: namely, that by tuning the behaviour of either or both the two additional scalar degrees of freedom, we can reconstruct any possible expansion history of the universe [81]. For instance, one can imagine fixing the form of the scalar potential and then reconstruct the form of $F(R)$ which gives the wanted evolution [encoded in $H(t)$ or, equivalently, $a(t)$]. Alternatively, one can start from a given form of $F(R)$ which might not admit the wanted evolution (e.g., matter dominated-like expansion followed by accelerated expansion) and reconstruct the form of the scalar potential which can allow for such expansion. It should also be remarked that any solution in conventional $F(R)$ gravity is also a solution in mimetic $F(R)$ gravity, but the converse is not true. In [81] specific solutions which allow unification of early-time inflation and late-time acceleration with intermediate matter domination era, as well as bouncing Universes, are studied and it is shown that they can be implemented in mimetic $F(R)$ gravity. Of course, the exact forms of the mimetic potential or the $F(R)$ function in these cases are quite complicated; nonetheless, the study serves as a proof of principle that, in such theories, one can realize any given expansion history of the universe without the need for dark components, which remains the main goal of modified theories of gravity.

Three further recent studies by Odintsov and Oikonomou [112, 120, 130] have demonstrated how one can, in mimetic

$F(R)$ gravity, realize inflationary cosmologies which are compatible with Planck and BICEP2/Keck Array constraints on the scalar spectral index and on the tensor-to-scalar ratio, n_s and r, respectively. Both the reconstruction (determination of the potential once the form of $F(R)$ and the evolutionary history of the universe are given) and the inverse reconstruction (determination of the form of $F(R)$ once the potential and the evolutionary history of the universe are given) are studied in detail and it is demonstrated that several viable options for realizing inflation compatibly with observational constraints are possible. However, the studies also point out a possible weakness of mimetic $F(R)$ gravity in this respect: namely, the forms of both the mimetic potential and the $F(R)$ function can become extremely complicated. The forms of both the mimetic potential and the Lagrange multiplier increase in complexity as the complexity of the $F(R)$ form increases. Therefore, these and other studies on mimetic $F(R)$ gravity and extensions thereof should not be viewed as the ultimate cosmological theory of everything, but rather as a proof of principle that within these theories one can reproduce basically any cosmological scenario and thus solve the "dark universe" problems, although this might come at the cost of sacrificing simplicity.

In addition, [112, 120, 130] also remarked that, although in principle the forms of the mimetic potential and the $F(R)$ function can be arbitrary, it must be kept in mind that, in order to realize viable inflation, a mechanism for graceful exit to the conventional radiation dominated era must be achieved. This entails ensuring that the theory contains unstable de Sitter vacua, which eventually becomes the cosmological attractor of the dynamical system. It is precisely the functional form of $F(R)$ which has to ensure that graceful exit takes place. Therefore, in mimetic gravity, although in principle the form of the potential is arbitrary, the same cannot be said about the functional form of $F(R)$, which has to be such as to ensure graceful exit from inflation. Therefore, in the interest of simplicity, a practical approach to constructing a minimal model of mimetic $F(R)$ gravity with the desired inflationary properties would be to choose the simplest possible functional form of $F(R)$ which ensures graceful exit from inflation, then performing the reconstruction technique to determine the form of the mimetic potential which allows the desired expansion history following inflation to be realized. Another possible solution, which we will discuss shortly, is to consider $F(R,\phi)$ inflation [95, 116], where a dynamical scalar field ϕ is coupled to gravity.

3.2.1. Late-Time Evolution in Mimetic Gravity. So far we have discussed mimetic gravity and variants thereof at early times, that is, at the epoch when primordial curvature perturbations were generated. However, it is also interesting to consider late-time evolution in mimetic gravity. The equations of motion are incredibly complex and in principle do not allow for analytical solutions. However, this complexity can be bypassed by means of the method of dynamical analysis (see, e.g., [391–394]), which gives information about the global behaviour of solutions. In particular, one proceeds by transforming the equations of motion into their autonomous form and extract the critical points. Subsequently perturbations are linearized around these critical points and expressed in terms of the perturbation matrix, the eigenvalues of which determine the type and stability of the critical points.

A detailed dynamical analysis of mimetic $F(R)$ gravity was presented in [86]. This type of analysis allows us to bypass the complexity of the equations of motion by extracting critical points and studying the corresponding observables, such as the energy densities of the various energy components, the corresponding EoS, and the deceleration parameter. In particular, the analysis finds that the only stable critical points, that is, those that can play the role of attractors at late times, are those that exist in $F(R)$ gravity as well. In other words, stable solutions in mimetic $F(R)$ gravity can only affect the expansion history of the universe at early and intermediate times, whereas at late times the expansion history has to coincide with that driven by conventional $F(R)$ gravity. An immediate implication of this finding is that, although mimetic $F(R)$ gravity could drive inflation differently from $F(R)$ gravity, the late-time acceleration of the universe in these theories has to coincide with the usual $F(R)$ gravity one [86]. However, these conclusions have been reached only by studying the theory at the level of the background. It is expected that different conclusions would be reached if the same study would be performed at the level of perturbations. This is true because the new terms present in the equations of motion of mimetic $F(R)$ gravity compared to conventional $F(R)$ gravity can contribute to the perturbation equations, although they do not contribute at the background level [86]. Finally, the energy conditions required to avoid the Dolgov-Kawasaki instability in mimetic gravity were studied in [108], which found that these are the same as in conventional $F(R)$ gravity.

As we mentioned above, the conclusions reached about the late-time evolution in mimetic $F(R)$ gravity hold only at the background level. However, in conventional $F(R)$ gravity, there exists a serious problem during the late-time evolution at the perturbation level, namely, that of dark energy oscillations [395] (see also, e.g., [396–399]). The degree of freedom associated with the modification of GR (that is, $dF(R)/dR$) leads to high frequency oscillations of the dark energy around the line of the phantom divide during matter era. As a consequence, some derivatives of the Hubble parameter may diverge and become singular, and the solution is unphysical. Usually in conventional $F(R)$ gravity the problem is solved by adding power-law modifications by hand.

In [153], it was instead argued that in mimetic $F(R)$ gravity it is possible to overcome the problem by a suitable choice of the potential. By appropriately choosing the potential and the Lagrange multiplier, it is possible to damp the oscillations within a mimetic $F(R)$ model whose corresponding conventional $F(R)$ model suffered from the oscillations problems. The oscillations die out for redshifts $z \leq 3$, so there is no issue with dark energy oscillations at our current epoch. Moreover, the values of the dark energy equation of state and the total equation of state are very close to the observed values. The model can in principle be discerned from $F(R)$ gravity in that the predicted growth factor is lower in magnitude, a very testable prediction in view of future experiments, further supporting the viability of mimetic $F(R)$ gravity as a cosmological framework.

3.2.2. Mimetic $F(R,\phi)$ Gravity. The question of constructing a theoretically motivated but at the same time simple model for mimetic $F(R)$ gravity which provides an early-time inflation epoch, but at the same time graceful exit from the latter, was addressed in [95]. Here, a model of mimetic $F(R,\phi)$ gravity was considered, where ϕ is a scalar field coupled to gravity. We will not go into the details of the work, for which we refer the reader to the original paper [95]. The basic idea is to use the $F(R)$ sector to reproduce a variety of cosmological scenarios: among the ones considered in the paper were accelerated cosmologies at high and low curvatures (thus unifying inflation and late-time acceleration), with Einstein gravity at intermediate curvatures. In particular, the accelerated cosmologies are realized by making use of a "switching-on" cosmological constant. The dynamical field ϕ evolves in such a way to allow for graceful exit from the inflationary period, thus making the vacua of the first de Sitter period (corresponding to inflation) unstable. Entry into the late-time accelerated epoch, represented by a stable de Sitter attractor, is also made possible by the dynamical field, which thus links all epochs of the expansion history of the universe in a unified way. In the minimal case studied in [95], the mimetic component ensures the presence of cosmological nonbaryonic dark matter although, as we have extensively discussed, it is possible by adding a suitable potential for the mimetic field to obtain similar solutions, but with a different form of $F(R)$.

3.2.3. Nonlocal Mimetic $F(R)$ Gravity. A further extension of mimetic $F(R)$ gravity was presented in [133], which embeds the theory into the framework of nonlocal theories of gravity. Recall that these theories were first presented in [373], inspired by quantum loop corrections. These theories feature nonlocal operators (i.e., inverse of differential operators) of the curvature invariants. The prototype of nonlocal mimetic $F(R)$ gravity is given by the action [133]:

$$I = \int d^4x\sqrt{-g}\left[R\left(1+f\left(\Box^{-1}R\right)\right) + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1\right) - V(\phi)\right], \tag{63}$$

where as usual ϕ is the mimetic field and the Lagrange multiplier term enforces the constraint on its gradient. It is actually more useful to introduce an additional scalar field ψ, which allows us to translate the action given by (63) to a local scalar-tensor form, as follows [133]:

$$I = \int d^4x\sqrt{-g}\left[R\left(1+f(\psi)\right) + \xi\left(\Box\psi - R\right) + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+1\right) - V(\phi)\right], \tag{64}$$

where ξ is an additional Lagrange multiplier which enforces the constraint on the scalar field ψ:

$$\Box\psi = R. \tag{65}$$

Aside from the two constraint equations obtained by varying the action with respect to the Lagrange multipliers, variation of the action with respect to the yields the gravitational field equations [133]:

$$\begin{aligned} &R_{\mu\nu}\left(1+f(\psi)-\xi\right) - \frac{1}{2}g_{\mu\nu}R\left(1+f(\psi)-\xi\right) \\ &\quad - \partial_\rho\xi\partial^\rho\psi \\ &\quad = \frac{1}{2}\left(\partial_\mu\xi\partial_\nu\psi + \partial_\mu\psi\partial_\nu\xi\right) \\ &\quad - \left(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu\right)\left(f(\psi)-\xi\right) - \lambda\partial_\mu\phi\partial_\nu\phi \\ &\quad - g_{\mu\nu}V(\phi). \end{aligned} \tag{66}$$

Instead, variation with respect to the two scalar fields leads to the following equations of motion [133]:

$$\begin{gathered} \Box\xi + f'(\psi)R = 0, \\ -\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\partial^\nu\phi\right) = \frac{1}{2}\frac{dV}{d\phi}. \end{gathered} \tag{67}$$

In [133], this model was studied in detail making use of the reconstruction technique. In particular, two forms for the $f(\psi)$ function have been studied: exponential and power-law. It was shown that appropriate choices for the mimetic potential, as expected, can give the desired expansion history of the universe, which in the cases studied included viable inflation unified with late-time acceleration with intermediate epoch compatible with Einstein's gravity and cosmological dark matter provided by the mimetic field, as well as solutions with cosmological bounces.

3.3. Unimodular Mimetic Gravity. As we have seen, mimetic gravity provides a geometric explanation for dark matter in the universe, with dark matter emerging as an integration constant as a result of gauging local Weyl invariance, without the need for additional fluids. An older theory, known as unimodular gravity [400] (see also [98, 401–419]), had instead been proposed much earlier to solve, in a geometrical fashion as well, one more of the conundrums of modern cosmology: the dark energy problem. In this framework, dark energy emerges in the form of a cosmological constant from the trace-free part of Einstein's field equations, with the trace-free part which results in turn by enforcing the condition that the square root of (minus) the determinant of the metric is equal to 1, or in general a constant. It would therefore be interesting to combine the two different approaches of mimetic gravity and unimodular gravity into a single framework which could geometrically explain both dark matter and dark energy by a vacuum theory, without need for additional fluids. This is the proposal of Nojiri et al. in [136].

In order to combine mimetic gravity and unimodular gravity it is necessary to enforce two constraints. The first is the constraint on the gradient of the mimetic field (3), whereas the second is the unimodular constraint:

$$\sqrt{-g} = 1. \tag{68}$$

In order to enforce these two constraints, it is conceptually simple to make use of two Lagrange multipliers. This approach has two advantages. First, it keeps the two concepts of mimetic and unimodular gravity separate and facilitates the extraction of physical information. Second, it is as usual more convenient to have the two constraints emerge from the equations of motion. The action for unimodular mimetic gravity thus reads [136]

$$I = \int d^4x \cdot \left[\sqrt{-g} \left(R - V(\phi) - \eta \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) - \lambda \right) + \lambda \right], \tag{69}$$

where variation with respect to the Lagrange multiplier η enforces the mimetic constraint (3), whereas variation with respect to the Lagrange multiplier λ enforces the unimodular constraint (68).

The equations of motion for the gravitational field are obtained by varying the action with respect to the metric and are given by [136]:

$$\begin{aligned} 0 = {} & \frac{1}{2} g_{\mu\nu} \left(R - V(\phi) - \eta \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 1 \right) - \lambda \right) \\ & - R_{\mu\nu} + \eta \partial_\mu \phi \partial_\nu \phi + \frac{1}{2} T_{\mu\nu}, \end{aligned} \tag{70}$$

whereas variation with respect to the mimetic field yields the usual equation of motion:

$$0 = 2\nabla^\mu \left(\lambda \partial_\mu \phi \right) - \frac{dV}{d\phi}. \tag{71}$$

Although we do not show the steps explicitly, for which we refer the reader to the original paper [136], in the usual FLRW setting it is possible to manipulate the Einstein equations in order to get the following reconstruction equation for the mimetic potential $V(\phi) = V(t)$ (as usual, the mimetic field can be identified with time):

$$\begin{aligned} V(\phi) = V(t) = {} & \frac{a(t)^2}{3} \int_0^\phi dt\, a(t)^{-2} \left[-6H(t)\, p(t) \right. \\ & - 2\frac{dp(t)}{dt} \\ & \left. + 2\left(-18H(t)^3 - 6H(t) \frac{dH(t)}{dt} + 4\frac{d^2H(t)}{dt^2} \right) \right]. \end{aligned} \tag{72}$$

The content of the above equation is clear: as in all extensions so far discussed of mimetic gravity, one can always reconstruct the potential for the mimetic field which can provide the desired expansion encoded in $H(t)$ or $a(t)$. The reconstruction technique is very powerful although, as we have seen, the corresponding potentials are complicated and somewhat hard to justify from first principles, although the reconstruction technique serves in this case as a proof of principle tool.

Having made this consideration, let us consider a few examples where the reconstruction technique is applied. Let us consider the following simple potential [136]:

$$V(\phi) = 12 e^{2H_0\phi} H_0^3 \phi. \tag{73}$$

It can be easily shown that it leads to the following solution for the scale factor $a(t)$ and the Hubble parameter $H(t)$:

$$\begin{aligned} a(t) &= e^{H_0 t}, \\ H(t) &= H_0, \end{aligned} \tag{74}$$

that is, a de Sitter cosmology. The functional forms of the Lagrange multiplier are also quite simple and are given by the following:

$$\begin{aligned} \lambda(t) &= 6H_0^2 \left(1 + 2e^{2H_0 t} H_0 t \right), \\ \eta(t) &= 3H_0^2. \end{aligned} \tag{75}$$

Another choice for the potential which leads to a physically interesting solution is the following [136]:

$$V(\phi) = -\frac{8\phi^{-2+4/3(1+w)} \left(1 + 5w + 2w^2 \right)}{9(1+w)^3}, \tag{76}$$

for which the scale factor and the Hubble parameter read

$$\begin{aligned} a(t) &= t^{2/3(1+w)}, \\ H(t) &= \frac{2}{3t(1+w)}. \end{aligned} \tag{77}$$

The above solution is that corresponding to an universe dominated by a fluid with EoS w. The two Lagrange multipliers are given by

$$\begin{aligned} \lambda(t) &= \frac{8\left(-3w(1+w) + t^{4/3(1+w)} \left(1 + 5w + 2w^2 \right) \right)}{9t^2(1+w)^3}, \\ \eta(t) &= \frac{4(2+w)}{3t^2(1+w)^2}. \end{aligned} \tag{78}$$

Thus, we see that with two relatively simple choices of potential it is possible to reconstruct two important expansion histories of the universe: the late-time de Sitter phase and the expansion dominated by a perfect fluid with arbitrary EoS. Although we will not discuss this case explicitly here, for which instead we redirect the reader to [136], it is possible by a choice of a more complicated potential, to realize a viable inflationary model within unimodular mimetic gravity, which is compatible with bounds from Planck and BICEP2/Keck Array. Moreover, it has been shown that graceful exit from these types of inflationary periods can be achieved by ensuring that the corresponding de Sitter vacua which drives the period of accelerated expansion is unstable.

Two further comments are in order here. First, it is possible to provide an effective fluid description of unimodular mimetic gravity [136]. Namely, manipulation of the Einstein equations shows that the contribution of the unimodular and mimetic parts of the action can be interpreted as that of a perfect fluid carrying energy density ρ and pressure p as follows:

$$\begin{aligned} \rho &= G - T - 4\widetilde{V}, \\ p &= -\widetilde{V}, \end{aligned} \tag{79}$$

where $\widetilde{V}$ is defined as

$$\widetilde{V} = -\lambda(t) - V(t). \tag{80}$$

Furthermore, the effective energy density and pressure defined as per above satisfy the continuity equation.

The second comment is related to the fact that the unimodular constraint is enforced in a noncovariant way (cf. the action given by (69), where one of the terms in λ is not multiplied by $\sqrt{-g}$). It is nonetheless possible to present a covariant formulation of unimodular mimetic gravity via the following action:

$$I = \int d^4x \left[\sqrt{-g} \left(R - \eta \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) - \lambda \right) + \lambda \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma} \right], \tag{81}$$

where $a_{\nu\rho\sigma}$ is a three-form field, variation of which gives the constraint $\partial_\mu \lambda = 0$, implying that the Lagrange multiplier λ is constant. On the other hand, the covariant version of the unimodular constraint is obtained by variation with respect to λ, from which one is left with

$$\sqrt{-g} = \epsilon^{\mu\nu\rho\sigma} \partial_\mu a_{\nu\rho\sigma}. \tag{82}$$

Further manipulation, for which we refer the reader to [136], shows that the Friedmann equations one obtains from the covariant version of unimodular mimetic gravity are equivalent to those of the noncovariant version, and thus one can reproduce precisely the same cosmological scenarios in both theories.

3.3.1. Unimodular Mimetic $F(R)$ Gravity. A minimal extension of the unimodular mimetic gravity framework we have discussed so far is to consider unimodular mimetic $F(R)$ gravity, which is described by the action [138]:

$$I = \int d^4x \cdot \left[\sqrt{-g} \left(F(R) - V(\phi) - \eta \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) - \lambda \right) + \lambda \right]. \tag{83}$$

As expected, the equations of motion are slightly more complicated than in the unimodular mimetic case, but no conceptual difficulty is added. Specifically, variation with respect to the two Lagrange multipliers enforces the usual unimodular and mimetic constraints, whereas variation with respect to the metric gives rise to the equations for the gravitational field [138]:

$$\frac{g_{\mu\nu}}{2} \left(F(R) - V(\phi) + \eta \left(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + 1 \right) - \lambda \right) - R_{\mu\nu} F'(R) - \eta \partial_\mu \phi \partial_\nu \phi + \nabla_\mu \nabla_\nu F'(R) - g_{\mu\nu} \Box F'(R) = 0. \tag{84}$$

Finally, variation with respect to the mimetic field yields the following equation:

$$0 = 2\nabla^\mu \left(\lambda \partial_\mu \phi \right) - \frac{dV}{d\phi}. \tag{85}$$

The more complex structure of the equations of motion complicates the reconstruction procedure, that is, the equivalent of (72), which now reads [138]

$$V(t) = \frac{a(t)^{3/2}}{2} \int dt\, a(t)^{-3/2} f(t), \tag{86}$$

where the function $f(t)$ is given by

$$\begin{aligned} f(t) = -\Bigg[& 18H(t) \left(\dot{H} + H^2 \right) F'(R) - 6H^2 \frac{dF'(R)}{dt} \\ & - 18\dot{H}H^3 + 6H \frac{d^3 F'(R)}{dt^3} + \left(6\ddot{H} + 2\dot{H}H \right) F'(R) \\ & + H\dot{H} \frac{dF'(R)}{dt} + 6H^2 \frac{dF'(R)}{dt} - 2H \frac{d^2 F'(R)}{dt^2} \\ & - 2\left(\ddot{H} + 6\dot{H}H \right) + 2 \frac{d^2 F'(R)}{dt^2} \Bigg]. \end{aligned} \tag{87}$$

Despite the increased complexity of the equations of motion, the same considerations apply as for unimodular mimetic gravity, as well as all extensions of mimetic gravity hereto considered. Namely, it is always possible to reconstruct any viable cosmological expansion scenario, including unification of inflation and late-time acceleration with intermediate radiation and matter domination, with graceful exit from inflation triggered by unstable de Sitter vacua. This can be achieved by appropriately choosing either or both the form of the mimetic potential or the function $F(R)$. The price to pay would eventually be a considerable complexity in the functional form of both, which of course does not represent a first principle obstacle [138].

3.4. Mimetic Horndeski Gravity. One can further consider more general scalar-tensor theories, which can be "mimetized" according to the procedures we have described so far, namely, through a singular disformal transformation or through a Lagrange multiplier term in the action enforcing the mimetic constraint. In fact, analogously to GR, one can show that the most general scalar-tensor model is invariant under disformal transformations, provided the latter is invertible. This has been shown in all generality in [104]. One can then "mimetize" such theories by considering the following action [104]:

$$\begin{aligned} I = \int d^4x \sqrt{-g} \Big[& \mathscr{L} \left(g_{\mu\nu}, \partial_{\lambda_1} g_{\mu\nu}, \ldots, \partial_{\lambda_1} \cdots \partial_{\lambda_p} g_{\mu\nu}, \phi, \right. \\ & \left. \partial_{\lambda_1} \phi, \ldots, \partial_{\lambda_1} \cdots \partial_{\lambda_q} \phi \right) + \lambda \left(g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + 1 \right) \Big], \end{aligned} \tag{88}$$

where $p, q \geq 2$ are integers and $\mathscr{L}$ is the Lagrangian density which is a function of the metric and the mimetic field. In general, the constraint enforced by the Lagrange multiplier can be generalized to $b(\phi) g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi = -1$, but for the sake of simplicity we will set $b(\phi) = 1$ here and redirect the reader to the work of [104] for more general discussions, and for the explicit form of the equations of motion. In fact, setting $b(\phi) \neq 1$ is basically equivalent to assigning a potential to the mimetic field. In [104] it was shown that the two approaches to mimetic Horndeski gravity, namely, singular disformal transformation and Lagrange multiplier, are equivalent.

Of course, the considerations made above can be applied in the case of a specific scalar-tensor model, namely, Horndeski gravity [275] (see, e.g., [420–438] for further recent work on the topic of Horndeski gravity and theories beyond Horndeski). Recall that Horndeski gravity is the most general 4D local scalar-tensor theory with equations of motion no higher than second order. The Horndeski action can be written as a sum of four terms:

$$I = \int d^4x\sqrt{-g}\mathscr{L}_H = \int d^4x\sqrt{-g}\sum_n \mathscr{L}_n, \tag{89}$$

where $\mathscr{L}_n$s read

$$\begin{aligned}
\mathscr{L}_0 &= K(X,\phi),\\
\mathscr{L}_1 &= -G_3(X,\phi)\Box\phi,\\
\mathscr{L}_2 &= G_{4,X}(X,\phi)\left[(\Box\phi)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2\right] + RG_4(X,\phi),\\
\mathscr{L}_3 &= -\frac{1}{6}G_{5,X}(X,\phi)\\
&\quad\cdot\left[(\Box\phi)^3 - 3\Box\phi\left(\nabla_\mu\nabla_\nu\phi\right)^2 + 2\left(\nabla_\mu\nabla_\nu\phi\right)^3\right]\\
&\quad + G^{\mu\nu}\nabla_\mu\nabla_\nu\phi G_5(X,\phi),
\end{aligned} \tag{90}$$

where $X \equiv -g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi/2$, $(\nabla_\mu\nabla_\nu\phi) \equiv \nabla_\mu\nabla_\nu\phi\nabla_\mu\nabla^\nu\phi$, $(\nabla_\mu\nabla_\nu\phi)^3 \equiv \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\rho\phi\nabla^\nu\nabla_\rho\phi$, the functions $K(X,\phi)$, $G_3(X,\phi)$, $G_4(X,\phi)$, $G_5(X,\phi)$ are free, and X denotes differentiation with respect to X.

The mimetic version of the above Horndeski model has been studied in a variety of papers recently (e.g., [104, 114, 128]). We report some particular cases taken into consideration. We remark that the freedom in the free functions K, G_3, G_4, and G_5, as well as in the function $b(\phi)$ (which, when $\neq 1$, is equivalent to providing a potential for the mimetic field), results in the possibility of reproducing basically any given expansion scenario of the universe. A specific model studied in [104] is one where the functions take the form:

$$\begin{aligned}
K(X,\phi) &= c_2X,\\
G_3(X,\phi) &= 0,\\
G_4(X,\phi) &= \frac{1}{2},\\
G_5(X,\phi) &= 0.
\end{aligned} \tag{91}$$

In this case, on a flat FLRW background the solution is given by

$$\begin{aligned}
a(t) &= t^{2/3(1+w)},\\
\phi(t) &= \pm\sqrt{-\frac{\alpha}{c_2}}\ln\left(\frac{t}{t_0}\right),\\
b(\phi) &= -\frac{1}{\dot{\phi}^2},
\end{aligned} \tag{92}$$

with t_0 and integration constant and $\alpha = -8w/3(1+w)^2$. Thus, we see that the scenario under consideration has reproduced the expansion history of a universe filled with a perfect fluid with EoS w [104]. Another case considered in [104] is the mimetic cubic Galileon model, where the functions take the following form:

$$\begin{aligned}
K(X,\phi) &= c_2X,\\
G_3(X,\phi) &= \frac{2c_3}{\widetilde{\Lambda}^3X},\\
G_4(X,\phi) &= \frac{1}{2},\\
G_5(X,\phi) &= 0,
\end{aligned} \tag{93}$$

where the cut-off scale is subsequently set to $\widetilde{\Lambda} = 1$. It is then found that the model can reproduce the expansion history of a universe filled with nonrelativistic matter, followed by a cosmological constant dominated expansion analogous to the late-time acceleration we are experiencing [104]. The case of a nonminimal coupling to the auxiliary metric, $\widetilde{g}_{\mu\nu}$ was examined as well, which we will not report on here and for which we will redirect the reader to the original paper [104].

To conclude, we report on the following specific case of mimetic Horndeski model which was studied in [114]. The Horndeski part of the action of the theory is given by

$$\begin{aligned}
I_H = \int d^4x\sqrt{-g}&\left[\alpha\left(XR + (\Box\phi)^2 - \nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\right)\right.\\
&\left. + \gamma\phi G_{\mu\nu}\nabla^\mu\nabla^\nu\phi - \beta\phi\Box\phi\right],
\end{aligned} \tag{94}$$

which corresponds to the following choice for the functions discussed previously:

$$\begin{aligned}
K(X,\phi) &= 0,\\
G_3(X,\phi) &= \beta\phi,\\
G_4(X,\phi) &= \alpha X,\\
G_5(X,\phi) &= 0.
\end{aligned} \tag{95}$$

A number of solutions, including cosmological bounces, inflation unified with late-time acceleration, and future singularities, have been discussed. As a specific example, on a flat FLRW background, the following choice of potential for the mimetic field [114],

$$V(\phi) = -\beta + 2V_0 + \frac{3V_0^2}{c_1}(\phi-\phi_0)^2, \tag{96}$$

gives rise to the following solution:

$$H(t) = \frac{V_0}{c_1}(t-t_0), \tag{97}$$

which represents a regular bounce solution. Another bounce solution can be obtained by considering the following potential [114]:

$$V(\phi) = b^2c_1\frac{\sinh^2 b\phi}{\cosh^2 b\phi}, \tag{98}$$

for which the scale factor and correspondingly the Hubble parameter read:

$$a(t) = a_0 \cosh bt,$$
$$H(t) = b\frac{\sinh bt}{\cosh bt}. \tag{99}$$

In Section 5 we will present a detailed discussion on a specific mimetic Horndeski model, constructed as an extension of a covariant Hořava-like theory of gravity.

3.5. Einstein-Aether Theories, Hořava-Lifshitz Gravity, and Covariant Renormalizable Gravity. In closing, we comment on some connections between mimetic gravity and other theories of modified gravity, such connections having been identified recently: namely, the scalar Einstein-aether theory, Hořava-Lifshitz gravity, and a covariant realization of the latter, that is, covariant renormalizable gravity.

3.5.1. Einstein-Aether Theories. An interesting connection which can be identified is that between mimetic gravity and Einstein-aether theories [276, 277] (see also [278–280, 439–444]). These are a class of Lorentz-violating generally covariant extensions of GR. By Lorentz-violating generally covariant we mean that Lorentz invariance is preserved at the level of the action, only to be broken dynamically. In fact, the theory contains a unit time-like vector u^μ (which is called the aether) whose norm is fixed by a Lagrange multiplier term in the action. This entails the fixing of a preferred rest frame at each space-time point. In fact, mimetic gravity itself dynamically violates Lorentz symmetry because the gradient of the mimetic field fixes a preferred direction in space-time.

To be precise, mimetic gravity is in correspondence with a particular version of the Einstein-aether theory, namely, the scalar Einstein-aether theory [77, 145]. In this theory, the role of aether is played by the gradient of a scalar field, precisely as occurs in mimetic gravity: the role of aether is played by the four-velocity vector which in turn is the gradient of the mimetic field. It should be noted that the scalar Einstein-aether theory is quite different from the original vector theory. Moreover, in the case where the potential for the potential for the scalar field in such theories (corresponding to the potential for the mimetic field in mimetic gravity) is constant, the model corresponds to the IR limit of projectable Hořava-Lifshitz gravity, which we will comment on further in Section 3.5.2.

3.5.2. Hořava-Lifshitz Gravity. Recall that Hořava-Lifshitz gravity [338] (HLG hereafter) is a framework and candidate theory of quantum gravity, wherein gravity is made power-counting renormalizable by altering the graviton propagator in the UV. This is achieved by abandoning Lorentz symmetry as a fundamental symmetry of nature, in favour of a Lifshitz anisotropic scaling in the UV. For an incomplete list of references concerning further work in Hořava-Lifshitz gravity, see, for instance, [339–363] and references therein. If the lapse function in HLG is only a function of time, that is, $N = N(t)$, the theory takes the name of projectable Hořava-Lifshitz gravity. Recall that in the Arnowitt-Deser-Misner decomposition of space-time [445]:

$$ds^2 = -N^2c^2dt^2 + g_{ij}\left(dx^i + N_i dt\right)\left(dx^j + N^j dt\right), \tag{100}$$

the function N takes the name of lapse.

Previous work has shown that the IR limit of the nonprojectable version of Hořava-Lifshitz gravity can be obtained from the Einstein-aether theory (the vector version) by requiring that the aether be hypersurface orthogonal: that is, $u_\mu = N\nabla_\mu T$, where T is a scalar field and N is chosen in such a way to ensure u_μ has unit norm. In this case T is responsible for the preferred foliation in Hořava gravity, and N is the lapse function. If T is set equal to coordinate time, the resulting action is that of nonprojectable Hořava gravity [357, 443].

Further requiring that $NdT = S$, where S is a scalar field, reduces the vector Einstein-aether theory to the scalar Einstein-aether theory [446]. This condition implies that $N = N(T)$, which upon identification of T with coordinate time corresponds exactly to the defining condition for projectable Hořava gravity. In this case, the unit norm constraint cannot be solved for a generic N but has to be imposed at the level of the action, for instance, via a Lagrange multiplier term $\propto \lambda(\nabla_\mu S\nabla_\mu S - 1)$. Therefore, the condition under which $dS = NdT$ is invariant is that S be invariant under a shift symmetry: $S \rightarrow S + dS$, which shows why the equivalence between scalar Einstein-aether theory and mimetic gravity fails if a nonzero potential for the mimetic field is included [446]. Notice also that, as is known, dark matter emerges as an integration constant in the IR limit of projectable Hořava-Lifshitz gravity [354]. Given the correspondence between this theory and mimetic gravity, then, it should not come as a surprise that dark matter emerges in a similar fashion within the framework of mimetic gravity, as a purely geometrical effect.

A complete proof of the equivalence between mimetic gravity and the IR limit of projectable Hořava-Lifshitz gravity was presented in [135]. In particular, it was shown that the action for the IR limit of projectable Hořava-Lifshitz gravity can be written as

$$S = S_{\mathrm{EH}} + \int d^4x\sqrt{-g}\left[\frac{\Sigma}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1\right) + \frac{\gamma}{2}\left(\Box\phi\right)^2\right], \tag{101}$$

which we immediately recognize as the action for mimetic gravity without potential, with the addition of a higher derivative term which, as we shall see shortly, was actually added shortly after the original formulation of the theory on purely phenomenological grounds, to cure some undesirable properties at the perturbation level. In Hořava-Lifshitz gravity, Σ is a Lagrange multiplier which had been introduced to enforce the projectability condition. This equivalence was demonstrated rigorously in Appendix A of [135]. Therefore, mimetic gravity appears in the IR limit of a candidate theory of quantum gravity.

3.5.3. Covariant Renormalizable Gravity. Recall that HLG achieves power-counting renormalizability by breaking diffeomorphism invariance. However, this breaking appears explicitly at the level of the action. This has been at the center of criticism, which has related this explicit breaking to the appearance of unphysical modes in the theory which is coupled strongly in the IR [350, 351, 357–359]. Therefore, in order to circumvent these issues it would be desirable to have a theory which preserves the renormalizability properties of Hořava gravity in the UV but retains diffeomorphism invariance at the level of the action. Therefore, diffeomorphism invariance should be broken dynamically in the UV.

An example of such theory has been presented by Nojiri and Odintsov [356, 361, 362] (see also [364–372]) and comes under the name of covariant renormalizable gravity (CRG henceforth). The theory features a nonstandard coupling of a perfect fluid to gravity. When considering perturbations around a flat background, this nonstandard coupling dynamically breaks diffeomorphism invariance. The price to pay is the presence of this exotic fluid, which could have a stringy origin. In particular, one formulation of CRG introduces the fluid via a Lagrange multiplier term, precisely as done in mimetic gravity. For this reason, in the limit where the nonstandard coupling of the fluid to gravity disappears, one recovers the action of mimetic gravity. In one of its equivalent formulations, the action of CRG is given by [356, 361, 362]

$$I = \int d^4x\sqrt{-g}\left\{\frac{R}{2} - \alpha\left[\left(\partial^\mu\phi\partial^\nu\phi\nabla_\mu\nabla_\nu + 2U_0\nabla^\rho\nabla_\rho\right)^n \cdot \left(\partial^\mu\phi\partial^\nu\phi R_{\mu\nu} + U_0 R\right)\right]^2 - \lambda\left(\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + U_0\right)\right\}, \tag{102}$$

where U_0 and α are constants and $U_0 = 1/2$ to recover the mimetic constraint. In particular, the above case corresponds to the $z = 2n + 2$ version of CRG, where z is the dynamical critical exponent which quantifies the degree of anisotropy between space and time in the UV regime of the theory.

Following the initial proposal by Nojiri and Odintsov, other CRG-like models were studied in recent years. For instance, one particular CRG-like model was studied by Cognola et al. in [367], where black hole and de Sitter solutions were also studied. The action of such CRG-like theory takes the following form [367]:

$$\begin{aligned} I = \frac{1}{2}\int d^4x\sqrt{-g}\Big\{&R - 2\Lambda \\ &- \alpha\left[\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right)\nabla_\mu\phi\nabla_\nu\phi\right]^n \\ &- \frac{\lambda}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 1\right) - V(\phi)\Big\}. \end{aligned} \tag{103}$$

In the above action, the Horndeski-like coupling of the scalar field to curvature (for the case $n = 1$) has been considered copiously in the literature; for an incomplete list see, for instance, [420, 426, 433, 447–455] and references therein.

In the continuation of our review, after a brief interlude on perturbations in mimetic gravity, we shall consider a case study of a mimetic-like theory. Our choice of case study will fall upon the CRG-like model of Cognola et al. as defined by the action in (103). We shall study in detail its cosmological solutions and perturbations around a flat background. A pathological behaviour of the model when considering perturbations around a flat FLRW background will be cured by appropriately modifying the model.

4. A Brief Interlude: Cosmological Perturbations in Mimetic Gravity

Before proceeding to the case study of a specific mimetic-like model, it is mandatory to discuss the issue of perturbations in mimetic gravity. Recall in Section 2 that we remarked that mimetic gravity does not properly belong to the class of modified theories of gravity with a full extra scalar degree of freedom. Instead, the additional scalar is constrained by the Lagrange multiplier, a situation quite different from, for instance, that of Horndeski models which have a proper scalar degree of freedom.

However, it is clear that the Lagrange multiplier kills the wave-like parts of the scalar degree of freedom: in other words, given that the constraint takes out any higher derivative, it is not possible to have oscillating (wave-like solutions). As a consequence, we can already envisage that the sound speed in the minimal mimetic gravity model will satisfy $c_s = 0$. This implies, as we have anticipated, that there are no propagating scalar degrees of freedom in the theory.

The property of vanishing sound speed in mimetic gravity can be rigorously demonstrated, by considering small longitudinal perturbations around a flat background [73]. The line element is then given by

$$\begin{aligned} ds^2 = &-(1 + 2\Phi(t,\mathbf{x}))\,dt^2 \\ &+ a^2(t)\,(1 - 2\Psi(t,\mathbf{x}))\,\delta_{ij}dx^i dx^j, \\ &\qquad i, j = 1, 2, 3. \end{aligned} \tag{104}$$

$\Phi \equiv \Phi(t,\mathbf{x})$ and $\Psi \equiv \Psi(t,\mathbf{x})$ are functions of the space-time coordinates such that $|\Phi(t,x)|, |\Psi(t,x)| \ll 1$, and $g^{00}(t,x) \simeq -1 + 2\Phi(t,x)$, $g^{11}(t,x) \simeq a(t)^{-2}(1 + 2\Psi(t,x))$. Here, we used the conformal Newtonian gauge. Moreover, from field equations we also have

$$\Phi(t,\mathbf{x}) = \Psi(t,\mathbf{x}). \tag{105}$$

Correspondingly, given that around a flat FLRW background the mimetic field plays the role of "clock," we perturb it as

$$\phi = t + \delta\phi(t,\mathbf{x}), \tag{106}$$

where $|\delta\phi| \equiv |\delta\phi(t,x)| \ll 1$, which together with (3) implies that

$$\Phi = \delta\dot{\phi}. \tag{107}$$

Then, by perturbing the $0i$ components of the Einstein equations, a relatively straightforward calculation [73] shows that the evolution equation for the perturbation to the mimetic field, $\delta\phi$, satisfies the following equation:

$$\delta\ddot{\phi} + H\delta\dot{\phi} + \dot{H}\delta\phi = 0. \tag{108}$$

The aforementioned pathological property of perturbations in mimetic gravity can be noticed from (108). The evolution equation for perturbations to the mimetic field does not depend on the Laplacian (or in general on spatial derivatives) of the latter. In other words, there is no term of the form $c_s^2\Delta\delta\phi$ in (108), which implies that the sound speed is identically zero. This means that, even when pressure is nonvanishing, the dust degree of freedom induced in mimetic gravity behaves as dust with zero sound speed, and as such quantum perturbations to the mimetic field cannot be defined in the usual fashion. Else, they would fail in providing the seeds for the observed large-scale structure of the universe which grow via gravitational instability. We remark once more that this behaviour is not unexpected, given that the condition enforced by the Lagrange multiplier eliminates wave degrees of freedom. Moreover, this fact has been shown in all generality for mimetic Horndeski models in [128].

In order to have a theory whose quantum perturbations can be defined in a sensible way, the minimal action for mimetic gravity has to be modified, for instance, by introducing higher derivative (HD) terms. As an example, consider the following action [73]:

$$I = d^4x\sqrt{-g}\left[R + \lambda\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1\right) - V(\phi) + \frac{1}{2}\gamma\left(\Box\phi\right)^2\right]. \tag{109}$$

The corresponding equations of motion read

$$G^\mu_\nu = \left[V + \gamma\left(\partial_\alpha\phi\partial^\alpha\chi\right)\right]\delta^\mu_\nu + 2\lambda\partial^\mu\phi\partial_\nu\phi - \gamma\left(\partial^\mu\chi\partial_\nu\phi + \partial^\mu\phi\partial_\nu\chi\right), \tag{110}$$

where $\chi \equiv \Box\phi$. Thus, identification of the mimetic field with time on a flat FLRW background implies that $\chi = 3H$. There are two main effects of the introduction of the specific HD term on the theory: one at the level of the background and one at the level of perturbations. At the level of the background, it can easily be shown that the Friedmann equation is modified from (32) to [73]

$$2\dot{H} + 3H^2 = \frac{2}{2-3\gamma}V, \tag{111}$$

which corresponds to a renormalization of the amplitude of the potential. At the level of perturbations, it can be shown that (108) is modified to the following [73]:

$$\ddot{\delta\phi} + H\dot{\delta\phi} - \frac{c_s^2}{a^2}\Delta\delta\phi + \dot{H}\delta\phi = 0, \tag{112}$$

where

$$c_s^2 = \frac{\gamma}{2-3\gamma}. \tag{113}$$

Therefore, the addition of the higher derivative term results in a small but nonvanishing sound speed, which implies that the behaviour of mimetic matter deviates from the usual perfect fluid dust. The nonvanishing sound speed also results in the possibility of defining quantum perturbations in a sensible way. It is beyond the scope of our review to provide further technical details on the matter, but it can be shown that the simple model we have just described is capable of producing red-tilted (i.e., $n_s < 1$) scalar perturbations which are enhanced over gravity waves (implying a small value of r), which is consistent with observations from Planck and BICEP2/Keck Array [73].

In concluding this brief interlude, let us also spare a few words on further modifications of mimetic gravity involving higher derivative terms. These have been studied in [84, 85], in particular by adding the previously discussed $(\Box\phi)^2$ term, as well as a term proportional to $\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi$. It was noticed that these terms affect the growth of perturbations below the sound horizon, in particular suppressing the growth of those with large momenta. The result is the presence of a cut-off in the matter power spectrum for perturbations below a certain wavelength. On larger scales, instead, the predictions for the matter power spectrum match those of collisionless cold dark matter.

The suppression of power on small scales is particularly intriguing in the light of the observation that the collisionless cold dark matter paradigm appears to suffer from a number of shortcomings on subgalactic scales. The core-cusp problem refers to the discrepancy between N-body simulations of collisionless cold dark matter, which predict a cuspy profile for the dark matter halo in galaxies, and observations which instead suggest a cored profile towards the center. The discrepancy is particularly large for dwarf galaxies, but indications that a cored profile is favoured for larger galaxies as well persist (see, e.g., [456–460]). Moreover, these same simulations predict an abundance of substructure which is approximately 10 times larger than what we actually observe, an issue which is referred to as the "missing satellites problem" (see, e.g., [461–463]). Although this problem is most acute for satellite galaxies, it exists for field galaxies as well (e.g., [464]). To make matters worse, the most massive subhalos in these simulations find no observed counterpart, despite one would expect star formation to be more efficient within them: this is known as the "too big to fail problem" (see, e.g., [465–468]). For an incomplete list of comprehensive reviews on these issues, refer, for instance, to [469, 470] and references therein.

Several approaches to solving these problems exist in the literature. If one insists that dark matter is cold and collisionless, then an important role must be played by the baryonic content of the universe. In fact, it has been argued in several works that baryonic feedback processes (see, e.g., [471–480]) due to supernovae or to the stellar and gaseous content of galaxies, or dynamical friction between dark matter and baryon clumps (see, e.g., [481, 482]), can in principle solve or at least alleviate these problems. Alternatively, the small-scale discrepancies might be taken as an indication that something is lacking in the collisionless cold dark matter picture, with DM possibly having sizeable self-interactions. This approach has been undertaken in a number of works; see, for instance, [483–492] and references therein. In particular, a possibility which has received a lot of attention recently is that where the paucity of structure on small scales is explained by modifying the properties of dark matter in such a way that the resulting matter power spectrum is suppressed at large wavenumbers, by coupling the dark matter to a bath of dark radiation (e.g., a massless or light dark photon) or to a light scalar, which delays kinetic decoupling; refer, for instance, to [493–507] and references therein.

A different mechanism, but with similar outcomes, occurs in mimetic gravity. Namely, the suppression of small-scale power, operated by the higher derivative terms, has the potential to solve the missing satellite problem and the too big to fail problem, as shown in [84]. Moreover, the same higher derivative terms could potentially also cure the caustic singularities from which mimetic gravity suffers. The reason for this is that the higher derivative terms effectively correspond to terms parametrizing dissipation, or viscosity, which emerges due to the fact that the velocity dispersion of the dust is now nonzero. Actually, in [84] it was also argued that these same dissipative terms could alleviate the core-cusp problem. Although these discussions remain preliminary, it is very interesting to note that modifying the action for mimetic gravity by the addition of higher derivative terms could provide a solution to the small-scale structure puzzles of collisionless cold dark matter.

5. A Case Study: Mimetic Horndeski Covariant Hořava-Like Gravity

Having discussed in detail the physics behind mimetic gravity, and many of its extensions, we now provide a detailed case study of a specific mimetic model. Our choice falls on the covariant Hořava-like theory of gravity first discussed by Cognola et al. [367], with action defined by (103). We will argue that this model can be viewed as a mimetic Horndeski model. We will study the model, its background solutions, and scalar perturbations in detail. Not unexpectedly, we will find that the sound speed of the minimal model is vanishing, and thus quantum perturbations cannot be defined in the usual way. To circumvent this problem, we modify the model by the addition of higher order terms, repeating the analysis of background solutions and scalar perturbations, and show that it will be necessary to go beyond the Horndeski framework in order to have a nonvanishing sound speed. The discussion in this section is largely based on the work of [131].

5.1. Mimetic Covariant Hořava-Like Gravity. Let us start from the action of the CRG-like model first discussed by Cognola et al. [367], which is given by the following:

$$\begin{aligned} I = \frac{1}{2}\int d^4x\sqrt{-g}\,\Big\{R - 2\Lambda & \\ - \alpha\left[\left(R^{\mu\nu} - \frac{\beta}{2}Rg^{\mu\nu}\right)\nabla_\mu\phi\nabla_\nu\phi\right]^n & \\ - \lambda\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + U_0\right)\Big\}, & \end{aligned} \tag{114}$$

where α, β are arbitrary constants, Λ is the cosmological constant, n is a natural number, λ is a Lagrange multiplier, and U_0 determines the constraint imposed on the gradient of the cosmological field ϕ. Recall that the Horndeski-like coupling in the action above, for $n = 1$, has been considered in several works (e.g., [420, 433, 447–455]). If we set $U_0 = 1/2$, we see that the constraint on the gradient of the scalar field corresponds precisely to that of mimetic gravity (3). Thus, we can extend the model to include the mimetic field by adding a potential for the scalar field, which is now made dynamical. The action now reads

$$\begin{aligned} I = \frac{1}{2}\int d^4x\sqrt{-g}\,\Big\{R - 2\Lambda & \\ - \alpha\left[\left(R^{\mu\nu} - \frac{R}{2}g^{\mu\nu}\right)\nabla_\mu\phi\nabla_\nu\phi\right]^n & \\ - \frac{\lambda}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + 1\right) - V(\phi)\Big\}, & \end{aligned} \tag{115}$$

where we have set the dimensionless parameter $\beta = 1$ and added a potential for the mimetic field, $V(\phi)$. Variation with respect to λ immediately leads to (3), whereas variation with respect to the field modifies (7) to

$$\begin{aligned} \frac{dV(\phi)}{d\phi} &= \nabla_\mu\left[\left(2n\alpha F^{n-1}G^{\mu\nu} + \lambda g^{\mu\nu}\right)\partial_\nu\phi\right] \\ &= \frac{1}{\sqrt{-g}}\partial_\mu\left\{\sqrt{-g}\left[\left(2n\alpha F^{n-1}G^{\mu\nu} + \lambda g^{\mu\nu}\right)\partial_\nu\phi\right]\right\}, \end{aligned} \tag{116}$$

where we define the following quantities:

$$\begin{aligned} F &\equiv T_{\mu\nu}R^{\mu\nu} - \frac{RT}{2}, \\ T_{\mu\nu} &\equiv \nabla_\mu\phi\nabla_\nu\phi,\quad T \equiv g^{\mu\nu}T_{\mu\nu} = -1. \end{aligned} \tag{117}$$

Finally, variation of the action with respect to the metric leads to the gravitational field equations, which for this theory read

$$\begin{aligned} G_{\mu\nu} &+ \Lambda g_{\mu\nu} + \frac{\alpha}{2}F^n g_{\mu\nu} \\ &= n\alpha F^{n-1}\left[R^\rho_\mu T_{\rho\nu} + R^\rho_\nu T_{\rho\mu} - \frac{1}{2}\left(TR_{\mu\nu} + RT_{\mu\nu}\right)\right] \\ &+ \frac{\lambda}{2}T_{\mu\nu} \\ &+ n\alpha\left[D_{\alpha\beta\mu\nu}\left(T^{\alpha\beta}F^{n-1}\right) - \frac{1}{2}D_{\mu\nu}\left(TF^{n-1}\right)\right] \\ &+ \Omega^{\alpha\beta}\frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}} - g_{\mu\nu}\frac{V(\phi)}{2}, \end{aligned} \tag{118}$$

where we have defined the differential operators:

$$\begin{aligned} D_{\alpha\beta\mu\nu} &\equiv \frac{1}{4}\Big[\left(g_{\mu\alpha}g_{\nu\beta} + g_{\nu\alpha}g_{\mu\beta}\right)\Box \\ &+ g_{\mu\nu}\left(\nabla_\alpha\nabla_\beta + \nabla_\beta\nabla_\alpha\right) \\ &- \left(g_{\mu\alpha}\nabla_\beta\nabla_\nu + g_{\nu\alpha}\nabla_\beta\nabla_\mu + g_{\mu\beta}\nabla_\alpha\nabla_\nu + g_{\nu\beta}\nabla_\alpha\nabla_\mu\right)\Big], \\ D_{\mu\nu} &\equiv g_{\mu\nu}\Box - \frac{1}{2}\left(\nabla_\mu\nabla_\nu + \nabla_\nu\nabla_\mu\right), \end{aligned} \tag{119}$$

where $\Box \equiv \nabla^i\nabla_i$ is the d'Alembertian operator. Note that in the above (118), $\Omega_{\mu\nu}$ is a tensor that will not play any role if $T_{\mu\nu}$ does not depend on the metric, which we assume is our case; thus, we can drop it from the gravitational field equations. Finally, the form of the Lagrange multiplier λ can be determined from the trace of (118):

$$\begin{aligned}
&-R+4\Lambda-\frac{\lambda}{2}T \\
&\quad=2\alpha F^n(n-1) \\
&\qquad+\frac{n\alpha}{2}\left(g_{\mu\nu}\Box+\nabla_\mu\nabla_\nu+\nabla_\nu\nabla_\mu\right)\left(T^{\mu\nu}F^{n-1}\right) \\
&\qquad-\frac{3n\alpha}{2}\Box\left(TF^{n-1}\right)-2V(\phi).
\end{aligned} \tag{120}$$

We will consider the case with $n = 1$, which is particularly interesting given that it is equivalent to a specific instance of a mimetic Horndeski model.

5.1.1. Cosmological Solutions. Let us now consider cosmological solutions, in particular considering a flat FLRW metric (10). If we take the hypersurfaces of constant time to be equal to those of constant ϕ, by making use of the mimetic constraint (3), we see that the field can be identified (up to an integration constant) with time:

$$\phi = t. \tag{121}$$

As in the original mimetic gravity, the scalar field can induce an effective cold dark matter component given that, from (7), one has $-(G-T) \propto a^3$, and recall that $-(G-T)$ gives the energy density of the perfect dust-like fluid induced by mimetic field (or, more precisely, by its gradient which plays the role of 4-velocity field). However, the introduction of additional terms depending on the field in the action (115) changes these features. Here, we would like to study the behaviour of mimetic field and the cosmological solutions of the more involved theory (115).

Let us begin by considering the tensor $T_{\mu\nu}$ in (117), which reads

$$\begin{aligned}
&T_{00}=\dot{\phi}^2=1, \\
&T_{0i}=T_{i0}=T_{ij}=0, \quad i,j=1,2,3.
\end{aligned} \tag{122}$$

The, from (116) we derive

$$\frac{1}{a^3}\partial_0\left[a^3\left(2n\alpha\left(3H^2\right)^n-\lambda\right)\right]=\frac{dV(\phi)}{d\phi}. \tag{123}$$

Given the mimetic constraint (3), this equation is a consequence of the two EOMs inferred from (118):

$$0=\Lambda-3H^2+\frac{\alpha}{2}(1-4n)\left(3H^2\right)^n+\frac{\lambda}{2}+\frac{V(\phi)}{2}, \tag{124}$$

$$\begin{aligned}
0&=\Lambda-3H^2-2\dot{H}+\frac{\alpha}{2}(1-2n)\left(3H^2\right)^n \\
&\quad+3^{n-1}\alpha n(1-2n)\dot{H}H^{2n-2}+\frac{V(\phi)}{2}.
\end{aligned} \tag{125}$$

In our analysis, as the second independent equation, together, with (123), we choose the trace equation (120): namely,

$$\begin{aligned}
\frac{\lambda}{2}&=6\dot{H}+12H^2-4\Lambda+\alpha(5n-2)\left(3H^2\right)^n \\
&\quad+3^n n\alpha(2n-1)H^{2n-2}\dot{H}-2V(\phi).
\end{aligned} \tag{126}$$

For the simplest case $V(\phi)=0$, from (123) we get

$$\left[2n\alpha\left(3H^2\right)^n-\lambda\right]=\frac{C_0}{a^3}, \tag{127}$$

where C_0 is an integration constant. Thus, in the limit $\alpha = 0$, we recover the original mimetic gravity model of [69], while when $\alpha \neq 0$ we may interpret this equation as a generalized Friedmann equation, with C_0 determining the amount of dark matter in the universe (given that its contribution scales with scale factor a as a^{-3}, as is expected for dust). Moreover, in the limits given by

$$\begin{aligned}
&1 \ll \alpha, \quad n=1, \\
&\frac{1}{\alpha^{n-1}} \ll R, \quad 1<n,
\end{aligned} \tag{128}$$

if we neglect the cosmological constant Λ, as well as the integration constant C_0, one gets from (126)

$$-\frac{\dot{H}}{H^2}=\frac{2}{n}. \tag{129}$$

Thus, if $1/\alpha^{n-1}$ is set at the scale of the early-time acceleration, models with $1 \ll n$ lead to cosmological solutions for inflation.

In the case where we set $n = 1$, the parameter α is dimensionless and the EOMs are second order, which is not surprising given that our model is a special case of Horndeski's theory. Therefore, we can incorporate the cosmological constant in the potential and obtain from (123), (126) that

$$\frac{dV}{dt}=\frac{1}{a^3}\partial_0\left[a^3\left(6\alpha H^2-\lambda\right)\right], \tag{130}$$

$$\lambda=6(2+\alpha)\dot{H}+6(4+3\alpha)H^2-4V(\phi), \tag{131}$$

where we have taken into account that $V \equiv V(\phi) = V(t)$ on the FLRW space-time, so that we can replace the potential derivative of the field with its time derivative.

Let us manipulate (125) further, which gives

$$2\dot{H}+3H^2=\frac{V}{(2+\alpha)}, \tag{132}$$

showing that the model we are considering is essentially equivalent to the model proposed in [73]. Since (130) is a consequence of (131) and (132), we may choose to infer the cosmological solutions from (132) only. This equation is a nonlinear Riccati type equation and can be transformed in the linear second-order differential equation:

$$\ddot{u}-\frac{3}{4(2+\alpha)}Vu=0, \tag{133}$$

by introducing the Sturm-Liouville canonical substitution

$$H = \frac{2}{3}\frac{\dot{u}}{u}, \qquad a = u^{2/3}. \tag{134}$$

Let us discuss some examples by starting from (133). If $V = V_0$ is constant, we recover the de Sitter solution:

$$u \sim \exp\left[\frac{3H_0 t}{2}\right], \qquad a \sim \exp\left[H_0 t\right], \qquad H_0 = \frac{2}{3}\sqrt{\frac{3V_0}{4(2+\alpha)}}. \tag{135}$$

Another well-motivated choice for the potential is a quadratic one:

$$V(\phi) = 3(2+\alpha)\left[H_0^2 + \beta^2(2\phi - \phi_0)\left(-H_0 + \frac{\beta^2}{4}(2\phi - \phi_0)\right) - \frac{2}{3}\beta^2\right], \tag{136}$$

where H_0 is a constant Hubble parameter and β, ϕ_0 are dimensional constants (in the specific case $[\beta] = [\phi_0^{-1}] = [H]$). After the identification $\phi = t$ the explicit solution for u is found to be

$$u(t) = u_0 e^{(3/2)H_0 t - (3\beta^2/4)t(t - 2t_0)}, \quad t_0 \equiv \frac{\phi_0}{2}, \tag{137}$$

with u_0 constant and t_0 a fixed time. The Hubble parameter is given by

$$H \equiv \frac{2}{3}\frac{\dot{u}}{u} = H_0 - \beta^2(t - t_0), \tag{138}$$

and we see that, for t close to t_0, one has a quasi-de Sitter expansion, while for large $t_0 \ll t$, the Hubble parameter tends to vanish. This solution corresponds to a Starobinsky-like accelerated expansion [188] in the Jordan frame and gives an interesting inflationary solution.

Let us provide one final example with a potential given by

$$V(\phi) = \frac{4A^2(2+\alpha)}{3}\frac{\cosh A\phi}{1 + \cosh A\phi}, \tag{139}$$

where $0 < A$ is a constant with mass-dimension 1. The corresponding solution is given by

$$u(t) = 1 + \cosh At, \qquad a = (1 + \cosh At)^{2/3}, \qquad H = \frac{2A}{3}\frac{\sinh At}{(1 + \cosh At)}. \tag{140}$$

This solution represents a cosmological bounce with $-\infty < t, \phi < +\infty$ and shows that the mimetic field may act as a phantom fluid.

5.1.2. Cosmological Scalar Perturbations. In this section, we will consider the scalar perturbations around the FLRW metric (10) in the model defined by (115) with $n = 1$, analyzed in the previous section. The perturbed metric in conformal Newtonian gauge is given by (104). Concerning perturbations to the mimetic field, we obtain once more (106) which leads to (107) once more. Note that in this case the identity (105), which is valid for the original mimetic dark matter model, is no longer true. Furthermore, we notice that

$$T_{00} = 1 + 2\delta\dot{\phi}, \qquad T_{0i} = \partial_i \delta\phi, \qquad T = -1 + \mathcal{O}\left(\Phi^2\right). \tag{141}$$

From the $(1,2)$-component of (118), we obtain

$$G_{12}\left(1 - \frac{\alpha}{2}\right) = \alpha D_{\alpha\beta 12} T^{\alpha\beta}, \tag{142}$$

with

$$G_{12} = -\partial_x \partial_y (\Phi - \Psi), \qquad D_{\alpha\beta 12} T^{\alpha\beta} = H\partial_x \partial_y \delta\phi + \partial_x \partial_y \delta\dot{\phi}. \tag{143}$$

Therefore, we obtain

$$\Psi = \Phi + \left(\frac{2\alpha}{2-\alpha}\right)\left(H\delta\phi + \delta\dot{\phi}\right). \tag{144}$$

From $(0,1)$-component of (118) we derive

$$G_{01}\left(1 + \frac{\alpha}{2}\right) = \alpha\dot{H}\partial_x \delta\phi + \frac{\lambda}{2}\partial_x \delta\phi + \alpha D_{\alpha\beta 01} T^{\alpha\beta}, \tag{145}$$

with

$$G_{01} = 2\partial_x\left(\dot{\Psi} + H\Phi\right), \qquad D_{\alpha\beta 01} T^{\alpha\beta} = -\left(H^2 + \dot{H}\right)\partial_x \delta\phi, \qquad \lambda = 6\alpha H^2 - 4\dot{H} - 2\alpha\dot{H}, \tag{146}$$

where the last equality is a consequence of (131), (132).

Finally, we can obtain a closed equation for $\delta\phi$, which reads

$$\delta\ddot{\phi} + H\delta\dot{\phi} + \dot{H}\delta\phi = 0. \tag{147}$$

From the above we immediately read that the sound speed is vanishing, given that there is no dependence on the Laplacian of $\delta\phi$. The implications are that in these kinds of models, scalar perturbations do not propagate (as in the original mimetic model of [73]), rendering the usual definition of quantum perturbations quite problematic. Given that this feature is rooted in the model being of the mimetic Horndeski form [128], to address the problem, either we must take $n \neq 1$ in (103) or we must modify the original action along the lines of [73, 84].

5.2. Modified Higher Order Mimetic Horndeski Model. With our goal being that of addressing the problem of scalar perturbations, we modify the model given by (115) for $n = 1$ as

$$I = \frac{1}{2}\int d^4x\sqrt{-g}\left[R\left(1 + ag^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi\right) - \frac{c}{2}\left(\Box\phi\right)^2 + \frac{b}{2}\left(\nabla_\mu\nabla_\nu\phi\right)^2 - \frac{\lambda}{2}\left(g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + 1\right) - V\left(\phi\right)\right]. \quad (148)$$

The original model given by (103) for $n = 1$ is recovered for $a = \alpha/2$ and $b = c = 2\alpha$ by using the following identity [508]:

$$-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi R + \left(\Box\phi\right)^2 - \left(\nabla_\mu\nabla_\nu\phi\right)^2 = G^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi + \text{total derivative}. \quad (149)$$

On the other hand, for generic values of a, b, c, the action (148) describes a higher order derivative model in the scalar sector with field equations at the fourth order; namely,

$$\begin{aligned}(1-a)\,G_{\mu\nu} &= \frac{1}{2}g_{\mu\nu}\left[\frac{b}{2}\phi^{\alpha\beta}\phi_{\alpha\beta} - \frac{c}{2}\left(\Box\phi\right)^2 - V\left(\phi\right)\right] \\ &+ \lambda\nabla_\mu\phi\nabla_\nu\phi - b\phi_{\mu\rho}\phi^\rho_\nu + \frac{b}{2}g^{\alpha\beta}\left[\nabla_\alpha\left(\phi_{\mu\nu}\nabla_\beta\phi\right)\right. \\ &\left. - \nabla_\alpha\left(\phi_{\mu\beta}\nabla_\nu\phi\right) - \nabla_\alpha\left(\phi_{\nu\beta}\nabla_\mu\phi\right)\right] + c\left[\phi_{\mu\nu}\right. \\ &\left. + g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\left(\Box\phi\nabla_\beta\phi\right) - \nabla_\mu\Box\phi\nabla_\nu\phi - \nabla_\nu\Box\phi\nabla_\mu\phi\right],\end{aligned} \quad (150)$$

where we adopted the notation

$$\phi_{\alpha\beta} \equiv \nabla_\alpha\nabla_\beta\phi. \quad (151)$$

Let us explore some cosmological applications. On a flat FRW metric (10), (132) is modified as follows:

$$2\dot{H} + c_h H^2 = c_v V\left(\phi\right), \quad (152)$$

where

$$c_h \equiv \frac{12a + 3b - 9c - 12}{4a + b - 3c - 4}, \qquad c_v \equiv \frac{2}{4 - 4a - b + 3c}. \quad (153)$$

It is understood that by setting $a = \alpha/2$ and $b = c = 2\alpha$, we recover (132). Similarly to how we proceeded in the previous section, we can rewrite (152) as

$$\ddot{u} - \frac{c_h c_v V\left(\phi\right)}{4}u = 0, \quad (154)$$

where we introduced the auxiliary function u:

$$H = \frac{2}{c_h}\frac{\dot{u}}{u}, \qquad a\left(t\right) = u^{2/c_h}. \quad (155)$$

By choosing the following potential:

$$V\left(\phi\right) = \frac{9}{c_h c_v}\left[H_0^2 + \beta^2\left(2\phi - \phi_0\right)\left(-H_0 + \frac{\beta^2}{4}\left(2\phi - \phi_0\right)\right) - \frac{2}{3}\beta^2\right], \quad (156)$$

with H_0, β, ϕ_0 dimensional constants, we recover the Starobinsky-like solution (137) and (138). If we choose the following potential:

$$V\left(\phi\right) = \frac{4A^2}{c_v c_h}\frac{\cosh A\phi}{1 + \cosh A\phi}, \quad (157)$$

with $0 < A$ being a constant, we recover the bounce solution (140). Therefore, we can recover all the solutions of the Horndeski-like model previously analyzed. We will now study the scalar perturbations around FRW space-time.

5.2.1. Cosmological Scalar Perturbations. If we consider the perturbed metric (104) and the perturbed field (106), we still recover (107) and (141). Therefore, from the (i, j)-components, $i, j = 1, 2, 3$ of (150), we obtain now

$$\Psi = \Phi + \frac{b}{2 - 2a}\left(\delta\dot{\phi} + H\delta\phi\right). \quad (158)$$

Moreover, from the components $(0, i)$ or $(i, 0)$, $i = 1, 2, 3$ of (150), we derive

$$\delta\ddot{\phi} + H\delta\dot{\phi} - \frac{c_s^2}{a^2}\nabla^2\delta\phi + \dot{H}\delta\phi = 0, \quad (159)$$

where the nonvanishing squared sound speed c_s^2 reads

$$c_s^2 \equiv \frac{b - c}{2c_2}, \quad c_2 \equiv \frac{\left(2 + b - 2a\right)\left(4 + 3c - 4a - b\right)}{4\left(a - 1\right)}. \quad (160)$$

We immediately note that $c_s^2 = 0$ when $b = c$ and we recover (147). Since only for $b = c$ does the model fall within the Horndeski class, we can consider $(b - c)$ as being the Horndeski breaking parameter. This also confirms that, in order to obtain a nonvanishing sound speed, it is necessary to go beyond the mimetic Horndeski framework: that is, $b \neq c$. The result is analogous to that obtained in [73, 84].

6. Spherically Symmetric Solutions in Mimetic Gravity

In this section, we will explore static spherically symmetric solutions (SSS) in mimetic gravity. To do so, let us return to the general formulation of mimetic gravity with action given by (11). Variation of the action with respect to the metric with the mimetic constraint (3) simply leads to

$$G_{\mu\nu} = \frac{\lambda}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}\frac{V\left(\phi\right)}{2}, \quad (161)$$

namely, (118) with $\alpha = 0$ and $\Lambda = 0$. The trace of this equation reads

$$R = \frac{\lambda}{2} + 2V\left(\phi\right). \quad (162)$$

Thus, from (161), we get

$$G_{\mu\nu} = \left(R - 2V\left(\phi\right)\right)\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}V\left(\phi\right). \quad (163)$$

The continuity equation of the mimetic field is derived as

$$\frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}\lambda\partial^\nu\phi\right) = \frac{dV\left(\phi\right)}{d\phi}, \quad (164)$$

which is automatically satisfied when (163) holds true.

In this chapter we will consider pseudo-SSS space-times, whose general topological formulation is given by

$$ds^2 = -a(r)^2 b(r) dt^2 + \frac{dr^2}{b(r)} + r^2 \left(\frac{d\rho^2}{1-k\rho^2} + \rho^2 d\phi^2 \right), \tag{165}$$

where $a(r)$, $b(r)$ are functions of the radial coordinate r and the manifold is a sphere when $k = 1$, a torus when $k = 0$, or a compact hyperbolic manifold when $k = -1$. The Ricci scalar in this case reads

$$R = -\frac{1}{r^2} \left[3r^2 b'(r) \frac{a'(r)}{a(r)} + r^2 b''(r) + 2r^2 b(r) \frac{a''(r)}{a(r)} + 4rb'(r) + 4rb(r) \frac{a'(r)}{a(r)} + 2b(r) - 2k \right], \tag{166}$$

where the prime denotes a derivative with respect to r. The symmetries of the EOMs require the mimetic field to be a function of r only: namely, $\phi \equiv \phi(r)$, and from (3) one has

$$\phi'(r) = \sqrt{-\frac{1}{b(r)}}, \tag{167}$$

leading to a pure imaginary expression for the field, which is to be expected from a time-like vector $\partial_\mu \phi$ with temporal component equal to zero. Therefore, it is clear that the correspondence with the dark matter is only formal, which justifies the introduction of the potential to make the mimetic field dynamical and possibly reproduce the dark matter phenomenology, since the four-velocity vector u_μ in (9) cannot be physical. This has been explored in [119] and will be discussed in Section 7.

The $(0,0)$- and $(1,1)$-components of the field equations (164) lead to (to derive the EOMs, one may plug the expression for the Ricci scalar directly into the action to obtain, after integration by parts [509], $\mathscr{L} = 2a(1 - rb'(r) - b(r)) + a(r)r^2\lambda(b(r)\phi'(r)^2 + 1) - a(r)r^2 V(\phi)$. Thus, the derivatives with respect to $a(r)$ and $b(r)$ with (3) lead to (168) and (170), while the derivation with respect to ϕ leads to (171)):

$$k - b'(r) r - b(r) = \frac{V(\phi) r^2}{2}, \tag{168}$$

$$\left(b'(r) r + 2r \frac{a'(r)}{a(r)} b(r) + b(r) - k \right) = \frac{\lambda}{2} b(r) r^2 \phi'(r)^2 - \frac{V(\phi) r^2}{2}, \tag{169}$$

where λ is given by (162).

We can also rewrite (169) with (167)-(168) as

$$4a'(r) b(r) = -\lambda a(r) r. \tag{170}$$

Finally, from (164), one finds the following:

$$\frac{d}{dr} \left(a(r) b(r) \lambda r^2 \phi' \right) = a(r) r^2 \frac{dV(\phi)}{d\phi}. \tag{171}$$

As a first investigation, we will consider the case where potential is equal to zero, that is, vacuum solutions.

6.1. Vacuum Solutions. In this subsection we set $V(\phi) = 0$. From (168) we immediately see that

$$b(r) = \left(k - \frac{r_s}{r} \right), \tag{172}$$

with r_s being a mass scale, either positive or negative, while the second field equation (169) with (162) and (166) leads to

$$a(r) = a_1 + \frac{a_2}{\sqrt{1 - kr_s/r}} \left[\left(\sqrt{1 - \frac{kr_s}{r}} \right) \cdot \log \left[\sqrt{\frac{r}{r_0}} \left(1 + \sqrt{1 - \frac{kr_s}{r}} \right) \right] - 1 \right], \quad k = \pm 1, \tag{173}$$

$$a(r) = a_1 + a_2 \left[2 \left(\frac{r}{r_0} \right)^{3/2} + 3 \right], \tag{174}$$

with a_1, a_2 being constants and r_0 a length scale. If $a_2 = 0$, namely, $R = 0$ and $\lambda = 0$, we recover the topological Schwarzschild solution of General Relativity. When $a_2 \neq 0$, we can pose $a_1 = 0$. Thus, the solution for flat topology ($k = 0$) is as follows:

$$ds^2 = -a_2^2 \left[2 \left(\frac{r}{r_0} \right)^{3/2} + 3 \right]^2 \left(\frac{\tilde{r}_s}{r} \right) dt^2 + \frac{dr^2}{(\tilde{r}_s/r)} + r^2 \left(d\rho^2 + \rho^2 d\phi^2 \right), \quad 0 < \tilde{r}_s, \tag{175}$$

where $\tilde{r}_s = -r_s$ has to be positive to preserve the metric signature. The Ricci scalar is nonzero and reads

$$R = -\frac{6\tilde{r}}{2r^3 + 3(r/r_0)^{3/2} r_0^3}. \tag{176}$$

This solution presents a naked singularity at $r = 0$, as for the corresponding topological case $k = 0$ of the Schwarzschild metric.

The spherical case ($k = 1$) is the more interesting and is given by

$$ds^2 = -a_2^2 \left[\left(\sqrt{1 - \frac{r_s}{r}} \right) \log \left[\sqrt{\frac{r}{r_0}} \left(1 + \sqrt{1 - \frac{r_s}{r}} \right) \right] - 1 \right]^2 dt^2 + \frac{dr^2}{(1 - r_s/r)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right), \tag{177}$$

where we have introduced the polar coordinates θ, ϕ in the angular part (see also [79]). In this case the Ricci scalar reads

$$R = -\frac{1}{r^2 \left[\left(\sqrt{1 - r_s/r} \right) \log \left[\sqrt{r/r_0} \left(1 + \sqrt{1 - r_s/r} \right) \right] - 1 \right]}. \tag{178}$$

For $r_s < 0$ the metric is regular everywhere, but when $0 < r_s$, the solution is regular and preserves the signature; only in the region $r_s < r$ the solution is regular and preserves the signature; since for $r < r_s$ the metric coefficient $g_{00}(r) \equiv -a^2(r)b(r)$ acquires an imaginary part. In this respect, we note that the special choice $r_s = r_0$ when $r < r_s$ leads to

$$\begin{aligned} g_{00}(r) &\equiv a(r)^2 b(r) \\ &= -a_2^2 \left(\sqrt{\frac{r_s}{r} - 1} \arctan\left[\sqrt{\frac{r_s}{r} - 1} \right] + 1 \right)^2, \end{aligned} \tag{179}$$

which is real and negative.

In general, we observe that when $0 < r_s$, the point $r = r_s$ cannot represent a horizon like for the Schwarzschild space-time, due to the fact that $g_{11}(r) \equiv 1/b(r)$ diverges but $g_{00}(r) = -a_2^2$ and the Ricci scalar are regular (we also cannot associate with $r = r_s$ any thermodynamical quantity such as temperature $T = (a(r)/2)db(r)/dr$ which diverges at $r = r_s$); moreover, for $r < r_s$ the metric becomes imaginary (or, in the special case $r_s = r_0$, it acquires the signature $(-{+}{+}{+})$) and does not allow a wormhole description.

Finally, the topological case $k = -1$ reads

$$\begin{aligned} ds^2 = {} & -a_2^2 \left[\left(\sqrt{1 + \frac{r_s}{r}} \right) \log \left[\sqrt{\frac{r}{r_0}} \left(1 + \sqrt{1 + \frac{r_s}{r}} \right) \right] \right. \\ & \left. - 1 \right]^2 dt^2 + \frac{dr^2}{(-1 - r_s/r)} + r^2 \left(\frac{d\rho^2}{1 + \rho^2} \right. \\ & \left. + \rho^2 d\phi^2 \right), \end{aligned} \tag{180}$$

and the Ricci scalar is given by

$$\begin{aligned} & R \\ & = -\frac{1}{r^2 \left[\left(\sqrt{1 + r_s/r} \right) \log \left[\sqrt{r/r_0} \left(1 + \sqrt{1 + r_s/r} \right) \right] - 1 \right]}. \end{aligned} \tag{181}$$

In this case, for $0 < r_s$, the metric coefficient $g_{11}(r) \equiv 1/b(r)$ is negative and the solution is unphysical. On the other hand, for $r_s < 0$, we obtain $0 < g_{11}(r)$ when $r < -r_s$, but $g_{00}(r) \equiv -a^2(r)b(r)$ becomes imaginary (except for the special choice $r_0 = -r_s$).

6.2. Nonvacuum Solutions. In this subsection, we will consider the case $V(\phi) \neq 0$. We immediately see that, for $V(\phi) = 2\Lambda$ with Λ a cosmological constant, one solution of (168) and (169) with $\Lambda = 0$ is the topological Schwarzschild de Sitter solution:

$$\begin{aligned} b(r) &= k - \frac{r_s}{r} - \frac{\Lambda r^2}{3}, \\ a(r) &= a_1, \end{aligned} \tag{182}$$

with r_s, a_1 being constants.

In considering other solutions, we will take the spherical case $k = 1$ in (165). By using (167) one readily obtains from (170) and (171) that

$$\begin{aligned} \phi(r) &= \pm i \int \frac{dr}{\sqrt{b(r)}}, \\ 4\frac{d}{dr}\left(a'(r) b(r)^{3/2} r \right) &= a(r) r^2 \sqrt{b(r)} \frac{dV(r)}{dr}, \end{aligned} \tag{183}$$

where we treat the potential as a function of r. These two equations with (168) can be used to reconstruct the potential when a choice for $b(r)$ is made. We will now provide some examples of the reconstruction technique.

Let us consider a linear modification to the Schwarzschild metric:

$$b(r) = \left(1 - \frac{r_s}{r} + \gamma r \right), \tag{184}$$

with r_s, γ being constants whose mass-dimension is positive, such that from (168) we obtain

$$V(r) = -\frac{4\gamma}{r}. \tag{185}$$

The corresponding solution for the mimetic field is found to be an elliptic function, and explicit expressions can be given only in limiting cases. When $r \approx r_s$, one can neglect the linear correction in (184) to recover the solution in (172) and (173) with $k = 1$. Thus, the mimetic field reads

$$\begin{aligned} & \phi\left(r \ll \sqrt{\frac{r_s}{\gamma}} \right) \\ & \simeq \pm i \left[r \sqrt{1 - \frac{r_s}{r}} + \frac{r_s}{2} \log \left[2r \left(1 + \sqrt{1 - \frac{r_s}{r}} \right) - r_s \right] \right], \end{aligned} \tag{186}$$

or, by expanding the result around $r = r_s$,

$$\begin{aligned} \phi(r \simeq r_s) &\simeq \phi_s \pm 2i \sqrt{r_s (r - r_s)}, \\ r &\simeq r_s - \frac{(\phi_s - \phi)^2}{4 r_s}, \end{aligned} \tag{187}$$

with $\phi_s = \pm(i r_s/2)\log(r_s)$. The explicit form of the potential at $r \simeq r_s$, $\phi \simeq \phi_s$ is found to be

$$V(\phi \simeq \phi_s) \simeq -\frac{4\gamma}{r_s} - \frac{\gamma(\phi_s - \phi)^2}{r_s^3}. \tag{188}$$

On the other hand, for large distances, we may ignore the Newtonian term in (184) and from the second equation in (183) we derive

$$\begin{aligned} & a\left(\sqrt{\frac{r_s}{\gamma}} \ll r \right) \\ & \simeq \frac{c_1(4 + 6\gamma r) + 3c_2\sqrt{1 + \gamma r} - c_2(2 + 3\gamma r)\arctan\left[\sqrt{1 + \gamma r} \right]}{\sqrt{1 + \gamma r}}, \end{aligned} \tag{189}$$

with $c_{1,2}$ dimensional constants. We can choose $c_1 = 1/4$ and $c_2 = 0$, and in the given limit the metric simply reads

$$ds^2\left(\sqrt{\frac{r_s}{\gamma}} \ll r\right) \simeq -\left(1+\frac{3\gamma r}{2}\right)^2 dt^2 + \frac{dr^2}{(1+\gamma r)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{190}$$

The corresponding expressions for the field and the potential are given by

$$\phi(r) \simeq \pm\frac{2i\sqrt{1+r\gamma}}{\gamma}, \quad r \simeq -\frac{4+\gamma^2\phi^2}{4\gamma},$$
$$V(\phi) \simeq \frac{16\gamma^2}{4+\gamma^2\phi(r)^2}. \tag{191}$$

Here, we must require $4/\gamma^2 < |\phi|^2$ in order to guarantee the positivity of r. The behaviour of the potential can be derived by interpolating the expressions in (188) and (191).

The metric under investigation reduces to the usual Schwarzschild space-time for short distances, while at large distances its 00-component behaves as

$$g_{00}\left(\sqrt{\frac{r_s}{\gamma}} \ll r\right) \simeq -\left(1+\frac{3\gamma r}{2}\right)^2. \tag{192}$$

Therefore, the corresponding Newtonian potential $\Phi(r) = -(g_{00}(r)+1)/2$ acquires linear and quadratic contributions with respect to the Newtonian solution. The quadratic correction can be viewed as a negative cosmological constant in the background and can be ignored if $\gamma^2 r^2$ is sufficiently small. On the other hand, the linear term (for $\gamma > 0$) could help in explaining the inferred flatness of galactic rotation curves, which has been interpreted as one of the key evidences for the presence of dark matter. We will return to this issue in the next section and analyze the problem more closely there, where we will provide a metric whose cosmological constant-like contribution is independent from the linear one.

As a second example of reconstruction procedure, we consider the following ansatz:

$$b(r) = 1 - \frac{r_s^2}{r^2}, \tag{193}$$

with r_s being once more a positive dimensional constant. From (168) we get

$$V(r) = -\frac{2r_s^2}{r^4}, \tag{194}$$

and the metric is fixed by making use of the second equation in (183) which leads to

$$a(r) = \frac{\left[c_1 + c_2 \arctan\left[r/\sqrt{r_s^2 - r^2}\right]\right]}{\sqrt{1 - r_s^2/r^2}}. \tag{195}$$

The metric signature for $r_s < r$ is preserved when $c_2 = 0$, and by choosing $c_1 = 1$ we obtain

$$ds^2 = -dt^2 + \frac{dr^2}{(1 - r_s^2/r^2)} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{196}$$

The field and the potential are found to be

$$\phi = \pm ir\sqrt{1 - \frac{r_s^2}{r^2}}, \quad r = \sqrt{r_s^2 - \phi^2},$$
$$V(\phi) = -\frac{2r_s^2}{\left(r_s^2 - \phi^2\right)^2}. \tag{197}$$

The radial coordinate is always real and positive.

The above metric (196) is very interesting, given that it may be used to describe a traversable wormhole (see, e.g., [510–534] and references therein; see also [535–549] for recent work on the subject), where the space is divided into two spherical holes connected by a "throat" located at $r = r_s$. Moreover, the metric satisfies the following requirements:

(1) $g_{00}(r)$ and $g_{11}^{-1}(r)$ are well defined for all $r_s \le r$.
(2) $g_{00}(r)$ is regular on the throat with $g_{00+}(r_s) = g_{00-}(r_s)$ and $g'_{00+}(r_s) = g'_{00-}(r_s)$.
(3) $g_{11}^{-1}(r_s) = 0$ and $0 < g_{11}^{-1}(r)$ for all $r_s < r$.
(4) Given $g_{11}(r)^{-1} = [1 - \tilde{b}(r)/r]$, we have $\tilde{b}'_+(r_s) = \tilde{b}'_-(r_s) < 1$.

The above correspond to the traversability conditions [512, 515], and thus we conclude that the wormhole described by our solution is traversable. We also note that our space-time is asymptotically flat.

In the next section we will use a different approach to fix $a(t)^2 b(r)$, which encodes the physical Newtonian potential, in order to apply mimetic gravity to reproduce the phenomenology of galactic rotation curves.

7. Rotation Curves of Galaxies in Mimetic Gravity

So far we have seen that, within mimetic gravity, dark matter emerges as a geometrical effect at a cosmological level, that is, in the form of a perfect fluid whose energy density decays as a^{-3}. However, one of the first (and probably most renown) clues as to the presence of dark matter came not from cosmological scales but from astrophysical ones. In fact, it was the observations of Vera Rubin, along with her colleagues Kent Ford, David Burstein, and Norbert Thonnard, that galactic rotation curves were asymptotically flat or even slightly growing with radius r, far beyond the region where luminous matter is present, which signalled the presence of something unexpected [4–6]. In fact, if luminous matter was the only thing responsible for the shape of the inferred rotation curves, simple classical mechanics calculations dictate that such curve should fall as $v_{\rm rot}(r) \propto 1/\sqrt{r}$, which clearly did not match what was observed. Therefore, it would be interesting to see whether it is possible, within mimetic gravity, to explain the shape of galactic rotation curves, thus fulfilling the requirements of a successful theory of dark matter on both cosmological and astrophysical scales.

The first solution to this problem was found in [119]. The key idea is closely related to that of MOND [550]: to introduce a new scale in the theory, relating it to either the scale where predictions of Newtonian gravity fail or to a scale intrinsically present in the rotation curves data. The work of [119] has drawn from two classes of examples in the literature: the static solution of conformal Weyl gravity [551], used in such context in [552–555], and solutions in $f(R) = R^n$ gravity [556], used in [557–559]. Let us begin by considering the first cases, which feature linear and quadratic corrections to the Schwarzschild metric.

For our purpose, it is convenient to redefine

$$\tilde{a}(r) = a(r)^2 b(r), \tag{198}$$

within the metric (165) with $k = 1$: namely,

$$ds^2 = -\tilde{a}(t)\,dt^2 + \frac{dr^2}{b(r)} + r^2\left(r^2 d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{199}$$

Since

$$\begin{aligned} a(r) &= \sqrt{\frac{\tilde{a}(r)}{b(r)}}, \\ a'(r) &= \frac{1}{2\sqrt{\tilde{a}(r)\,b(r)}}\left(\tilde{a}'(r) - \tilde{a}(r)\frac{b'(r)}{b(r)}\right), \end{aligned} \tag{200}$$

by using (168) the second equation in (183) reads

$$\begin{aligned} &\frac{d}{dr}\left[\left(\tilde{a}'(r)\,b(r) - \tilde{a}(r)\,b'(r)\right)\frac{r}{\sqrt{\tilde{a}(r)}}\right] \\ &\quad = \sqrt{\tilde{a}(r)}\left[-b''(r)\,r - \frac{2}{r}\left(1 - b(r)\right)\right]. \end{aligned} \tag{201}$$

We choose the following ansatz for $\tilde{a}(r)$:

$$\tilde{a}(r) = 1 - \frac{r_s}{r} + \gamma r - \lambda_0 r^2, \tag{202}$$

with r_s, λ_0, and γ being positive dimensional constants. Given the metric element $g_{00}(r) = -\tilde{a}(r)$, the Newtonian potential reads

$$\Phi(r) = -\frac{\left(g_{00}(r) + 1\right)}{2}. \tag{203}$$

Thus, the Newtonian potential associated with (202) can be described in the following way:

(1) At small distances, the metric leads to a classical Newtonian term r_s/r.

(2) At very large distances, the "cosmological constant" term $\lambda_0 r^2$ emerges, reflecting the fact that the metric is immersed in a cosmological de Sitter background.

(3) At intermediate distance the linear term γr appears, revealing a new feature with respect to the Schwarzschild de Sitter metric at galactic scales.

At intermediate galactic scales, we can safely assume that the $\lambda_0 r^2$ term is negligible; hence, the Newtonian potential (203) reads

$$\Phi(r) \simeq -\frac{r_s}{2r}\left(1 - \frac{\gamma r^2}{r_s}\right). \tag{204}$$

Correspondingly, the rotational velocity profile reads

$$v_{\text{rot}}^2 \simeq v_{\text{Newt}}^2 + \frac{\gamma c^2 r}{2}, \tag{205}$$

where we reintroduced the speed of light c and v_{Newt} is the contribution expected from the luminous matter component. Therefore, on sufficiently large scales, v_{rot} does not fall-off as per the Keplerian result $1/\sqrt{r}$ but increases slightly as $\sqrt{r}$. This is particularly true for galaxies where the falling Newtonian contribution cannot compete with the rising determined by the γ term (depending of course on the size of γ_0), which occurs for small and medium sized low surface brightness (LSB) galaxies. In fact rotation curve results for such galaxies exhibit precisely this behaviour (see, e.g., discussion in [554] in the context of conformal Weyl gravity): that is, the rotation curves of these galaxies start rising immediately.

The situation is different for sufficiently extended galaxies, for instance, large high surface brightness (HSB) galaxies. For these galaxies the Newtonian contribution might be sufficient to complete with the rising linear term, $\propto \gamma r$. This leads to a region of approximate flatness before any rise starts and is consistent with the data for such galaxies (see, e.g., [554]). Moreover, for these galaxies r might be sufficiently large that the de Sitter term $\propto r^2$ should be taken into account. Because of the negative sign, the effect of this term is to reduce the velocity; thus, the rotational velocity profile is given by

$$v^2 \simeq v_{\text{Newt}}^2 + \frac{\gamma c^2 r}{2} - \lambda_0 c^2 r^2. \tag{206}$$

Clearly, sufficiently far from the center of such galaxies, the quadratic term takes over and arrests the rising behaviour driven by the linear term. This is in perfect agreement with data from HSB galaxies, which are large enough to feel the effect of the de Sitter term. Moreover, the negative sign in front of this term has another important implication: given that v^2 cannot go negative, bound orbits are no longer possible on scales greater than $R \sim \gamma_0/2\lambda_0$. This could provide a dynamical explanation for the maximum size of galaxies, determined by the interplay between the linear (γ) and the quadratic (λ_0) terms.

Let us turn to the question of reproducing such behaviour in mimetic gravity. In order to reconstruct the complete form of the metric (199), we must use (201) to derive

$$\begin{aligned} &b(r) \\ &= \frac{\left(1 - r_s/r + \gamma r - \lambda_0 r^2\right)\left(1 - 3r_s/r + \gamma r/3 + c_0/r^2\right)}{\left(1 - 3r_s/2r + \gamma r/2\right)^2}, \end{aligned} \tag{207}$$

with c_0 being a constant. The (on-shell) form of the potential inferred from (168) is quite involved and results in

$$\begin{aligned} V(r) = &-\frac{2}{3r^2\left(2r - 3r_s + \gamma r^2\right)^3}\left[54 r_s^2 r - 27 r_s^3\right. \\ &+ 171\gamma r_s^2 r^2 - 8\gamma^2\lambda_0 r^7 + r^4\left(16\gamma + 7 r_s \gamma^2\right. \\ &\left.+ 324 r_s \lambda_0\right) + 4 r_s r^3\left(-17\gamma - 108 r_s \lambda_0\right) + r^6\left(\gamma^3\right. \end{aligned}$$

$$-44\gamma\lambda_0\Big)+6r^5\left(\gamma^2-12\lambda_0+12r_s\gamma\lambda_0\right)$$
$$-12c_0\left[-r_s+2r\left(1+r_s\gamma\right)+2r^3\left(\gamma^2+\lambda_0\right)\right.$$
$$\left.\left.-\gamma\lambda_0 r^4+3r^2\left(\gamma-3r_s\lambda_0\right)\right]\right]. \tag{208}$$

Moreover, one uses the first equation in (183) to recover the behaviour of the field which can be found only in the limiting cases [119].

In the limit $\gamma = \lambda_0 = 0$ (corresponding to small distances), one has

$$\begin{aligned}\tilde{a}(r) &\simeq 1-\frac{r_s}{r},\\ b(r) &\simeq \frac{4\left(c_0+r\left(r-3r_s\right)\right)\left(r-r_s\right)}{r\left(2r-3r_s\right)^2}.\end{aligned} \tag{209}$$

Thus, if we set

$$c_0=\frac{9r_s^2}{4}, \tag{210}$$

we have

$$\begin{aligned}\tilde{a}(r) &\simeq 1-\frac{r_s}{r},\\ b(r) &\simeq 1-\frac{r_s}{r};\end{aligned} \tag{211}$$

namely, we recover the vacuum Schwarzschild solution of General Relativity. Analogously to (186) and (187) we derive

$$\begin{aligned}&\phi\left(r\simeq r_s\right)\simeq\phi_s\pm 2i\sqrt{r_s\left(r-r_s\right)},\\ &r\simeq r_s-\frac{\left(\phi_s-\phi\right)^2}{4r_s},\end{aligned} \tag{212}$$

with $\phi_s = \pm(ir_s/2)\log[r_s]$. In this case the potential behaves as given in the following:

$$V\left(\phi\simeq\phi_s\right)\simeq-\frac{32\gamma}{3r_s}+\frac{13\gamma\left(\phi-\phi_s\right)^2}{r_s^3}. \tag{213}$$

In the limit $r_s = \gamma = c_0 = 0$ (cosmological scales) we obtain

$$\tilde{a}(r)=b(r)\simeq\left(1-\lambda_0 r^2\right), \tag{214}$$

which corresponds to the static patch of the de Sitter solution. Then, the field assumes the form

$$\phi\simeq\pm i\frac{\arcsin\left[\sqrt{\lambda_0}r\right]}{\sqrt{\lambda_0}},\quad r\simeq\pm\frac{\sin\left[\sqrt{\lambda_0}\left|\phi\right|\right]}{\sqrt{\lambda_0}}. \tag{215}$$

We note that $0 < r$ as long as $0 < r < H_0^{-1}$, where $H_0^{-1} = 1/\sqrt{\lambda_0}$ is the cosmological horizon of the de Sitter solution with positive cosmological constant. The corresponding behaviour of the potential is

$$\begin{aligned}V(\phi)&\\ &\simeq 6\lambda_0\\ &\quad\mp\frac{4\gamma}{3}\left(\frac{\sqrt{\lambda_0}}{\sin\left[\sqrt{\lambda_0}\left|\phi\right|\right]}+4\sqrt{\lambda_0}\sin\left[\sqrt{\lambda_0}\left|\phi\right|\right]\right).\end{aligned} \tag{216}$$

This result with $\gamma = 0$ is consistent with [73].

Finally, in the limit $r_s = \lambda_0 = c_0 = 0$ of galactic scales, the metric reads

$$\begin{aligned}\tilde{a}(r) &\simeq \left(1+\gamma r\right),\\ b(r) &\simeq \frac{4\left(1+\gamma r\right)\left(3+\gamma r\right)}{3\left(2+\gamma r\right)^2}.\end{aligned} \tag{217}$$

In this case, the field is given by

$$\begin{aligned}&\phi\simeq\pm\frac{i}{2\gamma}\sqrt{3\left(3+4\gamma r+\gamma^2 r^2\right)},\\ &r\simeq\frac{-6\mp\sqrt{9-12\gamma^2\phi^2}}{3\gamma},\end{aligned} \tag{218}$$

while the potential in (208) behaves as

$$V(r)\simeq-\frac{2\gamma\left(16+6\gamma r+\gamma^2 r^2\right)}{3r\left(2+\gamma r\right)^3}. \tag{219}$$

The explicit reconstruction of the potential at the galactic scale must take into account that only the solutions with the positive sign inside r yield a positive radius for $0 < \gamma$. Thus, we obtain

$$V(\phi)=-\frac{2\sqrt{3}\gamma^2\left(27-4\gamma^2\phi^2+2\sqrt{9-12\gamma^2\phi^2}\right)}{\left(3-4\gamma^2\phi^2\right)^{3/2}\left(-6+\sqrt{9-12\gamma^2\phi^2}\right)}. \tag{220}$$

In conclusion, we are able to describe, in limiting cases, the behaviour of the potential $V(\phi)$ which leads to the solution in (202) and (207) with the corresponding Newtonian potential in (204). The considered solution turns out to be the Schwarzschild solution at small distances, the static patch of the de Sitter space-time at cosmological distance, and, most intriguingly, presents a linear term at the galactic scales which can address the problem of galactic rotation curves in mimetic gravity. We note that, from the dependence of the potential on the field, the potential is always real, as it is required for consistency.

To fix the values of γ, λ in the metric, one needs the data from rotation curves of galaxies. Since at the galactic scale our metric reproduces the Newtonian potential (204), while asymptotically it turns out to coincide with the de Sitter metric, we can follow the analyses of [554, 555], where the same potential has been found as a result of the Riegert solution [551] of conformal Weyl gravity. Thus, we adopt the results in [554, 555], where the same parameters were fitted to rotation curves. The total sample fitted consists of 138 galaxies, 25 of them being dwarf galaxies. Some of the galaxies in the sample are sufficiently extended as to be sensitive to the de Sitter quadratic term. We refer the reader to [554, 555] and references therein for data on the galaxies included.

As per the analysis of [554, 555] the potential we reconstructed yields an excellent fit to the rotation curves, with a reduced χ^2 of ≈1. In fact, the linear and quadratic corrections to the Newtonian potential capture the falling and rising features in the rotation curves quantitatively rather than barely qualitatively. The best-fit values to our γ and λ_0 parameters in mimetic gravity (202) turn out to be [554, 555]:

$$\gamma \simeq 3.06 \times 10^{-30}\,\mathrm{cm}^{-1}, \qquad \lambda_0 \simeq 9.54 \times 10^{-54}\,\mathrm{cm}^{-2}. \tag{221}$$

The parameter λ_0 is best expressed as $\sim(100\,\mathrm{Mpc})^{-2}$, suggesting that it is most important on scales of large galaxies or clusters.

We previously mentioned that the idea adopted to fit rotation curves resembles that of MOND, that is, to introduce a new scale in the theory, which could be a scale intrinsically present in the data. Let us elaborate on this point more quantitatively. Considering the measured distance R_{last} and rotational velocity v_{last} of the outermost data in the rotation curves, it can be shown that the combination $\gamma_{\text{last}} \equiv v_{\text{last}}/c^2 R_{\text{last}}$ for each of the galaxies in the sample is, within better than an order of magnitude, very close to the best-fit value for γ. In other words, the rotation curve data contain a preferred scale, which we introduced through the non-Newtonian correction.

We conclude this chapter by considering, for completeness, the case of a general power-law correction:

$$\tilde{a}(r) = 1 - \frac{r_s}{r} + \gamma r^m - \lambda_0 r^2, \tag{222}$$

with m being a positive real parameter, with the full metric given by (201); namely,

$$b(r) = \frac{\left(4(2+m)r^2 - 12(2+m)r_s r + 9(2+m)r_s^2 - 4(m-2)\gamma r^{2+m}\right)\left(r - r_s + \gamma r^{1+m} - \lambda_0 r^3\right)}{(2+m)\,r\left(3r_s - 2r + (m-2)\gamma r^{1+m}\right)^2}. \tag{223}$$

In deriving this last expression we have set the integration constant in such a way as to recover the Schwarzschild solution in the limit $\gamma = \lambda_0 = 0$, namely, at short distances. On the other hand, at large distances, in the limit $r_s = \gamma = 0$, we once more find the static patch of the de Sitter solution. Instead, at galactic scales, in the limit $r_s = \lambda_0 = 0$, we obtain

$$\tilde{a}(r) = 1 + \gamma^m r, \qquad b(r) \simeq \frac{4(1+\gamma r^m)(2+m+2\gamma r^m - m\gamma r^m)}{(2+m)\left((m-2)\gamma r^m - 2\right)^2}. \tag{224}$$

The potential can be found only in an implicit way and its on-shell expression results in

$$V(r) = \frac{2m^2\gamma r^{m-2}\left(\gamma^2 m^2 r^{2m} + m\left(8 - 6\gamma r^m - 4\gamma^2 r^{2m}\right) + 4\left(2 + 3\gamma r^m + \gamma^2 r^{2m}\right)\right)}{(2+m)\left((m-2)\gamma r^m - 2\right)^3}. \tag{225}$$

The trivial case $m = 2$ corresponds to a de Sitter-like solution with negative cosmological constant (for $0 < \gamma$):

$$\tilde{a}(r) = b(r) = 1 + \gamma r^2, \quad m = 2. \tag{226}$$

In this case the field is given by $\phi = \pm i\,\mathrm{arcsinh}[\sqrt{\gamma}r]/\sqrt{\gamma}$ and the potential reads $V(\phi) \simeq -6\gamma$.

The non-Newtonian correction we have considered has recently been studied in the context of $f(R)$ gravity in [556]. In the low-energy limit of power-law $F(R) \propto R^n$ gravity, a r^m correction to the Newtonian potential emerges, with m related to the power n as

$$m = \frac{12n^2 - 7n - 1 - \sqrt{36n^4 + 12n^3 - 83n^2 + 50n + 1}}{6n^4 - 4n + 2}. \tag{227}$$

Stability of the potential at large distances and constraints from Solar System tests require $0 < m < 1$. In [556] the above model was fitted to 15 LSB galaxies, and an excellent fit is achieved, with best-fit value $m = 0.817$ (corresponding to $n = 3.5$). Interestingly, the study [557] performs a similar fit to two objects in [557]. The two objects, whose rotation curves cannot be explained successfully by particle dark matter, are the following: the dwarf galaxy Orion and the low luminosity spiral NGC 3198 (which had not been analyzed in [556]). In this case, an excellent fit is obtained for $m = 0.7$.

Let us conclude by making an important remark. Although the potential leading to the chosen non-Newtonian corrections has only been given implicitly, the form of $b(r)$ has been provided explicitly, demonstrating how the desired behaviour for galaxy rotation curves, consistent with data, can be obtained in mimetic gravity. This is a notable result, as it implies that it is possible to reproduce the behaviour of dark matter not only on cosmological scales in mimetic gravity, but on astrophysical scales as well, solving the galaxy rotation curves problem, which is one of the pillars of evidence for dark matter.

8. Conclusion

Mimetic gravity has emerged as an interesting and viable alternative to General Relativity, wherein the dark components of the universe (underlying dark matter, the late-time acceleration, and inflation) can find unified geometrical explanation and interpretation. The theory is related to General Relativity by a singular disformal transformation, which is the reason behind its exhibiting a wider class of solutions. Here, we have reviewed the main aspects of mimetic gravity, beginning by placing it in the wider context of theories of modified gravity. After having reviewed the underlying theory behind mimetic gravity, we have studied some of its solutions and extensions, such as mimetic $f(R)$ gravity, mimetic unimodular gravity, and others, focusing on the reconstruction technique which allows the realization of numerous wishful cosmological expansion scenarios. After having discussed the issue of perturbations within the theory, we have considered a specific mimetic-like model, namely, mimetic covariant Hořava-like gravity, wherein we applied the concepts discussed in the first part of the review. The final part of the review has been devoted to the study of static spherically symmetric solutions within mimetic gravity, and the application of these to the study of rotation curves within such theory.

The dark components of our universe remain as mysterious as ever. It is possible that we might shed light on the nature of dark matter, dark energy, and inflation, as more data from experiments and surveys pours in the coming years. Thus far, theories of modified gravity, despite their *prima facie* complications, appear as viable and theoretically well-motivated routes to pursue. This is true for mimetic gravity as well, and thus we expect it to remain an active arena of research in the field of modified gravity in the coming years.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Research Article

Gravitational Quasinormal Modes of Regular Phantom Black Hole

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We investigate the gravitational quasinormal modes (QNMs) for a type of regular black hole (BH) known as phantom BH, which is a static self-gravitating solution of a minimally coupled phantom scalar field with a potential. The studies are carried out for three different spacetimes: asymptotically flat, de Sitter (dS), and anti-de Sitter (AdS). In order to consider the standard odd parity and even parity of gravitational perturbations, the corresponding master equations are derived. The QNMs are discussed by evaluating the temporal evolution of the perturbation field which, in turn, provides direct information on the stability of BH spacetime. It is found that in asymptotically flat, dS, and AdS spacetimes the gravitational perturbations have similar characteristics for both odd and even parities. The decay rate of perturbation is strongly dependent on the scale parameter b, which measures the coupling strength between phantom scalar field and the gravity. Furthermore, through the analysis of Hawking radiation, it is shown that the thermodynamics of such regular phantom BH is also influenced by b. The obtained results might shed some light on the quantum interpretation of QNM perturbation.

1. Introduction

As a major topic in cosmology, the accelerated expansion of our universe has caused widespread concern in the scientific community. Since the effect of gravity causes the expansion speed to slow down, the accelerated expansion of the universe implies the existence of an unknown form of energy in the universe. The latter provides a repulsive force to push the expansion of the universe. Such unknown energy is called dark energy (DE). Subsequently, a large number of DE models have been proposed, among which the one with cosmological constant is the most famous. Even though the model of DE with the cosmological constant is reasonable in physical theory and consistent with most observations, two difficulties still remain unsolved, namely, how to derive "vacuum energy" from quantum field theory and why the magnitudes of present DE and dark matter are of the same order.

Many modern astrophysics observations indicated the possibility of pressure to density ratio $w < -1$. For example, a model-free data analysis from 172 type Ia supernovae (SNIa) resulted in a range of $-1.2 < w < -1$ for our present epoch [1]. According to the WMAP data during 7 years, $w = -1.10^{+0.14}_{-0.14}(1\sigma)$ [2]. By using the data from Chandra telescope, an analysis of the hot gas in 26 X-ray luminous dynamically relaxed galaxy clusters gives $w = -1.20^{+0.24}_{-0.28}$ [3]. The data on SNIa from the SNLS3 sample estimates $w = -1.069^{+0.091}_{-0.092}$ [4]. In fact, several DE models with a supernegative equation of state provide better fits to the above data [5–8]. And all these approaches are in favor of phantom DE scenario [9–13], in which a constant equation of state parameter is used [14, 15]. This implies that the phantom model might be meaningful for in-depth understanding of DE.

In the phantom model, the signature of the metric is +2, and the action of the model reads

$$S = \int \sqrt{-g} d^4 x \left[R + \epsilon g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 2V(\phi) \right], \tag{1}$$

where R is the scalar curvature, $V(\phi)$ is the potential of the scalar field, and $\epsilon = -1$ corresponds to a phantom scalar field while $\epsilon = +1$ is for a normal canonical scalar field.

Bronnikov and Fabris first investigated the properties of BH with phantom scalar field in vacuum and derived a phantom regular BH solution 10 years ago [16]. Inside the event horizon of such phantom BH there is no singularity similar to the case of regular BHs with nonlinear electrodynamics sources [17]. Outside the event horizon, the properties of a phantom BH are similar to those of a Schwarzschild BH. Due to the absence of the singularity, such phantom regular BH solution has attracted much attention from researchers.

On the other hand, the research of BH perturbation has always been an important issue in BH physics. The first work on QNM in AdS spacetime was about scalar wave in Schwarzschild-AdS spacetime [18], which is then followed by a study on scalar wave in topological AdS spacetime [19]. There are a large number of works on regular BH's QNMs [17, 20–25]. Among various types of perturbation, gravitational perturbation is generally considered to be the most important form due to its practical significance. The intrinsic properties and the stability of a BH can be unfolded through its corresponding gravitational perturbation. In the fifties of last century, Regge and Wheeler began to study the gravitational perturbations of static spherically symmetric BHs. It was pointed out later that [26] the higher dimensional gravitational perturbations can be classified into three types, namely, scalar-gravitational, vector-gravitational, and tensor-gravitational perturbations. The first two types are associated with odd (vector-gravitational) and even (scalar-gravitational) parity in accordance with the spatial inversion symmetry of the perturbations and are of great physical interest [27]. These findings significantly simplify the study of gravitational perturbation of BH. Subsequently, people developed many new methods, and further studies on the gravitational perturbations result in a large number of master equations for various forms of BHs in 4-dimensions [27–30], in higher dimensions [26], and for stationary BHs [31, 32]. In fact, gravitational perturbations of a BH may generate relatively strong gravitational waves (GWs). Recently, the GWs from a binary BH system have been detected by LIGO [33], so BH is proven to be the most probable source of GWs by modern technology. Meanwhile since many alternative theories of gravity can produce the same GW signal within the present accuracy in far field, the reported GW detection still leaves a window for alternative gravity theories [34], which included the theory of phantom BHs. Therefore, the properties of QNMs of gravitational perturbation near the horizon of phantom BH may provide us essential information on the underlying physics of gravity theory. This is the main purpose of the present study.

The thermodynamics of BHs is also an important subject in BH physics. Some works indicated that Hawking radiation can be considered as an effective quantum thermal radiation around the horizon [35, 36], where the corresponding Hawking temperature can be derived from the tunneling rate [35–40]. Furthermore, a natural correspondence between Hawking radiation and QNM has been established recently [35–37, 39, 41]. Therefore, in this work, we will also investigate the Hawking radiation of regular phantom BH.

The paper is organized as follows. In Section 2, we briefly review the regular phantom BH solutions and discuss their properties in three different spacetimes, namely, asymptotically flat, de Sitter (dS), and anti-de Sitter (AdS). In this work, we focus on the odd parity and even parity gravitational perturbations. As the main component of this paper, Section 3 includes two subsections. In Section 3.1, we derive the master equation for odd parity gravitational perturbation and analyze the corresponding temporal evolution of the perturbed metric; in Section 3.2, corresponding studies are carried out for the even parity gravitational perturbation. In Section 4, we calculate the Hawking radiation of the regular phantom BH. We summarize our results and draw concluding remarks in Section 5.

2. The General Metric for Regular Phantom Black Holes

In this section, we discuss the phantom ($\epsilon = -1$) regular BH solution by considering the following static metric with spherical symmetry:

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + p(r)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right). \tag{2}$$

According to the action, (1), the field equation for a self-gravitating minimally coupled scalar field with an arbitrary potential $V(\phi)$ can be expressed as

$$\widehat{G}_{\mu\nu} = R_{\mu\nu} - \frac{g_{\mu\nu}}{2}\left(R - \epsilon\phi^{;\alpha}\phi_{;\alpha} - 2V(\phi)\right) - \epsilon\phi_{;\mu}\phi_{;\nu} = 0. \tag{3}$$

By combining the scalar field equation,

$$\epsilon\phi^{;\alpha}_{;\alpha} - \frac{dV(\phi)}{d\phi} = 0, \tag{4}$$

a regular phantom BH solution can be obtained as

$$f(r) = \left\{\frac{c}{b^2} + \frac{1}{p^2(r)} + \frac{3m}{b^3}\left[\frac{br}{p^2(r)} + \arctan\left(\frac{r}{b}\right)\right]\right\} \cdot p^2(r), \tag{5}$$

where

$$p(r) = \sqrt{b^2 + r^2}, \tag{6}$$

$$V(\phi(r)) = -\frac{c}{b^2}\frac{p^2+2r^2}{p^2} - \frac{3m}{b^3}\left\{\frac{3br}{p^2} + \frac{p^2+2r^2}{p^2}\arctan\left(\frac{r}{b}\right)\right\}, \tag{7}$$
$$\phi(r) = \sqrt{2}\epsilon \arctan\left(\frac{r}{b}\right) + \phi_0,$$

m is the Schwarzschild mass defined in the usual way, and c and b are integration constant and scale parameter, respectively. Then it is necessary to determine the possible kinds of spacetime for such phantom BH, which can be classified as a regular infinity ($r \to \infty$) to be flat, de Sitter (dS), or anti-de Sitter (AdS). The corresponding parameters c, b, m should be restricted in each spacetime.

For the asymptotically flat spacetime, in accordance with (5), one has $c = -3\pi m/2b$ and

$$m = \frac{2b^3}{3\left[\pi b^2 - 2br_h + \pi r_h^2 - 2b^2 \arctan(r_h/b) - 2r_h^2 \arctan(r_h/b)\right]}. \tag{8}$$

In this case, the spacetime is asymptotically flat, namely, $r \to \infty$, $f(r) \to 1$. And r_h is the event horizon of the phantom BH. We note when $b \to 0$ that (5) becomes Schwarzschild flat spacetime.

For the de Sitter spacetime, one has

$$c = -\frac{b^2\left[\left(b^2+r_c^2\right)\arctan(r_c/b) - \left(b^2+r_h^2\right)\arctan(r_h/b) + b(r_c-r_h)\right]}{\left(b^2+r_c^2\right)\left(b^2+r_h^2\right)\arctan(r_c/b) - \left(b^2+r_c^2\right)\left(b^2+r_h^2\right)\arctan(r_h/b) + b(r_c-r_h)\left(b^2-r_cr_h\right)},$$
$$m = \frac{b^3\left(r_c^2-r_h^2\right)}{3\left[\left(b^2+r_c^2\right)\left(b^2+r_h^2\right)\arctan(r_c/b) - \left(b^2+r_c^2\right)\left(b^2+r_h^2\right)\arctan(r_h/b) + b(r_c-r_h)\left(b^2-r_cr_h\right)\right]}, \tag{9}$$

where r_c, r_h are the cosmological horizon and event horizon, respectively. We note when $b \to 0$ that (5) becomes Schwarzschild dS metric.

For the anti-de Sitter spacetime, we choose $f(r) \to r^2$ with $r \to \infty$ without loss of generality. By expanding (5) around infinity, one finds

$$c = \frac{2b^3 - 3\pi m}{2b},$$
$$m = \frac{2b^3\left(b^2+r_h^2+1\right)}{3\left[\pi b^2 - 2b^2\arctan(r_h/b) - 2r_h^2\arctan(r_h/b) - 2br_h + \pi r_h^2\right]}, \tag{10}$$

where r_h is the event horizon of AdS spacetime. We note when $b \to 0$ that (5) can be returned to Schwarzschild-AdS spacetime. In this context, the parameter b measures the coupling strength between phantom scalar field and the gravity for all three spacetimes.

Since the parameters c, m can be expressed in terms of b, r_h, and r_c (dS), the structures of the regular phantom BH spacetime are completely determined by b, r_h, and r_c (dS) (cf. Figure 1). One can readily verify that all the spacetimes are indeed nonsingular even at $r = 0$. As for any asymptotically flat spacetime, such flat regular phantom BH has a Schwarzschild-like structure. However, its tendency of approaching flat spacetime at infinity becomes slower with increasing b. In the dS case, the spacetime is bounded by two horizons, that is, $r_h < r < r_c$. There is a maximum for $f(r)$, and it decreases with increasing b. For the AdS phantom BH, $f(r) \to r^2$ when $r \to \infty$, and for larger b, the approach to the asymptotic solution becomes slower.

3. Gravitational Quasinormal Frequencies for Regular Phantom Black Holes

As proposed by Regge-Wheeler, two important gravitational perturbations are of odd and even parity. The perturbation gauge $h_{\mu\nu}$ for each type has its own definition. In this section, we choose the Regge-Wheeler-Zerilli gauge to discuss the master equation for each perturbation type. Here we take the magnetic quantum $M = 0$ to make φ disappear completely because all values of M lead to the same radial equation [27]. Since the total number of equations in the even parity case is bigger than that in odd parity case, the derivation of the master equation for even parity perturbation is thus more complicated. Once the master equation is derived, the effective potential and the corresponding quasinormal modes can be obtained. We will first discuss the odd parity gravitational quasinormal frequencies in asymptotically flat, dS, and AdS spacetimes and then study the case for even parity. In our work, we consider the metric perturbations not only of the Ricci curvature tensor and scalar curvature (the l.h.s. of the Einstein field equation), but also of the energy momentum tensor (the r.h.s. of the Einstein field equation). On the other hand, we will not consider the perturbations of the phantom scalar field. This is because such perturbations can be canceled out through an appropriate choice of V in the action (see Appendix for details).

In order to discuss gravitational perturbation, one may write down

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}, \tag{11}$$

where the small perturbation $h_{\mu\nu}$ will be divided into odd and even modes in the subsequent sections.

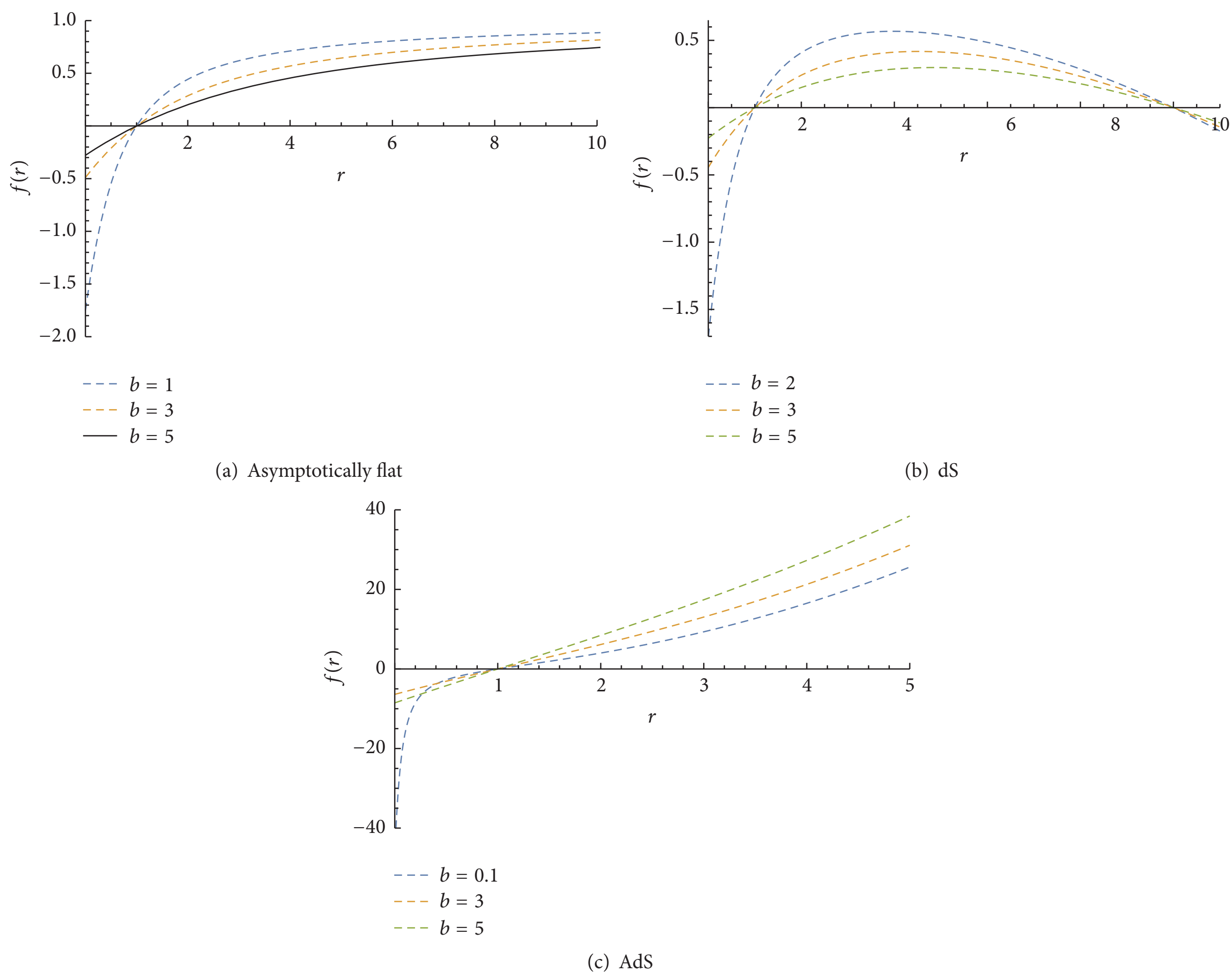

FIGURE 1: The structures of the regular phantom BH's metric function $f(r)$ for different values of b: (a) asymptotically flat spacetime with $r_h = 1$. (b) de Sitter spacetime with $r_h = 1$, $r_c = 10$; (c) anti-de Sitter spacetime with $r_h = 1$.

3.1. Master Equation and Quasinormal Modes for Odd Parity Perturbation. The odd parity perturbation $h_{\mu\nu}$ has the form as [27]

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & h_0(r,t) \\ 0 & 0 & 0 & h_1(r,t) \\ 0 & 0 & 0 & 0 \\ h_0(r,t) & h_1(r,t) & 0 & 0 \end{pmatrix} Q_p(\theta), \tag{12}$$

where $Q_p(\theta) = \sin\theta dP_l(\cos\theta)/d\theta$ ($P_l(\cos\theta)$ is the Legendre function), which satisfies

$$Q_p'' - \cot(\theta) Q_p'(\theta) = -kQ_p(\theta), \tag{13}$$

where $k = l(l+1)$ and l is the angular quantum number.

Then the separation of variables can be carried out by writing $h_0(r,t) = \exp(-i\omega t)h_0(r)$, $h_1(r,t) = \exp(-i\omega t)h_1(r)$. By substituting (5)–(7), (11)–(13) into the field equation (3) and only keeping the first order perturbation terms, we obtain the independent perturbation equations as follows:

$$\begin{aligned} \delta\widehat{G}_{13} = {} & h_0' - \frac{2h_0 p'}{p} - \frac{1}{\omega p^2}\Big\{ih_1\Big(-\omega^2 p^2 \\ & + f\left(-2 + l + l^2 + 2pf'p' + p^2\left(2V(\phi) + f''\right)\right) \\ & + f^2 p\left(\epsilon p \phi'^2 + 2p''\right)\Big)\Big\} = 0, \end{aligned} \tag{14}$$

$$\delta\widehat{G}_{23} = \frac{i\omega h_0}{f^2} + \frac{h_1 f'}{f} + h_1' = 0. \tag{15}$$

Equation (15) implies $h_0 = -(f^2/i\omega)(h_1 f'/f + h_1')$. Substituting h_0 into (14), we get the master equation for odd parity

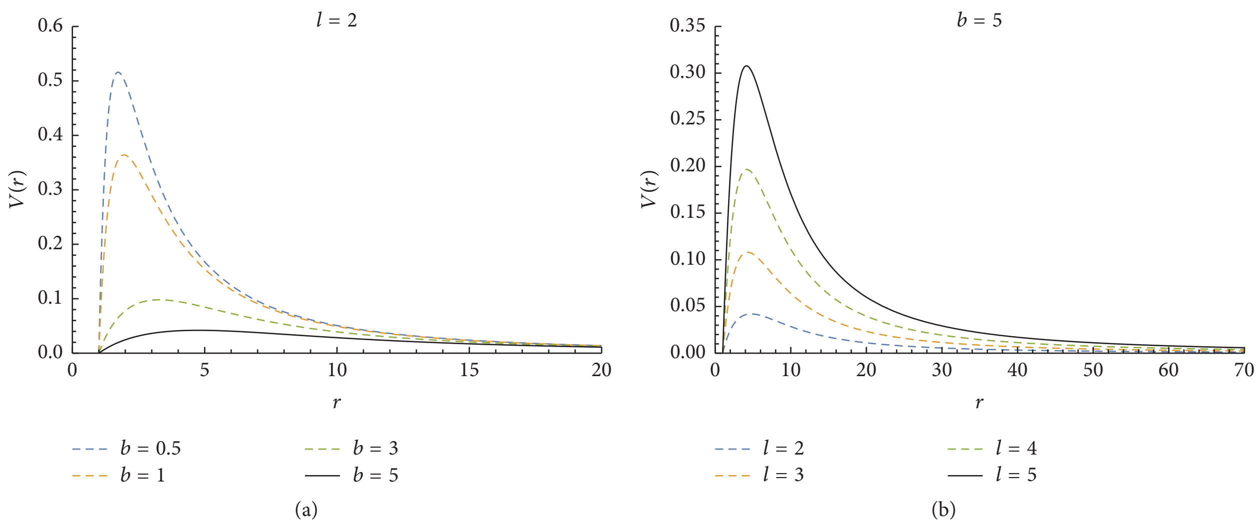

FIGURE 2: The effective potential for odd parity gravitational perturbation in asymptotically flat spacetime, with $r_h = 1,\ l = 2$ (a) and with $r_h = 1,\ b = 5$ (b).

perturbation:

$$h_1'' + \left(\frac{3f'}{f} - \frac{2p'}{p}\right)h_1' + \frac{h_1}{f^2p^2}\left\{p^2\left(\omega^2 + f'^2\right)\right. - f\left(-2 + l + l^2 + 2p^2V(\phi) + 4pf'p'\right) \left. - f^2p\left(\epsilon p\phi'^2 + 2p''\right)\right\} = 0. \tag{16}$$

Finally, we renormalize h_1 by

$$h_1(r) = B(r)\Phi(r); \quad B(r) = \frac{p(r)}{f(r)}. \tag{17}$$

By substituting (17) into the master equation, (16), and using a tortoise coordinate r_*, the Schrödinger-type wave equation for this case can be expressed as

$$\frac{d^2\Phi}{dr_*^2} + \left(\omega^2 - V_o(r)\right)\Phi = 0, \tag{18}$$

where the effective potential for odd parity perturbations $V_o(r)$ is

$$V_o(r) = \frac{f}{p^2}\left[-2 + l + l^2 + 2fp'^2 + p^2\left(2V(\phi) + \epsilon f\phi'^2 + f''\right) + p\left(f'p' + fp''\right)\right]. \tag{19}$$

Equation (19) can be used to describe the effective potential of "odd"-type perturbation V_o in different spacetimes and be utilized to discuss the relationship between the effective potential and model parameters such as the angular harmonic index l and the parameter b.

(I) Figures 2, 3, and 4 show the potential functions, temporal evolution of the gravitational perturbation, and quasinormal frequency obtained by WKB method in asymptotically flat spacetime.

(i) The form of the effective potential as a function of r for different values of l and b is shown in Figure 2. From the left plot, one sees that as b increases, the shape of the effective potential becomes smoother. The maximum of the effective potential decreases with increasing b and the position of the peak shifts to the right. From the right plot, we see that, with the increase of angular quantum number l, the maximum of the effective potential also increases. $V(r)$ is always found to be positive definite outside the event horizon, which indicates that the corresponding QNMs are likely to be stable.

(ii) We adopt the finite difference method to analyze the stability of such BH. By applying the coordinate transformation $(t, r) \rightarrow (\mu, \nu)$ with $\mu = t - r_*$, $\nu = t + r_*$ to (16) and integrating numerically using the finite difference method [24, 42–44], we obtain the differential equation for $h_1(\mu, \nu)$. Figure 3 shows the stability of a regular phantom BH in asymptotically flat spacetime with $r_h = 1$. The temporal evolution of each mode in Figure 3 corresponds to a corresponding case in Figure 2. Since $V(r)$ decreases significantly with the growth of b, the decay rate ($|\mathrm{Im}(\omega)|$) becomes smaller and oscillation frequency (i.e., $\mathrm{Re}(\omega)$) also drops; as l increases, the values of $V(r)$ are raised correspondingly, so that the oscillation frequency (i.e., $\mathrm{Re}(\omega)$) slightly increases together with the decay rate ($|\mathrm{Im}(\omega)|$).

(iii) We employ the WKB approximation [45, 46] to evaluate the quasinormal frequencies.

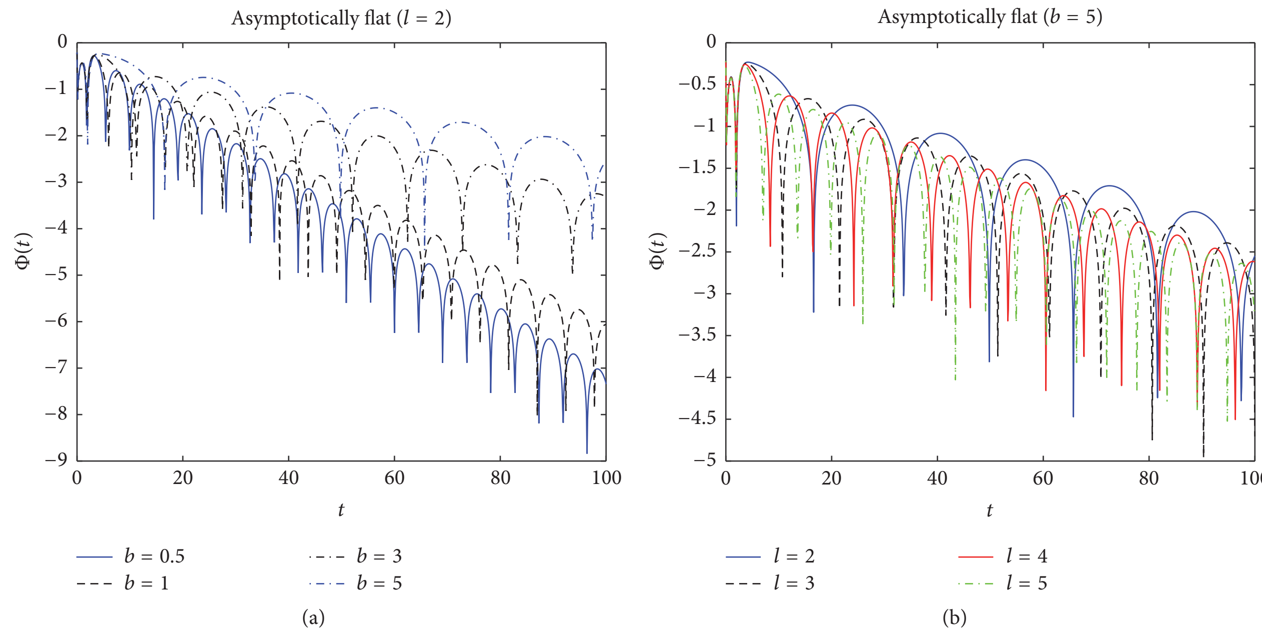

FIGURE 3: The temporal evolution of odd parity gravitational perturbation in asymptotically flat phantom BH, with $r_h = 1,\ l = 2$ (a) and with $r_h = 1,\ b = 5$ (b).

The complex frequency ω is determined by [47]

$$\omega^2 = \left[V_0 + \left(-2V_0''\right)^{1/2} P\right] - i\left(n + \frac{1}{2}\right)\left(-2V_0''\right)^{1/2}(1 + \Omega), \quad \text{(3rd order)},$$
$$i\frac{\omega^2 - V_0}{\sqrt{-2V_0''}} - P - \Omega - P_4 - P_5 - P_6 = n + \frac{1}{2}, \quad \text{(6th order)}, \tag{20}$$

where $V_0^{(n)} = (d^nV/dr_*^n)|_{r_*=r_*(r_h)}$, P, Ω, P_4, P_5, and P_6 are presented in [47, 48].

By making use of (19), we evaluate the QNM frequencies by employing the 6th order WKB method (see Figure 4). Figure 4 shows that the fundamental quasinormal modes ($n = 0$) have the smallest imaginary parts, as the modes decay the slowest. As n increases, the imaginary part of the corresponding quasinormal mode becomes bigger for given l and b. For a given principal quantum number n, both the real and the imaginary parts of the frequency decrease with increasing b; on the other hand, the real part of the frequency increases significantly with the angular quantum number b, while the imaginary part also increases slightly with increasing b.

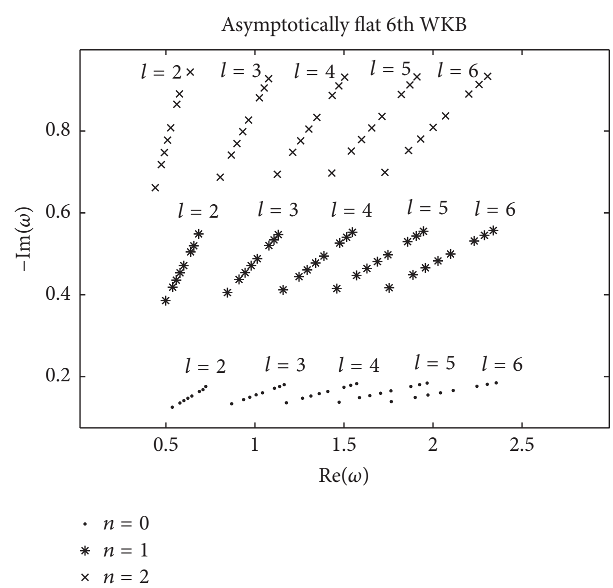

FIGURE 4: Calculated quasinormal frequencies of odd parity gravitational perturbation for $r_h = 1$ in asymptotically flat spacetime. Each group of dots, from top-right to bottom-left, corresponds to QNM frequencies obtained by assuming different values of b = 0.3, 0.4, 0.5, 0.7, 0.8, 0.9, 1.0, 1.2, respectively.

(II) Figures 5, 6, and 7 show the potential function between the event horizon r_h and cosmological horizon r_c, temporal evolution of the gravitational field, and quasinormal frequency obtained by WKB method in dS spacetime.

(i) The form of the effective potential in dS spacetime is similar to that in asymptotically flat spacetime. Figure 5 indicates that, for given l and r, the effective potential decreases with

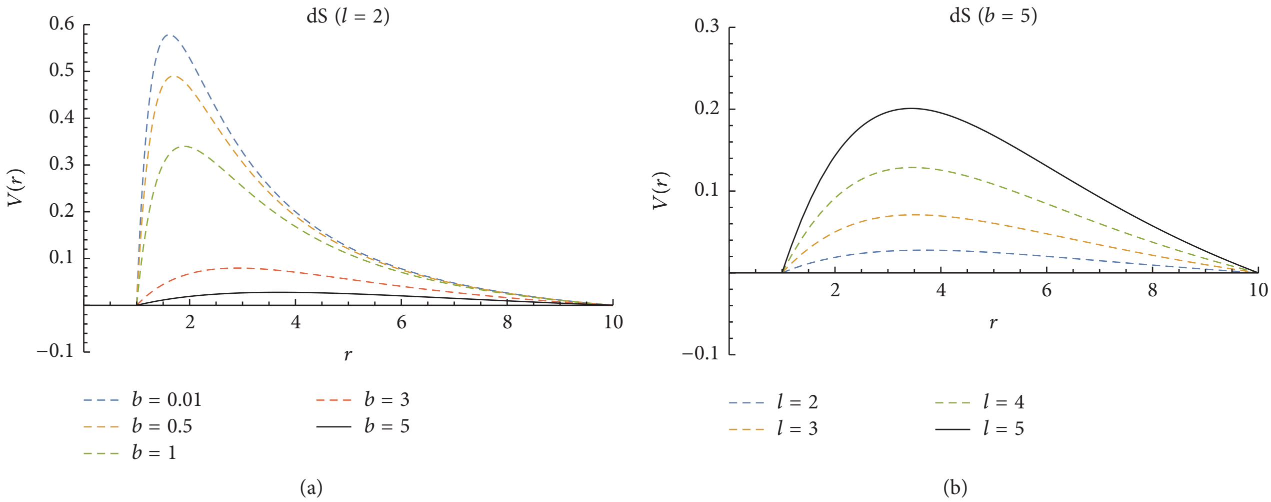

FIGURE 5: The effective potential for odd parity gravitational perturbation in dS spacetime with $r_h = 1,\ r_c = 10,\ l = 2$ (a) and with $r_h = 1,\ r_c = 10,\ b = 5$ (b).

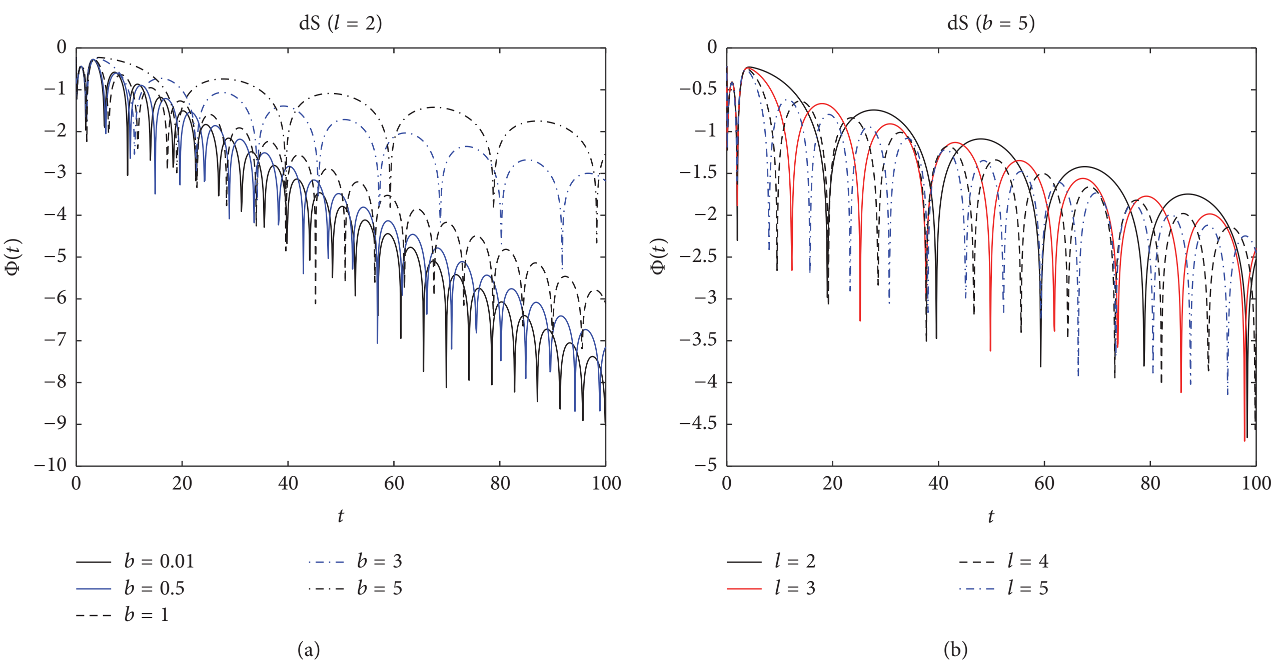

FIGURE 6: The temporal evolution of odd parity gravitational perturbation for dS phantom BH with $r_h = 1,\ l = 2$ (a), and with $r_h = 1,\ b = 5$ (b).

increasing b; and for given b and r, it increases with increasing angular quantum number l. In the range $r_h < r < r_c$, $V(r)$ is also positive definite, which implies that the corresponding QNM is likely to be stable.

(ii) Figure 6 studies the stability of a regular phantom with $r_h = 1,\ r_c = 10$. Similar to the case in asymptotically flat spacetime, it is found that, with increasing b, the oscillation frequency (i.e., $\mathrm{Re}(\omega)$) decreases, while the decay rate ($|\mathrm{Im}(\omega)|$) becomes smaller. Therefore, for smaller value of b, the BH returns to its stable state more quickly as small perturbation dies out faster.

(iii) In accordance with the results of finite difference method, the frequencies presented in Figure 7 by WKB method also show that, for given n and l, the decay rate of perturbation (i.e., $|\mathrm{Im}(\omega)|$) decreases with increasing b. Moreover, it illustrates that, for a given b, the fundamental mode can be found at $n = 0,\ l = 2$. Due to the smallness of the real parts of the frequencies, these modes of oscillation persist for longer time.

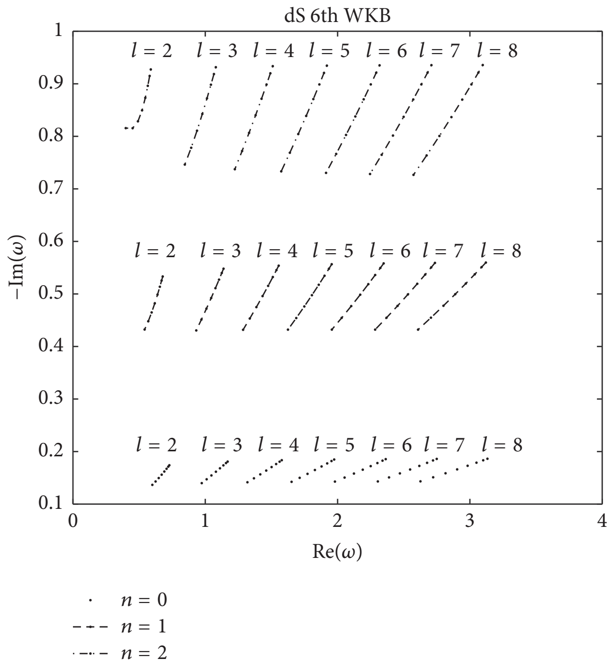

Figure 7: Calculated quasinormal frequencies of odd parity gravitational perturbation with $r_h = 1$, $r_c = 10$ in dS spacetime. Each group of dots, from top-right to bottom-left, corresponds to QNM frequencies obtained by assuming different values of $b = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, respectively.

(III) Figures 8 and 9 show the potential function beyond the event horizon $r > r_h$ and the temporal evolution of the gravitational perturbation in AdS spacetime.

(i) The form of the effective potential in AdS spacetime is quite different from that in asymptotically flat and dS spacetime. Figure 8 indicates that the value of $V(r)$ is divergent as $r \to \infty$. The effects of b and l on the effective potential are similar to previous cases; for a given r, smaller b or larger l leads to bigger V.

(ii) Figure 9 studies the stability of AdS regular phantom BH spacetime with $r_h = 1$. The results are consistent with the above calculated potential function. It is again inferred that the fundamental mode of gravitational perturbation in odd parity occurs for $l = 2$ and larger b, since such kind of QNM will take a longer time to be stable.

3.2. Master Equation and Quasinormal Modes for Even Parity Perturbation. Another canonical form for the gravitational perturbations is of even parity. After applying the separation of variables, it can be expressed as [27, 28]

$$h_{\mu\nu} = \exp(-i\omega t)\begin{pmatrix} H_0(r) f(r) & H_1(r) & 0 & 0 \\ H_1(r) & \dfrac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & p(r)^2 K(r) & 0 \\ 0 & 0 & 0 & p(r)^2 K(r)\sin^2\theta \end{pmatrix} P_l(\cos\theta), \tag{21}$$

where $H_0(r), H_1(r), H_2(r), K(r)$ are unknown functions for the even parity perturbation. It is noted that these functions are not independent.

Now we derive the first order perturbation equations by substituting (5)–(7), (11), and (21) into (3) and find the relationships among $H_0(r), H_1(r), H_2(r), K(r)$:

$$\delta\widehat{G}_{34} = 0 \Longleftrightarrow \qquad H_0 = H_2 \tag{22}$$

$$\delta\widehat{G}_{12} = 0 \Longleftrightarrow \qquad \frac{\chi_1(r)}{i\omega}H_1 - \frac{p'}{p}H_0 + K' + \left(-\frac{f'}{2f} + \frac{p'}{p}\right)K = 0, \tag{23}$$

$$\delta\widehat{G}_{11} = 0 \Longleftrightarrow \qquad -\chi_0(r) H_0 - \frac{p' H_0'}{p} - \frac{\left(-2 + l + l^2\right)K}{2fp^2} + \left(\frac{f'}{2f} + \frac{3p'}{p}\right)K' + K'' = 0, \tag{24}$$

$$\delta\widehat{G}_{23} = 0 \Longleftrightarrow \qquad -\frac{i\omega H_1}{f} - \frac{f' H_0}{f} - H_0' + K' = 0, \tag{25}$$

$$\delta\widehat{G}_{22} = 0 \Longleftrightarrow \qquad \frac{2i\omega H_1}{f} + H_0' - \left(1 + \frac{f' p}{2fp'}\right)K' + \frac{\left(-2 + l + l^2\right)f - 2\omega^2 p^2}{2f^2 pp'}K - \frac{-2 + l + l^2 + 2p^2 V(\phi)}{2fpp'}H_0 = 0, \tag{26}$$

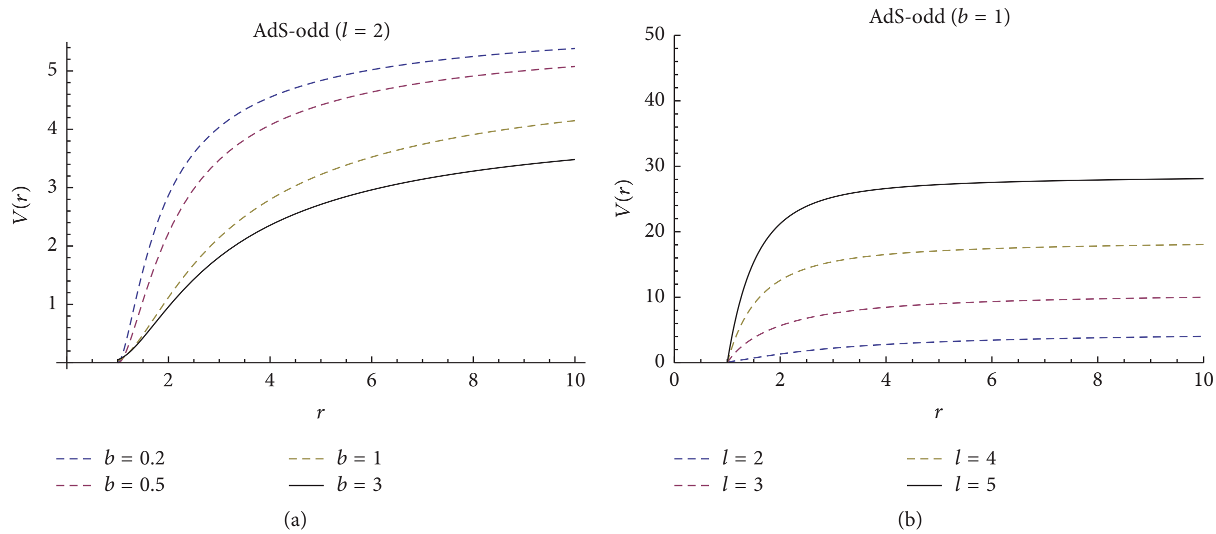

FIGURE 8: The effective potential for odd parity gravitational perturbation in AdS spacetime with $r_h = 1,\ l = 2$ (a) and with $r_h = 1,\ b = 1$ (b).

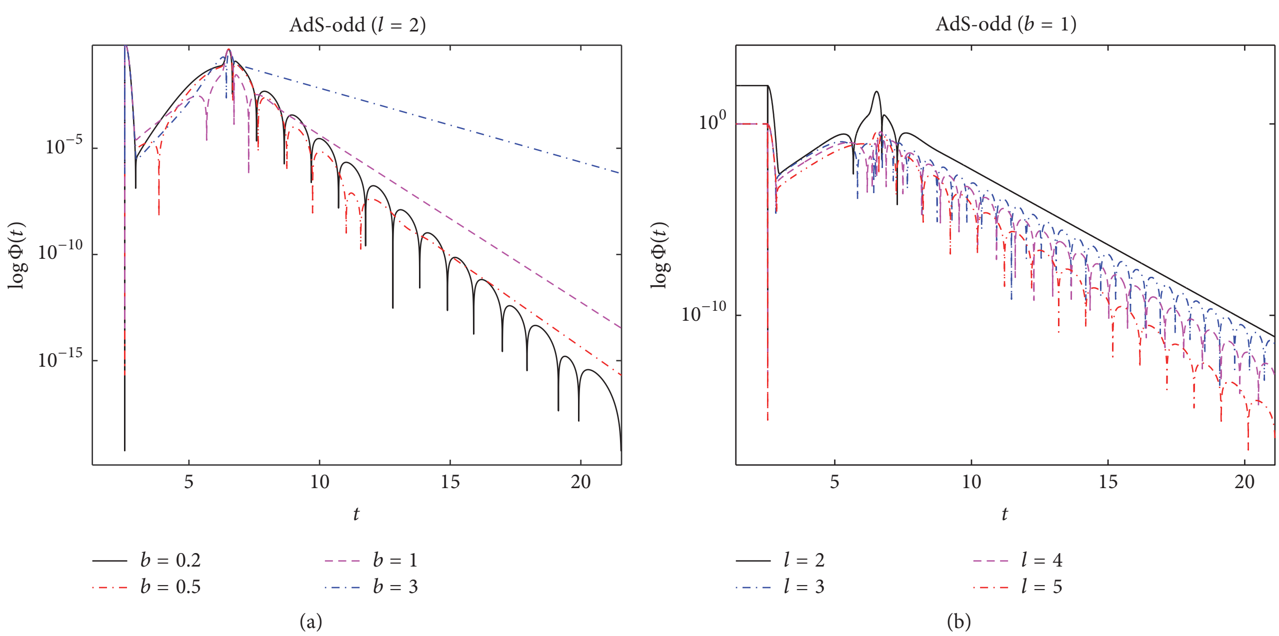

FIGURE 9: The temporal evolution of odd parity gravitational perturbation in AdS phantom BH spacetime with $r_h = 1,\ l = 2$ (a) and with $r_h = 1,\ b = 1$ (b).

where

$$\chi_1(r) = \frac{-2 + l + l^2 + 2fp'^2 + p^2\left(2V(\phi) + \epsilon f\phi'^2\right) + 2p\left(f'p' + 2fp''\right)}{2p^2},$$
$$\chi_0(r) = \frac{-2 + l + l^2 + 4fp'^2 + 2p^2\left(V(\phi) + \epsilon f\phi'^2\right) + 4p\left(f'p' + 2fp''\right)}{2fp^2}. \quad (27)$$

Solving (23), H_0 can be expressed as

$$H_0 = \frac{p\left\{\left(\chi_1(r)/i\omega\right)H_1 + K' - \left(f'/2f - p'/p\right)K\right\}}{p'}. \quad (28)$$

We define a function $\Psi(r)$ satisfying

$$H_1 = -i\omega\frac{r}{f}(\Psi + K). \quad (29)$$

By observing (22), (28), and (29), it turns out that we need to express K by Ψ in order to express all perturbation functions $H_0(r)$, $H_1(r)$, $H_2(r)$, and $K(r)$ in terms of Ψ. This can be achieved by substituting (28) and (29) into (25) and (26) and evaluating the subtraction (25)-(26), and one eventually obtains the following expression:

$$\begin{aligned}&\frac{\left(-2+l+l^2+2p^2V(\phi)+3pf'p'\right)K'}{2fp'^2}\\&\quad-\left(\frac{r\omega^2}{f^2}+\chi_2(r)\right)\Psi\\&\quad-\left(\frac{rp'(r)-p(r)}{f(r)^2p'(r)}\omega^2+\chi_3(r)\right)K=0,\end{aligned} \quad (30)$$

where

$$\begin{aligned}\chi_2(r)&=\frac{r\left(-2+l+l^2+2p^2V(\phi)+2pf'p'\right)}{2f^2p'^2}\chi_1(r),\\ \chi_3(r)&=\frac{1}{4f^2p'^2}\left(f'\left(-4fp'^2+l^2+l+2p^2V(\phi)-2\right)\right.\\&\quad+2r\chi_1\left(2pf'p'+l^2+l+2p^2V(\phi)-2\right)+2pf'^2p'\\&\quad\left.-4fpp'V(\phi)\right).\end{aligned} \quad (31)$$

Substituting (28)–(31) into (24), K can be solved as

$$K(r) = A_1(r)\Psi'(r) + A_2(r)\Psi(r), \quad (32)$$

where $A_1(r)$, $A_2(r)$ are

$$\begin{aligned}A_1(r)={}&-\left(2rfp^2\chi_1p'\left(l^2+l+2p^2V(\phi)+3pf'p'-2\right)\right)\left\{4p\chi_3p'^2\left(p''p+p'^2-p^2\chi_0\right)f^3-2p^2p'\left(\chi_0\left(l^2+l\right.\right.\right.\\&\left.+2p^2V(\phi)+3pf'p'-2\right)-2\chi_3\left(r\chi_1+f'\right)p'^2\Big)f^2+\Big(2V(\phi)\chi_0\left(2r\chi_1+f'\right)p^5+\Big(\chi_0\left(4\omega^2+3f'^2+6r\chi_1f'\right)p'\\&+2V(\phi)\left(p'\left(2r\chi_1'+f''\right)-f'p''+2\chi_1\left(p'-rp''\right)\right)\Big)p^4+\Big(\chi_0\left(-4r\omega^2p'^2+2\left(l^2+l-2\right)r\chi_1+\left(l^2+l-2\right)f'\right)\\&+p'\left(-4p''\omega^2+2V(\phi)\left(2r\chi_1+f'\right)p'+6rf'p'\chi_1'+3f'p'f''-3f'^2p''+6\chi_1f'\left(p'-rp''\right)\right)\Big)p^3+\Big(-4\omega^2p'^3\\&+3f'^2p'^3+4r\omega^2p''p'^2-2\left(l^2+l-2\right)V(\phi)p'+2l^2r\chi_1'p'+2lr\chi_1'p'-4r\chi_1'p'+l^2f''p'+lf''p'-2f''p'-l^2f'p''\\&-lf'p''+2f'p''+2\chi_1\left(3rf'p'^3+\left(l^2+l-2\right)p'-\left(l^2+l-2\right)rp''\right)\Big)p^2+2p'^2\Big(2r\omega^2p'^2+\left(l^2+l-2\right)r\chi_1\\&-\left(l^2+l-2\right)f'\Big)p-\left(l^2+l-2\right)^2p'\Big)f-p^2p'\Big(2r\chi_1\left(2p'\left(p-rp'\right)\omega^2+\left(l^2+l+2p^2V(\phi)-2\right)f'+3pf'^2p'\right)\\&\left.+f'\left(4p'\left(p-rp'\right)\omega^2+\left(l^2+l+2p^2V(\phi)-2\right)f'+3pf'^2p'\right)\Big)\right\}^{-1}\\ A_2(r)={}&-\Big(2p\Big(2\chi_2p'^2\left(p''p+p'^2-p^2\chi_0\right)f^3+2p\chi_2\left(r\chi_1+f'\right)p'^3f^2+\Big(2rV(\phi)\chi_0\chi_1p^4\\&+\left(3r\chi_0\chi_1f'p'+2V(\phi)\left(rp'\chi_1'+\chi_1\left(p'-rp''\right)\right)\right)p^3\\&+\left(r\chi_0\left(\left(l^2+l-2\right)\chi_1-2\omega^2p'^2\right)+p'\left(2rV(\phi)\chi_1p'+3f'\left(rp'\chi_1'+\chi_1\left(p'-rp''\right)\right)\right)\right)p^2\\&+\left(rp'\left(2p'p''\omega^2+\left(l^2+l-2\right)\chi_1'\right)+\chi_1\left(3rf'p'^3+\left(l^2+l-2\right)p'-\left(l^2+l-2\right)rp''\right)\right)p\\&+rp'^2\left(2\omega^2p'^2+\left(l^2+l-2\right)\chi_1\right)\Big)f+rpp'\left(2\omega^2f'p'^2-\chi_1\left(3pp'f'^2+\left(l^2+l+2p^2V(\phi)-2\right)f'-2r\omega^2p'^2\right)\right)\Big)\Big)\end{aligned}$$

$$\cdot\Big\{4p\chi_3 p'^2\left(p''p+p'^2-p^2\chi_0\right)f^3-2p^2p'\left(\chi_0\left(l^2+l+2p^2V(\phi)+3pf'p'-2\right)-2\chi_3\left(r\chi_1+f'\right)p'^2\right)f^2+\Big(2V(\phi)$$
$$\cdot\chi_0\left(2r\chi_1+f'\right)p^5+\left(\chi_0\left(4\omega^2+3f'^2+6r\chi_1 f'\right)p'+2V(\phi)\left(p'\left(2r\chi_1'+f''\right)-f'p''+2\chi_1\left(p'-rp''\right)\right)\right)p^4$$
$$+\Big(\chi_0\left(-4r\omega^2p'^2+2\left(l^2+l-2\right)r\chi_1+\left(l^2+l-2\right)f'\right)$$
$$+p'\left(-4p''\omega^2+2V(\phi)\left(2r\chi_1+f'\right)p'+6rf'p'\chi_1'+3f'p'f''-3f'^2p''+6\chi_1 f'\left(p'-rp''\right)\right)\Big)p^3+\Big(-4\omega^2p'^3$$
$$+3f'^2p'^3+4r\omega^2p''p'^2-2\left(l^2+l-2\right)V(\phi)p'+2l^2r\chi_1'p'+2lr\chi_1'p'-4r\chi_1'p'+l^2f''p'+lf''p'-2f''p'-l^2f'p''$$
$$-lf'p''+2f'p''+2\chi_1\left(3rf'p'^3+\left(l^2+l-2\right)p'-\left(l^2+l-2\right)rp''\right)\Big)p^2+2p'^2\Big(2r\omega^2p'^2+\left(l^2+l-2\right)r\chi_1$$
$$-\left(l^2+l-2\right)f'\Big)p-\left(l^2+l-2\right)^2p'\Big)f-p^2p'\left(2r\chi_1\left(2p'\left(p-rp'\right)\omega^2+\left(l^2+l+2p^2V(\phi)-2\right)f'+3pf'^2p'\right)\right.$$
$$\left.+f'\left(4p'\left(p-rp'\right)\omega^2+\left(l^2+l+2p^2V(\phi)-2\right)f'+3pf'^2p'\right)\right)\Big\}^{-1}. \tag{33}$$

The resulting master equation can be derived by evaluating

$$\delta\widehat{G}_{23}+\delta\widehat{G}_{22}=\frac{i\omega H_1}{f}+\frac{\left(-2+l+l^2\right)f-2\omega^2p^2}{2f^2pp'}K-\frac{pf'K'}{2fp'}-\frac{-2+l+l^2+2p^2V(\phi)+2pf'p'}{2fpp'}H_0=0. \tag{34}$$

By substituting (28), (29), and (32) into the above equation, the corresponding master equation is given by

$$\Psi''(r)+\Psi'(r)\left\{\frac{A_2}{A_1}+\frac{A_1'}{A_1}+\frac{-2r\chi_1-f'}{2f}+\frac{p'\left(p\left(2\omega^2+2fV(\phi)+r\chi_1 f'+f'^2\right)+2\left(-r\omega^2+ff'\right)p'\right)}{fP_v}\right\}+\Psi(r)$$
$$\cdot\left\{\frac{-r\left(1+A_2\right)\chi_1\left(P_v-pf'p'\right)}{A_1fP_v}\right.$$
$$\left.+\frac{1}{2A_1fP_v}\left(-4r\omega^2p'^2+A_2\left(-P_vf'+p'\left(p\left(4\omega^2+f'^2\right)-4r\omega^2p'\right)\right)+2f\left(P_vA_2'+2A_2p'\left(pV(\phi)+f'p'\right)\right)\right)\right\}=0, \tag{35}$$

where $P_v(r)=-2+l+l^2+2p^2V(\phi)+3pf'p'$. Finally, one defines $\Psi(r)=B_1(r)B_2(r)\Phi(r)$ and $f(r)=Q(r)F(r)$ (where $Q(r)$ is the coefficient of ω^2); the master equation can be simplified into the following equation:

$$\frac{d^2\Phi(r)}{dr_*^2}+\left\{\omega^2-V_e(r)\right\}\Phi(r)=0, \tag{36}$$

where $dr_*=dr/F(r)$,

$$V_e=\frac{F(r)\left(2Q(r)\left(F'(r)Q'(r)+F(r)Q''(r)\right)-F(r)Q'(r)^2\right)+4\widetilde{V}(r)}{4Q(r)^2}, \tag{37}$$

where

$$\begin{aligned}
Q(r) = {} & \frac{1}{2\sqrt{A_1 (3pf'p' + l^2 + l + 2p^2 V(\phi) - 2)^2}} \{4fp' (r (A_1' - 2) p' - pA_1' + A_2 (p - rp')) (3pf'p' + l^2 + l \\
& + 2p^2 V(\phi) - 2) + A_1 (r^2 \chi_1^2 (2pf'p' + l^2 + l + 2p^2 V(\phi) - 2)^2 \\
& + 2 (rp' (4 (l^2 + l - 2) fp'' - f'p' (2fp'^2 + 3 (l^2 + l - 2))) + p^3 (V(\phi) (6f'p' - 4fp'') + 4fp'\phi' V'(\phi)) \\
& + p (-2f (3rf''p'^3 + (l^2 + l - 2) p'' + 2rp'^3 V(\phi)) + f'p' (2fp' (3rp'' + p') + 3 (l^2 + l - 2)) - 8rf'^2 p'^3) \\
& + 2p^2 p' (f (p' (3f'' - 2r\phi' V'(\phi)) + 2V(\phi) (2rp'' + p')) + f'p' (4f' - 3rV(\phi)))))\}^{1/2}, \\
\widetilde{V}(r) = {} & \frac{1}{16A_1^2 (l^2 + l + 2p^2 V(\phi) + 3pf'p' - 2)^2} (-16 (p^2 (p'^2 - 2pp'') V(\phi)^2 + (2pf'p'^3 - (l^2 + l - p^2 f'' - 2) p'^2 \\
& - 4p^2 f'p''p' - (l^2 + l - 2) pp'') V(\phi) + p' ((2p'^3 - 3pp'p'') f'^2 - (p'V'(\phi) \phi' p^2 + 2 (l^2 + l - 2) p'') f' \\
& - (l^2 + l - 2) (pV'(\phi) \phi' + p'f''))) f^2 - 4 (12V(\phi)^2 f'' p^4 + 2f' (2p'V(\phi)^2 + (16p'f'' - f'p'') V(\phi) \\
& + f'p'V'(\phi) \Psi') p^3 + 2 (12f'^2 f'' p'^2 + V(\phi) (5f'^2 p'^2 + 6 (l^2 + l - 2) f'')) p^2 + f' (4f'^2 p'^3 \\
& + 2 (l^2 + l - 2) V(\phi) p' + 16 (l^2 + l - 2) f''p' - (l^2 + l - 2) f'p'') p + (l^2 + l - 2) f'^2 p'^2 + 3 (l^2 + l - 2)^2 f'') f \\
& + (8pp'f'^2 + 3 (l^2 + l + 2p^2 V(\phi) - 2) f')^2) A_1^2 + 4A_2 f (l^2 + l + 2p^2 V(\phi) + 3pf'p' - 2) (2pp'f'^2 + (l^2 + l - 4fp'^2 \\
& + 2p^2 V(\phi) - 2) f' - 4fpV(\phi) p') A_1 - 4f (l^2 + l + 2p^2 V(\phi) + 3pf'p' - 2) (A_1' f' (l^2 + l + 2p^2 V(\phi) + 2pf'p' \\
& - 2) + 2f (-A_1'' l^2 - A_1'' l - 2A_1' f'p'^2 + A_2'(r) (l^2 + l + 2p^2 V(\phi) + 3pf'p' - 2) - 2p^2 V(\phi) A_1'' + 2A_1'' \\
& - pp' (2V(\phi) A_1' + 3f'A_1''))) A_1 + 4A_2^2 f^2 (l^2 + l + 2p^2 V(\phi) + 3pf'p' - 2)^2 - 4f^2 A_1'^2 (l^2 + l + 2p^2 V(\phi) + 3pf'p' \\
& - 2)^2.
\end{aligned} \tag{38}$$

Here we study the QNMs for even parity perturbation in asymptotically flat, dS, and AdS spacetimes. Figures 10, 11, and 12 show the effective potential functions and corresponding temporal evolution of even parity perturbation in asymptotically flat, dS, and AdS spacetimes, respectively. We consider the potential in the range $r > r_h$ in asymptotically flat and AdS spacetimes and $r_h < r < r_c$ in dS spacetime. Here, the potential functions are all positive definite so that the corresponding regular phantom BHs are likely to be stable. Furthermore, in asymptotically flat and dS spacetimes, the relationships between $\mathrm{Re}(\omega)$, $\mathrm{Im}(\omega)$ and b, l are similar to those of odd parity; the differences are in details. However, the potential function in AdS spacetime is mostly a convex function, and therefore it approaches infinity faster than that of odd parity. Nevertheless, the resulting QNMs in AdS spacetime are also found to be stable.

4. Hawking Radiation of the Regular Phantom Black Hole

As mentioned before in Section 2, the interior structures of regular BHs are quite different from those of traditional ones. In order to achieve a better understanding of the properties of such regular phantom BHs, it is meaningful to investigate the corresponding Hawking radiation, which is considered to be a promising method to study the thermodynamics of BHs.

From a quantum mechanics viewpoint, QNM carries information on the quantization of the system around BH's horizon [35–37, 39, 41]. The Hawking radiation spectrum provides an interpretation of such quantization at an effective temperature [25, 35, 36]. In this paper, we adopt a simple and effective method (i.e., Hamilton-Jacobi equation) to analyze the Hawking radiation of the regular phantom BH. Based on the tunneling theory of Hawking radiation [49, 50], this

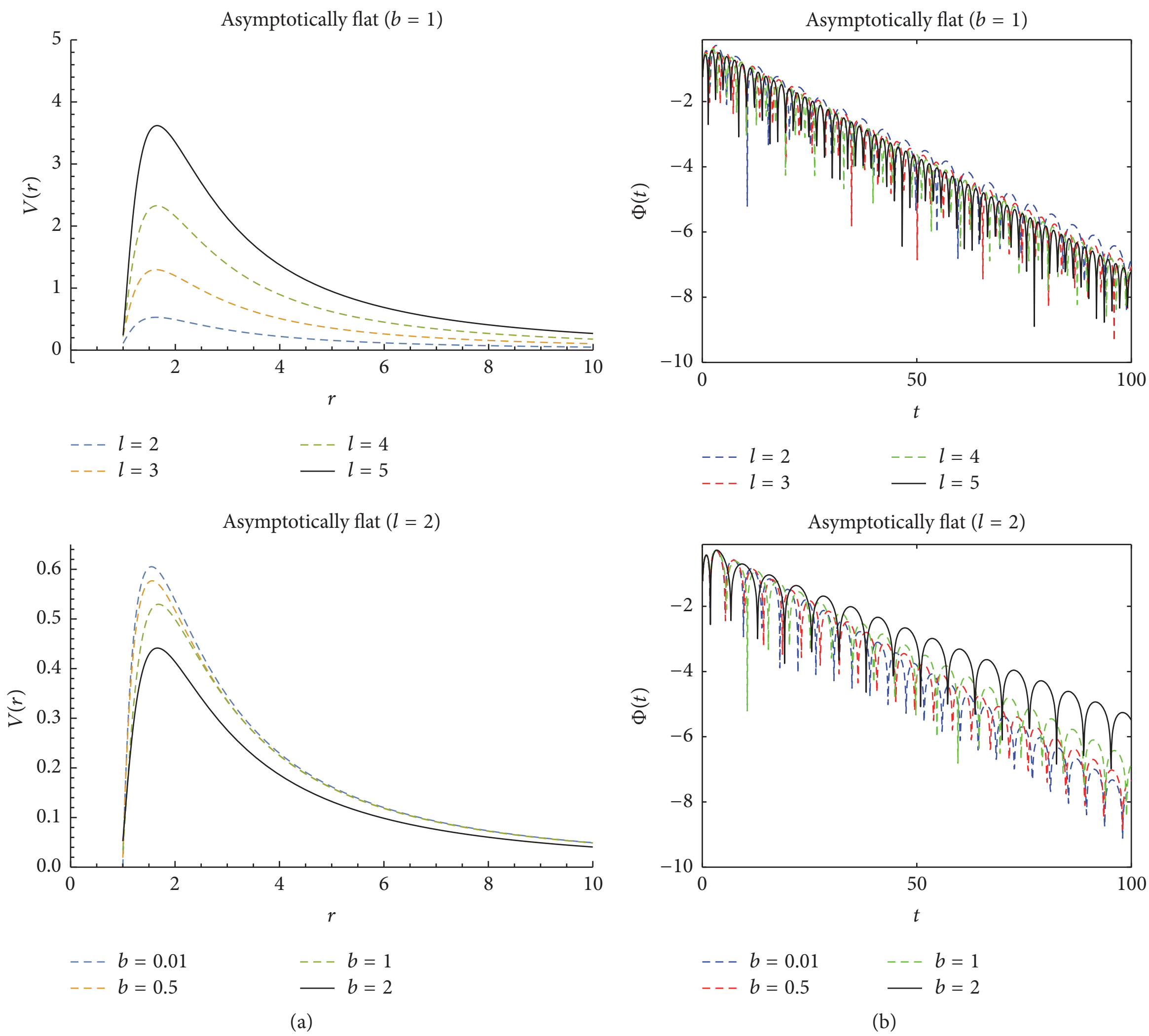

FIGURE 10: (a) The effective potential of even parity perturbation in asymptotically flat spacetime. (b) The corresponding temporal evolution of even parity gravitational perturbation field in asymptotically flat spacetime. Calculations are carried out by using $r_h = 1$.

method was first put forward by Srinivasan in 1999 [51] and plays an important role in recent studies [52, 53].

Up to this point, we have been using the natural units in the calculations of QNM. However, since the Planck constant $\hbar$ is very important in the study of tunneling radiation, we will explicitly write it down here in the field functions. In other words, h_0 and h_1 of (12) are now rewritten as $h_0(r,t) = \exp(-i\omega t/\hbar)h_0(r)$, $h_1(r,t) = \exp(-i\omega t/\hbar)h_1(r)$, and (21) becomes

$$h_{\mu\nu} = \exp\left(-\frac{i\omega t}{\hbar}\right)\begin{pmatrix} H_0(r) f(r) & H_1(r) & 0 & 0 \\ H_1(r) & \dfrac{H_2(r)}{f(r)} & 0 & 0 \\ 0 & 0 & p(r)^2 K(r) & 0 \\ 0 & 0 & 0 & p(r)^2 K(r)\sin^2\theta \end{pmatrix} P_l(\cos\theta). \quad (39)$$

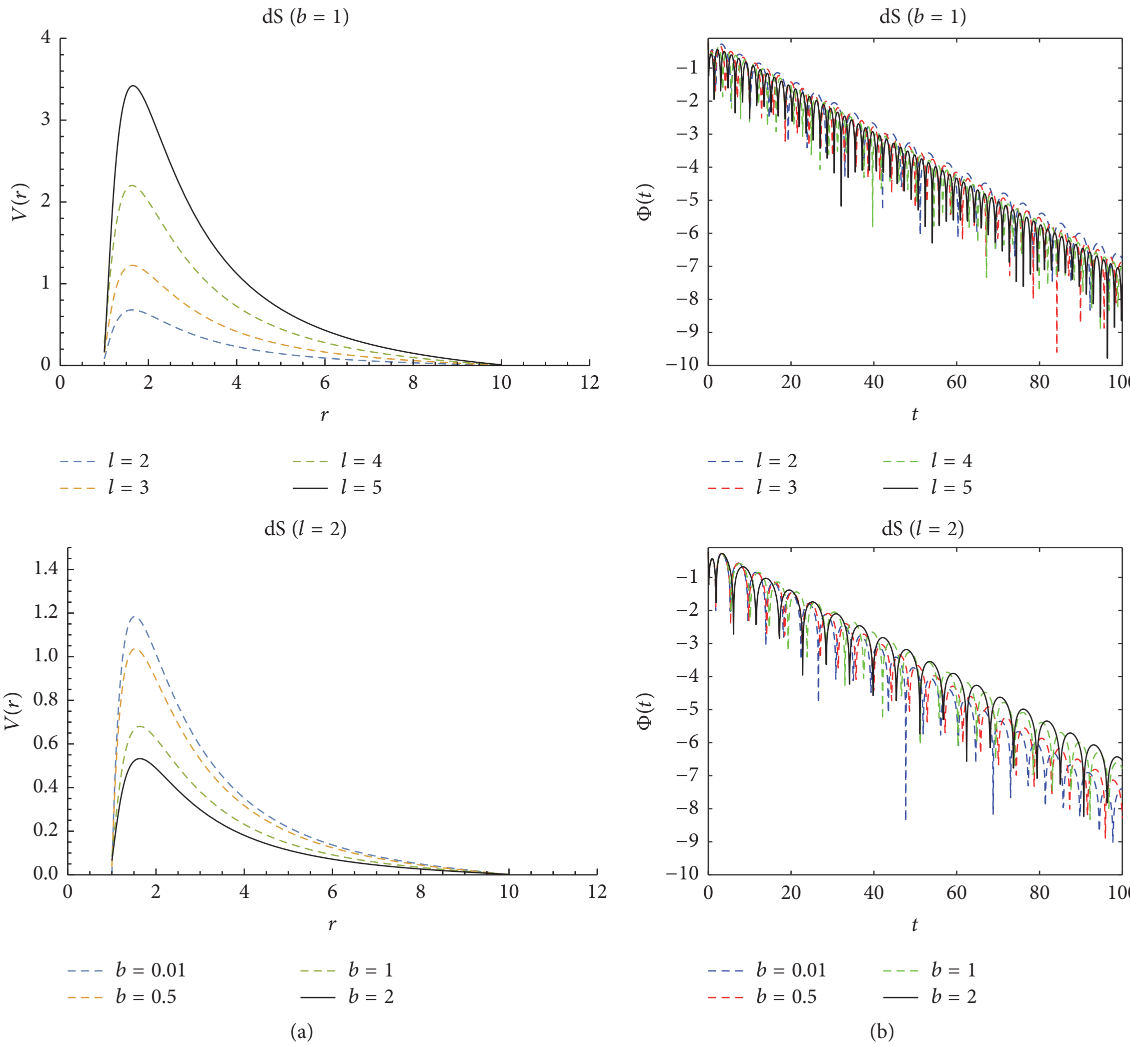

FIGURE 11: (a) The effective potential of even parity perturbation in dS spacetime. (b) The corresponding temporal evolution of even parity gravitational perturbation. In the calculations, we choose $r_h = 1,\ r_c = 10$.

The corresponding Schrödinger-type equation can be expressed as

$$\frac{d^2\Phi}{dr_*^2} + \left(\frac{\omega^2}{\hbar^2} - V(r)\right)\Phi = 0. \tag{40}$$

Since the Hawking radiation reflects BH's radial properties in the vicinity of the horizon, it is worthwhile to explicitly discuss radial field equations. According to (40), near the horizons $r \to r_h$, one has $V_o \to 0$ and $V_e \to 0$, so that the field equation can be simplified to

$$\frac{d^2\Phi(r)}{dr_*^2} + \frac{\omega^2}{\hbar^2}\Phi(r) = 0. \tag{41}$$

By introducing $\Phi(r) = Ce^{-(i/\hbar)R(r)}$ and using the semiclassical approximation to neglect the terms of $\mathcal{O}(\hbar)$, as $\hbar$ is small, one obtains

$$-f^2R'^2 + \omega^2 = 0. \tag{42}$$

Therefore, $R'(r) = \pm\omega/f$, where $\pm$ represents ingoing or outgoing mode. The above equation is no other than the Hamilton-Jacobi equation at the horizon. We note that the above argument also applies to the case of asymptotically dS spacetime, where the event horizon r_h and cosmological horizon r_c should be both considered. In fact, according to the Hamilton-Jacobi equation $g^{\mu\nu}(\partial S/\partial x^\mu)(\partial S/\partial x^\nu)+m^2 = 0$, the radial Hamilton-Jacobi equation can be transformed into (42) at the horizon, so we can use this result to study the Hawking tunneling radiation under semiclassical approximation [54–56].

Owing to the coordinate singularity at the horizon, the integration from inner surface of horizon to outer surface shall be carried out by using Residue Theorem:

$$R(r_h) = \pm\frac{i\pi\omega}{f'(r_h)}. \tag{43}$$

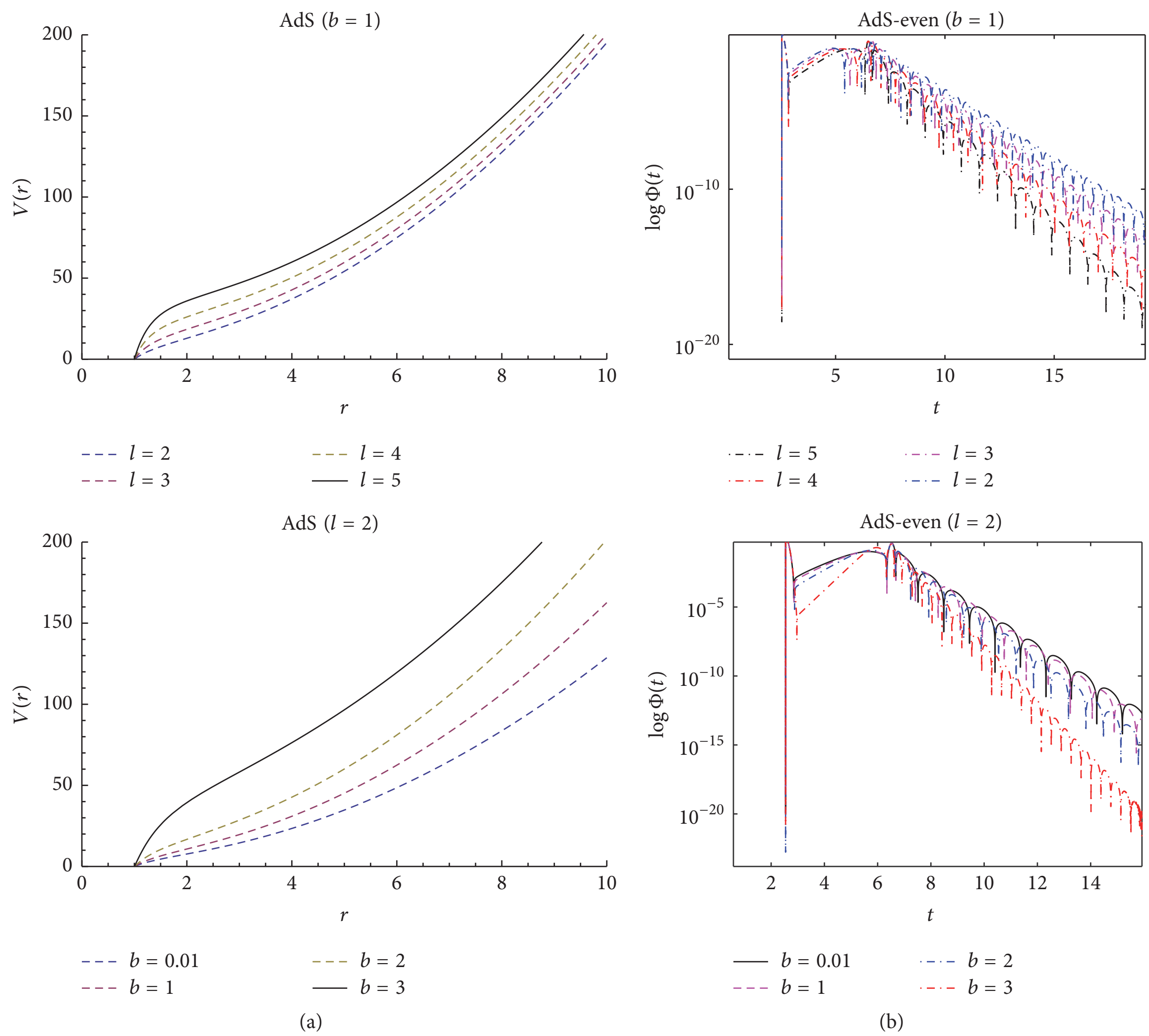

FIGURE 12: (a) The effective potential of even parity perturbation in AdS spacetime. (b) The corresponding temporal evolution of even parity gravitational perturbation in AdS spacetime. In the calculations, we choose $r_h = 1$.

Then, the tunneling rate of Hawking radiation is [57, 58]

$$\Gamma = \frac{\exp\left(-2\ \mathrm{Im}\left(R_{+}\right)\right)}{\exp\left(-2\ \mathrm{Im}\left(R_{-}\right)\right)} = \exp\left(-\frac{4\pi\omega}{f'\left(r_h\right)}\right). \tag{44}$$

According to the relationship between tunneling rate Γ and the Hawking temperature, the temperature of the BH in the vicinity of the event horizon r_h is

$$T_H = \frac{f'\left(r_h\right)}{4\pi}. \tag{45}$$

From the inside of the cosmological horizon r_c in dS spacetime, we consider the incident wave solution of Hawking radiation. Therefore, there is an extra minus sign in the resulting expression of the Hawking temperature in the neighborhood of the cosmological horizon [59] as follows:

$$T_c = -\frac{f'\left(r_c\right)}{4\pi}. \tag{46}$$

Figure 13 shows the impact of the parameter b on the thermodynamics of a regular phantom BH for different spacetimes. In asymptotically flat spacetime, the Hawking temperature T_H decreases monotonously with increasing b and a given r_h; and for a given b, T_H becomes smaller with increasing r_h. In asymptotically AdS spacetime, however, the Hawking temperature T_H increases monotonously with increasing b and a given r_h; and for a given b, T_H becomes larger with increasing r_h. In asymptotically dS spacetime, the relationship between T_H and the event horizon r_h is similar to that in asymptotically flat case, while the Hawking temperature T_C in the vicinity of the cosmological horizon h_c also decreases monotonously with increasing b, and it decays faster than other cases. The different dependence of Hawking temperature on b probably results from the distinctive spacetime structure of AdS phantom BH from others.

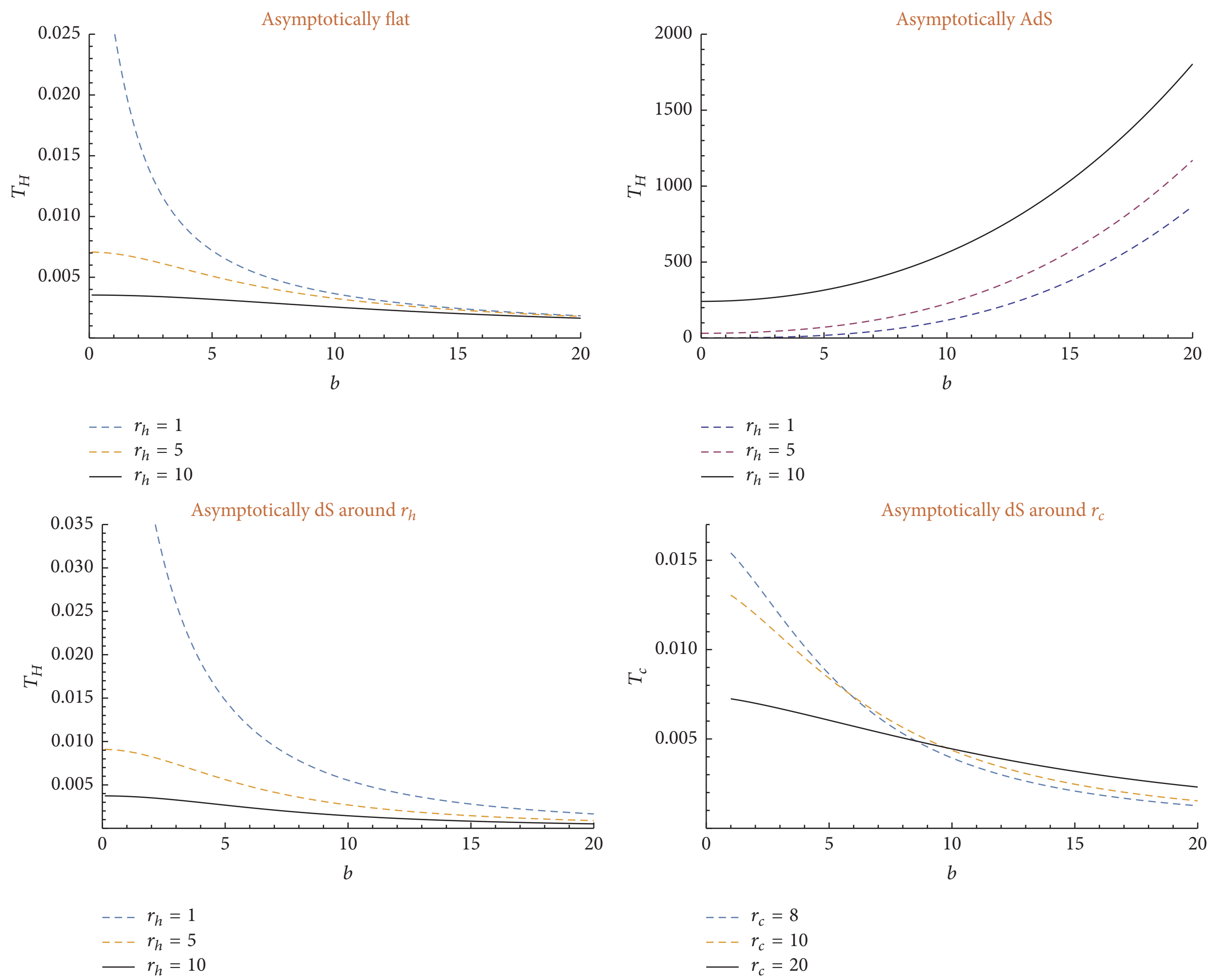

FIGURE 13: The Hawking temperature as a function of b in asymptotically flat and dS spacetimes.

5. Conclusions and Remarks

We studied static spherically symmetric solutions of regular phantom BH in asymptotically flat, dS, and AdS spacetimes and then investigated the QNMs of gravitational perturbations as well as Hawking radiations for these BH spacetimes. In our calculations, besides the metric perturbation $\delta g_{\mu\nu}$ of the Ricci curvature tensor and scalar curvature, its effect on the energy momentum tensor of the matter field is also considered. In the derivation of the master equation for even parity, we made use of the method proposed in [26]. However, the BH metric in this paper is a self-gravitating solution of a minimally coupled scalar field with an arbitrary potential rather than the Lovelock equations, so that it does not satisfy the relation of Eq. (5.4) in [26]. As a result, the obtained master equation is more complicated. It is found that the calculated effective potential of AdS spacetime in the limit $r \to \infty$ is very different from those of asymptotically flat and dS spacetimes. For the asymptotical AdS spacetime, the effective potentials diverge at infinity, which implies that the wave function Φ should vanish in this limit [60–64]. At the outside of the event horizon, on the other hand, the corresponding wave function must be an incoming wave. The distinct nature of AdS BH spacetime leads to the fact that some traditional numerical methods such as WKB approximation, continuous fraction method cease to be valid. In our calculations, we therefore employed the finite difference method to numerically calculate the temporal evolution of small gravitational perturbations. The relationships between Hawking temperature and parameters such as b, r_h in different spacetime are also studied.

Owing to the importance of the parameter b, which carries the physical content of the coupling strength between phantom scalar field and the gravity, we studied the dependence of various physical quantities on this specific parameter, among others. From the above calculated results, we draw the following conclusions. For odd parity, in asymptotic flat and dS spacetime, the perturbation frequency ($\mathrm{Re}(\omega)$) becomes smaller and the decay rate ($|\mathrm{Im}(\omega)|$) decreases when b increases. The perturbation frequency ($\mathrm{Re}(\omega)$) and the decay rate ($|\mathrm{Im}(\omega)|$) both increase with increasing l. For even parity, the shape of the effective potential in the vicinity of the event horizon is very different from that for odd parity. In particular, $V_e(r)$ decreases monotonically in the range $r > r_h$, which is caused by the metric transformation $F(r) = f(r)/Q(r)$. However, our numerical calculations show that the resultant gravitational perturbations are stable in all three different spacetime backgrounds. The Hawking temperature T_H (and T_c) of the phantom BH decreases with increasing b in asymptotically flat and dS spacetimes. On the contrary, in AdS spacetime, the temperature T_H increases with increasing b. These results reflect distinct natures of the horizons of different spacetimes.

Recently, another type of regular phantom BH, which is a solution of Einstein-Maxwell equations, was proposed by Lemos and Zanchin [65]. The properties of interior as well as exterior structures were investigated. It is meaningful to investigate the stability of such BH solution and compare it with the phantom BH in the present paper.

Appendix

We note that $V(\phi(r))$ in (7) is just the 0-order solution. However, when we investigate the perturbation in the BH background, the potential could, in principle, be expanded to higher order terms. Since the higher order terms of $V(\phi)$ have not been determined yet, we can freely choose the form of such higher order terms to cancel out the contributions of the perturbation of phantom scalar field. In the Appendix, we carry out explicit calculations to show that this is indeed possible.

We consider the following metric and phantom scalar field:

$$\begin{aligned} g_{\mu\nu} &= \overline{g}_{\mu\nu} + h_{\mu\nu}, \\ \phi &= \overline{\phi} + \phi_1, \end{aligned} \tag{A.1}$$

where $\overline{g}_{\mu\nu}$, $\overline{\phi}$ are background metric and phantom scalar field, respectively, and ϕ_1 represents the first order perturbation of phantom scalar field. Then

$$g^{\mu\nu} = \overline{g}^{\mu\nu} - h^{\mu\nu} + h^{\mu}_{\alpha} h^{\nu\alpha}. \tag{A.2}$$

The corresponding metric and potential of phantom scalar field can be expressed as

$$\begin{aligned} \sqrt{-g} &= \sqrt{-\overline{g}} + \sqrt{-g_1} + \sqrt{-g_2} + \cdots, \\ R &= \overline{R} + R_1 + R_2 + \cdots, \\ V(\phi) &= \overline{V} + V_1 + V_2 + \cdots, \end{aligned} \tag{A.3}$$

where the superscript – and subscripts 1, 2 represent the background quantities and the first and the second order perturbations, respectively. Therefore, the action of the gravity with the phantom scalar field, (1), can be expanded as

$$S = \overline{S} + S_1 + S_2 + \cdots, \tag{A.4}$$

where

$$\begin{aligned} \overline{S} &= \int d^4x \sqrt{-\overline{g}} \left(\overline{R} + \epsilon \overline{g}^{\mu\nu} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} - 2\overline{V}\right), \\ S_1 &= \int d^4x \left\{ \sqrt{-\overline{g}} \left[R_1 + \epsilon \left(2\overline{\phi}_{,\mu} \phi_1^{,\mu} - h^{\mu\nu} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} \right) \right.\right. \\ &\quad \left.\left. - 2V_1 \right] + \sqrt{-g_1} \left(\overline{R} + \epsilon \overline{g}^{\mu\nu} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} - 2\overline{V} \right) \right\}, \\ S_2 &= \int d^4x \left\{ \sqrt{-\overline{g}} \left[R_2 \right.\right. \\ &\quad \left. + \epsilon \left(\phi_{1,\mu} \phi_1^{,\mu} - 2h^{\mu\nu} \overline{\phi}_{,\mu} \phi_{1,\nu} + h^{\mu}_{\alpha} h^{\nu\alpha} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} \right) - 2V_2 \right] \\ &\quad + \sqrt{-g_1} \left[R_1 + \epsilon \left(2\overline{\phi}_{,\mu} \phi_1^{,\mu} - h^{\mu\nu} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} \right) - 2V_1 \right] \\ &\quad \left. + \sqrt{-g_2} \left(\overline{R} + \epsilon \overline{g}^{\mu\nu} \overline{\phi}_{,\mu} \overline{\phi}_{,\nu} - 2\overline{V} \right) \right\}. \end{aligned} \tag{A.5}$$

In the background equation, there is no effect of V_1 and V_2, while in the perturbed equation we can always choose the form of the perturbed potential V_1, V_2 as

$$\begin{aligned} V_1 &= \epsilon \overline{\phi}_{,\mu} \phi_1^{,\mu}, \\ V_2 &= \frac{\epsilon}{2} \phi_{1,\mu} \phi_1^{,\mu} - \epsilon h^{\mu\nu} \overline{\phi}_{,\mu} \phi_{1,\nu}, \end{aligned} \tag{A.6}$$

to cancel out the contributions from the phantom scalar field perturbation ϕ_1. Therefore, in the text, we do not explicitly consider the perturbation of the phantom field.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

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Research Article

Kodama-Schwarzschild versus Gaussian Normal Coordinates Picture of Thin Shells

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Geometry of the spacetime with a spherical shell embedded in it is studied in two coordinate systems: Kodama-Schwarzschild coordinates and Gaussian normal coordinates. We find explicit coordinate transformation between the Kodama-Schwarzschild and Gaussian normal coordinate systems. We show that projections of the metrics on the surface swept by the shell in the 4D spacetime in both cases are identical. In the general case of time-dependent metrics we calculate extrinsic curvatures of the shell in both coordinate systems and show that the results are identical. Applications to the Israel junction conditions are discussed.

1. Introduction

Thin shells provide a useful tool to study dynamics of collapsing body eventually forming black hole and Hawking radiation from the black hole [1, 2].

Dynamics of domain walls was studied by Israel [3], Poisson [4], Ipser and Sikivie [5], Berezin et al. [6], Blau et al. [7], Chowdhury [8], Gladush [9], Kraus and Wilczek [10], and many other authors.

There are two natural settings to study geometry of the spacetime with a spherical shell: that based on Kodama-Schwarzschild coordinates and that employing the Gaussian normal coordinate system.

Kodama found that in any (possibly time-dependent) spherically symmetric spacetime there exists a conserved vector which is timelike in the exterior of the shell [11]. Although the Kodama vector does not reduce to the Killing vector even in the static spacetime, it can be used to define a preferred "time coordinate" and to construct a geometrically preferred coordinate system for a spherically symmetric spacetime [11–15]. Because the Kodama vector is orthogonal to dr, one can construct the time coordinate t so that dr is orthogonal to dt [15]. Using the Schwarzschild radial coordinate r, one arrives at the diagonal, time-dependent spherically symmetric metric which in this parametrization in the (r,t) sector has the metric components $g_{rr} = (1 - 2m(t,r)/r)^{-1}$ and $g_{tt} = b^2(r,t)(1 - 2m(t,r)/r)$.

The Gaussian normal coordinate system in the neighborhood of the shell [7, 16] is constructed by using a family of nonintersecting geodesics orthogonal to the surface Σ swept by the shell. Coordinates of a point outside of the shell are introduced as the geodesic distance from the point to the shell along the geodesic orthogonal to the surface Σ and coordinates of the intersection point of the geodesic with Σ.

The aim of the present paper is to study connection between two approaches. We find explicit coordinate transformation between the Kodama -Schwarzschild and Gaussian normal coordinate systems. We show that projections of the metrics on the surface swept by the shell in the 4D spacetime in both cases are identical. In the general case of time-dependent metrics we calculate extrinsic curvatures of the shell in both coordinate systems and show that the results are identical. Applications of the above results to the Israel junction conditions are discussed.

2. Kodama-Schwarzschild Coordinates

The $(2+1)$ dimensional hypersurface Σ swept by a spherically symmetric shell divides 4D spacetime in two regions $V^{\pm}$. Any spherically symmetric metric in $D = 1 + 3$ spacetime has the general form

$$ds^2 = g_{a,b}(x)\, dx^a dx^b = {}^{(2)}g_{i,j}(x)\, dx^i dx^j + r^2(x)\, d\Omega^2. \tag{1}$$

Here $x^a = (x^i, x^\alpha)$, where x^i are coordinates in the base space and θ and φ are coordinates on the spherically symmetric fibers. For any spherically symmetric spacetime it is possible to introduce a vector $k^a = \varepsilon_\perp^{ab}\nabla_b r$ (Kodama vector) [11–15, 17], which lies in the radial-temporal plane, where $\varepsilon_\perp^{ab}$

$$\varepsilon_\perp^{ab} = \begin{pmatrix} \varepsilon_\perp^{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

By construction Kodama vector k is orthogonal to $\nabla_a r$. Choosing the time coordinate t so that $\partial_t \sim k$ [15], one obtains the metric in the diagonal form because k and dt are orthogonal to dr. In the parametrization through the time coordinate t and Schwarzschild radial coordinate r the metric can be expressed as

$$ds_\pm^2 = -b_\pm^2(r, t_\pm)\, f_\pm(r, t_\pm)\, dt_\pm^2 + \frac{dr^2}{f_\pm(r, t_\pm)} + r^2 d\Omega^2, \tag{3}$$

where $f_\pm(r, t_\pm) = (1 - 2m_\pm(r, t_\pm)/r)$. In this parametrization $m(r, t)$ is interpreted as the quasi-local mass (Misner-Sharp-Hernandez mass) [18, 19]. Note that in this parametrization the metric is diagonal.

Position of the surface is defined by parametric equations $r = R(\tau)$, $t_\pm = T_\pm(\tau)$. The metrics induced on the shell from the regions $V^\pm$ are

$$ds_\pm^2 = -b_\pm^2(R, T_\pm)\, f_\pm(R, T_\pm)\, dT_\pm^2 + \frac{dR^2}{f_\pm(R, T_\pm)} + R^2 d\Omega^2. \tag{4}$$

From the requirement that the metrics induced on Σ from both regions $V^\pm$ coincide (first Israel condition) it follows that

$$- b_+^2(R, T_+)\, f_+(R, T_+)\, \dot{T}_+^2 + f_+^{-1}(R, T_+)\, \dot{R}^2 = -b_-^2(R, T_-)\, f_-(R, T_-)\, \dot{T}_-^2 + f_-^{-1}(R, T_-)\, \dot{R}^2. \tag{5}$$

By choosing τ as the proper time on the surface, one obtains

$$-b_\pm^2(R, T_\pm)\, f_\pm(R, T_\pm)\, \dot{T}_\pm^{\,2} + f_\pm^{-1}(R, T_\pm)\, \dot{R}^2 = -1, \tag{6}$$

and projection of the metric on Σ is

$$ds^2 = -d\tau^2 + R^2(\tau)\, d\Omega^2. \tag{7}$$

3. Gaussian Normal Coordinates

Gaussian normal coordinate system in 4D spacetime in which a hypersurface swept by the spherical shell divides into two regions is introduced starting from a certain coordinate system $\hat{x}^\mu$ with a metric $\hat{g}_{\mu\nu}(\hat{x})$. The surface Σ is parametrized by coordinates $x^i = (\xi, \theta, \varphi)$ Consider a neighborhood of Σ with a system of geodesics orthogonal to Σ. The neighborhood is chosen so that the geodesics do not intersect; that is, any point in the neighborhood is located on one and only one geodesic. Let us consider a point in the neighborhood of Σ with the geodesic orthogonal to Σ which goes through this point. The new coordinate system x^μ is introduced in the following way. Three coordinates of the point coincide with the coordinates x^i of the point of intersection of the geodesic with Σ. The fourth coordinate of a point is equal to the proper geodesic distance along the geodesic from the point to Σ. The proper length along the geodesic is

$$\eta = \int_0^\eta d\eta' \sqrt{g_{\mu\nu}(x^\mu) \frac{dx^\mu}{d\eta'} \frac{dx^\nu}{d\eta'}}, \tag{8}$$

where η is the affine parameter along the geodesic. Expression (8) is invariant under the coordinate transformations with Jacobian equal to unity, and we can rewrite (8) through the new coordinates x^μ and the metric $g_{\mu\nu}(x)$. Taking the derivative over η from both sides of (8) over η, one has

$$g_{\mu\nu}(x) \frac{\partial x^\mu(\eta, x^i)}{\partial \eta} \frac{\partial x^\nu(\eta, x^i)}{\partial \eta} = 1, \tag{9}$$

or $g_{\eta\eta}(\eta, x^i) = 1$. Orthogonality condition of the tangent vector to geodesic to the tangent surface to Σ is

$$g_{\mu\nu}(x) \frac{\partial x^\mu(\eta, x^i)}{\partial \eta} \frac{\partial x^\nu(\eta, x^i)}{\partial x^i}\bigg|_{\eta=0} = 0, \tag{10}$$

or $g_{\eta i}(0, x^i) = 0$. The tangent vector $\partial x^\mu(\eta, x^i)/\partial\eta|_{\eta=0}$ is orthogonal to Σ and the vector $\partial x^\nu(0, x^i)/\partial x^i$ is in the plane tangent to Σ.

The metrics in $V^\pm$ are (below, to simplify formulas, we omit the subscript $\pm$ everywhere, where it does not lead to confusion)

$$ds_\pm^2 = d\eta^2 - p^2(\eta, \xi)\, d\xi^2 + 2q(\eta, \xi)\, d\tau d\eta + \rho^2(\eta, \xi)\, d\Omega^2\big|_\pm. \tag{11}$$

Because of condition (10), on the surface Σ the interval reduces to

$$ds^2 = -p^2(0, \xi)\, d\xi^2 + \rho^2(0, \tau)\, d\Omega^2. \tag{12}$$

On the surface Σ reparametrization of τ allows setting $p^2(0, \xi) = 1$, which is assumed in the following. It is seen that one can identify τ with ξ and $R(\tau)$ with $\rho(0, \xi)$. In the following we use the variable τ.

4. Transformation between the Coordinate Systems

Coordinate transformation $t_\pm = t_\pm(\eta, \tau)$ and $r_\pm = r(\eta, \tau)$ from Kodama-Schwarzschild coordinates $x^\mu = (t, r, \theta, \varphi)$ to Gaussian normal coordinates $\hat{x}^\mu = (\eta, \tau, \theta, \varphi)$ yields the following relations between the components of the metrics (6) and (11):

$$-b^2 f\dot{t}^2 + f^{-1}\dot{r}^2 = -p^2, \tag{13}$$

$$-b^2 f t'^2 + f^{-1} r'^2 = 1, \tag{14}$$

$$-b^2 f\dot{t}t' + f^{-1}\dot{r}r' = q, \tag{15}$$

where prime and dot denote derivatives over η and τ. On the surface Σ transformations (13)–(15) are of the same form with the substitution $\dot{t} \to \dot{T}$, $\dot{r} \to \dot{R}$ and $q = 0$, $p = 1$.

It is straightforward to obtain solution of systems (13)–(15) in the spacetime regions $V^{\pm}$ as $\dot{t} = \dot{t}(p,q,\dot{r})$, $t' = t'(p,q,\dot{r})$, and $r' = r'(p,q,\dot{r})$. Instead of writing this cumbersome and not instructive general solution, we consider the restriction of the transformation to the surface Σ which we use as follows:

$$\begin{aligned} \dot{t}^2\big|_{\Sigma} &= \dot{T}^2 = \frac{f(T,R) + \dot{R}^2}{b^2(T,R)\, f^2(T,R)}, \\ t'^2\big|_{\Sigma} &= \frac{\dot{R}^2}{b^2(T,R)\, f^2(T,R)}, \\ r'^2\big|_{\Sigma} &= f(T,R) + \dot{R}^2. \end{aligned} \tag{16}$$

Because Σ is orientable, on Σ a normal vector can be defined. In Kodama-Schwarzschild coordinates tangent, u_μ, and normal, n_μ, vectors to the surface Σ at either side of the surface are

$$u^{\mu}_{\pm} = \left(\dot{T}_{\pm}, \dot{R}, 0, 0\right), \quad u^2 = 1, \tag{17}$$

$$n_{\mu\pm} = \pm N\left(-\dot{R}, \dot{T}_{\pm}, 0, 0\right), \quad u^{\mu} n_{\mu} = 0, \tag{18}$$

$$n^{\mu}_{\pm} = \pm N\left(\frac{\dot{R}}{b^2_{\pm} f_{\pm}}, \dot{T}_{\pm} f_{\pm}, 0, 0\right). \tag{19}$$

Normalizing n^2 to unity, we obtain

$$n^2 = N^2\left(-\frac{\dot{R}^2}{b^2 f} + \dot{T}^2 f\right) = \frac{N^2}{b^2} = 1. \tag{20}$$

Transformations of the components of the tangent vector from Kodama-Schwarzschild coordinates (t,r) to Gaussian coordinates (τ,η) are

$$\begin{aligned} \hat{u}^{\pm}_{\eta} &= u^{\pm}_{\mu} \frac{\partial x^{\mu}_{\pm}}{\partial \eta}\bigg|_{\Sigma} = -b^2 f \dot{T} t' + \frac{\dot{R} r'}{f}\bigg|^{\pm}_{\Sigma} = 0, \\ \hat{u}^{\pm}_{\tau}\big|_{\Sigma} &= u^{\pm}_{\mu} \frac{\partial x^{\mu}_{\pm}}{\partial \tau}\bigg|_{\Sigma} = -b^2 f \dot{T}\dot{t} + \frac{\dot{R}\dot{r}}{f}\bigg|^{\pm}_{\Sigma} = -1. \end{aligned} \tag{21}$$

The corresponding transformations of the components of the normal vector are

$$\hat{n}^{\pm}_{\tau} = n^{\pm}_{\mu} \frac{\partial x^{\mu}_{\pm}}{\partial \tau}\bigg|_{\Sigma} = -\dot{R}\dot{t} + \dot{T}\dot{r}\Big|^{\pm}_{\Sigma} = 0, \tag{22}$$

$$\hat{n}^{\pm}_{\eta} = n^{\pm}_{\mu} \frac{\partial x^{\mu}_{\pm}}{\partial \eta}\bigg|_{\Sigma} = N\left(-\dot{R}t' + \dot{T}r'\right)\Big|^{\pm}_{\Sigma} = \pm 1. \tag{23}$$

In (22) and (23) we used expressions (16), where all the square roots for $\dot{T}, t'|_{\Sigma}$, and r'_{Σ} are taken with the same signs. The upper sign in (23) corresponds to the square roots taken with the sign (+).

Next, we consider another method to construct the explicit form of the coordinate transformation from Kodama-Schwarzschild coordinates to Gaussian normal coordinates. The problem can be solved in principle by solving the geodesic equations:

$$\frac{d^2 x^{\mu}}{d\lambda^2} + \Gamma^{\mu}_{\nu\lambda} \frac{dx^{\nu}}{d\lambda}\frac{dx^{\lambda}}{d\lambda} = 0. \tag{24}$$

In the general case with metric (3) depending on t the system of nonlinear differential equations is not tractable. Explicit relations can be obtained in the case of metric (3) with the components independent of t (in this case it is possible to set $b(r) = 1$: introducing new variable ρ by the relation $d\rho = dr/b(r)$ and denoting $k(\rho) = b^2 f(r(\rho))$, we obtain the metric (3) with $b = 1$.). In this case the geodesic equations in Kodama -Schwarzschild coordinates are

$$\begin{aligned} &\frac{d^2 t}{d\lambda^2} + \frac{f_{,r}}{f}\frac{dt}{d\lambda}\frac{dr}{d\lambda} = 0, \\ &\frac{d^2 r}{d\lambda^2} + \frac{f_{,r} f}{2}\left(\frac{dt}{d\lambda}\right)^2 - \frac{f_{,r}}{2f}\left(\frac{dr}{d\lambda}\right)^2 + fr\left(\frac{d\theta}{d\lambda}\right)^2 \\ &\qquad + fr\sin^2\theta\left(\frac{d\varphi}{d\lambda}\right)^2 = 0, \\ &\frac{d^2\theta}{d\lambda^2} = 0 = \frac{d^2\varphi}{d\lambda^2}, \end{aligned} \tag{25}$$

where λ is affine parameter. The first integrals of the system of equations are

$$\begin{aligned} \frac{dt}{d\lambda} &= \frac{C_t(\tau)}{f(r)}, \\ \frac{dr}{d\lambda} &= \pm\left(f(r) + C^2_t(\tau) - C_{\theta} f(r)\, r^2\right)^{1/2}, \\ \frac{d\theta}{d\lambda} &= C_{\theta}, \\ \frac{d\varphi}{d\lambda} &= C_{\varphi}. \end{aligned} \tag{26}$$

Here $r = r(\lambda,\tau)$, $t = t(\lambda,\tau)$. To maintain spherical symmetry, we take $C_{\theta} = C_{\varphi} = 0$. By construction the vector $l^{\mu}(\lambda,\tau) = (\partial t/\partial\lambda, \partial r/\partial\lambda, 0, 0)$ is tangent to the geodesic. Let us consider the geodesics orthogonal to Σ. In this case the affine parameter λ can be identified with the parameter η. From (26) it follows that the vector l^{μ} is normalized to unity. At the surface Σ the vector l^{μ} up to the sign coincides with the normal vector (19). Thus, at the surface Σ we have $C_t^{\,2} = \dot{R}^2$. For $l^{\mu}(\lambda,\tau)$ we obtain

$$l^{\mu} = \pm\left(\frac{\dot{R}}{f(r)}, \sqrt{f(r) + \dot{R}^2}, 0, 0\right). \tag{27}$$

On the surface Σ solution (27) coincides with formulas (16).

In Kodama-Schwarzschild parametrization the variables τ and η have a clear geometrical meaning: at the (t, r) plane τ varies along the trajectory of the shell $(R(\tau), T(\tau))$, and η varies along the geodesics orthogonal to the surface swept by the shell.

5. Extrinsic Curvature

The extrinsic curvatures at either side of Σ are

$$K_{ij}^{\pm} = \left(h_{\mu}^{\lambda} n_{\nu;\lambda} \frac{dx^{\mu}}{dx^{i}} \frac{dx^{\nu}}{dx^{j}} \right)_{\Sigma}^{\pm}, \tag{28}$$

where $x^i = (\tau, \theta, \varphi)$ are coordinates on Σ, $h_{\mu\nu} = g_{\mu\nu} - n_{\mu} n_{\nu}$, and (;) denote covariant derivative with respect to $g_{\mu\nu}^{\pm}$. In Kodama-Schwarzschild parametrization $x^{\mu}|_{\Sigma} = (T, R, \theta, \varphi)$. In Kodama-Schwarzschild parametrization the nonzero components of the extrinsic curvature are

$$\begin{aligned} K_{\tau\tau}^{\pm} &= h_{\mu}^{\lambda} n_{\nu;\lambda} u^{\mu} u^{\nu} \Big|^{\pm} = n_{\nu;\lambda} u^{\lambda} u^{\nu} \Big|^{\pm}, \\ K_{\theta\theta}^{\pm} &= h_{\mu}^{\lambda} n_{\nu;\lambda} \frac{dx^{\mu}}{d\theta} \frac{dx^{\nu}}{d\theta} \Big|^{\pm} = h_{\theta}^{\lambda} n_{\theta;\lambda} \Big|^{\pm} = n_{\theta;\theta}^{\pm} \Big|^{\pm} \\ &= - \Gamma_{\theta\theta}^{R} n_{R} \Big|^{\pm}, \\ K_{\varphi\varphi}^{\pm} &= n_{\varphi;\varphi}^{\pm} = -\Gamma_{\varphi\varphi}^{R} n_{R} \Big|^{\pm}. \end{aligned} \tag{29}$$

Using the identity $n_{\mu;\nu} u^{\mu} u^{\nu} = -n_{\mu} u_{;\nu}^{\mu} u^{\nu}$, we have $K_{\tau\tau} = -u^{\nu} u_{;\nu}^{\mu} n_{\mu}$. From the identity $u_{\nu} u^{\nu} = 1$ it follows that $u^{\mu} u_{\mu;\nu} u^{\nu} = 0$ or $u_T u_{;\nu}^{T} u^{\nu} + u_R u_{;\nu}^{R} u^{\nu} = 0$. Direct calculation yields

$$u_{;\nu}^{R} u^{\nu} = \ddot{R} + \dot{T}^2 \Gamma_{TT}^{R} + \dot{R}^2 \Gamma_{RR}^{R} + 2\dot{T}\dot{R}\Gamma_{RT}^{R}. \tag{30}$$

$K_{\tau\tau}$ is expressed as

$$\begin{aligned} K_{\tau\tau} &= -u^{\nu} u_{;\nu}^{\mu} n_{\mu} = u_{;\nu}^{R} u^{\nu} \left(n_T \frac{u_R}{u_T} - n_R \right) \\ &= -\frac{u_{;\nu}^{R} u^{\nu}}{\dot{T} b(R,T) f(R,T)}. \end{aligned} \tag{31}$$

Using the expressions of Section 4 for u_{μ} and n_{μ} with the upper signs, we obtain the extrinsic curvature in Kodama-Schwarzschild coordinates

$$\begin{aligned} K_{\tau\tau}^{\pm} = &-\frac{1}{\sqrt{\dot{R}^2 + f(R,T)}} \left(\ddot{R} + \frac{f_{,r}(R,T)}{2} \right. \\ &\left. + \left(f(R,T) + \dot{R}^2 \right) \frac{b_{,r}(R,T)}{b(R,T)} - \dot{R}\dot{T} \frac{f_{,t}(R,T)}{f(R,T)} \right) \Bigg|^{\pm}, \end{aligned} \tag{32}$$

$$K_{\theta\theta}^{\pm} = R\sqrt{\dot{R}^2 + f(R,T)} \Big|^{\pm}, \tag{33}$$

$$K_{\varphi\varphi}^{\pm} = R\sqrt{\dot{R}^2 + f(R,T)} \sin^2\theta \Big|^{\pm}. \tag{34}$$

Here transforming from (31) to (32) we substituted

$$\dot{T}_{\pm} b(R, T_{\pm}) f_{\pm}(R, T_{\pm}) = \left(f_{\pm}(R, T_{\pm}) + \dot{R}^2 \right)^{1/2} \tag{35}$$

which follows from (6).

In Gaussian normal coordinates the components of the extrinsic curvature are

$$\begin{aligned} \widehat{K}_{\tau\tau}^{\pm} &= \widehat{n}_{\tau;\tau}^{\pm} = -\Gamma_{\tau\tau}^{\eta} \widehat{n}_{\eta} \Big|_{\Sigma}^{\pm} = -\frac{1}{2} \left(p^2 \right)_{,\eta} (\tau, \eta) \widehat{n}_{\eta} \Big|_{\eta=0}^{\pm}, \\ \widehat{K}_{\theta\theta}^{\pm} &= \widehat{n}_{\theta;\theta}^{\pm} = - \Gamma_{\theta\theta}^{\eta} \widehat{n}_{\eta} \Big|_{\eta=0}^{\pm} = \frac{1}{2} r_{,\eta}^2 \widehat{n}_{\eta} \Big|_{\eta=0}^{\pm}, \\ \widehat{K}_{\varphi\varphi}^{\pm} &= \widehat{n}_{\varphi;\varphi}^{\pm} = -\Gamma_{\varphi\varphi}^{\eta} \widehat{n}_{\eta} \Big|_{\eta=0}^{\pm} = \frac{1}{2} r_{,\eta}^2 \widehat{n}_{\eta} \sin^2\theta \Big|_{\eta=0}^{\pm}. \end{aligned} \tag{36}$$

In the general case of the functions $f(t, r)$ depending on t calculation is straightforward but cumbersome. Below we perform calculation for the case of $f^{\pm}(r)$ independent of t. Using solutions (27),

$$\begin{aligned} t' &= \frac{\dot{R}}{f(r)}, \\ r' &= \sqrt{f(r) + \dot{R}^2}, \end{aligned} \tag{37}$$

we have

$$\begin{aligned} \dot{t}' \Big|_{\Sigma} &= \frac{\ddot{R} f(R) - f_{,r}(R) \dot{R}^2}{f^2(R)}, \\ \dot{r}' \Big|_{\Sigma} &= \frac{f_{,r}(R) \dot{R} + 2\dot{R}\ddot{R}}{2\sqrt{f(R) + \dot{R}^2}}. \end{aligned} \tag{38}$$

Using (13), we obtain

$$\begin{aligned} p_{,\eta}^2 \Big|_{\Sigma} = \left(f\dot{t}^2 - f^{-1}\dot{r}^2 \right)_{;\eta} \Big|_{\Sigma} = &\left[f_{,r}(R) r' \dot{T}^2 \right. \\ &\left. + 2f(R) \dot{T}\dot{t}' + \frac{f_{,r}(R) r' \dot{R}^2}{f^2(R)} - \frac{2\dot{R}\dot{r}'}{f(R)} \right]_{\Sigma}. \end{aligned} \tag{39}$$

Substituting expressions (38), we obtain

$$\begin{aligned} K_{\tau\tau}^{\pm} &= -\frac{2\ddot{R} + f_{,r}^{\pm}(R)}{2\sqrt{f^{\pm}(R) + \dot{R}^2}}, \\ \widehat{K}_{\theta\theta}^{\pm} &= R\sqrt{f^{\pm}(R) + \dot{R}^2}, \\ \widehat{K}_{\varphi\varphi}^{\pm} &= R\sqrt{f^{\pm}(R) + \dot{R}^2} \sin^2\theta. \end{aligned} \tag{40}$$

It is seen that extrinsic curvatures in both parametrizations coincide.

6. Israel Junction Conditions

Next, we consider the Einstein equations and the Israel junction conditions. The energy-momentum tensor is taken in the form

$$T_{\mu\nu} = \theta(\eta) S^{+}_{\mu\nu} + \theta(-\eta) S^{-}_{\mu\nu} + \delta(\eta) \delta^{i}_{\mu}\delta^{j}_{\nu} S_{ij}. \tag{41}$$

Because the values of the extrinsic curvatures at the opposite sides of Σ are different, the derivative of the extrinsic curvature through the surface Σ contains δ-singularity. From the singular part of the (ij) component of the Einstein equations projected on Σ,

$$\begin{aligned} &{}^{(3)}R^{i}{}_{j} - \delta^{i(3)}_{j} R - \left(K^{i}_{j} - \delta^{i}_{j} K\right)_{,\eta} - K K^{i}{}_{j} \\ &\quad + \left(K^{2} + K^{i}_{j} K^{j}_{i}\right)/2 = 8\pi G T^{i}{}_{j}, \end{aligned} \tag{42}$$

follow the relations,

$$\left[K^{i}{}_{j}\right] - \delta^{i}{}_{j}\left[K\right] = -8\pi G S^{i}{}_{j}, \tag{43}$$

where $[K^{i}{}_{j}] = K^{i+}{}_{j} - K^{i-}{}_{j}$ and $K = K^{i}{}_{i}$. The $(\eta\, i)$ component of the Einstein equations,

$$-K^{j}{}_{i|j} + K_{,i} = 8\pi G T^{\eta}{}_{i}, \tag{44}$$

(vertical bar stands for covariant derivative with respect to metric (7)) yields

$$K^{j\pm}{}_{i|j} - K^{\pm}_{i} = 8\pi G \theta(\pm\eta) S^{\eta\pm}_{i}. \tag{45}$$

From the $(\eta\ \eta)$ component of the Einstein equations,

$$-\frac{{}^{(3)}R}{2} - \frac{K^{i}_{j} K^{j}_{i}}{2} - \frac{K^{2}}{2} = 8\pi G T^{\eta}_{\eta}, \tag{46}$$

it follows that

$$-\frac{1}{2}\left(K^{i}_{j} K^{j}_{i} + K^{2}\right)^{\pm} = 8\pi G \theta(\pm\eta) S^{\eta\pm}{}_{\eta}. \tag{47}$$

Further restrictions on $S^{\pm}_{\mu\nu}$ follow from the conservation equations of the energy-momentum tensor [7].

Projections of the components of the bulk metric $g_{\mu\nu}$ defined as $\tilde{g}_{ij} = g_{\mu\nu}\partial x^{\mu}/\partial x^{i}\ \partial x^{\nu}/\partial x^{j}|_{\Sigma}$ are

$$\begin{aligned} \tilde{g}_{\tau\tau} &= -1, \\ \tilde{g}_{\theta\theta} &= R^{2}(\tau), \\ \tilde{g}_{\varphi\varphi} &= R^{2}(\tau)\sin^{2}\theta. \end{aligned} \tag{48}$$

Projections of the tangent vector $\tilde{u}_{i} = u_{\mu}\partial x^{\mu}/\partial x^{i}|_{\Sigma}$ are

$$\begin{aligned} \tilde{u}_{\tau} &= -1, \\ \tilde{u}_{\theta} &= \tilde{u}_{\varphi} = 0. \end{aligned} \tag{49}$$

Assuming that the energy-momentum tensor S_{ij} has the form $S_{ij} = \tilde{\sigma}\tilde{u}_{i}\tilde{u}_{j} + \tilde{\zeta}(\tilde{g}_{ij} - \tilde{u}_{i}\tilde{u}_{j})$, from (43), we obtain

$$\begin{aligned} S^{\tau}_{\tau} &= -\tilde{\sigma} + 2\tilde{\zeta}, \\ S^{\theta}_{\theta} &= S^{\varphi}_{\varphi} = \tilde{\zeta}. \end{aligned} \tag{50}$$

From Israel conditions (45) written as

$$\left[K^{i}{}_{j}\right] = -8\pi G\left(S^{i}{}_{j} - \frac{1}{2}\delta^{i}_{j} S\right) \tag{51}$$

we have

$$\begin{aligned} \left[K^{\tau}_{\tau}\right] &= 4\pi G\tilde{\sigma}, \\ \left[K^{\theta}_{\theta}\right] = \left[K^{\varphi}_{\varphi}\right] &= -8\pi G\left(\frac{\tilde{\sigma}}{2} - \tilde{\zeta}\right). \end{aligned} \tag{52}$$

In the case of metric (3) with the function f independent of t from (40) one obtains

$$\begin{aligned} K^{\pm\tau}_{\tau} &= \frac{1}{\dot{R}}\frac{d}{d\tau}\sqrt{f^{\pm}(R) + \dot{R}^{2}}, \\ K^{\pm\theta}_{\theta} = K^{\pm\varphi}_{\varphi} &= \frac{1}{R}\sqrt{f^{\pm}(R) + \dot{R}^{2}}. \end{aligned} \tag{53}$$

Israel conditions take a simple form in the case $\tilde{\zeta} = 0$. From the Israel conditions it follows that

$$\begin{aligned} \frac{1}{\dot{R}}\frac{d}{d\tau}\left[\sqrt{\dot{R}^{2} + f^{+}(R)} - \sqrt{\dot{R}^{2} + f^{-}(R)}\right] &= 4\pi\tilde{\sigma}, \\ \frac{1}{R}\left[\sqrt{\dot{R}^{2} + f^{+}(R)} - \sqrt{\dot{R}^{2} + f^{-}(R)}\right] &= -4\pi\tilde{\sigma}. \end{aligned} \tag{54}$$

Solving this system of equations, we find that $R\tilde{\sigma} = \text{const}$ and

$$R\left(\sqrt{\dot{R}^{2} + f^{+}(R)} - \sqrt{\dot{R}^{2} + f^{-}(R)}\right) = M = \text{const}. \tag{55}$$

Relation (55) can be rewritten as (cf.[8])

$$\dot{R}^{2} + f^{\pm}(R) - \frac{1}{M^{2}}\left[m_{+} - m_{-} \mp \frac{M^{2}}{2R}\right]^{2} = 0. \tag{56}$$

7. Conclusions

Geometry and connection between the two coordinate systems used to study dynamics of thin shells, Kodama-Schwarzschild coordinates and Gaussian normal coordinate system, were studied. Transformation between the coordinate systems is studied and explicitly constructed for the case of Kodama-Schwarzschild metric independent of time. Extrinsic curvatures of the surface swept by the shell in the ambient space are calculated for a general time-dependent metric in both Kodama-Schwarzschild and normal Gaussian parametrizations and are shown to give the same result. Application to the Israel junction conditions is discussed.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

Black Hole Entropy from Indistinguishable Quantum Geometric Excitations

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In loop quantum gravity the quantum geometry of a black hole horizon consists of discrete nonperturbative quantum geometric excitations (or punctures) labeled by spins, which are responsible for the quantum area of the horizon. If these punctures are compared to a gas of particles, then the spins associated with the punctures can be viewed as single puncture area levels analogous to single particle energy levels. Consequently, if we *assume* these punctures to be *indistinguishable*, the microstate count for the horizon resembles that of Bose-Einstein counting formula for gas of particles. For the Bekenstein-Hawking area law to follow from the entropy calculation in the large area limit, the Barbero-Immirzi parameter (γ) approximately takes a constant value. As a by-product, we are able to speculate the state counting formula for the SU(2) quantum Chern-Simons theory coupled to indistinguishable sources in the weak coupling limit.

1. Introduction

One of the prime achievements of canonical quantum gravity, more specifically *loop quantum gravity* (LQG), is the provision of a description of microstates of equilibrium black hole horizon, modeled as quantum *isolated horizon* (IH) [1, 2], leading to an *ab initio* quantum statistical derivation of entropy from first principles. The quantum geometry of a cross section of an IH is depicted as a topological 2-sphere with quantum degrees of freedom at localized points called *punctures*. These punctures *are* the quantum geometry and each puncture is associated with a quantum number j. If these punctures are considered to be distinguishable, then the microstate count for the quantum IH resembles that of Maxwell-Boltzmann (MB) counting for a gas of distinguishable particles [3–6]. In this case, if the microcanonical entropy has to be given by *Bekenstein-Hawking area law* (BHAL) (i.e., one-fourth of the area of the horizon divided by Planck length squared), then the *Barbero-Immirzi parameter* (γ) needs to take a certain fixed value.

However, recently there has been a trend of doing black hole thermodynamics in the quantum IH framework considering the punctures to be *indistinguishable* [7–10]. While in some cases Gibbs's approximation has been used in the MB counting to implement indistinguishability [10], the others use both Bose-Einstein (BE) and Fermi-Dirac (FD) statistics [7, 8] by treating the punctures *literally* as quantum mechanical particles with spins and thus categorizing the punctures with integral and half-integral spins into two different species which follow BE and FD statistics, respectively. Also, there is an instance where anyonic statistics have been used [9].

In this article, we shall revisit this issue of indistinguishability of punctures of quantum IH. We *assume* that the punctures are *indistinguishable*. Consequently, the microstate count now *resembles* that of BE statistics applied to a gas of particles. We explain that the resemblance between a quantum IH and a gas of particles follows if we view the quantum number j as denoting the area levels of an individual puncture, rather than "spin," similar to the energy levels of an individual particle in a gas. Automatically, this provides an explanation of how our work differs from other instances in literature where BE statistics have been discussed in relation to quantum IH by considering the quantum number j as "spins" associated with the punctures (e.g., see [7]). Also, we explain why we think it is more reasonable to consider the quantum number j as area levels of the individual punctures rather than "spin" analog of a particle by pointing out some fundamental difference between the physics associated with a quantum IH and a gas of particles, in spite of the structural similarity in their statistical mechanical framework. Then, we calculate the microcanonical entropy of an IH with a given classical area $A \gg \mathcal{O}(\ell_p^2)$ (where ℓ_p is the Planck length) and find the restrictions to be imposed on the BI parameter for the BHAL to follow. We find graphically that the BI parameter has an area dependence but approaches a constant value in the large area limit. Finally we end with some concluding remarks.

2. Microstate Count

An individual puncture of a quantum IH labeled with quantum number j contributes a quantum of area a_j to the quantum IH. If s_j denotes the number of punctures with label j, then, for a set $\{s_j\}$, the total quantum area is given by $A_{\text{quant}} = \sum_j s_j a_j$. Also, g_j is the degeneracy associated with a puncture labeled by quantum number j.

On the other hand, let us consider a gas of particles with total energy $\sum_i n_i \epsilon_i$, where $i \equiv$ *single particle energy level* (SPEL), $\epsilon_i \equiv$ energy of a particle in the ith level, and $n_i \equiv$ number of particles in the ith level. Let ω_i be the degeneracy associated with the ith level.

There is a manifest structural similarity between these two systems if we consider the following correspondence ($\mathcal{G}$):

$$\begin{aligned} j &\longleftrightarrow i, \\ s_j &\longleftrightarrow n_i, \\ a_j &\longleftrightarrow \epsilon_i, \\ g_j &\longleftrightarrow \omega_i. \end{aligned} \tag{1}$$

Thus, j can be called *single puncture area level* (SPAL) for a quantum IH.

As the underlying quantum theory of the gas of particles provides the details of ϵ_i, ω_i, and so forth, so does LQG for a quantum IH, namely, $a_j = 8\pi\gamma\ell_p^2\sqrt{j(j+1)}$, $g_j = (2j+1)$, and j can take values in the range $1/2, 1, 3/2, \ldots, A/8\pi\gamma\ell_p^2$ [1, 2].

Now, for a quantum IH, s_j can vary arbitrarily without any restrictions and the punctures are indistinguishable by assumption. Hence, considering the correspondence $\mathcal{G}$, the system has a structural similarity with a gas of indistinguishable particles where n_i can vary arbitrarily without any restrictions. The microstate count for a configuration $\{n_i\}$ is well known [11, 12]. By the correspondence $\mathcal{G}$, we can simply get the number of microstates corresponding to a configuration $\{s_j\}$ of a quantum IH, which is given by

$$\Omega\left[\left\{s_j\right\}\right] = \prod_j \frac{\left(s_j + g_j - 1\right)!}{s_j!\left(g_j - 1\right)!}. \tag{2}$$

(*Digression.* If we consider the punctures to be distinguishable and since any number of punctures can have any value of j, the microstate count for a set $\{s_j\}$ of punctures is given by $\Omega[\{s_j\}] = (\sum_k s_k)\prod_j (g_j^{s_j}/s_j!)$ [3, 5, 6], which resembles Maxwell-Boltzmann (MB) counting for a gas of particles considering the correspondence $\mathcal{G}$ [11, 12].)

The counting details are available in standard textbooks of statistical mechanics (e.g., see [11, 12]) and need not be discussed here unnecessarily. However, we shall cross-check the above formula with a simple example for a clarification. Let us consider that there are two punctures with $j = 1/2$ and three punctures with $j = 1$. So the corresponding degeneracies are $-1/2, 1/2$ and $-1, 0, 1$, respectively. Since the punctures are indistinguishable, the distinct microstates that can be constructed out of the two $j = 1/2$ punctures are $(1/2, 1/2)$, $(1/2, -1/2)$, and $(-1/2, -1/2)$. Similarly, for the three $j = 1$ punctures the distinct microstates are $(1,1,1)$, $(1,1,0)$, $(1,0,0)$, $(0,0,0)$, $(0,0,-1)$, $(0,-1,-1)$, $(-1,-1,-1)$, $(-1,-1,1)$, $(-1,1,1)$, and $(-1,0,1)$. Hence, there are $3 \times 10 = 30$ distinct microstates of the system of five indistinguishable punctures. Now, if we put $s_{1/2} = 2$, $g_{1/2} = 2$ and $s_1 = 3$, $g_1 = 3$, then (2) yields 30. So, the formula given by (2) stands justified.

This completes our effort to explain in what precise sense the punctures of a quantum IH obey BE statistics under the assumption of indistinguishability. However, we need to further clarify certain other issues regarding the analogy between a quantum IH and a gas of particles in order to differentiate our viewpoint from other instances in literature which discuss BE statistics in relation to quantum IH.

From the correspondence $\mathcal{G}$ between a quantum IH and a gas of particles, it is manifest that there is no corresponding analog of spin of a particle for a puncture of quantum IH. The quantum number j is called "spin" of a puncture in literature because it originates from the SU(2) spin representation carried by the edge of the bulk spin network graph that ends on the corresponding puncture [1, 2]. But we have already explained that j actually represents the SPAL of a quantum IH when it comes to microstate counting.

The spins of particles do not explicitly appear in the microstate counting. It only implicitly affects the counting by imposing restriction on n_i through spin-statistics connection. On the other hand, any number of punctures can be associated with integral or half-integral j-s in case of quantum IH. Alternatively, it can be said that there is no other independent quantum number (analog of particle spin) associated with the punctures of a quantum IH which imposes any restriction on s_j implicitly.

In spite of all the above facts one can still wish to treat the quantum number j as "spins" and punctures as particles. In that case, one needs to show how this treatment invokes the spin-statistics connection and affects the microstate count of a quantum IH. The main obstacle in taking this viewpoint is the deep mismatch between the fundamental natures of punctures of quantum IH and quantum mechanical particles. Particles are perturbative quantum excitations of matter fields propagating on smooth background geometry [13], whereas the punctures are nonperturbative background independent quantum geometric excitations of the IH [1, 2]. The spin-statistics connection heavily relies on local Lorentz invariance related to the background spacetime. But, punctures of quantum IH being themselves the representatives of the quantum geometry, the notion of spin-statistics connection is hitherto unknown in this scenario.

3. Entropy

Now, we shall calculate the entropy of an IH with classical area $A \gg \mathcal{O}(\ell_p^2)$. The physical quantum states of the quantum IH which are of interest in this particular calculation are such that the quantum area corresponding to each state is within $\pm\mathcal{O}(\ell_p^2)$ window of the classical area A; that is, in mathematical language $A_{\text{quant}}[\{s_j\}] = A \pm \mathcal{O}(\ell_p^2)$. As we are working with $A \gg \mathcal{O}(\ell_p^2)$, we can neglect $\pm\mathcal{O}(\ell_p^2)$ for all practical purposes. Thus, the relevant configurations that will contribute to the entropy must obey the constraint

$$C : 8\pi\gamma \sum_j s_j \sqrt{j(j+1)} = \mathscr{A}, \tag{3}$$

where $\mathscr{A} = A/\ell_p^2$ is the dimensionless area of the quantum IH. So the entropy of the IH, using Boltzmann entropy formula and setting Boltzmann constant to unity, is then given by

$$S = \log\left[\sum_{\{s_j\}} \Omega\left[\{s_j\}\right]\right], \tag{4}$$

where the argument of the logarithm represents the total number of microstates arising from all possible configurations, the sum being over all possible configurations constrained by C. *Assuming* that the basic postulates of equilibrium statistical mechanics are valid in the present scenario of quantum IH, there is one most probable configuration whose corresponding number of microstates is overwhelmingly large compared to any other configuration [11, 12] such that the entropy of the IH can be approximately given by the entropy of the most probable configuration alone; that is,

$$\begin{aligned} S = \log\left[\sum_{\{s_j\}} \Omega\left[\{s_j\}\right]\right] &= \log\Omega\left[\{s_j^\star\}\right] \\ &+ \text{contributions from the sub-dominant configurations} \\ &\simeq \log\Omega\left[\{s_j^\star\}\right]. \end{aligned} \tag{5}$$

The most probable configuration can be obtained by extremizing the entropy corresponding to a configuration subject to the area constraint C and hence it is the solution of the following equation:

$$\delta \log\Omega\left[\{s_j\}\right] - \lambda\delta\mathscr{A} = 0, \tag{6}$$

where δ represents arbitrary variation with respect to the variable s_j and λ is the Lagrange multiplier. The distribution function for the most probable configuration (or the most probable distribution) comes out to be

$$s_j^\star = \frac{(g_j - 1)}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1} = \frac{2j}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1}. \tag{7}$$

Considering $s_j \gg 1$ and consequently applying Stirling's approximation, the entropy is calculated to be

$$\begin{aligned} S &\simeq \log\Omega\left[\{s_j^\star\}\right] \\ &= \lambda\mathscr{A} - \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} 2j\log\left(1 - e^{-8\pi\lambda\gamma\sqrt{j(j+1)}}\right) + \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \psi_j, \end{aligned} \tag{8}$$

where the upper limit on j is written in terms of $\mathscr{A}$ which is $j_{\max} = A/8\pi\gamma\ell_p^2 = \mathscr{A}/8\pi\gamma$ and here

$$\begin{aligned} \psi_j &= \log(g_j - 1)! - (g_j - 1)\log(g_j - 1) + (g_j - 1) \\ &= \log(2j)! - 2j\log(2j) + 2j. \end{aligned} \tag{9}$$

The most probable distribution satisfies the area constraint which leads to the following equation:

$$\begin{aligned} \mathscr{A} &= 8\pi\gamma \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \frac{\sqrt{j(j+1)}(g_j - 1)}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1} \\ &= 16\pi\gamma \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \frac{j\sqrt{j(j+1)}}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1}. \end{aligned} \tag{10}$$

Besides this, summing over j gives the total number of punctures for the most probable distribution; that is,

$$\begin{aligned} N_0 := \sum_j s_j^\star &= \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \frac{(g_j - 1)}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1} \\ &= \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \frac{2j}{e^{8\pi\lambda\gamma\sqrt{j(j+1)}} - 1}. \end{aligned} \tag{11}$$

It may be noted that the allowed values of λ and γ must be such that $N_0 \gg 1$. This condition will be automatically satisfied if, at least, $s_j^\star|_{\max} \gg 1$ is assured for the allowed values of λ and γ. By $s_j^\star|_{\max}$ we have meant the peak values of the curve $s_j^\star$ versus j for given values of λ and γ. As we proceed, we shall see that there exist such values of λ and γ for which the condition is satisfied.

Remarks. At this point, we would like to make a few general remarks. We have calculated the entropy of an IH with a given area and hence only the area constraint has appeared along with its corresponding Lagrange multiplier λ. However, it is unlike the calculations made in the author's one of the earlier publications [6]. The article [6] was meant to show that the entropy calculations, resulting in the area law and the subleading logarithmic terms, can also be done with fixed number of punctures directly from the Chern-Simons state counting formula of [14]. Also there was an attempt to give a reason why the number of punctures may be kept fixed for the calculations. All these were specifically meant to clarify the ad hoc proposal of fixing the number of punctures and calculating the entropy without making a connection to the physical Hilbert space of the quantum IH done in [4]. Hence, the article [6] has some issues of interest in its own right in the context of entropy calculation of quantum IH with fixed number of punctures and was particularly of interest in the contemporary flow of literature. However, in course of time, our understanding of quantum IH has evolved and we firmly believe that the fixation of the number of punctures may lead to new mathematical exercises, but the physically correct description of quantum IH does not allow one to a priori fix the number of punctures. This is because *the full Hilbert space of an IH of a given area admits all possible number of punctures which can give rise to the quantum area within a Planck area window about the classical area* [1, 2]. Further, as we have already discussed in an earlier section, there is a structural similarity between the statistical mechanical framework of a quantum IH and a gas of particles if one considers a specific mapping. But this never implies that the punctures can be literally treated as particles. Indeed, the very notion of background independent approach of LQG differentiates the notion of punctures from the notion of particles arising from background dependent quantum field theory. Unlike what occurs in case of quantum field theory one can develop a notion of particle number from the number operator and, by showing it as a conserved quantity, there is no such construction in existence for the number of punctures of quantum IH in LQG framework. We hope these arguments clarify our shift in paradigm towards keeping the number of punctures arbitrary in case of entropy calculation of quantum IH.

4. The BHAL and the Barbero-Immirzi Parameter

The next step is to choose γ in such a way that (8) yields the BHAL. Hence, we demand that the entropy should be given by the BHAL; that is,

$$S = \frac{\mathscr{A}}{4}, \tag{12}$$

and then explore the consequences. Now, demanding (12) we obtain the following equation:

$$\lambda\mathscr{A} - \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} 2j \log\left(1 - e^{-8\pi\lambda\gamma\sqrt{j(j+1)}}\right) + \sum_{j=1/2}^{\mathscr{A}/8\pi\gamma} \psi_j = \frac{\mathscr{A}}{4}. \tag{13}$$

Our motto is to show that there exists certain value(s) of γ for which the BHAL can be derived from the indistinguishable punctures, which is equivalent to finding the solutions of (10) and (13). This can be accomplished by finding if there is any intersection of the 2D surfaces described by (10) and (13) in the 3D space spanned by $\mathscr{A}$, λ, and γ. Such a plot in Mathematica, shown in Figure 1, reveals that there indeed exist values of γ, which furthermore fulfill certain relevant criteria as follows:

(i) γ should lie in the range $0 < \gamma \leq \mathscr{A}/4\pi$ because $j_{\max} = \mathscr{A}/8\pi\gamma \geq 1/2$ and γ appears in the area spectrum as a multiplicative factor and hence needs to be positive definite.

(ii) The allowed values of γ are such that we have $\mathscr{A} \gg \mathscr{O}(1)$ and $\lambda > 0$ for $s_j^\star$ to be positive definite for all j (see (7)).

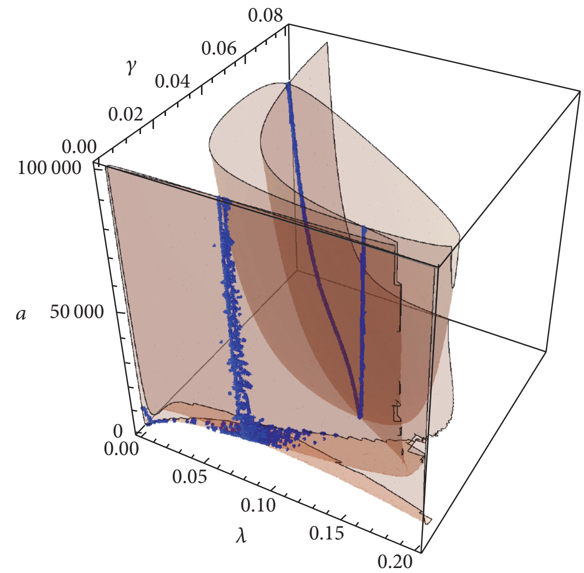

Figure 1: The blue curves are the solutions of (10) and (13). $\mathscr{A}$ is symbolized as "a" in the 3D plot. There are two branches of solutions, of which the U-shaped curve is excluded on physical grounds. The other curve is the only feasible solution which has a broadening due to graphical impreciseness of the plot. (See the Appendix.)

On physical grounds we shall exclude the U-shaped curve from consideration because it gives rise to multiple values of γ for a given $\mathscr{A}$ which imply the following unphysical result: for a given $\mathscr{A}$ we will have two different most probable spin distributions. So we are left with the other curve which will give us the value(s) of γ for which the BHAL is valid. From Figure 2, which gives us the view of 3D plot along the λ-axis, we see that with increasing area ($\mathscr{A}$) the curve straightens very rapidly and asymptotically approaches to some constant value γ_0 (e.g.). Besides that, from Figure 1 it was already evident that λ becomes independent of $\mathscr{A}$ with increasing $\mathscr{A}$. Hence, for $\mathscr{A} \gg \mathscr{O}(1)$, we have practically $\gamma = \gamma_0$ and $\lambda = \lambda_0$ for the BHAL to hold. In view of this, we can conclude that the BHAL follows from the LQG description of black hole horizon, with the same consequences whether we consider the punctures to be distinguishable or not. The only difference is the value of γ_0 which is of the order of 10^{-3} here but of the order of 10^{-1} in case of distinguishable punctures. It may be further noted that the order of magnitude of γ and λ along the curve ensures that at least $s_j^\star|_{\max} \gg 1$ so that Stirling's approximation is valid. One can check that to remain assured.

5. State Counting of Quantum SU(2) Chern-Simons Theory Coupled to Indistinguishable Sources: Some Speculations

In this section we discuss a few possible new ideas which germinate from this exercise of state counting and entropy calculation for quantum IH with indistinguishable punctures. To begin with, we briefly discuss the subject matter of [6]. In [6],

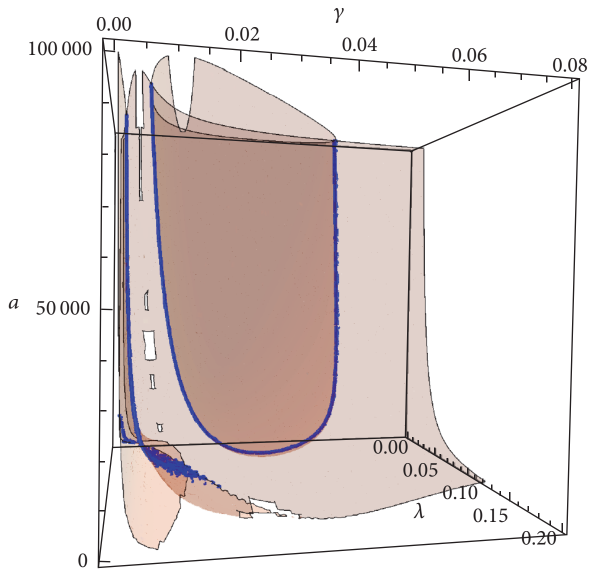

FIGURE 2: The view of the plot in Figure 1 along the λ-axis reveals the variation of γ with $\mathscr{A}$ (symbolized as a in the figure). As $\mathscr{A}$ increases, the value of γ asymptotically approaches a fixed value γ_0 (e.g.) of the order of 10^{-3}. (See the Appendix.)

we begin with the original formula of state count for a set of N punctures with spins $(j_1, j_2, \ldots, j_N)$ [14]. Then we do a sum over all possible spins and using multinomial expansion switch over to the expression of microstates in terms of the spin configurations. Then, entropy of an IH with given area is calculated by the method of most probable distribution for fixed number of punctures (and for arbitrary number of punctures see [15]). That the punctures were considered to be distinguishable during state counting becomes explicit when we switch over to the spin configuration language. The expression for the number of microstates for a given spin configuration $\{s_j\}$ is given by [6]

$$\Omega_{\text{dist.}}\left[\left\{s_j\right\}\right] = \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\underbrace{\frac{\left(\sum_l s_l\right)}{\prod_j s_j!}\prod_j g_j^{s_j}(a,k)}_{\text{MB statistics by }\mathscr{G}}, \quad (14)$$

where $g_j(a,k) = \{\sin(((2j+1)a\pi)/(k+2))/\sin((a\pi)/(k+2))\}$. The underbraced part clearly resembles the MB state counting for ideal gas if we consider the correspondence $\mathscr{G}$ and manifests the distinguishability of the punctures through the combinatorial expression. In fact, in the large k limit, converting the sum over a into an integration over a continuous variable, say θ, followed by a saddle point approximation about a value $\theta = \theta_0$, expression (14) reduces to

$$\Omega_{\text{dist.}}\left[\left\{s_j\right\}\right] \simeq \frac{\left(\sum_l s_l\right)!}{\prod_j s_j!}\prod_j g_j^{s_j}, \quad (15)$$

with $g_j = 2j+1$, as the zeroth-order (in the fluctuations about θ_0) term. This formula is nothing but the state count for distinguishable punctures with configuration $\{s_j\}$, obtained by ignoring the deeper details of the state counting associated with the internal symmetry of the quantum IH. It is the effective formula which leads to the leading term of the entropy which is proportional to area [6, 15]. Interestingly, it may be noted that the combinatorics involved in formula (15) is present in the exact form in the underbraced part of formula (14), except the fact that g_j differs.

Now, when we consider the punctures to be indistinguishable, we do not have a complete formula for the microstate count from which such a similar exercise can be performed. So, we try to go in the reverse direction in this case and look for the possible complete formula for the state count which takes into account the underlying symmetry of the IH and the associated Chern-Simons theory. Thus, the first step for the state counting for indistinguishable punctures is to ignore the minute details involved with the symmetries and so forth of the quantum IH and do the state counting for a configuration $\{s_j\}$ of indistinguishable punctures and this is what has resulted in formula (2). Further, we have also shown that this state count results in the BHAL, that is, the leading term of the entropy.

Henceforth, taking cue from the case of distinguishable punctures, we speculate that the combinatorial form of formula (2) for indistinguishable punctures should appear in the complete formula which takes into account the underlying symmetry of the quantum IH and the associated Chern-Simons theory, as follows:

$$\Omega_{\text{indist.}}\left[\left\{s_j\right\}\right] = \frac{2}{k+2}\sum_{a=1}^{k+1}\sin^2\frac{a\pi}{k+2}\underbrace{\prod_j\frac{\left(s_j+g_j(a,k)-1\right)!}{s_j!\left(g_j(a,k)-1\right)!}}_{\text{BE statistics by }\mathscr{G}}, \quad (16)$$

except the fact that the form of g_j differs. This above formula is what we expect to get while we do the state counting for the quantized SU(2) Chern-Simons theory, in the large k (weak coupling) limit, coupled to a configuration $\{s_j\}$ of indistinguishable sources. Although this is only a guess, it gives us a hint towards what the answer may look like in case of indistinguishable punctures when it will be derived from counting the dimensionality of the Hilbert space associated with SU(2) Chern-Simons theory coupled to indistinguishable sources like what was done for distinguishable sources in [14]. Further, if the above formula can be shown to be the correct and the complete one, then there may be a way to find some approximation to get a correction to formula (2) to obtain the subleading correction to the BHAL.

Apart from this, the problem becomes more interesting due to the fact that it has a direct link with the counting of the conformal blocks of Wess-Zumino-Witten model [16]. This may lead to some new physical problems in the context of conformal quantum field theories. Thus, the exercise presented here, which invokes the idea about how to implement the statistics in case of indistinguishable punctures of quantum IH correctly, may be considered as a small but crucial first step towards exploring potential new physics problems related to quantum IH with indistinguishable punctures and beyond.

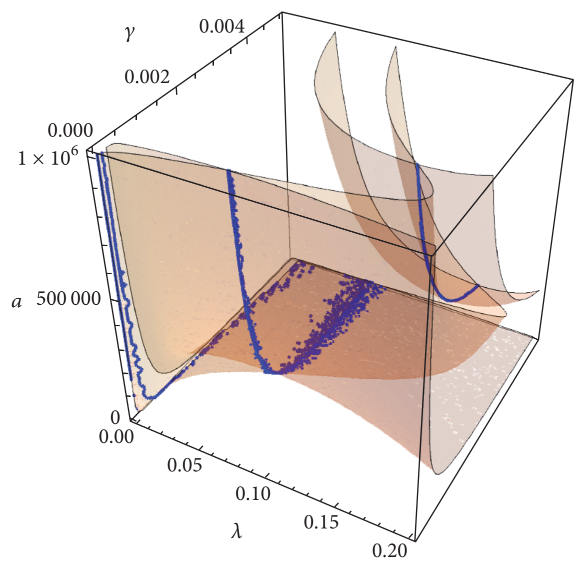

Figure 3: This is a magnified view of the plot in Figure 1 to focus on the curve which gives the solution (along midway of the λ-axis).

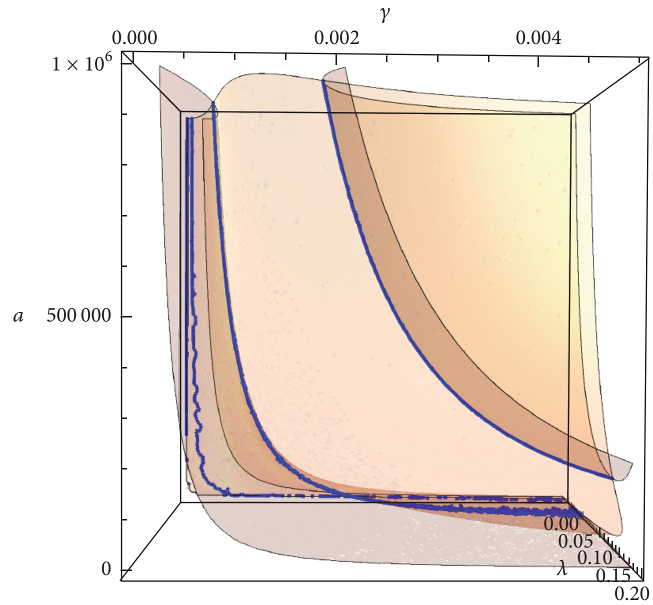

Figure 4: This is a magnified view of the plot in Figure 2 to show the view along λ-axis more closely.

6. Conclusion

What we have done here is the entropy calculation of the IH by assuming that the punctures of quantum IH are indistinguishable and we do not go into the debate whether this assumption is justified within this quantum IH framework. As far as the physical result is concerned, that is, the BHAL resulting from an *ab initio* statistical mechanical calculation from the quantum geometry of black hole horizon as laid down by LQG, there is no much difference whether we consider the punctures to be distinguishable or not. Given that the calculations are done for large area black holes, the BHAL is obtained for a constant value of γ, except the fact that order of magnitude of γ is different in case of distinguishable and indistinguishable punctures.

However, there is a very important issue which we are unable to address here in the context of the assumption of indistinguishability. The states of the quantum IH are actually given by that of the quantum SU(2) Chern-Simons theory coupled to the punctures [14]. Now, in the case of distinguishable punctures, it can be shown that the MB counting results from a zeroth-order approximation of the complete formula for the state counting [6]. The distinguishability is inherited by the fact that we consider a sum over all values of spins on the microstate count for a given set of spins $j_1, \ldots, j_N$ and arrive at the MB counting for a given spin configuration by the application of multinomial expansion [6]. The next order correction leads to the logarithmic correction [6, 15]. Now, when the punctures are indistinguishable, we can not carry out a naive sum over all values of spins for a given puncture data as this will count the microstates which are meant to be indistinguishable. Thus, under the assumption of indistinguishability of punctures, it will be an interesting problem to look upon how the BE counting formula, as discussed here, comes out as an approximation of the exact scenario of quantum Chern-Simons theory. Then it will also be clear what the next order correction is. Will it be the same known logarithmic correction $-3/2 \log \mathscr{A}$ [17]? We hope to investigate this problem in future.

Appendix

Here we have presented some magnified versions of the 3D plot in Figures 3 and 4.

Competing Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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Research Article

On the Jeans Theorem and the "Tolman–Oppenheimer–Volkoff Equation" in R^2 Gravity

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Corda, Mosquera Cuesta, and Lorduy Gòmez have shown that spherically symmetric stationary states can be used as a model for galaxies in the framework of the linearized R^2 gravity. Those states could represent a partial solution to the Dark Matter Problem. Here, we discuss an improvement of this work. In fact, as the star density is a functional of the invariants of the associated Vlasov equation, we show that any of these invariants is in its turn a functional of the local energy and the angular momentum. As a consequence, the star density depends only on these two integrals of the Vlasov system. This result is known as the "*Jeans theorem*." In addition, we find an analogy of the historical Tolman–Oppenheimer–Volkoff equation for the system considered in this paper. For the sake of completeness, in the final section of the paper, we consider two additional models which argue that Dark Matter could not be an essential element.

1. Introduction

The Dark Matter problem started in the 30s of the last century [1]. When one observes the Doppler shift of stars which move near the plane of our galaxy and calculates the velocities, one finds a large amount of matter inside the galaxy which prevents the stars from escaping. That (supposed and unknown) matter generates a very large gravitational force that the luminous mass in the galaxy cannot explain. In order to achieve the very large discrepancy, the sum of all the luminous components of the galaxy should be two or three times more massive. On the other hand, one can calculate the tangential velocity of stars in orbits around the galactic centre like a function of distance from the centre. The result is that stars which are far away from the galactic centre move with the same velocity independent of their distance from the centre.

These strange issues generate a portion of the Dark Matter problem. In fact, either the luminous matter is not able to correctly describe the radial profile of our galaxy or the Newtonian theory of gravitation cannot describe dynamics far from the galactic centre.

Other issues of the problem arise from the dynamical description of various self-gravitating astrophysical systems. Examples are stellar clusters, external galaxies, and clusters and groups of galaxies. In those cases, the problem is similar, as there is more matter arising from dynamical analyses with respect to the total luminous matter.

Zwicky [2] found that in the Coma cluster the luminous mass is too little to generate the gravitational force which is needed to hold the cluster together [2].

The more diffuse way to attempt to solve the problem is to assume that the Newtonian gravity holds at all scales that should exist and that they should exhibit nonluminous components which contribute to the missing mass. There are a lot of names which are used to define such nonluminous components. The MAssive Compact Halo Objects (MACHOs) are supposed to be bodies composed of normal baryon matter, which do not emit (or emit little) radiation and drift through interstellar space unassociated with planetary systems [3]. They could be black holes and/or neutron stars populating the outer reaches of galaxies. The Weakly Interacting Massive Particles (WIMPs) are hypothetical particles which do not interact with standard matter (baryons, protons, and neutrons) [4]. Hence, they should be particles outside the Standard Model of Particles Physics but they have not yet been directly detected. Dark Matter is usually divided into three different flavors: Hot Dark Matter (HDM) [5], Warm Dark Matter (WDM) [6], and Cold Dark Matter (CDM) [7]. HDM should be composed of ultrarelativistic particles like neutrinos. CDM should consist of MACHOs, WIMPs, and axions, which are very light particles with a particular behavior of self-interaction [8]. If we consider the standard model of cosmology, the most recent results from the Planck mission [9] show that the total mass energy of the known universe contains 4.9% ordinary matter, 26.8% Dark Matter, and 68.3% Dark Energy.

An alternative approach is to explain large-scale structure without dark components in the framework of Extended Theories of Gravity [10–17] and the references within. In other words, we call "Dark Matter" a gravitational effect that we do not yet understand as a modification to both Newtonian and Einsteinian gravity could be needed (see [10–17]). The underlying idea in Extended Theories of Gravity is that General Relativity is a particular case of a more general effective theory which comes from basic principles [10–17]. The standard Einstein-Hilbert action of General Relativity [18, 19] is modified by adding new degrees of freedom, like high order curvature corrections (the so-called $f(R)$ theories of gravity [10–17, 20]) and scalar fields (the generalization of the Nordström–Jordan–Fierz–Brans–Dicke theory of gravitation [21–25], which is known as scalar-tensor gravity [26–28]). In this different context, one assumes that gravity is not scale-invariant and takes into account only the "observed" ingredients, that is, curvature and baryon matter. Thus, it is not required to search for candidates for Dark Matter which have not yet been found [10–17, 20]. In this perspective, the gravitational wave astronomy should be the ultimate test for the physical consistency of General Relativity or of any other theory of gravitation [10].

2. R^2 Theory of Gravity and Vlasov System

In the framework of $f(R)$ theories of gravity, the R^2 theory is the simplest among the class of viable models with R^m terms. Those models support the acceleration of the universe in terms of cosmological constant or quintessence as well as early time inflation [11, 15, 20]. Moreover, they should pass the Solar System tests, as they have an acceptable Newtonian limit, no instabilities, and no Brans–Dicke problem (decoupling of scalar) in scalar-tensor version. We recall that the R^2 theory was historically proposed in [29] with the aim of obtaining the cosmological inflation.

The R^2 theory arises from the action [30]

$$S = \int d^4x\sqrt{-g}\left(R + bR^2 + \mathscr{L}_m\right), \quad (1)$$

where b represents the coupling constant of the R^2 term. In general, when the constant coupling of the R^2 term in the gravitational action (1) is much more minor than the linear term R, the variation from standard General Relativity is very weak and the theory can pass the Solar System tests [30]. In fact, as the effective scalar field arising from curvature is highly energetic, the constant coupling of the R^2 nonlinear term $\rightarrow 0$ [30]. In that case, the Ricci scalar, which represents an extra dynamical quantity in the metric formalism, should have a range longer than the size of the Solar System. This is correct when the effective length of the scalar field l is much shorter than the value of 0.2 mm [31]. Hence, this effective scalar field results to be hidden from Solar System and terrestrial experiments. By analysing the deflection of light by the sun in the R^2 theory through a calculation of the Feynman amplitudes for photon scattering, one sees that, to linearized order, the result is the same as in standard General Relativity [22]. By assuming that the dynamics of the matter (the stars making of the galaxy) can be described by the Vlasov system, a model of stationary, spherically symmetric galaxy can be obtained. This issue was described in detail in [30], in the framework of the R^2 theory. For the sake of completeness, in this section, we shortly review this issue.

In this paper, we consider Greek indices run from 0 to 3. By varying the action of (1) with respect to $g_{\mu\nu}$, one gets the field equations [30]

$$\begin{aligned} G_{\mu\nu} &+ b\left\{2R\left[R_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R\right] - 2R_{;\mu;\nu} + 2g_{\mu\nu}\Box R\right\} \\ &= T_{\mu\nu}^{(m)}. \end{aligned} \quad (2)$$

By taking the trace of this equation, the associated Klein–Gordon equation for the Ricci curvature scalar

$$\Box R = E^2\left(R + T\right) \quad (3)$$

is obtained [30], where $\Box$ is the d'Alembertian operator and the energy term, E, has been introduced for dimensional motivations [30]

$$E^2 \equiv \frac{1}{6b}. \quad (4)$$

Hence, b is positive [30]. $T_{\mu\nu}^{(m)}$ in (2) is the standard stress-energy tensor of the matter and General Relativity is easily reobtained for $b = 0$ in (2).

As we study interactions between stars at galactic scales, we consider the linearized theory in vacuum ($T_{\mu\nu}^{(m)} = 0$), which gives a better approximation than the Newtonian theory [30]. Calling $\widetilde{R}$ the linearized quantity which corresponds to R, considering the plane wave [30]

$$b\widetilde{R} = a\left(\vec{p}\right)\exp\left(iq^{\beta}x_{\beta}\right) + \text{c.c.} \quad (5)$$

with [30]

$$\begin{aligned} p^{\beta} &\equiv \left(\omega, \vec{p}\right), \quad \omega = p \equiv \left|\vec{p}\right|, \\ q^{\beta} &\equiv \left(\omega_E, \vec{p}\right), \quad \omega_E = \sqrt{E^2 + p^2}, \end{aligned} \quad (6)$$

one can choose a gauge for a gravitational wave propagating in the $+z$ direction in which a first-order solution of (2) with $T^{(m)}_{\mu\nu} = 0$ is given by the conformally flat line element [30]

$$ds^2 = \left[1 + b\widetilde{R}(t,z)\right]\left(dx^2 + dy^2 + dz^2 - dt^2\right). \tag{7}$$

We recall that the dispersion law for the modes of $b\widetilde{R}$, that is, the second of (6), is that of a wave packet [30]. The group velocity of a wave packet of $b\widetilde{R}$ centred in $\vec{p}$ is [30]

$$\overrightarrow{v_G} = \frac{\vec{p}}{\omega_E}. \tag{8}$$

From the second of (6) and (8), one gets [30]

$$v_G = \frac{\sqrt{\omega_E^2 - E^2}}{\omega_E}, \tag{9}$$

which can be rewritten as [30]

$$E = \sqrt{(1 - v_G^2)}\omega_E. \tag{10}$$

If one assumes that the dynamics of the stars making of the galaxy are described by the Vlasov system, the gravitational forces between the stars will be mediated by metric (7). Thus, the key assumption is that, in a cosmological framework, the wave packet of $b\widetilde{R}$ centred in $\vec{p}$, which is given by the (linearized) space-time curvature, governs the motion of the stars [30]. In this way, the "*curvature*" energy E is identified as the Dark Matter content of a galaxy of typical mass energy $E \simeq 10^{45}$ g, in ordinary c.g.s. units [30]. As $E \simeq 10^{45}$ g, from (4), one gets $b \simeq 10^{-34}\,\mathrm{cm}^4$ in natural units [30]. Hence, the constant coupling of the R^2 term in action (1) is much more minor than the linear term R and the variation from standard General Relativity is very weak. This implies that the theory can pass the Solar System tests as the effective length of the scalar field is $l \ll 0.2$ mm [30].

We can use a conformal transformation [30, 32] to rescale the line element (7) like

$$\widetilde{g}_{\alpha\beta} = e^{\Phi} g_{\alpha\beta}, \tag{11}$$

where we set [30]

$$\Phi \equiv b\widetilde{R}. \tag{12}$$

Thus, in the linearized theory, we get

$$e^{\Phi} = 1 + b\widetilde{R}. \tag{13}$$

Hence, it is the Ricci scalar, that is, the *scalaron* [29, 30], the scalar field which translates the analysis into the conformal frame, the Einstein frame [15, 30].

Particles in space-time make up an ensemble with no collisions and are governed by a line element like (7) if the particle density satisfies the Vlasov equation [30, 32–34]

$$\partial_t f + \frac{p^a}{p^0}\partial_{x^a} f - \Gamma^a_{\mu\nu}\frac{p^\mu p^\nu}{p^0}\partial_{p^a} f = 0, \tag{14}$$

where $\Gamma^\alpha_{\mu\nu}$ are the Christoffel coefficients, f is the particle density, and p^0 is given by p^a ($a = 1, 2, 3$), according to the relation [30, 32–34]

$$g_{\mu\nu} p^\mu p^\nu = -1. \tag{15}$$

Equation (15) means that the four-momentum p^μ lies on the mass shell of the space-time [30, 32–34].

In general, the Vlasov-Poisson system is introduced through the system of equations [30, 32–34]

$$\begin{gathered}\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \\ \nabla \cdot U = 4\pi\rho, \\ \rho(t,x) = \int dv f(t,x,v).\end{gathered} \tag{16}$$

In (16), t is the time and x and v are the position and the velocity of the stars, respectively. $U = U(t,x)$ is the average Newtonian potential generated by the stars. System (16) represents the nonrelativistic kinetic model for an ensemble of particles (stars in the galaxy) with no collisions. The stars interact only through the gravitational forces which they generate collectively and are considered pointlike particles, and we neglect the relativistic effects [30, 32–34]. The function $f(t,x,v)$ in the Vlasov-Poisson system (16) is nonnegative and gives the density on phase space of the stars within the galaxy [30].

In this approach, the first-order solutions of the Klein–Gordon equation (3) for the Ricci curvature scalar are considered like galactic high energy *scalarons*, which are expressed in terms of wave packets having stationary solutions within the Vlasov system [30]. The energy of the wave packet is interpreted like the Dark Matter component which guarantees the galaxy's equilibrium [30]. This approximation is not as precise as one would aspect [30], but here we consider it as the starting point of our analysis.

The analysis in [30] permits rewriting the Vlasov-Poisson system in spherical coordinates as

$$\begin{aligned}&-\frac{d^2 b\widetilde{R}}{dt^2} + \frac{1}{r^2}\frac{d}{dr}\left(\frac{d}{dr}b\widetilde{R}r^2\right) = \left(1 + 2b\widetilde{R}\right)\mu(t,r), \\ &\mu(t,r) = \int \frac{dp}{\sqrt{1+p^2}} f(t,x,p), \\ &\partial_t f + \frac{p}{\sqrt{1+p^2}} \cdot \partial_x f \\ &\quad - \left[\left(\frac{d}{dt}b\widetilde{R} + \frac{x \cdot p}{\sqrt{1+p^2}}\frac{1}{r}\frac{d}{dr}b\widetilde{R}\right)p \right. \\ &\quad \left. + \frac{x}{\sqrt{1+p^2}}\frac{1}{r}\frac{d}{dr}b\widetilde{R}\right] \cdot \partial_p f = 0.\end{aligned} \tag{17}$$

In (17), p denotes the vector $p = (p_1, p_2, p_3)$ with $p^2 = |p|^2$, and x denotes the vector $x_i = (x_1, x_2, x_3)$ [30].

As one is interested in stationary states, one calls λ the wavelength of the "*galactic*" *gravitational wave* (7), that is, the characteristic length of the gravitational perturbation [30]. One further assumes that $\lambda \gg d, d$ being the galactic scale of order $d \sim 10^5$ light-years [35, 36]. In other words, the gravitational perturbation can be considered "*frozen-in*" with respect to the galactic scale [30].

In that way, one can write down the system of equations defining the stationary solutions of (17) [30]:

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{d}{dr}b\widetilde{R}r^2\right) = \left(1+2b\widetilde{R}\right)\mu(r), \tag{18}$$

$$\mu(r) = \int \frac{dp}{\sqrt{1+p^2}} f(x,p), \tag{19}$$

$$p\cdot\partial_x f - \frac{1}{r}\frac{d}{dr}b\widetilde{R}\left[(p\cdot x)p + x\right]\cdot\partial_p f = 0. \tag{20}$$

Therefore, the idea in [30] is that the spin-zero degree of freedom arising from the R^2 term in the gravitational Lagrangian, that is, the scalaron, is a potential candidate for the Dark Matter. In this approach, the dominant contribution to the curvature within a galaxy comes from the scalaron field equation (3). That equation has a proper baryon source term. This enables the baryons themselves to evolve obeying a collisionless Boltzmann equation. Then, the baryons can propagate on the curvature generated by the scalaron.

3. The Jeans Theorem and the "Tolman–Oppenheimer–Volkoff Equation"

The following results will be obtained adapting the ideas introduced in [32–34, 37]. Let us start by recalling some important definitions in the conformal frame:

$$\begin{aligned}
P(r) &\equiv \int \frac{dp}{\sqrt{1+p^2}}\left(\frac{x\cdot p}{r}\right)^2 f(x,p) \\
&\qquad\qquad\text{radial pressure} \\
\rho(r) &\equiv (1+2bR)\int dp\sqrt{1+p^2}\, f(x,p) \\
&\qquad\qquad\text{mass-energy density} \\
P_T(r) &\equiv \int \frac{dp}{\sqrt{1+p^2}}\left|\frac{x\wedge p}{r}\right| f(x,p) \\
&\qquad\qquad\text{tangential pressure.}
\end{aligned} \tag{21}$$

Thus, (18) becomes

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{d}{dr}b\widetilde{R}r^2\right) = \rho(r) - P(r) - 2P_T(r). \tag{22}$$

Let us consider the stationary solution of our stellar dynamics model, that is, (18), (19), and (20). The particle density is a functional of the invariants of the Vlasov equation (14). The Jeans theorem states that any of these invariants must be a functional of the local energy and the angular momentum. In that way, the particle density depends only on these two integrals of the system under consideration. In order to prove these statements, one introduces the new coordinates

$$\begin{aligned}
r &= |x|, \\
Y &= \left(1+\frac{b}{2}\widetilde{R}\right)\frac{x\cdot p}{|x|}, \\
Z &= \left(1+b\widetilde{R}\right)\left[|x|^2 - |p|^2 - (x\cdot p)^2\right].
\end{aligned} \tag{23}$$

We note that, with f being spherically symmetric, one can write it as a function of r, Y, Z; that is, $f(x,p) \to f(r,Y,Z)$. Thus, (20) in the new coordinates reads

$$Y\partial_r f + \left[\frac{Z}{r^3} - \left(1+b\widetilde{R}\right)\frac{d}{dr}b\widetilde{R}\right]\partial_Y f = 0. \tag{24}$$

This equation has the same form of equation (2.11) in [37] with $m(r) = (1+b\widetilde{R})((d/dr)b\widetilde{R}r^2)$. Thus, by applying the result in [37], one obtains that f must have the form

$$f(r,Y,Z) = A\left(\overline{E},Z\right), \tag{25}$$

where

$$\begin{aligned}
\overline{E}(r,Y,Z) &= \frac{1}{2}Y^2 + \frac{1}{2}\frac{Z}{r^2} + \frac{1+b\widetilde{R}}{2} \\
&= \left(\frac{1+b\widetilde{R}}{2}\right)\left(1+p^2\right).
\end{aligned} \tag{26}$$

Returning to the system of (18), (19), and (20), one gets immediately

$$f(x,p) = A(E,X), \tag{27}$$

where

$$E = \left(1+\frac{b}{2}\widetilde{R}\right)\sqrt{1+p^2} \tag{28}$$

is the local energy of the particles and

$$X = \left(1+\frac{b}{2}R\right)|x\wedge p|^2 \tag{29}$$

is the modulus squared of the local angular momentum. Using (27) and a transformation of variables, system (21) becomes

$$\begin{aligned}
P(r) &= \frac{\pi}{r^2}\int dE\int dX A(E,X)\sqrt{E^2 - \frac{X}{r^2} - \left(1+\frac{b}{2}\widetilde{R}\right)}, \\
\rho(r) &= \frac{\pi}{r^2}\int dE E^2 dX \frac{A(E,X)}{\sqrt{E^2 - X/r^2 - \left(1+(b/2)\widetilde{R}\right)}}, \\
P_T(r) &= \frac{\pi}{2r^4}\int dE\int dX \frac{XA(E,X)}{\sqrt{E^2 - X/r^2 - \left(1+(b/2)\widetilde{R}\right)}}.
\end{aligned} \tag{30}$$

A direct computation permits obtaining

$$\frac{d}{dr}P(r) = -\left(1 + b\widetilde{R}\right)\frac{d}{dr}\rho(r) - \frac{2}{r}\left[P(r) - P_T(r)\right], \quad (31)$$

which is analogous to the historical Tolman–Oppenheimer–Volkoff equation [38, 39] for the system under analysis in this paper.

For the sake of completeness, we stress that the Jeans instability (gravitational stability) has been analysed in the context of modified theories of gravity for rotating/nonrotating configurations in [40–42]. Related questions on the Tolman–Oppenheimer–Volkoff equation and the Jeans instability (but not for plane waves) were also studied in [43–45]. The Tolman–Oppenheimer–Volkoff equation has been also analysed in the dimensional gravity's rainbow in the presence of cosmological constant in [46] and in the framework of dilaton gravity in [47].

4. Two Additional Models

For the sake of completeness, we play the devil's advocate and consider two models which argue that Dark Matter is not an essential element, even though popular models postulate that it comprises roughly a fourth of a universe. Our starting point is the relation [48, 49]

$$G = G_0\left(1 - \frac{t}{t_0}\right), \quad (32)$$

where G_0 is the present value of G, t_0 is the present age of the universe, and t is the time elapsed from the present epoch. Similarly, one could deduce that [49]

$$r = r_0\left(\frac{t_0}{t_0 + t}\right). \quad (33)$$

In this scheme, the gravitational constant G varies slowly with time. This is suggested by Sidharth's 1997 cosmology [48], which correctly predicted a Dark Energy driven accelerating universe at a time when the accepted paradigm was the Standard Big Bang cosmology in which the universe would decelerate under the influence of Dark Matter.

We reiterate the following: the problem of galactic rotational curves [49, 50]. We would expect, on the basis of straightforward dynamics, that the rotational velocities at the edges of galaxies would fall off according to

$$v^2 \approx \frac{GM}{r}. \quad (34)$$

However, it is found that the velocities tend towards a constant value:

$$v \sim 300\,\text{km/sec}. \quad (35)$$

This, as known, had led to the postulation of the as yet undetected additional matter, the so-called Dark Matter. We observe that from (33) it can be easily deduced that [51]

$$a \equiv (\ddot{r}_0 - \ddot{r}) \approx \frac{1}{t_0}\left(t\ddot{r}_0 + 2\dot{r}_0\right) \approx -2\frac{r_0}{t_0^2}, \quad (36)$$

as we are considering infinitesimal intervals t and nearly circular orbits. Equation (36) shows that there is anomalous inward acceleration, as if there is an extra attractive force, or an additional central mass [52].

Thus,

$$\frac{GMm}{r^2} + \frac{2mr}{t_0^2} \approx \frac{mv^2}{r}. \quad (37)$$

From (37), it follows that

$$v \approx \left(\frac{2r^2}{t_0^2} + \frac{GM}{r}\right)^{1/2}. \quad (38)$$

Equation (38) shows that, at distances within the edge of a typical galaxy, that is, $r < 10^{23}$ cm, (34) holds but as we reach the edge and beyond, that is, for $r \geq 10^{24}$ cm, we have $v \sim 10^7$ cm/sec, in agreement with (35).

Then, the time variation of G explains observation without invoking Dark Matter. It may also be mentioned that other effects like the Pioneer anomaly and shortening of the period of binary pulsars can be deduced [53], while new effects also are predicted.

Milgrom [54] approached the problem by modifying Newtonian dynamics at large distances. This approach is purely phenomenological. The idea was that perhaps standard Newtonian dynamics work at the scale of the Solar System but, at galactic scales involving much larger distances, the situation might be different. However, although it produces the asymptotically at rotation curves of galaxies, a simple modification of the distance dependence in the gravitation law, as pointed out by Milgrom, should be not sufficient. Such a law would predict the wrong form of the mass velocity relation. So, Milgrom suggested the following modification to Newtonian dynamics: a test particle at a distance r from a large mass M is subject to the acceleration a given by

$$\frac{a^2}{a_0} = MGr^{-2}, \quad (39)$$

where a_0 is acceleration such that standard Newtonian dynamics are a good approximation only for acceleration values much larger than a_0. The above equation however would be true when a is much less than a_0. Both statements can be combined in the heuristic relation

$$\mu\left(\frac{a}{a_0}\right)a = MGr^{-2}. \quad (40)$$

In (40), $\mu(x) \approx 1$ when $x \gg 1$, and $\mu(x) \approx x$ when $x \ll 1$. It is worthwhile to note that (39) and (40) are not deduced from any theory but rather are an ad hoc fit to explain observations. Interestingly, it must be mentioned that most of the implications of the Modified Newton Dynamics (MOND) do not depend strongly on the exact form of μ.

It can then be shown that the problem of galactic velocities is solved [55–59].

It is interesting to note that there is an interesting relationship between the varying G approach, which has a theoretical base, and the purely phenomenological MOND approach. Let us write

$$\beta\frac{GM}{r}=\frac{r^2}{t_0^2}$$
$$\text{or }\beta=\frac{r^3}{GMt_0^2}. \tag{41}$$

Hence,

$$\alpha_0=\frac{v^2}{r}=\frac{GM}{r^2}\alpha=\frac{r}{t_0^2}. \tag{42}$$

So,

$$\frac{\alpha}{\alpha_0}=\frac{r^3}{GMt_0^2}=\beta. \tag{43}$$

At this stage, we can see a similarity with MOND. If $\beta \ll 1$, we are with the usual Newtonian dynamics and if $\beta > 1$ then we get back to the varying G case exactly as with MOND.

5. Concluding Remarks

The results in [30] have shown that spherically symmetric stationary states can be used as a model for galaxies in the framework of the linearized R^2 gravity. Those states could, in principle, be a partial solution to the Dark Matter problem. In this paper, an improvement of this work has been discussed. As the star density is a functional of the invariants of the associated Vlasov equation, it has been shown that any of these invariants is in turn a functional of the local energy and the angular momentum. Then, the star density depends only on these two integrals of the Vlasov system. This result represents the so-called "Jeans theorem." In addition, an analogy of the historical Tolman–Oppenheimer–Volkoff equation [38, 39] for the system considered in this paper has been discussed. We tried this extension of previous work in [30] because, on the one hand, the Jeans theorem is important in galaxy dynamics and in the framework of molecular clouds [60]. On the other hand, the historical Tolman–Oppenheimer–Volkoff equation constrains the structure of a spherically symmetric body of isotropic material which is in static gravitational equilibrium, as modelled by metric theories of gravity, starting from the general theory of relativity [38, 39]. Thus, a viable extended theory of gravity, like the R^2 gravity, must show consistence with these two important issues.

For the sake of completeness, in Section 4 of this paper, two additional models which argue that Dark Matter could not be an essential element have been discussed. In fact, Dark Matter is considered a mysterious and controversial issue. There are indeed a minority of researchers who think that the dynamics of galaxies could not be determined by massive, invisible Dark Matter halos (see [10, 11, 51–59]). We think that, at the present time, there is not a final answer to the Dark Matter issue. In other words, it is undoubtedly true that the universe exhibits a plethora of mysterious phenomena for which many unanswered questions still exist. Dark Matter is an important part of this intriguing puzzle. Thus, when one works in classical, modern, and developing astrophysical and cosmological theories, it is imperative to repeatedly question their capabilities, identify possible shortcomings, and propose corrections and alternative theories for experimental submission. In the procedures and practice of scientific professionals, no such clues, evidence, or data may be overlooked.

Finally, we take the chance to stress that important future impacts which could help in a better understanding of the important Dark Matter issue could arise from the nascent gravitational wave astronomy [61]. The first direct detection of gravitational waves by the LIGO Collaboration, the so-called event GW150914 [61], represented a cornerstone for science and for astrophysics in particular. We hope that the gravitational wave astronomy will become an important branch of observational astronomy which will aim to use gravitational waves to collect observational data not only about astrophysical objects such as neutron stars and black holes, but also about the mysterious issues of Dark Matter and Dark Energy. In order to achieve this prestigious goal, a network including interferometers with different orientations is required and we hope that future advancements in ground-based projects and space-based projects will have a sufficiently high sensitivity [61–63]. For the benefits of the reader, we also signal two important works on self-gravitating systems [64] and on Jeans mass for anisotropic matter [65].

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Research Article

Ultrarelativistic Spinning Particle and a Rotating Body in External Fields

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We use the vector model of spinning particle to analyze the influence of spin-field coupling on the particle's trajectory in ultrarelativistic regime. The Lagrangian with minimal spin-gravity interaction yields the equations equivalent to the Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations of a rotating body. We show that they have unsatisfactory behavior in the ultrarelativistic limit. In particular, three-dimensional acceleration of the particle becomes infinite in the limit. Therefore, we examine the nonminimal interaction through the gravimagnetic moment κ and show that the theory with $\kappa = 1$ is free of the problems detected in MPTD equations. Hence, the nonminimally interacting theory seems a more promising candidate for description of a relativistic rotating body in general relativity. Vector model in an arbitrary electromagnetic field leads to generalized Frenkel and BMT equations. If we use the usual special-relativity notions for time and distance, the maximum speed of the particle with anomalous magnetic moment in an electromagnetic field is different from the speed of light. This can be corrected assuming that the three-dimensional geometry should be defined with respect to an effective metric induced by spin-field interaction.

1. Introduction

The problem of a covariant description of rotational degrees of freedom has a long and fascinating history [1–13]. Equations of motion of a rotating body in curved background were formulated usually in the multipole approach to description of the body; see [1] for the review. The first results were reported by Mathisson [2] and Papapetrou [3]. They assumed that the structure of test body can be described by a set of multipoles and have taken the approximation which involves only first two terms (the pole-dipole approximation). The equations are then derived by integration of conservation law for the energy-momentum tensor, $T^{\mu\nu}{}_{;\mu} = 0$. Manifestly covariant equations were formulated by Tulczyjew [4] and Dixon [5, 6]. In the current literature, they usually appear in the form given by Dixon (the equations (6.31)–(6.33) in [5]); we will refer to them as Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) equations. They are widely used now to account for spin effects in compact binaries and rotating black holes; see [14–16] and references therein.

Concerning the equations of spinning particle in electromagnetic field, maybe the best candidates are those of Frenkel [9, 10] and Bargmann, Michel, and Telegdi (BMT) [11]. Here, the strong restriction on possible form of semiclassical equations is that the reasonable model should be in correspondence with the Dirac equation. In this regard, the vector model of spin (see below) is of interest because it yields the Frenkel equations at the classical level and implies the Dirac equation after canonical quantization [17].

In this work, we study behavior of a particle governed by these equations (as well as by some of their generalizations) in the ultrarelativistic regime. To avoid the ambiguities in the passage from Lagrangian to Hamiltonian description and vice versa, and in the choice of possible form of interaction, we start in each case from an appropriate variational problem. The vector models of spin provide one possible way to achieve this (for early attempts to build a vector model, see review [18]). In these models, the basic variables in spin sector are ω^μ and π_μ, where ω^μ is non-Grassmann vector and π_μ

represents its conjugated momentum. The spin-tensor is a composite quantity constructed from these variables; $S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu)$. To have a theory with right number of physical degrees of freedom for the spin, certain constraints on the eight basic variables should follow from the variational problem. It should be noted that, even for the free theory in flat space, search for the variational problem represents rather nontrivial task (for the earlier attempts, see [19] and the review [18]).

To explain in a few words the problem which will be under discussion, we recall that typical relativistic equations of motion have singularity at some value of a particle speed. The singularity determines behavior of the particle in ultrarelativistic limit. For instance, the standard equations of spinless particle interacting with electromagnetic field in the physical-time parametrization $x^\mu(t) = (ct, \mathbf{x}(t))$,

$$\frac{d}{dt}\left(\frac{\dot{x}^\mu}{\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}}\right) = \frac{e}{mc^2}F^\mu{}_\nu\dot{x}^\nu, \tag{1}$$

become singular as the relativistic-contraction factor vanishes, $\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = c^2 - \mathbf{v}^2 = 0$. Rewriting the equations in the form of second law of Newton, we find an acceleration. For the case, the longitudinal acceleration reads $a_\parallel = \mathbf{va} = (e(c^2 - \mathbf{v}^2)^{3/2}/mc^3)(\mathbf{Ev})$; that is, the factor, elevated in some degree, appears on the right hand side of the equation and thus determines the value of velocity at which the longitudinal acceleration vanishes, $a_\parallel \xrightarrow{v\to c} 0$. For the present case, the singularity implies that, during its evolution in the external field, the spinless particle can not exceed the speed of light c.

In the equations for spinning particle, instead of the original metric ($\eta_{\mu\nu}$ in flat and $g_{\mu\nu}$ in curved space), emerges the effective metric $G_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$, with spin- and field-dependent contribution $h_{\mu\nu}$. This turns out to be true for both MPTD and Frenkel equations. This leads to (drastic in some cases) changes [20, 21] in behavior of spinning particle as compared with (1). The present work is devoted to detailed analysis of the behavior in ultrarelativistic regime.

We will use the following terminology. The speed v_{cr} that a particle can not exceed during its evolution in an external field is called critical speed (we prefer the term critical speed instead of maximum speed since v_{cr} generally is spin- and field-dependent quantity; see below). The observer-independent scale c of special relativity is called, as usual, the speed of light.

The work is organized as follows. In Section 2, we define three-dimensional acceleration (28) of a particle in an arbitrary gravitational field. The definition guarantees that massive spinless particle propagating along four-dimensional geodesic can not exceed the speed of light. Then, we obtain expressions (46) and (47) for the acceleration implied by equation of a general form (45). They will be repeatedly used in the subsequent sections. In Section 3, we shortly review the vector model of spin and present three equivalent Lagrangians of the free theory. In Section 4.1, we obtain equations of the particle minimally interacting with gravity starting from the Lagrangian action without auxiliary variables. The variational problem leads to the theory with fixed value of spin. In Section 4.2, we present the Lagrangian which leads to the model of Hanson-Regge type [22], with unfixed spin and with a mass-spin trajectory constraint. In Section 4.3, we present the MPTD equations in the form convenient for our analysis and show their equivalence with those obtained in Section 4.1. In Section 4.4, we discuss the problems arising in ultrarelativistic limit of MPTD equations. The first problem is the discrepancy between the critical speed and the speed of light. We should note that similar observations were mentioned in a number of works. The appearance of trajectories with space-like four-velocity was remarked by Hanson and Regge in their model of spherical top in electromagnetic field [22]. Space-like trajectories of this model in gravitational fields were studied in [23, 24]. The second problem is that the transversal acceleration increases with velocity and blows up in the ultrarelativistic limit.

In [25], Khriplovich proposed nonminimal interaction of a rotating body through the gravimagnetic moment κ. In Section 5.1, we construct the nonminimal interaction starting from the Hamiltonian variational problem and show (Section 5.2) that the model with $\kappa = 1$ has reasonable behavior in ultrarelativistic limit. The Lagrangian with one auxiliary variable for the particle with gravimagnetic moment is constructed in Section 5.3. In Section 6 we construct two toy models of spinless particle with critical speed different from the speed of light. In Section 7.1 we analyze generalization of the Frenkel equations to the case of a particle with magnetic moment in an arbitrary electromagnetic field in Minkowski space. Here, we start from the Lagrangian action with one auxiliary variable. In Section 7.2 we show that critical speed of the particle with anomalous magnetic moment is different from the speed of light, if we use the standard special-relativity notions for time and distance. In Section 7.3 we show that the equality between the two speeds can be preserved assuming that three-dimensional geometry should be defined with respect to effective metric arisen due to interaction of spin with electromagnetic field. We point out that a possibility of deformed relation between proper and laboratory time in the presence of electromagnetic field was discussed before by van Holten in his model of spin [26].

Notation. Our variables are taken in arbitrary parametrization τ, and then $\dot{x}^\mu = dx^\mu/d\tau$. Covariant derivative is $\nabla P^\mu = dP^\mu/d\tau + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha P^\beta$ and curvature is $R^\sigma{}_{\lambda\mu\nu} = \partial_\mu\Gamma^\sigma{}_{\lambda\nu} - \partial_\nu\Gamma^\sigma{}_{\lambda\mu} + \Gamma^\sigma{}_{\beta\mu}\Gamma^\beta{}_{\lambda\nu} - \Gamma^\sigma{}_{\beta\nu}\Gamma^\beta{}_{\lambda\mu}$. The square brackets mean antisymmetrization, $\omega^{[\mu}\pi^{\nu]} = \omega^\mu\pi^\nu - \omega^\nu\pi^\mu$. For the four-dimensional quantities, we suppress the contracted indexes and use the notations $\dot{x}^\mu G_{\mu\nu}\dot{x}^\nu = \dot{x}G\dot{x}$, $N^\mu{}_\nu\dot{x}^\nu = (N\dot{x})^\mu$, and $\omega^2 = g_{\mu\nu}\omega^\mu\omega^\nu$, $\mu, \nu = 0, 1, 2, 3$. Notations for the scalar functions constructed from second-rank tensors are $\theta S = \theta^{\mu\nu}S_{\mu\nu}$ and $S^2 = S^{\mu\nu}S_{\mu\nu}$.

When we work in four-dimensional Minkowski space with coordinates $x^\mu = (x^0 = ct, x^i)$, we use the metric $\eta_{\mu\nu} = (-,+,+,+)$, then $\dot{x}\omega = \dot{x}^\mu\omega_\mu = -\dot{x}^0\omega^0 + \dot{x}^i\omega^i$, and so on. Suppressing the indexes of three-dimensional quantities, we use bold letters: $v^i\gamma_{ij}a^j = \mathbf{v}\gamma\mathbf{a}$, $v^iG_{i\mu}v^\mu = \mathbf{v}Gv$, $i, j = 1, 2, 3$, and so on.

Electromagnetic field:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \left(F_{0i} = -E_i,\ F_{ij} = \epsilon_{ijk} B_k\right),$$
$$E_i = -\frac{1}{c}\partial_t A_i + \partial_i A_0, B_i = \frac{1}{2}\epsilon_{ijk} F_{jk} = \epsilon_{ijk}\partial_j A_k. \tag{2}$$

2. Three-Dimensional Acceleration of Spinless Particle in General Relativity

By construction of Lorentz transformations, the speed of light in special relativity is an observer-independent quantity. As we have mentioned in Introduction, the invariant scale is closely related with the critical speed in an external field. In a curved space, we need to be more careful since the three-dimensional geometry should respect the coordinate independence of the speed of light. To achieve this, we use below the Landau and Lifshitz procedure [27] to define time interval, three-dimensional distance, and velocity. Then, we introduce the notion of three-dimensional acceleration which guarantees that massive spinless particle propagating along four-dimensional geodesic can not exceed the speed of light. Expression (47) for longitudinal acceleration implied by equation of the form in (45) will be repeatedly used in subsequent sections.

Consider an observer that labels the events by the coordinates x^μ of pseudo Riemann space [27, 28]

$$\mathbf{M}^{(1,3)} = \left\{x^\mu, g_{\mu\nu}(x^\rho), \operatorname{sign} g_{\mu\nu} = (-,+,+,+)\right\}, \tag{3}$$

to describe the motion of a particle in gravitational field with metric $g_{\mu\nu}$. Formal definitions of three-dimensional quantities subject to the discussion can be obtained representing interval in $1+3$ block-diagonal form [27]

$$\begin{aligned} -ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -c^2 \left[\frac{\sqrt{-g_{00}}}{c}\left(dx^0 + \frac{g_{0i}}{g_{00}} dx^i\right)\right]^2 \\ &\quad + \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}\right) dx^i dx^j. \end{aligned} \tag{4}$$

This prompts introducing infinitesimal time interval, distance, and speed as follows:

$$dt = \frac{\sqrt{-g_{00}}}{c}\left(dx^0 + \frac{g_{0i}}{g_{00}} dx^i\right) \equiv -\frac{g_{0\mu} dx^\mu}{c\sqrt{-g_{00}}}. \tag{5}$$

$$dl^2 = \left(g_{ij} - \frac{g_{0i} g_{0j}}{g_{00}}\right) dx^i dx^j \equiv \gamma_{ij} dx^i dx^j,$$
$$v = \frac{dl}{dt}. \tag{6}$$

Therefore, the conversion factor between intervals of the world time dx^0/c and the time dt measured by laboratory clock is

$$\frac{dt}{dx^0} = \frac{\sqrt{-g_{00}}}{c}\left(1 + \frac{g_{0i}}{g_{00}}\frac{dx^i}{dx^0}\right). \tag{7}$$

Introduce also the three-velocity vector $\mathbf{v}$ with components

$$v^i = \left(\frac{dt}{dx^0}\right)^{-1}\frac{dx^i}{dx^0}, \tag{8}$$

or, symbolically, $v^i = dx^i/dt$. We stress that, contrary to d/dx^μ, the set $(d/dt, d/dx^i)$ is nonholonomic basis of tangent space (let $e_\mu = \tilde{a}^\alpha{}_\mu \partial_\alpha$ be a basis of tangent space and $e^\mu = a^\mu{}_\alpha dx^\alpha$, where $a^\mu{}_\alpha \tilde{a}^\alpha{}_\nu = \delta^\mu{}_\nu$, be the dual basis for e_μ; i.e., $e^\mu(e_\nu) = \delta^\mu{}_\nu$; e_μ is the holonomic basis (i.e., $e_\mu = \partial/\partial x'^\mu$ is tangent to some coordinate lines x'^μ) if $(e_\mu e_\nu - e_\nu e_\mu) f = 0$; for the matrix $a^\mu{}_\alpha$, which determines the dual basis e^μ, this condition reduces to the simple equation $\partial_\mu a^\alpha{}_\nu - \partial_\nu a^\alpha{}_\mu = 0$; for the matrix which determines our $1+3$ decomposition we have $a^0{}_\mu = -g_{0\mu}/c\sqrt{-g_{00}}$, $a^i{}_0 = 0$, and $a^i{}_j = \delta^i{}_j$; then, for instance, $\partial_\mu a^0{}_\nu - \partial_\nu a^0{}_\mu = -(1/c\sqrt{-g_{00}})(\partial_\mu g_{0\nu} - \partial_\nu g_{0\mu}) \neq 0$; so the set $(\partial/\partial t, \partial/\partial x^i)$ generally does not represent a holonomic basis). This does not represent any special problem for our discussion since we are interested in the differential quantities such as velocity and acceleration.

Equation (8) is consistent with the above definition of v: $v^2 = (dl/dt)^2 = \mathbf{v}^2 = v^i \gamma_{ij} v^j$. In the result, the interval acquires the form similar to special relativity (but now we have $\mathbf{v}^2 = \mathbf{v}\gamma\mathbf{v}$):

$$-ds^2 = -c^2 dt^2 + dl^2 = -c^2 dt^2 \left(1 - \frac{\mathbf{v}^2}{c^2}\right). \tag{9}$$

This equality holds in any coordinate system x^μ. Hence, a particle with the propagation law $ds^2 = 0$ has the speed $\mathbf{v}^2 = c^2$, and this is a coordinate-independent statement.

For the latter use we also introduce the four-dimensional quantity

$$v^\mu = \left(\frac{dt}{dx^0}\right)^{-1}\frac{dx^\mu}{dx^0} = \left(\left(\frac{dt}{dx^0}\right)^{-1}, \mathbf{v}\right). \tag{10}$$

Combining (8) and (7), we can present the conversion factor in terms of three-velocity as follows:

$$\left(\frac{dt}{dx^0}\right)^{-1} = v^0 = \frac{c}{\sqrt{-g_{00}}} - \frac{g_{0i} v^i}{g_{00}}. \tag{11}$$

These rather formal tricks are based [27] on the notion of simultaneity in general relativity and on the analysis of flat limit. Four-interval of special relativity has direct physical interpretation in two cases. First, for two events which occur at the same point, the four-interval is proportional to time interval; $dt = -ds/c$. Second, for simultaneous events, the four-interval coincides with distance; $dl = ds$. Assuming that the same holds in general relativity, let us analyze infinitesimal time interval and distance between two events with coordinates x^μ and $x^\mu + dx^\mu$. The world line $y^\mu = (y^0, \mathbf{y} = \text{const})$ is associated with laboratory clock placed at the spacial point $\mathbf{y}$. So, the time interval between the events $(y^0, \mathbf{y})$ and $(y^0 + dy^0, \mathbf{y})$ measured by the clock is

$$dt = -\frac{ds}{c} = \frac{\sqrt{-g_{00}}}{c} dy^0. \tag{12}$$

Consider the event x^μ infinitesimally closed to the world line $(y^0, \mathbf{y} = \text{const})$. To find the event on the world line which is simultaneous with x^μ, we first look for the events $y^\mu_{(1)}$ and $y^\mu_{(2)}$ which have null-interval with x^μ, $ds(x^\mu, y^\mu_{(a)}) = 0$. The equation $g_{\mu\nu}dx^\mu dx^\nu = 0$ with $dx^\mu = x^\mu - y^\mu$ has two solutions: $dx^0_\pm = g_{0i}dx^i/-g_{00} \pm \sqrt{d\mathbf{x}\gamma d\mathbf{x}}/\sqrt{-g_{00}}$; then $y^0_{(1)} = x^0 - dx^0_+$ and $y^0_{(2)} = x^0 - dx^0_-$. Second, we compute the middle point

$$y^0 = \frac{1}{2}\left(y^0_{(1)} + y^0_{(2)}\right) = x^0 + \frac{g_{0i}dx^i}{g_{00}}. \tag{13}$$

By definition, the event $(y^0, \mathbf{y})$ with the null coordinate (13) is simultaneous with the event $(x^0, \mathbf{x})$ (in the flat limit, the sequence $y^\mu_{(1)}$, x^μ, $y^\mu_{(2)}$ of events can be associated with emission, reflection, and absorption of a photon with the propagation law $ds = 0$; then the middle point in (13) should be considered simultaneous with x^0). By this way, we synchronized clocks at the spacial points $\mathbf{x}$ and $\mathbf{y}$. According to (13), the simultaneous events have different null-coordinates, and the difference dx^0 obeys the equation

$$dx^0 + \frac{g_{0i}dx^i}{g_{00}} = 0. \tag{14}$$

Consider a particle which propagated from x^μ to $x^\mu + dx^\mu$. Let us compute time interval and distance between these two events. According to (13), the event

$$\left(x^0 + dx^0 + \frac{g_{0i}dx^i}{g_{00}}, \mathbf{x}\right), \tag{15}$$

at the spacial point $\mathbf{x}$ is simultaneous with $x^\mu + dx^\mu$.

According to (12) and (13), the time interval between the events x^μ and (15) is

$$dt = \frac{\sqrt{-g_{00}}}{c}\left(dx^0 + \frac{g_{0i}}{g_{00}}dx^i\right). \tag{16}$$

Since the events $x^\mu + dx^\mu$ and (15) are simultaneous, this equation gives also the time interval between x^μ and $x^\mu + dx^\mu$. Further, the difference of coordinates between the events $x^\mu + dx^\mu$ and (15) is $dz^\mu = (-g_{0i}dx^i/g_{00}, dx^i)$. As they are simultaneous, the distance between them is

$$\begin{aligned} dl^2 = -ds^2 = g_{\mu\nu}dz^\mu dz^\nu &= \left(g_{ij} - \frac{g_{0i}g_{0j}}{g_{00}}\right)dx^i dx^j \\ &\equiv \gamma_{ij}dx^i dx^j. \end{aligned} \tag{17}$$

Since (15) occurs at the same spacial point as x^μ, this equation gives also the distance between x^μ and $x^\mu + dx^\mu$. Equations (16) and (17) coincide with the formal definitions presented above, in (5) and (6).

We now turn to the definition of three-acceleration. The spinless particle in general relativity follows a geodesic line. If we take the proper time to be the parameter, geodesics obey the system

$$\begin{aligned} \nabla_s\frac{dx^\mu}{ds} \equiv \frac{d^2x^\mu}{ds^2} + \Gamma^\mu{}_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} &= 0, \\ g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} &= -1, \end{aligned} \tag{18}$$

where

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}\left(\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}\right). \tag{19}$$

Due to this definition, system (18) obeys the identity $g_{\mu\nu}(dx^\mu/ds)\nabla_s(dx^\nu/ds) = 0$.

The system in this parametrization has no sense of the case we are interested in, $ds^2 \to 0$. So, we rewrite it in arbitrary parametrization τ.

$$\begin{aligned} &\frac{d\tau}{ds}\frac{d}{d\tau}\left(\frac{d\tau}{ds}\frac{dx^\mu}{d\tau}\right) + \left(\frac{d\tau}{ds}\right)^2\Gamma^\mu{}_{\alpha\beta}(g)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0, \\ &\frac{d\tau}{ds} = \frac{1}{\sqrt{-\dot{x}g\dot{x}}}; \end{aligned} \tag{20}$$

this yields the equation of geodesic line in reparametrization-invariant form

$$\begin{aligned} &\frac{1}{\sqrt{-\dot{x}g\dot{x}}}\frac{d}{d\tau}\left(\frac{\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}}\right) \\ &\quad = -\Gamma^\mu{}_{\alpha\beta}(g)\frac{\dot{x}^\alpha}{\sqrt{-\dot{x}g\dot{x}}}\frac{\dot{x}^\beta}{\sqrt{-\dot{x}g\dot{x}}}. \end{aligned} \tag{21}$$

Formalism (5)–(9) remains manifestly covariant under subgroup of spacial transformations $x^0 = x'^0$, $x^i = x^i(x'^j)$, and $\partial x^i/\partial x'^j \equiv a^i{}_j(x')$. Under these transformations, g_{00} is a scalar function and g_{0i} is a vector while g_{ij} and γ_{ij} are tensors. Since $g^{ij}\gamma_{jk} = \delta^i{}_k$, the inverse metric of γ_{ij} turns out to be $(\gamma^{-1})^{ij} = g^{ij}$. Introduce the covariant derivatives ∇_k of a vector field $\xi^i(x^0, x^k)$:

$$\nabla_k\xi^i = \partial_k\xi^i + \tilde{\Gamma}^i{}_{kj}(\gamma)\xi^j. \tag{22}$$

The three-dimensional Christoffel symbols $\tilde{\Gamma}^i{}_{jk}(\gamma)$ are constructed with help of three-dimensional metric $\gamma_{ij}(x^0, x^k)$ written in (6), where x^0 is considered as a parameter:

$$\tilde{\Gamma}^i{}_{jk}(\gamma) = \frac{1}{2}\gamma^{ia}\left(\partial_j\gamma_{ak} + \partial_k\gamma_{aj} - \partial_a\gamma_{jk}\right). \tag{23}$$

As a consequence, the metric γ is covariantly constant, $\nabla_k\gamma_{ij} = 0$.

The velocity in (8) behaves as a vector, $v^i(x^0) = a^i{}_j(x'^k(x^0))v'^j(x^0)$, so below we use also the covariant derivative

$$\nabla_0 v^i = \frac{dv^i}{dx^0} + \tilde{\Gamma}^i{}_{jk}(\gamma)\frac{dx^j}{dx^0}v^k. \tag{24}$$

We associated with $\mathbf{M}^{(1,3)}$ the one-parameter family of three-dimensional spaces $\mathbf{M}^3_{x^0} = \{x^k, \gamma_{ij}, \nabla_k\gamma_{ij} = 0\}$. Note that velocity has been defined above as a tangent vector to the curve which crosses the family and is parameterized by this parameter, $x^i(x^0)$.

To define an acceleration of a particle in the three-dimensional geometry, we need the notion of a constant vector field (or, equivalently, the parallel-transport equation). In the case of stationary field, $g_{\mu\nu}(x^k)$, we can identify the curve $x^i(x^0)$ of $\mathbf{M}^{(1,3)}$ with that of any one of $\mathbf{M}^3_{x^0} = \{x^k, \gamma_{ij}(x^k)\}$. So, we have the usual three-dimensional Riemann geometry, and an analog of constant vector field of Euclidean geometry is the covariantly constant field along the line $x^i(x^0)$, $\nabla_0\xi^i = 0$. For the field of velocity, its deviation from the covariant constancy is the acceleration [21]

$$a^i = \left(\frac{dt}{dx^0}\right)^{-1}\nabla_0 v^i = \left(\frac{dt}{dx^0}\right)^{-1}\frac{dv^i}{dx^0} + \widetilde{\Gamma}^i{}_{jk}v^j v^k. \tag{25}$$

To define an acceleration in general case, $\gamma_{ij}(x^0, x^i)$, we need to adopt some notion of a constant vector field along the trajectory $x^i(x^0)$ that crosses the family $\mathbf{M}^3_{x^0}$. We propose the definition which preserves one of basic properties of constant fields in differential geometry. In Euclidean and Minkowski spaces, the canonical scalar product of two constant fields does not depend on the point where it was computed. In (pseudo) Riemann space, constant vector field is defined in such a way that the same property holds [29]. In particular, taking the scalar product along a line $x^i(x^0)$, we have $(d/dx^0)(\xi, \eta) = 0$. For the constant fields in the three-dimensional geometry resulting after Landau-Lifshitz 1 + 3 decomposition, we demand the same (necessary) condition: $(d/dx^0)[\xi^i(x^0)\gamma_{ij}(x^0, x^i(x^0))\eta^i(x^0)] = 0$. Taking into account that $\nabla_k\gamma_{ij} = 0$, this condition can be written as follows:

$$\left(\nabla_0\xi + \frac{1}{2}\xi\partial_0\gamma\gamma^{-1}, \eta\right) + \left(\xi, \nabla_0\eta + \frac{1}{2}\gamma^{-1}\partial_0\gamma\eta\right) = 0. \tag{26}$$

This equation is satisfied, if we take the parallel-transport equation to be

$$\nabla_0\xi^i + \frac{1}{2}\left(\xi\partial_0\gamma\gamma^{-1}\right)^i = 0. \tag{27}$$

Deviation from the constant field is an acceleration. So we define acceleration with respect to physical time as follows:

$$a^i = \left(\frac{dt}{dx^0}\right)^{-1}\left[\nabla_0 v^i + \frac{1}{2}\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i\right]. \tag{28}$$

For the special case of stationary field, $g_{\mu\nu}(x^i)$, definition (28) reduces to (25) and to that of Landau and Lifshitz; see page 251 in [27].

The extra term that appeared in this equation plays an essential role in providing that for the geodesic motion we have $a_\parallel \xrightarrow{v\to c} 0$. As a consequence, geodesic particle in gravitational field can not exceed the speed of light. To show this, we compute the longitudinal acceleration $(\mathbf{v}\gamma\mathbf{a})$ implied by geodesic equation (21). Take $\tau = x^0$; then $\sqrt{-\dot{x}g\dot{x}} = (dt/dx^0)\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}$, and spacial part of (21) is

$$\left(\frac{dt}{dx^0}\right)^{-1}\frac{d}{dx^0}\frac{v^i}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} = \frac{f^i}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}, \tag{29}$$

where

$$\begin{aligned} f^i\left(v^\mu\right) = &-\left(\frac{dt}{dx^0}\right)^{-2}\Gamma^i{}_{00} - \Gamma^i{}_{jk}v^j v^k \\ &- 2\left(\frac{dt}{dx^0}\right)^{-1}\Gamma^i{}_{0k}v^k = -\Gamma^i_{\mu\nu}v^\mu v^\nu \end{aligned} \tag{30}$$

is nonsingular function as $v \to c$. Computing derivative on the l.h.s. of (29), we complete dv^i/dx^0 up to covariant derivative $\nabla_0 v^i$:

$$\begin{aligned} \frac{d}{dx^0}\frac{v^i}{\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}} = &\nabla_0 v^i - \widetilde{\Gamma}^i{}_{jk}\left(\gamma\right)v^j v^k\frac{dt}{dx^0} \\ &+ \frac{v^i}{2\left(c^2 - \mathbf{v}\gamma\mathbf{v}\right)}\frac{d}{dx^0}\left(\mathbf{v}\gamma\mathbf{v}\right). \end{aligned} \tag{31}$$

For the derivative contained in the last term we find, using covariant constancy of γ,

$$\begin{aligned} \frac{d}{dx^0}\left[\mathbf{v}\gamma\left(x^0, x^i\right)\mathbf{v}\right] &= 2\mathbf{v}\gamma\nabla_0\mathbf{v} + \mathbf{v}\partial_0\gamma\mathbf{v} + \mathbf{v}\nabla_0\gamma\mathbf{v} \\ &= 2\mathbf{v}\gamma\nabla_0\mathbf{v} + \mathbf{v}\partial_0\gamma\mathbf{v}. \end{aligned} \tag{32}$$

Then, (29) acquires the form

$$\begin{aligned} &\left(\frac{dt}{dx^0}\right)^{-1}\left[M^i{}_j\nabla_0 v^j + \frac{\left(\mathbf{v}\partial_0\gamma\mathbf{v}\right)}{2\left(c^2 - \mathbf{v}\gamma\mathbf{v}\right)}v^i\right] \\ &\quad = f^i + \widetilde{\Gamma}^i{}_{kl}v^k v^l, \end{aligned} \tag{33}$$

where

$$M^i{}_j = \delta^i{}_j + \frac{v^i\left(\mathbf{v}\gamma\right)_j}{c^2 - \mathbf{v}\gamma\mathbf{v}}. \tag{34}$$

We apply the inverse matrix

$$\widetilde{M}^i{}_j = \delta^i{}_j - \frac{v^i\left(\mathbf{v}\gamma\right)_j}{c^2} \tag{35}$$

and use the identity

$$\widetilde{M}^i{}_j v^j = \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2}v^i, \tag{36}$$

and then

$$\begin{aligned} &\left(\frac{dt}{dx^0}\right)^{-1}\left[\nabla_0 v^i + \frac{\left(\mathbf{v}\partial_0\gamma\mathbf{v}\right)}{2c^2}v^i\right] \\ &\quad = \widetilde{M}^i{}_j\left[f^j + \widetilde{\Gamma}^j{}_{kl}v^k v^l\right]. \end{aligned} \tag{37}$$

Next, we complete $\nabla_0 v^i$ up to acceleration (28). Then, (37) yields

$$a^i = \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}\left[\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i - \frac{(\mathbf{v}\partial_0\gamma\mathbf{v})}{c^2}v^i\right] + \widetilde{M}^i{}_j\left[-\Gamma^j_{\mu\nu}v^\mu v^\nu + \widetilde{\Gamma}^j{}_{kl}(\gamma)v^k v^l\right]. \quad (38)$$

Contracting this with $(\mathbf{v}\gamma)_i$, we use $(\mathbf{v}\gamma)_i\widetilde{M}^i{}_j = ((c^2-\mathbf{v}\gamma\mathbf{v})/c^2)(\mathbf{v}\gamma)_j$ and obtain longitudinal acceleration

$$\mathbf{v}\gamma\mathbf{a} = \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}\left[(\mathbf{v}\partial_0\gamma\mathbf{v}) - (\mathbf{v}\partial_0\gamma\mathbf{v})\frac{(\mathbf{v}\gamma\mathbf{v})}{c^2}\right] + \left(1-\frac{\mathbf{v}\gamma\mathbf{v}}{c^2}\right)(\mathbf{v}\gamma)_i\left[-\Gamma^i_{\mu\nu}v^\mu v^\nu + \widetilde{\Gamma}^i{}_{kl}(\gamma)v^k v^l\right]. \quad (39)$$

This implies $\mathbf{v}\gamma\mathbf{a} \to 0$ as $\mathbf{v}\gamma\mathbf{v} \to c^2$.

The last term in (28) yields the important factor $(\mathbf{v}\partial_0\gamma\mathbf{v})$ in (39). As equations of motion (38) and (39) do not contain the square root $\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}}$, they have sense even for $v > c$. Without this factor, we would have $\mathbf{v}\gamma\mathbf{a} \neq 0$ as $\mathbf{v}\gamma\mathbf{v} \to c^2$, so the particle in gravitational field could exceed c and then continue to accelerate. The same happens if we try to define acceleration using usual derivative instead of the covariant one. Indeed, instead of (28), let us define an acceleration according to the expression $a'^i = (dt/dx^0)^{-1}[dv^i/dx^0 + (1/2)(\mathbf{v}\partial_0\gamma\gamma^{-1})^i]$. Then for the geodesic particle we obtain, instead of (39), the longitudinal acceleration $\mathbf{v}\gamma\mathbf{a}' =$ [r.h.s. of (39)] $- (\mathbf{v}\gamma)_i\widetilde{\Gamma}^i{}_{jk}(\gamma)(dx^j/dx^0)v^k =$ [r.h.s. of (39)] $- (1/2)\partial_i\gamma_{jk}v^iv^jv^k$. The extra term does not involve the factor $c^2-\mathbf{v}\gamma\mathbf{v}$ and so does not vanish at $|v| = c$.

Let us confirm that c is the only special point of function (39) representing the longitudinal acceleration. Using (19), (6)–(10), (23), and the identities

$$\gamma_{ij}g^{jk} = \delta_i{}^k, \qquad \gamma_{ij}g^{j0} = -\frac{g_{0i}}{g_{00}}, \quad (40)$$

we can present the right hand side of (39) in terms of initial metric as follows:

$$\mathbf{v}\gamma\mathbf{a} = \frac{c^2-\mathbf{v}\gamma\mathbf{v}}{2c\sqrt{-g_{00}}}\left\{\frac{c}{\sqrt{-g_{00}}}\left[\left(\frac{dt}{dx^0}\right)^{-1}\partial_0 g_{00} + v^k\partial_k g_{00}\right] - \partial_0 g_{00}\left(\frac{dt}{dx^0}\right)^{-2} - 2\partial_0 g_{0k}\left(\frac{dt}{dx^0}\right)^{-1}v^k - \partial_0 g_{kl}v^kv^l\right\} \equiv \frac{c^2-\mathbf{v}\gamma\mathbf{v}}{2c\sqrt{-g_{00}}}\left\{\frac{c}{\sqrt{-g_{00}}}v^\mu\partial_\mu g_{00} - \partial_0 g_{\mu\nu}v^\mu v^\nu\right\}. \quad (41)$$

The quantity v^μ has been defined in (10). Excluding v^0 according to this expression, we obtain

$$\mathbf{v}\gamma\mathbf{a} = \frac{c^2-\mathbf{v}\gamma\mathbf{v}}{2\sqrt{-g_{00}}}\left\{\frac{v^k\partial_k g_{00}}{\sqrt{-g_{00}}} - 2\partial_0\left(\frac{g_{0i}}{\sqrt{-g_{00}}}\right)v^i - \frac{1}{c}\partial_0\gamma_{ij}v^iv^j\right\}. \quad (42)$$

For the stationary metric, $g_{\mu\nu}(x^k)$, (42) acquires a specially simple form:

$$\mathbf{v}\gamma\mathbf{a} = -\left(c^2-\mathbf{v}\gamma\mathbf{v}\right)\frac{v^k\partial_k g_{00}}{2g_{00}}. \quad (43)$$

This shows that the longitudinal acceleration has only one special point in the stationary gravitational field; $\mathbf{v}\gamma\mathbf{a} \to 0$ as $\mathbf{v}\gamma\mathbf{v} \to c^2$. Then, the same is true in general case (41), at least for the metric which is sufficiently slowly varied in time.

While we have discussed the geodesic equation, the computation which leads to formula (39) can be repeated for a more general equation. Let us formulate the result which will be repeatedly used below. Using the factor $\sqrt{-\dot{x}g\dot{x}}$, we construct the reparametrization-invariant derivative

$$D = \frac{1}{\sqrt{-\dot{x}g\dot{x}}}\frac{d}{d\tau}. \quad (44)$$

Consider the reparametrization-invariant equation of the form

$$DDx^\mu(\tau) = \mathcal{F}^\mu(Dx^\nu,\ldots) \quad (45)$$

and suppose that the three-dimensional geometry is defined by $g_{\mu\nu}$. Then, (45) implies the three-acceleration

$$a^i = \widetilde{M}^i{}_j\left[\left(c^2-\mathbf{v}\gamma\mathbf{v}\right)\mathcal{F}^j + \widetilde{\Gamma}^j{}_{kl}(\gamma)v^kv^l\right] + \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}\left[\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i - \frac{v^i}{c^2}(\mathbf{v}\partial_0\gamma\mathbf{v})\right] \quad (46)$$

and the longitudinal acceleration

$$\mathbf{v}\gamma\mathbf{a} = \frac{\left(c^2-\mathbf{v}\gamma\mathbf{v}\right)^2}{c^2}(\mathbf{v}\gamma\boldsymbol{\mathcal{F}}) + \frac{c^2-\mathbf{v}\gamma\mathbf{v}}{c^2}\left[(\mathbf{v}\gamma)_i\widetilde{\Gamma}^i{}_{kl}(\gamma)v^kv^l + \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}(\mathbf{v}\partial_0\gamma\mathbf{v})\right]. \quad (47)$$

The spacial part of the force is $\mathcal{F}^i = \mathcal{F}^i(v^\nu/\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}})$, where v^μ is given by (10) and the connection $\widetilde{\Gamma}^i{}_{kl}(\gamma)$ is constructed with help of the three-dimensional metric $\gamma_{ij} = (g_{ij} - g_{0i}g_{0j}/g_{00})$ according to (23). For the geodesic equation in this notation, we have $\mathcal{F}^i = -\Gamma^i{}_{\mu\nu}(v^\mu v^\nu/(c^2-\mathbf{v}\gamma\mathbf{v}))$. With this $\mathcal{F}^i$, (46) and (47) coincide with (38) and (39).

3. Vector Model of Relativistic Spin

The variational problem for vector model of spin interacting with electromagnetic and gravitational fields can be formulated with various sets of auxiliary variables [17, 30–33]. For the free theory in flat space, there is Lagrangian action without auxiliary variables. Configuration space consists of the position $x^\mu(\tau)$ and non-Grassmann vector $\omega^\mu(\tau)$ attached to the point x^μ. The action reads [20, 32]

$$S = -\frac{1}{\sqrt{2}}\int d\tau \sqrt{m^2c^2 - \frac{\alpha}{\omega^2}}\sqrt{-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}}. \quad (48)$$

The matrix $N_{\mu\nu}$ is the projector on the plane orthogonal to ω^ν:

$$N_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega_\mu\omega_\nu}{\omega^2},$$
$$\text{then } N_{\mu\alpha}N^{\alpha\nu} = N_\mu{}^\nu, \quad (49)$$
$$N_{\mu\nu}\omega^\nu = 0.$$

The double square-root structure in expression (48) seems to be typical for the vector models of spin [22, 34]. This yields the primary constraint T_4 in (62) and, at the end, supplementary spin condition (77). The parameter m is mass, while α determines the value of spin. The value $\alpha = 3\hbar^2/4$ corresponds to an elementary spin one-half particle. The model is invariant under reparametrizations and local spin-plane symmetries [35] (the reparametrizations are $\tau \to \tau'(\tau)$, $x'^\mu(\tau') = x^\mu(\tau)$, and $\omega'^\mu(\tau') = \omega^\mu(\tau)$; i.e., both x and ω are scalar functions; the local spin-plane transformations act in the plane determined by the vectors ω^μ and π^ν).

The spin is described by Frenkel spin-tensor [9]. In our model, this is a composite quantity constructed from ω^μ and its conjugated momentum $\pi_\mu = \partial L/\partial\dot{\omega}^\mu$ as follows:

$$S^{\mu\nu} = 2(\omega^\mu\pi^\nu - \omega^\nu\pi^\mu) = \left(S^{i0} = D^i,\ S_{ij} = 2\epsilon_{ijk}S_k\right), \quad (50)$$

and then $S_i = \epsilon_{ijk}\omega_j\pi_k = (1/4)\epsilon_{ijk}S_{jk}$. Here, S_i is three-dimensional spin-vector and D_i is dipole electric moment [12]. In contrast to its constituents ω^μ and π^ν, the spin-tensor is invariant under local spin-plane symmetry and thus represents an observable quantity. Canonical quantization of the model yields one-particle sector of the Dirac equation [17].

In formulation (48), the model admits minimal interaction with electromagnetic field and with gravity. This does not spoil the number and the algebraic structure of constraints presented in the free theory. To describe the spinning particle with magnetic and gravimagnetic moments, we will need the following two reformulations.

In the spinless limit, $\alpha = 0$ and $\omega^\mu = 0$, functional (48) reduces to the standard expression, $-mc\sqrt{-\dot{x}^\mu\dot{x}_\mu}$. The latter can be written in equivalent form using the auxiliary variable $\lambda(\tau)$ as follows: $(1/2\lambda)\dot{x}^2 - (\lambda/2)m^2c^2$. Similarly to this, (48) can be presented in the equivalent form

$$L = \frac{1}{4\lambda_1}\left[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega} - \sqrt{[\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}]^2 - 4(\dot{x}N\dot{\omega})^2}\right] - \frac{\lambda_1}{2}\left[(mc)^2 - \frac{\alpha}{\omega^2}\right]. \quad (51)$$

In this formulation, our model admits interaction of spin with an arbitrary electromagnetic field through the magnetic moment; see Section 7.1. Another form of the Lagrangian is

$$L = -\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}}\sqrt{(1-\lambda^2)^{-1}[-\dot{x}N\dot{x} - \dot{\omega}N\dot{\omega} + 2\lambda\dot{x}N\dot{\omega}]} \equiv \quad (52)$$

$$-\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}}\sqrt{-(N\dot{x}, N\dot{\omega})\begin{pmatrix}\frac{\eta}{1-\lambda^2} & \frac{-\lambda\eta}{1-\lambda^2}\\ \frac{-\lambda\eta}{1-\lambda^2} & \frac{\eta}{1-\lambda^2}\end{pmatrix}\begin{pmatrix}N\dot{x}\\ N\dot{\omega}\end{pmatrix}}. \quad (53)$$

Its advantage is that the expression under the square root represents quadratic form with respect to the velocities $\dot{x}$ and $\dot{\omega}$. To relate Lagrangians (48) and (52), we exclude λ from the latter. Computing variation of (52) with respect to λ, we obtain the equation

$$(\dot{x}N\dot{\omega})\lambda^2 - (\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega})\lambda + (\dot{x}N\dot{\omega}) = 0, \quad (54)$$

which determines λ:

$$\lambda_\pm = \frac{(\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega}) \pm \sqrt{(\dot{x}N\dot{x} + \dot{\omega}N\dot{\omega})^2 - 4(\dot{x}N\dot{\omega})^2}}{2(\dot{x}N\dot{\omega})}. \quad (55)$$

We substitute λ_+ into (52) and use $\lambda_+\lambda_- = 1$, and then (52) turns into (48). In formulation (52), our model admits interaction of spin with gravity through the gravimagnetic moment; see Section 5.3.

4. Minimal Interaction with an Arbitrary Gravitational Field

4.1. Lagrangian and Hamiltonian Formulations. The minimal interaction with gravitational field can be achieved by covariantization of the formulation without auxiliary variables. In expressions (48) and (49), we replace $\eta_{\mu\nu} \to g_{\mu\nu}$ and usual derivative by the covariant one; $\dot{\omega}^\mu \to \nabla\omega^\mu = d\omega^\mu/d\tau + \Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\omega^\beta$. Thus, our Lagrangian in a curved background reads [33]

$$\begin{aligned} L &= -\frac{1}{\sqrt{2}}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2} \\ &\cdot \sqrt{-\dot{x}N\dot{x} - \nabla\omega N\nabla\omega + \sqrt{[\dot{x}N\dot{x} + \nabla\omega N\nabla\omega]^2 - 4(\dot{x}N\nabla\omega)^2}} \\ &\equiv -\frac{1}{\sqrt{2}}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2} L_0. \end{aligned} \quad (56)$$

Velocities $\dot{x}^\mu$, $\nabla\omega^\mu$ and projector $N_{\mu\nu}$ transform like contravariant vectors and covariant tensor, so the action is manifestly invariant under general-coordinate transformations.

Let us construct Hamiltonian formulation of model (56). Conjugate momenta for x^μ and ω^μ are $p_\mu = \partial L/\partial\dot{x}^\mu$ and $\pi_\mu = \partial L/\partial\dot{\omega}^\mu$, respectively. Due to the presence of Christoffel symbols in $\nabla\omega^\mu$, the conjugated momentum p_μ does not transform as a vector, so it is convenient to introduce the canonical momentum

$$P_\mu \equiv p_\mu - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta; \quad (57)$$

the latter transforms as a vector under general transformations of coordinates. Manifest form of the momenta is as follows:

$$\begin{aligned} P_\mu &= \frac{1}{\sqrt{2}L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2}\left[N_{\mu\nu}\dot{x}^\nu - K_\mu\right], \\ \pi_\mu &= \frac{1}{\sqrt{2}L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2}\left[N_{\mu\nu}\nabla\omega^\nu - R_\mu\right], \end{aligned} \quad (58)$$

with

$$\begin{aligned} K_\mu &= T^{-1/2}\left[(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\dot{x})_\mu \right. \\ &\quad \left. - 2(\dot{x}N\nabla\omega)(N\nabla\omega)_\mu\right], \\ R_\mu &= T^{-1/2}\left[(\dot{x}N\dot{x} + \nabla\omega N\nabla\omega)(N\nabla\omega)_\mu \right. \\ &\quad \left. - 2(\dot{x}N\nabla\omega)(N\dot{x})_\mu\right]. \end{aligned} \quad (59)$$

These vectors obey the following remarkable identities:

$$\begin{aligned} K^2 &= \dot{x}N\dot{x}, \\ R^2 &= \nabla\omega N\nabla\omega, \\ KR &= -\dot{x}N\nabla\omega, \\ \dot{x}R + \nabla\omega K &= 0, \\ K\dot{x} + R\nabla\omega &= \sqrt{[\dot{x}N\dot{x} + \nabla\omega N\nabla\omega]^2 - 4(\dot{x}N\nabla\omega)^2}. \end{aligned} \quad (60)$$

Using (49), we conclude that $\omega\pi = 0$ and $P\omega = 0$; that is, we found two primary constraints. Using the relations in (60), we find one more primary constraint, $P\pi = 0$. At last, computing $P^2 + \pi^2$ given by (58); we see that all the terms with derivatives vanish, and we obtain the primary constraint

$$T_1 \equiv P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0. \quad (61)$$

In the result, action (56) implies four primary constraints, T_1 and

$$\begin{aligned} T_2 &\equiv \omega\pi = 0, \\ T_3 &\equiv P\omega = 0, \\ T_4 &\equiv P\pi = 0. \end{aligned} \quad (62)$$

The Hamiltonian is constructed excluding velocities from the expression

$$H = p_\mu\dot{x} + \pi\dot{\omega} - L + \lambda_i T_i \equiv P\dot{x} + \pi\nabla\omega - L + \lambda_i T_i, \quad (63)$$

where λ_i is the Lagrangian multipliers associated with the primary constraints. From (58), we observe the equalities $P\dot{x} = (\sqrt{2}L_0)^{-1}(m^2c^2 - \alpha/\omega^2)^{1/2}[\dot{x}N\dot{x} - \dot{x}K]$ and $\pi\nabla\omega = (\sqrt{2}L_0)^{-1}(m^2c^2 - \alpha/\omega^2)^{1/2}[\nabla\omega N\nabla\omega - \nabla\omega R]$. Together with (60), they imply $P\dot{x} + \pi\nabla\omega = L$. Using this in (63), we conclude that the Hamiltonian is composed of the primary constraints

$$\begin{aligned} H &= \frac{\lambda_1}{2}\left(P^2 + m^2c^2 + \pi^2 - \frac{\alpha}{\omega^2}\right) + \lambda_2(\omega\pi) \\ &\quad + \lambda_3(P\omega) + \lambda_4(P\pi). \end{aligned} \quad (64)$$

The full set of phase-space coordinates consists of the pairs x^μ, p_μ and ω^μ, π_μ. They fulfill the fundamental Poisson brackets $\{x^\mu, p_\nu\} = \delta^\mu_\nu$ and $\{\omega^\mu, \pi_\nu\} = \delta^\mu_\nu$ and then $\{P_\mu, P_\nu\} = R^\sigma{}_{\lambda\mu\nu}\pi_\sigma\omega^\lambda$, $\{P_\mu, \omega^\nu\} = \Gamma^\nu_{\mu\alpha}\omega^\alpha$, and $\{P_\mu, \pi_\nu\} = -\Gamma^\alpha_{\mu\nu}\pi_\alpha$. For the quantities x^μ, P^μ, and $S^{\mu\nu}$, these brackets imply the typical relations used by people for spinning particles in Hamiltonian formalism.

$$\begin{aligned} &\{x^\mu, P_\nu\} = \delta^\mu_\nu, \\ &\{P_\mu, P_\nu\} = -\frac{1}{4}R_{\mu\nu\alpha\beta}S^{\alpha\beta}, \\ &\{P_\mu, S^{\alpha\beta}\} = \Gamma^\alpha_{\mu\sigma}S^{\sigma\beta} - \Gamma^\beta_{\mu\sigma}S^{\sigma\alpha}, \\ &\{S^{\mu\nu}, S^{\alpha\beta}\} = 2\left(g^{\mu\alpha}S^{\nu\beta} - g^{\mu\beta}S^{\nu\alpha} - g^{\nu\alpha}S^{\mu\beta} + g^{\nu\beta}S^{\mu\alpha}\right). \end{aligned} \quad (65)$$

To reveal the higher-stage constraints and the Lagrangian multipliers, we study the equation $\dot{T}_i = \{T_i, H\} = 0$. T_2 implies the secondary constraint

$$\dot{T}_2 = 0 \Longrightarrow T_5 \equiv \pi^2 - \frac{\alpha}{\omega^2} \approx 0; \quad (66)$$

then T_1 can be replaced on $P^2 + m^2c^2 \approx 0$. Preservation in time of T_4 and T_3 gives the Lagrangian multipliers λ_3 and λ_4:

$$\begin{aligned} \lambda_3 &= 2a\lambda_1(\pi\theta P), \\ \lambda_4 &= -2a\lambda_1(\omega\theta P), \end{aligned} \quad (67)$$

where we have denoted

$$\theta_{\mu\nu} \equiv R_{\alpha\beta\mu\nu}S^{\alpha\beta}, \quad (68)$$

$$a = \frac{2}{16m^2c^2 + (\theta S)}. \quad (69)$$

Table 1: Algebra of constraints.

	T_1	T_5	T_2	T_3	T_4
$T_1 = P^2 + m^2c^2$	0	0	0	$\frac{1}{2}(\omega\theta P)$	$\frac{1}{2}(\pi\theta P)$
$T_5 = \pi^2 - \frac{\alpha}{\omega^2}$	0	0	$-2T_5$	$-2T_4$	$-\frac{2T_3}{\omega^2}$
$T_2 = \omega\pi$	0	$2T_5$	0	$-T_3$	T_4
$T_3 = P\omega$	$-\frac{1}{2}(\omega\theta P)$	$2T_4$	T_3	0	$-\frac{1}{8a}$
$T_4 = P\pi$	$-\frac{1}{2}(\pi\theta P)$	$\frac{2T_3}{\omega^2}$	$-T_4$	$\frac{1}{8a}$	0

Preservation in time of T_1 gives the equation $\lambda_3(\omega\theta P) + \lambda_4(\pi\theta P) = 0$ which is identically satisfied by virtue of (67). No more constraints are generated after this step. We summarize the algebra of Poisson brackets between the constraints in Table 1. T_3 and T_4 represent a pair of second-class constraints, while T_2, T_5, and the combination

$$T_0 = T_1 + 4a(\pi\theta P)T_3 - 4a(\omega\theta P)T_4 \tag{70}$$

are the first-class constraints. Taking into account that each second-class constraint rules out one phase-space variable, whereas each first-class constraint rules out two variables, we have the right number of spin degrees of freedom, $8-(2+4) = 2$.

It should be noted that ω^μ and π^μ turn out to be space-like vectors. Indeed, in flat limit and in the frame where $p^\mu = (p^0, \mathbf{0})$, the constraints $\omega p = \pi p = 0$ imply $\omega^0 = \pi^0 = 0$. This implies $\omega^2 \geq 0$ and $\pi^2 \geq 0$. Combining this with constraint (66), we conclude $\omega^2 > 0$ and $\pi^2 > 0$.

We point out that the first-class constraint $T_5 = \pi^2 - \alpha/\omega^2 \approx 0$ can be replaced on the pair

$$\begin{aligned} \pi^2 &= \text{const}, \\ \omega^2 &= \text{const}; \end{aligned} \tag{71}$$

this gives an equivalent formulation of the model. The Lagrangian which implies constraints (62) and (71) has been studied in [17, 30, 31]. Hamiltonian and Lagrangian equations for physical variables of the two formulations coincide [32], which proves their equivalence.

Using (67), we can present Hamiltonian (64) in the form

$$\begin{aligned} H &= \frac{\lambda_1}{2}\left(P^2 + m^2c^2 + 4a\left[(\pi\theta P)(P\omega) - (\omega\theta P)(P\pi)\right]\right) \\ &+ \frac{\lambda_1}{2}\left(\pi^2 - \frac{\alpha}{\omega^2}\right) + \lambda_2(\omega\pi). \end{aligned} \tag{72}$$

The dynamics of basic variables is governed by Hamiltonian equations $\dot{z} = \{z, H\}$, where $z = (x, p, \omega, \pi)$, and the Hamiltonian is given in (72). Equivalently, we can use the first-order variational problem equivalent to (56):

$$\begin{aligned} S_H = \int d\tau p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu & \\ - \left[\frac{\lambda_1}{2}\left(P^2 + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2}\right) + \lambda_2(\omega\pi)\right. & \\ \left. + \lambda_3(P\omega) + \lambda_4(P\pi)\right]. & \end{aligned} \tag{73}$$

Variation with respect to λ_i gives constraints (61) and (62), while variation with respect to x, p, ω, and π gives the dynamical equations. By construction of S_H, the variational equation $\delta S_\kappa/\delta p_\mu = 0$ is equivalent to $\dot{x}^\mu = \{x^\mu, H\}$ and so on. The equations can be written in a manifestly covariant form as follows:

$$\dot{x}^\mu = \lambda_1\left[P^\mu + 2a\left[(\pi\theta P)\omega^\mu - (\omega\theta P)\pi^\mu\right]\right], \tag{74}$$

$$\nabla P_\mu = R^\alpha{}_{\beta\mu\nu}\pi_\alpha\omega^\beta\dot{x}^\nu, \tag{75}$$

$$\begin{aligned} \nabla\omega^\mu &= -2\lambda_1 a(\omega\theta P)P^\mu + \lambda_2\omega^\mu + \lambda_1\pi^\mu, \\ \nabla\pi_\mu &= -2\lambda_1 a(\pi\theta P)P_\mu - \lambda_2\pi_\mu - \lambda_1\frac{\omega_\mu}{\omega^2}. \end{aligned} \tag{76}$$

According to general theory [29, 36, 37], neither constraints nor equations of motion determine the functions λ_1 and λ_2. Their presence in the equations of motion implies that evolution of our basic variables is ambiguous. This is in correspondence with two local symmetries presented in the model. The variables with ambiguous dynamics do not represent observable quantities, so we need to search for variables that can be candidates for observables. Consider antisymmetric tensor (50). As a consequence of $T_3 = 0$ and $T_4 = 0$, this obeys the Pirani supplementary condition [4, 5, 7]

$$S^{\mu\nu}P_\nu = 0. \tag{77}$$

Besides, the constraints T_2 and T_5 fix the value of square

$$S^{\mu\nu}S_{\mu\nu} = 8\alpha, \tag{78}$$

so we identify $S^{\mu\nu}$ with the Frenkel spin-tensor [9]. Equations (77) and (78) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle. Equations of motion for $S^{\mu\nu}$ follow from (76). Besides, we express (74) and (75) in terms of the spin-tensor. This gives the system

$$\dot{x}^\mu = \lambda_1 \left[P^\mu + a S^{\mu\beta} \theta_{\beta\alpha} P^\alpha \right], \tag{79}$$

$$\nabla P_\mu = -\frac{1}{4} R_{\mu\nu\alpha\beta} S^{\alpha\beta} \dot{x}^\nu \equiv -\frac{1}{4} \theta_{\mu\nu} \dot{x}^\nu, \tag{80}$$

$$\nabla S^{\mu\nu} = 2 \left(P^\mu \dot{x}^\nu - P^\nu \dot{x}^\mu \right), \tag{81}$$

where θ has been defined in (68). Equation (81), contrary to (76) for ω and π, does not depend on λ_2. This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (79)–(81), together with (77) and (78), form a closed system which determines evolution of a spinning particle.

To obtain the Hamiltonian equations, we can equally use the Dirac bracket constructed with help of second-class constraints:

$$\begin{aligned} \{A, B\}_{\mathrm{D}} &= \{A, B\} \\ &\quad - \frac{1}{8a} \left[\{A, T_3\} \{T_4, B\} - \{A, T_4\} \{T_3, B\} \right]. \end{aligned} \tag{82}$$

Since the Dirac bracket of a second-class constraint with any quantity vanishes, we can now omit T_3 and T_4 from (72); this yields the Hamiltonian

$$H_1 = \frac{\lambda_1}{2} \left(P^2 + m^2 c^2 \right) + \frac{\lambda_1}{2} \left(\pi^2 - \frac{\alpha}{\omega^2} \right) + \lambda_2 (\omega \pi). \tag{83}$$

Then, (74)–(76) can be obtained according to the rule $\dot{z} = \{z, H_1\}_{\mathrm{D}}$. The quantities x^μ, P^μ, and $S^{\mu\nu}$, being invariant under spin-plane symmetry, have vanishing brackets with the corresponding first-class constraints T_2 and T_5. So, obtaining equations for these quantities, we can omit the last two terms in H_1, arriving at the familiar relativistic Hamiltonian

$$H_2 = \frac{\lambda_1}{2} \left(P^2 + m^2 c^2 \right). \tag{84}$$

Equations (79)–(81) can be obtained according to the rule $\dot{z} = \{z, H_2\}_{\mathrm{D}}$. From (84), we conclude that our model describes spinning particle without gravimagnetic moment. The Hamiltonian with gravimagnetic moment κ has been proposed by Khriplovich [25] adding nonminimal interaction $(\lambda_1/2)(\kappa/16) R_{\mu\nu\alpha\beta} S^{\mu\nu} S^{\alpha\beta}$ to the expression for H_2. The corresponding Lagrangian formulation will be constructed in Section 5.1.

Let us exclude momenta P^μ and the auxiliary variable λ_1 from the Hamiltonian equations. This yields second-order equation for the particle's position $x^\mu(\tau)$. To achieve this, we observe that (79) is linear on P.

$$\dot{x}^\mu = \lambda_1 T^\mu{}_\nu P^\nu, \quad \text{with } T^\mu{}_\nu = \delta^\mu_\nu + a S^{\mu\alpha} \theta_{\alpha\nu}. \tag{85}$$

Using the identity

$$(S\theta S)^{\mu\nu} = -\frac{1}{2} (S\theta) S^{\mu\nu}, \quad \text{where } S\theta = S^{\alpha\beta} \theta_{\alpha\beta}, \tag{86}$$

we find inverse of the matrix $T^\mu{}_\nu$:

$$\begin{aligned} \widetilde{\mathcal{T}}^\mu{}_\nu &= \delta^\mu{}_\nu - \frac{1}{8m^2c^2} S^{\mu\sigma} \theta_{\sigma\nu}, \\ T^\mu{}_\alpha \widetilde{\mathcal{T}}^\alpha{}_\nu &= \delta^\mu_\nu, \end{aligned} \tag{87}$$

so (85) can be solved with respect to P^μ, $P^\mu = (1/\lambda_1) \widetilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu$. We substitute P^μ into the constraint $P^2 + m^2c^2 = 0$; this gives expression for λ_1:

$$\lambda_1 = \frac{\sqrt{-G_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}}{mc} \equiv \frac{\sqrt{-\dot{x} G \dot{x}}}{mc}. \tag{88}$$

We have introduced the effective metric

$$G_{\mu\nu} \equiv \widetilde{\mathcal{T}}^\alpha{}_\mu g_{\alpha\beta} \widetilde{\mathcal{T}}^\beta{}_\nu. \tag{89}$$

The matrix G is composed of the original metric $\eta_{\mu\nu}$ plus (spin- and field-dependent) contribution; $G_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(S)$. So, we call G the effective metric produced along the world line by interaction of spin with gravity. The effective metric will play the central role in our discussion of ultrarelativistic limit.

From (85) and (88), we obtain the final expression for P_μ,

$$\begin{aligned} P^\mu &= \frac{mc}{\sqrt{-\dot{x} G \dot{x}}} \widetilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu \\ &= \frac{mc}{\sqrt{-\dot{x} G \dot{x}}} \left[\dot{x}^\mu - \frac{1}{8m^2c^2} S^{\mu\nu} \theta_{\nu\sigma} \dot{x}^\sigma \right], \end{aligned} \tag{90}$$

and Lagrangian form of the Pirani condition,

$$S^\mu{}_\nu \dot{x}^\nu - \frac{1}{8(mc)^2} (SS\theta\dot{x})^\mu = 0. \tag{91}$$

Using (90) and (91) in (80) and (81), we finally obtain

$$\nabla \left[\frac{\widetilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu}{\sqrt{-\dot{x} G \dot{x}}} \right] = -\frac{1}{4mc} R^\mu{}_{\nu\alpha\beta} S^{\alpha\beta} \dot{x}^\nu, \tag{92}$$

$$\nabla S^{\mu\nu} = \frac{1}{4mc\sqrt{-\dot{x} G \dot{x}}} \dot{x}^{[\mu} S^{\nu]\sigma} \theta_{\sigma\alpha} \dot{x}^\alpha. \tag{93}$$

These equations, together with conditions (91) and (78), form closed system for the set $(x^\mu, S^{\mu\nu})$. The consistency of constraints (91) and (78) with the dynamical equations is guaranteed by Dirac procedure for singular systems.

4.2. Lagrangian Action of Spinning Particle with Unfixed Value of Spin. Lagrangians (48) and (56) yield the fixed value of spin (78); that is, they correspond to an elementary particle. Let us present the modification which leads to the theory with unfixed spin and similarly to Hanson-Regge approach [22], with a mass-spin trajectory constraint. Consider the following Lagrangian in curved background:

$$L = -\frac{mc}{\sqrt{2}}\sqrt{-\dot{x}N\dot{x} - l^2\frac{\nabla\omega N\nabla\omega}{\omega^2} + \sqrt{\left[\dot{x}N\dot{x} + l^2\frac{\nabla\omega N\dot{\omega}}{\omega^2}\right]^2 - 4l^2\frac{(\dot{x}N\nabla\omega)^2}{\omega^2}}}, \quad (94)$$

where l is a parameter with the dimension of length. Applying the Dirac procedure as in Section 4.1, we obtain the Hamiltonian

$$H = \frac{\lambda_1}{2}\left(P^2 + m^2c^2 + \frac{\pi^2\omega^2}{l^2}\right) + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi), \quad (95)$$

which turns out to be combination of the first-class constraints $P^2 + m^2c^2 + \pi^2\omega^2/l^2 = 0$ and $\omega\pi = 0$ and the second-class constraints $P\omega = 0$ and $P\pi = 0$. The Dirac procedure stops on the first stage; that is, there are no secondary constraints. As compared with (56), the first-class constraint $\pi^2 - \alpha/\omega^2 = 0$ does not appear in the present model. Due to this, square of spin is not fixed; $S^2 = 8(\omega^2\pi^2 - \omega\pi) \approx 8\omega^2\pi^2$. Using this equality, the mass-shell constraint acquires the string-like form

$$P^2 + m^2c^2 + \frac{1}{8l^2}S^2 = 0. \quad (96)$$

The model has four physical degrees of freedom in the spin-sector. As the independent gauge-invariant degrees of freedom, we can take three components S^{ij} of the spin-tensor together with any one product of conjugate coordinates, for instance, $\omega^0\pi^0$.

Using the auxiliary variable λ, we can rewrite the Lagrangian in the equivalent form

$$L = \frac{1}{2\lambda}\left[\dot{x}N\dot{x} + l^2\frac{\nabla\omega N\nabla\omega}{\omega^2} - \sqrt{\left[\dot{x}N\dot{x} + l^2\frac{\nabla\omega N\dot{\omega}}{\omega^2}\right]^2 - 4l^2\frac{(\dot{x}N\nabla\omega)^2}{\omega^2}}\right] - \frac{\lambda}{4} \cdot m^2c^2. \quad (97)$$

Contrary to (94), it admits the massless limit.

4.3. Mathisson-Papapetrou-Tulczyjew-Dixon (MPTD) Equations and Dynamics of Representative Point of a Rotating Body. In this section, we discuss MPTD equations of a rotating body in the form studied by Dixon (our S is twice of that of Dixon) (for the relation of the Dixon equations with those of Papapetrou and Tulczyjew, see page 335 in [5]),

$$\nabla P^\mu = -\frac{1}{4}R^\mu{}_{\nu\alpha\beta}S^{\alpha\beta}\dot{x}^\nu \equiv -\frac{1}{4}(\theta\dot{x})^\mu, \quad (98)$$

$$\nabla S^{\mu\nu} = 2\left(P^\mu\dot{x}^\nu - P^\nu\dot{x}^\mu\right), \quad (99)$$

$$S^{\mu\nu}P_\nu = 0, \quad (100)$$

and compare them with equations of motion of our spinning particle. In particular, we show that the effective metric $G_{\mu\nu}$ also emerges in this formalism. MPTD equations appeared in multipole approach to description of a body [1–6], where the energy-momentum of the body is modelled by a set of multipoles. In this approach, $x^\mu(\tau)$ is called representative point of the body; we take it in arbitrary parametrization τ (contrary to Dixon, we do not assume the proper-time parametrization; i.e., we do not add the equation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$ to the system above). $S^{\mu\nu}(\tau)$ is associated with inner angular momentum, and $P^\mu(\tau)$ is called momentum. First-order equations (98) and (99) appear in the pole-dipole approximation, while algebraic equation (100) has been added by hand (for geometric interpretation of the spin supplementary condition in the multipole approach, see [5]). After that, the number of equations coincides with the number of variables.

To compare MPTD equations with those of Section 4.1, we first observe some useful consequences of system (98)–(100).

Take derivative of the constraint, $\nabla(S^{\mu\nu}P_\nu) = 0$, and use (98) and (99); this gives the expression

$$(P\dot{x})P^\mu = P^2\dot{x}^\mu + \frac{1}{8}(S\theta\dot{x})^\mu, \quad (101)$$

which can be written in the form

$$P^\mu = \frac{P^2}{(P\dot{x})}\left(\delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu\right)\dot{x}^\nu \equiv \frac{P^2}{(P\dot{x})}\widetilde{\mathcal{T}}^\mu{}_\nu\dot{x}^\nu. \quad (102)$$

Contract (101) with $\dot{x}_\mu$. Taking into account that $(P\dot{x}) < 0$, this gives $(P\dot{x}) = -\sqrt{-P^2}\sqrt{-\dot{x}\widetilde{\mathcal{T}}\dot{x}}$. Using this in (102), we obtain

$$P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\widetilde{\mathcal{T}}\dot{x}}}\left(\widetilde{\mathcal{T}}\dot{x}\right)^\mu, \qquad \widetilde{\mathcal{T}}^\mu{}_\nu = \delta^\mu{}_\nu + \frac{1}{8P^2}(S\theta)^\mu{}_\nu. \quad (103)$$

For the latter use, we observe that in our model with composite $S^{\mu\nu}$ we used identity (86) to invert T^μ_ν; then Hamiltonian equation (79) has been written in the form of (90); the latter can be compared with (103).

Contracting (99) with $S_{\mu\nu}$ and using (100), we obtain $(d/d\tau)(S^{\mu\nu}S_{\mu\nu}) = 0$; that is, square of spin is a constant of motion. Contraction of (101) with P_μ gives $(PS\theta\dot{x}) = 0$. Contraction of (101) with $(\dot{x}\theta)_\mu$ gives $(P\theta\dot{x}) = 0$. Contraction of (98) with P_μ gives $(d/d\tau)(P^2) = -(1/2)(P\theta\dot{x}) = 0$; that is, P^2 is one more constant of motion, say k, $\sqrt{-P^2} = k = \text{const}$ (in our model this is fixed as $k = mc$). Substituting (103) into (98)–(100), we now can exclude P^μ from these equations, modulo to the constant of motion $k = \sqrt{-P^2}$.

Thus, square of momentum can not be excluded from system (98)–(101); that is, MPTD equations in this form do not represent a Hamiltonian system for the pair x^μ, P^μ. To improve this point, we note that (103) acquires a conventional form (as the expression for conjugate momenta of x^μ in the Hamiltonian formalism) if we add to system (98)–(100) one more equation, which fixes the remaining quantity P^2 (Dixon noticed this for the body in electromagnetic field; see his equation (4.5) in [6]). To see how the equation could look, we note that for nonrotating body (pole approximation) we expect equations of motion of spinless particle; $\nabla p^\mu = 0$, $p^\mu = (mc/\sqrt{-\dot{x}g\dot{x}})\dot{x}^\mu$, and $p^2 + (mc)^2 = 0$. Independent equations of system (98)–(101) in this limit read $\nabla P^\mu = 0$, $P^\mu = (\sqrt{-P^2}/\sqrt{-\dot{x}g\dot{x}})\dot{x}^\mu$. Comparing the two systems, we see that the missing equation is the mass-shell condition $P^2 + (mc)^2 = 0$. Returning to the pole-dipole approximation, an admissible equation should be $P^2 + (mc)^2 + f(S,\ldots) = 0$, where f must be a constant of motion. Since the only constant of motion in arbitrary background is S^2, we have finally

$$P^2 = -(mc)^2 - f\left(S^2\right). \tag{104}$$

With this value of P^2, we can exclude P^μ from MPTD equations, obtaining closed system with second-order equation for x^μ (so, we refer to the resulting equations as Lagrangian form of MPTD equations). We substitute (103) into (98)–(100); this gives

$$\nabla \frac{\left(\widetilde{\mathcal{T}}\dot{x}\right)^\mu}{\sqrt{-\dot{x}\widetilde{\mathcal{T}}\dot{x}}} = -\frac{1}{4\sqrt{-P^2}}(\theta\dot{x})^\mu, \tag{105}$$

$$\nabla S^{\mu\nu} = -\frac{1}{4\sqrt{-P^2}\sqrt{-\dot{x}\widetilde{\mathcal{T}}\dot{x}}}\dot{x}^{[\mu}(S\theta\dot{x})^{\nu]}, \tag{106}$$

$$(SS\theta\dot{x})^\mu = -8P^2(S\dot{x})^\mu, \tag{107}$$

where (104) is implied. They determine evolution of x^μ and $S^{\mu\nu}$ for each given function $f(S^2)$.

It is convenient to introduce the effective metric $\mathcal{G}$ composed of the "tetrad field" $\widetilde{\mathcal{T}}^\mu{}_\nu$:

$$\mathcal{G}_{\mu\nu} \equiv g_{\alpha\beta}\widetilde{\mathcal{T}}^\alpha{}_\mu\widetilde{\mathcal{T}}^\beta{}_\nu. \tag{108}$$

Equation (107) implies the identity

$$\dot{x}\widetilde{\mathcal{T}}\dot{x} = \dot{x}\mathcal{G}\dot{x}, \tag{109}$$

so we can replace $\sqrt{-\dot{x}\widetilde{\mathcal{T}}\dot{x}}$ in (105)–(107) by $\sqrt{-\dot{x}\mathcal{G}\dot{x}}$.

In resume, we have presented MPTD equations in the form

$$\begin{gathered} P^\mu = \frac{\sqrt{-P^2}}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}\left(\widetilde{\mathcal{T}}\dot{x}\right)^\mu, \\ \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu, \\ \nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \\ S^{\mu\nu}P_\nu = 0, \end{gathered} \tag{110}$$

$$P^2 + (mc)^2 + f\left(S^2\right) = 0, \tag{111}$$

$$S^2 \text{ is a constant of motion,} \tag{112}$$

with $\widetilde{\mathcal{T}}^\mu{}_\nu$ given in (103). Now, we are ready to compare them with Hamiltonian equations of our spinning particle, which we write here in the form

$$\begin{gathered} P^\mu = \frac{mc}{\sqrt{-\dot{x}G\dot{x}}}\left(\widetilde{T}\dot{x}\right)^\mu, \\ \nabla P^\mu = -\frac{1}{4}(\theta\dot{x})^\mu, \\ \nabla S^{\mu\nu} = 2P^{[\mu}\dot{x}^{\nu]}, \\ S^{\mu\nu}P_\nu = 0, \\ P^2 + (mc)^2 = 0, \\ S^2 = 8\alpha, \end{gathered} \tag{113}$$

with $\widetilde{\mathcal{T}}^\mu{}_\nu$ given in (87). Comparing the systems, we see that our spinning particle has fixed values of spin and canonical momentum, while for MPTD particle the spin is a constant of motion and momentum is a function of spin. We conclude that all the trajectories of a body with given m and $S^2 = \beta$ are described by our spinning particle with spin $\alpha = \beta/8$ and with the mass equal to $\sqrt{m^2 - f^2(\beta)/c^2}$. In this sense, our spinning particle is equivalent to MPTD particle.

We point out that our final conclusion remains true even if we do not add (104) to MPTD equations; to study the class of trajectories of a body with $\sqrt{-P^2} = k$ and $S^2 = \beta$, we take our spinning particle with $m = k/c$ and $\alpha = \beta/8$.

MPTD equations in the Lagrangian form in (105)–(107) can be compared with (91)–(93).

4.4. Ultrarelativistic Limit: The Problems with MPTD Equations. The equations for trajectory (92) and for precession of spin (93) became singular at critical velocity which obeys the equation

$$\dot{x}G\dot{x} = 0. \tag{114}$$

As we discussed in Introduction, the singularity determines behavior of the particle in ultrarelativistic limit. In (114), effective metric (89) appeared instead of the original metric $g_{\mu\nu}$. It should be noted that the incorporation of constraints

(62) and (66) into a variational problem, as well as the search for an interaction consistent with them, represents very strong restrictions on possible form of the Lagrangian. So, the appearance of effective metric seems to be unavoidable in a systematically constructed model of spinning particle. The same conclusion follows from our analysis of MPTD equations in Section 4.3.

The effective metric is composed of the original one plus (spin- and field-dependent) contribution; $G = g+h(S)$. So, we need to decide which of them the particle probes as the space-time metric. Let us consider separately the two possibilities.

Let us use g to define the three-dimensional geometry in (5)–(8). This leads to two problems. The first problem is that the critical speed turns out to be slightly more than the speed of light. To see this, we use the Pirani condition to write (114) in the form

$$-\left(\frac{dt}{dx^0}\right)^2 \dot{x}G\dot{x} = \left(c^2 - \mathbf{v}\gamma\mathbf{v}\right) + \frac{1}{\left(2m^2c^2\right)^2}\left(v\theta SS\theta v\right) = 0, \tag{115}$$

with v^μ defined in (10). Using the expression $S^{\mu\nu} = 2\omega^{[\mu}\pi^{\nu]}$, we obtain

$$-\left(\frac{dt}{dx^0}\right)^2 \dot{x}G\dot{x} = \left(c^2 - \mathbf{v}\gamma\mathbf{v}\right) + \frac{1}{\left(m^2c^2\right)^2}\left(\pi^2\left(v\theta\omega\right)^2 + \omega^2\left(v\theta\pi\right)^2\right) = 0. \tag{116}$$

As π and ω are space-like vectors (see the discussion below (70)), the last term is nonnegative; this implies $|\mathbf{v}_{\text{cr}}| \geq c$. Let us confirm that generally this term is nonvanishing function of velocity; then $|\mathbf{v}_{\text{cr}}| > c$. Assume the contrary that this term vanishes at some velocity; then

$$v\theta\omega = \theta_{0i}\omega^i + \theta_{i0}v^i\omega^0 = 0, \tag{117}$$

$$v\theta\pi = \theta_{0i}\pi^i + \theta_{i0}v^i\pi^0 = 0. \tag{118}$$

We analyze these equations in the following special case. Consider a space with covariantly constant curvature $\nabla_\mu R_{\mu\nu\alpha\beta} = 0$. Then, $(d/d\tau)(\theta_{\mu\nu}S^{\mu\nu}) = 2\theta_{\mu\nu}\nabla S^{\mu\nu}$, and using (93) we conclude that $\theta_{\mu\nu}S^{\mu\nu}$ is an integral of motion. We further assume that the only nonvanishing part is the electric [38] part of the curvature, $R_{0i0j} = K_{ij}$, with $\det K_{ij} \neq 0$. Then, the integral of motion acquires the form

$$\theta_{\mu\nu}S^{\mu\nu} = 2K_{ij}S^{0i}S^{0j}. \tag{119}$$

Let us take the initial conditions for spin such that $K_{ij}S^{0i}S^{0j} \neq 0$; then this holds at any future instant. Contrary to this, system (117) implies $K_{ij}S^{0i}S^{0j} = 0$. Thus, the critical speed does not always coincide with the speed of light and, in general case, we expect that $\mathbf{v}_{\text{cr}}$ is both field- and spin-dependent quantity.

The second problem is that acceleration of MPTD particle grows up in the ultrarelativistic limit. In the spinless limit (92) turn into the geodesic equation. Spin causes deviations from the geodesic equation due to right hand side of this equation, as well as due to the presence of the tetrad field $\widetilde{\mathcal{T}}^\mu{}_\nu$ and of the effective metric G in the left hand side. Due to the dependence of the tetrad field on the spin-tensor S, the singularity presented in (93) causes the appearance of the term proportional to $1/\sqrt{\dot{x}G\dot{x}}$ in the expression for longitudinal acceleration. In the result, the acceleration grows up to infinity as the particle's speed approximates to the critical speed. To see this, we separate derivative of $\widetilde{\mathcal{T}}^\mu{}_\nu$ in (92).

$$\nabla\left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}G\dot{x}}}\right] = -T^\mu{}_\alpha\left(\nabla\widetilde{\mathcal{T}}^\alpha{}_\beta\right)\frac{\dot{x}^\beta}{\sqrt{-\dot{x}G\dot{x}}} - \frac{1}{4mc}T^\mu{}_\nu\left(\theta\dot{x}\right)^\nu. \tag{120}$$

Using (93), we obtain

$$\left[\nabla\widetilde{\mathcal{T}}^\mu{}_\nu\right]\dot{x}^\nu = -\frac{S^{\mu\alpha}}{8m^2c^2}\left[\frac{R_{\alpha\nu\beta\sigma}\dot{x}^\beta\left(S\theta\dot{x}\right)^\sigma}{2mc\sqrt{-\dot{x}G\dot{x}}} + S^{\beta\sigma}\left(\nabla R_{\alpha\nu\beta\sigma}\right)\right]\dot{x}^\nu. \tag{121}$$

Using this expression together with the identity $(TS)^{\mu\nu} = 8m^2c^2aS^{\mu\nu}$, (120) reads

$$\frac{d}{d\tau}\left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}G\dot{x}}}\right] = \frac{f^\mu}{\sqrt{-\dot{x}G\dot{x}}}, \tag{122}$$

where we denoted

$$f^\mu \equiv aS^{\mu\alpha}\left[\frac{R_{\alpha\nu\beta\sigma}\dot{x}^\beta\left(S\theta\dot{x}\right)^\sigma}{2mc\sqrt{-\dot{x}G\dot{x}}} + S^{\beta\sigma}\left(\nabla R_{\alpha\nu\beta\sigma}\right)\right]\dot{x}^\nu - \left(\Gamma\dot{x}\dot{x}\right)^\mu - \frac{\sqrt{-\dot{x}G\dot{x}}}{4mc}\left(T\theta\dot{x}\right)^\mu. \tag{123}$$

It will be sufficient to consider static metric $g_{\mu\nu}(\mathbf{x})$ with $g_{0i} = 0$. Then three-dimensional metric and velocity are

$$\gamma_{ij} = g_{ij}, \qquad v^i = \frac{c}{\sqrt{-g_{00}}}\frac{dx^i}{dx^0}. \tag{124}$$

Taking $\tau = x^0$, the spacial part of (122) with this metric reads

$$\left(\frac{dt}{dx^0}\right)^{-1}\frac{d}{dx^0}\left[\frac{v^i}{\sqrt{-vGv}}\right] = \frac{f^i(v)}{\sqrt{-vGv}}, \tag{125}$$

with v^μ defined in (10), for the case

$$v^\mu = \left(\frac{c}{\sqrt{-g_{00}}}, \mathbf{v}\right), \qquad -vGv = -v\widetilde{\mathcal{T}}v = c^2 - \mathbf{v}g\mathbf{v} + \frac{(vS\theta v)}{8m^2c^2}. \tag{126}$$

In the result, we have presented the equation for trajectory in the form convenient for analysis of acceleration; see (29). Using the definition of three-dimensional covariant derivative (25), we present the derivative on the l.h.s. of (125) as follows:

$$\begin{aligned}\frac{d}{dx^0}\left[\frac{v^i}{\sqrt{-vGv}}\right] &= \frac{1}{\sqrt{-vGv}}\left[\mathscr{M}^i{}_k\nabla_0 v^k\right.\\ &\left.-\tilde{\Gamma}(\gamma)^i_{jk}v^jv^k\frac{dt}{dx^0}+\frac{Kv^i}{2(-vGv)}\right].\end{aligned} \tag{127}$$

We have denoted

$$\begin{aligned}K &= \left(\nabla_0 G_{\mu\nu}\right)v^\mu v^\nu - v^\mu G_{\mu 0}v^k\partial_k\ln\left(-g_{00}\right),\\ \mathscr{M}^i{}_k &= \delta^i{}_k - \frac{v^iv^\mu G_{\mu k}}{vGv}.\end{aligned} \tag{128}$$

The matrix $\mathscr{M}^i{}_k$ has the inverse

$$\begin{aligned}\widetilde{\mathscr{M}}^i{}_k &= \delta^i{}_k + \frac{v^iv^\mu G_{\mu k}}{v^\sigma G_{\sigma 0}v^0},\\ \text{then } \widetilde{\mathscr{M}}^i{}_kv^k &= v^i\frac{vGv}{v^\sigma G_{\sigma 0}v^0}.\end{aligned} \tag{129}$$

Combining these equations, we obtain the three-acceleration of our spinning particle:

$$\begin{aligned}a^i &= \left(\frac{dt}{dx^0}\right)^{-1}\nabla_0 v^i\\ &= \widetilde{\mathscr{M}}^i{}_k\left[f^k+\left(\tilde{\Gamma}vv\right)^k\right]+\frac{Kv^i}{2v^\sigma G_{\sigma 0}}.\end{aligned} \tag{130}$$

Finally, using manifest form of f^i from (123), we have

$$\begin{aligned}a^i &= \frac{\widetilde{\mathscr{M}}^i{}_k\widehat{S}^k}{\sqrt{-vGv}} - c^2\widetilde{\mathscr{M}}^i{}_k\frac{\gamma^{kj}\partial_j g_{00}}{2g_{00}}\\ &- \frac{\sqrt{-vGv}}{4mc}\widetilde{\mathscr{M}}^i{}_k(T\theta v)^k + \frac{Kv^i}{2v^\sigma G_{\sigma 0}}\\ &+ a\widetilde{\mathscr{M}}^i{}_kS^{k\alpha}R_{\alpha\nu\beta\sigma;\lambda}S^{\beta\sigma}v^\nu v^\lambda.\end{aligned} \tag{131}$$

The longitudinal acceleration is obtained by projecting a^i on the direction of velocity, that is,

$$\begin{aligned}(\mathbf{v}\gamma\mathbf{a}) &= \frac{ac\left(\mathbf{v}\gamma\widetilde{\mathscr{M}}\right)_k\widehat{S}^k}{2m\sqrt{-vGv}} - c^2\left(\mathbf{v}\gamma\widetilde{\mathscr{M}}\right)_k\frac{\gamma^{kj}\partial_j g_{00}}{2g_{00}}\\ &- \frac{\sqrt{-vGv}}{4mc}\left(\mathbf{v}\gamma\widetilde{\mathscr{M}}\right)_k(T\theta v)^k + \frac{K}{2v^\sigma G_{\sigma 0}}(\mathbf{v}\gamma\mathbf{v})\\ &+ a\left(\mathbf{v}\gamma\widetilde{\mathscr{M}}\right)_kS^{k\alpha}R_{\alpha\nu\beta\sigma;\lambda}S^{\beta\sigma}v^\nu v^\lambda,\end{aligned} \tag{132}$$

where $\widehat{S}^k = S^{k\mu}R_{\mu\nu\alpha\beta}v^\nu v^\alpha(S\theta v)^\beta$. As the speed of the particle gets closer to the critical velocity, the longitudinal acceleration diverges due to the first term in (132). In resume, assuming that MPTD particle sees the original geometry $g_{\mu\nu}$, we have a theory with unsatisfactory behavior in the ultrarelativistic limit.

Let us consider the second possibility; that is, we take $G_{\mu\nu}$ to construct three-dimensional geometry (5)–(8). With these definitions we have, by construction, $-\dot{x}G\dot{x} = (dt/dx^0)^2(c^2 - (\mathbf{v}\gamma\mathbf{v}))$, so the critical speed coincides with the speed of light. In the present case, the expression for three-acceleration can be obtained in closed form for an arbitrary curved background. Taking $\tau = x^0$, the spacial part of (122) implies

$$\left(\frac{dt}{dx^0}\right)^{-1}\frac{d}{dx^0}\left[\frac{v^i}{\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}}}\right] = \frac{f^i(v)}{\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}}}, \tag{133}$$

where, from (123), f^i is given by

$$\begin{aligned}f^i &\equiv aS^{i\alpha}\left[\frac{R_{\alpha\nu\beta\sigma}v^\beta(S\theta v)^\sigma}{2mc\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}}}+S^{\beta\sigma}\left(\nabla R_{\alpha\nu\beta\sigma}\right)\right]v^\nu\\ &- \Gamma^i{}_{\mu\nu}(G)v^\mu v^l\nu - \frac{\sqrt{c^2-\mathbf{v}\gamma\mathbf{v}}}{4mc}(T\theta v)^i.\end{aligned} \tag{134}$$

Equation (133) is of the form in (29), so the acceleration is given by (38) and (39), where, for the present case, $\gamma_{ij} = G_{ij} - G_{0i}G_{0j}/G_{00}$.

$$\begin{aligned}a^i &= \widetilde{M}^i{}_j\left[f^j+\tilde{\Gamma}^j{}_{kl}(\gamma)v^kv^l\right]+\frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}\\ &\cdot\left[\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i - \frac{(\mathbf{v}\partial_0\gamma\mathbf{v})}{c^2}v^i\right],\end{aligned} \tag{135}$$

$$\begin{aligned}\mathbf{v}\gamma\mathbf{a} &= \left(1-\frac{\mathbf{v}\gamma\mathbf{v}}{c^2}\right)\left[(\mathbf{v}\gamma)_i\left[f^i(v)+\tilde{\Gamma}^i{}_{kl}(\gamma)v^kv^l\right]\right.\\ &\left.+\frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}(\mathbf{v}\partial_0\gamma\mathbf{v})\right].\end{aligned} \tag{136}$$

With f^i given in (134), the longitudinal acceleration vanishes as $v \to c$.

Let us resume the results of this subsection. Assuming that spinning particle probes the three-dimensional space-time geometry determined by the original metric g, we have a theory with unsatisfactory ultrarelativistic limit. First, the critical speed, which the particle can not overcome during its evolution in gravitational field, can be more than the speed of light. Second, the longitudinal acceleration grows up to infinity in the ultrarelativistic limit. Assuming that the the particle sees the effective metric $G(S)$ as the space-time metric, we avoided the two problems. But the resulting theory still possesses the problem. Acceleration (135) contains the singularity due to $f^i \sim 1/\sqrt{c^2-(\mathbf{v}\gamma\mathbf{v})}$; that is, at $v = c$ the acceleration becomes orthogonal to the velocity but remains divergent. We conclude that MPTD equations do not seem promising candidate for description of a relativistic rotating body.

5. Nonminimal Interaction with Gravitational Field

Can we modify the MPTD equations to obtain a theory with reasonable behavior with respect to the original metric $g_{\mu\nu}$? In the previous section we have noticed that the bad behavior of acceleration originates from the fact that variation rate of spin (93) diverges in the ultrarelativistic limit, $\nabla S \sim 1/\sqrt{\dot{x}G\dot{x}}$, and contributes to expression for acceleration (132) through the tetrad field $\widetilde{\mathcal{T}}^{\mu}{}_{\nu}(S)$. To improve this, we note that MPTD equations result from minimal interaction of spinning particle with gravitational field. In this section, we demonstrate that vector model of spin admits also a nonminimal interaction which involves the interaction constant κ. By analogy with the magnetic moment, the interaction constant κ is called gravimagnetic moment [25]. In the resulting theory, the equation for precession of spin, $\nabla S \sim 1/\sqrt{-\dot{x}G\dot{x}}$, is replaced by $\nabla S \sim \sqrt{-\dot{x}g\dot{x}}$. This improves the bad behavior of MPTD equations. As it will be seen below, introducing κ we effectively change the supplementary spin condition and hence the definition of center of mass.

5.1. Hamiltonian Variational Problem. We add the term $(\lambda_1/2)\kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu \equiv (\lambda_1/2)(\kappa/16)\theta S$ into Hamiltonian action (73). Thus, we consider the variational problem

$$\begin{aligned} S_\kappa = \int d\tau p_\mu\dot{x}^\mu + \pi_\mu\dot{\omega}^\mu - \Big[\frac{\lambda_1}{2}\Big(P^2 \\ + \kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2}\Big) \\ + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \lambda_4(P\pi)\Big] \end{aligned} \tag{137}$$

on the space of independent variables x^μ, p_ν, ω^μ, π_ν, and λ_a. We have denoted $P_\mu \equiv p_\mu - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta$, $P^2 = g^{\mu\nu}P_\mu P_\nu$, and so on. Note also that the first two terms can be identically rewritten in the general-covariant form $p_\mu\dot{x}^\mu + \pi_\mu\dot{\omega}^\mu = P_\mu\dot{x}^\mu + \pi_\mu\nabla\omega^\mu$. Variation of the action with respect to λ_a gives the algebraic equations

$$P^2 + \kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0, \tag{138}$$

$$\begin{aligned} \omega\pi &= 0, \\ P\omega &= 0, \\ P\pi &= 0, \end{aligned} \tag{139}$$

while variations with respect to the remaining variables yield dynamical equations which can be written in the covariant form as follows (note that, by construction of S_κ, the variational equation $\delta S_\kappa/\delta p_\mu = 0$ is equivalent to $\dot{x}^\mu = \{x^\mu, H\}$ and so on):

$$\begin{aligned} &\frac{\delta S_\kappa}{\delta p_\mu} = 0 \Longleftrightarrow \\ &\dot{x}^\mu = \lambda_1 P^\mu + \lambda_3\omega^\mu + \lambda_4\pi^\mu, \end{aligned} \tag{140}$$

$$\begin{aligned} &\frac{\delta S_\kappa}{\delta x^\mu} = 0 \Longleftrightarrow \\ &\nabla P_\mu = -R_{\mu\nu\alpha\beta}\dot{x}^\nu\omega^\alpha\pi^\beta - \frac{1}{2}\lambda_1\kappa\nabla_\mu R_{\sigma\nu\alpha\beta}\omega^\sigma\pi^\nu\omega^\alpha\pi^\beta, \end{aligned} \tag{141}$$

$$\begin{aligned} &\frac{\delta S_\kappa}{\delta \pi_\mu} = 0 \Longleftrightarrow \\ &\nabla\omega^\mu = \lambda_1\pi^\mu - \lambda_1\kappa R^\mu{}_{\alpha\beta\nu}\omega^\alpha\omega^\beta\pi^\nu + \lambda_2\omega^\mu + \lambda_4 P^\mu, \end{aligned} \tag{142}$$

$$\begin{aligned} &\frac{\delta S_\kappa}{\delta \omega^\mu} = 0 \Longleftrightarrow \\ &\nabla\pi_\mu = -\frac{\lambda_1\alpha}{\omega^4}\omega_\mu - \lambda_1\kappa R_{\mu\nu\alpha\beta}\pi^\nu\omega^\alpha\pi^\beta - \lambda_2\pi_\mu - \lambda_3 P_\mu. \end{aligned} \tag{143}$$

Equation (140) has been repeatedly used to obtain the final form in (141)–(143) of the equations $\delta S_\kappa/\delta x^\mu = 0$, $\delta S_\kappa/\delta\pi_\mu = 0$, and $\delta S_\kappa/\delta\omega^\mu = 0$. Computing time derivative of algebraic equations (139) and using (140)–(143), we obtain the consequences

$$\pi^2 - \frac{\alpha}{\omega^2} = 0, \tag{144}$$

$$\begin{aligned} \lambda_3 &= 4a\lambda_1\Big[2(1-\kappa)R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\pi^\mu P^\nu \\ &\quad + \kappa\pi^\sigma\left(\nabla_\sigma R_{\mu\nu\alpha\beta}\right)\omega^\mu\pi^\nu\omega^\alpha\pi^\beta\Big], \\ \lambda_4 &= -4a\lambda_1\Big[2(1-\kappa)R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu P^\nu \\ &\quad + \kappa\omega^\sigma\left(\nabla_\sigma R_{\mu\nu\alpha\beta}\right)\omega^\mu\pi^\nu\omega^\alpha\pi^\beta\Big]. \end{aligned} \tag{145}$$

Here and below we use the following notation. In the equation which relates velocity and momentum will appear the matrix $\mathcal{T}^\mu_\nu$:

$$\begin{aligned} \mathcal{T}^\alpha{}_\nu &\equiv \delta^\alpha{}_\nu - (\kappa-1)aS^{\alpha\sigma}\theta_{\sigma\nu}, \\ a &= \frac{2}{16m^2c^2 + (\kappa+1)(S\theta)}. \end{aligned} \tag{146}$$

The matrix has an inverse given by

$$\begin{aligned} \widetilde{\mathcal{T}}^\alpha{}_\nu &\equiv \delta^\alpha{}_\nu + (\kappa-1)bS^{\alpha\sigma}\theta_{\sigma\nu}, \\ b &= \frac{1}{8m^2c^2 + \kappa(S\theta)}. \end{aligned} \tag{147}$$

The vector Z^μ is defined by

$$Z^\mu = \frac{b}{8c}S^{\mu\sigma}\left(\nabla_\sigma R_{\alpha\beta\rho\delta}\right)S^{\alpha\beta}S^{\rho\delta} \equiv \frac{b}{8c}S^{\mu\sigma}\nabla_\sigma(S\theta). \tag{148}$$

This vanishes in a space with homogeneous curvature, $\nabla R = 0$.

The time derivatives of (138), (144), and (145) do not yield new algebraic equations. Due to (144), we can replace constraint (138) on $P^2 + \kappa R_{\alpha\beta\mu\nu}\omega^\alpha\pi^\beta\omega^\mu\pi^\nu + (mc)^2 = 0$. The obtained expressions for λ_3 and λ_4 can be used to exclude these variables from (140)–(143).

The constraints and equations of motion do not determine the functions λ_1 and λ_2; that is, the nonminimal interaction preserves both reparametrization and spin-plane symmetries of the theory. The presence of λ_1 and λ_2 in (142) and (143) implies that evolution of the basic variables is ambiguous, so they are not observable. To find the candidates for observables, we note once again that (142) and (143) imply an equation for $S^{\mu\nu}$ which does not contain λ_2. So, we rewrite (140) and (141) in terms of spin-tensor and add to them the equation for $S^{\mu\nu}$; this gives the system

$$\dot{x}^\mu = \lambda_1 \left[\mathcal{T}^\mu{}_\nu P^\nu + \kappa \frac{ac}{b} Z^\mu \right], \tag{149}$$

$$\nabla P_\mu = -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\lambda_1 \kappa}{32}\nabla_\mu (S\theta), \tag{150}$$

$$\nabla S^{\mu\nu} = -\frac{\kappa\lambda_1}{4}(\theta S)^{[\mu\nu]} + 2P^{[\mu}\dot{x}^{\nu]}. \tag{151}$$

Besides, constraints (138), (139), and (144) imply

$$P^2 + \frac{\kappa}{16}\theta S + (mc)^2 = 0, \tag{152}$$

$$\begin{aligned} S^{\mu\nu}P_\nu &= 0, \\ S^{\mu\nu}S_{\mu\nu} &= 8\alpha. \end{aligned} \tag{153}$$

Equations (153) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle. Equation (151), contrary to the equations for ω and π, does not depend on λ_2. This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ_1 is related to reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Equations (149)–(151), together with (152) and (153), form a closed system which determines evolution of the spinning particle.

The gravimagnetic moment κ can generally take any value. When $\kappa = 0$, we recover the MPTD equations.

Let us exclude P^μ and λ_1 from (150) and (151). Using (147) we solve (149) with respect to P^μ. Using the resulting expression in constraint (152), we obtain λ_1:

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{m_r c}, \\ \text{with } m_r^2 &\equiv m^2 + \frac{\kappa}{16c^2}(S\theta) - \kappa^2 Z^2, \end{aligned} \tag{154}$$

where the effective metric now is given by

$$\mathcal{G}_{\mu\nu} = \widetilde{\mathcal{T}}^\alpha{}_\mu g_{\alpha\beta} \widetilde{\mathcal{T}}^\beta{}_\nu. \tag{155}$$

Then the expression for momentum in terms of velocity implied by (149) is

$$P^\mu = \frac{m_r c}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \widetilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu - \kappa c Z^\mu. \tag{156}$$

We substitute this P^μ into (150) and (151):

$$\begin{aligned} \nabla \left[\frac{m_r}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \widetilde{\mathcal{T}}^\mu{}_\nu \dot{x}^\nu \right] &= -\frac{1}{4c}\theta^\mu{}_\nu \dot{x}^\nu \\ &\quad - \kappa \frac{\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{32 m_r c^2}\nabla^\mu (S\theta) \\ &\quad + \kappa \nabla Z^\mu, \\ \nabla S^{\mu\nu} &= -\frac{\kappa\sqrt{-\dot{x}\mathcal{G}\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} \\ &\quad - \frac{2m_r c(\kappa - 1) b}{\sqrt{-\dot{x}\mathcal{G}\dot{x}}} \dot{x}^{[\mu}(S\theta\dot{x})^{\nu]} \\ &\quad + 2\kappa c \dot{x}^{[\mu} Z^{\nu]}. \end{aligned} \tag{157}$$

Together with (153), this gives us the Lagrangian equations for the spinning particle with gravimagnetic moment. Comparing our equations to those of spinning particle on electromagnetic background (see (226)–(228)), we see that the two systems have the same structure after the identification $\kappa \sim \mu$ and $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$, where μ is the magnetic moment. That is, a curvature influences trajectory of spinning particle in the same way as the electromagnetic field with the strength $\theta_{\mu\nu}$.

5.2. Ultrarelativistic Limit Requires the Value of Gravimagnetic Moment $\kappa = 1$. In the previous subsection, we have formulated Hamiltonian variational problem for spinning particle with gravimagnetic moment κ in an arbitrary gravitational background. The model is consistent for any value of κ. When $\kappa = 0$, our equations of motion (149)–(151) coincide with MPTD equations (110). As we have shown above, they have unsatisfactory behavior in ultrarelativistic limit. Consider now our spinning particle with gravimagnetic moment $\kappa = 1$. This implies $\widetilde{\mathcal{T}}^\mu{}_\nu = \delta^\mu{}_\nu$ and $\mathcal{G}_{\mu\nu} = g_{\mu\nu}$ and crucially simplifies the equations of motion. (Besides $S^{\mu\nu}P_\nu = 0$, there are other supplementary spin conditions [3–7]. In this respect, we point out that the MPTD theory implies this condition with certain P_ν written in (110). Introducing κ, we effectively changed P_ν and hence changed the supplementary spin condition. For instance, when $\kappa = 1$ and in the space with $\nabla R = 0$, we have $P^\mu = (\widetilde{m}c/\sqrt{-\dot{x}g\dot{x}})\dot{x}^\mu$ instead of (110).) Hamiltonian equations (149)–(151) read

$$\begin{aligned} \frac{m_r c}{\sqrt{-\dot{x}g\dot{x}}}\dot{x}^\mu &= P^\mu + cZ^\mu, \\ \nabla P_\mu &= -\frac{1}{4}\theta_{\mu\nu}\dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32 m_r c}\nabla_\mu (S\theta), \\ \nabla S^{\mu\nu} &= -\frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} + 2P^{[\mu}\dot{x}^{\nu]}, \end{aligned} \tag{158}$$

while the Lagrangian equations are composed now of the equation for trajectory

$$\nabla \left[\frac{m_r \dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}} \right] = -\frac{1}{4c}\theta^\mu{}_\nu \dot{x}^\nu - \frac{\sqrt{-\dot{x}g\dot{x}}}{32 m_r c^2}\nabla^\mu (S\theta) + \nabla Z^\mu \tag{159}$$

and by the equation for precession of spin-tensor:

$$\nabla S^{\mu\nu} = -\frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}(\theta S)^{[\mu\nu]} + 2c\dot{x}^{[\mu}Z^{\nu]}. \tag{160}$$

These equations can be compared with (92) and (93). In the modified theory, we have the following:

(1) Time interval and distance should be unambiguously defined within the original space-time metric $g_{\mu\nu}$. So, the critical speed is equal to the speed of light.

(2) Covariant precession of spin (160) has a smooth behavior; in particular, for homogeneous field, $\nabla R = 0$, we have $\nabla S \sim \sqrt{-\dot{x}g\dot{x}}$ contrary to $\nabla S \sim 1/\sqrt{-\dot{x}g\dot{x}}$ in (93).

(3) Even in homogeneous field we have modified dynamics for both x and S. Equation (159) in the space with homogeneous curvature has the structure similar to (1); hence, we expect that longitudinal acceleration vanishes as $v \rightarrow c$. Let us confirm this by direct computations.

To find the acceleration, we separate derivative of the radiation mass m_r and write (159) in the form

$$\frac{d}{d\tau}\left[\frac{\dot{x}^\mu}{\sqrt{-\dot{x}g\dot{x}}}\right] = \frac{f^\mu}{\sqrt{-\dot{x}g\dot{x}}}, \tag{161}$$

where the force is

$$\begin{aligned} f^\mu \equiv& -\Gamma^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta - \frac{\sqrt{-\dot{x}g\dot{x}}}{4m_r c}\theta^\mu{}_\nu\dot{x}^\nu + \frac{\dot{x}g\dot{x}}{32m_r^2c^2}\nabla^\mu(S\theta) \\ &+ \frac{\sqrt{-\dot{x}g\dot{x}}}{m_r}\nabla Z^\mu - \dot{x}^\mu\frac{\dot{m}_r}{m_r}. \end{aligned} \tag{162}$$

While this expression contains derivatives of spin due to $\dot{m}_r$-term, the resulting expression is nonsingular function of velocity because ∇S is a smooth function. Hence, contrary to (123), the force now is nonsingular function of velocity. We take $\tau = x^0$ in the spacial part of system (161); this gives

$$\left(\frac{dt}{dx^0}\right)^{-1}\frac{d}{dx^0}\left[\frac{v^i}{\sqrt{c^2-(\mathbf{v}\gamma\mathbf{v})}}\right] = \frac{f^i(v)}{\sqrt{c^2-(\mathbf{v}\gamma\mathbf{v})}}, \tag{163}$$

where $f^i(v)$ is obtained from (162) replacing $\dot{x}^\mu$ by v^μ of (10). This system is of the form in (29), so the acceleration is given by (38) and (39):

$$\begin{aligned} a^i =& \widetilde{M}^i{}_j\left[f^j + \widetilde{\Gamma}^j{}_{kl}(\gamma)v^kv^l\right] + \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1} \\ &\cdot\left[\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i - \frac{(\mathbf{v}\partial_0\gamma\mathbf{v})}{c^2}v^i\right], \end{aligned} \tag{164}$$

$$\begin{aligned} \mathbf{v}\gamma\mathbf{a} =& \left(1-\frac{\mathbf{v}\gamma\mathbf{v}}{c^2}\right)\left[(\mathbf{v}\gamma)_i\left[f^i(v) + \widetilde{\Gamma}^i{}_{kl}(\gamma)v^kv^l\right]\right. \\ &\left.+\frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}(\mathbf{v}\partial_0\gamma\mathbf{v})\right]. \end{aligned} \tag{165}$$

With the smooth f^i given in (162), and as $v \rightarrow c$, acceleration (164) remains finite while longitudinal acceleration (165) vanishes. Due to identity (36), we have $(\mathbf{v}\gamma)_i f^i \xrightarrow{v\rightarrow c} -(\mathbf{v}\gamma)_i\Gamma^i{}_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta$; that is, the trajectory tends to that of spinless particle in the limit.

In resume, contrary to MPTD equations, the modified theory is consistent with respect to the original metric $g_{\mu\nu}$. Hence, the modified equations could be more promising for description of the rotating objects in astrophysics. It would be interesting to see if the nonminimal interaction allows removing the space-like–time-like transitions observed for spinning particle in the Schwarzschild background [24].

We should note that MPTD equations follow from a particular form assumed for the multipole representation of a rotating body [1]. It would be interesting to find a set of multipoles which yields modified equations (157).

5.3. Lagrangian of Spinning Particle with Gravimagnetic Moment. We look for the Lagrangian which in the phase space implies variational problem (137). First, we note that the constraints $\omega\pi = P\omega = 0$ always appear from the Lagrangian which depends on $N\dot{x}$ and $N\dot{\omega}$. So, we omit the corresponding terms in (137). Second, we present the remaining terms in (137) in the form

$$\begin{aligned} S_\kappa =& \int d\tau p_\mu\dot{x}^\mu + \pi_\mu\dot{\omega}^\mu - \frac{\lambda_1}{2}(P,\pi)\begin{pmatrix} g & \lambda g \\ \lambda g & \sigma\end{pmatrix}\begin{pmatrix} P \\ \pi\end{pmatrix} \\ &- \frac{\lambda_1}{2}\left[(mc)^2 - \frac{\alpha}{\omega^2}\right], \end{aligned} \tag{166}$$

where we have introduced the symmetric matrix

$$\begin{aligned} \sigma^{\mu\nu} = g^{\mu\nu} + \kappa R_\alpha{}^\mu{}_\beta{}^\nu\omega^\alpha\omega^\beta, \\ \text{then } \sigma^{\mu\nu}\omega_\nu = \omega^\mu. \end{aligned} \tag{167}$$

The matrix which appeared in (166) is invertible; the inverse matrix is

$$\begin{pmatrix} K\sigma & -\lambda K \\ -\lambda K & K\end{pmatrix}, \quad \text{where } K = \left(\sigma - \lambda^2 g\right)^{-1}. \tag{168}$$

When $\kappa = 0$, we have $K^{\mu\nu} = (1-\lambda^2)^{-1}g^{\mu\nu}$, and (168) coincides with the matrix which appeared in (53). Third, we note that the Hamiltonian variational problem of the form $p\dot{q} - (\lambda_1/2)pAp$ follows from the reparametrization-invariant Lagrangian $\sqrt{\dot{q}A^{-1}\dot{q}}$. So, we tentatively replace the matrix which appeared in free Lagrangian (53) by (168) and switch on the minimal interaction of spin with gravity, $\dot{\omega} \rightarrow \nabla\omega$. This gives the following Lagrangian formulation of spinning particle with gravimagnetic moment:

$$L = -\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}}\sqrt{-(N\dot{x}, N\nabla\omega)\begin{pmatrix} K\sigma & -\lambda K \\ -\lambda K & K\end{pmatrix}\begin{pmatrix} N\dot{x} \\ N\nabla\omega\end{pmatrix}} = \tag{169}$$

$$-\sqrt{(mc)^2 - \frac{\alpha}{\omega^2}}\sqrt{-\dot{x}NK\sigma N\dot{x} - \nabla\omega NKN\nabla\omega + 2\lambda\dot{x}NKN\nabla\omega}. \tag{170}$$

Let us show that this leads to the desired Hamiltonian formulation (137). The matrixes σ, K, and N are symmetric and mutually commuting. Canonical momentum for λ vanishes and hence represents the primary constraint, $p_\lambda = 0$. In terms of the canonical momentum $P_\mu \equiv p_\mu - \Gamma^\beta_{\alpha\mu}\omega^\alpha\pi_\beta$, the expressions for conjugate momenta $p_\mu = \partial L/\partial\dot{x}^\mu$ and $\pi_\mu = \partial L/\partial\dot{\omega}^\mu$ read

$$\begin{aligned}P_\mu &= \frac{1}{L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2} \\ &\quad\cdot\left[(\dot{x}NK\sigma N)_\mu - \lambda(\nabla\omega NKN)_\mu\right], \\ \pi_\mu &= \frac{1}{\sqrt{2}L_0}\left[m^2c^2 - \frac{\alpha}{\omega^2}\right]^{1/2} \\ &\quad\cdot\left[(\nabla\omega NKN)_\mu - \lambda(\dot{x}NKN)_\mu\right],\end{aligned} \tag{171}$$

where L_0 is the second square root in (170). They immediately imply the primary constraints $\omega\pi = 0$ and $P\omega = 0$. From the expressions

$$\begin{aligned}P^2 &= \frac{1}{L_0^2}\left[(mc)^2 - \frac{\alpha}{\omega^2}\right]\left[(\dot{x}NK\sigma K\sigma N\dot{x})\right. \\ &\quad\left.+\lambda^2(\nabla\omega NKKN\nabla\omega) - 2\lambda(\dot{x}NK\sigma KN\nabla\omega)\right], \\ \pi\sigma\pi &= \frac{1}{L_0^2}\left[(mc)^2 - \frac{\alpha}{\omega^2}\right]\left[\lambda^2(\dot{x}NK\sigma KN\dot{x})\right. \\ &\quad\left.+(\nabla\omega NK\sigma KN\nabla\omega) - 2\lambda(\dot{x}NK\sigma KN\nabla\omega)\right], \\ 2\lambda P\pi &= \frac{1}{L_0^2}\left[(mc)^2 - \frac{\alpha}{\omega^2}\right]\left[-2\lambda^2(\dot{x}NK\sigma KN\dot{x})\right. \\ &\quad - 2\lambda^2(\nabla\omega NKKN\nabla\omega) + 2\lambda(\dot{x}NK\sigma KN\nabla\omega) \\ &\quad\left.+2\lambda^3(\dot{x}NKKN\nabla\omega)\right],\end{aligned} \tag{172}$$

we conclude that their sum does not depend on velocities and hence gives one more constraint:

$$P^2 + \pi\sigma\pi + 2\lambda P\pi = -\left[(mc)^2 - \frac{\alpha}{\omega^2}\right]. \tag{173}$$

Then, the Hamiltonian is $H = P\dot{x} + \pi\nabla\omega - L + \lambda_i T_i$. From (171), we obtain $P\dot{x} + \pi\nabla\omega = L$, so the Hamiltonian is composed of primary constraints:

$$\begin{aligned}H &= \frac{\lambda_1}{2}\left[P^2 + \kappa R_{\alpha\mu\beta\nu}\omega^\alpha\pi^\mu\omega^\beta\pi^\nu + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2}\right. \\ &\quad\left.+2\lambda(P\pi)\right] + \lambda_2(\omega\pi) + \lambda_3(P\omega) + \nu p_\lambda.\end{aligned} \tag{174}$$

This expression is equivalent to the Hamiltonian written in variational problem (137). Problems (137) and (174) yield the same equations for the set x^μ, P^μ, and $S^{\mu\nu}$.

6. Spinless Particle Nonminimally Interacting with Electromagnetic Field, Speed of Light, and Critical Speed

In this section, we consider examples of manifestly Poincare-covariant and reparametrization-invariant equations of spinless particle in Minkowski space which lead to critical speed different from the speed of light. We achieve this assuming a nonminimal interaction with electromagnetic field. These toy models confirm that the critical speed different from the speed of light does not contradict relativistic invariance (existence of the observer-independent scale c).

Let us denote $x^i(t)$, $i = 1, 2, 3$, physical dynamical variables describing trajectory of relativistic particle subject to an external force. In order to work with manifestly covariant quantities, we use parametric equations of the trajectory $x^\mu(\tau) = (x^0 \equiv ct(\tau),\ x^i(\tau))$, where τ is an arbitrary parameter along the world line. In this section, we use the usual special-relativity notions for time, three-dimensional distance, velocity, and acceleration as well as for the scalar product:

$$\begin{aligned}v^i &= \frac{dx^i}{dt}, \\ a^i &= \frac{dv^i}{dt}, \\ \mathbf{va} &= v^i a^i.\end{aligned} \tag{175}$$

Let us start from the standard Lagrangian of spinless particle in electromagnetic field. Using the auxiliary variable λ, the Lagrangian reads

$$L = \frac{1}{2\lambda}\dot{x}^2 - \frac{\lambda}{2}m^2c^2 + \frac{e}{c}A\dot{x}. \tag{176}$$

This implies the manifestly relativistic equations.

$$\left(\frac{\dot{x}^i}{\sqrt{-\dot{x}^2}}\right)^{\cdot} = \frac{e}{mc^2}F^\mu{}_\nu\dot{x}^\nu. \tag{177}$$

They became singular as $\dot{x}^2 \to 0$. Using the reparametrization-invariant derivative $D = (1/\sqrt{-\dot{x}^2})(d/d\tau)$, they read in manifestly reparametrization-invariant form

$$DDx^\mu = f^\mu \equiv \frac{e}{mc^2}F^\mu{}_\nu Dx^\nu. \tag{178}$$

Due to the identities

$$\begin{aligned}\dot{x}_\mu DDx^\mu &= 0, \\ \dot{x}_\mu f^\mu &= 0,\end{aligned} \tag{179}$$

the system contains only three independent equations. The first identity became more transparent if we compute derivative on the left hand side of (177); then the system reads

$$M^\mu{}_\nu\ddot{x}^\nu = \frac{e\sqrt{-\dot{x}^2}}{mc^2}F^\mu{}_\nu\dot{x}^\nu, \tag{180}$$

where $M^\mu{}_\nu$ turns out to be projector on the plane orthogonal to $\dot{x}^\mu$:

$$M^\mu{}_\nu = \delta^\mu{}_\nu - \frac{\dot{x}^\mu \dot{x}_\nu}{\dot{x}^2}, \qquad \dot{x}_\mu M^\mu{}_\nu = 0. \tag{181}$$

Using reparametrization invariance, we can take physical time as the parameter, $\tau = t$; this directly yields equations for observable dynamical variables $x^i(t)$. In the physical-time parametrization, we have $x^\mu = (ct, \mathbf{x}(t))$, $\dot{x}^\mu = (c, \mathbf{v}(t))$, and $1/\sqrt{-\dot{x}^2} = 1/\sqrt{c^2 - \mathbf{v}^2}$. Time-like component of system (177) reads

$$\frac{d}{dt}\frac{c}{\sqrt{c^2 - \mathbf{v}^2}} = \frac{e}{mc^2} F^0{}_i \dot{x}^i \tag{182}$$

and gives the value of acceleration along the direction of velocity:

$$\mathbf{va} = \frac{e\left(c^2 - \mathbf{v}^2\right)^{3/2}}{mc^3} (\mathbf{Ev}). \tag{183}$$

The longitudinal acceleration vanishes as $|\mathbf{v}| \to c$. Hence, the singularity in (177) implies that the particles speed can not exceed the value c.

Components of three-acceleration vector can be obtained from the spatial part of system (177):

$$\frac{\ddot{x}^i}{\sqrt{c^2 - \mathbf{v}^2}} + \dot{x}^i \frac{d}{dt}\frac{1}{\sqrt{c^2 - \mathbf{v}^2}} = \frac{e}{mc^2} F^i{}_\nu \dot{x}^\nu. \tag{184}$$

Using (182), we obtain

$$\mathbf{a} = \frac{\sqrt{c^2 - \mathbf{v}^2}}{mc}\left[e\mathbf{E} - \frac{e(\mathbf{Ev})}{c^2}\mathbf{v} + \frac{e}{c}\mathbf{v} \times \mathbf{B}\right]. \tag{185}$$

In accordance with degeneracy (179), scalar product of the spacial part with $\mathbf{v}$ gives time-like component (183).

Let us discuss two modifications which preserve both relativistic covariance and reparametrization invariance of equations of motion but could yield nonvanishing longitudinal acceleration as $|\mathbf{v}| \to c$.

First, in the presence of external fields we can construct an additional reparametrization invariant. For instance, we can use the derivative

$$D' \equiv \frac{1}{\sqrt{-\dot{x}g\dot{x}}}\frac{d}{d\tau}, \tag{186}$$

where the usual relativistic factor $\dot{x}\eta\dot{x}$ is replaced by

$$\begin{aligned} -\dot{x}^\mu g_{\mu\nu}\dot{x}^\nu &= -\dot{x}^2 - \epsilon k (\dot{x}FF\dot{x}) \\ &= c^2 - \mathbf{v}^2 - \epsilon k (\dot{x}FF\dot{x}), \quad \epsilon = \pm 1. \end{aligned} \tag{187}$$

The right dimension can be supplied by the constant k equal to e^6/m^4c^8 or $\hbar^3/m^4c^5$. Second, we consider nonminimal interaction with the force constructed from reparametrization-invariant quantities, $f^\mu(D'x, F, \partial F, \ldots) = f^\mu{}_0 + f^\mu{}_\nu D'x^\nu + \cdots$. Hence, let us consider the manifestly covariant and reparametrization-invariant equations

$$D'D'x^\mu = f^\mu, \qquad \text{or } M^\mu{}_\nu \ddot{x}^\nu = -(\dot{x}g\dot{x}) f^\mu + \frac{(\dot{x}\dot{g}\dot{x})}{2(\dot{x}g\dot{x})}\dot{x}^\mu, \tag{188}$$

where

$$M^\mu{}_\nu = \delta^\mu{}_\nu - \frac{\dot{x}^\mu (\dot{x}g)_\nu}{(\dot{x}g\dot{x})}, \qquad (\dot{x}g)_\mu M^\mu{}_\nu = 0, \qquad M^\mu{}_\nu \dot{x}^\nu = 0. \tag{189}$$

Due to noninvertibility of $M^\mu{}_\nu$, system (188) consists of three second-order equations and one equation of first order. Contracting (188) with $(\dot{x}g)_\mu$, we obtain the first-order equation

$$\dot{x}\dot{g}\dot{x} = 2(\dot{x}g\dot{x})(\dot{x}gf). \tag{190}$$

This is analog of $\dot{x}_\mu f^\mu = 0$ of the previous case in (179) and can be used to present f^0 through three-dimensional force. Using reparametrization invariance, we take $\tau = x^0$ and write (188) in the form

$$\frac{d}{dx^0}\frac{1}{\sqrt{-\dot{x}g\dot{x}}} = \sqrt{-\dot{x}g\dot{x}} f^0, \tag{191}$$

$$\frac{\ddot{x}^i}{\sqrt{-\dot{x}g\dot{x}}} + \dot{x}^i \frac{d}{dx^0}\frac{1}{\sqrt{-\dot{x}g\dot{x}}} = \sqrt{-\dot{x}g\dot{x}} f^i. \tag{192}$$

Using (191) in (192), we obtain

$$\frac{d^2x^i}{dx^{02}} = -\dot{x}g\dot{x}\left[f^i - f^0\frac{dx^i}{dx^0}\right], \qquad \dot{x}^\mu = \frac{dx^\mu}{dx^0}. \tag{193}$$

We take $x^0 = ct$ and then three-acceleration

$$\mathbf{a} = -\dot{x}g\dot{x}\left[\mathbf{f} - \frac{f^0}{c}\mathbf{v}\right], \qquad \dot{x}^\mu = \left(c, \mathbf{v} = \frac{d\mathbf{x}}{dt}\right). \tag{194}$$

Equations (190) and (194) are equivalent to initial system (188). Equation (194) implies the longitudinal acceleration:

$$\begin{aligned} \mathbf{va} &= -\dot{x}g\dot{x}\left[\mathbf{vf} - \frac{\mathbf{v}^2}{c}f^0\right] \\ &\equiv -\dot{x}g\dot{x}\left[\frac{c^2 - \mathbf{v}^2}{c}f^0 + \dot{x}_\mu f^\mu\right], \\ f^\mu &= f^\mu\left(\frac{\dot{x}}{\sqrt{-\dot{x}g\dot{x}}}, F, \partial F, \ldots\right). \end{aligned} \tag{195}$$

The acceleration vanishes at the values of speed which zero out r.h.s. of this equation. If in physical-time parametrization the four-force obeys the identity $\dot{x}_\mu f^\mu = 0$, we have two special points, $|\mathbf{v}| = c$ and v', determined from $\dot{x}g\dot{x} = 0$. In absence of the identity, and if $\dot{x}_\mu f^\mu \neq 0$ as $|\mathbf{v}| \to c$, the speed of light does not represent a special point of (195). Let us illustrate this equation with two examples.

Example 1. Consider the minimal interaction $f^\mu = (e/mc^2)F^\mu{}_\nu D'x^\nu$ and relativistic-contraction factor (187); then (195) reads

$$\mathbf{va} = \frac{e(\mathbf{vE})}{mc^3}\sqrt{-\dot{x}g\dot{x}}\left(c^2 - \mathbf{v}^2\right). \tag{196}$$

Besides the usual special point, $\mathbf{v}^2 = c^2$, there is one more, say $v' = |\mathbf{v}'|$, determined by $\dot{x}g\dot{x} = 0$. This surface is slightly different from the sphere $c^2 - \mathbf{v}^2 = 0$. So, the second special point generally differs from the speed of light. To see this, we compute the last term in (187):

$$-\dot{x}FF\dot{x} = c^2 E_i\left(\delta_{ij} - \frac{v_i v_j}{c^2}\right)E_j + \mathbf{v}^2 B_i N_{ij} B_j. \tag{197}$$

Here, $N_{ij} \equiv \delta_{ij} - v_i v_j/\mathbf{v}^2$ is projection operator on the plane orthogonal to the vector $\mathbf{v}$, so we can write $\mathbf{B}N\mathbf{B} = (N\mathbf{B})^2 = \mathbf{B}_\perp^2$. Then, factor (187) reads

$$-\dot{x}g\dot{x} = c^2 - \mathbf{v}^2 + \epsilon k\left[c^2\mathbf{E}\left(1 - \frac{\mathbf{vv}}{c^2}\right)\mathbf{E} + \mathbf{v}^2\mathbf{B}_\perp^2\right]. \tag{198}$$

The quantity $\delta_{ij} - v_i v_j/c^2$ turns into the projection operator N when $|\mathbf{v}| = c$. Hence,

$$-\dot{x}g\dot{x} \xrightarrow{|\mathbf{v}|\to c} \epsilon k c^2\left[\mathbf{E}_\perp^2 + \mathbf{B}_\perp^2\right]. \tag{199}$$

If $\mathbf{E}$ and $\mathbf{B}$ are not mutually parallel in the laboratory system, this expression does not vanish for any orientation of $\mathbf{v}$. This implies that factor (187) does not vanish at $|\mathbf{v}| = c$.

We confirmed that longitudinal acceleration generally vanishes at two different values of speed, c and v'. Then, (196) implies the following possibilities.

(A) Let $\epsilon = +1$; then from (198) we conclude $c < v'$, and speed of the particle approximates to c. The second special point v' turns out to be irrelevant. So, $v_{cr} = c$.

(B) Let $\epsilon = -1$; then $v' < c$, and the particle with small initial velocity will approximate to the critical velocity $v_{cr} = v' < c$. So, it never approximates to the speed of light.

Example 2 (possibility of superluminal motion?). Take relativistic-contraction factor (187) with $\epsilon = +1$ and nonparallel electric and magnetic fields. As we have seen above, this implies $c < v'$, where v' is a solution of $\dot{x}g\dot{x} = 0$. Consider the nonminimal interaction

$$f^\mu = \frac{e}{mc^2}F^\mu{}_\nu D'x^\nu + \tilde{f}^\mu,$$
$$\text{where } \tilde{f}^\mu = -\tilde{k}^2 D'x^\alpha\partial^\mu(FF)_{\alpha\beta}D'x^\beta. \tag{200}$$

We assume homogeneous and nonstationary fields with growing tension:

$$\partial^i\mathbf{E} = \partial^i\mathbf{B} = 0,$$
$$\frac{d}{dt}|\mathbf{E}| > 0, \tag{201}$$
$$\frac{d}{dt}|\mathbf{B}| > 0;$$

then $\tilde{f}^i = 0$, $\tilde{f}^0 = -(\tilde{k}^2/c\dot{x}g\dot{x})\dot{x}(\partial/\partial t)(FF)\dot{x}$. The longitudinal acceleration reads

$$\mathbf{va} = a_1(v) + a_2(v)$$
$$\equiv \frac{e(\mathbf{vE})}{mc^3}\sqrt{-\dot{x}g\dot{x}}\left(c^2 - \mathbf{v}^2\right) - \frac{\tilde{k}^2\mathbf{v}^2}{c^2}\dot{x}\frac{\partial}{\partial t}(FF)\dot{x}. \tag{202}$$

We have $a_1(c) = 0$, while $a_2(c)$ is positive according to (197) and (201). So, the particle overcomes the light barrier. In the region $c < v < v'$, we have $a_1(v) < 0$ and $a_2(v) > 0$, so the particle will continue to accelerate up to critical velocity v_{cr} determined by the equation $a_1 + a_2 = 0$. If $a_2 > |a_1|$ in the region, the particle will accelerate up to the value $v_{cr} = v'$. Above this velocity, (202) becomes meaningless.

The toy examples show that critical speed in a manifestly relativistic and reparametrization-invariant theory does not always coincide with the speed of light, if we assume the usual special-relativity definitions of time and distance. In general case, we expect that $\mathbf{v}_{cr}$ is both field- and spin-dependent quantity. In the next section, we repeat this analysis for more realistic case of a particle with spin.

7. Spinning Particle in an Arbitrary Electromagnetic Field

7.1. Lagrangian and Hamiltonian Formulations. In formulation (51), the vector model of spin admits interaction with an arbitrary electromagnetic field. To introduce coupling of the position variable x with electromagnetic field, we add the minimal interaction $(e/c)A_\mu\dot{x}^\mu$. As for spin, it couples with A^μ through the term

$$D\omega^\mu \equiv \dot{\omega}^\mu - \lambda\frac{e\mu}{c}F^{\mu\nu}\omega_\nu, \tag{203}$$

where the coupling constant μ is the magnetic moment. They are the only terms we have found compatible with symmetries and constraints which should be presented in the theory. Adding these terms to the free theory in (51), we obtain the action

$$S = \int d\tau\frac{1}{4\lambda}\left[\dot{x}N\dot{x} + D\omega N D\omega\right.$$
$$\left. - \sqrt{[\dot{x}N\dot{x} + D\omega ND\omega]^2 - 4(\dot{x}ND\omega)^2}\right] \tag{204}$$
$$- \frac{\lambda}{2}\left(m^2c^2 - \frac{\alpha}{\omega^2}\right) + \frac{e}{c}A\dot{x}.$$

In their work [22], Hanson and Regge analyzed whether the spin-tensor in (50) interacts directly with an electromagnetic field and concluded on impossibility to construct the interaction in closed form. In our model, an electromagnetic field interacts with the part ω^μ of the spin-tensor.

Let us construct Hamiltonian formulation of the model. The procedure which leads to the Hamiltonian turns out to be very similar to that described in Section 4.1, so we present only the final expression (for details, see [32]). Conjugate momenta for x^μ, ω^μ, and λ are denoted as p^μ, π^μ, and p_λ. We use also the canonical momentum $\mathscr{P}^\mu \equiv p^\mu - (e/c)A^\mu$. Contrary to p^μ, the canonical momentum $\mathscr{P}^\mu$ is $U(1)$ gauge-invariant quantity. With these notations, we obtain the Hamiltonian variational problem which is equivalent to (204):

$$\begin{aligned} S_H = \int d\tau p_\mu \dot{x}^\mu + \pi_\mu \dot{\omega}^\mu + p_\lambda \dot{\lambda} \\ - \left[\frac{\lambda}{2} \left(P^2 - \frac{e\mu}{2c} (FS) + (mc)^2 + \pi^2 - \frac{\alpha}{\omega^2} \right) \right. \\ \left. + \lambda_2 (\omega\pi) + \lambda_3 (P\omega) + \lambda_4 (P\pi) + \lambda_0 p_\lambda \right]. \end{aligned} \tag{205}$$

The expression in square brackets represents the Hamiltonian. By $\lambda_2, \lambda_3, \lambda_4$, and λ_0, we denoted the Lagrangian multipliers; they are written in front of the corresponding primary constraints. The fundamental Poisson brackets $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$ and $\{\omega^\mu, \pi^\nu\} = \eta^{\mu\nu}$ imply

$$\begin{aligned} \{x^\mu, \mathscr{P}^\nu\} = \eta^{\mu\nu}, \\ \{\mathscr{P}^\mu, \mathscr{P}^\nu\} = \frac{e}{c} F^{\mu\nu}, \end{aligned} \tag{206}$$

$$\{S^{\mu\nu}, S^{\alpha\beta}\} = 2\left(\eta^{\mu\alpha} S^{\nu\beta} - \eta^{\mu\beta} S^{\nu\alpha} - \eta^{\nu\alpha} S^{\mu\beta} + \eta^{\nu\beta} S^{\mu\alpha}\right). \tag{207}$$

According to (207), the spin-tensor is generator of Lorentz algebra $SO(1,3)$. As $\omega\pi$, ω^2, and π^2 are Lorentz-invariants, they have vanishing Poisson brackets with $S^{\mu\nu}$. To reveal the higher-stage constraints, we write the equations $\dot{T}_i = \{T_i, H\} = 0$. The Dirac procedure stops on third stage with the following equations:

$$\begin{aligned} p_\lambda = 0 \Longrightarrow \\ T_1 \equiv \mathscr{P}^2 - \frac{e\mu}{2c} (FS) + m^2 c^2 + \pi^2 - \frac{\alpha}{\omega^2} = 0 \Longrightarrow \\ \lambda_3 C + \lambda_4 D = 0, \end{aligned} \tag{208}$$

$$T_2 \equiv (\omega\pi) = 0 \Longrightarrow T_5 \equiv \pi^2 - \frac{\alpha}{\omega^2} = 0, \tag{209}$$

$$T_3 \equiv (\mathscr{P}\omega) = 0 \Longrightarrow \lambda_4 = -\frac{2\lambda c}{e} aC, \tag{210}$$

$$T_4 \equiv (\mathscr{P}\pi) = 0 \Longrightarrow \lambda_3 = \frac{2\lambda c}{e} aD. \tag{211}$$

We have denoted

$$\begin{aligned} C = -\frac{e(\mu-1)}{c} (\omega F \mathscr{P}) + \frac{e\mu}{4c} (\omega\partial)(FS), \\ D = -\frac{e(\mu-1)}{c} (\pi F \mathscr{P}) + \frac{e\mu}{4c} (\pi\partial)(FS). \end{aligned} \tag{212}$$

Besides, here and below we will use the following notation. In the equation which relates velocity and canonical momentum will appear the matrix $T^{\mu\nu}$:

$$\begin{aligned} T^{\mu\nu} = \eta^{\mu\nu} - (\mu-1) a (SF)^{\mu\nu}, \\ a = \frac{-2e}{4m^2c^3 - e(2\mu+1)(SF)}. \end{aligned} \tag{213}$$

Using the identity $S^{\mu\alpha} F_{\alpha\beta} S^{\beta\nu} = -(1/2)(S^{\alpha\beta} F_{\alpha\beta}) S^{\mu\nu}$, we find the inverse matrix

$$\begin{aligned} \tilde{\mathscr{T}}^{\mu\nu} = \eta^{\mu\nu} + (\mu-1) b (SF)^{\mu\nu}, \\ b = \frac{2a}{2 + (\mu-1) a (SF)} \equiv \frac{-2e}{4m^2c^3 - 3e\mu (SF)}. \end{aligned} \tag{214}$$

The vector Z^μ is defined by

$$Z^\mu = \frac{b}{4c} S^{\mu\sigma} \left(\partial_\sigma F_{\alpha\beta}\right) S^{\alpha\beta} \equiv \frac{b}{4c} S^{\mu\sigma} \partial_\sigma (FS). \tag{215}$$

This vanishes for homogeneous field; $\partial F = 0$. The last equation from (208) turns out to be a consequence of (210) and (211) and can be omitted. Due to the secondary constraint, T_5 appeared in (209); we can replace the constraint T_1 on the equivalent one:

$$T_1 \equiv \mathscr{P}^2 - \frac{e\mu}{2c} (FS) + m^2 c^2 = 0. \tag{216}$$

The Dirac procedure revealed two secondary constraints written in (216) and (209) and fixed the Lagrangian multipliers λ_3 and λ_4; the latter can be substituted into the Hamiltonian. The multipliers λ_0 and λ_2 and the auxiliary variable λ have not been determined. H vanishes on the complete constraint surface, as it should be in a reparametrization-invariant theory.

We summarized the algebra of Poisson brackets between constraints in Table 2. The constraints p_λ, T_1, T_2, and T_5 form the first-class subset, while T_3 and T_4 represent a pair of second class. The presence of two primary first-class constraints p_λ and T_2 is in correspondence with the fact that two Lagrangian multipliers remain undetermined within the Dirac procedure.

The evolution of the basic variables was obtained according the standard rule $\dot{z} = \{z, H\}$ (equivalently, we can look for

Table 2: Algebra of constraints.

	T_1	T_5	T_2	T_3	T_4
$T_1=\mathscr{P}^2-\frac{\mu e}{2c}(FS)+m^2c^2$	0	0	0	$-2C$	$-2D$
$T_5=\pi^2-\frac{\alpha}{\omega^2}$	0	0	$-2T_5\approx 0$	$-2T_4\approx 0$	$\frac{2\alpha}{(\omega^2)^2}T_3\approx 0$
$T_2=\omega\pi$	0	$2T_5\approx 0$	0	$-T_3\approx 0$	$T_4\approx 0$
$T_3=\mathscr{P}\omega$	$2C$	$2T_4\approx 0$	$T_3\approx 0$	0	$T_1+\frac{e}{2ca}\approx\frac{e}{2ca}$
$T_4=\mathscr{P}\pi$	$2D$	$-\frac{2\alpha}{(\omega^2)^2}T_3\approx 0$	$-T_4\approx 0$	$-T_1-\frac{e}{2ca}\approx-\frac{e}{2ca}$	0

the extremum of variational problem (205)). The equations read

$$\dot{x}^\mu=\lambda\left(T^\mu{}_\nu\mathscr{P}^\nu+\frac{\mu ca}{b}Z^\mu\right),$$
$$\dot{\mathscr{P}}^\mu=\frac{e}{c}(F\dot{x})^\mu+\lambda\frac{\mu e}{4c}\partial^\mu(FS), \tag{217}$$

$$\dot{\omega}^\mu=\lambda\frac{e\mu}{c}(F\omega)^\mu-\lambda\frac{2caC}{e}\mathscr{P}^\mu+\pi^\mu+\lambda_2\omega^\mu,$$
$$\dot{\pi}^\mu=\lambda\frac{e\mu}{c}(F\pi)^\mu-\lambda\frac{2caD}{e}\mathscr{P}^\mu-\frac{\alpha}{(\omega^2)^2}\omega^\mu-\lambda_5 2\pi^\mu. \tag{218}$$

The ambiguity due to the variables λ and λ_2 means that the interacting theory preserves both reparametrization and spin-plane symmetries of the free theory. As a consequence, all the basic variables have ambiguous evolution. x^μ and $\mathscr{P}^\mu$ have one-parametric ambiguity due to λ while ω and π have two-parametric ambiguity due to λ and λ_2. The variables with ambiguous dynamics do not represent observable quantities, so we look for the variables that can be candidates for observable quantities. We note that (218) implies an equation for $S^{\mu\nu}$ which does not contain λ_2:

$$\dot{S}^{\mu\nu}=\lambda\frac{e\mu}{c}(FS)^{[\mu\nu]}+2\mathscr{P}^{[\mu}\dot{x}^{\nu]}. \tag{219}$$

This proves that the spin-tensor is invariant under local spin-plane symmetry. The remaining ambiguity due to λ contained in (217) and (219) is related with reparametrization invariance and disappears when we work with physical dynamical variables $x^i(t)$. Thus, we will work with x^μ, $\mathscr{P}^\mu$, and $S^{\mu\nu}$.

The term $\alpha/2\omega^2$ in Lagrangian (204) provides the constraint T_5 which can be written as follows: $\omega^2\pi^2=\alpha$. Together with $\omega\pi=0$, this implies fixed value of spin:

$$S^{\mu\nu}S_{\mu\nu}=8\left(\omega^2\pi^2-(\omega\pi)^2\right)=8\alpha, \tag{220}$$

for any solution to the equations of motion. The constraints $\omega\mathscr{P}=\pi\mathscr{P}=0$ imply the Pirani condition for the spin-tensor in (50):

$$S^{\mu\nu}\mathscr{P}_\nu=0. \tag{221}$$

Equations (220) and (221) imply that only two components of spin-tensor are independent, as it should be for spin one-half particle.

Equations (217) and (219), together with (220) and (221), form a closed system which determines evolution of a spinning particle.

The quantities x^μ, P^μ, and $S^{\mu\nu}$, being invariant under spin-plane symmetry, have vanishing brackets with the corresponding first-class constraints T_2 and T_5. So, obtaining equations for these quantities, we can omit the corresponding terms in Hamiltonian (205). Further, we can construct the Dirac bracket for the second-class pair T_3 and T_4. Since the Dirac bracket of a second-class constraint with any quantity vanishes, we can now omit T_3 and T_4 from (205). Then, the relativistic Hamiltonian acquires an expected form:

$$H=\frac{\lambda}{2}\left(\mathscr{P}^2-\frac{e\mu}{2c}(FS)+m^2c^2\right). \tag{222}$$

Equations (217) and (219) follow from this H with use of Dirac bracket, $\dot{z}=\{z,H\}_{\text{DB}}$.

We can exclude the momenta $\mathscr{P}$ and the auxiliary variable λ from the equations of motion. This yields second-order equation for the particle's position. To achieve this, we solve the first equation from (217) with respect to $\mathscr{P}$ and use the identities $(SFZ)^\mu=-(1/2)(SF)Z^\mu$ and $\widetilde{\mathscr{T}}^\mu{}_\nu Z^\nu=(b/a)Z^\mu$; this gives $\mathscr{P}^\mu=(1/\lambda)\widetilde{\mathscr{T}}^\mu{}_\nu\dot{x}^\nu-\mu cZ^\mu$. Then, the Pirani condition reads $(1/\lambda)(S\widetilde{\mathscr{T}}\dot{x})^\mu=\mu c(SZ)^\mu$. Using this equality, $\mathscr{P}^2$ can be presented as $\mathscr{P}^2=(1/\lambda^2)(\dot{x}G\dot{x})+\mu^2c^2Z^2$, where the symmetric matrix appeared:

$$G_{\mu\nu}=\left(\widetilde{\mathscr{T}}^T\widetilde{\mathscr{T}}\right)_{\mu\nu}$$
$$=\left[\eta+b(\mu-1)(SF+FS)+b^2(\mu-1)^2FSSF\right]_{\mu\nu}. \tag{223}$$

The matrix G is composed of the Minkowsky metric $\eta_{\mu\nu}$ plus (spin- and field-dependent) contribution; $G_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}(S)$. So, we call G the effective metric produced along the world line by interaction of spin with electromagnetic field. We substitute $\mathscr{P}^2$ into constraint (216); this gives expression for λ:

$$\lambda=\frac{\sqrt{-\dot{x}G\dot{x}}}{m_rc},$$
$$m_r^2=m^2-\frac{\mu e}{2c^3}(FS)-\mu^2Z^2. \tag{224}$$

This shows that the presence of λ in (203) implies highly nonlinear interaction of spinning particle with electromagnetic field. The final expression of canonical momentum through velocity is

$$\mathcal{P}^{\mu} = \frac{m_r c}{\sqrt{-\dot{x}G\dot{x}}}\widetilde{\mathcal{T}}^{\mu}{}_{\nu}\dot{x}^{\nu} - \mu c Z^{\mu}. \quad (225)$$

Using (224) and (225), we exclude $\mathcal{P}^{\mu}$ and λ from Hamiltonian equations (217), (219), and (221). This gives closed system of Lagrangian equations for the set x, S. We have the dynamical equations

$$D\left[m_r\left(\widetilde{\mathcal{T}}Dx\right)^{\mu}\right] = \frac{e}{c^2}(FDx)^{\mu} + \frac{\mu e}{4m_r c^3}\partial^{\mu}(SF) + \mu DZ^{\mu}, \quad (226)$$

$$DS^{\mu\nu} = \frac{e\mu}{m_r c^2}(FS)^{[\mu\nu]} - 2bm_r c(\mu-1)Dx^{[\mu}(SFDx)^{\nu]} + 2\mu c Dx^{[\mu}Z^{\nu]}, \quad (227)$$

the Lagrangian counterpart of Pirani condition

$$S^{\mu\nu}\left[\left(\widetilde{\mathcal{T}}\dot{x}\right)_{\nu} - \frac{\mu\sqrt{-\dot{x}G\dot{x}}}{m_r}Z_{\nu}\right] = 0, \quad (228)$$

as well as to the value-of-spin condition; $S^{\mu\nu}S_{\mu\nu} = 8\alpha$. In the approximation $O^3(S, F, \partial F)$ and when $\mu = 1$, they coincide with Frenkel equations; see [31].

Equation (226) shows how spin modifies Lorentz-force equation (1). In general case, the Lorentz force is modified due to the presence of (time-dependent) radiation mass m_r (224), the tetrad field $\widetilde{\mathcal{T}}$, and the effective metric G and due to two extra terms on right hand side of (226). Contribution of anomalous magnetic moment $\mu \neq 1$ to the difference between $\dot{x}^{\mu}$ and $\mathcal{P}^{\mu}$ in (225) is proportional to $J/c^3 \sim \hbar/c^3$, while the term with a gradient of field is proportional to $J^2/c^3 \sim \hbar^2/c^3$.

Consider the homogeneous field;

$$\partial_{\alpha}F^{\mu\nu} = 0, \qquad Z^{\mu} = 0. \quad (229)$$

Then, contraction of (228) with $F_{\mu\nu}$ yields $(SF)^{\cdot} = 0$; that is, $S^{\mu\nu}F_{\mu\nu}$ turns out to be the conserved quantity. This implies $\dot{m}_r = \dot{a} = \dot{b} = 0$. Hence, the Lorentz force is modified due to the presence of time-independent radiation mass m_r, the tetrad field $\widetilde{\mathcal{T}}$, and the effective metric G.

Consider the "classical" value of magnetic moment $\mu = 1$. Then, $\widetilde{\mathcal{T}}^{\mu\nu} = \eta^{\mu\nu}$ and $G_{\mu\nu} = \eta_{\mu\nu}$. The Lorentz force is modified due to the presence of time-dependent radiation mass m_r and two extra terms on right hand side of (226).

Let us specify the equation for spin precession to the case of uniform and stationary field, supposing also $\mu = 1$ and taking physical time as the parameter; $\tau = t$. Then, (228) reduces to the Frenkel condition, $S^{\mu\nu}\dot{x}_{\nu} = 0$, while (227) reads $\dot{S}^{\mu\nu} = (e\sqrt{-\dot{x}^2}/m_r c^2)(FS)^{[\mu\nu]}$. We decompose spin-tensor on electric dipole moment $\vec{D}$ and Frenkel spin-vector $\vec{S}$ according to (50); then $\vec{D} = -(2/c)\vec{S}\times\vec{v}$, while precession of $\vec{S}$ is given by

$$\frac{d\vec{S}}{dt} = \frac{e\sqrt{c^2 - \vec{v}^2}}{m_r c^3}\left[-\vec{E}\times\left(\vec{v}\times\vec{S}\right) + c\vec{S}\times\vec{B}\right]. \quad (230)$$

7.2. Ultrarelativistic Limit within the Usual Special-Relativity Notions. After identification of $\theta_{\mu\nu} \equiv R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$ and $\kappa \sim \mu$, equations of motion in electromagnetic and in gravitational fields acquire the similar structure. Equations (217) and (219) can be compared with (149)–(151) and (226)-(227) with (157). In particular, in the Lagrangian equations with anomalous magnetic moment ($\mu \neq 1$) in Minkowski space also appeared effective metric (223). So, we need to examine the ultrarelativistic limit. In this section, we do this under the usual special-relativity notions; that is, we suppose that the particle probes three-dimensional geometry (175). We show that the critical speed turns out to be different from the speed of light while an acceleration, contrary to Section 4.4, vanishes in ultrarelativistic limit. It will be sufficient to estimate the acceleration in uniform and stationary field (229). We take $\tau = t$ in (226)–(228) and compute the time derivative on l.h.s. of (226) with $\mu = 1, 2, 3$. Then, the equations read

$$a^i - \frac{v^i}{2(-vGv)}\frac{d}{dt}(-vGv) = T^i{}_{\nu}\left[\frac{e\sqrt{-vGv}}{m_r c^2}(Fv)^i - \frac{d}{dt}\widetilde{\mathcal{T}}^{\nu}{}_{\alpha}v^{\alpha}\right], \quad (231)$$

$$\frac{d}{dt}S^{\mu\nu} = \frac{e\mu\sqrt{-vGv}}{m_r c^2}(FS)^{[\mu\nu]} - \frac{2bm_r c(\mu-1)}{\sqrt{-vGv}}v^{[\mu}(SFv)^{\nu]}, \quad (232)$$

$$(Sv)^{\mu} + b(\mu-1)(SSFv)^{\mu} = 0, \quad (233)$$

where $v^{\mu} = (c, \mathbf{v})$. Equations (233) and (223) imply

$$-vGv = -v\widetilde{\mathcal{T}}v = c^2 - \mathbf{v}^2 - (\mu-1)b(vSFv). \quad (234)$$

We compute the time derivatives in (231):

$$\frac{d}{dt}(-vGv) = -2(\mathbf{va}) - (\mu-1)b\left\{[v(FS+SF)]_i a^i + \frac{e\mu\sqrt{-vGv}}{m_r c^2}[(vFFSv) + (vFSFv)] - \frac{2bm_r c(\mu-1)}{\sqrt{-vgv}}\left[v^2(vFSFv) - (vSFv)(vFv)\right]\right\}, \quad (235)$$

$$
\begin{aligned}
&-T^i{}_\nu \frac{d}{dt}\widetilde{\mathcal{T}}^\nu{}_\alpha v^\alpha = -\frac{e\sqrt{-vGv}}{m_r c^2}\left\{\mu(\mu-1)b(FSFv)^i\right. \\
&\quad \left. -\mu(\mu-1)a(SFFv)^i - \mu(\mu-1)^2 ab(SFFSFv)^i\right\} \\
&\quad + \frac{2bm_r c(\mu-1)}{\sqrt{-vGv}} T^i{}_\nu \left[v^\nu(vFSFv) - (SFv)^\nu(vFv)\right].
\end{aligned} \tag{236}
$$

We note that all the potentially divergent terms (two last terms in (235) and in (236)), arising due to the contribution from $\dot{S} \sim 1/\sqrt{-vGv}$, disappear on the symmetry grounds. We substitute nonvanishing terms into (231) obtaining the expression

$$
\begin{aligned}
M^i{}_j a^j &= \frac{e\sqrt{-vGv}}{m_r c^2}\left\{(Fv)^i - \mu(\mu-1)b(FSFv)^i\right. \\
&\quad + (\mu-1)^2 a(SFF[\eta + \mu bSF]v)^i \\
&\quad \left. - v^i \frac{\mu(\mu-1)b}{2(-vGv)}(vFFSv)\right\},
\end{aligned} \tag{237}
$$

where the matrix

$$
\begin{gathered}
M^i{}_j = \delta^i{}_j + \frac{v^i v^\mu \Omega_{\mu j}}{2(-vGv)}, \\
\text{with } \Omega_{\mu j} = 2\delta_{\mu j} + (\mu-1)b(FS+SF)_{\mu j}
\end{gathered} \tag{238}
$$

has the inverse

$$
\widetilde{M}^i{}_j = \delta^i{}_j - \frac{v^i v^\mu \Omega_{\mu j}}{2c^2 - (\mu-1)bv^\mu(FS+SF)_{\mu 0}v^0}, \tag{239}
$$

with the property

$$
\widetilde{M}^i{}_j v^j = v^i \frac{2(-vGv)}{2c^2 - (\mu-1)bv^\mu(FS+SF)_{\mu 0}v^0}. \tag{240}
$$

Applying the inverse matrix, we obtain the acceleration

$$
\begin{aligned}
a^i &= \frac{e\sqrt{-vGv}}{m_r c^2}\left\{\widetilde{M}^i{}_j\left[(Fv)^j - \mu(\mu-1)b(FSFv)^j\right.\right. \\
&\quad \left. + (\mu-1)^2 a(SFF[\eta+\mu bSF]v)^j\right] - v^i \\
&\quad \left. \cdot \frac{\mu(\mu-1)b(vFFSv)}{2c^2 - (\mu-1)bv^\mu(FS+SF)_{\mu 0}v^0}\right\}.
\end{aligned} \tag{241}
$$

For the particle with nonanomalous magnetic moment ($\mu = 1$), the right hand side reduces to the Lorentz force, so the expression in braces is certainly nonvanishing in the ultrarelativistic limit. Thus, the acceleration vanishes only when $v \to v_{cr}$, where the critical velocity is determined by the equation $vGv = 0$.

Let us estimate the critical velocity. Using the consequence $(\dot{x}SF\dot{x}) = -b(\mu-1)(\dot{x}FSSF\dot{x})$ of the Pirani condition and the expression $S^\mu{}_\alpha S^{\alpha\nu} = -4[\pi^2\omega^\mu\omega^\nu + \omega^2\pi^\mu\pi^\nu]$, we write

$$
\begin{aligned}
&-(\dot{x}G\dot{x}) \\
&\quad = c^2 - \mathbf{v}^2 \\
&\qquad + 4b^2(\mu-1)^2\left[\pi^2(\omega F\dot{x})^2 + \omega^2(\pi F\dot{x})^2\right].
\end{aligned} \tag{242}
$$

As π and ω are space-like vectors (see the discussion below (70)), the last term is nonnegative, so $v_{cr} \geq c$. We show that generally this term is nonvanishing function of velocity; then $v_{cr} > c$. Assume the contrary that this term vanishes at some velocity; then

$$
\begin{aligned}
\omega F\dot{x} &= -\omega^0(\mathbf{E}\mathbf{v}) + (\boldsymbol{\omega}, c\mathbf{E} + \mathbf{v}\times\mathbf{B}) = 0, \\
\pi F\dot{x} &= -\pi^0(\mathbf{E}\mathbf{v}) + (\boldsymbol{\pi}, c\mathbf{E} + \mathbf{v}\times\mathbf{B}) = 0.
\end{aligned} \tag{243}
$$

This implies $c(\mathbf{DE}) + (\mathbf{D}, \mathbf{v}\times\mathbf{B}) = 0$. Consider the case $\mathbf{B} = 0$; then it should be $(\mathbf{DE}) = 0$. On the other hand, for the homogeneous field, the quantity $S^{\mu\nu}F_{\mu\nu} = 2[(\mathbf{DE}) + 2(\mathbf{SB})] = 2(\mathbf{DE})$ is a constant of motion. Hence, we can take the initial conditions for spin such that $(\mathbf{DE}) \neq 0$ at any instant; this implies $v_{cr} > c$.

7.3. Ultrarelativistic Limit within the Geometry Determined by Effective Metric. According to the previous section, if we insist on preserving usual special-relativity definitions of time and distance (175), the speed of light does not represent special point of the equation for trajectory. Acceleration of the particle with anomalous magnetic moment generally vanishes at the speed slightly higher than the speed of light. Hence, we arrive at a rather surprising result that speed of light does not represent maximum velocity of manifestly relativistic equation (237). This state of affairs is unsatisfactory because the Lorentz transformations have no sense above c, so two observers with relative velocity $c < v < v_{cr}$ will not be able to compare results of their measurements.

To keep the condition $v_{cr} = c$, we use formal similarity of the matrix G which appeared in (223) with space-time metric. Then, we can follow the general-relativity prescription of Section 2 to define time and distance in the presence of electromagnetic field; that is, we use G of (223) to define three-dimensional geometry (5)–(8). The effective metric depends on x^i via the field strength $F(x^0, x^i)$ and on x^0 via the field strength as well as via the spin-tensor $S(x^0)$. So, the effective metric is time-dependent even in stationary electromagnetic field. With these definitions we have, by construction, $-\dot{x}G\dot{x} = (dt/dx^0)^2(c^2 - (\mathbf{v}\gamma\mathbf{v}))$, so the critical speed coincides with the speed of light. The intervals of time and distance are given now by (5) and (6); they slightly differ from those in empty space.

In the present case, the expression for three-acceleration can be obtained in closed form in an arbitrary electromagnetic field. We present (226) in the form in (44):

$$DDx^{\mu} = \mathscr{F}^{\mu} = -Dx^{\mu}\frac{Dm_r(S)}{m_r} - T^{\mu}{}_{\nu}D\widetilde{\mathscr{T}}^{\nu}{}_{\alpha}(S)\,Dx^{\alpha} + T^{\mu}{}_{\nu}\left\{\frac{e}{m_rc^2}(FDx)^{\nu} + \frac{\mu e}{4m_r^2c^3}\partial^{\nu}(SF) + \frac{\mu}{m_r}DZ^{\nu}\right\}. \qquad (244)$$

Then, the acceleration is given by (46). The first two terms on right hand side of (244) give potentially divergent contributions arising from the piece $\dot{S} \sim 1/\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}$ of (227). In the previous section, we have seen that the dangerous contribution contained in the second term disappears. To analyze the first term, we substitute $\mathscr{F}^i$ from (244) into (46). Using the property $\widetilde{M}^i{}_j v^j = v^i((c^2 - \mathbf{v}\gamma\mathbf{v})/c^2)$, we obtain the acceleration

$$\begin{aligned} a^i = {} & \left(c^2 - \mathbf{v}\gamma\mathbf{v}\right)\left[-v^i\frac{\dot{m}_r}{m_rc^2} - \frac{\widetilde{M}^i{}_jT^j{}_{\nu}\dot{\widetilde{T}}^{\nu}{}_{\alpha}v^{\alpha}}{c^2 - \mathbf{v}\gamma\mathbf{v}} + \widetilde{M}^i{}_jT^j{}_{\nu}\left\{\frac{e}{m_rc^2\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}(Fv)^{\nu} + \frac{\mu e}{4m_r^2c^3}\partial^{\nu}(SF) + \frac{\mu}{m_r\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}}\dot{Z}^{\nu}\right\}\right] \\ & + \widetilde{M}^i{}_j\widetilde{\Gamma}^j{}_{kl}(\gamma)\,v^kv^l + \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}\left[\left(\mathbf{v}\partial_0\gamma\gamma^{-1}\right)^i - \frac{v^i}{c^2}(\mathbf{v}\partial_0\gamma\mathbf{v})\right], \end{aligned} \qquad (245)$$

so the divergency due to $\dot{m}_r \sim 1/\sqrt{c^2 - \mathbf{v}\gamma\mathbf{v}}$ is cancelled by the factor in front of this term. In the result, the acceleration is finite as $v \to c$. Besides, taking into account the property $(\mathbf{v}\gamma)_i\widetilde{M}^i{}_j = (\mathbf{v}\gamma)_j((c^2 - \mathbf{v}\gamma\mathbf{v})/c^2)$, we conclude that longitudinal acceleration (47),

$$\mathbf{v}\gamma\mathbf{a} = \frac{\left(c^2 - \mathbf{v}\gamma\mathbf{v}\right)^2}{c^2}(\mathbf{v}\gamma\mathscr{F}) + \frac{c^2 - \mathbf{v}\gamma\mathbf{v}}{c^2}\left[(\mathbf{v}\gamma)_i\widetilde{\Gamma}^i{}_{kl}(\gamma)\,v^kv^l + \frac{1}{2}\left(\frac{dt}{dx^0}\right)^{-1}(\mathbf{v}\partial_0\gamma\mathbf{v})\right], \qquad (246)$$

vanishes in this limit.

8. Conclusion

In this work we have studied behavior of ultrarelativistic spinning particle in external fields. To construct interaction of spin with external fields and to analyze its influence on the trajectory of the particle, we used the vector model of spin. Minimal interaction with gravity was formulated starting from the Lagrangian variational problem without auxiliary variables (56). The nonminimal interaction with gravity through the gravimagnetic moment κ [25] has been achieved in the Lagrangian with one auxiliary variable (170).

The variational problems imply the fixed value of spin (78); that is, they correspond to an elementary spin one-half particle. The vector model also allowed us to construct Lagrangian action (94) with unfixed spin and with a mass-spin trajectory constraint, that is, with the properties of Hanson-Regge relativistic top [22]. In this model appeared the fundamental length scale and spin has four physical degrees of freedom. At last, interaction of spinning particle with magnetic moment μ with an arbitrary electromagnetic field was achieved in the Lagrangian action with one auxiliary variable (204). Equations of motion of minimally interacting spinning particle (i.e., with $\kappa = 0$) have been identified with Mathisson-Papapetrou-Tulczyjew-Dixon equations. They are widely used in the current literature for description of rotating bodies in general relativity. To study the class of trajectories of a body with fixed integrals of motion $\sqrt{-P^2} = k$ and $S^2 = \beta$, we can use our spinning particle with $m = k/c$ and $\alpha = \beta/8$.

To study our general-covariant equations in the laboratory frame, we used the Landau-Lifshitz 1 + 3-splitting formalism of four-dimensional pseudo Riemann space, where the basic structure is a congruence of one-dimensional time-like curves identified with world lines of the laboratory clocks. This formalism allows one to determine the time interval, distance, and then velocity between two infinitesimally closed points x^{μ} and $x^{\mu} + \delta x^{\mu}$ of the particle's world line. The basic requirement for definition of the three-dimensional quantities is that speed of light should be a coordinate-independent notion. Due to the decomposition of space time into time + space, one manipulates only time-varying vector and tensor fields. In the resulting three-dimensional geometry with Riemannian scalar product, we asked about the notion of a constant vector field. We have suggested notion (27) which follows from the geometric requirement that scalar product of constant fields does not depend on the point where it was computed. For the vector field of velocity, its deviation from the constant field has given us acceleration (28). Then, we showed that the definition adopted is consistent with the basic principle of general relativity: massive spinless particle, propagating in a gravitational field along a four-dimensional geodesic, can not exceed the speed of light. With this definition at hand, we analyzed ultrarelativistic behavior of the spinning particle in external fields.

Evolution of the fast MPTD particle in the laboratory frame was studied on the base of Lagrangian equations (92) and (93). In these equations, we observed the emergence of effective metric (89) instead of the original one. We have examined the two metrics as candidates for construction of three-dimensional space-time geometry (5)–(8) probed by the particle. In both cases, the MPTD equations have unsatisfactory behavior in the ultrarelativistic limit. In particular, three-dimensional acceleration (28) increases with velocity and becomes infinite in the limit.

Further, we showed that spinning particle with $\kappa = 1$ is free of the problems detected in MPTD equations. For this value of gravimagnetic moment, the effective metric does not appear and the three-dimensional geometry should be defined, unambiguously, with respect to the original metric. Critical velocity of the theory coincides with the speed of light and three-dimensional acceleration vanishes as $v \to c$. So, the spinning particle with gravimagnetic moment $\kappa = 1$ seems a more promising candidate for the description of a relativistic rotating body in general relativity. An interesting property of the resulting equations is that spin ceases to affect the trajectory in ultrarelativistic limit; the trajectory of spinning particle becomes more and more close to that of spinless particle as $v \to c$. Besides, the spin precesses with finite angular velocity in this limit.

Equations in electromagnetic and in gravitational fields become very similar after the identification of $\mu \sim \kappa$ and $R_{\mu\nu\alpha\beta}S^{\alpha\beta} \sim F_{\mu\nu}$. In particular, interaction of spin with electromagnetic field in Minkowski space also produces effective metric (223) for the particle with anomalous magnetic moment $\mu \neq 1$. If we insist on the usual special-relativity notions of time and distance, the critical speed turns out to be more than the speed of light. To preserve the equality $v_{\text{cr}} = c$, we are forced to assume that particle in electromagnetic field probes the three-dimensional geometry determined with respect to the effective metric instead of the Minkowski metric. In the result, we have rather unusual picture of the Universe filled with spinning matter. Since G depends on spin, in this picture there is no unique space-time manifold for the Universe of spinning particles; each particle will probe its own three-dimensional geometry. In principle, this could be an observable effect. With effective metric (223), (5) implies that the time of life of muon in electromagnetic field and in empty space should be different.

Competing Interests

The authors declare that they have no competing interests.

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Research Article

Holographic Phase Transition Probed by Nonlocal Observables

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From the viewpoint of holography, the phase structure of a 5-dimensional Reissner-Nordström-AdS black hole is probed by the two-point correlation function, Wilson loop, and entanglement entropy. As the case of thermal entropy, we find for all the probes that the black hole undergoes a Hawking-Page phase transition, a first-order phase transition, and a second-order phase transition successively before it reaches a stable phase. In addition, for these probes, we find that the equal area law for the first-order phase transition is valid always and the critical exponent of the heat capacity for the second-order phase transition coincides with that of the mean field theory regardless of the size of the boundary region.

1. Introduction

Phase transition is a ubiquitous phenomenon for garden-variety thermodynamic systems. Due to the pioneering work by Hawking [1, 2], a black hole is also a thermodynamic system. Such a fact is further supported by AdS/CFT correspondence [3–5], where a black hole in the AdS bulk is dual to a thermal system without gravity. So one can naturally expect that a black hole can also undertake some interesting phase transitions as the general thermodynamic system. Actually it has been shown that a charged AdS black hole undergoes a Hawking-Page phase transition [6, 7], which is interpreted as the confinement/deconfinement phase transition in the dual gauge field theory [8] and a van der Waals-like phase transition before it reaches the stable state [9]. The Hawking-Page phase transition implies that the thermal AdS is unstable and it will transit to the stable Schwarzschild AdS black hole lastly. The van der Waals-like phase transition has been observed till now in many circumstances. The first observation was contributed by [9] in the T-S plane. Specifically speaking, in a fixed charge ensemble, for a black hole endowed with small charge, there is an unstable black hole interpolating between the stable small hole and stable large hole, and the small stable hole will undertake a first-order phase transition to the large stable hole as the temperature of the black hole reaches a critical temperature. As the charge increases to the critical charge, the small hole and the large hole merge into one and squeeze out the unstable phase so that an inflection point emerges and the phase transition is second order. When the charge exceeds the critical charge, the black hole is always stable. Recently in the extended phase space, where the negative cosmological constant is treated as the pressure while its conjugate acts as the thermodynamical volume, the van der Waals-like phase transition has also been observed in the P-V plane [10–16]. In addition, it was shown in [17] that the van der Waals-like phase transition also shows up in the Q-Φ plane. Particularly, in the Gauss-Bonnet gravity, it is found that the Gauss-Bonnet coupling parameter α also affects the phase structure of the space time, and in the T-α plane, a 5-dimensional neutral Gauss-Bonnet black hole also demonstrates the van der Waals-like phase transition [18].

In this paper, we intend to probe the Hawking-Page phase transition and van der Waals-like phase transition appeared in a 5-dimensional Reissner-Nordström-AdS black hole by the geodesic length, minimal area surface, and minimal surface area in the bulk, which are dual to the nonlocal observables on the boundary theory by holography, namely, the two-point correlation function, Wilson loop, and entanglement entropy, individually (recently these nonlocal observables have been used to probe the nonequilibrium thermalization process, and it has been found that all of them have the same effect [19–25]). In fact, there have been some similar works to probe the phase structure by holographic entanglement entropy. In [26], the phase structure of entanglement entropy is studied in the T-S plane for both a fixed charge ensemble and a fixed chemical potential ensemble, and it is found that the phase structure of entanglement entropy is similar to that of the thermal entropy. In particular, the entanglement entropy is found to demonstrate the same second-order phase transition at the critical point as the thermal entropy. Soon after, it is found that the entanglement entropy can also probe the van der Waals-like phase transition in the P-V plane [27]. In [28], Nguyen has investigated exclusively the equal area law of holographic entanglement entropy and found that the equal area law holds regardless of the size of the entangling region. Very recently [29] investigated entanglement entropy for a quantum system with infinite volume; their result showed that the entanglement entropy also exhibits the same van der Waals-like phase transition as the thermal entropy. They also checked the equal area law and obtained the critical exponent of the heat capacity near the critical point.

In this paper, we will further investigate whether one can probe the phase structure by two-point correlation function and Wilson loop besides the entanglement entropy. We intend to explore whether they exhibit the similar van der Waals-like phase transition as the entanglement entropy and thermal entropy. In addition, we also want to check whether these nonlocal observables can probe the Hawking-Page phase transition between the AdS black hole and thermal gas so that we can get a complete picture about the phase transition of the black holes in the framework of holography.

This paper is organized as follows. In Section 2, we will discuss the thermal entropy phase transition of a 5-dimensional Reissner-Nordström-AdS black hole in the T-S plane in a fixed charge ensemble. Then in Section 3, we will probe these phase transitions by geodesic length, Wilson loop, and holographic entanglement entropy individually. In each subsection, the equal area law is checked and the critical exponent of the heat capacity is obtained for different sizes of the boundary region. Section 4 is devoted to discussions and conclusions.

2. Thermodynamic Phase Transition of the 5-Dimensional Reissner-Nordström-AdS Black Hole

Starting from the action

$$S = \frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}\left[R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{n(n-1)}{l^2}\right], \quad (1)$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and l is the AdS radius, we shall focus on the case of $n = 4$, in which the charged Reissner-Nordström-AdS black hole can be written as [9]

$$\begin{aligned} ds^2 &= -f(r)\,dt^2 + f^{-1}(r)\,dr^2 \\ &\quad + r^2\left[d\phi^2 + \sin^2\phi\left(d\theta^2 + \sin^2\theta d\psi^2\right)\right], \end{aligned} \quad (2)$$

where $\phi \in (0,\pi)$, $\theta \in (0,\pi)$, and $\psi \in (0,2\pi)$ are hyperspherical coordinates for the 3 spheres, and

$$f(r) = 1 + \frac{r^2}{l^2} - \frac{8M}{3\pi r^2} + \frac{4Q^2}{3\pi^2 r^4}, \quad (3)$$

with M and Q being the mass and charge of the black hole. Whence we can get the Hawking temperature of this space time as

$$T = \left.\frac{f'(r)}{4\pi}\right|_{r_+} = \frac{3\pi^2 r_+^6 - 8l^2\left(Q^2 - M\pi r_+^2\right)}{6l^2\pi^3 r_+^5}. \quad (4)$$

In addition, it follows from the Bekenstein-Hawking formula that the entropy of the black hole is given by

$$S = \frac{\pi^2 r_+^3}{2}, \quad (5)$$

where r_+ is the outer event horizon of the black hole, namely, the largest root of the equation $f(r_+) = 0$. With this, the mass of the back hole can thus be expressed as the function of the event horizon:

$$M = \frac{4l^2Q^2 + 3l^2\pi^2 r_+^4 + 3\pi^2 r_+^6}{8l^2\pi r_+^2}. \quad (6)$$

Substituting (5) and (6) into (4), we can get the relation between the temperature T and entropy S of the 5-dimensional Reissner-Nordström-AdS black hole:

$$T = \frac{12S^2 + l^2\left(-2\pi^2 Q^2 + 32^{1/3}\pi^{4/3}S^{4/3}\right)}{62^{2/3}l^2\pi^{5/3}S^{5/3}}. \quad (7)$$

In addition, with the relation $F = M - TS$, the Helmholtz free energy can be expressed as

$$F = \frac{5Q^2}{6\pi r_+^2} - \frac{1}{8}\pi r_+^2\left(-1 + r_+^2\right). \quad (8)$$

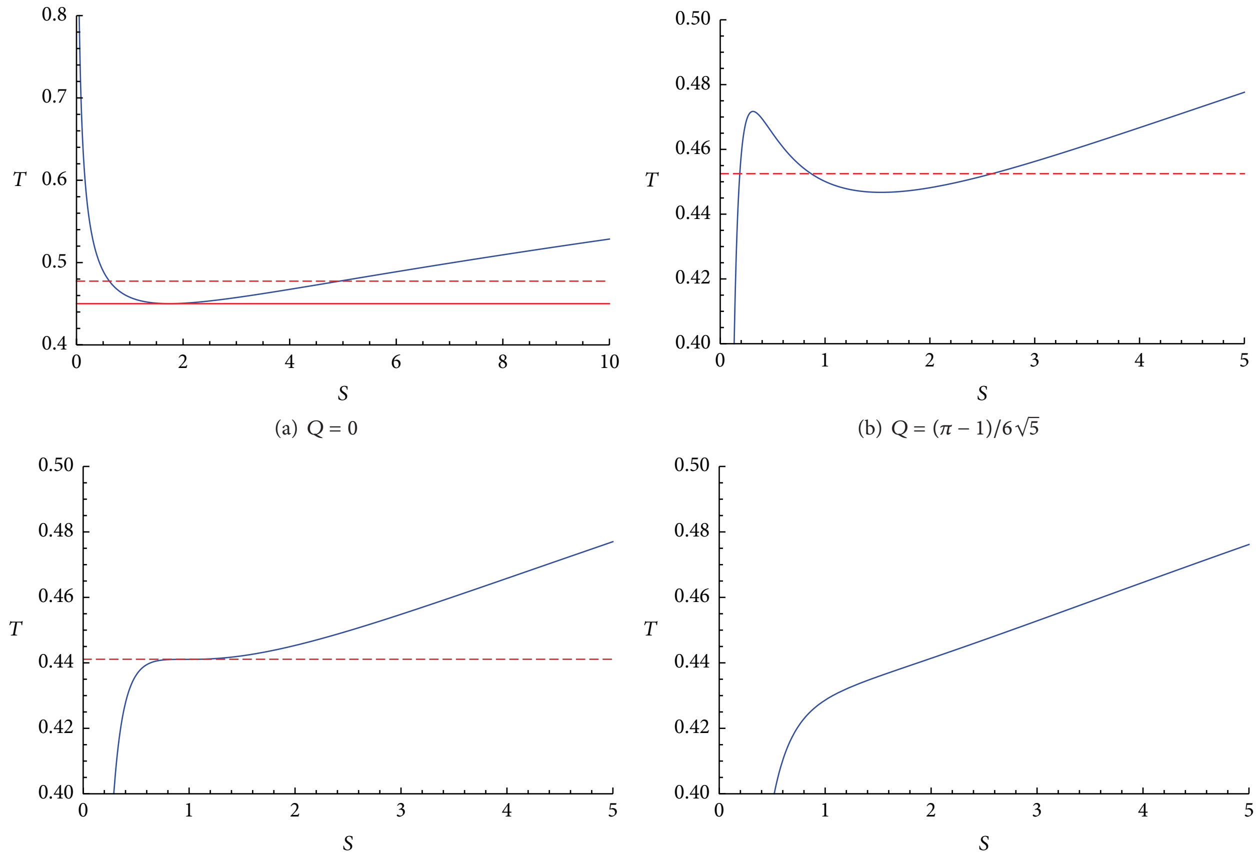
Note that this formula for our free energy has implicitly chosen the pure AdS as the reference space time because the free energy vanishes for pure AdS by this formula. Now let us review the relevant phase transitions in the fixed charge ensemble by (7) and (8) in the T-S plane.

To achieve this, we should first find the critical charge by the following equations:

$$\left(\frac{\partial T}{\partial S}\right)_Q = \left(\frac{\partial^2 T}{\partial S^2}\right)_Q = 0. \quad (9)$$

Inserting (7) into (9), we can get the values for the critical charge and critical entropy:

$$\begin{aligned} Q_c &= \frac{l^2\pi}{6\sqrt{5}}, \\ S_c &= \frac{l^3\pi^2}{6\sqrt{3}}. \end{aligned} \quad (10)$$



(a) $Q = 0$ (b) $Q = (\pi - 1)/6\sqrt{5}$ (c) $Q = \pi/6\sqrt{5}$ (d) $Q = (\pi + 1)/6\sqrt{5}$

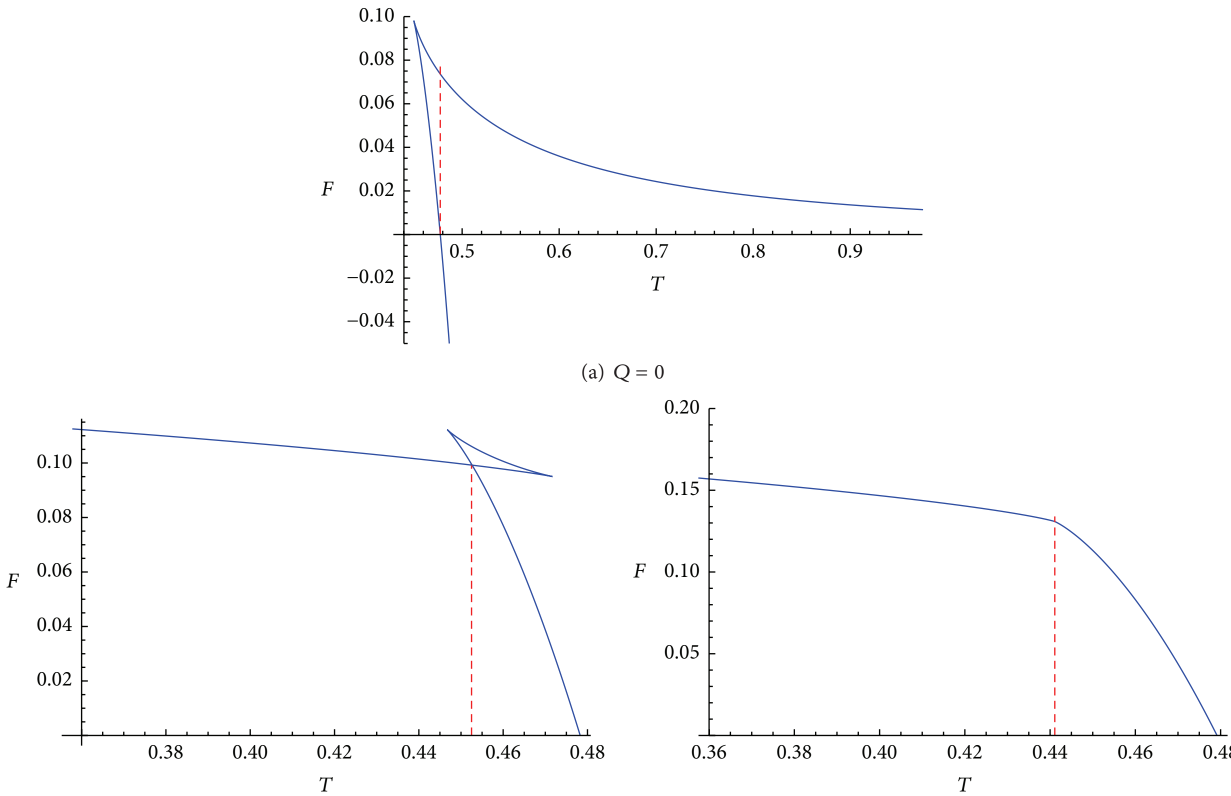
Figure 1: Relation between the entropy and temperature for different charges in the fixed charge ensemble. The red solid line corresponds to the minimum temperature of space time and the red dashed lines in (a), (b), and (c) correspond to the locations of Hawking-Page phase transition, first-order phase transition, and second-order phase transition individually.

Substituting these critical values into (7), we can get the critical temperature:

$$T_c = \frac{\sqrt{3}\,(3+5)}{10 l \pi}. \tag{11}$$

We plot the discharge curves for different charges in Figure 1. For the case $Q = 0$, there is a minimum temperature $T_0 = \sqrt{2}/\pi$ [30], which is indicated by the red solid line in (a). When the temperature is lower than T_0, we have only a thermal AdS. When the temperature is higher than T_0, there are two additional black hole branches. The small branch is unstable while the large branch is stable. This can be justified by checking the corresponding heat capacities, which is related to their slopes. The Hawking-Page phase transition occurs at the temperature given by $T_1 = 3/2\pi$ [30], which is higher than T_0 and indicated by the red dashed line. This can be observed by the F-T relation in Figure 2(a), where T_0 is the horizontal coordinate of the cusp and T_1 is the horizontal coordinate for the intersection of the stable branch and the horizontal axis. Obviously, when the temperature is lower than T_1, the thermal AdS is the most stable state. While when the temperature is higher than T_1, the most stable state is taken over by the large black hole branch.

For the case $Q \neq 0$, the phase structure is similar to that of the van der Waals phase transition. That is, for a small charge, there is an unstable black hole interpolating between the stable small hole and stable large hole. The small stable hole will jump to the large stable hole at the critical temperature $T_\star$, which is labeled by the red dashed line in Figure 1(b). As the charge increases to the critical charge, the small hole and the large hole merge into one and squeeze out the unstable phase. So there is an inflection point in Figure 1(c). The heat capacity is divergent in this case; the phase transition is therefore second order. As the charge exceeds the critical charge, we simply have one stable black hole at each temperature, which can be justified by the slope of the curve in Figure 1(d). The van der Waals-like phase transition can also be observed from the F-T relation. From Figure 2(b), we see a swallowtail structure, which corresponds to the unstable phase in Figure 1(b). The critical temperature $T_\star = 0.4526$ for the phase transition is apparently read off by the horizontal coordinate of the junction between the small black hole and the large black hole. As the temperature is lower than the critical temperature $T_\star$, the free energy of the small black hole is lowest, so the small hole is stable. As the temperature is higher than $T_\star$,



(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

(c) $Q = \pi/6\sqrt{5}$

Figure 2: Relation between the free energy and temperature for different charges. The horizontal coordinates of the red dashed lines correspond to the temperatures of the Hawking-Page phase transition, first-order phase transition, and second-order phase transition.

the free energy of the large black hole is lowest, so the large hole dominates thereafter. The nonsmoothness of the junction indicates that the phase transition is first order. When the charge is arriving at the critical charge Q_c, the swallowtail structure in Figure 2(b) shrinks into a point as is shown in Figure 2(c). The horizontal coordinate of the inflection point corresponds to the critical temperature T_c of the second-order phase transition, which is consistent with the analytical result in (11).

For the first-order phase transition in Figure 1(b), we would like to check whether Maxwell's equal area law holds with the following formula:

$$A_1 \equiv \int_{S_1}^{S_3} T(S,Q)\, dS = T_\star (S_3 - S_1) \equiv A_3, \qquad (12)$$

in which $T(S,Q)$ is defined in (7); S_1 and S_3 are the smallest and largest roots of the equation $T(S,Q) = T_\star$. After a simple calculation, we find $S_1 = 0.186987$ and $S_3 = 2.60575$. With these values, we find that A_1 and A_3 in (12) equal 1.09481 and 1.09482, respectively. So the equal area law in the T-S plane holds within our numerical accuracy.

For the second-order phase transition in Figure 1(c), we are interested in the critical exponent associated with the heat capacity:

$$C_Q = T \left. \frac{\partial S}{\partial T} \right|_Q . \qquad (13)$$

Near the critical point, writing the entropy as $S = S_c + \delta$ and expanding the temperature in terms of small δ, we find

$$\begin{aligned} T - T_c &= \frac{\left(220 l^2 \pi^2 Q^2 - 212^{1/3} l^2 \pi^{4/3} S^{4/3} + 30 S^2\right)}{2432^{2/3} l^2 \pi^{5/3} S^{14/3}} \left(S - S_c\right)^3, \end{aligned} \qquad (14)$$

in which we have used (9). In this case, (13) further implies $C_Q \sim (T - T_c)^{-2/3}$; namely, the critical exponent is $-2/3$, which is the same as the one from the mean field theory. In addition, taking logarithm to (14), we have a linear relation:

$$\log \left|T - T_c\right| = 3 \log \left|S - S_c\right| + \text{constant}, \qquad (15)$$

with 3 the slope. In what follows, we will use this logarithm to check the critical exponent for the analogous heat capacities in the framework of holography.

It is noteworthy that by holography the whole phase structure described above is not only for the bulk black hole but also for the dual boundary system, where the thermal entropy is simply given by the black hole entropy, and so on.

3. Phase Transition in the Framework of Holography

In this section we shall investigate the phase structures of some nonlocal observables such as two-point correlation function, Wilson loop, and entanglement entropy in the dual fled theory by holography to see whether they have the same phase structure as the thermal entropy.

3.1. Phase Transition of Two-Point Correlation Function. According to the AdS/CFT correspondence, if the conformal dimension Δ of scalar operator $\mathcal{O}$ of dual field theory is large enough, the equal time two-point correlation function can be holographically approximated as [31]

$$\left\langle \mathcal{O}\left(t_0, x_i\right) \mathcal{O}\left(t_0, x_j\right)\right\rangle \approx e^{-\Delta L}, \tag{16}$$

where L is the length of the bulk geodesic between the points (t_0, x_i) and (t_0, x_j) on the AdS boundary. Taking into account the spherical symmetry of the 5-dimensional Reissner-Nordström-AdS black hole, we can simply choose ($\phi = \pi/2$, $\theta = \theta_0$, $\psi = 0$) and ($\phi = \pi/2$, $\theta = \theta_0$, $\psi = \pi$) as the two boundary points. Then with θ to parameterize the trajectory, the proper length is given by

$$\begin{gathered} L = 2\int_0^{\theta_0} \mathscr{L}\left(r\left(\theta\right), \theta\right) d\theta, \\ \mathscr{L} = \sqrt{\frac{\dot{r}^2}{f\left(r\right)} + r^2}, \end{gathered} \tag{17}$$

in which $\dot{r} = dr/d\theta$. Imagining θ as time, and treating $\mathscr{L}$ as the Lagrangian, one can get the equation of motion for $r(\theta)$ by making use of the Euler-Lagrange equation; that is

$$0 = \dot{r}^2 f'\left(r\right) - 2f\left(r\right)\ddot{r} + 2rf\left(r\right)^2, \tag{18}$$

which can be solved by imposing the following boundary conditions:

$$\begin{gathered} \dot{r}\left(0\right) = 0, \\ r\left(0\right) = r_0. \end{gathered} \tag{19}$$

To explore whether the size of the boundary region affects the later phase structure, we here choose $\theta_0 = 0.14, 0.2$ as two examples. Note that, for a fixed θ_0, the geodesic length is divergent, so it should be regularized by subtracting off the geodesic length in pure AdS with the same boundary region, denoted by L_0. To achieve this, we are required to set a UV cutoff for each case, which is chosen to be $r(0.139)$ and $r(0.199)$, respectively, for our two examples. In this paper, we obtain L_0 also by numerics though there is an analytical result for $r(\theta_0)$ for pure AdS in Einstein gravity. We label the regularized geodesic length as $\delta L \equiv L - L_0$.

We plot the relation between T and δL for different θ_0 in Figures 3 and 4. As shown in Figures 3 and 4, δL demonstrates a similar phase structure as the thermal entropy. Moreover, we find that the minimum temperature T_0 as well as Hawking-Page phase transition temperature T_1 in (a), the first-order phase transition temperature $T_\star$ in (b), and second-order phase transition temperature T_c in (c) are exactly the same as those in T-S plane, which justifies our notation. To be more specific, it is easy to check T_0 by locating the position of local minimum. But in order to confirm $T_\star$ and T_c, we are required to examine the equal area law for the first-order phase transition and obtain $-2/3$ as the critical exponent for the second-order phase transition, which are documented as follows.

In the δL-T plane, we define the equal area law as

$$A_1 \equiv \int_{\delta L_1}^{\delta L_3} T\left(\delta L\right) d\delta L = T_\star\left(\delta L_3 - \delta L_1\right) \equiv A_3, \tag{20}$$

in which $T(\delta L)$ is an Interpolating Function obtained from the numeric result and δL_1 and δL_3 are the smallest and largest roots of the equation $T(\delta L) = T_\star$. For the case $\theta_0 = 0.14$, we find $\delta L_1 = 0.0000556147$, $\delta L_3 = 0.000194497$. Substituting these values into (20), we find $A_1 = 0.0000628401$, $A_3 = 0.0000628583$. For the case $\theta_0 = 0.2$, after simple calculation, we find $A_1 = 0.0000108924$, $A_3 = 0.0000108875$. It is obvious that for different θ_0, A_1, and A_3 are equal within our numeric accuracy. Thus, the equal area law also holds in the δL-T plane.

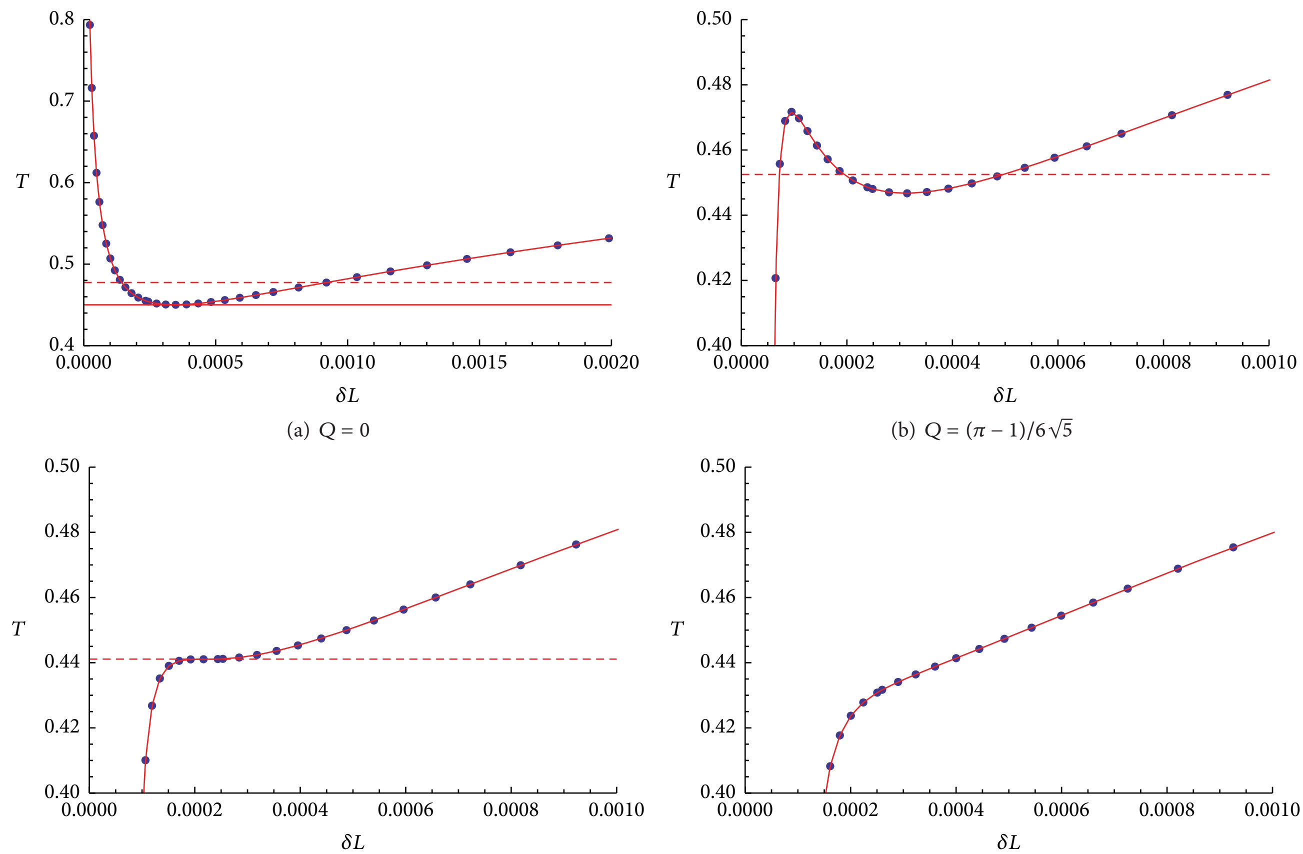
In addition, in order to investigate the critical exponent for the analogous heat capacity of the geodesic length. We are interested in the logarithm of the quantities $T - T_c$, $\delta L - \delta L_c$, in which T_c is the critical temperature defined in (11), and L_c is obtained numerically by the equation $T(\delta L) = T_c$. We plot the relation between $\log|T-T_c|$ and $\log|\delta L-\delta L_c|$ for different θ_0 in Figure 5, where these straight lines can be fitted as

$$\begin{aligned} &\log\left|T - T_c\right| \\ &= \begin{cases} 23.2318 + 3.06832\log\left|\delta L - \delta L_c\right|, & \text{for } \theta_0 = 0.14, \\ 31.9841 + 3.00077\log\left|\delta L - \delta L_c\right|, & \text{for } \theta_0 = 0.2. \end{cases} \end{aligned} \tag{21}$$

It is obvious that the slope is about 3, which indicates that the critical exponent is $-2/3$ for the analogous heat capacity and the phase transition is also second order at T_c for the geodesic length.

3.2. Phase Transition of Wilson Loop. In this subsection, we are going to study the phase structure of the Wilson loop, which in the bulk corresponds to the minimal area surface by holography. Wilson loop operator is defined as a path ordered integral of gauge field over a closed contour, and its expectation value is approximated geometrically by the AdS/CFT correspondence as [32]

$$\left\langle W\left(C\right)\right\rangle \approx e^{-A_\Sigma/2\pi\alpha'}, \tag{22}$$



(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

(c) $Q = \pi/6\sqrt{5}$

(d) $Q = (\pi + 1)/6\sqrt{5}$

FIGURE 3: Relation between the geodesic length and temperature in the fixed charge ensemble for different charges at $\theta_0 = 0.14$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.

where C is the closed contour, Σ is the minimal bulk surface ending on C with A its minimal area, and α' is the Regge slope parameter. Next we choose the line with $\phi = \pi/2$ and $\theta = \theta_0$ as our loop. Then we can employ (θ, ψ) to parameterize the minimal area surface, which is invariant under the ψ-direction by our rotational symmetry. Thus, the corresponding minimal area surface can be expressed as

$$A = 2\pi \int_0^{\theta_0} r \sin\theta \sqrt{\frac{\dot{r}^2}{f(r)} + r^2} d\theta, \tag{23}$$

in which $\dot{r} = dr/d\theta$. Making use of the Euler-Lagrange equation, one can get the equation of motion for $r(\theta)$. Then with the boundary conditions $r'(0) = 0$, $r(0) = r_0$, we can further get the numeric result of $r(\theta)$. Similar to the case of geodesic length, we choose $\theta_0 = 0.14, 0.2$ as two examples and the corresponding UV cutoffs are set to be $r(0.139)$, $r(0.199)$. We label the regularized minimal area surface as $\delta A \equiv A - A_0$, where A_0 is the minimal area in pure AdS with the same boundary region. We plot the relation between δA and T for different θ_0 in Figures 6 and 7. Comparing Figure 6 with Figure 7, we find they are the same nearly besides the scale of the horizontal coordinate. In other words, θ_0 affects only the value but not the phase structure of minimal area surface in the T-δA plane. The result tells us that the similar phase structure also shows up for the minimal surface area. Here we concentrate only on scrutinizing the equal area law for the first-order phase transition and the critical exponent of the analogous heat capacity for the second-order phase transition.

First, in the δA-T plane, the equal area law can be similarly defined as

$$A_1 \equiv \int_{\delta A_1}^{\delta A_3} T(\delta A)\, d\delta A = T_\star \left(\delta A_3 - \delta A_1\right) \equiv A_3, \tag{24}$$

in which $T(\delta A)$ is an Interpolating Function obtained from our numeric result and δA_1 and δA_3 are the smallest and largest roots of the equation $T(\delta A) = T_\star$, respectively. As the same as that of the geodesic length, for a fixed θ_0, we first obtain δA_1 and δA_3 and then substitute these values into (24) to produce A_1, A_3. The concrete values are listed in Table 1. Obviously, for both θ_0, A_1 and A_3 are equal within the reasonable numeric accuracy. The equal area law thus holds in the δA-T plane, which reinforces the fact that the

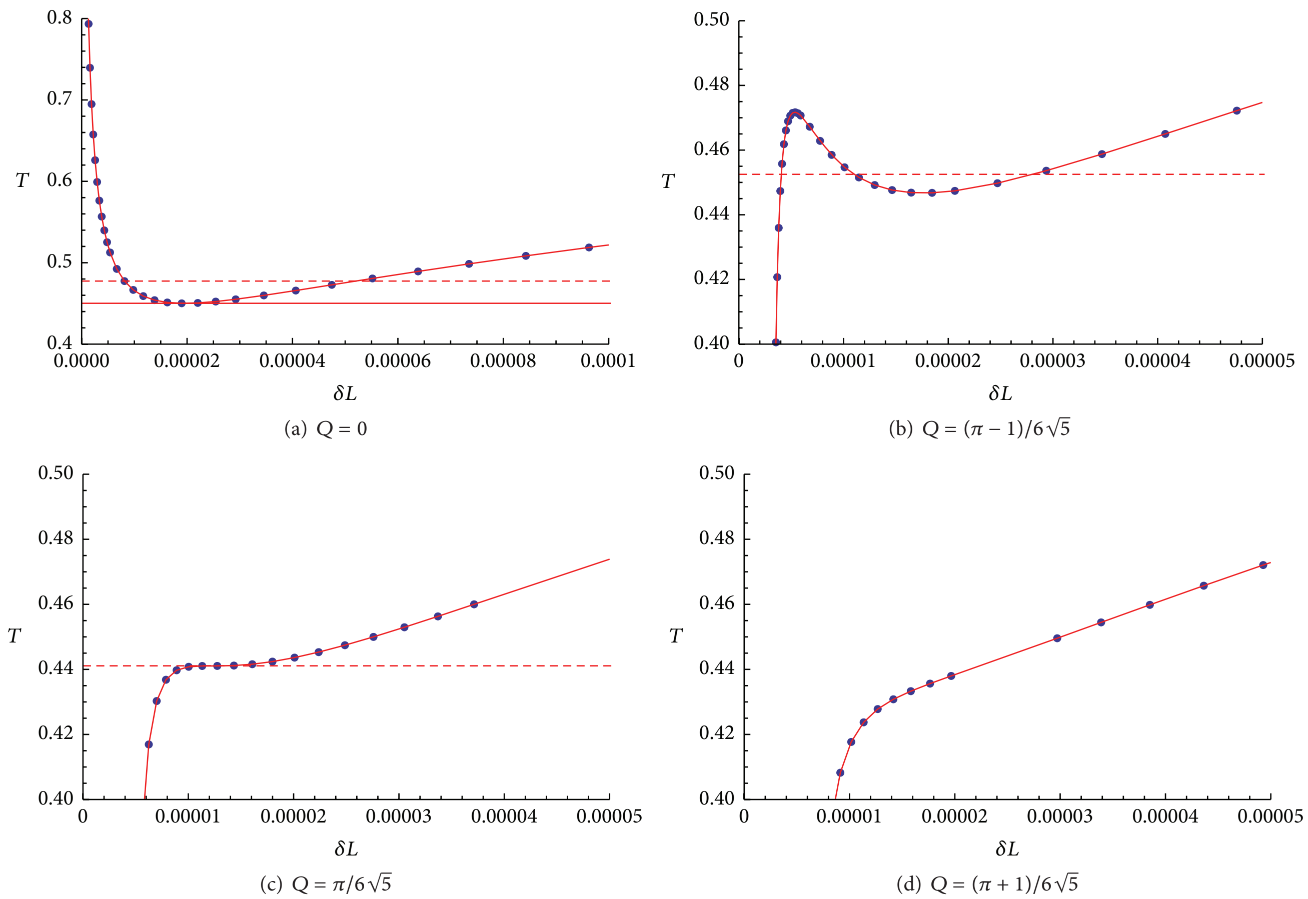


FIGURE 4: Relation between the geodesic length and temperature in the fixed charge ensemble for different charges at $\theta_0 = 0.2$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.

minimal surface area has the same first-order phase transition behavior as that of the thermal entropy.

Second, in order to check whether the minimal surface area also demonstrates the same second-order phase transition as the thermal entropy, we would like to evaluate the critical exponent of the analogous heat capacity at the critical point in the δA-T plane. To this end, we plot the relations between $\log|T - T_c|$ and $\log|\delta A - \delta A_c|$ in Figure 8. The numerical results for these curves can be fitted as

$$\log|T - T_c| = \begin{cases} 27.5226 + 3.04698 \log|\delta A - \delta A_c|, & \text{for } \theta_0 = 0.14, \\ 24.692 + 3.00462 \log|\delta A - \delta A_c|, & \text{for } \theta_0 = 0.2. \end{cases} \tag{25}$$

With 3 the slope, we can conclude that the minimal surface area also has the same second-order phase transition as the thermal entropy.

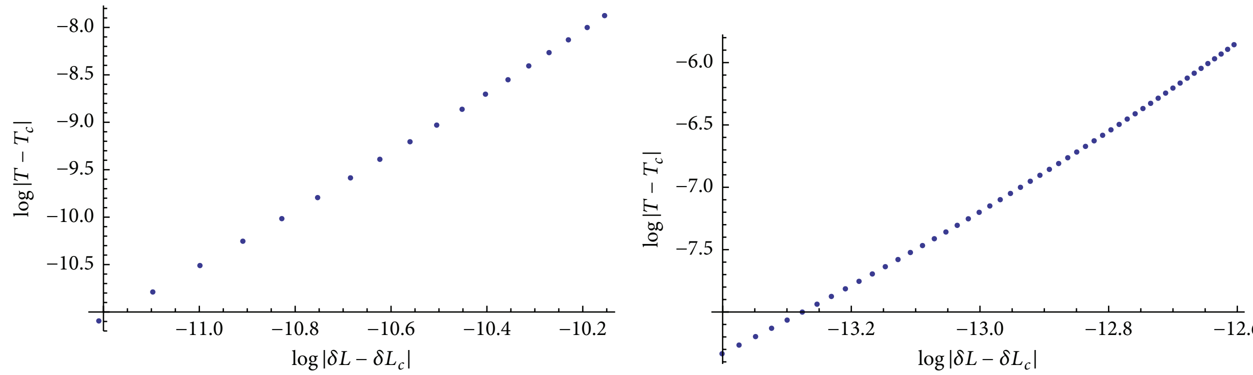
3.3. Phase Transition of Entanglement Entropy. Holographic entanglement entropy is another nonlocal observable, and it has been used extensively to probe the superconductivity phase transition besides the thermalization process recently [33–40]. In this subsection, we intend to employ it to probe the phase structure of a 5-dimensional Reissner-Nordström-AdS black hole. According to the formula in [41, 42], holographic entanglement entropy can be given by the area A_Σ of a minimal surface Σ anchored on the boundary entangling surface $\partial\Sigma$; namely,

$$S = \frac{A_\Sigma(t)}{4G}. \tag{26}$$

For simplicity, we choose $\phi = \phi_0$ as our entangling surface and employ (ϕ, θ, ψ) to parameterize the minimal surface. But with the symmetry of (2), (26) can be rewritten as

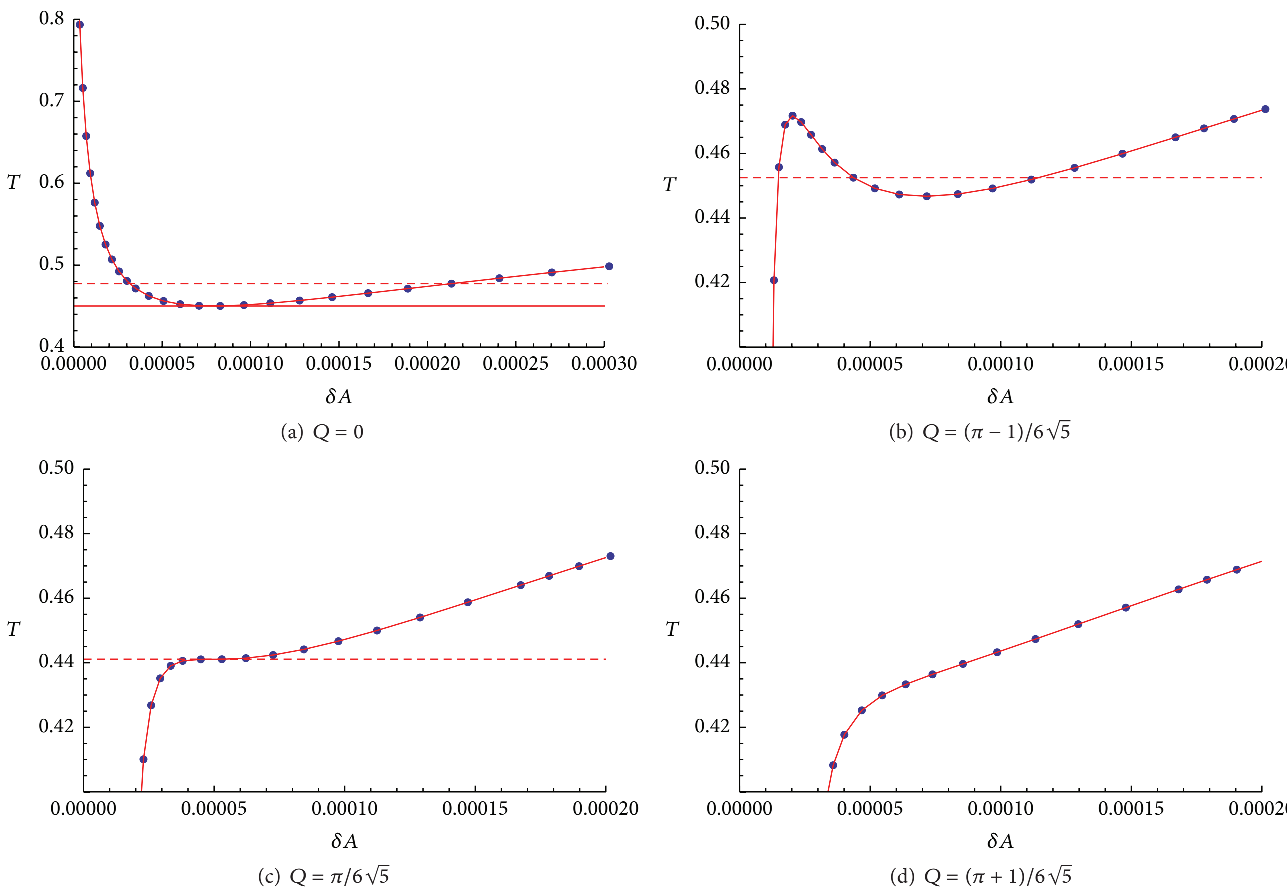
$$S = 4\pi \int_0^{\phi_0} r^2 \sin^2\phi \sqrt{\frac{\dot{r}^2}{f(r)} + r^2 d\phi} \tag{27}$$

with $\dot{r} = dr/d\phi$. Similarly, we can solve the equation of motion for $r(\phi)$ numerically and eventually obtain the regularized entanglement entropy δS. We plot the relation



(a) $\theta_0 = 0.14$ (b) $\theta_0 = 0.2$

FIGURE 5: Relation between $\log|T - T_c|$ and $\log|\delta L - \delta L_c|$ near the critical point of second-order phase transition for different θ_0.



(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

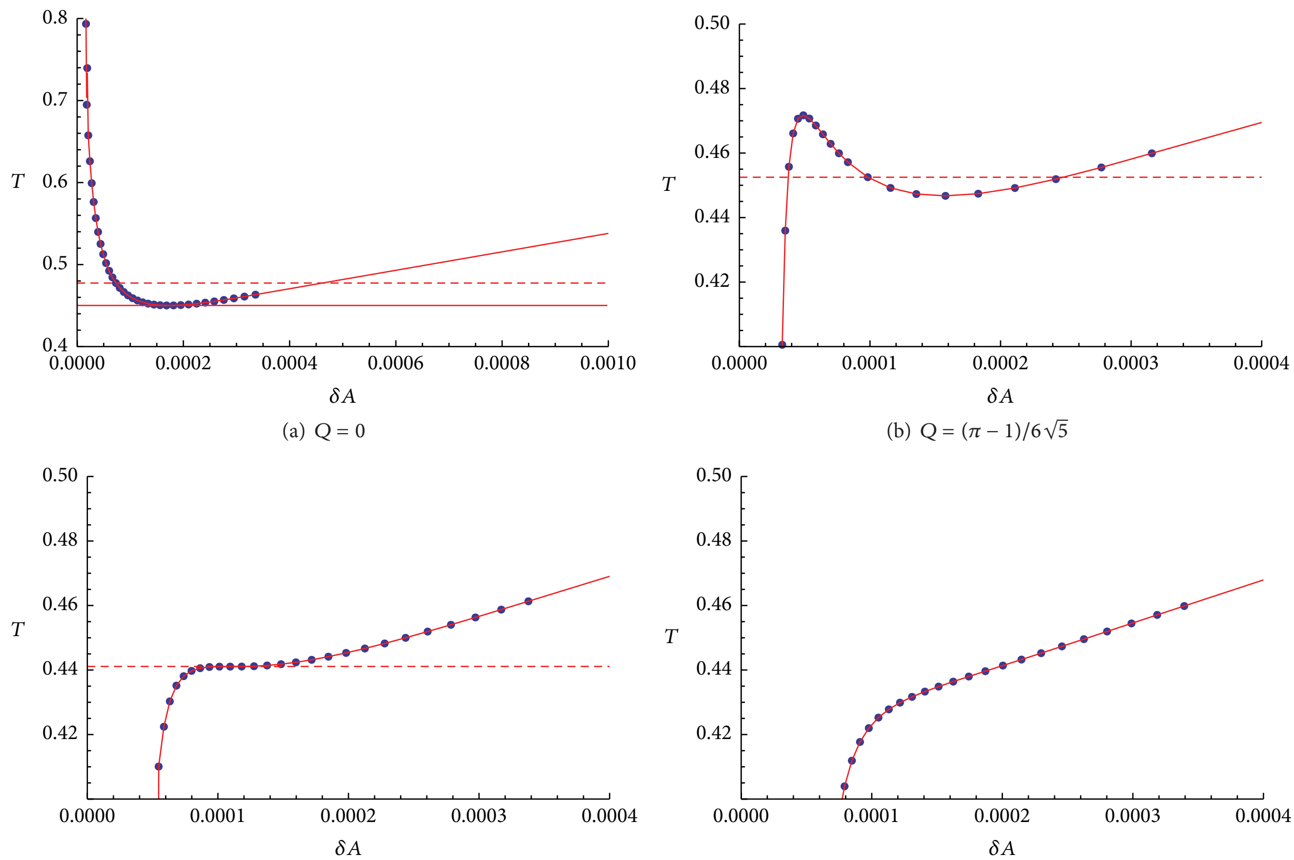
(c) $Q = \pi/6\sqrt{5}$

(d) $Q = (\pi + 1)/6\sqrt{5}$

FIGURE 6: Relation between the minimal area surface and temperature in the fixed charge ensemble for different charges at $\theta_0 = 0.14$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.

TABLE 1: Check of the equal area law in the T-δA plane for different θ_0.

$\theta_0 = 0.14$	$\theta_0 = 0.2$
$T_\star = 0.4526$	$T_\star = 0.4526$
$\delta A_1 = 0.0000434217 \mid \delta A_3 = 0.000114835$	$\delta A_1 = 0.0000374703 \mid \delta A_3 = 0.000248923$
$A_1 = 0.0000320531 \mid A_3 = 0.0000323218$	$A_1 = 0.0000957474 \mid A_3 = 0.0000957035$



(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

(c) $Q = \pi/6\sqrt{5}$

(d) $Q = (\pi + 1)/6\sqrt{5}$

FIGURE 7: Relation between the minimal area surface and temperature in the fixed charge ensemble for different charges at $\theta_0 = 0.2$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.

between δS and T for $\phi_0 = 0.14, 0.2$ in Figures 9 and 10, respectively. As one can see, it exhibits a similar behavior as the thermal entropy. To be more precise, we would like to check the equal area law with the following equation:

$$A_1 \equiv \int_{\delta S_1}^{\delta S_3} T(\delta S)\, d\delta S = T_\star (\delta S_3 - \delta S_1) \equiv A_3, \qquad (28)$$

in which $T(\delta S)$ is an Interpolating Function obtained from the numeric result and δS_1 and δS_3 are the smallest and largest roots of the equation $T(\delta S) = T_\star$. For different $\phi_0 s$, the results of δS_1, δS_3 and A_1, A_3 are listed in Table 2. It is obvious that A_1 nearly equals A_3 regardless of the choice of ϕ_0. That is, the equal area law is also valid for the entanglement entropy.

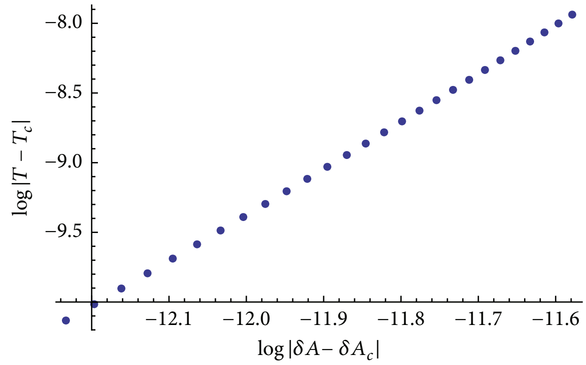
To get the critical exponent of second-order phase transition of entanglement entropy, we should find the slope of a linear function represented by $\log|T - T_c|$ and $\log|\delta S - \delta S_c|$, in which S_c is the critical entropy obtained numerically by the equation $T(\delta S) = T_c$. The numeric results for different ϕ_0 are plotted in Figure 11. The results for these curves can be further fitted as

$$\log|T - T_c| = \begin{cases} 26.653 + 3.00107 \log|\delta S - \delta S_c|, & \text{for } \phi_0 = 0.14 \\ 21.2674 + 2.92789 \log|\delta S - \delta S_c|, & \text{for } \phi_0 = 0.2. \end{cases} \qquad (29)$$

One can see that the slope is always about 3 for different ϕ_0. So we can conclude that the entanglement entropy also has the same second-order phase transition as the thermal entropy.

4. Concluding Remarks

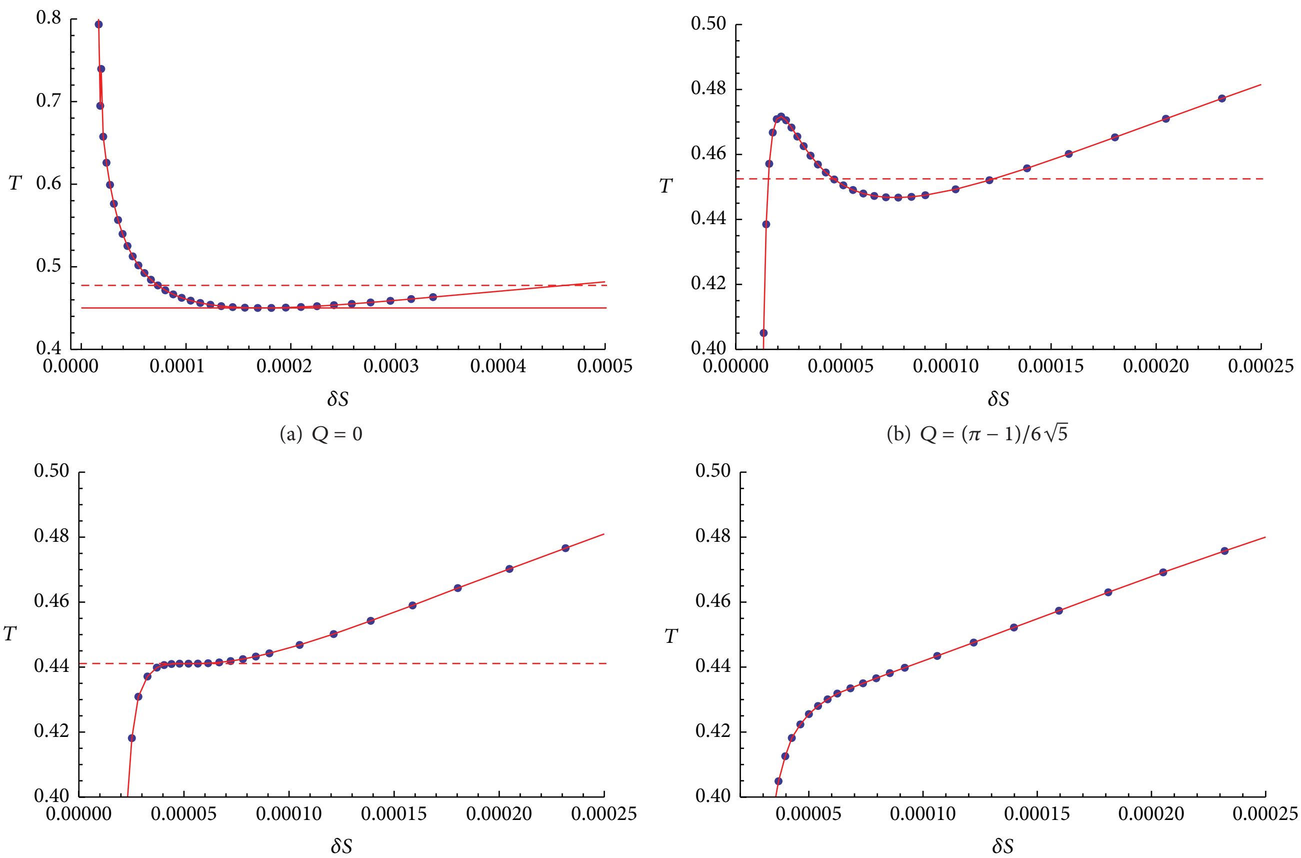
Investigation on the phase transition of the black holes is important and necessary. On the one hand, it is helpful for us to understand the structure and nature of the space time. On the other hand, it may uncover some phase transitions of the realistic physics in the conformal field theory according to the AdS/CFT correspondence. It is well known now that the Hawking-Page phase transition in the gravity system



(a) $\theta_0 = 0.14$

(b) $\theta_0 = 0.2$

FIGURE 8: Relation between $\log |T - T_c|$ and $\log |\delta A - \delta A_c|$ near the critical point of second-order phase transition for different θ_0.



(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

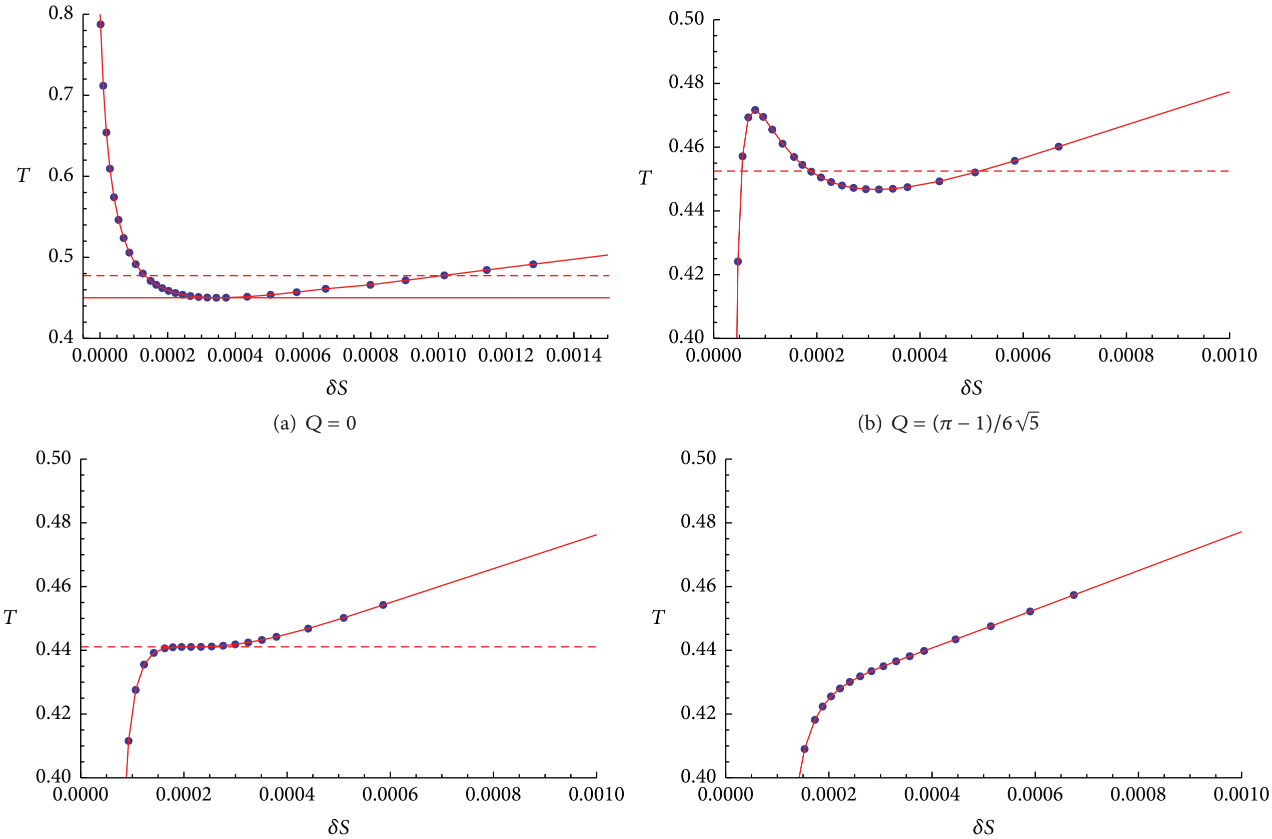
(c) $Q = \pi/6\sqrt{5}$

(d) $Q = (\pi + 1)/6\sqrt{5}$

FIGURE 9: Relation between the entanglement entropy and temperature in the fixed charge ensemble for different charges at $\phi_0 = 0.14$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.

TABLE 2: Check of the equal area law in the T-δS plane for different ϕ_0.

$\phi_0 = 0.14$	$\phi_0 = 0.2$
$T_\star = 0.4526$	$T_\star = 0.4526$
$\delta S_1 = 0.0000155355 \mid \delta S_3 = 0.000123195$	$\delta S_1 = 0.000186918 \mid \delta S_3 = 0.000517552$
$A_1 = 0.0000487431 \mid A_3 = 0.0000487266$	$A_1 = 0.000148384 \mid A_3 = 0.000149645$



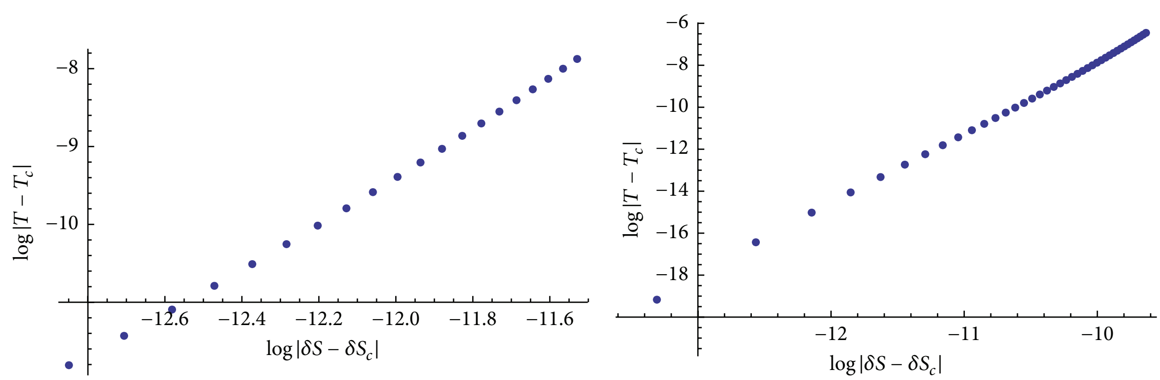
(a) $Q = 0$

(b) $Q = (\pi - 1)/6\sqrt{5}$

(c) $Q = \pi/6\sqrt{5}$

(d) $Q = (\pi + 1)/6\sqrt{5}$

FIGURE 10: Relation between the entanglement entropy and temperature in the fixed charge ensemble for different charges at $\phi_0 = 0.2$. The red solid line corresponds to the location of the minimum temperature T_0; the dashed lines in (a), (b), and (c) correspond individually to the locations of Hawking-Page phase transition T_1, first-order phase transition $T_\star$, and second-order phase transition T_c.



(a)

(b)

FIGURE 11: Relation between $\log|T - T_c|$ and $\log|\delta S - \delta S_c|$ near the critical point of second-order phase transition for different ϕ_0.

is dual to the confinement/deconfinement phase transition, and the phase transition of a scalar field is dual to the superconductivity phase transition in the dual conformal field theory.

In this paper, we investigated the van der Waals-like phase transition in the framework of holography so that we can explore whether there is a realistic similar phase transition in physics. Taking the 5-dimensional Reissner-Nordström-AdS black hole as the gravity background, we investigated the phase structure of the two-point correlation function, Wilson loop, and holographic entanglement entropy. For all the nonlocal observables, we observed that the black hole undergoes a van der Waals-like phase transition. This conclusion is reinforced by the investigation of the equal area law and critical exponent of the analogous heat capacity in which we found that the equal area law is valid always and the critical exponent of the heat capacity coincides with that of the mean field theory regardless of the size of the boundary region. In addition, we found the black hole undergoes a Hawking-Page phase transition before the van der Waals-like phase transition for all the nonlocal observables. We also obtained the minimum temperature and Hawking-Page phase transition temperature. Our investigation thus provides a complete picture depicting the phase transition of charged AdS black hole in the framework of holography.

Competing Interests

The authors declare that they have no competing interests.

Acknowledgments

The authors would like to thank Rong-Gen Cai for his discussions. This work is supported by the National Natural Science Foundation of China (Grants nos. 11405016 and 11575270), China Postdoctoral Science Foundation (Grant no. 2016M590138), Natural Science Foundation of Education Committee of Chongqing (Grant no. KJ1500530), and Basic Research Project of Science and Technology Committee of Chongqing (Grant no. cstc2016jcyja0364).

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Research Article

Thermodynamic Partition Function from Quantum Theory for Black Hole Horizons in Loop Quantum Gravity

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We establish the link between the thermodynamics and the quantum theory of black hole horizons through the construction of the thermodynamic partition function, partly based on some physically plausible arguments, by beginning from the description of quantum states of the horizon, considering loop quantum gravity (LQG) as the underlying theory. Although the effective "thermalized" form of the partition function has been previously used in the literature to study the effect of thermal fluctuations of the black hole horizon, nonetheless the direct link to any existing quantum theory (which is here taken to be LQG), especially a derivation of the partition function from the quantum states of the horizon, appears to be hitherto absent. This work is an attempt to bridge this small, but essential, gap that appears to be present between the existing literature of quantum theory and thermodynamics of black holes. Furthermore, it may be emphasized that this work is *only* concerned with the *metric independent* approaches to black hole thermodynamics.

1. Introduction

The thermodynamic properties associated with spacetime geometry are intimately associated with the presence of a physical horizon. In the Euclidean quantum gravity approach, it had been actually proven that the entropy of the spacetime geometry vanishes in the absence of a horizon [1]. Similarly, using certain properties of Canonical quantum gravity, within certain approximations, it has been shown that the partition function of a spacetime reduces to that of the horizon only [2, 3]. The physical significance of the presence of the horizon looks even more compelling when we find that (i) the entropy of the black hole is given by its horizon area instead of the volume of the region behind the horizon [4, 5], (ii) the thermal radiation of the black hole is emitted at the temperature which is the one associated with the horizon [6, 7], and (iii) there are laws of thermodynamics associated with the horizon [8]. All these findings compel us to think that there may be some quantum theory of the horizon geometry itself which describes the independent quantum degrees of freedom of the horizon and then only we can have a proper understanding and logical reasoning behind the emergent thermodynamic properties of the horizon manifested in its own right. The knowledge of such quantum theory of the horizon will also enable us to formulate a spacetime-metric or back ground independent horizon thermodynamics. In fact, in recent literature there have been some serious attempts to study horizon thermodynamics without referring to the bulk spacetime by constructing the partition function for the horizon only and investigating the relevant thermodynamic properties.

Thermal fluctuations of black hole horizons have been studied in various details by several authors in the recent past leading to interesting results regarding black hole thermodynamics [2, 3, 9–14]. In [2, 3, 14] thermal stability of black holes under thermal fluctuations was studied in canonical and grand canonical ensembles leading to a stability criterion for black holes which successfully explained the stabilities and instabilities of certain known black hole solutions. On the other hand, the issues of canonical and grand canonical entropy of black holes in presence of thermal fluctuations have been studied in [9–13]. The common and unique property of all these works is the use of the horizon partition function, whose construction is based on some heuristic modeling or assumptions or approximations regarding the underlying quantum structure of the horizon geometry, without referring to any classical background geometric structures, such as the metric and hence valid for arbitrary black hole horizons independent of the structure of the associated bulk spacetime. The thermodynamic properties associated with black hole horizons urge one to think of an inherent quantum description of the horizon geometry which will give rise to the notion of microstates so as to have a physical meaning of the associated entropy and hence thermality. This is the underlying physical essence of this particular statistical mechanical approach to horizon thermodynamics presented and investigated in [2, 3, 9–14].

However, none of these works [2, 3, 9–14] presents a logical straightforward derivation of the partition function based on the fundamental quantum structure of black hole horizons, which would have made the link between the microscopic quantum theory of the horizon and the associated thermodynamics as had been urged by the works in [1, 4–8]. This crucial direct link still goes begging, even though there is a well defined quantum theory for the horizon geometry present in the literature for several years [15, 16]. Perhaps the clearest quantum description of the geometry of a black hole horizon is provided by loop quantum gravity [15, 16], where the quantum degrees of freedom of the horizon belong to the Hilbert space of a quantum Chern-Simons theory coupled to point like sources (to be explained in detali later).

The theory, thus, provides a self-contained platform for the application of the statistical mechanical techniques so as to unravel the corresponding thermodynamic properties of the black hole horizon. This paper is aimed at providing a direct link between the quantum theory of horizon geometry and the associated thermodynamics through a derivation of the thermodynamic partition function associated with the horizon. This bridges the gap between the quantum geometric framework of equilibrium horizons laid down in [15, 16] and the thermodynamic aspects of the horizon studied in [2, 3, 9–14], which in turn answers the issues motivated by the works in [1, 4–8] as discussed earlier. This is precisely the important value addition of this work to the extant literature and hopefully may be of interest to the concerned readers. The structure of the paper can be debriefed as follows.

2. Equilibrium Horizon and Quantum Fluctuations: Microcanonical Ensemble

In modern day literature, a black hole horizon in gravitational and thermal equilibrium with ambient matter and radiation is described by the notion of an isolated horizon (IH) [17–19], which is a 2 + 1-dimensional null inner boundary of a 3 + 1-dimensional spacetime. An IH is a generalization of an event horizon of a stationary black hole spacetime to a more realistic scenario. As opposed to the global notion of the event horizon, an IH is defined locally, without any reference of the ambient bulk spacetime and hence can admit matter and radiation arbitrarily close to itself, whereas an event horizon cannot. The laws of black hole mechanics are completely realizable in the local framework of IH [19–21]. The mass and the surface gravity associated with the IH are also defined from a completely local perspective [19]. The strength of the IH framework is its local, background independent description in terms of the connection variables, which has led to rigorous analysis of the connection dynamics, symplectic structure, and so forth of the IH. The analysis reveals that the symplectic structure of the IH is that of an SU(2) Chern-Simons (CS) theory [17, 18] (a concerned reader may look into [22] where, taking the event horizon in the Schwarzschild spacetime as an example (which is a special case of an IH), it has been shown that there is indeed an SU(2) CS theory on the event horizon). In the quantum theory, the states of a QIH are given by that of the CS theory coupled to the edges of the bulk spin network which span the bulk quantum geometry and intersect the IH at specific points called punctures [15, 16].

The Hilbert space of a quantum spacetime admitting QIH as an inner boundary is given by $\mathscr{H} = \mathscr{H}_V \otimes \mathscr{H}_S$ modulo gauge transformations, where V denotes bulk and S denotes boundary (QIH) at a particular time slice [15, 16]. Mathematically, if the 4d spacetime ($\mathbf{R} \times \Sigma$) admits a 3d IH (Δ) as null inner boundary, then $S \equiv \Delta \cap \Sigma$ denotes a cross section of the IH [17, 18]. Hence, a generic quantum state of the spatial geometry of such a spacetime can be written as $|\Psi\rangle = |\Psi_V\rangle \otimes |\Psi_S\rangle$, where $|\Psi_V\rangle$ is the wave function corresponding to the volume (V) or bulk states represented by an oriented graph, say Γ, consisting of edges and vertices [23] and $|\Psi_S\rangle$ denotes a generic quantum state of the QIH. $|\Psi_S\rangle \in \mathscr{H}_S \equiv$ the Hilbert space of the CS theory coupled to the punctures $\{\mathscr{P}\}$ made by the bulk spin network Γ with the IH endowing them with the spin representations carried by the respective piercing edges which are solely responsible for all the relevant features of the QIH, the most important being the quantum area spectrum of the QIH. To be precise, for a given N number of punctures, with spins $(j_1, \ldots, j_N)$, the QIH Hilbert space is given by $\mathscr{H}_S \equiv \mathrm{Inv}(\bigotimes_{i=1}^{N} \mathscr{H}_{j_i})$ where "Inv" denotes the invariance under the local SU(2) gauge transformations on the QIH. Now, as it is seen that at the quantum level the full Hilbert space is the direct product space of the bulk and boundary Hilbert spaces, a generic quantum state of the QIH (boundary) can be written in terms of basis states on $\mathscr{H}_S$, independent of the bulk wave function. Hence, one should understand that a basis state of the QIH Hilbert space is actually a generic quantum state of the full Hilbert space, since the bulk part of the wave function is a linear combination of the basis states of the bulk geometry. In other words, a given *spin configuration* on the QIH admits all possible graphs (Γ-s) in the bulk consistent with the given configuration. This spin configurations provide the area eigenstate basis, which is the all important material in the context of QIH entropy. Such a basis state of the QIH Hilbert space is denoted by the ket $|\{s_j\}\rangle$. This is an eigenstate of the area operator associated with the QIH, having the area eigenvalue given by $\widehat{A}_S|\{s_j\}\rangle = 8\pi\gamma\ell_p^2 \sum_{j=1/2}^{k/2} s_j \sqrt{j(j+1)}|\{s_j\}\rangle$. Such a spin configuration (eigenstate) has a $(N!/\prod_j s_j!)$-fold degeneracy due to the possible arrangement of the spins yielding the same area eigenvalue. Hence, a generic quantum state of the QIH can be written as

$$|\Psi_S\rangle = \sum_{\{s_j\}} c\left[\{s_j\}\right]\left|\{s_j\}\right\rangle, \tag{1}$$

where $|c[\{s_j\}]|^2 = \omega[\{s_j\}]$ (say) is the probability that the QIH is found in the state $|\{s_j\}\rangle$.

We would like to mention that the topological structures, that is, the punctures, arise only in the quantum theory and we can consider it to be a macroscopic parameter only if we deal with an ensemble of QIHs rather than an ensemble of IHs. Thus, the Hilbert space of a QIH for a given N should be appropriately written with proper designation as $\mathscr{H}_S^{k,N} = \mathrm{Inv}(\bigotimes_{l=1}^N \mathscr{H}_{j_l})$. But, the full Hilbert space of a classical IH designated by the corresponding CS level k takes into account all possible sets of punctures [15, 16] and is given by

$$\mathscr{H}_S^k = \bigoplus_{\{\mathscr{P}\}} \mathrm{Inv}\left(\bigotimes_{l=1}^N \mathscr{H}_{j_l}\right), \tag{2}$$

where

$$\bigoplus_{\{\mathscr{P}\}} \equiv \bigoplus_{N;1/2\le j_l\le k/2\ \forall l\in[1,N]\ni\sum_{l=1}^N \sqrt{j_l(j_l+1)}=k/2\pm\mathscr{O}(1/8\pi\gamma)} \tag{3}$$

and γ is the Barbero-Immirzi parameter. The CS level (k) is defined as $k \equiv A/4\pi\gamma\ell_p^2$, where A is the classical area of the IH [15]. In the quantum theory, the CS coupling k is an integer which is a necessary prequantization condition for the quantization of the classical IH [16]. Since we shall begin with the application of quantum mechanical ensemble theory to the QIH, the macroscopic parameters will be considered to be k and N [24] and the microstates resulting in the quantum degeneracy for a given k and N will give rise to the statistical or microcanonical entropy as a function of k and N [24, 25].

3. Quantum Fluctuations: The Microcanonical Ensemble

The number of microstates for a QIH with CS level k and number of punctures N are given by the dimension of the Hilbert space $\mathscr{H}_S^{k,N} = \mathrm{Inv}(\bigotimes_{l=1}^N \mathscr{H}_{j_l})$, where $1/2 \le j_l \le k/2\ \forall l \in [1, N]$. The formula for the number of microstates for an arbitrary sequence of spins $(j_1, \ldots, j_N)$ on N number of punctures on the QIH is given by [26]

$$\Omega(j_1,\ldots,j_N) = \frac{2}{k+2} \cdot \sum_{a=1}^{k+1} \frac{\sin(a\pi(2j_1+1)/(k+2))\cdots\sin(a\pi(2j_N+1)/(k+2))}{(\sin(a\pi/(k+2)))^{N-2}}. \tag{4}$$

To obtain the total number of microstates of a QIH for given k and N, that is, the dimensionality of $\mathscr{H}_S^{k,N}$, we must consider all possible values of spin from $1/2$ to $k/2$ for each puncture. This is the precise argument which was provided in [24] in the other way round so as to render k and N to be macroscopic parameters designating a macroscopic state of a QIH that is taking the sum over all spins; the only parameters which are left bare are k and N. Hence we can write $\Omega(k, N) = \sum_{j_1,\ldots,j_N} \Omega(j_1, \ldots, j_N)$, where each spin is summed from $1/2$ to $k/2$. In principle and by definition, the microcanonical entropy of the QIH is given by $S_{\mathrm{MC}} = \log \Omega(k, N)$, where we have set the Boltzmann constant to unity. In practice [24], the calculation of $\Omega(k, N)$ is performed only approximately (the approximation is good enough in the limit $k, N \to \infty$ which is the appropriate for QIHs with large classical area and large number of punctures) by finding the *most probable distribution* through the application of the method of Lagrange undetermined multipliers. The calculation has been extensively carried out in [24] and is needless to repeat here. What we shall discuss here is the physical essence of the method alongside mentioning the crucial steps of the calculation. Firstly, one can express the sum over spin values as sum over spin configurations by the application of the multinomial theorem [24]; that is,

$$\Omega(k,N) = \sum_{j_1,\ldots,j_N} \Omega(j_1,\ldots,j_N) = \sum_{\{s_j\}} \Omega\left[\{s_j\}\right], \tag{5}$$

where

$$\Omega\left[\{s_j\}\right] = \frac{N!}{\prod_j s_j!} \frac{2}{k+2} \sum_{a=1}^{k+1} \sin^2\frac{a\pi}{k+2} \cdot \prod_j \left\{\frac{\sin(a\pi(2j+1)/(k+2))}{\sin(a\pi/(k+2))}\right\}^{s_j}. \tag{6}$$

This puts forward a very clear picture of the physical scenario which manifests the quantum jumps of QIH state from one area eigenstate $|\{s_j\}\rangle$ to another, since the spin configurations are the eigenstates of the area operator of the QIH as discussed earlier. The variation of spin values all over the punctures of the QIH is the quantum fluctuations of the system. Now, any puncture can take any spin value from $1/2$ to $k/2$ with equal a priori probability and hence all the spin sequences are equally probable. But when we look at the spin configurations, the one allowing the maximum number of possible equi-probable microstates (spin sequences) that is maximally degenerate is the most probable spin configuration. Hence quantum state corresponding to the most probable configuration is the highest entropy state. At par with the basic postulates of equilibrium statistical mechanics, the degeneracy corresponding to this state overwhelmingly outnumbers the degeneracies corresponding to the other subdominant configurations which may be regarded as negligible quantum fluctuations. Thus, it is a good enough approximation to consider $\Omega(N,k) \simeq \Omega[\{s_j^\star\}]$, where $s_j^\star$ is the distribution corresponding to the most probable spin configuration and comes out to be

$$s_j^\star \simeq N(2j+1)\exp\left[-\lambda\sqrt{j(j+1)} - \sigma\right]. \tag{7}$$

It may be mentioned that the classical area (A_{cl}) is related to k as $k = A/4\pi\gamma\ell_p^2$. It is straightforward to show that the microcanonical entropy comes out to be [24, 25]

$$S_{\mathrm{MC}} = \frac{\lambda k}{2} + N\sigma, \tag{8}$$

where λ and σ are the Lagrange multipliers and we have neglected the subleading terms in the entropy [24] that are irrelevant in the present analysis. The Lagrange multipliers are solvable in terms of k and N from the following equations [24, 25]:

$$e^{\sigma} = \frac{2}{\lambda^2}\left(1 + \frac{\sqrt{3}}{2}\lambda\right)e^{-(\sqrt{3}/2)\lambda}, \tag{9a}$$

$$\frac{k}{N} = 1 + \frac{2}{\lambda} + \frac{4}{\lambda\left(\sqrt{3}\lambda + 2\right)}. \tag{9b}$$

It should be noted that the most probable spin configuration is determined by the given set of values of k and N.

The physical essence of the microcanonical ensemble analysis is that the thermal fluctuations of the macroscopic variables are completely disallowed and what we deal with are purely quantum mechanical fluctuations of the system. It is easy to understand because we define the microcanonical ensemble by fixing the macroscopic variables, which are here k and N, and do not allow them to fluctuate. Thus the classically behaving macroscopic variables remain constant and the QIH jumps randomly from one quantum mechanical state to another which gives rise to the quantum uncertainty and hence the microcanonical entropy. In that sense, the resulting entropy is purely statistical. As we shall pass on to the grand canonical ensemble scenario, the effects of the thermal fluctuations will be taken into account because now the fixed macroscopic variables will be allowed to fluctuate.

Note. At this point, having known the degeneracy corresponding to given k and N, we can write down the canonical partition function as

$$Z_C(\beta, N) = \sum_k \Omega(k, N)\exp\left[-\beta E(k, N)\right] \tag{10}$$

and the grand canonical partition function as

$$Z_G(\beta, \mu) = \sum_{k,N}\Omega(k, N)\exp -\beta\left[E(k, N) - \mu N\right], \tag{11}$$

where $E(k, N)$ is the mean energy or the statistical mean of the Hamiltonian operator for the QIH (although the classical energy associated with the classical IH [19] has not been quantized till date, there has been a proposal of the most general possible structure of the Hamiltonian operator associated with the QIH [27, 28], which satisfies the properties of the classical energy in the correspondence limit) with given k and N. This is like considering several different copies of microcanonical ensembles of QIHs designated by different sets of values of k and N and then dealing with the effective classical behaviour of the relevant quantities, the quantum fluctuations being averaged out [29]. So (10) and (11) can be regarded as the "thermalized" forms of the quantum mechanical canonical and grand canonical partition functions, respectively, since the many-body structure of the QIH is not manifested at all in these above forms and is glossed out by the classically behaving macroscopic variables. *Similar* forms of the partition functions have been extensively used to investigate the thermodynamics of black holes in the IH framework previously in literature [2, 3, 9, 11–14]. However, there are several technical caveats in common which plague the approaches which also curtains some important physical issues regarding the quantum statistics and thermodynamics of black holes. In the forthcoming sections, we shall arrive at the above "thermalized" forms of the partition functions by beginning from the treatment with quantum mechanical ensembles of QIHs to get an in depth understanding of the thermodynamical issues based on the underlying quantum structures. Remarkably, albeit expectedly, the technical caveats of the earlier approaches to black hole thermodynamics get removed automatically, alongside the fact that the associated physical understandings become far more transparent, which will be discussed in detail in a separate section.

4. Thermalization: Other Quantum Mechanical Ensembles

Having reviewed the microcanonical ensemble results for a QIH, now we shall investigate the other quantum mechanical ensembles, namely, canonical and grand canonical. In the following paragraphs we shall explore in detail the construction of the canonical and grand canonical partition functions beginning from the very basic definitions and arrive at their respective "thermalized" forms.

4.1. Quantum Mechanical Canonical Partition Function. The Hilbert space structure $\mathscr{H}_S^{k,N}$ of a QIH with given k and N emulates that of a gas of particles [30] in the sense that every physical phenomenon concerning a QIH is a collective many-body effect, a puncture being an individual body. It is also evident from the quantum area of QIH that is $8\pi\gamma\ell_p^2\sum_{l=1}^{N}\sqrt{j_l(j_l+1)}$ [15, 16] or the quantum energy of the QIH proposed in [27, 28]. Following this quantum "particle-like" structure of the QIH, the quantum mechanical canonical partition function can be written as

$$Z_C(\beta, N) = \sum_{\{s_j\}}\Omega\left[\{s_j\}\right]\exp\left[-\beta E\left[\{s_j\}\right]\right], \tag{12}$$

where $E[\{s_j\}] = \sum_j \epsilon_j s_j$ is the quantum energy of the QIH corresponding to the spin configuration $\{s_j\}$ and ϵ_j is the energy contribution from a single puncture carrying spin j and for any configuration, the total number of punctures must add up to N since it is kept fixed in the canonical ensemble.

Now, from (7), considering that λ and σ are solvable in terms of k and N from (9a) and (9b), it is manifested that there is $s_j^\star$ for every k and N. Thus, considering a fixed value of N, a different choice of k will give a different $s_j^\star$. It is then completely physical to argue that the most probable distributions corresponding to $k-1$ and $k+1$ are the closest less probable distributions corresponding to the most probable distribution for k. Likewise, the other distributions in (12) are the most probable distributions for some k. Thus, for fixed N, in the quantum mechanical canonical ensemble, the sum over spin configurations can be replaced by sum over k; that is,

$$Z_C(\beta, N) = \sum_k \Omega(k, N) \exp\left[-\beta E(k, N)\right], \tag{13}$$

where β is the temperature of the heat bath with which the QIH is considered to be in thermal contact. Since k is an integer and since we deal with the large k regime only the above expression can be well approximated by replacing with the summation by an integration as follows:

$$Z_C(\beta, N) \simeq \int dk\, \Omega(k, N) \exp\left[-\beta E(k, N)\right]. \tag{14}$$

Equation (14) is the expression for the canonical partition function for a QIH which is suitable for studying the thermal fluctuations. This may be regarded as the "thermalized" form of the quantum mechanical canonical partition function given by (12). The information about the many-body quantum structure of the QIH is glossed by the classical behaviour of the the macroscopic variables.

4.2. Quantum Mechanical Grand Canonical Partition Function. Now, we shall allow the macroscopic variable N to vary along side the energy. So, let us consider a grand canonical ensemble of QIHs where the number of punctures N is now allowed to vary along side the energy. The quantum mechanical grand canonical partition function can be written as

$$Z_G(\beta, \mu) = \sum_{N=1}^{\infty} \exp\left[\beta\mu N\right] Z_C(\beta, N), \tag{15}$$

where Z_C is the partition function for a quantum mechanical canonical ensemble of QIHs. μ is some fictitious parameter conjugate to the macroscopic variable N, playing a role *analogous* to the chemical potential in case of a gas of particles. Let us rewrite (15) manifesting the many-particle structure of the QIH as

$$\begin{aligned} &Z_G(\beta, \mu) \\ &\quad = \sum_{\{s_j\}} \Omega\left[\{s_j\}\right] \exp\left[-\beta E\left[\{s_j\}\right] + \beta\mu N\left[\{s_j\}\right]\right]. \end{aligned} \tag{16}$$

Now, we can argue in a similar way as in the case of the canonical ensemble. From (7), considering that λ and σ are solvable in terms of k and N from (9a) and (9b), it is manifested that there is $s_j^\star$ for every k and N. Thus, different choices of k and N will give different $s_j^\star$. Based on a physical argument similar to that of the quantum mechanical canonical ensemble, in the quantum mechanical grand canonical ensemble, the sum over spin configurations can be replaced by sum over k and N; that is,

$$Z_G(\beta, \mu) = \sum_{k,N} \Omega(k, N) \exp -\beta\left[E(k, N) - \mu N\right]. \tag{17}$$

k, N are like "effective" quantum numbers for the energy and they are equispaced as they are integers. Since we are working in the large k and large N limits, the above discrete sum can be well approximated by replacing the summation by integration:

$$\begin{aligned} &Z_G(\beta, \mu) \\ &\quad \simeq \int dk\, dN\, \Omega(k, N) \exp -\beta\left[E(k, N) - \mu N\right]. \end{aligned} \tag{18}$$

Equation (18) is the expression for the canonical partition function for a QIH which is suitable for studying the thermal fluctuations. This may be regarded as the "thermalized" form of the quantum mechanical grand canonical partition function given by (15). The information about the many-body quantum structure of the QIH is glossed by the classical behaviour of the the macroscopic variables.

5. The Chemical Potential: IH Thermodynamics from QIH Thermodynamics

It is worth mentioning that the most crucial consequence of all the above analysis, from the thermodynamic perspective, is that the grand canonical partition function of QIH with $\mu = 0$ is the canonical partition function for IH; that is, $Z_G^{\mathrm{QIH}}(\beta, 0) = Z_C^{\mathrm{IH}}(\beta)$. This is nothing surprising because looking at the structure of the full Hilbert space of an IH one would naturally consider the number of punctures to be unconserved for a given CS level (k). Thus, in the quantum mechanical canonical partition function of an IH, the sum over all possible quantum states underlying a classical IH implies sum over *all possible* spin sequences which give rise to area eigenvalues within $\mathcal{O}(\ell_p^2)$ of the classical area irrespective of the number of punctures. In fact the partition function written in [16], although in the area ensemble, takes a sum over number of punctures also. But the significance of QIH and quantum hair N remained unnoticed in [16]. In that sense it is more reasonable to consider N as a macroscopic variable and introduce μ on the first hand and then shift to the asymptotic view where the effect of the quantum hair disappears and μ is zero.

However, one must *not* confuse the analysis and the arguments regarding the chemical potential (μ) of a QIH with that of black holes having $\Phi \neq 0$ corresponding to any classical nongravitational charge, for example, Reissner-Nordstrom. One should remember that in this case there is an interplay between gravitational and nongravitational fields to attain the equilibrium. Hence, it is obvious that Φ need not vanish to attain the equilibrium from the asymptotic viewpoint. The most important point to be noted is that one can tune the parameter Φ from outside by controlling the nongravitational field, whereas μ cannot be controlled externally. The parameter Φ actually couples a nongravitational variable (Q) to the partition function but μ does so in the case of N which is purely (quantum) gravitational macroscopic variable. Although Φ and μ appear to be playing the same role in different cases, basically they are very different as far as their physical implications are concerned. Further, Φ has nothing to do with quantum gravity, whereas μ comes into play only when we consider the quantum gravity effects because the corresponding "charge" N is a *quantum hair* [24, 30].

Finally, let us write down the IH partition function explicitly, which can be used directly to study the effects of Gaussian thermal fluctuations on the stability of a black hole [2]. The chemical potential μ is defined as

$$\mu = -\frac{\beta^{-1}\partial S_{\mathrm{MC}}}{\partial N} = -\frac{\sigma}{\beta}. \tag{19}$$

Thus, from the asymptotic viewpoint, as discussed above, μ vanishes and consequently $\sigma = 0$. Now, using $\sigma = 0$ in (9a), one obtains

$$\begin{aligned} 1 &= \frac{2}{\lambda^2}\left(1+\frac{\sqrt{3}}{2}\lambda\right)e^{-(\sqrt{3}/2)\lambda} \Longrightarrow \\ \lambda &= \lambda_0. \end{aligned} \tag{20}$$

The above solution has been obtained graphically in [25] which gives the estimate $\lambda_0 = 1.2$. Consequently, the microcanonical entropy is then given by

$$\begin{aligned} S_{\mathrm{MC}} &= \lambda_0 \frac{k}{2} \\ &= \lambda_0 \frac{A_{\mathrm{cl}}}{8\pi\gamma\ell_p^2} \quad \text{using } k = \frac{A_{\mathrm{cl}}}{4\pi\gamma\ell_p^2} \\ &= \frac{A_{\mathrm{cl}}}{4\ell_p^2} \quad \text{by choosing } \gamma = \frac{\lambda_0}{2\pi}. \end{aligned} \tag{21}$$

One may note that now we have $k = A_{\mathrm{cl}}/2\lambda_0\ell_p^2$. Also, as a parallel consequence, using $\lambda = \lambda_0$ in (9b), one obtains

$$\frac{k}{N} = 1 + \frac{2}{\lambda_0} + \frac{4}{\lambda_0\left(\sqrt{3}\lambda_0+2\right)} = c_0\,(\text{say}). \tag{22}$$

So, to calculate $Z_{\mathrm{IH}}(\beta) = Z_G^{\mathrm{QIH}}(\beta, 0)$, it will not suffice to put $\mu = 0$ in (18), but we also have to impose the relation between k and N in (22) by using a Dirac delta function, namely, $\delta(k/c_0, N)$. Thus, (18) now gets modified as

$$\begin{aligned} & Z_G^{\mathrm{QIH}}(\beta,0) \\ &\quad \simeq \int dk\,dN\,\Omega(k,N)\exp -\beta E(k,N)\,\delta\left(\frac{k}{c_0},N\right) \\ &\quad = \frac{1}{c_0}\int dk\,\Omega(k)\exp -\beta E(k) \\ &\quad = \frac{1}{2c_0\lambda_0}\int dA_{\mathrm{cl}}\,\Omega(A_{\mathrm{cl}})\exp -\beta E(A_{\mathrm{cl}}), \end{aligned} \tag{23}$$

where we have set $\ell_p^2 = 1$ for convenience. Thus, we obtain the canonical partition function for the IH that is $Z_C^{\mathrm{IH}}(\beta)$ beginning from the fundamental quantum structures of the QIH. This is the same partition function which was studied in [2] and generalised to charged horizon in [3], to study the effect of Gaussian thermal fluctuations on the stability of black holes. But, as has been pointed out earlier, the derivations of the partition functions were based on some approximations and failed to establish a clear and direct link with the undelying quantum theory. One can see Appendix B for an elaborate discussion in this issue.

6. Discussion

The prime motive of the paper was the exact derivation of the horizon partition function using the fundamental quantum theory of the horizon geometry, which, we hope, has been elaborately worked out here up to a satisfactory level. The fact that the entropy of spacetime geometry vanishes in the absence of a horizon [1] strongly motivates one to think of independent quantum degrees of freedom of the horizon giving rise to the thermodynamic properties uniquely associated with the horizon; and the derivation of the thermodynamic partition function right from the scratch, which has been presented in this work, satisfactorily quenches the urge by establishing the all needed direct link between the quantum and thermodynamic aspects of the horizon. Hence, the major acquisition one can have by studying this work is the logical step by step derivation of the theromodynamic partition functions of the black hole horizon based on the underlying quantum geometric framework and a clear understanding about the role of quantum and thermal fluctuations in the corresponding thermodynamics. Following this, the "thermalized" form of the partition function derived here, which was previously used in the literature to study the thermal fluctuations of the black hole horizons, now has a clear meaning and stands on the firm ground provided by the quantum geometric description of black hole horizon [15, 16]. This can be considered as an important value addition to the literature of black hole horizon thermodynamics. Apart from this, an important clarification made in this paper is that the QIH thermodynamics is delicately different from IH thermodynamics due to the role of the quantum variable N which is present in the quantum scenario, but absent in the classical thermodynamic picture. In fact, by beginning from IH framework one completely misses out the essence of the underlying quantum structure, resulting in the incorporation of ad hoc assumptions and unnecessary approximations in the procedure. As has been pointed out earlier, the identification of N as a macroscopic variable for QIH is the difference. Now, looking at the full Hilbert space of an IH which allows arbitrary number of punctures for a given k, one may be tempted to ask that why at all shall we consider N as a macroscopic variable a priori. The answer to this question may be that the Hamiltonian operator [27, 28] commutes with the number operator for punctures which renders N to be a constant of motion or, in other words, apparently there is no dynamics that that can change the number of punctures. On the other hand, there is no role of N

at the classical level which is manifested by the absence of any $\mu\delta N$ term in the first law derived from the classical theory of IH [19]. Hence, it is most appropriate to begin from the study of quantum mechanical ensembles of QIHs by considering N as a macroscopic variable and take the limit $\mu = 0$ to arrive at the IH scenario, as shown in this paper. However, all the arguments regarding the role of μ in the local and asymptotic views remain at the qualitative level, the only understanding being the redshift factor as the cause of the behaviour of μ. A quantitative understanding of how μ vanishes at the asymptopia, by an LQG calculation, may bring forth some new insights. Finally, it may be pointed out that although we restrict our analysis to purely gravitational case and stop at canonical partition function of the IH, one can trivially consider nongravitational charge and extend the analysis to the grand canonical partition function of charged IHs. One can look into [3] for this extension to charged horizons. But, one must be alert that there will be tricky issues involved if one tries to generalize the analysis to rotating horizons, because we still do not have a proper quantum theory of rotating IH and this may be considered as a challenging research problem with potential consequences.

Appendix

A. Some Physical Speculations from the Chemical Potential

The chemical potential for QIH is purely of quantum origin and is absent in the classical theory. This makes the role of the chemical potential quite subtle to understand and of course very different from our usual notion of chemical potential occurring in thermodynamics of ordinary systems. It allows us to make some interesting speculations regarding the possible role of the chemical potential μ as far as the physical issues regarding the QIH are concerned and the possible new physics which may follow, in the following sections.

A.1. Quantum Interconversion of Geometry and Matter. In case of a system of gas of particles, we have an easy understanding of the physical meaning to the chemical potential μ. It governs the physical exchange of particles between the system (the gas) and the ambient reservoir, both of which are 3 + 1-dimensional systems. Hence, most crucially, the particles retain their identity whether in the system or in the reservoir and the scenario can be well visualized. But the scenario of QIHs is very much different. First of all the QIH is the quantum structure of the 2 + 1-dimensional IH, whereas the heat bath that is the exterior spacetime is 3 + 1-dimensional. The punctures, which are topological defects on the QIH, are describable only as long as they are on the QIH. They are not defined elsewhere off the QIH. Thus there is no possible physical reservoir of punctures with which the QIH can physically *exchange* punctures. In this case the reservoir in contact with the QIH carries an abstract sense. Well, it can be argued that the punctures get detached to yield open ends in the bulk quantum geometry which are often considered as matter excitations [31], but still they are not punctures which are quantum geometric excitations. The physical process in either direction is a conversion between two *different* objects: (i) detachment of punctures: conversion of horizon geometry to bulk matter; (ii) attachment of punctures: conversion of bulk matter to horizon geometry. Hence, the scenario of exchange of similar objects does not appear here. The very concept of "exchange" gets replaced by the concept of "conversion." To be more precise and to make comparison (or rather distinction) between the two apparently similar scenarios we can say that the fluctuations of N imply "exchange of particles" for a gas and "mutual conversion of bulk matter and horizon geometry at the quantum level" in case of a QIH. It is quite clear that in case of QIHs the physical meaning of the parameter μ is drastically different from that in the case of a gas of particles. The situation urges us to understand the precise quantum dynamics which converts a geometrical excitation on the horizon into an excitation of matter field in the bulk.

A.2. Quantum Topology of the Isolated Horizon: Local versus Asymptotic View. What we have been dealing with in this work is pure gravity and there is no nongravitational field in this scenario. In the LQG framework, the degrees of freedom of the QIH are completely described by CS theory coupled to the topological defects on the horizon which act as sources (topological charges) [15, 16]. The macroscopic variables k and N are well defined as far as LQG is concerned and are pure gravity variables. Both the dynamical and the equilibrium states of the horizon are completely governed by purely quantum gravitational effects, devoid of any nongravitational fields in this scenario. Thus, as opposed to the "tunable" potential corresponding to any nongravitational charges, the "chemical potential" of a QIH corresponding to the topological charge N cannot be tuned externally. What we can do is to observe the behaviour of the parameter and understand its consequences as far as the thermodynamics of a QIH is concerned. Hence, even though we define the grand canonical partition function for a QIH by fixing β and μ, it may happen that a particular value μ is allowed for the equilibrium.

At thermodynamic equilibrium, the macroscopic variables of the QIH are related by an equation of state. No matter crosses the horizon. But, even in this thermodynamic equilibrium condition a nonzero "chemical potential" that is $\mu \neq 0$ will imply a tendency of fluctuation of N about the mean value N_0 which is still dynamically present there. Since the horizon has attained equilibrium that is isolated, the area is now fixed; the fluctuations of N which can now take place are only those which can keep the area fixed. A bit of thinking will allow one to realize that these fluctuations of N are nothing but the changes in the quantum topology of the QIH (e.g., *one* puncture of area a get replaced by *two* punctures of area $a/2$) and hence can be appropriately called quantum topological fluctuations. Thus, in spite of the fixed area at equilibrium, there will be puncture dynamics going on for a nonzero chemical potential which gives rise to the visualization of the quantum description of radiation and accretion from the horizon [32–34].

However, tracing back to the original proposal of the existence of quantum hair N in [30], it should be remembered that these quantum topological fluctuations will only be observable for a local observer close to the horizon and the effects must vanish for an asymptotic observer who can only see the classical IH, the existence of punctures, and corresponding dynamics being glossed out at infinity. This explanation is supported by the fact that the first law of IH thermodynamics [19] does not contain any $\mu\delta N$ term as opposed to that of a QIH [30]. Thus what is observable at asymptopia is quantum topological equilibrium that is the smoothness of the topology of the horizon $S^2 \times R$. Such a quantum topological equilibrium will be attained only for $\mu = 0$.

But for a local observer very close to the horizon, the quantum topological fluctuations must be observable and in that case a positive chemical potential [25] must be there which will result in a quantum topological fluctuation going on for a fixed classical area of the IH. This is like quantum dynamics underlying a classical equilibrium. The situation gives rise to the scenario that an infalling observer just before hitting the horizon will see a region of quantum fuzziness, which must be present there due to the quantum uncertainties evoked by the quantum structure of the IH. However, this explanation is provided on intuitive grounds and remains very much speculative unless one explores the scenario with LQG dynamics.

B. Value Addition to Earlier Literature: Bridging the Gap

Here we discuss the different background independent statistical mechanical approaches to black hole horizon thermodynamics that has gradually evolved step by step in literature, which will finally give us an understanding about the shortcomings of these earlier approaches and how the related technical caveats get eliminated by the present work based on the QIH framework. This will reveal the value addition of the present work to the existing literature.

As far as our knowledge is concerned, a generic statistical mechanical approach without prior use of any background metric was used to study the canonical ensemble scenario with the aim to investigate the effects of thermal fluctuations for black holes in [9–11]. The canonical partition function was written as

$$Z_C = \sum_i g(E_i) \exp -\beta E_i \simeq \int g(E) \exp -\beta E \, dE, \tag{B.1}$$

for a black hole (assuming in the hindsight). This is tantamount to *model* the quantum theory of the black hole as that of a single particle, alike the harmonic oscillator, whose energy spectrum is characterized by a single quantum number and also equispaced. Without the equispaced energy spectrum, the discrete sum cannot be approximated as an integration in any limit. The usual thermodynamic results following from the effective single particle model were applied to selected black holes and relevant conclusions were drawn. The above partition function, if not stated to be only a heuristic *model*, has hardly any relationship with actual black hole quantum states originating from the theory of QIH. The above *model* got improved a bit and some properties of QIH were incorporated in [12]. The canonical partition function was written as

$$\begin{aligned} Z_C &= \sum_i g[E(A_i)] \exp -\beta E(A_i) \\ &\simeq \int dx\, g[E(A(x))] \exp -\beta E[A(x)], \end{aligned} \tag{B.2}$$

considering that the energy is a function of the area of the horizon and then, assuming that all the punctures on the horizon carry spin half (neglecting the effects of other spins), the area spectrum of the horizon is made equispaced; that is, $A \propto N$. The energy states are now characterized by a single quantum number but the energy spectrum is no more equispaced. Thus, the effective single particle model for black holes has got more generalized and for large horizons with sufficiently densely packed quantum states the discrete sum over the quantum number is replaced by an integration over a continuous variable which represents the quantum number in the continuum limit. It was only in [2] that a logical explanation of the partition function was given starting from the IH framework, albeit in a superficial approach using the tools of LQG. The partition function was no more any model but a true partition function derived from the fundamental theory of IH. But still the technical assumptions and approximations regarding the linearized area spectrum and so forth plagued the calculations resulting in the effective one-particle picture.

Similar to the canonical ensemble study of the effective one-particle model, the fluctuations of *nongravitational* charges were studied in the grand canonical ensemble in [11, 13], considering that the effective one particle carries a quantized charge representing the nongravitational charge of the black hole. The partition function is written as

$$\begin{aligned} Z_G &= \sum_i g\left[E(A_i), Q_j\right] \exp -\beta E\left(A_i, Q_j\right) + \beta\Phi Q_j \\ &\simeq \int dx\, dy\, g\left[E(A(x)), Q(y)\right] \\ &\cdot \exp -\beta E(A(x), Q(y)) + \beta\Phi Q(y), \end{aligned} \tag{B.3}$$

using similar assumptions and approximations in the appropriate limits as discussed above for the canonical case, where Φ is the chemical potential corresponding to the nongravitational charge Q. In [3] the partition function was derived in the IH framework, albeit in a superficial approach using the tools of LQG. The explanation of the Hamiltonian associated with the horizon was improved and the structure of the formalism was put on a stronger ground than ever. The partition function was a generic one for charged IH. But still the technical assumptions and approximations regarding the linearized area spectrum and so forth plagued the calculations resulting in the effective one-particle picture. It is worth explaining that the assumption of the linearized quantum area spectrum is tantamount to disregard the role of

N as an independent macroscopic parameter for QIH. Hence, the existence of the macroscopic variable at the quantum level continued to be opaqued by the assumption of the linearized quantum area spectrum.

The relevance of the many-particle nature of the QIH remained unnoticed and hence the canonical ensemble of QIHs and IHs has not been differentiated. It was in [30] that the crucial observation was made and was justified with simple reasonings in [24] that the number of punctures on the QIH should be considered to be an independent macroscopic variable which characterizes a macrostate of a QIH. Hence, it is worth investigating the crucial difference between the canonical ensemble thermodynamics of QIH and classical IH. The canonical partition function is defined for a fixed N for QIH, whereas for classical IH there is no concept of N. That is why the canonical partition function in (14) is tagged by the parameter N, whereas there is no such tag for the canonical partition function of IH in [2]. In case of IH the canonical partition function is defined for fixed nongravitational charges [2] which are allowed to fluctuate in the corresponding grand canonical ensemble [3]. But for the QIH, the many-body quantum structure compels us to treat the complete thermodynamics in the grand canonical ensemble even in the absence of nongravitational charges.

Furthermore, there are some very important consequences of the study of the thermodynamics of the black holes beginning from the fundamental quantum structure of QIH rather than from the classical IH framework. For a canonical ensemble of IH [2] the energy is considered to be a function of area which is again assumed to be equispaced so that the summation can be replaced by integration. This assumption is severe and requires that only similar spins can be considered to be there at each puncture, preferably spin half, which is not true, although statistically spin half is the majority as found from the most probable distribution in the microcanonical ensemble. In the usual background independent approach to study the horizon thermodynamics in [2, 3, 12–14] it is considered that "the energy is a function of area." In a precise sense, this is a very confusing statement. The statement should be made very precise; otherwise the underlying essence is not captured. Suppose one says the classical energy of an IH is the function of its classical area (or equivalently the CS level), which is fine. But this is not any sort of energy spectrum but rather the mean energy from the perspective of the QIH scenario [27, 28]. Thus the statement "equispaced area spectrum" does not have a clear meaning in the context of *classical* IH and hence approximations and assumptions had to be incorporated in [2, 3, 12–14].

In this work, as we begin from the very fundamental quantum structure of the IH that is the QIH framework, the physical and technical aspects of the calculations are logical and straightforward, devoid of any assumption and approximation. How the mean energy shows up in the partition function even though we begin from the quantum mechanical ensemble using the Hamiltonian operator for the QIH and hence the true energy spectrum of the same is very much transparent. Moreover, without having to make any assumption, we have the mean energy as the function of k and N, which are integers and hence equispaced. Thus the thermalized forms of the partition functions automatically reduce to the convenient forms which can further be approximated by replacing summation by integration.

Competing Interests

The author declares that they have no competing interests.

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Research Article

Thermodynamics of Acoustic Black Holes in Two Dimensions

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It is well-known that the thermal Hawking-like radiation can be emitted from the acoustic horizon, but the thermodynamic-like understanding for acoustic black holes was rarely made. In this paper, we will show that the kinematic connection can lead to the dynamic connection at the horizon between the fluid and gravitational models in two dimensions, which implies that there exists the thermodynamic-like description for acoustic black holes. Then, we discuss the first law of thermodynamics for the acoustic black hole via an intriguing connection between the gravitational-like dynamics of the acoustic horizon and thermodynamics. We obtain a universal form for the entropy of acoustic black holes, which has an interpretation similar to the entropic gravity. We also discuss the specific heat and find that the derivative of the velocity of background fluid can be regarded as a novel acoustic analogue of the two-dimensional dilaton potential, which interprets why the two-dimensional fluid dynamics can be connected to the gravitational dynamics but it is difficult for four-dimensional case. In particular, when a constraint is added for the fluid, the analogue of a Schwarzschild black hole can be realized.

1. Introduction

The concept of acoustic black holes, based on the kinematical analogue between the motion of sound wave in a convergent fluid flow and the motion of a scalar field in the background of Schwarzschild spacetime, had been suggested initially in 1981 [1], and then many physical systems had been used to present a similar concept, such as Bose-Einstein condensates (BEC) [2], superfluid He [3], slow light [4], electromagnetic waveguide [5], and light in a nonlinear liquid [6]; see review [7]. Recently, the observation of the acoustic horizon had been made in different physical systems [8–11]. But the Hawking-like radiation [12, 13] is still difficult to be observed due to the small radiation temperature, while the recent report [14] on the observation of Hawking-like radiation was related to a self-amplifying mechanism called black hole laser suggested by Corley and Jacobson [15]. Thus, the analogue of black holes provides a prospective avenue to observe the Hawking radiation experimentally, since the direct observation from astrophysical black holes is nearly impossible due to the temperature that is several orders of magnitude lower than the cosmic microwave background.

As stated above, the acoustic black hole is based on only the kinematical analogue, irrelevant to the dynamics. But for Schwarzschild black holes, once the radiation starts, it will get hotter and hotter by losing energy, which is evident from the relation that the temperature is inversely proportional to the mass. This description is beyond kinematics and is mainly based on thermodynamics that is definitely dependent on the dynamics [16, 17]. Recently, it was found that when the back-reaction [18, 19] was considered (some different views for this were seen in [20–22], but how to discuss the back-reaction exactly is still open now [7]), the acoustic black hole will not present similar thermodynamic behaviors of Schwarzschild black holes while it looks more like a near-extremal Reissner-Nordström black hole. This is an important step for the possible thermodynamics of an acoustic black hole. Next, as for thermodynamics of a gravitational black hole, the description usually has to be made with the help of proper expressions for the mass and the entropy of the acoustic black hole. Some recent developments had shown that the entropy could be endowed to an acoustic black hole by understanding the microscopical modes in some situations [23–26]. But the mass is definitely relevant to the classical dynamics, and thus its definition requires an analogue for gravity's equation. An interesting work [27] had been attempted for this, in which the mass and the entropy were defined by using an analogue between two-dimensional (2D) dilaton black holes and the acoustic black holes but at the same time the fluid dynamics had been fixed by the two-dimensional dilaton gravity [28]. Thus, by the analogue for gravitational dynamics, Cadoni [27] realized the thermodynamics for an acoustic black hole in two dimensions.

In the earlier discussion about thermodynamics of acoustic black holes, the dynamics is fixed in advance. In this paper, we want to study whether the dynamics of fluid can support the description of thermodynamics spontaneously only if the acoustic horizon has formed. This reminds us of a method which is well-known in gravity; that is, Einstein's equation can be derived from the thermodynamics plus the knowledge from the black hole physics [29, 30]. Thus, Einstein's equation could be regarded as a thermodynamic identity; see the review paper [31]. That means that, at the horizon, Einstein's equation is equivalent to the thermodynamic first law, and the thermodynamic quantities such as the mass and the entropy can be read off directly from Einstein's equation. Then, one might ask whether this can be extended to the fluid model, or whether it is permitted to read off the mass and the entropy of an acoustic black hole from the fluid equation. It is feasible if the kinematical analogue can give the connection between two kinds of dynamics. Moreover, the application of the method should be able to give consistent results with those obtained by other methods [18, 19, 27]. In this paper, we will try to explore these from the equation of motion for the fluid, but our discussion is limited in two dimensions. Once the mass and the entropy of the acoustic black hole are given, we can estimate the thermodynamic evolution using the specific heat, as used usually for the Schwarzschild black hole. By the estimation, we are able to find how the elements in a fluid model influence the evolution of the acoustic black hole.

The structure of the paper is as follows. We will first revisit the concept of the acoustic black hole with the model used in the initial paper of Unruh and explain the Hawking-like radiation with the perturbed action. In Section 3, we will analyze the background fluid equations and construct the kinematical connection between acoustic black holes and 2D black holes, which leads to the dynamic connection between them. Then, we identify the corresponding thermodynamic quantities by a similar method for Einstein's equation. We also discuss the same identification from the equations for BEC as an acoustic analogue of black holes. Section 4 contributes to the thermodynamic stability of the acoustic black hole with the aid of the specific heat. Finally, we discuss and summarize our results in Section 5.

2. Acoustic Black Hole

Consider an irrotational, barotropic fluid that was also considered in the seminal paper [1] of Unruh with the following action [19]:

$$S = -\int d^4x \left[\rho\dot{\psi} + \frac{1}{2}\rho\left(\vec{\nabla}\psi\right)^2 + u\left(\rho\right)\right], \qquad (1)$$

where ρ is the mass density, ψ is the velocity potential, that is, $\vec{v} = \vec{\nabla}\psi$, and $u(\rho)$ is the internal energy density. In this paper, we only involve the linear perturbation for the derivation of the acoustic metric, so action (1) is enough for our discussion.

The variation of S will give the equations of motion, and one of them is the Bernoulli equation:

$$\dot{\psi} + \frac{1}{2}\vec{v}^2 + \mu\left(\rho\right) = 0, \qquad (2)$$

where $\mu(\rho) = du/d\rho$, and the other one is the continuity equation:

$$\dot{\rho} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0. \qquad (3)$$

We linearize the density and the velocity potential by replacing $\psi \to \psi + \phi$ and $\rho \to \rho + \sigma$, where ψ and ρ are related to the background fluid determined by the equations of motion, and ϕ and σ are small perturbations. When we put the new velocity potential and mass density after perturbations into the initial action (1), a new added term, up to quadratic order, appears as

$$S_p = -\frac{1}{2}\int d^4x \left[\rho\left(\vec{\nabla}\phi\right)^2 - \frac{\rho}{c_s^2}\left(\dot{\phi} + \vec{v}\cdot\vec{\nabla}\phi\right)^2\right], \qquad (4)$$

where the speed of sound c_s is defined as $c_s^2 = \rho(d\mu/d\rho)$ and the equation of motion for ϕ, $\phi + (\rho/c_s^2)(\dot{\phi} + \vec{v}\cdot\vec{\nabla}\phi) = 0$, is used. It is noted that, for some cases such as presented in Unruh's original model [1], a completely appropriate approximation can be taken for the speed of sound as a position-independent constant, but there are also some other cases such as BEC [2, 32] which did not take the speed of sound as a constant. In this paper, we take the speed of sound as constant, in addition to the special case where the nonconstant speed of sound will be pointed out.

The propagation of sound wave on the background fields can be obtained by a wave equation, $\nabla^2\phi = (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi)$, where $g_{\mu\nu}$ can be read off from the so-called acoustic metric:

$$ds^2 = \frac{\rho}{c_s}\left[-c_s^2dt^2 + \left(d\vec{x} - \vec{v}dt\right)^2\right]. \qquad (5)$$

This metric, if considered as a metric of acoustic black hole, can be understood from a model called the river model of black holes [33, 34]: the fluid as the background is flowing along a direction to the region beyond the Newtonian escape velocity (i.e., the local velocity of sound), and the point where the velocity of background fluid equals the sound velocity represents the horizon of the black hole. From this description, it is noted that the essence of the acoustic black hole, in particular, Hawking radiation which propagates against the fluid, can be understood with a two-dimensional model. So in the paper we will consider only the case of two dimensions, for which we take the direction of fluid flowing as the x-axis and thus the components of the speed $\vec{v}$ along the other two directions will be suppressed naturally. Moreover, one can also reduce the dimensions by the spherical symmetry, in which this analogue will reproduce the results for the so-called "dirty black holes" [35].

Furthermore, if the scalar field ϕ is quantized, a similar effect to Hawking radiation appears at the horizon with the temperature which can be expressed as

$$T_a = \frac{\hbar}{4\pi c_s k_B} \left. \frac{d\left(c_s^2 - v^2\right)}{dx} \right|_{v=-c_s} = \frac{\hbar}{2\pi k_B} v'\left(x_h\right), \qquad (6)$$

where the horizon is located at $x = x_h$ that is determined by $v(x_h) = -c_s$ and the prime is the derivative with regard to the coordinate x. A better interpretation for the origin of Hawking radiation can be from the analysis for the corrected action S_p [22], in which some extra terms have to be added in order to ensure the positivity of perturbation modes and differentiability of the classical ground state profile (i.e., $\partial_x \phi$), and thus the superluminal modes that can overcome the frame-dragging speed and approach the horizon from the inside will be ripped apart at the horizon so that the part with positive frequencies escapes into the exterior region and the others drop again into the interior domain. This interpretation is consistent with that from the quantization, but it evidently draws support from the dispersion relation that is necessary to be considered in the realistic experiment of simulation about black/white hole horizon and that can also avoid the trans-Planckian puzzle [22].

3. Thermodynamic Description

At first, let us review how the Einstein equation is identified with the thermodynamic relation [31, 36, 37]. Start with the stationary metric in ADM form [38]:

$$ds^2 = -N_t(r)^2 dt^2 + \left[dr + N_r(r)\,dt\right]^2 + r^2 d\Omega^2, \qquad (7)$$

where $N_t(r)$ and $N_r(r)$ are the lapse and shift functions, respectively. The metric is well behaved on the horizon, and for a four-dimensional spherical Schwarzschild solution, $N_t = 1$ and $N_r = \sqrt{2M/r}$ (M is the mass of the black hole); for a four-dimensional Reissner-Nordström solution, $N_t = 1$ and $N_r = \sqrt{2M/r - Q^2/r^2}$ (M is the mass and Q is the charge of the black hole). In particular, it is noted that the acoustic metric can be obtained by taking $N_t = -c_s$ and $N_r = -v$ up to a conformal factor ρ/c_s.

For the metric (7), the horizon, $r = r_h$, is determined from the condition $N_t(r_h) - N_r(r_h) = 0$. The temperature associated with this horizon is $k_B T_H = (\hbar c/4\pi)(d/dr)(N_t^2(r) - N_r^2(r))|_{r=r_h} = (\hbar c N_t(r_h)/2\pi)(N_t'(r_h) - N_r'(r_h))$, where the prime is the derivative with regard to the coordinate r. Consider the (r, r) components of the vacuum Einstein equation, $R_{rr} = 0$, and evaluate it at the horizon; this gives

$$N_t\left(r_h\right)\left[N_t'\left(r_h\right) - N_r'\left(r_h\right)\right] - \frac{1}{2} = 0. \qquad (8)$$

Then, multiplying the equation by an imaginary displacement dr_h of the horizon and introducing some constants, we can rewrite it as

$$\underbrace{\frac{\hbar c N_t\left(r_h\right)}{2\pi}\left[N_t'\left(r_h\right) - N_r'\left(r_h\right)\right]}_{k_B T_H} \underbrace{\frac{c^3}{G\hbar} d\left(\frac{1}{4} 4\pi r_h^2\right)}_{dS} - \underbrace{\frac{1}{2}\frac{c^4}{G} dr_h}_{dE} = 0 \qquad (9)$$

and read off the expressions

$$S = \frac{1}{4 l_P^2}\left(4\pi r_h^2\right) = \frac{A_H}{4 l_P^2}; \qquad E = \frac{c^4}{2G} r_h = \frac{c^4}{G}\left(\frac{A_H}{16\pi}\right)^{1/2}, \qquad (10)$$

where A_H is the horizon area and $l_P^2 = G\hbar/c^3$. If the source of Einstein's equation is considered, a term PdV will be added to (9) as in [36, 37], which will give an exact and strict form for the first law of thermodynamics by including the matter's influence. Here we ignore this term only for the brevity. In particular, the method is classical although we insert the Planck constant in the expressions of the temperature and the entropy by hand, which showed that the single quantity should stem from the quantum statistical mechanics. Along this line, many different situations had been discussed, which indicates the universality of this kind of identification; see review [39].

3.1. Two-Dimensional Kinematical Connection. Before applying the thermodynamic identification to acoustic black holes, we have to be sure whether there is the dynamic connection between the fluid and gravitational models only by the kinematical analogue. Now we give some direct kinematical relations between 2D black holes and acoustic black holes, which were once given in [27] but the dynamics is fixed in advance.

Generally, 4D Einstein gravity can decay into 2D by the method of spherical reduction if the line element can be written as $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta + e^{-2\Phi(x^\alpha)} d\Omega^2$, where $\alpha, \beta = 0, 1$ and $\Phi(x^\alpha)$ is the dilaton field. In particular, 2D gravity is also called dilaton gravity [28]. Start with the action considered in [27, 28]

$$S_g = \frac{1}{2}\int d^2x \sqrt{-g}\left(\Phi R + \lambda^2 V(\Phi)\right), \qquad (11)$$

where Φ is a scalar (the dilaton), λ is a parameter for the balance of the dimension, and $V(\Phi)$ is the dilaton potential. The equation of motion related to the variable $g_{\alpha\beta}$ is

$$R = -\lambda^2 \frac{dV}{d\Phi}. \qquad (12)$$

The model admits black hole solutions with the form in Schwarzschild gauge as [40]

$$ds^2 = -\left(J(\Phi) - \frac{2M}{\lambda}\right)d\tau^2 + \left(J(\Phi) - \frac{2M}{\lambda}\right)^{-1} dr^2, \qquad \Phi = \lambda r, \tag{13}$$

where M is the mass of the black hole and $J = \int V\, d\Phi$. The location r_h of the black hole horizon is determined by $J(\lambda r_h) = 2M/\lambda$. The metric is also gotten by taking $N_t = 1$ and $N_r = \sqrt{1 - J - 2M/\lambda}$ from the ADM form (7), but its relation to the 2D section of Schwarzschild black hole is subtle, which will be involved later.

Comparing the metric (13) with the acoustic metric (5) in two dimensions, the kinematical relation can be gotten as

$$\begin{aligned} r &\sim \int \rho\, dx, \\ \tau &\sim t + \int dx \frac{v}{c_s^2 - v^2}, \\ J &\sim \frac{2M}{\lambda} + \frac{\rho}{c_s}\left(c_s^2 - v^2\right), \end{aligned} \tag{14}$$

as presented in [27].

When we endow the thermodynamic-like description to the acoustic black hole, the role played by relations (14) will be seen. Now we turn to another kinematical relation. As stated in [16, 17], Hawking radiation is purely kinematic effect, and so is the temperature. Thus, one can relate the temperature $T_b = (\lambda/4\pi)V(\lambda r_h)$ for the 2D black holes [41, 42] with that in (6) for the acoustic black hole by

$$\frac{\hbar}{2\pi k_B} v'(x_h) \sim \frac{\lambda}{4\pi} V(\lambda r_h), \tag{15}$$

which, together with the fluid's equation at the horizon where the two equations of motion for the fluid have the same form, gives a constraint for the fluid dynamics at the horizon. But it has to be stressed that this is not evident for 4D situation, since the conformal factor (i.e., irrelevant to the kinematical results but can influence the dynamics) is not treated properly in kinematical analogue there. Therefore, one has to be careful about 4D situation, but at least in two dimensions, we can proceed to the discussion about thermodynamics for acoustic black holes.

3.2. Correspondence between Full Actions. As well known, the acoustic metric is obtained through the linear perturbation, so in this subsection we will give a brief proof that, at the perturbative level, the two cases are still equivalent. For the acoustic fluid, the corrected action is given by (4), and in two dimensions, the expression is

$$S_P = -\frac{1}{2}\int d^2x A\left[\rho\left(\partial_x\phi\right)^2 - \frac{\rho}{c_s^2}\left(\partial_t\phi + v\cdot\partial_x\phi\right)^2\right], \tag{16}$$

where A is the cross-sectional area of fluid and ϕ is the perturbative field whose equation of motion is given by

$$\nabla^2\phi = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\phi\right) = 0. \tag{17}$$

For 2D dilaton gravity, the corrected action can be given by the Polyakov-Liouville action (see [28] and the references therein):

$$I_{\mathrm{PL}} = -\frac{\alpha}{2\pi}\int d^2x\sqrt{-g}\left(\frac{(\nabla\varphi)^2}{2} + \varphi R\right), \tag{18}$$

where α is quantum coupling parameter related to the number of conformal scalar fields, R is 2D scalar curvature, and φ field is related to the back-reaction of spacetime caused by Hawking radiation. The equation of motion for the φ field is

$$\nabla^2\varphi = R. \tag{19}$$

Within the Schwarzschild gauge,

$$ds^2 = -X(r)\,d\tau^2 + \frac{1}{X(r)}dr^2, \tag{20}$$

(17) becomes

$$-\frac{1}{X}\partial_\tau^2\phi + \frac{dX}{dr}\frac{d\phi}{dr} + X\partial_r^2\phi = 0 \tag{21}$$

and (19) becomes

$$-\frac{1}{X}\partial_\tau^2\varphi + \frac{dX}{dr}\frac{d\varphi}{dr} + X\partial_r^2\varphi = R = -\frac{d^2X}{dr^2}. \tag{22}$$

Then, we can relate the two equations with $\phi = \varphi + \ln X$. Since X is independent of the time, the two fields ϕ and φ have the same behaviors at the thermodynamic or dynamic level. Furthermore, we can see the equivalence by the respective numbers of degree of freedom; that is, for the fluid model, there are three parameters—the fluid speed v, the fluid density ρ, and the perturbative field ϕ; for 2D dilaton gravity, there are also three parameters—the dilaton field Φ, the metric field $g_{\mu\nu}$ (only one component in the Schwarzschild gauge), and the Polyakov-Liouville field φ. But for 4D situation, there is never so nice correspondence, and even for the 4D Schwarzschild black holes, the gravitational parameters can not be modeled by the fluid parameters directly, which requires conformal parameters to relate them, but the conformal parameter is dependent on the time, so one must be careful to treat 4D situation, in particular at the level of thermodynamics.

3.3. Thermodynamic Identification. As far as we know, the method of thermodynamic identification has not been applied for 2D gravity. Here we give a brief implementation for this. At the horizon of black hole (13), the field equation for 2D dilaton gravity model (11) becomes

$$\left.\frac{dJ}{dr}\right|_{r_h} = \lambda V(\lambda r_h). \tag{23}$$

When the imaginary displacement dr_h of the horizon is included, the equation is reexpressed with the form

$$\underbrace{\frac{\hbar\lambda}{4\pi}V\left(\lambda r_h\right)}_{k_B T_b}\underbrace{\frac{2\pi\lambda}{\hbar}dr_h}_{dS_b}-\underbrace{\frac{\lambda}{2}dJ\left(\lambda r_h\right)}_{dE_b}=0. \tag{24}$$

From this, one can read off

$$S_b=\frac{2\pi\lambda}{\hbar}r_h,$$
$$E_b=\frac{\lambda}{2}J\left(\lambda r_h\right), \tag{25}$$

up to an integral constant, respectively. In particular, they are consistent with the expressions of entropy and energy obtained by using other methods [41, 42].

Now we begin to discuss the thermodynamics for an acoustic black hole. Taking the derivative with regard to x for the steady Bernoulli equation (2), we have Euler's equation in two dimensions:

$$\rho v\frac{dv}{dx}+c_s^2\frac{d\rho}{dx}=0, \tag{26}$$

which is one of the equations of motion in two dimensions but it is consistent with the other one at the horizon. Considering the expression of the temperature (6), we can rewrite the equation at the horizon as

$$\underbrace{\frac{\hbar}{2\pi}v'\left(x_h\right)}_{k_B T_a}\underbrace{\frac{2\pi}{\hbar K}\rho\left(x_h\right)dx_h}_{dS_a}-\underbrace{\frac{c_s}{K}d\rho\left(x_h\right)}_{dE_a}=0, \tag{27}$$

where K is introduced in order to balance the dimension. Then, we read off

$$S_a=\frac{2\pi}{\hbar K}\int^{x_h}\rho\left(x\right)dx;$$
$$E_a=\frac{c_s}{K}\int^{x_h}\rho'\left(x\right)dx, \tag{28}$$

where the integral is made near the horizon. Then, we attempt to understand these identifications (28). The mass can be interpreted with a force-density term in Euler's equation [7]; that is, $c_s^2(d\rho/dx)=\delta E_a/\delta x=F_x\equiv-\rho(d\mu/d\rho)(d\rho/dx)$. Here it is also regarded as the acoustic analogue of gravitational mass of black holes. The entropy can be interpreted as from the property of the horizon, but it can also be interpreted with other ways, that is, brick wall model [23–26] that assumes the black hole is under an equilibrium state with the thermal gas surrounding it. Note that our identification for the entropy of the acoustic black hole is formally different from that from brick wall model [24, 25], but one can obtain a consistent form from the two methods when the system's evolution is considered; that is, $\dot{S}_a\propto\kappa$ (i.e., the surface gravity of acoustic black holes via $T=\kappa/2\pi$) by using the results of [19] for back-reaction expression of ρ or v, in line with [23, 24].

It is evident that the thermodynamics of acoustic black holes cannot be Schwarzschild-like, since the Schwarzschild analogue usually takes $\rho(x)\propto x^{-3/2}$ and $v(x)\propto x^{-1/2}$ up to a conformal difference. One might suspect that this inconsistency is due to the one-dimensional linear fluid model that we take approximately (see the discussion below metric (5)), but the same mathematical forms will also be gotten by reducing action (1) with the consideration of spherical symmetry, since $\vec{\nabla}\psi=(\partial\psi/\partial x)\widehat{x}+(\partial\psi/\partial y)\widehat{y}+(\partial\psi/\partial z)\widehat{z}=(\partial\psi/\partial r)\widehat{r}+(1/r)(\partial\psi/\partial\theta)\widehat{\theta}+(1/r\sin\theta)(\partial\psi/\partial\phi)\widehat{\phi}$ which reduces to $\vec{\nabla}\psi=(\partial\psi/\partial x)\widehat{x}$ for the case discussed in the paper or $\vec{\nabla}\psi=(\partial\psi/\partial r)\widehat{r}$ when the velocity is suppressed in two angular directions. Thus, if a two-dimensional fluid model is considered, it might not be related to the simple reduction of four-dimensional gravity, described by (8), so its thermodynamics does not have to be Schwarzschild-like.

As a consistent check, we will see that the kinematical relation (14) can give the thermodynamic connection between the 2D dilaton black hole and the acoustic black hole; that is, taking $\lambda\sim 1/K$, the relation $r\sim\int\rho\,dx$ indicates $S_b\sim S_a$; from the relation $J\sim 2M/\lambda+(\rho/c_s)(c_s^2-v^2)$, one has $dJ\sim 2\rho dv\sim 2c_s d\rho$ at the horizon, which indicates $E_b\sim E_a$. Therefore, the kinematical relations lead to a direct connection between the thermodynamics of 2D dilaton black hole and acoustic black hole, under the condition that any relation between fluid dynamics and gravitational dynamics is not assumed in advance.

In order to present the universal property of thermodynamic identification (28), we will also include the external potential explicitly in the discussion of acoustic black hole with another kind of popular model that is related to BEC. For the analogue of BEC, the corresponding Euler equation can be expressed as [2, 32]

$$\rho_B v_B\frac{dv_B}{dx}+\frac{\rho_B}{m}\frac{dV_{\text{ext}}}{dx}+c_B^2\frac{d\rho_B}{dx}=0. \tag{29}$$

This can be obtained from the Gross-Pitaevskii equation, $i\hbar\partial_t\Psi=(-(\hbar^2/2m)\nabla^2+V_{\text{ext}}+(4\pi a\hbar^2/m)|\Psi|^2)\Psi$, where the condensate is considered in the dilute gas approximation and nearly all atoms are in the same single-particle quantum state $\Psi(x,t)$, m is the mass of individual atoms, a is the scattering length, and $V_{\text{ext}}(x)$ is the external potential that trapped these bosons. In particular, if a background stationary state, $\Psi_B(x,t)=\sqrt{\rho_B(x)}e^{i\psi_B(x)}e^{-i\omega t}$, is considered, the propagation of small perturbations of the condensate around this background can be calculated to get the acoustic metric, as presented in (5). The velocity of the background is given by $v_B=(\hbar/m)\nabla\psi_B(x)$ and the local velocity of sound by $c_B(x)=(\hbar/m)\sqrt{4\pi a\rho_B(x)}$. Moreover, the quantum pressure term $Q(\vec{x})=-(\hbar^2/2m)(\nabla^2\sqrt{\rho_B(\vec{x})}/\sqrt{\rho_B(\vec{x})})$ is ignored since BEC for the analogue of acoustic black holes always works within the regime of validity of Thomas-Fermi approximation [43] that the condensate does not vary on length scales shorter than the healing length $\xi=1/\sqrt{8\pi\rho_B a}$.

With the temperature of the acoustic black hole known in advance (note that the speed of sound here is not constant), (29) can be rewritten at the horizon as

$$\underbrace{\frac{\hbar}{2\pi}\left(v_B'(x_h)+c_B'(x_h)\right)}_{k_BT}\underbrace{\frac{2\pi}{\hbar K}\rho_B(x_h)\,dx}_{dS_B} - \underbrace{\left(\frac{\rho_B(x_h)}{Kmc_B(x_h)}dV_{\text{ext}}(x_h)+\frac{3c_B(x_h)}{2K}d\rho_B(x_h)\right)}_{dE_B} = 0. \quad (30)$$

It is easily seen that the identification of the entropy is the same with that in (28). The mass is expressed as

$$E_B = \frac{1}{K}\int^{x_h}\left[\frac{\rho_B(x_h)}{mc_B(x_h)}dV_{\text{ext}}(x_h)+\frac{3c_B(x_h)}{2}d\rho_B(x_h)\right], \quad (31)$$

where the expression of E_B is different from that of E_a due to the nonconstant speed of sound and the external potential. But it is noted that the expression of entropy is model-independent. This can also be understood by the entropic gravity [44], which assumed that the gravity is derived from the change of entropy with the form $\Delta S = 2\pi k_B(\Delta x/\lambda)$. Thus, if we take the proper form for the constant K and the relation $r \sim \int \rho\, dx$, the entropy in (28) is consistent with ΔS, which shows again that the thermodynamic-like description for analogous black hole is feasible at least in 2D.

4. Specific Heat

From the discussion above, we have known the expressions of thermodynamic quantities of acoustic black holes which conform to the first law of thermodynamics. Actually, more importantly, we want to discuss how to use these quantities to describe the evolution of acoustic physical systems, that is, the change caused by emission of thermal radiation from the acoustic horizon. The calculation of back-reaction provided a fundamental method to answer this question, but here we want to estimate it via specific heat of an acoustic black hole, which will give the information about the change of temperature during the radiation. Generally, the temperature in black hole theory is a geometric quantity related closely to the spacetime background, so the change of temperature will indicate the change of spacetime background that is also called back-reaction if the change is not so violent.

According to our results in (6) and (28), the specific heat of an acoustic black hole can be written as

$$C_a = \frac{dE_a}{dT_a} = \frac{dE_a/dx}{dT_a/dx} = -\frac{2\pi k_B\rho(x_h)}{\hbar K}\frac{v'(x_h)}{v''(x_h)}. \quad (32)$$

It is easily seen that the sign of specific heat is dependent on the first and second derivatives of the velocity with regard to the coordinate x. If we take the model of Laval nozzle that is described in [19], which gave $v'(x_h) > 0$ and $v''(x_h) < 0$, we find a positive specific heat. This means that the temperature will decrease after the radiation is emitted, and such behavior of acoustic black holes resembles a near-extremal Reissner-Nordström black hole but not a Schwarzschild black hole, which is consistent with the result of back-reaction analysis [19]. Thus, it indicates that the thermodynamics of acoustic black holes is model-dependent. The extension to the case where the speed of sound is not constant is possible and will not change our conclusion about the model-dependent thermodynamic-like behaviors for the corresponding acoustic black holes.

The specific heat of 2D dilaton black holes is also easy to be gotten:

$$C_b = \frac{dE_b}{dT_b} = 2\pi\frac{V(\Phi_h)}{dV(\Phi_h)/d\Phi}, \quad (33)$$

which is consistent with that obtained in [45, 46]. In particular, the specific heat is dependent on the dilaton potential which can lead to a model similar to the near-extremal Reissner-Nordström black hole under some conditions. Then, whether the two specific heats have any relation under the kinematical analogue, a straight use of kinematical relations in (15) and $r \sim \int \rho\, dx$ shows $C_a \sim C_b$, which is further evidence for the connection between the two dynamics with only the kinematical analogue made in advance.

From the analysis above, it is seen that the derivative of the parameter v in the fluid can be regarded as the analogue of dilaton potential, which is equivalent to such analogue, $v \sim M$, where M is the mass of 2D dilaton black hole. This is different from the usual kinematical analogue for 2D sections of Schwarzschild black hole; that is, $v \sim \sqrt{2M_H/r_H}$, where M_H and r_H are the mass and the radial coordinate of Schwarzschild black hole, respectively. In fact, a specific situation for 2D dilaton black hole, that is, $V(\Phi) = 1/\sqrt{2\Phi}$, can be taken in the Schwarzschild form by making the transformations $g_{\mu\nu} \rightarrow (1/\sqrt{2\Phi})g_{\mu\nu}$ and $r \rightarrow (\lambda/2)r_H^2$ for the metric (13). This means the kinematical relation in (15) becomes $(\hbar/2\pi k_B)v'(x_h) \sim (\lambda/4\pi)(1/\sqrt{2\Phi_h}) \sim (\hbar/8\pi G k_B M)$, where the speed of light is taken as $c = 1$. As discussed in the last section, the kinematical relations can lead to the relation between the mass of acoustic black hole and the gravitational black hole, so we have $v'(x_h) \sim 1/4GE_a \sim 1/-4\int^{x_h}\rho(x)v'(x)dx$. Thus, the relation for the parameter v in the fluid is

$$-4v'(x_h)\int^{x_h}\rho(x)\,v'(x)\,dx \sim 1, \quad (34)$$

which is necessary to model the thermal behavior of a Schwarzschild black hole. With this relation, it is easy to confirm that $C_a < 0$. Of course, one can also obtain a similar relation for the other physical systems like BEC to attempt to find the Hawking-like radiation as emitted from a Schwarzschild black hole. Further, as seen from the transformation for 2D Schwarzschild metric into dilaton black hole metric, a conformal factor related to the dilaton potential is involved. Thus, it is understood why the acoustic black holes always present directly the behaviors of 2D dilaton black holes, while an extra constraint is required for the simulation of Schwarzschild-like black holes.

5. Conclusion

In this paper, we have reinvestigated the concept of acoustic black holes and discussed background fluid equations. Even though the fluid equations cannot give the complete expressions for each field without any extra knowledge besides the equations of motion, they still include a wealth of information. Via fluid's equation of motion as a thermodynamic identity, we have read off the mass and the entropy for acoustic black holes with the temperature known in advance. This identification is guaranteed by the kinematical connection which can lead to the dynamic connection between the fluid and gravitational models in two dimensions. In particular, through the analysis for the kinematical relations, we have found that the thermodynamics of acoustic black holes reproduces the thermodynamics of two-dimensional dilaton black holes exactly, and so the fluid can be regarded as a natural analogue of two-dimensional dilaton gravity, which is significant for many ongoing related experimental observations. Novelly, we have also found that the entropy for acoustic black holes is model-independent and has an interpretation similar to the entropic gravity. Moreover, it is found that the derivative of velocity v of the background fluid is a nice analogue of the dilaton potential, from the kinematical and thermodynamic perspective, respectively, which also interprets why it is difficult for the 4D kinematical analogue to lead to any dynamic analogue, due to conformal difference.

With the mass and the temperature identified by thermodynamics, we have proceeded to get the specific heat for the acoustic black hole and found the sign of the specific heat is model-dependent, which means that some extra knowledge besides the equations of motion must be given to estimate the thermodynamic stability. However, when we want to model some kind of black holes, that is, Schwarzschild black holes, the relation about the parameters in the fluid can be known only by the fluid equation and the thermodynamic correspondence, as made in (34). Finally, all our analysis is made for 2D model, so whether they can be extended to 4D or higher dimensions is unclear now. It might be possible, however, to relate 2D with 4D cases by taking into account properly the behavior of 2D dilaton gravity under Weyl rescaling of the metric [45], which deserves a further investigation.

Competing Interests

The author declares that they have no competing interests.

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Research Article

Thermodynamic Product Relations for Generalized Regular Black Hole

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We derive thermodynamic product relations for four-parametric regular black hole (BH) solutions of the Einstein equations coupled with a nonlinear electrodynamics source. The four parameters can be described by the mass (m), charge (q), dipole moment (α), and quadrupole moment (β), respectively. We study its complete thermodynamics. We compute different thermodynamic products, that is, area product, BH temperature product, specific heat product, and Komar energy product, respectively. Furthermore, we show some complicated function of horizon areas that is indeed *mass-independent* and could turn out to be *universal*.

1. Introduction

The thermodynamic product for Reissner Nordstrøm (RN) BH, Kerr BH, and Kerr-Newman (KN) BH [1] has been examined; a simple area product is sufficient to draw a conclusion that the product of area (or entropy) is a *universal* quantity. The universal term is used here to describe when the product of any thermodynamic quantities is simply *mass-independent*. Alternatively mass-dependent thermodynamic quantities imply that they are *not* universal quantity. This is strictly followed throughout the work.

In case of RN-AdS [2], Kerr-AdS, and KN-AdS BH [3] one can not find a simple area product relation; instead one could find a complicated function of event horizon ($\mathscr{H}^+$) area and Cauchy horizon area ($\mathscr{H}^-$) that might be universal. Very recently, we derived for a regular Ayón-Beato and García BH (ABG) [4] the function of $\mathscr{H}^+$ area and $\mathscr{H}^-$ area is

$$f\left(\mathscr{A}_+, \mathscr{A}_-\right) = 384\pi^3 q^6, \tag{1}$$

where the function should read

$$\begin{aligned} f\left(\mathscr{A}_+, \mathscr{A}_-\right) &= \mathscr{A}_+\mathscr{A}_-\left(\mathscr{A}_+ + \mathscr{A}_-\right) + 24\pi q^2 \mathscr{A}_+\mathscr{A}_- \\ &- 256\pi^4 q^8 \left(\frac{\mathscr{A}_+ + \mathscr{A}_-}{\mathscr{A}_+\mathscr{A}_-}\right) \\ &- \frac{\mathscr{A}_+\mathscr{A}_+}{\mathscr{A}_+ + \mathscr{A}_+ + 4\pi q^2} \left[\left(\mathscr{A}_+ + \mathscr{A}_-\right)^2 \right. \\ &+ 24\pi q^2 \left(\mathscr{A}_+ + \mathscr{A}_-\right) - \mathscr{A}_+\mathscr{A}_- - \frac{256\pi^4 q^8}{\mathscr{A}_+\mathscr{A}_-} \\ &\left. + 176\pi^2 q^4 \right]. \end{aligned} \tag{2}$$

It indicates a very complicated function of horizons area that turns out to be universal. But it is not a simple area product of horizon radii as in RN BH, Kerr BH, and KN BH. This has been very popular topic in recent years in the GR (General Relativity) community [1] as well as in the String community [5] (see also [6–10]). These universal relations are particularly interesting because they could hold in more general situations like when a BH space-time is perturbed by surrounding matter. For example, KN BH surrounded by a ring of matter the universal relation does indeed holds [1].

Recently, Page and Shoom [8] have given a heuristic argument for the universal area product relation of a four-dimensional adiabatically distorted charged rotating BH. They in fact showed that the product of outer horizon area and inner horizon area could be expressed in terms of a polynomial function of its charge, angular momenta, and inverse square root of cosmological constant. It has been argued by Cvetič et al. [5] that if the cosmological parameter is quantized, the product of $\mathscr{H}^+$ area and $\mathscr{H}^-$ area could provide a "looking glass for probing the microscopy of general BHs".

However, in this work, we would like to evaluate the thermodynamic product formula for a generalized regular (singularity-free) BH described by the four parameters, namely, m, q, α, and β [11]. This class of BH is a solution of Einstein equations coupled with a nonlinear electrodynamics source. We examine complete thermodynamic properties of this BH. We show some complicated function of physical horizon areas that is indeed mass-independent but it is not a simple area product as in Kerr BH or KN BH. We also compute the specific heat to examine the thermodynamic stability of the BH. Finally we compute the Komar energy of this BH.

It should be noted that Smarr's mass formula and Bose-Dadhich identity do not hold for $\alpha = 3$ and $\beta = 3$ when one has taken into account the nonlinear electrodynamics [12–15]. It also should be mentioned that, for some regular BHs [16], once the entropy is taken to be the Bekenstein-Hawking entropy [17, 18] the first law of BH thermodynamics is no longer established because there is an inconsistency between the conventional 1st law of BH mechanics and Bekenstein-Hawking area law. The authors [16] also showed the corrected form of the first law of BH thermodynamics for this class of regular BHs. We should expect that this is also true for regular ABG BH and it should be valid for arbitrary values of α and β.

The plan of the paper is as follows. In Section 2, we have described the basic properties of the generalized regular BH and computed various thermodynamic properties. Finally, we conclude our discussions in Section 3.

2. Generalized Regular BH Solution

The gravitational field around the four-parametric regular BH solution is described by the metric

$$ds^2 = -\mathscr{B}(r)\,dt^2 + \frac{dr^2}{\mathscr{B}(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (3)$$

where the function $\mathscr{B}(r)$ is defined by

$$\mathscr{B}(r) = 1 - \frac{2mr^{\alpha-1}}{(r^2+q^2)^{\alpha/2}} + \frac{q^2 r^{\beta-2}}{(r^2+q^2)^{\beta/2}}. \quad (4)$$

And the electric field is given by

$$\mathscr{E}(r) = q\left[\frac{m\alpha\left\{5r^2 - (\alpha-3)\,q^2\right\} r^{\alpha-1}}{(r^2+q^2)^{\alpha/2+2}} + \frac{\left\{4r^4 - (7\beta-8)\,q^2r^2 + (\beta-1)(\beta-4)^4\right\} r^{\beta/2}}{4\,(r^2+q^2)^{\beta/2+2}}\right], \quad (5)$$

where the parameters are described previously. This is a class of regular (curvature free) BH solution in GR. The source is nonlinear electrodynamics. In the weak field limits the nonlinear electrodynamics becomes Maxwell field.

In the asymptotic limit, the above solution behaves as

$$\begin{aligned} -g_{tt} &= 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \alpha\frac{mq^2}{r^3} - \beta\frac{q^4}{2r^4} + \mathscr{O}\left(\frac{1}{r^5}\right), \\ \mathscr{E}(r) &= \frac{q}{r^2} + \alpha\frac{5m}{2r^3} - \beta\frac{9q^3}{4r^4} + \mathscr{O}\left(\frac{1}{r^5}\right). \end{aligned} \quad (6)$$

It may be noted that, up to $\mathscr{O}(1/r^3)$, we recover the charged BH behavior and the parameters m and q are related to the mass and charges, respectively. From the electric field behavior we can say that α and β are associated with the electric dipole moment and electric quadrupole moments, respectively. It also should be noted that, in the limit $\alpha = \beta = 0$, we obtain the RN BH and, in the limits $\alpha = 3$ and $\beta = 4$, we recover the ABG BH. This BH solution can be treated as a generalization of ABG BH [19]. The above metric as well as the scalar invariants $R, R_{ab}R^{ab}, R_{abcd}R^{abcd}$ and the electric field are regular everywhere in the space-time. Hence, in this sense it is called a regular BH in Einstein-Maxwell gravity. The first regular model is discovered by Bardeen [20] in 1968.

The BH horizons can be obtained by setting $\mathscr{B}(r) = 0$; that is,

$$\left(r^2+q^2\right)^{(\alpha+\beta)/2} - 2mr^{\alpha-1}\left(r^2+q^2\right)^{\beta/2} + q^2r^{\beta-2}\left(r^2+q^2\right)^{\alpha/2} = 0. \quad (7)$$

In this work we restrict our case $\alpha \geq 3$ and $\beta \geq 4$.

Case 1. We have set $\alpha = 4$ and $\beta = 5$. In this case the horizon equation is found to be

$$\begin{aligned} & r^{10} - 2mr^9 + \left(4m^2 + 5q^2\right) r^8 - 6mq^2r^7 \\ & \quad + \left(4m^2q^2 + 9q^4\right) r^6 - 6mq^4r^5 + 10q^6r^4 \\ & \quad - 2mq^6r^3 + 5q^8r^2 + q^{10} = 0. \end{aligned} \quad (8)$$

To find the roots we apply Vieta's theorem; we find

$$\begin{aligned} \sum_{i=1}^{10} r_i &= 2m, \\ \sum_{1\leq i<j\leq 10} r_ir_j &= 4m^2 + 5q^2, \\ \sum_{1\leq i<j<k\leq 10} r_ir_jr_k &= 6mq^2, \end{aligned}$$

$$\sum_{1\le i<j<k<l\le 10} r_i r_j r_k r_l = 4m^2q^2 + 9q^4,$$
$$\sum_{1\le i<j<k<l<p\le 10} r_i r_j r_k r_l r_p = 6mq^4,$$
$$\sum_{1\le i<j<k<l<p<s\le 10} r_i r_j r_k r_l r_p r_s = 10q^6,$$
$$\sum_{1\le i<j<k<l<p<s<t\le 10} r_i r_j r_k r_l r_p r_s r_t = 2mq^6,$$
$$\sum_{1\le i<j<k<l<p<s<t<u\le 10} r_i r_j r_k r_l r_p r_s r_t r_u = 5q^8,$$
$$\prod_{i=1}^{10} r_i = q^{10}. \tag{9}$$

Eliminating mass parameter we find the following set of equations:

$$\begin{gathered}
\sum_{1\le i<j\le 10} r_i r_j - \left(\sum_{i=1}^{10} r_i\right)^2 = 5q^2,\\
\sum_{1\le i<j<k\le 10} r_i r_j r_k = 3q^2 \sum_{i=1}^{10} r_i,\\
\sum_{1\le i<j<k<l\le 10} r_i r_j r_k r_l - q^2\left(\sum_{i=1}^{10} r_i\right)^2 = 9q^4,\\
\sum_{1\le i<j<k<l<p\le 10} r_i r_j r_k r_l r_p = 3q^4 \sum_{i=1}^{10} r_i,\\
\sum_{1\le i<j<k<l<p<s<t\le 10} r_i r_j r_k r_l r_p r_s r_t = q^6 \sum_{i=1}^{10} r_i.
\end{gathered} \tag{10}$$

By further elimination in (10), finally we find the following mass-independent relations:

$$\begin{aligned}
&\sum_{1\le i<j<k<l\le 10} r_i r_j r_k r_l - q^2 \sum_{1\le i<j\le 10} r_i r_j = 4q^4,\\
&3\sum_{1\le i<j<k<l<p<s<t\le 10} r_i r_j r_k r_l r_p r_s r_t\\
&\quad = q^2 \sum_{1\le i<j<k<l<p\le 10} r_i r_j r_k r_l r_p,\\
&\sum_{1\le i<j<k<l<p<s\le 10} r_i r_j r_k r_l r_p r_s = 10q^6,\\
&\sum_{1\le i<j<k<l<p<s<t<u\le 10} r_i r_j r_k r_l r_p r_s r_t r_u = 5q^8,\\
&\prod_{i=1}^{10} r_i = q^{10}.
\end{aligned} \tag{11}$$

In terms of area $\mathcal{A}_i = 4\pi r_i^2$, the mass-independent relations are

$$\begin{aligned}
&\sum_{1\le i<j<k<l\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l} - 4\pi q^2 \sum_{1\le i<j\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j}\\
&\quad = \left(8\pi q^2\right)^2,\\
&3\sum_{1\le i<j<k<l<p<s<t\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s\mathcal{A}_t}\\
&\quad = 4\pi q^2 \sum_{1\le i<j<k<l<p\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p},\\
&\sum_{1\le i<j<k<l<p<s\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s} = 640\pi^3 q^6,\\
&\sum_{1\le i<j<k<l<p<s<t<u\le 10} \sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s\mathcal{A}_t\mathcal{A}_u}\\
&\quad = 1280\pi^4 q^8,\\
&\prod_{i=1}^{10} \sqrt{\mathcal{A}_i} = \left(4\pi q^2\right)^5.
\end{aligned} \tag{12}$$

From this relation one can obtain the mass-independent entropy relation by substituting $\mathcal{A}_i = 4\mathcal{S}_i$. Thus the mass-independent complicated function of horizon areas that we have found could turn out to be a *universal* quantity.

Case 2. Now we have set $\alpha = 5$ and $\beta = 6$. In this case the horizon equation is given by

$$\begin{aligned}
&r^{12} - \left(4m^2 - 8q^2\right) r^{10} - \left(4m^2q^2 - 22q^4\right) r^8 + 26q^6 r^6\\
&\quad + 17q^8 r^4 + 6q^{10} r^2 + q^{12} = 0.
\end{aligned} \tag{13}$$

Let us put $r^2 = x$; then the equation reduces to sixth-order polynomial equation:

$$\begin{aligned}
&x^6 - \left(4m^2 - 8q^2\right) x^5 - \left(4m^2q^2 - 22q^4\right) x^4 + 26q^6 x^3\\
&\quad + 17q^8 x^2 + 6q^{10} x + q^{12} = 0.
\end{aligned} \tag{14}$$

Again we apply Vieta's theorem; we get

$$\sum_{i=1}^{6} x_i = 4m^2 - 8q^2,$$
$$\sum_{1\le i<j\le 6} x_i x_j = 22q^4 - 4m^2q^2,$$

$$\sum_{1\le i<j<k\le 6} x_i x_j x_k = -26q^6,$$
$$\sum_{1\le i<j<k<l\le 6} x_i x_j x_k x_l = 17q^8,$$
$$\sum_{1\le i<j<k<l<p\le 6} x_i x_j x_k x_l x_p = -6q^{10},$$
$$\prod_{i=1}^{6} x_i = q^{12}. \tag{15}$$

Eliminating mass parameter, we obtain the mass-independent relation:

$$\begin{aligned}
&\sum_{1\le i<j\le 6} x_i x_j + q^2 \sum_{i=1}^{6} x_i = 14q^4,\\
&\sum_{1\le i<j<k\le 6} x_i x_j x_k = -26q^6,\\
&\sum_{1\le i<j<k<l\le 6} x_i x_j x_k x_l = 17q^8,\\
&\sum_{1\le i<j<k<l<p\le 6} x_i x_j x_k x_l x_p = -6q^{10},\\
&\prod_{i=1}^{6} x_i = q^{12}.
\end{aligned} \tag{16}$$

In terms of area $\mathcal{A}_i = 4\pi r_i^2 = 4\pi x_i$ the above mass-independent relation could be written as

$$\begin{aligned}
&\sum_{1\le i<j\le 6} \mathcal{A}_i \mathcal{A}_j + 4\pi q^2 \sum_{i=1}^{6} \mathcal{A}_i = 14\left(4\pi q^2\right)^2,\\
&\sum_{1\le i<j<k\le 6} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k = -26\left(4\pi q^2\right)^3,\\
&\sum_{1\le i<j<k<l\le 6} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k \mathcal{A}_l = 17\left(4\pi q^2\right)^4,\\
&\sum_{1\le i<j<k<l<p\le 6} \mathcal{A}_i \mathcal{A}_j \mathcal{A}_k \mathcal{A}_l \mathcal{A}_p = -6\left(4\pi q^2\right)^5,\\
&\prod_{i=1}^{6} \mathcal{A}_i = \left(4\pi q^2\right)^6.
\end{aligned} \tag{17}$$

Again we have found mass-independent complicated function of horizon areas that could turn out to be universal in nature.

Case 3. Now we have set $\alpha = 4$ and $\beta = 6$. In this case the horizon equation is

$$\begin{aligned}
&r^{10} - 2mr^9 + 6q^2r^8 - 6mq^2r^7 + 12q^4r^6 - 6mq^4r^5\\
&\quad + 11q^6r^4 - 2mq^6r^3 + 5q^8r^2 + q^{10} = 0.
\end{aligned} \tag{18}$$

Again we apply Vieta's theorem; we get

$$\begin{aligned}
&\sum_{i=1}^{10} r_i = 2m,\\
&\sum_{1\le i<j\le 10} r_i r_j = 6q^2,\\
&\sum_{1\le i<j<k\le 10} r_i r_j r_k = 6mq^2,\\
&\sum_{1\le i<j<k<l\le 10} r_i r_j r_k r_l = 12q^4,\\
&\sum_{1\le i<j<k<l<p\le 10} r_i r_j r_k r_l r_p = 6mq^4,\\
&\sum_{1\le i<j<k<l<p<s\le 10} r_i r_j r_k r_l r_p r_s = 11q^6,\\
&\sum_{1\le i<j<k<l<p<s<t\le 10} r_i r_j r_k r_l r_p r_s r_t = 2mq^6,\\
&\sum_{1\le i<j<k<l<p<s<t<u\le 10} r_i r_j r_k r_l r_p r_s r_t r_u = 5q^8,\\
&\prod_{i=1}^{10} r_i = q^{10}.
\end{aligned} \tag{19}$$

Eliminating mass parameter we have found the following set of mass-independent relation:

$$\begin{aligned}
&3 \sum_{1\le i<j<k<l<p<s<t\le 10} r_i r_j r_k r_l r_p r_s r_t\\
&\quad = q^2 \sum_{1\le i<j<k<l<p\le 10} r_i r_j r_k r_l r_p,\\
&\sum_{1\le i<j\le 10} r_i r_j = 6q^2,\\
&\sum_{1\le i<j<k<l\le 10} r_i r_j r_k r_l = 12q^4,\\
&\sum_{1\le i<j<k<l<p<s\le 10} r_i r_j r_k r_l r_p r_s = 11q^6,\\
&\sum_{1\le i<j<k<l<p<s<t<u\le 10} r_i r_j r_k r_l r_p r_s r_t r_u = 5q^8,\\
&\prod_{i=1}^{10} r_i = q^{10}.
\end{aligned} \tag{20}$$

If we are working in terms of area then the above mass-independent relation could be written as

$$3\sum_{1\leq i<j<k<l<p<s<t\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s\mathcal{A}_t} = 4\pi q^2 \sum_{1\leq i<j<k<l<p\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p}, \quad (21)$$

$$\sum_{1\leq i<j\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j} = 24\pi q^2,$$

$$\sum_{1\leq i<j<k<l\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l} = 3\left(8\pi q^2\right)^2,$$

$$\sum_{1\leq i<j<k<l<p<s\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s} = 11\left(4\pi q^2\right)^3, \quad (22)$$

$$\sum_{1\leq i<j<k<l<p<s<t<u\leq 10}\sqrt{\mathcal{A}_i\mathcal{A}_j\mathcal{A}_k\mathcal{A}_l\mathcal{A}_p\mathcal{A}_s\mathcal{A}_t\mathcal{A}_u} = 5\left(4\pi q^2\right)^4,$$

$$\prod_{i=1}^{10}\sqrt{\mathcal{A}_i} = \left(4\pi q^2\right)^5.$$

Once again these are mass-independent relations that could turn out to be universal. So, we can compute different thermodynamic product relations for different values of α and β which are mass-independent.

The Hawking [18] temperature on $\mathcal{H}^i$ reads off

$$T_i = \frac{\kappa_i}{2\pi} = \frac{1}{4\pi}\left[\alpha\frac{r_i}{\left(r_i^2+q^2\right)} - \frac{(\alpha-1)}{r_i} + \frac{q^2(\alpha-\beta)r_i^{\beta-1}}{\left(r_i^2+q^2\right)^{\beta/2+1}} + \frac{q^2(\beta-\alpha-1)r_i^{\beta-2}}{\left(r_i^2+q^2\right)^{\beta/2}}\right]. \quad (23)$$

It should be noted that the Hawking temperature product depends on the mass parameter and thus it is not a universal quantity.

Another important parameter in BH thermodynamics that determines the thermodynamic stability of BH is defined by

$$C_i = \frac{\partial m}{\partial T_i}. \quad (24)$$

For generalized regular BH, it is found to be

$$C_i = \frac{(\partial m/\partial r_i)}{(\partial T_i/\partial r_i)}, \quad (25)$$

where

$$\frac{\partial m}{\partial r_i} = \frac{\left[\alpha r_i^{\alpha}\left(r_i^2+q^2\right)^{\alpha/2-1} - (\alpha-1)r_i^{\alpha-2}\left(r_i^2+q^2\right)^{\alpha/2}\right]}{2r_i^{2(\alpha-1)}} + \frac{q^2\left[(\beta-\alpha-1)r_i^{\beta-\alpha-2}\left(r_i^2+q^2\right)^{(\beta-\alpha)/2} - (\beta-\alpha)r_i^{\beta-\alpha}\left(r_i^2+q^2\right)^{(\beta-\alpha-2)/2}\right]}{\left(r_i^2+q^2\right)^{\beta-\alpha}},$$

$$\frac{\partial T_i}{\partial r_i} = \frac{1}{4\pi}\left[\frac{\alpha-1}{r_i^2} - \frac{\alpha\left(r_i^2-q^2\right)}{\left(r_i^2+q^2\right)^2} + q^2(\alpha-\beta)\frac{\left\{(\beta-1)r_i^{\beta-2}\left(r_i^2+q^2\right)^{\beta/2+1} - (\beta+2)r_i^{\beta}\left(r_i^2+q^2\right)^{\beta/2}\right\}}{\left(r_i^2+q^2\right)^{\beta+2}}\right] + \frac{1}{4\pi}\left[q^2(\beta-\alpha-1)\frac{\left\{(\beta-2)r_i^{\beta-3}\left(r_i^2+q^2\right)^{\beta/2} - \beta r_i^{\beta-1}\left(r_i^2+q^2\right)^{\beta/2-1}\right\}}{\left(r_i^2+q^2\right)^{\beta}}\right]. \quad (26)$$

The BH undergoes a second-order phase transition when $\partial T_i/\partial r_i = 0$. In this case the specific heat diverges.

Finally, Komar [21] energy computed at $\mathscr{H}^i$ is given by

$$E_i = \frac{1}{2}\left[\alpha\frac{r_i^3}{\left(r_i^2+q^2\right)} - (\alpha-1)\,r_i + \frac{q^2\,(\alpha-\beta)\,r_i^{\beta+1}}{\left(r_i^2+q^2\right)^{\beta/2+1}} + \frac{q^2\,(\beta-\alpha-1)\,r_i^{\beta}}{\left(r_i^2+q^2\right)^{\beta/2}}\right]. \tag{27}$$

In the limits $\alpha = 3$ and $\beta = 4$, one obtains the result of ABG BH.

3. Discussion

In this work, we examined thermodynamic product relations for generalized regular (curvature-free) BH. Generalized BH means the BHs represented by the four parameters, that is, m, q, α, and β. We determined different thermodynamic product particularly area products for different values of α and β. We showed that there is some complicated function of horizon areas indeed mass-independent that could turn out to be universal. We also derived the specific heat to determine the thermodynamic stability of the BH. Finally, we computed Komar energy for this generalized BH.

Competing Interests

The author declares that they have no competing interests.

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Research Article

Solution of Deformed Einstein Equations and Quantum Black Holes

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Recently, one- and two-parameter deformed Einstein equations have been studied for extremal quantum black holes which have been proposed to obey deformed statistics by Strominger. In this study, we give a deeper insight into the deformed Einstein equations and consider the solutions of these equations for the extremal quantum black holes. We then represent the implications of the solutions, such that the deformation parameters lead the charged black holes to have a smaller mass than the usual Reissner-Nordström black holes. This reduction in mass of a usual black hole can be considered as a transition from classical to quantum black hole regime.

1. Introduction

Recently, one- and two-parameter deformed Einstein equations, which are thought to describe the gravitational fields of extremal quantum black holes, have been studied in the framework of entropic gravity proposal [1]. Extremal black holes form by a process in which mass of a charged black hole decreases due to the Hawking radiation. Mass of the black hole reaches a minimum value proportional to its charge and this value is equal to $Q/\sqrt{G}$ (or QM_{Planck}) [2, 3]. On the other hand, a black hole is a structure where mass or energy should be concentrated at a region in which an object must have a velocity above the speed of light in order to escape from the gravitational field of that mass. Then, the radius of that region is the Schwarzschild radius $R_S = 2Gm/c^2$.

On the other hand, quantum mechanically, mass can only be localized into a region, reduced Compton wavelength $\lambda = \hbar/mc$. When a mass is localized into the reduced Compton wavelength, then it automatically contains the Schwarzschild radius. This means that localizing a mass into the reduced Compton wavelength creates a black hole since it is concentrated in a region whose radius is smaller than the Schwarzschild radius. Therefore, the mass concentrated into the reduced Compton wavelength is a quantum black hole.

These extremal black holes with a possible minimum mass are quantum mechanically stable objects and are useful for studying the quantum mechanics of black holes [4].

Extremal black holes are used to study the quantum mechanics of black holes. For large Q the black holes are macroscopic and for small Q the black holes are microscopic so that the quantum gravity is needed. In order to obtain the quantum field theoretical description of black holes, extremal black holes are considered to be as point particles [5].

One of the ways of studying quantum mechanics of black holes is the scattering of black holes to investigate whether they are bosons, fermions, or something else [4, 5]. Understanding the quantum statistics obeyed by the black holes is a good idea for solving the quantum black hole puzzle. The leading studies have shown that the statistical description of quantum black holes obeys neither Bose nor Fermi statistics. Instead, the quantum black holes obey infinite statistics or more generally deformed statistics, since infinite statistics firstly introduced by Greenberg [6, 7] is the special case of deformed Bose and Fermi algebra [8].

Therefore, the extremal quantum black holes can be considered as deformed bosons or fermions and the statistics obeyed by the extremal quantum black holes is deformed statistics. Moreover, the statistical mechanics of the deformed bosons and fermions have been studied in the literature through recent years [9–15]. For a particular class of quantum black holes, one type of deformed gas model can be accompanied according to the physical specifications of the black hole and deformed gas model. Arbitrarily, two different deformed gas models have been devoted to the different family of extremal quantum black holes in two recent studies [1].

q-deformed Bose gas model and (q, p)-deformed Fermi gas model have been taken into account as the quantum black holes. Then, the q-deformed and (q, p)-deformed Einstein equations have been obtained as the gravitational field equations for these deformed gas models. To obtain the deformed Einstein equations, Verlinde's entropic gravity approach [16] has been applied to the deformed entropy of the considered gas model. Verlinde connects the entropy of a source mass to gravitational field equations with a statistical description and reformulated the equations by an entropy-area law. Verlinde's statistical description of gravity has also been inspiring to more studies on modifications of Einstein equations [17–33].

Here, we firstly give a brief summary of one- and two-parameter deformed Einstein equations and then the solutions of the deformed Einstein equations for charged black holes. Since the solutions of standard Einstein equations for charged black holes are the Reissner-Nordström solutions in classical gravity, the solutions of the deformed Einstein equations for charged black holes can be considered in quantum gravity. Lastly, the implications of the solutions are represented. These are that the deformation parameters lead the charged black holes to have a smaller mass than the usual Reissner-Nordström black holes. This reduction in mass of a usual black hole can be considered as a transition from classical to quantum black hole regime.

2. Deformed Einstein Equations

By using the entropy of the deformed gas models in Verlinde's entropic gravity approach, the deformed Einstein equations are obtained to describe the gravitational fields of these deformed objects. For a q-deformed Bose gas model, we identify its quantum algebraic structure by the q-deformed boson algebra [34]:

$$\begin{aligned} a_2 a_2^* - q^2 a_2^* a_2 &= 1, \\ a_1 a_1^* - q^2 a_1^* a_1 &= 1 + \left(q^2 - 1\right) a_2^* a_2, \\ a_2 a_1 &= q a_1 a_2, \\ a_2 a_1^* &= q a_1^* a_2. \end{aligned} \tag{1}$$

Here, a and a^* represent the deformed annihilation and creation operators, respectively. q is also a real deformation parameter with $0 \leq q < \infty$. The grand partition function of the q-deformed boson model is [34]

$$Z = \prod_k \sum_{m=0}^{\infty} (m+1) e^{-\beta \varepsilon_k \{m\}} z^m, \tag{2}$$

where $\beta = 1/kT$ and k is the Boltzmann constant, $z = e^{\beta\mu}$ is the fugacity, ε_k is the energy of the single-particle state, m is the occupation number of the single-particle state, and $\{m\}$ is the deformed occupation number and is given by

$$\{m\} = \frac{1 - q^{2m}}{1 - q^2}. \tag{3}$$

The deformed entropy of the model is also given as

$$\begin{aligned} S = \frac{4\pi V (2m)^{3/2}}{h^3 T} E^{5/2} \left[\frac{5\sqrt{\pi}}{4} z + \frac{5\sqrt{\pi}}{2} \delta(q) z^2 \right. \\ \left. - \frac{\sqrt{\pi}}{2} z \ln z - 2\sqrt{\pi} \delta(q) z^2 \ln z + \cdots \right], \end{aligned} \tag{4}$$

where $E = kT$ is the average energy of single particle, V is the volume enclosed by the deformed bosons, m is the mass of deformed bosons, T is the temperature of the model, and $\delta(q) = (1/4) \cdot \{[3/(1+q^2)^{3/2}] - (1/\sqrt{2})\}$ [34]. The deformed entropy in (4) is used to obtain the one-parameter deformed or equivalently the q-deformed Einstein equations for q-deformed bosons.

On the other hand, to obtain the two-parameter deformed Einstein equations, it is suitable to introduce the (q, p)-deformed Fermi gas model whose quantum algebraic structure is given by the equations

$$\begin{aligned} c_i c_j &= -\frac{q}{p} c_j c_i, \quad i < j, \\ c_i c_j^* &= -qp c_j^* c_i, \quad i \neq j, \\ c_i^2 &= 0, \\ c_1 c_1^* + p^2 c_1^* c_1 &= p^{2\widehat{N}}, \\ c_i c_i^* + q^2 c_i^* c_i &= c_{i+1} c_{i+1}^* + p^2 c_{i+1}^* c_{i+1}, \\ & i = 1, 2, \ldots, d-1, \\ q^{2\widehat{N}} &= c_d c_d^* + q^2 c_d^* c_d, \end{aligned} \tag{5}$$

where c_i and c_i^* are fermion annihilation and creation operators, respectively, and the total deformed number operator is

$$\sum_{i=1}^{d} c_i^* c_i = \left[\widehat{N}_1 + \widehat{N}_2 + \cdots + \widehat{N}_d\right] = \left[\widehat{N}\right]. \tag{6}$$

Eigenvalue spectrum of total number operator is given by the following generalized Fibonacci basic integers:

$$[n] = \frac{q^{2n} - p^{2n}}{q^2 - p^2}, \tag{7}$$

where q and p are the real positive independent deformation parameters [35]. The deformed entropy of the model is

$$\begin{aligned} S = {} & \frac{(2\pi m)^{3/2} V}{h^3 T} \\ & \cdot E^{5/2} \left[\frac{5}{2} f_{5/2}(z, q, p) - f_{3/2}(z, q, p) \ln z \right], \end{aligned} \tag{8}$$

where

$$f_n(z,q,p) = \frac{1}{|\ln(q^2/p^2)|}\left[\sum_{l=1}^{\infty}(-1)^{l-1}\frac{(q^2z)^l}{l^{n+1}} - \sum_{l=1}^{\infty}(-1)^{l-1}\frac{(p^2z)^l}{l^{n+1}}\right]. \quad (9)$$

This deformed entropy in (8) is also used to obtain the two-parameter deformed or equivalently the (q,p)-deformed Einstein equations for (q,p)-deformed fermions.

In order to construct the deformed Einstein equations from the entropies in (4) and (8), Verlinde's proposal is applied to the deformed gas models. The fundamental notion needed to derive the gravity is information in Verlinde's proposal. It is formally the amount of information associated with the matter and its location, measured in terms of entropy. When matter is displaced in space due to a reason, the result is a change in the entropy and this change causes a reaction force. This force is the gravity being an entropic force as an inertial reaction against the force causing the increase of the entropy [16].

The source of gravity is energy or matter and it is distributed evenly over the degrees of freedom in space-time. The existence of energy or matter in space-time causes a temperature in the space-time. The product of the change of entropy during the displacement of source and the temperature is in fact the work and this work is originally led by the force which is known to be gravity [16].

By using Verlinde's idea, one- and two-parameter deformed Einstein equations are recently derived from the deformed entropies (4) and (8) of the q-deformed Bose gas model and (q,p)-deformed Fermi gas model, respectively [1]. Eventually, the q-deformed Einstein equation is given as [1]

$$\frac{10\pi V(2mE)^{3/2}}{h^3}g(z,q)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}, \quad (10)$$

where

$$g(z,q) = \left[\frac{5\sqrt{\pi}}{4}z + \frac{5\sqrt{\pi}}{2}\delta(q)z^2 - \frac{\sqrt{\pi}}{2}z\ln z - 2\sqrt{\pi}\delta(q)z^2\ln z + \cdots\right]. \quad (11)$$

Then, the (q,p)-deformed Einstein equation is similarly given as

$$\frac{5V(2\pi mE)^{3/2}}{2h^3}F(z,q,p)\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}, \quad (12)$$

where

$$F(z,q,p) = \frac{5}{2}f_{5/2}(z,q,p) - f_{3/2}(z,q,p)\ln z. \quad (13)$$

The equations in (10) and (12) are one- and two-parameter deformed Einstein equations, respectively, and they are assumed to describe the gravitational fields generated by the extremal quantum black holes which obey the statistics of deformed particles in accordance with Strominger's proposal.

In the next section, we solve one- and two-parameter deformed Einstein equations for a charged extremal black hole and investigate the implications of the solutions.

3. Solution of Deformed Einstein Equations

Since the underlying statistics of the extremal quantum black holes is known to be the deformed statistics, we admit the particles forming deformed gas models to be the quantum black holes and the corresponding deformed Einstein equations for these deformed particles are assumed to describe the gravitational fields of the quantum black holes of these deformed particles.

We know that the extremal quantum black holes should be charged, because the mass of them should decrease to the minimum value proportional to the charge. The classical charged black holes are treated by the standard Einstein equations and the classical solutions of the standard Einstein equations for the charged black holes are known as the Reissner-Nordström solutions. Here, we obtain the quantum analogs of the solutions of the Einstein equations for these classical charged black holes.

Deformed version of the Einstein field equations is assumed to describe the geometry of the space-time surrounding a charged spherical quantum black hole. Therefore, we need to solve the deformed Einstein-Maxwell equations for the charged quantum black holes. Because of the spherical symmetry, the generic form for the metric in 4 dimensions is [36]

$$ds^2 = -e^{2\alpha(r,t)}dt^2 + e^{2\beta(r,t)}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2. \quad (14)$$

The deformed Einstein equation for the charged spherical quantum black hole is

$$\Psi^{q,p}\left(R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R\right) = 8\pi G T_{\mu\nu}, \quad (15)$$

where

$$\Psi^{q,p} = \begin{cases} \dfrac{10\pi V(2mE)^{3/2}}{h^3}g(z,q) \\ \dfrac{5V(2\pi mE)^{3/2}}{2h^3}F(z,q,p) \end{cases} \quad (16)$$

for q-deformed and (q,p)-deformed Einstein equations, respectively. The energy-momentum tensor $T_{\mu\nu}$ here is one for electromagnetism in this problem and

$$T_{\mu\nu} = F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \quad (17)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor [36]. Also, trace of $T_{\mu\nu}$ for $F_{\mu\nu}$ is

$$T = g^{\mu\nu}T_{\mu\nu} = g^{\mu\nu}F_{\mu\rho}F_{\nu}^{\rho} - \frac{1}{4}g^{\mu\nu}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = 0, \quad (18)$$

since $g^{\mu\nu}g_{\mu\nu} = 4$ in 4 dimensions. Taking the trace of (15) gives $\Psi^{q,p}R = -8\pi GT$ and then by using this and (18) in (15) gives

$$\Psi^{q,p}R_{\mu\nu} = 8\pi GT_{\mu\nu}. \tag{19}$$

Since there is spherical symmetry and only electric charge for our quantum black hole, the electromagnetic field strength tensor has no magnetic field components and the only nonzero component of electric field is radial component which should be independent of θ and ϕ. Then, the radial electric field component is in the form of

$$E_r = F_{tr} = -F_{rt} = f(r,t). \tag{20}$$

The nonzero components of the Ricci tensor for metric (14) are given as [36]

$$\begin{aligned}
R_{tt} &= \partial_t^2\beta + (\partial_t\beta)^2 - (\partial_t\alpha)(\partial_t\beta) \\
&\quad + e^{2(\alpha-\beta)}\left[\partial_r^2\alpha + (\partial_r\alpha)^2 - (\partial_r\alpha)(\partial_r\beta) + \frac{2}{r}\partial_r\alpha\right], \\
R_{rr} &= -\partial_r^2\alpha - (\partial_r\alpha)^2 + (\partial_r\alpha)(\partial_r\beta) + \frac{2}{r}\partial_r\beta \\
&\quad + e^{-2(\alpha-\beta)}\left[\partial_t^2\beta + (\partial_t\beta)^2 - (\partial_t\alpha)(\partial_t\beta)\right], \\
R_{tr} &= \frac{2}{r}\partial_t\beta, \\
R_{\theta\theta} &= e^{-2\beta}\left[r(\partial_r\beta - \partial_r\alpha) - 1\right] + 1, \\
R_{\phi\phi} &= R_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{21}$$

Also, the corresponding nonzero components of the energy-momentum tensor, which is obtained by (17) and (20), are given as [36]

$$\begin{aligned}
T_{tt} &= \frac{f(r,t)^2}{2}e^{-2\beta}, \\
T_{rr} &= -\frac{f(r,t)^2}{2}e^{-2\alpha}, \\
T_{tr} &= 0, \\
T_{\theta\theta} &= \frac{r^2 f(r,t)^2}{2}e^{-2(\alpha+\beta)}, \\
T_{\phi\phi} &= T_{\theta\theta}\sin^2\theta.
\end{aligned} \tag{22}$$

By using the two sets of equations in (21) and (22), it is also obtained that $\beta(r,t) = \beta(r)$ and

$$\alpha(r,t) = \alpha(r) = -\beta(r). \tag{23}$$

Now, the solutions of the Maxwell equations $g^{\mu\nu}\nabla_\mu F_{\nu\sigma} = 0$ and $\nabla_{[\mu}F_{\nu\rho]} = 0$ are needed to determine the components of the electromagnetic field strength tensor, $f(r,t)$, in (20). Solving the Maxwell equations for (20) gives

$$f(r,t) = f(r) = \frac{Q}{\sqrt{4\pi}}\frac{1}{r^2}. \tag{24}$$

The final step to obtain the solution of the deformed Einstein equations for a charge Q quantum black hole is to find the remaining unknown variable $\alpha(r)$ appearing in metric (14) for the space-time which is curved by the charged quantum black hole. To this end, one equation is enough to determine the unknown variable. It can be the $\theta\theta$ component of the deformed Einstein equation (15):

$$\begin{aligned}
\Psi^{q,p}R_{\theta\theta} &= 8\pi GT_{\theta\theta}, \\
\partial_r\left(re^{2\alpha}\right) &= 1 - \frac{1}{\Psi^{q,p}}\frac{GQ^2}{r^2}.
\end{aligned} \tag{25}$$

The solution is found to be

$$e^{2\alpha} = 1 - \frac{R_S}{r} + \frac{1}{\Psi^{q,p}}\frac{GQ^2}{r^2}, \tag{26}$$

where R_S is the integration constant and is known to be the Schwarzschild radius $R_S = 2Gm$. Rewriting metric (14) with (26) gives

$$ds^2 = \Delta dt^2 + \Delta^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2, \tag{27}$$

where

$$\Delta = 1 - \frac{2Gm}{r} + \frac{1}{\Psi^{q,p}}\frac{GQ^2}{r^2}. \tag{28}$$

The singularities and the event horizons for these black holes are determined by the function Δ and the radius r. There is a true curvature singularity at $r = 0$, since the metric goes to infinity for this value. The coordinate singularity also occurs at $\Delta = 0$ and the conditions giving this singularity occur from the solution of $\Delta = 0$, such as

$$r_\pm = Gm \pm \sqrt{G^2m^2 - \frac{GQ^2}{\Psi^{q,p}}}. \tag{29}$$

This implies that for some suitable cases we can have the event horizons r_+ and r_- which determine the place of the coordinate singularity in the space-time. Equation (29) constitutes three cases of solutions such that $Gm^2 < Q^2/\Psi^{q,p}$, $Gm^2 > Q^2/\Psi^{q,p}$, and $Gm^2 = Q^2/\Psi^{q,p}$.

The first case $Gm^2 < Q^2/\Psi^{q,p}$ is unphysical since this solution states that the total energy of the black hole is less than the energy of the electromagnetic contribution. Also, this condition makes Δ different from zero, which makes the first case invalid.

The second case $Gm^2 > Q^2/\Psi^{q,p}$ implies a physical situation since the energy of electromagnetic field is less than the total energy. Two event horizons r_+ and r_- also make $\Delta = 0$.

Finally, the third case $Gm^2 = Q^2/\Psi^{q,p}$ gives the extremal charged black hole solution, since the mass of the black hole decreases to the minimum value from the second case $Gm^2 > Q^2/\Psi^{q,p}$. This minimum mass solution for extremal black holes remains stationary for all times. Also, this case makes $\Delta = 0$ at a single radius $r_{\pm} = Gm$ and this states a single event horizon. This deformed case solution $Gm^2 = Q^2/\Psi^{q,p}$ is the analog of classical Reissner-Nordström solution $m = Q/\sqrt{G}$ which is often examined in the studies of quantum gravity. In the second case, the mass of the black hole is allowed to be in very large classical scales due to the capability of getting bigger values than the charge, implied in the inequality $Gm^2 > Q^2/\Psi^{q,p}$, whereas the mass of the deformed black hole is allowed to decrease by very small values which could fall into the quantum regime, because the decrease of the mass is governed by a very small term $1/\Psi^{q,p}$ being order of $h^{6/7}$ in the right hand side of the third-case equation $Gm^2 = Q^2/\Psi^{q,p}$.

In our deformed case, this decrease in mass of black hole which is controlled by the term $1/\Psi^{q,p}$ is different from the classical Reissner-Nordström solution. We now discuss the effects of this extra term on mass reduction.

We investigate the reduction of the mass with respect to the classical Reissner-Nordström case, for the q-deformed and (q, p)-deformed Einstein cases. From (16), we have two $1/\Psi^{q,p}$ values for the mass of the extremal quantum black hole in the third case of $Gm^2 = Q^2/\Psi^{q,p}$; that is,

$$m^q = \left(\frac{h^3}{10\pi V (2E)^{3/2}} \frac{1}{g(z,q)}\right)^{2/7} \left(\frac{Q^2}{G}\right)^{2/7}, \qquad (30)$$

$$m^{(q,p)} = \left(\frac{2h^3}{5V (2\pi E)^{3/2}} \frac{1}{F(z,q,p)}\right)^{2/7} \left(\frac{Q^2}{G}\right)^{2/7}. \qquad (31)$$

While the minimum mass of an extremal quantum black hole for the q-deformed case is (30), it is (31) for the (q, p)-deformed case. However, the minimum mass of a classical Reissner-Nordström black hole is given as $m = Q/\sqrt{G}$. When we compare the minimum masses of deformed quantum case and the classical Reissner-Nordström cases, we obtain

$$m^q = \left(\frac{1}{10\pi V (2E)^{3/2} Q^{3/2}}\right)^{2/7} \left(\frac{h^3 G^{3/4}}{g(z,q)}\right)^{2/7} m, \qquad (32)$$

$$m^{(q,p)} = \left(\frac{2}{5V (2\pi E)^{3/2} Q^{3/2}}\right)^{2/7} \left(\frac{h^3 G^{3/4}}{F(z,q,p)}\right)^{2/7} m. \qquad (33)$$

Equations (32) and (33) imply that the mass of the charged extremal black hole in the deformed quantum case can decrease to a smaller value than that of the classical Reissner-Nordström case. To understand the decrease in the mass, we examine the behaviors of the factors $(h^3G^{3/4}/g(z,q))^{2/7}$ and $(h^3G^{3/4}/F(z,q,p))^{2/7}$ in front of the classical mass of the charged black hole in (32) and (33), respectively. Therefore, the behavior of $(h^3G^{3/4}/g(z,q))^{2/7}$ with respect to z and q in Figures 1 and 2 is represented, for $q < 1$ and $1 < q$, respectively. We also represent the behavior of $(h^3G^{3/4}/F(z,q,p))^{2/7}$ with respect to z, q, and p in Figures 3 and 4, for $q, p < 1$ and $1 < q, p$, respectively.

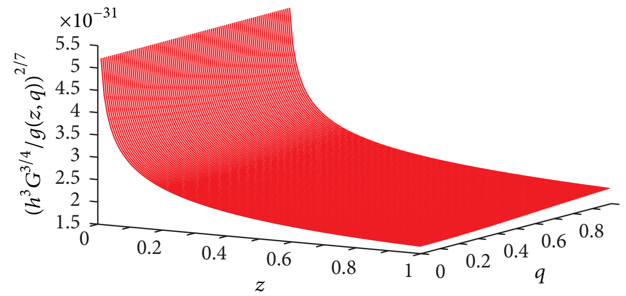

Figure 1: The q-deformed mass reduction factor $(h^3G^{3/4}/g(z,q))^{2/7}$ for $q < 1$.

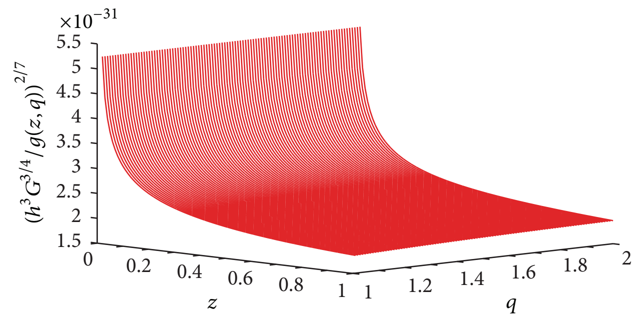

Figure 2: The q-deformed mass reduction factor $(h^3G^{3/4}/g(z,q))^{2/7}$ for $q > 1$.

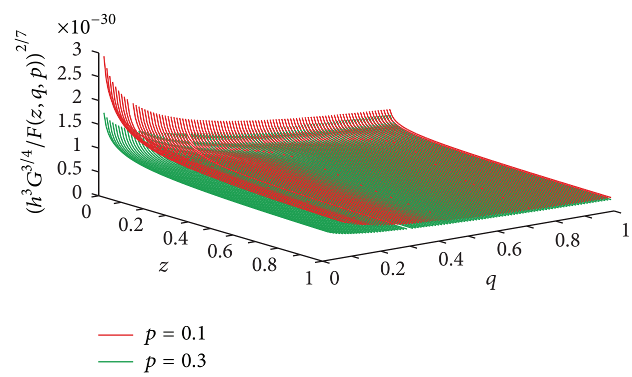

Figure 3: The q, p-deformed mass reduction factor $(h^3G^{3/4}/F(z, q, p))^{2/7}$ for various values of the deformation parameters p and $q < 1$.

4. Conclusions

Recently, q-deformed and (q, p)-deformed Einstein equations have been proposed for the investigations of charged extremal quantum black holes, in the framework of entropic gravity approach [1]. In this study, we give a review and deeper meaning to the deformed Einstein equations, which is based on Strominger's idea, such that the quantum black holes obey the deformed statistics. We then consider the solutions of these equations for the charged extremal quantum black holes. We analyze the obtained solutions for q-deformed and (q, p)-deformed cases, separately.

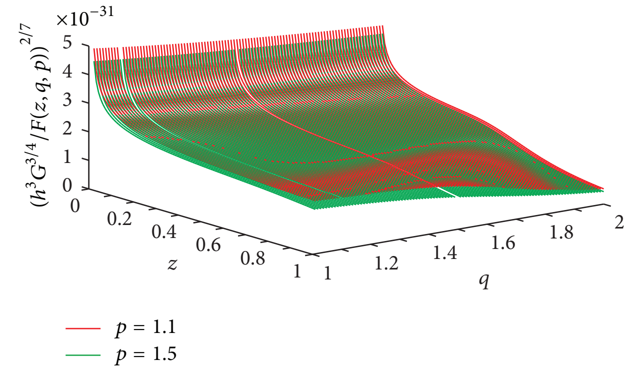

FIGURE 4: The q, p-deformed mass reduction factor $(h^3G^{3/4}/F(z,q,p))^{2/7}$ for various values of the deformation parameters p and $q > 1$.

We represent the true and coordinate singularities from the solutions for quantum black holes. Also, the event horizons for these singularities are mentioned briefly. We also investigate the possible decrease in mass via Hawking radiation to a minimum value which is determined by the charge of quantum black hole. The difference in the decrease in classical black holes and quantum black holes is obvious from (30), (31), and $m = Q/\sqrt{G}$. According to this difference, the reduced quantum and classical mass of the extremal black holes are represented in (32) and (33).

We illustrate the decreases in quantum masses m^q and $m^{q,p}$ in Figures 1–4 with respect to the classical mass m. According to Figures 1 and 2, the mass of the quantum black hole m^q in (32) is at least 10^{-30} times smaller than the classical black hole mass m, in the q-deformed case. After considering the inverse of the volume, charge, and energy factors, mass m^q gets smaller than $10^{-30}m$. We again see a similar situation for the mass of the quantum black hole $m^{q,p}$ in (33) from Figures 3 and 4. $m^{q,p}$ is at least 10^{-30} times smaller than the classical black hole mass m, in the (q,p)-deformed case. Considering the inverse of the volume, charge, and energy factors, mass $m^{q,p}$ similarly gets smaller than $10^{-30}m$ in the (q,p)-deformed case.

Since the theoretical possibility of concentrating a mass into its reduced Planck mass gives a radius containing the Schwarzschild radius and the obtained quantum masses of the extremal black holes in (32) and (33) are at least 10^{-30} times smaller than the classical masses due to possible Hawking radiation, the solutions of the deformed Einstein equations imply that all the propositions and ideas considered here seem to be consistent with each other. Three independent ideas have been used to obtain these equations. Verlinde's proposition is on gravity having an entropic origin, Strominger's proposition is on the type of the underlying statistics obeyed by the quantum black holes, and our idea is to get the gravitational field equations for these quantum black holes from Verlinde's proposition, by considering the quantum black holes as the deformed bosons or fermions due to Strominger's statement that the statistics obeyed by the quantum black holes is deformed statistics.

Competing Interests

The authors declare that there are no competing interests regarding the publication of this paper.

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Research Article

The Effect of de-Sitter Like Background on Increasing the Zero Point Budget of Dark Energy

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During this work, using subtraction renormalization mechanism, zero point quantum fluctuations for bosonic scalar fields in a de-Sitter like background are investigated. By virtue of the observed value for spectral index, $n_s(k)$, for massive scalar field the best value for the first slow roll parameter, ϵ, is achieved. In addition, the energy density of vacuum quantum fluctuations for massless scalar field is obtained. The effects of these fluctuations on other components of the universe are studied. By solving the conservation equation, for some different examples, the energy density for different components of the universe is obtained. In the case which all components of the universe are in an interaction, the different dissipation functions, $\widetilde{Q}_i$, are considered. The time evolution of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ shows that $\widetilde{Q} = 3\gamma H(t)\rho_m$ has the best agreement in comparison to observational data including CMB, BAO, and SNeIa data set.

1. Introduction

For a de-Sitter like background, to investigate the effects of the quantum fluctuations on the energy budget of the universe both massive and massless scalar fields are considered. Because of the appearance of the first slow roll parameter, the relation between power spectral index and slow roll parameters leads to a way to connect theoretical results with observations. In fact one can estimate the best value of the first slow roll parameter based on observations, for instance, Planck 2013 [1]. From this perspective, for zero point quantum fluctuations both energy density and pressure are calculated. Interestingly it is observed that the contribution of zero point energy is increased against the results for the normal de-Sitter ones [2, 3]. The importance of this note is that whereas the zero point contribution of dark energy potentially is detectable, therefore the possibility of dark energy detection is increased. Also it should be stressed, due to time dependency of Hubble parameter which appeared in energy density of zero point quantum fluctuations, energy can transfer between different components of the universe. By considering this concept one can propose different manners to investigate interaction between zero point fluctuations and other sectors. In a general case where all components are in an interaction, three different dissipation functions $\widetilde{Q}$ are considered. For this special case time evolution of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ shows that $\widetilde{Q} = 3\gamma H(t)\rho_m$ is in best agreement in comparison to observations including CMB, BAO, and SNeIa data set. Furthermore, energy density of matter, ρ_m, has some deviations in comparison to the result of ordinary de-Sitter one. In fact in ρ_m, beside a^{-3}, some extra terms appeared, where these new terms could be interpreted as new source for cold dark matter risen from interaction between matter and quantum fluctuations. In fact as it was discussed in [3] zero point quantum fluctuations could be considered as subdark energy, and these extra terms in question can be proposed as subdark matter. The scheme of this paper is as follows.

In Section 2 the general framework of this work including the mathematical calculations is discussed. In Section 3 the cosmological role for zero point energy density is investigated, and the results of this work are compared with previous works. In Section 4 to estimate the amount of subdark energy and also subdark matter, the bounds which are risen from time evolution of dark energy are considered. And at last, we have conclusion.

2. Massive Scalar Field and Slow Roll Parameters

To study the effect of zero point quantum fluctuations let us consider a real minimally coupled bosonic scalar field Φ in a semiclassical general relativity mechanism [4]. In such scenario the geometry is not quantized but the energy momentum tensor related to the scalar field is calculated by means of the vacuum expectation value concept. To begin we consider the action

$$S = S_m + \int d^4x \frac{\sqrt{-g}}{\kappa^2}\left(R - 2\Lambda + \kappa^2 \mathscr{L}_\Phi\right), \tag{1}$$

where S_m is the matter action, g is the determinant of the metric, R is the Ricci scalar, Λ is the Einstein's cosmological constant, and $\mathscr{L}_\Phi$ is defined as

$$\mathscr{L}_\Phi = -\frac{1}{2}g^{\mu\nu}\left(\nabla_\mu\Phi\nabla_\nu\Phi\right) - V(\Phi), \tag{2}$$

where $V(\Phi)$ is the potential of the model. Variation of the action (1) with respect to metric yields

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + g_{\mu\nu}\Lambda = \kappa^2\left(T^M_{\mu\nu} + \left\langle 0\left|T^\Phi_{\mu\nu}\right|0\right\rangle_{\text{ren}}\right), \tag{3}$$

where $T^M_{\mu\nu} = (-2/\sqrt{-g}) \times (\delta S_m/\delta g^{\mu\nu})$ and

$$T^\Phi_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \frac{1}{2}g_{\mu\nu}\left(\partial_\alpha\Phi\partial^\alpha\Phi + 2V(\Phi)\right). \tag{4}$$

Using the above equation, the 00 and ii components of energy-momentum tensor read

$$\begin{aligned} T^{\Phi 0}_{\ 0} &= -\frac{1}{2}\left[\dot{\Phi}^2 + \frac{\Phi^2_{,i}}{a^2(t)} + 2V(\Phi)\right], \\ \frac{1}{3}T^{\Phi i}_{\ i} &= \frac{1}{2}\left[\dot{\Phi}^2 - \frac{1}{3}\frac{\Phi^2_{,i}}{a^2(t)} - 2V(\Phi)\right]. \end{aligned} \tag{5}$$

It is also obvious that variation (1) with respect to Φ yields

$$\square\Phi - V_{,\Phi}(\Phi) = 0, \tag{6}$$

where $\square$ is d'Alembert operator. As it is mentioned above in semiclassical approach one can quantize the scalar field and therefore the vacuum expectation value of energy density and pressure could be obtained. For this purpose, the quantized scalar field is defined as

$$\begin{aligned} &\Phi(t, \mathbf{x}) \\ &= \int \frac{d^3k}{(2\pi)^3\sqrt{2k}}\left[a_\mathbf{k}\phi(t)e^{-j\mathbf{k}\cdot\mathbf{x}} + a^\dagger_\mathbf{k}\phi^*(t)e^{j\mathbf{k}\cdot\mathbf{x}}\right], \end{aligned} \tag{7}$$

where $\phi(t)$ is a function which should be determined, $a_\mathbf{k}$ ($a^\dagger_\mathbf{k}$) is annihilation (creation) operators and $\sqrt{-1} = j$ [5, 6]. Now introducing (7) to (6), it expresses

$$\ddot{\phi} + 3H(t)\dot{\phi} + \frac{k^2}{a^2}\phi + V_{,\phi} = 0, \tag{8}$$

where $H = \dot{a}/a$, and $a(t)$ is the scale factor of the universe, overdot denotes derivation with respect to the cosmic time, and the $k^2\phi/a^2$ term appeared due to scalar field's spatial dependency. For more convenience one can use conformal time $\eta = \int dt/a(t)$, and therefore (8) could be rewritten as

$$\phi'' + 2\mathscr{H}\phi' + k^2\phi + a^2\frac{V'(\phi)}{\phi'} = 0, \tag{9}$$

in which overprime indicates derivation with respect to η, and $\mathscr{H} = a'/a = \dot{a}$ is the conformal, comoving, Hubble parameter. Now, by defining $\chi := \phi/a$, using a power law potential, $V(\phi) = m^2\phi^2/2$, relation (9) reads

$$\chi'' + \left(k^2 + m^2a^2 - \frac{a''}{a}\right)\chi = 0. \tag{10}$$

It should be stressed for de-Sitter like background in question that the scale factor is taken as $a(t) = t^{1/\epsilon}$, where $\epsilon = -\dot{H}/H^2 \ll 1$ is the first slow roll parameter [7, 8]. Using definitions of scale factor (i.e., $a(\eta) \simeq -(1+\epsilon)/H\eta$) and second slow roll parameter, $\tau = m^2/3H^2$, one can attain

$$-m^2a^2 + \frac{a''}{a} \simeq \frac{1}{\eta^2}\left(2 + 3(\epsilon - \tau)\right). \tag{11}$$

By introducing $\tilde{\nu} := (\epsilon - \tau) + 3/2$ and keeping only up to first orders of the first and second slow roll parameters (10) could be rewritten as

$$\chi'' + \left(k^2 - \frac{\tilde{\nu}^2 - 1/4}{\eta^2}\right)\chi = 0, \tag{12}$$

where $\tilde{\nu}^2 \simeq 3(\epsilon - \tau) + 9/4$. Solving the above Bessel like equation, the magnitude of ϕ can be achieved as

$$|\phi| \simeq \frac{H(1-\epsilon)}{k}\left(\frac{k}{aH(1-\epsilon)}\right)^{3/2-\tilde{\nu}}. \tag{13}$$

To estimate the best value of ϵ, one can use the power spectrum concept for instance in light of Planck 2013 [1]. It is obvious that, by considering the Fourier transformation for an arbitrary function as

$$\widetilde{\Phi}_\mathbf{k}(t, \mathbf{r}) = \int \frac{d^3k}{(2\pi)^3\sqrt{2k}}e^{j\mathbf{k}\cdot\mathbf{r}}\tilde{\phi}_\mathbf{k}(t), \tag{14}$$

where $\mathbf{k}$ is comoving momentum and $\mathbf{r}$ is spatial vector, the power spectrum can be defined as

$$\left\langle\tilde{\phi}_\mathbf{k}(t)\tilde{\phi}^*_{\mathbf{k}'}(t)\right\rangle = \frac{2\pi^2}{k^2}\rho_S(2\pi)^3\delta^3\left(\mathbf{k} - \mathbf{k}'\right), \tag{15}$$

where ρ_S is power spectrum in question and $\langle\widetilde{\phi}_{\mathbf{k}}(t)\widetilde{\phi}^*_{\mathbf{k}'}(t)\rangle$ indicates the mean square value of $\widetilde{\phi}_{\mathbf{k}}(t)$. By combining (7), (14), and (15) the power spectrum can be achieved as

$$\rho_S = \frac{k^2}{(2\pi)^2}\left|\phi(t)\right|^2. \tag{16}$$

To achieve this result the relation between annihilation and creation operators reads $[a_{\mathbf{k}}, a^{\dagger}_{\mathbf{k}'}] = (2\pi)^3\delta^3(\mathbf{k}-\mathbf{k}')$. At last if one considers the relation (13), the power spectrum could be rewritten as

$$\rho_S = (1-2\epsilon)H^2\left(\frac{k}{aH(1-\epsilon)}\right)^{3-2\widetilde{\nu}}. \tag{17}$$

Also another important quantity is notable that is the spectral index which plays a crucial role in inflationary investigations and is defined as

$$n_S - 1 = \frac{d\ln\rho_S}{d\ln k}. \tag{18}$$

Substituting (17) into the above equation, one has

$$n_S - 1 = (1-2\epsilon)(3-2\widetilde{\nu}), \tag{19}$$

and at last after some manipulations the spectral index is obtained as $n_S - 1 = -2\epsilon + 2\tau$. By means of observed value of spectral index, $n_S < 0.9675$, risen from Planck 2013 [1], the best value of the first slow roll parameter is $\epsilon \simeq 0.02$.

2.1. Typical Example: Massless Scalar Field. In this case we want to consider massless scalar field. Therefore if, in action (1), one assumes $V(\Phi) = 0$, solving (9), for positive modes, is attained as

$$\phi(\eta) = \frac{1}{a(\eta)}\left(1-\frac{i\xi}{k\eta}\right)e^{(-ik\eta)}, \tag{20}$$

where $\xi = (2+3\epsilon)/2$. In this case, as massive ones, the first slow roll parameter is considered only up to the first order. For this typical case we want to estimate the zero point quantum fluctuation contribution in the energy budget of the universe. To begin we have to calculate the vacuum expectation value of the scalar field energy-momentum tensor. By means of (5) and (20), one has

$$\begin{aligned}\rho_{\text{vac}} &= \left\langle 0\left|T^{\phi}_{00}\right|0\right\rangle \\ &= \frac{1}{2}\int_0^{a(t)\Lambda_c}\frac{d^3k}{(2\pi)^3 2k}\left[\left|\dot{\phi}\right|^2 + \frac{k^2}{a^2(t)}\left|\phi\right|^2\right], \\ P_{\text{vac}} &= \frac{1}{3}\Sigma_i\left\langle 0\left|T^{\phi i}_{i}\right|0\right\rangle \\ &= \frac{1}{2}\int_0^{a(t)\Lambda_c}\frac{d^3k}{(2\pi)^3 2k}\left[\left|\dot{\phi}\right|^2 - \frac{k^2}{3a^2(t)}\left|\phi\right|^2\right],\end{aligned} \tag{21}$$

where ρ_{vac} and P_{vac}, indicate the energy density and pressure of the vacuum quantum fluctuations, respectively. According to quantum field theory the cutoff Λ_c should be considered greater than physical momenta $k/a(t)$. By considering (20) and definition of scale factor for a de-Sitter like background, one has

$$\begin{aligned}\left(\left|\dot{\phi}\right|^2 + \frac{k^2}{a^2(t)}\left|\phi\right|^2\right) &= \frac{1}{a^4}\left[2k^2 + a^2H^2(1+9\epsilon)\right] \\ &\quad + \mathcal{O}\left(\epsilon^2\right), \\ \left(\left|\dot{\phi}\right|^2 - \frac{k^2}{3a^2(t)}\left|\phi\right|^2\right) &= \frac{2}{3a^4}\left[k^2 - \frac{a^2H^2}{2}(1-3\epsilon)\right] \\ &\quad + \mathcal{O}\left(\epsilon^2\right).\end{aligned} \tag{22}$$

By virtue of (22), solving (21) leads to

$$\begin{aligned}\rho_{\text{vac}} &= \frac{\Lambda_c^4}{16\pi^2} + (1+9\epsilon)\frac{H^2(t)\Lambda_c^2}{16\pi^2}, \\ P_{\text{vac}} &= \frac{\Lambda_c^4}{48\pi^2} - (1-3\epsilon)\frac{H^2(t)\Lambda_c^2}{48\pi^2}.\end{aligned} \tag{23}$$

The first terms in (23) are the contribution of the energy density and pressure for Minkowskian space time; and because of the cutoff dependency the latter terms are well known bare quantities. To get rid of quartic divergencies the subtraction mechanism is a good suggestion, which is close to Casimir approach [9]. The base of the Casimir effect is on the subtraction mechanism. In this effect the energy that gives rise to observable effects is the difference between vacuum energy risen from two parallel conducting plates, for example, for electromagnetic field and vacuum energy computed in Minkowskian space-time in an infinite volume. Therefore when two infinite values for energy of space time and plates are subtracted one can obtain a finite quantity. Therefore by means of Arnowitt-Deser-Misner (ADM) approach [2] and subtraction mechanism one concludes that the vacuum energy to an asymptotically flat space-time with metric $g_{\mu\nu}$ can be achieved as $E = H_{\text{GR}}(g_{\mu\nu}) - H_{\text{GR}}(\eta_{\mu\nu})$, where H_{GR} refers to the Hamiltonian which is calculated in general relativity. This equation indicates that flat space-time does not gravitate and the contribution of the energy which is obtained in Minkowski space-time can be subtracted from related quantity in curved background [2, 9–11]. Thence, because flat space-time does not gravitate one is able to subtract the contribution of quartic terms in Minkowski space from the same terms in FriedmannLemaitreRobertsonWalker (FLRW) space-time. Also it should be stressed that the results which were obtained for de-Sitter like scenario have some notable differences with normal de-Sitter ones. The first is, for de-Sitter like investigations, the Hubble parameter is not a constant and this causes the appearance of the first slow roll parameter in the model which could be considered to investigate the accuracy of this model. As a second note, the coefficients which appeared in energy density and pressure cause the increase of the zero point quantum fluctuations contribution in the dark energy. Now let us investigate the bare quantities which are

$$\rho_{\text{bare}} = (1+9\epsilon)\frac{H^2(t)\Lambda_c^2}{16\pi^2},$$
$$P_{\text{bare}} = -(1-3\epsilon)\frac{H^2(t)\Lambda_c^2}{48\pi^2}. \quad (24)$$

Following [3] one can introduce counter terms for energy density and pressure, respectively, as follows:

$$\rho_{\text{count}} = -(1+9\epsilon)\frac{H^2(t)\Lambda_c^2}{16\pi^2} + \rho_Z, \quad (25)$$

$$P_{\text{count}} = (1-3\epsilon)\frac{H^2(t)\Lambda_c^2}{48\pi^2} + P_Z, \quad (26)$$

where

$$\rho_Z = (1+9\epsilon)\frac{H^2(t)M^2}{16\pi^2}, \quad (27)$$

$$P_Z = -(1-3\epsilon)\frac{H^2(t)M^2}{48\pi^2}, \quad (28)$$

where subscript Z refers to the zero point and M is in order of Planck mass. The interesting result which could be realized from (27) and (28) is that they appear to contain a mix of ultraviolet (UV) cutoff, M as Planck mass, and infrared (IR) cutoff, and $H(t)$ as the inverse of horizon size. Based on quantum field theory concepts, we know the widely separated energy scales are decoupled; but one can find that the origin of this mix of UV-IR cutoffs arises from classical subtraction procedure, that is, $E = H_{GR}(g_{\mu\nu}) - H_{GR}(\eta_{\mu\nu})$. In fact the origin of this mix goes back to (21), where it involves $T^{\phi}_{\mu\nu}$ and $\langle 0|T^{\phi}_{\mu\nu}|0\rangle$ as local and global quantities, respectively. For more discussion we refer the reader to [2, 11, 12]. Using (27) and (28), the equation of state (EoS) parameter for the vacuum fluctuations could be expressed as

$$\omega_Z = \frac{P_Z}{\rho_Z} = \frac{-1}{3} + 4\epsilon. \quad (29)$$

This relation indicates that the EoS of zero point quantum fluctuations is dependent on the first slow roll parameter. It should be noted also that whereas this approach is similar to Casimir mechanism both positive and negative signs for energy density are acceptable. To consider this fact one can consider coefficient $\sigma = \pm 1$, for (27), and redefines it as

$$\rho_Z = \sigma(1+9\epsilon)\frac{H^2(t)M^2}{16\pi^2}. \quad (30)$$

The positive sign causes an attractive force and negative ones are related to the repulsive case. Now for more discussions about time dependency of energy density of vacuum fluctuations, one is able to redefine it based on critical energy density of the universe. Hence considering definition of critical energy density ($\rho_{\text{cri}} = 3H^2(t)/8\pi G$) and by means of definition of Planck mass $M_{\text{Pl}} = 1/\sqrt{G}$, ($G$ is the Newtonian constant), the energy density of zero point fluctuations could be rewritten as

$$\rho_Z = \Omega_Z\rho_{\text{cri}} + \mathcal{O}\left(\epsilon^2\right) = \beta\frac{1+9\epsilon}{\epsilon^2}a^{-2\epsilon}(t), \quad (31)$$

where $\Omega_Z = \beta(1+9\epsilon)/M_{\text{Pl}}^2$ and $\beta = \sigma M^2/16\pi^2$. Because of the time dependency of $\rho_Z(t)$, the conservation equation for ρ_Z is not satisfied. Hence using $\dot{\rho}_Z = -2\epsilon H(t)\rho_Z$ and (31) one has

$$\dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) = Q, \quad (32)$$

where Q is dissipation function and it could be obtained as

$$Q = 2H(t)\rho_Z(1+5\epsilon). \quad (33)$$

Therefore the energy density of quantum fluctuations is capable of exchanging energy with other components of the universe. To investigate the transformation of energy we consider some different cases as follows.

3. Transformation of Energy between Different Components of the Universe

3.1. Transformation of Energy between Zero Point Fluctuations and Matter. In this case one has

$$\dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) = 2H(t)\rho_Z(1+5\epsilon),$$
$$\dot{\rho}_m + 3H(t)\rho_m = -2H(t)\rho_Z(1+5\epsilon), \quad (34)$$

and therefore the combination of the two sections of the above equation yields

$$\dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) + \dot{\rho}_m + 3H(t)\rho_m = 0, \quad (35)$$

which indicates that the conservation equation in general is satisfied. Therefore by means of (31) and (35), ρ_m could be achieved as

$$\rho_m = \Psi a^{-2\epsilon}(t) + \widetilde{\Psi}a^{-3}(t), \quad (36)$$

where $\Psi = -2\beta(1+44\epsilon/3)/3\epsilon^2$ and $\widetilde{\Psi}$ is integration constant. From (36) it is realized that in our model the matter density equation is modified, where the first term indicates the matters which are risen from interaction of quantum fluctuations with matter and the latter indicates the remaining contribution of matter, namely, ordinary cold dark matter.

3.2. Transformation of Energy between Zero Point Energy and the Remnant Components of Dark Energy. Assume there is an internal interaction between different components of dark energy, namely, ρ_Λ and ρ_Z, where ρ_Λ indicates energy density of cosmological constant. In this case, one can suppose that $\omega_\Lambda = -1$ and therefore the conservation equation reads

$$\dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) + \dot{\rho}_\Lambda = 0. \quad (37)$$

By virtue of $\dot{\rho}_Z = -2\beta(1+3\epsilon)/(\epsilon^2 a^{3\epsilon}(t))$ and definition of scale factor in de-Sitter like background, one has

$$\rho_\Lambda = \frac{\beta}{\epsilon}(1+14\epsilon) H^2(t) + C_0, \quad (38)$$

where C_0 is integration constant. In addition it is obvious that because $\dot{\rho}_Z$ is proportional to $\dot{\rho}_\Lambda$ the Big Bang Nucleosynthesis (BBN) constraint which has been discussed in [2] could be considered to estimate the upper bound on $\Omega_Z = \beta(1+9\epsilon)/M_{\rm Pl}^2$. Also it should be stressed that, by comparing ρ_Z with the one in normal de-Sitter model, it is clear that the coefficient $(1+9\epsilon)$ causes the increase of the magnitude of zero point energy density.

3.3. Transformation of Energy between All Components of the Universe. In this stage, one can consider a general case where all components of the universe are in an interaction. Therefore the conservation equations could be written as

$$\dot{\rho}_m + 3H(t)\rho_m = \widetilde{Q}_i, \quad (39)$$

$$\dot{\rho}_\Lambda + \dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) = -\widetilde{Q}_i, \quad (40)$$

where $\widetilde{Q}_i$ are dissipation functions and are defined as follows:

(i) $\widetilde{Q}_1 = 3\kappa H(t)\rho_\Lambda$,

(ii) $\widetilde{Q}_2 = 3\gamma H(t)\rho_m$,

(iii) $\widetilde{Q}_3 = 3\theta H(t)\rho_Z$.

A also κ, γ, and θ indicate the strength of the interaction between different components of the universe [13].

3.3.1. Solving Conservation Equation for $\widetilde{Q}_1$. By virtue of (40) and dissipation function $\widetilde{Q}_1$, the conservation equation for dark energy components of the universe is as follows:

$$\dot{\rho}_\Lambda + \dot{\rho}_Z + 3\kappa H(t)\rho_\Lambda + 3H(t)\rho_Z(1+\omega_Z) = 0. \quad (41)$$

Based on (31) and (29), (41) could be rewritten as

$$\dot{\rho}_\Lambda + 3\kappa H(t)\rho_\Lambda + 2\beta(1+14\epsilon)H^3(t) = 0, \quad (42)$$

and hence solving this differential equation yields

$$\rho_\Lambda = \frac{-\breve{D}}{(3\kappa - 2\epsilon)} a^{-2\epsilon}(t) + \widetilde{c} a^{-3\kappa}, \quad (43)$$

where $\breve{D} = 2(1+14\epsilon)\beta/\epsilon^2$ and $\widetilde{c}$ is integration constant. By substituting (43) into (39) one can attain ρ_m as follows

$$\rho_m = \frac{-\breve{D}}{(3-2\epsilon(1-1/\kappa))} a^{-2\epsilon}(t) + \frac{\kappa\widetilde{c}}{1-\kappa} a^{-3\kappa} + \breve{B} a^{-3}(t). \quad (44)$$

In the above equation $\breve{B}$ is integration constant. From this relation one can conclude that, in matter equation, only ordinary cold dark matter does not appear; rather, an extra term appears which is risen from interaction of matter and quantum fluctuations, namely, subdark matter. In the following, we will come back to this issue.

3.3.2. Solving Conservation Equation for $\widetilde{Q}_2$. By rewriting (39) and (40) for $\widetilde{Q}_2$, one arrives to

$$\dot{\rho}_m + 3H(t)\rho_m = 3\gamma H(t)\rho_m, \quad (45)$$

$$\dot{\rho}_\Lambda + \dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) = -3\gamma H(t)\rho_m, \quad (46)$$

and thus (45) could be considered as

$$\dot{\rho}_m + 3H(t)(1-\gamma)\rho_m = 0, \quad (47)$$

and after solving, one can attain

$$\rho_m = \rho_{m0} a^{-3(1-\gamma)}(t). \quad (48)$$

Substituting (48) into (46) yields

$$\rho_\Lambda = \frac{\beta(1+14\epsilon)}{\epsilon} H^2(t) + \frac{\rho_{m0}\gamma}{3(1-\gamma)} a^{-3(1-\gamma)} + \mathfrak{M}, \quad (49)$$

where $\mathfrak{M}$ is the integration constant.

3.3.3. Solving Conservation Equation for $\widetilde{Q}_3$. In this stage, one can suppose that the interaction between different components of the universe is determined by virtue of $\widetilde{Q}_3$. Thence (39) and (40) are rearranged as

$$\dot{\rho}_m + 3H(t)\rho_m = 3\theta H(t)\rho_Z, \quad (50)$$

$$\dot{\rho}_\Lambda + \dot{\rho}_Z + 3H(t)\rho_Z(1+\omega_Z) = -3\theta H(t)\rho_Z. \quad (51)$$

Using definition of scale factor in de-Sitter like background, one is allowed to rewrite (50) as

$$\dot{\rho}_m + \frac{3}{\epsilon}\rho_m a^{-\epsilon} - \frac{3\theta}{\epsilon}\left(\frac{\beta(1+3\epsilon)}{\epsilon^2}\right) a^{-3\epsilon} = 0, \quad (52)$$

and solving this equation for ρ_m yields

$$\rho_m = \breve{A} a^{-2\epsilon} + \breve{K} a^{-3}(t), \quad (53)$$

where $\breve{A} \simeq 3\theta\beta(1+8\epsilon)/\epsilon^2$ and $\breve{K}$ is the integration constant. Hence, to attain ρ_Λ, from (51) one has

$$\dot{\rho}_\Lambda + \frac{\beta}{\epsilon^3}[2+3\theta+27\epsilon(1+\theta)] a^{-3\epsilon}(t) = 0, \quad (54)$$

and solution of this equation for ρ_Λ reads

$$\rho_\Lambda = \frac{3\beta}{\epsilon}[2+3\theta+27\epsilon(1+\theta)] H^2(t). \quad (55)$$

4. Bounds Which Are Risen from Time Evolution of Dark Energy

In this section we want to compare the results of this work with results which are risen from standard ΛCDM model. For this end, one can start from the Friedmann equation. Therefore, the ratio of dark energy density and critical energy density in the standard model as a function of red shift parameter, z, is obtained as

$$\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} = \frac{\Omega_\Lambda (1+z)^{3(1+\omega_\Lambda)}}{\Omega_M (1+z)^3 + \Omega_\Lambda}. \tag{56}$$

It should be emphasized in the above equation that one can get $\Omega_\Lambda = 0.73$, $\Omega_M = 0.23$, and $\omega_\Lambda = -0.98$, which are obtained from a combination of CMB, BAO, and SNeIa data sets [2, 14–16]. Whereas, in this work, the components of dark energy are quantum fluctuations and cosmological constant, thence the Friedmann equation is obtained as

$$\begin{aligned}\frac{3H^2(t)}{8\pi G} &= \left(\rho_m(t) + \rho_{\mathrm{DE}}\right)\\ &= \left(\rho_m(t) + \rho_\Lambda(t) + \Omega_Z \rho_{\mathrm{cri}}(t)\right),\end{aligned} \tag{57}$$

where subscript DE indicates dark energy. By substituting (31) into (57), the Friedmann equation could be rewritten as

$$H^2(t) = \frac{H_0^2(t)}{1-\Omega_z}\left(\Omega_m(t) + \Omega_\Lambda(t)\right), \tag{58}$$

where $\Omega_i(t) = \rho_i(t)/\rho_{\mathrm{cri}}(0)$, and i refers to m, Λ, and Z, respectively; in addition $\rho_{\mathrm{cri}}(0)$ denotes the critical energy density in present epoch. It should be noted that the energy densities of curvature and radiation are neglected. From (31) and definition of dimensionless energy density parameters, one gets

$$\rho_{\mathrm{DE}} = \Omega_Z \rho_{\mathrm{cri}}(t) + \rho_{\mathrm{cri}}(0)\,\Omega_\Lambda. \tag{59}$$

By virtue of definition of $\rho_{\mathrm{cri}}(t)$, (58) could be rearranged as

$$\rho_{\mathrm{cri}}(t) = \frac{\rho_{\mathrm{cri}}(0)}{1-\Omega_Z}\left[\Omega_m(t) + \Omega_\Lambda\right]. \tag{60}$$

By dividing ρ_{DE} and $\rho_{\mathrm{cri}}(t)$, one arrives to

$$\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} = \Omega_Z + \frac{(1-\Omega_Z)\,\Omega_\Lambda}{\Omega_m(t) + \Omega_\Lambda}. \tag{61}$$

Whereas different equations for ρ_m (based on different conditions for interaction) are attained, $\Omega_m(t)$ in (60) gets different forms. Thus by considering (36), (44), (53), and (61) it could be rewritten, respectively, as follows.

(a) From (36), consider

$$\begin{aligned}\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} &= \Omega_Z\\ &+ \frac{(1-\Omega_Z)(\Omega_{\mathrm{DE}}-\Omega_Z)}{\Omega_m(1+z)^3 + \widetilde{\Omega}_m(1+z)^{2\epsilon} + (\Omega_{\mathrm{DE}}-\Omega_Z)},\end{aligned} \tag{62}$$

where $\widetilde{\Omega}_m = \Psi/\rho_{\mathrm{cri}}(0)$ and $\Omega_m = \widetilde{\Psi}/\rho_{\mathrm{cri}}(0)$. In Figure 1, the evolution of (62) versus z parameter shows a deviation in comparison to the ordinary $\rho_{\mathrm{DE}}(t)/\rho_{\mathrm{cri}}(t)$, (56) that illuminates the effects of Ω_Z and ϵ in the evolution of this function.

(b) From (44), consider

$$\begin{aligned}\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} &= \Omega_Z\\ &+ \frac{(1-\Omega_Z)(\Omega_{\mathrm{DE}}-\Omega_Z)}{\Omega_m(1+z)^3 + \overline{\Omega}_m(1+z)^{2\epsilon} + \Omega_m^*(1+z)^{3\kappa} + (\Omega_{\mathrm{DE}}-\Omega_Z)},\end{aligned} \tag{63}$$

where $\overline{\Omega}_m = -\breve{D}/\rho_{\mathrm{cri}}(0)(3-2\epsilon(1-1/\kappa))$ and $\Omega_m^* = \kappa\widetilde{c}/\rho_{\mathrm{cri}}(0)(1-\kappa)$. Considering the above equation, it is found out that the $\widetilde{Q}_1$ case does not lead to physical result, because the equation for ρ_m can not satisfy observations as well.

(c) From (48), consider

$$\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} = \Omega_Z + \frac{(1-\Omega_Z)(\Omega_{\mathrm{DE}}-\Omega_Z)}{\Omega_m(1+z)^{3(1-\gamma)} + (\Omega_{\mathrm{DE}}-\Omega_Z)}, \tag{64}$$

where $\Omega_m(1+z)^{3(1-\gamma)}$ indicates dark matter component of the universe. Figure 2 shows a deviation with respect to ordinary $\rho_{\mathrm{DE}}(t)/\rho_{\mathrm{cri}}(t)$ (56) where in comparison to Figure 1 it is realized that Q_2 function has better results.

(d) From (53), consider

$$\begin{aligned}\frac{\rho_{\mathrm{DE}}(t)}{\rho_{\mathrm{cri}}(t)} &= \Omega_Z\\ &+ \frac{(1-\Omega_Z)(\Omega_{\mathrm{DE}}-\Omega_Z)}{\Omega_m(1+z)^3 + \Omega_m^\sharp(1+z)^{2\epsilon} + (\Omega_{\mathrm{DE}}-\Omega_Z)},\end{aligned} \tag{65}$$

where $\Omega_m = \breve{K}/\rho_{\mathrm{cri}}(0)$ and $\Omega_m^\sharp = \breve{A}/\rho_{\mathrm{cri}}(0)$. The energy density parameter in this case behaves as the case related to (38). This manner is similar to case (a) that is (62), and therefore it does not need to plot its evolution. It should be noted that the behaviour of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ based on different quantities for Ω_Z is plotted in Figure 3.

As it is obvious, Figure 3 illuminates the evolution of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ versus z (62) for three different values $\Omega_Z = 0.16, 0.05$, and -0.16. The results are shown with green dashed, red dotted, and blue solid lines, respectively. It is clear that $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ decreases with decreasing Ω_Z.

5. Conclusion and Discussion

Vacuum quantum fluctuations in a de-Sitter like background for both massive and massless bosonic scalar fields have been investigated. In light of Planck database 2013 we have estimated the best value for the first slow roll parameter and using such quantity the different components of the Universe's energy budget have been calculated. It should be stressed that the scalar fields have been quantized in a FLRW framework and it has been shown that the contribution of vacuum fluctuations has increased in such background in comparison

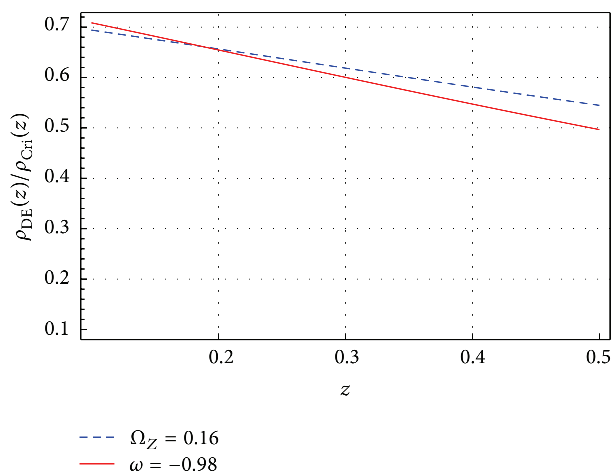

FIGURE 1: $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ versus z have been shown with dashed-blue line (62) and solid-red line, respectively (56). The auxiliary parameters are $\Omega_Z = 0.16$, $\Omega_m = 0.17$, $\widetilde{\Omega}_m = 0.1$, $\Omega_{\mathrm{DE}} = 0.73$, $\epsilon = 0.02$, and $\omega_\Lambda = -0.98$.

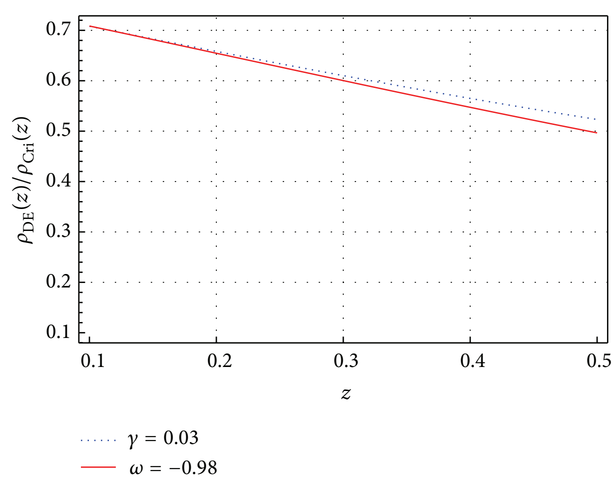

FIGURE 2: $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ versus z have been shown with dotted line (64) and solid line (56). The auxiliary parameters are $\Omega_Z = 0.16$, $\Omega_m = 0.23$, $\Omega_{\mathrm{DE}} = 0.73$, $\gamma = 0.03$, and $\omega_\Lambda = -0.98$.

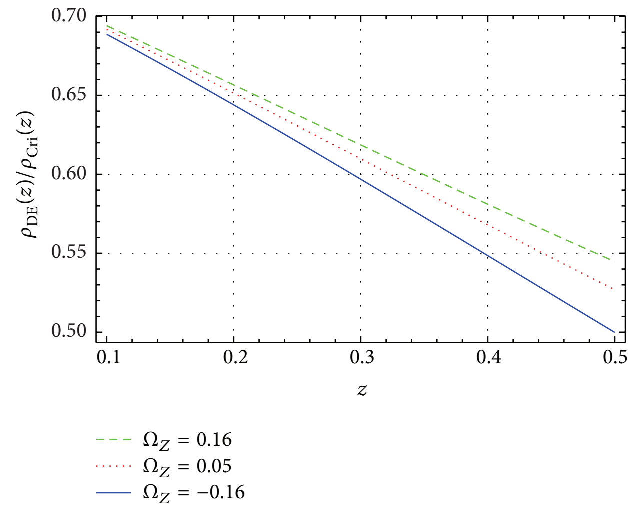

FIGURE 3: $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ versus z (62) for three different values $\Omega_Z = 0.16$, $\Omega_Z = 0.05$, and $\Omega_Z = -0.16$. They have been shown with dashed-green, dotted-red, and blue (solid) lines, respectively.

with normal de-Sitter case. It should be emphasized that the subtraction approach has been used to eliminate the infinities which appeared in the calculations. Incidentally, using the physical energy density of zero point quantum fluctuation, it has been realized that this component of the universe has to have an interaction with other components of the universe. In addition, when the energy density of matter is achieved, it has been found that beside ordinary dark matter there exist matter components which were created due to interaction with zero point quantum fluctuations. Also whereas zero point energy density was time dependent, the transformation of energy between different ingredients of the universe has been investigated. It was considerable that, for the state in which all components of the universe exchange energy between themselves, time evolution of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ has shown that $Q_2 = 3\gamma H\rho_m$ is in best agreement in comparison with observational database and also the interaction term $\widetilde{Q}_1 = 3\kappa H(t)\rho_\Lambda$ had not any physical results. At last for more details, the bounds which have risen from time evolution of dark energy density in comparison to standard Λ cosmology have been investigated. To compare the results of this work with observational data, we have regarded the time evolution of $\rho_{\mathrm{DE}}(z)/\rho_{\mathrm{cri}}(z)$ which was concluded from a combination of CMB, BAO, and SNeIa data sets. From Figures 2 and 3, the evolution of (62) and (64) versus z in comparison to observational results has been illustrated.

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this paper. Haidar Sheikhahmadi declares that the received funding mentioned in the "Acknowledgments" section did not lead to any conflict of interests regarding the publication of this paper.

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